# On the geometry of locally conformal almost Kähler Manifolds

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By

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# Dedication

This work is dedicated to my son Ntokozo Junior Amukelani Khuzwayo and my Fiancé Phendukile Sanele Mhlongo, the year 2019 was very difficult for you but you showed me the qualities of a strong women, the love and strength you shared with me inspired me to strive for excellence.

# Preface

The work described in this dissertation was carried out in the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Pietermaritzburg, from February 2019 to May 2020 and remotely from March 2020 to September 2020, under the supervision of Prof. Fortuné Massamba. This study represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

Mr Ntokozo Sibonelo Khuzwayo.

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I am grateful to all the people whose support was a breakthrough in the completion of this project. Lastly, I thank the Lord for keeping me and my family alive to see this day.

# Abstract

In this work, we study a class of almost Hermitian manifolds called locally conformal almost Kähler manifolds. These are almost Hermitian manifolds which contains an open cover $\{U_t\}_{t\in I}$ and a family of $\mathcal{C}^\infty$ functions $f_t : U_t \to \mathbb{R}$ such that each conformal metric $g_t$ on $U_t$ is an almost Kähler metric. Locally conformal almost Kähler manifolds also falls under a class of locally conformal symplectic manifolds. More precisely, locally conformal almost Kähler manifolds are manifolds whose fundamental 2-form is locally conformal symplectic. We first recall some of the existing geometric properties of almost Hermitian manifolds. Then further use these properties to derive those of locally conformal almost Kähler manifolds. A new example of a locally conformal almost Kähler manifold is given. We further investigate the relationship between the covariant derivative and the Nijenhuis tensor on a locally conformal almost Kähler manifold. The equivalence of the Nijenhuis tensor defined on each $U_t$ and the one defined globally is also proven.

The relationship between the curvature tensors induced by the two conformal metrics on a locally conformal Kähler manifolds are considered. In particular, we show that a locally conformal almost Kähler manifold is an almost Kähler manifold under some curvature conditions. To achieve our goal, we first prove the relation between scalar curvatures $\tau^t$ and $\tau$ together with the corresponding scalar $*$-curvatures $\tau^{t^*}$ and $\tau^*$ of a locally conformal almost Kähler manifold.

Moreover, among other results, we also investigate the canonical foliations of locally conformal almost Kähler manifolds. Accurately, we give necessary and sufficient conditions for the metric on a locally conformal almost Kähler manifolds to be a bundle-like for foliations $\mathcal{F}$.

# Contents

Chapter One

# Introduction

## 1.1 Introduction

The study of manifolds whose metric is locally conformal to an almost Kähler metric is considered as one of the most interesting studies in the field of differential geometry [4]. This is because of its richness in the theory that is applicable in physics, algebraic geometry, symplectic geometry, etc. To our knowledge, locally conformal (almost) Kähler structures were first studied by P. Libermann [13] in the 1950s. In 1966, A Gray [5] also contributed to the study by considering (almost) Hermitian manifolds whose metric is conformal to a local (almost) Kähler metric. However, globally conformal (almost) Kähler manifolds share the same topological properties with locally conformal (almost) Kähler manifolds [21]. It is therefore provocative to consider those almost Hermitian manifolds whose metric is locally conformal to an almost Kähler metric. The difference between locally conformal Kähler manifolds and locally conformal almost Kähler manifolds is the condition of integrability of an almost complex structure, this is equivalent to an almost complex structure being parallel with respect to a globally defined connection or the vanishing of a Nijenhuis tensor. Therefore, the geometric properties which do not depend on the almost complex structure will apply to both of these manifolds.

Libermann defined a locally conformal (almost) Kähler metric as a metric $g$ at which in the neighborhood of each point of a $2n$-dimensional almost Hermitian manifold, it is conformal to an (almost) Kähler metric. To be more specific, a metric $g$ is locally conformal (almost) Kähler if there is an open cover $\{U_t\}_{t\in I}$ of an (almost) Hermitian manifold and a family $\{f_t\}_{t\in I}$ of smooth functions $f_t : U_t \to \mathbb{R}$ such that each metric

$$g_t = \exp(-f_t)g_{|U_t},$$

is a local (almost) Kählerian metric. An (almost) Hermitian manifold equipped with this metric is a locally conformal (almost) Kähler manifold. Another way these manifolds are characterized is by a globally defined 1-form $\omega$ on a Hermitian manifold satisfying

$$d\Omega = \omega \wedge \Omega \text{ and } d\omega = 0, \tag{1.1}$$

where $\Omega$ is a Kähler form of an (almost) Hermitian manifold [21]. The closed 1-form $\omega$ is known as the Lee form as it was introduced by Lee [12]. In the symplectic viewpoint, locally conformal (almost) Kahler manifold $(M, J, g)$ is a locally conformal symplectic manifold $(M, \Omega)$ endowed with an (almost) complex structure and an (almost) Hermitian metric corresponding to the locally conformal symplectic form. Of course, we recall that a locally conformal symplectic manifold is a $2n$-dimensional connected paracompact manifold equipped with a non-degenerate 2-form $\Omega$, so that at every point in an open neighborhood $U$, we have $d(exp(-f_t)\Omega|_{U_t}) = 0$. Equivalently, $(M, \Omega)$ is locally conformal symplectic if there exist a Lee form $\omega$ such that 1.1 holds [25].

## 1.2 Main results

Here is the summary of some results found on locally conformal almost Kähler manifolds. Let $(M, J, g)$ be a $2n$-dimensional almost Hermitian manifold.

(1) The almost Hermitian manifold $(M, J, g)$ is locally conformal almost Kähler if and only if

$$(\nabla^b_X J)Y = (\nabla_X J)Y + \frac{1}{2}\Big\{(\omega \circ J)(Y)X - \omega(Y)JX + g(X, Y)JB - \Omega(X, Y)B\Big\},$$

for any vector field $X, Y$ tangent to $M$.

(2) If the manifold $(M, J, g)$ is locally conformal almost Kähler, then the scalar curvatures $\tau^t$ and $\tau$ of $M$ are related by

$$\exp(-f_t)\tau_t = \tau + (2n-1)\left\{\text{div}B - \frac{1}{2}(1-n)||B||^2\right\}.$$

Also the scalar $*$-curvatures $\tau^{t*}$ and $\tau^*$ are related by

$$\exp(-f_t)\tau^{t*} = \tau^* + \text{div}B + (n-1)||B||^2.$$

(3) Let $(M, J, g)$ be a $2n$-dimensional compact locally conformal almost Kähler manifold with $n > 1$ and contained in $\mathcal{L}_1$. If

$$\tau^* = \tau,$$

then $(M, J, g)$ is an almost Kähler manifold.

(4) For a locally conformal almost Kähler manifold admitting a foliation $\mathcal{F}$, the following are equivalent:

(i) The foliation $\mathcal{F}$ is Riemannian.

(ii) The Lee vector field $B$ is auto-parallel with respect to $\nabla$, that is,

$$\nabla_B B = B\left(\ln(||B||)\right) B.$$

Moreover, the leaves of the distribution $D$ are hypersurfaces with mean curvature vector field

$$H' = \frac{1}{2n-1}\left(\mathrm{div}_{|_{M'}} B\right) B.$$

Moreover, they are totally geodesic hypersurfaces if and only if the dual vector field $B$ of $\omega$ preserves their metrics.

Also these integral manifolds are minimal if and only if the dual vector field incompressible along the manifolds.

## 1.3 Organization of the thesis

This dissertation is organized in the following way: The second chapter introduces some basic principles and definitions of almost complex manifolds, complex manifolds, almost Hermitian manifolds, Hermitian manifolds, almost Kähler manifolds, Kähler manifolds, almost symplectic manifolds and symplectic manifolds. An almost complex structure will be defined and its integrability conditions will be provided (see [18], for more details). We will introduce the connections on almost Hermitian manifolds together with its properties established by A. Gray[5].

The third chapter defines a locally conformal almost Kähler structures (see [21] and references therein for more details) supported by an example. Furthermore, we outline some of the geometric properties of underlying manifolds by considering the connections $\nabla^t$ and $\nabla$ induced by the conformal metrics $g^t$ and $g$, respectively. Using the fact that in such ambiant manifold, the almost complex structure is not parallel with respect to the Weyl connection $\nabla^b$, we follow the work of Dragomir [[4], corollary 1.1] to give a representation of a Weyl connection with respect to the almost complex structure. We will then end the chapter by giving some representation of the connections $\nabla$ and $\nabla^t$ with regards to the Nijenhuis tensors $N$ and $N^t$, respectively.

In Chapter 4, we investigate the curvature properties of locally conformal almost Kähler manifolds. More precisely, we establish the relation between the scalar curvatures and $\tau$ and $\tau^t$, together with the corresponding scalar $*$curvatures $\tau^*$ and $\tau^{*t}$. We prove that such ambiant manifold is a subclass of an almost Hermitian manifold defined by the curvature identities introduced by A. Gray [6], then $(M, J, g)$ is an almost Kähler manifold. We also focus on foliations that arise naturally when the fact the Lee form is nowhere vanishing. We define a bundle like metric for a foliation $\mathcal{F}$ on a Riemannian manifold. Furthermore, we give a characterization for the bundle-like metric. We also prove that the minimality of the leaves coincides with the incompressibility of the Lee vector field.

Finally, in Chapter 5, we conclude on the results obtained in this dissertation and we provide future research directions.

# PRELIMINARIES

## 2.1 Almost Hermitian manifolds

Almost Hermitian manifolds are important in differential geometry. They mode a huge class of generalized Kähler manifolds. The rest of generalized Kähler manifolds like almost Kähler, nearly Kähler, semi Kähler, and quasi Kähler manifolds fall under the class of almost Hermitian manifolds (for more details, see [5] and [6]).

Let $M$ be a $2n$-dimensional $\mathcal{C}^\infty$ (smooth) manifold. Next, let $TM$ be a tangent bundle of $M$. That is,

$$TM = \bigcup_{x \in M} T_x M,$$

where $T_xM = \{V : \exists\, \alpha : (-\varepsilon, \varepsilon) \to M,\ \alpha(0) = x,\ \alpha'(0) = V\}$.

Furthermore, let $J : TM \to TM$ be an $\mathbb{R}$-linear endomorphism such that $J^2 = -\mathbb{I}_{TM}$. In particular, $J$ can be viewed as a linear operator given by

$$J = \begin{bmatrix} 0 & \mathbb{I}_{TM} \\ \mathbb{I}_{TM} & 0 \end{bmatrix}.$$

An almost complex manifold of dimension $n$ is a real $2n$-dimensional manifold endowed with an almost complex structure $J$. We will use $(M, J)$ to denote an almost complex manifold. It is known that an almost complex manifold is orientable [11]. Let $(M, J)$ be a $2n$-dimensional almost complex manifold. Then the real tangent bundle $TM$ has a local basis

$$\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{2n}}; \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{2n}} \right\}.$$

Also, the real cotangent bundle $T^*M$ has a local basis

$$\{dx_1, \ldots, dx_{2n}; dy_1, \ldots, dy_{2n}\}.$$

Next, we consider the complexified cotangent bundle $T^*M \otimes \mathbb{C}$ given by

$$T^*M \otimes \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M,$$

where $T^{(1,0)}M$ and $T^{(0,1)}M$ are $i-$eigenbundle and $-i$-eigenbundle, respectively.

Also, we have a natural splitting

$$\Lambda^k T^*M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{(p,q)}M,$$

with

$$\Lambda^{(p,q)}(M) = \Lambda^p_{\mathbb{C}} T^{*(0,1)}M \otimes \Lambda^q_{\mathbb{C}} T^{*(0,1)}M.$$

**Definition 2.1.1.** *The sections of $\Lambda^{(p,q)}(M)$ are called (p,q)-forms.*

Now let us consider an open set $U \subset M$ and a map $\psi : U \to \mathbf{U}$, where $\mathbf{U}$ is an open set in $\mathbb{R}^{2n}$. Then $(U, \psi) = (U, (x^1, x^2, \ldots, x^{2n}))$ defines a chart on $M$.

In particular, if $(U_a, \psi_a)$ and $(U_b, \psi_b)$ are two charts on $M$, we can construct the transitional maps as illustrated in Figure 2.1 below:

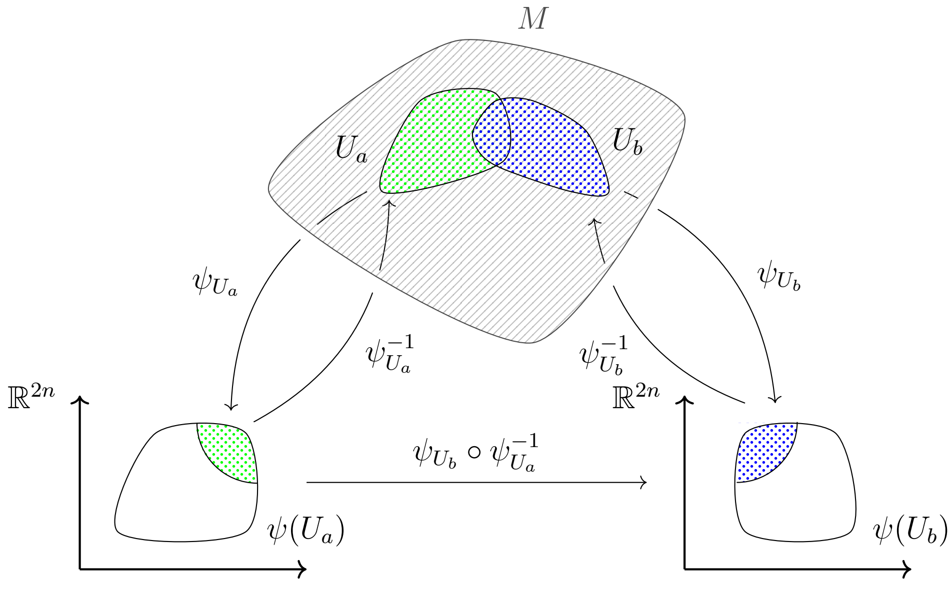


***Figure 2.1: Transitional functions.***

On an almost complex manifold, the transition map $\psi_{U_b} \circ \psi_{U_a}^{-1}$ is holomorphic, i.e. it is a complex-valued function.

Any $X \in \mathcal{C}^\infty(TM)$, is locally represented by $X = \sum_{i=1}^{2n} X^i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^{2n} X^j \frac{\partial}{\partial x_j}$, where $X^i : M^{2n} \to \mathbb{R}$. We define the Lie bracket of $X$ and $Y$ by

$$[X, Y] = X(Y) - Y(X).$$

**Definition 2.1.2.** *An almost complex structure $J$ is integrable if*

$$[T^{(1,0)}M, T^{(0,1)}M] \subset T^{(1,0)}M.$$

In addition, for all vector field $X$ and $Y$ in $M$, the Nijenhuis tensor field $N$ of $J$ is given by

$$N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]. \tag{2.1}$$

The well-known theorem of Newlander and Nirenberg [18] assets that the $2n$-dimensional almost complex manifold $(M, J)$ is a complex manifold if and only if the Nijenhuis tensor $N$ vanishes or equivalently if the almost complex structure $J$ is integrable.

Next, we define a Riemannian metric $g$ on a $\mathcal{C}^\infty$-real differentiable manifold $M$ as a $\mathcal{C}^\infty$-family of inner product $g_x : T_xM \times T_xM \to \mathbb{R}$ on $T_xM$.
Then [11]

(1) $g(X,Y) = g(Y,X)$, i.e. $g$ is symmetric

(2) $g(X,X) \geq 0$, i.e. $g$ is positive define and

(3) $g(X,X) = 0 \iff X = 0$,

for all vector fields $X$ and $Y$ on $M$.

**Definition 2.1.3.** *[21] A Hermitian metric is a Riemannian metric $g$ on an almost complex manifold $(M, J)$ such that for each point $x$ in $M$ and each $X, Y$ tangent to $M$, we have*

$$g(X,Y) = g(JX, JY), \tag{2.2}$$

*for all vector fields $X$ and $Y$ on $M$.*

Using the above definition, it follows that an almost Hermitian manifold $(M, J, g)$ is an almost complex manifold $(M, J)$ together with a Hermitian metric $g$. Furthermore, by using the Newlander and Nirenberg Theorem [18], we deduce that if $N$ vanishes, then $(M, J, g)$ is a Hermitian manifold.
Every (almost) complex manifold admits a Hermitian metric. This can be verified by noting that (almost) complex manifolds are paracompact by definition and as a result, they admit a Riemannian metric which we shall denote by $g_1$. Now let us define $g$ by

$$g(X,Y) = g_1(X,Y) + g_1(JX, JY), \tag{2.3}$$

for any vector fields $X$ and $Y$ on $M$, then one has that $g$ is a Hermitian metric [26].
To further outline the geometry of (almost) Hermitian manifolds, we must consider

a tensor $\Omega$ induced by the metric $g$. This is a 2-form known as the (almost) Kähler form of an (almost) Hermitian manifold $M$ and satisfies

$$\Omega(X,Y) = g(JX,Y) \text{ and } \Omega(JX,JY) = \Omega(X,Y), \tag{2.4}$$

for all $X, Y \in TM$.

Next, we introduce a well-defined method used to differentiate vector fields and any other tensors. Let $\Gamma(TM)$ be the space of vector fields in $M$. Then

**Definition 2.1.4.** *An affine connection on $M$ is a mapping*

$$\begin{array}{rccl} \nabla: & \Gamma(TM)\times\Gamma(TM) & \longrightarrow & \Gamma(TM) \\ & (X,Y) & \longmapsto & \nabla_X Y, \quad \textit{for all } X,Y\in\Gamma(TM). \end{array} \tag{2.5}$$

*We say $\nabla_X Y$ is a covariant derivative of $Y$ in the direction of $X$.*

Let us consider a $\mathcal{C}^\infty$-function $f : M \to \mathbb{R}$ and for any vector fields $X$ and $Y$ on $M$, then the relations [11]

(1) $\nabla_{fX}Y = f\nabla_X Y$ and

(2) $\nabla_X(fY) = df(X)Y + f\nabla_X Y$,

hold.

**Definition 2.1.5.** *The gradient of a $\mathcal{C}^\infty$-function $f$ is a vector field* grad $f$ *such that*

$$X(f) = g(\text{grad } f, X), \tag{2.6}$$

*for all $X \in \Gamma(TM)$. In addition* grad $f = (df)^\sharp$, *where $\sharp$ denotes the raising of indices.*

**Definition 2.1.6.** *The Levi-Civita connection is an affine connection $\nabla$ such that*

- $\nabla g = 0$, *that is, $\nabla$ preserves the metric,*
- $\nabla_X Y - \nabla_Y X = [X,Y]$, *i.e $\nabla$ is torsion-free, for all $X$, $Y \in \Gamma(TM)$.*

The existence and uniqueness of the Levi-Civita connection on any $\mathcal{C}^\infty$ manifold is known as the Fundamental Theorem of Riemannian Geometry [11].

Let $\nabla$ be the Levi-Civita connection on $M$ and $\sigma$ be the coderivative of $\Omega$, then we can get many other tensor fields that can be used to define different classes of almost Hermitian manifolds. Among these tensor fields, we mention the following.

Let $X, Y$, and $Z$ be tangent to an almost Hermitian manifold $M$. Then [7]

(1) $(\nabla_X\Omega)(Y,Z) = g(Y,(\nabla_X J)Z) = g((\nabla_X J)Y,Z)$ and $(\nabla_X\Omega)(Y,Z) = (\nabla_X\Omega)(JY,JZ)$ defines the covariant derivative.

(2) $3d\Omega(X,Y,Z) = \hat{\bigoplus}_{XYZ}(\nabla_X\Omega)(Y,Z)$ defines the exterior derivative and $\hat{\bigoplus}_{XYZ}$ denotes the cyclic sum over $X, Y$ and $Z$.

(3) $(\sigma\Omega)(X) = \sum_{i=1}^{n}\Big((\nabla_{E_i}\Omega)(E_i,X)+(\nabla_{JE_i}\Omega)(JE_i,X)\Big)$ defines the codifferential of $\Omega$ where $\{E_1,\dots,E_n,JE_1,\dots,JE_n\}$ is the local orthogonal frame.

(4) $\omega(X) = \frac{1}{n-1}\sigma\Omega(JX)$ is a 1-form an almost Hermitian manifolds and is called the Lee form.

**Definition 2.1.7.** *An almost Kähler manifold is an almost Hermitian manifold $(M,J,g)$ such that the associated almost Hermitian form $\Omega$ is closed, that is,*

$$d\Omega = 0.$$

The condition for an almost Hermitian manifold to be an almost Kähler manifold is equivalent to

$$g((\nabla_X J)Y,Z) + g((\nabla_Y J)Z,X) + g((\nabla_Z J)X,Y) = 0. \tag{2.7}$$

If the almost complex structure $J$ is integrable, or equivalently, if $N = 0$ if then $(M,J,g)$ is said to be a Kähler manifold. For a Kähler manifold, the integrability of almost complex structure $J$ and the vanishing of the Nijenhuis tensor $N$ can be characterized by a single condition $\nabla J = 0$ [2].

Next, we consider any connected manifold $M$ of dimension $2n$ such that its fundamental form $\Omega$ does not necessarily depend on the metric and the almost complex structure.

**Definition 2.1.8.** *The pair $(M,\Omega)$ is called a symplectic manifold if $d\Omega = 0$. Moreover, if $d\Omega \neq 0$, then $(M,\Omega)$ is an almost symplectic manifold.*

It is worth noting that in both symplectic and almost symplectic manifolds, the 2-form $\Omega$ is non-degenerate.

# Locally conformal almost Kähler manifolds

In this chapter, we wish to define a locally conformal almost Kähler manifold and give its characterization together with its geometric properties. An example of a locally conformal almost Kähler manifold will be constructed by considering the Cartesian product of two manifolds. We will give a characterization of locally conformal almost Kähler manifolds using the covariant derivatives and the Nijenhuis tensors by following the properties and charecterizations established in the case of locally conformal Kähler manifolds [4].

## 3.1 Geometric properties of locally conformal almost Kähler manifolds

Let $M$ be a $2n$-dimensional almost Hermitian manifold with the metric $g$ and the almost complex structure $J$ satisfying

$$J^2 = -\mathbb{I}, \quad g(JX, JY) = g(X, Y),$$

for any vector fields $X$ and $Y$ tangent to $M$, where $\mathbb{I}$ stands for the identity transformation of tangent bundle $TM$. Then for any vector fields $X$ and $Y$, the tensor

$$\Omega(X, Y) = g(X, JY), \tag{3.1}$$

defines the fundamental 2-form of $M$ which is non-degenerate and gives an almost symplectic structure on $M$.

**Definition 3.1.1.** *Let $(M, J, g)$ be a $2n$-dimensional almost Hermitian manifold. Then $(M, J, g)$ is a locally conformal almost Kähler manifold if there is an open covering $\{U_t\}_{t\in I}$ of $M$ and a family $\{f_t\}_{t\in I}$ of $\mathcal{C}^\infty$-functions $f_t : U_t \to \mathbb{R}$ such that, for any $t \in I$, the metric form*

$$g_t = \exp(-f_t) g_{|_{U_t}}, \tag{3.2}$$

*is almost Kähler metric.*

The metric $g_{|U_t}$ is given by $g_{|U_t} = \iota_t^* g$, where $\iota_t : U_t \to M$ defines the inclusion. Likewise, $(M, J, g)$ is *globally conformal almost Kähler* if there is a $\mathcal{C}^\infty$-function $f : M \longrightarrow \mathbb{R}$ so that the metric

$$\exp(-f)g,$$

is almost Kähler metric.

Another way to define a locally conformal almost Kähler manifolds is to mimic the approach to locally conformal symplectic manifolds [21]. In symplectic viewpoint, a locally conformal symplectic manifold is a $\mathcal{C}^\infty$ manifold $M$ which has an open cover $\{U_t\}_{t\in I}$ and a family of $\mathcal{C}^\infty$ functions $f_t : U_t \to \mathbb{R}$ such that the fundamental mental form $\Omega_t = \exp(f_t)\Omega|_{U_t}$ is symplectic, for any $t \in I$ [25]. Hence one can easily deduce that a locally conformal almost Kähler manifold is a locally conformal symplectic manifold that admits an almost Hermitian structure.

Let $\Omega_t$ be the 2-form associated with $(J, g_t)$. Then (3.2) leads, for any vector fields $X$ and $Y$ on $M$

$$\begin{aligned} \Omega_t(X,Y) &= g_t(X, JY) \\ &= \exp(-f_t)g(X, JY) \\ &= \exp(-f_t)\Omega(X, Y). \end{aligned} \tag{3.3}$$

Another equivalent way to characterize the locally conformal almost Kähler manifolds is the one established by H.C. Lee [12]. Lee noticed that on a locally conformal symplectic manifold, the 1-forms $df_t$ fits together into a globally defined 1-form $\omega$. For locally conformal almost Kähler manifolds, the Lee form $\omega$ is given by

$$\omega = \frac{1}{n-1}(\delta\Omega) \circ J, \tag{3.4}$$

where $\delta$ denotes the the formal adjoint of exterior differentiation operator $d$ with respect to $g$ [21].

The existence of a Lee form was defined by H.C. Lee [12], hence the name Lee form.

The Lee form is important because it characterizes locally conformal almost Kähler manifolds. To be more specific, we give the following theorem:

**Theorem 3.1.1.** *[4] The almost Hermitian manifold $(M, J, g)$ is a locally conformal almost Kähler manifold if and only if there exists a globally defined 1-form $\omega$ such that*

$$d\Omega = \omega \wedge \Omega, \quad d\omega = 0. \tag{3.5}$$

*Proof.* Let $(M, J, g)$ be a locally conformal almost Kähler manifold. Then equation (3.2) holds. In fact, since $g_t(X, JY) = \Omega_t(X, Y)$ and $\Omega_t$ is symplectic on each open set $U_t$ of $M$, we have

$$0 = d\Omega_t = \exp(-f_t)df_t \wedge \Omega + \exp(-f_t)d\Omega,$$

so that $d\Omega = df_t \wedge \Omega$ on $U_t$. Now on the overlaps $U_{tr} = U_t \cap U_r$, we have

$$df_t \wedge \Omega = df_r \wedge \Omega.$$

Therefore,

$$(df_t - df_r) \wedge \Omega = 0,$$

and since $\Omega$ is nondegenerate we have that $df_t = df_r$ on $U_{tr}$. Hence the local 1-forms $df_t$ glue up to a globally defined 1-form $\omega$ on $M$. That is, $\omega$ satisfies the exactness property $\omega_{|U_t} = df_t$.

Conversely, if let $\omega$ be a closed satisfying (3.5). Then the Poincaré lemma [11] asserts that there is an open cover $\{U_t\}_{t\in I}$ and smooth real functions $(f_t : U_t \to \mathbb{R})_{t\in I}$ such that for each $t \in I$, we have $\omega_{|U_t} = df_t$. Since $d\Omega = \omega \wedge \Omega$, the restriction $d\Omega = df_t \wedge \Omega$ on each $U_t$ gives

$$d\Omega_t = \exp(-f_t)\{-df_t \wedge \Omega + d\Omega\} = 0.$$

Thus $d(\exp(-f_t)\Omega) = 0$. Hence $(M, J, g)$ is a locally conformal almost Kähler manifold. □

**Example 3.1.1.** We consider the 4-dimensional manifold

$$M^4 = \{p \in \mathbb{R}^4 | x_1 \neq 0,\ x_2 > 0\},$$

where $p = (x_1, x_2, y_1, y_2)$ are the standard coordinates in $\mathbb{R}^4$. The vector fields,

$$X_i = x_2 \frac{\partial}{\partial x_i}, \quad Y_i = \frac{1}{x_2^3}\frac{\partial}{\partial y_i}, \quad \text{for } \ i = 1, 2,$$

are linearly independent at each point of $M$. Let $g$ be the Riemannian metric on $M$ defined by $g(X_i, X_j) = g(Y_i, Y_j) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol, $g(X_i, Y_j) = 0$. That is, the form of the metric becomes

$$g = \frac{1}{x_2^2}(dx_1^2 + dx_2^2) + x_2^6(dy_1^2 + dy_2^2).$$

Let $J$ be the $(1, 1)$-tensor field defined by, $JX_1 = Y_1$, $JX_2 = -Y_2$, $JY_2 = X_2$, $JY_1 = -X_1$. Thus, $(J, g)$ defines an almost Hermitian structure on $M^4$. The non-zero component of the fundamental 2-form $J$ is

$$\Omega(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}) = -\frac{1}{x_2^2} \ \text{ and } \ \Omega(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}) = \frac{1}{x_2^2}$$

and we have

$$\Omega = \frac{1}{x_2^2}\left\{-dx_1 \wedge dy_1 + dx_2 \wedge dy_2\right\}.$$

Its differential gives

$$d\Omega = \frac{2}{x_2^3} dx_1 \wedge dy_1 \wedge dx_2.$$

By letting

$$\omega = -\frac{1}{x_2} dx_2,$$

we have,

$$d\Omega = 2\omega \wedge \Omega.$$

It is easy to see that $d\omega = 0$ and the dual vector field $B$ is given by

$$B = -X_2.$$

Let us consider the open neighborhood $U$ of $M$ given by $U = \{p \in M^4 | x_2 > 0\}$, and there exists a differentiable function $f$ on $U$ such that $\omega = d\, f$, where $f = -\ln(x_2)$. By the characterization given in Theorem 3.1.1 above-mentioned, $(M^4, J, g)$ is a locally conformal almost Kähler manifold.

Also, the Lee form $\omega$ analogous to $g$ has a metrically corresponding vector field $B = \omega^\sharp$ called *the Lee vector field*, where $\sharp$ denotes the operation of lifting indices by a metric $g$. The Lee vector field is defined by $g(B, X) = \omega(X)$ and locally by

$$B = \frac{1}{n-1} J\left(\sum_{j=1}^{n}((\nabla_{E_j} J)E_j) + (\nabla_{JE_j})JE_j\right),$$

where $\{E_j, JE_j\}_{j\in\{1,\ldots n\}}$ denotes the local orthonormal $J$-frame.

**Remark 3.1.1.** The existence of the Lee form $\omega$ such that equation (3.5) holds implies that $d\omega = 0$ for $n \geq 3$ . However, this is true except on complex surfaces, i.e. for $n = 2$, the relation (3.5) holds true but the Lee form is not generally closed [24]. If we consider $n = 1$, then $\omega = 0$ and $(M, J, g)$ is always an almost Hermitian manifold.

Now we give another typical example of a locally conformal almost Kähler manifold.

**Example 3.1.2.** Let $\tilde{N}$ be an almost Kähler manifold whose corresponding 2- form $\Theta$ is exact, i.e $\Theta = d\lambda$ where $\lambda$ is a 1-form on $\tilde{N}$. Next, denote by $\tilde{M}$ a locally conformal symplectic manifold. Then the 2-form $\Phi$ of $\tilde{M}$ satifies $d\Phi = \omega \wedge \Phi$ , where $d\omega = 0$ and $\omega$ is its Lee form. Let $N = \tilde{N} \times \tilde{M}$ be the Cartesian product of both manifolds. We define a 2-form $\Omega$ on $N$ by

$$\Omega = d\lambda + \Phi + \lambda \wedge \omega. \tag{3.6}$$

It was proven in [17] that $\Omega$ is non-degenerate. Taking the exterior derivative in (3.6), we get

$$\begin{aligned} d\Omega &= d(d\lambda) + d\Phi + d\lambda \wedge \omega - \lambda \wedge d\omega \\ &= \omega \wedge \Phi + d\lambda \wedge \omega \\ &= \omega \wedge (d\lambda + \Phi + \lambda \wedge \omega) \\ &= \omega \wedge \Omega. \end{aligned}$$

Hence by Theorem 3.1.1, $N$ is a locally conformal almost Kähler manifold.

Next, wish to study the relationship of the Levi-Civita connections induced by the locally conformal Kahler metric $g_t$ and $g$. Hence we start by the following results due to Sorin and Ornea [4].

**Theorem 3.1.2.** *[4] The Levi-Civita connection $\nabla^t$ of the almost Kähler metric $g_t$ on $M$ satisfies*

$$\nabla^t_X Y = \nabla_X Y - \frac{1}{2}\big(\omega(X)Y + \omega(Y)X - g(X,Y)B\big), \tag{3.7}$$

*for all $X, Y \in TM$, where $\nabla$ is the Levi-Civita connection with respect to $g$.*

*Proof.* Suppose $M$ is a $\mathcal{C}^\infty$ manifold and let $g_t = \exp(-f_t)g$ be two conformally related metrics on $M$. Since any two connections differ by tensor, the Levi-Civita connections $\nabla^t$ and $\nabla$ of conformally related Riemannian metrics $g_t$ and $g$, respectively, satisfies

$$\nabla^t_X Y = \nabla_X Y + \alpha_X Y, \tag{3.8}$$

for any vector fields $X$ and $Y$ on $M$. Next, we compute

$$\begin{aligned} g_t(\nabla^t_X Y, Z) &= \exp(-f_t)g(\nabla^t_X Y, Z) \\ &= \exp(-f_t)(\nabla_X Y + \alpha_X Y, Z) \\ &= \exp(-f_t)\{g(\nabla_X Y, Z) + g(\alpha_X Y, Z)\}. \end{aligned} \tag{3.9}$$

Let us consider the Koszul formula [11] given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Z,Y) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \end{aligned} \tag{3.10}$$

Now for the Levi-Civita connection $\nabla^t$, induced by the metric $g_t$, we can write the Koszul formula as

$$\begin{aligned} 2\exp(-f_t)g(\nabla^t_X Y, Z) &= X(\exp(-f_t)g(Z,Y)) + Y(\exp(-f_t)g(Z,X)) \\ &\quad - Z(\exp(-f_t)g(X,Y)) - \exp(-f_t)g(X,[Y,Z]) \\ &\quad + \exp(-f_t)g(Y,[Z,X]) + \exp(-f_t)g(Z,[X,Y]). \end{aligned} \tag{3.11}$$

Next, using the product rule, we evaluate the derivatives

$$X(\exp(-f_t)) = -\exp(-f_t)X(f), \;\; Y(\exp(-f_t)) = -\exp(-f_t)Y(f)$$

and

$$Z(\exp(-f_t)) = -\exp(-f_t)Z(f).$$

Thus

$$\begin{aligned} 2\exp(-f_t)g(\nabla^t_X Y, Z) = {} & \exp(-f_t)Xg(Z,Y)) + \exp(-f_t)Yg(Z,X)) \\ & - (\exp(-f_t)Zg(X,Y)) - \exp(-f_t)g(Z,Y)X(f) \\ & - \exp(-f_t)g(Z,X)Y(f) + \exp(-f_t)g(X,Y)Z(f) \\ & - \exp(-f_t)g(X,[Y,Z]) + \exp(-f_t)g(Y,[Z,X]) \\ & + \exp(-f_t)g(Z,[X,Y]). \end{aligned} \tag{3.12}$$

Now we substitute (3.12) into (3.9), we get

$$\begin{aligned} 2\exp(-f_t)\{g(\nabla_X Y, Z) + g(\alpha_X Y, Z)\} = {} & \exp(-f_t)Xg(Z,Y)) + \exp(-f_t)Yg(Z,X)) \\ & - (\exp(-f_t)Zg(X,Y)) - \exp(-f_t)g(Z,Y)X(f) \\ & - \exp(-f_t)g(Z,X)Y(f) + \exp(-f_t)g(X,Y)Z(f) \\ & - \exp(-f_t)g(X,[Y,Z]) + \exp(-f_t)g(Y,[Z,X]) \\ & + \exp(-f_t)g(Z,[X,Y]). \end{aligned} \tag{3.13}$$

Furthermore, we divide by $\exp(-f_t)$ since it is a strictly positive function, to get

$$\begin{aligned} 2(g(\nabla_X Y, Z) + g(\alpha_X Y, Z)) = {} & Xg(Z,Y) + Yg(Z,X) \\ & - Zg(X,Y) - g(Z,Y)X(f) \\ & - g(Z,X)Y(f_t) + g(X,Y)Z(f_t) \\ & - g(X,[Y,Z]) + g(Y,[Z,X]) \\ & + g(Z,[X,Y]). \end{aligned} \tag{3.14}$$

Now using the Koszul formula again, we have

$$\begin{aligned} 2(g(\nabla_X Y, Z) + g(\alpha_X Y, Z)) = {} & -g(Z,Y)X(f_t) - g(Z,X)Y(f) + g(X,Y)Z(f_t) \\ & + 2g(\nabla_X Y, Z). \end{aligned}$$

Now we shall transpose the term containing $\nabla_X Y$ and use the symmetric property of $g$ to get

$$2g(\alpha_X Y, Z)) = -g(Y,Z)X(f_t) - g(X,Z)Y(f_t) + g(X,Y)Z(f_t),$$

which implies

$$2g(\alpha_X Y, Z)) = -g(X(f_t)Y + Y(f_t)X, Z) + g(X, Y)Z(f_t).$$

However, we know by Theorem 3.1.1 that the Lee form $\omega$ is closed, hence exact on $U_t$, i.e. $df_t = \omega$. As a result, we shall use the definition of a gradient 2.1.5 to write $X(f_t) = df_t(X) = g(\omega^\sharp, X)$ . Using this information in terms of a vector field $Z$, we get

$$2g(\alpha_X Y, Z)) = -g(X(f_t)Y + Y(f_t)X, Z) + g(X, Y)g(\omega^\sharp, Z).$$

Since $g$ is non-degenerate, we get a representation of $\alpha_X Y$ at which

$$\alpha_X Y = -\frac{1}{2}\big(\omega(X)Y + \omega(Y)X - g(X, Y)\omega^\sharp\big). \tag{3.15}$$

Finally, substituting (3.15) in (3.8) and using the fact that $B = \omega^\sharp$, we get

$$\nabla^t_X Y = \nabla_X Y - \frac{1}{2}\big(\omega(X)Y + \omega(Y)X - g(X, Y)B\big), \tag{3.16}$$

as required. □

The connection $\nabla^t$ defined by Theorem 3.1.2 is globally defined. In fact, S. Dragomir and L. Ornea [4] proved that the Levi-Civita connections $\nabla^t$ of the local almost Kähler metrics $\{g_t\}_{t\in I}$ glue up to a globally defined torsion-free linear connection $\nabla^b$ on $M$ given by

$$\nabla^b_X Y = \nabla_X Y - \frac{1}{2}\Big(\omega(X)Y + \omega(Y)X - g(X, Y)B\Big). \tag{3.17}$$

Moreover, $\nabla^b$ satisfies

$$\nabla^b g = \omega \otimes g. \tag{3.18}$$

Indeed, for any vector fields $X$, $Y$ and $Z$ on $U_t$, we have

$$\begin{aligned}(\nabla^b_X g)(Y, Z) &= Xg(Y, Z) - g(\nabla^b_X Y, Z) - g(Y, \nabla^b_X Z) \qquad (3.19)\\ &= X(\exp(f_t)g_t(Y, Z)) - \exp(f_t)g_t(\nabla^b_X Y, Z) - \exp(f_t)g_t(Y, \nabla^b_X, Z)\\ &= X(f_t)\exp(f_t)g_t(Y, Z) + \exp(f_t)Xg_t(Y, Z) - \exp(f_t)g(\nabla^b_X Y, Z)\\ &- \exp(f_t)g_t(Y, \nabla^b_X Z)\\ &= X(f_t)\exp(f_t)g_t(Y, Z) + \exp(f_t)\big\{Xg_t(Y, Z) - g_t(\nabla^b_X Y, Z)\\ &- g_t(\nabla^b_X Y, Z).\big\}\end{aligned}$$

However,

$$Xg_t(Y, Z) - g_t(\nabla^b_X Y, Z) - g_t(\nabla^b_X Y, Z) = 0.$$

Thus, we have

$$(\nabla^b_X g)(Y, Z) = X(f_t)\exp(f_t)g_t(Y, Z).$$

Finally, since $X(f_t) = df_t(X)$ and $g = \exp(f_t)g_t(Y,Z)$, we have

$$(\nabla^b_X g)(Y,Z) = df_t(X)g(Y,Z).$$

Moreover, we know form Theorem 3.1.1 that $df_t = \omega$. Hence we conclude

$$(\nabla^b_X g)(Y,Z) = \omega(X)g(Y,Z).$$

That is

$$\nabla^b g = \omega \otimes g.$$

**Definition 3.1.2.** *The globally defined torsion-free linear connection $\nabla^b$ given in (3.17) is called the Weyl connection.*

**Remark 3.1.2.** The Lee form $\omega$ of a Weyl connection $\nabla^b$ induced by $\exp(-f_t)g$ measures the difference between the Weyl connection and the Levi-Civita connection.

## 3.2 Connections on a locally conformal almost Kähler manifold

If the almost complex structure $J$ is parallel with respect to the connection $\nabla^b$, i.e., $\nabla^b J = 0$, then $(M, J, g)$ is a locally conformal Kähler manifold [4]. In the case of locally conformal almost Kähler manifolds, we generalize the results in [4] in the following way.

**Theorem 3.2.1.** *The almost Hermitian manifold $(M, J, g)$ is locally conformal almost Kähler if and only if*

$$(\nabla^b_X J)Y = (\nabla_X J)Y + \frac{1}{2}\Big\{(\omega\circ J)(Y)X - \omega(Y)JX + g(X,Y)JB - \Omega(X,Y)B\Big\}, \quad (3.20)$$

*for any vector field $X, Y$ tangent to $M$.*

*Proof.* We shall first take the covariant derivative of $JY$ with respect to the linear connection $\nabla_X$ which yields

$$(\nabla_X J)Y = \nabla_X(JY) - J(\nabla_X Y), \quad (3.21)$$

which is also true for $\nabla^t_X$ because it is also a linear connection. Now we use (3.7) in the above equation to get

$$\begin{aligned}(\nabla^b_X J)Y &= \nabla_X(JY) \\ &\quad - \frac{1}{2}\omega(JY)X + \frac{1}{2}g(X,JY)B - J(\nabla_X Y) + \frac{1}{2}\omega(Y)JX - \frac{1}{2}g(X,Y)JB \\ &= \nabla_X(JY) - J(\nabla_X Y) \\ &\quad - \frac{1}{2}\omega(JY)X + \frac{1}{2}g(X,JY)B + \frac{1}{2}\omega(Y)JX - \frac{1}{2}g(X,Y)JB,\end{aligned}$$

which by using (3.21) again simplifies to

$$(\nabla^b_X J)Y = (\nabla_X J)Y + \frac{1}{2}\Big\{(\omega \circ J)(Y)X - \omega(Y)JX + g(X,Y)JB - \Omega(X,Y)B\Big\},$$

which completes the proof. □

Next, we consider the representation of the the covariant derivative $\nabla$ with respect to the metric $g$ of a locally conformal almost Kähler manifold. S. Kobayashi and K. Nomizu [11] proved that in an almost Hermitian manifold, we have

$$g((\nabla_X J)Y, Z) = 3d\Omega(X, JY, JZ) - 3d\Omega(X, Y, Z) + g(N(Y, Z), JX).$$

In the case of locally conformal almost Kähler manifold, we have the following:

**Theorem 3.2.2.** *Let $(M, J, g)$ be a locally conformal almost Kähler manifold. Then*

$$g((\nabla_X J)Y, Z) = \frac{1}{2}\Omega(N(Y, Z), X),$$

*for all $X, Y$ tangent to $M$, where $N$ and $\Omega$ denotes the Nijenhuis tensor and the non-degenarate 2-form, respectively.*

*Proof.* We recall from (2.1) that the Nijenhius tensor of an almost Hermitian manifold is defined by

$$N(Y, Z) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY],$$

for all $X, Y \in TM$. Also, by S. Kobayashi and K. Nomizu [11], the differential of a 2-form $\Omega$ on $M$ is given by

$$\begin{aligned} 3d\Omega(X, Y, Z) &= X\Omega(Y, Z) + Y\Omega(Z, X) + Z\Omega(X, Y) \\ &\quad - \Omega([X, Y], Z) - \Omega([Z, X], Y) - \Omega([Y, Z], X). \end{aligned} \tag{3.22}$$

Now if we replace $Y$ by $JY$ and $Z$ by $JZ$ in Equation (3.22), we get

$$\begin{aligned} 3d\Omega(X, JY, JZ) &= X\Omega(JY, JZ) + JY\Omega(JZ, X) + JZ\Omega(X, JY) \\ &\quad - \Omega([X, JY], JZ) - \Omega([JZ, X], JY) - \Omega([JY, JZ], X). \end{aligned} \tag{3.23}$$

Next, we use (3.21) to compute

$$\begin{aligned} g((\nabla_X J)Y, Z) &= g(\nabla_X(JY) - J(\nabla_X Y), Z) \\ &= g(\nabla_X JY, Z) - g(J(\nabla_X Y), Z) \\ &= g(\nabla_X JY, Z) + g(\nabla_X Y, JZ). \end{aligned} \tag{3.24}$$

We now multiply by 2 on both sides of (3.24) and using (3.4) together with the Koszul formula (3.10), this yields

$$\begin{aligned}
2g((\nabla_X J)Y, Z) &= 2g(\nabla_X JY, Z) + 2g(\nabla_X Y, JZ) \\
&= X(g(JY, Z)) + JYg(Z, X) - Zg(X, JY) \\
&+ g([X, JY], Z) + g([Z, X], JY) + g([JY, Z], X) \\
&+ X(g(Y, JZ)) + Yg(JZ, X) - JZg(X, Y) \\
&+ g([X, Y], JZ) + g([JZ, X], Y) + g([Y, JZ], X) \\
&= -X(\Omega(Y, Z)) - JY\Omega(JZ, X) - Z\Omega(X, Y) \\
&- \Omega([X, JY], JZ) + \Omega([Z, X], Y) + g(J[JY, Z], JX) \\
&+ X\Omega(JY, JZ) - Y\Omega(Z, X) - JZ\Omega(JX, Y) \\
&+ \Omega([X, Y], Z) - \Omega([JZ, X], JY) - g(J[Y, JZ], JX) \\
&+ \Big\{\Omega([Y, Z], X) - g([Y, Z], JX)\Big\} \\
&- \Big\{\Omega([JY, JZ, X) - g([JY, JZ], JX)\Big\} \\
&= X(\Omega(JY, JZ)) - JY\Omega(JZ, X) - JZ\Omega(JX, Y) \\
&- \Omega([X, JY], JZ) - \Omega([JZ, X], JY) - \Omega([JY, JZ], X) \\
&- X\Omega(Y, Z) - Y\Omega(Z, X) - Z\Omega(X, Y) \\
&+ \Omega([X, Y], Z) + \Omega([Z, X], Y) + \Omega([Y, Z], X) + g([JY, JZ, JX) \\
&- g([Y, Z], JX) - g(J[JY, Z], JX) - g(J[Y, JZ], JX) \\
&= 3d\Omega(X, JY, JZ) - 3d\Omega(X, Y, Z) + g(N(Y, Z), JX).
\end{aligned}$$

Hence

$$g((\nabla_X J)Y, Z) = \frac{3}{2}d\Omega(X, JY, JZ) - \frac{3}{2}d\Omega(X, Y, Z) + \frac{1}{2}g(N(Y, Z), JX). \tag{3.25}$$

Now since $(M, J, g)$ is locally conformal Kähler, there exist a closed 1-form $\omega$ such that $d\Omega = \omega \wedge \Omega$, where $(\omega \wedge \Omega)(X, Y, Z) = \hat{\bigoplus}_{X,Y,Z}\omega(X)\Omega(Y, Z)$. Therefore (3.25) becomes

$$\begin{aligned}
g((\nabla_X J)Y, Z) &= \frac{3}{2}\Big\{(\omega \wedge \Omega)(X, JY, JZ) - (\omega \wedge \Omega)(X, Y, Z)\Big\} + \frac{1}{2}g(N(Y, Z), JX) \\
&= \frac{3}{2}\hat{\bigoplus}_{X,Y,Z}\Big\{\omega(X)\Omega(JY.JZ) - \omega(X)\Omega(Y, Z)\ \Big\} + \frac{1}{2}g(N(Y, Z), JX).
\end{aligned}$$

Moreover, by the almost Hermitian property of the metric $g$ defined in Equation (2.4), the term $\omega(X)\Omega(JY.JZ) - \omega(X)\Omega(Y, Z)$ will vanish. Hence,

$$g((\nabla_X J)Y, Z) = \frac{1}{2}g(N(Y, Z), JX) = \frac{1}{2}\Omega(N(Y, Z), X),$$

as required. □

Next, we consider the underlying geometric properties with regard to local almost Kähler metric $g_t$ in relation to Theorem 3.2.2 on each $U_t$. To achieve this, we shall first understand the relation between the Nijenhius tensors $N^t$ and $N$ corresponding to $g_t$ and $g$, respectively.

**Proposition 3.2.1.** *Let $(M, J, g)$ be a locally conformal almost Kähler manifold. Then the Nijehnuis tensor $N^t$ and $N$ corresponding to $g_t$ and $g$, respectively, satisfies*

$$N^t(X, Y) = N(X, Y). \tag{3.26}$$

*Proof.* We first note that since $J_t = J$, we have $\Omega_t(X, Y) = g_t(X, J_tY) = g_t(X, JY)$. Now using the torsion free property of $\nabla^t$ we have

$$[X, Y]^t = \nabla^t_X Y - \nabla^t_Y X,$$

which implies that

$$[JX, JY]^t = \nabla^t_{JX} JY - \nabla^t_{JY} JX,$$

$$\begin{aligned} J[JX, Y]^t &= J(\nabla^t_{JX} Y - \nabla^t_Y JX) \\ &= J\nabla^t_{JX} Y - J\nabla^t_{JY} X \end{aligned}$$

and

$$\begin{aligned} J[X, JY]^t &= J\nabla^t_X JY - \nabla^t_{JY} X \\ &= J\nabla^t_X JY - J\nabla^t_{JY} X. \end{aligned}$$

Therefore, the Nijenhuis tensor on each $U_t$ is given by

$$\begin{aligned} N^t(X, Y) &= [JX, JY]^t - J[JX, Y]^t - J[X, JY]^t - [X, Y]^t. \\ &= \nabla^t_{JX} JY - \nabla^t_{JY} JX - (J\nabla^t_{JX} Y - J\nabla^t_{YJ} X) \\ &- (J\nabla^t_X JY - J\nabla^t_{JY} X) - (\nabla^t_X Y - \nabla^t_Y X) \\ &= (\nabla^t_X J)(JY) + (\nabla^t_{JX} J)Y - (\nabla^t_Y J)(JX) - (\nabla^t_{JY} J)X. \end{aligned}$$

From Equation of $JY$ with respect to $\nabla^t$, we get

$$(\nabla^t_X J)Y = \nabla^t_X (JY) - J(\nabla^t_X Y). \tag{3.27}$$

On the other hand, we know from Equation (3.21) that

$$(\nabla_X J)Y = \nabla_X (JY) - J(\nabla_X Y). \tag{3.28}$$

Now if we replace $Y$ by $JY$ in the in the above equation we get

$$(\nabla^t_X J)(JY) = -\nabla_X Y - J(\nabla^t_X JY). \tag{3.29}$$

Also, we replace $X$ by $JX$ in (3.28), we get

$$(\nabla_{JX}J)Y = \nabla_{JX}(JY) - J(\nabla_{JX}Y). \tag{3.30}$$

Now substituting (3.7) into (3.27), we obtain

$$\begin{aligned}(\nabla^t_X J)Y = \nabla_X(JY) &- \frac{1}{2}\omega(X)JY - \frac{1}{2}\omega(JY)X + \frac{1}{2}g(X, JY)B \\ &- J(\nabla_X Y) + \frac{1}{2}\omega(X)JY + \frac{1}{2}\omega(Y)JX - \frac{1}{2}g(X,Y)JB.\end{aligned}$$

Thus, one has

$$\begin{aligned}(\nabla^t_X J)Y = \nabla_X(JY) &- \frac{1}{2}\omega(JY)X + \frac{1}{2}g(X, JY)B - J(\nabla_X Y) + \frac{1}{2}\omega(Y)JX \\ &- \frac{1}{2}g(X,Y)JB.\end{aligned} \tag{3.31}$$

Let us replace $Y$ by $JY$ in (3.31) to get

$$\begin{aligned}(\nabla^t_X J)(JY) = -\nabla_X Y &+ \frac{1}{2}\omega(Y)X - \frac{1}{2}g(X,Y)B - J(\nabla_X(JY)) + \frac{1}{2}\omega(JY)JX \\ &- \frac{1}{2}g(X, JY)JB\end{aligned} \tag{3.32}$$

and replacing $X$ by $JX$ on (3.31) we get

$$\begin{aligned}(\nabla^t_{JX} J)Y = \nabla_{JX}(JY) &- \frac{1}{2}\omega(JY)JX + \frac{1}{2}g(X,Y)B - J(\nabla_{JX})Y - \frac{1}{2}\omega(Y)X \\ &- \frac{1}{2}g(JX,Y)JB.\end{aligned} \tag{3.33}$$

We then add (3.32) and (3.33) to obtain

$$\begin{aligned}(\nabla^t_X J)(JY) + (\nabla^t_{JX} J)Y &= -\nabla_X Y + \frac{1}{2}\omega(Y)X - \frac{1}{2}g(X,Y)B - J(\nabla_X(JY)) \\ &+ \frac{1}{2}\omega(JY)JX - \frac{1}{2}g(X, JY)JB + \nabla_{JX}(JY) \\ &- \frac{1}{2}\omega(JY)JX + \frac{1}{2}g(X,Y)B - J(\nabla_{JX})Y \\ &- \frac{1}{2}\omega(Y)X - \frac{1}{2}g(JX,Y)JB,\end{aligned}$$

which gives

$$(\nabla^t_X J)(JY) + (\nabla^t_{JX} J)Y = -\nabla_X Y - J(\nabla_X(JY)) + \nabla_{JX}(JY) - J(\nabla_{JX})Y. \tag{3.34}$$

Therefore, by using Equation (3.29) and (3.30) we have

$$(\nabla^t_X J)(JY) + (\nabla^t_{JX} J)Y = (\nabla_X J)(JY) + (\nabla_{JX} J)Y,$$

which implies

$$N^t(X,Y) = (\nabla_X J)(JY) + (\nabla_{JX})Y - (\nabla_Y J)(JX) - (\nabla_{JY} J)X.$$

Hence,

$$N^t(X,Y) = N(X,Y),$$

which completes the proof. □

Therefore, in relation to Theorem 3.2.2, with regards to $g_t$, we have

**Corollary 3.2.1.** *Let $(M,J,g)$ be a locally conformal almost Kähler manifold, on each $U_t$, we have*

$$g_t((\nabla^t_X J)Y,Z) = \frac{1}{2}\Omega_t(N^t(Y,Z),X).$$

*Proof.* Using (3.24) for $\nabla^t$, we have

$$2g_t((\nabla^t_X J)Y,Z) = 2g_t(\nabla^t_X JY,Z) + 2g_t(\nabla^t_X Y, JZ).$$

In particular, we compute

$$2g_t((\nabla^t_X J)Y,Z) = 2\exp(-f_t)\Big\{g(\nabla^t_X JY,Z) + g(\nabla^t_X Y,JZ)\Big\}. \tag{3.35}$$

That is,

$$\begin{aligned}
2g_t((\nabla^t_X J)Y,Z) &= \exp(-f_t)\Big\{Xg(JY,Z) + JYg(Z,X) - Zg(X,JY)\\
&+ g([X,JY],Z) + g([Z,X],JY) + g([JY,Z],X)\\
&+ Xg(Y,JZ) + Yg(JZ,X) - JZg(X,Y)\\
&+ g([X,Y],JZ) + g([JZ,X],Y) + g([Y,JZ],X)\Big\}\\
&= \exp(-f_t)\Big\{ - X\Omega(Y,Z) - JY\Omega(JZ,X) - Z\Omega(X,Y)\\
&- \Omega([X,JY],JZ) + \Omega([Z,X],Y) + g(J[JY,Z],JX)\\
&+ X\Omega(JY,JZ) - Y\Omega(Z,X) - JZ\Omega(JX,Y)\\
&+ \Omega([X,Y],Z) - \Omega([JZ,X],JY) - g(J[Y,JZ],JX)\\
&+ \Big(\Omega([Y,Z],X) - g([Y,Z],JX)\Big)\\
&- \Big(\Omega([JY,JZ,X) - g([JY,JZ],JX)\Big)\Big\}\\
&= \exp(-f_t)\Big\{X\Omega(JY,JZ) - JY\Omega(JZ,X) - JZ\Omega(JX,Y)\\
&- \Omega([X,JY],JZ) - \Omega([JZ,X],JY) - \Omega([JY,JZ],X)\\
&- X\Omega(Y,Z) - Y\Omega(Z,X) - Z\Omega(X,Y)\\
&+ \Omega([X,Y],Z) + \Omega([Z,X],Y) + \Omega([Y,Z],X) + g([JY,JZ,JX)\Big\}\\
&- g([Y,Z],JX) - g(J[JY,Z],JX) - g(J[Y,JZ],JX)\\
&= 3d\Omega_t(X,JY,JZ) - 3d\Omega_t(X,Y,Z) + g_t(N^t(Y,Z),JX).
\end{aligned}$$

Now since $(M, J, g)$ is a locally conformal almost Kähler manifold, $d\Omega_t = 0$. Hence $g_t((\nabla^t_X J)Y, Z) = \frac{1}{2}\Omega_t(N^t(Y, Z), JX)$. □

**Remark 3.2.1.** Proposition 3.2.1 and Corollary 3.2.1 implies that $g_t((\nabla^t_X J)Y, Z) = \frac{1}{2}\Omega_t(N(Y, Z), JX)$.

The following was proved by Vaisman [21]. In this work, we prove that the result holds on $U_t$ as well.

**Theorem 3.2.3.** *Let $(M, J, g)$ be an almost Hermitian manifold. Then $(M, J, g)$ is a locally conformal almost Kähler manifold if and only if the Nijenhuis tensor $N$ of $M$ satisfies*

$$\hat{\bigoplus}_{XYZ}\Omega_t(N^t(Y, Z), X) = 0, \tag{3.36}$$

*on each $U_t$, where $\hat{\bigoplus}$ denote the cyclic sum over $X, Y$ and $Z$.*

*Proof.* To prove that the imposed assertion is true, we use Corollary 3.2.1 to compute

$$\begin{aligned}\hat{\bigoplus}_{XYZ}\Omega_t(N^t(Y, Z), X) &= \Omega_t(N^t(Y, Z), X) + \Omega_t(N^t(Z, X), Y) + \Omega_t(N^t(X, Y), Z)\\ &= g_t(N^t(Y, Z), JX) + g_t(N^t(Z, X), JY) + g_t(N^t(X, Y), JZ)\\ &= g_t((\nabla^t_X J)Y, Z) + g_t((\nabla^t_Y J)Z, X) + g_t((\nabla^t_X J)Y, Z).\end{aligned}$$

Since $g_t$ is almost Kähler, Equation (2.7) implies that

$$g_t((\nabla^t_X J)Y, Z) + g_t((\nabla^t_Y J)Z, X) + g_t((\nabla^t_X J)Y, Z) = 0.$$

Hence,

$$\hat{\bigoplus}_{XYZ}\Omega_t(N^t(Y, Z), X) = 0,$$

as required. □

**Remark 3.2.2.** Theorem 3.2.3 and Proposition 3.2.1 implies that

$$\hat{\bigoplus}_{XYZ}\Omega_t(N(Y, Z), X) = 0.$$

# Curvature criteria for locally conformal almost Kähler manifolds

This chapter is devoted to the curvature properties of locally conformal almost Kähler manifolds. We give a characterization of locally conformal almost Kähler manifolds in terms of the Ricci curvatures $\tau^t$ and $\tau$ together with the corresponding Ricci $*$-curvatures. Using a $(0,2)$ tensor $P$ which exist on every locally conformal almost Kähler manifold with curvature $R$ (see [19], for more details), we establish a condition for a locally conformal almost Kähler manifold to be an almost Kähler manifold. On the last section, we will consider the canonical foliations of locally conformal almost Kähler manifolds.

## 4.1 Curvature relations of locally conformal almost Kähler metrics

In general, the curvature is used to determine the curve direction changes on a sufficiently small distance from one point to another on any curve. However, for higher dimensions (greater than 2), it is not easy to determine a single number at any given point. As a consequence, Riemann invented a way to define the curvature in higher dimensions.

Let $(M, J, g)$ be a $2n$-dimensional almost Hermitian manifold. Here we keep the formalism of local transformations and others formulas defined in the previous chapter.

For the Riemann curvature $R$ of a metric $g$, we use the following convention

$$R(X, Y, Z, W) = g(R(X, Y)Z, W), \tag{4.1}$$

where

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{4.2}$$

for any vector field $X$, $Y$ and $Z$ on $M$.

Let $\{E_i\}_{1\le i\le 2n}$ be the orthonormal basis with respect to $g$. The Ricci curvature tensor $\rho$ and the scalar curvature $\tau$ are given by

$$\rho(X,Y)=\sum_{i=1}^{2n}R(E_i,X,Y,E_i) \text{ and } \tau=\sum_{i=1}^{2n}\rho(E_i,E_i). \tag{4.3}$$

Now we consider the Ricci $*$-curvature tensor $\rho^*$ and the scalar $*$-curvature $\tau^*$ defined by

$$\rho^*(X,Y)=\sum_{i=1}^{2n}R(E_i,X,JY,JE_i) \text{ and } \tau^*=\sum_{i=1}^{2n}\rho^*(E_i,E_i). \tag{4.4}$$

Similarly, the curvatures corresponding to the metric $g_t$ will be denoted by $R^t$, $\rho^t$, $\tau^t$, $\rho^{t*}$ and $\tau^{t*}$, respectively.

**Lemma 4.1.1.** *Let $(M,J,g)$ be a locally conformal almost Kähler manifold. Then the curvature tensors $R^t$ and $R$ with respect to the metrics $g_t$ and $g$, respectively, are related as*

$$\begin{aligned}R^t(X,Y)Z&=R(X,Y)Z+\frac{1}{2}\left\{(\nabla_Y\omega)Z+\frac{1}{2}\omega(Y)\omega(Z)\right\}X\\&-\frac{1}{2}\left\{(\nabla_X\omega)Z+\frac{1}{2}\omega(X)\omega(Z)\right\}Y+\frac{1}{2}g(Y,Z)\left\{\nabla_XB+\frac{1}{2}\omega(X)B\right\}\\&-\frac{1}{2}g(X,Z)\left\{\nabla_YB+\frac{1}{2}\omega(Y)B\right\}-\frac{||B||^2}{4}\left\{g(Y,Z)X-g(X,Z)Y\right\},\end{aligned} \tag{4.5}$$

*where $||B||^2=g(B,B)$.*

*Proof.* Using the convention in (4.2) for the curvature tensors $R^t$ and $R$ and the relation (3.7), and for any vector fields $X$, $Y$ and $Z$ on $M$, the expressions

$$\begin{aligned}\nabla^t_X\nabla^t_YZ&=\nabla_X\nabla_YZ-\frac{1}{2}\omega(X)\nabla_YZ-\frac{1}{2}\omega(\nabla_YZ)+\frac{1}{2}g(X,\nabla_YZ)B\\&-\frac{1}{2}X(\omega(Y))Z-\frac{1}{2}\omega(Y)\nabla_XZ+\frac{1}{4}\omega(X)\omega(Y)Z+\frac{1}{4}\omega(Y)\omega(Z)X\\&-\frac{1}{4}\omega(Y)g(X,Z)B-\frac{1}{2}X(\omega(Z))Y-\frac{1}{2}\omega(Z)\nabla_XY+\frac{1}{4}\omega(X)\omega(Z)Y\\&+\frac{1}{4}\omega(Y)\omega(Z)X-\frac{1}{4}\omega(Z)g(X,Y)B+\frac{1}{2}X(g(Y,Z))B+\frac{1}{2}g(Y,Z)\nabla_XB\\&-\frac{1}{4}||B||^2g(Y,Z)X,\end{aligned} \tag{4.6}$$

Also,

$$\begin{aligned}\nabla^t_Y\nabla^t_X Z &= \nabla_Y\nabla_X Z - \frac{1}{2}\omega(Y)\nabla_X Z - \frac{1}{2}\omega(\nabla_X Z) + \frac{1}{2}g(Y,\nabla_X Z)B\\ &- \frac{1}{2}Y(\omega(X))Z - \frac{1}{2}\omega(X)\nabla_Y Z + \frac{1}{4}\omega(Y)\omega(X)Z + \frac{1}{4}\omega(X)\omega(Z)Y\\ &- \frac{1}{4}\omega(X)g(Y,Z)B - \frac{1}{2}Y(\omega(Z))X - \frac{1}{2}\omega(Z)\nabla_Y X + \frac{1}{4}\omega(Y)\omega(Z)X\\ &+ \frac{1}{4}\omega(X)\omega(Z)Y - \frac{1}{4}\omega(Z)g(X,Y)B + \frac{1}{2}Y(g(X,Z))B + \frac{1}{2}g(X,Z)\nabla_Y B\\ &- \frac{1}{4}||B||^2 g(X,Z)Y. \qquad (4.7)\end{aligned}$$

It is worth noting that

$$\nabla^t_{[X,Y]}Z = \nabla_{[X,Y]}Z - \frac{1}{2}\omega([X,Y])Z - \frac{1}{2}\omega(Z)[X,Y] + \frac{1}{2}g([X,Y],Z)B. \qquad (4.8)$$

Putting the pieces (4.6), (4.7) and (4.8) together, one obtains

$$\begin{aligned}R^t(X,Y)Z &= R(X,Y)Z + \frac{1}{2}\left\{(\nabla_Y\omega)Z + \frac{1}{2}\omega(Y)\omega(Z)\right\}X\\ &- \frac{1}{2}\left\{(\nabla_X\omega)Z + \frac{1}{2}\omega(X)\omega(Z)\right\}Y + \frac{1}{2}g(Y,Z)\left\{\nabla_X B + \frac{1}{2}\omega(X)B\right\}\\ &- \frac{1}{2}g(X,Z)\left\{\nabla_Y B + \frac{1}{2}\omega(Y)B\right\} - \frac{||B||^2}{4}\left\{g(Y,Z)X - g(X,Z)Y\right\},\end{aligned} \qquad (4.9)$$

which completes the proof. □

Next, from the above Lemma, we define $(0,2)$-tensor field $P$ by

$$P(X,Y) = (\nabla_X\omega)Y + \frac{1}{2}\omega(X)\omega(Y) - \frac{1}{4}||B||^2 g(X,Y), \qquad (4.10)$$

and this trace is given by

$$\mathrm{trace}P = \mathrm{div}B - \frac{1}{2}(1-n)||B||^2. \qquad (4.11)$$

**Lemma 4.1.2.** *The $(0,2)$-tensor field $P$ is symmetric.*

*Proof.* For any vector fields $X$ and $Y$ on $M$ and since $\omega$ is closed, we have

$$\begin{aligned}P(X,Y) &= (\nabla_Y\omega)X + \frac{1}{2}\omega(X)\omega(Y) - \frac{1}{4}||B||^2 g(X,Y)\\ &= Y(\omega(X)) - \omega(\nabla_Y X) + \frac{1}{2}\omega(X)\omega(Y) - \frac{1}{4}||B||^2 g(X,Y)\\ &= Y(\omega(X)) - \omega([Y,X]) - \omega(\nabla_X Y) + \frac{1}{2}\omega(X)\omega(Y) - \frac{1}{4}||B||^2 g(X,Y)\\ &= (\nabla_X\omega)Y + \frac{1}{2}\omega(X)\omega(Y) - \frac{1}{4}||B||^2 g(X,Y),\end{aligned}$$

which completes the proof. □

The Lie derivative $g$ with respect to the vector field $B$ gives, for any vector fields $X$ and $Y$,

$$\begin{aligned}(L_B g)(X,Y) &= X(g(B,Y)) - g([B,X],Y) - g(X,[B,Y]) \\ &= (\nabla_X \omega)Y + (\nabla_Y \omega)X \\ &= 2(\nabla_X \omega)Y. \end{aligned} \tag{4.12}$$

The last equality of (4.12) follows from the fact that the smooth 1-form $\omega$ is closed.

**Definition 4.1.1.** *Let $\omega$ be a Lee form such that $\nabla\omega = 0$, then $\omega$ is said to be parallel with respect to $\nabla$ or it is $\nabla$-parallel.*

**Lemma 4.1.3.** *The dual vector field $B$ of $\omega$ preserves the metric $g$ if and only if the Lee form $\omega$ is $\nabla$-parallel.*

*Proof.* Suppose that the vector field $B$ preserves the metric $g$. Then by Equation (4.12), we have $(L_B g)(X,Y) = 2(\nabla_X \omega)Y = 0$. Hence, by Definition 4.1.1 the Lee form is $\nabla$-parallel. Conversely, if the Lee form $\omega$ is $\nabla$-parallel, then by Equation (4.12), we have $2(\nabla_X\omega)Y = (L_B g)(X,Y) = 0$, as required. $\square$

The Riemannian curvatures are related by, for any $X$, $Y$, $Z$ and $W$ on $M$,

$$\begin{aligned}\exp(f_t)R^t(X,Y,Z,W) = R(X,Y,Z,W) &+ \frac{1}{2}\left\{g(X,W)P(Y,Z) - g(Y,W)P(X,Z)\right\} \\ &+ \frac{1}{2}\left\{g(Y,Z)P(X,W) - g(X,Z)P(Y,W)\right\}. \end{aligned} \tag{4.13}$$

Let $\{E_i\}$ be the orthonormal basis with respect to $g$. Then, we have

$$g(E_i, E_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let $E_i^t = \exp(f_t)^{\frac{1}{2}} E_i$, for any $i = 1, 2, \cdots, 2n$. Therefore, we have the following.

**Lemma 4.1.4.** *The frame $\{E_i^t\}_{1 \le i \le 2n}$ is the orthonormal basis with respect to $g_t$.*

The following identities generalize the ones given in [19, p.216].

**Lemma 4.1.5.** *The Ricci curvature tensors $\rho^t$ and $\rho$ with respect to $g_t$ and $g$, respectively, are related by*

$$\rho^t(X,Y) = \rho(X,Y) + (n-1)P(X,Y) + \frac{1}{2}g(X,Y)\,\mathrm{trace}\,P. \tag{4.14}$$

*Proof.* Using the Lemma 4.1.4 and for any vector fields $X$ and $Y$ on $M$, one has

$$\begin{aligned}
\rho^t(X,Y) &= \sum_{i=1}^{2n} R^t(E_i^t, X, Y, E_i^t) = \sum_{i=1}^{2n} \exp(f_t) R^t(E_i, X, Y, E_i) \\
&= \sum_{i=1}^{2n} R(E_i, X, Y, E_i) + \frac{1}{2}\left\{\sum_{i=1}^{2n} g(E_i,E_i)P(X,Y) - \sum_{i=1}^{2n} g(X,E_i)P(E_i,Y)\right\} \\
&+ \frac{1}{2}\left\{\sum_{i=1}^{2n} g(X,Y)P(E_i,E_i) - \sum_{i=1}^{2n} g(E_i,Y)P(X,E_i)\right\} \\
&= \rho(X,Y) + (n-1)P(X,Y) + \frac{1}{2} g(X,Y)\operatorname{trace} P,
\end{aligned}$$

which completes the proof. □

Also, corresponding Ricci $*$-curvatures $\rho^{t*}$ and $\rho^*$ are related by

$$\rho^{t*}(X,Y) = \rho^*(X,Y) + \frac{1}{2}\left\{P(X,Y) + P(JX,JY)\right\}. \tag{4.15}$$

**Corollary 4.1.1.** *The scalar curvatures $\tau^t$ and $\tau$ are related by*

$$\exp(-f_t)\tau_t = \tau + (2n-1)\left\{\operatorname{div} B - \frac{1}{2}(1-n)||B||^2\right\}. \tag{4.16}$$

*Proof.* Using the Lemma 4.1.4, the scalar curvature $\tau^t$, we have

$$\tau^t = \sum_{i=1}^{2n} \rho^t(E_i^t, E_i^t) = \exp(f_t)\sum_{i=1}^{2n} \rho^t(E_i, E_i). \tag{4.17}$$

Then, applying Equation (4.14) into (4.17), we get

$$\begin{aligned}
\exp(-f_t)\tau^t &= \sum_{i=1}^{2n} \rho^t(E_i, E_i) \\
&= \sum_{i=1}^{2n} \rho(E_i,E_i) + (n-1)\sum_{i=1}^{2n} P(E_i,E_i) + n\operatorname{trace} P \\
&= \tau + (2n-1)\operatorname{trace} P \\
&= \tau + (2n-1)\left\{\operatorname{div} B - \frac{1}{2}(1-n)||B||^2\right\}.
\end{aligned}$$

Therefore,

$$\exp(-f_t)\tau^t = \tau + (2n-1)\left\{\operatorname{div} B - \frac{1}{2}(1-n)||B||^2\right\},$$

which completes the proof. □

Now if we consider a relation between the scalar $*$-curvature $\tau^{t*}$ and $\tau^*$, we get the following.

**Corollary 4.1.2.** *The scalar $*$-curvatures $\tau^{t*}$ and $\tau^*$ are related by*

$$\exp(-f_t)\tau^{t*} = \tau^* + \mathrm{div}B + (n-1)||B||^2. \tag{4.18}$$

*Proof.* The scalar $*$-curvature $\tau^{t*}$ is given by

$$\tau^{t*} = \sum_{i=1}^{2n} \rho^{t*}(E_i^t, E_i^t) = \exp(f_t) \sum_{i=1}^{2n} \rho^{t*}(E_i, E_i). \tag{4.19}$$

Now applying the relation (4.15) into (4.19), we compute

$$\begin{aligned}
\exp(-f_t)\tau^{t*} &= \sum_{i=1}^{2n} \rho^{t*}(E_i, E_i) \\
&= \sum_{i=1}^{2n} \rho^*(X, Y) + \frac{1}{2}\sum_{i=1}^{2n} \{P(E_i, E_i) + P(JE_i, jE_i)\} \\
&= \tau^* + \mathrm{div}B + (n-1)||B||^2.
\end{aligned}$$

Hence,

$$\exp(-f_t)\tau^{t*} = \tau^* + \mathrm{div}B + (n-1)||B||^2,$$

as required. □

Gray in [6] considered some curvature identities for Hermitian and almost Hermitian manifolds. Let $\mathcal{L}$ be the class of almost Hermitian manifolds as defined in [6]. Then the manifold under consideration is an element of the class $\mathcal{L}$. Now consider as in [6] the curvature operator $R^t$ of a locally conformal almost Kähler manifold $M$:

(1) $R^t(X, Y, Z, W) = R^t(X, Y, JZ, JW)$,

(2) $R^t(X, Y, Z, W) - R^t(JX, JY, Z, W) = R^t(JX, Y, JZ, W) + R^t(JX, Y, Z, JW)$,

(3) $R^t(X, Y, Z, W) = R^t(JX, JY, JZ, JW)$,

for any $X$, $Y$, $Z$ and $W$ on $M$.

The item (1) is called Kähler identity if $M$ is locally conformal Kähler manifold (see [6] for more details and reference therein).

We will focus, throughout the rest of this thesis, on the item (1). Items (2) and (3) are considered in a further work whose details are given in [10] (*in preparation*).

Using further notations as in [6], we denoted by $\mathcal{L}_i$ the subclass of manifolds whose curvature operator $R^t$ satisfies identity (i). Here (i) may be either the item (1), (2) or (3) above. As in [6], it is easy to see that

$$\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_3 \subseteq \mathcal{L}.$$

Therefore we have the following result.

**Lemma 4.1.6.** *If a locally conformal almost Kähler manifold is in a class $\mathcal{L}_1$, then the equality holds:*

$$\tau^* - \tau = 2(n-1)\operatorname{trace} P. \tag{4.20}$$

*Proof.* The proof follows from a straightforward calculation using the fact that, for any vector fields $X$ and $Y$ on $M$, we have

$$\begin{aligned}
\rho^t(X,Y) &= \sum_{i=1}^{2n} \exp(f_t) R^t(E_i, X, Y, E_i) \\
&= \sum_{i=1}^{2n} \exp(f_t) R^t(E_i, X, JY, JE_i) \\
&= \rho^{*t}(X,Y),
\end{aligned} \tag{4.21}$$

which leads to

$$(\rho^* - \rho)(X,Y) = (n - \frac{3}{2})P(X,Y) + \frac{1}{2}g(X,Y)\operatorname{trace} P - \frac{1}{2}P(JX,JY).$$

This completes the proof. □

The relation (4.21) leads to

$$\begin{aligned}
\tau^t &= \sum_{i=1}^{2n} \rho^t(E_i,E_i) = \sum_{i=1}^{2n}\sum_{j=1}^{2n} R^t(E_j,E_i,E_i,E_j) \\
&= \sum_{i=1}^{2n}\sum_{j=1}^{2n} R^t(E_j,E_i,JE_i,JE_j) = \sum_{i=1}^{2n} \rho^{t*}(E_i,E_i) \\
&= \tau^{t*}.
\end{aligned}$$

**Theorem 4.1.1.** *Let $(M,J,g)$ be a $2n$-dimensional compact locally conformal almost Kähler manifold with $n > 1$ and contained in $\mathcal{L}_1$. If*

$$\tau^* = \tau,$$

*then $(M,J,g)$ is an almost Kähler manifold.*

*Proof.* By Lemma 4.1.6, we have $\tau^* - \tau = 2(n-1)\operatorname{trace} P$, with $\operatorname{trace} P = \operatorname{div} B - \frac{1}{2}(1-n)||B||^2$. Taking into account this, integrating the relation (4.20) and using Green's Theorem, we have

$$0 = \int_M \{\tau^* - \tau\} = (n-1)^2 \int_M ||B||^2.$$

Hence, under our assumption, we obtain $B = 0$. Therefore $\omega = 0$ identically on $M$. Hence $(M.J,g)$ is an almost Kähler manifold. □

As an example for this Theorem, we have compact flat locally almost Kähler manifolds. For compact flat manifolds have been detailed in [3] and reference therein.

## 4.2 Lee form and canonical foliations

As mentioned in the previous Chapter, an almost Hermitian manifold $(M, J, g)$ with almost Kähler form $\Omega$ satisfying $d\Omega = \omega \wedge \Omega$ and $d\omega = 0$ is a locally conformal almost Kähler manifold.

Now let $(M, J, g)$ be a locally conformal almost Kähler manifold and assume that the Lee form $\omega$ is never vanishing on $M$. Then $\omega = 0$ defines on $M$ an integrable distribution, and hence a foliation $\mathcal{F}$, on $M$ (see [8] for more details and reference therein).

Let $D := \ker \omega$ be the distribution on $M$ and $D^\perp$ be the distribution spanned the vector field $B$.Then, we have the following decomposition

$$TM = D \oplus D^\perp, \tag{4.22}$$

where $\oplus$ denotes the orthogonal direct sum. By the decomposition (4.22), any $X \in \Gamma(TM)$ is written as

$$X = QX + Q^\perp X, \tag{4.23}$$

where $Q$ and $Q^\perp$ are the projection morphisms of $TM$ into $D$ and $D^\perp$, respectively. Here, it is easy to see that $Q^\perp X = \frac{1}{||B||^2}\omega(X)B$ and

$$X = QX + \frac{1}{||B||^2}\omega(X)B.$$

Let $\mathcal{F}$ be a foliation on a locally conformal almost Kähler manifold $(M, J, g)$ of codimension 1. The metric $g$ is said to be *bundle-like* for the foliation $\mathcal{F}$ if the induced metric on the transversal distribution $D^\perp$ is parallel with respect to the intrinsic connection on $D^\perp$. This is true if and only if the Levi-Civita connection $\nabla$ of $(M, J, g)$ satisfies (see [?] and [20] for more details):

$$g(\nabla_{Q^\perp Y} QX, Q^\perp Z) + g(\nabla_{Q^\perp Z} QX, Q^\perp Y) = 0, \tag{4.24}$$

for any $X$, $Y$, $Z \in \Gamma(TM)$. If for a given foliation $\mathcal{F}$, the Riemannian metric $g$ on $M$ is bundle-like for $\mathcal{F}$, then we say that $\mathcal{F}$ is a *Riemannian foliation on* $(M, J, g)$.

Let $\mathcal{F}^\perp$ be the orthogonal complementary foliation generated by $B$. Now we provide necessary and sufficient conditions for the metric on an locally conformal almost Kähler manifold to be bundle-like for foliations $\mathcal{F}$ and $\mathcal{F}^\perp$. Therefore

**Theorem 4.2.1.** *Let $(M, J, g)$ be a locally conformal almost Kähler manifold and let $\mathcal{F}$ be a foliation on $M$ of codimension 1. Then the following assertions are equivalent:*

*(i) The foliation $\mathcal{F}$ is Riemannian.*

*(ii) The Lee vector field $B$ is auto-parallel with respect to $\nabla$, that is,*

$$\nabla_B B = B\left(\ln(||B||)\right) B.$$

*Proof.* For any $X$, $Y$, $Z \in \Gamma(TM)$, we have $Q^{\perp}Y = \frac{1}{||B||^2}\omega(Y)B$, $Q^{\perp}Z = \frac{1}{||B||^2}\omega(Z)B$ and the left-hand side of (4.24) gives

$$g(\nabla_{Q^{\perp}Y}QX, Q^{\perp}Z) + g(\nabla_{Q^{\perp}Z}QX, Q^{\perp}Y) = \frac{2}{||B||^2}\omega(Y)\omega(Z)\omega(\nabla_B QX),$$

for which the equivalence follows. □

Let $M'$ be a leaf of the distribution $D$. Since $M'$ is a submanifold of $M$ and for any $X$, $Y \in \Gamma(TM')$, we have

$$\nabla_X Y = \nabla'_X Y + \alpha(X, Y), \tag{4.25}$$

$$\nabla_X B = -A_B X + \nabla'^{\perp}_X B, \tag{4.26}$$

where $\nabla'$ and $\alpha$ are the Levi-Civita connection and the second fundamental form of $M'$, respectively. Here $A_B$ is the shape operator with respect to $B$. On the other hand, we have $g(\nabla_X B, B) = X(\omega(B)) - g(\nabla_X B, B)$, hence

$$g(\nabla'^{\perp}_X B, B) = \frac{1}{2}X(\omega(B)),$$

for any $X \in \Gamma(TM')$. Therefore, the Weingartem formula (4.26) becomes

$$\nabla_X B = -A_B X + \frac{1}{2}X(\omega(B))B. \tag{4.27}$$

**Proposition 4.2.1.** *Let $(M, J, g)$ be a locally conformal almost Kähler manifold. Then, the mean curvature vector field $H'$ of the leaves of the integrable distribution $D$ defined in (4.22) is given by*

$$H' = \frac{1}{2n-1}\left(\mathrm{div}_{|_{M'}} B\right) B.$$

*Moreover, these leaves are totally geodesic hypersurfaces of $M$ if and only if the dual vector field $B$ of $\omega$ preserves their metrics.*

*Proof.* Let $M'$ be a leaf of the integrable distribution $D$. Using (4.25) and (4.27), the second fundamental form of $M'$ gives

$$\alpha(X, Y) = g(A_B X, Y)B = g(\nabla_X B, Y)B,$$

for any $X$, $Y \in \Gamma(TM')$. Fixing a local orthonormal frame $\{e_1, \cdots, e_{2n-1}\}$ in $TM'$, one has,

$$H = \frac{1}{2n-1}\sum_{i=1}^{2n-1}\alpha(e_i, e_i) = \frac{1}{2n-1}\left(\mathrm{div}_{|_{M'}} B\right) B.$$

The last assertion follows and this completes the proof. □

The cosymplectic version of a such result was found by Massamba and Maloko Mavambou in [16, Theorem 3.8]. Therefore we have the following results.

**Corollary 4.2.1.** *Let $(M, J, g)$ be a locally conformal almost Kähler manifold. Then, the leaves $M'$ of the distribution $D$ in (4.22) are minimal if and only if the dual vector field $B$ is incompressible along $M'$.*

Chapter Five

# Conclusion and Perspectives

We have investigated the concept of conformality in almost Kähler structures which is characterized by a smooth 1-form $\omega$ on the underlying manifolds satisfying the following conditions:

$$d\Omega = 2\omega \wedge \Omega \text{ and } d\omega = 0.$$

Here $\Omega$ stands as the second fundamental 2-form.

We established the relation between the scalar curvatures and $\tau$ and $\tau^t$, together with the corresponding scalar $*$curvatures $\tau^*$ and $\tau^{t^*}$. We proved that under some conditions the ambiant manifold is part of the class of almost Kähler manifolds. Focusing on canonical foliations that arise in locally conformal almost Kähler manifolds, we also proved the geometric configuration of the Lee vector field depends on the bundle-like condition of the metric for a foliation $\mathcal{F}$. We showed that the leaves of this foliation are locally conformal almost Kähler hypersurfaces with a mean curvature vector

$$H' = \frac{1}{2n-1}\left(\mathrm{div}_{|_{M'}} B\right) B.$$

These leaves are totally geodesic hypersurfaces if and only if the dual vector field $B$ of the Lee form $\omega$ preserves their metrics. Moreover they are minimal as a submanifold immersed in the locally conformal almost Kähler manifolds if and only the dual vector field $B$ is incompressible along the leaves.

The relationship of locally conformal almost Kähler manifolds with other manifolds discussed in this work is summarized by the inclusion diagram below.

$$\begin{tikzcd}
& & \text{Kähler} \arrow[d] \arrow[lldd] \arrow[rrdd] & & \\
& & \text{Symplectic} & & \\
\text{Locally conformal Kähler} \arrow[rrd] \arrow[rrdd] & & & & \text{Almost Kähler} \arrow[llu] \arrow[lld] \arrow[lldd] \\
& & \text{Locally conformal symplectic} \arrow[uu] & & \\
& & \text{Locally conformal almost Kähler} \arrow[u] & &
\end{tikzcd}$$

One of the principal problems in the geometry of the manifold under consideration is to classify those admitting some (almost) Kähler metric. Even though we have obtained the results in terms of curvature properties, it appears that there are a lot of ways to approach this problem. For instance, in the case of locally conformal Kähler manifolds, I. Vaisman [23] proved that compact locally conformal Kähler manifolds which admit some global Kähler metric are globally conformal Kähler. We wish to test this assertion in the case of locally conformal almost Kähler manifolds and derive more conditions. Our approach would be to start by studying topological properties of almost Kähler manifolds.

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