MATRICES OF GRAPHS AND DESIGNS WITH EMPHASIS ON THEIR INTEGRAL EIGEN-PAIR BALANCE CHARACTERISTIC

by

CAROL LYNNE JESSOP

Submitted in fulfilment of the academic
requirements for the degree of
Master of Science
in the
School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal

Durban
March 2014
Final version

As the candidate's supervisor, I have approved this dissertation for submission.

Signed: ____________________ Name: ____________________ Date: ________________

ABSTRACT

The interplay between graphs and designs is well researched. In this dissertation we connect designs and graphs entirely through their associated matrices – the *incidence matrix* for designs and the *adjacency matrix* for graphs. The *properties* of graphs are immediately adopted by their associated designs, and the *linear algebra* of the common matrix, will apply to both designs and graphs sharing this matrix.

We apply various techniques of finding the eigenvalues of the matrices associated with graphs/designs, to determine the eigenvalues of well-known classes of graphs, such as complete graphs, complete bipartite graphs, cycles, paths, wheels, stars and hypercubes.

Graphs which are well connected, or *edge-balanced*, in terms of a centrally defined set of vertices, appear to give rise to a *conjugate pair* of eigenvalues.

The association of integers, conjugate pairs and edge-balance with the eigenvalues of graphs provide the motivation for the new concepts of *eigen-sum* and *eigen-product* balanced properties of classes of graphs and designs. We combine these ideas by considering *eigen bi-balanced* classes of graphs, where *robustness* and the *reciprocity* of the eigen-pair a, b allowed for the *ratio* of the eigen-pair sum $a+b$ to the eigen-pair product ab, and the *asymptotic* behaviour of this ratio (in terms of large values of the size of the graph/designs). The product of the average degree of a graph with the Riemann integral of the eigen bi-balanced ratio of the class of graphs is introduced as the *area* of a class of graphs/designs associated with the eigen-pair. We observe that unique area of the class of complete graphs appears to be the largest. Also, the interval of asymptotic convergence of the eigen bi-balanced ratio, of uniquely eigen-bi-balanced classes of graphs, appears to be $[-1,0]$.

We construct a new class of graphs, called *q-cliqued* graphs, involving *q* maximal cliques of size *q*, connected, and hence edge-balanced, to a central vertex. We apply the eigenvector method to find a general *conjugate eigen-pair* associated with the *q*-cliqued graphs and then determine the eigen-pair characteristics above for this class of graphs. The eigen-bi-balanced ratio associated with a conjugate pair of eigenvalues of the class of *q*-cliqued graphs, is the same as the eigen-bi-balanced ratio of the class of the complements of these graphs.

The *q*-cliqued graphs are also designs, and we use the case $q=10$ as an application of a hypothetical entomological experiment involving 10 treatments and 10 blocks. We use the design's graphical characteristics to determine a possible scheduling situation which involves the chromatic number of its associated graph.

PREFACE

The experimental work described in this dissertation was carried out in the School of Mathematics, University of Natal, Durban, from February 2011 to November 2013, under the supervision of Dr Paul August Winter.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

COLLEGE OF AGRICULTURE, ENGINEERING AND SCIENCE

DECLARATION 1 - PLAGIARISM

I, Carol Lynne Jessop, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

3. This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

4. This thesis does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:
   a. Their words have been re-written but the general information attributed to them has been referenced
   b. Where their exact words have been used, then their writing has been placed in italics and inside quotation marks, and referenced.

5. This thesis does not contain text, graphics or tables copied and pasted from the Internet, unless specifically acknowledged, and the source being detailed in the thesis and in the References sections.

Signed: ………………………………………………………… Date:…………………………...

COLLEGE OF AGRICULTURE, ENGINEERING AND SCIENCE

DECLARATION 2 – PUBLICATIONS

Publication 1 (*As a preprint*)

Winter, P.A. and Jessop C.L. Integral Eigen-Pair Balanced Classes of Graphs Ratios, Asymptotes, Density and Areas. Combinatorics and Graph Theory, viXra:1305.0050. (2013)

(50% contribution from each author)

Presentation of chapters 3,4 and 5 to this preprint archive, while presenting same chapters for publication in peer reviewed journal.

Publication 2

Winter, P.A. and Jessop C.L. Integral Eigen-Pair Balanced Classes of Graphs Ratios, Asymptotes, Density and Areas. The Indian Journal of Pure and Applied Mathematics, (2013).

(50% contribution from each author)

We have submitted material based on Chapters 3, 4 and 5 of this dissertation, and the preprint above, to this journal, and await their response.

Signed: ……………………………Name: ……………………..……Date: ………….…………..

Signed: ……………………………Name: ……………………..……Date: ………….…………..

LIST OF CONTENTS

ACKNOWLEDGEMENTS

To my late father, Stephen Oates, for inspiring me to always seek perfection.

To my mother, Sheilah Oates, for teaching me the value of education.

To my husband, Craig Jessop, for giving me this unique opportunity to pursue my love and passion for Mathematics, and for encouraging me to follow my dreams. I have certainly rekindled my love and awe for mathematics through this process!

To my son Julian Jessop, for his pride in his mother and for always trying his best in whatever he embarks on.

To my son Thomas Jessop, for attempting to share in the wonder of the construction and properties of the q-cliqued graphs, and for his wonderful sense of humour.

To my son Robert Jessop, for his inquisitive mind in understanding how things work, and his never-ending perseverance.

Special thanks go to my supervisor, Dr Paul Winter, for his encouragement, guidance and assistance during the course of my research, and during the preparation of my dissertation. It was a great privilege to work with him, after being one of his undergraduate students more than twenty years ago! I enjoyed working with him, learning from his amazing insight into the subject, and sharing his passion for mathematics. I appreciate his patience with me, especially during the initial months when I needed to refresh my understanding of some basic concepts. Thank you very much Paul!

Lastly, thanks to Dr Costas Zachariades, of the Agricultural Research Institute of South Africa, for his contribution to the entomological experiment.

LIST OF SYMBOLS

C	set of complex numbers
Q	set of rational numbers
$\Re$	set of real numbers
Z	set of integers
$Q[x]$	the rings of polynomials over the rationals
$Z[x]$	the rings of polynomials over the integers
U_n	the *n*th roots of unity
$O(n)$	of order n
I	the general identity matrix, consisting of 1's down the diagonal, and 0's elsewhere, where the dimensions are not explicitly specified
$I_{n,n}$	the nxn identity matrix consisting of 1's down the diagonal, and 0's elsewhere
$J_{s,t}$	the sxt matrix containing all 1's
$diag(a_1,...,a_n)$	diagonal matrix of size nxn, with $(a_1,...,a_n)$ down the diagonal, all other entries 0
$\underline{1_n}$	the $nx1$ vector of 1's
$G=(V,E)$	graph G with vertex set V and edge set E
m	number of edges in G
n	number of vertices in G
$t(G)$	number of spanning trees of a connected graph
$deg(v)$	the number of edges incident with v in G
$deg\text{-}(G)$	a listing of the degrees of vertices in the graph G, in descending order
$\in(v)$	eccentricity of vertex v, i.e., the greatest distance between v and any other vertex in G
$\chi(G)$	chromatic number of G
$eccG(u)$	the maximum distance between u and any other vertex of G
$rad(G)$	the radius of G
$diam(G)$	the diameter of G
$shad(G)$	the shadow number of a graph G
$\Delta(G)=\Delta$	the maximum degree of the vertices in G
$\delta(G)=\delta$	the minimum degree of the vertices in G
$\overline{G}$	the complement of G
c	the size of co-clique
$N_G(v)$	the neighbourhood of a vertex $v \in V$

$G \coprod H$	the disjoint union of graphs G and H
$G \oplus H$	the join of G and H
$G \setminus x$	the graph obtained from G by removing vertex x
$G \setminus xy$	the graph obtained from G by removing adjacent vertices x and y.
N_{vb}	incidence matrix of a block design, having v treatments, and b blocks
N_v^{Ω}	incidence matrix of size vxv called the *full-max incidence matrix*
$\sum_{i=1}^{n} x_i$	the sum of all x_i, where $1 \le i \le n$
$\prod_{i=1}^{n} x_i$	the product of all x_i, where $1 \le i \le n$
$tr(A)$	the trace of matrix A
$\det(A)$	the determinant of matrix A
$cof(A)$	the cofactor of matrix A
A^T	the transpose of matrix A
A^{-1}	the inverse of matrix A
f	the number of distinct eigenvalues of a matrix
$A(G)$	the adjacency matrix for graph G
$P_{A(G)}$	the characteristic polynomial of the adjacency matrix of G
λ	an eigenvalue of a matrix
NB(i)	the neighbourhood block of i
BM(G)	the block matrix
Spec(D)	the spectrum of a design D
$G_{K_q}{}^{*}$	q-cliqued graph on (q^2+1) vertices, with $\frac{q(q^2+1)}{2}$ edges
K_n	complete graph on n vertices with $\frac{n(n-1)}{2}$ edges
$K_{s,t}$	complete bipartite graph on $(s+t)$ vertices and st edges
C_n	cycle graph on n vertices, with n edges
P_n	path graph on n vertices, with $(n-1)$ edges
W_n	wheel graph on n vertices, with $(n-1)$ spokes
X_{m+n}	generalised wheel graph on $(m+n)$ vertices, with mn spokes
Y_{m+n}	generalised complete wheel graph on $(m+n)$ vertices, with mn spokes
$S_{1,mP_{k+1}}$	star graph with m rays of length $(k+1)$, on $(km+1)$ vertices, with km edges

G_n	graph on n vertices, obtained from a cycle C_{n-1} on $(n-1)$ vertices, connected to a pendant vertex v by a single edge
H_p	p-regular hypercube, on 2^p vertices and with $p2^{p-1}$ edges
D_n	dumbbell graph on $2n$ vertices and with $(2n+1)$ edges, obtained from two copies of the complete graph on n vertices, where one vertex of the first complete graph is connected to one vertex in the second complete graph by a single edge
$\Im$	class of graphs with certain properties
$r = r(a\Im b)$	eigen-bi-balanced ratio of the class of graphs $\Im$, associated with the eigen-pair (a,b)
$A(\Im)^{a,b}$	eigen-bi-balanced ratio area of the class of graphs $\Im$, with respect to the eigen-pair (a,b)
H	height of the graph
$\Omega_r(\Im)$	eigen-pair density of a class of eigen-bi-balanced graphs with asymptote r
$r(a\Im b)^\infty$ or $asymp(r)$	asymptotic eigen-bi-balanced ratio
C_r^∞ A	r-asymptotic eigen-bi-balanced matrix associated with the adjacency matrix
$E^{C_r^\infty}$	energy of the r-asymptotic eigen-bi-balanced matrix C_r^∞, associated with the graph G
CSET(A)	the set of all eigenvalues of matrix A
DS	determined by spectrum

CHAPTER 1

INTRODUCTION AND DEFINITIONS

1.1 Introduction

The interplay between graphs and designs is well documented (see e.g. Rudvalis [42], Bose [9], and Haemers [28]). In this dissertation we connect designs and graphs entirely through their associated matrices – the incidence matrix for designs and the adjacency matrix for graphs. The properties of graphs are immediately adopted by their associated designs, and the linear algebra of the common matrix then applies to both designs and graphs sharing this matrix.

The purpose of Chapter 1 is to define the important terms that will be required in this thesis. We define some basic terms in graph theory and design theory, and then connect graphs and designs through their matrices. We finally define some basic linear algebra of matrices. Terms not defined in this chapter will be defined in subsequent chapters as they are required.

In Chapter 2, we apply various techniques of finding eigenvalues of a matrix to determine the eigenvalues of well-known classes of graphs, namely graphs with circulant adjacency matrices, complete graphs, cycles, paths, complete bipartite graphs, graphs which are the join of two graphs whose adjacency matrices are both circulant matrices, wheel graphs, star graphs, graphs with a pendant vertex, and hypercubes. In some cases, we use more than one technique to verify the eigenvalues.

In Chapter 3, we define eigen-sum and eigen-product balanced properties, as well as eigen-bi-balance, critically eigen-bi-balanced, and the eigen-bi-balanced ratio associated with classes of graphs. We then investigate the asymptotic behaviour of the eigen-bi-balanced ratio, and define the area and density of classes of graphs/designs. We consider these attributes for the common classes of graphs as in Chapter 2.

In Chapter 4, we define the construction of the class of q-cliqued graphs, and prove that these graphs are design graphs. We determine various characteristics and ratios for this class of graphs, and investigate the associated Laplace matrix. We finally review the linear algebra of the distance matrices of reduced q-clique design graphs.

In Chapter 5, we determine the general form of a conjugate eigen-pair of the *q*-cliqued graphs as defined in Chapter 4. We determine the eigen-bi-balanced ratio and area of the class of *q*-cliqued graphs. We finally determine that the complement of the *q*-cliqued graph is connected, and that the class of the complement of the *q*-cliqued graph has the same eigen-bi-balanced ratio as the class of *q*-cliqued graphs.

In Chapter 6, we apply the 3-colouring of the 3-cliqued design graph associated with the 3-cliqued-block graph, to an entomological experiment which involves the study of the interaction between insects and plants.

In Chapter 7, we present a summary of the findings of this research thesis, and suggest areas for future research.

We now define the important terms that will be used in this thesis:

1.2 Graph theory

1.2.1 Graphs

We shall use the notation of Harris, Hirst and Mossinghoff [30], to define a graph *G*. A *graph* $G=(V,E)$ consists of a finite non-empty set *V* of elements called *vertices*, and a (possibly empty) set *E* of 2-element subsets of *V* called edges. The number *n* of elements in *V* is called the *order* and the number *m* of elements in *E* is called the *size* of *G*. All graphs which we shall consider will be finite, simple and undirected.

If G has only one vertex, then G is trivial; otherwise G is non-trivial.

Let $e=\{u,v\}\in E(G)$. Then we say *u* and *v* are adjacent, while *e* is incident with *u* and *v*. We also say that *e* joins *u* and *v*. Instead of writing $e=\{u,v\}$, we can also write $e=uv$.

The degree of a vertex *v* in *G*, denoted *deg(v),* is the number of edges incident with *v*. By *deg(G)* we mean a listing of the degrees of the graph *G* in descending order. A vertex of degree 1 is called an *end-vertex*. A *k-regular* graph is a graph where each vertex has degree *k*.

The *maximum (minimum) degree* $\Delta(G)=\Delta$ ($\delta(G)=\delta$) of G is the maximum (minimum) of the degrees of the vertices in G.

The *neighbourhood* $N_G(v)$ of a vertex $v\in V$, is the set of all adjacent vertices to *v* in *G*.

$G\backslash x$ is obtained from G by removing vertex x, and $G\backslash xy$ is obtained from graph G by removing adjacent vertices x and y.

1.2.2 Complement, complete, bipartite and sub-graph

The *complement* $\overline{G}$ of a graph G is a graph whose vertex set is the same as G, but if there is no edge between vertices u and v in G, then there is an edge between u and v in $\overline{G}$.

The *complete graph* is a graph in which every pair of vertices in G is adjacent in G.

A *bipartite* graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V; that is, U and V are each co-cliques of G. A *complete bipartite* graph (or *bi-clique*) is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set.

A *sub-graph* of a graph G is a graph whose vertex set is a subset of that of G, and whose adjacency relations are a subset of that of G, restricted to the vertices in this subset.

1.2.3 Walk, trail, path, cycle, circuit, length and tree

A walk W in a graph G is an alternating sequence $W: v_0, e_1, v_1, e_2, ..., v_{r-1}, e_r, v_r$ of vertices and edges (not necessarily distinct) such that $e_i = v_{i-1}v_i \;\; for \;\; i = 1, ..., r.$ Since the vertices that appear in a walk determine the edges in the walk, we can omit the edges in the description of a walk, and denote the walk W by $v_0, v_1, ..., v_{r-1}, v_r$. We say that r is the length of W, and that W begins at v_0 and ends at v_r .

If all the edges of the walk are distinct, then the walk is called a *trail*. If all the vertices of the walk are distinct, then the walk is called a *path*. Therefore every path is a trail, but not every trail is a path.

A *closed path* or *cycle* is a path $v_1, v_2, ..., v_{k-1}, v_k, \;\; for \;\; k \geq 3,$ together with edge $v_k v_1.$ Similarly, a trail that begins and ends at the same vertex is called a *closed trail* or a *circuit*. The *length* of a path (or trail, cycle or circuit) is its number of edges, including any repetitions.

A *tree* is a connected graph, which does not contain a cycle as a sub-graph.

1.2.4 Connectivity

The *distance* between two vertices *u* and *v*, $d(u,v)$, is the length of the shortest $u-v$ path in G.

A graph G is *connected* if every pair of vertices of G is joined by a path and a *component* of G is a maximal connected sub-graph of G. If there is no path connecting the two vertices, i.e., if they belong to different connected components, then conventionally the distance is defined as *infinite*. In this thesis, all graphs are assumed to be connected, unless otherwise stated.

A *cut-vertex* of a connected graph G is a vertex whose removal from V increases the number of components in *G*.

A *block* of a connected graph G is a maximal sub-graph of G that does not have a cut-vertex.

The *eccentricity,* $eccG(u)$, of a vertex *u* of G is the maximum distance between *u* and any other vertex of G.

The *radius,* $rad(G)$, of G is the minimum of all the eccentricities of the vertices of G and the *diamete*r, $diam(G)$, is the maximum of all eccentricities of the vertices of G.

A *central vertex* in a graph of radius r is a vertex whose eccentricity is *r* - that is, a vertex that achieves the radius.

The *centre* of G is the collection of vertices whose eccentricities equal the radius of G , while *periphery* of G is the collection of vertices whose eccentricities equal the diameter of G.

1.2.5 Clique, maximal clique, strong cliques, co-cliques and colouring

A complete sub-graph of a graph G is called a *clique* of G. The order of the largest clique in G is called the *clique number* of G.

A *maximal clique* of a graph G, is a clique of G, which is not a subset of a larger clique of G.

A *strong clique* is a sub-graph of *G* which is a maximal clique and has at least one cut-vertex.

A set of vertices of G which are non-adjacent in G is a *co-clique* of G. The order of the largest *co-clique* of G is called the *co-clique number* of G.

A *proper colouring* of the vertices of a graph G is an assignment of colours to the vertices so that no two adjacent vertices receive the same colour. The least number of colours required to form a proper colouring of a graph is called the *chromatic number* of G and is denoted by $\chi(G)$.

1.2.6 Clique (or chromatic) invariance

A graph on *n* vertices with clique number *d* (or chromatic number *d*) is *clique* (*or chromatic) invariant* if:

$$\frac{n-1}{d} = d.$$

For example, the path on 5 vertices is clique invariant since its clique number is 2 and

$$\frac{n-1}{d} = \frac{5-1}{2} = 2 = d.$$

It is also chromatic invariant since its chromatic number is 2.

Also, the star graph on 5 vertices has clique number and chromatic number 2 and

$$\frac{n-1}{d} = \frac{5-1}{2} = 2 = d,$$

so that it is clique and chromatic invariant.

1.2.7 Co-clique invariance

A graph on *n* vertices with clique number *q* and co-clique number *c* is said to be *co-clique invariant* if:

$$\frac{n+2q}{c} = c.$$

For example, the path on 5 vertices has clique number $q = 2$ and co-clique number $c = 3$ so that

$$\frac{n+2q}{c} = \frac{5+4}{3} = 3.$$

So, P_5 is co-clique invariant – but this graph is not regular nor a design-graph, which will defined later.

The star graph on 5 vertices has clique number 2 and co-clique number 4 so that
$\frac{n+2q}{c} = \frac{5+4}{4} = 2.25$

and S_5 is not co-clique invariant.

1.2.8 Adjacency matrix

The adjacency matrix of a graph G, denoted by $A(G)$, is an *nxn* matrix where the *ij*th entry of $A(G)$ is 1 if vertices v_i v_j are adjacent in G, or 0 otherwise. $A(G)$ is symmetric, and has 0 in each entry in its main diagonal.

1.2.9 Laplace and signless Laplace matrix

Given a graph G with *n* vertices, the *degree matrix* $D(G)$ is an *nxn* diagonal matrix defined as

$$d_{ij} = \begin{cases} \deg(v_i) & if \;\; i = j \\ 0 & otherwise \end{cases}$$

The *Laplace* matrix of G is defined as the nxn matrix $L(G)$, where

$$L(G) = D(G) - A(G),$$

i.e., it is the difference between the degree matrix $D(G)$ and the adjacency matrix $A(G)$ of the graph. From the definitions it follows that:

$$l_{ij} = \begin{cases} \deg(v_i) & if \;\; i = j \\ -1 & if \;\; i \neq j, \;\; and \;\; v_i \;\; adjacent \;\; to \;\; v_j \\ 0 & otherwise \end{cases}$$

where $\deg(v_i)$ is degree of the vertex v_i.

The matrix $A(G) + D(G)$ is called the *signless Laplace* matrix.

1.2.10 Join of graphs

Let G and H be two graphs, not necessarily connected. Then the join of G and H, denoted by $G \oplus H$, is formed when every vertex in G is joined to every vertex in H.

1.2.11 Class of graphs

A class of graphs is a collection of graphs that satisfies a well-defined property. For example, the class of complete graphs is a class of graphs where each graph in the class has the property that each and every pair of vertices is adjacent. We are interested in classes of graphs, whose significant property can be described in terms of their order n.

1.2.12 Strongly regular graphs

A *strongly regular* graph with parameters (v, k, λ, μ) (often denoted by $srg(v, k, \lambda, \mu))$ is a simple graph of order n satisfying:

(i) each vertex is adjacent to k $(1 \leq k \leq n-2)$ vertices;
(ii) for each pair of adjacent vertices there are λ vertices adjacent to both; and
(iii) for each pair of non-adjacent vertices there are μ vertices adjacent to both.

1.3 Design theory

1.3.1 Block designs

We will consider *block designs* D, which can take on one or more of the following descriptions: incomplete, proper, equireplicate, binary and symmetric block designs - see Dey [21].

We let the number of treatments of the design be v, where the treatments are elements of $V = \{v_1, v_2, ...v_v\}$; $v \geq 2$. The number of blocks b_i (subsets of V) is b, with no treatment repeated in any block, and each block has size $k_i \neq v$ (*incomplete*) and each treatment v_i occurs r_i times in the design. The design is *proper* if $k_i = k$ and *equireplicate* if $r_i = r$.

We associate the *incidence matrix* $N = N_{vb}$ with this block design where n_{ij} is the number of times the ith treatment appears in the jth block. A block design is called *binary* if $n_{ij} = 0$ *or* 1. The block design is said to be *symmetric* of size v if $b=v$ so that $r=k$ (see section 1.3.1.2).

1.3.1.1 Full design and trivial design

If we take all possible blocks of maximum size $v-1$ (called the *full design*) we will create the incidence matrix of size vxv called the *full maximum incidence matrix* $N_v{}^{\Omega}$ which is the square vxv binary matrix with 0's down the main diagonal and 1's everywhere else.

The full design, with $v=2$ treatments, is called the trivial design.

We shall omit *v* from the notation of this matrix in its general form. Note that each treatment occurs $r=v-1$ times in the design, and that each block has size $k=v-1$. This incidence matrix is symmetric.

The incidence matrix N^{Ω} of the full block design D $(v \geq 3)$ is the same as the adjacency matrix A of the complete graph G on v vertices. Any two blocks of the full design intersect in $\lambda = v-2$ treatments, this is equivalent to given any pair of vertices of $G,$ both these vertices are joined to λ vertices. This is the *balance factor* of the complete graph.

1.3.1.2 Symmetric designs

If the design is symmetric it does not necessarily mean that N is symmetric:

$$N = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

is a non-symmetric incidence matrix of a design on 5 treatments, with 5 blocks, each treatment occurring twice in the design and each block having size 2 so that $r=k=2$.

The transpose of N above is an incidence matrix of the *dual* of the original design. If N and N^T are the same, then the design is *self-dual*. The full matrix is self-dual.

In Koolen and Moultron [35], the incidence graph of a design is defined as the graph with $(v+b)x(v+b)$ symmetric adjacency matrix

$$\begin{bmatrix} 0 & N \\ N^T & 0 \end{bmatrix}.$$

1.3.2 Strongly symmetric or PIEBS designs

For the purposes of this dissertation we connect the incidence matrix of a self-dual design directly to an adjacency matrix of a graph by introducing the following:

A *strongly symmetric* design will imply:

(1) N is *symmetric*, so that $v = b$ and $N = N^T$:
$n_{ij} = 1$ or 0 respectively $\Leftrightarrow n_{ji} = 1$ or 0 respectively, $i \neq j$
and $v_i \in b_j \Leftrightarrow v_j \in b_i$;

(2) $v_i \notin b_i \forall i$ (zero diagonal condition) and

(3) The number of treatments v and block size k cannot both be odd together. (Note that (1) implies this condition)

If all the above conditions hold, the design is said to be a *PIEBS* (proper, incomplete, equireplicate, binary and symmetric) design, or simply *strongly symmetric* design.

There is a great interest in Intra-Block Analysis to experimental situations - see Dey [21].

1.3.3 The Laplace and signless Laplace matrix of a strongly symmetric or PIEBS block design

The Laplace matrix of a strongly symmetric or PIEBS block design is defined as:

$$L_{i,j} = \begin{cases} r \quad for \;\; i = j \\ -1 \;\; if \;\; v_i \;\; is \;\; in \;\; b_j \\ 0 \;\; otherwise \end{cases}$$

$1 \leq i \leq v$ and $1 \leq j \leq b \;\; (v = b)$

i.e., $L = D - N$ where D is the diagonal matrix with diagonal entry *i* being the sum of the *i*th row of A.

The signless Laplace matrix is

$$Q_{i,j} = \begin{cases} r \quad for \;\; i = j \\ 1 \;\; if \;\; v_i \;\; is \;\; in \;\; b_j \\ 0 \;\; otherwise \end{cases}$$

for $1 \leq i \leq v$ and $1 \leq j \leq b$.

i.e., $Q = D + N$.

1.3.4 C-matrix associated with PIEBS designs

In this section, we use standard matrix definitions and notation – refer to Anton [3], or any other standard linear algebra text book.

Using our PIEBS designs, the following linear model is defined

Definition 1.3.4.1

$$Y_{iju} = \mu + \tau_i + \beta_j + \varepsilon_{ij}$$

where

- $Y_{ij\mu}$ is the observable random variable corresponding to the *u*th observation in the *(i,j)*th cell defined by the *i*th treatment and the *j*th block;
- μ is a general mean;
- τ_i is the effect of the *i*th treatment;
- β_j is the effect on the *j*th block; and
- ε_{ij} is the random error component, assumed to be mutually uncorrelated, with zero means and constant finite variance σ^2.

See Dey [21].

If $n_{dij} = 0$ for some pair (i, j), then there is no observation in that cell.

If we let *n*= *kv*, (noting that *v*=*b*), then D_1 (respectively, D_2) denotes the *vxn* (respectively *bxn*) treatments (respectively, blocks) *versus* the observational incidence matrix, i.e., the (α, β)th element of D_1 (respectively D_2) is 1 if the β th observation comes from the α th treatment (respectively, α th block), and is zero otherwise.

We rewrite Definition 1.3.4.1 as

$$Y = \mu 1_n + D_1^T \tau + D_2^T \beta + \varepsilon$$

where A^T is the transpose of *A* and 1_n is the $nx1$ vector of 1's.

Note that:

$1_n^T Y$ is the general total of observations.

$B = D_2 Y$ is the vector block totals.

$T = D_1 Y$ is the vector treatment totals.

It can be verified that:

$D_1 D_1^T = R = diag(r, r, ..., r)$ the diagonal matrix of replication numbers (r=k for our designs)

$D_2 D_2^T = K = diag(k, k, ..., k)$ the diagonal matrix of block sizes.

Thus $R = K$ and $K^T = K$.

Also, $D_1 D^T{}_2 = N,$ which is the symmetric incidence matrix of the design, i.e., $N = N^T$.

We define $C = K - NK^{-1}N,$ which is referred to as the *'C-matrix'* of the design. The C-matrix is of fundamental importance in the analysis of block designs, and is also referred to as the *information* matrix. The row sums are zero for each row in the C-matrix. Note that $C\tau = T - NK^{-1}B,$ which is referred to as the vector of *adjusted treatment totals*. Refer to Dey [21] for more detail on the C-matrix.

The definition of the C-matrix is relevant in this thesis, as we determine the eigenvalues of the C-matrix in Section 2.12.

1.3.5 Balanced PIEBS designs

If we introduce balance into a PIEBS block design, i.e., each pair of distinct treatments occurs in exactly λ blocks, then we have a *symmetric balanced incomplete block (BIB)* design, denoted by $\lambda - (v, b, k).$ This is equivalent to any 2 blocks intersecting in exactly λ treatments - see Dey [21]. The value λ is referred to as the *balance factor* of the design.

We can describe balance in the PIEBS design as follows:

$$vr = bk; \ \ \lambda(v-1) = r(k-1)$$

Since $v = b, r = k$ and $\lambda = \dfrac{k(k-1)}{v-1}$, both *v* and *k* cannot be odd. Also, since N must be symmetric, with 0's down the main diagonal, this imposes strong constraints on the design.

For example, does a $2-(7,7,4)$ strongly symmetric design exist? This question can be answered by considering regular graphs as in Bose and Shrikhande [10].

For this dissertation, we shall focus on PIEBS designs, with the balanced designs the full designs (see section 1.3.1.1) so that the designs become *graphic designs*. This type of design lends itself to the ideas of *adjacency* and *connectivity*.

Since the designs will have an associated symmetric incidence matrix, $i \in b_j \Leftrightarrow j \in b_i$, $i \neq j$, b_i and b_j are adjacent in the design via the edge $e_{i,j}$, $i \neq j$, when the *ij*th entry of the incidence matrix is 1, otherwise they are non-adjacent.

The set of edges in a design is denoted by E. A (b_1, b_k)-*walk* in a design is a sequence of (not necessarily distinct) blocks: $b_1, b_2, \ldots, b_k$ such that edges $e_{i,i+1}$ exist for $i = 1,2,\ldots,k-1$, b_1 and b_k are the *end vertices* of the walk. If the blocks are all distinct then the walk is called a (b_1, b_k)-*path*. If the edges of the walk are distinct the path is called a (b_1, b_k)-*trail*. The length of the walk, path or trail is the number of edges, counting repetitions.

Two blocks b_i and b_j are *(block) connected in the design* if there is a (b_i, b_j)-path in the design, and the length of its *shortest* (b_i, b_j)-path is the *distance* between the two blocks.

Once we have the idea of adjacency and that of distance between blocks, we can apply the graph-theoretical definitions of the degree of a block and the completeness, sub-design, radius, diameter, girth, chromatic number, tightness of the first and second type of strongly symmetric designs.

A complete sub-design of a strongly symmetric design is called a *clique* of the design and the clique number is the size of the largest clique in the design, and a maximal proper sub-collection of *q* blocks in a connected design whose elements are pairwise non-adjacent is called a *q-co-clique (q-independent vertex set)* of the design and *q* the co-clique number.

Note that if *N* is the incidence matrix of a PIEBS design with each block having size k and balance factor λ then:

$$NN^T = N^2 = \begin{bmatrix} k & \lambda & \lambda & \cdots & \lambda \\ \lambda & k & \lambda & \cdots & \lambda \\ \lambda & \lambda & k & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \vdots & \lambda & \cdots & k \end{bmatrix}$$

λ is also the number of different walks of length 2 between blocks b_i and b_j.

This matrix is the same as the product of the incidence matrix and its transpose mentioned in Bruck and Ryser [13] and Chowla and Ryser [14], with $\lambda = 1$. In this paper, if π is a finite projective plane, then there exists a positive integer Q such that each line of π contains exactly $Q+1$ distinct points, and each point of π is incident with exactly $Q+1$ distinct lines. Moreover π has exactly Q^2+Q+1 distinct points and Q^2+Q+1 distinct lines. *N* is the incidence matrix of a finite projective plane defined in this paper. This projective plane has at least 3 point on a line, the *i*th point on the *j*th line implying the *j*th point is on the *i*th line and the *i*th point is not incident with the *i*th line. The product of *N* with its transpose N^T yields the same result as the incidence matrix of our PIEBS design with $Q+1=k$, $Q^2+Q+1=v=b$, and $\lambda=1$. Thus the existence or non-existence of such projective planes would imply the existence or non-existence of our PIEBS design.

Also the *C* matrix is

$$
\begin{aligned}
C \quad &= K - NK^{-1}N^T \\
&= K - N^2K^{-1} \\
&= \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k \end{bmatrix} - \begin{bmatrix} k & \lambda & \lambda & \cdots & \lambda \\ \lambda & k & \lambda & \cdots & \lambda \\ \lambda & \lambda & k & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \vdots & \lambda & \cdots & k \end{bmatrix} \begin{bmatrix} \frac{1}{k} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{k} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{k} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{k} \end{bmatrix}
\end{aligned}
$$

$$
= \begin{bmatrix} k & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k \end{bmatrix} - \begin{bmatrix} 1 & \frac{\lambda}{k} & \frac{\lambda}{k} & \cdots & \frac{\lambda}{k} \\ \frac{\lambda}{k} & 1 & \frac{\lambda}{k} & \cdots & \frac{\lambda}{k} \\ \frac{\lambda}{k} & \frac{\lambda}{k} & 1 & \cdots & \frac{\lambda}{k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda}{k} & \frac{\lambda}{k} & \frac{\lambda}{k} & \cdots & 1 \end{bmatrix}
$$

$$= \begin{bmatrix} k-1 & \frac{-\lambda}{k} & \frac{-\lambda}{k} & \cdots & \frac{-\lambda}{k} \\ \frac{-\lambda}{k} & k-1 & \frac{-\lambda}{k} & \cdots & \frac{-\lambda}{k} \\ \frac{-\lambda}{k} & \frac{-\lambda}{k} & k-1 & \cdots & \frac{-\lambda}{k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-\lambda}{k} & \frac{-\lambda}{k} & \frac{-\lambda}{k} & \cdots & k-1 \end{bmatrix}$$

So, each row sum of C is, with $\lambda = \dfrac{k(k-1)}{v-1}$,

$$(k-1) + (v-1)\left(\frac{-\lambda}{k}\right) = (k-1) + \left(\frac{(v-1)}{k}\right)\left(\frac{-k(k-1)}{v-1}\right) = 0.$$

1.3.6 K-lantern property of designs

Since the blocks of the design are *distinct*, no two blocks can be adjacent to the same k-collection of blocks. This is equivalent to excluding the case where 2 rows (columns) of the incidence matrix of the design are identical, which we shall relate to a structure which we shall refer to as a k-lantern structure (see definition of a q-lantern sub-graph below).

1.4 Graphs and designs connected through matrices

1.4.1 Lantern graph

A k-lantern (denoted by $S(G)$), is a sub-graph of a k-regular graph, where $S(G)$ is a tri-partite graph with 3 disjoint sets of vertices – u (singleton), S (set of k vertices) and v (singleton) – u and v are non-adjacent and each is adjacent to the k vertices in S. The vertices in S can be adjacent in G.

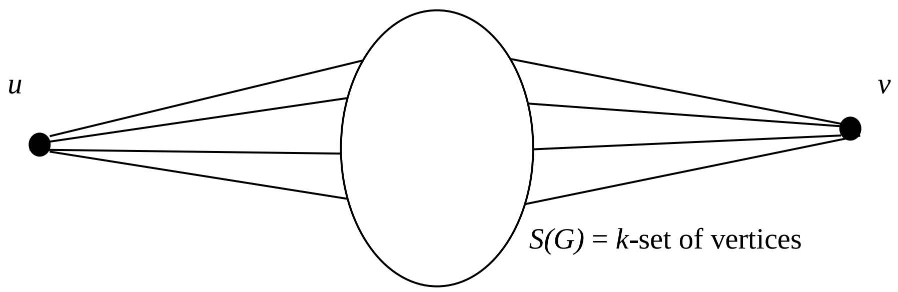

Figure 1.4.1.1: K-lantern sub-graph $S(G)$

If a regular graph G does not have a q-lantern sub-graph, then no two rows (or columns) of its adjacency matrix will be identical. The vertices u and v are also called twin vertices (see Kotlov and Lov'asz [36]). This property is necessary for the matrix to be that of a design (see Theorem 1.4.2.1).

There is a need to construct designs (see Dey [21]) and in this dissertation we use graphs to construct a large class of designs via their adjacency matrices with the advantage that any property of the graphs can be inherited by the associated design – especially the linear algebra of their associated matrices.

1.4.2 Graphic designs and design graphs

We want the incidence matrix of a PIEBS block design on v treatments, each block containing k treatments, to be identical to the adjacency matrix of a graph G – such designs are called *graphic designs* and the associated graph G of the adjacency matrix is called the *design graph*.

We need the condition $v = b$, and the size k of each block is the same as the number of times each treatment occurs in the design.

Using the adjacency matrix, we have a design condition D1:

D1. Each treatment i cannot occur in block $i = b_i$ (0's down main diagonal in both matrices).

Using the design, each row has k entries, thus a graph restriction is:

G1. The graph G must be regular of degree k.

Thus from the adjacency matrix of G:

D2. $v.k = 2m$ where the right hand side is twice the number of edges of G so that D2 implies both n and k cannot be odd.

The incidence matrix is symmetric, and D1 shows that $n.k$ must be even, so that D1 implies D2. Since the blocks of the design must be distinct, we have another graph condition which is necessary for constructing the blocks from the graph's vertex adjacency relationships (see below):

G2. No two non-adjacent vertices of G can have the same neighbour set, i.e., G must not have a k-lantern sub-graph.

These are the conditions we will need to establish a one-to-one correspondence between the adjacency matrix of a *k*-regular graph G on *n* vertices, with the incidence matrix of a PIEBS design, with $v = n$ treatments, and each block containing *k* treatments.

The only graphic designs on 2 or 3 treatments are the full designs. On 4 treatments we have only the trivial and the full design - see argument below and on 5 treatments, we have the full design and the design associated with the 5–cycle (see arguments after Theorem 1.4.2.1).

Once we have the incidence matrix of a PIEBS design we have the adjacency matrix of some *k*-regular graph.

How do we go from a graph to a design?

The paper on graphs and polarities of designs by Rudvalis [42], looks at a one-to-one correspondence between graphs and incomplete designs. We shall show how we can associate a regular graph with a PIEBS design through matrices.

Let G be a regular graph of degree *k* on *v* vertices which does not contain a *k*-lantern sub-graph. We label the vertices $1,2,...,v.$ We associate with each vertex *i* the *neighbourhood block* of *i*, denoted by $NB(i)$, which is the set of labels $\{i_{1,}i_2,...,i_k\}$ which are the labels of the vertices *adjacent* to vertex *i*, i.e., the neighbourhood of *i*. Obviously, no two adjacent vertices will have the same neighbourhood, and we insist that two non-adjacent vertices do not have the same neighbourhood so that all the $NB(i)$ are distinct, i.e., G does not have a *k*-lantern sub-graph. This is called the *exclude-k-lantern condition:*

$$NB(i) \neq NB(j),\ \ i \neq j\ .$$

Clearly $i \notin NB(i)$ and furthermore:

$$j \in NB(i) \Leftrightarrow i \in NB(j)$$

which is called the *adjacency* condition.

The corresponding adjacency matrix of G will be a vxv symmetric matrix with *k* entries in each row and column and 0's down the main diagonal.

Since $v.k$ is twice the number of edges and hence both *v* and *k* cannot be odd together.

We now create the *block matrix* $BM(G)$ of G as follows: The labels $1,2,\ldots,v$ are the rows (treatments) and the columns will be (blocks) $NB(1), NB(2),\ldots, NB(v)$. We insert a 1 in entry *i,j* if $j \in NB(i) \Leftrightarrow i \in NB(j)$, 0 otherwise. Since $i \notin NB(i)$ and the adjacency condition holds, along with the exclude-*k*-lantern condition (no two columns are the same), $BM(G)$ is a symmetric matrix with 0's down the main diagonal.

$BM(G)$ is therefore an incidence matrix of some strongly symmetric design D with treatments $1,2,3,\ldots,v$ and blocks $NB(i)$ with the condition $i \notin NB(i)$ for all i (similar to the absolute polarity condition of Hubaut [32]).
Such a design associated through such a graph is called a *graphic design.*
Note that the graph we use to construct the design is not the same as the concurrence graph constructed in Bailey and Cameron [7].

If a design with treatments *i* and blocks b_i has its incidence matrix satisfying $i \notin b_i$ for all *i*, we say the design is *non-absolute*. Thus:

Theorem 1.4.2.1

A connected *k*-regular graph *G* has an associated graphic design *D* (strongly symmetric) iff *G* has no *k*-lantern sub-graph.

Given a PIEBS design with treatments $i = 1,2,\ldots,v$ and blocks b_i of size k (with the conditions $i \notin b_i \;\; for \;\; all \;\; i,$ and $j \in b_i \Leftrightarrow i \in b_j$ and *v* and *k* not both being odd together), there is a regular graph (the *design graph*) associated with the design where the treatments are the vertices and in which the *i*th block b_i associated with this vertex (*i*=vertex) denotes the adjacency to the *k* neighbours of *i*; i.e., the adjacency matrix of this block graph is the same as the incidence matrix of the design it was constructed from.

If we apply the results to a design on 4 treatments, with blocks of size 2, it is impossible to construct a symmetric matrix of size 4 with 2 entries in each row and column with 0's down the main diagonal; i.e., no such non-absolute design exists and therefore no graph exists.

The only regular graph of degree 2, on 4 vertices, is isomorphic to the cycle. Note that the adjacency matrix will have columns 2 and 4 being identical - i.e., two vertices 2 and 4 are adjacent to the same neighbour vertices 1 and 3 which will not allow for the formation of a design associated with the graph.

The cycle on 5 vertices labelled 1,2,3,4,5,1 translates to the design with treatments 1,2,3,4 and 5: vertex 1 has neighbours $b_1 = 2,5,$ vertex 2 neighbours $b_2 = 1,3,$

vertex 3 neighbours $b_3 = 2,4,$, vertex 4 neighbours $b_4 = 3,5,$ and vertex 5 has neighbours 1,4= $b_5 = 1,4.$ The incidence matrix of this design is identical to the adjacency matrix of the 5 cycle.

Since one cannot have a 3-regular graph on 5 vertices the only other graphic design is the full design on 5 vertices.

The spectrum of a design *D*, written Spec(*D*), is the spectrum of the associated incidence matrix. If all designs which have the same spectrum as a design *D* are isomorphic, then the design *D* is said to be *determined by its spectrum (DS)* - see Dam and Haemers [20].

Once a graph is determined to be a design graph then all the properties of the graph are inherited by the associated design, such as diameter, radius, and chromatic number.

We can introduce balance into the full design and the associated block graph as follows. Any 2 blocks of the design must intersect in λ treatments which is equivalent to the associated block graph having every pair of vertices being adjacent to the same λ vertices. This must not violate the condition that 2 non-adjacent vertices must not have the same neighbourhood set.

For a graph with general "balance" see Bose [9].

1.4.3 Graphs which cannot be design graphs

We want a *k*-regular sub-graph on $(k+2)$ vertices which has *k*-lantern sub-graphs:

For $k = 2$ we get the cycle on 4 vertices which is a 2-lantern graph.

For $k = 3$ it is not possible, as we have 5 vertices and a 3-regular graph so that 5.3=15, which is not even.

For $k = 4$ we have a 4-regular graph on 6 vertices – add edges to the middle set S on the tri-partition of the 4-lantern graph to get a 4-regular graph with a 4-lantern sub-graph.

For $k = 5$ G is on 7 vertices – not possible.

So we can construct a *k*-regular graph on $(k+2)$ vertices, where *k* is even, as follows:

Let v and w be a pair of vertices, and connect each of v and w to a set S consisting of *k* disconnected vertices. Then *v* and *w* will have degree *k* and all the vertices in S will have degree 2. Add $\frac{k(k-2)}{2}$ edges between the vertices in S, so that each vertex in S has degree k. The resulting graph is *k*-regular, and has a *k*-lantern sub-graph, so cannot be a design graph.

The complete bipartite graph, on $n+n=2n$ vertices, is not a design graph as it contains an *n*-lantern sub-graph.

1.5 Linear algebra of matrices

1.5.1 Inverse of N^Ω

We now determine the inverse of N^Ω, entirely in terms of N^Ω and $I_{v,v}$. We use this inverse to create a quadratic equation, for which N^Ω is a solution. This quadratic equation has significance in terms of finding the eigenvalues of N^Ω.

Theorem 1.5.1.1

The inverse of N^Ω is

$$(N^\Omega)^{-1} = \frac{1}{r}N^\Omega - \frac{r-1}{r}I_{v,v}, \text{ where } r = v-1 \text{ and } v \geq 2.$$

Proof

$$(N^\Omega)^2 - (r-1)N^\Omega$$

$$= \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix} - (r-1) \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} v-1 & v-2 & v-2 & \cdots & v-2 \\ v-2 & v-1 & v-2 & \cdots & v-2 \\ v-2 & v-2 & v-1 & \cdots & v-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v-2 & v-2 & v-2 & \cdots & v-1 \end{bmatrix} - (v-2) \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} v-1 & 0 & 0 & \cdots & 0 \\ 0 & v-1 & 0 & \cdots & 0 \\ 0 & 0 & v-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v-1 \end{bmatrix}$$

$$= rI_{v,v}$$

So $(N^{\Omega})^2 - (r-1)N^{\Omega} = rI_{v,v}$

$$\Rightarrow N^{\Omega}\left(N^{\Omega} - (r-1)I_{v,v}\right) = rI_{v,v}$$

$$\Rightarrow N^{\Omega}\left(\frac{1}{r}N^{\Omega} - \frac{(r-1)}{r}I_{v,v}\right) = I_{v,v}$$

$$\Rightarrow \left(N^{\Omega}\right)^{-1} = \frac{1}{r}N^{\Omega} - \frac{(r-1)}{r}I_{v,v}$$ □

Corollary 1.5.1

From the above theorem, we determine the quadratic:

$$N^{\Omega}(N^{\Omega})^{-1} = N^{\Omega}(\frac{1}{r}N^{\Omega} - \frac{r-1}{r}I_{v,v}) = I_{v,v}$$

$$\Rightarrow \quad (N^{\Omega})^2 - (r-1)N^{\Omega} - rI_{v,v} = 0$$

where $r = v - 1 = k$.

1.5.2 Minor, cofactor, cofactor of A, orthogonal, orthonormal and diagonal matrices

The *minor* of the entry in the *i*-th row and *j*-th column (also called the *(i,j) minor*, or a *first minor*) is the determinant of the sub-matrix formed by deleting the *i*-th row and *j*-th column of *A*. This number is often denoted $M_{i,j}$. The *(i,j) cofactor* is obtained by multiplying the minor by $(-1)^{i+j}$.

The *cofactor* of *A*, denoted by $cof(A)$, is the sum of the cofactors of each entry of *A*.

An *orthogonal* matrix *A* is a symmetric matrix, where $A^{-1} = A^T$, where the columns are orthogonal, and have unit length.

A *diagonal* matrix *A* is a square matrix, such that if there are any non-zero entries, they will only occur on the main diagonal, and is denoted by $diag(a_{1,1}, a_{2,2,}, ..., a_{n,n})$.

1.5.3 Characteristic polynomial

The *characteristic polynomial* of matrix $A(G)$ is denoted by $P_{A(G)}(\lambda)$, and is expressed in terms of λ.

$$P_{A(G)}(\lambda) = \det(\lambda I - A)$$

where A is the adjacency matrix of graph *G* and $\det(\lambda I - A)$ is the determinant of $(\lambda I - A)$.

1.5.4 Eigenvalues, eigenvectors and conjugate pairs

An eigenvector of a square matrix *A*, associated with a graph and/or design *G*, is a non-zero vector $\underline{v}$ that, when the matrix is multiplied by $\underline{v}$, yields a constant multiple of $\underline{v}$, the multiplier being commonly denoted by λ. That is:

$$A\underline{v} = \lambda \underline{v}$$

The number λ is called the *eigenvalue* of *A* corresponding to the *eigenvector* $\underline{v}$. Let CSET(*A*) denote a complete set of eigenvectors of matrix *A* which contains n independent eigenvectors of *A*. If *Q* is the matrix, whose columns are the orthonormal eigenvectors of *A*, then $Q^{-1}AQ = diag(\lambda_1, \lambda_2, ..., \lambda_n)$ where $\lambda_1, \lambda_2, ..., \lambda_n$ are the (not necessarily distinct) eigenvalues of *A*.

The eigenvalues are therefore the solution to the characteristic polynomial $P_{A(G)}(\lambda) = \det(\lambda I - A) = 0$. The Cayley Hamilton theorem states that the matrix *A* is also a solution of this characteristic equation - See Brouwer and Haemers [12].

If *G* is *k*-regular then the largest eigenvalue of the adjacency matrix is *k*, and if *d* is the diameter of any graph *G*, then the number of distinct eigenvalues is at least $(d+1)$, with equality for a certain class of graphs - see Cohen, Brouwer and Neumaier [15]. The chromatic number, $\chi(G) \le \lambda + 1$, where λ is the largest eigenvalue of the adjacency matrix of *G* – see Brouwer and Haemers [12].

It is a well-known result that the complete graph on *n* vertices has eigenvalues *n-1* and -1 - see Brouwer and Haemers [12]. The *spectral radius* of a graph *G* is the largest of all the absolute values of the eigenvalues of the adjacency matrix associated with the graph, and therefore the spectral radius of the complete graph on *n* vertices is $(n-1)$. Since the complete graph is $(n-1)$-regular, its largest eigenvalue is $(n-1)$, and hence this value is the largest of all spectral radii associated with regular graphs. In general, the spectral radius is at most $n-1$, where $n-1$ is the maximum degree of a vertex of a graph on *n* vertices.

There is also interest in the *second largest* eigenvalue of a *k*-regular graph - see Lubotzky, Phillips and Sarnak [39].

There is a relationship between the largest eigenvalue and the clique number *q* of a graph as in Nikiforov [40]:

$$\lambda^2 \le \frac{2m(q-1)}{q}.$$

A *conjugate eigenpair* is a pair of eigenvalues which are of the form $\lambda = a \pm \sqrt{b}$ where $a, b \in \Re,\ b > 0$. The λ stands for the pair of eigenvalues λ_1 and λ_2.

Regarding the conjugate pairs as roots of a quadratic equation, it is a well-known result, that if the roots of the quadratic equation $x^2 + cx + d = 0$ are *e* and *f*, then the *sum* $(e+f) = -c$ and *product* $(ef) = d$.

1.5.5 Spectrum of A

We list the eigenvalues of a matrix (the *spectrum*) in descending order, and their multiplicity as follows: $(\lambda_1)^{m_1}, (\lambda_2)^{m_2}, \ldots$

The spectrum of a graph *G*, written Spec(*G*), is the spectrum of the associated adjacency matrix. If all graphs which have the same spectrum as a graph *G* are isomorphic, then the graph *G* is said to be *determined by its spectrum (DS* for short*)* – see Dam and Haemers [20].

For example paths, cycles, complete graphs and complete bipartite graphs on *2n* vertices and their complements are all *DS*.

1.5.6 Tightness of the first and second type of *G*

If G has diameter d and maximum vertex degree Δ, largest eigenvalue λ_1 and f distinct eigenvalues, then:

- The tightness of G of the first type is $f\Delta$; and
- The tightness of G of the second type is $(d+1)\lambda_1$.

In the recent literature it was suggested that graphs with a small tightness of the first type are good models for certain networks - see Cvetkovi and Davidsovi [18].

"Tightness" in the general sense shall refer to the strength of "connectivity" of graphs such as its *robustness* (see definition below).

1.5.7 Eigen-co-cliqued ratio

The diameter of a graph is used as a bound on the number of distinct eigenvalues of matrices associated with graphs (see Brouwer and Haemers [12]) while the number of end-points of a tree are used as a bound for the multiplicity of eigenvalues of a tree (see Brouwer [11]). In Dam [21], the Delsarte (Hoffman) ratio (bound) is considered involving a *k*-regular graph's co-clique number, the order of the graph, an eigenvalue and *k*. We consider the multiplicities of eigenvalues and introduce our own ratio definition involving the co-clique of a graph and obtain equality for some graphs.

The *eigen-co-cliqued ratio* of the multiplicity of an eigenvalue λ of classes of the matrices associated with the graphs (designs) with co-clique number c is $\dfrac{n+\lambda}{c}$.
This ratio is *strict* if we have equality with the multiplicity of the eigenvalue, otherwise *non-strict.*

For example, the complete bipartite graph on *2n* vertices has co-clique number *n* and eigenvalue –*n* (of multiplicity 1). Therefore the eigen-co-clique ratio is $\dfrac{2n-n}{n}=1$, which is the same as its multiplicity, so that its ratio is strict.

The 5-cycle (which has an associated design) has eigenvalue 2 of multiplicity 1 and co-clique number 2. The eigen-co-clique ratio is $\dfrac{5-2}{2}=1.5$ which does not equal the multiplicity of the eigenvalue 2. Therefore the eigen-co-clique ratio for the eigenvalue 2 is non-strict, but is greater than its multiplicity.

1.5.8 Circulant matrices

For any given $a_0, a_1, ..., a_{n-1} \in C$, the circulant matrix $C = (a_{i,j})_{nxn}$ is defined by $a_{i,j} = a_{j-i(\mathrm{mod}\, n)}$ for $1 \le i \le n$ and $1 \le j \le n$.

$$C = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$$

If C is the adjacency matrix of an associated graph, then C is symmetric, so that

$$a_{n-1} = a_1;\ a_{n-2} = a_2;\ \ldots;\ a_1 = a_{n-1}.$$

Other linear algebra definitions can be found in Brouwer and Haemers [12].

1.5.9 Robustness or tightness of networks and the reciprocal of eigenvalues

Denoting the eigenvalues of the Laplace matrix associated with a graph G by $\theta_1 \le \theta_2 \le ... \le \theta_n$, we note that G is connected if and only if $\theta_1 = 0 \ne \theta_2$. Thus a connected graph G always has a Laplace eigenvalue equal to 0, and $\theta_2 > 0$.

There are many complex networks in large-scale engineering, biological, and social systems. The second smallest eigenvalue of the Laplace matrix $\theta_2 > 0$ is called the *algebraic connectivity* of a graph, and is a measure of speed of solving consensus problems in networks – see Olfati-Saber [41].

We have the inequality $\theta_2 \le \upsilon(G) \le \eta(G)$, where $\upsilon(G)$ denotes the vertex connectivity (the minimum number of vertices required to be removed to disconnect graph *G*) and $\eta(G)$ denotes the edge connectivity (the minimum number of edges required to be removed to disconnect *G*). According to this inequality, a network with a relatively high algebraic connectivity is necessarily robust to both vertex (node) failures and edge failures and θ_2 is a lower bound on this degree of robustness.

Assuming that the time delay in all links in a network (represented by a graph G) is equal to τ, it can be shown that $\tau \le \dfrac{\pi}{2\theta_n}$, so that the higher θ_n is, the smaller the time delay. The the reciprocal of the eigenvalue θ_n is a measure of *robustness to delay* for reaching a consensus in a network.

If G is *k*-regular then $\theta_n = (k - \lambda)$, where λ is an eigenvalue of the adjacency matrix of *G*. Thus measure of robustness to delay is associated with the reciprocal of the difference $(k - \lambda)$. For example, the eigenvalues of the adjacency matrix of the complete graph G, are -1 and $n-1$. Thus the time delay τ satisfies

$$\tau \leq \frac{\pi}{2[(n-1)-(-1)]} = \frac{\pi}{2n}$$

where -1 is an eigenvalue of the adjacency matrix of G, and $k = n-1$.

For large n we have $n \approx n-1$ so that the *reciprocal* of the largest *eigenvalue* $n-1$ of G affects the time delay, and hence robustness, of the large associated network - see Olfati-Saber [41].

The above ideas and results provide the motivation for the idea of *area* of a graph as defined in Chapter 3.

1.6 Conclusion

This concludes the general definitions required for this thesis. Specific definitions are included in the chapters that follow.

CHAPTER 2

TECHNIQUES FOR FINDING EIGENVALUES

The history of the linear algebra of matrices associated with graphs/designs is a colourful one, and many techniques, both elegant and novel, have evolved for finding the eigenvalues of different types of graphs. In this chapter, we illustrate a few techniques of finding eigenvalues for graphs by applying them to determine the eigenvalues of the following classes of graphs:

- Graphs where their adjacency matrix is a circulant matrix;
- Complete graphs;
- Cycle graphs;
- Path graphs;
- Complete bipartite graphs;
- Eigenvalues of the adjacency matrix associated with a graph which is the join of two graphs whose adjacency matrices are both circulant matrices;
- Wheel graphs;
- Star graphs;
- Graphs with a pendant vertex; and
- Hypercube graphs.

This section is a combination of original work and work referenced from other sources. Where there is no specific reference given, the associated theorems and proofs are original. In many cases, the external source has stated a specific result, and the proof has been developed during this research thesis.

The software packages Bluebit Matrix calculator and Mathematica were used extensively in the research and verification of the characteristic polynomial and the eigenvalues of numerous graphs. These packages were used to verify certain conjectures for specific examples of graphs.

The application of eigenvalues to the real world is vast. One of the examples, is the Hückel method or Hückel molecular orbital method (HMO), proposed by Erich Hückel as far back as 1930. Within HMO theory, the total energy of π-electrons is equal to the sum of the energies of all π-electrons in the considered molecule, and can be calculated from the eigenvalues of the underlying molecular graph. See Adiga, Bayad, Gutman and Srinivas [1].

A benzene molecular ring can be “mapped” onto a graph consisting of a cycle with 6 vertices – the vertices represent the atoms, and the edges the bonds between the atoms. Hückel molecular theory then allows the energy of the ring to be associated with the sum of the absolute value of each the eigenvalues which arise from the adjacency matrix associated with the graph.

We will use various methods of finding the eigenvalues of the graphs, and in some cases, more than one method, to determine the eigenvalues for classes of graphs. However, we first need Lemmas 2.1, 2.2 and 2.3. These lemmas give us insight into the characteristic polynomial of the union of two disjoint graphs, the characteristic polynomial of a graph in terms of removing a vertex, and finally a specific case of the characteristic polynomial of a graph in terms of removing a vertex of degree one.

Lemma 2.1

Let $G \coprod H$ denote the disjoint union of graphs G and H.

Then

$$P_{A(G \coprod H)}(\lambda) = P_{A(G)}(\lambda).P_{A(H)}(\lambda)$$

i.e., the characteristic polynomial of the union of disjoint graphs is the product of the characteristic polynomial of each of the disjoint graphs.

See Averbouch [5].

Proof

Note that $\det\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A).\det(B)$ for any square matrices A and B, not necessarily of the same order. Now the characteristic polynomial of $G \coprod H$ is obtained as follows:

$$\begin{aligned} P_{A(G \coprod H)}(\lambda) &= \det(\lambda I - (G \coprod H)) \\ &= \begin{vmatrix} \lambda I - G & 0 \\ 0 & \lambda I - H \end{vmatrix} \\ &= \det(\lambda I - G).\det(\lambda I - H) \\ &= P_{A(G)}(\lambda).P_{A(H)}(\lambda) \end{aligned}$$

□

Lemma 2.2

Let G be a tree, and let x be a vertex of G. Then the characteristic polynomial of G is obtained by,

$$P_{A(G)}(\lambda) = \lambda P_{A(G\backslash x)}(\lambda) - \sum_{all\ x\ adj\ y} P_{A(G\backslash xy)}(\lambda)$$

for all vertices y of G, where x is adjacent to y in G.

See Brouwer and Haemers [12].

Lemma 2.3

Let x_i be a vertex of degree one in the graph G on n vertices, and let x_j be the vertex adjacent to x_i. Let G_1 be the sub-graph obtained from G by deleting the vertex x_i, and let G_2 be the sub-graph obtained from G by deleting both vertices x_i and x_j. Then

$$P_{A(G)}(\lambda) = \lambda P_{A(G_1)}(\lambda) - P_{A(G_2)}(\lambda)$$

See Bian [8].

Proof

Without loss of generality, let $i<j$, so row i comes before row j in $P_{A(G)}(\lambda) = \det(\lambda I - A(G))$. Then we have,

$$(\lambda I - A(G))_{i,i} = \lambda$$
$$(\lambda I - A(G))_{i,j} = -1$$
$$(\lambda I - A(G))_{i,k} = 0 \text{ for } 1 \le k \le n \text{ and } k \ne i \text{ and } k \ne j$$

Expand the determinant of $(\lambda I - A(G))$ along the ith row, where there are only two non-zero entries as defined above. Then

$$P_{A(G)}(\lambda) = (-1)^{i+i}(\lambda)M_{i,i} \det(\lambda I - A(G_1)) + (-1)^{i+j}(-1)M_{i,j}$$

Now $M_{i,i} = \det(\lambda I - A(G_1)$, so

$$P_{A(G)}(\lambda) = \lambda P_{A(G_1)}(\lambda) + (-1)^{i+j+1} M_{i,j}$$

Now expand $M_{i,j}$ along the *i*th column, which only has one non-zero entry of -1 in the *(j-1)*th row as x_i has degree one and is only adjacent to x_j. So,

$$\begin{aligned}P_{A(G)}(\lambda) &= \lambda P_{A(G_1)}(\lambda)+(-1)^{i+j+1}(-1)^{i+j-1}(-1)\det(\lambda I - A(G_2)) \\ &= \lambda P_{A(G_1)}(\lambda)-P_{A(G_2)}(\lambda)\end{aligned}$$

□

2.1 Eigenvalues of graphs having circulant adjacency matrices

The following Lemmas 2.1.1 – 2.1.5 are required for the proof of the main theorem on eigenvalues of adjacency matrices of graphs, where the adjacency matrices are of the form of a circulant matrix, as defined in section 1.5.8. Lemma 2.1.1 is a well known result and is merely stated, whereas Lemma 2.1.2 – Lemma 2.1.5 comprise original work.

Lemma 2.1.1

Let $n \in Z$ be an integer such that *n>0.*

Let $z \in C$ be a complex number such that $z^n = 1$.

Then

$U_n = \left\{ e^{\frac{2\pi i j}{n}},\ 0 \le j \le n-1 \right\}$ where U_n is the *n*th roots of unity and $i = \sqrt{-1}$.

That is $\quad z \in \left\{ 1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \ldots, e^{\frac{2(n-1)\pi i}{n}} \right\}$.

Thus for every positive integer *n*, the number of *n*th roots of unity is *n*.

The root $e^{\frac{2\pi i}{n}}$ is known as the first *n*th root of unity.

Lemma 2.1.2

Let W be the (nxn) circulant matrix with the first row (0 1 0 0 ... 0),

$$W = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{nxn}$$

Then W^m is obtained from W by shifting each '1' entry in each row by *(m-1)* steps to the right, for $m \geq 2.$

Proof

$$W^2 \quad = W \times W$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{nxn} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{nxn}$$

$$= \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}_{nxn}$$

For this step, W^2 is found from W by shifting each '1' entry in each row to the right by one step.

Assume the hypothesis it true for $k \leq m,$ i.e., that W^k is obtained from W by shifting each '1' entry in each row to the right by $(k-1)$ steps for $k \leq m.$

Then

$$W^k = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{nxn}$$

Where $a_{1,k+1} = 1,\ a_{2,k+2} = 1,$ etc.

Then $W^{k+1} = W^k W$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{nxn} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{nxn}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}_{nxn}$$

where $a_{1,k+2} = 1,\ a_{2,k+3} = 1,$ etc. This is equivalent to taking W and shifting each '1' entry in each row to the right by k steps.

Therefore, Lemma 2.1.2 is proved by induction. □

Lemma 2.1.3

Let F_n be a (nxn) matrix, for $n \geq 2,$ where

$$F_n = \begin{bmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

Then $\det(F_n) = \lambda^n$.

Proof

For $n = 2$, we have $F_2 = \begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix}$

Then $\det(F_2) = (\lambda)^2 - 0 = \lambda^2$

For $n = k$, $F_k = \begin{bmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{kxk}$

Assume the hypothesis it true for $k \leq n$, i.e., $\det(F_k) = \lambda^k$ for $k \leq n.$

Now F_{k+1} is the matrix as follows:

$$F_{k+1} = \begin{bmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{(k+1)x(k+1)}$$

Then $\det(F_{k+1}) = \det\begin{bmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{(k+1)x(k+1)}$

$$= \lambda \det(F_k) - (-1)\det\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{kxk}$$

$$= \lambda\lambda^k + 1.0$$
$$= \lambda^{k+1}$$

We have therefore proved by induction that $\det(F_n) = \lambda^n$ for all $n \geq 2$. □

Lemma 2.1.4

Let R_n be a (*nxn*) matrix, for $n \geq 2$, where

$$R_n = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{nxn}$$

Then $\det(R_n) = -1$.

Proof

For $n = 2$, $R_n = \begin{bmatrix} 0 & -1 \\ -1 & \lambda \end{bmatrix}$

Then $\det(R_n) = 0 - (-1)^2$
$= -1$

For $n = k$, $R_k = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{kxk}$

Assume the hypothesis it true for $k \le n$, i.e., $\det(R_k) = -1$ for $k \le n.$

Now R_{k+1} is the matrix as follows:

$$R_{k+1} = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{(k+1)x(k+1)}$$

Then $\det(R_{k+1}) = \det \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{(k+1)x(k+1)}$

Then expanding along the first row, with all zeroes other than the second column, we get

$$\begin{aligned} \det(R_{k+1}) &= (-1)(-1)\det \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{kxk} \\ &= \det(R_k) \\ &= -1 \end{aligned}$$

We have therefore proved by induction that $\det(R_n) = -1$ where $n \ge 2.$ □

Lemma 2.1.5

Let W be the *(nxn)* circulant matrix, for $n \geq 2,$ with the first row (0 1 0 0 ... 0),

$$W = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{nxn}$$

Then the eigenvalues of W are the *n* roots of $\lambda^n = 1$.

Proof

For $n = 2$, $\det(\lambda I - W) = \det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}$

$$= \lambda^2 - 1$$

To find the eigenvalues, we set

$\lambda^2 - 1 = 0$

$\Rightarrow \lambda^2 = 1$

$\Rightarrow \lambda = \pm\sqrt{1}$

To find the eigenvalues of W_k we calculate

$$\det(\lambda I - W_k) = \begin{bmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{kxk}$$

$$= \lambda \det\begin{bmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{kxk} - (-1)\det\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ -1 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{kxk}$$

Then from Lemma 2.1.4 and Lemma 2.1.5

$$\begin{aligned}\det(\lambda I - W_k) &= \lambda \det(F_{k-1}) + \det(R_{k-1}) \\ &= \lambda\lambda^{k-1} - 1 \\ &= \lambda^k - 1\end{aligned}$$

To find the eigenvalues of W_k, we set $\det(\lambda I - W_k) = 0$

So $\lambda^k - 1 = 0$
$\Rightarrow \lambda^k = 1$
$\Rightarrow \lambda = \sqrt[k]{1}$

We have therefore proved that, in the general case, the eigenvalues of W_n are the *n*th roots of $\lambda^n = 1$, for $n \geq 2.$ □

The following theorem is a key result to be used in the determination of the eigenvalues of the adjaceny matrix associated with a number of types of graphs, for example, complete graphs and cycle graphs.

Theorem 2.1.1

Let $A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}_{nxn}$

be a (nxn) circulant matrix, for $n \geq 2.$

Then the eigenvectors of the circulant matrix *A* are given by:
$\underline{v_j} = \left(1, \rho_j, \rho_j{}^2, ..., \rho_j{}^{n-1}\right)^T, \quad j = 0,1,...,n-1$

where $\rho_j = \exp\left(\frac{2\pi i j}{n}\right)$ are the *n*th roots of unity and $i = \sqrt{-1}$ is the imaginary unit.

The corresponding eigenvalues are then given by

$$\lambda_j = a_0 + a_1\rho_j + a_2\rho_j{}^2 + ... + a_{n-1}\rho_j{}^{n-1}, \; j = 0,...,n-1$$

See Gray [27].

Proof

1. Let W be the (nxn) circulant matrix with the first row $(010...0)$ and let $q(W) = a_0 I_{n,n} + a_1 W + a_2 W^2 + ... + a_{n-1} W^{n-1}$ where $a_0, a_1, ..., a_{n-1}$ are the entries in the first row of circulant matrix *A* above.

   Then, using Lemma 2.1.2 above,

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$$

$$= a_0 I_{n,n} + a_1 \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix} + ... +$$

$$a_{n-1} \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= a_0 I_{n,n} + a_1 W + a_2 W^2 + \ ... \ + a_{n-1} W^{n-1}$$
$$= q(W).$$

2. If λ is an eigenvalue of *W*, then λ^n is an eigenvalue of W^n

   If $\quad W\underline{x} = \lambda \underline{x}$ then

$$W^2 \underline{x} = W(W\underline{x}) = W\lambda \underline{x} = \lambda W \underline{x} = \lambda(\lambda \underline{x}) = \lambda^2 \underline{x}$$

   and

$$W^k \underline{x} = W^{k-1}(W\underline{x}) = W^{k-1}\lambda x = \lambda(W^{k-1}\underline{x}) = ... = \lambda^{k-1}(\lambda \underline{x}) = \lambda^k \underline{x}$$

3. If λ is an eigenvalue of W with corresponding eigenvector $\underline{x}$, then

Then

$$\begin{aligned}
A\underline{x} &= q(W)\underline{x} \\
&= \left(a_0 I_{n,n} + a_1 W + a_2 W^2 + \ldots + a_{n-1} W^{n-1}\right)\underline{x} \\
&= a_0\,\underline{x} + a_1 W\,\underline{x} + a_2 W^2\,\underline{x} + \ldots + a_{n-1} W^{n-1}\,\underline{x} \\
&= a_0\,\underline{x} + a_1 \lambda \underline{x} + a_2 \lambda^2\,\underline{x} + \ldots + a_{n-1} \lambda^{n-1}\,\underline{x} \\
&= \left(a_0 + a_1 \lambda + a_2 \lambda^2 + \ldots + a_{n-1} \lambda^{n-1}\right)\underline{x} \\
&= q(\lambda)\underline{x}
\end{aligned}$$

So, $q(\lambda)$ is an eigenvalue of $q(W) = A$.

4. Now $A = q(W) = a_0 I_{n,n} + a_1 W + a_2 W^2 + \ldots + a_{n-1} W^{n-1}$

and $\rho_j = \exp\left(\frac{2\pi i j}{n}\right), j = 0, \ \ldots \ , n-1$ is an eigenvalue of W, by Lemma 2.1.5.

Then from 1, 2 and 3 above,

$$\begin{aligned}
\lambda_j &= q(\rho_j) \\
&= a_0 + a_1 \rho_j + a_2 {\rho_j}^2 + \ldots + a_{n-1} {\rho_j}^{n-1}, \ j = 0,1,\ldots, n-1
\end{aligned}$$

is an eigenvalue of $A = q(W)$.

5. Let $\underline{v_j} = \left(1, \rho_j, {\rho_j}^2, \ldots, {\rho_j}^{n-1}\right)^T, \quad j = 0,1,\ldots, n-1.$ Then for $0 \le j \le n-1,$ we have

$$A\underline{v_j} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}_{nxn} \begin{bmatrix} 1 \\ \rho_j^1 \\ \rho_j^2 \\ \vdots \\ \rho_j^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} a_0 + a_1\rho_j^1 + a_2\rho_j^2 + \ldots + a_{n-1}\rho_j^{n-1} \\ a_{n-1} + a_0\rho_j^1 + a_1\rho_j^2 + \ldots + a_{n-2}\rho_j^{n-1} \\ \vdots \\ a_{n-t} + \ldots + a_0\rho_j^t + a_1\rho_j^{t+1} + \ldots + a_{n-(t+1)}\rho_j^{n-t} \\ \vdots \\ \vdots \\ a_1 + a_2\rho_j^1 + a_3\rho_j^2 + \ldots + a_0\rho_j^{n-1} \end{bmatrix}$$

From 2nd row of $A\underline{v}_j$, factorise out ρ_1^1 to get

$\rho_1^1\left(\rho_1^{-1}a_{n-1} + a_0 + a_1\rho_1^1 + a_2\rho_1^2 \ldots + a_{n-2}\rho_1^{n-2}\right)$.

Now $\rho_j^{-1} = e^{\frac{2\pi i j(-1)}{n}} = e^{\frac{-2j\pi i}{n}} = e^{\frac{(n-1)2j\pi i}{n}} = \rho_j^{n-1}$

since $\dfrac{-1}{n}2j\pi = \left(\dfrac{(n-1)-n}{n}\right)2j\pi = \dfrac{n-1}{n}2j\pi - 2j\pi = \dfrac{n-1}{n}2\pi$ because of 2π periodicity.

Thus, the second row of $A\underline{v}_j$ has been reduced to

$= \rho_j^1\left(\rho_j^{-1}a_{n-1} + a_0 + a_1\rho_j^1 + a_2\rho_j^2 \ldots + a_{n-2}\rho_j^{n-2}\right)$

$= \rho_j^1\left(\rho_j^{n-1}a_{n-1} + a_0 + a_1\rho_j^1 + a_2\rho_j^2 \ldots + a_{n-2}\rho_j^{n-2}\right)$

$= \rho_j^1\left(1st\ \ row\ \ of\ \ A\underline{v}_j\right)$

$= \rho_j^1$ (1st row of $A\underline{v}_j$).

Generally, from the $(t+1)$th row of $A\underline{v}_j$

$\left(a_{n-t} + a_{n-(t-1)}\rho_j^1 \ldots + a_0\rho_j^t + a_1\rho_j^{t+1} + a_2\rho_j^{t+2} \ldots + a_{n-(t+1)}\rho_j^{n-1}\right)$

$= \rho_j^t\left(a_{n-t}\rho_j^{-t} + a_{n-(t-1)}\rho_j^{1-t} + \ldots + a_0 + a_1\rho_j^1 + \ldots + a_{n-(t+1)}\rho_j^{n-(t+1)}\right)$

Since $\rho_j^{-t} = e^{\frac{2\pi i j}{n}(-t)} = e^{\left(\frac{2\pi i j}{n}(n-t) + \frac{2\pi i j n}{n}\right)} = \rho_j^{n-t}$ so that

$\rho_j^{-t} = \rho_j^{n-t} \Rightarrow \rho_j^{1-t} = \rho_j^{-(t-1)} = \rho_j^{n-(t-1)}$ etc.

Hence the $(t+1)$th row

$= \rho_j^t$ (1st row of $A\underline{v_j}$).

$= \rho_j^t \lambda_j$

Therefore we have

$$A\underline{v_j} = \begin{bmatrix} \lambda_j \\ \rho_j^1 \lambda_j \\ \rho_j^2 \lambda_j \\ \vdots \\ \rho_j^{n-1} \lambda_j \end{bmatrix} = \begin{bmatrix} \lambda_j \\ \rho_j^1 \lambda_j \\ \rho_j^2 \lambda_j \\ \vdots \\ \rho_j^{n-1} \lambda_j \end{bmatrix} = \lambda_j \begin{bmatrix} 1 \\ \rho_j^1 \\ \rho_j^2 \\ \vdots \\ \rho_j^{n-1} \end{bmatrix} = \lambda_j \underline{v_j}$$

Therefore, we have shown that v_j is an eigenvector of *A* with eigenvalue λ_j for $1 \le j \le n.$ □

The result of this key theorem allows us to easily calculate the eigenvalues of many circulant matrices which are adjacency matrices of corresponding graphs.

Corollary 2.1.1

Let $\underline{v_j} = \left[1, \rho_j, \rho_j{}^2, \ldots, \rho_j{}^{n-1}\right]^T, \quad j = 0,1,\ldots,n-1$ be the eigenvector as defined in Theorem 2.1.1. Then $\sum_{k=0}^{n-1} \rho_j^k = 0; \quad j \ne 0.$

Proof

It is a well-known result, and proved in 4 different ways in Section 2.2, that an eigenvalue of the complete graph is -1 for $j \ne 0$, with eigenvector

$$\underline{v_j}^T = \left[1, \rho_j, \rho_j{}^2, \ldots, \rho_j{}^{n-1}\right] \quad j \ne 0.$$

Therefore $A(K_n)\left[1, \rho_j, \rho_j{}^2, \ldots, \rho_j{}^{n-1}\right]^T = (-1)\left[1, \rho_j, \rho_j{}^2, \ldots, \rho_j{}^{n-1}\right]^T$

Expanding the first row of LHS we get: $\sum_{k=1}^{n-1} \rho_j^k; \quad j \neq 0$ since the first row of $A(K_n)$ is $(0111...111)$. The first row of RHS is -1, so $\sum_{k=1}^{n-1} \rho_j^k = -1; \quad j \neq 0.$ Adding 1 to both sides, we get $\sum_{k=0}^{n-1} \rho_j^k = 0; \quad j \neq 0.$ □

2.2 Eigenvalues of complete graphs

In this section, we use four different techniques to find the eigenvalues of the complete graph K_n, on *n* vertices, namely:

- Induction;
- Proof using $J_{n,n}$ matrix;
- Proof using diagonalisation method; and
- Inverse method.

The following example is a complete graph K_6 on 6 vertices. Note that each vertex is joined to every other vertex in the complete graph.

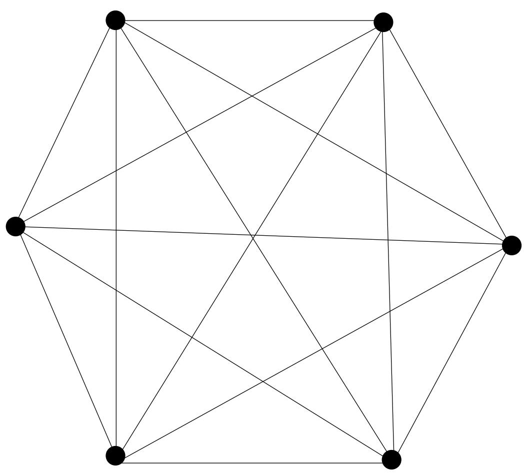

Figure 2.2.1: Complete graph K_6

The following lemma is an important result, as it is used in the proof of Theorem 2.2.1 using the induction method. In this lemma, we calculate the determinant of the matrix which is used in the expansion of the determinant of the adjacency matrix of the complete graph. It enables us to use the inductive method for the proof of the eigenvalues of the adjacency matrix of the complete graph. (This proof is entirely original work).

Lemma 2.1.1

If $H_n = \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{nxn}$, where H_n is a nxn matrix, with $n \geq 2$,

then $\det H_n = \det \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{nxn}$

$= (-1)(\lambda + 1)^{n-1}$.

Proof (by induction)

For $n = 2$, $H_2 = \begin{bmatrix} -1 & -1 \\ -1 & \lambda \end{bmatrix}$

$$\begin{aligned} \det(H_2) &= -\lambda - 1 \\ &= -(\lambda + 1) \\ &= -(\lambda + 1)^1 \end{aligned}$$

$$\det(H_3) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix}$$

$$= -\det\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} + \det\begin{vmatrix} -1 & -1 \\ -1 & \lambda \end{vmatrix} - \det\begin{vmatrix} -1 & \lambda \\ -1 & -1 \end{vmatrix}$$

$$= -1\det\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} + \det\begin{vmatrix} -1 & -1 \\ -1 & \lambda \end{vmatrix} - \det\begin{vmatrix} -1 & \lambda \\ -1 & -1 \end{vmatrix}$$

$$\begin{aligned}
&= -1(\lambda^2 - 1) + 2(-\lambda - 1) \\
&= -(\lambda+1)(\lambda-1) - 2(\lambda+1) \\
&= -(\lambda+1)(\lambda-1+2) \\
&= -(\lambda+1)^2
\end{aligned}$$

$$\det(H_4) \quad = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & \lambda & -1 & -1 \\ -1 & -1 & \lambda & -1 \\ -1 & -1 & -1 & \lambda \end{bmatrix}$$

$$= -\det\begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} + 3\det\begin{vmatrix} -1 & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix}$$

$$= -\det(A(K_3) + 3\det(H_3))$$

$$\begin{aligned}
&= -(\lambda\det(A(K_2)) + 2\det(H_2)) + 3\det(H_3) \\
&= -(\lambda(\lambda^2 - 1) - 2(\lambda+1)) + 3(\lambda+1)^2 \\
&= -(\lambda(\lambda+1)(\lambda-1) - 2(\lambda+1)) + 3(\lambda+1)^2 \\
&= -(\lambda+1)(\lambda^2 - \lambda - 2) + 3(\lambda+1)^2 \\
&= -(\lambda+1)(\lambda+1)(\lambda-2) + 3(\lambda+1)^2 \\
&= -(\lambda+1)^2((\lambda-2)+3) \\
&= -(\lambda+1)^3
\end{aligned}$$

$$\begin{aligned}
\det(H_5) \quad &= -\det(A(K_4) + 4\det(H_4)) \\
&= -(\lambda\det A(K_3) + 3\det(H_3)) + 4\det(H_4)) \\
&= -(\lambda(\lambda\det A(K_2) + 2\det(H_2)) + 3\det(H_3)) + 4\det(H_4)) \\
&= -\left(\lambda^2(\lambda^2 - 1) - \lambda 2(\lambda+1) - 3\det(\lambda+1)^2\right) - 4(\lambda+1)^3 \\
&= -\left((\lambda+1)\lambda(\lambda^2 - \lambda - 2) - 3\det(\lambda+1)^2\right) - 4(\lambda+1)^3 \\
&= -(\lambda+1)[\lambda(\lambda+1)(\lambda-2) - 3\det(\lambda+1)] - 4(\lambda+1)^3
\end{aligned}$$

$$= -(\lambda+1)^2\left[\lambda^2 - 2\lambda - 3\right] - 4(\lambda+1)^3$$
$$= -(\lambda+1)^2\left[(\lambda+1)(\lambda-3)\right] - 4(\lambda+1)^3$$
$$= -(\lambda+1)^3(\lambda-3) - 4(\lambda+1)^3$$
$$= -(\lambda+1)^3(\lambda-3+4)$$
$$= -(\lambda+1)^3(\lambda+1)$$
$$= -(\lambda+1)^4$$

Assume the hypothesis it true for all $k \le n$, i.e., $\det(H_k) = -(\lambda+1)^{k-1}$ for all $k \le n$.

Then, for $n = k+1$,

$$\det H_{k+1} = \det\begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{(k+1)x(k+1)}$$

Then, expanding along the first row,

$$\det H_{k+1}$$
$$= (-1)\det\begin{bmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk} + (-1)(-1)[(k+1)-1]\det\begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk}$$

The first term is obtained from the expansion of the first column (in the first row) and the second terms is from the $((k+1)-1)$ identical terms obtained from the expansion of the 2nd to $((k+1)-1)$th columns.

Let $\det(A_k) = \lambda I - A(K_k) = \begin{bmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk}$ and

$$H_k = \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk} .$$

Then,

$$\begin{aligned}
&\det(H_{k+1}) \\
&= (-1)\det(A(K_k)) + k\det(H_k) \\
&= (-1)\{\lambda\det(A(K_{k-1})) + (k-1)\det(H_{k-1})\} + k\det(H_k) \\
&= (-1)\{\lambda(\lambda\det(A_{k-2}) + (k-2)\det(H_{k-2})) + (k-1)\det(H_{k-1})\} + k\det(H_k) \\
&= (-1)\{\lambda^2\det(A_{k-2}) + \lambda(k-2)\det(H_{k-2}) + (k-1)\det(H_{k-1})\} + k\det(H_k) \\
&= (-1)\{\lambda^2(\lambda\det(A_{k-3}) + (k-3)\det(H_{k-3})) + \lambda(k-2)\det(H_{k-2}) + (k-1)\det(H_{k-1})\} \\
&\qquad + k\det(H_k) \\
&= (-1)\{\lambda^3\det(A_{k-3}) + \lambda^2(k-3)\det(H_{k-3}) + \lambda(k-2)\det(H_{k-2}) + (k-1)\det(H_{k-1})\} \\
&\qquad + k\det(H_k)
\end{aligned}$$

Now, the leading λ must have power $(k-2)$ so that we get $\det(A_{k-(k-2)})$ and $\det(H_{k-(k-2)})$ which are both known. So, continuing,

$$\begin{aligned}
&\det(H_{k+1}) \\
&= (-1)[\ \lambda^{k-2}\det(A_{k-(k-2)}) + \lambda^{k-3}2\det(H_2) + \lambda^{k-4}3\det(H_3) + \lambda^{k-5}4\det(H_4) + \ldots \\
&\qquad + \lambda^2(k-3)\det(H_{k-3}) + \lambda(k-2)\det(H_{k-2}) + (k-1)\det(H_{k-1})\ ] + k\det(H_k)
\end{aligned}$$

Substituting $\det(A(K_2)) = (\lambda^2 - 1) = (\lambda+1)(\lambda-1)$ and $\det(H_k) = -(\lambda+1)^{k-1}$ for all $k \le n,$ we get

$$\begin{aligned}
&\det(H_{k+1}) \\
&= (-1)[\ \lambda^{k-2}(\lambda+1)(\lambda-1) - \lambda^{k-3}2(\lambda+1) - \lambda^{k-4}3(\lambda+1)^2 - \lambda^{k-5}4(\lambda+1)^3 + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-4} - \lambda(k-2)(\lambda+1)^{k-3} - (k-1)(\lambda+1)^{k-2}\ \Big] - k(\lambda+1)^{k-1}
\end{aligned}$$

Factorising $(\lambda+1)$ out of the k terms in the square brackets, we get

$$\begin{aligned}
&\det(H_{k+1}) \\
&= (-1)(\lambda+1)[\ (\ \lambda^{k-2}(\lambda-1) - \lambda^{k-3}2 - \lambda^{k-4}3(\lambda+1)^1 - \lambda^{k-5}4(\lambda+1)^2 + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-5} - \lambda(k-2)(\lambda+1)^{k-4} - (k-1)(\lambda+1)^{k-3}\ )\ ] - k(\lambda+1)^{k-1}
\end{aligned}$$

Working with the first two terms in square brackets, we get

$$\begin{aligned}
&\det(H_{k+1}) \\
&= (-1)(\lambda+1)[\ (\ \lambda^{k-3}(\lambda^2 - \lambda) - \lambda^{k-3}2 - \lambda^{k-4}3(\lambda+1)^1 - \lambda^{k-5}4(\lambda+1)^2 + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-5} - \lambda(k-2)(\lambda+1)^{k-4} - (k-1)(\lambda+1)^{k-3}\ )\ ] - k(\lambda+1)^{k-1}
\end{aligned}$$

$$\begin{aligned}
&= (-1)(\lambda+1)[\ (\ \lambda^{k-3}(\lambda^2 - \lambda - 2) - \lambda^{k-4}3(\lambda+1)^1 - \lambda^{k-5}4(\lambda+1)^2 + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-5} - \lambda(k-2)(\lambda+1)^{k-4} - (k-1)(\lambda+1)^{k-3}\ )\ ] - k(\lambda+1)^{k-1}
\end{aligned}$$

$$\begin{aligned}
&= (-1)(\lambda+1)[\ (\ \lambda^{k-3}(\lambda+1)(\lambda-2) - \lambda^{k-4}3(\lambda+1)^1 - \lambda^{k-5}4(\lambda+1)^2 + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-5} - \lambda(k-2)(\lambda+1)^{k-4} - (k-1)(\lambda+1)^{k-3}\ )\ ] - k(\lambda+1)^{k-1}
\end{aligned}$$

Taking out the next factor of $(\lambda+1)$ from inside the square brackets, we get

$$\begin{aligned}
&\det(H_{k+1}) \\
&= (-1)(\lambda+1)^2[\ (\ \lambda^{k-3}(\lambda-2) - \lambda^{k-4}3 - \lambda^{k-5}4(\lambda+1) + \ldots \\
&\qquad - \lambda^2(k-3)(\lambda+1)^{k-6} - \lambda(k-2)(\lambda+1)^{k-5} - (k-1)(\lambda+1)^{k-4}\ )\ ] - k(\lambda+1)^{k-1}
\end{aligned}$$

Working with the first two terms in square brackets, we get (1)

$$\begin{aligned}
&\det\left(H_{k+1}\right)\\
&=(-1)(\lambda+1)^2\Big[\ \Big(\ \lambda^{k-4}\left(\lambda^2-2\lambda\right)-\lambda^{k-4}3-\lambda^{k-5}4(\lambda+1)+\ldots\\
&\qquad -\lambda^2(k-3)(\lambda+1)^{k-6}-\lambda(k-2)(\lambda+1)^{k-5}-(k-1)(\lambda+1)^{k-4}\ \Big)\ \Big]-k(\lambda+1)^{k-1}
\end{aligned}$$

$$\begin{aligned}
&=(-1)(\lambda+1)^2\Big[\ \Big(\ \lambda^{k-4}\left(\lambda^2-2\lambda-3\right)-\lambda^{k-5}4(\lambda+1)+\ldots\\
&\qquad -\lambda^2(k-3)(\lambda+1)^{k-6}-\lambda(k-2)(\lambda+1)^{k-5}-(k-1)(\lambda+1)^{k-4}\ \Big)\ \Big]-k(\lambda+1)^{k-1}
\end{aligned}$$

$$\begin{aligned}
&=(-1)(\lambda+1)^2\Big[\ \Big(\ \lambda^{k-4}(\lambda+1)(\lambda-3)-\lambda^{k-5}4(\lambda+1)+\ldots\\
&\qquad -\lambda^2(k-3)(\lambda+1)^{k-6}-\lambda(k-2)(\lambda+1)^{k-5}-(k-1)(\lambda+1)^{k-4}\ \Big)\ \Big]-k(\lambda+1)^{k-1}
\end{aligned}$$

Note that the first term in the square brackets comprises of $(\lambda+1)\lambda^{k-t}(\lambda-(t-1))$.

We do the step (1) above a total of $(k-3)$ times, taking out the factor $(\lambda+1)^{k-3}$ to get

$$\begin{aligned}
&\det\left(H_{k+1}\right)\\
&=(-1)(\lambda+1)^{k-3}\Big[\ \lambda(\lambda+1)(\lambda-(k-2))-(k-1)(\lambda+1)\ \Big]-k(\lambda+1)^{k-1}
\end{aligned}$$

Note that the power of lambda in the first term in the square brackets is $(k-2)-(k-3)=1$ and the power of $(\lambda+1)$ in the second term in the square brackets is also $(k-2)-(k-3)=1$. Simplifying, we get

$$\begin{aligned}
&\det\left(H_{k+1}\right)\\
&=(-1)(\lambda+1)^{k-3}\Big[\ (\lambda+1)\left(\lambda^2-\lambda(k-2)-(k-1)\right)\ \Big]-k(\lambda+1)^{k-1}\\
&=(-1)(\lambda+1)^{k-2}\Big[\ \left(\lambda^2-\lambda(k-2)-(k-1)\right)\ \Big]-k(\lambda+1)^{k-1}\\
&=(-1)(\lambda+1)^{k-2}\Big[\ (\lambda+1)(\lambda-(k-1))\ \Big]-k(\lambda+1)^{k-1}\\
&=(-1)(\lambda+1)^{k-1}\Big[\ (\lambda-(k-1))\ \Big]-k(\lambda+1)^{k-1}\\
&=(-1)(\lambda+1)^{k-1}\Big[\ (\lambda-(k-1)+k)\ \Big]\\
&=(-1)(\lambda+1)^{k-1}\Big[\ (\lambda+1)\ \Big]\\
&=(-1)(\lambda+1)^{k}
\end{aligned}$$

This concludes the proof, by induction, that $\det H_n=(-1)(\lambda+1)^{n-1}$, for all $n\geq 2$. □

This is a key result to be used in the inductive proof of Theorem 2.2.1 below.

Theorem 2.2.1

Let $A(K_n)$ be the adjacency matrix of the complete graph K_n on *n* vertices.

Then $A(K_n)=\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{nxn}$

has eigenvalue $(n-1)$ with multiplicity 1, and eigenvalue -1 with multiplicity $(n-1)$.
Hence $\det(\lambda I - A(K_n)) = (\lambda+1)^{n-1}\{\lambda-(n-1)\}$

Proof of Theorem 2.2.1 (by induction)

For $n=2$, $A(K_2)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{aligned} \det(\lambda I - A(K_2)) &= \det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} \\ &= \lambda^2 - 1 \\ &= (\lambda+1)(\lambda-1) \\ &= (\lambda+1)(\lambda-1). \end{aligned}$$

Note that the eigenvalues of A(K_2) are λ = -1 (1 time) and λ=1 (once).

Assume the hypothesis it true for $k \le n$, i.e.,

$$\det(\lambda I - A(K_k)) = \det\begin{bmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk}$$

$$= (\lambda+1)^{k-1}\{\lambda-(k-1)\} \text{ for } k \le n$$

i.e., $\lambda = -1$ $(k-1)$ times, and $\lambda = (k-1)$ once.

Then, for $n = k+1$,

$$\det[\lambda I - A(K_{k+1})]$$

$$= \lambda \det \begin{bmatrix} \lambda & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk} + k \det \begin{bmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & -1 & \cdots & -1 \\ -1 & -1 & \lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{bmatrix}_{kxk}$$

$$= \lambda \det(A(K_k)) + k \det(H_k)$$

Now applying the inductive hypothesis for $\det(A(K_k))$, and Lemma 2.1.1 for $\det(H_k)$, we get

$$\begin{aligned} \det[\lambda I - A(K_{k+1})] &= \lambda(\lambda+1)^{k-1}\{\lambda - (k-1)\} + k(-1)(\lambda+1)^{k-1} \\ &= (\lambda+1)^{k-1}\{\lambda^2 - \lambda(k-1) - k\} \\ &= (\lambda+1)^{k-1}(\lambda+1)(\lambda-k) \\ &= (\lambda+1)^{k}(\lambda-k) \end{aligned}$$

i.e., $\lambda = -1$ k times and $\lambda = k$ once.

So we have proved that the eigenvalues of the adjacency matrix of the complete graph $A(K_n)$ are $\lambda = -1$ and $\lambda = n-1$, and that the characteristic polynomial is $P_{A(K_n)}(\lambda) = (\lambda+1)^{n-1}(\lambda-(n-1))$. The two factors $(\lambda+1)$ and $(\lambda-(n-1))$ give rise to the quadratic $\lambda^2 - (n-2)\lambda - (n-1)$ which has the associated conjugate pairs $\lambda = \frac{(n-2)}{2} \pm \sqrt{\frac{(2-n)^2 + 4(n-1)}{4}}$. □

Proof of Theorem 2.2.1 (using $J_{n,n}$ matrix)

The complete graph K_n has adjacency matrix $A(K_n) = J_{n,n} - I_{n,n}$. The rank of $J_{n,n}$ is 1 i.e., there is one nonzero eigenvalue for $J_{n,n}$ equal to *n* (with an eigenvector $\underline{1}_n^T = (1,1,...,1)^T$. As $\sum_{i=1}^{n} \lambda_i = tr(J) = \sum_{i=1}^{n} J_{i,i} = n$, all the remaining eigenvalues are 0. Hence, the eigenvalues of $J_{n,n}$ are *n* (once) and 0 $(n-1)$ times.

With eigenvalue *n*, and eigenvector $\underline{x} = \underline{1}_n^T$,

$$\begin{aligned} A(K_n)\underline{x} &= (J_{n,n} - I_{n,n})\underline{x} \\ &= J_{n,n}\underline{x} - I_{n,n}\underline{x} \\ &= n\underline{x} - \underline{x} \\ &= (n-1)\underline{x} \end{aligned}$$

With eigenvalue 0, and eigenvector $\underline{x}$

$$\begin{aligned} A(K_n)\underline{x} &= (J_{n,n} - I_{n,n})\underline{x} \\ &= J_{n,n}\underline{x} - I_{n,n}\underline{x} \\ &= -\underline{x} \end{aligned}$$

Therefore, the eigenvalues of $A(K_n)$ are $(n-1)$ (with multiplicity 1) and -1 (with multiplicity $(n-1)$. □

Proof of Theorem 2.2.1 (using diagonalisation method)

Let Q be the orthogonal matrix, which diagonalises $J_{n,n}$,

then $Q^{-1}J_{n,n}Q = diag(n,0,0,...,0)$.

Then, since $A(K_n) = J_{n,n} - I_{n,n}$

$$\begin{aligned} Q^{-1}A(K_n)Q &= Q^{-1}(J_{n,n} - I_{n,n})Q \\ &= Q^{-1}J_{n,n}Q - Q^{-1}I_{n,n}Q \\ &= diag(n,0,0,...,0) - diag(1,1,1,...,1) \\ &= diag(n-1,-1,-1,...,-1) \end{aligned}$$

which implies that Q diagonalises $A(K_n)$ with eigenvalues $(n-1)$ once, and -1 repeated $(n-1)$ times. □

Proof of Theorem 2.2.1 (inverse method)

Since $A(K_n)$ is the same as the full matrix N^{Ω} of a strongly symmetric design of v treatments, with $v = n$, we get from section 1.5.1

$$(N^{\Omega})^{-1} = \frac{1}{r}N^{\Omega} - \frac{r-1}{r}I_{v,v}; \quad r = v-1$$

and hence we obtain the quadratic

$$(N^{\Omega})^2 - (r-1)N^{\Omega} - rI_{v,v} = 0$$

Hence, in terms of graphs we have

$$(A(K_n))^2 - (r-1)A(K_n) - rI_{v,v} = 0$$

Letting $r = n-1$, the roots of the above quadratic are

$$\frac{(n-2) \pm \sqrt{(n-2)^2 + 4(n-1)}}{2} = \frac{(n-2) \pm \sqrt{n^2}}{2}$$

Thus the two roots are $(n-1)$ and -1.

From the Cayley-Hamilton theorem, these roots are the eigenvalues of $A(K_n)$. The rank of the adjacency matrix of the complete graph is n, and since the rank is equivalent to the number of non –zero eigenvalues, the matrix must therefore have n non zero eigenvalues. Since the complete graph is $n-1$ regular, there can only be one eigenvalue of value $(n-1)$. Therefore, the multiplicity of the remaining eigenvalue of value -1 is $(n-1)$.

2.3 Eigenvalues of cycles

Let C_n be a cycle graph on *n* vertices. Then, C_8 is an example of a cycle graph.

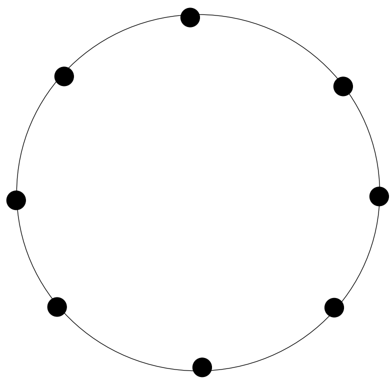

Figure 2.3.1: Cycle graph C_8

We first need Definition 2.3.1 and Lemma 2.3.1 as follows:

Definition 2.3.1: Order *d* linear homogeneous recurrence relation

An order *d linear homogeneous recurrence relation with constant coefficients* is an equation of the form

$$x_n = c_1 x_1 + c_2 x_2 + ... + c_d x_d$$

where the *d* coefficients c_i (for all *i*) are constants.

Lemma 2.3.1

The linear homogeneous recurrence relation with constant coefficients

$$x_{i-1} + x_{i+1} = \lambda x_i \text{ for } 2 \leq i \leq n-1,$$

and initial conditions $x_0 = x_n$ and $x_{n+1} = x_1$

has solutions

$$\lambda_j = 2\cos\left(\frac{2\pi j}{n}\right); \; 0 \leq j \leq= n-1.$$

Proof

$$x_{i-1} + x_{i+1} = \lambda x_i$$

Replacing $i+1$ with *n*, we get:

$$x_{n-2} + x_n = \lambda x_{n-1}$$

$$\Rightarrow x_n = \lambda x_{n-1} - x_{n-2}$$

so that $c_{n-1} = \lambda$ and $c_{n-2} = -1$ as in Definition 2.3.1.

More precisely, this is an infinite list of simultaneous linear equations, one for each $n > (d-1)$. A sequence which satisfies a relation of this form is called a **linear recurrence sequence** or LRS. There are *d* degrees of freedom for LRS, i.e., the initial values $c_1,\ldots,c_d$ can be taken to be any values but then the linear recurrence determines the sequence uniquely.

The same coefficients yield the characteristic polynomial (also "auxiliary polynomial")

$$p(t) = t^d - c_1 t^{d-1} - c_2 t^{d-2} - \ldots - c_d$$

whose *d* roots play a crucial role in finding and understanding the sequences satisfying the recurrence. If the roots $r_1, r_2, \ldots$ are all distinct, then the *solution to the recurrence* takes the form

$$x_n = k_1 r_1^n + k_2 r_2^n + \ldots + k_d r_d^n .$$

Thus for $x_n = \lambda x_{n-1} - x_{n-2}$, we have $d = 2,$ $c_1 = \lambda$ and $c_2 = -1.$

Then $p(t) = t^2 - \lambda t + 1$

with roots $r_1, r_2 = \dfrac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$

Thus we have

$$x_n = k_1 r_1^{\,n} + k_2 r_2^{\,n} \text{ with } r_1, r_2 = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

By calculation, the sum of the roots of the quadratic $p(t) = t^2 - \lambda t + 1$ is λ and the product of the roots of this quadratic is 1. Therefore,

$r_1 + r_2 = \lambda$ and $r_1 r_2 = 1.$ (1)

Then from (1) we have

$r_1 + r_2 = \lambda$

Substituting $r_2 = \dfrac{1}{r_1}$ we get

$$r_1 + \frac{1}{r_1} = \lambda$$

The initial conditions give:

1. $x_{n+1} = x_1$

$$\Rightarrow k_1 r_1^{\,n+1} + k_2 r_2^{\,n+1} = k_1 r_1 r_1^{\,n} + k_2 r_2 r_2^{\,n} = k_1 r_1 + k_2 r_2$$

$$\Rightarrow r_1^{\,n} = 1 \text{ and } r_2^{\,n} = 1$$

2. $x_n = x_0$

$$\Rightarrow k_1 r_1^{\,n} + k_2 r_2^{\,n} = k_1 + k_2$$

$$\Rightarrow r_1^{\,n} = 1 \text{ and } r_2^{\,n} = 1$$

Recalling from Lemma 2.1.1, that if $r^n = 1$ then the solutions are $U_n = \left\{ e^{\frac{2\pi i k}{n}} = \cos(2\pi k) + i \sin(2\pi k);\ 0 \le k \le n-1 \right\}$ where U_n is the nth root of unity and $i = \sqrt{-1}$.

Therefore, $r_1^{\,n} = 1$ has solution set $U_n = \left\{ e^{\frac{2\pi i j}{n}} = \cos\left(\frac{2\pi j}{n}\right) + i \sin\left(\frac{2\pi j}{n}\right);\ 0 \le j \le n-1 \right\}$ where U_n is the nth root of 1.

Then from (1) we have

$$\lambda_k = r_1 + \frac{1}{r_1}$$

$$= e^{\frac{2\pi ij}{n}} + e^{-\frac{2\pi ij}{n}} \ ; \ 0 \le j \le n-1$$
$$= \cos\left(\frac{2\pi j}{n}\right) + i\sin\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{-2\pi j}{n}\right) + i\sin\left(\frac{-2\pi j}{n}\right) \ ; \ 0 \le j \le n-1$$
$$= \cos\left(\frac{2\pi j}{n}\right) + i\sin\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{2\pi j}{n}\right) - i\sin\left(\frac{2\pi j}{n}\right) ; \ 0 \le j \le n-1$$
$$= 2\cos\left(\frac{2\pi j}{n}\right); \ 0 \le j \le n-1. \qquad \square$$

Theorem 2.3.1

Let C_n be a cycle graph on *n* vertices. Then the adjacency matrix $A(C_n)$ is

$$A(C_n) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and has eigenvalues

$$\lambda_j = 2\cos\left(\frac{2\pi j}{n}\right); \ 0 \le j \le n-1 \ \text{ and for } \ n \ge 3.$$

Proof of Theorem 2.3.1 (using circulant matrices)

$A(C_n)$ is a circulant matrix, so as per Theorem 2.1.1, the eigenvalues of $A(C_n)$ are

$$\lambda_j = q(\rho_j)$$
$$= 0 + 1\rho_j + 0\rho_j^2 + \ldots + 1\rho_j^{n-1}, where \ \rho_j = e^{\frac{2\pi ij}{n}}, \ 0 \le j \le n-1$$
$$= \rho_j + \rho_j^{n-1}, \ \text{where} \ \rho_j = e^{\frac{2\pi ij}{n}} \ \text{and} \ n \ge 3.$$

Therefore,

$$\lambda_j \quad = e^{\frac{2\pi ij}{n}} + e^{\frac{2(n-1)\pi ij}{n}}$$

$$= \cos\left(\frac{2\pi j}{n}\right) + i\sin\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{2(n-1)\pi j}{n}\right) + i\sin\left(\frac{2(n-1)\pi j}{n}\right)$$
$$= \cos\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{2n\pi j}{n} - \frac{2\pi j}{n}\right) + i\left(\sin\left(\frac{2\pi j}{n}\right) + \sin\left(\frac{2n\pi j}{n} - \frac{2\pi j}{n}\right)\right)$$

$$= \cos\left(\frac{2\pi j}{n}\right) + \left\{\cos\left(\frac{2n\pi j}{n}\right)\cos\left(\frac{2\pi j}{n}\right) + \sin\left(\frac{2n\pi j}{n}\right)\sin\left(\frac{2\pi j}{n}\right)\right\} +$$
$$i\left(\sin\left(\frac{2\pi j}{n}\right) + \left\{\sin\left(\frac{2n\pi j}{n}\right)\cos\left(\frac{2\pi j}{n}\right) - \cos\left(\frac{2n\pi j}{n}\right)\sin\left(\frac{2\pi j}{n}\right)\right\}\right)$$

$$= \cos\left(\frac{2\pi j}{n}\right) + \left\{(1)\cos\left(\frac{2\pi j}{n}\right) + 0\right\} + i\left(\sin\left(\frac{2\pi j}{n}\right) + \left\{0 + (-1)\sin\left(\frac{2\pi j}{n}\right)\right\}\right)$$

$$= \cos\left(\frac{2\pi j}{n}\right) + \cos\left(\frac{2\pi j}{n}\right) + i\left(\sin\left(\frac{2\pi j}{n}\right) - \sin\left(-\frac{2\pi j}{n}\right)\right)$$

$$= 2\cos\left(\frac{2\pi j}{n}\right)$$

So, we have shown that, the eigenvalues of the adjacency matrix of the cycle graph $A(C_n)$, are

$$\lambda_j = 2\cos\left(\frac{2\pi j}{n}\right); \; 1 \le j \le n-1. \qquad \square$$

To illustrate specific examples, we have for $j = 0$ and all $n \ge 3$:

$$\lambda_0 = e^{\frac{2\pi i.0}{n}} + e^{\frac{2(n-1)\pi i.0}{n}}$$
$$= 1 + 1$$
$$= 2 \text{ for all } n \ge 3.$$

Also, for n=3, we have

$$\lambda_0 = 2$$
$$\lambda_1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$$
$$= -0.5 + \frac{\sqrt{3}}{2}i - 0.5 - \frac{\sqrt{3}}{2}i$$

$$= -1$$

$$\begin{aligned}\lambda_2 &= 2\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) + \cos\left(\frac{8\pi}{3}\right) + i\sin\left(\frac{8\pi}{3}\right)\\ &= -0.5 - \frac{\sqrt{3}}{2}i - 0.5 + \frac{\sqrt{3}}{2}i\\ &= -1\end{aligned}$$

Thus the eigenvalues of the adjacency matrix associated with the cycle graph on 3 vertices are $\lambda = (-1)$ twice, and $\lambda = 2$ once.

Proof of Theorem 2.3.1 (eigenvector method)

For $n = 3$, C_3 is the cycle graph on 3 vertices. The adjacency matrix $A(C_3)$ of cycle C_3 is:

$$A(C_3) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The eigenvalues of $A(C_3)$ are 2 and -1 (by calculation using Bluebit Online Matrix Calculator).

Now for $j = 0,1,2$

$2\cos\left(\frac{2\pi j}{3}\right)$ is:

$$2\cos(0) = 2$$

$$2\cos\left(\frac{2\pi}{3}\right) = -1$$

$$2\cos\left(\frac{4\pi}{3}\right) = -1$$

For $n = 4$, then C_4 is the cycle graph on 4 vertices. The adjacency matrix, $A(C_4)$, of C_4 is:

$$A(C_3) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues of $A(C_4)$ are 2, 0 (twice) and -2 (by calculation using Bluebit Online Matrix Calculator).

Now for $j = 0,1,2,3$

$2\cos\left(\frac{2\pi j}{4}\right)$ is:

$$2\cos(0) = 2$$

$$2\cos\left(\frac{2\pi}{4}\right) = 0$$

$$2\cos\left(\frac{4\pi}{4}\right) = -2$$

$$2\cos\left(\frac{6\pi}{4}\right) = 0$$

For the general case, C_n is the cycle graph on *n* vertices. The adjacency matrix of C_n is:

$$A(C_n) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Let $\underline{x_n} = (x_1, x_2, ..., x_n)$ be the eigenvector.

Then, $A(C_n)\underline{x_n} = \lambda_n \underline{x_n}$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \lambda_n \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_2 + x_n \\ x_1 + x_3 \\ x_2 + x_4 \\ x_3 + x_5 \\ \vdots \\ x_{n-3} + x_{n-1} \\ x_{n-2} + x_n \\ x_1 + x_{n-1} \end{bmatrix} = \lambda_n \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}$$

i.e., $x_{i-1} + x_{i+1} = \lambda_n x_i$, for $1 \le i \le n$ and $x_0 = x_n$ and $x_{n+1} = x_1$

From Lemma 2.3.1, this linear homogeneous recurrence relation with constant coefficients has solution

$$\lambda_j = 2\cos\left(\frac{2\pi j}{n}\right);\ \ 0 \le j \le n-1$$

Therefore, the eigenvalues of the adjacency matrix of the cycle graph C_n on *n* vertices are

$$\lambda_j = 2\cos\left(\frac{2\pi j}{n}\right);\ \ 0 \le j \le n-1 \text{ for } n \ge 3.$$

□

2.4 Eigenvalues of paths

Let P_n be a cycle graph on *n* vertices. Then, P_6 is an example of a path graph.

Figure 2.4.1: Path graph P_6

We first need Lemma 2.4.1. The proof of this lemma is similar to that of Lemma 2.3.1, but the solutions are different as there are different initial conditions.

Lemma 2.4.1

The linear homogeneous recurrence relation with constant coefficients

$$x_{i-1} + x_{i+1} = \lambda x_i$$

for $2 \le i \le n-1$, and $x_0 = 0$ and $x_{n+1} = 0$

has solutions

$$\lambda_j = 2\cos\left(\frac{\pi j}{n+1}\right), j = 1,\ldots,n.$$

Proof

The initial conditions have $x_0 = 0$ and $x_{n+1} = 0$ to indicate that the path starts and ends with x_1 and x_n respectively.

Refer to Lemma 2.3.1 to obtain the following:

$$x_n = k_1 r_1{}^n + k_2 r_2{}^n \quad \text{with} \quad r_1^n, r_2^n = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}$$

By calculation, the sum of the roots of the quadratic $p(t) = t^2 - \lambda t + 1$ is λ and the product of the roots of this quadratic is 1. Therefore,

$$r_1 + r_2 = \lambda \quad \text{and} \quad r_1 r_2 = 1. \tag{1}$$

Then from (1) we have

$$r_1 + r_2 = \lambda$$

Substituting $r_2 = \dfrac{1}{r_1}$ we get

$$r_1 + \frac{1}{r_1} = \lambda$$

The initial conditions give:

1. $x_0 = 0$

$$\Rightarrow k_1 r_1^{\,0} + k_2 r_2^{\,0} = k_1 + k_2 = 0$$
$$\Rightarrow k_1 = -k_2$$
$$\Rightarrow x_n = k_1(r_1^{\,n} - r_2^{\,n}); \quad k_1 \neq 0$$

2. $x_{n+1} = 0$

$$\Rightarrow k_1 r_1^{\,n+1} + k_2 r_2^{\,n+1} = 0$$
$$\Rightarrow r_1^{\,n+1} - r_2^{\,n+1} = 0$$
$$\Rightarrow r_1^{\,n+1} = r_2^{\,n+1}$$

From $r_1 r_2 = 1$

$$\Rightarrow (r_1 r_2)^{n+1} = 1^{n+1} = 1$$
$$\Rightarrow r_1^{\,n+1} r_2^{\,n+1} = 1^{n+1} = 1$$
$$\Rightarrow r_1^{\,n+1} r_1^{\,n+1} = 1$$
$$\Rightarrow r_1^{\,2n+2} = 1$$

i.e., r_1 is the $(2n+2)$th root of unity.

Recalling from Lemma 2.1.1, that if $r^n = 1$ then the solutions are $U_n = \left\{ e^{\frac{2\pi i k}{n}} = \cos(2\pi k) + i\sin(2\pi k); \; 0 \le k \le n-1 \right\}$ where U_n is the nth root of unity and $i = \sqrt{-1}$.

Therefore, $r_1^{\,2n+2} = 1$ has solution set $U_{2n+2} = \left\{ e^{\frac{2\pi i k}{2n+2}} = \cos\left(\frac{2\pi k}{2n+2}\right) + i\sin\left(\frac{2\pi k}{2n+2}\right); \; 0 \le k \le n-1 \right\}$ where U_{2n+2} is the $(2n+2)$th root of 1.

Then from (1) we have

$$\lambda_k \quad = r_1 + \frac{1}{r_1}$$

$$= e^{\frac{2\pi i k}{2n+2}} + e^{-\frac{2\pi i k}{2n+2}} \; ; \; 0 \le k \le n-1$$

$$= \cos\left(\frac{2\pi k}{2n+2}\right) + i\sin\left(\frac{2\pi k}{2n+2}\right) + \cos\left(\frac{-2\pi k}{2n+2}\right) + i\sin\left(\frac{-2\pi k}{2n+2}\right)$$

$; \; 0 \le k \le n-1$

$$= \cos\left(\frac{2\pi k}{2n+2}\right) + i\sin\left(\frac{2\pi k}{2n+2}\right) + \cos\left(\frac{2\pi k}{2n+2}\right) - i\sin\left(\frac{2\pi k}{2n+2}\right)$$

$; \; 0 \le k \le n-1$

$$= 2\cos\left(\frac{2\pi k}{2n+2}\right); \; 0 \le k \le n-1$$

$$= 2\cos\left(\frac{\pi k}{n+1}\right); \; 0 \le k \le n-1.$$

Setting $j = k+1$, we get

$$\lambda_j \quad = 2\cos\left(\frac{\pi j}{n+1}\right); \; 1 \le j \le n. \qquad \square$$

Theorem 2.4.1

Let P_n be a path graph on *n* vertices. Then the adjacency matrix $A(P_n)$ is

$$A(P_n) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and has eigenvalues

$$\lambda_j = 2\cos\left(\frac{\pi j}{n+1}\right); \; 1 \le j \le n \quad \text{and } n \ge 2.$$

Proof (eigenvector method)

For $n=2$, let P_2 be the path graph on 2 vertices. The adjacency matrix $A(P_2)$ of path P_2 is

$$A(P_2)=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of $A(P_2)$ are 1 and -1 (by calculation using Bluebit Online Matrix Calculator).

Now $2\cos\left(\frac{\pi j}{3}\right)$, where $j=1,2$ is:

$$2\cos\left(\frac{\pi}{3}\right)=1$$

$$2\cos\left(\frac{2\pi}{3}\right)=-1$$

For $n=3$, P_3 is the path graph on 3 vertices. The adjacency matrix $A(P_3)$ of P_3 is

$$A(P_3)=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of $A(P_3)$ are $+\sqrt{2}$, $-\sqrt{2}$ and 0 (by calculation using Bluebit Online Matrix Calculator).

Now $2\cos\left(\frac{\pi j}{4}\right)$, where $j=1,2,3$ is:

$$2\cos\left(\frac{\pi}{4}\right)=\sqrt{2}$$

$$2\cos\left(\frac{2\pi}{4}\right)=0$$

$$2\cos\left(\frac{3\pi}{4}\right)=-\sqrt{2}$$

For $n = 4$, P_4 is the path graph on 4 vertices. The adjacency matrix $A(P_4)$ of P_4 is

$$A(P_4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of $A(P_4)$ are 1.618034, 0.618034, -0.618034, and -1.618034 (by calculation using Bluebit Online Matrix Calculator).

Now $2\cos\left(\frac{\pi j}{5}\right)$, where $j = 1,2,3,4$ is:

$$2\cos\left(\frac{\pi}{5}\right) = 1.618034$$

$$2\cos\left(\frac{2\pi}{5}\right) = 0.618034$$

$$2\cos\left(\frac{3\pi}{5}\right) = -0.618034$$

$$2\cos\left(\frac{4\pi}{5}\right) = -1.618034$$

For the general case, P_n, is the path on n vertices. The adjacency matrix of P_n is:

$$A(P_n) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Let $\underline{x_n} = (x_1, x_2, \ldots, x_n)$ be the eigenvector.

Then, $A(P_n)\underline{x_n} = \lambda_n \underline{x_n}$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \lambda_n \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_2 \\ x_1 + x_3 \\ x_2 + x_4 \\ x_3 + x_5 \\ \vdots \\ x_{n-3} + x_{n-1} \\ x_{n-2} + x_n \\ x_{n-1} \end{bmatrix} = \lambda_n \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}$$

$\Rightarrow x_{i-1} + x_{i+1} = \lambda_n x_i, \quad for \;\; 1 \le i \le n$ and $x_0 = 0$ and $x_{n+1} = 0.$

From Lemma 2.4.1, this linear homogeneous recurrence relation with constant coefficients has solution

$$\lambda_j = 2\cos\left(\frac{\pi j}{n+1}\right); \; j = 1,\dots,n$$

Therefore, the eigenvalues of the adjacency matrix of the cycle graph P_n on *n* vertices are

$$\lambda_j = 2\cos\left(\frac{\pi j}{n+1}\right); \; j = 1,\dots,n \;\; n \ge 2 \qquad \square$$

From Lemma 2.2, where x_i is an end vertex of P_n, we also get

$$P_{A(P_n)}(\lambda) = \lambda P_{A(P_{n-1})}(\lambda) - P_{A(P_{n-2})}(\lambda)$$

$P_{A(P_2)}(\lambda) = \lambda^2 - 1$ by calculation of $\det(\lambda I - A(P_2))$

$P_{A(P_3)}(\lambda) = \lambda^3 - 2\lambda$ by calculation of $\det(\lambda I - A(P_3))$

$$\begin{aligned}
P_{A(P_4)}(\lambda) &= \lambda P_{A(P_3)}(\lambda) - P_{A(P_2)}(\lambda) \\
&= \lambda(\lambda^3 - 2\lambda) - (\lambda^2 - 1) \\
&= \lambda^4 - 3\lambda^2 + 1
\end{aligned}$$

$$\begin{aligned}
P_{A(P_5)}(\lambda) &= \lambda P_{A(P_4)}(\lambda) - P_{A(P_3)}(\lambda) \\
&= \lambda(\lambda^4 - 3\lambda^2 + 1) - (\lambda^3 - 2\lambda) \\
&= \lambda^5 - 3\lambda^3 + \lambda - \lambda^3 + 2\lambda \\
&= \lambda^5 - 4\lambda^3 + 3\lambda
\end{aligned}$$

$$\begin{aligned}
P_{A(P_6)}(\lambda) &= \lambda P_{A(P_5)}(\lambda) - P_{A(P_4)}(\lambda) \\
&= \lambda(\lambda^5 - 4\lambda^3 + 3\lambda) - (\lambda^4 - 3\lambda^2 + 1) \\
&= \lambda^6 - 4\lambda^4 + 3\lambda^2 - \lambda^4 + 3\lambda^2 - 1 \\
&= \lambda^6 - 5\lambda^4 + 6\lambda^2 - 1 \qquad \text{etc.}
\end{aligned}$$

Also, from Theorem 2.4.1,

$$\begin{aligned}
P_{A(P_n)}(\lambda) &= \prod_{j=1}^{n}\left(\lambda - 2\cos\left(\frac{\pi j}{n+1}\right)\right) \\
&= \lambda\prod_{j=1}^{n-1}\left(\lambda - 2\cos\left(\frac{\pi j}{n}\right)\right) - \prod_{j=1}^{n-2}\left(\lambda - 2\cos\left(\frac{\pi j}{n-1}\right)\right)
\end{aligned}$$

So for $n = 4$,

$$P_{A(P_4)}(\lambda)$$

$$= \lambda\prod_{j=1}^{3}\left(\lambda - 2\cos\left(\frac{\pi j}{4}\right)\right) - \prod_{j=1}^{2}\left(\lambda - 2\cos\left(\frac{\pi j}{3}\right)\right)$$

$$=\left[\lambda\left(\lambda-2\cos\left(\frac{\pi}{4}\right)\right)\left(\lambda-2\cos\left(\frac{2\pi}{4}\right)\right)\left(\lambda-2\cos\left(\frac{3\pi}{4}\right)\right)\right]$$
$$-\left[\left(\lambda-2\cos\left(\frac{\pi}{3}\right)\right)\left(\lambda-2\cos\left(\frac{2\pi}{3}\right)\right)\right]$$

$$=\left(\lambda\left(\lambda-\sqrt{2}\right)(\lambda-0)\left(\lambda+\sqrt{2}\right)\right)-((\lambda-1)(\lambda+1))$$

$$=\left(\lambda\left(\lambda-\sqrt{2}\right)(\lambda-0)\left(\lambda+\sqrt{2}\right)\right)-((\lambda-1)(\lambda+1))$$

$$=\lambda^2\left(\lambda^2-2\right)-\left(\lambda^2-1\right)$$

$$=\lambda^4-3\lambda^2+1$$

2.5 Eigenvalues of bipartite graphs

Let $K_{s,t}$ be the complete bipartite graph on $(s+t)$vertices, with partition (A,B), where $|A|=s$ and $|B|=t$. Then, $K_{3,4}$ is an example of a bipartite graph.

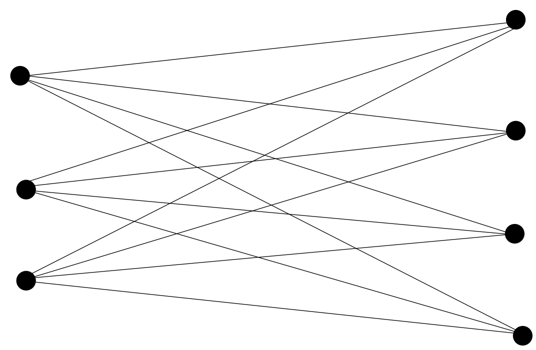

Figure 2.5.1: Bipartite graph $K_{3,4}$

Theorem 2.5.1

The eigenvalues of $K_{s,t}$ are 0, with multiplicity $s+t-2$, and $+\sqrt{st}$ and $-\sqrt{st}$.

See Fox [24].

Proof

We can write the adjacency matrix of $K_{s,t}$ as the block matrix

$$A(K_{s,t}) = \begin{bmatrix} 0_{s,s} & J_{s,t} \\ J_{t,s} & 0_{t,t} \end{bmatrix}$$

This matrix has rank 2, since the first *s* rows are the same, the last *t* rows are the same, and the first and last rows are linearly independent. Thus $K_{s,t}$ has nullity $(s+t-2)$, and hence has eigenvalue 0 with multiplicity $s+t-2$. See Anton [3].

Now let $\underline{v} \in R^{s+t}$ be the vector whose first *s* entries are *x* and last *t* entries are *y*, i.e., $\underline{v} = (x, x, ..., x, y, y, ...y)$ with *x* occurring *s* times, and *y* occurring *t* times. The edges joining the two partitions of the bipartite graph are significant in determining the eigenvalues, and suggest the splitting of the eigenvector into two parts relating to the bipartition. This definition of $\underline{v}$ facilitates finding the eigenvalues as follows

$A(K_{s,t})\underline{v} = \begin{bmatrix} 0_{s,s} & J_{s,t} \\ J_{t,s} & 0_{t,t} \end{bmatrix} \underline{v} = (ty, ..., ty, sx, ..., sx)$ with *ty* occurring *x* times, and *sx* occurring *t* times. To get eigenvlaues, we solve $A\underline{v} = \lambda\underline{v}$. So,

$$(ty, ..., ty, sx, ..., sx) = \lambda(x, ..., x, y, ..., y) = (\lambda x, ..., \lambda x, \lambda y, ...\lambda y)$$

with λx occurring *s* times, and λy occurring *t* times.

Therefore $ty = \lambda x$ and $sx = \lambda y$

So $t\left(\frac{sx}{\lambda}\right) = \lambda x$

$\Rightarrow \lambda^2 = st$

$\Rightarrow \lambda = \pm\sqrt{st}$

Hence the eigenvalues of $K_{s,t}$ are 0 with multiplicity $s+t-2$, and $+\sqrt{st}$ and $-\sqrt{st}$, each with multiplicity 1. □

2.6 Eigenvalues of the adjacency matrix associated with a graph which is the join of two graphs whose adjacency matrices are both circulant matrices

The following theorem gives the eigenvectors and eigenvalues of a matrix which is the adjacency matrix of the join of two graphs, whose adjacency matrices are both circulant matrices. This proof is key, as it can be applied to easily obtain the eigenvalues of a number of different graphs, which are made up of the join of two graphs whose adjacency matrices are circulant matrices, for example, the join of the complement of the complete graph on one vertex and a cycle (the wheel graph), the join of the complement of the complete graph on more than one vertex and a cycle (the generalised wheel graph) and the join of the complement of the complete graph on more than one vertex and a complete graph (the generalised complete wheel graph).

Theorem 2.6.1

Let $U_k^T = \left[1 \rho_{m,k}^1 \quad \rho_{m,k}^2 \quad \cdots \quad \rho_{m,k}^{(m-1)}\right]$ and $V_j^T = \left[1 \rho_{n,j}^1 \quad \rho_{n,j}^2 \quad \cdots \quad \rho_{n,j}^{(n-1)}\right]$

where $\rho_{m,k} = e^{\frac{2\pi k}{m}}$ for $1 \le k \le m-1$ and $\rho_{n,j} = e^{\frac{2\pi j}{n}}$ for $1 \le j \le n-1$.

Let square matrices $A = [a_1, a_2, ..., a_m]$ and $B = [b_1, b_2, ..., b_n]$ be two circulant matrices.

Then

1. $A \oplus B = \begin{bmatrix} A & J_{mn} \\ J_{nm} & B \end{bmatrix}$

2. CSET($A \oplus B$)= $\{W_1, W_2, ..., W_{m+n}\}$

   where

   $W_k^T = \left[0_{1,m}, V_k^T\right]$ if $1 \le k \le n-1$;
   $W_{n+k}^T = \left[U_k^T, 0_{1,n}\right]$ if $1 \le k \le m-1$;

   $\{W_n^T, W_{n+m}^T\} = \left\{ \left[J_{1,m}, \alpha J_{1,n}\right] \,\middle|\, n\alpha^2 + \alpha(d_A - d_B) - m = 0 \right\}$,

   where $d_A = a_1 + a_2 + ... + a_m$
   and $d_B = b_1 + b_2 + ... + b_n$

3. The eigenvalues λ_k of $A \oplus B$ are given by:

$$\lambda_k = b_1 + b_2\rho_n^k + b_3\rho_n^{2k} + \ldots + b_n\rho_n^{(n-1)k} \text{ for } 1 \le k \le n-1;$$

$$\lambda_{n+k} = a_1 + a_2\rho_m^k + a_3\rho_m^{2k} + \ldots + a_m\rho_m^{(m-1)k} \text{ for } 1 \le k \le m-1;$$

$$\{\lambda_n, \lambda_{n+m}\} = \left\{ n\alpha + d_A \middle| \; n\alpha^2 + \alpha(d_A - d_B) - m = 0 \right\}$$

See Lee and Yeh [37].

Proof

To show that $W_k^T = \left[0_{1,m}, V_k^T\right]$ is not an eigenvector for $k = 0,$ we let

$C = A \oplus B = \begin{bmatrix} A & J_{mn} \\ J_{nm} & B \end{bmatrix}$ so that

$$CW_0^T = \begin{bmatrix} A & J_{mn} \\ J_{nm} & B \end{bmatrix}\left[0_{1,m}, V_0^T\right] = \begin{bmatrix} A & J_{mn} \\ J_{nm} & B \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} m \\ m \\ . \\ d_b \\ d_b \\ . \\ d_b \end{bmatrix} = \lambda\left[0_{1,m}, V_0^T\right] = \lambda\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

which means $m = 0,$ which is impossible. Therefore $W_k^T = \left[0_{1,m}, V_k^T\right]$ is not an eigenvector for $k = 0.$

To show that $W_j^T = \left[0_{1,m}, V_j^T\right] = \left[000\ldots001\rho_{n,j}^1 \quad \rho_{n,j}^2 \quad \ldots \quad \rho_{n,j}^{(n-1)}\right]^T;\ 1 \le j \le n-1$ is an eigenvector of C consider:

$$CW_j^T = \begin{bmatrix} A & J_{mn} \\ J_{nm} & B \end{bmatrix}\left[000\ldots01\rho_{n,j}^1 \quad \rho_{n,j}^2 \quad \ldots \quad \rho_{n,j}^{(n-1)}\right]^T$$

The first *m* rows look like: $\sum_{k=0}^{n-1}\rho_{n,j}^{k}\ \sum_{k=0}^{n-1}\rho_{n,j}^{k}\ ...\sum_{k=0}^{n-1}\rho_{n,j}^{k}$. Then, from Corollary 2.1.1,

$\sum_{k=0}^{n-1}\rho_{n,j}^{k} = 0,$ so the first *m* rows of CW_j^T are 0.

The next *n* rows look like: $b_1 + b_2\,\rho_{n,j}^1 + \quad b_3\rho_{n,j}^2 + \quad ... \quad + b_n\rho_{n,j}^{(n-1)}$; $1 \le j \le n-1$ which, from Theorem 2.1.1, is the eigenvalue corresponding to eigenvector $\left[1, \rho_{n,j}^1, \quad \rho_{n,j}^2, \quad ... \quad , \rho_{n,j}^{(n-1)}\right]$ for $1 \le j \le n-1.$

Thus we have proved that

$$W_j^T = \left[0_{1,m}, V_j^T\right] = \left[000...001\rho_{n,j}^1 \quad \rho_{n,j}^2 \quad ... \quad \rho_{n,j}^{(n-1)}\right]^T \quad 1 \le j \le n-1.$$

is an eigenvector of $C = A \oplus B.$

Applying the same method, we can show that $W_{n+j}^T = \left[U_j^T, 0_{1,n}\right]$ for $1 \le j \le m-1$ are eigenvectors of $C = A \oplus B.$

To determine the first set of eigenvalues of $A \oplus B,$ we set $\underline{v} = \left(0_{1,m}, \underline{x}^T,\right)^T$ where $\underline{x}^T = (x_1, .x_2, ..., x_n)$ and solve for $(A \oplus B)\underline{v} = \lambda\underline{v}$. We specifically select $\underline{v}$ to be of this form as we understand the join of sub-graphs *A* and *B* and this vector isolates the edges in *B*.

Solving $(A \oplus B)\underline{v} = \lambda\underline{v},$ we get

$$\begin{bmatrix} A_{m,m} & J_{m,n} \\ J_{n,m} & B_{n,n} \end{bmatrix} \begin{bmatrix} 0_{m,1} \\ \underline{x}_{n,1} \end{bmatrix} = \lambda \begin{bmatrix} 0_{m,1} \\ \underline{x}_{n,1} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} J_{m,n}\,\underline{x}_{n,1} \\ B_{mn,n}\,\underline{x}_{n,1} \end{bmatrix} = \lambda \begin{bmatrix} 0_{nx1} \\ \underline{x}_{nx1} \end{bmatrix}$$

Solving $B\underline{x} = \lambda\underline{x}$ we get the eigenvalues of B, which are, as per Theorem 2.1.1, $\lambda_k = b_1 + b_2\rho_j + b_3{\rho_j}^2 + ... + b_m{\rho_j}^{n-1},\ 1 \le k \le n-1.$

To determine the next set of eigenvalues of $A \oplus B,$ we set $\underline{v} = \left(\underline{x}^T, 0_{1,n}\right)^T$ where $\underline{x}^T = \left(x_1, .x_2, ..., x_{m,}\right)$ and solve for $(A \oplus B)\underline{v} = \lambda\underline{v}$. We specifically select $\underline{v}$ to be of this form as we understand the join of sub-graphs *A* and *B*, and this vector isolates the edges in *A*.

Solving $(A \oplus B)\underline{v} = \lambda \underline{v},$ we get

$$\begin{bmatrix} A_{m,m} & J_{m,n} \\ J_{n,m} & B_{n,n} \end{bmatrix} \begin{bmatrix} \underline{x}_{m,1} \\ 0_{n,1} \end{bmatrix} = \lambda \begin{bmatrix} \underline{x}_{m,1} \\ 0_{n,1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_{m,m}\underline{x}_{m,1} \\ J_{n,m}\underline{x}_{m,1} \end{bmatrix} = \lambda \begin{bmatrix} \underline{x}_{m,1} \\ 0_{n,1} \end{bmatrix}$$

Solving $A\underline{x} = \lambda \underline{x}$ we get the eigenvalues of A, which are, as per Theorem 2.1.1, $\lambda_{n+k} = a_1 + a_2\rho_j + a_3\rho_j^{\,2} + \ldots + a_m\rho_j^{\,m-1},\ 1 \le k \le m-1.$

To find the eigenvalues λ_n and λ_{n+m} of $A \oplus B,$ we solve: $(A \oplus B)\underline{v} = \lambda \underline{v}$, where $\underline{v} = \left(J_{1,m}, \alpha J_{1,n}\right)^T$. The edges between the two graphs *A* and *B*, which form the join between the sub-graphs, are significant in the determination of the conjugate eigen-pair of the adjacency matrix of the resultant graph. We use the factor of α in the vector $\underline{v}$ to assist in obtaining the conjugate eigenvalues as follows:

$$\begin{bmatrix} A_{m,m} & J_{m,n} \\ J_{n,m} & B_{n,n} \end{bmatrix} \begin{bmatrix} J_{mx1} \\ \alpha_{nx1} \end{bmatrix} = \lambda \begin{bmatrix} J_{m,1} \\ \alpha_{n,1} \end{bmatrix}$$

$$\begin{bmatrix} (d_A + n\alpha)_{m,1} \\ (m + \alpha d_B)_{n,1} \end{bmatrix} = \lambda \begin{bmatrix} J_{m,1} \\ \alpha J_{n,1} \end{bmatrix}$$

$$d_A + n\alpha = \lambda$$
$$m + \alpha d_B = \lambda\alpha$$

Therefore,

$$m + \alpha d_B = (d_A + n\alpha)\alpha$$
$$n\alpha^2 + \alpha(d_A - d_B) - m = 0$$

$$\alpha = \frac{-(d_A - d_B) \pm \sqrt{(d_A - d_B)^2 + 4nm}}{2n}$$

So, the conjugate pair of eigenvalues are

$$\lambda = n\left[\frac{-(d_A - d_B) \pm \sqrt{(d_A - d_B)^2 + 4nm}}{2n}\right] + d_A \qquad \square$$

2.7 Eigenvalues of wheel graphs

2.7.1 Wheel graph

Let W_n be a wheel graph on *n* vertices, with $(n-1)$ spokes, and with the central vertex labelled v_1 and the outer vertices $v_2, v_3, ..., v_n$.

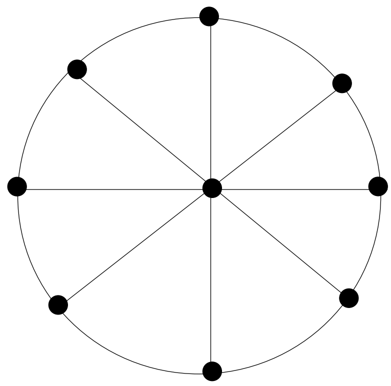

Figure 2.7.1.1: Wheel graph C_9

Then the adjacency matrix of W_n is

$$A(W_n) = \overline{K_1} \oplus C_{n-1} = \begin{bmatrix} A & J_{1,n-1} \\ J_{1,n-1}{}^T & B \end{bmatrix}$$

where

- *A* is a *1x1* matrix of a single vertex (the number zero), so $m = 1$ in this example
- *B* is an $(n-1)x(n-1)$ adjacency matrix of a cycle on $(n-1)$ vertices.

So, from Theorem 2.6.1, W_n has eigenvalues:

$$\lambda_k \quad = b_1 + b_2\rho_n^k + b_3\rho_n^{2k} + \ldots + b_n\rho_n^{(n-1)k} \text{ for } 1 \le k \le n-2;$$

$$= \rho_n^k + \rho_n^{(n-1)k} \text{ where } \rho_n = \exp\left(\frac{2\pi i}{n}\right);$$

$$\lambda_{n+k} \quad = a_1 + a_2\rho_n^k + a_3\rho_n^{2k} + \ldots + a_m\rho_n^{(m-1)k} \text{ for } 1 \le k \le m-1;$$

which gives no eigenvalues as $m-1=0$

$\{\lambda_{n-1}, \lambda_n\}$

$$= \{(n-1)\alpha + d_A\}$$

where $(n-1)\alpha^2 + \alpha(d_A - d_B) - 1 = 0$ and $d_A = 0, d_B = 2$

i.e., $(n-1)\alpha^2 - 2\alpha - 1 = 0$

$$\Rightarrow \alpha = \frac{2 \pm \sqrt{4 + 4(n-1)}}{2(n-1)}.$$

Therefore,

$$\{\lambda_{n-1}, \lambda_n\} = \{(n-1)\alpha + d_A\}$$

$$= \left\{(n-1)\left(\frac{2 \pm \sqrt{4+4(n-1)}}{2(n-1)}\right) + 0\right\}$$

$$= \left\{\frac{2 \pm \sqrt{4+4(n-1)}}{2}\right\}$$

$$= \{1 \pm \sqrt{n}\}$$

Example 2.7.1.1

For the case $n = 4$, we have the adjacency matrix of W_4 as

$A(W_4)$

$$= \overline{K_1} \oplus C_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of W_4 are

$$\lambda_k = \rho_3^k + \rho_3^{2k} \text{ where } \rho_3 = \exp\left(\frac{2\pi i}{3}\right); \text{ for } 1 \le k \le 2$$

$$\begin{aligned}
\lambda_1 &= \rho_3^1 + \rho_3^2 \\
&= \left[\exp\left(\frac{2\pi i}{3}\right)\right]^1 + \left[\exp\left(\frac{2\pi i}{3}\right)\right]^2 \\
&= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \\
&= -0.5 + 0.5i - 0.5 - 0.5i \\
&= -1
\end{aligned}$$

$$\begin{aligned}
\lambda_2 &= \rho_3^2 + \rho_3^4 \\
&= \left[\exp\left(\frac{2\pi i}{3}\right)\right]^2 + \left[\exp\left(\frac{2\pi i}{3}\right)\right]^4 \\
&= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) + \cos\left(\frac{8\pi}{3}\right) + i\sin\left(\frac{8\pi}{3}\right) \\
&= -0.5 - \frac{\sqrt{3}}{2}i - 0.5 + \frac{\sqrt{3}}{2}i \\
&= -1
\end{aligned}$$

$$\begin{aligned}
\{\lambda_3, \lambda_4\} &= \{1 \pm \sqrt{4}\} \\
&= \{3, -1\}.
\end{aligned}$$

So W_4 has eigenvalues 3 (of multiplicity 1) and (-1) of multiplicity 3.

2.7.2 Generalised wheel graph

Let X_{m+n} be a generalised wheel graph on $(m+n)$ vertices, with (mn) spokes. Then $X_{m+n} = \overline{K_m} \oplus C_n$, and X_{4+5} is an example of a generalised wheel graph.

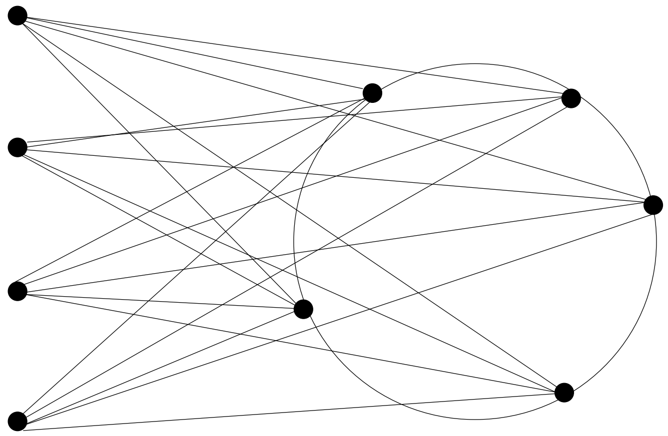

Figure 2.7.2.1: Generalised wheel graph X_{4+5}

Let the central vertices be labelled $v_1, v_2, ..., v_m$ and the outer vertices form a cycle on *n* vertices labelled $v_{m+1}, v_{m+2}, ..., v_{m+n}$.

Then the adjacency matrix of X_{m+n} is

$$A(X_{m+n}) = A(\overline{K_m} \oplus C_n) = \begin{bmatrix} A & J_{m,n} \\ J_{m,n}{}^T & B \end{bmatrix}$$

where

- A is an mxm matrix of 0's; and
- B is an nxn matrix of a cycle on *n* vertices.

So, from Theorem 2.6.1, $A(X_{m+n})$ has eigenvalues

$$\begin{aligned} \lambda_k &= b_1 + b_2\rho_n^k + b_3\rho_n^{2k} + ... + b_n\rho_n^{(n-1)k} \text{ for } 1 \le k \le n-1 \\ &= \rho_n^k + \rho_n^{(n-1)k} \text{ where } \rho_n = \exp\left(\frac{2\pi i}{n}\right) \end{aligned}$$

$$\begin{aligned} \lambda_{n+k} &= a_1 + a_2\rho_n^k + a_3\rho_n^{2k} + ... + a_m\rho_n^{(m-1)k} \text{ for } 1 \le k \le m-1 \\ &= 0 \text{ for } 1 \le k \le m-1 \end{aligned}$$

$\{\lambda_n, \lambda_{n+m}\} = \{n\alpha + d_A\}$ where $n\alpha^2 + \alpha(d_A - d_B) - m = n\alpha^2 - 2\alpha - m = 0.$

So, $n\alpha^2 - 2\alpha - m = 0$

$$\Rightarrow \alpha = \frac{2 \pm \sqrt{4 + 4mn}}{2n}$$

$$= \frac{1 \pm \sqrt{1 + mn}}{n}.$$

Therefore,

$$\{\lambda_n, \lambda_{n+m}\} = \left\{1 \pm \sqrt{(1 + mn)}\right\}.$$

Example 2.7.2.1

For the case m=2 and n=4, we have the adjacency matrix of X_{2+4} as $A(X_{2+4})$

$$A(X_{2+4}) = \overline{K_2} \oplus C_4 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of $A(X_{2+4})$ are

$\lambda_k = \rho_4^k + \rho_4^{3k}$ where $\rho_4 = \exp\left(\frac{2\pi i}{4}\right), k = 1,2,3$

$\lambda_1 = \rho_4^1 + \rho_4^3$
$= 0$

$\lambda_2 = \rho_4^2 + \rho_4^6$

$= -2$

$\lambda_3 = \rho_4^3 + \rho_4^9$
$= 0$

$\lambda_5 = 0$

$\{\lambda_4, \lambda_6\} = \left\{1 \pm \sqrt{1 + (4)(2)}\right\}$
$= \{4, -2\}$

So $A(X_{2+4})$ has eigenvalues 4 (of multiplicity 1), 0 (of multiplicity 3), and -2 (of multiplicity 2).

2.7.3 Generalised complete wheel graph

Let Y_{m+n} be a generalised complete wheel graph on $(m+n)$ vertices, consisting of *mn* spokes. Then $Y_{m+n} = \overline{K_m} \oplus K_n$, and Y_{3+5} is an example of a generalised complete wheel graph.

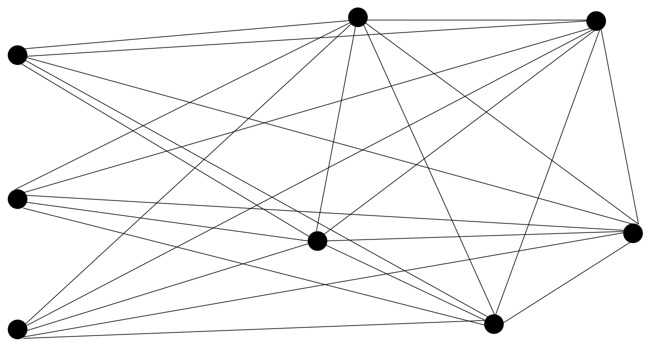

Figure 2.7.3.1: Generalised complete wheel graph Y_{3+5}

Label the centre *m* vertices $v_1 v_2, \ldots, v_m$, and let the outer vertices form a complete graph on *n* vertices, labelled $v_{m+1}, v_{m+2}, \ldots, v_{m+n}$. Then the adjacency matrix Y_{m+n} is

$$A(Y_{m+n}) = A(\overline{K_m} \oplus K_n) = \begin{bmatrix} A & J_{m,n} \\ J_{m,n}{}^T & B \end{bmatrix}$$

Where

- *A* is an *mxm* matrix of 0's
- *B* is an *nxn* matrix of a complete graph on *n* vertices.

So, from Theorem 2.6.1 $A(Y_{m+n})$ has eigenvalues

$$\begin{aligned} \lambda_k &= b_1 + b_2\rho_n^k + b_3\rho_n^{2k} + \ldots + b_n\rho_n^{(n-1)k} \text{ for } 1 \le k \le n-1; \\ &= \rho_n^k + \rho_n^{2k} + \ldots + \rho_n^{(n-1)k} \text{ where } \rho_n = \exp\left(\frac{2\pi i}{n}\right) \end{aligned}$$

$$\begin{aligned}\lambda_{n+k} &= a_1 + a_2\rho_n^k + a_3\rho_n^{2k} + \ldots + a_m\rho_n^{(m-1)k} \text{ for } 1 \le k \le m-1;\\ &= 0 \text{ for } 1 \le k \le m-1\end{aligned}$$

$$\{\lambda_n, \lambda_m\} = \{n\alpha + d_A\}$$

where $n\alpha^2 + \alpha(d_A - d_B) - m = 0$ and $m \ne 0, n \ne 0, d_A = 0$ and $d_B = (n-1)$.

So $n\alpha^2 + \alpha[0 - (n-1)] - m = 0$

$$\Rightarrow n\alpha^2 - (n-1)\alpha - m = 0$$

$$\Rightarrow \alpha = \frac{(n-1) \pm \sqrt{(n-1)^2 + 4nm}}{2n}.$$

Therefore,

$$\{\lambda_n, \lambda_m\} = \left\{n\alpha + d_A = n * \frac{(n-1) \pm \sqrt{(n-1)^2 + 4mn}}{2n} + 0\right\}$$

$$= \left\{n\left[\frac{(n-1) \pm \sqrt{(n-1)^2 + 4mn}}{2n}\right]\right\}$$

$$= \left\{\frac{(n-1) \pm \sqrt{(n-1)^2 + 4mn}}{2}\right\}.$$

Example 2.7.3.1

For the case *m* = 2 and *n* = 4, we have the adjacency matrix of Y_{2+4} as

$$A(Y_{2+4}) = \overline{K_2} \oplus K_4 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of $A(Y_{2+4})$ are:

$$\lambda_k = \rho_4^k + \rho_4^{2k} + \rho_4^{3k} \text{ where } \rho_4 = \exp\left(\frac{2\pi i}{4}\right); \text{ for } 1 \le k \le 3$$

$$\begin{aligned}
\lambda_1 &= \rho_4^1 + \rho_4^2 + \rho_4^3 \\
&= \exp\left(\frac{2\pi i}{4}\right) + \left[\exp\left(\frac{2\pi i}{4}\right)\right]^2 + \left[\exp\left(\frac{2\pi i}{4}\right)\right]^3 \\
&= (0+i) + (-1+0) + (0-i) \\
&= -1
\end{aligned}$$

$$\begin{aligned}
\lambda_2 &= \rho_4^2 + \rho_4^4 + \rho_4^6 \\
&= \left[\exp\left(\frac{2\pi i}{4}\right)\right]^2 + \left[\exp\left(\frac{2\pi i}{4}\right)\right]^4 + \left[\exp\left(\frac{2\pi i}{4}\right)\right]^6 \\
&= (-1+0) + (1+0) + (-1+0) \\
&= -1
\end{aligned}$$

$$\begin{aligned}
\lambda_3 &= \rho_4^3 + \rho_4^6 + \rho_4^9 \\
&= \left[\exp\left(\frac{2\pi i}{4}\right)\right]^3 + \left[\exp\left(\frac{2\pi i}{4}\right)\right]^6 + \left[\exp\left(\frac{2\pi i}{4}\right)\right]^9 \\
&= (0-i) + (-1+0) + (0+i) \\
&= -1
\end{aligned}$$

$\lambda_5 = 0$

$$\begin{aligned}
\{\lambda_4, \lambda_6\} &= \left\{\frac{3 \pm \sqrt{9 + 4.2.4}}{2}\right\} \\
&= \left\{\frac{3 \pm \sqrt{41}}{2}\right\}
\end{aligned}$$

So $A(Y_{2+4})$ has eigenvalues -1 (of multiplicity 3), 0 (of multiplicity 1) and $\frac{3 \pm \sqrt{41}}{2}$ (each of multiplicity 1).

2.8 Eigenvalues of star graphs

In this section, we determine the eigenvalues of the adjacency matrices of various examples of star graphs. In section 2.8.1, we use the fact that the star graph with *m* rays of length 1, is also a bipartite graph, to determine the eigenvalues of the adjacency matrix of this graph. In section 2.8.2, we use the determinant method, and expand the determinant to determine the eigenvalues of the adjacency matrix of a star graph with *m* rays of length 2. Finally, in section 2.8.3, we remove a vertex, and apply Lemma 2.2 to determine the eigenvalues of the adjacency matrix of the star graph with *m* rays of length 2. We obtain the same results as in section 2.8.2 and show that there is often more than one method to obtain the eigenvalues of the adjacency matrix of a graph.

2.8.1 Star graph with *m* rays of length 1

Take *m* copies of the path P_{k+1}, join the paths at their end vertices, in the central vertex *u*: denote the graph on $km+1$ vertices by $S_{1,mP_{k+1}}$; $m \geq 2, k \geq 2$

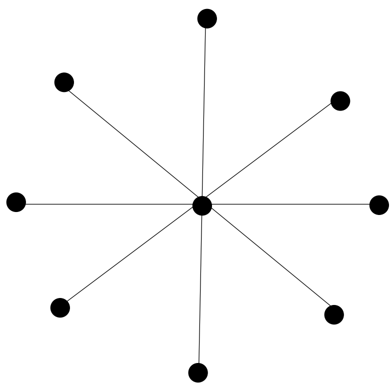

Figure 2.8.1.1: Star graph $S_{1,8P_2}$

If $k=1$ and $m \geq 3$, then we get the star graph $K_{1,m}$ with *m* rays of length 1 which has eigenvalues of 0 (multiplicity $m-1$) and $\pm\sqrt{m}$ (from section 2.5 above).

2.8.2 Star graph with *m* rays of length 2

If $k=2$ and $m \geq 2$, we have the star graph S_{1,mP_3}, where $S_{1,8P_3}$ is drawn below.

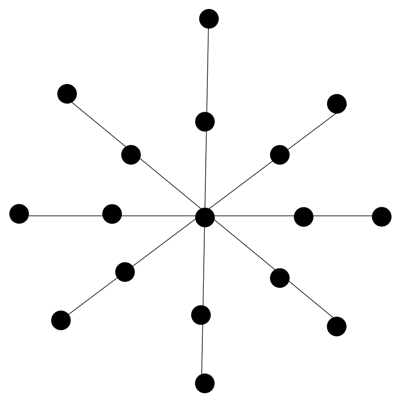

Figure 2.8.2.1: Star graph $S_{1,8P_3}$

We label the vertices of the star graph S_{1,mP_3} with m rays of length 2 on $2m+1$ vertices as follows:

The central vertex is called u, the sets of m vertices a distance 1 and 2 respectively from u are labelled $V_1 = \{v_1^1, v_2^1, ..., v_m^1\}$ and $V_2 = \{v_1^2, v_2^2, ..., v_m^2\}$ respectively.

For constructing the adjacency matrix A of the S_{1,mP_3} we label the central vertex u as v_1, the vertices of V_1 and V_2 as $V_1 = \{v_2, v_3, ..., v_{m+1}\}$ and $V_2 = \{v_{m+2}, v_{m+3}, ..., v_{2m+1}\}$ respectively.

For $k=2, m=2$ we have the path P_5 with adjacency matrix

$$A(S_{1,2P_3}) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(\lambda I - A(S_{1,2P_3})) = \det\begin{bmatrix} \lambda & -1 & -1 & 0 & 0 \\ -1 & \lambda & 0 & -1 & 0 \\ -1 & 0 & \lambda & 0 & -1 \\ 0 & -1 & 0 & \lambda & 0 \\ 0 & 0 & -1 & 0 & \lambda \end{bmatrix}_{5x5}$$

and expanding using the first row,

$$\det(\lambda I - A(S_{1,2P_3})) \quad = \lambda\begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix} + \begin{vmatrix} -1 & 0 & -1 & 0 \\ -1 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix}$$

$$-\begin{vmatrix} -1 & \lambda & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix}.$$

Now expanding the last 2 determinants about the 3rd and 4th rows respectively, we get

$$\det(\lambda I - A(S_{1,2P_3})) = \lambda\begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix} + \lambda\begin{vmatrix} -1 & 0 & 0 \\ -1 & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} - \lambda\begin{vmatrix} -1 & \lambda & -1 \\ -1 & 0 & 0 \\ 0 & -1 & \lambda \end{vmatrix}.$$

Finally, expanding the last two determinants about the 1st and 2nd rows respectively, we get

$$\det(\lambda I - A(S_{1,2P_3})) = \lambda\begin{vmatrix} \lambda & 0 & -1 & 0 \\ 0 & \lambda & 0 & -1 \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix} - \lambda\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} - \lambda\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix}.$$

The first determinant involves the circulant matrix with solutions:

$$\left[\exp(\frac{2\pi ij}{4})\right]^2 = \exp(\pi ij); \quad j = 0,1,2,3.$$

The second determinant involves the circulant matrix with solutions:

$$\exp(\frac{2\pi ij}{2}) = \exp(\pi ij); \; j = 0,1.$$

Thus the characteristic polynomial is:

$$\begin{aligned}
&\lambda(\lambda-1)^2(\lambda+1)^2 - 2\lambda(\lambda-1)(\lambda+1) \\
&= \lambda(\lambda-1)(\lambda+1)[(\lambda-1)(\lambda+1)-2] \\
&= \lambda(\lambda-1)(\lambda+1)(\lambda^2-3)
\end{aligned}$$

So, the eigenvalues of the adjacency matrix of a star graph, with 2 rays of length 2, are $0,\ 1,\ -1,\ \pm\sqrt{3},$ each of multiplicity 1.

We generalize this finding to the adjacency matrix of the star graph on $2m+1$ vertices with m rays of length 2. For $m \geq 2$ and $k = 2$ we have $2m+1$ vertices:

$u, v_1^1, v_2^1, \ldots, v_m^1, v_1^2, v_2^2, \ldots, v_m^2$, and the star graph S_{1,mP_3}

The adjacency matrix is

$$A(S_{1,mP_3}) = \begin{bmatrix} 0_{1,1} & J_{1,m} & 0_{1,m} \\ J_{m,1} & 0_{m,m} & I_{m,m} \\ 0_{m,1} & I_{m,m} & 0_{m,m} \end{bmatrix}_{(2m+1)x(2m+1)} .$$

The characteristic polynomial is

$$\det(\lambda I - A(S_{1,mP_3}))$$

$$= \det\begin{bmatrix} \lambda & -J_{1,m} & 0_{1,m} \\ -J_{m,1} & \lambda I_{m,m} & -I_{m,m} \\ 0_{m,1} & -I_{m,m} & \lambda I_{m,m} \end{bmatrix}_{(2m+1)x(2m+1)}$$

$$== \det\begin{bmatrix} \lambda & -1 & -J_{1,m-1} & 0 & 0_{1,m-1} \\ -1 & \lambda & 0_{1,m-1} & -1 & 0_{1,m-1} \\ -J_{m-1,1} & 0_{m-1,1} & \lambda I_{m-1,m-1} & 0_{m-1,1} & -I_{m-1,m-1} \\ 0 & -1 & 0_{1,m-1} & \lambda & 0_{m-1,1} \\ 0_{m-1,1} & 0_{m-1,1} & -I_{m-1,m-1} & 0_{m-1,1} & \lambda I_{m-1,m-1} \end{bmatrix}$$

Expanding the determinant using the first row, we get:

$$\det(\lambda I - A(S_{1,mP_3}))$$
$$= \lambda \det\begin{bmatrix} \lambda I_{m,m} & -I_{m,m} \\ -I_{m,m} & \lambda I_{m,m} \end{bmatrix}_{2mx2m} + m\det\begin{bmatrix} -1 & 0_{1,m-1} & -1 & 0_{1,m-1} \\ -J_{m-1,1} & \lambda I_{m-1,m-1} & 0_{m-1,1} & -I_{m-1,m-1} \\ 0 & 0_{1,m-1} & \lambda & 0_{m-1,1} \\ 0_{m-1,1} & -I_{m-1,m-1} & 0_{m-1,1} & \lambda I_{m-1,m-1} \end{bmatrix}$$

There are m occurrences of the second term in the expression above, as the expansion of all the m non-zero entries in the first row yield the same minor as above, with alternating signs.

Now expanding the determinant of the second term using the $(m+1)$th row, we get:

$$\det(\lambda I - A(S_{1,mP_3}))$$
$$= \lambda \det\begin{bmatrix} \lambda I_{m,m} & -I_{m,m} \\ -I_{m,m} & \lambda I_{m,m} \end{bmatrix}_{2mx2m} + m(-1)^{m+1+m+1}\lambda \det\begin{bmatrix} -1 & 0_{1,m-1} & 0_{1,m-1} \\ -J_{m-1,1} & \lambda I_{m-1,m-1} & -I_{m-1,m-1} \\ 0_{m-1,1} & -I_{m-1,m-1} & \lambda I_{m-1,m-1} \end{bmatrix}_{2m-1,2m-1}$$

Now expanding the determinant in the second term using the first row, we get:

$$\det(\lambda I - A(S_{1,mP_3}))$$
$$= \lambda \det\begin{bmatrix} \lambda I_{m,m} & -I_{m,m} \\ -I_{m,m} & \lambda I_{m,m} \end{bmatrix}_{2mx2m} + m(-1)^{m+1+m+1}\lambda(-1)\det\begin{bmatrix} \lambda I_{m-1,m-1} & -I_{m-1,m-1} \\ -I_{m-1,m-1} & \lambda I_{m-1,m-1} \end{bmatrix}_{2m-2,2m-2}$$

$$= \lambda \det\begin{bmatrix} \lambda I_{m,m} & -I_{m,m} \\ -I_{m,m} & \lambda I_{m,m} \end{bmatrix}_{2mx2m} - m\lambda \det\begin{bmatrix} \lambda I_{m-1,m-1} & -I_{m-1,m-1} \\ -I_{m-1,m-1} & \lambda I_{m-1,m-1} \end{bmatrix}_{2m-2,2m-2}$$

The determinant in the first term comes from the circulant matrix with eigenvalues

$$\exp\left(\frac{2\pi ij}{2m}\right)^m;\ \ 0 \le j \le 2m-1,$$

which yields eigenvalues $\lambda = 1$ repeated m times and eigenvalue $\lambda = -1$ repeated m times.

The determinant in the second term comes from the circulant matrix with eigenvalues

$$[\exp(\frac{2\pi ij}{2m-2})]^{m-1};\ 0 \le j \le 2m-1,$$

which yields eigenvalues $\lambda = 1$ repeated $(m-1)$ times and eigenvalue $\lambda = -1$ repeated $(m-1)$ times.

This yields the characteristic polynomial:

$$\begin{aligned}
&\det(\lambda I - A(S_{1,mP_3})) \\
&= \lambda(\lambda-1)^m(\lambda+1)^m - m\lambda(\lambda-1)^{m-1}(\lambda+1)^{m-1} \\
&= \lambda(\lambda-1)^{m-1}(\lambda+1)^{m-1}[(\lambda-1)(\lambda+1)-m] \\
&= \lambda(\lambda-1)^{m-1}(\lambda+1)^{m-1}[(\lambda^2-(m+1)]
\end{aligned}$$

Thus the eigenvalues of the adjacency matrix $A(S_{1,mP_3})$ of a star graph with *m* rays of length 2, are:

- $\lambda = 1$ with multiplicity $m-1$;
- $\lambda = -1$ with multiplicity $m-1$;
- $\lambda = 0$ with multiplicity 1; and
- $\lambda = \pm\sqrt{m+1},$ each with multiplicity 1.

2.8.3 Star graph with *m* rays of length 2 (removing a vertex)

Let *G* be a star graph with central vertex *x* and *m* rays of length 2. Let *x* be adjacent to *y*.
From Lemma 2.2, we get

$$P_{A(G)}(\lambda) = \lambda P_{A(G\backslash x)}(\lambda) - \sum_{all\ x\ adj\ y} P_{A(G\backslash xy)}(\lambda)$$

If we remove *x* from *G*, we get *m* copies of $K_2,$ each having eigenvalues 1 and -1. So

$$\lambda\ P_{A(G\backslash x)}(\lambda) = \lambda(\lambda-1)^m(\lambda+1)^m$$

If we remove *x* and its adjacent vertex *y* from *G*, we get $m-1$copies of K_2 and a single vertex. Thus, since there are *m* vertices adjacent to *x*, we get

$$\sum P_{A(G\backslash xy)}(\lambda) = m\lambda(\lambda-1)^{m-1}(\lambda+1)^{m-1}$$

Therefore,

$$\begin{aligned}
&P_{A(G)}(\lambda)\\
&\quad=\lambda(\lambda-1)^m(\lambda+1)^m-m\lambda(\lambda-1)^{m-1}(\lambda+1)^{m-1}\\
&\quad=\lambda(\lambda-1)^{m-1}(\lambda+1)^{m-1}[(\lambda-1)(\lambda+1)-m]\\
&\quad=\lambda(\lambda-1)^{m-1}(\lambda+1)^{m-1}[\lambda^2-1-m]\\
&\quad=\lambda(\lambda-1)^{m-1}(\lambda+1)^{m-1}[\lambda^2-(m+1)]
\end{aligned}$$

Thus the eigenvalues of a star graph on $(2m+1)$ vertices of length 2 are:

- $\lambda=1$ with multiplicity $m-1$;
- $\lambda=-1$ with multiplicity $m-1$;
- $\lambda=0$ with multiplicity 1; and
- $\lambda=\pm\sqrt{m+1}$, each with multiplicity 1.

The star graph on $(m+1)$ vertices with *m* rays of length 2, can be extended to the star graph with *m* rays of length *k*. This is not included in this thesis.

2.9 Eigenvalues of graphs with a pendant vertex

Using Lemma 2.3, we can easily obtain the characteristic polynomial of a cycle graph which is connected to a pendant vertex via a single edge. Let G_n be a graph on n vertices, such that G_n is obtained from a cycle C_{n-1} on $(n-1)$ vertices, connected to a pendant vertex x, via a single edge. Then, from Lemma 2.3,

$$P_{A(G_n)}(\lambda)=\lambda P_{A(C_{n-1})}(\lambda)-P_{A(P_{n-2})}(\lambda).$$

Applying Theorem 2.3.1 and Theorem 2.4.1, we get

$$P_{A(G_n)}(\lambda)=\lambda\prod_{j=0}^{n-2}\left[\lambda-2\cos\left(\frac{2\pi j}{n-1}\right)\right]-\prod_{k=1}^{n-2}\left[\lambda-2\cos\left(\frac{\pi k}{n-1}\right)\right].$$

2.10 Eigenvalues of hypercubes

The class of *p*-regular hypercubes H_p, on 2^p vertices and $p2^{p-1}$ edges, can be obtained as the one-dimensional skeleton of the geometric hypercube; for instance, Q_3 is the graph formed by the 8 vertices and 12 edges of a three-dimensional cube. The hypercube has eigenvalues:

$$(p-2k)^{\binom{p}{k}}; k = 0,1,2,\ldots,p\,.$$

See Brouwer and Haemers [12].

2.11 Eigenvalues of the complement of *G*

It is interesting to determine the eigenvalues of the complement of a graph. In particular, we look at the the eigenvalues of the complement of the *k*-regular graph, but first we need Lemma 2.11.1 and Lemma 2.11.2.

Lemma 2.11.1

If *A* is symmetric, and *Q* is orthogonal, then $Q^{-1}AQ$ is symmetric.

Proof

If *A* is symmetric, then $A^T = A$ and if *Q* is othogonal, then $Q^T = Q^{-1}$. Also if *Q* is orthonormal, then so is Q^{-1}, and then $\left(Q^{-1}\right)^T = \left(Q^{-1}\right)^{-1} = Q$.

So,

$$\left(Q^{-1}AQ\right)^T = Q^T A^T \left(Q^{-1}\right)^T = Q^{-1}AQ$$

Therefore, we have proved that $Q^{-1}AQ$ is symmetric. □

Lemma 2.11.2

Let G a graph of order *n* which is regular of degree *k*, and let the eigenvalues of G be $\lambda_n,\ldots,\lambda_2,\lambda_1 = k$. Let $A(G)$ be the adjacency matrix of G . Then there is an orthogonal matrix Q of eigenvectors of $A(G)$, where the first column is $\frac{1}{\sqrt{n}}(1,1,\ldots,1) = pJ_{n,1}{}^T$, where $p = \frac{1}{\sqrt{n}}$, such that $Q^{-1}A(G)Q = diag(k,\lambda_1,\lambda_2,\ldots,\lambda_n)$.
Then Q also diagonalises $J_{n,n}$ and $Q^{-1}J_{n,n}Q = diag(n,0,\ldots,0)$.

Proof

Let $\overline{G}$ be the compliment of $G,$ and let $A'(\overline{G})$ be the adjacency matrix of $\overline{G}.$ Then

$$A'(\overline{G}) = J_{n,n} - I_{n,n} - A(G)$$

and

$$Q^{-1}A'(\overline{G})Q = Q^{-1}J_{n,n}Q - I_{n,n} - Q^{-1}A(G)Q$$

As $\overline{G}$ is $n-1-k$ regular, the sum of the entries in each row of $A'(\overline{G})$ is $(n-1-k)$, and the sum of the entries in each column of $A'(\overline{G})$ is $(n-1-k)$. Therefore, the first entry of LHS is

$$\frac{1}{\sqrt{n}}(1,1,\ldots,1)A'(\overline{G})\frac{1}{\sqrt{n}}(1,1,\ldots,1) = \frac{1}{n}n(n-1-k) = (n-1-k).$$

The first entry of $I_{n,n}$ is 1, and the first entry of $Q^{-1}A(G)Q$ is *k*, so then the first entry of $Q^{-1}J_{n,n}Q$ is n.

For $n = 3$,

let $Q = \begin{bmatrix} p & a & d \\ p & b & e \\ p & c & f \end{bmatrix}$.

Since Q is orthogonal, $Q^{-1} = Q^T = \begin{bmatrix} p & p & p \\ a & b & c \\ d & e & f \end{bmatrix}$ where $p = \frac{1}{\sqrt{3}}$.

Then

$$Q^{-1}J_{n,n}Q = \begin{bmatrix} p & p & p \\ a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p & a & d \\ p & b & e \\ p & c & f \end{bmatrix} = \begin{bmatrix} p & p & p \\ a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 3p & a+b+c & d+e+f \\ 3p & a+b+c & d+e+f \\ 3p & a+b+c & d+e+f \end{bmatrix}$$

Since (a,b,c) and (d,e,f) are perpendicular to (p,p,p) we have $p(a+b+c)=0$ and $p(d+e+f)=0$, therefore

$$Q^{-1}J_{n,n}Q \quad = \begin{bmatrix} p & p & p \\ a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 3p & 0 & 0 \\ 3p & 0 & 0 \\ 3p & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3p(3p) & 0 & 0 \\ 3p(a+b+c) & 0 & 0 \\ 3p(d+e+f) & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9p^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the first entry in the matrix is 3, (as shown above), we have proved the result for $n=3$.

This approach can be generalised for all *n*.

Let Q be an orthogonal matrix of eigenvectors of A, with the first columns $p.J_{n,1}{}^T$ such that $Q^{-1}AQ = diag(k, \lambda_2, ..., \lambda_n)$.

Let $Q = \begin{bmatrix} p & q_{1,2} & q_{1,3} & \cdots & q_{1,n} \\ p & q_{2,2} & q_{2,3} & \cdots & q_{2,n} \\ p & q_{3,2} & q_{3,3} & \cdots & qn \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p & q_{n,2} & q_{n,3} & \cdots & q_{n,n} \end{bmatrix}$

and then, since Q is orthogonal, $Q^{-1} = Q^T = \begin{bmatrix} p & p & p & \cdots & p \\ q_{1,2} & q_{2,2} & q_{3,2} & \cdots & q_{n,2} \\ q_{1,3} & q_{2,3} & q_{3,3} & \cdots & q_{n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{1,n} & q_{2,n} & q_{3,n} & \cdots & q_{n,n} \end{bmatrix}$.

Then

$Q^{-1}J_{n,n}Q$

$$= \begin{bmatrix} p & p & p & \cdots & p \\ q_{1,2} & q_{2,2} & q_{3,2} & \cdots & q_{n,2} \\ q_{1,3} & q_{2,3} & q_{3,3} & \cdots & q_{n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{1,n} & q_{2,n} & q_{3,n} & \cdots & q_{n,n} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p & q_{1,2} & q_{1,3} & \cdots & q_{1,n} \\ p & q_{2,2} & q_{2,3} & \cdots & q_{2,n} \\ p & q_{3,2} & q_{3,3} & \cdots & q_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p & q_{n,2} & q_{n,3} & \cdots & q_{n,n} \end{bmatrix}$$

$$= \begin{bmatrix} p & p & p & \cdots & p \\ q_{1,2} & q_{2,2} & q_{3,2} & \cdots & q_{n,2} \\ q_{1,3} & q_{2,3} & q_{3,3} & \cdots & q_{n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{1,n} & q_{2,n} & q_{3,n} & \cdots & q_{n,n} \end{bmatrix} \begin{bmatrix} np & \sum q_{i,2} & \sum q_{i,3} & \cdots & \sum q_{i,n} \\ np & \sum q_{i,2} & \sum q_{i,3} & \cdots & \sum q_{i,n} \\ np & \sum q_{i,2} & \sum q_{i,3} & \cdots & \sum q_{i,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ np & \sum q_{i,2} & \sum q_{i,3} & \cdots & \sum q_{i,n} \end{bmatrix}$$

Since the vectors in Q are perpendicular to $J_{n,1}{}^T$ we have $\sum q_{i,2} = 0, \sum q_{i,3} = 0, \ldots, \sum q_{i,n} = 0$, and therefore

$$Q^{-1}J_{n,n}Q = \begin{bmatrix} p & p & p & \cdots & p \\ q_{1,2} & q_{2,2} & q_{3,2} & \cdots & q_{n,2} \\ q_{1,3} & q_{2,3} & q_{3,3} & \cdots & q_{n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{1,n} & q_{2,n} & q_{3,n} & \cdots & q_{n,n} \end{bmatrix} \begin{bmatrix} np & 0 & 0 & \cdots & 0 \\ np & 0 & 0 & \cdots & 0 \\ np & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ np & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$Q^{-1}J_{n,n}Q = \begin{bmatrix} n^2p^2 & 0 & 0 & \cdots & 0 \\ np\sum q_{i,2} & 0 & 0 & \cdots & 0 \\ np\sum q_{1,3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ np\sum q_{i,n} & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} n^2p^2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Substituting $p = \frac{1}{\sqrt{n}}$, we have proved the result for the general case, i.e.,

$$Q^{-1}J_{n,n}Q = \begin{bmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

i.e., Q diagonalises $J_{n,n}$ such that $Q^{-1}J_{n,n}Q = diag(n,0,0,...,0)$. □

Theorem 2.11.1

If G is of order *n*, and is *k*-regular, and has eigenvalues $k, \lambda_1, \lambda_2, ..., \lambda_{n-1}$ then the eigenvalues of its complement graph, $\overline{G}$ are: $n-1-k$ and $-1-\lambda_i$; $i = 1,2,...,n-1$. See Fox [24].

Proof

Since G has *n* vertices, and is *k*-regular, then $\overline{G}$ is $n-1-k$ regular, and has eigenvalue $n-1-k$ as any *k*-regular graph has *k* as an eigenvalue.

Let $A(G)$ be the adjacency matrix of G, then $(J_{n,n} - I_{n,n} - A(G))$ is the adjacency matrix of $\overline{G}$.

There is an orthogonal matrix Q of eigenvectors of $A(G)$, with first column $pJ_{n,1}$; *where* $p = \frac{1}{\sqrt{n}}$, such that $Q^{-1}A(G)Q = diag(k, \lambda_1, \lambda_2, ..., \lambda_{n-1})$.

Then, from Lemma 2.11.2,

$$
\begin{aligned}
&Q^{-1}(J_{n,n} - I_{n,n} - A(G))Q \\
&= Q^{-1}J_{n,n}Q - Q^{-1}I_{n,n}Q - Q^{-1}A(G)Q \\
&= diag(n,0,0,...,0) - I_{n,n} - diag(k, \lambda_{1,} \lambda_2, ..., \lambda_{n-1}) \\
&= diag(n-1-k, -1-\lambda_1, -1-\lambda_2, ..., -1-\lambda_{n-1})
\end{aligned}
$$

□

2.12 Eigenvalues of the *C*-matrix

As per Section 1.3.4, the *C*-matrix is defined as $C = K - NK^{-1}N$.

For design graphs, the eigenvalues of N are the same as the eignevalues of the adjacency matrix $A(G)$ of the associated graph G.

$$K = kI_{n,n}; \quad K^{-1} = \frac{1}{k} I_{n,n}$$

Now let Q diagonalise N , i.e., $Q^{-1}NQ = diag(\lambda_1, \lambda_2, ..., \lambda_n)$.

Therefore,

$$
\begin{aligned}
&Q^{-1}CQ \\
&= Q^{-1}(K - NK^{-1}N)Q \\
&= Q^{-1}kI_{n,n}Q - Q^{-1}Nk^{-1}I_{n,n}NQ \\
&= kI_{n,n} - k^{-1}I_{n,n}Q^{-1}NQQ^{-1}NQ \\
&= kI_{n,n} - k^{-1}I_{n,n}diag(\lambda_1, \lambda_2, ..., \lambda_n).diag(\lambda_1, \lambda_2, ..., \lambda_n) \\
&= kI_{n,n} - k^{-1}I_{n,n}[diag(\lambda_1{}^2, \lambda_2{}^2, ..., \lambda_n{}^2)] \\
&= diag(k - \frac{\lambda_1{}^2}{k}, k - \frac{\lambda_2{}^2}{k}, ..., k - \frac{\lambda_n{}^2}{k})
\end{aligned}
$$

So, the eigenvalues of the *C*-matrix are $(k - \frac{\lambda_1{}^2}{k}, k - \frac{\lambda_2{}^2}{k}, ..., k - \frac{\lambda_n{}^2}{k})$.

2.13 Conclusion

In this chapter, we presented different techniques for finding eigenvalues for certain classes of graphs. Some methods use known results, whilst others hinge on the definition of the eigenvalue of the adjacency matrix of a graph. One obviously prefers the shorter, more elegant proofs; however some of the longer proofs illustrate the combinatorial aspects associated with determining eigenvalues of adjacency matrices associated with graphs.

The most significant technique is that of the eigenvector method. In this method, the choice of the form of the eigenvector is often determined by the edge connectivity of the graph involved, and results in determining a pair of conjugate eigenvalues. For the complete graph, each vertex is connected to every other vertex and the K_n graph is $(n-1)$ regular. It is noted that $(n-1)$ is also one of the conjugate eigenvalues of the adjacency matrix associated with the graph K_n. When considering the complete bipartite graph, the strong connectivity between the two partite sets suggested dividing the eigenvector into two separate parts to mimic this characteristic. We noted the following interesting observations:

- For the complete graph K_n, all vertices are of maximum degree of $(n-1)$. We can therefore regard each vertex as a central vertex, and it appears that this gives rise to the conjugate pair of eigenvalues;
- For the bipartite graph, the two disjoint sets of vertices are ‘strongly’ connected to each other, so that each set can be regarded as a central aspect of the graph contributing to its conjugate eigen-pair;
- For the wheel graph, the connectivity of the central vertex to every other vertex led to the formation of a vector that resulted in a conjugate pair of eigenvalues;
- For the join of two graphs, these graphs, by definition of the join of two graphs, involve a ‘strong’ connection between the two graphs. This connection allowed for the generation of the conjugate eigen-pair; and
- For the star graph, the central vertex is at the end of each of the rays of the graph and there is a conjugate pair of eigenvalues of the adjacency matrix associated with the star graph.

For each of the classes of graphs above, there exists a conjugate pair of eigenvalues whose sum and product are integral. Graphs which are well connected, or *edge-balanced*, in terms of a centrally defined set of vertices, appear to give rise to a *conjugate pair* of eigenvalues. This significant characteristic is used in the definitions in the next chapter. The edges incident with the vertex/vertices having the ‘central vertex’ attribute appear to stabilize or ‘balance’ the graph; hence the idea of balance was adopted in Chapter 3.

Also note that the cycle graphs and the path graphs are not well connected, and do not have a central vertex. They also do not have a conjugate pair of eigenvalues.

CHAPTER 3

INTEGRAL EIGEN-PAIR BALANCED CLASSES OF GRAPHS

In this chapter, the significance of integers and eigenvalues allows us to define *eigen-sum* and *eigen-product* balanced properties of classes of graphs and designs involving a non-zero pair (a,b) of eigenvalues. Using the non-zero property of the eigen-pair, and the idea of robustness, we consider the ratio of the eigen-pair sum to the eigen-pair product, and the *asymptotic* behaviour of this ratio (in terms of large values of the order of the graph/designs). This may have significance in networks as they involve a large number of vertices.

The product of the average degree of a graph with the Riemann integral of the eigen-bi-balanced ratio is introduced as the eigen-bi-balanced ratio *area* of classes of graphs/designs providing a further dimension to the robustness associated with graphs. We observe that the area of the class of complete graphs appears to be the largest. Also, the interval of asymptotic convergence of the unique eigen-bi-balanced ratio of classes of graphs appears to be $[-1,0]$.

We also define new concepts of eigen-bi-balanced density, energy and asymptotes, and finally define a matrix eigen-bi-balanced ratio.

The definitions of the above eigen-bi-balanced properties of classes of graphs are original, and therefore this section comprises original work. In most of the examples, the eigenvalues as determined in Chapter 2 have been used to determine the associated eigen-bi-balanced properties of the associated classes of graphs.

3.1 Integers, conjugate pairs and eigenvalues of a graph

There has been much work done in the analysis of eigenvalues of matrices which are adjacency matrices of associated graphs. The following are some examples of these findings, with the references as specified:

- There has been interest in classes of graphs whose *pairs* of eigenvalues satisfy certain conditions. In Sarkar and Mukherjee [43], graphs are considered with reciprocal *pairs* of eigenvalues $(\lambda, \frac{1}{\lambda})$ whose *product* is the integer 1;

- *Pairs* of eigenvalues (1,-1), summing to 0, and whose product is -1, are considered in Dias [23];

- *Summing* the eigenvalues of the adjacency matrix of a graph is connected to the *energy* of physical structures - see Aimei and Feng [2]; and

- In the paper by van Dam [19], on regular graphs with 4 eigenvalues, he considers the eigenvalue pair of real conjugates $\frac{a \pm \sqrt{b}}{2}$ and shows that if a matrix has an eigenvalue $\frac{a+\sqrt{b}}{2}$, then it has an eigenvalue $\frac{a-\sqrt{b}}{2}$ of the same multiplicity, and vice versa. Adding the pair of conjugates $\frac{a+\sqrt{b}}{2}$ and $\frac{a-\sqrt{b}}{2}$, we obtain the integer *a*. Their product is $\frac{a^2-b}{4}$ which is integral, provided the numerator is a multiple of 4. The paper shows that there are graphs whose matrices have conjugate pairs of eigenvalues whose sum does not necessarily sum to the *same* integer a.

The following references show other areas of research which, together with the results on eigenvalues, provide motivation for the new definitions which are contained in this chapter.

- There has been interest in the importance of pairs of numbers, whose sum and product produce the same integral constant, and this exists outside the linear algebra of matrices - see, for example, Dettmann and Morris [22];

- In Brouwer and Haemers [12], integral trees (where the eigenvalues of trees are integral**)** are investigated;

- In the cryptography article, Hamada [29] considers the conjugate code *pair c*onsisting of linear codes $[n,k']$ and $[n,k'']$ satisfying the constant (integral) sum term $k'+k''=n+k$ where *n* is the dimension of the vector space involved and *k* is the *k*-digit secret information sent;

- In the paper by Kadin [34], he investigates the Cooper *pair* of opposite wave vectors *k* and –*k* which *balance* by summing to 0 and whose product is $-k^2$;

- Hinch and Leal [31] consider the notion of an isolated particle in the absence of rotary Brownian motion, under the condition that the hydrodynamic and external field *couples* exactly *balance* one another; and

- Armstrong [4] investigates the importance of the quadratic part of a characteristic equation which has the form: $x^2 - \tau x + \delta$. This quadratic gives rise to the two eigenvalues $a,b=\dfrac{\tau\pm\sqrt{\tau^2-4\delta}}{2}$. The *sum* and *product* (τ, and δ respectively), are often referred to as the eigen-pair, but we shall focus on the *pair of eigenvalues* (a,b) as the eigen-pair.

Generally, there often exist two eigenvalues (associated with the adjacency matrix of a graph) whose sum or product is integral. In chapter 2, we found conjugate eigen-pairs associated with the adjacency matrix of certain classes of graphs, such as complete graphs, wheel graphs and complete bipartite graphs. Adding or multiplying the pair of eigenvalues results in an integer. It is therefore possible to get the same integer when adding or multiplying two distinct, non-zero eigenvalues. This integer is either a fixed constant, or a function of an inherent property of the graph, for all graphs belonging to a certain class of graphs. For example, complete graphs K_n on n vertices, have a pair of eigenvalues with sum of $f(n)=n-2$, and product of $g(n)=1-n$ for each $n\geq 2$, and the complete bipartite graphs $K_{\frac{n}{2},\frac{n}{2}}$ on n vertices have eigen-pair sum (of non-zero eigenvalues) of 0 and product of $\dfrac{-n^2}{4}$.

Definition 3.1.1: Function $f(p)$ of a member of a class of graphs

We define a *function of a member* belonging to a class of graphs, as a real function $f(p)$ of an inherent property p of the member in the class, such as the number of vertices or the clique number of a graph, etc.

In this chapter, we combine the ideas of a pair of eigenvalues and their balanced integral sum and product with respect to a class of graphs, to introduce a definition which is a form of *integral-eigenvalue balance* associated with classes of graphs. We investigate *classes of graphs* on *n* vertices with pairs (a,b) of distinct non-zero eigenvalues such that $a+b=s$ or $ab=t$ where *s,t* are the *same integer* (respectively) for each graph in the class **or** the same *function* for each graph in the class.

3.2 Integral sum eigen-pair balanced classes of graphs

Definition 3.2.1: Sum*(s)*eigen-pair (integral) balanced

The class $\Im$ of connected graphs on n elements is said to be *sum*(s)*eigen-pair (integral) balanced* if there exists a pair (a,b) of *distinct non-zero* eigenvalues of the matrices associated with each class of the structures such that $a+b=s$ is the **same integer** as a fixed constant for each member in the class, or s is the **same integer** as a **function of each member** in the class. The sum balance is *exact,* if *s* is the same integer as a fixed constant for each member in the class, or otherwise it is *non-exact*.

The following are some examples of such classes of graphs, noting that $\text{sum}(a,b) = a+b$ in the examples below.

3.2.1 Complete graphs

As per Theorem 2.1.1, the distinct eigenvalues of the complete graph K_n are -1 and $(n-1)$, with the sum of the eigenvalues being $(n-2)$. Therefore the class of complete graphs K_n is non-exact sum*$(n-2)$*eigen-pair balanced, for $n \geq 3$.

Complete graphs are also design graphs.

3.2.2 Complete bipartite graphs

As per Theorem 2.5.1, the class of complete bipartite graphs $K_{p,p}$ on $n = 2p$ vertices has as its associated eigenvalues $p, -p$ and 0, so it is exact sum*(0)*eigen-pair balanced.

The class of complete bipartite graphs $K_{p,k}$ on $p+k$ vertices, $p \neq k$, have eigenvalues $-\sqrt{pk}$, $\sqrt{pk}$, 0 (as per Theorem 2.5.1), so it is exact sum*(0)*eigen-pair balanced (this includes the star graphs with radius 1).

For the case $p = k$ the complete bipartite graph is not a design graph as it contains a *p*-lantern sub-graph. For the case $p \neq k$, the complete bipartite graph is not a design graph as it is not *p*-regular or *k*-regular.

3.2.3 Cycle graphs

As per Theorem 2.3.1, the cycle C_n on n vertices and *n* edges has eigenvalues $2\cos\left(\frac{2\pi j}{n}\right)$, $0 \leq j \leq n-1$.

The 3-cycle (complete graph on 3 vertices) has distinct eigenvalues -1 and 2.

The 4-cycle has distinct eigenvalues 2, 0, -2.

The 5-cycle has distinct eigenvalues $2, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$.

The 6-cycle has distinct eigenvalues $2,\ 1,\ -1,\ -2$.

The 7-cycle has distinct eigenvalues 2, 1.247, -0.445, -1.802.

For the 4- and 5-cycle, there exist two distinct non-zero eigenvalues whose sum is 0. For the 3- and 6-cycle, there exist two distinct non-zero eigenvalues whose sum is 1. For the 7-cycle, there are no two distinct eigenvalues whose sum is 0 or 1. Therefore, the class of cycles is neither sum*(0)*eigen-pair balanced nor sum*(1)*eigen-pair balanced.

However, even cycles are sum*(0)*eigen-pair balanced, since if $n = 2k$,

$$2\cos\left(\frac{2\pi j}{n}\right) = 2\cos\left(\frac{\pi j}{k}\right), \quad 0 \le j \le 2k-1.$$

Then $j = 0$ and $j = k$ yield eigenvalues

$$(a,b) = \left(2\cos\left(\frac{\pi.0}{k}\right), 2\cos\left(\frac{\pi.k}{k}\right)\right) = (2,-2)$$

And $\text{sum}(a,b) = 2 - 2 = 0.$

All cycle graphs, except for the one on 4 vertices, are design graphs.

3.2.4 Path graphs

As per Theorem 2.4.1, the path P_n on $n \ge 2$ vertices and $(n-1)$ edges has eigenvalues

$$2\cos\left(\frac{\pi j}{n+1}\right), \; 1 \le j \le n.$$

Note that:

$$\cos\left(\frac{n\pi}{n+1}\right) = \cos\left(\pi - \frac{\pi}{n+1}\right) = \cos\pi\cos\left(\frac{\pi}{n+1}\right) + \sin\pi\sin\left(\frac{\pi}{n+1}\right) = -\cos\left(\frac{\pi}{n+1}\right)$$

So that the non-zero pair, with $j = n$ and $j = 1$,

$$(a,b) = \left(2\cos\left(\frac{n\pi}{n+1}\right), 2\cos\left(\frac{\pi}{n+1}\right)\right)$$

has the sum

$$\text{sum}(a,b) = 2\cos\left(\frac{n\pi}{n+1}\right) + 2\cos\left(\frac{\pi}{n+1}\right) = -2\cos\left(\frac{\pi}{n+1}\right) + 2\cos\left(\frac{\pi}{n+1}\right) = 0$$

So, the class of path graphs is exact sum*(0)*eigen-pair balanced.

Path graphs are not design graphs, as they are not *k*-regular.

3.2.5 Graph which is the join of two graphs whose adjacency matrices are both circulant matrices

As per Theorem 2.6.1, the conjugate eigenvalues of the join of two circulant matrices of graphs are

$$(a,b)=n\left[\frac{-(d_A-d_B)\pm\sqrt{(d_A-d_B)^2+4nm}}{2n}\right]+d_A.$$

So the sum of the eigenvalues is:

$$\begin{aligned}
\text{Sum}(a,b) &= n\left[\frac{-(d_A-d_B)+\sqrt{(d_A-d_B)^2+4nm}}{2n}\right]+d_A \\
&\quad + n\left[\frac{-(d_A-d_B)-\sqrt{(d_A-d_B)^2+4nm}}{2n}\right]+d_A \\
&= n\left[\frac{-2(d_A-d_B)}{2n}\right]+2d_A \\
&= -(d_A-d_B)+2d_A \\
&= d_A+d_B
\end{aligned}$$

so the class of graphs, which are the join of two graphs whose adjacency matrices are circulant, are sum* (d_A+d_B)*eigen-pair balanced.

3.2.6 Wheel graphs

3.2.6.1 Cycle wheel graphs

As per section 2.7.1, the cycle wheel graph W_n on n vertices, and with $(n-1)$ spokes, has conjugate eigenvalues

$$(a,b)=\frac{2\pm\sqrt{4+4(n-1)}}{2}.$$

The sum of the conjugate eigenvalues is therefore

$$\begin{aligned}
sum(a,b) &= \frac{2+\sqrt{4+4(n-1)}}{2}+\frac{2-\sqrt{4+4(n-1)}}{2} \\
&=2
\end{aligned}$$

so the class of cycle wheel graphs is exact sum*(2)*eigen-pair balanced.

3.2.6.2 Generalised cycle wheel graphs

As per section 2.7.2, the generalized cycle wheel graph X_{n+m} on $(m+n)$ vertices and with mn spokes has conjugate eigenvalues

$$(a,b)=\frac{2\pm\sqrt{4+4mn}}{2}.$$

The sum of the conjugate eigenvalues is therefore

$$\begin{aligned} sum(a,b) &= \frac{2+\sqrt{4+4mn}}{2} + \frac{2-\sqrt{4+4mn}}{2} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

so the class of generalized cycle wheel graphs is exact sum*(2)*eigen-pair balanced.

3.2.6.3 Generalised complete wheel graphs

As per section 2.7.3, the generalized complete wheel graph Y_{n+m} on $(m+n)$ vertices and with mn spokes has conjugate eigenvalues

$$(a,b)=\frac{(n-1)\pm\sqrt{(n-1)^2+4mn}}{2}.$$

The sum of the conjugate eigenvalues is therefore

$$\begin{aligned} sum(a,b) &= \left[\frac{(n-1)+\sqrt{(n-1)^2+4mn}}{2}\right] + \left[\frac{(n-1)-\sqrt{(n-1)^2+4mn}}{2}\right] \\ &= \left[\frac{2(n-1)}{2}\right] \\ &= n-1 \end{aligned}$$

so the class of generalized complete wheel graphs is non-exact sum*$(n-1)$*eigen-pair balanced.

3.2.7 Strongly regular graphs

Theorem 3.2.7.1

If a connected regular graph G of degree k is strongly regular (as per section 1.2.12), then $A(G)$ has at least 3 different eigenvalues. The eigenvalues are:

$$k,\ \frac{(\lambda-\mu)+\sqrt{(\lambda-\mu)^2-4(k-\mu)}}{2} \text{ and } \frac{(\lambda-\mu)-\sqrt{(\lambda-\mu)^2-4(k-\mu)}}{2}$$

See Spielman [44].

Proof

We will consider the adjacency matrices of strongly regular graphs. Let A be the adjacency matrix of a strongly regular graph with parameters *(k, λ, μ)*. We already know that *A* has an eigenvalue of *k* with multiplicity 1. We will now show that A has just two other eigenvalues.

To prove this, first observe that the *(u; v)* entry of A^2 is the number of common neighbours of vertices *u* and *v*. For $u=v$, this is just the degree of vertex u. We will use these facts to write A^2 as a linear combination of A, I and J.

The adjacency matrix of the complement of *A* is $(J_{n,n}-I_{n,n}-A)$.

So,

$$\begin{aligned} A^2 &= \lambda A+\mu(J_{n,n}-I_{n,n}-A)+kI_{n,n} \\ &= (\lambda-\mu)A+\mu J_{n,n}+(k-\mu)I_{n,n}. \end{aligned}$$

For every vector $\underline{v}$ orthogonal to 1,

$$\begin{aligned} A^2\underline{v} &= (\lambda-\mu)A\underline{v}+\mu J_{n,n}\underline{v}+(k-\mu)I_{n,n}\underline{v} \\ &= (\lambda-\mu)A\underline{v}+(k-\mu)I_{n,n}\underline{v}. \end{aligned}$$

So, every eigenvalue θ of *A*, other than *k*, satisfies

$$\theta^2=(\lambda-\mu)\theta+(k-\mu).$$

The eigenvalues of *A*, other than *k*, are those θ that satisfy this quadratic equation, and so are given by

$$\theta=\frac{(\lambda-\mu)\pm\sqrt{(\lambda-\mu)^2-4(k-\mu)}}{2}. \qquad \square$$

The complement of an $srg(v,k,\lambda,\mu)$ is also strongly regular. It is an $srg(v, v-k-1, v-2-2k+\mu, v-2k+\lambda)$.

Note that if we ignore the largest eigenvalue *k* of strongly regular graphs, adding the remaining two eigenvalues yields the integer $(\lambda-\mu)$ so the class of strongly regular graphs with parameters *(v, k, λ, μ)* is non-exact sum*(λ-μ)*eigen-pair balanced - see Godsil and Royle [25] for more on strongly regular graphs.

Strongly regular graphs are not design graphs.

3.2.8 Divisible design graphs

Definition 3.2.8.1: Divisible design graph

A *k*-regular graph is a *divisible design graph* if the vertex set can be partitioned into m classes of size n, such that two distinct vertices from the same class have exactly λ_1 common neighbours, and two vertices from different classes have exactly λ_2 common neighbours.

The eigenvalues of divisible design graphs are provided in Haemers [28] – there are 5 distinct eignvalues. Two of the eigenvalues are

$$(a,b) = \pm\sqrt{k-\lambda_1}\,,$$

so the sum of the eigen-pair is

$$sum(a,b) = \left(\sqrt{k-\lambda_1}\right) + \left(-\sqrt{k-\lambda_1}\right) = 0\,.$$

Therefore, the class of divisible design graphs is exact sum*(0)*eigen-pair balanced.

3.2.9 Bipartite graphs with four distinct eigenvalues

The incidence graphs of symmetric $2-(v,k,1)$ designs are examples of bipartite graphs with four distinct eigenvalues. It is proven by Cvetkovic, Doob and Sachs [17] that these are the only examples, i.e., a connected bipartite regular graph with four distinct eigenvalues must be the incidence graph of a symmetric $2-(v,k,1)$ design. Moreover its spectrum is:

$$(k)^1, (\sqrt{k-\lambda})^{v-1}, (-\sqrt{k-\lambda})^{v-1}, (-k)^1.$$

Note that this class of graphs are exact sum*(0)*eigen-pair balanced.

3.2.10 Hypercube graphs

As per section 2.10, the *p*-regular hypercube on 2^p vertices and $p2^{p-1}$ edges has eigenvalues

$$(p-2k)^{\binom{p}{k}};\ 0 \le k \le p.$$

Using the eigenvalues p and $p-2k$, for $p \neq 2k$ and $k \neq 0$, this class of graphs will be sum* $(2p-2k)$ *eigen-pair balanced.

3.2.11 Eigenvalue pair of real conjugates

By $Z[x]$ and $Q[x]$ we denote the rings of polynomials over the integer and rational numbers, respectively.

Lemma 3.2.11.1

If a monic polynomial $p(x) \in Z[x]$ has a monic divisor $q(x) \in Q[x]$, then also $q(x) \in Z[x]$.

Lemma 3.2.11.2

If $\left(a+\sqrt{b}\right)$, with $a, b \in Q$, is an irrational root of a polynomial $p(x) \in Q[x]$, then so is $\left(a-\sqrt{b}\right)$, with the same multiplicity.

The characteristic polynomial $P_{A(G)}(\lambda)$ of the adjacency matrix of a graph is monic and has integral coefficients. Using Lemmas 3.2.11.1 and 3.2.11.2 we now obtain the following results.

Corollary 3.2.11.1

Every rational eigenvalue of a graph is integral.

Corollary 3.2.11.2

If $\dfrac{a+\sqrt{b}}{2}$ is an irrational eigenvalue of a graph, for some $a, b \in Q$, then so is $\dfrac{a-\sqrt{b}}{2}$, with the same multiplicity, and $a, b \in Z$.

See van Dam [20].

Adding the pair of real conjugates $(a,b)=\left(\frac{a+\sqrt{b}}{2},\frac{a-\sqrt{b}}{2}\right)$, we obtain the integer a. Therefore, if the real conjugate pairs are eigenvalues associated with the adjacency matrix of all graphs belonging to a class of graphs, then the class of graphs is sum*(a)*eigen-pair balanced.

3.3 Integral product eigen-pair balanced classes of graphs

Definition 3.3.1: Product*(t)*eigen-pair (integral) balanced

A class $\Im$ of connected graphs on n elements is said to be *product*(t)*eigen-pair (integral) balanced* if there exists a pair of (a,b) of *distinct non-zero* eigenvalues (counting eigenvalues only once i.e., ignoring multiplicities) of the matrices associated with each class of the structures such that $(ab=t)$is the same integer as a fixed constant for *each* member in the class, or *t* is the same integer as a function of each member in the class. The product balance is *exact*, if *t* is the same integer as a fixed constant for each member in the class, otherwise it is *non-exact*.

The following are some examples of such classes of graphs, noting that product$(a,b)=ab$ in the examples below.

3.3.1 Complete graphs

As per Theorem 2.2.1, the complete graph K_n has distinct eigenvalues -1 and $(n-1)$ for $n\geq 3$. Therefore, the class of complete graphs K_n is non-exact product*$(1-n)$*eigen-pair balanced for $n\geq 3$.

3.3.2 Complete bipartite graphs

As per Theorem 2.5.1, the class of complete bipartite graphs $K_{p,p}$ on $n=2p$ vertices, has as its associated eigenvalues p, $-p$ and 0, so that they are non-exact product*$\left(-p^2\right)$*eigen-pair balanced.

As per Theorem 2.5.1, the class of complete bipartite graphs $K_{p,k}$ on $(p+k)$ vertices, $p\neq k$, has distinct eigenvalues $-\sqrt{pk},\sqrt{pk}$, and 0, so that it is non-exact product*$(-pk)$*eigen-pair balanced (this includes the star graphs with radius 1).

3.3.3 Cycle graphs

As per Theorem 2.3.1, cycles graphs on *n* vertices have associated eigenvalues $2\cos\left(\frac{2\pi j}{n}\right),\ 0 \le j =\le n-1.$

C_3 on 3 vertices, has eigenvalues 2 and -1 so that the eigen-pair product is -2.

C_4 on 4 vertices, has eigenvalues 2, 0 and -2 so that the eigen-pair product is -4.

C_5 on 5 vertices, has eigenvalues 2, $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$ with conjugate eigen-pair product -1.

C_6 on 6 vertices, has eigenvalues 2, 1, -1 and -2. Possible eigen-pair products are -2, 1, -2, and -4.

The 7-cycle has eigenvalues 2, 1.247, -0.445, -1.802. No product of two eigenvalues yields an integer!

Therefore, the class of cycles is not eigen*(*k*)*product balanced for any integer *k*.

However, the class of even cycles is product*(-4)*eigen-pair balanced, since if $n = 2k$ then: $2\cos\left(\frac{2\pi j}{n}\right) = 2\cos\left(\frac{2\pi j}{2k}\right);\ 0 \le j \le 2k-1,$

so that for j=0 we get eigenvalue 2 and $j = k$ we get eigenvalue -2, with eigen-pair product -4.

3.3.4 Path graphs

As per Theorem 2.4.1, paths graphs on *n* vertices have eigenvalues

$$2\cos\left(\frac{\pi j}{n+1}\right),\ 1 \le j \le= n.$$

From section 3.2.4, $\cos\left(\frac{\pi n}{n+1}\right) = -\cos\left(\frac{\pi}{n}\right)$, so that with $j = 1$ and $j = n$,

$$(a,b) = \left(2\cos\left(\frac{\pi}{n+1}\right), 2\cos\left(\frac{\pi n}{n+1}\right)\right) = \left(2\cos\left(\frac{\pi}{n+1}\right), -2\cos\left(\frac{\pi}{n+1}\right)\right),\ \text{and so}$$

$$\text{product}(a,b) = 2\cos\left(\frac{\pi}{n+1}\right).-2\cos\left(\frac{\pi}{n+1}\right)$$

$$= -4\left(\cos\left(\frac{\pi}{n+1}\right)\right)^2$$

which is a function of *n* but is not integral in general.

3.3.5 Graph which is the join of two graphs whose adjacency matrices are both circulant matrices

As per Theorem 2.6.1, the conjugate eigenvalues are

$$(a,b) = n\left[\frac{-(d_A-d_B)\pm\sqrt{(d_A-d_B)^2+4nm}}{2n}\right]+d_A.$$

So the product of the eigenvalues is:

$$\text{product}(a,b) = \left\{n\left[\frac{-(d_A-d_B)+\sqrt{(d_A-d_B)^2+4nm}}{2n}\right]+d_A\right\}$$

$$\left\{n\left[\frac{-(d_A-d_B)-\sqrt{(d_A-d_B)^2+4nm}}{2n}\right]+d_A\right\}$$

$$=$$

$$n^2\left[\frac{(d_A-d_B)^2-\left[(d_A-d_B)^2+4nm\right]}{4n^2}\right]-2d_A(d_A-d_B)+(d_A)^2$$

$$=\left[\frac{-(4nm)}{4}\right]-2d_A(d_A-d_B)+(d_A)^2$$

$$=-mn-2d_A(d_A-d_B)+(d_A)^2$$

so the class of graphs whose adjacency matrix is the join of two graphs whose adjacency matrices are circulant matrices is non-exact product* $\left(-mn-2d_A(d_A-d_B)+(d_A)^2\right)$*eigen-pair balanced.

3.3.6 Wheel graphs

3.3.6.1 Cycle wheel graphs

As per section 2.7.1, the cycle wheel graph W_n on n vertices, and with $(n-1)$ spokes has conjugate eigenvalues

$$(a,b) = \frac{2 \pm \sqrt{4+4(n-1)}}{2}.$$

The product of the conjugate eigenvalues is:

$$\begin{aligned}
\text{product}(a,b) &= \left[\frac{2+\sqrt{4+4(n-1)}}{2}\right]\left[\frac{2-\sqrt{4+4(n-1)}}{2}\right] \\
&= \frac{4-[4+4(n-1)]}{4} \\
&= -(n-1)
\end{aligned}$$

so the class of cycle wheel graphs is non-exact product* $(1-n)$ *eigen-pair balanced.

3.3.6.2 Generalised cycle wheel graphs

As per section 2.7.2, the generalized cycle wheel graph X_{m+n} on $(m+n)$ vertices and with mn spokes has conjugate eigenvalues

$$(a,b) = \frac{2 \pm \sqrt{4+4mn}}{2}.$$

The product of the conjugate eigenvalues is

$$\begin{aligned}
\text{product}(a,b) &= \left[\frac{2+\sqrt{4+4mn}}{2}\right]\left[\frac{2-\sqrt{4+4mn}}{2}\right] \\
&= \frac{4-(4+4mn)}{4} \\
&= -mn
\end{aligned}$$

so the class of generalized wheel graphs is non-exact product* $(-mn)$ *eigen-pair balanced.

3.3.6.3 Generalised complete wheel graphs

As per section 2.7.3, the generalized complete wheel graph Y_{m+n} on $(m+n)$ vertices and with mn spokes has conjugate eigenvalues

$$(a,b)=\frac{(n-1)\pm\sqrt{(n-1)^2+4mn}}{2}.$$

The product of the conjugate eigenvalues is

product(a,b)

$$=\left[\frac{(n-1)+\sqrt{(n-1)^2+4mn}}{2}\right]\left[\frac{(n-1)-\sqrt{(n-1)^2+4mn}}{2}\right]$$
$$=\left[\frac{(n-1)^2-\left[(n-1)^2+4mn\right]}{4}\right]$$
$$=\frac{-4mn}{4}$$
$$=-mn$$

so the class of generalized complete wheel graphs is non-exact product* $(-mn)$*eigen-pair balanced.

3.3.7 Strongly regular graphs

As per Theorem 3.2.7.1, the conjugate eigen-pair of strongly regular graphs is

$$(a,b)=\frac{(\lambda-\mu)\pm\sqrt{(\lambda-\mu)^2-4(k-\mu)}}{2}.$$

If we multiply the two conjugate pairs of strongly regular graphs we obtain the integer $(\mu-k)$, so that the class of strongly regular graphs is non-exact product* $(\mu-k)$ *eigen-pair balanced.

3.3.8 Divisible design graphs

As per Section 3.2.8, two fo the eigenvalues of a divisible design graph are

$$(a,b)=\pm\sqrt{k-\lambda_1}.$$

This class of graphs has eigen-pair product

product$(a,b)=\left(\sqrt{k-\lambda_1}\right)\left(-\sqrt{k-\lambda_1}\right)=-(k-\lambda_1)=\lambda_1-k$

Therefore, the class of divisible design graphs is non-exact product*(λ_1-k) *balanced.

3.3.9 Bipartite graphs with four distinct eigenvalues

As per Section 3.2.9, the eigenvalues of a bipartite graph with four distinct eigenvalues are

$(a,b)=\pm\sqrt{k-\lambda}$ and $(c,d)=\pm k.$

Then product$(a,b)=-(k-\lambda)$ and product$(c,d)=-k^2$.

Therefore, incidence graphs of symmetric 2-(*v*, *k*, l) designs are product *t*eigen-pair balanced for $t=(\lambda-k)$and $t=-k^2$ of the non-exact kind.

3.3.10 Hypercube graphs

As per Section 2.10, the class of *p*-regular hypercubes on 2^p vertices and $p2^{p-1}$ edges has eigenvalues:

$(p-2k)^{\binom{p}{k}},\ 0\le k\le p.$

Using the eigenvalues *p* and $p-2k$, for $p\ne 2k$, $k\ne 0$, this class of graphs is product* (p^2-2pk) *eigen-pair balanced.

3.3.11 Eigenvalue pair of real conjugates

The product of the real conjugate pair of eigenvalues $\dfrac{a+\sqrt{b}}{2}$ and $\dfrac{a-\sqrt{b}}{2}$ is $\dfrac{a^2-b}{4}$. This is integral, provided the numerator is a multiple of 4.

Therefore, if the real conjugate pairs are eigenvalues associated with the adjacency matrix of all graphs belonging to a class of graphs, then the class of graphs is product*$\left(\dfrac{a^2-b}{4}\right)$*eigen-pair balanced, provided a^2-b is a multiple of 4.

3.4 Eigen-bi-balanced classes of graphs

Definition 3.4.1: Eigen-bi-balanced classes of graphs

Classes of graphs, which are both sum and product eigen-pair balanced, are said to be *eigen-bi-balanced* with respect to the eigen-pair (a,b). If this pair is unique to the class, then it is *uniquely eigen-bi-balanced*. For example, the class of complete graphs is uniquely eigen-bi-balanced with respect to the eigen-pair $(n-1,-1)$.

The largest eigenvalue occurs in the eigen-pair associated with some classes of graphs discussed above. We observe the following:

- The only regular eigen-pair balanced graphs on 2 and 3 vertices are K_2 and K_3;
- The 4-cycle is the same as the complete bipartite graph $K_{2,2}$, which is sum and product eigen-pair balanced;
- The only other regular graph on 4 vertices is K_4;
- The 5-cycle has eigenvalues $(2)^1, \left(\frac{-1+\sqrt{5}}{2}\right)^2, \left(\frac{-1-\sqrt{5}}{2}\right)^2$ which is not sum or product eigen-pair balanced when the largest eigenvalue is included in the eigen-pair; and
- The only other regular graph on 5 vertices is K_5.

Thus we have the following theorem:

Theorem 3.4.1

The only regular graphs on *n* vertices, where $2 \le n \le 5$, belonging to eigen-pair balanced classes of graphs, where the eigen-pair contains the largest eigenvalue, are K_2, K_3, K_4, K_5 and $K_{2,2}$.

3.5 Eigen-bi-balanced classes of graphs – criticality, ratios, asymptotes and area

If a class of graphs $\Im$ are *both* sum and product eigen-pair balanced with respect to the eigen-pair (a,b), they have been defined above as *eigen-bi-balanced* with respect to (a,b). The class of complete graphs *G* is eigen-bi-balanced with the property that the removal of any vertex *v* from *G* results in a complete graph, which belongs to the same class of complete graphs, which is eigen-bi-balanced. The same holds for complete bipartite graphs except for star graphs. Such graphs are said to be *stable* eigen-bi-balanced.

Definition 3.5.1: Critically eigen-bi-balanced classes of graphs

If G belongs to a class $\Im$ of eigen-bi-balanced graphs, and there exists a vertex v of G, such that $G-v$ belongs to a class $\Im'$ of graphs which is *not* eigen-bi-balanced, we say that $\Im$ is *critically* eigen-bi-balanced with respect to v.

Wheels on p spokes are eigen-bi-balanced and the removal of the central vertex results in p-cycles, which are not eigen-bi-balanced. Therefore, the class of wheel graphs are critically eigen-bi-balanced with respect to their central vertex. This suggests that the central vertex is essential to the eigen-bi-balanced characteristic of wheels.

The reciprocals of eigenvalues are connected to the idea of *robustness* or *tightness* of graphs - see section 1.5.9 and Brouwer and Haemers [12]. Since a and b are non-zero, the sum of their reciprocals is defined. Therefore we have the following definition.

Definition 3.5.2: Eigen-bi-balanced ratio of classes of graphs

The *eigen-bi-balanced ratio* of the class of graphs (with respect to the eigen-pairs (a,b)) is

$$r(a\Im b)=\frac{1}{b}+\frac{1}{a}=\frac{a+b}{ab}$$

As a and b are non-zero, the product ab is never zero, and so this ratio will always be defined.

Definition 3.5.3: Eigen-bi-balanced ratio asymptote of classes of graphs

If this ratio is a function $f(n)$ of the order n of the graph, and has a horizontal asymptote, we call this asymptote the *eigen-bi-balanced ratio asymptote* with respect to the eigen-pair (a,b) and is denoted by:

$$r(a\Im b)^{\infty} \ \ or \ \ asymp(r(a\Im b)).$$

This asymptote can be seen as describing the behavior of the ratio as the order of the graph becomes increasingly large.

The "area" term n^2 can be found in the following relation involving spanning trees. Let D be the matrix composed of the degrees of G in the main diagonal – form D' by adding 1 to each entry of D. We then form the *shadow number* of a graph defined by

$shad(G) = \det(D' - A)$, where A is the adjacency matrix of G. We then have the combinatorial result

$$n^2 = \frac{shad(G)}{t(G)} = \frac{\det(D'-A)}{t(G)}$$

where $t(G)$ is the number of spanning trees of a connected graph *G*.

Also, the number of spanning trees of a connected graph *G* is associated with the Laplacian eigenvalues, $\theta_n \geq \theta_{n-1} \geq ... > \theta_1 = 0$, of the graph by the following:

$$\prod_{j=2}^{n} \theta_j = nt(G).$$

This excludes the first Laplacian eigenvalue. Thus, as in the case of the complete graph, the eigenvalue $n-1$ of the adjacency matrix associated with *G* will not be taken into account when considering spanning trees.

Eigenvalues have been associated with the *expansion* of graphs (see Brouwer and Haemers [12]), which motivates the idea of *areas* associated with a class of graphs.

If the eigen-bi-balanced ratio of a class of graphs is a function of *n*, then we are able to integrate it with respect to *n*, which leads to the following definition.

Definition 3.5.4: Eigen-bi-balanced ratio area of classes of graphs

We define the *eigen-bi-balanced ratio area* of the class of graphs with respect to the eigen-pair (a,b) as:

$$Ar(\Im)^{a,b} = \begin{cases} \dfrac{2m}{n}\left|\displaystyle\int \frac{a+b}{ab}\,dn\right| & if \quad a+b \neq 0 \\ \\ \dfrac{2m}{n}\left|\displaystyle\int_a^b dn\right| = \dfrac{2m}{n}|2b| & if \quad a+b = 0 \end{cases}$$

where *m* is the number of edges and *n* is the number of vertices,
and $Ar(\Im)^{a,b} = 0$ when *n*=0,1 or 2.

Now we define *breadth,* denoted by *B,* as

$$B = \frac{2m}{n}$$ i.e., the average degree of the vertices in G,

and define *height*, denoted by *H,* as

$$H = \begin{cases} \left| \int \frac{a+b}{ab} dn \right| & if \quad a+b \neq 0 \\ \left| \int_a^b dn \right| = \frac{2m}{n} |2b| & if \quad a+b = 0 \end{cases}$$

so $Ar(\Im)^{a,b} = B.H.$

If there is more than one pair giving rise to such area, then the area of the class is $\max Ar(\Im)^{a_i,b_i}$ for all pairs (a_i, b_i). If there is only one eigen-pair associated with the class of graphs that gives rise to the area, then the area is *unique.*

The height involves binding the sum of the reciprocals of the eigen-pair by its integration, and we multiply this height by the average degree. This involves one of the most basic, yet important combinatorial aspects of the graph, and results in the term n^2 appearing in the eigen-bi-balanced ratio area of some classes of graphs.

3.6 Examples of eigen-bi-balanced classes of graphs

When we refer to a graph *G* having eigen-pair balanced properties such as sum, product, bi-balanced, ratio, asymptote, etc. we imply that *G* belongs to a class of graphs having such eigen-pair properties. We will now look at various examples of eigen-bi-balanced classes of graphs:

3.6.1 Complete graphs

The complete graph on *n* vertices has the unique eigen-bi-balanced ratio of:

$$\begin{aligned} r((n-1)K_n(-1)) &= \frac{(n-1)+(-1)}{(n-1)(-1)} \\ &= \frac{n-2}{1-n} \end{aligned}$$

This depends on the order of the graph and has the unique eigen-bi-balanced ratio asymptote:

$$r((n-1)K_n(-1))^{\infty} = -1$$

and eigen-bi-balanced ratio area:

$$Ar(K_n)^{-1,n-1} \qquad = \frac{2m}{n}\left|\int \frac{n-2}{1-n}dn\right|$$

$$= \frac{2\frac{n(n-1)}{2}}{n}\left|\int [\frac{2}{n-1} - \frac{n}{n-1}]dn\right|$$

$$= (n-1)\left|\int [\frac{2}{n-1} - \frac{n}{n-1}]dn\right|$$

$$= (n-1)B$$

where $B = \left|\int \frac{2}{n-1} - \frac{n-1}{n-1} - \frac{1}{n-1}\right| dn = n - \ln|n-1| + c.$

When n=0 we have A=0 so c=0 so that its area is

$$Ar(K_n)^{-1,n-1} = (n-1)(n - \ln(n-1)) = (n-1)H.$$

Note that the length of the longest path for the complete graph is $n-1$, so that H in the above expression can be regarded as the *height* of the graph. Also, the term $\ln(n-1)$ occurs as part of the upper bound of the *diameter* of a graph involving the second largest eigenvalue - see Brouwer and Haemers [12].

Is this area the maximum for all classes of eigen-bi-balanced graphs?

3.6.2 Complete bipartite graphs

The complete bipartite graph $K_{s,t}$ on $s+t$ vertices has the eigen-bi-balanced ratio of

$$r(\sqrt{st}K_{s,t} - \sqrt{st}) = \frac{\sqrt{st} - \sqrt{st}}{-st} = 0$$

which is independent of the size of the graph.

Its area is:

$$Ar(K_{s,t})^{-\sqrt{st},\sqrt{st}} = \frac{2st}{(s+t)}\left|2\sqrt{st}\right|$$

$$= 4\frac{(st)^{\frac{3}{2}}}{s+t}.$$

This attains its maximum when $s = t = \frac{n}{2}$, then the graph (the *complete split bipartite* graph on *n* vertices) is *s*-regular and the area is

$$Ar(K_{\frac{n}{2},\frac{n}{2}})^{-\frac{n}{2},\frac{n}{2}} = \frac{n}{2}\left|\int_{-n/2}^{n/2} dn\right| = \frac{n^2}{2}$$

3.6.3 Wheel graphs

Wheels on n vertices, containing $(n-1)$ spokes and $2(n-1)$ edges, have eigen-bi-balanced ratio

$$r\left(\frac{2+\sqrt{4+4(n-1)}}{2} W_n \frac{2-\sqrt{4+4(n-1)}}{2}\right) = \frac{-2}{(n-1)}.$$

This depends on the size of the graph, so they have an eigen-bi-balanced ratio asymptote of 0 and eigen-bi-balanced ratio area of:

$$\begin{aligned}
Ar(W_n)^{\frac{2+\sqrt{4+4(n-1)}}{2},\frac{2-\sqrt{4+4(n-1)}}{2}} &= \frac{2m}{n}\left|\int \frac{a+b}{ab}\, dn\right| \\
&= \frac{2m}{n}\left|\int \frac{2}{1-n}\, dn\right| \\
&= \frac{4(n-1)}{n}(\ln|n-1| + c) \\
&= \frac{4(n-1)}{n}(\ln|n-1| + c)
\end{aligned}$$

Now $Ar(W_n)^{\frac{2+\sqrt{4+4(n-1)}}{2},\frac{2-\sqrt{4+4(n-1)}}{2}}$ = 0 when *n*=2 so that *c*=0.

So $Ar(W_n)^{\frac{2+\sqrt{4+4(n-1)}}{2},\frac{2-\sqrt{4+4(n-1)}}{2}} = \frac{4(n-1)}{n}(\ln|n-1|)$.

3.6.4 Star graphs

As per section 2.8.2, star graphs with *m* rays of length 2 have eigenvalues $0,\ -1,\ 1,\ \pm\sqrt{m+1}$.

Using the pair $(a,b)=(-1,1)$ we obtain the ratio

$$r(-1S_{1,mP_3}1) = \frac{0}{-1} = 0$$

and using the pair $(a,b)=\left(\sqrt{m+1},-\sqrt{m+1}\right)$ we get

$$r\left(\sqrt{m+1}S_{1,mP_3} - \sqrt{m+1}\right) = \frac{0}{-(m+1)} = 0.$$

Using the class of graphs where $(m+1)=t^2$, and eigen-pair $(a,b)=\left(\sqrt{m+1},1\right)$, we have the ratio

$$r(\sqrt{m+1}S_{1,mP_3}1) = \frac{\sqrt{m+1}+1}{\sqrt{m+1}} = \frac{t+1}{t} \ .$$

Therefore the class of star graphs with *m* rays of length 2 does not have a unique eigen-bi-balanced ratio.

The area with respect to the pair $(a,b)=(-1,1)$ is

$$Ar(S_{1,mP_3})^{-1,1} = \frac{2m}{n}|2b| = \frac{2.2m}{2m+1}|2| = \frac{8m}{2m+1}$$

and with respect to the pair $(c,d)=\left(\sqrt{m+1},-\sqrt{m+1}\right)$ is

$$Ar(S_{1,mP_3})^{\sqrt{m+1},-\sqrt{m+1}} = \frac{2m}{n}|2b| = \frac{4m}{2m+1}2\sqrt{m+1}$$

Since $m=\frac{(n-1)}{2}$, the areas are, respectively:

$$Ar(S_{1,mP_3})^{-1,1} = \frac{8m}{2m+1} = \frac{4(n-1)}{n-1+1} = \frac{4(n-1)}{n} \text{ and}$$

$$Ar(S_{1,mP_3})^{-\sqrt{m+1},\sqrt{m+1}} = \frac{4m}{2m+1}2\sqrt{m+1} = \frac{2(n-1)}{n}2\sqrt{\frac{n+1}{2}} = \frac{2\sqrt{2}(n-1)}{n}\sqrt{n+1}$$

The greater of the two gives the area of the class of graphs, i.e.,

$$Ar(S_{1,mP_3})^{-\sqrt{m+1},\sqrt{m+1}} = \frac{2\sqrt{2}(n-1)}{n}\sqrt{n+1}.$$

3.6.5 Hypercube graphs

The p-regular hypercube has eigenvalues $\lambda = (p-2k)^{\binom{p}{k}}$; $0 \le k \le p$, and eigen-bi-balanced ratio (k fixed, n varying, $p \ne 2k$ and $k \ne 0$):

$$\frac{2p-2k}{p^2-2pk} = 2\frac{\frac{\ln n}{\ln 2}-k}{\frac{\ln^2 n}{\ln^2 2}-2k\frac{\ln n}{\ln 2}}$$

$$= 2\ln 2\frac{\ln n - k\ln 2}{\ln^2 n - 2k\ln 2\ln n}$$

where, since $n = 2^p$, we have $p = \frac{\ln n}{\ln 2}$.

Then $asymp\left(\frac{2p-2k}{p^2-2pk}\right) = 0.$

For $k = 1$ and using the eigenvalue pair p and p-2 , the eigen-bi-balanced ratio area is

$$Ar(\Im)^{p,p-2} = \int \frac{2p-2}{p^2-2p}d(2^p)$$

$$= 2\ln 2\int \frac{\ln n - \ln 2}{\ln^2 n - 2\ln 2\ln n}dn \qquad (1)$$

Setting $u = \ln^2 n - 2\ln 2\ln n$ and re-arranging, we get

$$\ln^2 n - 2\ln 2\ln n - u = 0$$

So $\ln n = \frac{2\ln 2 \pm \sqrt{(2\ln 2)^2 + 4u}}{2}$

$$= \ln 2 + \sqrt{\ln^2 2 + u}$$

$$\Rightarrow n = e^{\ln 2+\sqrt{\ln^2 2+u}} = 2e^{\sqrt{\ln^2 2+u}}$$

$$\Rightarrow e^{\sqrt{\ln^2 2+u}} = \frac{n}{2} \qquad (2)$$

Also, for $u = \ln^2 n - 2\ln 2 \ln n,$ we have

$$\frac{du}{dn} = [\frac{2\ln n}{n} - \frac{2\ln 2}{n}] = \frac{2}{n}(\ln n - \ln 2)$$

$$\Rightarrow du = \frac{2}{n}(\ln n - \ln 2)dn$$

Now substituting above results into (1), we get

$$\begin{aligned} Ar(\Im)^{p,p-2} &= 2\ln 2 \int \frac{\ln n - \ln 2}{\ln^2 n - 2\ln 2 \ln n} dn \\ &= 2\ln 2 \int \frac{n}{2u} du \\ &= \ln 2 \int \frac{n}{u} du \\ &= \ln 2 \int \frac{e^{\ln 2 + \sqrt{\ln^2 2 + u}}}{u} du \qquad (3) \end{aligned}$$

$u = \ln^2 n - 2\ln 2 \ln n,$

Now $u > 1$ and $e^{\ln 2 + \sqrt{\ln^2 2 + u}} < e^u$ so that:

$$\ln 2 \int \frac{e^{\ln 2 + \sqrt{\ln^2 2 + u}}}{u} du < \ln 2 \int e^u du = \ln 2 e^u = \ln 2 e^{\ln^2 n - 2\ln 2 \ln n}$$

Although this is not a good approximation for the area of a hypercube class of graphs, it may suggest that the area of such a class of graphs is greater than that of complete graphs.

3.6.6 Join of two graphs

Taking the join of the complement of the complete graph on 2 vertices and the complete graph on n vertices, we get from section 2.7.3, that the resulting adjacency matrix of the graph has the conjugate pair of eigenvalues

$$(a,b) = \frac{(n-1) \pm \sqrt{(n-1)^2 + 8n}}{2}$$

so that their eigen-bi-balanced ratio is

$$r = \frac{\left(\frac{(n-1)+\sqrt{(n-1)^2+8n}}{2}\right)+\left(\frac{(n-1)-\sqrt{(n-1)^2+8n}}{2}\right)}{\left(\frac{(n-1)+\sqrt{(n-1)^2+8n}}{2}\right)\left(\frac{(n-1)-\sqrt{(n-1)^2+8n}}{2}\right)}$$

$$= \frac{\frac{2(n-1)}{2}}{\frac{(n-1)^2-(n-1)^2-8n}{4}}$$

$$= \frac{n-1}{-2n}$$ which tends to -1/2.

3.6.7 Cycle graphs

Conjecture 3.6.7.1

The only class of regular graphs which are neither sum nor product eigen-pair balanced are cycles.

3.6.8 Dumbbell graphs

Note that cycles are neither sum nor product eigen-pair balanced. Are there any other classes of graphs which are neither sum nor product eigen-pair balanced?

Let us consider the dumbbell graphs D_n (two copies of K_n joined by a single edge).

For $n = 3$, let D_3 be the dumbbell graph with 2 copies of K_3, joined by a single edge. Then D_3 is

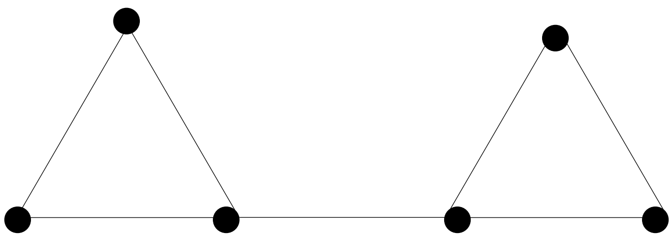

Figure 3.6.9.1: Diagram of dumbbell D_3

Then the adjacency matrix of D_3 is

$$A(D_3)=\begin{bmatrix} 0 & 1 & 1 & 1 & & \\ 1 & 0 & 1 & & & \\ 1 & 1 & 0 & & & \\ 1 & & & 0 & 1 & 1 \\ & & & 1 & 0 & 1 \\ & & & 1 & 1 & 0 \end{bmatrix},$$

with all blank entries containing zero. Then $A(D_3)$ has eigenvalues -0.41421 and 2.41421, whose sum is 2, and whose product is -1.

For $n=4$, let D_4 be the dumbbell graph with 2 copies of K_4, joined by a single edge. Then D_4 is

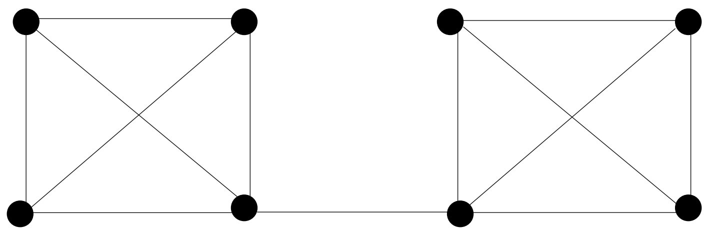

Figure 3.6.9.2: Diagram of dumbbell D_4

Then the adjacency matrix of D_4 is

$$A(D_4)=\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & & & \\ 1 & 0 & 1 & 1 & & & & \\ 1 & 1 & 0 & 1 & & & & \\ 1 & 1 & 1 & 0 & & & & \\ 1 & & & & 0 & 1 & 1 & 1 \\ & & & & 1 & 0 & 1 & 1 \\ & & & & 1 & 1 & 0 & 1 \\ & & & & 1 & 1 & 1 & 0 \end{bmatrix},$$

with all blank entries containing zero. Then $A(D_4)$ has eigenvalues 3 and -1, whose sum is 2, and whose product is -3.

For $n = 5$, let D_5 be the dumbbell graph with 2 copies of K_5, joined by a single edge.

Then, the adjacency matrix of D_5 has eigenvalues 3.82843 and -1.82843 whose sum is 2, and whose product is -7.

So this class of graphs may be sum eigen-pair balanced, but is not product eigen-pair balanced.

3.7 Eigen-bi-balanced properties of the class of complements of graphs

Theorem 3.7.1

Let $\Im$ be a class of eigen-bi-balanced, *k*-regular graphs with eigen-pair a,b; $a,b \neq k$ or -1. Let $\overline{\Im}$ be the class of graphs consisting of the complement, $\overline{G}$, of graphs $G \in \Im$, where $\overline{G}$ is connected. Then for all $\overline{G} \in \overline{\Im}$, $\overline{G}$ is $(n-1-k)$-regular, eigen-bi-balanced with eigen-pair $(c,d) = (-1-a, -1-b)$, and the eigen-bi-balanced ratio of $\overline{\Im}$ is $r(c\overline{\Im}d) = \dfrac{c+d}{cd} = \dfrac{-2-(a+b)}{1+(a+b)+ab}$. Therefore, the class $\overline{\Im}$ of graphs is eigen-bi-balanced.

Proof

As per Theorem 2.11.1, if a,b are eigenvalues of G, and $a,b \neq k$, or -1, then $c = (-1-a)$, and $d = (-1-b)$ are eigenvalues of $\overline{G}$.

Therefore,

$$\begin{aligned}
\text{Sum of the eigen-pair } (c,d) \text{ of } \overline{G} &= c + d \\
&= (-1-a) + (-1-b) \\
&= -2 - (a+b)
\end{aligned}$$

$$\begin{aligned}
\text{Product of the eigen-pair } (c,d) \text{ of } \overline{G} &= cd \\
&= (-1-a) \times (-1-b) \\
&= 1 + (a+b) + ab
\end{aligned}$$

Since $G \in \Im$ is eigen-bi-balanced, then $(a+b)$ and (ab) are constant integers. Therefore, $\overline{G} \in \overline{\Im}$ is eigen-bi-balanced.

Therefore,

$$r(c\overline{\Im}d) = \frac{c+d}{cd} = \frac{-2-(a+b)}{1+(a+b)+ab} = \text{ratio of integers}$$

Therefore the class $\overline{\Im}$ of graphs is eigen-bi-balanced. □

Corollary 3.7.1

Let $\Im$ be a class of eigen-bi-balanced, *k*-regular graphs with eigen-pair $a,b;\ a,b \neq k$ or -1. Let $\overline{\Im}$ be the class of graphs consisting of the complement, $\overline{G},$ of graphs $G \in \Im$, where $\overline{G}$ is connected. If the asymptote of the ratio $\dfrac{a+b}{ab} = t;\ \ a,b \neq k\ \ or -1$ and $\lim\limits_{n\to\infty} ab = \infty$ then

$$r(c\overline{\Im}d)^{\infty} = asymp(r(c\overline{\Im}d)) = \frac{-t}{t+1}.$$

Proof

$$r(c\overline{\Im}d) = \frac{c+d}{cd}$$

$$= \frac{-2-(a+b)}{1+(a+b)+ab}$$

$$= \frac{\dfrac{-2}{ab} - \dfrac{(a+b)}{ab}}{\dfrac{1}{ab} + \dfrac{(a+b)}{ab} + 1}$$

Now $\lim\limits_{n\to\infty} ab = \infty$

$$\Rightarrow r(c\overline{\Im}d)^{\infty} = asymp(r(c\overline{\Im}d)) = \lim_{n\to\infty}\left(\frac{\dfrac{-2}{ab} - t}{\dfrac{1}{ab} + t + 1}\right) = \frac{-t}{t+1}$$ □

Corollary 3.7.2

Let $\Im$ be a class of eigen-bi-balanced *k*-regular graphs with eigen-bi-balanced ratio asymptote $f(t)=t$ with respect to pair a,b *where* $a,b \neq k$ or -1. Let $\overline{\Im}$ be the class of graphs consisting of the complement, $\overline{G},$ of graphs $G \in \Im$, where $\overline{G}$ is connected. Then the eigen-bi-balanced ratio asymptote of $\overline{\Im}$ with respect to eigen-pair (c,d), is $g(t)=\dfrac{-t}{t+1}$ which is an involution.

Proof

As per Corollary 3.7.1, let $g(t)=asymp\left(r\left(c\overline{\Im}d\right)\right)=\dfrac{-t}{t+1}$.

Then,

$$g\left(\frac{-t}{t+1}\right)=\frac{-\left(\dfrac{-t}{t+1}\right)}{\left(\dfrac{-t}{t+1}\right)+1}=\frac{\dfrac{t}{t+1}}{\dfrac{1}{t+1}}=t$$

So that $g[g(t)]=t$ which implies that $g(t)=g^{-1}(t)$.

Functions, which are equal to their own inverse, are called involutions, so that $g(t)$ is an involution. □

Corollary 3.7.3

The involution $g(t)=\dfrac{-t}{(t+1)}$ is a solution of the differential equation

$$\frac{dg(t)}{dt}=\frac{1}{t(t+1)};\quad g\left(\frac{-1}{2}\right)=1.$$

Proof

Since

$$g'(t)=\frac{-1(t+1)+t}{(t+1)^2}=\frac{-1}{(t+1)^2}$$

then

$$g'(g(t))=\frac{-1}{\left(\dfrac{-t}{(t+1)}+1\right)^2}=-(t+1)^2 \qquad (1)$$

Also, $g(t)=\dfrac{-t}{t+1}$,

$\Rightarrow t+(t+1)g(t)=0$

$\Rightarrow f(t)+(t+1)g(t)=0$

Differentiating both sides, we get

$$f'(t)+g'(t)(t+1)+g(t)=0 \qquad (2)$$

Since $g(t)$ is an involution, $g(g(t))=t$, so differentiating both sides and recalling $f(t)=t$ we get

$$g'(g(t))g'(t)=f'(t)=1. \qquad (3)$$

Substituting (1) and (3) into (2), we get:

$f'(t)+g'(t)(t+1)+g(t)=0$

$\Rightarrow g'(g(t))g'(t)+g'(t)(t+1)+g(t)=0$

$\Rightarrow -(t+1)^2 g'(t)+g'(t)(t+1)+g(t)=0$

$\Rightarrow g'(t)[-(t+1)^2+(t+1)]+g(t)=0$

$\Rightarrow g'(t)[-t(t+1)]+g(t)=0$

$\Rightarrow \dfrac{dg(t)}{dt}=\dfrac{g(t)}{t(t+1)}$

$$\Rightarrow \frac{dy}{y}=\frac{dt}{t(t+1)}, \qquad (4)$$

where $y=g(t)$.

Solving this variable separable equation we get:

$$\ln|y|=\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt=\ln|t|-\ln|t+1|+c=\ln\frac{|t|}{|t+1|}+c$$

$\Rightarrow y=k\dfrac{|t|}{|t+1|}$ is a general solution of (4).

If $t\le 0$ *and* $t+1>0$ then the solution is:

$y = \frac{-kt}{t+1}$;

and if $y\left(\frac{-1}{2}\right) = 1$ then $k = 1$, so

$y = \frac{-t}{t+1}$. □

Recall from Section 3.6.6, that $-\frac{1}{2}$ is the eigen-bi-balanced ratio asymptote of the class of graphs comprising of the join of the complement of the complete graph on 2 vertices, and the complete graph on *n* vertices.

Note: There are other possible solutions of the general differential equation $\frac{dg(t)}{dt} = \frac{1}{t(t+1)}$; $g(p) = q$.

For example, when $t \geq 0 \Rightarrow t+1 \geq 0$ so that $y = \frac{t}{t+1}$ is a solution of the differential equation $\frac{dg(t)}{dt} = \frac{1}{t(t+1)}$; $g(1) = \frac{1}{2}$.

3.8 Properties of eigen-bi-balanced classes of graphs

Theorem 3.8.1

If a class of non-complete graphs $\Im$ is eigen-bi-balanced with respect to the eigen-pair (a,b), and (a,b) are conjugate eigen-pairs arising from the quadratic: $\lambda^2 + s\lambda + t'$, with at least one of *a,b* positive and of the form $n + c$ (*n* an integer and *c* negative), and the ratio $r(a\Im b) = \frac{a+b}{ab}$ is a function of n, then t' is negative and the eigen-pair balanced ratio asymptote lies on the interval [-1,0].

Proof

Let the conjugate eigen-pair (a,b) arise from the roots of the quadratic $\lambda^2 + s\lambda + t'$; i.e., $a,b = \frac{-s \pm \sqrt{s^2 - 4t'}}{2}$.

Assume $t' > 0$, then $s \geq 2\sqrt{t'}$

$\Rightarrow$ $t' \geq 0$ and $s \geq 0$

$\Rightarrow$ $a < 0$ and $b < 0$ which is a contradiction of the assumption that at least one of a,b is positive.

Therefore, we have shown that $t' \leq 0$.

If $t' = 0$, then

$$a,b = \frac{-s \pm \sqrt{s^2}}{2}$$

$\Rightarrow$ either $a = 0$ or $b = 0$, which is not allowed.

Therefore, we have shown that $t' < 0$.

So let $t' = -t$; where $t > 0$.

We have $a \leq n-1$ and $b \leq n-1$ (as these are conjugate eigenvalues of the class of non-complete graphs $\Im$), and $a+b = -s$; and $ab = -t$ and $r(a\Im b) = \dfrac{a+b}{ab} = \dfrac{s}{t}$. (1)

If $a = -b$ then the ratio $r = \dfrac{a+b}{ab} = 0$. However, we are given that $r(a\Im b) = \dfrac{a+b}{ab}$ is a function of *n*, so then we must have $a \neq -b$.

If a and b are both fixed constants, then the ratio is not a function of n, so we can't have a and b both fixed constants.

From (1) above, $r = \dfrac{s}{t}$. If $t = f(n)$, and $f(n)$ is of order *n*, and s is a fixed constant c, then $asymp(r(a\Im b)) = asymp\left(\dfrac{s}{t}\right) = asymp\left(\dfrac{c}{f(n)}\right) = 0$.

If $s = a+b$ is a function of n, so will $t = -(ab)$ be a function of n.

If both a and b are functions of n then $a+b$ is $O(n^p)$ and ab has $O(n^q)$ where $q \geq p$.

Therefore, $asymp(r(a\Im b)) = 0$.

Now let us assume that $a = n + c > 0$ and $b = k$, k negative (as per conditions in Theorem 3.8.1).

Then if $s = n + c'$; $t = kn + c''$; $(k < 0)$; c', c'' are constant.

$$\Rightarrow \ asymp(r(a \Im b)) = \frac{1}{k} < 0$$

Since a is an integer, $b = k$ must be an integer too (as they are conjugate pairs), so therefore $k \le -1$. Therefore,

$$asymp(r(a \Im b)) = \frac{1}{k} \ge -1.$$

So we have proved that $-1 \le asymp(r(a \Im b)) < 0.$

We have therefore shown separately that $asymp(r(a \Im b)) = 0$ and $-1 \le asymp(r(a \Im b)) < 0$ under different conditions.

Therefore $asymp(r(a \Im b)) \in [-1,0].$ □

For the complete graph K_n, the quadratic for the complete graph, with eigenvalues $(-1, n-1)$, sum = $n-2$ and product $-(n-1)$, is:

$$\lambda^2 - (n-2)\lambda - (n-1) \Rightarrow \lambda = \frac{(n-2) \pm \sqrt{(n-2)^2 + 4(n-1)}}{2}$$

So, $asymp(r(-1 K_n n-1)) = \dfrac{n-2}{-(n-1)} = -1.$

So the eigen-bi-balanced ratio asymptote of the complete graphs is -1, which is the same as one of the eigen-pairs. If a class of graphs is eigen-bi-balanced with respect to the pair (a, b) and its asymptote is the same as one of the eigen-pairs, then it is said to be *asymptotically closed* with respect to the pair (a, b). Therefore, the class of complete graphs is asymptotically closed with respect to the pair $(-1, n-1)$.

Theorem 3.8.2

The eigen-bi-balanced ratio areas of complete bipartite graphs, wheel graphs and star graphs with rays of length 2, are each bounded above by the area of the complete graph.

Proof

As per Section 3.6.1, the eigen-bi-balanced ratio area of the complete graph is

$$Ar(K_n)^{-1,n-1} = (n-1)(n-\ln(n-1)).$$

As per Section 3.6.2, the eigen-bi-balanced ratio area of the split complete bipartite graph is

$$Ar\left(K_{\frac{n}{2},\frac{n}{2}}\right)^{\frac{n}{2},\frac{n}{2}} = \frac{n^2}{2}.$$

As per Section 3.6.3, the eigen-bi-balanced ratio area of the wheel graph is

$$Ar(W_n)^{\frac{2+\sqrt{4+4(n-1)}}{2},\frac{2-\sqrt{4+4(n-1)}}{2}} = \frac{4(n-1)}{n}(\ln|n-1|).$$

As per Section 3.6.4, the eigen-bi-balanced ratio area of the star graph with rays of length 2 is

$$Ar(S_{1,mP_3})^{\sqrt{m+1},-\sqrt{m+1}} = \frac{2\sqrt{2}(n-1)}{n}\sqrt{n+1},\ \text{where } m = \frac{(n-1)}{2} \text{ and } n \geq 3.$$

1. Considering the eigen-bi-balanced ratio area of the complete graph and the complete split bipartite graph, we get

N	**Complete graph**	**Complete split bi-partite graph**
2	2	2
3	4.61	4.5
4	8.7	8
5	14.45	12.5
100	9445.08	5 000
1 000	992 100.15	500 000
10 000	99 897 906.81	50 000 000

Table 3.8.1: Eigen-bi-blanced area of complete and complete split bi-partite graphs for values on *n*

Now, for large values of *n*,

$Ar(K_n)^{-1,n-1} = (n-1)(n-\ln(n-1))$, behaves like n^2 and is an increasing function of *n*, and

$$Ar\left(K_{\frac{n}{2},\frac{n}{2}}\right)^{\frac{n}{2},\frac{n}{2}} = \frac{n^2}{2}.$$

So we can conclude that

$$Ar(K_n)^{-1,n-1} > Ar\left(K_{\frac{n}{2},\frac{n}{2}}\right)^{\frac{n}{2},\frac{n}{2}} \text{ for large } n.$$

i.e., the area of the complete graphs is greater than or equal to the area of the split complete bipartite graph.

2. Considering the eigen-bi-balanced ratio area of the complete graph and the wheel graph for $n \geq 3,$ we get

N	**Complete graph**	**Wheel graph**
3	4.61	1.85
4	8.7	3.3
5	14.45	4.4
100	9445.08	18.2
1 000	992 100.15	27.6
10 000	99 897 906.81	36.84

Table 3.8.2: Eigen-bi-blanced area of complete and wheel graphs for values on *n*

Now, for large values of *n*,

$Ar(K_n)^{-1,n-1} = (n-1)(n-\ln(n-1))$, behaves like n^2 and is an increasing function of *n*, and

$$Ar(W_n)^{\frac{2+\sqrt{4+4(n-1)}}{2},\frac{2-\sqrt{4+4(n-1)}}{2}} = \frac{4(n-1)}{n}(\ln|n-1|) < n.$$

So we can conclude that

$$Ar(K_n)^{-1,n-1} > Ar(W_n)^{\frac{2+\sqrt{4+4(n-1)}}{2},\frac{2-\sqrt{4+4(n-1)}}{2}} \text{ for large } n.$$

i.e., The eigen-bi-balanced ratio area of the complete graphs is greater than the area of the wheel graph for $n \geq 3.$

3. Considering the eigen-bi-balanced ratio area of the complete graph and the star graph with rays of length 2, for $n \geq 3$, we get

N	**Complete graph**	**Star graph with rays of lenth 2**
3	4.61	3.77
5	14.45	5.54
7	31.25	6.86
101	9 639.48	28.28
1001	99 4092.24	89.44
10001	999 17896.60	282.84

Table 3.8.2: Eigen-bi-blanced area of complete and complete split bi-partite graphs for values on *n*

Now, for large values of *n,*

$Ar(K_n)^{-1,n-1} = (n-1)(n-\ln(n-1))$, behaves like n^2 and is an increasing function of *n,* and

$Ar(S_{1,mP_3})^{-\sqrt{m+1},\sqrt{m+1}} = \dfrac{2\sqrt{2}(n-1)}{n}\sqrt{n+1},$ where $m = \dfrac{(n-1)}{2}$ behaves like $2\sqrt{2n}$ and is an increasing function of *n.*

So we can conclude that

$Ar(K_n)^{-1,n-1} > Ar(S_{1,mP_3})^{-\sqrt{m+1},\sqrt{m+1}},$ where $m = \dfrac{(n-1)}{2}$

i.e., the eigen-bi-balanced ratio area of the complete graph is greater than the area of the star graph with rays of length 2, for $n \geq 3$.

So we can conclude that the eigen-bi-balanced areas of complete bipartite graphs, wheel graphs and star graphs with rays of length 2, are each bounded above by the area of the complete graph. □

Theorem 3.8.3

If a class of graphs has eigen-bi-balanced ratio

$$r = r(a\Im b) = \frac{a+b}{ab}$$

then $ar \neq 1$ and $br \neq 1$.

Also, if r is non-zero, the elements of the eigen-pair (a,b) cannot both be $\frac{1}{r}$.

Proof

Let $\frac{a+b}{ab} = r(a\Im b) = r \Rightarrow a+b = rab.$

If we let $ab = y,$ we get $a+b = rab = ry$

so

$$a + \frac{y}{a} = ry$$

$$\Rightarrow a^2 + y = ray$$

$$\Rightarrow y = \frac{-a^2}{1-ar} = ab$$

$$\Rightarrow b = \frac{a}{ar-1}, \text{ for } ar \neq 1.$$

Swopping the roles of *a* and *b* we get the desired result. □

The following theorem can be derived from Lee and Yeh [37]:

Theorem 3.8.4

Define the class of graphs

$\Im = \overline{K_k} \oplus K_n$

where k is fixed, and n, which varies and is greater than 1.

Then this class has eigen-pair $a,b = \dfrac{(n-1) \pm \sqrt{(n-1)^2 + 4nk}}{2}$

with eigen-bi-balanced ratio $r(a\Im b) = \dfrac{n-1}{-nk}$

with eigen-bi-balanced ratio asymptote $asymp(r(a\Im b)) = \dfrac{-1}{k}$

and area $Ar(\Im)^{a,b} = \left(\dfrac{n(n-1)+2kn}{n+k}\right)\left(\dfrac{n}{k} - \dfrac{1}{k}\ln(n+1)\right)$

Proof

From section 2.7.3, the eigenvalue conjugate pair associated with this join is:

$$(a,b) = \frac{(n-1) \pm \sqrt{(n-1)^2 + 4nk}}{2}$$

Then sum $(a,b) = \dfrac{2(n-1)}{2} = (n-1)$ and

and product $(a,b) = \dfrac{-4nk}{4} = -kn$,

and $r(a\Im b) = \dfrac{n-1}{-nk}$

and $asymp(r(a\Im b)) = \dfrac{-1}{k}$ as n becomes increasingly large.

The eigen-bi-balanced ratio area is (with average degree B):

$$\begin{aligned} Ar(\Im)^{a,b} &= B\left|\int \frac{1-n}{kn}\, d(n+k)\right| \\ &= \frac{n(n-1)+2kn}{(n+k)}\left|\int [\frac{1}{kn} - \frac{1}{k}\, d(n)]\right| \end{aligned}$$

$$=\frac{n(n-1+2k)}{(n+k)}\left(\frac{n}{k}-\frac{1}{k}\ln n+c\right) \tag{1}$$

With $k=1$, the area must be that of the complete graph on $(n+1)$ vertices, which is, as per section 3.6.1,

$$\begin{aligned}Ar(K_{n+1})^{-1,n} &= (n+1-1)(n+1-\ln(n+1-1)) \\ &= (n)(n+1-\ln(n)).\end{aligned}$$

Hence, from (1) with $k=1$,

$$Ar(\Im)^{a,b}=\frac{n(n-1+2)}{n+1}\left(\frac{n}{1}-\frac{1}{1}\ln n+c\right)=Ar(K_{n+1})^{-1,n}=(n)(n+1-\ln(n))$$

$$\begin{aligned}&\Rightarrow \frac{n(n+1)}{n+1}(n-\ln n+c)=(n)(n+1-\ln(n)) \\ &\Rightarrow \frac{n(n+1)}{n+1}(n-\ln n+c)=(n)(n-\ln(n)+1) \\ &\Rightarrow c=1\end{aligned}$$

So,

$$Ar(\Im)^{a,b}=\left(\frac{n(n-1)+2kn}{n+k}\right)\left(\frac{n}{k}-\frac{1}{k}(\ln n)+1\right) \qquad \square$$

Note that the complement $\overline{G}$ of any $G\in\Im$ above, is not connected, and therefore cannot be eigen-bi-balanced as per the definition. Hence the class of graphs $\overline{G}\in\overline{\Im}$ is not eigen-bi-balanced. This result applies for the class of complete graphs, complete bipartite graphs, wheels and star graphs.

Alternatively, we could have formed the join (with *n* vertices) $\Im=\overline{K_k}\oplus K_{n-k}$, which, by substituting *n* with $(n-k)$ into Theorem 3.8.4, has conjugate pairs

$$(a,b)=\frac{n-k-1\pm\sqrt{(n-k-1)^2+4k(n-k)}}{2}$$

with ratio

$$r(a\Im b)=\frac{k+1-n}{k(n-k)}=\frac{-(n-k)+1}{k(n-k)}$$

which has asymptote as before $asymp(r(a\Im b))=\dfrac{-1}{k}$.

The area is

$$Ar(\Im)^{a,b} = B\left|\int[-\frac{1}{k}dn+\frac{1}{k(n-k)}]dn\right|$$

$$= B(\frac{n}{k}-\frac{1}{k}\ln(n-k)+c)$$

$$= \left(\frac{(n-k)(n-k-1)+2k(n-k)}{n}\right)\left(\frac{n}{k}-\frac{1}{k}\ln(n-k)+c\right)$$

When $k=1$, we must get the area the of the complete graph on *n* vertices, so that,

$$Ar(K_n)^{-1,n-1} = (n-1)(n-\ln(n-1))$$

$$= \left(\frac{(n-1)(n-1-1)+2(n-1)}{n}\right)\left(\frac{n}{1}-\frac{1}{1}\ln(n-1)+c\right)$$

$$= \left(\frac{(n-1)(n-2)+2(n-1)}{n}\right)\left(n-\ln(n-1)+c\right)$$

$$= (n-1)(n-\ln(n-1)+c) \text{ which gives } c=0.$$

Hence $A(\Im)^{a,b} = \left(\frac{(n-k)(n-k-1)+2k(n-k)}{n}\right)\left(\frac{n}{k}-\frac{1}{k}(\ln(n-k))\right)$

where $\Im = \overline{K_k} \oplus K_{n-k}$.

Conjecture 3.8.1

The maximum eigen-bi-balanced ratio area of classes of graphs on at least 6 vertices is that of the complete graph and is equal to $(n-1)\,(n-\ln(n-1))$.

Remark: The height $H=n$ of the complete split bipartite graph is greater than the height $H=n-\ln(n-1)$ of complete graphs. However multiplying by the average degree results in the complete graph having the greater area

i.e., $Ar(K_n)^{-1,n-1} = (n-1)(n-\ln(n-1)) > \frac{n^2}{2} = Ar(K_{\frac{n}{2},\frac{n}{2}})^{-\frac{n}{2},\frac{n}{2}}$

A trivial association with spanning trees and areas is given below:

From section 3.6.2, the complete split-bipartite graph has area:

$$Ar(G)=\frac{n^2}{2}=\frac{shad(G)}{2t(G)}$$

From section 3.6.1, for the complete graph the area, in respect of the eigenvalues -1 and $n-1$, is:

$$\begin{aligned}Ar(K_n)^{-1,n-1} &= (n-1)(n-\ln(n-1)) \\ &= n^2-n\ln(n-1)-n+\ln(n-1) \\ &< n^2+\ln(n-1)\end{aligned}$$

Now $\ln(n-1)<\ln(n)<n<2n+1$,

so $Ar(K_n)^{-1,n-1}<n^2+2n+1=(n+1)^2=\dfrac{shad(K_{n+1})}{t(K_{n+1})}$.

Note that for wheels with *n* spokes, the conjugate eigen-pair, as per Section 2.7.1, is

$(a,b)=\dfrac{2\pm\sqrt{4+4n}}{2}$ so that

$$\begin{aligned}|a+b|+|ab| &= \left|\frac{2+\sqrt{4+4n}}{2}+\frac{2-\sqrt{4+4n}}{2}\right|+\left|\frac{2+\sqrt{4+4n}}{2}.\frac{2-\sqrt{4+4n}}{2}\right| \\ &= 2+n\end{aligned}$$

For the join of two cycles of length *n*, there exists a pair of eigenvalues $(a,b)=2\pm n$ so that

$$\begin{aligned}&|a+b|+|ab| \\ &=|2+n+2-n|+|(2+n)(2-n)| \\ &=4+\left|4-n^2\right| \\ &=n^2\end{aligned}$$

See Lee and Yeh [37].

Note that for $c(d)$ respectively being the maximum (minimum) degrees of the vertices in the respective graphs, the wheel graph above has $|a+b|+|ab|=(2+n)\leq 3(n-1)=cd$ and the join of two cycles of length *n* has $|a+b|+|ab|=n^2\leq (n+2)^2=cd$. This suggests part (i) of the following conjecture.

For star graphs with *m* rays of length 2, we have eigen-pair $(a,b)=\left(\sqrt{m+1},-\sqrt{m+1}\right)$, and $|a+b|=\left|\sqrt{m+1}-\sqrt{m+1}\right|=0$ and $|ab|=\left|\left(\sqrt{m+1}\right)\left(-\sqrt{m+1}\right)\right|=(m+1)$. Also $c=m$ and $d=1$, so $\dfrac{|ab|}{d}=\dfrac{m+1}{1}=c+1$. So, the star graph with *m* rays of length 2, suggests part (ii) of the conjecture below.

Conjecture 3.8.2

If a class of non-complete graphs, is eigen-bi-balanced with associated eigen-pair (a,b) of a member of the class, the member having maximum (minimum) degree *c(d)* respectively, then

(i) *if* $a+b\neq 0$ then $|a+b|+|ab|\leq cd$

(ii) *if* $a+b=0$ then $\dfrac{|ab|}{d}\leq c+1.$

3.9 Eigen-bi-balanced classes of graphs – density

Definition 3.9.1: Eigen-bi-balanced density

The interval [-1,0] is more convenient if it is a positive interval: we define the *eigen-bi-balanced density* of a class of eigen-bi-balanced graphs with asymptote $asymp(r(a\Im b))$ as

$$\Omega_r(\Im)=|asymp(r(a\Im b))|$$

so that the complete graph has eigen-bi-balanced density 1, which we propose is the *largest* density of all possible eigen-bi-balanced graphs (the maximum density of a class of graphs will be the largest of its densities over all its possible ratios).

Conjecture 3.9.1

The asymptotic eigen-bi-balanced ratio of uniquely eigen-bi-balanced classes of graphs lies in the interval [-1,0] and the complete graph is the only graph which is asymptotically closed with respect to its unique eigen-pair. The density lies on the interval [0,1] with the largest density that of complete graphs, which equals 1.

3.10 Eigen-bi-balanced classes of graphs – energy and asymptotes

There is much research on the *energy* of a graph - it is related to the total π-electron energy in a molecule represented by a (molecular) graph.

Definition 3.10.1: Energy of a graph

The *energy of a graph* with adjacency matrix A with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ is:

$$E^A = \sum_{i=1}^{n} |\lambda_i|$$

See Stevanoviç [45].

If we have a class of eigen-bi-balanced graphs, is there a way of determining if the asymptotic ratio has an effect on the energy of a graph? It may be possible by assigning this asymptotic value to the vertices of the graph as, for example, a weight of a loop on a vertex – see Adiga, Bayad, Gutman and Srinivas [1]. This suggests the following definition:

Definition 3.10.2: R-asymptotic eigen-bi-balanced matrix

The *r-asymptotic eigen-bi-balanced matrix* $C_r^{\infty} = (c_{ij})$, associated with the adjacency matrix $A = (a_{ij})$ of *G* on n vertices with an eigen-bi-balanced ratio asymptote $r(a\Im b)^{\infty}$, is defined as:

$$c_{ij} = \begin{cases} a_{ij}; i \neq j \\ \deg(a_j) + r(a\Im b)^{\infty}; \ \ i = j \end{cases}$$

If G is *k*-regular and A has eigenvalues $k = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$,

then the eigenvalues of C_r^{∞} are:

$$2k + r(a\Im b) = \lambda_1 \geq \lambda_2 + k + r(a\Im b) \geq \ldots \geq \lambda_n + k + r(a\Im b)$$

In particular, if $r = 0$ the C_0^{∞} is the signless Laplacian matrix.

Definition 3.10.3: *R*-asymptotic eigen-bi-balanced matrix

The *energy* of the *r-asymptotic eigen-bi-balanced matrix* $C_r^{\infty} = C,$ associated with the graph G on n vertices and m edges, with eigenvalues of $C_r^{\infty} = C$ being $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n,$ is:

$$E^{C_r^{\infty}} = \sum_{i=1}^{n} \left| \lambda_i - \frac{2m}{n} \right|$$

See Stevanoviç [45].

If $r = 0$ then we get the energy of the signless Laplacian matrix.

If $r \neq 0,$ such as for the complete graph K_n on n vertices and $\frac{n(n-1)}{2}$ edges, then its (-1)-asymptotic eigen-bi-balanced energy is found as follows:

As per Theorem 2.2.1, the eigenvalues of K_n are $(n-1)^1; (-1)^{n-1}$ so that the eigenvalues of C_{-1}^{∞} are:

$\lambda_1 = 2(n-1) - 1 = 2n - 3$ once

$\lambda_2 = -1 + (n-1) - 1 = n - 3$ $(n-1)$times

so that the *r*-asymptotic eigen-bi-balanced energy of G (with eigen-pair (a, b)) is:

$$\begin{aligned}
E^{C_{-1}^{\infty}} &= \sum_{i=1}^{n} \left| \lambda_i - \frac{2m}{n} \right| \\
&= \left| (2n-3) - \frac{2n(n-1)}{2n} \right| + (n-1)\left| (n-3) - \frac{2n(n-1)}{2n} \right| \\
&= \left| (2n-3) - (n-1) \right| + (n-1)\left| (n-3) - (n-1) \right| \\
&= \left| (n-2) \right| + (n-1)\left| (-2) \right| \\
&= n - 2 + 2n - 2 \\
&= 3n - 4
\end{aligned}$$

This energy is greater than the normal energy $E^A = 2n - 2$ of a complete graph on a large number of vertices. This asymptotic energy can be regarded as the eigen-pair balanced energy associated with the graph G as the order of G becomes increasingly large.

3.11 Eigen-bi-balanced classes of graphs – matrix ratio

Definition 3.11.1: Matrix eigen-bi-balanced ratio equation

Let A be the adjacency matrix of a graph $G \in \Im$, where $\Im$ is a eigen-bi-balanced class of graphs. If the eigenvectors $\underline{v_1}, \underline{v_2}$, associated with the eigen-pair (a,b), have *unit length*, then we have the *matrix eigen-bi-balanced ratio equation:*

$$\frac{\underline{v_1}^t A\underline{v_1} + \underline{v_2}^t A\underline{v_2}}{\underline{v_1}^t A\underline{v_1}\underline{v_2}^t A\underline{v_2}} = \frac{a+b}{ab}$$

For example, if $\Im$ is the class of complete graphs, K_n, then as per Theorem 2.1.1, $A(K_n)$ has eigenvalues $n-1$ and -1, with eigenvectors of unit length $\underline{v_1} = \frac{1}{\sqrt{n}}(1,1,\ldots,1)$ and $\underline{v_2}$ (which is the unit eigenvector associated with the second eigenvalue of -1), then

$$\frac{\underline{v_1}^t A\underline{v_1} + \underline{v_2}^t A\underline{v_2}}{\underline{v_1}^t A\underline{v_1}\underline{v_2}^t A\underline{v_2}} = \frac{n-1+(-1)}{(n-1)(-1)} = \frac{n-2}{1-n}.$$

This is an original definition, and is interesting, along with other ratios of matrices which have been defined and investigated over time, for example, the Rayleigh ratio of $R(H,\underline{x}) = \frac{.\underline{x}^T H \underline{x}}{\underline{x}^T \underline{x}}$, for any (hxh) matrix H and any $(hx1)$ vector $\underline{x}$.

3.12 Conclusion

In this chapter, we used the ideas of integral eigenvalues and conjugate eigen-pairs to introduce the new idea of eigen-sum and eigen-product balanced properties of graphs, involving a pair of non-zero distinct eigenvalues *a* and *b*. The fact that these attributes were non-zero, together with the idea of robustness, provided the motivation for the definition of the eigen-bi-balanced ratio of classes of graphs, which allowed for the definitions of area and asymptotic ratio of classes of graphs. We found areas and aysmptotes of known classes of graphs and it appears that complete graphs have the largest area and the asymptotes of all graphs may belong to the interval [-1,0].

Since, in this chapter, we considered known classes of graphs, such as complete graphs, bipartite graphs, wheel graphs, it is natural to determine other classes of graphs which are eigen-bi-balanced. In Chapter 4, we construct a new class of *q*-regular graphs (called *q*-

cliqued graphs) on $\left(q^2+1\right)$ vertices, involving exactly q cliques, each of order q, together with a central vertex. In Theorem 5.1 we show that its conjugate eigenvalues are a function of q, i.e., $\lambda = \dfrac{-1 \pm \sqrt{1+4(q-1)}}{2}$. We shall then determine the eigen-bi-balanced properties of the new class of q-cliqued graphs.

Since this chapter contains new eigen-bi-balanced definitions, the work and results in this chapter are entirely original.

CHAPTER 4

Q-CLIQUED GRAPHS/DESIGNS

There is much interest in considering graphs which have sub-graphs of a particular kind, such as cliques – see Babat and Sivasubramaniam [6], Graham and Hoffman [26], and Liazi, Milis, Pascual and Zissimopoulos [38]. In this chapter, we consider graphs which have cliques of order q as sub-graphs, and which are also design graphs. We will define the construction of a class of graphs called *q-cliqued* graphs on q^2+1 vertices, and then prove that these q-cliqued graphs are design graphs. We determine various characteristics of q-cliqued graphs, namely chromatic number, co-clique number, radius, diameter, etc. Finally, we investigate linear algebra of distance matrices of reduced q-cliqued design graphs.

This chapter consists entirely of original work, and contains the general construction of q-cliqued graphs. The q-cliqued graphs are constructed and illustrated specifically for the cases $q = 2,3,4,$ and 5, and then the generalised construction is defined. A number of characteristics of the q-cliqued graph are then determined.

This entire chapter is original work, applying the definitions in Chapter 1 to the newly defined q-cliqued graph.

4.1 Construction of *q*-cliqued graphs

In this section, for $q \geq 2,$ we construct a q-clique design graph, labelled ${G_{K_q}}^*$, and find its associated adjacency matrix. We take q copies of the complete graph on q vertices K_q, together with a single vertex v. Generally, we label the vertices of the ith copy of $(K_q)^i$ as $v_1^i, v_2^i, \ldots, v_q^i$, for $1 \leq i \leq q.$

4.1.1 For $q = 2$, the graph ${G_{K_2}}^*$:

For $q = 2$, take 2 copies of K_2, namely $(K_2)^i$; $i = 1,2$ together with a single vertex $v.$ Join v to v_1^i; $i = 1,2$, so that v has degree 2. More generally, join v to v_1^i; $1 \leq i \leq q.$ so that v has degree q generally.

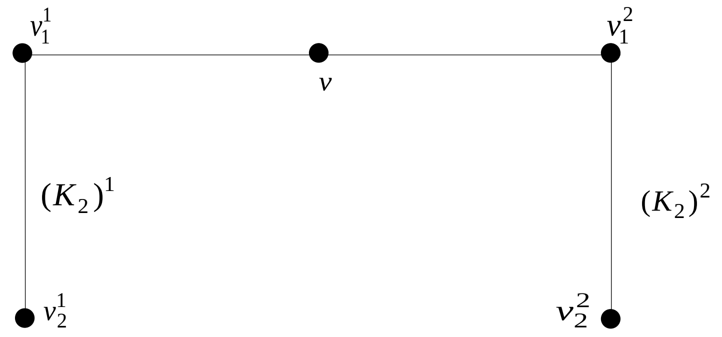

Figure 4.1.1.1: Construction of $G_{K_2}{}^*$ - (a)

Finally, join vertices v_2^1 *and* v_2^2 of $(K_2)^1$ and $(K_2)^2$ to form a 5-cycle.

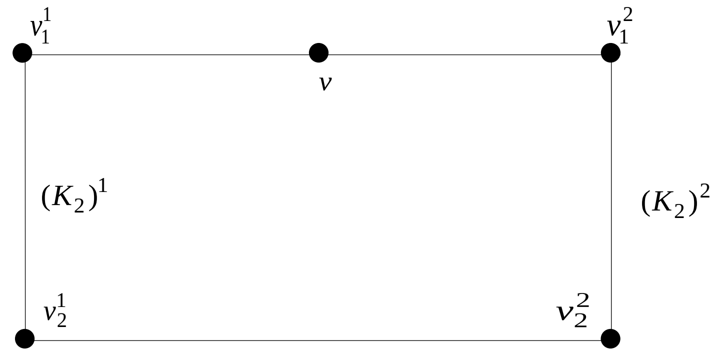

Figure 4.1.1.2: Construction of $G_{K_2}{}^*$ - (b)

Label vertex v as vertex v_1, and then for each sub-clique, label the vertices starting from $v_1^1 = v_2, v_2^1 = v_3,$ and $v_1^2 = v_4, v_2^2 = v_5$.

This graph does not contain a 2-lantern sub-graph so it is a design graph, namely a 2-cliqued design graph.

Then the *5x5* adjacency matrix of $G_{K_2}{}^*$, where the rows are $v_1,...,v_5$ and the columns are $v_1,...,v_5$, is:

$$A\left(G_{K_2}{}^*\right) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

By definition of $\det\left(\lambda I - A\left(G_{K_2}{}^{*}\right)\right)$, the characteristic polynomial of $A\left(G_{K_2}{}^{*}\right)$ is $\lambda^5 - 5\lambda^3 + 5\lambda - 2.$

The eigenvalues of this adjacency matrix are: 2 (once); $\frac{-1+\sqrt{5}}{2}$ (twice) and $\frac{-1-\sqrt{5}}{2}$ (twice). The conjugate eigen-pairs are $\frac{-1\pm\sqrt{5}}{2}$. The diameter is 2, the tightness of type 1 and tightness of type 2 is 6. This graph and its associated design are DS, as they are cycles.

4.1.2 For $q=3$, the graph $G_{K_3}{}^{*}$:

For $q=3$, we take 3 copies of K_3, namely $(K_3)^1$, $(K_3)^2$, and $(K_3)^3$ together with a single central vertex *v*. Join *v* to v_1^i; $i=1,2,3$:

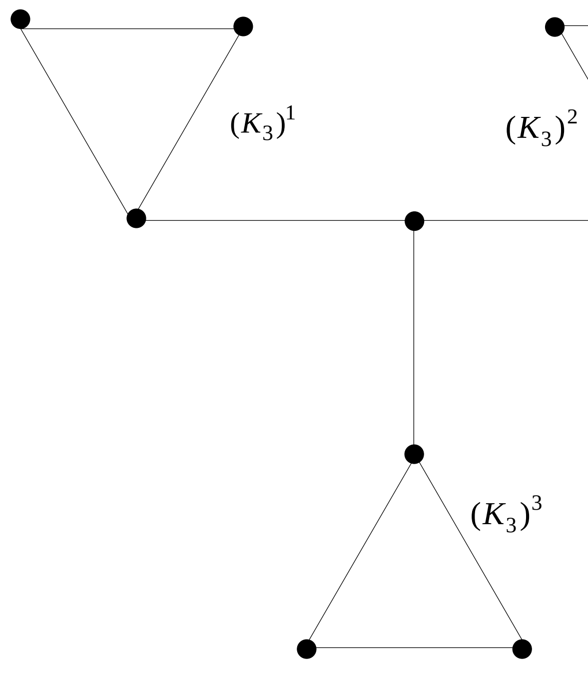

Figure 4.1.2.1: Construction of $G_{K_3}{}^{*}$ - (a)

Join the remaining vertices of the 3 copies of K_3 to form 3 5-cycles. i.e., v_3^1 and $v_{2;}^2$ v_3^2 and v_2^3; v_3^3 and v_2^1.

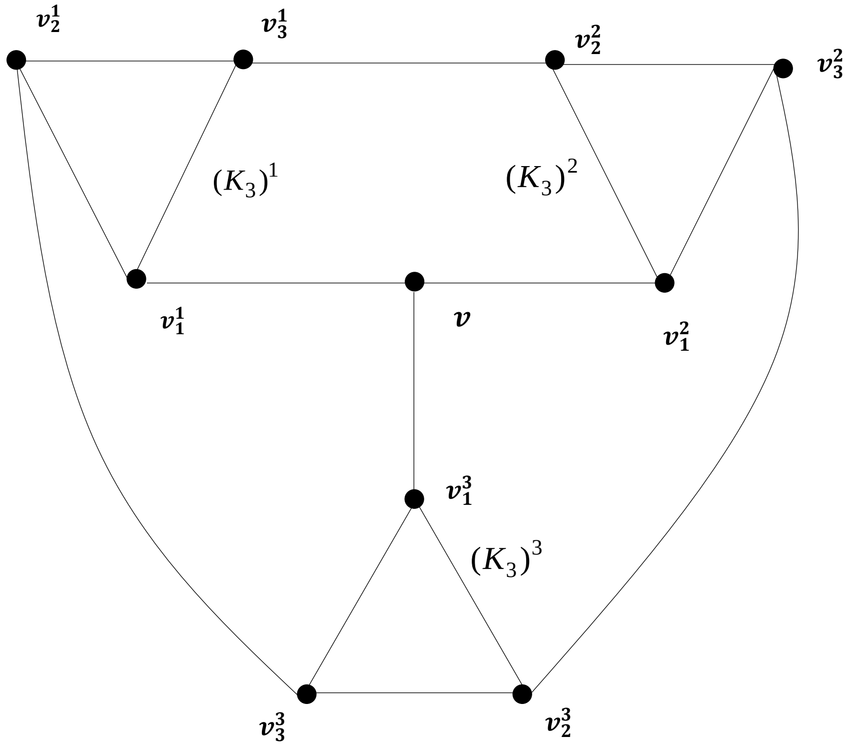

Figure 4.1.2.2: Construction of $G_{K_3}{}^*$ - (b)

Label central vertex *v* as vertex v_1, and then for each sub-clique, label the vertices starting from $v_1^1 = v_2$, $v_2^1 = v_3$, $v_3^1 = v_4$, and $v_1^2 = v_5$, $v_1^2 = v_6$, $v_1^2 = v_7$, and $v_1^3 = v_8$, $v_1^3 = v_9$, $v_1^3 = v_{10}$.

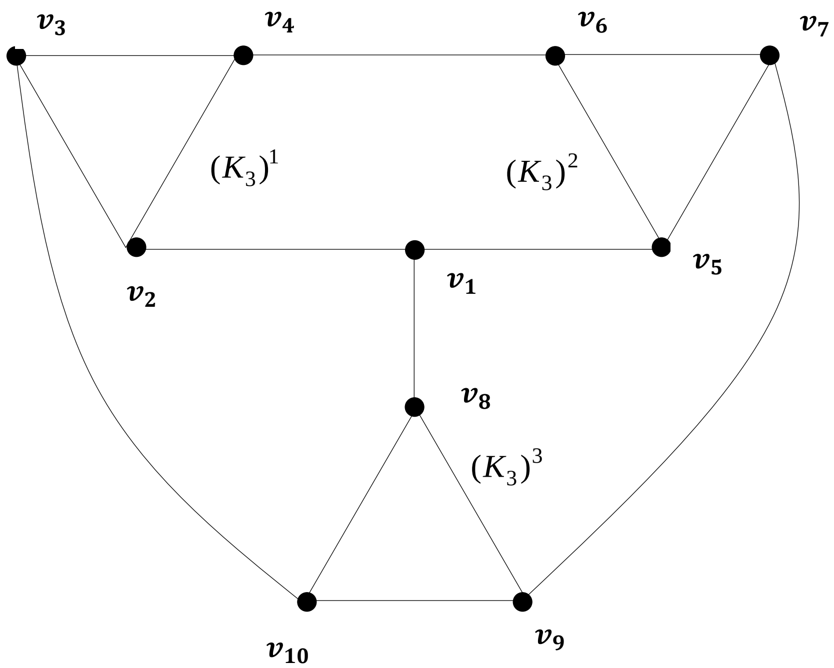

Figure 4.1.2.2: Construction of $G_{K_3}{}^{*}$ - (b)

This is called the 3-cliqued-block-design graph.

Then the 10*x*10 adjacency matrix of $G_{K_3}{}^{*}$, where the rows are $v_{1,}, v_2, \ldots, v_{10}$ and the columns are $v_{1,}, v_2, \ldots, v_{10}$ is:

$$A\left(G_{K_3}{}^{*}\right) = \begin{bmatrix} 0 & 1 & 0 & & 1 & & & 1 & & \\ 1 & 0 & 1 & 1 & & & & & & \\ & 1 & 0 & 1 & & & & & & 1 \\ & 1 & 1 & 0 & & 1 & & & & \\ 1 & & & & 0 & 1 & 1 & & & \\ & & & 1 & 1 & 0 & 1 & & & \\ & & & & 1 & 1 & 0 & & 1 & \\ 1 & & & & & & & 0 & 1 & 1 \\ & & & & & & 1 & 1 & 0 & 1 \\ & & 1 & & & & & 1 & 1 & 0 \end{bmatrix}$$

All blank elements are zero. Since no two columns are the same, the exclude 3-lantern condition holds.

The characteristic polynomial of the adjacency matrix for $q=3$ is:

$$\lambda^{10}-15\lambda^{8}-6\lambda^{7}+75\lambda^{6}+48\lambda^{5}-144\lambda^{4}-114\lambda^{3}+75\lambda^{2}+68\lambda+12$$

The eigenvalues of this adjacency matrix are: 3, 1, -2, -2, 1.879, 1.879, -0.347, -0.347, -1.532, -1.532 .

The conjugate eigen-pair is $\frac{-1\pm\sqrt{9}}{2}$.

The number of distinct eigenvalues is 6, and the number of integer eigenvalues is 4. The radius is 2, and the diameter (maximum eigenvalue) is 3, the tightness of type 1 is 18, and tightness of type 2 is 12.

4.1.3 For $q=4$, the graph $G_{K_4}{}^{*}$:

For $q=4$, take 4 copies of K_4, namely $(K_4)^1$, $(K_4)^2$, $(K_4)^3$, and $(K_4)^4$, together with a central vertex v. Join v to v_1^i; $i=1,2,3,4.$ Label each of the vertices within each copy of K_4 clockwise, starting with $v_1^i;v_2^i;v_3^i;v_4^i;\ \ i=1,2,3,4.$

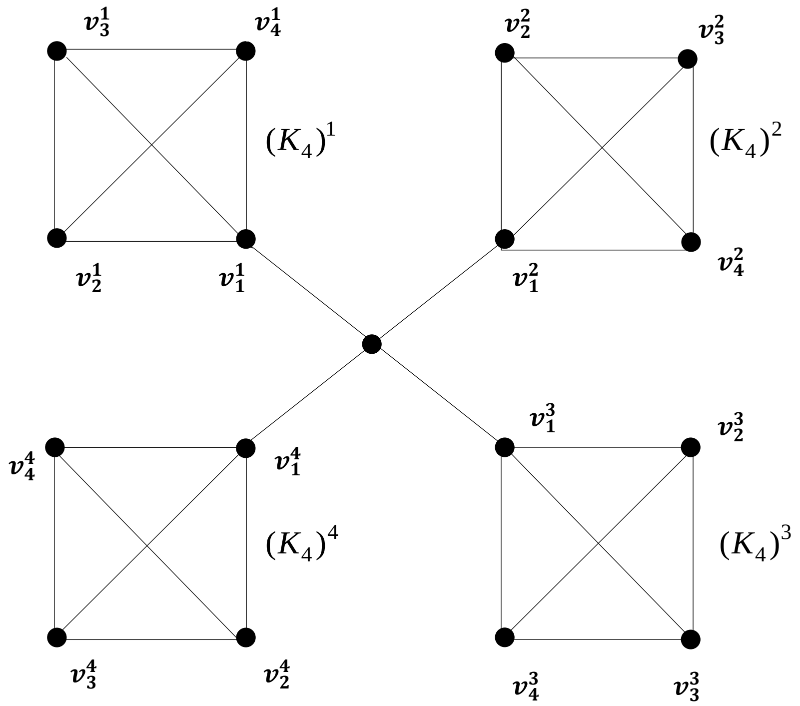

Figure 4.1.3.1: Construction of $G_{K_4}{}^{*}$ - (a)

Join vertices v_4^i to v_2^{i+1} for $1 \le i \le n$ where $v_2^{n+1} = v_2^1$ to form 4 5-cycles. Join vertex v_3^i to v_3^{i+1} for $1 \le i \le n$ where *i* is odd.

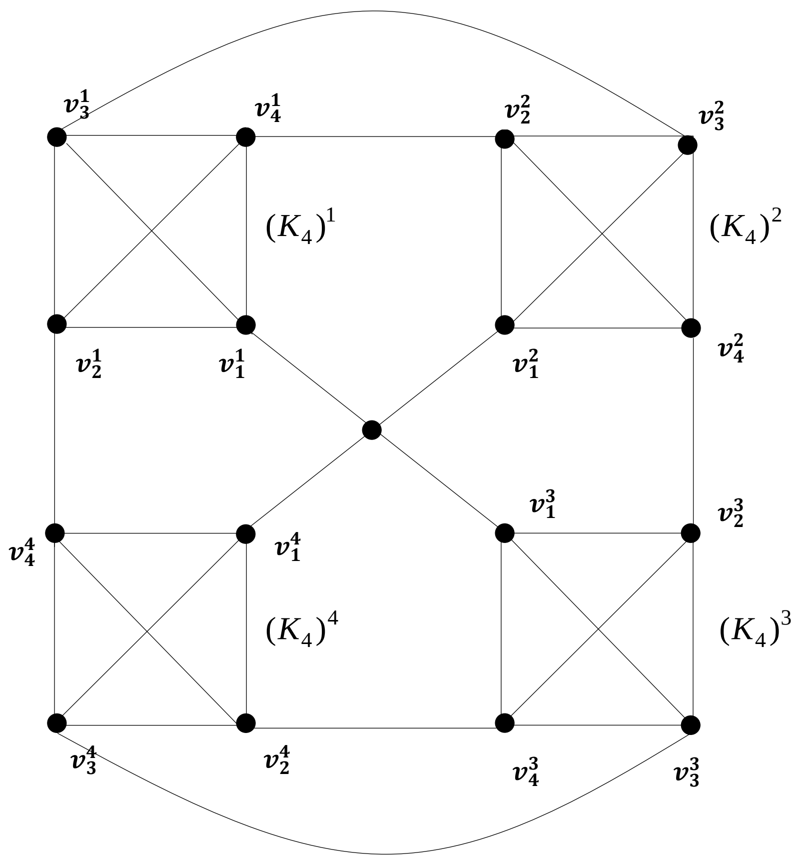

Figure 4.1.3.1: Construction of $G_{K4}{}^*$ - (b)

Label vertex *v* as vertex v_1, and then for each sub-clique, label the vertices clockwise for each sub-clique, starting from $v_1^1 = v_2$, $v_2^1 = v_3$, $v_3^1 = v_4$, $v_4^1 = v_5$ and $v_1^2 = v_6$, $v_2^2 = v_7$, $v_3^2 = v_8$, $v_4^2 = v_9$, and $v_1^3 = v_{10}$, $v_2^3 = v_{11}$, $v_3^3 = v_{12}$, $v_4^3 = v_{13}$, and $v_1^4 = v_{14}$, $v_2^4 = v_{15}$, $v_3^4 = v_{16}$, $v_4^4 = v_{17}$.

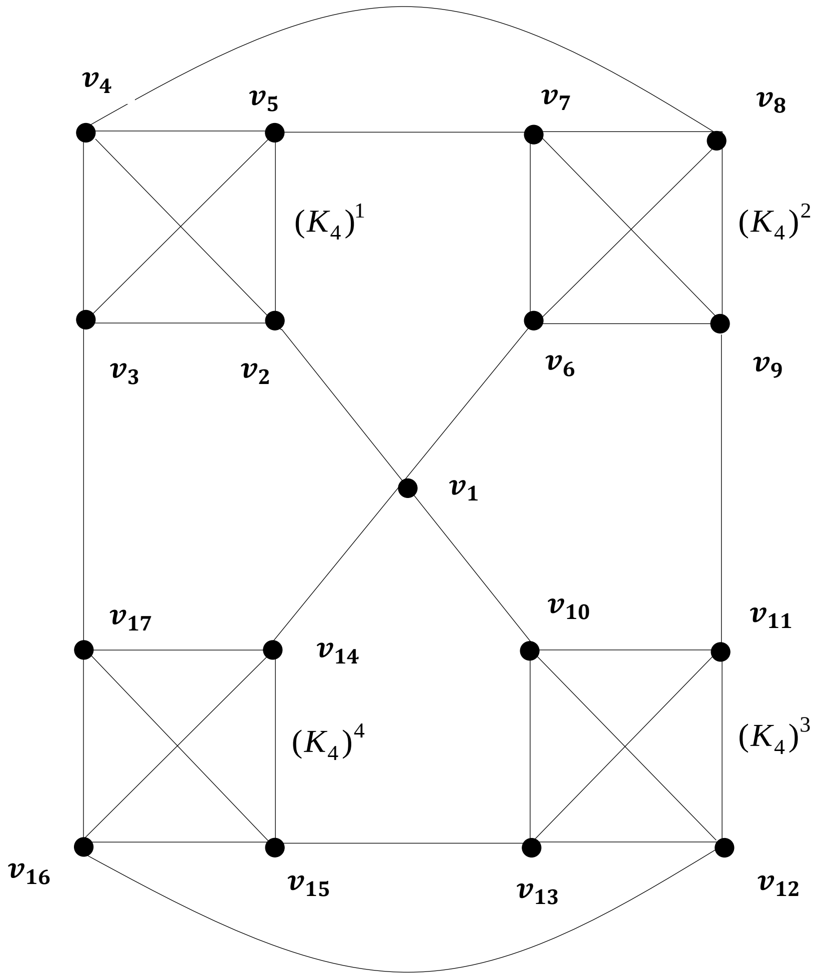

Figure 4.1.3.1: Construction of $G_{K_4}{}^{*}$ - (c)

Then the $17x17$ adjacency matrix of $G_{K_4}{}^{*}$, where the rows are $v_1, v_2, \ldots, v_{17}$ and the columns are $v_1, v_2, \ldots, v_{17}$ is:

$$A\left(G_{K_4}{}^{*}\right) = \begin{bmatrix}
0 & 1 & & & & 1 & & & & 1 & & & & 1 & & & \\
1 & 0 & 1 & 1 & 1 & & & & & & & & & & & & \\
 & 1 & 0 & 1 & 1 & & & & & & & & & & & & 1 \\
 & 1 & 1 & 0 & 1 & & & 1 & & & & & & & & & \\
 & 1 & 1 & 1 & 0 & & 1 & & & & & & & & & & \\
1 & & & & & 0 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 1 & 1 & 0 & 1 & 1 & & & & & & & & \\
 & & & 1 & & 1 & 1 & 0 & 1 & & & & & & & & \\
 & & & & & 1 & 1 & 1 & 0 & & 1 & & & & & & \\
1 & & & & & & & & & 0 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 1 & 1 & 0 & 1 & 1 & & & & \\
 & & & & & & & & & 1 & 1 & 0 & 1 & & & 1 & \\
 & & & & & & & & & 1 & 1 & 1 & 0 & & 1 & & \\
1 & & & & & & & & & & & & & 0 & 1 & 1 & 1 \\
 & & & & & & & & & & & & 1 & 1 & 0 & 1 & 1 \\
 & & & & & & & & & & & 1 & & 1 & 1 & 0 & 1 \\
 & & 1 & & & & & & & & & & & 1 & 1 & 1 & 0
\end{bmatrix}$$

All blank entries are zero. The eigenvalues for this adjacency matrix are:

-2.303, 1.303, 4 , 3.403, 2.935, 2.303, -0.463, -0.684, -1.303, -1.719, -1.473, -2, -2, -2, 0, 0, 0. The conjugate eigen-pair is $\frac{-1 \pm \sqrt{13}}{2}$. The number of distinct eignvalues is 13, and the number of integer eigenvalues is 7. The radius is 2, and the diameter (maximum eigenvalue) is 4, the tightness of type 1 is $13 \times 4 = 52$, and tightness of type 2 is $(4+1) \times 4 = 20$.

4.1.4 For $q = 5$, the graph $G_{K_5}{}^*$:

For $q = 5$, $G_{K_5}{}^*$ is:

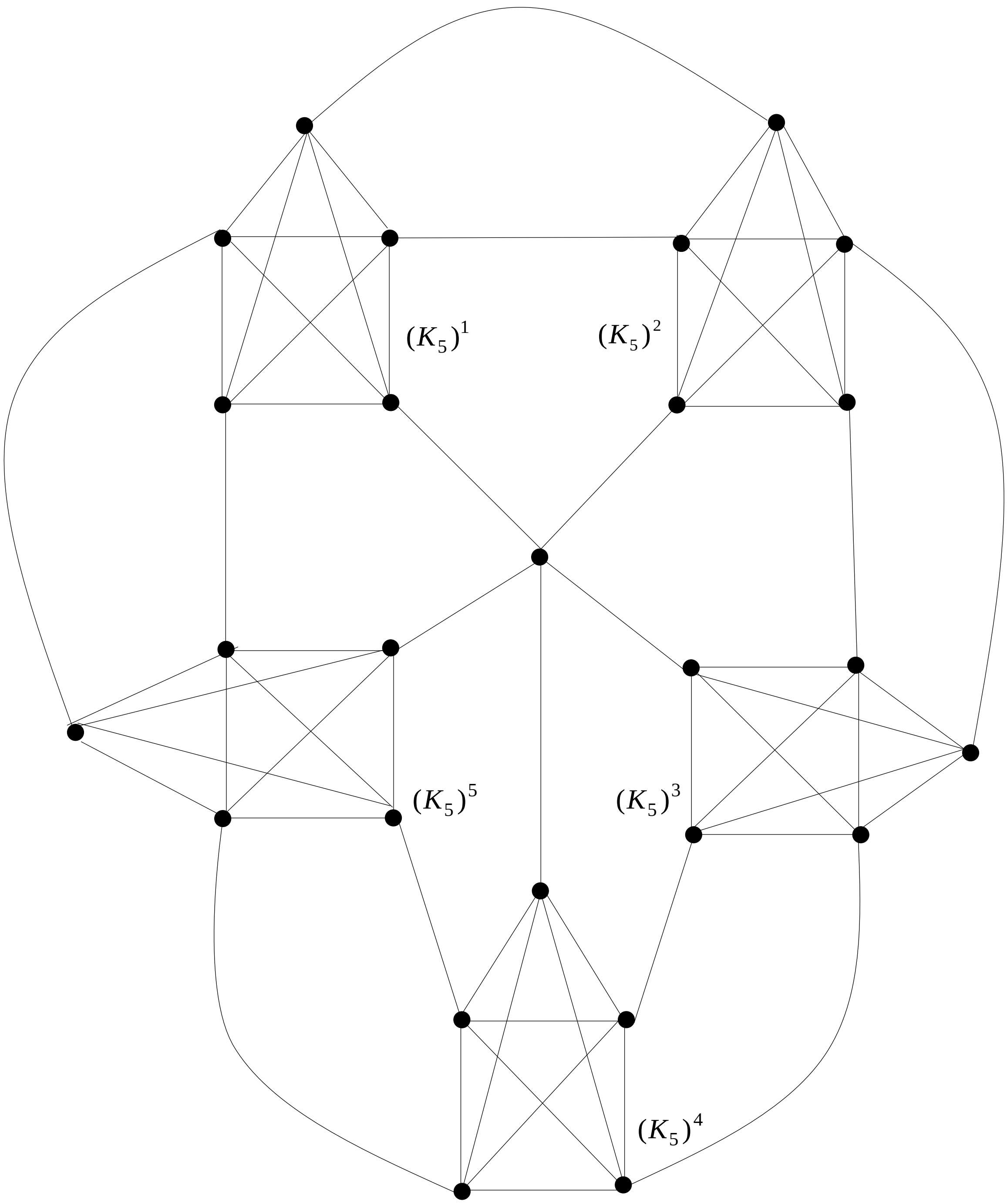

Figure 4.1.3.1: Construction of $G_{K_4}{}^*$

Label vertex v as vertex $v_{1,}$ and then for each sub-clique, label the vertices clockwise starting from $v_j^1,\ 1 \le j \le 5,\ v_j^2,\ 1 \le j \le 5,\ v_j^3,\ 1 \le j \le 5,$ $v_j^4,\ 1 \le j \le 5,\ v_j^5,\ 1 \le j \le 5.$ The resultant *26x26* adjacency matrix of $G_{K_5}{}^{*}$, where the rows are $v_1, v_2, \ldots, v_{26}$ and the columns are $v_1, v_2, \ldots, v_{26}$ is:

$$A\left(G_{K_5}{}^{*}\right) = \begin{bmatrix}
0&1&&&&&1&&&&&1&&&&&1&&&&&1&&&& \\
1&0&1&1&1&1&&&&&&&&&&&&&&&&&&&& \\
&1&0&1&1&1&&&&&&&&&&&&&&&&&&&&1 \\
&1&1&0&1&1&&&&&&&&&&&&&&&&&&&1& \\
&1&1&1&0&1&&&1&&&&&&&&&&&&&&&&& \\
&1&1&1&1&0&&1&&&&&&&&&&&&&&&&&& \\
1&&&&&&0&1&1&1&1&&&&&&&&&&&&&&& \\
&&&&&1&1&0&1&1&1&&&&&&&&&&&&&&& \\
&&&&1&&1&1&0&1&1&&&&&&&&&&&&&&& \\
&&&&&&1&1&1&0&1&&&1&&&&&&&&&&&& \\
&&&&&&1&1&1&1&0&&1&&&&&&&&&&&&& \\
1&&&&&&&&&&&0&1&1&1&1&&&&&&&&&& \\
&&&&&&&&&&1&1&0&1&1&1&&&&&&&&&& \\
&&&&&&&&&1&&1&1&0&1&1&&&&&&&&&& \\
&&&&&&&&&&&1&1&1&0&1&&&1&&&&&&& \\
&&&&&&&&&&&1&1&1&1&0&&1&&&&&&&& \\
1&&&&&&&&&&&&&&&&0&1&1&1&1&&&&& \\
&&&&&&&&&&&&&&&1&1&0&1&1&1&&&&& \\
&&&&&&&&&&&&&&1&&1&1&0&1&1&&&&& \\
&&&&&&&&&&&&&&&&1&1&1&0&1&&&1&& \\
&&&&&&&&&&&&&&&&1&1&1&1&0&&1&&& \\
1&&&&&&&&&&&&&&&&&&&&&0&1&1&1&1 \\
&&&&&&&&&&&&&&&&&&&&1&1&0&1&1&1 \\
&&&&&&&&&&&&&&&&&&&1&&1&1&0&1&1 \\
&&&1&&&&&&&&&&&&&&&&&&1&1&1&0&1 \\
&&1&&&&&&&&&&&&&&&&&&&1&1&1&1&0
\end{bmatrix}$$

All blank entries are zero.

Eigenvalues for $q = 5$:

-2.562, 1.562, 5.00, 4.381, 4.381, 3.447, 3.447, -1.662, -1.662, -1.272, -1.272, -0.719, -0.719, -0.174, -0.174, 0, 0, 0, 0, 0, -2, -2, -2, -2, -2, -2 . The conjugate eigen-pair is $\frac{-1 \pm \sqrt{17}}{2}$. The number of distinct eignvalues is 11, and the number of integer eigenvalues is 3. The radius is 2, and the diameter (maximum eigenvalue) is 5, the tightness of type 1 is $11 \times 5 = 55,$ and tightness of type 2 is $(4+1) \times 5 = 25.$

4.1.5 For $q = n$, the general construction of graph ${G_{K_n}}^*$:

The general construction of the $(1+n^2) x (1+n^2)$ adjacency matrix of ${G_{K_n}}^*$ where the rows are $v_1, v_2, \ldots, v_{1+n^2}$ and the columns are $v_1, v_2, \ldots, v_{1+n^2}$ is as follows:

$a_{i,i} = 0;\ 1 \le i \le (1+n^2)$

Join v to $v_1^i;\ 1 \le i \le n$:

$a_{1,1+\lambda n+1} = 1;\ 0 \le \lambda \le n-1$

$a_{1+\lambda n+1,1} = 1;\ 0 \le \lambda \le n-1$

Sub-cliques:

$a_{1+\lambda n+k,1+\lambda n+l} = 0;\ 0 \le \lambda \le n-1;\ 1 \le k \le n;\ 1 \le l \le n;\ \ k = l$

$a_{1+\lambda n+k,1+\lambda n+l} = 1;\ 0 \le \lambda \le n-1;\ 1 \le k \le n;\ 1 \le l \le n;\ \ k \ne l$

v_n^i of clique *i* (the *n*th vertex in clique i) joins to v_2^{i+1} (the *2*nd vertex in clique $(i+1)$):

$a_{1+\lambda n+n,1+(\lambda+1)n+2} = 1;\ 0 \le \lambda \le n-1;$

$a_{1+(\lambda+1)n+2,1+\lambda n+n} = 1;\ 0 \le \lambda \le n-1;$

v_j^i of clique *i* (the *j*th vertex in clique i) joins to v_{j-1}^{i+1} (the *(j-1)*th vertex in clique $(i+1)$):

$a_{1+\lambda n+j,1+(\lambda+1)n+(j-1))} = 1;\ \ 0 \le \lambda \le n-1;\ \ 4 \le j \le n-1;\ \ \ j\ \ even$

$a_{1+(\lambda+1)n+(j-1)),1+\lambda n+j,} = 1;\ \ 0 \le \lambda \le n-1;\ \ 4 \le j \le n-1;\ \ \ j\ \ even$

v_j^i of clique i (the *j*th vertex in clique *i*) joins to v_j^{i+1} (the *j*th vertex in clique $(i+1)$), $j = n-1;\ \ n\ \ even,\ \ j\ \ odd,\ \ i\ \ odd$**:**

$a_{1+\lambda n+j,1+(\lambda+1)n+j} = 1;\ \ 0 \le \lambda \le n-1;\ \ \ j = n-1,\ \ \lambda\ \ even$

$a_{1+(\lambda+1)n+j,1+\lambda n+j} = 1;\ \ 0 \le \lambda \le n-1;\ \ \ j = n-1,\ \ \lambda\ \ even$

If for $a_{i,j}\ \ i > \left(1+n^2\right)\ then\ \ i = i - \left(1+n^2\right)$ and if for $a_{i,j}\ \ j > \left(1+n^2\right)\ then\ \ j = j - \left(1+n^2\right)$

$a_{i,j} = 0;\ \ 1 \le i \le \left(1+n^2\right),\ \ 1 \le j \le \left(1+n^2\right)$ otherwise.

The results of this general definition of the construction of a *q*-cliqued graph have been verified using a generic Excel spreadsheet, for the cases $q = 4{,}5,$ and *6*.

4.2 *Q*-cliqued graphs are design graphs

Theorem 4.2.1

The *q*-cliqued graphs, denoted by $G_{K_q}{}^{*}$, and as constructed in Section 4.2, (for $q \ge 2$), are design graphs, that is

1. They are *q*-regular;
2. The number of vertices $\left(q^2+1\right)$ and *q* cannot both be odd; and
3. The blocks of the design must be distinct i.e., no two non-adjacent vertices in the graph are adjacent to the same set of vertices, i.e., the graph does not contain a *q*-lantern sub-graph.

Proof:

1. By definition of construction, the *q*-cliqued graphs are *q*-regular.
2. The number of vertices in a *q*-cliqued graph is q^2+1. If *q* is even, then q^2+1 is odd, and if *q* is odd, then q^2+1 is even. So condition 2 holds for all $q \ge 2$.

3. The condition 3 is true for $q = 2,3,4,$ and 5 since the constructed graphs do not contain q-lantern sub-graphs. This is also evident from the associated adjacency matrices, as no two columns are the same.

   For $q = n, n \geq 6$, consider the following cases of pairs of non-adjacent matrices in ${G_{K_n}}^*$:

   - Take two non-adjacent vertices v_i^a and v_j^b where v_i^a and v_j^b belong to different complete sub-graphs (each of order q) K_a and K_b of ${G_{K_n}}^*$. Then v_i^a is adjacent to $q-1$ vertices in K_a i.e., v_i^a is adjacent to $v_k^a;\ 1 \leq k \leq q,$ where $k \neq i$, and v_b^j is adjacent to $q-1$ vertices in K_b i.e., v_j^b is adjacent to all $v_l^b;\ 1 \leq l \leq q,$ where $l \neq j.$ Also, by definition of the construction of ${G_{K_n}}^*$, v_i^a in K_a is not adjacent to ALL $v_l^b;\ 1 \leq l \leq q,$ where $l \neq j$, and v_j^b in K_b is not adjacent to ALL $v_k^a;\ 1 \leq k \leq q,$ where $k \neq i.$ From this, v_i^a and v_j^b do not have the same neighbourhood set in ${G_{K_n}}^*$. Therefore v_i^a *and* v_j^b do not form twin vertices of a q-lantern sub-graph of ${G_{K_n}}^*$.
   - Take two non-adjacent vertices v_i^a *and* v_j^b where v_i^a ,$1 \leq i \leq q,$ is a vertex in the complete sub-graph K_a, of order q, and v is the central vertex of ${G_{K_n}}^*$. Then v_i^a is adjacent to $q-1$ vertices in K_a ie v_i^a is adjacent to $v_k^a;\ 1 \leq k \leq q,$ where $k \neq i$, and by construction, v is only adjacent to one of $v_k^a;\ 1 \leq k \leq q,$ where $k \neq i$. Therefore, v_i^a and v do not have the same neighbourhood set in ${G_{K_n}}^*$. Therefore v_i^a *and* v_j^b do not form the twin vertices of a q-lantern sub-graph of ${G_{K_n}}^*$.

Therefore the 3 conditions required for ${G_{K_n}}^*$ to be a design graph, have been met, so our q-cliqued graphs as defined in Section 4.1 are design graphs. □

4.3 Chromatic number of q-cliqued graphs

Theorem 4.3.1

The chromatic number of q-cliqued graphs ${G_{K_q}}^*$ is q, where $q \geq 3$.

Proof

For $q = 3$, ${G_{K_3}}^*$ is:

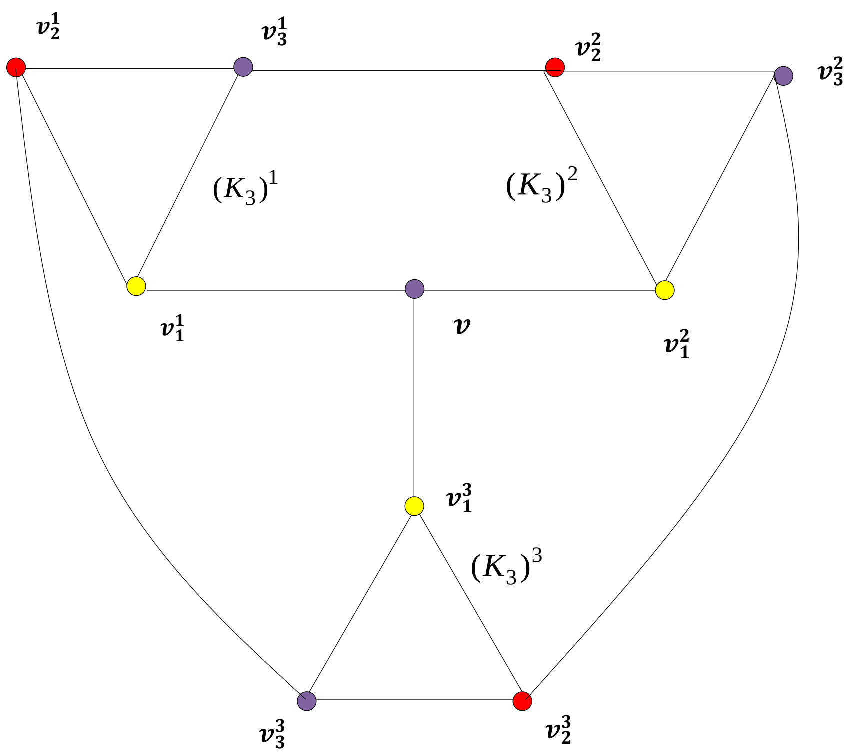

Figure 4.3.1: Colouring of ${G_{K_3}}^*$

Colour the central vertex v with colour 1 (purple), and v_1^i; $i = 1,2,3$ with the second colour (yellow). Colour v_2^i; $i = 1,2,3$ with the third colour (red), and colour v_3^i; $i = 1,2,3$ with the first colour (purple). No two adjacent vertices have the same colour, and no fewer than 3 colours could have been used (as each sub-clique requires at least 3 unique colours). So the chromatic number of ${G_{K_3}}^*$ is 3.

For $q = 4$, ${G_{K_4}}^*$ is:

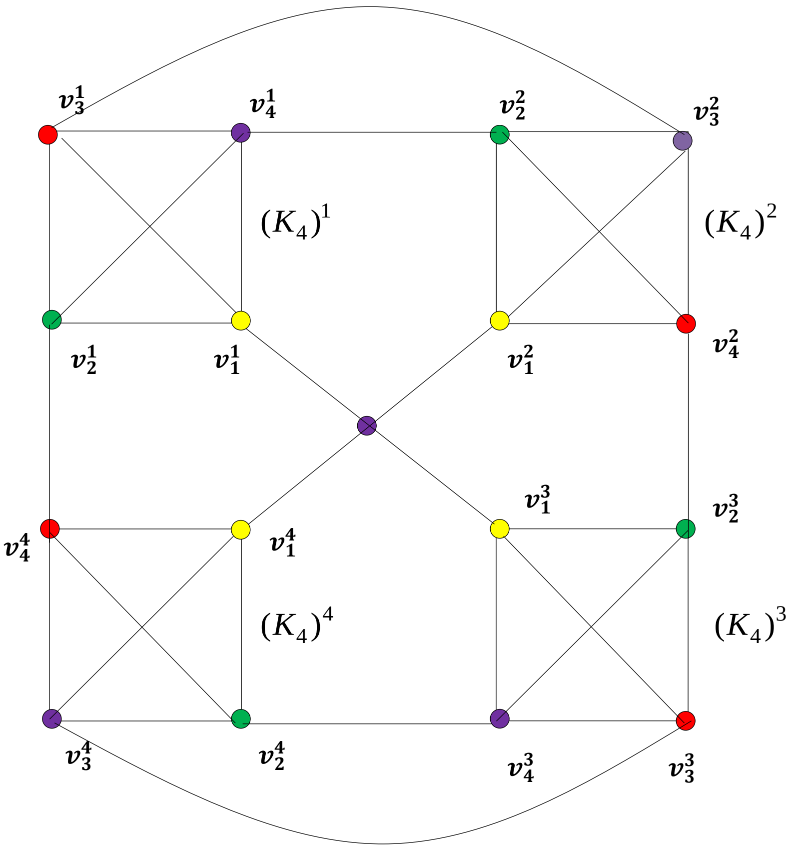

Figure 4.3.2: Colouring of ${G_{K_4}}^*$

Colour the central vertex v with colour 1 (purple), and $v_1^i;\ \ i = 1,2,3,4$ with the second colour (yellow). Colour $v_2^i;\ \ i = 1,2,3,4$ with the third colour (green) and colour $v_3^i;\ \ i = 1,2,3,4$ with the fourth colour (red). Colour $v_4^i;\ \ i = 1,2,3,4$ with the first colour (purple). No two adjacent vertices have the same colour, and no fewer than 4 colours could have been used (as each sub-clique requires at least 4 unique colours). So the chromatic number of ${G_{K_4}}^*$ is 4.

For $q = 5$, $G_{K_5}{}^*$ is:

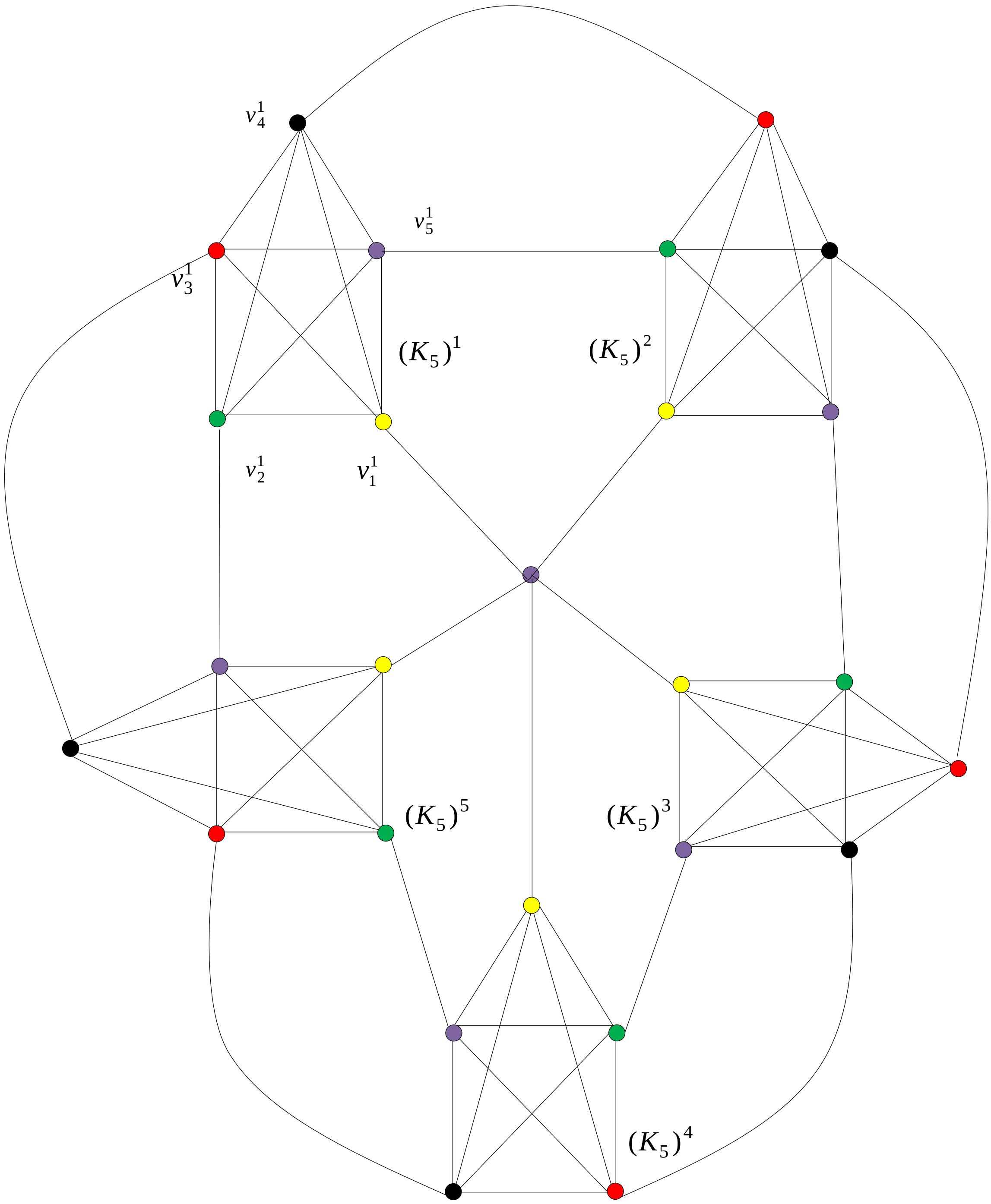

Figure 4.3.3: Colouring of $G_{K_5}{}^*$

Colour the central vertex v with colour 1 (purple), and $v_1^i;\ \ i=1,2,3,4,5$ with the second colour (yellow). Colour $v_2^i;\ \ i=1,2,3,4,5$ with the third colour (green), colour $v_3^i;\ \ i=1,2,3,4,5$ with the fourth colour (red), colour $v_4^i;\ \ i=1,2,3,4,5$ with the fifth colour (black), and $v_5^i;\ \ i=1,2,3,4,5$ with the first colour (purple). No two adjacent vertices have the same colour, and no fewer than 5 colours could have been used (as each sub-clique requires at least 5 unique colours). So the chromatic number of $G_{K_5}{}^*$ is 5.

For $q = 6$, $G_{K_6}{}^{*}$ is:

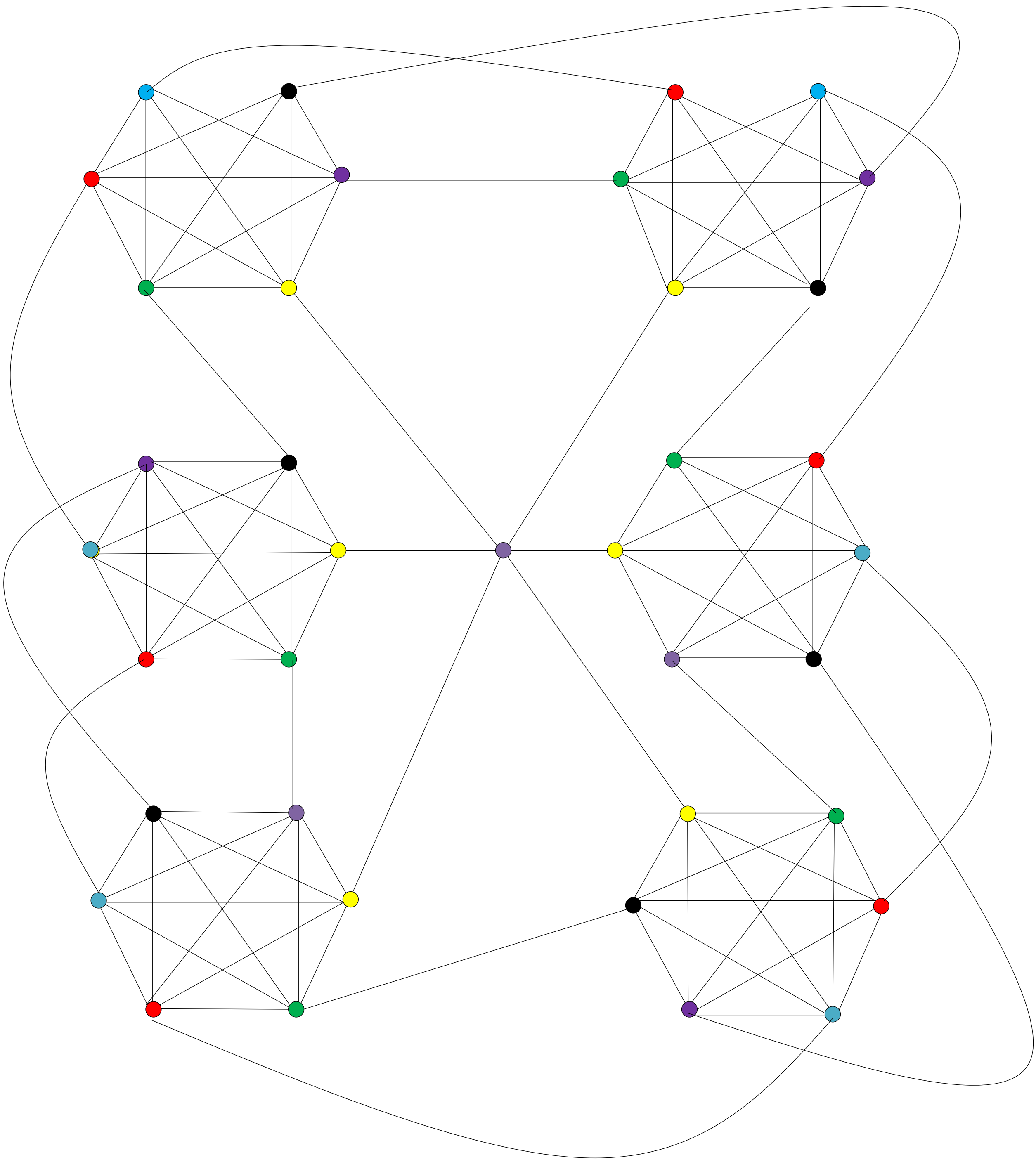

Figure 4.3.4: Colouring of $G_{K_6}{}^{*}$

Colour the central vertex v with colour 1 (purple), and $v_1^i;\ i=1,2,3,4,5,6$ with the second colour (yellow). Colour $v_2^i;\ i=1,2,3,4,5,6$ with the third colour (green), colour $v_3^i;\ i=1,2,3,4,5,6$ with the fourth colour (red), colour $v_4^i;\ i=1,2,3,4,5,6$ with the fifth colour (light blue).

- For i odd, and colour $v_5^i;\ 1\le i\le 6$ with the sixth colour (black) and colour $v_6^i;\ 1\le i\le 6$ with the first colour (purple).
- For i even, and colour $v_5^i;\ 1\le i\le 6$ with the first colour (purple) and colour $v_6^i;\ 1\le i\le 6$ with the sixth colour (black).

No two adjacent vertices have the same colour, and no fewer than 6 colours could have been used (as each sub-clique requires at least 6 unique colours). So the chromatic number of $G_{K_6}{}^*$ is 6.

For $q=n$, we take $G_{K_n}{}^*$. Colour the central vertex with colour 1. Colour $v_1^i;\ 1\le i\le n$ with the 2nd colour, $v_2^i;\ 1\le i\le n,$ with the third colour, and continue sequentially until you colour $v_{n-2}^i;\ 1\le i\le n,$ with the $(n-1)$th colour.

If n is even, then:

- when i is odd, colour $v_{n-1}^i;\ 1\le i\le n,$ with the nth colour and colour $v_n^i;\ 1\le i\le n,$ with the 1st colour.
- where i is even, colour $v_{n-1}^i;\ 1\le i\le n,$ with the 1st colour and colour $v_n^i;\ 1\le i\le n,$ with the nth colour.

If n is odd, then:

- for all i, colour $v_{n-1}^i;\ 1\le i\le n,$ with the nth colour and colour $v_n^i;\ 1\le i\le n,$ with the 1st colour .

With the above colouring, no two adjacent vertices will have the same colour (by the construction of the colouring). As it is not possible to colour the $G_{K_n}{}^*$ graph with fewer than n colours (since $G_{K_n}{}^*$ containes n cliques) such that all adjacent vertices have different colours, then the chromatic number of $G_{K_n}{}^*$ is n. □

4.4 Co-clique number of *q*-cliqued graphs

Theorem 4.4.1

The co-clique number of *q*-cliqued graphs ${G_{K_q}}^*$ is $q+1$, for $q \geq 3$.

Proof

For $q=3$, ${G_{K_3}}^*$ is:

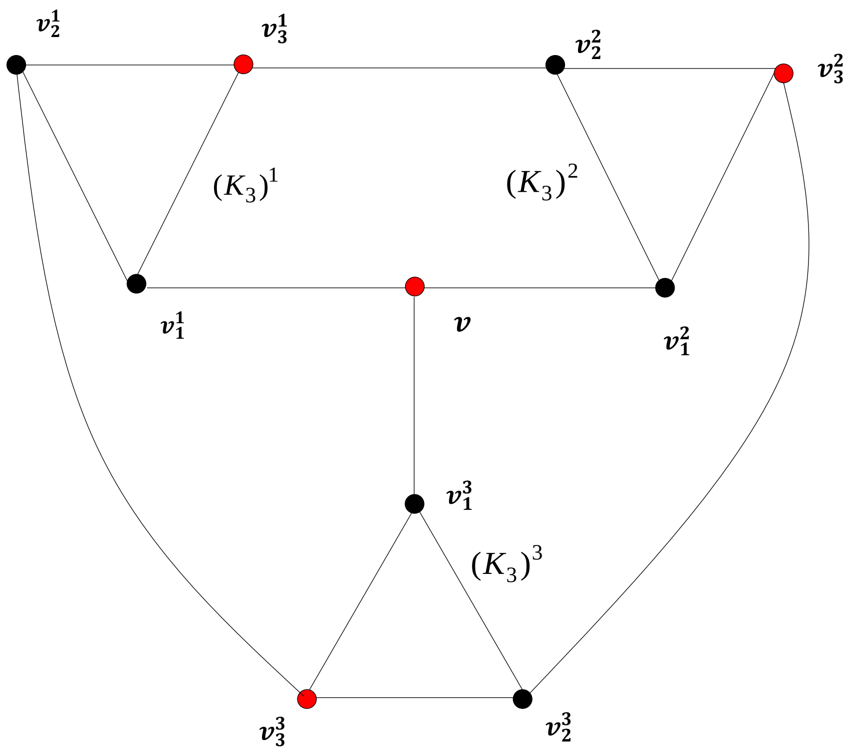

Figure 4.4.1: Co-clique number of ${G_{K_3}}^*$

From Figure 4.4.1, the largest set of non-adjacent vertices are shaded in red, and therefore the co-clique number of ${G_{K_3}}^*$ is 4. The shaded vertices are $v_3^i;\ 1 \leq i \leq 3,$ together with the central vertex. The co-clique number is therefore equal to $q+1$.

For $q = 4$, $G_{K_4}{}^{*}$ is:

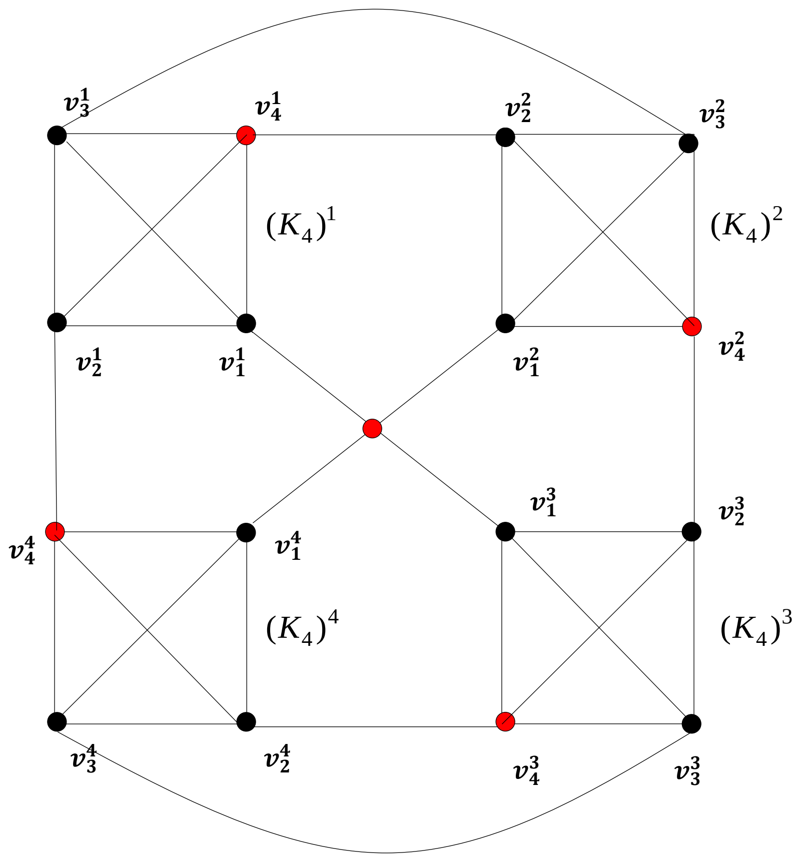

Figure 4.4.2: Co-clique number of $G_{K_4}{}^{*}$

From Figure 4.4.2, the largest set of non-adjacent vertices are shaded in red, and therefore the co-clique number of $G_{K_4}{}^{*}$ is 5. The shaded vertices are $v_4^i;\ 1 \le i \le 4,$ together with the central vertex. The co-clique number is therefore equal to $q+1$.

For $q = 5$, $G_{K_5}{}^{*}$ is:

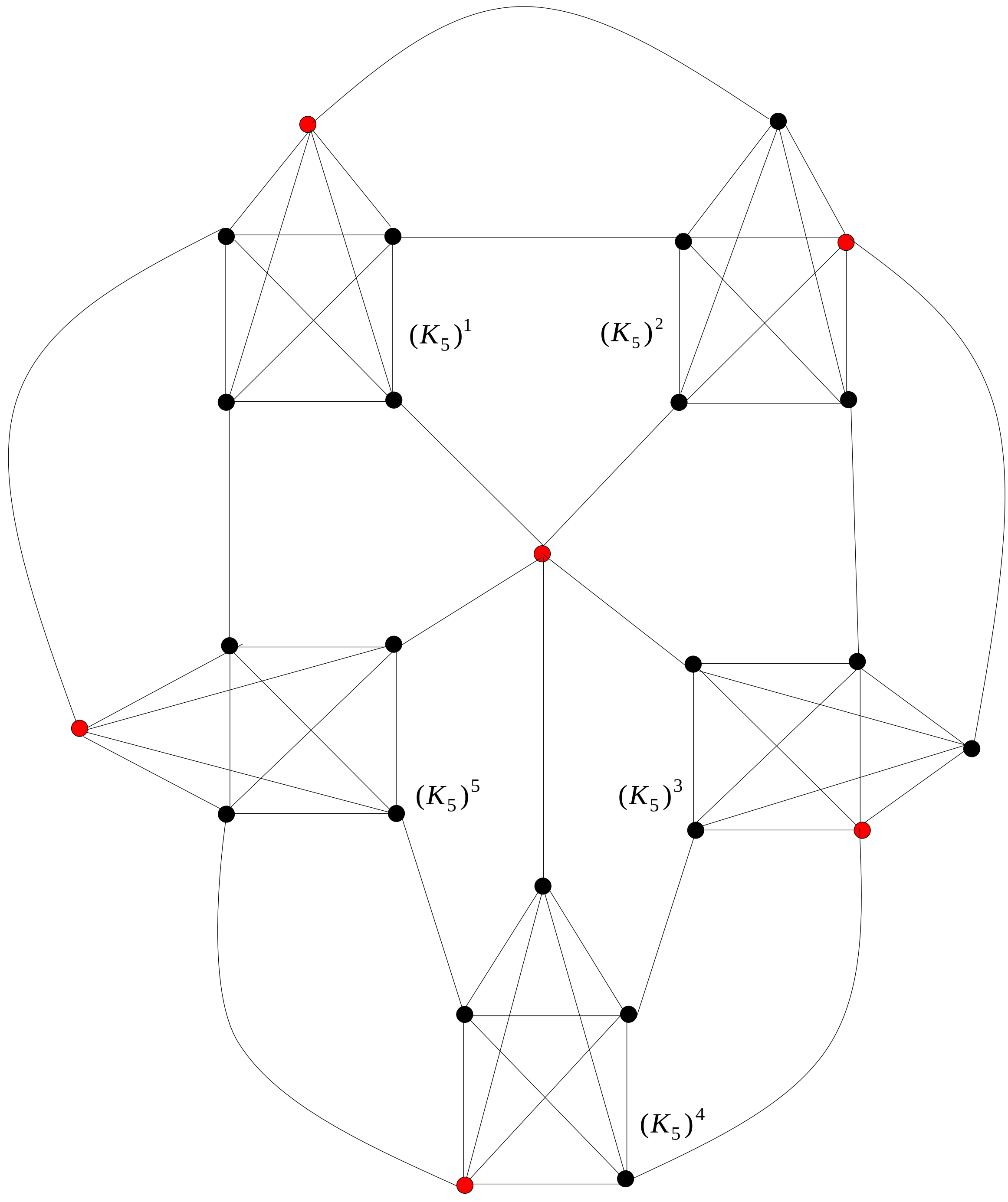

Figure 4.4.3: Co-clique number of $G_{K_5}{}^{*}$

From Figure 4.4.3, the largest set of non-adjacent vertices are shaded in red, and therefore the co-clique number of $G_{K_5}{}^*$ is 6. The shaded vertices are $v_4^i;\ 1 \le i \le 5,$ as well as the central vertex. The co-clique number is therefore equal to $q+1$.

In the general case of $G_{K_n}{}^*$, where $n \ge 4,$ the largest set of non-adjacent vertices will be obtained as follows:

Shade the vertices $v_4^i;\ 1 \le i \le n,$ together with the central vertex. The shaded vertices are all non-adjacent and this is a set of non-adjacent vertices of $G_{K_n}{}^*$. The co-clique number is therefore equal to at least $(n+1)$.

We shall now prove that there is no set of $(q+2)$ vertices, which are a subset of $G_{K_q}{}^*$, and which are disconnected. You can only select one vertex from each of the cliques, as each clique is a complete sub-graph, and therefore each vertex is connected to every other vertex in the clique. Therefore, as there are q cliques in $G_{K_q}{}^*$, at most $(q+1)$ vertices can be selected i.e., one vertex from each clique together with the central vertex. Therefore, there is no set of $(q+2)$ vertices which is a subset of $G_{K_q}{}^*$ and which is disconnected.

We have therefore proved that the co-clique number of q-cliqued graphs $G_{K_q}{}^*$ is $q+1$, for $q \ge 3$. □

4.5 Radius, diameter and tightness of *q*-cliqued graphs

Theorem 4.5.1

Let G be a q-cliqued graph. Then:

1. Each q-cliqued design graph has a 5-cycles incident with its central vertex.
2. The largest eigenvalue of a q-cliqued graph is q.
3. The radius of q-cliqued graphs is 2.
4. The diameter of q-cliqued graphs is:
   - For $q=2$, diameter = 2;
   - For $q=3$, diameter = 3; and
   - For $q \ge 4$, diameter = 4.
5. Tightness of type 2 $=5q;$ for $q \ge 4$.

Proof

1. By construction, each q-cliqued design graph contains 5-cycles incident with its central vertex.

2. A k-regular graph has k as its largest eigenvalue with a corresponding eigenvector $J_{n,1}^T = (1,1,...,1)^T$. Since a q-cliqued graph is q-regular, the largest eigenvalue is q.

3. The central vertex v of the q-cliqued graph has eccentricity 2, as the distance between v and any other vertex in the q-cliqued graph is at most 2. This is the minimum eccentricity of all vertices in G, and therefore it is the radius of G.

4. For $q = 2$, the central vertex of a q-cliqued graph has eccentricity 2. There is no other vertex with an eccentricity greater than 2, and therefore the diameter is 2.

   For $q = 3$, the vertices in the sub-cliques which are not connected to the central vertex, have eccentricity 3. There is no other vertex with an eccentricity greater than 3, and therefore the diameter is 3.

   For $q \geq 4$, the vertices in the sub-cliques which are not connected to the central vertex, have eccentricity 4. There is no other vertex with an eccentricity greater than 4, and therefore the diameter is 4.

5. As per Section 1.5.6, the tightness of type 2 for $q \geq 4$ is

   $$(d+1)\lambda = (4+1)q = 5q. \qquad \square$$

4.6 Ratios of *q*-cliqued graphs

As defined in Section 1.2.6 and Section 1.2.7, the q-cliqued graphs are important in that they are associated with non-balanced designs. The fact that these are on (q^2+1) vertices lends itself to the idea of ratio invariance of their properties such as clique number, chromatic number and co-clique number.

As per Section 1.2.6, $\frac{n-1}{d} = \frac{q^2+1-1}{q} = \frac{q^2}{q} = q,$ where d is the clique number, so q-cliques are clique number invariant.

As per section 1.2.6, $\frac{n-1}{d} = \frac{q^2+1-1}{q} = \frac{q^2}{q} = q$ where d is the chromatic number, so q-cliques are chromatic invariant.

As per Section 1.2.7, $\frac{n+2q}{c}=\frac{q^2+1+2q}{q+1}=\frac{(q+1)^2}{q+1}=q+1,$ where c is the co-clique number, so q-cliques are co-clique invariant.

4.7 Eigen-co-cliqued ratio

The eigenvalue of -2 is associated with the q-cliqued graphs for $q=3,4,5$. As per Section 1.5.7, the eigen-co-cliqued ratio of the q-cliqued class of graphs for $(\lambda=-2)$ is

$$\frac{n+\lambda}{c}=\frac{(q^2+1)+(-2)}{(q+1)}=\frac{q^2-1}{q+1}=q-1,$$ where c is the co-clique number.

For $q=3,$ the eigen-co-cliqued ratio is equal to the multiplicity of the eigenvalue of -2, and hence the ratio is strict.

4.8 Laplace and signless Laplace matrix of *q*-cliqued graphs

As per Section 1.5.9, the largest eigenvalue of the Laplacian matrix associated with a graph, is used in the definition of the robustness to delay for reaching consensus in a network. Also, as per Section 3.5, the eigenvalues of the Laplacian matrix are used in determining the number of spanning trees of a connected graph. In this section, we determine the Laplace matrix associated with the q-cliqued graph, for $q=3$.

$$L\left(G_{K_3}{}^*\right)=\begin{bmatrix} 3 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 3 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 3 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 3 \end{bmatrix}$$

The eigenvalues of the Laplace matrix of the 3-cliqued graph are:
$(5)^2,(4.532)^2,(3.347)^2,(2),(1.121)^2,(0).$

See Theorem 5.2.1 (6) for the general form of the eigenvalues of the Laplace matrix of *q*-cliqued graphs.

The signless Laplace matrix is:

$$Q\left(G_{K_3}{}^{*}\right)=\begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

and has eigenvalues $(6),(4.879)^2,(4),(2.653)^2,(1.468)^2,(1)^2$.

See Theorem 5.2.1 (6) for the general form of the eigenvalues of the signless Laplace matrix of *q*-cliqued graphs.

4.9 The linear algebra of distance matrices of reduced *q*-cliqued design graphs

Definition 4.9.1: Distance matrix

We define the *distance matrix* of a graph *G* by first assigning the weight 1 to each edge of *G*, i.e., $w(e)=1$ for all edges *e* of *G*. The distance $d_{ij\,j}$ from v_i to v_j, is defined as:

$d_{ij}=\min\limits_{P(v_i v_j)} w(P(v_i v_j))$ where $P(v_i v_j)$ ranges over all paths from v_i to v_j. The *distance matrix* $D(G)$ of a graph *G* is the square matrix which has d_{ij} as its *(i,j)*th entry.

Theorem 4.9.1

If G is a connected graph with r strong cliques G_i, which have adjacency matrices $A(G_i)$, and $D(G)$ is the distance matrix of G and $D(G_i)$ is the distance matrix of the strong clique G_i, then, recalling definitions in Section 1.5.2 and Section 1.5.3, we get

$$cof\,(D(G)) = \prod_{i=1}^{r} cof\,(D(G_i))$$

and

$$\det(D(G)) = \sum_{i=1}^{r} \det(D(G_i)) \prod_{j=1, j\neq i}^{r} cof\,(D(G_j)).$$

See Graham [26].

For example, G_i are copies of K_2 so we can use the alternate form:

$$\frac{\det(D(G))}{cof\,(D(G))} = \frac{\sum_{i=1}^{r} \det(D(G_i)) \prod_{j=1, j\neq i}^{r} cof\,(D(G_j))}{\prod_{i=1}^{r} cof\,(D(G_i))} = \sum_{i=1}^{r} \frac{\det(D(G_i))}{cof\,(D(G_i))}.$$

Since the distance matrix for K_2 is

$D(K_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so that $\det(D(K_2)) = -1,$ and $cof\,(D(K_2)) = -2,$ if we consider the tree $G = T_n$ with n vertices, $n-1$ edges, and distance matrix $D(G)$, we have,

$$cof\,(D(T_n)) = cof\,(D(K_2))^{n-1} = (-2)^{n-1},$$

$$\det(D(T_n)) = cof\,(A(G)) \sum_{1}^{n-1} \frac{\det(A(G_i))}{cof\,(A(G_i))} = (-2)^{n-1}(n-1)\frac{1}{2} = (-1)^{n-1}(2)^{n-2}(n-1)$$

which is independent of the structure of the tree.

For our q-cliqued graphs $G^*_{K_q}$, they are not strong clique graphs, as they do not contain cut-vertices. The removal of the inserted edges E between the cliques in our construction results in the cliques of the new graph $G-E$ becoming strong cliques. This graph is denoted by $(G^*_{K_q})'$ and has adjacency matrix $A((G^*_{K_q})')$.

For example, for $q = 2$ remove the edge e joining vertices v_3 and v_5 from the 2-cliqued graph.

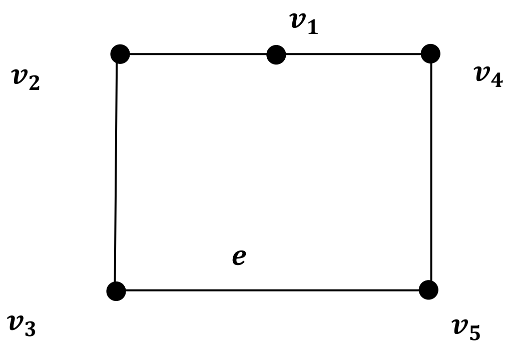

Figure 4.9.1: Diagram of ${G_{K_2}}^*$

This results in the following path on 5 vertices:

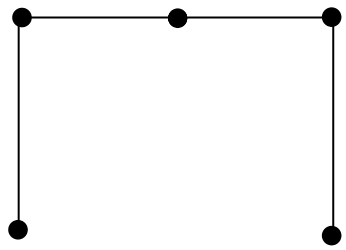

Figure 4.9.2: Diagram of $\left({G_{K_2}}^*\right)'$

Assigning weight 1 to each edge and applying Theorem 4.9.1 to the distance matrix $D(G)$ of $\left(G^*_{K_2}\right)'$ which is a tree.

$$D\left(\left(G^*_{K_2}\right)'\right) = \begin{bmatrix} 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 3 & 4 \\ 1 & 2 & 3 & 0 & 1 \\ 2 & 3 & 4 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} cof\left(D\left((G^*_{K_2})'\right)\right) &= \left(cof\,(A(K_2))\right)^{5-1} \\ &= (-2)^{5-1} \\ &= 16 \end{aligned}$$

$$\det\left(D\left((G_{K_2}^{*})'\right)\right) = cof\left(D\left((G_{K_2}^{*})'\right)\right)\sum_{i=1}^{5-1}\frac{\det(A(K_2))}{cof\,(A(K_2))}$$

$$=16(5-1)\left(\frac{-1}{-2}\right)$$

$$=32$$

For $q=3$, remove the edges between the sub-cliques.

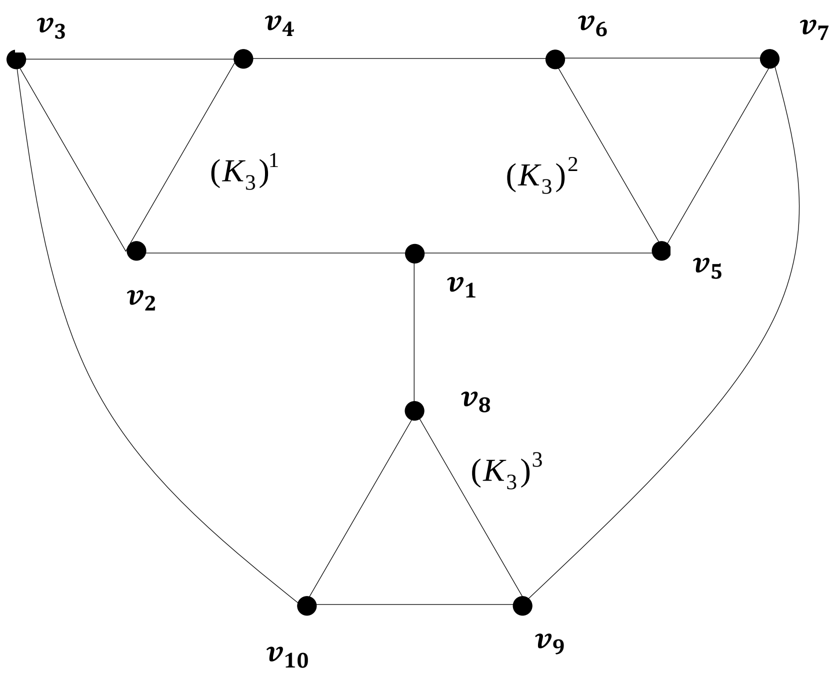

Figure 4.9.3: Diagram of $G_{K_3}{}^{*}$

Remove the edges between vertices 4,6 and 3,10 and 7,9 to get $\left(G_{K_3}{}^{*}\right)'$.

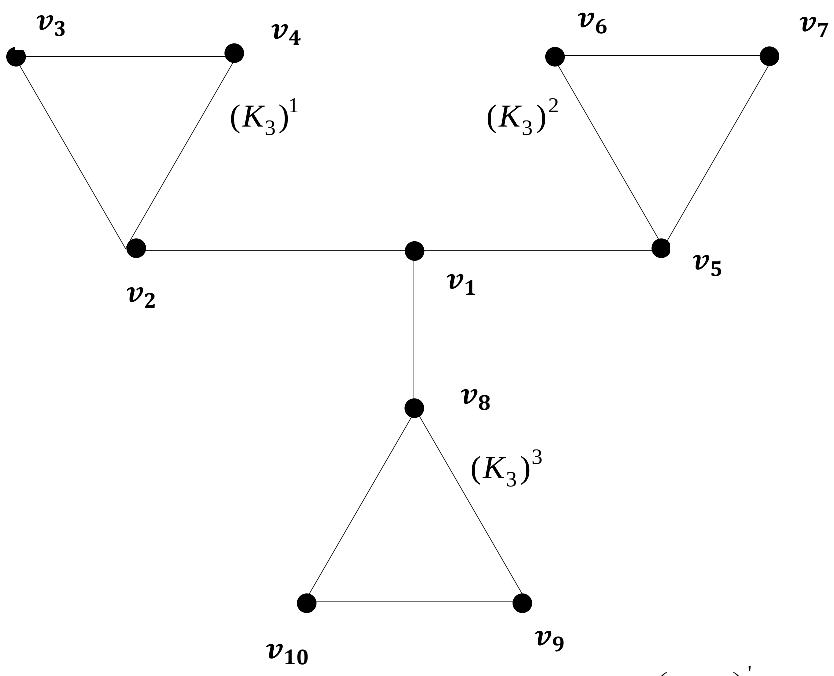

Figure 4.1.2.2: Diagram of $\left(G_{K3}{}^{*}\right)'$

Assign weight 1 to the remaining edges to get the distance matrix

$$A' = D\left(\left(G_{K_3}^{*}\right)'\right) = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 \\ 2 & 1 & 0 & 1 & 3 & 4 & 4 & 3 & 4 & 4 \\ 2 & 1 & 1 & 0 & 3 & 4 & 4 & 3 & 4 & 4 \\ 1 & 2 & 3 & 3 & 0 & 1 & 1 & 2 & 3 & 3 \\ 2 & 3 & 4 & 4 & 1 & 0 & 1 & 3 & 4 & 4 \\ 2 & 3 & 4 & 4 & 1 & 1 & 0 & 3 & 4 & 4 \\ 1 & 2 & 3 & 3 & 2 & 3 & 3 & 0 & 1 & 1 \\ 2 & 3 & 4 & 4 & 3 & 4 & 4 & 1 & 0 & 1 \\ 2 & 3 & 4 & 4 & 3 & 4 & 4 & 1 & 1 & 0 \end{bmatrix}$$

The reduced 3-cliqued graph $\left(G_{K_3}{}^{*}\right)'$, has 3 blocks consisting of 3 copies of K_3 and 3 blocks consisting of 3 copies of K_2.

The determinant of the distance matrix of K_3 is 2 and of K_2 is -1, and the $cof\left(K_3\right) = 3$ and $cof\left(K_2\right) = -2$ so that from Theorem 4.9.1:

$$\sum_{i=1}^{r} \frac{\det(D(G_i))}{cof\,(D(G_i))} = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{-1}{-2} + \frac{-1}{-2} + \frac{-1}{-2} = \frac{7}{2}$$

and $cof\left(D\left(\left(G_{K3}^{*} \right)' \right) \right) = \prod_{i=1}^{r} cof\,(A_i) = 3^3(-2)^3 = (27)(-8) = -216$

So that:

$$\det\left(D\left(\left(G_{K3}^{*} \right)' \right) \right) = cof\left(D\left(\left(G_{K3}^{*} \right)' \right) \right).\sum_{i=1}^{r} \frac{\det(D(G_i))}{cof\,(D(G_i))} = (-216)\left(\frac{7}{2} \right) = -756$$

Using the result of Theorem 4.9.1, we now determine the determinant and cofactor of the complete graph K_q on q vertices.

Theorem 4.9.2

For the complete graph on q vertices,

1. $\det(A(K_q)) = (q-1)(-1)^{q-1}$
2. $cof\,(A(K_q)) = q(-1)^{q-1}$

Proof

1. To prove: $\det(A(K_q)) = (q-1)(-1)^{q-1}$

For $q = 1$:

$$\begin{aligned} &\det(A(K_2)) \\ &= -1 \;\; by \;\; calculation \\ &= (2-1)(-1)^{2-1} \\ &= (q-1)(-1)^{q-1} \end{aligned}$$

Let us assume that

$$\det(A(K_k)) = (k-1)(-1)^{k-1} = \det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{(kxk)} \quad \text{for all } k \le q. \qquad (1)$$

Then, by expanding the determinant along the first row,

$$\text{RHS of (1)} \;= 0\times\det\begin{bmatrix}0&1&1&\cdots&1\\1&0&1&\cdots&1\\1&1&0&\cdots&1\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&1&1&\cdots&0\end{bmatrix}_{(k-1)x(k-1)}$$

$$-1\det\begin{bmatrix}1&1&1&\cdots&1\\1&0&1&\cdots&1\\1&1&0&\cdots&1\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&1&1&\cdots&0\end{bmatrix}_{(k-1)x(k-1)}$$

$$+1\det\begin{bmatrix}1&0&1&\cdots&1\\1&1&1&\cdots&1\\1&1&0&\cdots&1\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&1&1&\cdots&0\end{bmatrix}_{(k-1)x(k-1)}$$

$$-\ldots(-1)^{k-1}\det\begin{bmatrix}1&1&1&\cdots&1\\1&0&1&\cdots&1\\1&1&0&\cdots&1\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&1&1&\cdots&1\end{bmatrix}_{(k-1)x(k-1)}$$

$$=(-1)(k-1)\det\begin{bmatrix}1&1&1&\cdots&1\\1&0&1&\cdots&1\\1&1&0&\cdots&1\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&1&1&\cdots&0\end{bmatrix}_{(k-1)x(k-1)},$$

as all $k-1$ terms can be expressed in terms of $\det\begin{bmatrix}1&1&1&\cdots&1\\1&0&1&\cdots&1\\1&1&0&\cdots&1\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&1&1&\cdots&0\end{bmatrix}$.

So, comparing LHS and RHS of (1), we have

$$\det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{(k-1)x(k-1)} = (-1)^{k-2}.$$

So let $S_k = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{kxk}$. Then $\det(S_k) = (-1)^{k-1}$.

Now, by expanding the determinant of $\det(A(K_{k+1}))$ along the first row,

$$\det(A(K_{k+1})) = \det\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{(k+1)x(k+1)}$$

$$= 0\times\det\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{kxk} - 1\det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{kxk} + 1\det\begin{bmatrix} 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{kxk}$$

$$+ \ldots + (-1)^{k+2}\det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{kxk}$$

$$= (-1)(k)\det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{kxk}$$

… as this can be simplified to k identical terms

$$= (-1)(k)(-1)^{k-1}$$

… by the inductive hypothesis

$$= (k)(-1)^k.$$

So we have proved $\det(A(K_q)) = (q-1)(-1)^{q-1}$ by induction.

2. To prove: $cof\,(A(K_q)) = q(-1)^{q-1}$

For $q = 1$, we have

$$cof\,(A(K_2))$$
$$= -2$$
$$= 2(-1)^{2-1}$$
$$= q(-1)^{q-1}$$

Now for $k \geq 2$ we have, by definition of $cof\,(A(K_{k+1}))$ as per Section 1.5.2, and by definition of K_n,

$$cof\,(A(K_{k+1}))$$

$$= cof\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{(k+1)x(k+1)}$$

$$= (k+1)\{\det(A(K_k)) - \{(k+1)-1\}\det(S_k)\}$$ by definition of $cof(A)$

$$= (k+1)\{(k-1)(-1)^{k-1} - k(-1)^{k-1}\}$$ by Theorem 4.9.2 (1)

$$= (k+1)(-1)^{k-1}\{(k-1) - k\}$$

$$= (-1)^k (k+1)$$

So we have proved that $cof(K_q) = q(-1)^{q-1}$, for $q \geq 2$. □

We now apply the above result, together with Theorem 4.9.1, to determine the determinant of the distance matrix of $\left(G_{K_q}{}^*\right)'$, without knowing the actual format of this distance matrix!

Theorem 4.9.3

For our q-cliqued graphs on q^2+1 vertices, and for $\left(G_{K_q}{}^*\right)'$ as defined above, let $D\left(\left(G_{K_3}{}^*\right)'\right)$ be the distance matrix of $(G^*_{K_q})'$, and let $D(K_q)$ be the distance matrix of K_q.

Then

$$\det\left(D\left(\left(G_{K_q}{}^*\right)'\right)\right) = \left[q(-1)^{q-1}\right]^q \left[\sum_{i=1}^{q} \frac{-1}{-2} + \sum_{j=1}^{q} \frac{(q-1)(-1)^{q-1}}{q(-1)^{q-1}}\right]$$

$$= \left[-2q(-1)^{q-1}\right]^q \left[\frac{3q+2}{2}\right]$$

Proof

$\left(G_{K_q}{}^*\right)'$ is a connected graph with q copies of K_q and q copies of K_2. Then from Theorem 4.9.1,

$$\det\left(D\left(\left(G_{K_q}{}^*\right)'\right)\right) = cof\left(D\left(\left(G_{K_3}{}^*\right)'\right)\right)\left(\sum_{i=1}^{q} \frac{\det(D(K_q))}{cof(D(K_q))} + \sum_{i=1}^{q} \frac{\det(D(K_2))}{cof(D(K_q))}\right).$$

Also, from Theorem 4.9.2,

$$cof\left(D\left(\left(G_{Kq}{}^*\right)'\right)\right) = \prod_{i=1}^{q} cof(D(K_q)) \cdot \prod_{j=1}^{q} cof(D(K_2))$$

$$= \left[q(-1)^{q-1}\right]^q [-2]^q.$$

For the q blocks of K_q,

$$\sum_{i=1}^{q}\frac{\det\left(D\left(K_q\right)\right)}{cof\left(D\left(K_q\right)\right)}=\sum_{i=1}^{q}\frac{(q-1)(-1)^{q-1}}{q(-1)^{q}}.$$

For the q blocks of K_2,

$$\sum_{i=1}^{q}\frac{\det\left(D(K_2)\right)}{cof\left(D\left(K_2\right)\right)}=\sum_{i=1}^{q}\frac{-1}{-2}.$$

So,

$$\begin{aligned}\det\left(D\left(\left({G_{K_q}}^{*}\right)'\right)\right) &= \left[q(-1)^{q-1}\right]^{q}[-2]^{q}\left(\sum_{i=1}^{q}\frac{(q-1)(-1)^{q-1}}{q(-1)^{q}}+\sum_{i=1}^{q}\frac{-1}{-2}\right)\\ &=\left[-2q(-1)^{q-1}\right]^{q}\left[-(q-1)+\frac{q}{2}\right]\\ &=\left[-2q(-1)^{q-1}\right]^{q}\left[\frac{-q+2}{2}\right].\end{aligned}$$ □

4.10 Conclusion

We constructed the q-cliqued graphs specifically for q=2,3,4 and 5, and then defined the general construction for all q-cliqued graphs. We proved that q-cliqued graphs are design graphs, and then determined a number of properties of the q-cliqued graphs. In Chapter 6, we will apply the chromatic number of the 3-cliqued graph to solve a potential scheduling problem in a hypothetical entomological experiment.

We noticed that the q-cliqued graphs have a central vertex joined to q sub-cliques. The wheel and the star graphs have a central vertex, and the class of the wheel graphs and the class of the star graphs are eigen-bi-balanced. This suggests that the class of q-cliqued graphs could also be eigen-bi-balanced.

In the next chapter, we will determine three specific eigenvalues of the adjacency matrix of the q-cliqued graph, (including a conjugate eigen-pair), without determining ALL the eigenvalues – we will use the eigenvector method to do so. We will then determine the associated eigen-bi-balanced properties for the class of q-cliqued graphs. Finally, we will investigate the class of complements of the q-cliqued graphs and obtain an interesting result!

The construction and results in this chapter are entirely original.

CHAPTER 5

EIGENVALUES OF *Q*-CLIQUED GRAPHS AND EIGEN-BI-BALANCED PROPERTIES

In this chapter, we focus on the *q*-cliqued graphs as constructed in Chapter 4. We show that the *q*-cliqued graphs have eigenvalue *q* and conjugate eigen-pairs

$$\lambda = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}.$$

The determination of the conjugate eigen-pairs is equivalent to showing that the cubic

$$\lambda^3 - \lambda^2(q-1) - \lambda q - \lambda(q-1) + q(q-1) \quad = (\lambda - q)(\lambda^2 + \lambda - (q-1))$$

is a factor of the characteristic equation determined by $A\left(G_k{}^*\right)\underline{x} = \lambda \underline{x}$ where $A\left(G_k{}^*\right)$ is the adjacency matrix of the *q*-cliqued graph. The proof requires a number of specific definitions of vertices within the *q*-clique graph, and we use the connectivity between the first clique, the second to last clique, and the last clique in the proof of the conjugate eigen-pairs. The central vertex also plays a key role in this proof, as each sub-clique of K_q is connected to the central vertex. The proof of determining the conjugate eigen-pairs and the associated eigenvectors, is first determined explicitly for the cases *q*=3,4, and 5, and then generalized for the *q*-cliqued graph.

Once we have proved the conjugate eigen-pairs of the *q*-cliqued graph, we then determine the eigen-bi-balanced properties of the class of *q*-cliqued graphs associated with this eigen-pair. The values of all the newly defined eigen-bi-balanced properties, as defined in Chapter 3, are easily determined for this class of graphs. We finally investigate the eigen-bi-balanced properties of the class of the complement of the *q*-cliqued graphs, and show that the class of *q*-cliqued graphs and the class of the complement of the *q*-cliqued graphs have the same eigen-bi-balanced ratio of $\frac{1}{(q-1)}$!

This chapter consists entirely of original work, and contains some of the key results of the creative work done during this research thesis. The eigenvalues of the *q*-clique graphs for the cases *q*=3,4, and 5 were verified using Bluebit online matrix calculator and Mathematica, and then the general form of the conjugate eigen-pairs was hypothetised and proved.

The results in this chapter for q-cliqued graphs and for the eigen-bi-balanced properties for the class of q-cliqued graphs associated with the conjugate eigen-pair are indeed fascinating!

5.1 Conjugate eigen-pair of *q*-cliqued graphs

Theorem 5.1

The q-cliqued graphs, as constructed in Chapter 4, have eigenvalues $\lambda = q$ (and the q-cliqued graph is q-regular) and conjugate eigen-pairs

$$\lambda = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}.$$

The conjugate pairs arise out of the "tightness" of the connection between the central vertex and the cliques, and between two adjacent cliques – for convention we chose the second last and last clique.

Proof of Theorem 5.1

We will illustrate Theorem 5.1 for $q = 2,3,4,$ and $5,$ and then give the proof for all $q \geq 6.$ First, we need the following definitions.

5.1.1 Vertex notation convention

Several vertices will be important in the proof, and hence we will give them special labels as follows:

1. First vertex (central vertex), x_1;
2. Second vertex, x_2;
3. Third vertex, x_3;
4. Vertices in first clique = $\{x_{2,} x_3, ..., x_q, x_{q+1}\}$;
5. Anchor vertex of each clique = vertex in each clique which is joined to the first vertex x_1
6. Anchor vertex of the last clique, $x_a = x_{2+q(q-1)}$;
7. Switching pair of vertices, $x_{q^2-1} = x_{l-2}$ (third last vertex) and $x_{q^2} = x_{l-1}$ (second last vertex);
8. Last vertex, $x_l = x_{q^2+1}$.

The following definitions are also required for this proof:

Let $T = \{x_1, x_2\}$
and T' = {the set of vertices of the second last clique which are adjacent to vertices in the last clique}.

Then $T' = \{x_{k_1}, x_{k_2}, ..., x_{k_t}\}$, where $t = \dfrac{q-1}{2}$, q odd, or $t = \dfrac{q}{2}$, q even, and

S = the generating set of vertices

$= T \cup T'$

Also, if $S = \{x_1, x_2, ..., x_k\}$, then we define $\sum S = \sum_{i=1}^{k} x_i$.

5.1.2 The two main equations that generate the conjugate eigen-pairs

We will use the relationship $A\underline{x} = \lambda \underline{x}$ to determine the two main equations that generate the conjugate eigen-pairs:

$$\sum S = \lambda^2 x_l - q x_l \tag{1}$$

and

$$\lambda \sum S = (q-1)\sum S + (q-1)x_l$$

$$\Rightarrow \sum S = \frac{(q-1)x_l}{(\lambda - (q-1))} \tag{2}$$

Substituting (2), into (1) we get:

$$\frac{(q-1)\lambda x_l}{\lambda - (q-1)} = \lambda^2 x_l - q x_l; \quad \lambda \neq q-1$$

so that:

$$(q-1)\lambda = \lambda^2(\lambda-(q-1)) - q(\lambda-(q-1))$$
$$\Rightarrow \lambda^3 - \lambda^2(q-1) - q\lambda + q(q-1) - \lambda(q-1) = 0$$
$$\Rightarrow (\lambda - q)(\lambda^2 + \lambda - (q-1)) = 0$$

This gives us three eigenvalues:

- $\lambda = q;$ and
- the conjugate eigen-pair $\lambda = \dfrac{-1 \pm \sqrt{1+4(q-1)}}{2}$.

5.1.3 The case $q = 2$

We first take the case $q = 2$, and determine the eigenvalues of the *2*-cliqued graph, using the eigenvector method. So,

$$A\left(G_{K_2}{}^*\right)\underline{x} = \lambda \underline{x}$$
$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

This gives the following simultaneous equations:

$$x_2 + x_4 = \lambda x_1$$
$$x_1 + x_3 = \lambda x_2$$
$$x_2 + x_5 = \lambda x_3$$
$$x_1 + x_5 = \lambda x_4$$
$$x_3 + x_4 = \lambda x_5.$$

Taking the neighbours of x_3 and x_4 we get

$$(x_2 + x_5) + (x_1 + x_5) = \lambda(x_3 + x_4)$$
$$\Rightarrow (x_1 + x_2) + 2x_5 = \lambda(\lambda x_5)$$
$$\Rightarrow (x_1 + x_2) = \lambda^2 x_5 - 2x_5$$

Let $S = \{x_1, x_2\}$. Then we get

$$\sum S = (x_1 + x_2) = \lambda^2 x_5 - 2x_5 \qquad (1)$$

This verifies equation (1) of Section 5.1.2 for the case $q = 2$.

Taking the neighbours of the vertices in $S = \{x_1, x_2\}$ we get

$$
\begin{aligned}
& (x_2 + x_4) + (x_1 + x_3) = \lambda(x_1 + x_2) \\
\Rightarrow\ & (x_1 + x_2) + (x_3 + x_4) = \lambda(x_1 + x_2) \\
\Rightarrow\ & (x_1 + x_2) + \lambda x_5 = \lambda(x_1 + x_2) \\
\Rightarrow\ & (\lambda - 1)(x_1 + x_2) = \lambda x_5 \\
\Rightarrow\ & (x_1 + x_2) = \frac{\lambda}{\lambda - 1} x_5 \\
\Rightarrow\ & \sum S = \frac{\lambda}{\lambda - 1} x_5 \qquad (2)
\end{aligned}
$$

This verifies equation (2) of Section 5.1.2 for the case $q = 2$.

Substitute (2) into (1) to get

$$\frac{\lambda x_5}{\lambda - 1} = \lambda^2 x_5 - 2x_5;\ \ \lambda \neq 1$$

$$
\begin{aligned}
\Rightarrow\ & \lambda = \lambda^2(\lambda - 1) - 2(\lambda - 1) \\
\Rightarrow\ & \lambda^3 - \lambda^2 - 3\lambda + 2 = 0 \\
\Rightarrow\ & (\lambda - 2)(\lambda^2 + \lambda - 1) = 0
\end{aligned}
$$

So, solving this equation, we have eigenvalues $\lambda = 2$, (which is the same as the degree of the vertices in the *q*-cliqued graph), and the conjugate eigen-pairs $\lambda = \frac{-1 \pm \sqrt{1+4}}{2}$.

5.1.4 The case $q = 3$

We now take the case $q = 3$, and determine the eigenvalues of the *3*-cliqued graph, using the eigenvector method. So,

$$A\left(G_{K_3}{}^*\right)\underline{x} = \lambda \underline{x}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} x_2 + x_5 + x_8 \\ x_1 + x_3 + x_4 \\ x_2 + x_4 + x_{10} \\ x_2 + x_3 + x_6 \\ x_1 + x_6 + x_7 \\ x_4 + x_5 + x_7 \\ x_5 + x_6 + x_9 \\ x_1 + x_9 + x_{10} \\ x_7 + x_8 + x_{10} \\ x_3 + x_8 + x_9 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix}$$

This gives the following equations taking the neighbours of x_{10} :

$$x_3 + x_8 + x_9 = \lambda x_{10}$$

$$\Rightarrow \quad (x_2 + x_4 + x_{10}) + (x_1 + x_9 + x_{10}) + (x_7 + x_8 + x_{10}) = \lambda(\lambda(x_{10}))$$

$$\Rightarrow \quad x_1 + x_2 + x_4 + x_7 + (x_8 + x_9) = \lambda^2 x_{10} - 3x_{10}$$

Let $S = \{x_1, x_2, x_7\}$. Set $x_4 = -x_8$ and $x_9 = 0$, then we get

$$x_1 + x_2 - x_8 + x_7 + (x_8 + 0) = \lambda^2 x_{10} - 3x_{10}$$

$$\Rightarrow \quad x_1 + x_2 + x_7 = \lambda^2 x_{10} - 3x_{10}$$

$$\Rightarrow \quad \sum S = \lambda^2 x_{10} - 3x_{10} \tag{1}$$

This verifies equation (1) of Section 5.1.2 for the case $q = 3$.

Taking the neighbours of the vertices in $S = \{x_1, x_2, x_7\}$ we get

$$(x_2 + x_5 + x_8) + (x_1 + x_3 + x_4) + (x_5 + x_6 + x_9) = \lambda(x_1 + x_2 + x_7)$$

Set $x_4 = -x_8$ and $x_9 = 0$, from above, and set

$x_3 = x_1; x_2 = 0; x_6 = 2x_7; x_5 = \lambda x_{10}$.

Then $(2x_1 + 2x_2 + 2x_7 + 2x_5) = \lambda(x_1 + x_2 + x_7)$

$$\Rightarrow \quad 2(x_1 + x_2 + x_7) + 2\lambda x_{10} = \lambda(x_1 + x_2 + x_7)$$

$$\Rightarrow \quad x_1 + x_2 + x_7 = \frac{2\lambda x_{10}}{\lambda - 2}$$

$$\Rightarrow \quad \sum S = \frac{2\lambda x_{10}}{\lambda - 2} \tag{2}$$

This verifies equation (2) of Section 5.1.2 for the case $q = 3$.

Substitute (2) into (1) to get

$$\frac{2\lambda x_{10}}{\lambda - 2} = \lambda^2 x_{10} - 3x_{10}; \quad \lambda \neq 1$$

$$\Rightarrow \quad 2\lambda = \lambda^2(\lambda - 2) - 3(\lambda - 2)$$

$$\Rightarrow \quad \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow \quad (\lambda - 3)(\lambda^2 + \lambda - 2) = 0$$

$$\Rightarrow \quad \lambda = 3 \text{ or } \lambda = \frac{-1 \pm \sqrt{1-(4.-2)}}{2} = \frac{-1 \pm \sqrt{9}}{2}$$

So, solving this equation, we have eigenvalues $\lambda = 3$, (which is the same as the degree of the vertices in the *3*-cliqued graph), and the conjugate eigen-pairs $\lambda = \dfrac{-1 \pm \sqrt{9}}{2}$.

Let $\underline{x} = [x_1, x_2, ..., x_{10}]^T$. Then $A\left(G_{K_3}{}^*\right)\underline{x} = \lambda \underline{x}$ gives

$$\begin{bmatrix} x_2 + x_5 + x_8 \\ x_1 + x_3 + x_4 \\ x_2 + x_4 + x_{10} \\ x_2 + x_3 + x_6 \\ x_1 + x_6 + x_7 \\ x_4 + x_5 + x_7 \\ x_5 + x_6 + x_9 \\ x_1 + x_9 + x_{10} \\ x_7 + x_8 + x_{10} \\ x_3 + x_8 + x_9 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix}$$

With the values set as above, this becomes

$$\begin{bmatrix} \lambda x_{10} + x_8 \\ 2x_1 - x_8 \\ -x_8 + x_{10} \\ x_1 + 2x_7 \\ x_1 + 3x_7 \\ -x_8 + \lambda x_{10} + x_7 \\ 2x_7 + \lambda x_{10} \\ x_1 + x_{10} \\ x_7 + x_8 + x_{10} \\ x_1 + x_8 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ 0 \\ x_1 \\ -x_8 \\ \lambda x_{10} \\ 2x_7 \\ x_7 \\ x_8 \\ 0 \\ x_{10} \end{bmatrix} \qquad \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{pmatrix}$$

We will now verify equations (1) and (2) in this section, using the definition of the eigenvector above.

We use the generating set $S = \{x_1, x_2, x_7\}$ with its sum $\sum S = \sum x_1 + x_2 + x_7$. Now, using equations (3), (4) and (9) in the above, and noting that the variable x_2 is 0:

$$\begin{aligned} \lambda \sum S \quad &= \lambda(x_1 + x_2 + x_7) \\ &= (\lambda x_{10} + x_8) + (2x_1 - x_8) + (2x_7 + \lambda x_{10}) \\ &= 2x_1 + 2.0 + 2x_7 + 2\lambda x_{10} \\ &= 2\sum S + 2\lambda x_{10} \end{aligned}$$

$$\Rightarrow \sum S = \frac{2\lambda x_{10}}{\lambda - 2} \tag{13}$$

This is the same result as equation (2) above.

We now verify equation (1) above using the definition of the eigenvector.

Using equation (12), $\lambda\lambda x_{10} = \lambda x_1 + \lambda x_8$, and using equation (5) and (10) in RHS:

$$\begin{aligned} \lambda^2 x_{10} \quad &= \lambda x_1 + \lambda x_8 \\ &= (-x_8 + x_{10}) + (x_1 + x_{10}) \end{aligned} \tag{14}$$

We need an x_7 so we use equation (11) to substitute for x_8 which gives us

$$-x_8 = x_7 + x_{10} \tag{15}$$

Substituting (15) into (14) gives us:

$$\begin{aligned}
&\lambda^2 x_{10} \quad = (x_7 + x_{10} + x_{10}) + (x_1 + x_{10}) \\
&\Rightarrow \quad \lambda^2 x_{10} = x_1 + x_2 + x_7 + 3x_{10} \\
&\Rightarrow \quad \sum S = \lambda^2 x_{10} - 3x_{10} \qquad (16)
\end{aligned}$$

This is the same result as equation (1) above.

So we have verified both equations (1) and (2) by using the definition of the eigenvector.

5.1.5 The case $q = 4$

Step 1: Write down first equation using last vertex:
$x_3 + x_{14} + x_{15} + x_{16} = \lambda x_{17}$

Expand left hand side with their neighbors to get vertices belonging to set S:

$$\begin{aligned}
&(x_2 + x_4 + x_5 + x_{17}) + (x_1 + x_{15} + x_{16} + x_{17}) + \\
&(x_{13} + x_{14} + x_{16} + x_{17}) + (x_{12} + x_{14} + x_{15} + x_{17}) \\
&= \lambda(x_3 + x_{14} + x_{15} + x_{16})
\end{aligned}$$

$$x_1 + x_2 + x_4 + x_5 + 2(x_{14} + x_{15} + x_{16}) + x_{12} + x_{13} + 4x_{17} = \lambda(\lambda x_{17})$$

Step 2: Put $x_{16} = -x_{15}$ (second and third largest have opposite signs and are called the switching pair) – this guarantees $x_{15}, x_{16} \notin S$.

Set $T = \{x_1, x_2\}$ and $T' = \{$all vertices in S that belong to the second last clique, and which are neighbours of the last clique $\} = \{x_{12}, x_{13}\}$. Then the generating set $S = T \cup T' = \{x_1, x_2\} \cup \{x_{12}, x_{13}\} = \{x_1, x_2, x_{12}, x_{13}\}$.

Then we have

$$x_1 + x_2 + x_4 + x_5 + 2(x_{14}) + x_{12} + x_{13} + 4x_{17} = \lambda(\lambda x_{17})$$

Step 3: Set $x_4 = x_5 = x_{14} = 0$;

$$\begin{aligned}
&\quad x_1 + x_2 + x_{12} + x_{13} + 4x_{17} = \lambda^2 x_{17} \\
&\Rightarrow x_1 + x_2 + x_{12} + x_{13} = \lambda^2 x_{17} - 4x_{17}
\end{aligned}$$

$$\Rightarrow \sum S = \lambda^2 x_{17} - 4x_{17} \qquad (1)$$

This verifies equation (1) of Section 5.1.2 for the case $q = 4$.

Step 4: Taking the neighbours of the vertices in $S = \{x_1, x_2, x_{12}, x_{13}\}$ we get

$$(x_2 + x_6 + x_{10} + x_{14}) + (x_1 + x_3 + x_4 + x_5) +$$
$$(x_{10} + x_{11} + x_{13} + x_{16}) + (x_{10} + x_{11} + x_{12} + x_{15}) = \lambda(x_1 + x_2 + x_{12} + x_{13})$$

From above, $x_4 = x_5 = x_{14} = 0; x_{15} = -x_{16}$

$$x_2 + x_6 + x_{10} + x_1 + x_3 + x_{10} + x_{11} + x_{13} + x_{10} + x_{11} + x_{12} = 0$$
$$x_1 + x_2 + x_3 + x_6 + 3x_{10} + 2x_{11} + x_{12} + x_{13} = 0$$

$Set\ x_{10} = \lambda x_{17};$
$Set\ x_3 = 2x_1$
$Set\ x_{11} = x_2$
$x_{12} = 0$
$Set\ 2x_{13} = x_6$

$$x_1 + x_2 + 2x_1 + 3\lambda x_{17} + 2x_2 + 3x_{12} + 3x_{13} = \lambda(x_1 + x_2 + x_{12} + x_{13})$$
$$3(x_1 + x_2 + x_{12} + x_{13}) + 3\lambda x_{17} = \lambda(x_1 + x_2 + x_{12} + x_{13})$$
$$\Rightarrow \quad x_1 + x_2 + x_{12} + x_{13} = \frac{3\lambda x_{17}}{\lambda - 3}$$
$$\Rightarrow \quad \sum S = \frac{3\lambda x_{17}}{\lambda - 3} \qquad (2)$$

This verifies equation (2) of Section 5.1.2 for the case $q = 3$.

Step 5: Substitute (2) into (1) to we get

$$\frac{3\lambda x_{17}}{\lambda - 3} = \lambda^2 x_{17} - 4x_{17}$$
$$\Rightarrow \quad \lambda^2(\lambda - 3)x_{17} - 4(\lambda - 3)x_{17} = 3\lambda x_{17}$$
$$\Rightarrow \quad \lambda^3 x_{17} - 3\lambda^2 x_{17} - 4\lambda x_{26} + 12x_{17} - 3\lambda x_{17} = 0$$
$$\Rightarrow \quad \lambda^3 x_{17} - 3\lambda^2 x_{17} - 7\lambda x_{26} + 12x_{17} = 0$$
$$\Rightarrow \quad (\lambda - 4)(\lambda^2 + \lambda - 3)x_{17} = 0$$
$$\Rightarrow \quad \lambda = 4 \text{ or } \lambda = \frac{-1 \pm \sqrt{1 - (4)(-3)}}{2} = \frac{-1 \pm \sqrt{13}}{2}$$

So, solving this equation, we have eigenvalues $\lambda = 4$, (which is the same as the degree of the vertices in the *4*-cliqued graph), and the conjugate eigen-pairs $\lambda = \dfrac{-1 \pm \sqrt{13}}{2}$.

Let $\underline{x} = [x_1, x_2, ..., x_{17}]^T$. Then $A\left(G_{K_4}{}^*\right)\underline{x} = \lambda \underline{x}$ gives

$$\begin{bmatrix} x_2 + x_6 + x_{10} + x_{14} \\ x_3 + x_4 + x_5 + x_1 \\ x_2 + x_4 + x_5 + x_{17} \\ x_3 + x_5 + x_2 + x_8 \\ x_2 + x_3 + x_4 + x_7 \\ x_1 + x_7 + x_8 + x_9 \\ x_5 + x_6 + x_8 + x_9 \\ x_4 + x_6 + x_7 + x_9 \\ x_6 + x_7 + x_8 + x_{11} \\ x_1 + x_{11} + x_{12} + x_{13} \\ x_9 + x_{10} + x_{12} + x_{13} \\ x_{10} + x_{11} + x_{13} + x_{16} \\ x_{10} + x_{11} + x_{12} + x_{15} \\ x_1 + x_{15} + x_{16} + x_{17} \\ x_{13} + x_{14} + x_{16} + x_{17} \\ x_{12} + x_{14} + x_{15} + x_{17} \\ x_3 + x_{14} + x_{15} + x_{16} \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \end{bmatrix}$$

$$
= \begin{bmatrix} x_2 + 2x_{13} + \lambda x_{17} + 0 \\ 2x_1 + 0 + 0 + x_1 \\ x_2 + 0 + 0 + x_{17} \\ 2x_1 + 0 + x_2 + x_8 \\ x_2 + 2x_1 + 0 + x_7 \\ x_1 + x_7 + x_8 + x_9 \\ 0 + 2x_{13} + x_8 + x_9 \\ 0 + 2x_{13} + x_7 + x_9 \\ 2x_{13} + x_7 + x_8 + x_2 \\ x_1 + x_2 + x_{13} \\ x_9 + \lambda x_{17} + x_{13} \\ \lambda x_{17} + x_2 + x_{13} + x_{16} \\ \lambda x_{17} + x_2 + 0 - x_{16} \\ x_1 + 0 + x_{17} \\ x_{13} + 0 + x_{16} + x_{17} \\ -x_{16} + x_{17} \\ 2x_1 + 0 + 0 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ 2x_1 \\ 0 \\ 0 \\ 2x_{13} \\ x_7 \\ x_8 \\ x_9 \\ \lambda x_{17} \\ x_2 \\ 0 \\ x_{13} \\ 0 \\ -x_{16} \\ x_{16} \\ x_{17} \end{bmatrix} \qquad \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \end{pmatrix}
$$

Step 6: We will now verify equations (1) and (2) in this section, using the definition of the eigenvector above.

We use the generating set $S = \{x_1, x_2, x_{12}, x_{13}\}$ with its sum $\sum S = x_1 + x_2 + x_{12} + x_{13}$. Now, using equations (4),(5) and (15) and(16) in the above, and noting that the variable x_2 is 0:

$$
\begin{aligned}
\lambda \sum S \quad &= \lambda(x_1 + x_2 + x_{12} + x_{13}) \\
&= (x_2 + 2x_{13} + \lambda x_{17}) + (2x_1 + x_1) + (\lambda x_{17} + x_2 + x_{13} + x_{16}) \\
&\quad + (\lambda x_{17} + x_2 - x_{16}) \\
&= 3(x_1 + x_2 + x_{12} + x_{13}) + 3\lambda x_{17} \\
&= 3\sum S + 2\lambda x_{17}
\end{aligned}
$$

$$
\Rightarrow \quad \sum S = \frac{3\lambda x_{17}}{\lambda - 3} \tag{21}
$$

This is the same result as equation (2) above.

Step 7: We now verify equation (1) in this section, using the definition of the eigenvector above.

Next: equation 20 and 6 gives:

$$\lambda\lambda x_{17} = \lambda(2x_1) = x_2 + x_{17} + 0 + 0 + 0 + 0$$

Equation 17 gives $x_1 + x_{17} = 0$ and 18 +19 gives: $x_{13} + 2x_{17} = 0$ and $x_{12} = 0$, so that:

$$\lambda^2 x_{17} = x_1 + x_2 + x_{12} + x_{13} + 4x_{17}$$

$$\Rightarrow \sum S = \lambda^2 x_{17} - 4x_{17} \qquad (22)$$

This is the same result as equation (1) above.

So we have verified both equations (1) and (2) by using the definition of the eigenvector.

5.1.6 The case $q = 5$

Step 1: Write down first equation using last vertex:

$$x_3 + x_{22} + x_{23} + x_{24} + x_{25} = \lambda x_{26}$$

Expand left hand side with their neighbors to get vertices belonging to set S:

$$\begin{aligned}&(x_2 + x_4 + x_5 + x_6 + x_{26}) + (x_1 + x_{23} + x_{24} + x_{25} + x_{26})\\ &+ (x_{21} + x_{22} + x_{24} + x_{25} + x_{26}) + (x_{20} + x_{22} + x_{23} + x_{25} + x_{26})\\ &+ (x_4 + x_{22} + x_{23} + x_{24} + x_{26})\\ &= \lambda(x_3 + x_{22} + x_{23} + x_{24} + x_{25})\end{aligned}$$

$$\begin{aligned}&x_1 + x_2 + 2x_4 + x_5 + x_6 + x_{20} + x_{21} + 3x_{22} + 3x_{23} + 3x_{24} + 3x_{25} + 5x_{26}\\ &= \lambda(\lambda x_{26})\end{aligned}$$

Step 2: Put $x_{25} = -x_{24}$ (second and third largest have opposite signs and are called the switching pair) – this guarantees no $x_{24}, x_{25} \notin S$.

Set $T = \{x_1, x_2\}$ and $T' = \{$all vertices in S that belong to the second last clique, and which are neighbours of the last clique $\} = \{x_{20}, x_{21}\}$. Then the generating set $S = T \cup T' = \{x_1, x_2\} \cup \{x_{20}, x_{21}\} = \{x_1, x_2, x_{20}, x_{21}\}$

Then we have

$$x_1 + x_2 + 2x_4 + x_5 + x_6 + x_{20} + x_{21} + 3x_{22} + 3x_{23} + 5x_{26} = \lambda^2 x_{26}$$

Step 3: Put

$$x_4 = x_5 = 0; x_6 = -3x_{22}; x_{23} = 0$$
$$x_1 + x_2 + x_{20} + x_{21} + 5x_{26} = \lambda^2 x_{26}$$
$$\Rightarrow x_1 + x_2 + +x_{20} + x_{21} = \lambda^2 x_{26} - 5x_{26}$$
$$\Rightarrow \sum S = \lambda^2 x_{26} - 5x_{26} \qquad (1)$$

This verifies equation (1) of Section 5.1.2 for the case $q = 4$.

Step 4: Taking the neighbours of the vertices in $S = \{x_1, x_2, x_{20}, x_{21}\}$ we get

$$\left(x_2 + x_7 + x_{12} + x_{17} + x_{22}\right) + \left(x_1 + x_3 + x_4 + x_5 + x_6\right)$$
$$+ \left(x_{17} + x_{18} + x_{19} + x_{21} + x_{24}\right) + \left(x_{17} + x_{18} + x_{19} + x_{20} + x_{23}\right)$$
$$= \lambda(x_1 + x_2 + x_{20} + x_{21})$$

Switching pair: $x_{24} = -x_{25}$ and set $x_4 = x_5 = 0; x_6 = -3x_{22}; x_{23} = 0$

$$3x_1 = x_7; x_{12} = 4\lambda x_{26}$$
$$x_3 = 3x_2$$
$$x_{18} = \frac{3}{2}x_{20}; \quad x_{19} = \frac{3}{2}x_{21}$$

$$x_4 = x_5 = x_{17} = 0$$
$$x_{25} = -2x_{22} = -x_{24}$$

$$\left(x_2 + 3x_1 + 4x_{26} + 0 + x_{22}\right) + \left(x_1 + 3x_2 + 0 + 0 - 3x_{22}\right)$$
$$+ \left(0 + \frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{21} + x_{24}\right) + \left(0 + \frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{20} + 0\right)$$
$$= 4(x_1 + x_2 + x_{20} + x_{21}) + 4x_{26}$$
$$= \lambda(x_1 + x_2 + x_{20} + x_{21})$$

Therefore

$$\Rightarrow x_1 + x_2 + x_{20} + x_{21} = \frac{4x_{26}}{\lambda - 4}$$

$$\Rightarrow \sum S = \frac{4x_{26}}{\lambda - 4} \qquad (2)$$

This verifies equation (2) of Section 5.1.2 for the case $q = 4$.

Step 5: Substitute (2) into (1) to we get

$$\frac{4\lambda x_{26}}{\lambda - 4} = \lambda^2 x_{26} - 5x_{26}$$

$$\Rightarrow \quad \lambda^2(\lambda - 4)x_{26} - 5(\lambda - 4)x_{26} = 4\lambda x_{26}$$

$$\Rightarrow \quad \lambda^3 x_{26} - 4\lambda^2 x_{26} - 5\lambda x_{26} + 20x_{26} - 4\lambda x_{26} = 0$$

$$\Rightarrow \quad \lambda^3 x_{26} - 4\lambda^2 x_{26} - 9\lambda x_{26} + 20x_{26} = 0$$

$$\Rightarrow \quad (\lambda - 5)(\lambda^2 + \lambda - 4)x_{26} = 0$$

$$\Rightarrow \quad \lambda = 5 \text{ or } \lambda = \frac{-1 \pm \sqrt{1 - (4.-4)}}{2} = \frac{-1 \pm \sqrt{17}}{2}$$

So, solving this equation, we have eigenvalues $\lambda = 5,$ (which is the same as the degree of the vertices in the *5*-cliqued graph), and the conjugate eigen-pairs $\lambda = \frac{-1 \pm \sqrt{17}}{2}$.

Let $\underline{x} = [x_1, x_2, ..., x_{26}]^T$. Then $A\left(G_{K_5}{}^*\right)\underline{x} = \lambda \underline{x}$ gives

$$\begin{bmatrix} x_2 + x_7 + x_{12} + x_{17} + x_{22} \\ x_1 + x_3 + x_4 + x_5 + x_6 \\ x_2 + x_4 + x_5 + x_6 + x_{26} \\ x_2 + x_3 + x_5 + x_6 + x_{25} \\ x_2 + x_3 + x_4 + x_6 + x_9 \\ x_2 + x_3 + x_4 + x_5 + x_8 \\ x_1 + x_8 + x_9 + x_{10} + x_{11} \\ x_6 + x_7 + x_9 + x_{10} + x_{11} \\ x_5 + x_7 + x_8 + x_{10} + x_{11} \\ x_7 + x_8 + x_9 + x_{11} + x_{14} \\ x_7 + x_8 + x_9 + x_{10} + x_{13} \\ x_1 + x_{13} + x_{14} + x_{15} + x_{16} \\ x_{11} + x_{12} + x_{14} + x_{15} + x_{16} \\ x_{10} + x_{12} + x_{13} + x_{15} + x_{16} \\ x_{12} + x_{13} + x_{14} + x_{16} + x_{19} \\ x_{12} + x_{13} + x_{14} + x_{15} + x_{18} \\ x_1 + x_{18} + x_{19} + x_{20} + x_{21} \\ x_{16} + x_{17} + x_{19} + x_{20} + x_{21} \\ x_{15} + x_{17} + x_{18} + x_{20} + x_{21} \\ x_{17} + x_{18} + x_{19} + x_{21} + x_{24} \\ x_{17} + x_{18} + x_{19} + x_{20} + x_{23} \\ x_1 + x_{23} + x_{24} + x_{25} + x_{26} \\ x_{21} + x_{22} + x_{24} + x_{25} + x_{26} \\ x_{20} + x_{22} + x_{23} + x_{25} + x_{26} \\ x_4 + x_{22} + x_{23} + x_{24} + x_{26} \\ x_3 + x_{22} + x_{23} + x_{24} + x_{25} \end{bmatrix} = \begin{bmatrix} x_2 + 3x_1 + 4\lambda x_{26} + x_{22} \\ x_1 + 3x_2 - 3x_{22} \\ x_2 - 3x_{22} + x_{26} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ x_{15} + 0 + \frac{3}{2}x_{20} + x_{20} + x_{21} \\ \frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{21} + 2x_{22} \\ \frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{20} + 0 \\ x_1 + 0 + x_{26} \\ x_{21} + x_{22} + 0 + x_{26} \\ x_{20} + x_{22} + 2x_{22} + x_{26} \\ 0 + x_{22} - 2x_{22} + x_{26} \\ x_3 + x_{22} + 0 + 0 + 0 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \\ x_{18} \\ x_{19} \\ x_{20} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \\ x_{25} \\ x_{26} \end{bmatrix}$$

$$\begin{bmatrix} x_2 + 3x_1 + 4\lambda x_{26} + x_{22} \\ x_1 + 3x_2 - 3x_{22} \\ x_2 - 3x_{22} + x_{26} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ x_{15} + 0 + \frac{3}{2}x_{20} + x_{20} + x_{21} \\ \frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{21} + 2x_{22} \\ \frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{20} + 0 \\ x_1 + 0 + x_{26} \\ x_{21} + x_{22} + 0 + x_{26} \\ x_{20} + x_{22} + 2x_{22} + x_{26} \\ 0 + x_{22} - 2x_{22} + x_{26} \\ x_3 + x_{22} + 0 + 0 + 0 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \\ x_{18} \\ x_{19} \\ x_{20} \\ x_{21} \\ x_{22} \\ 0 \\ -x_{25} \\ x_{25} \\ x_{26} \end{bmatrix} \qquad \begin{pmatrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \\ 25 \\ 26 \\ 27 \\ 28 \\ 29 \end{pmatrix}$$

Step 6: We will now verify equation (2) in this section, using the definition of the eigenvector above.

We use the generating set $S = \{x_1, x_2, x_{21}, x_{22}\}$ with its sum $\sum S = x_1 + x_2 + x_{21} + x_{22}$. Now, using equations (4), (5), (23) and (24) in the above, and noting that the variable $x_2 = 0$:

$$\lambda \sum S = \lambda(x_1 + x_2 + x_{21} + x_{22})$$

$$= (x_2 + 3x_1 + 4\lambda x_{26} + x_{22}) + (x_1 + 3x_2 - 3x_{22}) + \left(\frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{21} + 2x_{22}\right)$$

$$+ \left(\frac{3}{2}x_{20} + \frac{3}{2}x_{21} + x_{20} + 0\right)$$

$$= 4(x_1 + x_2 + x_{20} + x_{21}) + 4\lambda x_{26}$$

$$= 4\sum S + 4\lambda x_{26}$$

$$\Rightarrow \quad \sum S = \frac{4\lambda x_{26}}{\lambda - 4} \tag{30}$$

This is the same result as equation (2) above.

Step 7: We now verify equation (1) in this section, using the definition of the eigenvector above.

(24) +(25) yield:

$$x_{20} + 2x_{22} + 2x_{26} = 0 \tag{31}$$

From (29) we get

$$x_3 + x_{22} = \lambda\ x_{26}$$

$$\Rightarrow \lambda^2 x_{26} = \lambda x_3 + \lambda x_{22}$$

Substituting (6) and (25), we get

$$\lambda^2 x_{26} \quad = (x_2 - 3x_{22} + x_{26}) + (x_1 + 0 + x_{26}) \tag{32}$$

Adding (31) and (26) to (32) we get

$$\lambda^2 x_{26} \quad = (x_2 - 3x_{22} + x_{26}) + (x_1 + 0 + x_{26}) + (x_{20} + 2x_{22} + 2x_{26}) + (x_{21} + x_{22} + 0 + x_{26})$$

$$= (x_1 + x_2 + x_{20} + x_{21}) + 5x_{26}$$

$$\Rightarrow \sum S = \lambda^2 x_{26} - 5x_{26} \tag{33}$$

This is the same result as equation (1) above.

So we have verified both equations (1) and (2) by using the definition of the eigenvector.

5.1.7 Eigenvalues of general case

Refer to Section 5.1.1 for the vertex notation and definitions. We require the following additional definitions to clarify the proof for the general case, where $q \geq 6$.

1. x_1 is the first vertex (central vertex);
2. x_2 is the second vertex;
3. x_3 is the third vertex;
4. Vertices in first clique = $\{x_{2,} x_3, ..., x_q, x_{q+1}\}$;
5. Vertices in last clique = $\{x_{a,} x_{a+1}, ..., x_{l-3}, x_{l-2}, x_{l-1}, x_l\}$;
6. Anchor vertex of clique is the vertex in each clique which is joined to the first vertex x_1;
7. Anchor vertex of the last clique, $x_a = x_{2+q(q-1)}$;
8. Switching pair of vertices are $x_{q^2-1} = x_{l-2}$ (third last vertex) and $x_{q^2} = x_{l-1}$ (second last vertex);
9. $x_l = x_{q^2+1}$ is the last vertex;
10. λx_l is the sum of the neighbours of x_l i.e. $\lambda x_l = x_3 + x_a + x_{a+1} + x_{a+2} + ...x_{l-3} + x_{l-2} + x_{l-2}$
11. Q is the set of vertices in the last clique which give ''0'' equations, i.e., $Q = \{x_{a+1}, x_{a+2}, ..., x_{l-3}\}$ and $\{x_{l-2}, x_{l-1}, x_l\} \notin Q$
12. Neighbours of $x_a = \{x_1, x_{l-2}, x_{l-1}, x_l\}$ and all other neighbours of x_a from from Q (which are 0)
13. Neighbours of $x_l = N(x_l)$
$$= \{x_l^1, x_l^2, ..., x_l^q\}$$
$$= \{x_3, x_a, x_{a+1}, x_{a+2}, ..., x_{l-3}, x_{l-2}, x_{l-1}\}$$
14. The sum of the neigbours of $x_l^i;\ 1 \leq i \leq q$ is $\lambda(\lambda x_l)$
15. The set T' consists of vertices from $\lambda(\lambda x_l)$ which belong to the second last ($(q-1)$th) clique, which are neighbours of the vertices from the last (q th) clique

i.e., $T' = \{x_{k_1}, x_{k_2}, ..., x_{k_t}\}$ where $t = \begin{cases} \dfrac{q-1}{2}; & q\ odd \\ \dfrac{q}{2}; & q\ even \end{cases}$.

16. $T = \{x_1, x_2\}$

17. Let $S =$ the generating set of vertices then $S = T \cup T'$.

18. P = the set of vertices in the second last clique, excluding the anchor vertex, which are not neighbours of the last clique, and are therefore not in T' as defined above

i.e., $P = \{x_{p1}, x_{p2}, ..., x_{pq-1-t}\}$ where $t = \begin{cases} \frac{q-1}{2}; & q \ odd \\ \frac{q}{2}; & q \ even \end{cases}$.

19. Q' is a subset of Q, whose vertices join backwards to vertices of T'. All vertices in Q' are in the last clique.

20. If $S = \{x_1, x_2, ..., x_k\}$, then we define $\sum S = \sum_{i=1}^{k} x_i$.

Step 1- write down the first equation using the last vertex:

$\lambda^2 x_l \ = \lambda(\lambda x_l)$

$= x_1$ central vertex

$+ x_2 + x_4 + x_5 + ... + x_q + x_{q+1}$ all vertices in first clique

$+ (q-2)x_a + (q-2)x_{a+1}, (q-2)x_{a+2} + ... + (q-2)x_{a+t} + ... + (q-2)x_{l-3}$

$+ (q-2)x_{l-2} + (q-2)x_{l-1} + qx_l + (x_{k_1} + x_{k_2} + ... + x_{k_t})$

$+ (x_{p1} + x_{p2} + ... + x_{pq-1-t})$

Step 2: Set $x_{l-1} = -x_{l-2}$ switching vertices

Step 3: Put $x_4 = x_5 = ... = x_q = 0; \quad Q = \{0,0,...,0\};$

$x_{q+1} = -(q-2)x_a;$

Then,

$\lambda^2 x_l = \ x_1 + x_2 + 0 + 0 + ... + 0$

$+ 0 + 0 + ... + 0 + 0$

$+ (q-2)x_{l-2} - (q-2)x_{l-2} + qx_l$

$+ (x_{k_1} + x_{k_2} + ... + x_{k_t})$

$+ (0 + 0 + ... + 0)$

$\Rightarrow \lambda^2 x_l = x_1 + x_2 + qx_l + (x_{k_1} + x_{k_2} + ... + x_{k_t})$

$$\Rightarrow \lambda^2 x_l - qx_l = x_1 + x_2 + \left(x_{k_1} + x_{k_2} + \ldots + x_{k_t}\right)$$
$$\Rightarrow \lambda^2 x_l - qx_l = \sum S$$
$$\Rightarrow \sum S = \lambda^2 x_l - qx_l \tag{1}$$

Step 4:

Now we look at the neighbors of the generating set *S*:

$S = T \cup T' = \{x_1, x_2\} \cup \{x_{k_1}, x_{k_2}, \ldots, x_{k_t}\}$

where $t = \dfrac{q-1}{2}$, q odd, and $t = \dfrac{q}{2}$, q even.

Neighbours of $x_1 : x_2, x_{2+q}, x_{2+2q}, \ldots., x_{2+q(q-1)} = x_a$

Neighbours of $x_2 : x_1, x_3, x_{4,}, \ldots, x_{q+1}$

Sum of neighbours of $T' = (t-1)\sum T' + t\sum P + \sum Q'$

Then the sum of the neighbors of the elements of *S*:

$$\begin{aligned}
\lambda \sum S &= (x_2 + x_{2+q} + x_{2+2q} + \ldots. + x_{2+q(q-1)}) \\
&\quad + (x_1 + x_3 + x_4 + \ldots + x_q + x_{q+1}) \\
&\quad + (t-1)\sum T' + t\sum P + \sum Q'
\end{aligned}$$

From before:

Put $x_4 = x_5 = \ldots = x_q = 0$; $Q = \{0,0,\ldots,0\}$; $x_{q+1} = -(q-2)x_a ; x_{l-1} = -x_{l-2}$,

$$\begin{aligned}
\lambda \sum S &= (x_2 + x_{2+q} + x_{2+2q} + \ldots. + x_{2+q(q-1)}) \\
&\quad + (x_1 + x_3 + 0 + \ldots + 0 - (q-2)x_a) \\
&\quad + (t-1)\sum T' + t\sum P + x_{l-2} \\
&= x_1 + x_2 + x_3 - (q-2)x_a + x_{2+q} + x_{2+2q} + \ldots. + x_{2+q(q-1)} \\
&\quad + (t-1)\sum T' + t\sum P + x_{l-2}
\end{aligned}$$

Set

$$\begin{aligned}
&x_3 = (q-2)x_2; \\
&x_{2+q} = (q-2)x_1 \\
&x_{2+2q} = (q-1)\lambda x_l \\
&x_{a-q} = x_{2+q(q-2)} = 0 \\
&x_{l-2} = (q-3)x_a = -x_{l-1}
\end{aligned}$$

$$\begin{aligned}\lambda\sum S &= x_1 + x_2 + (q-2)x_2 - (q-2)x_a + (q-2)x_1 + (q-1)\lambda x_l + \ldots. \\ &+ 0 + x_a + (t-1)\sum T' + t\sum P + (q-3)x_a \\ &= (q-1)x_1 + (q-1)x_2 + (q-1)\lambda x_l + (t-1)\sum T' + t\sum P\end{aligned}$$

Set

$x_{p_1} = \dfrac{q-t}{t} x_{k_1}$;

$x_{p_2} = \dfrac{q-t}{t} x_{k_2}$;

...

$x_{p_t} = \dfrac{q-t}{t} x_{k_t}$

and

$x_{p(t+1)} = 0$ if q is even, as P has one more vertex than T' when q is even.

Then,

$$\begin{aligned}\lambda\sum S &= (q-1)x_1 + (q-1)x_2 + (q-1)\lambda x_l \\ &+ (t-1)(x_{k_1} + x_{k_{2+}} + \ldots + x_{k_t}) + t\left[\frac{q-t}{t}(x_{k_1} + x_{k_2} + \ldots + x_{k_t})\right] \\ &= (q-1)x_1 + (q-1)x_2 + (q-1)\lambda x_l + (q-1)(x_{k_1} + x_{k_{2+}} + \ldots + x_{k_t}) \\ &= (q-1)(x_1 + x_2 + x_{k_1} + x_{k_2} + \ldots + x_{k_t}) + (q-1)\lambda x_l \\ &= (q-1)\sum S + (q-1)\lambda x_l\end{aligned}$$

$$\Rightarrow (\lambda - (q-1))\sum S = (q-1)\lambda x_l$$

$$\Rightarrow \sum S = \frac{(q-1)\lambda x_l}{\lambda - (q-1)} \tag{2}$$

Substituting (2) into (1), we get

$$\frac{(q-1)\lambda x_l}{\lambda - q} = \lambda^2 x_l - q x_l$$

$$\begin{aligned}&\Rightarrow \quad \lambda^2(\lambda - (q-1))x_l - q(\lambda - (q-1))x_l = (q-1)\lambda x_l \\ &\Rightarrow \quad \lambda^3 x_l - (q-1)\lambda^2 x_l - q\lambda x_l + q(q-1)x_l - (q-1)\lambda x_l = 0 \\ &\Rightarrow \quad \lambda^3 x_l - (q-1)\lambda^2 x_l - (2q-1)\lambda x_l + q(q-1)x_l = 0 \\ &\Rightarrow \quad (\lambda - q)(\lambda^2 + \lambda - (q-1))x_l = 0\end{aligned}$$

$$\Rightarrow \quad \lambda = q \text{ or } \lambda = \frac{-1 \pm \sqrt{1-(-4(q-1))}}{2} = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}$$

So, solving this equation, we have eigenvalues $\lambda = q$, (which is the same as the degree of the vertices in the q-cliqued graph), and the conjugate eigen-pairs $\lambda = \dfrac{-1 \pm \sqrt{1+4(q-1)}}{2}$.

5.1.8 General eigenvector

Let $\underline{x} = \left[x_1, x_2, \ldots x_{q^2+1} \right]^T$ be an eigenvector of $G_{K_q}{}^*$. Then, from applying the construction of the q-cliqued graphs and the anaylsis in the preceding sections, we have:

$$\begin{aligned}
x_1 &= x_2 + x_{2+q} + x_{2+2q} + \ldots. + x_{a-q} + x_a \\
x_2 &= x_1 + x_3 + x_4 + \ldots + x_{q+1} \\
x_3 &= x_2 + x_4 + x_5 + x_6 + \ldots + x_{q+1} + x_l \\
&= (q-1)x_2
\end{aligned}$$

$$x_4, x_5, \ldots, x_q = 0$$

$$x_{q+1} = -(q-2)x_a$$

$$x_{2+q} = (q-1)x_1$$

$$x_{2+2q} = \lambda(q-1)x_l$$

$$x_a = x_{l-(q-1)} = x_1 + x_{a+1} + x_{a+2} + \ldots. + x_{a+t} + \ldots + x_{l-2} + x_{l-1} + x_l$$

$$x_{p_1} = \frac{q-t}{t} x_{k_1};$$

$$x_{p_2} = \frac{q-t}{t} x_{k_2};$$

...

$$x_{p_t} = \frac{q-t}{t} x_{k_t}$$

and

$x_{p(t+1)} = 0$ if q is even, as P has one more vertex than T' when q is even.

$$\begin{aligned}
x_{q^2-1} &= x_{l-2} \\
&= x_a + (q-3)x_a + x_\alpha + x_l = \lambda x_{l-2},
\end{aligned}$$

where $x_\alpha \in T'$ and is connected to switching vertex x_{l-2}

$$
\begin{aligned}
x_{q^2} &= x_{l-1} \\
&= x_a - (q-3)x_a + x_l
\end{aligned}
$$

$$x_{l-2} = (q-3)x_a = -x_{l-1}$$

$$x_{l-1} = -x_{l-2}$$

$$
\begin{aligned}
x_{q^2+1} &= x_l \\
&= x_3 + x_a
\end{aligned}
$$

The general eigenvector will have $q-4-(t-1)$ entries which contain $x_a + x_l + x_{l-1} + x_{l-2}$.

Zero equations (obtained from all vertices in the last clique, which connect backwards to the $(q-1)$ clique, i.e., to the vertices of $T'\backslash\{x_\alpha\}$. $(t-1)$ of these such equations

$x_a + x_{\beta_1} + (x_{l-1} + x_{l-2}) + x_l = x_a + x_{k_\beta} + 0 + x_l$ $(t-1)$ of these such equations where $1 \le \beta \le t$, and $x_{k_\beta} \ne x_\alpha$.

Sum of generating set T' without x_α :

$(t-2)T'\backslash\{x_{k_i}\} + (t-1)P + (t-1)x_{a-q} = \lambda x_{k_i}$;

Equation for x_α in generating set: $T'\backslash\{x_\alpha\} + P + x_{a-q} + x_{l-2}$

5.1.9 The final general equations

As in the specific cases for $q = 4$ and 5, we need to verify the following two equations using the values of the entries in the eigenvector:

$$\sum S = \lambda^2 x_l - q x_l. \tag{1}$$

and

$$\lambda \sum S = (q-1)\sum S + (q-1)x_l \tag{2}$$

We shall now prove that equation (1) holds for values of the eigenvector:

The last equation in $A\left(G_{K_n}{}^*\right)\underline{x} = \lambda \underline{x}$ yields

$x_3 + x_a = \lambda x_l$

$\Rightarrow \lambda x_3 + \lambda x_a = \lambda^2 x_l$

Substituting ath and 3rd equations of $A\left(G_{K_n}{}^*\right)\underline{x} = \lambda \underline{x}$ we get

$$\begin{aligned}
\lambda^2 x_l &= \lambda x_3 + \lambda x_a \\
&= \left(x_2 + x_4 + x_5 + x_6 + \ldots + x_{q+1} + x_l\right) \\
&\qquad + \left(x_1 + x_{a+1} + x_{a+2} + \ldots. + x_{a+t} + \ldots + x_{l-2} + x_{l-1} + x_l\right) \\
&= x_1 + x_2 + \left(x_4 + x_5 + x_6 + \ldots + x_{q+1}\right) \\
&\qquad + \left(x_{a+1} + x_{a+2} + \ldots. + x_{a+t} + \ldots + x_{l-2} + x_{l-1} + 2x_l\right)
\end{aligned}$$

Setting $x_4 = x_5 = \ldots = x_q = 0,$ and $x_{q+1} = -(q-2)x_a,$ we get

$$\lambda^2 x_l = x_1 + x_2 - (q-2)x_a + \left(x_{a+1} + x_{a+2} + \ldots. + x_{a+t} + \ldots + x_{l-2} + x_{l-1} + 2x_l\right)$$

Now, adding the switching vertices, we get

$$x_{l-1} + x_{l-2} = \left(x_a - (q-3)x_a + x\right) + \left(x_a + (q-3)x_a + x_\alpha + x_l\right) = 0$$

$$\Rightarrow 2x_a + x_\alpha + 2x_l = 0$$

Adding the 0 equations yields: $(t-1)x_a + (t-1)x_l + x_{a-1} + x_{a-2} + .. + x_{a-(t-1)}$

Adding the other 0 equations yield: $q - 4 - (t-1)$ of $x_a + x_l$

This all yields:

$$\begin{aligned}
\lambda^2 x_l &= x_1 + x_2 - (q-2)x_a \\
&\quad + \left(x_{a+1} + x_{a+2} + \ldots. + x_{a+t} + \ldots + x_{l-2} + x_{l-1} + 2x_l\right) \\
&\quad + 2x_a + x_\alpha + 2x_l \\
&\quad + (t-1)x_a + (t-1)x_l + x_{a-1} + x_{a-2} + .. + x_{a-(t-1)} \\
&\quad + (q-4-(t-1))[x_a + x_l] \\
&= x_1 + x_2 - (q-2)x_a + 2x_l + 2x_a + x_\alpha + 2x_l \\
&\quad + (t-1)x_a + (t-1)x_l + x_{a-1} + x_{a-2} + .. + x_{a-(t-1)} \\
&\quad + (q-4-(t-1))[x_a + x_l]
\end{aligned}$$

= sum of elements from generating set $+ qx_l$

Therefore,

$\sum S = \lambda^2 x_l - qx_l$, which is equation (1) above.

Using the vector values as per 5.1.8, and referring to section 5.1.7, we have verified that

$$\Rightarrow \sum S = \frac{(q-1)\lambda x_l}{\lambda-(q-1)} \qquad (2)$$

So we have verified both equations (1) and (2) by using the general definition of the eigenvector. Substituting (2) into (1), we solve for the conjugate eigen-pair as per section 5.1.7. □

This concludes the proof of the conjugate eigen-pair of the adjacency matrix associated with the *q*-cliqued graphs, as constructed in section 4. It is interesting to note that the conjugate eigen-pairs are a function of the clique number of the graph.

In the next section, we determine the eigen-bi-balanced properties of *q*-cliqued graphs associated with the conjugate eigen-pair $\lambda = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}$.

5.2 Eigen-bi-balanced properties of *q*-cliqued graphs

Now that we have determined the conjugate eigen-pair for the class of *q*-cliqued graphs, we can determine the eigen-bi-blanced properties as defined in Chapter 3, for this newly defined class of graphs. We recall from Section 5.1 that the conjugate eigen-pair is $(a,b) = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}$ for all *q*-cliqued graphs as defined in Section 4.1. We will determine the eigen-bi-balanced properties of the class of *q*-cliqued graphs, associated with this conjugate eigen-pair. We note the importance of the central vertex, which is connected to the anchor vertex of each of the *q* sub-cliques in the *q*-cliqued graphs.

Theorem 5.2.1

For the class of *q*-cliqued graphs and the conjugate eigen-pair

$$(a,b) = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}$$

1. The class of *q*-cliqued graphs is sum*(-1)*eigen-pair balanced with respect to the conjugate eigen-pair $(a,b) = \frac{-1 \pm \sqrt{1+4(q-1)}}{2}$;

2. The class of q-cliqued graphs is product*$(1-q)$*eigen-pair balanced with respect to the conjugate eigen-pair $(a,b)=\dfrac{-1\pm\sqrt{1+4(q-1)}}{2}$;

3. The class of q-cliqued graphs has eigen-bi-balanced ratio

$$r\left(\frac{-1+\sqrt{1+4(q-1)}}{2}G_{K_q}{}^{*}\frac{-1-\sqrt{1+4(q-1)}}{2}\right)=\frac{1}{(q-1)}\ ,$$

with eigen-bi-balanced ratio asymptote

$$r\left(\frac{-1+\sqrt{1+4(q-1)}}{2}G_{K_q}{}^{*}\frac{-1-\sqrt{1+4(q-1)}}{2}\right)^{\infty}=0\text{, and}$$

density

$$\Omega_r(G_{K_q}{}^{*})=\left|asymp\left(r\left(\frac{-1+\sqrt{1+4(q-1)}}{2}G_{K_q}{}^{*}\frac{-1-\sqrt{1+4(q-1)}}{2}\right)\right)\right|=0;$$

4. The class of q-cliqued graphs has eigen-bi-balanced ratio area

$$Ar\left(G_{K_q}{}^{*}\right)^{\frac{-1+\sqrt{1+4(q-1)}}{2},\frac{-1-\sqrt{1+4(q-1)}}{2}}=\sqrt{n-1}(4\sqrt{n-1}+4\ln\left|\sqrt{n-1}-1\right|);\text{ and}$$

5. The class of q-cliqued graphs has $|a+b|+|ab|=q$ with respect to the conjugate eigen-pair $(a,b)=\dfrac{-1\pm\sqrt{1+4(q-1)}}{2}$.

6. The eigenvalues of the Laplace matrix of the q-cliqued graphs are

$$(c,d)=q-\left(\frac{-1\pm\sqrt{1+4(q-1)}}{2}\right)$$

and the eigenvalues of the signless Laplace matrix of the q-cliqued graphs are

$$(e,f)=q+\left(\frac{-1\pm\sqrt{1+4(q-1)}}{2}\right).$$

Proof

1. The sum of the conjugate eigen-pair (a,b) is

$$\text{sum}\left(\frac{-1+\sqrt{1+4(q-1)}}{2},\frac{-1-\sqrt{1+4(q-1)}}{2}\right)$$
$$=\frac{-1+\sqrt{1+4(q-1)}}{2}+\frac{-1-\sqrt{1+4(q-1)}}{2}$$
$$=-1$$

Therefore, the class of *q*-cliqued graphs is exact sum*(-1)*eigen-pair balanced. It is interesting that it is the conjugate pair of eigenvalues that satisfy the sum*(-1)*eigen-pair balanced criteria.

2. The product of the conjugate eigen-pair (a,b) is

$$\text{product}\left(\frac{-1+\sqrt{1+4(q-1)}}{2},\frac{-1-\sqrt{1+4(q-1)}}{2}\right)$$

$$=\frac{(-1)^2-(1+4(q-1))}{4}$$
$$=-(q-1)$$

We have shown that the product of the conjugate eigen-pair is an *integral function of q i.e.,* $f(q)=-(q-1)$ where $q-1$ is also the degree of the vertices in a complete graph of order *q*. These eigenvalues are therefore non-exact product*$(1-q)$*eigen-pair balanced.

3. The eigen-bi-balanced ratio is

$$r\left(\frac{-1+\sqrt{1+4(q-1)}}{2}G_{K_q}{}^{*}\frac{-1-\sqrt{1+4(q-1)}}{2}\right)$$
$$=\frac{-1}{-(q-1)}$$
$$=\frac{1}{(q-1)}$$

Note that the eigen-bi-balanced ratio is equal to the negative of the reciprocal of the product of the conjugate pairs. The asymptote of this ratio is 0, as the value of q increases. So

$$r\left(\frac{-1+\sqrt{1+4(q-1)}}{2}G_{K_q}{}^{*}\frac{-1-\sqrt{1+4(q-1)}}{2}\right)^{\infty}=0\text{, and}$$

$$\Omega_r(G_{K_q}{}^{*})=\left|asymp\left(r\left(\frac{-1+\sqrt{1+4(q-1)}}{2}G_{K_q}{}^{*}\frac{-1-\sqrt{1+4(q-1)}}{2}\right)\right)\right|=0;$$

4. The eigen-bi-balanced ratio area is

$$\begin{aligned}
Ar\left(G_{K_q}{}^{*}\right)^{\frac{-1+\sqrt{1+4(q-1)}}{2},\frac{-1-\sqrt{1+4(q-1)}}{2}} &= \frac{2m}{n}\left|\int\frac{a+b}{ab}dn\right| \\
&= \frac{q(q^2+1)}{q^2+1}\left|\int\frac{-1}{-(q-1)}dn\right| \\
&= 2q\left|\int\frac{1}{\sqrt{n-1}-1}dn\right| \\
&= 4q\left|\int\frac{udu}{u-1}\right| \\
&= 4q\left|\int\frac{u-1}{u-1}+\frac{1}{u-1}du\right| \\
&= \sqrt{n-1}\left(4\sqrt{n-1}+4\ln\left|\sqrt{n-1}-1\right|\right)+c
\end{aligned}$$

When $n=1$ we have $Ar=0$ so that $c=0$.

So

$$Ar\left(G_{K_q}{}^{*}\right)^{\frac{-1+\sqrt{1+4(q-1)}}{2},\frac{-1-\sqrt{1+4(q-1)}}{2}}=\sqrt{n-1}(4\sqrt{n-1}+4\ln\left|\sqrt{n-1}-1\right|)$$

5. $|a+b|+|ab|$

$$=\left|\frac{-1+\sqrt{1+4(q-1)}}{2}+\frac{-1-\sqrt{1+4(q-1)}}{2}\right|+\left|\frac{-1+\sqrt{1+4(q-1)}}{2}*\frac{-1-\sqrt{1+4(q-1)}}{2}\right|$$

$$=\left|\frac{-2}{2}\right|+\left|\frac{1-\left(1+4(q-1)\right)}{4}\right|$$

$$=1+(q-1)$$

$$=q$$

Then, as per Conjecture 3.8.2, $|a+b|+|a.b|=q<q^2=cd$ for $q\geq 2,$ where c is the maximum degree in $G_{K_q}{}^*$ and d is the minimum degree in $G_{K_q}{}^*$ i.e., $c=d=q.$

6. Recall the definition of the Laplace matrix and the signless Laplace matrix from Section 1.2.9, and note that q-cliqued graphs are q-regular. We also note from Brouwer and Haemers [12], that if G is k-regular and has eigenvalues $\lambda_1,\lambda_2,\lambda_3,...,\lambda_n,$ then the Laplace matrix has eigenvalues $k-\lambda_1,k-\lambda_2,k-\lambda_3,...,k-\lambda_n$ and the signless Laplace matrix has eigenvalues $k+\lambda_1,k+\lambda_2,k+\lambda_3,...,k+\lambda_n.$ So it follows that

$$(c,d)=q-\left(\frac{-1\pm\sqrt{1+4(q-1)}}{2}\right)$$

are eigenvalues of the Laplace matrix of the q-cliqued graphs and that

$$(e,f)=q+\left(\frac{-1\pm\sqrt{1+4(q-1)}}{2}\right)$$

are eigenvalues of the signless Laplace matrix of the q-cliqued graphs. □

Conjecture 5.2.1

The class of *q*-cliqued graphs is not critically eigen-bi-balanced with respect to the central vertex.

If we take $q = 2$, and remove the central vertex from the *q*-cliqued graph ${G_{K_2}}^*$, we obtain a connected graph G'_2 on 4 vertices.

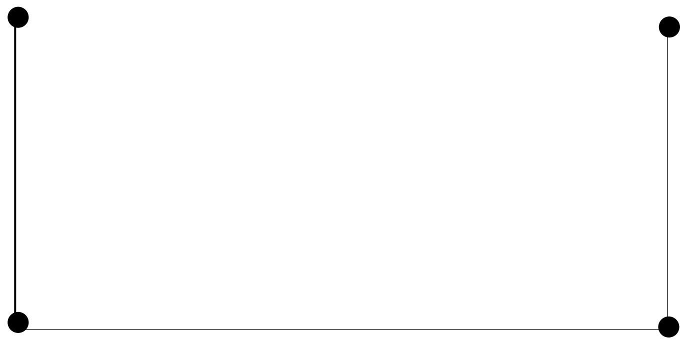

Figure 5.2.1: G_2' on 4 vertices

This is equivalent to the path on 4 vertices, i.e. P_4. As per section 2.4, the eigenvalues of the adjacency matrix of P_4 are 1.618034, 0.618034, -0.618034, and -1.618034. The pair of eigenvalues (1.618034, -0.618034) has sum of 1 and product of -1, therefore G' is sum*(1)*eigen-pair balanced and product *(-1)*eigen-pair balanced with respect to the eigen-pair (1.618034,-0.618034).

If we take $q = 3$, and remove the central vertex from the *q*-cliqued graph ${G_{K_3}}^*$, we obtain a connected graph G'_3 on 9 vertices.

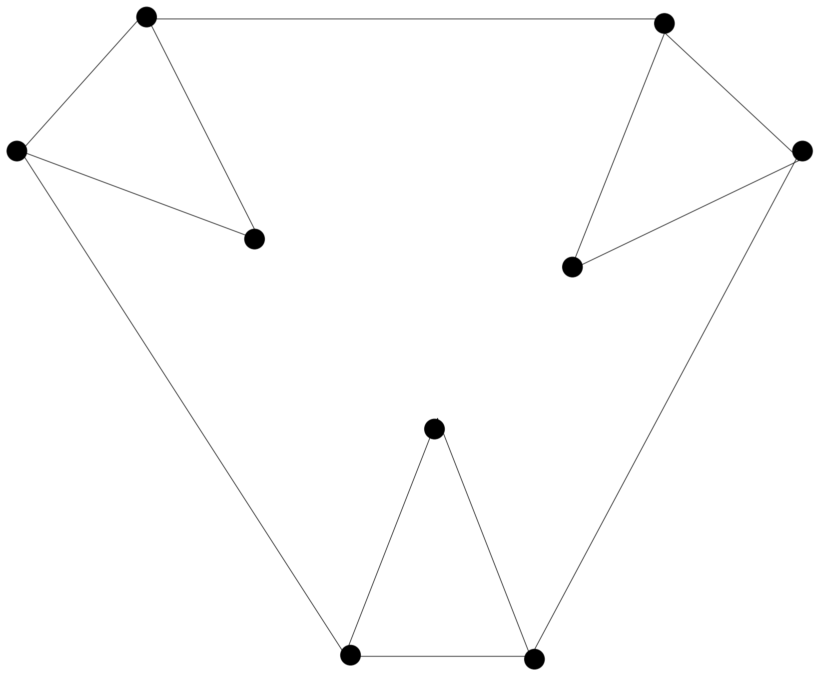

Figure 5.2.2: Diagram of G_3' on 9 vertices

G'_3 has adjacency matrix

$$A(G'_3)=\begin{bmatrix} 0 & 1 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & & 1 \\ 1 & 1 & 0 & & 1 & & & & \\ & & & 0 & 1 & 1 & & & \\ & & 1 & 1 & 0 & 1 & & & \\ & & & 1 & 1 & 0 & & 1 & \\ & & & & & & 0 & 1 & 1 \\ & & & & & 1 & 1 & 0 & 1 \\ & 1 & & & & & 1 & 1 & 0 \end{bmatrix}$$

where all blanks are zero in the above matrix. This matrix has eigenvalues (2.73205), $(1.87939)^2$, $(-0.34730)^2$, (-0.73205), $(-1.53209)^2$, (-2). The pair of eigenvalues (2.73205,-0.73205) has sum of 2 and product of -2, therefore G' is sum*(2)*eigen-pair balanced and product *(-2)*eigen-pair balanced with respect to the eigen-pair (2.73205, -0.73205).

If we take $q = 4$, and remove the central vertex from the q-cliqued graph $G_{K_4}{}^{*}$, we obtain a connected graph G'_4 on $q^2 = 16$ vertices.

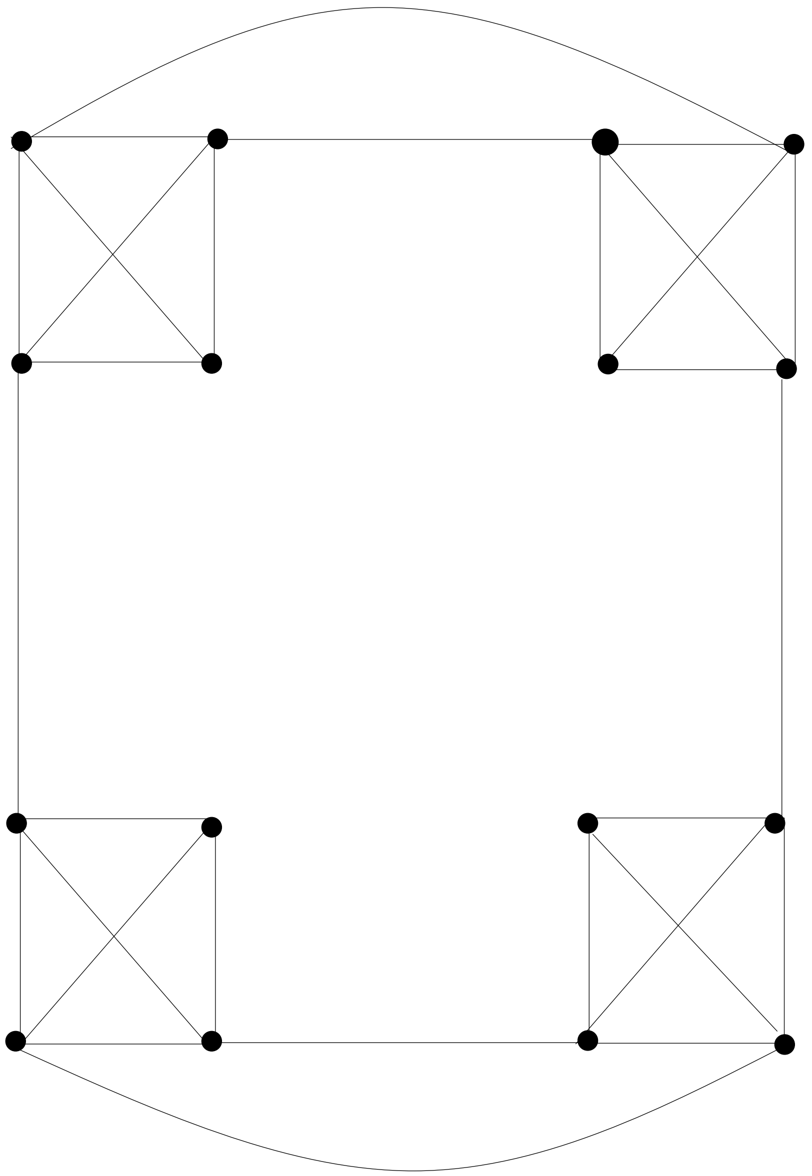

Figure: 5.2.3: Diagram of graph G_4' on 16 vertices

.

G'_4 has adjacency matrix:

$$A\left(G'_4\right)=\begin{bmatrix}
0 & 1 & 1 & 1 & & & & & & & & & & & & \\
1 & 0 & 1 & 1 & & & & & & & & & & & & 1 \\
1 & 1 & 0 & 1 & & & 1 & & & & & & & & & \\
1 & 1 & 1 & 0 & & 1 & & & & & & & & & & \\
 & & & & 0 & 1 & 1 & 1 & & & & & & & & \\
 & & & 1 & 1 & 0 & 1 & 1 & & & & & & & & \\
 & & 1 & & 1 & 1 & 0 & 1 & & & & & & & & \\
 & & & & 1 & 1 & 1 & 0 & & 1 & & & & & & \\
 & & & & & & & & 0 & 1 & 1 & 1 & & & & \\
 & & & & & & & 1 & 1 & 0 & 1 & 1 & & & & \\
 & & & & & & & & 1 & 1 & 0 & 1 & & & 1 & \\
 & & & & & & & & 1 & 1 & 1 & 0 & & 1 & & \\
 & & & & & & & & & & & & 0 & 1 & 1 & 1 \\
 & & & & & & & & & & & 1 & 1 & 0 & 1 & 1 \\
 & & & & & & & & & & 1 & & 1 & 1 & 0 & 1 \\
 & 1 & & & & & & & & & & & 1 & 1 & 1 & 0
\end{bmatrix}$$

where all blanks are zero in the above matrix. This matrix has eigenvalues (3.791288), (-0.791288) whose sum is 3 and whose product is -3. Therefore G' is sum*(3)*eigen-pair balanced and product *(-3)*eigen-pair balanced with respect to the eigen-pair (3.791288), (-0.791288).

We suspect that the removal of the central vertex from the *q*-cliqued class of graphs, results in a connected graph G', on q^2 vertices, which is sum*$(q-1)$*eigen-pair balanced and product*$(1-q)$*eigen-pair balanced, as the examples for *q*=2,3 and 4 above show.

Therefore the *q*-clique graphs may not be critically eigen-bi-balanced with respect to the central vertex.

5.3 Complement of *q*-cliqued graphs

The complement of a *q*-cliqued graph is easily defined as per the definition in section 1.2.2, and the class of the complements of *q*-cliqued graphs provides interesting analysis in terms of the eigen-bi-balanced characteristics. In the following theorem, we determine the conjugate eigen-pair of the complement of the *q*-cliqued graphs, and calculate the eigen-bi-balanced ratio of the class of the complements of *q*-cliqued graphs.

Theorem 5.3.1

Let $\overline{G}_{K_q}{}^*$, be the complement of the *q*-cliqued graph $G_{K_q}{}^*$, for $q \geq 2$. Then

1. $\overline{G}_{K_q}{}^*$ is connected;
2. The eigenvalues of $\overline{G}_{K_q}{}^*$ are $\dfrac{-1 \pm \sqrt{1+4(q-1)}}{2}$
3. The class of graphs $\overline{G}_{K_q}{}^*$ has eigen-bi-balanced ratio $\dfrac{1}{(q-1)}$, which is the same as that of the class of *q*-cliqued graphs $G_{K_q}{}^*$.

Proof

1. In $\overline{G}_{K_q}{}^*$, the central vertex v is adjacent to all vertices $v_i^j, 1 \leq i \leq q, i \neq 1, and\ 1 \leq j \leq q$, and has degree $q(q-1)$. Each vertex in a sub-clique, $v_1^j, 1 \leq j \leq q$, is connected to $v_i^k, 1 \leq i \leq q, j \neq k, 1 \leq k \leq q$. So v_1^j is connected to the central vertex via a path of length 2 via all vertices in the sub-cliques other than $v_1^j, 1 \leq i \leq q$. Hence we can deduce that this graph is connected i.e., all vertices are connected to the central vertex via a path of length 1 or 2. Also note that by definition $\overline{G}_{K_q}{}^*$, any two vertices are connected to each other via a path of maximum length 2.

2. Let $(a,b) = \dfrac{-1 \pm \sqrt{1+4(q-1)}}{2}$ be the conjugate eigen-pairs of $G_{K_q}{}^*$. Then, from Theorem 3.7.1, the eigenvalues of $\overline{G}_{K_q}{}^*$ are

$$c = -1 - a$$
$$= -1 - \frac{-1 + \sqrt{1+4(q-1)}}{2}$$
$$= \frac{-1 - \sqrt{1+4(q-1)}}{2}$$

and

$$\begin{aligned} d &= -1-b \\ &= -1-\frac{-1-\sqrt{1+4(q-1)}}{2} \\ &= \frac{-1+\sqrt{1+4(q-1)}}{2} \end{aligned}$$

i.e., we have been able to determine the eigenvalues of the adjacency matrix associated with the complement of the *q*-cliqued graph, without determining the adjacency matrix of the complement of the *q*-cliqued graph!

3. From Theorem 3.7.1, the eigen-bi-balanced ratio of $\overline{G}_{K_q}{}^*$ is

$$\begin{aligned} & r\left(\frac{-1-\sqrt{1+4(q-1)}}{2}\,\overline{G}_{K_q}{}^*\,\frac{-1+\sqrt{1+4(q-1)}}{2}\right) \\ &= \frac{-2-(a+b)}{1+(a+b)+ab} \\ &= \frac{-2-(-1)}{1+(-1)+(1-q)} \\ &= \frac{-1}{(1-q)} \\ &= \frac{1}{(q-1)} \end{aligned}$$

which is the same as the eigen-bi-balanced ratio of the class of graphs $G_{K_q}{}^*$ for the conjugate eigen-pair $\frac{-1\pm\sqrt{1+4(q-1)}}{2}$.

i.e.,
$$\begin{aligned} & r\left(\frac{-1+\sqrt{1+4(q-1)}}{2}\,\overline{G}_{K_q}{}^*\,\frac{-1-\sqrt{1+4(q-1)}}{2}\right) \\ &= \frac{1}{(q-1)} \\ &= r\left(\frac{-1-\sqrt{1+4(q-1)}}{2}\,G_{K_q}{}^*\,\frac{-1+\sqrt{1+4(q-1)}}{2}\right). \end{aligned}$$

Therefore we have proved that the eigen-bi-balanced ratio associated with a conjugate eigen-pair of eigenvalues of the class of graphs $G_{K_q}{}^*$, is the same as the eigen-bi-balanced ratio of the class of its complements, associated with the corresponding conjugate eigen-pair of $\overline{G}_{K_q}{}^*$. □

The above result is very interesting, given that we have not even constructed the adjacency matrix of the complement of the q-cliqued graph!

5.4 Conclusion

Having constructed the q-regular q-cliqued graphs, we noticed that they have a central vertex joined to cliques. Wheels and star graphs also have central vertices, and they are eigen-bi-balanced. This suggested that our q-cliqued graphs could also be eigen-bi-balanced.

In this chapter, we used the eigenvector method and the connectivity of the central vertex to form a cubic equation to find the eigenvalue of q, and a conjugate pair of eigenvalues. We analysed the specific cases for $q = 2,3,4,$ and 5, and then provided the generalized proof for finding these eigenvalues for all q-cliqued graphs. We also defined some of the values of the entries in the associated eigenvectors. The conjugate eigen-pair provides sum and product eigen-bi-balance of this class of graphs.

In this chapter, we calculated some of the eigen-bi-balanced properties of the class of q-cliqued graphs, and determined some properties of the class of complements of the q-cliqued graphs. The results of the eigen-bi-balanced properties of the class of complement of the q-cliqued graphs are very interesting!

The work and results in this chapter are entirely original.

CHAPTER 6

ENTOMOLOGICAL EXPERIMENT

The study of the interaction between insects and host-specific plants is important in bio-control situations and is well documented - see Jans and Nylin [33]. Many such experiments use block designs (see, for example, Coll [16]) and optimal scheduling would be advantageous when there is the occurrence of large number of treatments and blocks. Since we have a graph which is a block design graph, any application of graph theory to our graphs can be applied to its associated design, and in particular to experiments where block designs can be used to study the interaction of insects and plants. One of the important studies in graph theory is vertex colourings of graphs. It can be shown that a graph's chromatic number is greater or equal to the order of its largest clique, since a complete graph on *n* vertices requires *n* colours for a proper colouring.

Thus for our *q*-cliqued block graphs, their chromatic number is greater than or equal to *q*. We showed in Theorem 4.3.1 that $\chi({G_{K_q}}^*) = q$. We apply 3-colouring to the design associated with the 3-cliqued block graph relating to an entomological experiment as follows:

6.1 Experiment

We investigate the effect of 3 different species of insects on 10 different types of leaves (plants). We will have 10 cages containing the leaves and the insects, and they will be labelled as Cage 1, …, Cage 10.

We have 3 sets of leaves, each containing 10 different leaves. These leaves are to be divided (arbitrarily) into 10 cages, each cage labelled Cage 1, Cage 2, ..., Cage 10. Thus each type of leaf must appear 3 times in the experiment so that we need 3 sets of the 10 leaves.

The effect of the insects (using 10 insects per species) on the leaves in each cage will be studied. The application of the 3 different insects to the mini-groups (cages) must be done in the smallest number of time sessions, such that the following conditions hold:

A1. Each mini-group of triple leaves must be exposed to 3 different insects.

A2. An arbitrary mini-group of leaves will be called the *central-trial set* or *central cage,* and denoted by v_1.

A3. There must be 3 groups of 3-cliques P, Q and R of cages not containing the central trial set.

A4. Each cage in a clique cannot receive insects at the same time.

A5. Exactly one member from each different clique must receive a 3-set of insects at the same time, as well as not at the same time as the central cage receives its 3-set of insects.

A6. Exactly one member of each different clique, different from the cages in A5, must not receive a 3-set of insects at the same time.

A7. The three clique groups receiving the insects must be interchangeable (permutable) so that each clique can be exposed to all 3 insects other than the control.

These requirements can be depicted in a *3*-cliqued graph, where its central vertex is the central-trial set. The 10 vertices (labeled 1 to 10) represent the 10 cages each containing a set of 3 leaves, the 3 leaves in each cage (vertex) having their labels from the neighbour of the vertex (this is the block of the associated design).

The edges (adjacent cages) of the 3-cliqued graph represent tubes connected to the cages (vertices) with the condition that the tube cannot be open at both ends at the same time - forcing the insect into only one cage incident with the edge at a time.

The 3-cliqued graph has 15 edges, each vertex incident with 3 edges so that three different insect sets of 10 insects will be used. The proper colouring of the graph will refer to the time sessions when the insects can be released subject to conditions A1 – A7.

The chromatic number 3 refers to the condition where we require the smallest number of time sessions so that conditions A1 – A7 hold.

The 10 blocks containing 3 different leaves from the 10 different leaves will be:

1. {2,5,8};
2. {1,3,4};
3. {2,4,10};
4. {2,3,6};
5. {1,6,7};
6. {4,5,7};
7. {5,6,9};
8. {1,9,10};
9. {7,8,10};
10. {3,8,9}.

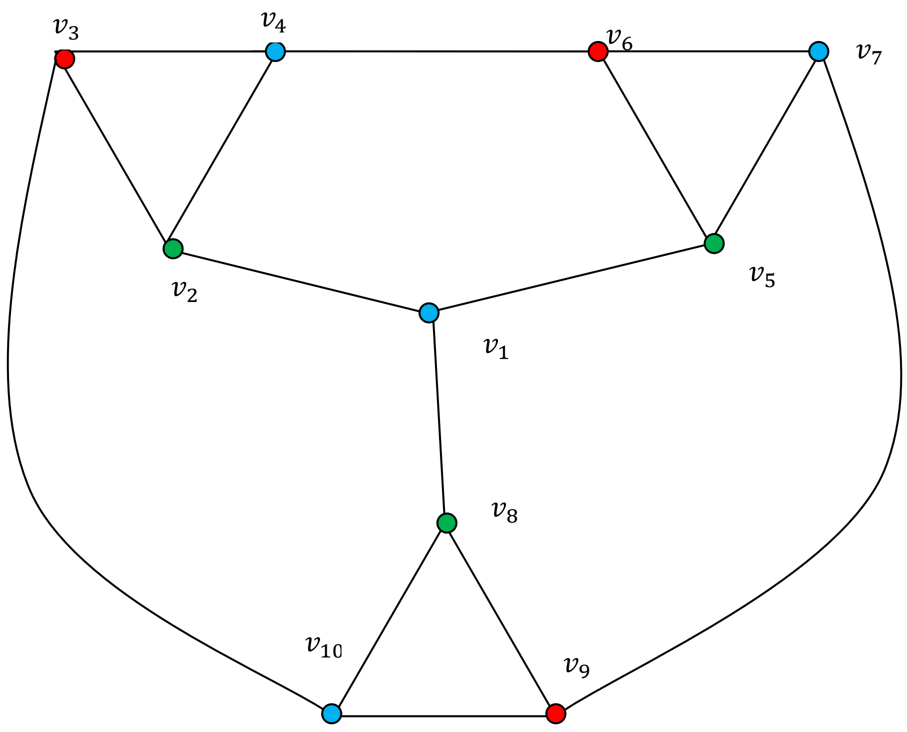

Figure 6.1: 3-cliqued graph with 3-colouring

Put 3 colours red, green and blue – vertex 1 coloured blue, vertices 2,5,8 coloured green, vertices 4,7,10 coloured blue, vertices 3,6,9 coloured red.

Label the insects *i*(1), *i*(2),…,*i*(30), and allocate them as follows:

1. The trial-set is the (arbitrary) block 1={2,5,8} – this block contains leaves 2,5 and 8 and is coloured blue . The other blocks which are coloured blue are: block 4={2,3,6}; block 7={5,6,9}; block 10={3,8,9}. We release insects *i*(1), *i*(2), i(3) into cage 1, *i*(4), *i*(5), *i*(6) into cage 4, *i*(7), *i*(8), *i*(9) into cage 7 and *i*(10), *i*(11), *i*(12) into cage 10 (we only open the side incident with these vertices).
2. For the vertices 2={1,4,3}; 5={1,6,7}; 8={1,9,10} coloured green we release the next 9 insects (3 per vertex): *i*(13) to *i*(21).
3. For the remaining 3 vertices 3={2,4,10}; 6={4,5,7}; 9={7,8,10} coloured red,we release the remaining 9 insects (3 for each vertex): *i*(22) to *i*(30).

With this assignment of colours in $G_{K_3}^{\ *}$, we will now show that the 7 conditions are satisfied.

We have now released all the insects in the least number of time sessions of 3, each cage being exposed to 3 different insects, satisfying A1.

The central cage receives insects at a different time from a block from each clique, and these respective blocks receive insects at the same time, satisfying A5.

The 3 cliques P, Q and R each do not have their 3 blocks receiving insects at the same time (all blocks are adjacent in each clique) and do not contain the central cage, satisfying conditions A3 and A4.

The edges between the cliques allow condition A7 to be satisfied.

Three 5-cycles through the central cage are each coloured with 3 colours representing the central cage not receiving insects at the same time as required in A5. 2 cages from 2 separate cliques do receive insects at the same time and 2 cages from the same separate cliques do not.

Once we have applied the insects with 3 different time sessions, we keep the vertices fixed and rotate the vertices (cages) of each clique once keeping the edges (tubes) fixed releasing 27 (fresh insects other than those released into vertex 1). For example, the block represented by vertex 2 with colour green, has edges (insects) $i(13)$, $i(14)$, and $i(15)$. These insects remain connected to the tubes when we rotate, but vertex 4 will replace vertex 2 or vertex 3 will replace vertex 2. This rotation allows each block of the clique to receive each of the 3 (edges of the triangle) of the clique. Keeping the edges fixed of each clique and rotating the vertices of each clique (not the colours of the vertices), and doing this for two sessions on 3-time intervals, each block of each clique will then have been exposed to the 9 insects connected to each clique.

After the first two time sessions, we fix the edges (tubes) and we move the whole cliques (as vertices) around without changing the vertex colouring, so that conditions A1, A2 still hold, and each block other than the trial block, is exposed to all 27 insects involved in the 3 cliques. Thus condition A5 holds without violating any other condition.

To determine the C matrix from $C = K - NK^{-1}N^{T} = K - N^{2}K^{-1}$, we first determine

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

So,

$$N^2 = \begin{bmatrix} 0&1&0&0&1&0&0&1&0&0\\ 1&0&1&1&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&1\\ 0&1&1&0&0&1&0&0&0&0\\ 1&0&0&0&0&1&1&0&0&0\\ 0&0&0&1&1&0&1&0&0&0\\ 0&0&0&0&1&1&0&0&1&0\\ 1&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1&0&1\\ 0&0&1&0&0&0&0&1&1&0 \end{bmatrix} \begin{bmatrix} 0&1&0&0&1&0&0&1&0&0\\ 1&0&1&1&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&1\\ 0&1&1&0&0&1&0&0&0&0\\ 1&0&0&0&0&1&1&0&0&0\\ 0&0&0&1&1&0&1&0&0&0\\ 0&0&0&0&1&1&0&0&1&0\\ 1&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&1&0&1\\ 0&0&1&0&0&0&0&1&1&0 \end{bmatrix}$$

$$= \begin{bmatrix} 3&0&1&1&0&1&1&0&1&1\\ 0&3&1&1&1&1&0&1&0&1\\ 1&1&3&1&0&1&0&1&1&0\\ 1&1&1&3&1&0&1&0&0&1\\ 0&1&0&1&3&1&1&1&1&0\\ 1&1&1&0&1&3&1&0&1&0\\ 1&0&0&1&1&1&3&1&0&1\\ 0&1&1&0&1&0&1&3&1&1\\ 1&0&1&0&1&1&0&1&3&1\\ 1&1&0&1&0&0&1&1&1&3 \end{bmatrix}$$

$$C = K - NK^{-1}N' = K - N^2K^{-1}$$

$$= 3.I_{10,10} - \begin{bmatrix} 3&0&1&1&0&1&1&0&1&1\\ 0&3&1&1&1&1&0&1&0&1\\ 1&1&3&1&0&1&0&1&1&0\\ 1&1&1&3&1&0&1&0&0&1\\ 0&1&0&1&3&1&1&1&1&0\\ 1&1&1&0&1&3&1&0&1&0\\ 1&0&0&1&1&1&3&1&0&1\\ 0&1&1&0&1&0&1&3&1&1\\ 1&0&1&0&1&1&0&1&3&1\\ 1&1&0&1&0&0&1&1&1&3 \end{bmatrix} \frac{1}{3} I_{10,10}$$

$$=\begin{bmatrix} 3 &&&&&&&&& \\ & 3 &&&&&&&& \\ && 3 &&&&&&& \\ &&& 3 &&&&&& \\ &&&& 3 &&&&& \\ &&&&& 3 &&&& \\ &&&&&& 3 &&& \\ &&&&&&& 3 && \\ &&&&&&&& 3 & \\ &&&&&&&&& 3 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 3 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 3 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 3 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 3 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 3 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 2 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 2 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 2 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 2 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 2 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 2 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 2 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 2 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 2 \end{bmatrix}$$

This C matrix is the information matrix of the design. Note that the row sums are all 0.

The eigenvalues of N are: 3, 1, -2, -2, 1.879, 1.879, -0.347, -0.347, -1.532, -1.532 , which includes the eigen-pair $\frac{-1 \pm \sqrt{9}}{2}$.

Recalling from Section 2.12,

$$
\begin{aligned}
Q^{-1}CQ &= Q^{-1}(K - NK^{-1}N)Q \\
&= Q^{-1}kI_{n,n}Q - Q^{-1}Nk^{-1}I_{n,n}NQ \\
&= kI_{n,n} - k^{-1}I_{n,n}Q^{-1}NQQ^{-1}NQ \\
&= kI_{n,n} - k^{-1}I_{n,n}diag(\lambda_1, \lambda_2, ..., \lambda_n).diag(\lambda_1, \lambda_2, ..., \lambda_n) \\
&= kI_{n,n} - k^{-1}I_{n,n}[diag(\lambda_1{}^2, \lambda_2{}^2, ..., \lambda_n{}^2)] \\
&= diag(k - \frac{\lambda_1{}^2}{k}, k - \frac{\lambda_2{}^2}{k}, ..., k - \frac{\lambda_n{}^2}{k})
\end{aligned}
$$

The eigenvalues of C are: 0, 2.667, 1.667 (twice), 2.959 (twice), 2.217 (twice) and 1.823 (twice). These indeed satisfy the formula above for the calculation of the eigenvalues of C, given the eigenvalues of N.

So, to determine eigenvalues of C, we determine $k - \frac{\lambda^2}{k}$, for all eigenvalues λ of N, namely, 3, 1, -2, -2, 1.879, 1.879, -0.347, -0.347, -1.532, -1.532.

$$3 - \frac{3^2}{3} = 0$$

$$3 - \frac{1^2}{3} = 2.667$$

$$3 - \frac{(-2)^2}{3} = 1.667$$

$$3 - \frac{1.879^2}{3} = 1.823$$

$$3 - \frac{(-0.347)^2}{3} = 2.959$$

$$3 - \frac{(-1.532)^2}{3} = 2.217$$

Note that the eigenvalues of C are either zero, or all positive, and non-integral.

Let the eigen-pair (a,b) be the conjugate eigen-pair $\left(\frac{-1+\sqrt{9}}{2}, \frac{-1-\sqrt{9}}{2}\right) = (1,-2)$

Then, from Theorem 5.2, the eigen-bi-balanced ratio area of the design above relative to the conjugate eigen-pair $(1,-2)$ is:

$$
\begin{aligned}
Ar(G_{K_3}{}^{*})^{1,-2} &= \sqrt{n-1}(4\sqrt{n-1}+4\ln\left|\sqrt{n-1}-1\right|) \qquad \text{where } n = 10\\
&= \sqrt{10-1}(4\sqrt{10-1}+4\ln\left|\sqrt{10-1}-1\right|)\\
&= 3(4.3+4\ln|3-1|)\\
&= 3(12+4\ln 2)\\
&= 44.317
\end{aligned}
$$

The sum of all the eigenvalues of $A(G_{K_3}{}^{*})$, energy, is

$$
\begin{aligned}
E^{A(G_{K_3}{}^{*})} &= 0+2.667+1.667+1.667+1.823+1.823+2.959+2.959\\
&\quad +2.217+2.217\\
&= 20
\end{aligned}
$$

We observe that $E^{A(G_{K_3}{}^{*})} < \dfrac{A(G_{K_3}{}^{*})^{1,-2}}{2}$ i.e., the energy is less than half the eigen-bi-balanced ratio area of the class of 3-cliqued graphs.

6.2 Conclusion

In this chapter, we used the case of q=3 for a hypothetical application of an entomological experiment, using the graph theoretical property of graph colouring to solve a possible scheduling problem in this experiment. We also investigated some other aspects of this 3-cliqued graph, such as the matrix C from design theory, and its energy.

CHAPTER 7

CONCLUSION

7.1 Summary

We began this thesis with the discussion of graphs and designs, linking these two structures together via matrices, so that we could apply graph-theoretical results to designs.

In Chapter 2, we presented different methods for finding eigenvalues of adjacency matrices associated with graphs, with the purpose of identifying some characteristic of the graph connected with the method and/or the resulting eigenvalues. The most significant technique is that of the eigenvector method. In this method, the choice of the form of the eigenvector is often determined by the edge connectivity of the graph involved, and results in determining a pair of conjugate eigenvalues.

We recall from Chapter 2, that:

- For the complete graph K_n, all vertices are of maximum degree of $(n-1)$. We can therefore regard each vertex as a central vertex, and it appears that this gives rise to the conjugate pair of eigenvalues;
- For the bipartite graph, the disjoint sets of vertices are 'strongly' connected to each other, so that each set can be regarded as a central aspect of the graph contributing to its conjugate eigen-pair;
- For the wheel graph, the connectivity of the central vertex to every other vertex led to the formation of a vector that resulted in a conjugate pair of eigenvalues;
- For the join of two graphs, these graphs, by definition of a join of two graphs, involve a 'strong' connection between the two graphs. This connection allowed for the generation of the conjugate eigen-pair; and
- For the star graph, the central vertex is at the end of each of the rays of the graph and there is a conjugate pair of eigenvalues of the adjacency matrix associated with the star graph.

Hence we noted that for the classes of the complete graphs, the complete bipartite graphs, the wheel graphs, the star graphs, and the join of two graphs, there is a form of a 'central vertex', which is well connected to other vertices, which gives rise to a conjugate pair of eigenvalues whose sum and product are integral. The sum and product being integral provided the motivation for the definitions of sum and product balance in Chapter 3.

We also noted that the cycle graphs and the path graphs are not well connected, and do not have a central vertex. They also do not have a conjugate pair of eigenvalues.

Classes of graphs, which are both eigen-sum-balanced and eigen-product-balanced with respect to eigen-pair (a,b) are significant, and were defined as eigen-bi-balanced. The non-zero property of the pair (a,b) of conjugate eigenvalues, belonging to a class of eigen-bi-balanced graphs, together with the possible association of robustness to the reciprocal of each of the members of the pair of eigenvalues of the adjacency matrix of graphs, allowed for the development of the eigen-bi-balanced ratio $\left(\frac{a+b}{ab}\right)$ associated with the class of eigen-bi-balanced graphs. This new idea led to the development of eigen-bi-balanced ratio asymptote of classes of graphs which may have relevance to networks on a large number of vertices. The eigen-bi-balanced ratio area of classes of graphs was introduced to possibly provide a further dimension to the robustness associated with eigen-bi-balanced classes of graphs.

These newly defined eigen-bi-balanced definitions were applied to known classes of graphs such as complete graphs, complete bipartite graphs, wheel graphs, star graphs, etc. We believed that it was necessary to have another class of graphs, different to the well-known classes of graphs discussed above, which is eigen-bi-balanced. We hence constructed a new class of graphs, called *q*-cliqued graphs, which have the desired property of a central vertex, which is well connected to an anchor vertex of each of the sub-cliques. We used the eigenvector method, focused on the connectivity of the central vertex, to determine a conjugate pair of eigenvalues which are both sum and product balanced. This class of graphs is significant in that each member of the pair of the conjugate eigenvalues is a function of the clique number of the graphs belonging to this class.

We found the eigen-bi-balanced ratio area, asymptotes, etc. and showed that a complement of a graph belonging to this class was connected and also eigen-bi-balanced.

We showed these graphs are also design graphs and used the case $q=3$ for a hypothetical application of an entomological experiment, using the graph theoretical property of colouring to solve a possible scheduling problem involved with this experiment. We also investigated some other aspects of this 3-cliqued graph, such as the C-matrix from design theory and its energy.

7.2 Future research

During this research thesis, the following questions have been posed:

1. The eigen-bi-balanced ratio area of complete graphs was found to be greater that the area of other well known classes of graphs. Is this the upper limit for all classes of eigen-bi-balanced graphs?
2. The eigen-bi-balanced ratio asymptote appears to lie on the interval [0,1]. This needs to be verified.
3. The asymptote -1 of complete graphs belongs to the eigen-pair associated with this class of graphs – is this the only class of graphs with this "closed" property?
4. Are cycles the only regular class of graphs which are neither sum or product eigen-balanced?
5. Does the eigen-bi-balanced ratio asymptote contribution to the energy of eigen-bi-balanced graphs result in compete graphs having maximum such energy compared to all other classes of eigen-bi-balanced graphs?
6. Which other classes of graphs exist, which have the property that the class of the complement of each graph in the class, is also eigen-bi-balanced? In this thesis, we have found this to be true for the class of *q*-cliqued graphs.
7. Is the class of *q*-cliqued graphs the only class whose eigen-bi-balanced ratio asymptote is the same as the eigen-bi-balanced ratio asymptote of the class comprising of the complement of each of the graphs in that class?
8. The energy of the complete graph K_n is $E^{K_n} = (2n-2)$ and the eigen-bi-balanced ratio area is $Ar(K_n)^{-1,n-1} = (n-1)(n-\ln(n-1))$ Thus, for large *n*, the energy behaves like $2n$ and the eigen-bi-balanced ratio area like n^2. Does this mean that the eigen-bi-balanced ratio area is always greater than the energy by a factor of $\frac{n}{2}$?
9. The complete split-bipartite graph has energy $\frac{n}{2}+\frac{n}{2}=n$ and eigen-bi-balanced ratio area $\frac{n^2}{2}$. Multiplying energy by $\frac{n}{2}$ gives us area, as in the case of the complete graph. Does this factor $\frac{n}{2}$ have any significance, when considering the energy and eigen-bi-balanced ratio area of other eigen-bi-balanced classes of graphs?

The research required to answer these questions could form the basis for additional research on topics covered in this thesis.

BIBLIOGRAPHY

[1] Adiga, C. Bayad, A. Gutman, I Srinivas S. The Minimum Covering Energy of a Graph. Kragujevac J. Sci. 34, 39-56 (2012).

[2] Aimei, Y. Mei, L. Feng, T. New Upper Bounds for the Energy of Graphs. Orsc.Edu.cn. (2005).

[3] Anton, H. Elementary Linear Algebra. John Wiley and Sons, Inc. (1977).

[4] Armstrong, J. Eigenpairs: The Unapologetic Mathematician. Unapologetic.worldpress.com. (2009).

[5] Averbouch, I., Charateristic Polynomial and Matching Polynomial Graph Polynomials, 238900-05/6 Lecture 3-4.

[6] Babat, R. B. Sivasubramanian, S. Inverse of the Distance Matrix of a Block Graph. Linear and Multilinear Algebra. Vol 59, No. 12, 1393-1373. (2011).

[7] Bailey, R.A. Cameron, P. J. Combinatorics of Optimal Designs. Surveys in Combinatorics. (2009).

[8] Bian, H.-Q. Isomorphic Lollipop Graphs are Cospectral with Respect to the Adjacency. Int. J. Contemp. Math. Sciences, Vol. 3, no. 36, 1783 – 1788. (2008).

[9] Bose, R. C. Strongly Regular Graphs, Partial Geometries and Partially Balanced Designs. Pacific Journal of Mathematics. Vol. 13, Number 2, 389-419. (1963).

[10] Bose, R. C. Shrikhande, S. S. Graphs in which Each Pair of Vertices is Adjacent to the Same Number *d* of Other Vertices. Institute of Statistics Mimeo Series No. 600.6. (1969).

[11] Brouwer, A. E. Distance Regular Graphs. Linear and Multilinear Algebra. Vol. 59, Issue 12, 1393-1397. (2011).

[12] Brouwer, A. E. Haemers, W. H. Spectra of Graphs. Springer, New York. (2011).

[13] Bruck, R. H. Ryser, H. J. The Nonexistence of Certain Finite Projective Planes. Canadian Journal of Mathematics, 83-92. (1940).

[14] Chowla, S. Ryser, H. J. Combinatorial Problems. Canadian Journal of Mathematics. 98-99. (1950).

[15] Cohen, A. M. Brouwer, A. E. Neumaier, A. Distance Regular Graphs. Springer-Verlag, New York. (1989).

[16] Coll, M. Feeding and Ovipositing on Plants by an Omnivorous Insect Predator. Oecologia. Volume 105, Number 2, 214-220. (1996).

[17] Cvetkovic, D. M. Doob, M. Sachs, H. Spectra of Graphs: Theory and Application, Academic press, New York, San Francisco, London, Mathematics (1980).

[18] Cvetkovi, D. Davidsovi, T. Well Suited Multiprocesssor Topologies with Small Number of Processors. Novi Sad J. Math, Vol. 38, No 3, 209-217. (2008).

[19] Dam, E.R. van**.** Regular graphs with four eigenvalues. Open Access publications from Tilburg University urn:nbn:nl:ui:12-80122, Tilburg University. (1995).

[20] Dam, E. R. van. Haemers, W. H. Which graphs are determined by their spectrum? Linear Algebra and its Applications, 373, 241-272. (2003).

[21] Dey, A. Incomplete Block Designs. Hindustan Book Agency, London. (2010).

[22] Dettmann, C. P. Morris, G. P. Proof of Lyapunov exponent pairing for system at constant kinetic energy. Phys. Rev E 53, 5545-5548. (1996).

[23] Dias, J. R. Constant-Isomer Benzenoid Series and Their Topological Characteristics. Theor. Chim. Acta, 143-162. (1990).

[24] Fox, J. Lecture 18: Spectral graph theory 1. Eigenvalues of graphs math.mit.edu/~fox/MAT307-lecture18.pdf.

[25] Godsil, C. Royle, G. Algebraic Graph Theory. New York: Springer-Verlag. (2004).

[26] Graham, R. L. Hoffman, A. J. On the Distance Matrix of a Directed Graph. Journal of Graph Theory, Vol. 1, 85-88. (1977).

[27] Gray, R. Toeplitz and Circulant Matrices: A Review. Now Publishers Inc, Boston-Delft, (2005).

[28] Haemers, W. H. Matrices for graphs, designs and codes. Department of Econometrics and OR, Tilburg University, The Netherlands.

[29] Hamada, M. Conjugate Codes and Applications to Cryptology. arXiv:quant-ph/0610193. (2006).

[30] Harris, J. M. Hirst, J. L. Mossinghoff, M. Combinatorics and Graph theory. Springer, New York. (2008).

[31] Hinch, E. J. Leal, L. G. Note on the Rheology of a Dilute Suspension of Dipolar Spheres with Weak Brownian Couple. J. of Fluid Mech. 56, 803. (1972).

[32] Hubaut, X. L. Strongly Regular Graphs. Discrete Mathematics, Vol. 13, Issue 4, 367-384. (1975).

[33] Jans, N. Nylin, S. The Role of Female Search Behaviour in Determining Host Plant Range in Plant Feeding Insects: a Test of the Information Processing Hypothesis. Proc. R. Soc. Lond. 701-707. (1997).

[34] Kadin, A. M. Spatial Structure of the Cooper Pair. Princeton Junction, New Jersey 08550, USA. (2005).

[35] Koolen, J. H. Moultron, V. Maximal Energy Bipartite Graphs. Graphs and Combinatorics. Vol. 19, 131-135. (2003).

[36] Kotlov, A. Lov'asz, L. The Rank and Size of Graphs. Department of Mathematics, Yale University, New Haven, CT 06520. Journal of Graph Theory. Vol 26 Issue 1, 185-189 (1996).

[37] Lee, S. L. Yeh, Y. Eigenvalues and Eigenvectors of graphs. Journal of Mathematical Chemistry. Vol 12, 121-135. (1993).

[38] Liazi, M. Milis, I. Pascual, F. Zissimopoulos ,V. The densest *k*-subgraph problem on clique graphs, Journal of Combinatorial Optimization, Vol. 14, No 4, 465-474. (2007).

[39] Lubotzky, A. Phillips, R. Sarnak, P. Ramanujan graphs. Combinatorica. Springer. (1988).

[40] Nikiforov, V. Some Inequalities for the Largest Eigenvalue of a Graph. J. Combinatorics, Probability and Computing, Vol. 11 Issue 2, 179 – 189. (2002).

[41] Olfati-Saber, R. Ultrafast Consensus in Small-World Networks. Control and Dynamical Systems, California Institute of Technology, olfati@cds.caltech.edu (2005).

[42] Rudvalis, A. (v,k,λ)-Graphs and Polarities of (v,k,λ)-Designs. Mathematische Zeitschrift. Vol. 120, Number 3, 224-230. (1971).

[43] Sarkar, J. Mukherjee, A. K. Graphs with reciprocal pairs of eigenvalues. Molecular Physics: an International Journal at the interface Between Chemistry and Physics; Vol. 90, Issue 6, 903-908. (1997).

[44] Spielman, D. A. Spectral Graph Theory Lecture 23, Strongly Regular Graphs, part 1, (2009).

[45] Stevanoviç, D. Energy of Graphs. A few open problems and some suggestions. dragance106@yahoo.com. University of Niˇs, Serbia (2006).