

THE
PARADIGMS OF MECHANICS

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A Symmetry Based Approach

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Dedicated to Johannes Kepler

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DECLARATION

I, Ryan Lee Lemmer, affirm that the material contained in this dissertation has not (to my knowledge) been published elsewhere except where due reference has been made in the text and that this dissertation is not being and has not been used for the award of any other degree or diploma in any university or other institution.

A handwritten signature in black ink, appearing to read 'R L Lemmer', with a stylized, flowing script.

R L Lemmer

November 1996

SUMMARY

An overview of the historical developments of the paradigms of classical mechanics, the free particle, oscillator and the Kepler problem, is given *ito* (in terms of) their conserved quantities.

Next, the orbits of the three paradigms are found from quadratic forms. The quadratic forms are constructed using first integrals found by the application of Poisson's theorem. The orbits are presented *ito* expanding surfaces defined by the quadratic forms.

The Lie and Noether symmetries of the paradigms are investigated. The free particle is discussed in detail and an overview of the work done on the oscillator and Kepler problem is given. The Lie and Noether theories are compared from various aspects.

A technical description of Lie groups and algebras is given. This provides a basis for a discussion of the historical development of the paradigms of mechanics *ito* their group properties.

Lastly the paradigms are discussed *ito* of Quantum Mechanics.

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PREFACE

The notion of conserved quantities was around long before differential equations were defined. This approach was entirely geometric in nature and became less effective as the systems studied became more complicated. By way of example consider the Kepler problem which was solved by use of its conserved quantities. Once the problem was modelled *ito* a differential equation, the results obtained by consideration of conserved quantities were confirmed.

Differential equations provided the required breakthrough since any system could be expressed in this form and one could then look to find a method of solution, many of which were devised in an *ad hoc* fashion. However, there was no general method of solution.

Lie recognised the close relationship between conserved quantities and infinitesimal transformations when he set out to find a general method of solution of differential equations. He limited his discussion to geometrically based transformations *ie* point and contact transformations and introduced the concept of *symmetry* of a differential equation.

The limitation on the type of infinitesimal transformation has since been removed. This has led to generalised symmetries and nonlocal symmetries. A symmetry may be described as the generator of infinitesimal transformations which leave the differential equation invariant. In turn, the infinitesimal transformations are associated with conserved quantities and, in this way, the symmetries of a differential equation are closely related to the conserved quantities of the associated physical system.

Lie's work on symmetries was neglected for several decades until L V Ovsiannikov extended the theory to partial differential equations of arbitrary order.

Emmy Noether formulated her celebrated theorem in the framework of Lagrangian Mechanics. She considered infinitesimal transformations which leave the integral of the Lagrangian (an Action Integral) invariant. This led her to *Noether symmetries* together with an explicit formula which gives, for each symmetry, an associated first integral.

The next major development in the study of physical systems (and of differential

equations) was with the introduction of group theoretical considerations. The set of first integrals of a system may be viewed *ito* of a Lie group or algebra. The group properties inform us of the global characteristics of a system whereas algebraic properties may only be interpreted locally. In classical mechanics the Lie algebra is defined with the Poisson Bracket as multiplication operator. The transition to Quantum Mechanics requires that we use the Dirac Commutator instead.

Symmetry groups of dynamical systems have been extensively studied in the literature. The early discussions were concerned with purely geometric symmetry *eg* rotational invariance. However, the apparent geometric symmetry did not explain the existence of degeneracies in spectra. This led to so called *dynamical* symmetry groups which were not obviously geometric in nature.

A set of integrals with zero Poisson Bracket (or commutator in Quantum Mechanics) with the Hamiltonian of a system is referred to as an *invariance* algebra. The group realisation (in the cases it exists) is called an invariance group. Dynamical symmetry groups are of this type. Invariance groups attracted most of the attention until the importance of larger groups, *noninvariance* groups, became apparent. A noninvariance group contains the invariance group of a Hamiltonian. However, it also contains integrals which have nonzero Poisson Bracket with the Hamiltonian and hence explicitly depend on time. Invariance groups provide us with the orbit while noninvariance groups yield the time-evolution of a system (provided there are sufficiently many first integrals).

A symmetry exhibited by a physical system may be described by the invariance of the equation of motion under the corresponding transformation group. Hence much attention has been given to *complete* symmetry groups: the largest symmetry group of transformations admitted by the equation of motion. Furthermore, the group (and algebraic) properties of a differential equation provide a means of solution and, more importantly, facilitate the classification of differential equations on the basis of group theoretical properties.

There is no real relation between the group that the symmetries form and that of the first integrals of the associated physical system. Nonetheless, the group theoret-

ical approach plays a very important role in both applications.

Initially the applications of Lie's theory were confined to point transformations and, in rarer cases, velocity dependent transformations. It is only quite recently that some authors have questioned the artificial constraints on the type of transformation used.

With each new development in the methods of solution, the paradigms of mechanics have been intensively re-investigated. It is for this reason that we limit our discussion to the Kepler problem, oscillator and free particle. The free particle and oscillator are the paradigms of linear systems and the Kepler problem the paradigm of nonlinear systems.

Applications of group theory in Quantum Mechanics are legion. Due to the close link between quantum systems and their classical counterparts, the study of classical systems has often led to solutions in quantum mechanics.

Chapter 1

Historical Development of the Paradigms of Mechanics

1.1 Introduction

Due to the apparent simplicity of the free particle, there is relatively little reference to it in the literature [35, 14, 1]. Despite the simplicity of this problem, we will later show that it has much in common with the other paradigms, both in terms of its conserved quantities and group theory considerations. However, there is much to be said about the Kepler problem and oscillator. In the early development of the solutions to these problems the focus was on finding the respective conserved quantities. For the Kepler problem the important conserved quantities are the angular momentum and Laplace-Runge-Lenz (hereafter termed LRL) [6, 13, 20, 49, 30] vector. The oscillator has an LRL type tensor known as the Jauch-Hill-Fradkin tensor ¹. According to Goldstein [9, 10] the earliest mention of angular momentum was by Laplace in 1798/9. However, angular momentum is implicit in Kepler's laws of planetary motion. We now describe the paths to the discovery of these vectors.

¹Once the general nature of the LRL became apparent attempts were made to generalise it to a conserved quantity for all central force problems[8].)

1.2 The Kepler Problem

Kepler based his studies of the motion of the planets on Tycho Brahe's and his own observations. In 1609 he published his first two laws governing planetary motion in his *Astronomia Nova*. The third law appeared in *Harmonice Mundi* [17, 18] in 1619. Kepler's monumental achievements² laid the foundations upon which Newton could construct his mechanics which would in turn verify Kepler's observational laws.

In the August of 1684 Newton accepted Christopher Wren's challenge [51] to derive the shape of the planets' orbits. Newton's quest for the solution led him to his three laws of motion and of universal gravitation. In the span of two years his work had evolved into his masterpiece *Philosophiæ Naturalis Principia Mathematica*. Newton used these laws to show that the centripetal force holding a planet in its orbit is inversely proportional to the square of its distance from the sun. In this he confirmed Kepler's laws. Newton is, however, falsely credited for proving the converse of the above *ie* that an inverse square law leads to orbits which are conic sections [52, 53]. In 1785 Coulomb's discovery of the inverse square law describing electrostatic forces reaffirmed Newton's gravitational law.

The earliest study of the direct method for finding the orbits given an inverse square law appears to have been by Jakob Hermann³ [13] (a disciple of the Bernoullis) in 1710. Hermann related his results in a letter to Bernoulli, who in turn generalised Hermann's results to allow for an arbitrary orientation of the orbit in the plane. His results led him to the first integral of motion which is today generally known as the Laplace-Runge-Lenz (LRL) vector.

Laplace rediscovered the LRL vector in his *Traité de Mécanique Céleste* of 1799

²Kepler's genius is often grossly understated. He was on the forefront of virtually every field in which he worked. He wrestled with the concept of infinitesimals, long before Newton, while developing his first law (the areal velocity of a planet in its orbit is a constant). Although he was preceded by Archimedes (a fact he admits), he was the first to recognise the use of infinitesimals in physics.

³The spelling of the name varies in the literature. There are at least three different versions: Hermann, Herman and Ermano.

[20]. He derived seven first integrals for the Kepler problem. Three of them were the components of the LRL vector, another three were the components of the angular momentum and the remaining one was related to the energy. He realised that there could be only five independent conserved quantities and found two relationships between the seven integrals. He also showed that the conservation of angular momentum explained why the motion was confined to a plane.

In July 1847 Hamilton [12] published his discovery of a new conserved vector which is perpendicular to the LRL vector.

The solution of the Kepler problem can be described *ito* Hamilton's vector, the LRL vector and the angular momentum. Conservation of angular momentum defines the plane in which the motion takes place while Hamilton's vector and the LRL vector are in the directions of the *latus rectum* and major axis respectively.

Gibbs was the first to present the LRL vector in the form in which we know it today. It appeared in modern vector notation in the Gibbs and Wilson textbook *Vector Analysis* of 1901. Interest in the vector was revived in 1924 in a paper by Lenz [30]. Lenz forged the connection between the LRL vector and Quantum Mechanics when he calculated the energy levels of the perturbed Kepler problem using nonrelativistic quantum mechanics.

1.3 The Oscillator

According to legend Galileo's attention was caught by a swinging lamp whilst attending Mass in Pisa. Using his pulse as a timing device he noticed that, even though the amplitude of the oscillations diminished, the period of the oscillation remained constant.

The oscillator next appeared in the literature in 1670 when Robert Hooke discovered the law of elasticity (stating that the stretching of a body is directly proportional to the force applied). This law was later restated *ito* the harmonic oscillator.

Newton treated the oscillator in the context of central force problems in his *Principia* (1687) [41]. In the next two centuries improvements were made in the mathe-

mathematical formulation of the equation of motion and in its solution in terms of parametric functions. Another important advance was made in 1880 when the quantal oscillator was used to model atomic spectra. This was before quantum mechanics was invented!

In 1901 Planck explained the observed distribution of energy in the black-body radiation spectrum. His theory was based on a statistical distribution of energy amongst a set of simple linear harmonic oscillators. The idea that the energies of the oscillators take on a discrete set of values (as opposed to varying continuously) gave rise to quantum theory. More than a decade later (1913) Bohr applied these ideas to the Rutherford model of the hydrogen atom using a Kepler-Coulomb potential. He arrived at a theoretical formula (which agreed with his observations) for the wavelengths of the atomic spectrum.

The time-dependent harmonic oscillator with equation of motion

$$\ddot{q} + \omega^2(t)q = 0$$

was first notably introduced at the 1911 Solvay Conference [36, 54] during which Lorentz conjectured that

$$I := \frac{\dot{q}^2}{\omega t} + \omega q^2$$

was an adiabatic invariant (provided $\omega(t)$ was a slowly varying function of time). Littlewood [36, 37] proved the conjecture 50 years later. The study of the one-dimensional time-dependent oscillator was greatly advanced by Lewis [31] when he found the exact invariant ⁴ (using Kruskal's asymptotic method [19])

$$I = \frac{1}{2} \left[(\rho \dot{q} - \dot{\rho} q)^2 + \left(\frac{q^2}{\rho^2} \right) \right],$$

where ρ is any solution of the Ermakov-Pinney equation [5, 45]

$$\ddot{\rho} + \omega^2 \rho = \rho^{-3}.$$

⁴This is a term often used in the physics literature to describe a function with the property that its total derivative *wrt* (with respect to) time is zero when the equation of motion is taken into consideration.

His interpretation of the invariant was that it is, up to an arbitrary multiplicative constant, the most general homogeneous quadratic invariant possible⁵ for the Hamiltonian of the form

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2(t)q^2.$$

Günther and Leach [11] generalised the Lewis invariant for the three-dimensional time-dependent harmonic oscillator (TDHO).

In 1940 Jauch and Hill [15] discovered four components of a conserved tensor for the two dimensional isotropic harmonic oscillator. Fradkin [7] generalised this composite invariant to the three-dimensional case and it is now known as the Jauch-Hill-Fradkin tensor. This tensor is the oscillator analogue of the LRL vector in the Kepler problem in the sense that its contraction with the angular momentum is zero and that both these vectors lead to the respective orbits. Günther and Leach [11] provided a class of invariants for the 3-D TDHO which is a generalisation of the Jauch-Hill-Fradkin tensor for the time-independent case.

1.4 The Laplace-Runge-Lenz Vector

Since the Jauch-Hill-Fradkin tensor may be considered as the oscillator analogue of the LRL vector for the Kepler problem, the question arises whether there is a general LRL-type conserved quantity for all central potential problems. Peres [44] attempted a generalisation by postulating a general form for the LRL vector. His result reflected what Fradkin [7] had known before him which was that any generalisation of the LRL vector could be only ‘piecewise conserved’. In particular Peres found that his ‘conserved quantity’ was singular and changed direction at the turning points of the orbit⁶. None the less, both Fradkin and Peres remained convinced that their

⁵It has since been shown that this is not the case [23]. The one dimensional autonomous oscillator has three quadratic invariants, $I_1 = \frac{1}{2}(p^2 + q^2)$, $I_2 = \frac{1}{2}(p^2 - q^2)\cos 2t + pq\sin 2t$ and $I_3 = \frac{1}{2}(p^2 - q^2)\sin 2t - pq\cos 2t$. Since the one dimensional time-dependent oscillator is related to the autonomous oscillator by $Q = q/\rho$, $P = \rho p - \dot{\rho}q$ it must also have three quadratic invariants.

⁶The Kepler problem was the exception since the perihelion, aphelion and centre of force are collinear.

generalisations of the LRL vector were truly first integrals. Fradkin justified this belief by the fact that the Poisson Bracket of his generalised LRL vector with the Hamiltonian was zero.

Holas and March [14] resolved the issue by pointing out that Fradkin's result did not determine the derivative at the turning points of the orbit and hence the quantity was only piecewise conserved. Furthermore they showed that the assumption that the generalised LRL vector is a first integral (in the normal sense) leads to a closed orbit. Conversely, for closed orbits which have n perihelions and aphelions, the n perihelion vectors and their corresponding n th-rank tensor are all invariants of the motion.

Chapter 2

The Orbits of the Paradigms of Mechanics

Summary The orbits of the free particle, oscillator and Kepler problem are found from quadratic forms which are constructed using first integrals found by the application of Poisson's theorem.

2.1 Introduction

Treatments of the orbits of the two great paradigms of mechanics, the Kepler problem and the simple harmonic oscillator, are often given in terms of conserved vectors in the case of the former and conserved rank two tensors in the case of the latter.

The oscillator, with equation of motion

$$\ddot{\mathbf{r}} = -\mathbf{r}, \quad (2.1.1)$$

(ω^2 can be taken to be one without loss of generality) has conserved angular momentum tensor

$$L_{ij} = x_i \dot{x}_j - \dot{x}_i x_j \quad (2.1.2)$$

and Jauch-Hill-Fradkin tensor [15, 7]

$$A_{ij} = \dot{x}_i \dot{x}_j + x_i x_j \quad (2.1.3)$$

(amongst others). The orbit equation is given by the quadratic form

$$\mathbf{r}^T (2EI - A) \mathbf{r} = L^2, \quad (2.1.4)$$

where E is the energy ($= \frac{1}{2} \text{Tr} (A)$).

The algebraic properties of the two problems are characterized by the Poisson bracket relationships of the conserved quantities. The invariance algebras are well known to be $so(4)$ for the Kepler problem (for $E < 0$; for $E > 0$ it is $so(3, 1)$ and for $E = 0$ it is $e(3)$) and $su(3)$ for the oscillator.

The value of the conserved quantities for each problem made the construction of similar quantities for *all* central force problems an attractive proposition although the utility of the results is not so obvious [8, 44, 14, 2].

Although one does not normally do so, the equations of motion of the Kepler problem and the oscillator can be considered as the equation for a free particle with a (presumably) symmetry breaking term attached. This suggests that the free particle should have similar conserved vectors and tensors. In this chapter we examine the properties of these conserved quantities and a few more, the construction of which is suggested naturally. We find a relation between the conserved vectors and tensors.

2.2 The Free Particle

2.2.1 The Conserved Quantities

The free particle has equation of motion

$$\ddot{\mathbf{r}} = \mathbf{0}$$

and Hamiltonian

$$H = \frac{1}{2} \mathbf{p} \cdot \mathbf{p}, \quad (2.2.1)$$

where $\mathbf{p} = \dot{\mathbf{r}}$. The natural analogue of Hamilton's vector is

$$\mathbf{K}_1 = \mathbf{p} \quad (2.2.2)$$

and the corresponding LRL vector is

$$\mathbf{J}_1 = \mathbf{p} \times \mathbf{L}. \quad (2.2.3)$$

However, there is a second vector of Hamilton's type

$$\mathbf{K}_2 = t\mathbf{p} - \mathbf{r} \quad (2.2.4)$$

for which the LRL vector is

$$\mathbf{J}_2 = (t\mathbf{p} - \mathbf{r}) \times \mathbf{L}. \quad (2.2.5)$$

The Poisson Bracket of two quantities A and B is defined as

$$[A, B] := \frac{\partial A}{\partial q_\alpha} \frac{\partial B}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha} \frac{\partial B}{\partial q_\alpha} \quad (2.2.6)$$

(where summation over the α is implied). Poisson's theorem ¹ [55] states that if A and B are two conserved quantities of a system, then the quantity C given by

$$C = [A, B]$$

is also a constant of the motion. In this way Poisson's theorem potentially provides a method for finding new conserved quantities for a system. Note that the theorem does not guarantee that the quantity C is functionally independent of existing integrals *ie* that we will find a new first integral.

The Poisson Brackets of the components of the Hamilton-like vectors are

$$\begin{aligned} [K_{1i}, K_{1j}] &= \frac{\partial K_{1i}}{\partial q_\alpha} \frac{\partial K_{1j}}{\partial p_\alpha} - \frac{\partial K_{1i}}{\partial p_\alpha} \frac{\partial K_{1j}}{\partial q_\alpha} \\ &= 0 \frac{\partial K_{1j}}{\partial p_\alpha} - \frac{\partial K_{1i}}{\partial p_\alpha} 0 \\ &= 0, \\ [K_{2i}, K_{2j}] &= \frac{\partial K_{2i}}{\partial q_\alpha} \frac{\partial K_{2j}}{\partial p_\alpha} - \frac{\partial K_{2i}}{\partial p_\alpha} \frac{\partial K_{2j}}{\partial q_\alpha} \\ &= -\delta_{i\alpha} t \delta_{j\alpha} - t \delta_{i\alpha} (-\delta_{j\alpha}) \\ &= -t \delta_{ij} + t \delta_{ij} \\ &= 0 \end{aligned}$$

¹Poisson discovered this theorem in 1809 and it first appeared in Journal de l' École polytechnique VIII (1809) p266. Poisson used the theorem to find new integrals from known ones. Some thirty years later Hamilton stated the theorem *ito* his mechanics.

and

$$\begin{aligned}[K_{1i}, K_{2j}] &= -\delta_{i\alpha}(-\delta_{j\alpha}) \\ &= \delta_{ij}.\end{aligned}$$

Similarly for the LRL vectors, we have

$$\begin{aligned}[J_{1i}, J_{1j}] &= (2E) \epsilon_{ijk} L_k \\ [J_{2i}, J_{2j}] &= I_3 \epsilon_{ijk} L_k \\ [J_{1i}, J_{2j}] &= 2L^2 \delta_{ij} + I_2 \epsilon_{ijk} x_i p_j - L_i L_j,\end{aligned}$$

where E is the value of the Hamiltonian (2.2.1) and the conserved quantities I_2 and I_3 are defined below. For the K_i s and J_i s we have

$$\begin{aligned}[K_{1i}, J_{1j}] &= A_{ij} - 2E\delta_{ij} \\ [K_{2i}, J_{1j}] &= B_{ij} - I_2\delta_{ij} + \frac{3}{2}\epsilon_{ijk} L_k \\ [K_{1i}, J_{2j}] &= B_{ij} - I_2\delta_{ij} - \frac{3}{2}\epsilon_{ijk} L_k \\ [K_{2i}, J_{2j}] &= C_{ij} - I_3\delta_{ij},\end{aligned}$$

where

$$\begin{aligned}A_{ij} &= p_i p_j \\ B_{ij} &= t p_i p_j - \frac{1}{2}(x_i p_j + x_j p_i) \\ C_{ij} &= t^2 p_i p_j - t(x_i p_j + x_j p_i) + x_i x_j\end{aligned}$$

are the components of the rank two conserved tensors and

$$\begin{aligned}I_2 &= Tr(B_{ij}) \\ I_3 &= Tr(C_{ij}).\end{aligned}$$

The tensor A_{ij} has the form of the Jauch-Hill-Fradkin tensor of the oscillator. Tensors similar to B_{ij} and C_{ij} were reported for the oscillator by Leach [23]. It is interesting that the conserved *tensors* follow from the application of Poisson's theorem to the conserved *vectors*.

2.2.2 Conserved Quantities and the Orbit

Since the angular momentum is conserved, the orbit is in a plane and may be described in terms of plane polar co-ordinates, (r, θ) . If θ is measured from \mathbf{J}_1 , the scalar product of \mathbf{J}_1 with \mathbf{r} gives the orbit equation

$$r = \frac{L^2}{J_1 \cos \theta}$$

(a straight line) just as the LRL vector does for the Kepler problem. It is well-known that the Jauch-Hill-Fradkin tensor is used to give the orbit of the oscillator in terms of the quadratic form [8]

$$\mathbf{r}^T (2EI - A) \mathbf{r} = L^2. \quad (2.2.7)$$

It so happens that the same result applies for the free particle. In order to find the quadratic form we calculate

$$\mathbf{r}^T A \mathbf{r} = r_i p_i p_j r_j = (\mathbf{r} \cdot \mathbf{p})^2. \quad (2.2.8)$$

Since

$$\begin{aligned} L^2 &= \mathbf{L} \cdot \mathbf{L} \\ &= (\mathbf{r} \times \mathbf{p}) \cdot (\mathbf{r} \times \mathbf{p}) \\ &= \mathbf{r} \cdot [\mathbf{p} \times (\mathbf{r} \times \mathbf{p})] \\ &= \mathbf{r} \cdot [(\mathbf{p} \cdot \mathbf{p}) \mathbf{r} - (\mathbf{p} \cdot \mathbf{r}) \mathbf{p}] \\ &= r^2 p^2 - (\mathbf{r} \cdot \mathbf{p})^2 \end{aligned}$$

we have

$$\mathbf{r}^T A \mathbf{r} = (2E) r^2 - L^2 \quad (2.2.9)$$

which is just the same form as (2.2.7).

To simplify the quadratic form

$$\mathbf{q}^T M \mathbf{q} = c \quad (2.2.10)$$

(where c is a constant) we define new coordinates in the following way. An $n \times n$ matrix M is always diagonalisable if it is real symmetric or Hermitian. In these cases

there are n linearly independent eigenvectors \mathbf{v}_i corresponding to the eigenvalues λ_i and

$$P^{-1}MP = \text{diag}(\lambda_1, \dots, \lambda_n) = D, \quad (2.2.11)$$

where

$$P = (\mathbf{v}_1, \dots, \mathbf{v}_n).$$

Solving for M from (2.2.11) and substituting into (2.2.10) we find that

$$(\mathbf{q}^T P) D (P^{-1} \mathbf{q}) = c. \quad (2.2.12)$$

With the definition of the new coordinates by

$$\mathbf{Q} = P^{-1} \mathbf{q}$$

(2.2.12) becomes

$$\mathbf{Q}^T D \mathbf{Q} = c \quad (2.2.13)$$

(since P is orthonormalised *ie* $P^{-1} = P^T$).

The new coordinates are given by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{q} \\ \mathbf{v}_2 \cdot \mathbf{q} \\ \vdots \\ \mathbf{v}_n \cdot \mathbf{q} \end{pmatrix}$$

ie the new coordinates are the projections of \mathbf{q} onto the eigenvectors. This means that the new coordinate axes are in the direction of the eigenvectors.

The Conserved Tensor A_{ij}

The eigenvalues of the matrix in (2.2.7) are $2E$, $2E$ and 0 . The two eigenvectors corresponding to the repeated eigenvalue, $2E$, are vectors \mathbf{u} such that

$$(2EI - A) \mathbf{u} = (2E) \mathbf{u}$$

which means that

$$A \mathbf{u} = 0$$

ie

$$(p_i p_j) u_j = 0 \quad i = 1, 3$$

(where summation over j is implied). This gives

$$\mathbf{u} \cdot \mathbf{p} = 0$$

ie the eigenvectors are any two linearly independent vectors in the plane normal to \mathbf{p} (and so may be taken as mutually perpendicular). The third eigenvector, corresponding to the eigenvalue zero, is \mathbf{p} ($= \mathbf{K}_1$). In terms of the new coordinates the quadric surface is given by

$$Q_1^2 + Q_2^2 = \frac{L^2}{2E}$$

which is a cylinder with axis of symmetry given by \mathbf{K}_1 . The orbit is the intersection of this surface with the plane defined by the constancy of the angular momentum and is, naturally, a straight line (see Fig 2.1).

The Conserved Tensor B_{ij}

The quadratic form associated with B is found from

$$\begin{aligned} \mathbf{r}^T B \mathbf{r} &= r_i B_{ij} r_j = r_i \left(t p_i p_j - \frac{1}{2} (r_i p_j + r_j p_i) \right) r_j \\ &= t (\mathbf{r} \cdot \mathbf{p})^2 - \frac{1}{2} [(\mathbf{r} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{p})] \\ &= t (\mathbf{r} \cdot \mathbf{p})^2 - (\mathbf{r} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{p}) \\ &= t (r^2 p^2 - L^2) - r^2 (\mathbf{r} \cdot \mathbf{p}) - r^2 (\mathbf{r} \cdot \mathbf{p}) \\ &= (t p^2 - \mathbf{r} \cdot \mathbf{p}) r^2 - t L^2 \\ &= I_2 r^2 - t L^2. \end{aligned}$$

Hence

$$\mathbf{r}^T (I_2 I - B) \mathbf{r} = t L^2$$

ie the quadric surface expands in time. The eigenvalues are $\frac{1}{2} \left(I_2 \pm \sqrt{I_2^2 + L^2} \right)$ and I_2 . The eigenvector corresponding to the third of these is \mathbf{L} . Those corresponding

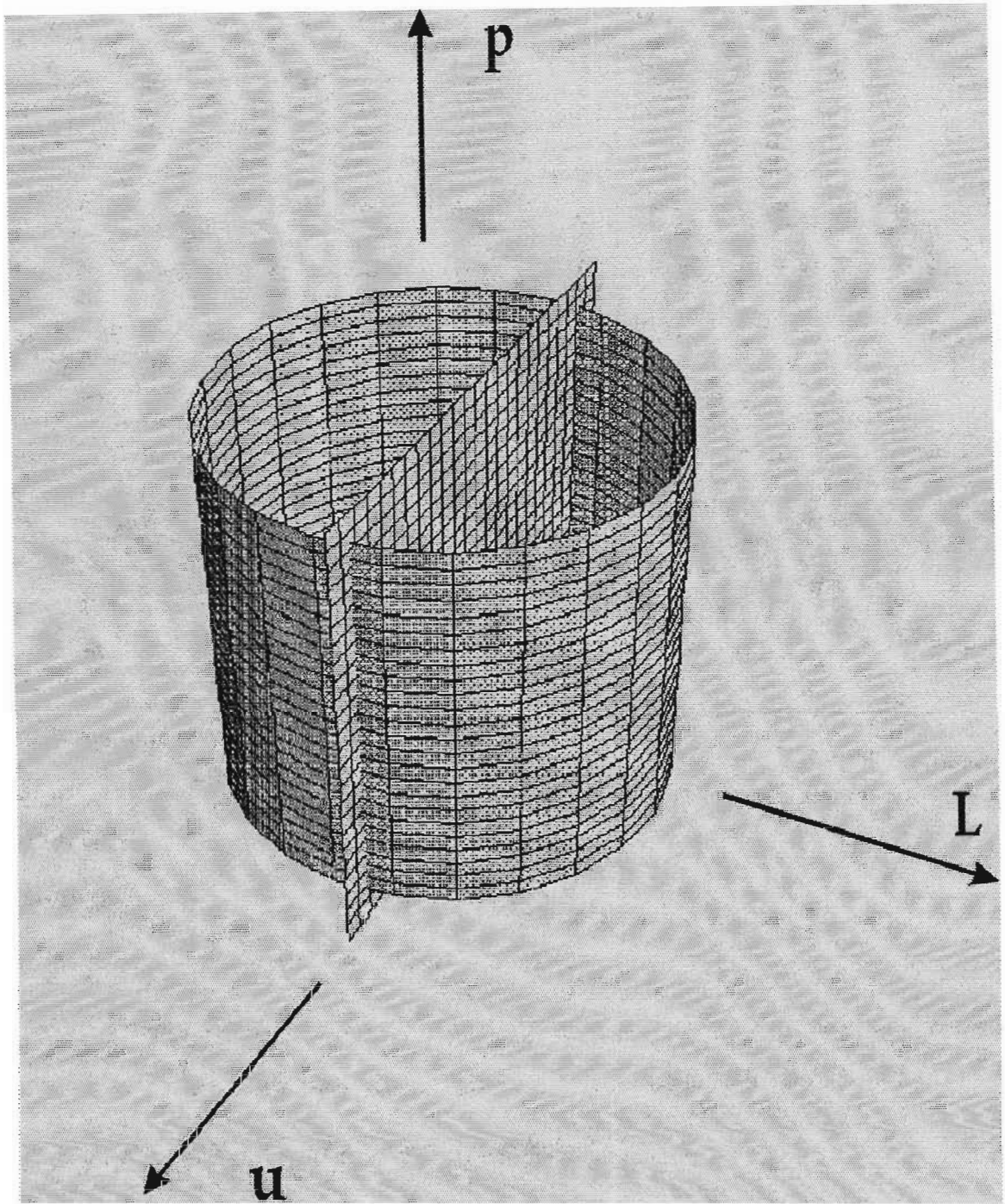


Figure 2.1: Since the eigenvectors corresponding to the eigenvalues $2E$ are any two vectors perpendicular to \mathbf{p} , we may choose one of them to be \mathbf{L} . Conservation of the magnitude of angular momentum defines a plane perpendicular to \mathbf{L} . The orbit is given by the intersection of the plane and cylinder *ie* two straight lines. The initial conditions will determine which line the particle is on.

to the first two are

$$\mathbf{u}_{\pm} = - \begin{pmatrix} \frac{1}{4}L_3L_1 + \frac{1}{2} \left(-I_2 \pm \sqrt{I_2^2 + L^2} \right) B_{13} \\ \frac{1}{4}L_3L_2 + \frac{1}{2} \left(-I_2 \pm \sqrt{I_2^2 + L^2} \right) B_{23} \\ \frac{1}{2} (L_3^2 - L^2) + \frac{1}{2} \left(-I_2 \pm \sqrt{I_2^2 + L^2} \right) B_{33} \end{pmatrix}$$

and are in the plane normal to \mathbf{L} . In terms of principal axes the quadric surface is

$$\frac{1}{2} \left(I_2 - \sqrt{I_2^2 + L^2} \right) Q_1^2 + \frac{1}{2} \left(I_2 + \sqrt{I_2^2 + L^2} \right) Q_2^2 + I_2 Q_3^2 = tL^2$$

or

$$-\frac{Q_1^2}{2 \left(I_2 + \sqrt{I_2^2 + L^2} \right)} - \frac{Q_2^2}{2 \left(I_2 - \sqrt{I_2^2 + L^2} \right)} + \frac{I_2^2}{L^2} Q_3^2 = t. \quad (2.2.14)$$

The presence of Q_3 is only nominal as that axis is in the direction of \mathbf{L} and the motion is normal to it. Consequently it is easier to picture the evolution in time of the intersection of the quadric surface with the $Q_1 - Q_2$ plane. Three stages are depicted in the figures (see Figs 2.2-2.8). For $t > 0$ (2.2.14) represents an hyperboloid of one sheet and for $t < 0$ an hyperboloid of two sheets. For $t = 0$ it represents a cone.

The Conserved Tensor C_{ij}

In the case of C the quadric surface is

$$\mathbf{r}^T (I_3 I - C) \mathbf{r} = t^2 L^2,$$

with eigenvalues I_3 , I_3 and zero. The eigenvectors corresponding to the double eigenvalue are given by the two linearly independent vectors, \mathbf{u} , in the plane defined by $\mathbf{u} \cdot \mathbf{K}_2 = 0$. The third vector is in the direction of \mathbf{K}_2 . In terms of principal axes we have

$$Q_1^2 + Q_2^2 = \frac{t^2 L^2}{I_3}.$$

The surface represents a cylinder with axis of symmetry given by \mathbf{K}_2 . The radius of the cylinder increases linearly in time so that the particle is always on its surface.

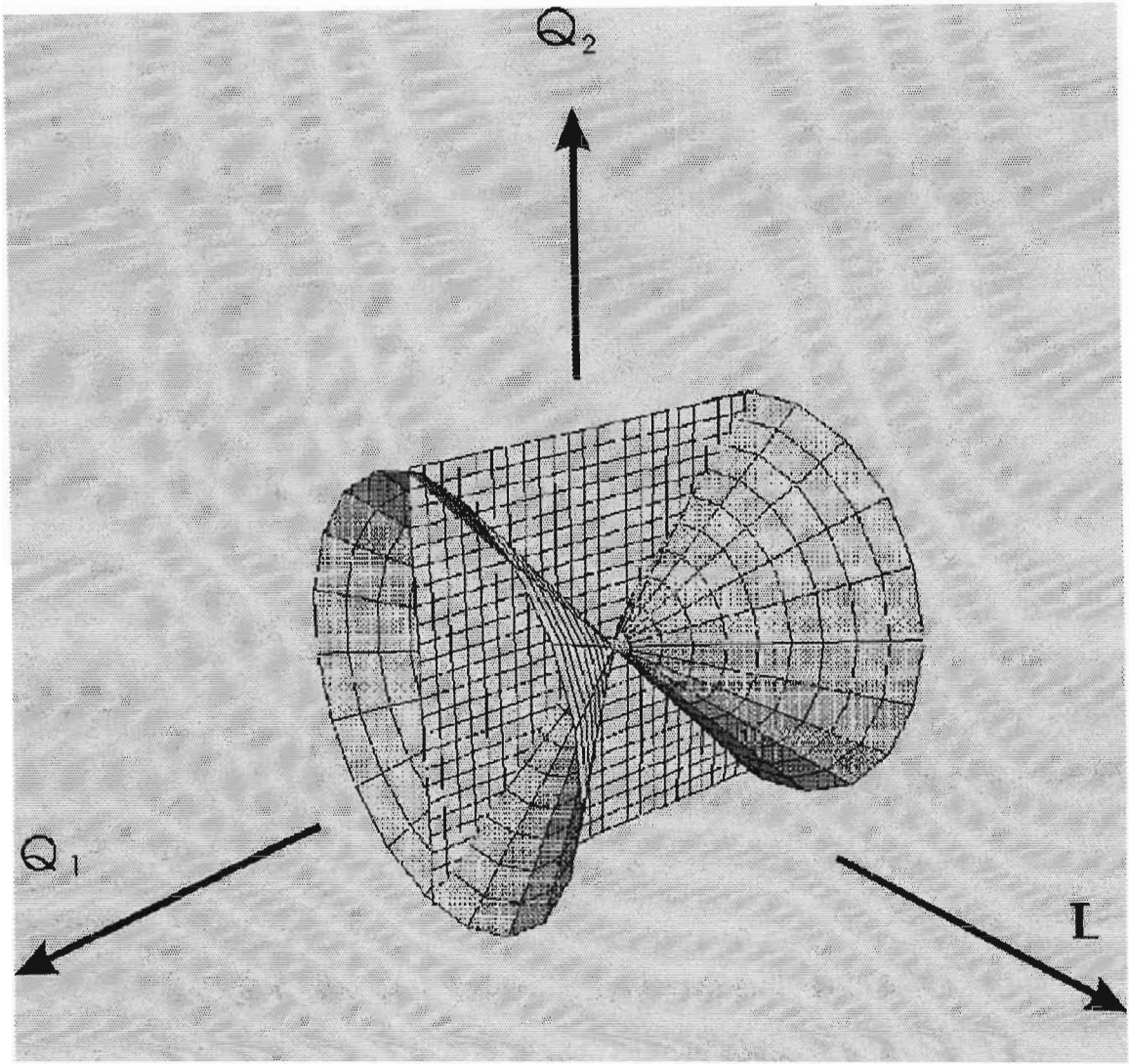


Figure 2.2: $t = 0$ The plane is defined by the constancy of the magnitude of angular momentum. It separates the cone into two symmetric parts.

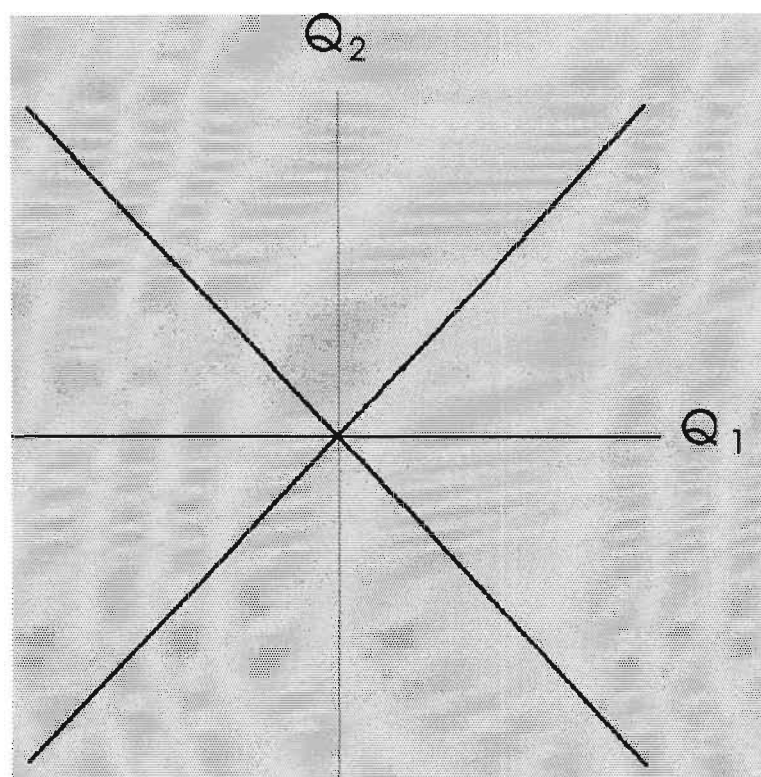


Figure 2.3: $t = 0$ The intersection of the plane and cone consists of two straight lines.

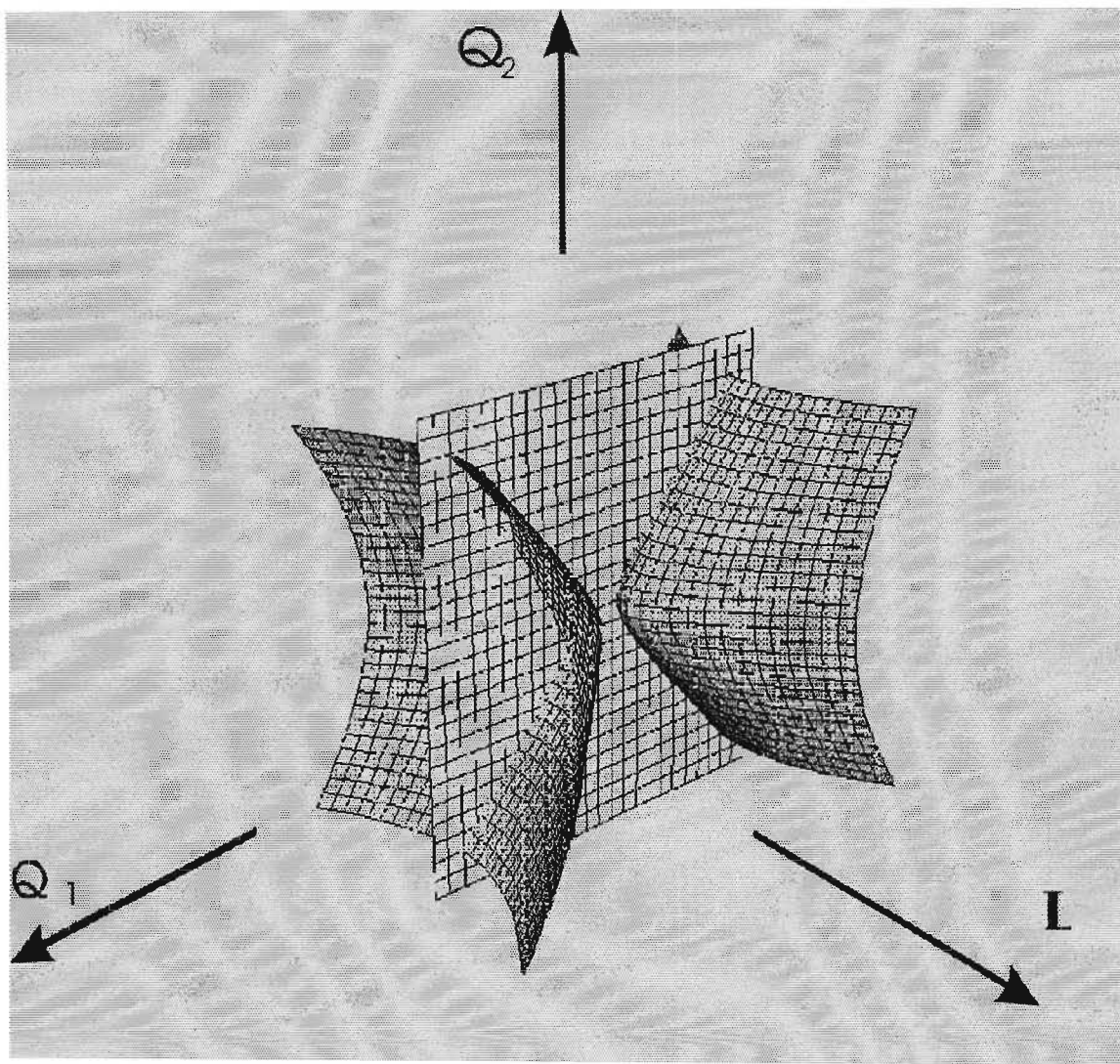


Figure 2.4: $t < 0$ The plane is defined by the constancy of the magnitude of angular momentum. It separates the hyperboloid of two sheets into two symmetric parts.

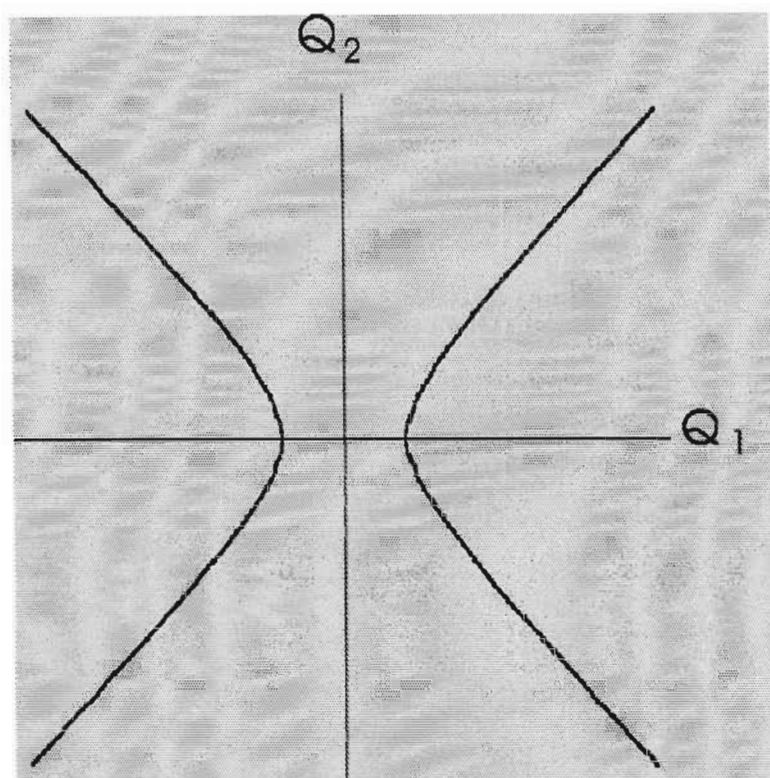


Figure 2.5: $t < 0$ The intersection of the plane and the hyperboloid of two sheets consists of two branches of an hyperbola with the Q_2 -axis as axis of symmetry.

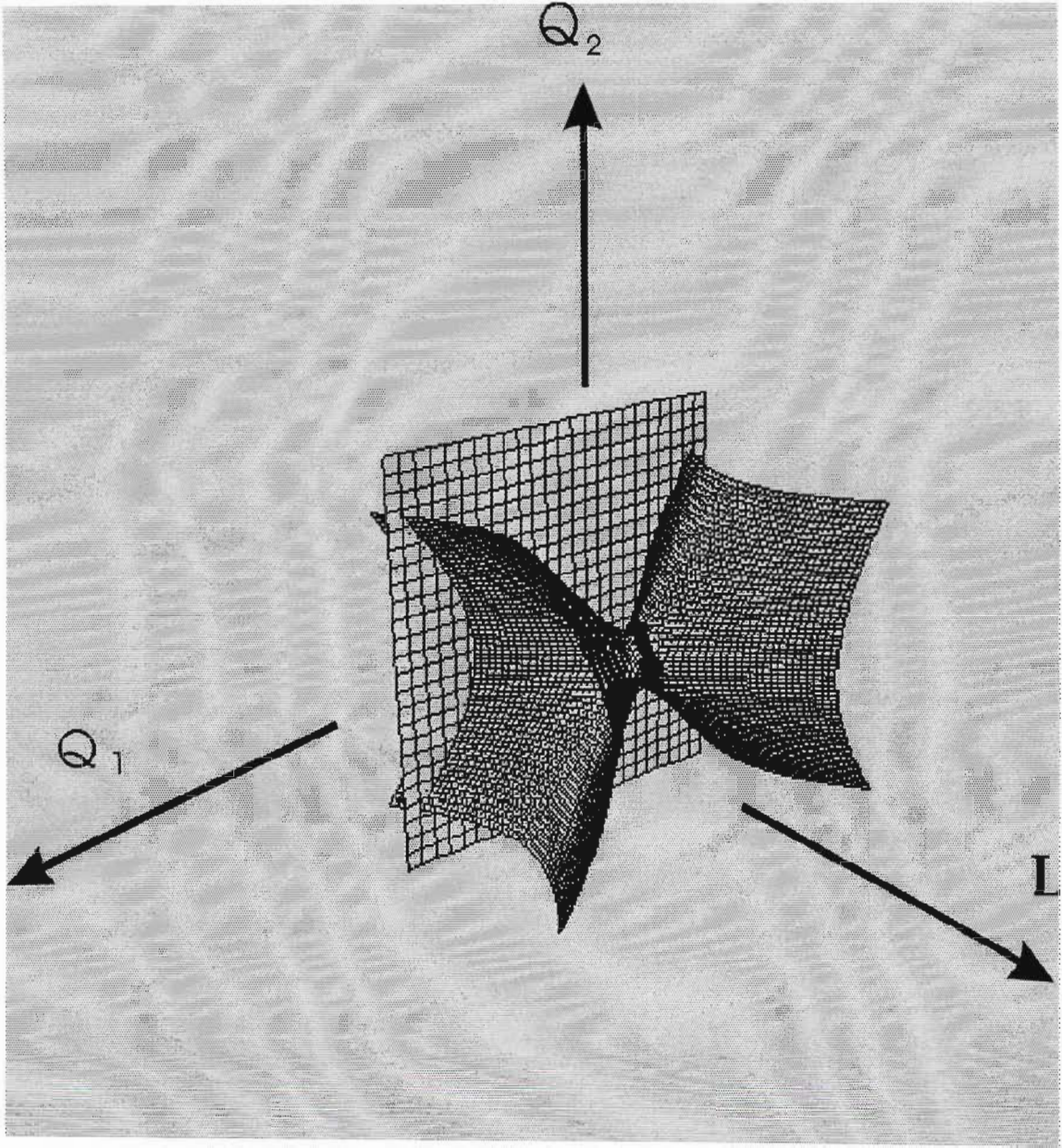


Figure 2.6: $t > 0$ The plane is defined by the constancy of the magnitude of angular momentum. It separates the hyperboloid of one sheet into two parts.

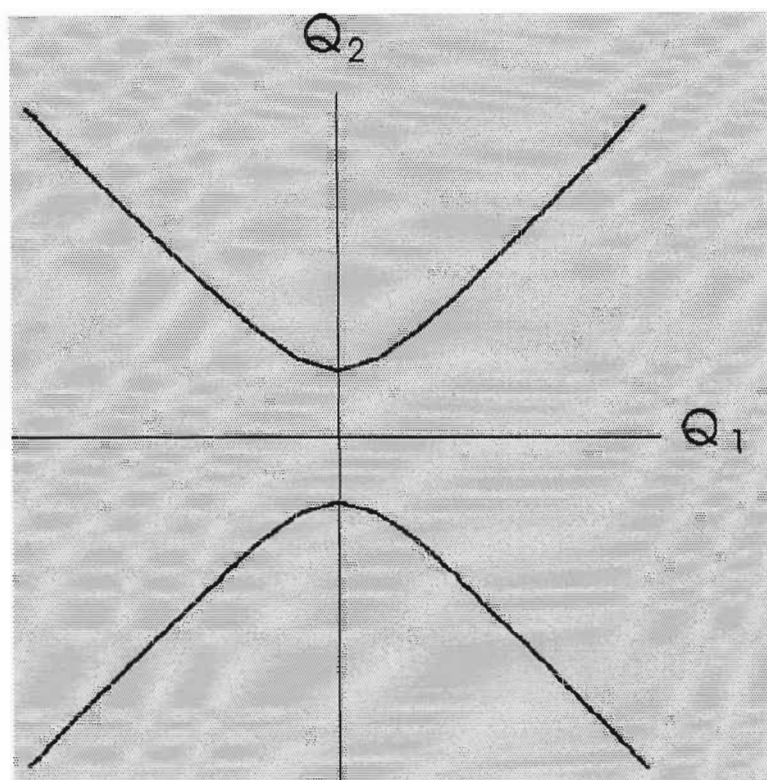


Figure 2.7: $t > 0$ The intersection of the plane and the hyperboloid of one sheet consists of two branches of an hyperbola with the Q_1 -axis as axis of symmetry.

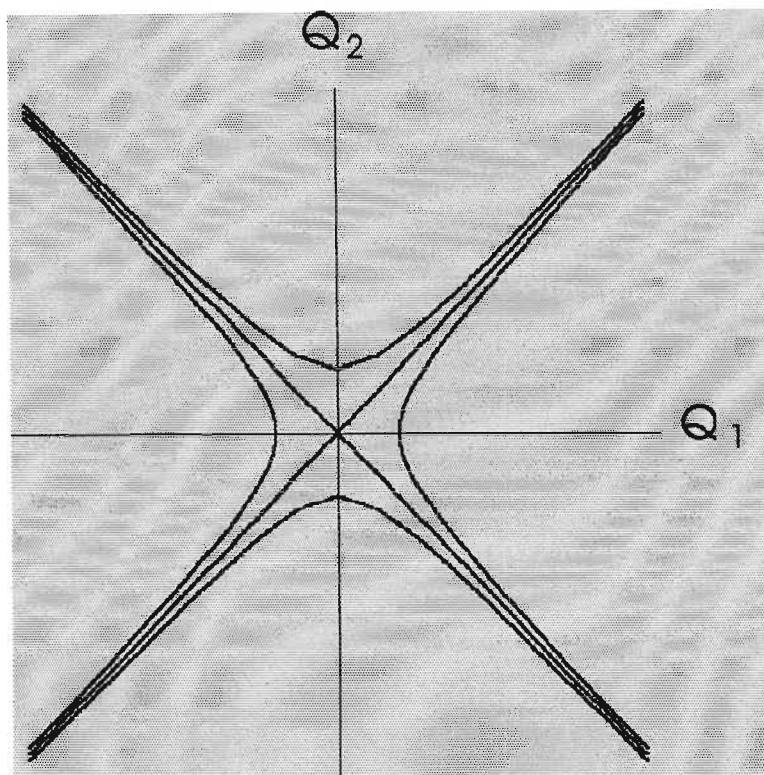


Figure 2.8: Both branches of the hyperbola move towards the asymptotes as positive time tends to zero and become coincident with the asymptotes when $t = 0$. They pass through the asymptote into the upper branches of the hyperbola. Hence for $t < 0$ the particle is on (say) the left branch of the hyperbola, at $t = 0$ the hyperbola degenerates to its asymptotes and for $t > 0$ the particle is on (say) the upper branch of the hyperbola.

2.2.3 Discussion

The first integrals of rank two tensor form of the free particle arise naturally from the conserved vectors once the analogues of the LRL vector are introduced. While the scalar products of the Hamilton and LRL type vectors lead to the quadratic integrals E , I_2 and I_3 , it is the Poisson Bracket relations which give rise to the conserved tensors. Although the tensors yield the orbit of the free particle in a nonstandard way, the method we use is both natural and general. The Poisson Bracket relations among the tensors A_{ij} , B_{ij} and C_{ij} yield nothing new as the brackets close. In fact the real pivot of the connection of the vectors and tensors is the introduction of the LRL analogues.

2.3 The Oscillator

2.3.1 Introduction

Summary

The two classes of explicitly time-dependent conserved second order tensors for the autonomous isotropic oscillator are both shown to give rise to orbit equations of the form of pulsating hyperbolæ. The noninvariance algebra of these two sets of integrals is the noncompact $su(2,1)$ in contrast to the compact $su(3)$ of the invariance algebra generated by the components of the angular momentum and the Jauch-Hill-Fradkin tensor.

It is well-known that the orbit of the time-independent isotropic oscillator can be described [7] in terms of a quadratic form constructed out of the elements of the Jauch-Hill-Fradkin tensor [15, 7] and the angular momentum, *viz*

$$A_{ij} = p_i p_j + q_i q_j \quad (2.3.1)$$

$$L_k = \varepsilon_{ijk} q_i p_j. \quad (2.3.2)$$

The Hamiltonian of the system, which is the conserved energy, is

$$H = E = \frac{1}{2} Tr(A_{ij}). \quad (2.3.3)$$

The invariance group of the oscillator, SU_3 (for three dimensions), can be found from a consideration of the algebra of these first integrals under the operation of taking the Poisson Bracket [7]. The time-dependent isotropic oscillator has the same properties [11] except that the algebra is not an invariance algebra of the Hamiltonian (since

$$\frac{\partial I}{\partial t} \neq 0 \Rightarrow [I, H] \neq 0)$$

but of the Ermakov-Lewis invariant [5, 31].

In addition to the autonomous integrals of angular momentum and Jauch-Hill-Fradkin tensor the time-independent isotropic oscillator has the explicitly time-dependent conserved rank two tensors [23]

$$B_{ij} = (p_i p_j - q_i q_j) \sin 2t - (q_i p_j + q_j p_i) \cos 2t \quad (2.3.4)$$

$$C_{ij} = (p_i p_j - q_i q_j) \cos 2t + (q_i p_j + q_j p_i) \sin 2t \quad (2.3.5)$$

which have no particular names. It is convenient to introduce the scalar integrals

$$J = \frac{1}{2} \text{Tr} (B_{ij}) \quad (2.3.6)$$

$$K = \frac{1}{2} \text{Tr} (C_{ij}). \quad (2.3.7)$$

These time-dependent tensors have nonzero Poisson Brackets with the Hamiltonian and so make no contribution to the invariance algebra of the isotropic oscillator. However, it was reported [23] that they do play roles in the invariance algebras for repulsors which are obtained from (2.3.3) by means of a time-dependent linear canonical transformation.

In this section we show that the quadratic forms associated with B_{ij} and C_{ij} provide information about the orbit in configuration space.

2.3.2 The Conserved Tensor B_{ij}

A quadratic form is associated with B_{ij} by double contraction with \mathbf{r} . We see that

$$\begin{aligned} \mathbf{r}^T B \mathbf{r} &= r_i B_{ij} r_j \\ &= J r^2 - L^2 \sin 2t, \end{aligned} \quad (2.3.8)$$

where L is the magnitude of the angular momentum. Hence

$$\mathbf{r}^T (JI - B) \mathbf{r} = L^2 \sin 2t, \quad (2.3.9)$$

is the quadratic form associated with B . The eigenvalues of the matrix $(JI - B)$ are J and $\frac{1}{2} (J \pm \sqrt{J^2 + 4L^2})$. The eigenvector of the eigenvalue, J , is \mathbf{L} and we use the constancy of the angular momentum to write the quadratic form in two-dimensions using the principal axes given by the eigenvectors corresponding to $\frac{1}{2} (J \pm \sqrt{J^2 + 4L^2})$ as

$$\left[J - \sqrt{J^2 + 4L^2} \right] \bar{Q}_1^2 + \left[J + \sqrt{J^2 + 4L^2} \right] \bar{Q}_2^2 = 2L^2 \sin 2t \quad (2.3.10)$$

which, in a more standard form, becomes

$$-\frac{2\bar{Q}_1^2}{J + \sqrt{J^2 + 4L^2}} - \frac{2\bar{Q}_2^2}{J - \sqrt{J^2 + 4L^2}} = \sin 2t. \quad (2.3.11)$$

(We have used the notation (\bar{Q}_1, \bar{Q}_2) for the variables along the principal axes for $(2JI - B)$ in the plane of the motion. We reserve Q_1 and Q_2 for the principal axes of $(2EI - A)$ which gives the orbit as

$$\frac{Q_1^2}{E + \sqrt{E^2 - L^2}} + \frac{Q_2^2}{E - \sqrt{E^2 - L^2}} = 1, \quad (2.3.12)$$

the standard equation of a geometric centred ellipse.)

Since the surface defined by (2.3.11) is much the same as the one defined by (2.2.14), we omit the description. Note, however, that although the curve specified by (2.3.11) is instantaneously an hyperbola, the path taken by the particle is in conformity with the elliptical orbit of (2.3.12).

2.3.3 The Conserved Tensor C_{ij}

The quadratic form associated with (C_{ij}) leads to the equation

$$\mathbf{r}^T (KI - C) \mathbf{r} = L^2 \cos 2t. \quad (2.3.13)$$

The eigenvalues of the matrix $(KI - C)$ are K and $\frac{1}{2} (K \pm \sqrt{K^2 + 4L^2})$. The eigenvector corresponding to K is \mathbf{L} . In terms of principal axes $(\bar{\bar{Q}}_1, \bar{\bar{Q}}_2)$ in the plane of

the motion the orbit is given by

$$-\frac{2\bar{Q}_1^2}{K + \sqrt{K^2 + 4L^2}} - \frac{2\bar{Q}_2^2}{K - \sqrt{K^2 + 4L^2}} = \cos 2t. \quad (2.3.14)$$

For fixed t , $t \neq \pi/4$, (2.3.14) describes an hyperbola. For $t = \pi/4$ it represents two straight lines. (The situation is much the same as described by Figs 2.2-2.8.) The hyperbola pulsates in the same way as that corresponding to (B_{ij}) except that the phase with respect to principal axes appropriate to (C_{ij}) is out compared with that for (B_{ij}) and its principal axes. The manner of pulsating of the hyperbolæ is the same as for (B_{ij}) and the actual orbit is as described in the previous section.

2.3.4 Noninvariance Algebra

Under the operation of taking the Poisson Bracket the components of \mathbf{L} possess the Lie algebra, $so(3)$, representing rotations in three dimensions. When the elements of the Jauch-Hill-Fradkin tensor, A_{ij} , are added, the Hamiltonian (2.3.3) possesses the eight element invariance algebra, $su(3)$. (The number of elements of A independent of H is five since $H = \frac{1}{2}Tr(A_{ij})$).

Without writing down the details of the Poisson Brackets we use the notation $[X, Y] \rightarrow Z$ to indicate that the Poisson Bracket of an integral of type X with one of type Y gives a linear combination of type Z integrals. Thus for the invariance algebra we have

$$[L, L] = L \quad [L, A] = A \quad [A, A] = L \quad (2.3.15)$$

and this is $su(3)$. It is not possible to introduce (B_{ij}) into (L_i) and (A_{ij}) by itself. (C_{ij}) must also be introduced because the bracket relations are

$$[A, B] = C \quad [B, C] = A \quad [C, A] = B. \quad (2.3.16)$$

The elements of L, A, B and C constitute the noncompact algebra $su(3, 2)$ under the operation of taking the Poisson Bracket.

However, if B (or C) replaces A , there is a closed algebra since

$$[L, L] = L \quad [L, B] = B \quad [B, B] = L \quad (2.3.17)$$

and

$$[L, L] = L \quad [L, C] = C \quad [C, C] = L. \quad (2.3.18)$$

In both cases the algebra is the noncompact $su(2, 1)$.

2.3.5 Discussion

The three classes of quadratic first integrals for the isotropic oscillator plus the angular momentum provide orbit equations for the classical motion of the particle. Angular momentum and the Jauch-Hill-Fradkin tensor provide the conventional elliptical orbit and the integrals have the compact algebra $su(3)$. The orbit is equally obtainable using the time-dependent integrals, (B_{ij}) or (C_{ij}) , and the angular momentum. The ‘pulsating hyperbolæ’ which B and C generate are orbit equations. In fact in extended configuration space they would provide the actual trajectory whereas the Jauch-Hill-Fradkin tensor gives a right elliptical cylinder on which the particle moves, but further information is required to locate the precise position of the particle.

The contrasting nature of the orbits given by A on the one hand and B and C on the other is reflected in the algebras of (L, A) and $(L, B \text{ or } C)$. The former is the compact $su(3)$ and the latter is the noncompact $su(2, 1)$. The compact algebra is associated with the geometrically compact ellipse and the noncompact algebra with the geometrically unconfined hyperbola.

2.4 The Kepler Problem

2.4.1 The Conserved Quantities, Energy and Orbits

The Kepler problem, with equation of motion

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3}, \quad (2.4.1)$$

has conserved angular momentum vector

$$\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}} \quad (2.4.2)$$

(as do all central force problems) which implies a planar orbit.

In plane polar coordinates Hamilton's vector has the form

$$\mathbf{K} = \dot{\mathbf{r}} - \frac{\mu}{L} \hat{\boldsymbol{\theta}}.$$

In coordinate-free form it is

$$\mathbf{K} = \dot{\mathbf{r}} - \frac{\mu}{rL^2} \mathbf{L} \times \mathbf{r}.$$

The LRL vector [6, 13, 20, 49, 30] is related to \mathbf{K} and \mathbf{L} by

$$\begin{aligned} \mathbf{J} &= \mathbf{K} \times \mathbf{L} \\ &= \dot{\mathbf{r}} \times \mathbf{L} - \mu \hat{\mathbf{r}}. \end{aligned} \tag{2.4.3}$$

If θ is the angle between \mathbf{J} and the radius vector, the equation of the orbit,

$$r = \frac{L^2}{\mu + J \cos \theta}, \tag{2.4.4}$$

follows from the scalar product of \mathbf{J} with \mathbf{r} .

The energy

$$E = \frac{1}{2} \mathbf{p} \cdot \mathbf{p} - \frac{\mu}{r}$$

is related to the LRL and Hamilton's vectors by

$$K^2 = 2E + \frac{\mu^2}{L^2} \tag{2.4.5}$$

$$J^2 = 2EL^2 + \mu^2. \tag{2.4.6}$$

From (2.4.6) we see that $E < 0$ implies that $J < \mu$ and hence (2.4.4) represents an ellipse. For $E = 0$ ($J = \mu$) the orbit is a parabola and for $E > 0$ ($J > \mu$) an hyperbola. Note that we have a bound orbit only when $E < 0$.

2.4.2 Poisson Bracket Relations

We consider the group structures of the conserved quantities. The components of the angular momentum, the energy E ($=H$) and the LRL vector have the Poisson

Bracket relations

$$\begin{aligned}
[L_i, L_j] &= \varepsilon_{ijk} L_k \\
[L_i, J_j] &= \varepsilon_{ijk} J_k \\
[L_i, H] &= 0 \\
[J_i, H] &= 0 \\
[J_i, J_j] &= (-2E) \varepsilon_{ijk} L_k.
\end{aligned}$$

The algebra depends on the value of the energy in the sense that for $E > 0$ the algebra is $so(3, 1)$, for $E < 0$ it is $so(4)$ and for $E = 0$ it is $e(3)$.

For Hamilton's vector we have

$$\begin{aligned}
[L_i, L_j] &= \varepsilon_{ijk} L_k \\
[L_i, K_j] &= \varepsilon_{ijk} K_k \\
[L_i, H] &= 0 \\
[K_i, H] &= 0 \\
[K_i, K_j] &= \left(\frac{\mu^2}{L^4} \right) \varepsilon_{ijk} L_k
\end{aligned}$$

which is isomorphic to the algebra $so(4)$. Fradkin [8] obtained this result in a similar fashion. It is interesting that he did not point out that the algebra is independent of the energy. This is in stark contrast to the LRL vector for which the algebra differs depending on the value of the energy (and hence the type of orbit). Interesting as the result may be, the geometry suggests that it should not be too surprising. (See Fig 2.9.)

For the oscillator we obtained new conserved quantities from Poisson's Theorem. A similar approach for the Kepler problem leads to a conserved tensor which follows from the Poisson Brackets of the components of \mathbf{K} and \mathbf{J} . From the Poisson Brackets

$$\begin{aligned}
[K_i, K_j] &= \left(\frac{\mu^2}{L^4} \right) \varepsilon_{ijk} L_k \\
[K_i, L_j] &= \varepsilon_{ijk} K_k
\end{aligned}$$

we may calculate $[K_i, J_j]$ from

$$[K_i, J_j] = [K_i, \varepsilon_{abj} K_a L_b]$$

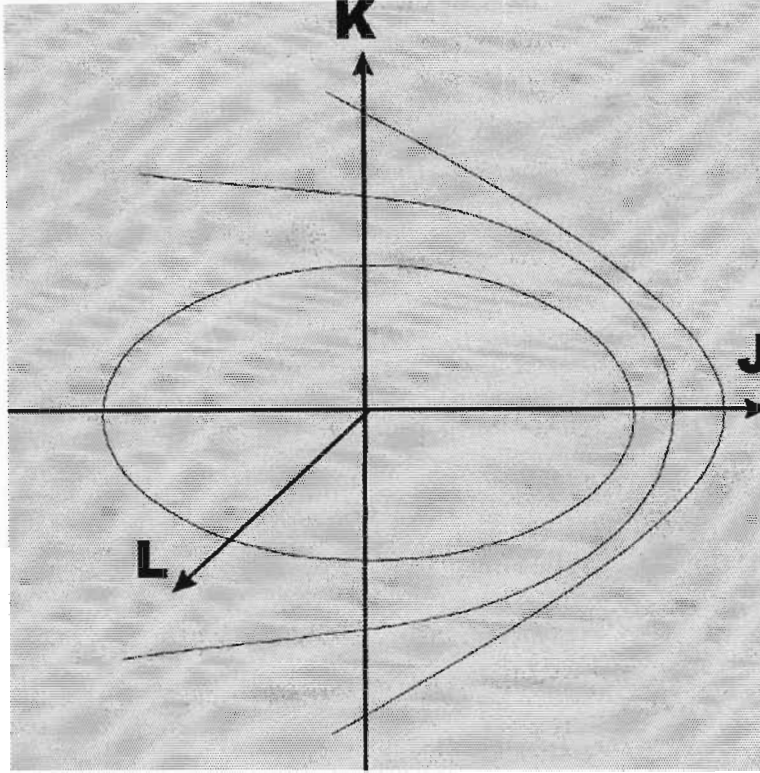


Figure 2.9: The particle is in the plane perpendicular to \mathbf{L} . The orbits are shown for different values of the energy E . \mathbf{K} intersects the orbits at two points for all values of the energy. The LRL vector intersects the ellipse twice and the parabola and hyperbola once. The geometry strongly supports the result that the group properties of the LRL vector should depend upon the value of the energy whereas those of Hamilton's vector do not.

$$= \varepsilon_{abj} [K_i, K_a] L_b + \varepsilon_{abj} [K_i, L_b] K_a.$$

The result yields the conserved tensor

$$R_{ij} := \left(\frac{\mu^2}{L^2} - K^2 \right) \delta_{ij} + K_i K_j - \left(\frac{\mu^2}{L^4} \right) L_i L_j.$$

2.4.3 The Conserved Tensor R_{ij}

As for the oscillator we seek a quadratic form which will produce the orbit.

$$\begin{aligned} \mathbf{r}^T R \mathbf{r} &= r_i R_{ij} r_j \\ &= \left(\frac{\mu^2}{L^2} - K^2 \right) r^2 + (\mathbf{r} \cdot \mathbf{K})^2 \end{aligned}$$

gives the quadratic form

$$\mathbf{r}^T (N I - R) \mathbf{r} = L^2 \tag{2.4.7}$$

where

$$N = (p^2 - 2E).$$

It would make more sense if we could write N in terms of the orbit $r(t)$ so that the quadratic form would depend on time implicitly. This can be done since

$$E = \frac{1}{2} \mathbf{p} \cdot \mathbf{p} - \frac{\mu}{r}.$$

This gives

$$N = \frac{2\mu}{r}.$$

The dependence on time is through the displacement $r(t)$ of the particle from the origin. For any given time we have a curve, a geometric centred ellipse for $E < 0$, to which the particle is confined. As the curve evolves with time the particle remains on the curve, but describes a focus centered ellipse. To illustrate this we need to write the quadratic form *ito* principal axes.

It is usefull to first define

$$S = \frac{1}{2} Tr(R).$$

Then the eigenvalues of the matrix in (2.4.7) are $N - S$, $N - \mu^2/L^2$ and $N + K^2$ and hence depend on time through $r(t)$. One may expect that the eigenvectors may also

be implicitly time-dependent and hence the transformation to principle axes will vary. However, this is not the case since the dependence on $r(t)$ in the matrix in (2.4.7) occurs in the diagonal terms only.

The eigenvalues turn out to be the triad of \mathbf{J}, \mathbf{K} and \mathbf{L} . The corresponding eigenvalues are $N - S, N - \mu^2/L^2$ and $N + K^2$ respectively. Since \mathbf{L} is conserved, we may neglect the corresponding coordinate and write the new quadratic form as

$$(N - S) Q_1^2 + \left(N - \frac{\mu^2}{L^2} \right) Q_2^2 = L^2. \quad (2.4.8)$$

Depending upon the value of r , (2.4.8) represents either an ellipse, hyperbola or two straight lines. When $r < 2L^2/\mu$ the instantaneous curve is an ellipse, for $r > 2L^2/\mu$ it is an hyperbola and when $r = 2L^2/\mu$ it represents two lines parallel to the y -axis. Figs 2.10-2.17 illustrate how we may obtain the Kepler orbit from (2.4.8).

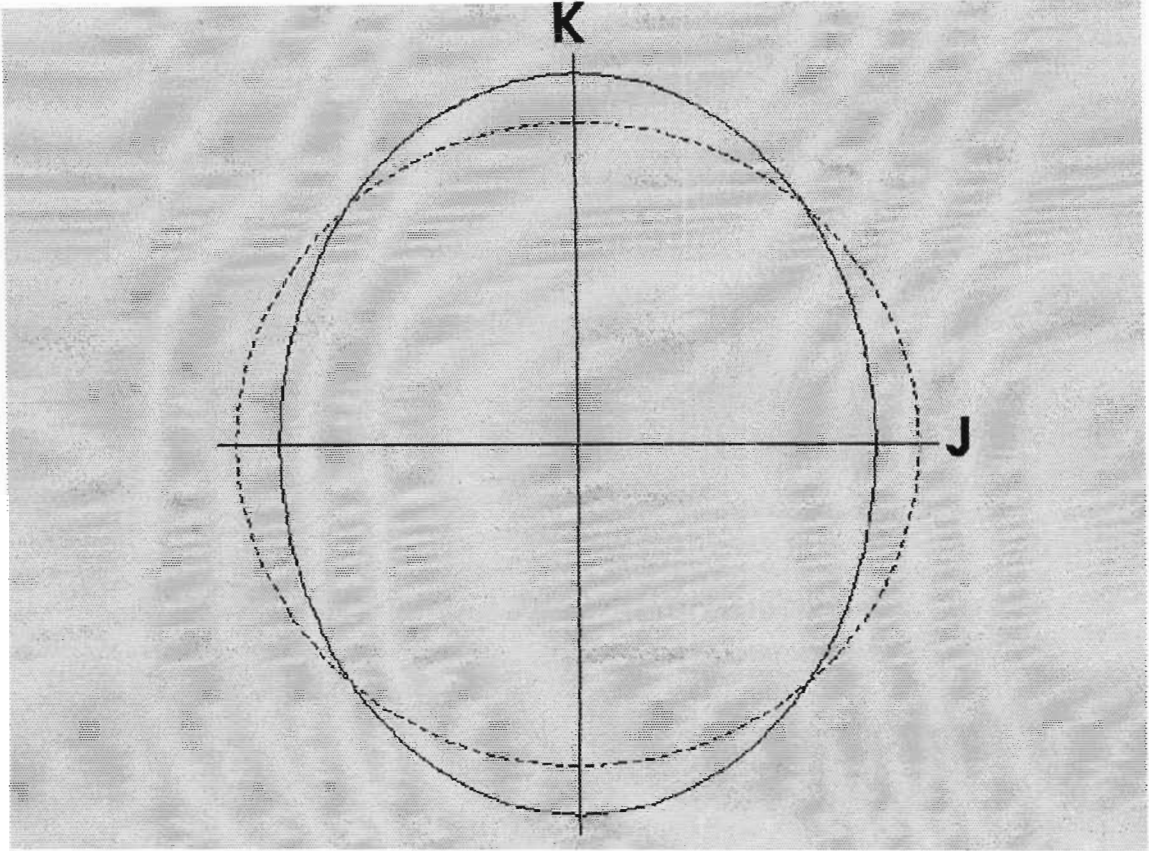


Figure 2.10: $E < 0$. The dashed curve is a circle which indicates the distance from the origin. The other curve is the instantaneous ellipse given by the quadratic form where r has the value of the radius of the circle and $r < L^2/\mu$.

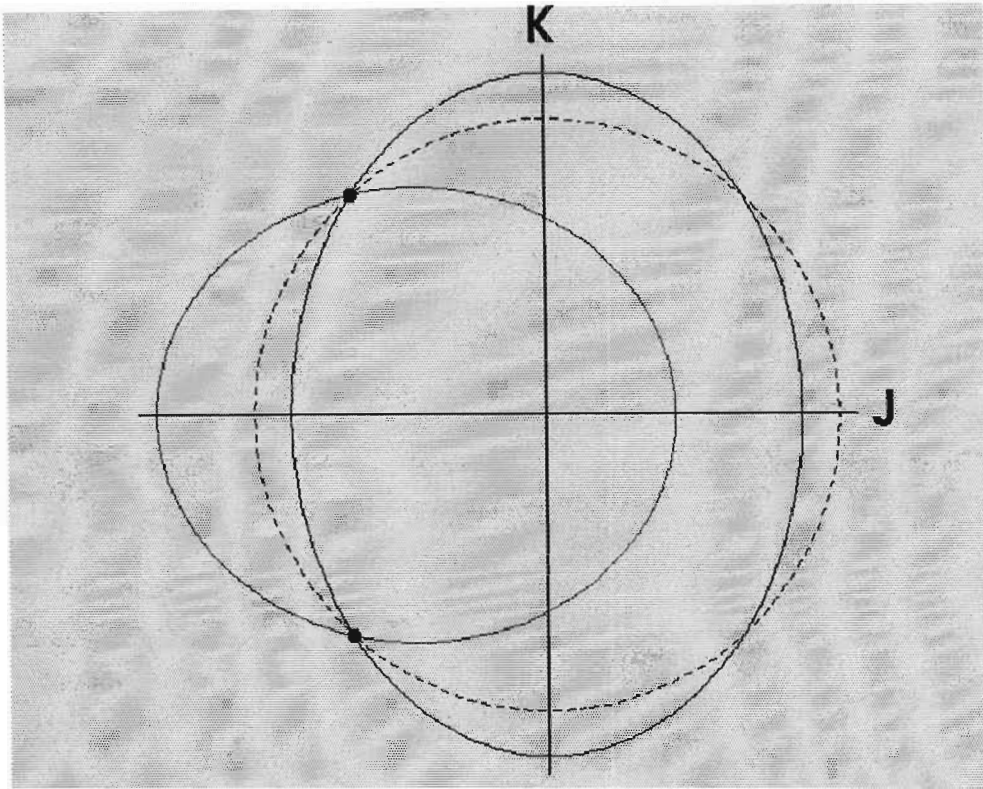


Figure 2.11: The Kepler orbit for the previous figure has been inserted. The points illustrate the intersection of the instantaneous ellipse and the circle (*ie* the positions on the instantaneous ellipse which are at a distance r from the origin). Note that the intersection points coincide with the orbit. This is so since the orbit may be defined by the instantaneous ellipses.

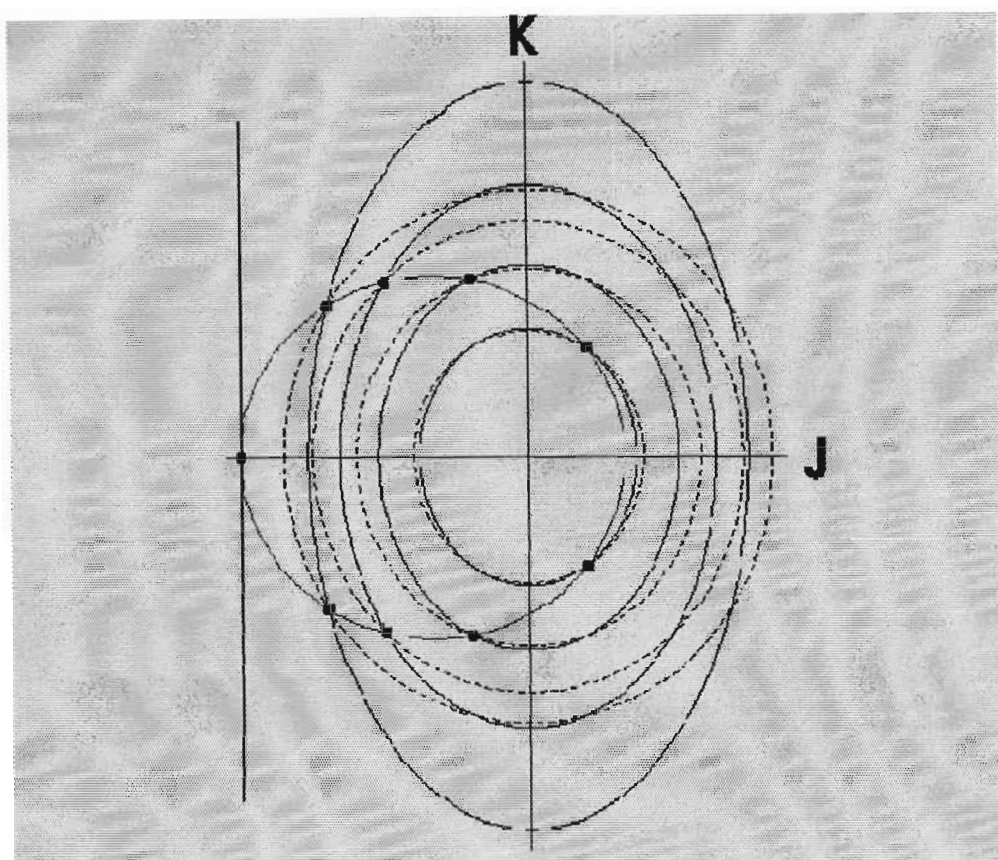


Figure 2.12: Several instantaneous ellipses have been plotted where $r < L^2/\mu$. The quadratic form gives the straight line for $r = L^2/\mu$. This value of r is reached at the point furthest from the origin. Hence there are no hyperbolæ necessary to describe the Kepler orbit for $E < 0$.

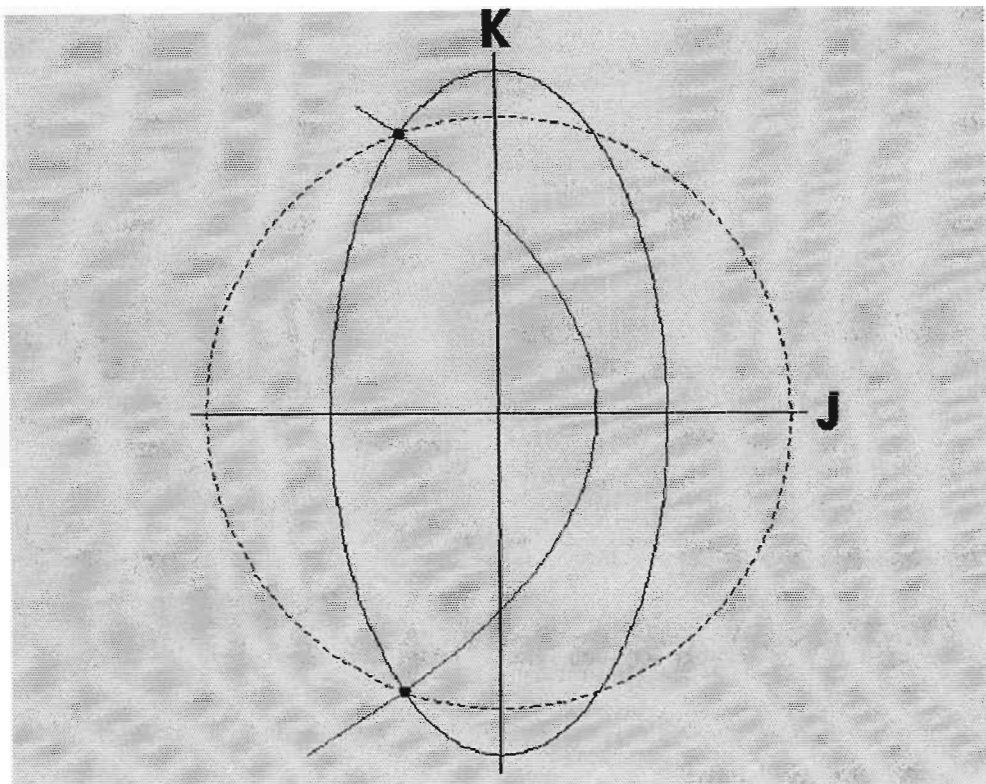


Figure 2.13: $E = 0$. The Kepler orbit is now a parabola. As for the $E < 0$ case the instantaneous ellipse coincides with the orbit at a distance r . The instantaneous curve in the figure is an ellipse since $r < L^2/\mu$.

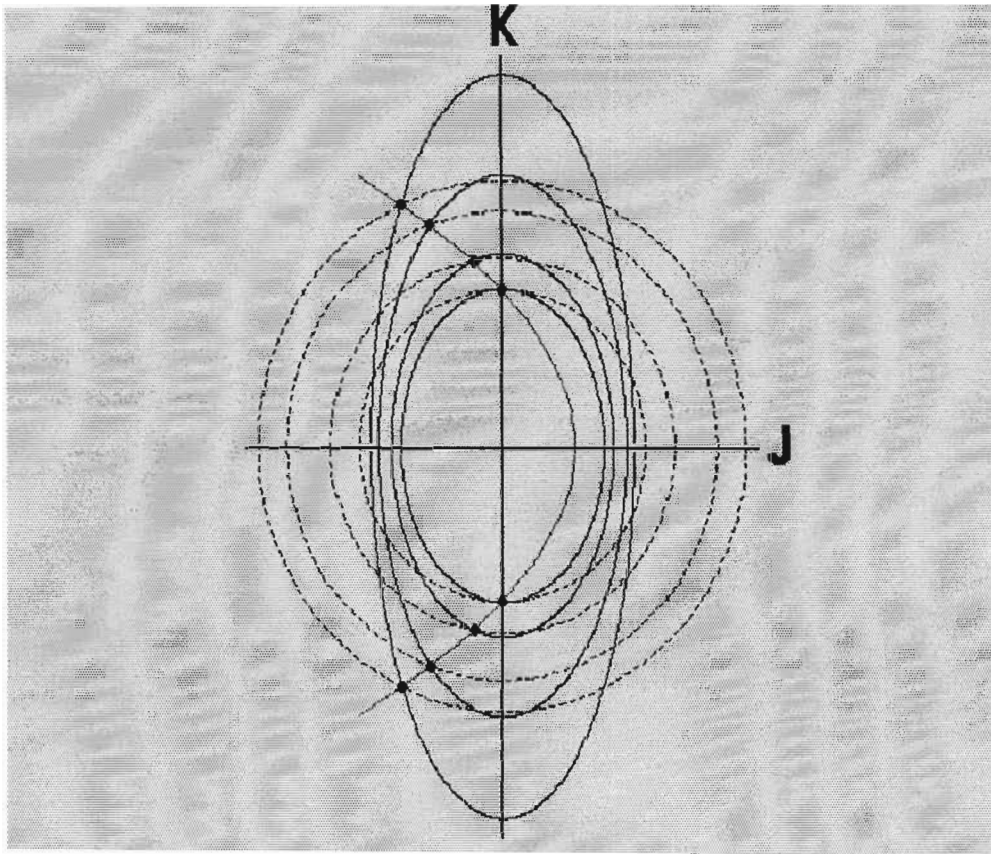


Figure 2.14: Several instantaneous ellipses have been added to the previous figure. The value of r is less than L^2/μ for all of them.

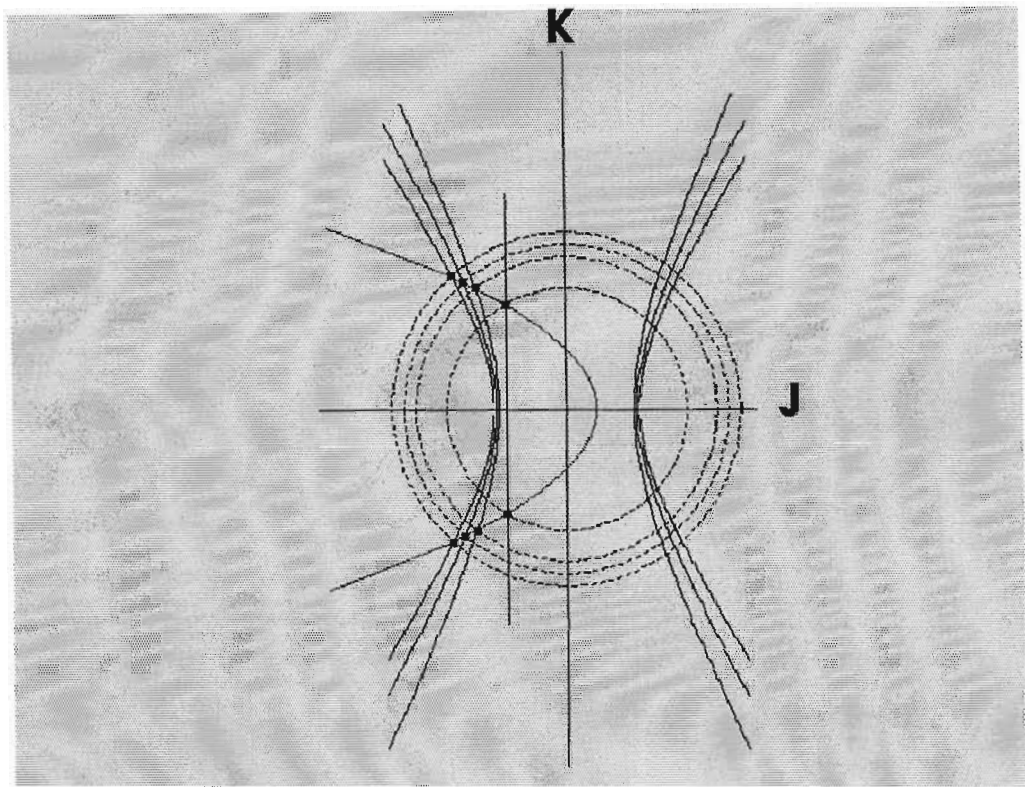


Figure 2.15: At $r = L^2/\mu$ the instantaneous curve is a straight line. The instantaneous hyperbolas for several values of $r > L^2/\mu$ are also indicated.

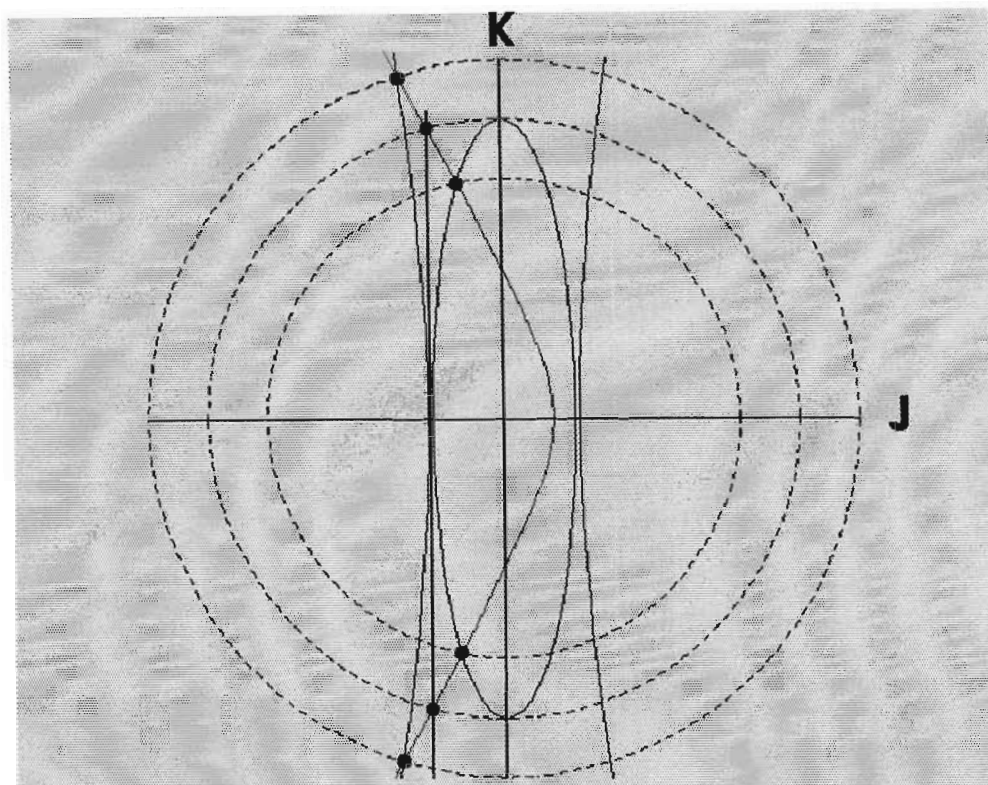


Figure 2.16: $E > 0$. The Kepler orbit is now an hyperbola. The figure shows only the left branch since the instantaneous curves do not give the right branch. All three types of instantaneous curves are indicated.

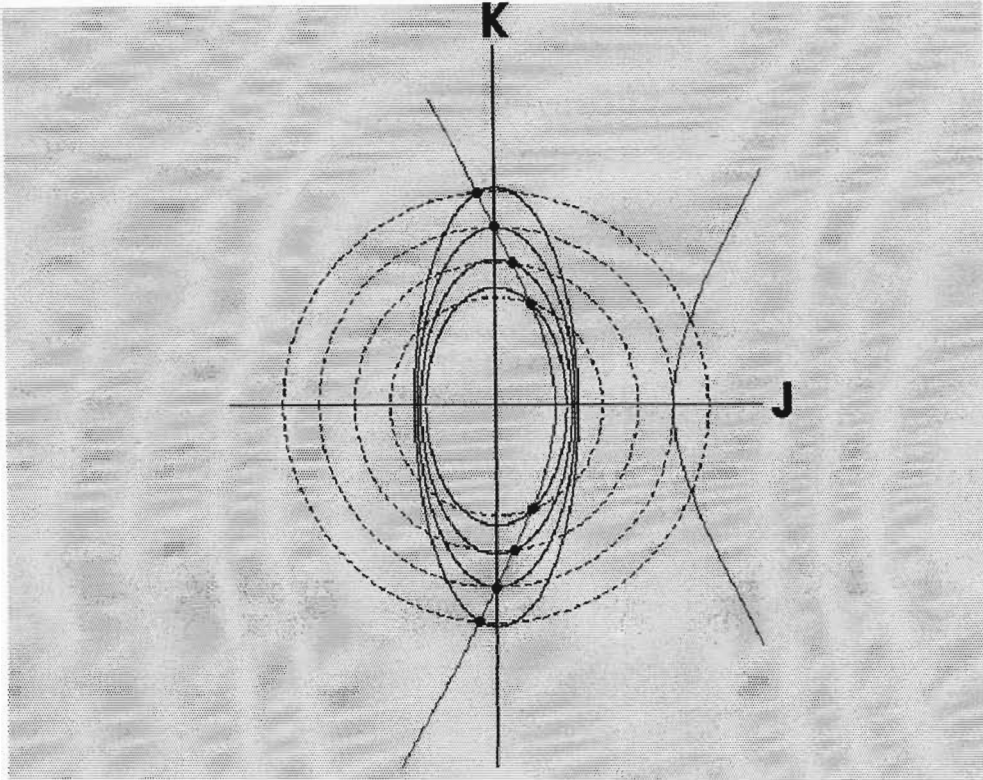


Figure 2.17: Several instantaneous ellipses illustrate how the orbit (for the sake of an uncluttered graph we show only instantaneous ellipses) may be found from the quadratic form.

Chapter 3

Lie and Noether Symmetries

3.1 Part I: Lie and Noether Symmetries

3.1.1 Lie Symmetries

Consider the general n^{th} order ordinary differential equation

$$E(x, y, y', \dots, y^{(n)}) = 0. \quad (3.1.1)$$

We are interested in those infinitesimal transformations which leave the equation invariant. The general infinitesimal point transformation may be written as

$$\begin{aligned} \bar{x} &= x + \epsilon \xi(x, y) \\ \bar{y} &= y + \epsilon \eta(x, y). \end{aligned} \quad (3.1.2)$$

For the following discussion we need not specify the variable dependence form of η and ξ . However, we do need to investigate how the derivatives transform. As a first step consider the transformed first derivative

$$\begin{aligned} \frac{d\bar{y}}{d\bar{x}} &= \frac{d(y + \epsilon \eta)}{d(x + \epsilon \xi)} \\ &= \frac{dy + \epsilon d\eta}{dx + \epsilon d\xi} \\ &= \frac{\frac{dy}{dx} + \epsilon \frac{d\eta}{dx}}{1 + \epsilon \frac{d\xi}{dx}} \end{aligned}$$

$$\begin{aligned}
&= (y' + \varepsilon \eta') (1 + \varepsilon \xi')^{-1} \\
&= (y' + \varepsilon \eta') (1 - \varepsilon \xi' + \varepsilon^2 \xi'^2 \dots) \\
&= y' + \varepsilon (\eta' - y' \xi'), \tag{3.1.3}
\end{aligned}$$

where we disregard terms of $O(\varepsilon^2)$ since ε is an infinitesimal. For the second derivative we have

$$\begin{aligned}
\frac{d^2 \bar{y}}{d\bar{x}^2} &= \frac{d}{d\bar{x}} \left(\frac{d\bar{y}}{d\bar{x}} \right) \\
&= \frac{d[y' + \varepsilon (\eta' - y' \xi')]}{d(x + \varepsilon \xi)} \\
&= \frac{dy' + \varepsilon d(\eta' - y' \xi')}{dx + \varepsilon d\xi} \\
&= \frac{\frac{dy'}{dx} + \varepsilon \frac{d}{dx} (\eta' - y' \xi')}{1 + \varepsilon \xi'} \\
&= [y'' + \varepsilon (\eta'' - y'' \xi' - y' \xi'')] [1 + \varepsilon \xi']^{-1} \\
&= y'' + \varepsilon (\eta'' - 2y'' \xi' - y' \xi''). \tag{3.1.4}
\end{aligned}$$

In general we have the recursive formula

$$\frac{d^n \bar{y}}{d\bar{x}^n} = y^{(n)} + \varepsilon \zeta_n, \tag{3.1.5}$$

where

$$\zeta_n = \zeta'_{n-1} - \xi' y^{(n)} \quad n \geq 1$$

and $\zeta_0 = \eta$. The $'$ denotes the total derivative *wrt* x . The complexity of the derivative obviously depends on the forms that η and ξ take. In the case of dependence on x and y only, we have point symmetries. When derivatives are included, we have Lie-Bäcklund symmetries. If the coefficient functions depend on integrals, the associated symmetries are known as nonlocal symmetries.

The requirement for the equation

$$E(x, y, y', \dots, y^{(n)}) = 0$$

to be invariant under the transformation (3.1.2) is

$$\begin{aligned} E(\bar{x}, \bar{y}, \bar{y}', \dots, \bar{y}^{(n)}) &= E(x + \epsilon \xi, y + \epsilon \eta, y' + \epsilon \zeta_1, \dots, y^{(n)} + \epsilon \zeta_n) \\ &= E(x, y, y', \dots, y^{(n)}) \end{aligned} \quad (3.1.6)$$

$$+ \epsilon \left(\xi \frac{\partial E}{\partial x} + \eta \frac{\partial E}{\partial y} + \zeta_1 \frac{\partial E}{\partial y'} + \dots + \zeta_n \frac{\partial E}{\partial y^{(n)}} \right) \quad (3.1.7)$$

$$= E(x, y, y', \dots, y^{(n)}) \quad (3.1.8)$$

to $O(\epsilon)$, ie

$$G^{[n]} E|_{E=0} = 0, \quad (3.1.9)$$

where $G^{[n]}$ is given by

$$G^{[n]} = \xi \frac{\partial}{\partial x} + \sum_{i=0}^n \zeta_i \frac{\partial}{\partial y^{(i)}}. \quad (3.1.10)$$

It is important to note that the equation itself must be brought into consideration when calculating the symmetries of that equation.

We say that

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \quad (3.1.11)$$

is the generator of the infinitesimal transformation (3.1.2). The generator of an infinitesimal transformation which leaves a quantity invariant is called a symmetry of that quantity. Furthermore, a generator is a symmetry of a quantity *only* if the associated transformation leaves the quantity invariant. Thus the terms *generator* and *symmetry* are not interchangeable.

The generalisation to higher dimensions is quite straightforward. An n -dimensional system (of sodes)

$$\mathbf{E}(t, \mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}) = 0 \quad (3.1.12)$$

has the symmetry

$$G = \xi \frac{\partial}{\partial t} + \eta_i \frac{\partial}{\partial r_i}$$

if

$$G^{[2]} = \xi \frac{\partial}{\partial t} + \eta_i \frac{\partial}{\partial r_i} + (\eta'_i - \xi' r'_i) \frac{\partial}{\partial r'_i} + (\eta''_i - 2\xi' r''_i - \xi'' r'_i) \frac{\partial}{\partial r''_i}$$

applied to (3.1.12) gives zero when (3.1.12) is taken into account.

Example

Returning to the one dimensional case we consider the free particle with equation of motion

$$\ddot{x} = 0.$$

Eq (3.1.9) now gives (we use ‘.’ for derivatives of time and ‘’ for space derivatives)

$$\eta'' - 2\ddot{x}\xi' - \dot{x}\xi'' = 0$$

which becomes (we consider only point transformations)

$$\frac{\partial^2 \eta}{\partial t^2} + 2\dot{x} \frac{\partial^2 \eta}{\partial t \partial x} + \frac{\partial^2 \eta}{\partial x^2} \dot{x}^2 = \dot{x} \left(\frac{\partial^2 \xi}{\partial t^2} + 2\dot{x} \frac{\partial^2 \xi}{\partial t \partial x} + \frac{\partial^2 \xi}{\partial x^2} \dot{x}^2 \right).$$

Separation by coefficients of distinct powers of \dot{x} gives

$$\begin{aligned} \frac{\partial^2 \eta}{\partial t^2} &= 0 \\ \frac{\partial^2 \xi}{\partial x^2} &= 0 \\ \frac{\partial^2 \eta}{\partial t \partial x} &= \frac{1}{2} \frac{\partial^2 \xi}{\partial t^2} \\ \frac{\partial^2 \eta}{\partial x^2} &= 2 \frac{\partial^2 \xi}{\partial t \partial x}. \end{aligned}$$

The first equation gives

$$\eta = a(x)t + b(x)$$

and the third yields

$$\xi = \dot{a}(x)t^2 + c(x)t + d(x).$$

From the two remaining determining equations we find that

$$\begin{aligned} a(x) &= A_0 + A_1 x \\ b(x) &= (2A_3 + A_2)x + A_4 x^2 + A_7 \\ c(x) &= A_3 + A_4 x \\ d(x) &= A_5 + A_6 x. \end{aligned}$$

Substituting these into ξ and η and setting, in turn, each of the arbitrary constants to unity while setting the remaining ones to zero, we find the 8 symmetries

$$G_1 = t^2 \frac{\partial}{\partial t} + xt \frac{\partial}{\partial x}$$

$$\begin{aligned}
G_2 &= t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} \\
G_3 &= xt \frac{\partial}{\partial t} + x^2 \frac{\partial}{\partial x} \\
G_4 &= \frac{\partial}{\partial t} \\
G_5 &= x \frac{\partial}{\partial x} \\
G_6 &= x \frac{\partial}{\partial t} \\
G_7 &= t \frac{\partial}{\partial x} \\
G_8 &= \frac{\partial}{\partial x}.
\end{aligned}$$

Determination of First Integrals

In general a scalar ordinary differential equation of the n th order

$$E(x, y, y', \dots, y^{(n)}) = 0$$

has a first integral

$$I = f(x, y, y', \dots, y^{(n-1)})$$

associated with the symmetry

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$

if

$$G^{[n-1]}f = 0$$

and

$$\left. \frac{df}{dx} \right|_{E=0} = 0.$$

The dependence of f on $y^{(n-1)}$ must be nontrivial.

The general conclusion is that associated with a symmetry of an n th order differential equation there exist $(n - 1)$ first integrals. The action of the symmetry requires that

$$\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta_1 \frac{\partial f}{\partial y'} + \zeta_2 \frac{\partial f}{\partial y''} + \dots + \zeta_{n-1} \frac{\partial f}{\partial y^{(n-1)}} = 0$$

which has the associated Lagrange's system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\zeta_1} = \dots = \frac{dy^{(n-1)}}{\zeta_{n-1}}$$

and this has n characteristics u_1, u_2, \dots, u_n . The first integral requirement is

$$u'_1 \frac{\partial f}{\partial u_1} + u'_2 \frac{\partial f}{\partial u_2} + \dots + u'_n \frac{\partial f}{\partial u_n} = 0$$

with associated Lagrange's system

$$\frac{du_1}{u'_1} = \frac{du_2}{u'_2} = \dots = \frac{du_n}{u'_n}$$

in which $y^{(n)}$ is replaced from $E = 0$. This leads to $n-1$ characteristics v_1, v_2, \dots, v_{n-1} . Since the characteristics are independent, we are guaranteed that the integrals are all independent.

For a system of n second order equations

$$\mathbf{E}(t, \mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}) = 0$$

a similar result holds. Associated with a symmetry

$$G = \xi \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y_i}$$

there are $2n - 1$ independent first integrals. Even if there are more symmetries, the number of independent integrals does not exceed $2n$. It may be that there is a repetition of integrals or, more commonly, that there are functional relations between the integrals. There is, of course, no guarantee that one can actually integrate the first order equation to find explicit formulæ for the first integrals. This is a definite disadvantage of the Lie method.

Example

Consider the simple, yet relevant, example concerning the free particle in one dimension

$$\ddot{x} = 0$$

which has the symmetry

$$G = \frac{\partial}{\partial t}$$

(due to the equation of motion being invariant under time translation). We seek the associated first integral of the form

$$I = f(t, x, \dot{x}).$$

The first requirement is that

$$G^{[1]}f = 0$$

ie

$$\frac{\partial f}{\partial t} = 0$$

which has the associated Lagrange's system

$$\frac{dt}{1} = \frac{dx}{0} = \frac{d\dot{x}}{0}.$$

This system has the characteristics

$$u = x \qquad v = \dot{x}.$$

This means our integral is of the form

$$I = f(x, \dot{x}).$$

The second requirement for I is

$$\dot{I} = \frac{\partial f}{\partial u}\dot{u} + \frac{\partial f}{\partial v}\dot{v} = 0$$

which has the associated Lagrange's system

$$\frac{du}{\dot{x}} = \frac{dv}{0}$$

ie

$$I = f(\dot{x}).$$

We may take the arbitrary function f to be the identity so that the integral is

$$I = \dot{x}.$$

3.1.2 Noether's Theorem

Calculus of Variations

Consider the functional

$$A = \int_{x_0}^{x_1} L(x, y, y') dx,$$

where x is the independent variable and y the dependent variable and L is an analytic function of x , y and y' . The value of A depends upon the functional dependence of y upon x . We seek the function $y = y(x)$ such that A takes a stationary value. Suppose that y is varied infinitesimally

$$\bar{y} = y + \varepsilon \zeta(x),$$

where $\zeta(x_0) = 0$ and $\zeta(x_1) = 0$ (so that there is no variation at the endpoints). Then the variation in A is

$$\delta A = \int_{x_0}^{x_1} L(x, \bar{y}, \bar{y}') dx - \int_{x_0}^{x_1} L(x, y, y') dx.$$

The transformed Lagrangian is

$$\begin{aligned} L(x, \bar{y}, \bar{y}') &= L(x, y + \varepsilon \zeta, y' + \varepsilon \zeta') \\ &= L(x, y, y') + \varepsilon \zeta \frac{\partial L}{\partial y} + \varepsilon \zeta' \frac{\partial L}{\partial y'} \end{aligned}$$

to the leading term in ε . The variation of the functional A becomes

$$\begin{aligned} \delta A &= \int_{x_0}^{x_1} \left(\varepsilon \zeta \frac{\partial L}{\partial y} + \varepsilon \zeta' \frac{\partial L}{\partial y'} \right) dx \\ &= \varepsilon \left[\int_{x_0}^{x_1} \zeta \frac{\partial L}{\partial y} dx + \int_{x_0}^{x_1} \zeta' \frac{\partial L}{\partial y'} dx \right] \\ &= \varepsilon \left[\int_{x_0}^{x_1} \zeta \frac{\partial L}{\partial y} dx + \left(\zeta \frac{\partial L}{\partial y'} \right) \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \zeta \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) dx \right] \\ &= \varepsilon \int_{x_0}^{x_1} \zeta \left\{ \frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) \right\} dx. \end{aligned}$$

For A to take a stationary value we must have $\delta A = 0$ ie

$$\int_{x_0}^{x_1} \zeta \left\{ \frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) \right\} dx = 0 \quad (3.1.13)$$

and that this be true for arbitrary ζ . This gives the Euler-Lagrange equation of the Calculus of Variations

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = 0. \quad (3.1.14)$$

Note that this equation follows from the requirement that the functional take a stationary value while the nature of the stationary value was not specified.

The nature of the stationary value is determined by the second order term in the variation. This is

$$\delta^2 A = \varepsilon^2 \int_{x_0}^{x_1} \left\{ \zeta^2 \frac{\partial^2 L}{\partial y^2} + 2\zeta\zeta' \frac{\partial^2 L}{\partial y \partial y'} + \zeta'^2 \frac{\partial^2 L}{\partial y'^2} \right\} dx.$$

The sign of the $\delta^2 A$ depends upon the Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 L}{\partial y^2} & \frac{\partial^2 L}{\partial y \partial y'} \\ \frac{\partial^2 L}{\partial y \partial y'} & \frac{\partial^2 L}{\partial y'^2} \end{pmatrix}.$$

If it is positive definite, the stationary value is a minimum. If it is negative definite, it is a maximum. If it is indefinite, there is a saddle point.

Maupertius believed [3] that the functional A was *the* functional which nature sought to minimise. In his own words, “It is the quantity of action which is Nature’s true storehouse, and which it economises as much as possible in the motion of light.”. Although he verified his postulate for the Newtonian law of propagation and the law of refraction, he was convinced that he had discovered the one action which was the underlying quantity to be minimised in all motions.

Formulation of Noether’s Theorem

Instead of transforming only the y as

$$y(x) \longrightarrow y(x) + \varepsilon \eta(x),$$

consider the more general transformation

$$\begin{aligned} x &\longrightarrow x + \varepsilon \xi \\ y &\longrightarrow y + \varepsilon \eta, \end{aligned}$$

where ε is again the parameter of smallness. We make no assumptions about the forms of the functions ξ and η . In the previous section we transformed only the dependent variable and obtained Lagrange's equation of motion. In this case, however, we transform the independent variable also and obtain an equation which involves ξ , η and the Lagrangian. This equation defines the Noether symmetries of the system at hand.

Our aim is to find functions ξ and η such that under the infinitesimal transformation the value of A is changed by a constant, *ie*

$$\int_{x_0}^{x_1} L(\bar{x}, \bar{y}, \bar{y}') d\bar{x} - \int_{x_0}^{x_1} L(x, y, y') dx = \varepsilon C, \quad (3.1.15)$$

where εC represents an infinitesimal constant¹. We observe that

$$\begin{aligned} L(\bar{x}, \bar{y}, \bar{y}') d\bar{x} &= L(x + \varepsilon\xi, y + \varepsilon\eta, y' + \varepsilon\zeta) d(x + \varepsilon\xi) \\ &= \left(L(x, y, y') + \varepsilon\xi \frac{\partial L}{\partial x} + \varepsilon\eta \frac{\partial L}{\partial y} + \varepsilon\zeta \frac{\partial L}{\partial y'} \right) (dx + \varepsilon\xi' dx) \\ &= \left[L + \varepsilon \left(\xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + \zeta \frac{\partial L}{\partial y'} + \xi' L \right) \right] dx \end{aligned}$$

to $O(\varepsilon)$ where $\zeta = \eta' - \xi'y'$. Then (3.1.15) is

$$\begin{aligned} \int_{x_0}^{x_1} \left[L + \varepsilon \left(\xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + \zeta \frac{\partial L}{\partial y'} + \xi' L \right) \right] dx - \int_{x_0}^{x_1} L dx &= \varepsilon C \\ \Leftrightarrow \int_{x_0}^{x_1} \left(\xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + \zeta \frac{\partial L}{\partial y'} + \xi' L \right) dx &= C \end{aligned}$$

which means that the integrand is a total derivative *ie* it makes zero contribution to the variation. Hence

$$\xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + \zeta \frac{\partial L}{\partial y'} + \xi' L = f' \quad (3.1.16)$$

where

$$\int_{x_0}^{x_1} f' dx = C.$$

The function f is called the *guage function*.

Example

¹We require that it be infinitesimal so that the identity infinitesimal transformation is the true identity.

Consider as an example the oscillator with Lagrangian

$$L = \frac{1}{2}(\dot{y}^2 - y^2).$$

Assume that ξ and η depend upon x and y only (ie we consider only point transformations). Now we have

$$-\eta y + \left(\frac{\partial \eta}{\partial x} + \dot{y} \frac{\partial \eta}{\partial y} \right) \dot{y} - \left(\frac{\partial \xi}{\partial x} + \dot{y} \frac{\partial \xi}{\partial y} \right) \dot{y}^2 + \frac{1}{2}(\dot{y}^2 - y^2) \left(\frac{\partial \xi}{\partial x} + \dot{y} \frac{\partial \xi}{\partial y} \right) = \frac{\partial f}{\partial x} + \dot{y} \frac{\partial f}{\partial y}.$$

In the same manner as with the Lie method, we separate by coefficients of powers of \dot{y} , viz

$$\begin{aligned} \dot{y}^3 : \quad & -\frac{\partial \xi}{\partial y} + \frac{1}{2} \frac{\partial \xi}{\partial y} = 0 \\ \dot{y}^2 : \quad & \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x} + \frac{1}{2} \frac{\partial \xi}{\partial x} = 0 \\ \dot{y}^1 : \quad & \frac{\partial \eta}{\partial x} - \frac{1}{2} y^2 \frac{\partial \xi}{\partial y} = \frac{\partial f}{\partial y} \\ \dot{y}^0 : \quad & -\eta y - \frac{1}{2} y^2 \frac{\partial \xi}{\partial x} = \frac{\partial f}{\partial x}. \end{aligned} \tag{3.1.17}$$

From (3.1.17)(a) we have

$$\xi = a(x)$$

and from (3.1.17)(b)

$$\eta = \frac{1}{2} \dot{a}(x) y + b(x).$$

Then (3.1.17)(c) becomes

$$\frac{\partial f}{\partial y} = \frac{1}{2} \ddot{a} y + \dot{b}$$

ie

$$f = \frac{1}{4} \ddot{a} y^2 + \dot{b} y + c(x)$$

and (3.1.17)(d) yields

$$-(\frac{1}{2} \dot{a} y + b) y - \frac{1}{2} y^2 \dot{a} = \frac{1}{4} \ddot{a} y^2 + \ddot{b} y + \dot{c}.$$

Separation by powers of y gives

$$\begin{aligned} y^2 : \quad & \ddot{a} + 4\dot{a} = 0 \\ y^1 : \quad & \ddot{b} + \dot{b} = 0 \\ y^0 : \quad & \dot{c} = 0. \end{aligned} \tag{3.1.18}$$

Eq (3.1.18)(c) implies

$$c = C_0$$

which can be ignored since it is an additive constant to f and does not appear in the ξ and η . From (3.1.18)(a) we have

$$a = A_0 + A_1 \sin 2x + A_2 \cos 2x$$

and from (3.1.18)(b)

$$b = B_1 \sin x + B_2 \cos x.$$

This gives

$$\xi = A_0 + A_1 \sin 2x + A_2 \cos 2x$$

$$\eta = A_1 y \cos 2x - A_2 y \sin 2x + B_1 \sin x + B_2 \cos x$$

$$f = B_1 y \cos x - B_2 y \sin x - A_1 y^2 \sin 2x - A_2 y^2 \cos 2x.$$

The five Noether symmetries are

$$\begin{aligned} A_0 : \quad G_1 &= \frac{\partial}{\partial x} & f &= 0 \\ A_1 : \quad G_2 &= \sin 2x \frac{\partial}{\partial x} + y \cos 2x \frac{\partial}{\partial y} & f &= -y^2 \sin 2x \\ A_2 : \quad G_3 &= \cos 2x \frac{\partial}{\partial x} - y \sin 2x \frac{\partial}{\partial y} & f &= -y^2 \cos 2x \\ B_1 : \quad G_4 &= \sin x \frac{\partial}{\partial y} & f &= y \cos x \\ B_2 : \quad G_5 &= \cos x \frac{\partial}{\partial y} & f &= -y \sin x \end{aligned}$$

in contrast to the eight Lie symmetries obtained by the Lie analysis.

Recall the defining equation for the Noether symmetries

$$\xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + (\eta' - y' \xi') \frac{\partial L}{\partial y'} + \xi' L = f'.$$

Taking total derivatives to the left side we obtain

$$\begin{aligned} f' &= \xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + (\eta' - y' \xi') \frac{\partial L}{\partial y'} + \xi' L \\ f' - (\xi L)' &= -\xi \left(y' \frac{\partial L}{\partial y} + y'' \frac{\partial L}{\partial y'} \right) + (\eta' - y' \xi') \frac{\partial L}{\partial y'} + \eta \frac{\partial L}{\partial y} \end{aligned}$$

$$\begin{aligned}
&= - \left[\xi \left(y' \frac{\partial L}{\partial y} + y'' \frac{\partial L}{\partial y'} \right) + y' \xi' \frac{\partial L}{\partial y'} \right] + \eta \frac{\partial L}{\partial y} + \eta' \frac{\partial L}{\partial y'} \\
&= - \left[\left(\xi y' \frac{\partial L}{\partial y'} \right)' - \xi y' \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) + \xi y' \frac{\partial L}{\partial y} \right] \\
&\quad + \left(\eta \frac{\partial L}{\partial y'} \right)' - \eta \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) + \eta \frac{\partial L}{\partial y}.
\end{aligned}$$

Again taking more total derivatives to the left side we have

$$\begin{aligned}
f' - (\xi L)' - \left[(\eta - y' \xi) \frac{\partial L}{\partial y'} \right]' &= (\eta - y' \xi) \frac{\partial L}{\partial y} - (\eta - y' \xi) \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) \\
&= (\eta - y' \xi) \left(\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) \right) \\
&= 0
\end{aligned}$$

due to the Euler-Lagrange equation. Hence

$$\left\{ f - \left[(\xi L) + (\eta - y' \xi) \frac{\partial L}{\partial y'} \right] \right\}' = 0,$$

ie

$$I = f - \left[(\xi L) + (\eta - y' \xi) \frac{\partial L}{\partial y'} \right]$$

is a first integral. We now have a precise expression for the first integral associated with a Noether symmetry. We may now formulate Noether's theorem as follows.

If the functional

$$A = \int_{x_0}^{x_1} L(t, y, y') dx$$

is varied by an infinitesimal constant under the transformation generated by the symmetry

$$G = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y},$$

then Lagrange's equation of motion admits the first integral

$$I = f - \left[(\xi L) + (\eta - y' \xi) \frac{\partial L}{\partial y'} \right].$$

An interesting property of a Noether transformation is that it leaves the corresponding first integral invariant [38]. This manifests itself in the form

$$G^{[1]} I = 0.$$

The same result applies to Lie symmetries. However, in the Lie case it is obvious since this is partly the definition of the associated first integral. It is interesting that the result also holds for Noether symmetries considering the very different approach. The Noether symmetry

$$G = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial y}$$

has associated conserved quantity

$$I = f - \left[\xi L + (\eta - \xi \dot{y}) \frac{\partial L}{\partial \dot{y}} \right].$$

Assuming the identity

$$G^{[1]} \left(\frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial}{\partial \dot{y}} (G^{[1]} L) + \left(\frac{\partial \xi}{\partial t} - \frac{\partial \eta}{\partial y} + 2 \frac{\partial \xi}{\partial y} \dot{y} \right) \frac{\partial L}{\partial \dot{y}}$$

and the equation

$$G^{[1]} L + \xi' L = f'$$

it is easy to show that

$$G^{[1]} I = 0$$

when expanding the derivatives

$$\begin{aligned} \xi' &= \frac{\partial \xi}{\partial t} + \dot{y} \frac{\partial \xi}{\partial y} \\ \eta' &= \frac{\partial \eta}{\partial t} + \dot{y} \frac{\partial \eta}{\partial y}. \end{aligned}$$

Returning to our example we calculate the associated first integrals. They are

$$\begin{aligned} I_1 &= \dot{y}^2 + y^2 \\ I_2 &= (\dot{y}^2 - y^2) \sin 2x - 2y\dot{y} \cos 2x \\ I_3 &= (\dot{y}^2 - y^2) \cos 2x + 2y\dot{y} \sin 2x \\ I_4 &= -2\dot{y} \sin x + 2y \cos x \\ I_5 &= -2\dot{y} \cos x - 2y \sin x. \end{aligned}$$

Derivative Dependent Symmetries

In the formulation of Noether's theorem there is no comment made about the variables appearing in ξ and η . The examples so far have contained just x and y and so the symmetries have been point symmetries. For a Lagrangian, $L(x, y, y')$, which leads to an Euler-Lagrange equation of second order it is possible to include y' in ξ and η .

The equation for the symmetry separates into y'' and non- y'' terms. The y'' part is

$$\frac{\partial f}{\partial y'} = \frac{\partial \xi}{\partial y'} L + \left(\frac{\partial \eta}{\partial y'} - y' \frac{\partial \xi}{\partial y'} \right) \frac{\partial L}{\partial y'}$$

and the rest is

$$\frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} = \xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + \left(\frac{\partial \eta}{\partial x} + y' \frac{\partial \eta}{\partial y} - y' \frac{\partial \xi}{\partial x} - y'^2 \frac{\partial \xi}{\partial y} \right) \frac{\partial L}{\partial y'} + \left(\frac{\partial \xi}{\partial x} + y' \frac{\partial \xi}{\partial y} \right) L.$$

The formula for the integral remains unchanged. The problem is that one cannot separate by powers of y' and hence there are infinitely many symmetries and no systematic method of finding them. One solution to this problem is to postulate a form for the symmetries. We will return to derivative dependent transformations when we investigate the paradigms of mechanics.

The Gauge Function

In some statements of Noether's theorem the gauge function is taken to be zero. We now calculate the Noether symmetries of the oscillator ($L = \frac{1}{2}(y'^2 - y^2)$) disregarding the gauge function.

The defining equation for the symmetries is

$$0 = \xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + (\eta' - y' \xi') \frac{\partial L}{\partial y'} + \xi' L.$$

Expanding the derivatives and assuming point symmetries we have

$$0 = -\eta y + y' \left(\frac{\partial \eta}{\partial x} + y' \frac{\partial \eta}{\partial y} \right) - y'^2 \left(\frac{\partial \xi}{\partial x} + y' \frac{\partial \xi}{\partial y} \right) + \frac{1}{2}(y'^2 - y^2) \left(\frac{\partial \xi}{\partial x} + y' \frac{\partial \xi}{\partial y} \right).$$

Separation by powers of y' yields

$$\begin{aligned} y'^3 &: \quad \xi = a(x) \\ y'^2 &: \quad \eta = \frac{1}{2}\dot{a}(x)y + b(x) \\ y' &: \quad 0 = \frac{1}{2}\ddot{a}(x)y + \dot{b}(x) \\ - &: \quad 0 = -\eta y - \frac{1}{2}y^2 \frac{\partial \xi}{\partial x}. \end{aligned}$$

The third equation gives

$$a(x) = A_0 + A_1 x$$

$$b(x) = B_0$$

and the coefficient of the zeroth power of y' gives

$$a(x) = A_1$$

$$b(x) = 0$$

ie there is only one symmetry instead of the five we get when the gauge function is present. The surviving symmetry is

$$G = \frac{\partial}{\partial x}$$

and corresponds to the Noether symmetry (found from the normal calculation with nonzero gauge function) for which the gauge function is zero.

Clearly it makes no sense to leave out the gauge function [27].

The Euler-Lagrange Equation for a Higher Order Functional

Consider the functional

$$A = \int_{x_0}^{x_1} L(x, y, y', y'') dx$$

which depends on the function $y(x)$. We transform y as

$$\bar{y} = y + \varepsilon \zeta,$$

where ε is an infinitesimal and $\zeta(x)$ is a differentiable function which is zero at the endpoints. Then the variation in A is

$$\begin{aligned}\delta A &= \int_{x_0}^{x_1} \{L(x, y + \varepsilon\zeta, y' + \varepsilon\zeta', y'' + \varepsilon\zeta'') - L(x, y, y', y'')\} dx \\ &= \int_{x_0}^{x_1} \left\{ L + \varepsilon \left[\zeta \frac{\partial L}{\partial y} + \zeta' \frac{\partial L}{\partial y'} + \zeta'' \frac{\partial L}{\partial y''} \right] - L \right\} dx \\ &= \varepsilon \int_{x_0}^{x_1} \left[\zeta \frac{\partial L}{\partial y} + \zeta' \frac{\partial L}{\partial y'} + \zeta'' \frac{\partial L}{\partial y''} \right] dx.\end{aligned}$$

We note that

$$\begin{aligned}\int_{x_0}^{x_1} \zeta' \frac{\partial L}{\partial y'} dx &= \left(\zeta \frac{\partial L}{\partial y'} \right) \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \zeta \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) dx \\ \int_{x_0}^{x_1} \zeta'' \frac{\partial L}{\partial y''} dx &= \left(\zeta' \frac{\partial L}{\partial y''} \right) \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \zeta' \frac{d}{dx} \left(\frac{\partial L}{\partial y''} \right) dx \\ &= \left(\zeta' \frac{\partial L}{\partial y''} \right) \Big|_{x_0}^{x_1} - \left(\zeta \frac{d}{dx} \left(\frac{\partial L}{\partial y''} \right) \right) \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} \zeta \frac{d^2}{dx^2} \left(\frac{\partial L}{\partial y''} \right) dx \\ &= \int_{x_0}^{x_1} \zeta \frac{d^2}{dx^2} \left(\frac{\partial L}{\partial y''} \right) dx,\end{aligned}$$

where we require that

$$\zeta'(x_0) = 0 \quad \zeta'(x_1) = 0,$$

ie the functions $y(x)$ and $\bar{y}(x)$ are tangential at the endpoints. Then

$$\delta A = \varepsilon \int_{x_0}^{x_1} \zeta \left[\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial L}{\partial y''} \right) \right] dx.$$

The function $y(x)$ for which the functional takes a stationary value is that for which $\delta A = 0$. This implies

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial L}{\partial y''} \right) = 0$$

which is the Euler-Lagrange equation for $L(x, y, y', y'')$. In general the equation is

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) + \cdots + (-)^n \frac{d^n}{dx^n} \left(\frac{\partial L}{\partial y^{(n)}} \right) = 0$$

for the Lagrangian $L(x, y, y', \dots, y^{(n)})$.

Noether's Theorem for Higher Order

To the second order in the dependent variable, the infinitesimal transformations are

$$\begin{aligned}\bar{x} &= x + \varepsilon\xi \\ \bar{y} &= y + \varepsilon\eta \\ \bar{y}' &= y' + \varepsilon\zeta_1, & \zeta_1 &= \eta' - y'\xi' \\ \bar{y}'' &= y'' + \varepsilon\zeta_2, & \zeta_2 &= \eta'' - 2y''\xi' - y'\xi''.\end{aligned}$$

The transformed functional is

$$\bar{A} = \int_{\bar{x}_0}^{\bar{x}_1} L(\bar{x}, \bar{y}, \bar{y}', \bar{y}'') d\bar{x}.$$

We require that the end points be unchanged ie that $\bar{x}_0 = x_0$, $\bar{x}_1 = x_1$. The functional transforms as

$$\begin{aligned}\bar{A} &= \int_{x_0}^{x_1} L(x + \varepsilon\xi, y + \varepsilon\eta, y' + \varepsilon\zeta_1, y'' + \varepsilon\zeta_2) d(x + \varepsilon\xi) \\ &= \int_{x_0}^{x_1} \left\{ L + \varepsilon \left[\xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + \zeta_1 \frac{\partial L}{\partial y'} + \zeta_2 \frac{\partial L}{\partial y''} + \xi' L \right] \right\} dx\end{aligned}$$

to first order in ε . Hence for the variation

$$\delta A = \varepsilon \int_{x_0}^{x_1} \left(\xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + \zeta_1 \frac{\partial L}{\partial y'} + \zeta_2 \frac{\partial L}{\partial y''} + \xi' L \right) dx$$

to be a constant infinitesimal we require that

$$\xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + \zeta_1 \frac{\partial L}{\partial y'} + \zeta_2 \frac{\partial L}{\partial y''} + \xi' L = f',$$

where f' denotes the total derivative of a function f . This is the condition for

$$A = \int_{x_0}^{x_1} L(x, y, y', y'') dx$$

to possess a Noether symmetry.

The generalisation is simple. For the transformations

$$\begin{aligned}\bar{x} &= x + \varepsilon\xi \\ \bar{y} &= y + \varepsilon\eta\end{aligned}$$

$$\begin{aligned}
\bar{y}' &= y' + \varepsilon \zeta_1 \\
\bar{y}'' &= y'' + \varepsilon \zeta_2 \\
&\vdots \\
\bar{y}^{(n)} &= y^{(n)} + \varepsilon \zeta_n
\end{aligned}$$

to leave the functional

$$A = \int_{x_0}^{x_1} L(x, y, y', \dots, y^{(n)}) dx$$

invariant we require

$$\xi \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + \zeta_1 \frac{\partial L}{\partial y'} + \dots + \zeta_n \frac{\partial L}{\partial y^{(n)}} + \xi' L = f',$$

where

$$\begin{aligned}
\zeta_1 &= \eta' - \xi' y' \\
\zeta_n &= \zeta'_{n-1} - \xi' y^{(n)}.
\end{aligned}$$

Higher Dimensional Systems

Noether's theorem applies to Lagrangians of systems of more than one degree of freedom. For an n -dimensional system the equation for the symmetries and gauge function is

$$f' = \xi \frac{\partial L}{\partial x} + \eta_i \frac{\partial L}{\partial y_i} + (\eta'_i - y'_i \xi') \frac{\partial L}{\partial y'_i} + \xi' L \quad i = 1, n$$

if the Lagrangian is of the form $L(x, \mathbf{y}, \dot{\mathbf{y}})$. The first integral associated with the symmetry

$$G = \xi \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y_i}$$

is given by

$$I = f - \left[\xi L + (\eta_i - y'_i \xi) \frac{\partial L}{\partial y'_i} \right],$$

where repeated indices indicate summation.

Example

Consider the free particle (in two dimensions) with the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2).$$

The determining equation for point symmetries is

$$\begin{aligned} \frac{\partial f}{\partial t} + \dot{x} \frac{\partial f}{\partial x} + \dot{y} \frac{\partial f}{\partial y} = & \left(\frac{\partial \eta}{\partial t} + \dot{x} \frac{\partial \eta}{\partial x} + \dot{y} \frac{\partial \eta}{\partial y} - \dot{x} \left(\frac{\partial \xi}{\partial t} + \dot{x} \frac{\partial \xi}{\partial x} + \dot{y} \frac{\partial \xi}{\partial y} \right) \right) \dot{x} \\ & + \left(\frac{\partial \zeta}{\partial t} + \dot{x} \frac{\partial \zeta}{\partial x} + \dot{y} \frac{\partial \zeta}{\partial y} - \dot{y} \left(\frac{\partial \xi}{\partial t} + \dot{x} \frac{\partial \xi}{\partial x} + \dot{y} \frac{\partial \xi}{\partial y} \right) \right) \dot{y}, \end{aligned}$$

where $\eta_1 = \eta$ and $\eta_2 = \zeta$.

Separation by powers of \dot{x} and \dot{y} at the third order gives

$$\begin{aligned} \dot{x}^3 : \quad & -\frac{\partial \xi}{\partial x} = 0 \\ \dot{x}^2 \dot{y} : \quad & -\frac{\partial \xi}{\partial y} = 0 \\ \dot{x} \dot{y}^2 : \quad & -\frac{\partial \xi}{\partial x} = 0 \\ \dot{y}^3 : \quad & -\frac{\partial \xi}{\partial y} = 0 \end{aligned}$$

which imply that $\xi = a(t)$. Next consider the squares. The coefficient of \dot{x}^2 gives

$$\eta = \dot{a}x + b(y, t),$$

that of $\dot{x}\dot{y}$ gives ζ as

$$\zeta = -\frac{\partial b}{\partial y}x + c(y, t)$$

and that of \dot{y}^2 gives

$$c = \dot{a}y + d(t) \quad b = e(t)y + g(t).$$

So far we have

$$\xi = a(t)$$

$$\eta = \dot{a}x + ey + g$$

$$\zeta = -ex + \dot{a}y + d.$$

The coefficient of \dot{x} gives f as

$$f = \frac{1}{2}\ddot{a}x^2 + \dot{e}xy + \dot{g}x + K(y, t).$$

The coefficient of \dot{y} requires

$$\dot{e}x + \frac{\partial K}{\partial y} = -\dot{e}x + \ddot{a}y + \dot{d}$$

which implies that

$$\dot{e} = 0 \quad K = \frac{1}{2}\ddot{a}y^2 + \dot{d}y + h(t).$$

The remaining term requires

$$\frac{1}{2}\ddot{a}x^2 + \ddot{g}x + \frac{1}{2}\ddot{a}y^2 + \ddot{d}y + \dot{h} = 0$$

which gives

$$a = A_0 + A_1t + A_2t^2$$

$$g = G_0 + G_1t$$

$$d = D_0 + D_1t$$

$$h = H_0$$

(we ignore h as it is an additive constant to f).

The coefficient functions are

$$\xi = A_0 + A_1t + A_2t^2$$

$$\eta = (A_1 + 2A_2t)x + E_0y + G_0 + G_1t$$

$$\zeta = -E_0x + (A_1 + 2A_2t)y + D_0 + D_1t$$

and the gauge function is

$$f = A_2x^2 + G_1x + A_2y^2 + D_1y.$$

We obtain three symmetries from a , viz

$$G_1 = \frac{\partial}{\partial t}$$

$$G_2 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$G_3 = t^2\frac{\partial}{\partial t} + 2t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$$

which form the algebra $sl(2, R)$, one from e , viz

$$G_4 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$$

which is $so(2)$, and four from g and d , viz

$$G_5 = \frac{\partial}{\partial x}$$

$$G_6 = t \frac{\partial}{\partial x}$$

$$G_7 = \frac{\partial}{\partial y}$$

$$G_8 = t \frac{\partial}{\partial y}$$

which is $4A_1$.

3.2 Part II: The Paradigms of Mechanics

The oscillator is the paradigm of linear systems and the free particle the simplest.

The Kepler problem is the paradigm of nonlinear systems.

Several methods have been devised to calculate the symmetries and first integrals of these systems. The different methods are based on different ideas of what the basic definition of a dynamical system is. Three possible formulations of a dynamical system are

- Newton's equation of motion,
- the Lagrangian formulation and
- the Hamiltonian formulation.

Noether's theorem deals with those transformations which leave the functional invariant. The Lie method is more general in the sense that it may be formulated *ito* any of the above mentioned expressions for dynamical systems.

Newtonian Formulation

The commonest formulation of the Lie method is that for the Newtonian equation of motion. Here we are simply interested in those transformations that leave the equation of motion invariant *ie* we have to consider how the second derivative transforms. We say that

$$G = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q}$$

is a symmetry of

$$E(t, q, \dot{q}, \ddot{q}) = 0$$

if

$$G^{[2]} = \xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial q} + (\dot{\eta} - \xi \dot{q}) \frac{\partial}{\partial \dot{q}} + (\ddot{\eta} - 2\dot{\xi} \ddot{q} - \xi \ddot{q}) \frac{\partial}{\partial \ddot{q}}$$

applied to E gives zero when $E = 0$ is taken into account.

Lagrangian Formulation

The Lagrangian formulation deals with those transformations which transform the Lagrangian in such a way as to leave the form of the equation of motion unchanged in the sense that [38]

$$\frac{d}{dT} \left(\frac{\partial \bar{L}}{\partial \dot{Q}} \right) - \frac{\partial \bar{L}}{\partial Q} = M(Q, \dot{Q}, T) \left[\frac{d}{dT} \left(\frac{\partial L}{\partial \dot{Q}} \right) - \frac{\partial L}{\partial Q} \right].$$

(The approach should not be confused with Noether's Theorem in which the Action Integral, rather than the equation of motion, is transformed.) Since we transform the Lagrangian, we need only include the first extension of the symmetry G . This simplifies only the initial part of the calculation since the system of partial differential equations that arises is the same as for the standard Lie calculation.

Hamiltonian Formulation

Leach [26] proposed the following formulation of the Lie method *ito* a Hamiltonian framework² The operator

$$G(q_i, p_i, t) = \xi(q_i, p_i, t) \frac{\partial}{\partial t} + \eta_j(q_i, p_i, t) \frac{\partial}{\partial q_j} + \zeta_j(q_i, p_i, t) \frac{\partial}{\partial p_j}$$

²Leach does not allow for ξ and η_i to depend on the momentum p_i . This restriction is not necessary and we do not impose it here.

is a (contact) symmetry of the system

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}\end{aligned}\tag{3.2.1}$$

if it leaves (3.2.1) invariant. The momenta are considered as independent variables and the symmetries are contact symmetries of the associated Newtonian equations of motion. We need only consider the first extension of G to include \dot{p}_i and \dot{q}_i .

3.2.1 The Lie Symmetries of the Free Particle

Lie [35] calculated the symmetries for the free particle in one dimension. Consider the free particle in n dimensions. The Lie symmetries are

$$\begin{aligned}G_1 &= \frac{\partial}{\partial t} \\ G_2 &= t \frac{\partial}{\partial t} \\ G_3 &= t^2 \frac{\partial}{\partial t} + t \left(\sum_{j=1}^n q_j \frac{\partial}{\partial q_j} \right) \\ G_{4i} &= \frac{\partial}{\partial q_i} \quad i = 1, n \\ G_{5i} &= t \frac{\partial}{\partial q_i} \quad i = 1, n \\ G_{6i} &= q_i \frac{\partial}{\partial t} \quad i = 1, n \\ G_{7i} &= t q_i \frac{\partial}{\partial t} + q_i \left(\sum_{j=1}^n q_j \frac{\partial}{\partial q_j} \right) \quad i = 1, n \\ G_{ij} &= q_i \frac{\partial}{\partial q_j} \quad i \neq j \text{ and } i, j = 1, n \\ G_{ii} &= q_i \frac{\partial}{\partial q_i} \quad i = 1, n.\end{aligned}$$

There are $n^2 + 4n + 3$ symmetries and the algebra is $sl(n+2, R)$.

Recall that the conserved vectors are

$$\begin{aligned}\mathbf{L} &= \mathbf{r} \times \dot{\mathbf{r}} \\ \mathbf{K}_1 &= \mathbf{p}, \quad \mathbf{J}_1 = \mathbf{p} \times \mathbf{L} \\ \mathbf{K}_2 &= t\mathbf{p} - \mathbf{r}, \quad \mathbf{J}_2 = (t\mathbf{p} - \mathbf{r}) \times \mathbf{L}.\end{aligned}$$

We now determine which of these integrals is associated with each of the given symmetries.

Consider the symmetry

$$G_1 = \frac{\partial}{\partial t}.$$

The associated first integral $I = f(t, \mathbf{q}, \dot{\mathbf{q}})$ is defined by

$$\begin{aligned} G_1^{[1]} I &= 0 \\ \dot{I} &= 0. \end{aligned}$$

The first equation has the associated Lagrange's system

$$\frac{dt}{1} = \frac{dq_i}{0} = \frac{dp_i}{0} \quad i = 1, n.$$

The characteristics are

$$u_i = q_i \quad v_i = p_i.$$

The second condition $\dot{I} = 0$ is

$$\frac{\partial I}{\partial u_i} \dot{u}_i + \frac{\partial I}{\partial v_i} \dot{v}_i = 0$$

with associated Lagrange's system

$$\frac{du_i}{v_i} = \frac{dv_i}{0} \quad i = 1, n$$

ie

$$\frac{du_1}{v_1} = \frac{du_2}{v_2} = \dots = \frac{du_n}{v_n} = \frac{dv_1}{0} = \frac{dv_2}{0} = \dots = \frac{dv_n}{0}.$$

At this stage, for the sake of simplicity, we discard the general approach and revert to three dimensions. The characteristics are

$$p_i \quad i = 1, 3$$

$$L_3 = q_1 p_2 - q_2 p_1$$

$$L_1 = q_2 p_3 - q_3 p_2.$$

In general I is of the form

$$I = f(\mathbf{L} \text{ (less one component)}, \mathbf{p}).$$

Hence the integrals associated with G_1 are

$$\begin{aligned}\mathbf{K}_1 &= \mathbf{p} \\ \mathbf{L} &\quad (\text{less one component}).\end{aligned}$$

The conserved quantities associated with

$$G_2 = t \frac{\partial}{\partial t}$$

are

$$\mathbf{K}_2 = t\mathbf{p} - \mathbf{r}$$

and $(n-1)$ integrals of the form

$$\frac{p_i}{p_j}.$$

The third symmetry

$$G_3 = t^2 \frac{\partial}{\partial t} + t \left(\sum_{j=1}^n q_j \frac{\partial}{\partial q_j} \right)$$

leads to \mathbf{K}_2 and \mathbf{L} (less one component). Note the similarity with G_1 which is to be expected since they are related by a simple transformation, *viz* $T = -1/t$, $Q_i = q_i/t$, which leaves $\ddot{\mathbf{r}} = 0$ invariant. G_{4i} is associated with

$$\begin{aligned}\mathbf{K}_1 &= \mathbf{p} \\ \mathbf{K}_2 &= t\mathbf{p} - \mathbf{r} \quad (\text{less one component}).\end{aligned}$$

G_{5i} gives \mathbf{K}_2 and \mathbf{K}_1 less one component.

G_{6i} gives $(\mathbf{K}_2)_i$, \mathbf{L} (less one component) and $\frac{p_i}{p_j}$ (where $i \neq j$).

The calculation for G_{7i} is a fine illustration of the fact that finding the first integrals associated with a Lie symmetry can be quite nontrivial. The symmetry is

$$G_{7i} = tq_i \frac{\partial}{\partial t} + q_i \left[\sum_{j=1}^n q_j \frac{\partial}{\partial q_j} \right] \quad i = 1, n.$$

We consider the $n = 2$ case first. One of the symmetries (G_{71}) is then

$$G = tx \frac{\partial}{\partial t} + x \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right).$$

The associated Lagrange's system of

$$G^{[1]}I = 0$$

is

$$\frac{dt}{tx} = \frac{dx}{x^2} = \frac{dy}{xy} = \frac{d\dot{x}}{\dot{x}(x - t\dot{x})} = \frac{d\dot{y}}{\dot{x}(y - t\dot{y})}. \quad (3.2.2)$$

The two obvious characteristics are

$$u = \frac{x}{t}$$

$$v = \frac{y}{t}.$$

The first and fourth parts of (3.2.2) give

$$\begin{aligned} \frac{dt}{tx} &= \frac{d\dot{x}}{\dot{x}(x - t\dot{x})} \\ &= \frac{d\dot{x}}{\dot{x}t(u - \dot{x})} \end{aligned}$$

ie

$$\begin{aligned} \frac{dt}{t} &= \frac{u}{\dot{x}(u - \dot{x})} d\dot{x} \\ &= \left(\frac{1}{\dot{x}} + \frac{1}{u - \dot{x}} \right) d\dot{x}. \end{aligned}$$

Integration yields

$$\begin{aligned} w &= \frac{\dot{x}}{t(u - \dot{x})} \\ &= \frac{\dot{x}}{x - t\dot{x}}. \end{aligned} \quad (3.2.3)$$

The first and fifth parts of (3.2.2) give

$$\begin{aligned} \frac{dt}{tx} &= \frac{d\dot{y}}{\dot{x}(y - t\dot{y})} \\ &= \frac{d\dot{y}}{\dot{x}t(v - \dot{y})} \end{aligned}$$

ie

$$\frac{dt}{t} = \frac{u}{\dot{x}(v - \dot{y})} d\dot{y}.$$

From (3.2.3) we have

$$\dot{x} = \frac{xw}{1 + wt}$$

and together with $x = ut$ the equation becomes

$$\frac{dt}{t} = \frac{(1 + wt)u}{utw(v - \dot{y})} d\dot{y}$$

or

$$\frac{w}{1 + tw} dt = \frac{d\dot{y}}{v - \dot{y}}.$$

Integration yields

$$\begin{aligned} s &= (1 + tw)(v - \dot{y}) \\ &= \left(1 + \frac{t\dot{x}}{x - t\dot{x}}\right) \left(\frac{y}{t} - \dot{y}\right) \\ &= \left(\frac{x}{x - t\dot{x}}\right) \left(\frac{y - t\dot{y}}{t}\right) \\ &= \left(\frac{x}{t}\right) \left(\frac{y - t\dot{y}}{x - t\dot{x}}\right). \end{aligned}$$

The first integral is now of the form

$$I = I(u, v, w, s).$$

The condition $\dot{I} = 0$ yields

$$\frac{\frac{du}{\dot{x} - \frac{x}{t^2}}}{\frac{dv}{\dot{y} - \frac{y}{t^2}}} = \frac{dw}{0} = \frac{ds}{\left(\frac{\dot{x}}{t} - \frac{x}{t^2}\right) \frac{y - t\dot{y}}{x - t\dot{x}}}.$$

In simpler form this is

$$\frac{du}{-\frac{1}{t^2}(x - t\dot{x})} = \frac{dv}{-\frac{1}{t^2}(y - t\dot{y})} = \frac{dw}{0} = \frac{ds}{-\frac{1}{t^2}(y - t\dot{y})}.$$

The second and fourth parts lead to

$$\begin{aligned} q &= v - s \\ &= \frac{x\dot{y} - \dot{x}y}{x - t\dot{x}}. \end{aligned}$$

w is again a characteristic and from parts one and four we have

$$\begin{aligned} du &= \left(\frac{x - t\dot{x}}{y - t\dot{y}}\right) ds \\ &= \frac{ds}{s} u \end{aligned}$$

so that

$$\begin{aligned} r &= \frac{s}{u} \\ &= \frac{y - ty}{x - tx}. \end{aligned}$$

The first integrals are

$$\begin{aligned} I_{11} &= \frac{\dot{x}}{x - tx} \\ I_{12} &= \frac{y - ty}{x - tx} \\ I_{13} &= \frac{x\dot{y} - \dot{x}y}{x - tx}. \end{aligned}$$

Instead of generalising this result using direct methods we use the following approach.

Theorem: The first integrals associated with

$$G_i = tq_i \frac{\partial}{\partial t} + q_i \left(\sum_{j=1}^n q_j \frac{\partial}{\partial q_j} \right) \quad i = 1, n$$

are

$$\begin{aligned} I_{ii} &= \frac{(\mathbf{K}_1)_i}{(\mathbf{K}_2)_i} \\ I_{ij} &= \frac{(\mathbf{K}_2)_j}{(\mathbf{K}_2)_i} \quad j \neq i \\ \bar{I}_{ij} &= \frac{L_{ij}}{(\mathbf{K}_2)_i} \quad j \neq i, \end{aligned}$$

where

$$L_{ij} = q_i p_j - q_j p_i.$$

Proof: Clearly these expressions are first integrals in the sense that

$$\dot{I}_{ii} = \dot{I}_{ij} = \dot{\bar{I}}_{ij} = 0.$$

We now have to show that

$$G_i^{[1]} \dot{I}_{ii} = G_i^{[1]} \dot{I}_{ij} = G_i^{[1]} \dot{\bar{I}}_{ij} = 0$$

ie that G_i leaves the integrals invariant. For I_{ii} the calculation is

$$\begin{aligned}
G_i^{[1]} I_{ii} &= \left(tq_i \frac{\partial}{\partial t} + q_i q_j \frac{\partial}{\partial q_j} + p_i (q_j - tp_j) \frac{\partial}{\partial p_j} \right) \frac{p_i}{q_i - tp_i} \\
&= tq_i \left[\frac{p_i^2}{(q_i - tp_i)^2} \right] + q_i^2 \frac{-p_i}{(q_i - tp_i)^2} + p_i (q_i - tp_i) \left[\frac{(q_i - tp_i) - p_i(-t)}{(q_i - tp_i)^2} \right] \\
&= \frac{tq_i p_i^2 - q_i^2 p_i + q_i p_i (q_i - tp_i)}{(q_i - tp_i)^2} \\
&= 0.
\end{aligned}$$

For I_{ij} we have

$$\begin{aligned}
G_i^{[1]} I_{ij} &= \left(tq_i \frac{\partial}{\partial t} + q_i q_k \frac{\partial}{\partial q_k} + p_i (q_k - tp_k) \frac{\partial}{\partial p_k} \right) \frac{q_j - tp_j}{q_i - tp_i} \\
&= tq_i \left[\frac{-p_j (q_i - tp_i) - (q_j - tp_j)(-p_i)}{(q_i - tp_i)^2} \right] + q_i^2 \frac{-(q_j - tp_j)}{(q_i - tp_i)^2} \\
&\quad + q_i q_j \frac{1}{q_i - tp_i} + p_i (q_i - tp_i) \frac{t(q_j - tp_j)}{(q_i - tp_i)^2} + p_i (q_j - tp_j) \frac{-t}{(q_i - tp_i)} \\
&= \frac{1}{(q_i - tp_i)^2} \left[(p_i q_j - q_i p_j) tq_i - q_i^2 (q_j - tp_j) + q_i q_j (q_i - tp_i) \right. \\
&\quad \left. + tp_i (q_i - tp_i) (q_j - tp_j) - tp_i (q_j - tp_j) (q_i - tp_i) \right] \\
&= 0.
\end{aligned}$$

The third calculation is much the same and is therefore omitted. Lastly we point out that there are $(2n - 1)$ (independent) integrals associated with G_i and hence we have listed *all* possible first integrals.

Returning to the other symmetries we note that G_{ii} is associated with $n - 1$ components of \mathbf{p} , the integral

$$\frac{\mathbf{K}_{2i}}{p_i}$$

and $n - 1$ integrals of the form

$$\frac{\mathbf{L}_j}{p_i} \quad j = 1, n \text{ and } j \neq i.$$

The final set of symmetries G_{ij} ($i \neq j$) lead to the angular momentum component

$$q_j p_i - q_i p_j,$$

\mathbf{p} and \mathbf{K}_2 .

It is interesting to note that some integrals yield physically significant conserved quantities (the last symmetry for instance) and others do not (eg G_2).

3.2.2 The Noether Symmetries of the Free Particle

The free particle has Lagrangian

$$L = \frac{1}{2} (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}).$$

For the sake of simplicity we consider the one-dimensional case. The Lagrangian is

$$L = \frac{1}{2} \dot{x}^2$$

and the equation determining the Noether point symmetries is

$$\xi \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial x} + (\dot{\eta} - \dot{\xi} \dot{x}) \frac{\partial L}{\partial \dot{x}} + \xi L = \dot{f}(t, x),$$

where the symmetry we seek is of the form

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}.$$

When we expand the total derivatives, the defining equation becomes

$$\left(\frac{\partial \eta}{\partial t} + \dot{x} \frac{\partial \eta}{\partial x} \right) \dot{x} - \frac{1}{2} \dot{x}^2 \left(\frac{\partial \xi}{\partial t} + \dot{x} \frac{\partial \xi}{\partial x} \right) = \frac{\partial f}{\partial t} + \dot{x} \frac{\partial f}{\partial x}.$$

Separation by powers of \dot{x} yields

$$\dot{x}: \quad \frac{\partial \eta}{\partial t} = \frac{\partial f}{\partial x} \tag{3.2.4}$$

$$\dot{x}^2: \quad \frac{\partial \eta}{\partial x} = \frac{1}{2} \frac{\partial \xi}{\partial t} \tag{3.2.5}$$

$$\dot{x}^3: \quad \frac{\partial \xi}{\partial x} = 0 \tag{3.2.6}$$

$$-: \quad \frac{\partial f}{\partial t} = 0. \tag{3.2.7}$$

The third equation gives

$$\xi = a(t).$$

Upon substitution into (3.2.5) we have

$$\eta = \frac{1}{2} \dot{a}(t)x + b(t).$$

The first equation now yields

$$f = \frac{1}{4} \ddot{a}(t)x^2 + \dot{b}(t)x + c(t)$$

and from (3.2.7)

$$a(t) = C_1 t^2 + C_2 t + C_3$$

$$b(t) = C_4 t + C_5$$

$$c(t) = C_6.$$

This leads to the functions

$$\xi = C_1 t^2 + C_2 t + C_3$$

$$\eta = C_1 t x + \frac{1}{2} C_2 x + C_4 t + C_5$$

$$f = \frac{1}{2} C_1 x^2 + C_4 x + C_6.$$

The five Noether symmetries are

$$G_1 = t^2 \frac{\partial}{\partial t} + t x \frac{\partial}{\partial x}, \quad f = \frac{1}{2} x^2$$

$$G_2 = t \frac{\partial}{\partial t} + \frac{1}{2} x \frac{\partial}{\partial x}, \quad f = 0$$

$$G_3 = t \frac{\partial}{\partial t}, \quad f = 0$$

$$G_4 = t \frac{\partial}{\partial x}, \quad f = x$$

$$G_5 = \frac{\partial}{\partial x}, \quad f = 0.$$

The first integrals associated with these symmetries are given by

$$I = f - \left[\xi L + (\eta - \dot{x} \xi) \frac{\partial L}{\partial \dot{x}} \right]$$

and for the above symmetries we have

$$I_1 = \frac{1}{2} \dot{x}^2$$

$$I_2 = (x - t \dot{x}) \dot{x}$$

$$I_3 = \frac{1}{2} (t \dot{x} - x)^2$$

$$I_4 = \dot{x}$$

$$I_5 = x - t \dot{x}.$$

Note that of these five integrals only two are independent.

In general the symmetries we seek are of the form

$$G = \xi \frac{\partial}{\partial t} + \boldsymbol{\eta} \cdot \frac{\partial}{\partial \mathbf{q}}$$

and are determined by the equation

$$\xi \frac{\partial L}{\partial t} + \boldsymbol{\eta} \cdot \frac{\partial L}{\partial \mathbf{q}} + (\dot{\boldsymbol{\eta}} - \dot{\xi} \dot{\mathbf{q}}) \cdot \frac{\partial L}{\partial \dot{\mathbf{q}}} + \dot{\xi} L = \dot{f}(t, \mathbf{q})$$

and the corresponding first integrals given by

$$I = f - \left[\xi L + (\boldsymbol{\eta} - \dot{\mathbf{q}} \xi) \cdot \frac{\partial L}{\partial \dot{\mathbf{q}}} \right].$$

In n dimensions the symmetries are

$$\begin{aligned} G_1 &= \frac{\partial}{\partial t} \\ G_2 &= t \frac{\partial}{\partial t} + \frac{1}{2} \sum_{j=1}^n q_j \frac{\partial}{\partial q_j} \\ G_3 &= t^2 \frac{\partial}{\partial t} + t \sum_{j=1}^n q_j \frac{\partial}{\partial q_j} \\ G_{4i} &= \frac{\partial}{\partial q_i} \\ G_{5i} &= t \frac{\partial}{\partial q_i} \\ G_{6ij} &= q_i \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial q_i} \quad i \neq j, i < j. \end{aligned}$$

The first three symmetries form the algebra $\mathfrak{sl}(2, R)$. G_{4i} and G_{5i} are solution symmetries and G_{6ij} is associated with the angular momentum. There are $\frac{1}{2}n(n-1) + 2n + 3$ Noether symmetries compared to the $n^2 + 4n + 3$ Lie symmetries. The first integrals take the forms

$$\begin{aligned} I_1 &= \frac{1}{2} \mathbf{p} \cdot \mathbf{p} \\ I_2 &= (\mathbf{q} - t\mathbf{p}) \cdot \mathbf{p} \\ I_3 &= \frac{1}{2} (t\mathbf{p} - \mathbf{q}) \cdot (t\mathbf{p} - \mathbf{q}) \\ I_{4i} &= p_i \\ I_{5i} &= (t\mathbf{p} - \mathbf{q})_i \\ I_{6ij} &= L. \end{aligned}$$

Noether's theorem produces the physically significant integrals quite neatly and performs reasonably well when compared with the Lie method. As we shall see, this good show does not persist as the physical systems become more complicated. The Noether theory has the advantage of producing the integrals by use of an explicit formula. The Lie method produces more symmetries, but finding the associated first integrals can be nontrivial. An easily overlooked limitation of Noether's theory is that the system at hand must have a Lagrangian formulation. However, this does not prove to be a problem in practice.

The Lie method is clearly superior to the Noether one. The following table compares the two methods symmetry by symmetry (the cases where not all of the components of a conserved vector were found for the Lie method are not indicated here).

Lie	Generators	Noether
\mathbf{L}, \mathbf{p}	$\frac{\partial}{\partial t}$	$\frac{1}{2}\mathbf{p} \cdot \mathbf{p}$
\mathbf{K}_2, \mathbf{L}	$t^2 \frac{\partial}{\partial t} + t \sum q_j \frac{\partial}{\partial q_j}$	$\frac{1}{2}(t\mathbf{p} - \mathbf{q}) \cdot (t\mathbf{p} - \mathbf{q})$
\mathbf{K}_2, \mathbf{p}	$\frac{\partial}{\partial q_i}$	\mathbf{p}_i
$\mathbf{K}_2, \mathbf{K}_1$	$t \frac{\partial}{\partial q_i}$	$(t\mathbf{p} - \mathbf{q})_i$
$\mathbf{K}_2, \frac{p_i}{p_j}, \mathbf{p}, \frac{\mathbf{K}_{2i}}{p_i}, \frac{\mathbf{L}_j}{p_i}$	$t \frac{\partial}{\partial t}, q_j \frac{\partial}{\partial q_j} \longrightarrow t \frac{\partial}{\partial t} + \frac{1}{2} q_j \frac{\partial}{\partial q_j}$	$(\mathbf{q} - t\mathbf{p}) \cdot \mathbf{p}$
$L_{ij} = q_i p_j - q_j p_i$	$q_i \frac{\partial}{\partial q_j} \longrightarrow q_i \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial q_i}$	\mathbf{L}

3.2.3 The Lie and Noether Symmetries of the Oscillator

The time independent case

The basic dynamical expressions for the time-independent oscillator are

- the Newtonian equation of motion

$$\ddot{\mathbf{q}} + \mathbf{q} = 0,$$

- the Lagrangian

$$L = \frac{1}{2} (\dot{\mathbf{q}} \cdot \dot{\mathbf{q}} - \mathbf{q} \cdot \mathbf{q}) \quad (3.2.8)$$

and finally

- the Hamiltonian

$$H = \frac{1}{2} (\mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q}). \quad (3.2.9)$$

Lie Symmetries

Anderson and Davidson [1] found the eight Lie symmetries for the one-dimensional oscillator. The algebra and global Lie group was later identified by Wulfman and Wybourne [56] who found that the algebra was a noncompact realisation of Cartan's A_2 algebra and that it contained a 3-parameter compact subalgebra which generates the compact group $SO(3)$. Since Cartan's A_2 algebra can only generate the three Lie groups $SU(3)$, $SU(2, 1)$ and $SL(3, R)$ and since only $SL(3, R)$ is both noncompact and contains an $SO(3)$ subgroup they concluded that the global Lie group was $SL(3, R)$.

Noether Symmetries

Lutzky [38] showed that Noether's theorem (with point symmetries) produces a five parameter subgroup of $SL(3, R)$. Although there is no need for the group of Noether symmetries to be a subgroup of the Lie symmetries, it happens to be the case here. The three symmetries omitted by Noether's theorem proved to be necessary for describing certain features of the motion. These symmetries leave Lagrange's equations of motion invariant but not the functional. In order to investigate how the Lagrangian transforms under those transformations not produced by Noether's theorem, Lutzky recalculated the Lie symmetries of the oscillator using the Lagrangian

formulation of the Lie method. He found three alternative Lagrangians for the oscillator which were equivalent (*ie* differ only by a total derivative of time) to the standard Lagrangian.

It has been shown that the group of Lie symmetries produce all of the useful (in the sense of producing the orbit) conserved quantities. In particular this includes the angular momentum

$$L_{ij} = q_i p_j - q_j p_i$$

and the Jauch-Hill-Fradkin tensor

$$A_{ij} = q_i q_j + p_i p_j.$$

Noether's theorem, however, does not produce the Jauch-Hill-Fradkin tensor from *point* transformations. When velocity dependent transformations are considered, Noether's theorem does produce the Jauch-Hill-Fradkin tensor [29].

The time-dependent case

The time-dependent oscillator, described by any of

•

$$\ddot{\mathbf{q}} + \omega^2(t)\mathbf{q} = 0, \tag{3.2.10}$$

•

$$L = \frac{1}{2} \left(\dot{\mathbf{q}} \cdot \dot{\mathbf{q}} - \omega^2(t)\mathbf{q} \cdot \mathbf{q} \right) \text{ or} \tag{3.2.11}$$

•

$$H = \frac{1}{2} \left(\mathbf{p} \cdot \mathbf{p} + \omega^2(t)\mathbf{q} \cdot \mathbf{q} \right) \tag{3.2.12}$$

was approached quite differently.

Lewis [31] applied Kruskal's asymptotic method [19] to construct the exact invariant

$$I = \frac{1}{2} \left[\frac{q^2}{\rho^2} + (\rho \dot{q} - \dot{\rho} q)^2 \right] \tag{3.2.13}$$

(for the one-dimensional time-independent harmonic oscillator), where ρ satisfies the Ermakov-Pinney equation [5, 45]

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}.$$

Note that the Lewis invariant is equally valid for the Lagrangian and Hamiltonian (one need only replace \dot{q} by p). Both Eliezer [4] and Lutzky [39] obtained the Lewis invariant from Noether's Theorem. Lewis and Reisenfeld [34] extended the application of the Lewis invariant to quantum mechanics.

Günther and Leach [11] generalised the work of Lewis and Reisenfeld [34] and found an invariant for the three-dimensional time-dependent harmonic oscillator by use of a time-dependent linear canonical transformation from (3.2.12) to the time-dependent Hamiltonian (3.2.9). The generalised invariant is given by

$$I_{mn} = \frac{1}{2} \left[\rho^{-2} q_m q_n + (\rho p_m - \dot{\rho} q_m) (\rho p_n - \dot{\rho} q_n) \right]$$

which reduces to the Jauch-Hill-Fradkin tensor for time-independent systems. Together with the angular momentum

$$L_{ij} = q_i p_j - q_j p_i,$$

I_{mn} provides a representation for the Lie algebra $su(3)$. The group $SU(3)$ is a noninvariance symmetry group for the three-dimensional time-dependent harmonic oscillator. Hence I_{mn} has the same role as the Jauch-Hill-Fradkin tensor for time-independent systems.

The Lie symmetries and $s\ell(3, R)$

Leach [24] was the first to make a study of the complete symmetry group and invariants of the one-dimensional time-dependent harmonic oscillator. The term *complete* indicates that the group is the largest admitted by the problem. His approach was Hamiltonian and we describe it briefly. Recall that I is considered an invariant of the Hamiltonian H if

$$\dot{I} = [I, H]_{PB} + \frac{\partial I}{\partial t} = 0$$

(this means that I may well depend on time). In practice, however, this does not provide us with enough information to determine the integrals. In order to calculate the integral, one has to also postulate a form for it. This method led Leach to five integrals for the time-independent case. Using the time-dependent canonical transformation of [11], he transformed these integrals to their time-dependent counterparts. For each of these integrals he calculated the corresponding group generators. These generators were the same as the five Noether symmetries found by Lutzky [38] for the time-independent case. The remaining three symmetries were those leaving the Newtonian equations of motion invariant. These symmetries have the same commutator relations as those found for the time-independent case (leaving Lagrange's equations of motion invariant). The complete group of eight symmetries has the same commutator properties as the Lie symmetries for the time-independent case, which we already know to be $SL(3, R)$.

Prince and Eliezer [47] considered the complete symmetry group of the n -dimensional oscillator and found that it was $SL(n+2, R)$. Noether's theorem gives $\frac{1}{2}(n^2 + 3n + 6)$ symmetries and the Lie method $(n^2 + 4n + 3)$, the difference being $\frac{1}{2}n(n + 5)$ (the same result as for the free particle).

Leach [26] calculated the Lie symmetries of the three dimensional time-independent harmonic oscillator by casting the Lie method in a Hamiltonian framework. The symmetries he found form the group $SL(3 + 2, R)$, the same as for the time-dependent oscillator. The time-invariance symmetry $\partial/\partial t$ is both a Lie and Noether symmetry. The difference in the two approaches is illustrated by the integrals each of them associate with this symmetry. Noether's theorem associates the energy E with this symmetry while the Lie method produces a far richer result. Instead of simply the energy the Lie method gives the Jauch-Hill-Fradkin tensor

$$A_{ij} = q_i q_j + p_i p_j$$

which contains the energy in the sense that

$$E = \frac{1}{2} Tr(A_{ij}).$$

The symmetry is furthermore associated with the angular momentum

$$L_{ij} = q_i p_j - q_j p_i.$$

Noether's theorem is constrained by the fact that it can produce only one integral (albeit easily) per symmetry while the Lie method does not suffer from this restriction and produces a set of $(2n - 1)$ integrals. One should point out though that we are merely interested in those conserved quantities that produce the trajectory and not every possible integral. The Jauch-Hill-Fradkin tensor of the orbit-producing variety and hence the fact that Noether's theorem does not produce it cannot be overlooked. produced.

3.2.4 The Lie and Noether Symmetries of the Kepler Problem

Just as Noether's theorem (with point symmetries) does not provide [26] the Jauch-Hill-Fradkin tensor for the oscillator, it does not produce the LRL vector for the Kepler problem. In order to overcome this shortcoming it was proposed that velocity dependent transformations be used.

Noether symmetries

LeBlond [29] considered the transformation

$$\bar{q}_i = q_i + \varepsilon \eta_i(q_j, \dot{q}_j, t) \quad (3.2.14)$$

and devised what he called a 'generalised' form for Noether's theorem. At first it may appear that he did not understand Noether's paper [42] at that stage since he made no mention of transforming the independent variable. It made be shown, however, that there is no loss of generality when time is not transformed. His approach led to the LRL vector and we briefly describe it. For the functional to be 'conserved' (under a transformation of the dependent variable and its derivatives) the Lagrangian must be transformed into an equivalent Lagrangian *ie*

$$\bar{L} = L + \varepsilon \dot{F}.$$

In general the Lagrangian varies by

$$\begin{aligned}\delta L &= \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \\ &= \left(\frac{\partial L}{\partial \dot{q}_i} \right)' \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \quad (\text{LEOM}) \\ &= \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]'.\end{aligned}$$

Since

$$\begin{aligned}\delta q_i &= \varepsilon \eta_i \\ \delta L &= \varepsilon \dot{F},\end{aligned}$$

we have

$$\left[\frac{\partial L}{\partial \dot{q}_i} \eta_i - F \right]' = 0$$

and the first integral associated with the transformation (3.2.14) is

$$I = \frac{\partial L}{\partial \dot{q}_i} \eta_i - F.$$

The classical Kepler problem has Lagrangian (in suitable units)

$$L = \frac{1}{2} \dot{\mathbf{q}} \cdot \dot{\mathbf{q}} + q^{-1}$$

and equation of motion

$$\ddot{\mathbf{q}} = -\frac{\mathbf{q}}{q^3}.$$

Consider the particular velocity dependent transformation given by the variation

$$\delta q_i = \varepsilon \left(\dot{q}_i q_k - \frac{1}{2} q_i \dot{q}_k - \frac{1}{2} \mathbf{q} \cdot \dot{\mathbf{q}} \delta_{ik} \right) \quad i = 1, 2, 3 \text{ and } k \text{ fixed}$$

and the corresponding variation of the velocities

$$\delta \dot{q}_i = \frac{1}{2} \varepsilon \left(\dot{q}_i \dot{q}_k - \dot{\mathbf{q}} \cdot \dot{\mathbf{q}} \delta_{ik} - \frac{q_i q_k}{q^3} + \frac{\delta_{ik}}{q} \right) \quad i = 1, 2, 3 \text{ and } k \text{ fixed}.$$

It turns out that the Lagrangian variation is indeed a total derivative given by

$$\begin{aligned}\delta L &= \varepsilon \left[\frac{\dot{q}_k}{q} - \frac{\mathbf{q} \cdot \dot{\mathbf{q}}}{q^3} q_k \right] \\ &= \varepsilon \frac{d}{dt} \left(\frac{q_k}{q} \right)\end{aligned}$$

and hence the generalised Noether's theorem is applicable and yields the conserved quantity

$$J_k = \dot{\mathbf{q}} \cdot \dot{\mathbf{q}} q_k - \mathbf{q} \cdot \dot{\mathbf{q}} \dot{q}_k - \frac{q_k}{q} \quad k = 1, 2, 3.$$

This is the Laplace-Runge-Lenz vector

$$\mathbf{J} = \dot{\mathbf{q}} \times (\mathbf{q} \times \dot{\mathbf{q}}) - \left(\frac{\mathbf{q}}{q}\right).$$

Lie Symmetries

In the already mentioned paper by Leach [26] he also calculated the Lie symmetries of the Kepler problem using the Hamiltonian formulation described earlier.

The classical Kepler problem has Hamiltonian

$$H = \frac{1}{2} \mathbf{p} \cdot \mathbf{p} - \frac{\mu}{r}.$$

The symmetries were calculated as

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t} \\ Y_2 &= t \frac{\partial}{\partial t} + \frac{2}{3} q_j \frac{\partial}{\partial q_j} - \frac{1}{3} p_j \frac{\partial}{\partial p_j} \\ Y_{3ij} &= q_i \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial p_i} \quad i \neq j = 1, 3. \end{aligned}$$

There are five symmetries since the third is skew symmetric. Y_{3ij} is associated with the integrals

$$\begin{aligned} L_{ij} &= q_i p_j - q_j p_i \\ E = H &= \frac{1}{2} \mathbf{p} \cdot \mathbf{p} - \frac{\mu}{r}. \end{aligned}$$

The second symmetry Y_2 corresponds to the LRL vector given component wise by

$$J_i = q_i p_j p_j - q_j p_i p_j - \mu \frac{q_i}{r}.$$

The first symmetry is the generator of time-translations and is therefore the one which yields all of the time-independent integrals. For the Kepler problem $\partial/\partial t$

leads to

$$\begin{aligned} L_{ij} &= q_i p_j - q_j p_i \\ E &= \frac{1}{2} \mathbf{p} \cdot \mathbf{p} - \frac{\mu}{r} \\ J_i &= q_i p_j p_j - q_i p_i p_j - \mu \frac{q_i}{r}. \end{aligned}$$

Chapter 4

Lie Algebras and Groups

4.1 Introduction

In the first chapter we discussed the historical development of the paradigms of mechanics *ito* of the conserved quantities used in their solution. We did not dwell on the group theoretical aspect of their development since such a discussion would assume knowledge of Lie groups and algebras. A brief overview is now presented on which we base an overview of the historical development of the paradigms *ito* group theoretical considerations.

4.2 Lie Groups and Algebras: A Brief Course

In the literature the theory of Lie groups and algebras is invariably presented in the context of manifold theory. For our purposes this is an unnecessarily complicated framework in which to convey the relevant ideas. The following discussion makes use of the concepts of group, vector space and algebra. Their definitions are given below.

4.2.1 Abstract Groups, Vector Spaces and Algebras

Groups

A group G is a set of elements on which a binary operation (which we will call addition) $+$ is defined such that:

G1 *Closure*: If $a, b \in G$, then so is $a + b$.

G2 *Associativity*: If a, b and $c \in G$, then

$$a + (b + c) = (a + b) + c.$$

G3 *Identity*: There exists an element 0 of G such that for all $a \in G$ we have $0 + a = a$.

G4 *Inverse*: For all $a \in G$ there exists an element denoted by $-a$ such that

$$a + (-a) = 0.$$

A *subgroup* H of G is a subset of G with the properties that $H + H$ is mapped into H and every element in H has an inverse in H . By defining scalar multiplication on an abelian ($a + b = b + a$) group we obtain a vector space.

Vector Spaces

A vector space V over a field F ¹ consists of the set V , which forms an abelian group under addition, together with scalar multiplication defined on the set V such that:

for all $a, b \in V$ and $\alpha, \beta \in F$ we have

$$V1 \quad \alpha a \in V$$

$$V2 \quad \alpha(\beta a) = (\alpha\beta)a$$

$$V3 \quad (\alpha + \beta)a = \alpha a + \beta a$$

$$V4 \quad \alpha(a + b) = \alpha a + \alpha b$$

$$V5 \quad 1a = a \text{ where } 1 \text{ is the multiplicative identity of the field } F.$$

¹For our purposes we do not require the rigorous definition of a field since we shall simply use the set of reals with the normal operations of addition and multiplication.

By defining a ‘multiplication operator’ on a vector space we obtain an algebra.

Algebras

An algebra consists of a vector space V over a field (of scalars) F together with a binary operation² of multiplication on the set of vectors such that for all $\alpha \in F$ and $x, y, z \in V$ we have

$$A1 \quad (\alpha x)y = \alpha(xy) = x(\alpha y)$$

$$A2 \quad (x + y)z = xz + yz$$

$$A3 \quad x(y + z) = xy + xz.$$

4.2.2 Groups of Transformations

A discrete set of transformations is one that is denumerable *ie* may be labeled by the set of integers. An example is the set with elements

$$\bar{x} = x + n, \tag{4.2.1}$$

where n is an integer.

In order to have a group of transformations we need to define a binary operation on the set. Let the binary operation on two transformations be successive application of the transformations. Hence the set of transformations (4.2.1) forms a group with identity $\bar{x} = x$ and inverse $-\bar{x} = x - n$. G1 and G2 follow from the fact that the set of integers forms a group under addition.

Lie Groups

A *Lie group* is a continuous group of transformations. We impose continuity on a set of transformations by defining continuity on the set of parameters. In the above example the set of parameters is the set of integers and hence continuity makes no sense here. Consider the set of transformations with elements

$$\bar{x} = x + r,$$

²The definition of a binary operation states that the set on which the operation acts be closed under the operation. (With this definition G1 is redundant.)

where r is any real number. We can define continuity on the set by letting the distance between two transformations be the absolute difference of their parameters. Hence the above set forms a Lie group.

4.2.3 Infinitesimal Transformations

The general one-parameter Lie group³ has elements

$$\bar{x} = f(x, a), \quad (4.2.2)$$

where f is an analytic function of the parameter a . Let the identity element be given by the value a_0 ie

$$x = f(x, a_0). \quad (4.2.3)$$

Variation of the parameter a by the infinitesimal ε gives

$$\bar{x} = f(x, a + \varepsilon).$$

Since f is an analytic function of the parameter, we may expand f in the neighbourhood of a_0 so that

$$\bar{x} = f(x, a_0) + \varepsilon f'(x, a_0) + O(\varepsilon^2).$$

Ignoring infinitesimals of order two and above and applying (4.2.3) we have

$$\bar{x} = x + \varepsilon \xi(x), \quad (4.2.4)$$

where

$$\xi(x) := f'(x, a_0).$$

The transformation (4.2.4) is the infinitesimal transformation associated with the one-parameter Lie group (4.2.2).

In general for an n -parameter Lie group with elements

$$\bar{x} = f(x, a_1, \dots, a_n) \quad (4.2.5)$$

³In order to simplify the discussion we consider only finite *point* transformations.

there is an associated set, $\{\bar{x}_i\}$, of n infinitesimal transformations

$$\bar{x}_i = x + \varepsilon \xi_i(x) \quad i = 1, n, \quad (4.2.6)$$

where

$$\xi_i(x) = \left. \frac{\partial f}{\partial a_i} \right|_{a_i=a_{i0}}. \quad (4.2.7)$$

The set of infinitesimal transformations does not form a group since there is no guarantee that the inverse of any transformation is in the set. However, using the notation of infinitesimal operators, we will show that the set of infinitesimal transformations may be enlarged to form a group.

4.2.4 Infinitesimal Operators

It is of interest to find those functions left invariant under the infinitesimal transformation (4.2.6). Referring to §3.1.1, we know that a function f is left invariant under the transformation iff

$$G_i f = 0,$$

where

$$G_i = \xi_i \frac{\partial}{\partial x}. \quad (4.2.8)$$

Hence the infinitesimal transformation

$$\bar{x}_i = x + \varepsilon \xi_i(x)$$

is associated with the infinitesimal operator (or symmetry) (4.2.8) by

$$\bar{x}_i = (1 + \varepsilon G_i)x.$$

We now return to the issue of forming a group from the set $\{\bar{x}_i\}$ associated with the n -parameter Lie group (4.2.5). Due to more convenient notation we will rather show how the set of infinitesimal *operators*, $\{G_i\}$, may be enlarged to form a group.

If we define the operation of addition on the set $\{G_i\}$ as

$$(G_i + G_j)f = G_i f + G_j f,$$

then the binary operation on two transformations may be expressed *ito* the associated symmetries as

$$\begin{aligned}\bar{x}_i \circ \bar{x}_j &= (x + \varepsilon \xi_i) \circ (x + \varepsilon \xi_j) \\ &= x + \varepsilon(\xi_i + \xi_j) \\ &= (1 + \varepsilon(G_i + G_j)) x\end{aligned}$$

(where we have neglected $O(\varepsilon^2)$). This means that the operation of successive application of infinitesimal transformations is abelian and clearly also admits associativity. Since the sum of two symmetries corresponds to the successive transformations of the associated infinitesimal transformations, the operation of addition of symmetries is abelian and associative *ie* the set $\{G_i\}$ obeys

$$\begin{aligned}G_i + G_j &= G_j + G_i \\ (G_i + G_j) + G_k &= G_i + (G_j + G_k)\end{aligned}$$

under the operation of addition. Note that there is no guarantee of closure under addition.

We define scalar multiplication on $\{G_i\}$ by

$$(\alpha G_i)f = \alpha(G_i f)$$

(where α is any real number). Then the inverse of G_i is simply $(-1)G_i$ (since $G_i - G_i = 0$, G corresponds to the identity transformation).

Consider the set V of all linear combinations of $\{G_i\}$. By definition, V is closed under addition and contains the identity element and the inverse of each of its elements. Hence V is an abelian group under the operation of addition. Furthermore, it is a vector space with scalar multiplication defined as above.

Lie algebra

In order to turn V into an algebra we need to define a multiplication operator on the set. This leads us to the Lie Bracket of two symmetries defined by

$$[G_i, G_j] = G_i G_j - G_j G_i.$$

The Lie Bracket satisfies A1-A3 ie

$$\begin{aligned} [\alpha G_i, G_j] &= \alpha [G_i, G_j] = [G_i, \alpha G_j] \\ [G_i + G_j, G_k] &= [G_i, G_k] + [G_j, G_k] \\ [G_i, G_j + G_k] &= [G_i, G_j] + [G_i, G_k]. \end{aligned}$$

Furthermore, it may be shown that the Lie Bracket of two symmetries may be expressed as a linear combination of symmetries in $\{G_i\}$, ie

$$[G_i, G_j] = C_{ij}^k G_k.$$

This means that the Lie Bracket is a binary operation. Hence the vector space V , together with the Lie Bracket defined on it, forms a Lie algebra.

The constants C_{ij}^k , referred to as the *structure constants* of the Lie algebra, completely define the algebra and obey Jacobi's Identity ie

$$C_{ij}^k + C_{jk}^i + C_{ki}^j = 0$$

and

$$C_{ij}^k = -C_{ji}^k.$$

These identities serve as constraints on the form that a Lie algebra can have.

We have shown⁴ that for every Lie group (of finite continuous transformations) there exists a Lie algebra (of infinitesimal transformations) which describes the group locally in a neighbourhood of the identity element.

4.2.5 Exponentiation

Given the n -parameter Lie group

$$\bar{x} = f(x, a_1, \dots, a_n),$$

we can find the associated Lie algebra from the infinitesimal transformations defined by

$$\xi_i(x) = \left. \frac{\partial f}{\partial a_i} \right|_{a=a_0}. \quad (4.2.9)$$

⁴The results generalise in a natural way to n parameters and higher dimensions.

Consider the inverse problem of finding the Lie group given the Lie algebra or, equivalently, finding the function f given ξ_i . We cannot use (4.2.9) since the derivative on the RHS is evaluated at a point. Even in cases for which

$$\left. \frac{\partial f}{\partial a_i} \right|_{a=a_0} = \frac{\partial f}{\partial a_i}$$

we would need to write $a_i = a_i(x)$ to find f ie we would need the Lie group!

Fortunately there is another route one can follow. Given the infinitesimal transformation

$$\bar{x}_i = x + \varepsilon \xi_i$$

we can find a one-parameter Lie group from the calculation (where we have substituted $\varepsilon = da$)

$$\begin{aligned} dx &= \xi_i da \\ \int_x^{\bar{x}} \frac{dx}{\xi_i} &= a - a_0. \end{aligned} \tag{4.2.10}$$

The process of finding the continuous transformation from the infinitesimal transformation is known as *exponentiation*. The integration in (4.2.10) can be done when the transformations are of point or contact type. However, for more general transformations, success is not guaranteed. This means that an infinitesimal transformation may not have an associated continuous transformation and hence a Lie algebra may not have an associated Lie group.

The structure constants of the Lie algebra describe the Lie group locally in the neighbourhood of the identity element. Since this is not the case for more general transformations, one should be careful when inferring group (global) properties from algebraic (local) properties.

A physical interpretation of this would be that the set of continuous transformations which leave a system invariant can often (not always) be represented by infinitesimal transformations of some abstract Lie group.

4.3 The Paradigms of Mechanics

4.3.1 Introduction

Given a set of first integrals we can form a group (under normal addition) in the same way as we did for infinitesimal transformations. Using the Poisson Bracket as multiplication operator we can extend the group to an algebra. As in the case of infinitesimal transformations, the algebra describes a system locally whereas a group is a global property.

When considering group and algebraic properties of the paradigms it is natural to include both the Classical and Quantum cases in the discussion. The strength of the group theoretical approach became apparent in the study of Quantum Mechanical systems, when it was realised that the problem of degeneracy is related to the existence of groups of transformations which leave the Hamiltonian invariant [15]. These transformations each have associated integrals which represent the symmetry of a system *ito* the Poisson Bracket. Since for a first integral we have

$$\dot{I} = [I, H] + \frac{\partial I}{\partial t} = 0,$$

if an integral does not explicitly depend on time it will have zero Poisson Bracket with the Hamiltonian.

If all the integrals in a set have zero Poisson Bracket with the Hamiltonian, then we refer to an *invariance* group (or algebra). Initially, invariance groups took centre stage in the investigation of group theoretical properties of physical systems.

Later on more consideration was given to larger symmetry groups, known as *non-invariance* [40, 22, 11] groups. A noninvariance group contains the usual invariance group as a subgroup *ie* the symmetry group of the Hamiltonian in a proper subgroup.

The time-evolution of a system may be expressed *ito* its time-dependent integrals. Since it is these integrals that yield the noninvariance group (as they have nonzero Poisson Bracket with the Hamiltonian), we say that the time-evolution of a system is given by its noninvariance algebra (provided there are sufficiently many integrals).

The largest symmetry group admitted by a physical system is referred to as the

complete symmetry group of the system. Interest in such groups gained popularity [24, 25] soon after the uses of noninvariance groups were established. Leach [25] showed that the complete symmetry group of an n -dimensional linear system, without damping, forcing or coupling terms, is $sl(n + 2, R)$. The nonlinear Kepler problem has the complete symmetry group $SO(3) + A_2$ [26].

In order to gain a better understanding of the more complicated Quantum Mechanical systems, the Classical and Quantum paradigms were extensively re-investigated.

Results for the free particle are sparse and we concentrate on the Kepler problem and the oscillator.

4.3.2 The Kepler Problem and $SO(4)$

Leach [26] calculated the Lie symmetries of the Kepler problem using a Hamiltonian formulation. The classical Kepler problem has Hamiltonian

$$H = \frac{1}{2} \mathbf{p} \cdot \mathbf{p} - \frac{\mu}{r}.$$

The symmetries are

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial t} \\ Y_2 &= t \frac{\partial}{\partial t} + \frac{2}{3} q_j \frac{\partial}{\partial q_j} - \frac{1}{3} p_j \frac{\partial}{\partial p_j} \\ Y_{3ij} &= q_i \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial p_i} \quad i \neq j = 1, 3. \end{aligned}$$

There are five symmetries (since the third is skew symmetric) and they form the algebra $so(3) + A_2$. The associated integrals are the angular momentum, energy and the LRL vector

$$\begin{aligned} L_{ij} &= q_i p_j - q_j p_i \\ E = H &= \frac{1}{2} \mathbf{p} \cdot \mathbf{p} - \frac{\mu}{r} \\ J_i &= q_i p_j p_j - q_i p_i p_j - \mu \frac{q_i}{r}. \end{aligned}$$

In two dimensions the Kepler problem has rotational symmetry $SO(3)$ (for bound states) [15]. In three dimensions the angular momentum \mathbf{L} also has the symmetry

group $SO(3)$. Fradkin [8] showed that the LRL vector provides additional symmetry and, together with the angular momentum, they generate an algebra $o(4)$ ⁵ which realises to the invariance Lie group $O(4)$. The Lie algebra is given *ito* Poisson Brackets (in Quantum Mechanics we use commutators). In the Quantum Mechanical case, the additional symmetries given by the LRL tensor give rise to extra degeneracies in the quantal solutions [14]. It may be shown that the Classical and Quantum cases have the same additional invariance.

4.3.3 The Oscillator

The time-independent case

The Jauch-Hill-Fradkin[15, 7] tensor (which applies to the three-dimensional case) provides symmetry additional to that given by the angular momentum alone and, together with the angular momentum, gives a suitable representation for the invariance algebra $su(3)$ [7]. The infinitesimal operators associated with these two tensors form the Lie group $SU(3)$. Since the invariants for the oscillator are isomorphic to the infinitesimal transformations of the $SU(3)$, we have a direct physical interpretation of the transformations *ito* a physical system.

In n dimensions [11] the invariance group for the time-independent harmonic oscillator is $SU(n)$.

The time-dependent case

Lewis [31] applied Kruskal's asymptotic method [19] to construct the exact invariant

$$I = \frac{1}{2} \left[\frac{q^2}{\rho^2} + (\rho\dot{q} - \dot{\rho}q)^2 \right]$$

(for the one-dimensional time-dependent harmonic oscillator), where ρ satisfies the

⁵It is alarming how indiscriminately the algebras $o(4)$ and $so(4)$ have been used in the literature. The 's' indicates the special group of rotations in four dimensions. It is special in the sense that the determinant of the matrix representation is 1. For $o(4)$ it can be -1 also.

Ermakov-Pinney equation [5, 45]

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}.$$

Lewis and Reisenfeld [34] extended the application of the Lewis invariant to quantum mechanics.

Günther and Leach [11] generalised the work of Lewis and Reisenfeld [34] and found an invariant for the three-dimensional time-dependent harmonic oscillator given by

$$I_{mn} = \frac{1}{2} \left[\rho^{-2} q_m q_n + (\rho p_m - \dot{\rho} q_m)(\rho p_n - \dot{\rho} q_n) \right].$$

It is a generalisation of the Jauch-Hill-Fradkin tensor for the time-independent case and plays a similar role *to* the symmetry group of the oscillator. Together with the angular momentum,

$$L_{ij} = q_i p_j - q_j p_i,$$

I_{mn} provides a suitable representation for the Lie algebra $su(3)$.

The group $SU(3)$ (compact) or $SU(2, 1)$ (noncompact) is the noninvariance symmetry group for the three-dimensional time-dependent harmonic oscillator [40, 50]. The noninvariance group contains the invariance group $S(U(2) \oplus U(1))$ which also happens to be the maximal compact subgroup.

The generalisation to n dimensions [11] is as expected. For the n -dimensional time-dependent harmonic oscillator the noninvariance group $SU(n)$ or $SU(n - 1, 1)$ has maximal compact subgroup $S(U(n - 1) \oplus U(1))$ which is the invariance group of the Hamiltonian.

The Lie symmetries and $sl(3, R)$

The algebra and global Lie group for the one-dimensional time-independent oscillator was identified by Wulfman and Wybourne [56] who found that the global Lie group was $SL(3, R)$. Lutzky [38] showed that Noether's theorem (with point symmetries) produces a five parameter subgroup $SL(3, R)$ of symmetries. The three symmetries omitted by Noether's theorem leave Lagrange's equations of motion invariant, but not the Action integral.

Leach [24] was the first to make a study of the complete symmetry group and invariants of the one-dimensional time-dependent harmonic oscillator. A Hamiltonian approach led Leach to five integrals for the time-independent case. The associated symmetries have the same commutator relations as those found for the time-independent case (leaving Lagrange's equations of motion invariant). The complete group of eight symmetries has the same commutator properties as the Lie symmetries for the time-independent case, *ie* the group is $SL(3, R)$.

Prince and Eliezer [47] generalised the results of Leach by considering the complete symmetry group of the n -dimensional oscillator. They found that it was $SL(n+2, R)$.

4.3.4 Discussion

The LRL vector for the Kepler problem and the Jauch-Hill-Fradkin type tensor for the oscillator both extend the symmetry of the respective problems.

Fradkin [8] proposed the existence of a generalisation of the LRL vector and Jauch-Hill-Fradkin tensor valid for all central potential problems. He found a quantity which was piecewise conserved and proceeded to show that it produced, together with the conserved angular momentum, suitable representations for the algebras $o(4)$ and $su(3)$ for all central force problems. In the literature these algebras are often associated with the observation that the problem at hand has closed orbits. As we showed in Chapter 2, this is not the case since, for the Kepler problem, Hamilton's vector gives the invariance algebra $o(4)$ for all values of the energy.

Since Fradkin was dealing with algebras, which may or may not lead to a realisation of the associated group *ito* finite continuous transformations, one cannot infer any group properties for central force problems from his results.

Bacry [2] clarified the matter by showing that $so(4)$ and $su(3)$ are both symmetry algebras for all problems with three degrees of freedom, even if the Hamiltonian is not associated with a central potential. His result follows from the fact that an invariance algebra of a Hamiltonian on a six-dimensional phase space is an invariance algebra for any other Hamiltonian. From this fact one may infer that all three-dimensional problems have the $so(3)$ symmetry algebra since any central force problem has this

as invariance algebra. However, it is obvious that not all three-dimensional problems have rotational symmetry! It is important to distinguish between the group (a global property) and algebraic (a local property) symmetry of a problem.

Bacry defines a *dynamical symmetry group* as a group of transformations which acts on a set of motions of a given energy. For different values of the energy the groups may either be isomorphic or distinct. For example the oscillator has the dynamical symmetry group $SU(3)$ for all values of the energy whereas for the Kepler problem the group depends on the energy. The groups are $SO(4)$ ($E < 0$), $SO(3,1)$ ($E > 0$) and the Euclidian group $e(3)$ ($E = 0$).

Finally, Bacry states that the only spherical potential with $SO(4)$ as symmetry *group* is the Kepler potential.

Chapter 5

Symmetry in Quantum Mechanics

5.1 Introduction

In the study of an atomic system we are interested in the symmetry group of the Hamiltonian invariant. The existence of a symmetry group raises the possibility of degeneracy¹. Hence consideration of the symmetry group of a quantum mechanical system reveals to us a great deal about the degeneracy to be expected.

In quantum mechanics a physical system is described *ito* the quantum mechanical states that it can occupy. The state of a system can change under the influence of external forces. These changes of state are represented by operators which act on the states. We may say that these operators represent the variables and conserved quantities of Classical Mechanics.

In Classical Mechanics we define a Lie algebra (consisting of a set of integrals) using the Poisson Bracket as multiplication operator. The Poisson Bracket of two quantities is defined by

$$[A, B] = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_i} \right).$$

In Quantum Mechanics the commutator plays the role of the Poisson Bracket in Classical Mechanics. In fact, the commutator and Poisson Bracket are directly related

¹Degeneracy occurs when several states of a system leads to the same energy value.

by

$$[\hat{A}, \hat{B}] = i\hbar [A, B]_{PB},$$

where A and B are classical quantities and \hat{A} and \hat{B} their Quantum Mechanical counterparts (*ie* operators). This provides a means of obtaining the quantum conditions in the case of operators that have classical analogues. (There are of course many quantum mechanical systems for which there are no classical analogues. In such cases one has to revert to more general methods.)

Poisson's Theorem [55] states that the a set of invariants is closed under the operation of the Poisson Bracket *ie*

$$[I_j, I_k] = C_{jk}^m I_m,$$

where there is summation over repeated indices and C_{jk}^m are constants. In order that there be an algebraic correspondence with the classical case, we require that the same result holds for the commutator of Quantum Mechanics. It is anticipated the constants have the same value as in the classical case [15].

In Classical Mechanics we may define a first integral, I , by

$$\dot{I} = \frac{\partial I}{\partial t} + [I, H].$$

It may be shown that the associated operator, \hat{I} , is also an invariant *ie*

$$\dot{\hat{I}} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar} [\hat{I}, H],$$

where the square brackets denote the commutator.

The overlap of two Quantum Mechanical states represents an *amplitude* and the *probability* of any of those states occurring is given by the square of the amplitude. A fundamental characteristic of Quantum Mechanics is that one can merely predict the *possibility* of a measurement yielding a result. This is in stark contrast with Classical Mechanics where it is possible to predict the state of a system accurately.

Physical quantities (or *observables*) that can be simultaneously determined correspond to Quantum Mechanical operators that commute *wrt* the commutator. Hence uncertainty in measurement arises only when two operators do not commute. It is an

important problem in Quantum Mechanics to find all of the compatible observables (*ie* those that can be measured simultaneously).

Consider the classical Poisson Bracket relations

$$[q_i, q_j] = 0$$

$$[p_i, p_j] = 0$$

$$[q_i, p_j] = \delta_{ij}$$

which give

$$[\hat{q}_i, \hat{q}_j] = 0$$

$$[\hat{p}_i, \hat{p}_j] = 0$$

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}.$$

The implications of this is quite profound since it implies that

$$[\hat{q}_i, \hat{p}_i] \neq 0$$

ie that position and momentum cannot be simultaneously determined. Also, from the Classical Poisson Brackets

$$[L_i, L_j] = \epsilon_{ijk}L_k$$

we infer that in Quantum Mechanics one cannot determine different components of angular momentum simultaneously. However, since

$$[L, L_i] = 0$$

one can measure the total angular momentum and one component simultaneously.

The above examples illustrate why the study of the group properties of physical quantities in Classical Mechanics (*ito* the Poisson Bracket) is of great consequence in Quantum Mechanics.

If \hat{T} is a symmetry operator of a system and commutes with the Hamiltonian,

$$[\hat{T}, \hat{H}] = 0,$$

then \hat{T} is a constant of the motion. This implies that constants of motion ² are compatible (*ie* can be measured simultaneously) with the energy of a system.

Despite their fundamental differences, a close analogy can be established between the quantum and classical models of a physical system.

5.2 The Paradigms in Quantum Mechanics

The use of invariance in the solution of problems in Quantum Mechanics has been well founded [34, 50, 40]. Lewis [31, 32, 33] illustrated that the invariants of a system may be used to solve the time-dependent Schrödinger equation for that system. Closely related to this is the topic of symmetry groups of dynamical systems. Initially the discussions pertained to purely geometrical symmetries *eg* the rotational symmetry group. When the need arose to explain the existence of degeneracies in spectra (which could not be explained by geometrical symmetries alone), the discussion widened to include symmetries which do not have an *obvious* geometrical basis³.

Jauch and Hill [15] showed that degeneracy in Quantum Mechanics is related to groups of transformations which leave the Hamiltonian invariant. Holas and March demonstrates that oscillator (time-independent case), Coulomb (Kepler problem) and constant (free particle) potentials are special cases of the class of problems with rotationally invariant time-independent potentials in the sense that their degeneracy algebras are all larger than $o(3)$. They are, respectively, $su(3)$, $o(4)$ and $su(3)$.

The classical results for the paradigms, listed in the previous chapter, carry over

²Note that a constant of motion is *not* necessarily a first integral.

³It is often apparent in the literature that the concept of the geometric nature of a transformation was not well understood. Some authors considered a transformation to be geometric as long as it did not contain velocity (and higher derivatives of position) and yet it has been shown that velocity dependent transformations yield well known conserved quantities [29]. However, care needs to be taken with this idea since, for example, the LRL vector which is commonly regarded as the cause of a dynamical symmetry [50] is equally associable with the point symmetries $\partial/\partial t$ and $t\partial/\partial t + 2/3 r \partial/\partial r$ [26]. The point is that in some cases point and dynamical symmetries provide us with the same conserved quantities.

to Quantum Mechanics quite naturally and with no surprises. The transcription to Quantum Mechanics requires replacing first integrals by their operator counterparts and the Poisson Bracket by the commutator.

For example, the conserved quantity

$$I_{mn} = \frac{1}{2} \left[\rho^{-2} q_m q_n + (\rho p_m - \dot{\rho} q_m) (\rho p_n - \dot{\rho} q_n) \right]$$

of Günther and Leach [11] becomes a tensor operator with the same form. The function ρ is the same as the classical function. I_{mn} has the same algebraic properties as the classical invariant.

In Classical Mechanics the noninvariance group transforms orbits with one energy into orbits with different energies while invariance groups transform orbits into other orbits of similar energy. On the other hand, in Quantum Mechanics, the concept of invariance groups is meaningful in the sense that it describes those operators which can be measured simultaneously with the energy.

As a final note we mention the importance of group theory in the classification of elementary particles and the quantisation of Quantum Mechanical systems. A brief survey of these applications would not be informative given the vast amount of published work on the matter.

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