AN ANALYSIS OF ERRORS MADE BY LEARNERS IN SOLVING LINEAR EQUATIONS AT GRADE 9 LEVEL IN SOUTH AFRICA

by

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DECLARATION

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This study commenced from June 2022 to November 2024 under supervision of Professor Sarah Bansilal of the School of Education, Mathematics and Computer Science Education Cluster, University of KwaZulu-Natal, Edgewood Campus.

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LIST OF ABBREVIATIONS

CAPS : Curriculum and Assessment Policy Statements

DBE : Department of Basic Education

FET : Further Education and Training

GET : General education and training

INT : Interviewer

L : Learner

LHS : Left Hand Side

NSC : National Senior Certificate

NSCEDR : National Senior Certificate Examination Diagnostic Report

RHS : Right Hand Side

TIMSS : Trends in International Mathematics and Science Study

ABSTRACT

The South African and Curriculum Assessment Policy Statements (CAPS) document, for the General Education and Training Phase (GET) Mathematics Grades 7 – 9 (2011) shows that learning of algebra permeates secondary school mathematics. As a result, for learners to achieve success in mathematics in secondary school, they need to become proficient at algebra, particularly in solving linear equations. It is thus imperative for mathematics educators to identify areas where learners have difficulty understanding and applying their knowledge to solving linear equations.

This study aims to explore the common errors made by Grade 9 learners in solving linear equations, and to interrogate possible reasons for these errors. 155 Learners from high schools in the Ntuzuma area of KwaZulu–Natal, South Africa, participated in the study. A qualitative research approach was used to analyse errors in solving linear equations. This was followed by the qualitative phase, in which a 12-item activity sheet was administered to 155 learners. Learners' common errors were identified and grouped according to error type. Learners were then selected to participate in semi-structured interviews, based on identified common errors. The interviews focused on those errors with a view to establishing the possible reasons for making them.

The study identified six common errors in solving linear equations which are : Based on these findings, it is recommended that further research be done to examine the extent to which current teaching approaches affect students' error formulation, as well as teachers' pedagogical content knowledge in teaching linear equations. Furthermore, in teaching linear equations, greater attention should be given to cognitive shifts required by learners in solving equations with letters on both sides of the equal sign.

KEY TERMS

Linear Equations; Algebra; Errors; Misconceptions; Grade 9 mathematics

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CHAPTER 1

INTRODUCTION

1.1 OVERVIEW

In this chapter, the background of and purpose for conducting this research study are presented. The aim, objectives and research questions in the study are introduced, followed by the significance of the results of the study. The anticipated benefits in the South African mathematical landscape are also highlighted. Later in this chapter, key terms and summaries of successive chapters are presented.

1.2 BACKGROUND

Algebra is a powerful tool with applications in many subjects, as well as in real life. It can be seen as a universal language in which global discussions can take place – with prescribed notations and rules. We need to learn these rules to communicate effectively using this language. One common algebra application is that of modelling the relationship between quantities and variables. As stated by Makonye (2015, p. 289), “algebra forms the basis of high school mathematics and various topics and concepts are dependent on it.” Seng (2010, p. 10) opine that “algebra is used generally in all branches of mathematics and science.” According to Star et al. (2015), ‘‘understanding algebra is a key for success in future mathematics courses, including geometry, trigonometry and calculus.” Many mathematics experts also consider algebra knowledge and skills important for post-secondary success as well as for producing a skilled workforce for scientific and technical careers.

Grade 12 learners need to master algebra to succeed in mathematics in South Africa. The South African National Senior Certificate (NSC) Mathematics Paper One consists mainly of algebra, with sections of examination including Algebra & Equations; Functions & Graphs; and Differential Calculus. These topics are categorised as being primarily algebraic and contribute about 32% of the entire Mathematics Paper One. It is significant to note that proficiency in algebraic is not limited to Paper One; Mathematics Paper two, also requires that learners have sound background knowledge of algebra as most problems require its application. For example, in Euclidean Geometry and Trigonometry, problems involving proving riders, proving the identities and finding the general solutions of trigonometric equations, require competence in the algebraic concepts of adding/subtracting like terms as well as factorisation of quadratics.

Understanding and learning to apply basic algebraic concepts is essential to the development of the mathematics skills learners require as they progress in their schooling. This reflects the position of the DBE (2011a, p. 10), which emphasises that, In the FET Phase, learners should be exposed to mathematical experiences that give them many opportunities to develop their mathematical reasoning and creative skills in preparation for more abstract mathematics in Higher/Tertiary Education institutions.'' The challenge for learners is that algebra needs to be learnt as a language, with its own notation system and requires knowledge of basic rules, such as $x + x = 2x$ and $x \times x = x^2$. Learners who are new to algebra may find these rules random and confusing, leaving them feeling disconnected from the subject. Research into the learning of early algebra (Booth, 1988; Kieran, 1992; Herscovics & Lincheviski, 1994; Bell, 1995; Lincheviski & Herscovics, 1996) suggests that algebra is viewed as a source of confusion and is regarded as a common problem section for learners. Kieran (1992) considers students' inability to acquire an in-depth sense of the structural aspects of algebra to be the main obstacle to understanding algebra. The main challenge in algebra is that learners fail to grasp fundamental algebraic concepts, resulting in difficulties in mathematics generally.

A key aspect of algebra is solving equations, which are mathematical statements showing that two mathematical expressions are equal. For instance, $2x + 1 = 7$ is an equation, in which $2x + 1$ and 7 are two mathematical expressions separated by an 'equal sign'. To solve an equation, one needs to find the value of the variable that makes the equation true. A linear equation is one in which the highest power of the variable is 1, for example $2x + 3 = 5$. By contrast, a quadratic equation is one in which the highest power of variables is 2. An example of this is $2x^2 + 5x + 3 = 0$.

Knowledge and understanding of algebra is crucial to solving equations in mathematics. Solving linear equations is part of algebraic activity that Kieran (2004) describe as 'transformational algebra'. This may involve working with different equivalent expressions, for example, transforming the equation $3(x + 3) + 5(x - 3) = 10$ into $8x = 16$, enabling one to find the solution to the equation, namely $x = 2$. Another example may involve transforming $2x^2 + 5x + 3 = 0$ into $(2x + 3)(x + 1) = 0$ by using the factorised form of the quadratic expression.

"Algebra includes many examples of transformational activity which are integral to a wide variety of algebraic, geometric and trigonometric problems." (Hall, 2002, p. 4). For example, algebra may be used to solve an equation to determine the size of angles in a triangle, or the lengths of the sides of a triangle. A linear equation may be used to find the simple interest rate in financial mathematics, or to find the number of terms in a number pattern added to get a particular total. The general solutions of trigonometric equations also require competence in linear equations.

In Grade 9 and later, linear equations serve as the basis for various types of equations. According to Hall (2002), linear equations are employed to address issues related to the physical science (or "physics") in particular, in addition to other sciences and social sciences. For instance, density can be expressed using the straightforward linear equation = M/V, which is defined as mass divided by volume. The foundation of physics is made up of these and other homogeneous equations, such as S = D/T (speed, distance, and time), I = W/V (power equation for determining fuses in an electrical circuit), and R = V/I (Ohm's Law). These kinds of equations effectively use mathematics as a tool by requiring the use of mathematical knowledge to address real-world situations.

1.3 MOTIVATION AND RATIONALE FOR THE STUDY

Having taught high school mathematics extensively, I have observed certain common mistakes learners make. The nature of these mistakes indicates that learners are struggling with algebra in particular. Ncube (2016, p. 1) opines that "Learners struggle to manipulate algebraic concepts according to accepted rules, procedures or algorithms. This, in turn, affects their performance in Mathematics, as success in this subject is largely affected by understanding of concepts in algebra."

The national pass rate for the class of 2023 was 82.9%, an increase of 2.8% from 80.1% in 2022. However, in 2023, the National Senior Certificate Examination Diagnostic Report (NSCEDR, 2023, hereinafter referred to as 'the NSCEDR') indicated that grade 12 learners had performed poorly in mathematics (Table 1.1) when compared with other gateway subjects. An analysis of the performance in gateway subjects from 2020 to 2023 shows that mathematics, of which algebra is a major part, was the lowest-performing subject with a pass rate of 53.8% in 2020, 57.6% in 2021, 55.0% in 2022 and 63.5% in 2023.

Table 1.1: Comparison of Performance in Gateway Subjects – 2020 to 2023

Subject Description	2020	2021	2022	2023
Accounting	75.5	74.7	75.4	76.8
Agricultural Sciences	72.7	75.4	75.8	80.5
Business Studies	77.9	80.5	76.7	81.8
Economics	68.8	67.9	71.5	74.5
Geography	75.3	74.3	81.3	86.2
History	92.1	89.5	88.2	87.7
Life Sciences	71.0	71.5	71.5	75.6
Mathematical Literacy	80.8	74.5	85.7	82.3
Mathematics	53.8	57.6	55.0	63.5
Physical Sciences	65.8	69.0	74.6	76.2

Table 1.2: Overall Achievement Rates in Mathematics

Year	No. Wrote	No. achieved at 30% and above	% achieved at 30% and above	No. achieved at 40% and above	% achieved at 40% and above
2019	222 034	121 179	54.6	77 751	35.0
2020	233 315	125 526	53.8	82 964	35.6
2021	259 143	149 177	57.6	97 561	37.6
2022	269 734	148 346	55.0	97 041	36.0
2023	262 016	166 337	63.5	114 311	43.6

Table 1.2 indicates a significant improvement in the mathematics pass rate in 2023, with candidates who passed at the 30% level increasing from 55% in 2022 to 63.5%. Correspondingly, the pass rate at the 40% level increased from 36% to 43.6% over the past two years. This achievement is, however, diminished considering that the number of learners who wrote the mathematics examination in 2023 decreased by 7 718 compared to 2022.

In this regard, “it is important to note that the South African education system, makes mathematics compulsory for all learners up to grade 9; thereafter, learners can either continue with mathematics or take up mathematical literacy.” (Mbatha, 2022, p. 6) It appears that the number of learners taking mathematical literacy is increasing, while those taking pure mathematics is declining. This is probably due to the preconception that pure mathematics is more difficult than mathematical literacy.

According to NSCEDR (2023) summary of candidates' performance in Grade 12 papers 1 and 2, candidates' algebraic abilities were lacking. The majority of applicants lacked foundational and elementary mathematical skills that they ought to have learned in lower school. When responding to complex questions, this becomes a barrier. The report goes on to say that learners should be careful when using algebraic manipulation techniques because failing to follow some fundamental guidelines can result in the unnecessary loss of marks (NSCEDR, 2023).

These observations triggered my interest in understanding the errors learners make when attempting to solve linear equations, leading me to embark on this study. Gaining a solid understanding of algebra in Grade 9 is vital for learners' mathematical understanding, as algebra forms the basis for mathematics and mathematical literacy in subsequent years. Luneta & Makonye (2010) emphasise the importance of teachers' handling of situations in which learners make errors or form misconceptions. Teachers must understand the nature of errors and the areas where learners face challenges in order to address them productively as a teaching resource.

1.4AIMS, OBJECTIVES AND RESEARCH QUESTIONS

The overall aim of the study is to explore the common errors made by Grade 9 learners in solving linear equations. The specific objectives are

1. To identify common Grade 9 learners' errors in solving linear equations; and
2. To explore the possible reasons for these errors.

The underlying research questions are:

1. What errors do Grade 9 learners commonly attempting to solve linear equations?
2. What are the possible reasons for these errors?

1.5 SIGNIFICANCE OF THE STUDY

As mentioned, Grade 9 algebra learning, in particular linear equations, forms the basis for mathematics and mathematical literacy in the years that follow. To support learners in gaining a strong basic foundation in algebra, teachers must better understand the specific areas learners find challenging at this level. It is hoped that this study will provide valuable insights into these challenges, and contribute to teachers' understanding of problem areas so that they are better equipped to support learners.

It is also suggested that when teachers interpret learners' errors as the latters' rational and meaningful attempts to comprehend and cope with new mathematical concepts, rather than simply making silly mistakes, teachers can alter their approach to dealing with these challenges.

Errors can be viewed as opportunities to develop more effective teaching methods and strategies.

On a more targeted level, it is hoped that this study will provide insight to assist the KwaZulu-Natal Department of Education in their curriculum planning, and to assist mathematics subject advisors plan their strategies.

1.6 DEFINITION OF KEY TERMS

Algebra

Algebra can be seen as a universal language in which global discussions can take place – with prescribed notations and rules. These rules are to be learned in order to communicate using this language. Sfard (1995, p. 18) defines algebra as a "science of generalised computations", using the term with respect to any kind of mathematical endeavour concerned with generalised computational processes, whatever the tools used to convey this generality. In mathematics, something that is said to happen 'in general' if it always happens. A central part of outcome for Patterns, functions and algebra in the curriculum "is for learners to achieve efficient manipulative skills in the use of algebra." (DoE, 2002, p. 88)

Misconceptions

Hansen et al., (as cited in Mamba, 2012, p. 16) acknowledges a misconception as "the misapplication of a rule, an over- or under-generalization, or an alternative conception of a situation." A misconception is frequently exposed when a teacher is alerted to a persistent mistake committed by a learner/s in reaction to a certain circumstance. According to Barmby et al. (2009), misconceptions are unavoidable since they arise naturally and the errors they lead to are generally predictable.

When solving linear equations, inverse operations are sometimes applied as a technique to isolate the unknown so as to generate a solution for the equation. Hall (2002) used the term *inverse errors* when learners use the incorrect inverse operation. For example, given $4x = 9$, a learner may subtract 4 from both sides instead of dividing by 4, resulting to $x = 5$ as a solution. We can see that the additive inverse was used rather than multiplicative inverse. In Arithmetic $4+\frac{1}{2}=4\frac{1}{2}$ but in algebra $4+x\neq 4x$. The experiences in arithmetic may be wrongly applied and $4x = 9$ is incorrectly taken as the same as $4 + x = 9$, hence the learners use the additive inverse rather than the multiplicative inverse.

Errors

Mamba (2012, p. 16) defines errors as “mistakes, slips, blunders, oversight or deviations from accuracy that are unintended and show no pattern in their occurrence.” Learners may commit errors when they fail to apply algebraic rules or are unable to connect new topics successfully with their existing knowledge. Errors made by learners may be classified as systematic, involving a repeated incorrect process of thinking, usually linked to misconceptions. By contrast, non – systematic errors occur where errors are random, with no evidence of a recurring incorrect way of thinking.

Below is the written solution demonstrating non-systematic errors taken from work of Lima and Tall (2007):

$$\begin{aligned}&3x - 1 = 3 + x\\&3x - 3x = +1\\&0 = \frac{1}{0} = 0\end{aligned}$$

In this excerpt if the learner has “shifted over” the – 1 on the left to give +1 on the right, however the $3 + x$ on the right has become $-3x$ on the left. The $3x - 3x$ was replaced by 0, then the ‘= +1’ was written as ‘= 1,’ and then ‘0’is written below. The solution was completed by writing an ‘answer’ equal to zero. Here there was no evidence in the response that the learners exhibited any understanding of the underlying mathematical structure.

Equation

An equation describes the numeric relationship between quantities. The following example is instructive: there are two boxes of balls, and one box contains 10 more balls than the other. If x indicates the number of balls in the first box, there are $x + 10$ balls in the second box. To find the total number of balls, we use an equation: total = $x + (x + 10)$. If we know that there are a total of 46 balls in both boxes, the number of balls per box can be determined by modifying the equation to $2x + 10 = 46$. Solving this equation, we get $x = 18$. This indicates that there are 18 balls in one box and 28 in the other. Sanders (2017) define an equation as created when two expressions are equated to each other.

A linear equation is where the highest power of the variable is 1.

1.7 DISSERTATION OUTLINE

The study is divided into five chapters.

The first chapter provides the background of the study, the statement of the problem, and study's aims and objectives, as well as research questions used. The significance of the study is also set out in this chapter.

Chapter Two sets out the basis of the study by examining previous research undertaken in the field of algebra. It discusses teaching and learning of algebra and linear equations, exploring research on errors and misconceptions related to solving linear equations. In particular, it discusses pertinent literature regarding the theoretical framework of constructivism underpinning the study. Literature review is used to assist in identifying and describing various categories of errors apparent in learning to solve linear equations.

Chapter Three discusses the methodological construct of the research. It details the paradigm, research approach, research style, sampling procedures, data collection and analysis procedures used. It also presents the research instruments, validity and reliability issues, and ethical considerations involved.

Chapter Four presents the analysis of data collected. The different errors are summarised, quantified and illustrated using learner responses. In certain instances, supporting evidence from interviews is used.

Chapter Five focuses on the discussion of the findings of the study. It provides answers to the two research questions stated above. The findings on the errors identified, which are related to the use of formal/informal methods of solving linear equations, variables and negative signs, are discussed. This chapter also discusses possible reasons for the errors identified. Finally, the chapter discusses the study's limitations and presents recommendations based on the insights gained.

1.8 CHAPTER SUMMARY

This chapter has outlined the background of the study, the statement of the problem, the aim of the study and its accompanying objectives, together with the applicable research questions.

The chapter has also set out the significance of the study and provided the chapter outline. The next chapter discusses a review of the related literature with a view to gaining a deeper understanding of the problem, and theoretical framework.

CHAPTER 2

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

2.1 INTRODUCTION

Numerous studies have explored the teaching and learning of mathematics, with particular emphasis on algebra and equations. Some of this research highlights the various misconceptions or preconceptions learners bring to the classroom, often shaped by their everyday experiences. These preconceived notions can sometimes hinder their understanding of mathematics which is the reason why I undertook this study.

This chapter begins by examining problems in learner performance in mathematics in South Africa. Following this is a discussion of some of the challenges faced with a particular focus on algebra, before focusing on South African studies in this area. The study moves on to an overall review of issues related to the learning and teaching of linear equations and thereafter a discussion of the notion of errors and misconceptions. Finally, the theoretical framework of constructivism, which informs the way in which errors and misconceptions are viewed in this study, is discussed. The framework used uses existing literature to identify and describe various categories of errors that occur in learning to solve linear equations.

2.2 LEARNERS PERFORMANCE IN MATHEMATICS IN SOUTH AFRICA

In the South African education system, mathematics is valued for its role in preparing learners for real-life challenges, empowering them to become active global citizens and to enter science or mathematical-related careers, should they so choose.

However, “the poor mathematics performance of Grade 12 learners has long been a concern to mathematics teachers, subject advisors, district senior managers and the nation at large.” (Ncube, 2016, p. 2). According to the NSCEDR (2023), Grade 12 mathematics results reflect a pass rate of less than 65% from 2020 to 2023. Moodley (2014) notes that South Africa’s mathematical performance is significantly inferior to the rest of the globe.

Focus on Grade 9 learners reveals a concerning performance in both national and international assessments. In the Annual National Assessment (ANA) tests for Grade 9, the pass rates were 13% in 2012, 14% in 2013, and 10% in 2014. These low pass rates suggest that Grade 9 learners are generally struggling with the mathematics that they have been taught.

The Trends in International Mathematics and Science Study (TIMSS) is an assessment of mathematics and science knowledge up to Grade 8 level in various countries worldwide. However, South Africa opted to use Grade 9 learners for these assessments because, according to Spaull (2013), the assessments were too difficult for the country's Grade 8 learners. Notwithstanding this, the TIMSS 2011 results showed that South Africa's (Grade 9) learners were still amongst the lowest-performing countries, which included Botswana and Honduras (Reddy, 2012). Singapore (average score of 621), the Republic of Korea (606), Chinese Taipei (599), Hong Kong SAR (594), and Japan (587) were the top five countries in terms of TIMSS performance in 2015. The five countries with the worst results were Saudi Arabia (368), South Africa (372), Morocco (384), Jordan (386), and Botswana (391). Out of these five countries, Botswana was the only one whose score differed from South Africa's in a statistically significant way (Vilakazi, 2021). These studies emphasise the poor outcomes in mathematics which has motivated me to conduct the study to learn more about why learners perform poorly in mathematics, particularly in algebra.

2.3 CHALLENGE OF LEARNING MATHEMATICS (SPECIFICALLY ALGEBRA)

Learners' performance in mathematics globally and locally is a major concern. Over the past 30 years, there has been extensive effort directed at, "firstly, understanding children's difficulties in mathematics and, secondly, examining ways of tackling these difficulties. Despite this, there is no evidence that this work has had any significant effect in improving either attainment or engagement in mathematics." (Hodgen et at. 2018, p 36) Acharya's (2017) study on the factors affecting difficulties in learning mathematics, identifies several key issues. These include teachers' educational backgrounds and their failure to connect mathematical concepts with previously learned mathematics structures. According to Bansilal et al., (2014, p. 3) "many studies point to the problem of poor mathematics content knowledge of teachers." Lack of regular school assessment and unfavourable economic conditions were also identified as contributing to difficulties in learning mathematics (Acharya, 2017).

Despite the essential role it plays in the development of many mathematics concepts, learners continue to have considerable difficulties with algebra and multiplicative reasoning in particular (Brown et al., 2008; Watson, 2007). Algebra instruction and learning have long been viewed as challenging, and learners face comparable difficulties in all grade levels, which has caused significant worry among all parties involved in the mathematics community (Lima & Tall, 2006). According to Mhakure et al., (2014), many learners find algebra to be one of the most difficult mathematical subjects. Nonetheless, it is crucial that learners get an understanding of algebra as without this algebraic ability, they would not be able to see its potential.

A major part of mathematics is generalising from the specific instance to wider cases. The power of algebra is that it has tools which allow us to express general relationships using the notion of variables. The table and explanation below illustrate how generalisation can be useful.

The results of an experiment that measured two quantities *x* and *y* were:

x	*Y*
2	6
3	9
4	12
5	15

Upon careful examination of this table, a pattern appears: $2 \times 3 = 6$, $3 \times 3 = 9$, $4 \times 3 = 12$ and $5 \times 3 = 15$. We can express this pattern verbally by saying that to arrive at the quantity in the second column, one needs to take the corresponding quantity in the first column and multiply it by 3. As such, the first column represents different values of the variable *x* and, each of them is related to the corresponding value of the second variable in the second column. In each case, the value of y is 3 times the value of *x*. Using algebra, we could express this relationship as $y = 3x$; this formula describes what is seen in the table. It is a generalisation of the pattern that appears in the table, set out in a simple and elegant manner. Expressing generalisations such as this lies at the heart of algebra.

Another perspective of algebra is that it can be seen as generalised arithmetic. In arithmetic, numbers and operations on numbers yield numbers. In algebra, variables and groups of variables are called algebraic expressions. There are rules which govern how operations on the variable or algebraic expressions can be carried out. The challenge for learners during the beginning stages of studying algebra is that it functions at a more general level than arithmetic, where the addition of whole numbers relates to physical movements. Learners are taught how to operate on these algebraic expressions using the rules of algebra to yield further algebraic expressions. According to Foster (2007), the use of variables and signs makes algebra abstract and difficult. This is because algebraic ideas are based on general ideas rather than real facts or events (Ncube, 2016). Generalisations are made in all fields of mathematics including algebra, statistics, geometry, trigonometry, and calculus, etc.

Knowledge in algebra is built upon previously acquired arithmetic knowledge (Booth, 1988; Kieran, 1989); hence, if learners possess a good arithmetic background, they are less likely to face challenges in algebra. Conversely, difficulties in algebra are, in part, attributable to their lack of understanding of various structural notions in arithmetic. Their experience of arithmetic directly impacts learners' conceptions of algebra; ways of operating that have been seen to work at one time are brought to mind in attempting to make sense of a new context.

Macgregor and Stacey (1997) conducted a series of studies to investigate the origins of learners' misinterpretations of letter usage in algebra. They deduced that learners can ignore letters while some of them associate them with numerical values. For example, in arithmetic a child knows '3 + 5 = 8, but this fact may sometimes be incorrectly applied or expressed in algebra when they try to combine letters and numbers inappropriately. For instance, when writing $4 + 3n$, if a learner does not know what the variable n is, or that it represents another (as yet unknown) number, they may use experience from arithmetic to add together the recognisable part of the equation, namely the numbers 4 + 3, leaving the variable n as it is, to arrive at $7n$. This also illustrates the point that children believe the operation can never be left in an answer (Booth, 1995; Kieran 1992), and is indicative of the struggle learners face in using and performing operations with variables.

Some authors (Anderson, 1997; Lima & Tall, 2008) show clearly how experiences in arithmetic – such as rules that apply in fraction operations – may trigger an incorrect application in algebra. For example, the fraction bar in 12/6 is associated with the process of dividing the number at the top by the number at the bottom' which gives the correct answer 2. This experience can trigger the incorrect operation in algebra, in which learners attempt to solve the problem a^{12}/a^{6} to arrive at the erroneous result a^{2}. This example can be used to explain the construct of a "met-before" (Lima & Tall, 2008, p. 6), which is a "mental construct that an individual uses at a given time based in experiences they have met before". 'Met-befores' may influence current learning both negatively or positively and can cause problems when used outside their domain of validity (Bansilal & Pillay, 2014), as illustrated in the above example $a^{12}/a^6 = a^2$.

The section above has underscored the need to focus a study on underlying reasons why learners struggle with algebra which helps provide a background to this study.

The discussion now turns to South African studies which have focused on learners' difficulties in negotiating algebraic notation and procedures.

2.4 SOUTH AFRICAN STUDIES ON ALGEBRA

Algebra is a fundamental topic in the school mathematics syllabus around the world, as well as in South Africa. The five content areas of Numbers, Operations and Relationships; Patterns, Functions and Algebra; Space and Shape (Geometry); Measurement; and Data Handling comprise the framework of the South African mathematics curriculum statement. The content area of Patterns, Functions, and Algebra carries a 35% weighting of the year's work in Grade 9, making it the most prominent of the five areas. This renders it even more problematic that, according to Baidoo (2019), Marange & Adendorff (2021) and Pournara (2020), algebra is one of the topics that learners in South Africa most struggle with. Lourens (2022) notes that the content area, 'Patterns, functions and algebra', which aims to provide guidelines for the teaching of early algebra in South African early years classrooms, seems to be inadequate for the implementation of early algebra in early years classrooms.

A study by Musi (2023) examined common challenges in algebraic expression in Grade 10 mathematics. The findings revealed several common issues relating to algebraic expressions, including the gap between algebra and arithmetic; inability of learners to represent word expression in algebraic format; inability of learners to manipulate algebraic expressions; and improper use of mathematical vocabulary/expression. Additionally, it was noted that some challenges stemmed from issues relating to mathematics teachers, in particular, teachers' lack of pedagogical knowledge and their inability to explain algebraic concepts in-depth.

In a study examining beliefs around algebra, Van Laren & Moore-Russo (2014) explored the perceptions of South African teachers from historically disadvantaged backgrounds. The findings suggest that teachers may not focus on how algebra can be used to describe patterns or how it can be used to identify types of change, thus failing to emphasise the key concepts that would enable learners to take full advantage of the power of algebra. Instead, the authors noted that “teachers might focus on the rules and procedures for moving about symbols without addressing the underlying conceptual context or possible applications” (Van Laren & Moore-Russo, 2014, p. 45).

It is essential that learners develop a strong basic foundation of algebraic knowledge before teachers progress to new or more advanced topics. Darmayanti et al., (2023) found that when learners fail to master basic algebraic concepts, they tend to experience difficulties in understanding the fundamental concepts of all other algebra topics. In order to achieve algebra success in the secondary grades of mathematics teaching and learning, it is becoming more widely recognized that it is crucial to introduce algebra concepts and skills in the early years of mathematics education.

In a related area, a study, undertaken by Afonso (2019) aimed to explore how two different mathematics curricula, namely that of South Africa and that of Singapore, provided opportunity for the development of algebraic thinking in the foundation phase. The motivation behind selecting the two countries in question stems from South Africa’s consistently low mathematics performance ranking in the TIMSS studies compared to Singapore’s consistent high performance. The findings show that both curriculum documents define what is required in terms of patterns and algebra, including a section where the content is explained and serves as a guide for teachers. In regard to learner performance, the findings indicated that although South African learners have the potential to think algebraically, they are not always offered the opportunity to do so Afonso (2019). In regard to teachers, the findings highlighted the need for teacher professional development relating to early algebra and functional thinking. Two types of content knowledge were identified as areas in which teachers’ capabilities could be strengthened, namely specialised teacher content knowledge and horizon teacher knowledge. According to Hunter (2016), focusing on strengthening these areas allows teachers to select or design appropriate tasks and identify spontaneous opportunities within the class to develop learners’ algebraic thinking and understanding. This would assist learners in understanding how algebra applies in a real-world context, and how these activities benefit them in later years.

In looking more closely at learners’ algebraic thinking, a study conducted by Pournara et al., (2016) to investigate how the kinds of algebraic errors learners make change over time. The performance of a group of 250 South African learners was tracked across their Grade 9, 10, 11 years of schooling. The findings showed that learners in grade 11 still encountered difficulties with aspects of basic algebra. This reinforces comments made consistently in the annual NSCEDR regarding the performance of Grade 12 learners on the National Senior Certificate Paper 1. For example, in 2023, examiners reported that:

> The algebraic skills of the candidates are poor. Most candidates lacked fundamental and basic mathematical competencies which should have been acquired in the lower grades. This becomes an impediment to candidates when answering complex questions (Department of Basic Education Diagnostic Report, 2023, p. 213).

These comments again emphasise the importance of basic algebraic skills; without these skills, learners are unable to apply algebra to solve more complex problems in many of the strands of mathematics, such as trigonometry, functions, calculus, probability, geometry as well as coordinate geometry. Vilakazi (2021) noted that almost 32% of the National Senior Certificate Paper 1 is based on algebraic topics such as equations, functions, graphs and calculus. A large part of the remaining National Senior Certificate mathematics examinations also rely on the application of algebra in the various topics, some of which are mentioned above.

Challenges associated with basic algebra are also evident at higher institutions of learning. Msomi & Bansilal (2022) conducted a study on Analysis of Students' Errors and Misconceptions in Solving Linear Ordinary Differential Equations Using the Method of Laplace Transform. The study participants were 81 students enrolled in an engineering mathematics course at a University of Technology in South Africa. The results showed that students experienced most problems when working in the first few steps, owing to their lack of skill in manipulation of algebraic expressions.

In 2011, the UKZN School of Mathematics, Sciences and Technology Education reflected on the poor performance of students in their first-year mathematics module. Analysis of the findings indicates that students lack basic skills and knowledge, which are the prerequisite for the study of first-year university mathematics modules. These basic skills and knowledge were identified as belonging to the following mathematics sections: (1) algebra, (2) functions, and (3) reasoning using symbols and connectives (Maharaj et al., *2015*). This implies that students who have not studied algebra or who lack proficiency in algebra, struggle to complete particular courses (Vilakazi, 2021).

As shown in the preceding studies, algebra presents a particular problem to learners in South Africa, making it urgent for us to study possible reasons for this problem. This study is on the section in algebra which focuses on solving linear equations.

2.5 RESEARCH REGARDING LINEAR EQUATIONS

One of algebra’s most useful features is its ability to provide systematic methods and procedures for solving various equations. Without algebra, the world would be vastly different, as equations would likely be solved only through trial and error.

The development of algorithms to solve different kinds of equations marked a milestone in the development of algebra. Solving linear equations can be done simply and elegantly using the tools of algebra. However, in order to solve linear equations, one needs to know the rules and notation governing the simplification and factorisation of algebraic expressions.

Various studies (Lima & Tall, 2008; Lima, 2010; Pournara, 2020; Linchevski & Herscovics; 1996; Sfard & Linchevski, 1994) have focused on solving linear equations. As mentioned, learners who lack a strong foundation in basic algebraic concepts, and mathematics in general, experience equations as difficult and confusing. Their incomplete understanding leads to errors being made when trying to solve linear equations.

Linchevski and Herscovis (as cited in Maharaj, 2008) point out that many high school learners, face cognitive obstacles in perceiving an equation as a mathematical object on which they can perform operations. These obstacles include the meaning of the equal sign, interpreting the structure of equations and constructing meaning for formal solution procedures. Research evidence indicates that a structural conception of a given equation is ‘a prerequisite for the comprehension of the approach that must be used’ (Sfard & Linchevski, 1994:211). Kieran (1992) classifies approaches to solve linear equations as formal and informal, where the latter includes undoing or working backwards, as well as trial and error substitution. The former requires that an equation be treated as a mathematical object and includes transposing of terms, and the balance method, which involves performing the same operation on both sides of the equation. To use an appropriate algorithm, a learner needs to first analyse the structure of the equation.

The next section focuses on the interpretation of the structure of the equation, which is central to this study.

2.5.1 Interpreting the Structure of the Equation and Constructing Meaning for Formal Solution Procedures

In a study by Pournara (2020), Grade 9 learners' responses to problems on linear equations suggest that, although they can solve equations of the form $ax + b$ using informal methods, they struggle with equations where the unknown appears on both sides, such as $ax + b = cx + d$.

An equation such as $2x - 3 = 5$ can be solved arithmetically by posing a series of questions. Firstly, one might ask, 'What number gives you the answer of 5 when 3 is subtracted from it?' The answer is 8. A follow-up would be 'What number multiplied by 2 gives you 8?" The answer is 4, which is the solution to the equation. Here, it is easy to see (using working backwards, substitution or inspection) that the solution is 4. However, this approach cannot be applied to equations of the form $ax + b = cx + d$. Filloy and Rojano (1989), explain when solving equations of the form $ax + b = c$, the process involves only operations on numbers. By contrast, for equations that take the form $ax + b = cx + d$, it is necessary to manipulate the unknowns as well.

Filloy and Rojano (1989), add that there is an increasing level of difficulty when moving from solving equations with the unknown only on one side of an equation, to those where the unknown occurs on both sides of the equation. They refer to this transition as the 'Didactic Cut', (Filloy& Rojano, 1989, p. 22).

Some learners attempted this arithmetic approach to solve equations in the form $ax + b = cx + d$, seemingly not appreciating the need for a different approach to solve equations with letters on both sides (Pournara, 2020). According Pournara (2020), this suggests evidence of the didactic cut - that these learners are not operating on the terms with letters in order to solve the equations.

A study by Lima & Healy (2010) found that learners do not solve equations by undoing the operations, or by performing the same operation on both sides. Rather, Lima and Tall (2008) highlight the use of procedural embodiments, which involves learners 'passing a number to the other side of the equation' with the magic of 'changing signs' or 'putting (the number) underneath' [the number e.g. in the equation $5x=2$ becomes $x=2/5$]. Some learners using these procedural embodiments do not understand what the procedures mean or why they work. To illustrate this, Lima and Healy (2010), present the following extract from an interview with a learner in which he explains his solving method for $3x - 1 = x + 3$.

S:	*I copied 3x, passed x that was in the left side to the other side of the equals sign, and it was positive, it got negative*
R:	*Why?*
S:	*Because when you change sides, the sign gets changed.*
R:	*Let me ask you, why does the sign change?*
S:	*I don't know why, I just know that it changes.*

Extract (Lima & Healy, 2010, p 6)

The above extract shows that the learner is following specific rules which he has been taught in solving the equation. Teachers develop rules for solving equations without explaining how or why they work, and learners are expected to know and use them (Powell, 2012). Freitas (2002) found that procedures related to phrases such as 'change side, change sign' (referred to as 'symbol shifting') are usually meaningless to students and often result in mistakes. If learners were given the explanations for the rules, they may forget these and the conditions under which the rules apply, leading to incorrect applications.

If symbol shifting is not set within a meaningful context, the manipulation may be seen as moving symbols around with a kind of additional 'magic' to get the correct answer. Additionally, whereas symbol shifting is effective for simplifying expressions such as $4x + 3y + 2x$, where the term $2x$ can be shifted next to the $4x$ and combined to give $6x + 3y$, it fails for equations where shifting terms to the other side requires additional actions such as 'change signs' or 'divide by the number'.

The evidence from Lima and Healy's study (2010) does not indicate the presence of a didactic cut. Learners who solved the two equations ($5t – 3 = 8$ and $3x -1 = 3 + x$) use similar methods, and the difference between the number of learners who correctly solved $5t – 3 = 8$ (28 learners) and those who correctly solved $3x – 1 = 3 + x$ (22 learners) is minimal. If the learners had experienced the didactic cut effect, they would've found solving the second type of equation significantly more difficult than the first type (Lima & Healy, 2010).

2.5.2 The Limited Meaning of the Equal Sign

Another widely recognised source of errors in solving equations is the understanding of the equal sign. Mamba (2012) notes that a lack of understanding of the equal sign and its properties is a major problem that hinders learners from solving equations correctly. The equal sign is considered one of the core algebraic ideas needed for the successful transition from arithmetic to algebra (Essien & Setati, 2006; Knuth et al., 2008). "It is vitally important that the equal sign is viewed as a symbol of equivalence when solving equations." (Sanders, 2017, p. 19).

According to Vermeulen & Meyer (2017), the '=' symbol representing the mathematical equal sign is a crucial concept for learners to solve numerical and algebraic equations. The shift from operating on arithmetic and on linear expressions to working with linear equations involves a further change in meaning. Some authors (Kieran, 1981; Tall, 2007) report that 13-year-old learners use the equals sign not to indicate equivalence, but as an operation sign that entices them to perform the operation before the sign and put the solution after it. They tend to expect that the right hand side ('RHS') must be the answer, for example, $5 + 2 = 7$. Hence these learners perceive that decomposition, such as $9 = 4 + 5$, is written backwards (Hescovics & Lincheviski, 1994). When learners are only exposed to equations such as $ax + b = c$ that operate from left to right with an answer to the right of the equal sign, they may mistakenly conclude that all equations should be read from left to right and calculations performed in that direction. Some researchers point out that this limited understanding is not only common in early elementary years but also in late elementary, middle school and high school years (Behr et al., 1980; Saenzo-Lundlow & Walgamut, 1998; Herscovics & Kieran, 1980, Kieran, 1981a). This results in Grade 9 learners writing linear equations like $19 = 3x + 4$ as $3x + 4 = 19$.

Some learners see the equal sign as a symbol separating two different expressions on either side, which don't necessarily have to be related. They do not see it as a symbol showing equivalence between the RHS and the left hand side ('the LHS') of the equation. They are reluctant to accept statements like $5 + 2 = 6 + 1$. When faced with a problem such as $5 + 2 = \ldots + 1$, learners assume that the answer is 7, because they believe that the equal sign signals them to complete the operation on the LHS of the equation. They would simply break it up and write $5 + 2 = 7 + 1$. With this level of understanding, the equal sign is nothing more than an operator that connects a problem with its answer (Kieran, 1981). They do not recognise it as a symbol that expresses an equality relationship between two expressions.

Other learners will simply disregard the placement of the equal sign in an equation and perform an operation on all of the numbers or terms. For example, when asked to fill in the missing number in the statement $4 + 3 = \ldots + 5$, learners add the given numbers, $4 +3 + 5 = 12$, to determine the value of the unknown. In this conception, the equal sign means the 'total' (McNeil & Alibali, 2005). In linear equations, when such learners are asked to solve $3x + 6 =$

$2x - 10 - 7x$, they write $3x + 6 + 2x + 10 + 7x$ and give the answer as $12x + 16$. This is particularly problematic because learners do not understand that equal sign describes a relationship between two expressions. Learners' understanding of the equal sign is clearly crucial to solving equations correctly.

As evident from the above, misconceptions and errors in algebra among high school learners have been widely documented. Various studies have examined this with a view to enhancing students' understanding of mathematics (Luneta & Makonye, 2010; Molefe & Brodie, 2010; Swan, 2002). Research conducted by Seng (2010) and Egodawatte (2011) found that high school learners have misconceptions (and make systematic errors) with fundamental algebra concepts such as variables, exponents, signs, algebraic expressions, equations and word problems.

A study by Ncube (2016) examined errors made by learners in simplifying algebraic expressions. Six common types of errors were identified in learners' responses. The main errors observed were conjoining, misapplications of rules, misinterpretation of symbolic notation, misusing the distributive property, sign error and the error of substituting letters by numeric values. A study by Mamba (2012) on learners' errors when solving algebraic tasks also indicated that the main reason for misconceptions seems to be the lack of understanding of the basic concepts including numbers and numerical operations; functions; the order of operations; equality; algebraic symbolism; algebraic equations; expressions and inequalities; and the difference between equations, expressions and inequalities.

The research discussed in the section above have been used to develop a framework for studying errors in solving equations at Grade 9 level which now follows:

2.6 A FRAMEWORK TO UNDERSTAND ERRORS AND MISCONCEPTIONS IN LINEAR EQUATIONS

2.6.1 Using Constructivism as a Lens to Understand Errors and Misconceptions

The aim of this study is to explore learners' errors in solving linear equations using the theory of constructivism. Constructivism refers to the process in which learners actively construct their understanding and knowledge of the world by experiencing things and reflecting on those experiences. According to Piaget (1970), learning is not a simple passive process of receiving from the surrounding environment. It is a dynamic process of an individual involving interaction between the individual’s existing knowledge and new ideas. Learners do not come to class with blank minds; rather, they bring ideas and facts shaped by their everyday experiences, and these ideas and facts can significantly influence what they gain from the classroom learning experience (Resnick, 1983; Ncube, 2016). These ideas and facts, having been actively constructed, provide everyday functionality to make sense of the world (Mestre, 1987). Learners develop their own understandings of mathematical concepts, which can be influenced by their peers, so that they adopt the meaning advocated by their peers. Some conceptions are related to the everyday language of certain terms, while others arise due to faulty previous instruction.

These conceptions can lead to misconceptions, resulting in consistent errors when solving mathematical problems. Learners attempt to associate existing knowledge with new information, sometimes linking unrelated information, which results in errors. They often make generalisations about certain rules and apply them in areas in which they do not apply. For example, they apply the fact that multiplication results in a larger number when two positive whole numbers are multiplied, to fractions, where this rule does not hold true.

In most cases, misconceptions result from over- or under-generalisation of a concept or rule, or being presented with a different conception of the situation (Hansen et al., 2017). Research indicates that it is impossible to avoid errors and misconceptions while teach; learners will always make errors and incorrect generalisations. However, misconceptions can be valuable resources for effective learning if they are identified and properly addressed; failing to do so can result in severe problems. It is important that errors or misconceptions are viewed as part of the process of learning, rather than something that should be eliminated (Mahlabela & Bansilal, 2015; Lannin, Barker & Townsend, 2007).

As learners are introduced to the concept of solving linear equations and learn the various procedures associated with solving equations, they often display errors and misconceptions (Lima & Healey, 2010). Within a constructivist perspective, learners’ errors and misconceptions in algebra influence their understanding of mathematics and other related subjects (Pournara et al., 2016). Swan (2002) suggests that learning would be more effective if learners are encouraged to discuss and resolve their misconceptions. Teachers play an indispensable role in this process; they need to understand the nature of misconceptions to effectively guide discussions addressing them. Discussions around the actual misconception should be used productively as a learning tool.

Although for the sake of completeness, both errors and misconceptions are discussed in this section, the focus of this study is primarily on errors made by learners in solving linear equations. In learning to solve linear equations, learners go through similar experiences as described above. Sometimes, they develop misconceptions which lead to consistent errors in solving linear equations. The next section of this study outlines various categories of errors, as derived from research. These categories serve as the conceptual framework for this study.

2.6.2 Categories of Learner Errors in Solving Linear Equations

This section discusses categories of common errors that impact on learners' performance in solving linear equations. These categories identified by research, provides the basis for the study's analysis of errors in chapters 4 and 5.

First, the five main categories of errors are distinguished, and then further subcategorised. The five categories are as follows:

Equation Errors: these occur when learners incorrectly apply formal methods of solving equations;

Variable Errors: these involve inappropriate manipulations of expressions;

Sign Errors: these relate to errors involving misuse of negative signs;

Slips: these are careless errors made by both novices and experts; and

Other Errors: these refer to incorrect responses that don't seem to have a consistent, identifiable error pattern.

Each of these categories is now elaborated on, with relevant subcategories identified and described.

2.6.2.1 Equation Errors

Equation errors occur when learners incorrectly apply formal methods which they were taught to solve the equation. For example, learners apply inverse operations to group like terms and constants on opposite sides of the equal sign. They attempt to isolate the unknown and find the solution while maintaining balance (Pournara, 2020).

These key steps used in solving a linear equation have guided the identification of categories to analyse the errors that relate to equality. In this study, errors involving equations are divided into the following seven sub-categories: forcing a familiar format; applying the incorrect inverse operation; division error; incomplete process error; transposition of terms error; changing equation to expression (where the equal sign is ignored); and substitution error.

2.6.2.1.1 Forcing a Familiar Format

This error is only observed in the last two lines of a response, where learners 'force' their answer to fit the form $x = k$ by eliminating additional letters as necessary (Pournara, 2020). For example, a learner may manipulate an equation to obtain $2x = 6x$, but ignores the second appearance of the variable x, solving the equation as $2x = 6$. Their solution would then be $x = 3$ rather than $x = 3x$. Sanders (2017) opines that this is done because the equation $x = 3$ looks familiar.

2.6.2.1.2 Applying the Incorrect Inverse Operation Errors

In these types of equation errors, learners use the incorrect inverse operation, for instance, applying the additive inverse when the multiplicative inverse is required, or vice versa (Sanders, 2017). For example, given $9x = 2$, a learner may subtract 9 from both sides instead of dividing by 9 i.e. $x = 2 - 9$. The answer arrived at is $x = -7$, which is incorrect.

Similar inverse errors reported by Lima and Tall (2008) are closely related to the inverse errors discussed above. The extract below (Lima & Tall, 2008) demonstrates these errors. A learner might begin the solution of the equation with the following sequence:

$$3x - 1 = 3 + x$$

$$3x - x = 3 + 1$$

$$2x = 4$$

Different students will then take various final steps, including:

$$x = 4 - 2 \qquad (1)$$

$$x = \frac{4}{-2} \qquad (2)$$

$$x = \frac{2}{4} \qquad (3)$$

The initial steps represent the 'change sign' technique. The final step in each case reveals a 'mal-rule', which involves applying a specific rule with some erroneous variation (Tall, 2007), as follows: (1) passes the 2 over the other side and changes the sign; (2) correctly 'shifting the two over and putting it underneath', but also changes the sign; and (3) shifting the 2 over and putting the 4 underneath (writing the quotient upside down). In all cases, the errors are likely to have arisen from the misapplication of existing knowledge (Tall, 2007).

Another error of this type occurs in the application of the additive inverse. Some learners perceive the equal sign as a symbol separating two different expressions which may be related, but not necessarily so. The following extract of a learner's work taken from a study done by Byers and Herscovics (Kieran pg. 399), illustrates such thinking:

$$
\begin{array}{lll}
\textit{Solve for } x & 2x+3=5+x & \\
 & 2x+3-3=5+x & \textit{Step } 1 \\
 & 2x=5+x-x-3 & \textit{Step } 2 \\
 & 2x-x=5-3 & \textit{Step } 3 \\
 & x=2 & \textit{Step } 4
\end{array}
$$

Figure 2.1: An Example of Incorrect Application of Additive Inverse

While the learner's intentions are clear, and the final answer is correct, the operation used demonstrates a limited understanding of the equal sign. The learner sees the equal sign as separating two sides. For example, in step 1, the additive inverse of + 3 is applied only to the LHS of the equation and later in step 2 the additive is also applied to the RHS to maintain balance.

2.6.2.1.3 Division Error

The division error is closely linked to the inverse error because both occur when learners face equations of the form $Ax = B$, and where they need to isolate the variable x. After reaching the step of the form $Ax = B$, learners correctly apply the multiplicative inverse operation by dividing by A on both sides of the equation, but are confused when they can't find the non–integer solution. As a result, they write a number unrelated to the operation B/A. For example, when learners reach the step $3x = 10$, they may write the answer as $x = 3.1$ *or* $x = 3$. Some learners give incomplete answers, such as $x =$, possibly because these learners did not have calculators and because they were afraid of writing wrong answers (Hall, 2002).

The division error may seem relatively unimportant in the context of solving linear equations. However according to Hall (2002, p. 28), “until division is mastered, learners without calculators may be unable to find non-integer solutions to linear equations”

2.6.2.1.4 Incomplete Structure Error

This error is characterised by an incomplete response where a learner does not produce an answer in the form $x = k$ but leaves the answer in the form $Ax = Bx$ or $Ax = B$. The high prevalence of incomplete responses is closely associated with the familiar format error. In many of these cases, learners did not manipulate their equations to produce a familiar format, rather ceasing the operation with forms such as $Ax = B$, $x = Ax$ or $\frac{Ax}{A} = \frac{B}{A}$ (Pournara, 2020). According to Tall (2002), these errors may arise because the learners do not recognise what should be done i.e. even towards the end of the question, they may not realise that they need to do the same to both sides. Tall (2002) assign this error as The inability to Isolate the Variable error.

2.6.2.1.5 Transposition Error

In algebra, transposition of terms refers to the process of moving a term from one side of an equation to the other side by changing its sign (Sanders, 2019). For example, if given equation $x + 3 = 9$, changes are made such that $x = 9 - 3$, then 3 is said to be transposed; or in the equation $3x - 2 = 7 + x$, changes are made such that $3x - x = 7 + 2$, then x and 2 are said to be transposed. According to Hall (2002, ‘Transposition, as its name suggests, is a change side – change sign technique’.

In the second example above, a transposition error would if – 2 and x were moved to either side of the equal sign with no corresponding change in their sign. The equation $3x - 2 = 7 + x$ would then be incorrectly written as $3x + x = 7 - 2$. Pournara (2020) refers to this type of error, where learners ‘move’ a term to the other side of the equal sign without changing the sign of that term, as a ‘Moving Error’. and He distinguishes two types of this error: moving constants and moving a letter term.

In one case, a learner demonstrated some understanding of transposition initially, but failed to apply this correctly in simplifying the equation. The equation was given as $3x - 2 + 4 = x + 7$ and simplified to $3x - x = 7 + 2 - 4$, showing an understanding of transposition by writing 2 and 4 on the RHS of the equation with the corresponding change in signs. However, we when the letter x is transposed to the LHS, it was added instead of subtracted. So in this study I did not classify this as transposition error because there was a correct application of transposition, although the operation lacked consistency.

2.6.2.1.6 Changing an Equation into an Expression (Ignoring Equal Sign)

This type of error involves learners changing the equation to an expression by ignoring the equal sign and proceeding as if they were simplifying an expression. Figure 2.1 below demonstrates this error:

$5a - 3 = 6 + a$

$5a - 3 + 6 + a$

$5a + a - 3 + 6$

$6a - 9$

$15a$

Figure 2.2: An Example of Changing an Equation to an Expression

The equation $5a - 3 = 6 + a$ is converted to an expression by removing the equal sign and writing $5a - 3 + 6 + a$. This would be followed by grouping and adding/conjoining of terms. In this conception, the equal sign means “total” (McNeil & Alibali, 2005). This kind of error indicates that the learner is unable to distinguish between an expression and an equation (Hall, 2002).

2.6.2.1.7 Substitution Error

The trial-and-error method is another commonly known strategy for solving linear equations. An equation such as $2x + 7 = 15$ can be solved arithmetically by asking what number, multiplied by 2 and adding 7, arrives at the answer of 15. Here it is easy to see that the solution is 4. An equation error not reported in literature is what I refer to as ‘substitution error’ which occurs when using the trial and error strategy This error occurs if the substituted x value does not yield the results on the RHS.

2.6.2.2 Variable Errors

Variable errors involve inappropriate operations on terms with variables. There are three relevant sub-errors involving variables, namely conjoining errors, errors involving inappropriate application of exponential laws, and bracket errors. Bracket errors are classified as variable errors as they always involve simplification of expressions containing variables.

2.6.2.2.1 Conjoining Errors

This type of error involves incorrectly reducing two terms (like or unlike), that are being added or subtracted, to a single term (Pournora, 2020). For example, given the equation $6a-2=3+a$ the LHS $6a-2$ is reduced to $4a$ by subtracting 2 from 6. The RHS of the equation $3+a$ is reduced to $3a$ by placing a next to 3, and the equation is reduced to $4a=3a$. Kieran (1992) attributes this type of error to the overgeneralisation (or false generalisation) of certain mathematically valid operations. The source of this error can be traced to arithmetic, where simplification results in a single numerical value. For example, a learner faced with the expression $6a – 2$ (the LHS of the equation above), may not know what ‘a’ is, but recognises 6 – 2. Relying on existing arithmetic knowledge, the learner subtracts 2 from 6, ignoring the coefficient a. Once the subtraction is done, the ‘a’ is then put back next to the numerical result, giving an answer of $4a$. Another conjoining error is when earners reduce the expression $3 + a$ (the RHS above) to $3a$, they respond to this problem as if they were answering $3\times a$.

Hall (2002) identifies another type of conjoining referred to as ‘deletion error’. This involves learners isolating letters or numbers using unknown rules of deleting the same letters or numbers in both like or unlike terms. Below are two examples that illustrate this error:

- $3x – x = 3$
- $3x – 3 = x$

In the first example, $3x – 3$, learners delete the letter x to arrive at an answer 3. In the second example, learners delete the numbers to arrive at an answer x.

2.6.2.2.2 Inappropriate Application of Exponential Laws

Experiences in algebraic exponents, such as simplifying $5x^2\times 4x^3$ to give $20x^5$ by ‘multiplying the coefficients and adding the powers’ can be misapplied in the simplification of expression where like terms should be added. For example, the expression $4x^2 + 3x^2$ can be wrongly simplified as $7x^4$. Although the focus was on both the exponents and coefficients, the exponential laws are applied inappropriately. Mbewe (2013) opines that learners misuse previously learnt procedures and rules in situations where they are not appropriate. For example, when simplifying the expression $4a + a$, instead of giving $5a$ as the answer most learners gave $4a^2$ as their solution. Learners multiply terms instead of adding. In this case, it shows learners failed to differentiate $a + a$ form $a \times a$.

In Grade 9, exponents are taught before addition and subtraction of like terms. Subsequently, when addition and subtraction of like terms are introduced, learners confuse expressions like $x \times x$ and $x + x$, where $x \times x$ simplifies to x^2 and $x + x$ simplifies to $2x$. When expressions like $x + x$ are then encountered, learners make errors in their manipulations, based on existing knowledge. As mentioned above, this interference of new knowledge with previously acquired knowledge is referred to Lima and Tall (2007) as ‘Met-Befores’

2.6.2.2.3 Bracket Errors

The inability to use brackets appropriately is one of the most significant challenges I have encountered when teaching algebra to Grade 8 and 9 learners. For some learners, this problem persists up to Grade 12 level. Research (Ncube, 2016) attributes invalid distribution of brackets to learners’ failure to master basic facts and concepts. Learners inappropriately use the distributive property in a variety of ways. They may expand one part and leave the other part unchanged, for instance, expanding $3(x+2)$ to arrive at $3x+2$ as their solution. Others do not know the limit of the pre multiplier (Ncube, 2016). For example, when simplifying the expression $3(x-5)+4$, they multiply the contents inside the bracket, as well as the expression outside the brackets, by 3 and give the answer of $3x-15+12$.

Another error in this category involves learners introducing brackets that do not exist into the equation. For example, given $3x + 2 = x + 4$, learners transform this equation to an expression by writing $(3x + 2)(x + 4)$. Learners are no longer solving for x, but are rather incorrectly applying the distributive property of brackets to simplify the expression $(3x + 2)(x + 4)$ to $3x^2 + 4x + 2x + 8$.

2.6.2.3 Sign Errors

In this study, sign errors refer to errors made when dealing with the negative symbol. Research has identified (Vlassis, 2004) a range of errors associated with the minus symbol.

I have identified four sub-categories of errors involving the negative, including the minus sign used incorrectly; the minus sign ignored; the minus sign recognised but not used; and the failure to combine signs and directions. These sub-categories are defined below.

2.6.2.3.1 Minus Sign used Incorrectly (Right to Left Reasoning)

Right to left reasoning, referred to by Vlassis (2004), involves incorrectly using the minus sign by subtracting the smaller coefficient from the larger coefficient, working from right to left. For example, when given $2x - 7x$, learners read from right to left as $7x - 2x$ to reach an answer of $5x$.

2.6.2.3.2 Minus Sign Ignored

This error occurs when the minus sign disappears from one line of working to the next. For example, the expression $-2x - 7x$ would be simplified to $9x$. The minus sign seems to be completely ignored, as if learners are simplifying the expression $2x + 7x$ to arrive at $9x$.

2.6.2.3.3 Minus Sign Recognised but Not Used

This error is characterised by numbers being added as though they were positive, but the result is given as negative. Examples of variations of this type of error, as illustrated by Sanders (2017), are provided below:

- $-8+3=-11$
- $3-8=-11$
- $-8-3=-5$
- $8 - x = -8x$

Vlassis (2004) attributes this error to brackets reasoning, a common type of reasoning that learners apply to equations. This reasoning involves putting imaginary brackets around like terms when preceded by the minus sign. For example, the expression $20 + 8 - 7n + 5n$ is treated as $20 + 8 - (7n + 5n)$, which is reduced to $28 - 12n$. The minus sign is acknowledged as it appears in the final answer, but it is not actually applied in the equation. This reasoning is frequently used in polynomials where like terms are presented in pairs (Vlassis, 2004).

2.6.2.3.4 Failure to Combine Signs and Directions

This error occurs when the minus sign is presented in front of a set of brackets. For example, the expression $-2(3x+1)$ would be simplified as $-6x+2$. Although the distributive property was correctly applied, both terms inside the brackets having been multiplied by 2, the negative sign in front of the brackets is used incorrectly.

2.6.2.4 Slips

Slips are careless errors that can be made by both experts and novices. They are easily detected and corrected.

2.6.2.5 Other Error

This category covers errors where no common error patterns are identified in learners' responses.

2.7 CHAPTER SUMMARY

This chapter has presented a review of research about the challenges of learning mathematics (specifically algebra) and learner performance in South Africa. This chapter has been able to present the research findings of the studies conducted on linear equations. This chapter has been able to use Literature as part of framework to describe and identify various categories of errors that are displayed in the learning of the solutions of linear equations.

CHAPTER 3

METHODOLOGY

3.1 INTRODUCTION

This chapter focuses on discussing the research methodology used in this study. According to Cohen et al. (2007), the purpose of research methodology is to clarify the process of scientific inquiry (as opposed to the products of the inquiry) in the broadest possible terms. The chapter presents the research paradigm, approach, and design adopted. This is followed by details of the participants, the data collection process, and the data analysis procedures used. Lastly, the chapter describes how reliability and validity issues were established for both data collection and instruments, as well as how ethical considerations were dealt with.

3.2 RESEARCH PARADIGM

Bertram and Christiansen (2014, p. 24) define research paradigm as “a particular worldview that defines, for the researchers who hold this view, what is acceptable to research and how this should be done.” Kivunja and Kuyini (2017, p. 26) specify that “this worldview is the perspective, or thinking, or school of thought, or set of shared beliefs, that informs the meaning or interpretation of research data.”

In this study, the interpretivist paradigm is used. According to Bertram and Christiansen (2014), researchers within the interpretivist paradigm “do not aim to predict what people do, but rather to describe and understand how people make sense of their worlds, and how they make meaning of their particular actions.” This paradigm is best suited to this study, as its purpose is to explore common errors made by learner in solving linear equations i.e. to identify errors made and to understand what led to those errors made by the learners.

Furthermore, this paradigm aligns with the data collection methods used in this study, including written documents and interviews. These methods were used to ensure that the study’s findings reflect participants’ understanding of how to solve linear equations based on their experience. According to Thorne (2016), these approaches offer a greater chance for dialogue between the participant and the researcher in order to create a meaningful reality in the participants' context.

3.3 RESEARCH APPROACH

According to Durrheim (2002), a research approach is the process of gathering and evaluating the data needed to address the research questions. According to Creswell (2014), a researcher's choice of research methodology is influenced by the research paradigm as well as the important questions the study aims to answer.

Williams (2007) states that three primary research methodologies are frequently used by scholars. First, researchers have the option of using a quantitative method based on positivism or post-positivism. This method primarily tests or validates hypotheses and assumptions using numerical and statistical descriptions (Creswell, 2009). Second, a qualitative method based on the interpretivism paradigm can be employed by researchers. "Understanding, exploring, discovering, and clarifying situations, feelings, perceptions, attitudes, values, beliefs, and experiences of an individual or group of people is the main focus in qualitative research," according to Kumar (2011, p. 104). According to Merriam (1992), the goal of the qualitative technique is to gain a thorough grasp of a particular phenomenon and go deeper into a scenario to uncover strong information for comprehending the phenomenon being studied. Williams (2007) proposed a third strategy called Mixed Methods study, which can be applied to study issues that call for both textual and numerical data.

This study aims to identify errors commonly made by learners in solving linear equations and to gain a deep understanding of why these errors are made. No statistical analysis was used. In this context, a qualitative approach within an interpretative paradigm is appropriate.

3.4 RESEARCH DESIGN

According to Bertram and Christiansen (2014), research topics, data collection techniques, and the researcher's working paradigm all typically have an impact on the research design that is employed when performing any kind of study. Rule and John (2011) define a case study as an in-depth, methodical investigation of a particular instance in its context. The case may involve an individual (a learner, instructor, or parent) or a group of individuals (a community or class of students). When a researcher examines a single case within its actual setting, constrained by time and activity, case studies are helpful. It addresses current affairs and asks how and why things occur. To put it another way, the researcher examines the case in great detail in order to document the participants' actual lived experiences and opinions regarding a certain circumstance (Cohen et al., 2011).

This study used a case study to investigate the typical mistakes made by ninth-grade students when attempting to solve linear equations. At this instance, the issue involved Grade 9 learners who were enrolled at two schools in KwaZulu-Natal's Pinetown District for the 2023 school year. Guthrie (2010) asserts that while case studies do not accurately reflect the population, well-chosen findings can be applied in different contexts.

3.5 SAMPLING PROCEDURE AND PARTICIPANTS

A sample, according to McMillan and Schumacher (2010), is a group of people from whom information is gathered. As a result, a sample is typically representative of the entire group being studied and is a portion of the overall population. Cohen et al. (2007) define sampling as the process of choosing which people, environments, activities, or events to study. According to Patton (2002), there are no set guidelines for the sample size of a qualitative study; instead, it is influenced by the goal of the investigation, the stakes, what is practical, what will be credible, and what can be accomplished with the time and resources at hand. To make conclusions, the researcher must, however, decide on a sample size and population characteristics.

The targeted population in this study was Grade 9 learners from Ntuzuma Township under Pinetown District of KwaZulu-Natal. Given the size of this population, drawing a sample was appropriate. Two schools were selected as a sample for the entire population. The two schools were selected based on convenience and an already established working relationship with them. Each school had two classes of Grade 9 learners. The total number of learners officially enrolled for the 2023 academic year was 196. Of these, 102 attended school A and 94 learners attended school B.

I visited the schools and explained to the teachers and the learners the purpose of the research study. All 196 learners were given an equal opportunity to participate in this study. Invitation Letters were distributed to learners inviting them to participate. Informed consent forms were included for parents or to guardians' consent to their children's participation. Grade 9 learners from the two schools who were willing to participate and provided completed the informed consent forms, formed the study sample. A total of 155 learners participated.

An analysis of learners' written responses to the Structured Activity Sheet (detailed in the next section) and identified errors was conducted. On this basis, nine learners were purposively selected to participate in the semi-structured interviews. Each interview was 40 to 45 minutes in duration and focused on each identified common error with a view to gaining a thorough understanding of the possible reasons for the errors.

3.6 DATA COLLECTION METHODS AND INSTRUMENTS

Data was collected in this study in two phases, the first being the completion of a Structured Activity Sheet (‘the Activity Sheet/s’) and the second being semi-structured interviews.

In the first phase, participants were given the Activity Sheet based on solving linear equations to complete in one hour under the supervision of a teacher. Participants’ responses were analysed to identify common errors, and responses grouped according to error type.

In the second phase, semi-structured interviews were conducted, focusing on each identified common error to gain an understanding of the possible reasons for the errors. Semi-structured interviews are flexible and allow the interviewer to go into as much depth as he or she requires or to clear up any misunderstandings (Vilakazi, 2021). Henning (2004, p. 6) expands on the benefits of semi-structured interviews, asserting that they “elicit thick data, which is the kind of data that gives an account of the phenomenon that is coherent; giving more than facts and empirical content; as well as interpreting the information in the light of other empirical information in the same study as well as from the basis of a theoretical framework that locates the study.” This kind of interview was appropriate for this study as it provided an opportunity to get to become closely familiar with the level of knowledge and understanding of the study participants.

The semi-structured interviews were held two days after the Activity Sheets were administered, so that participants could remember how they had come up with their solutions. The participants were selected for interviews based on the responses received from the activity sheets. They were first asked a general question regarding their attitudes towards and perceptions of mathematics. Thereafter, their written responses to the activity sheet items were discussed. They were asked to explain the procedures that they had followed in reaching their solutions (as explained in detail in Section 3.6.2 below). Interviews conducted were audio-recorded to capture as much accurate information as possible for further analysis. Ethical issues were taken into consideration during the course of the interviews as per the contents of the informed consent forms. All the interviews were conducted within the school buildings attended by the participants.

3.6.1 Structured Activity Sheet

In this study, a self-designed Activity Sheet (Table 3.1), was used to collect data to address research questions. The items included in the Activity Sheet were developed by me in consultation with my supervisor, guided by items appearing in previous research, as well as one item (item 2.10) from my Honours study (Gasa, 2021). The Activity Sheet comprised 12 items with varying difficulty levels, selected for research due to their potential to reveal a variety of equation errors.

Table 3.1 Structured Activity Sheet

Solve for *x* by inspection	
1.1	$\frac{x}{4} = 2$
1.2	$2x + 1 = 7$
Solve for *x*	
2.1	$3x + 7 = 35$
2.2	$12 = 3x$
2.3	$3x - 2 = 4 + x$
2.4	$2 - 3x = 7 - x$
2.5	$\frac{3x}{4} = 6$
2.6	$\frac{x-2}{3} - \frac{x+4}{6} = 2$
2.7	$5(x + 2) = 20$

2.8	$4(2x-1)+7=35$
2.9	$3(x-1)-4x=5-2(x+1)$
2.10	$(x-2)^2+3x-2=(x+3)^2$

3.6.2 Semi-structured Interviews

The second instrument is a semi-structured interview schedule consisting of three general questions applicable to all interviewed participants. Thereafter, specific questions based on their responses to the Activity Sheet were directed to each participant. The semi-structured interview guide is set out in Table 3.2 below

Table 3.2 Semi-structured Interview Guide

Process	Interview Question
1. Read	Please go to the first item and read the question.
2. Comprehension	What does the question mean?
3. Strategy Selection	How do you solve the equation?
4. Process and Explanation	Look at how you solved the equation in the activity and please explain how you came up with the solution

3.7 DATA ANALYSIS

According to Henning (2004, p. 33), data analysis is “a relatively systematic process of coding, categorising, summarising and interpreting data to provide an explanation of a single phenomenon of interest.” Below is a detailed discussion of how the data collected was analysed to address the research questions of this study.

3.7.1 Analysis of Items from the Structured Activity Sheet

Participants' Activity Sheets were analysed. The common errors identified were grouped according to error types, as identified by research. The error types were then quantified according to occurrence frequencies.

There were five main categories of errors that were identified, and these were discussed in detail in Chapter 2 above. Firstly, many errors arise as learners try to carry out the steps related to the procedure of solving the equation that they have learnt- their goal is to solve the equation, by executing the formal methods they know, but when faced with certain situations they choose an incorrect technique or step which is referred to as an Equation errors and it has seven subcategories. A second category of errors emerge when learners apply incorrect rules or techniques to simplify expressions containing the variable (unknown), referred to as variable errors, and this includes three subcategories. The third category relates to errors in the use of the negative sign and is called sign errors. The fourth category (slips) entails those errors due to small slips that could be easily corrected, while the fifth category includes those which did not fit into any of the preceding ones because there were no discernable patterns and are referred to as the other.

Each of the error categories was subdivided and these subcategories were allocated specific codes, mainly for ease of capturing learners' responses. 13 Error codes were used, and two additional category codes allocated to account for correct responses and no response (blank) code. The summary of the coding system is provided in Table 3.3 below.

Table 3.3: Coding and Categorisation of Errors

Number	Category	Code
1	Correct responses	1
2	No responses (Blanks)	2
3	Equation Errors:	
	Forcing Familiar Format	3

	Division Error	4
	Incomplete Structure	5
	Inverse Error	6
	Transposition Error	7
	Equation changed to Expression	8
	Substitution Error	9
4	Variable Errors:	
	Conjoining Errors	10
	Error with Exponents	11
	Bracket Errors	12
5	Sign Errors	13
6	Slip Errors	14
7	Other (unknown errors)	15

The additional two categories coded 1 and 2 in Table 3.3 above, are described below.

Correct responses:

This category covers responses where the procedure, strategies, and method used are mathematically correct. There were no errors relating to numbers, signs, simplifying expressions, adding or subtracting unlike terms or the equal sign.

No responses (Blanks):

This category covered instances where the learners did not attempt the question; they left it blank.

In classifying errors, certain errors were found to fit into multiple categories, such as variable and sign errors. For example, if $2 - 6x$ was simplified to $4x$, there are two different errors being made, namely adding unlike terms and misusing the minus sign. In this study, it was decided to only assign one code per error so that the number of errors attributed to learners made is not exaggerated.

These types of errors are classified as sign errors because, according to research (Vlassis, 2004), learners struggling with negative signs will provide the same answer (5) for 7 – 2 and 2 – 7. This indicates that learners understand that they need to subtract but don't know what to subtract. Therefore, whether the question is 2 – 7 or $2 - 7x$, 5 or $5x$ will be given as the answer, regardless of whether the terms in the expression are like terms. The focus of the learner is on subtraction; consequently, this example would be classified as a sign area as opposed to a variable error.

It is also possible for a participant's overall response to be attributed multiple error codes. This occurs where there are errors in different steps, in contrast to the abovementioned situation where one step could only be assigned one code. Consequently, the number of errors found exceeds the number of incorrect responses.

Additionally, there are cases where a single error occurs more than once in a single response. For example, if a learner made a right-to-left reasoning error on both the LHS and RHS of an equation in one response, this was counted as a single error.

3.7.2 Analysis of Semi Structured Interviews

The nine selected learners were interviewed one by one. During the interviews, learners were asked to explain the procedures that they had followed in reaching their solutions. To ensure that every aspect of the semi-structured interview was captured, the interviews were audio-recorded before being transcribed verbatim. The learners' explanations expressed in their home language were translated to English.

Semi-structured interviews revealed learners’ thought processes that were unclear in their written responses. Chapter 4 of this study presents a full analysis of the learners’ interview responses.

3.8 TIME FRAME

The study was carried out in the second term of the year. At that time, learners had covered most of the algebraic aspects including linear equations. The process of completing the Activity Sheets and conducting the interviews took two weeks.

3.9 VALIDITY AND RELIABILITY ISSUES

According to Winter (2000), validity refers to whether an account accurately represents features it is intended to describe, explain or theories. Guba and Lincoln (2005) suggest that the key validity criteria in qualitative research are credibility, transferability, dependability, and confirmability. These criteria are described below together with an account of how they were achieved in this research.

3.9.1 Credibility

The purpose is to establish confidence that the results are true, credible and believable.

This was done by adhering closely to participants’ written responses and comments given during interviews. Presenting some of the participants’ written work in the data analysis section also assisted in ensuring credibility.

3.9.2 Transferability

Transferability is the ability to apply research results to another context similar to the one in which the research was conducted (Koul, 2008). In this study, a description of the settings, including the targeted population, is provided to allow other researchers to apply the findings to similar settings. The findings are described in greater detail below and include direct quotes taken from the participants’ interviews.

3.9.3 Dependability

Dependability focuses on whether the research is transparent and clearly written. Howitt (2007) posits that it must be possible to trace the data, and that the findings must be consistent. The participants' original written responses and audio-recorded interviews can be used to confirm the dependability of this research.

3.9.4 Confirmability

Confirmability is another fundamental aspect of qualitative research to ensure the trustworthiness of the study is established (Nguyen et al., 2021). The purpose of confirmability is to provide substantial means to backtrack the findings of research to its original data sources (Betram & Christiansen, 2014). This study was conducted in a way that ensured that the findings were confirmable.

3.10 ETHICAL CONSIDERATIONS

In this study, learners were informed that participating in the test and interviews was entirely voluntary and they were free to withdraw from the study at any stage. Gatekeepers' Letters giving permission to conduct the study were obtained from the principals of the two schools and circuit manager. Invitation letters were distributed to learners inviting participation in the study and Informed Consent letters were distributed to learners for parents or guardians to give informed consent. The anonymity of the participants was ensured by using pseudonyms in all reports and analyses and at no time were results linked to the individuals. **Appendix G** is an example of the Informed Consent Form provided to parents or guardians to consent to their children's participation. A copy of the letter to the Department of Education requesting permission to conduct this research is included. A copy of confirmation of ethical clearance from the UKZN Research Office is included in **Appendix A**.

3.11 CHAPTER SUMMARY

This chapter presents the research procedures used in this study. Using an interpretivist paradigm, a qualitative case study approach is employed to explore the common errors made by Grade 9 learners when solving linear equations. In the data collection stage, a convenient sampling procedure was used to select learners to complete the Activity Sheet, and a purposive sampling procedure was used to select learners for semi-structured interviews. The data was analysed using learners' written responses and was further analysed through verbatim transcription of semi-structured interviews. Lastly, ethical procedures and the trustworthiness of the study were discussed. The next chapter analyses and presents the key findings of the study.

CHAPTER 4

RESULTS AND DATA ANALYSIS

4.1 INTRODUCTION

This chapter presents an analysis of the 155 learners' written responses to the 12 linear equation problems ('items') set out in the Activity Sheet. The first phase of this chapter presents an overall summary of the results in terms of correct, incorrect and blank results. Following this, the study goes into greater detail about the various categories of errors described in Chapter 3.

The second phase of this chapter presents an analysis of learners' responses to semi-structured interviews.

As the participants in the study are anonymous, no gender references are made.

4.2 OVERAL RESULTS

A summary of the results is presented in Table 4.1 below. It is important to note that a total of 1860 responses were analysed, consisting of 155 responses each addressing the 12 linear equations set out in the Activity Sheet. Of these, 526 were correct, while 1266 were incorrect and 68 were blank.

Table 4.1: Showing Frequency of Correct, Incorrect and Blank Responses

	Correct	Incorrect	Blanks	Total
Frequency	526	1266	68	1860
Percentage	28	68	4	100

In terms of trying to quantify the errors, some responses revealed more than one error. Hence the total number of identified errors was 1773. This overall total number of errors identified added to more than the total number of incorrect responses which was1266. In light of this, when the percentages of errors are reflected, they reflect the proportion of errors in relation to the total number of errors, and not the total number of incorrect responses.

The participants' solutions for each item in the Activity Sheet were meticulously analysed to determine the type of errors made. This analysis indicates that learners' errors in solving linear equations stem primarily from difficulties in manipulating algebraic expressions, variables, and dealing with negative values. These issues cause more errors than those resulting from executing formal methods of solving equations, including performing the same operation on both sides of the equation, transposition and substitution.

On comparison of the number of correct and incorrect responses in relation to the total number of attempted responses, it is noteworthy that of 1792 attempts (see Table 4.2 below) made by participants to solve linear equations, only 526 (i.e. just above a quarter) arrived at the correct answers. Based on these numbers, it is clear that while many participants attempted to solve the equations, errors encountered in the process resulted in only a few arriving at the correct answers.

Table 4.2: Frequency of Attempted, Correct and Incorrect Answers per Linear Equation

No	Equation	Attempt	Correct	Incorrect
1	$\frac{x}{4}=2$	152	100	52
2	$2x+1=7$	154	100	54
3	$3x+7=35$	152	41	111
4	$12=3x$	147	82	65
5	$3x-2=4+x$	149	33	115
6	$2-3x=7-x$	146	18	131
7	$\frac{3x}{4}=6$	145	3	108
8	$\frac{x-2}{3}-\frac{x+4}{6}=2$	151	5	145

9	$5(x+2)=20$	152	54	98
10	$4(2x-1)+7=35$	147	38	109
11	$3(x-1)-4x=5-2(x+1)$	149	16	133
12	$(x-2)^2+3x-2=(x+3)^2$	148	3	145
	TOTAL	1792	526	1266

On a closer examination of the number of correct responses per item, it is important to note that the number of correct responses decreases as one moves down Table 4.2. The last item shows 148 attempts, but only three correct responses. There is a noticeable decline in performance from the first equation to the last, with the results from equation five to equation 12 showing the lowest performance. The decline seen from equation five suggests that learners experience more significant challenges in working with equations including brackets and letters to both sides of the equation. These difficulties are compounded by the presence of additional negatives in equation 6, particularly given the variables being subtracted from both sides. Fractions in equation 7 and 8 make the situation more complicated. The last equation, $(x - 2)^2 + 3x - 2 = (x + 3)^2$ has a square binomial on both sides of the equation, which needs to be simplified before the application of formal methods to solve the equation.

As discussed, competence to solve linear equations is particularly important in algebra. However, learners with a poor foundation in basic algebraic and mathematic concepts and procedures (Larino, 2018) find equations extremely confusing and difficult to solve.

The next sub-section of the study analyses errors made with reference to the categories and sub-categories of errors discussed in Chapter 2 of the study.

4.2.1 Equation Errors

Equation errors are those errors that occur as learners incorrectly apply formal methods to solve equations, for example, applying inverse operations to collect like terms and constants on opposite sides of the equal sign in an attempt to isolate the unknown and determine the solution.

As mentioned in Chapter 3, the seven sub-categories of equations errors are: forcing a familiar format; applying the incorrect inverse operation; division errors; incomplete process error; transposition of terms errors; equation changed into expression (by ignoring the equal sign) errors and substitution errors. Of these, the equation changed into expression error sub–category (where learners ignore the equal sign), demonstrates the highest number of errors, contributing 18% of all errors identified in this study. The conjoining of terms sub–category was the second most dominant error, contributing 16% of all errors made.

4.2.1.1 Forcing a Familiar Format Error

There were 18 responses in which learners faced with the step $Ax = Bx$ manipulated the statement to arrive at their final answer by forcing a solution that takes a familiar format. These learners manipulated their procedure by dropping the x on one side to end up with a solution that looks familiar. The extracts in Figure 4.1 below, taken from the responses from three learners' (L18, L23 and L151) Activity Sheets, demonstrate how this error is made.

L18 (first extract, Figure 4.1) responds to item 2.3 by first conjoining the unlike terms on both sides of the equation to obtain $x = 4x$. Seemingly, the learner is aware that this cannot be the final answer, then decided to divide both sides by x, but only 'cancelled' the *x's* in the numerator and denominator on the RHS of the equation, so this left. Findings by Pournara (2020) suggest that learners "force" their answer to fit the form $x = k$ by elimination additional letters as necessary (Pournara, 2020). Hence it may be that this student did not cancel the LHS because s/he wanted to leave the answer in a familiar format $x = 4$.

Similarly, L23 (second extract, Figure 4.1) applies the additive inverse operation by adding 2 on both sides of the equation, i.e., $3x – 2 + 2 = 4 + x + 2$, to maintain the balance. The learner then conjoins the unlike terms $4 + x + 2$ to obtain $6x$, arriving at $3x = 6x$, that is, L23 seemed to add the coefficients of $3x$ and $6x$ to obtain $x = 9x$, and then divided both sides by x before arriving at $x = 9$. Here too, the cancelling was done on one side only.

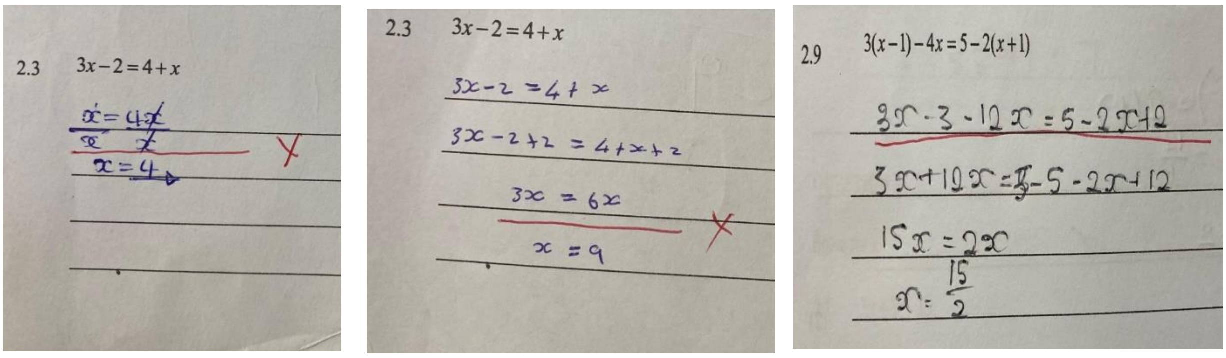


Figure 4.1: Responses of L18, L23 and L151 showing Forcing Familiar Format Errors

Another example of forcing a familiar format is shown above, where L151 (third extract, Figure 4.1) responds to item 2.9 by incorrectly applying the distributive property and then incorrectly manipulating the equation to arrive at $15x = 2x$. This learner seems to have divided both sides of the equation by *2x* and chose to write the answer as $x = \frac{15}{2}$ by cancelling the *x* that was next to 15. It is likely that the three learners of the extracts shown above knew what they wanted the structure of the final answer to look like, hence they chose to cancel letter *x* on one side only.

This approach concurs with Sanders (2017) opinion that they do these steps to arrive at the equations $x = 4$, $x = 9$ and $x = \frac{15}{2}$ because they look familiar.

This error accounts for 1% of mistakes identified in the study, occurring 18 times by different learners in various instances. It ranks as the least frequent mistake made within the sub-categories of equation errors.

4.2.1.2 Inverse Errors

In these types of equation errors, learners don’t know what inverse should be applied. Consequently, they apply additive inverse when multiplicative inverse is required, or vice versa. The extracts in Figure 4.2 below, taken from the responses from two learners’ (L1 and L2) Activity Sheets, illustrate how this error is made.

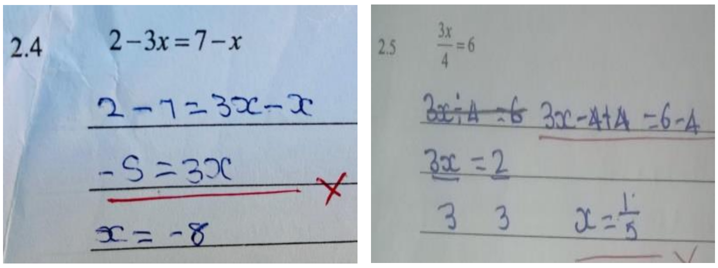


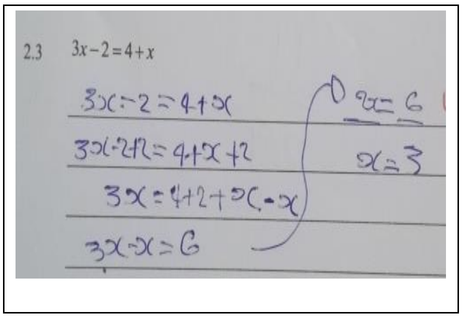
Figure 4.2: Responses of L1 and L2 showing Inverse Errors

L1 (first extract, Figure 4.2) responds to item 2.4 by grouping like terms on each side of the equation using transposition, reaching $-5=3x$ in the next step. The learner then writes the final answer as $x=-8$. Seemingly, the learner applies the additive inverse by subtracting 3 on both sides of the equation to obtain -8, instead of dividing by 3 on both sides to arrive at the correct answer $x=-\frac{5}{3}$. Hall (2002) identified this type of error as *other inverse's* error where learners use the incorrect inverse operation. This learner treated $-5 = 3x$ as $-5 = 3 + x$, hence 3 on the RHS was subtracted on both sides of the equation resulting in the answer $x = -8$. Hansen et al., (as cited in Mamba, 2012, p. 16) acknowledges a misconception as "the misapplication of a rule, an over- or under-generalization, or an alternative conception of a situation." These errors were also observed in the work done by Freitas (2002) and Lima and Hall (2002), and reveal the application of a 'mal-rule', i.e. applying a specific rule with some erroneous variation.

In contrast, L2 responds to $\frac{3x}{4}=6$ by writing her first step as $3x-4+4=6-4$, subtracting 4 on both sides of the equation instead of multiplying by 4. It appears that the learner was responding to $3x+4=6$, where they needed to apply the additive inverse of 4 to both sides of the equation to obtain $3x=2$ as the next step. The learner would've then applied the multiplicative inverse of 3 to both sides to obtain $x=\frac{2}{3}$. However, the answer arrived at is $x=\frac{1}{5}$.

Another type of inverse error occurs when learners incorrectly apply the additive inverse. Here, learners don't see the equal sign as a symbol showing the equivalence between each side of the equal sign. As a result, learners were subtracting or adding terms on one side of the equation and in the following step, doing the same thing on the other side.

The extract below, taken from the response of one learner's (L72) Activity Sheet, demonstrates how an additive inverse error is committed.



2.3 $3x-2=4+x$

$3x-2=4+x$

$3x-2+2=4+x+2$

$3x=4+2+x-x$

$3x-x=6$

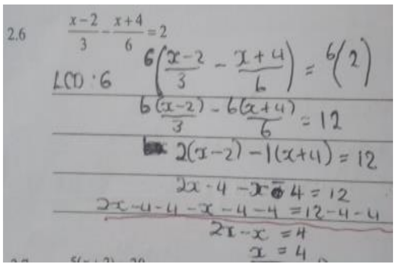
$2x=6$

$x=3$

Figure 4.3: L72's Response showing Additive Inverse Error

L72 responds to item 2.3 by adding 2 to both sides of the equation to maintain balance, getting to $3x-2+2=4+x+2$. In the second step, the learner subtracts x on the RHS and follows this by subtracting x on the LHS. Although the learners' intentions are clear and their answer is correct, this shows a lack of understanding of the equal sign. The equal sign is incorrectly perceived as a symbol separating two different expressions, which do not necessarily have to be related.

Another example illustrating the additive inverse error is shown in the extract below (Figure 4.4), taken from a learner's (L12) response to item 2.6 of their Activity Sheet. The learner eliminates the denominators by multiplying by the lowest common denominator and then removes the brackets correctly, arriving at $2x-4-x-4=12$. They then write $2x-4-4-x-4-4$ on the LHS. Even though they subtracted 4 from both sides of the equation to maintain balance, it seems that the learner thought that the additive inverse of -4 is -4. This is apparent from the following step, which is written as $2x-x=4$; the RHS of this equation should have been 12 + 4 + 4. Based on this, the learner arrives at a final answer $x = 4$, which is incorrect. Possibly, for this learner to recognise their mistakes, they must understand that their answer does not satisfy the original equation.



2.6 $\frac{x-2}{3}-\frac{x+4}{6}=2$

LCD : 6 $6\left(\frac{x-2}{3}-\frac{x+4}{6}\right)=6(2)$

$\frac{6(x-2)}{3}-\frac{6(x+4)}{6}=12$

$2(x-2)-1(x+4)=12$

$2x-4-x-4=12$

$2x-4-4-x-4-4=12-4-4$

$2x-x=4$

$x=4$

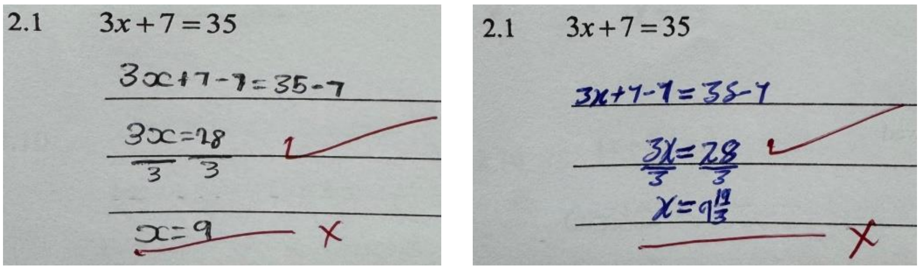
Figure 4.4: L12's Response showing Additive Inverse Error

The inverse error accounted for 3% of errors identified in the study.

4.2.1.3 Division Errors

Division error occurs when learners are faced with equations of the form *Ax* = *B,* then correctly apply the multiplicative inverse by dividing by A on both sides of the equation, but are confused when they are unable to find non-integer solution. As a result, they write a number unrelated to the operation $\frac{B}{A}$. the extracts in Figure 4.5 below, taken from the responses of two learners – (L46 and L40) activity sheets, illustrate how this error occurs.

L46 (first extract, Figure 4.5) responds to item 2.1 by first subtracting 7 from both sides of the equation to get *3x = 28*. To isolate the variable *x*, the learner divides both sides by 3, but is unable to find the correct answer *x* = 9.3, writing 9 as the quotient of 28 and 3. This approach is similar to that described by Hall, when he noted that learners get confused when they cannot find the non-integer solution, and as a result they write a number unrelated to the operation $\frac{B}{A}$.



2.1 $3x+7=35$

$3x+7-7=35-7$

$\frac{3x}{3}=\frac{28}{3}$

$x=9$

2.1 $3x+7=35$

$3x+7-7=35-7$

$\frac{3x}{3}=\frac{28}{3}$

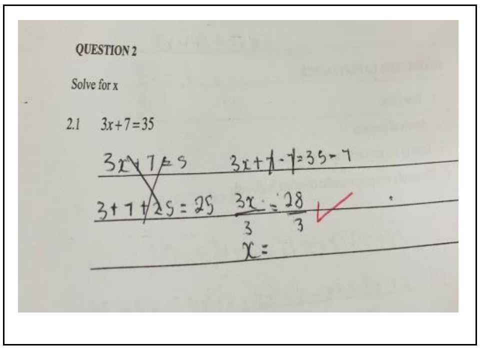
$x=9\frac{19}{3}$

Figure 4.5: Responses of L46 and L40 showing Division Errors

Similarly, L40 (second extract, Figure 4.5) responds to item 2.1 by dividing both sides of the equation after reaching the statement *3x = 28*. It seems that the learner knew that 3 divides into 28 nine times, but was unable to obtain the correct remainder, resulting in the incorrect answer $x = 9\frac{19}{3}$. From this, it can be deduced that the learner divided 28 by 3 to obtain 9.

Being aware that there is a remainder, an unknown rule was applied to get a remainder in fraction form. A possible explanation for this response is that after obtaining 9, when dividing 28 by 3, the learner subtracted 9 from 28 to obtain 19 and then wrote the remainder as $\frac{19}{3}$.

Some learners, when encountering a problem in which they make a division error, opt to leave the solution incomplete. Below is an extract (Figure 4,6) taken from the responses of one learner's (L28) Activity Sheet, illustrating this.



QUESTION 2

Solve for x

2.1 $3x+7=35$

~~$3x+7=35$~~ $3x+\cancel{7}-\cancel{7}=35-7$

~~$3+7+25=25$~~ $\frac{3x}{3}=\frac{28}{3}$ ✓

$x=$

Figure 4.6 Showing L28's Incomplete Response following a Division Error

After correctly applying known strategies, including multiplicative inverse in the last step, to solve the equation, the learner was unable to determine what should be written on the RHS of the equal sign, i.e. the value of *x*. Hall (2002) suggests that learners may give incomplete answers in response to this type of problem as they do not have calculators and are afraid of writing wrong answers.

In other instances, learners multiply or subtract instead of doing division on the RHS of the equation. The extracts in Figure 4.7 below, taken from responses of two learners' (L100 and L144) Activity Sheets, demonstrate this type of division error.

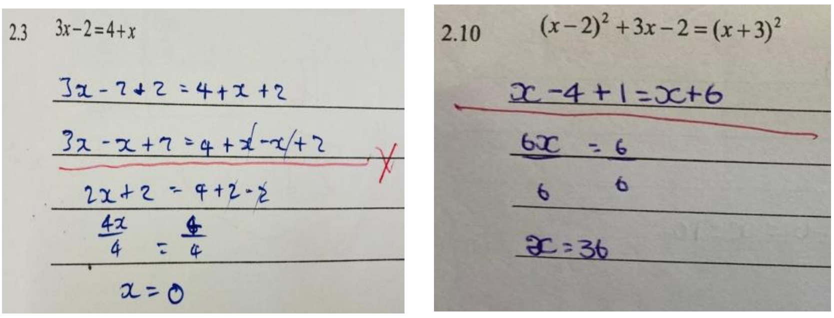


Figure 4.7: Responses of L100 and L144 showing Division Error

After reaching step $\frac{4x}{4}=\frac{4}{4}$ in response to item 2.3, L100 (first extract, Figure 4.7) attempts to isolate the variable x by dividing by 4 on both sides of the equation, instead of dividing 4 by 4 on the RHS to reach $x=1$. The learner then appears to subtract the numerator from the denominator to write $x=0$ as an answer. This indicates a lack of understanding of the meaning of $\frac{4}{4}$. A similar error is made by L144 (second extract, Figure 4.7). After reaching $6x=6$, the learner divides both sides of the equation by 6 in trying to isolate the variable x i.e. $\frac{6x}{6}=\frac{6}{6}$. The learner arrives at a final answer $x=36$. It can be deduced from this that the learner understood $\frac{6}{6}$ to mean 6 × 6.

These types of division errors were made by 47 learners, accounting for 3% of errors identified in this study.

4.2.1.4 Incomplete Procedure Errors

This type of error occurs when learners fail to manipulate an equation beyond the form $Ax = Bx$ or $Ax = B$ to reach $x = k$. In other words, learners start by correctly applying various strategies, but when they reach the step $Ax = Bx$ or $Ax = B$, they stop at this step. It is possible that this occurs because they did not realise that the usual strategies still apply, even at this step.

The extracts in Figure 4.8 below, taken from responses of three learners' (L111, L129 and L135) Activity Sheets, illustrate this error.

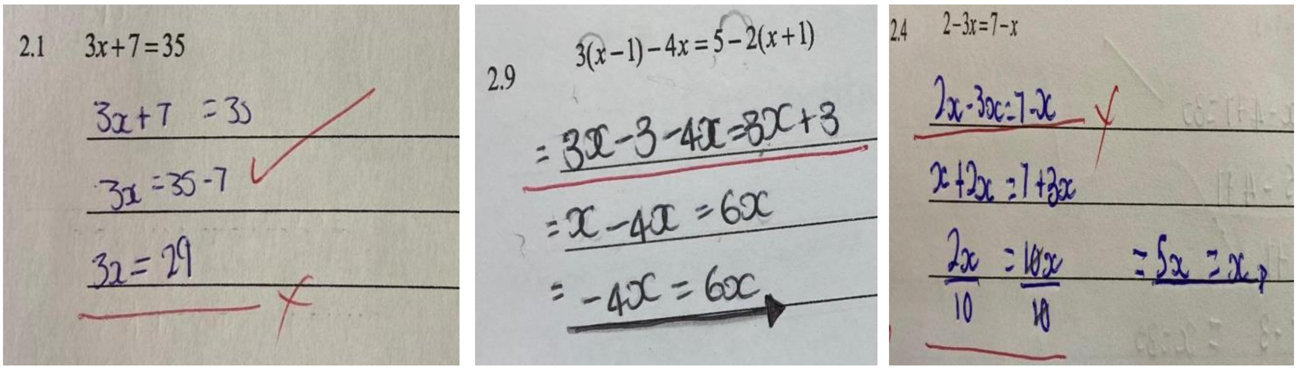


Figure 4.8: Responses of L111, L129 and L135 showing Incomplete Procedure

L111 (first extract, Figure 4.8) begins solving item 2.1 by transposing 7 to the RHS and writing $3x = 35 - 7$.This is then incorrectly simplified to $3x = 29$. The learner is then expected to divide both sides by 3 to isolate the variable, but stops at $3x = 29$. This approach is consistent with findings made by Tall (2002), that learners correctly reached the $Ax = B$ step but stop there. Tall (2002) opines that these errors may arise because learners do not realise what must be done. It appears that they didn't realise that the multiplicative inverse still applied even at the last step. Possible this learner was unable to see that $3x = 29$ is the same as '3 times x' so was unable to identify the operation of multiplication and hence could not apply the opposite operation. Similarly, L129 (second extract, Figure 4.8) manipulates the equation to reach the step $-4x = 6x$, but cannot complete the solution.

85 Incomplete responses are recorded in this study, with 47 of these left in the form $Ax = Bx$, as shown in Figure 4.8. The learners underlined their last step, possibly to indicate it as their final answer in responding to the question (solve for x). This suggests that learners do not understand the meaning of the final answer, as their solutions show no actual value of x. Possibly, learners are unable to identify what step or strategy should be implemented to reduce the equation to one variable only, x, on one side of the equal sign.

4.2.1.5 Equation Changed to Expression Errors

This type of error involves learners changing the equation to an expression by ignoring the equal sign and proceeding as if they're simplifying an expression. The study recorded 317 instances of this, where learners did not follow rules, instead changing the equation into an expression and conjoining terms. The extracts below, taken from the responses of three learners' (L42, L122 and L127) Activity Sheets, illustrate this type of error.

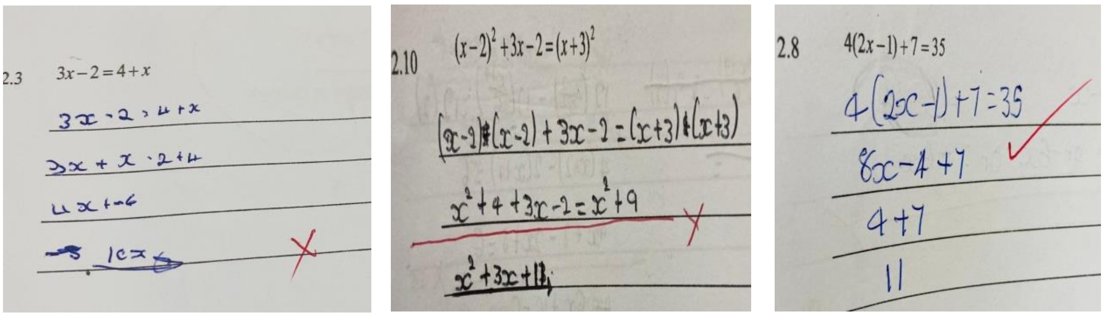


Figure 4.9: Responses of L42, L122 and L 127 showing Changing of an Equation to an Expression

L42 (first extract, Figure 2.3) writes $3x+x-2+4$ as the first step in response to item 2.3. From this, it seems that the learner applies associative property and groups like terms as if simplifying the expression $3x-2+4+x$. Then, the learner conjoins unlike terms, reaching a final answer of $10x$. In this process, they ignore the equal sign and treat the equation as an expression. This lack of understanding is particularly problematic, as learners don't seem to grasp the need to solve for x, and don't recognise that the equal sign describes a relationship between two expressions. In this conception, the equal sign means 'total' as expressed by McNeil and Alibali (2005).

In another example (second extract, Figure 4.9), L122 responds to item 2.10 by first providing their answer as an expression containing three terms. The learner incorrectly simplifies the square binomials on both sides of the equation $(x-2)^2+3x-2=(x+3)^2$, using unknown rules to reach $x^2+4+3x-2=x^2+9$. The equation is then converted to an expression $x^2+3x+11$. This kind of error indicates a lack of understanding of the difference between an expression and an equation (Hall, 2002). The learner groups the like terms from both sides by first writing $3x$ next to x^2, leaving out the other x^2 from the RHS, followed by adding all the constants and getting 11. Again, as in the first example above, the learner does not seem to be solving for x, but only simplifying the expression. Similarly, L127 (third extract, Figure 4.9) responds to item 2.8 by ignoring the RHS of the equation and works with the LHS only. L127 begins $4(2x-1)+7=35$ by removing the brackets and writes $8x-4+7$. The 35 on the RHS is left out, and the learner conjoins $8x$ and 4, writing the next step as $4+7$ and the final answer as 11. This suggests that the learner does not recognise the importance of the equal sign. This was the most prevalent error found in this study, accounting for 18% of the errors identified.

4.2.1.6 Transposition Errors

Transposition error is one of the most significant errors made. When learners make transposition errors they continue working without realising they made a mistake and spend much time producing an incorrect answer. There were 47 cases where this error was made by different learners, and it contributed 3% of the total errors identified in the study. Learners who made this error did not change the sign when ‘moving the numbers and letters’ across the equal sign.

The extracts below, taken from the responses of two learners’ (L1 and L117) Activity Sheets, demonstrate this type of error. In the first example, L1 responds to item 2.3 (first extract, Figure 4.10) by moving -2 from the LHS to the RHS of the equal sign, with no corresponding change in sign. The same thing was done to x from the RHS to the LHS. This accords with the findings of Freitas (2002), who noted that procedures related to phrases such as “change side, change sign” are meaningless to learners and often result in mistakes. In the second example, L117 (second extract, Figure 4.10) responds to item 2.10 by writing $x^2-4+3x-2=x^2+9$. The learner moves constants -4 and -2 to the RHS without changing signs, resulting in $x^4-3x=9-4-2$.

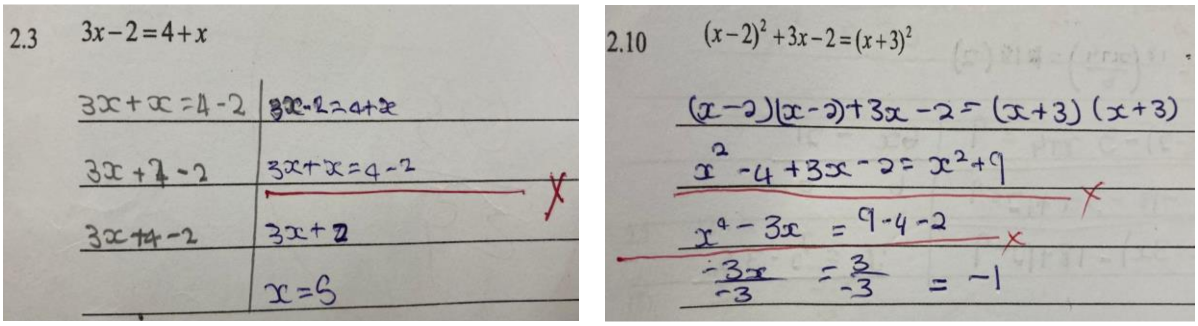


Figure 4.10: Responses of L1 and L117 showing Transposition Errors

The application of commutative properties is cited by Araya et al (2010) as one source of these errors in the learners' work. It is sometimes wrongly used, though in its own right, it is good to teach; the commutative and associative properties are helpful in solving equations because they allow changing the appearance of the equation without changing the relationships between the components. For instance, consider the equation $4 + 5x + 7 = 9$. Addition is commutative, so we can rewrite $5x+7$ as $7+5x$, and the above equation as $4+7+5x=9$. This assists in adding the like terms, 4 and 7. However these properties are not applicable when terms are shifted across the equal sign.

Furthermore, commutative and associative properties do not apply to subtraction. For example, in the extract below, taken from the responses of one learner's (L20) Activity Sheet, the Learner changes the equation $2-3x=7-x$ to $3x-2=x-7$ in the second step. This action changes the relationship between the components making the statement incorrect.

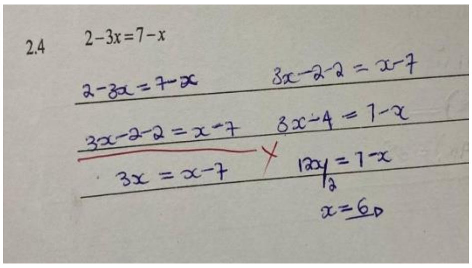


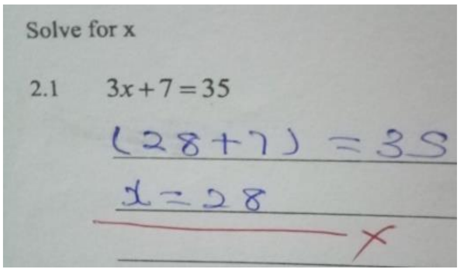
Figure 4.11: Response of L20 showing Transposition Error

Research (Kieran, 2002) suggests that the emphasis on symmetry is absent in the procedure of transposing. Learners who use transposition do not operate on the equation as a mathematical object, but rather blindly apply the change side, change sign rule.

4.2.1.7 Substitution Error

This type of error is observed when a learner is not using an algebraic approach to solve equations, but uses arithmetic to determine *x* and substituted different values of *x*, which will maintain equivalence. The error occurs when substituting the values of *x* that don’t make the LHS equal to the RHS.

The extract below, taken from the responses of one learner’s (L47) Activity Sheet, demonstrates this type of error.



Solve for x

2.1 $3x+7=35$

$(28+7)=35$

$x=28$

Figure 4.12: Response of L47 showing Substitution Error

Seemingly, in responding to item 2.1, the learner understands that by adding seven to $3x$ an answer of 35 is reached. They determine that the number is 28, but give this as the final answer i.e., $x = 28$, as if the equation being solved is $x + 7 = 35$. The number next to *x* is disregarded.

In a similar situation, L66 (first extract, Figure 4.13) responds to $3x + 7 = 35$ by substituting the *x* and writing $3(9) + 7 = 35$ in the second line. The learner multiplies 9 and 3, writing the next step written as $27 + 7 = 35$. Although this learner considered the coefficient of *x* by multiplying 3 and 9, the outcome is incorrect because $27 + 7$ does not satisfy the equation $3x + 7 = 35$. Rather than writing the final answer as $x = 9$, the learner write $x = 27$. This indicates a lack of understanding of the structure of the equation. Another possible reason for the error is that the answer is a fraction.

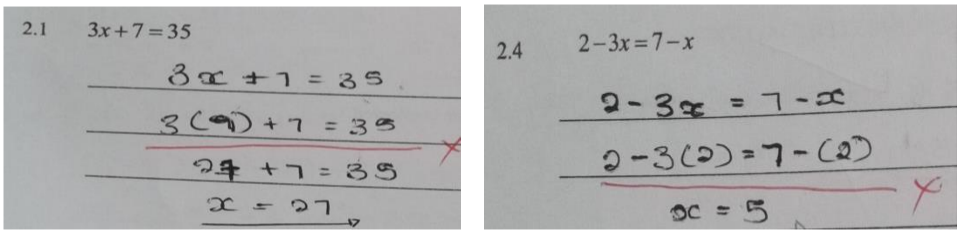


Figure 4.13: L66's Response showing Substitution Errors

In another example, L66 responds to item 2.4 (second extract, Figure 2.4) with variables on both sides using substitution strategy to find the solution, writing *2 - 3(2) = 7 - (2)*. Although the learner appears to comprehend that the letter on the LHS and RHS of the equation has the same value, they don't seem to understand that the two sides need to be equal. Substituting *x = 2* on this item does not maintain balance and is not a solution to this equation.

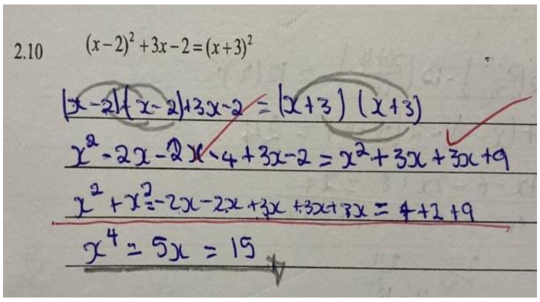
According to Lima and Tall (2007), linear equations with unknowns on sides, such as *3x + 2 = 8 + x*, can't be solved by 'undoing' the arithmetic operations. The increased level of difficulty in solving these equations, known as the 'didactic cut' (Filloy and Rojano, 1989), is founded on the premise that it is more difficult for learners to find the required value because they have to operate on the unknown and not just numbers (as they may have done with evaluation equations).

With regard to the frequency of occurrence, it is unsurprising that Substitution Error is among the most prevalent errors recorded in this study, accounting for 14% of the errors identified. On the other hand, the substitution strategy led to the most correct answers on items with variables on one side (e.g. items 1.1 and 1.2) of the equation. However, when learners attempt to apply the strategy to find solutions to items with variables on both sides, more substitution errors occur. The findings of this study indicate that learners find it more difficult to solve equations with letters on both sides of the equation, supporting Filloy and Rojano's (1989) views in so far as the didactic cut is concerned. 250 Learners made substitution errors in trying to find solutions to linear equations using arithmetic to determine and substitute the values for *x*.

4.2.2 Variable Errors

4.2.2.1 Errors with Exponents

This sub-category of variable errors involves the inappropriate application of exponential laws in the simplification process. The extract below, taken from the responses of one learner's (L138) Activity Sheet, illustrates this type of error.



2.10 $(x-2)^2+3x-2=(x+3)^2$

$(x-2)(x-2)+3x-2=(x+3)(x+3)$

$x^2-2x-2x+4+3x-2=x^2+3x+3x+9$

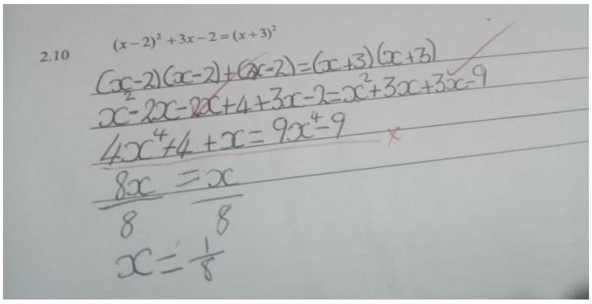
$x^2+x^2-2x-2x+3x+3x+3x=4+2+9$

$x^4=5x=15$

Figure 4.14: L138's Responses showing Exponent Error

In this instance, the learner responds to item 2.10 by correctly simplifying the square binomials on each side of the equation, writing $x^2 - 2x - 2x + 4 + 3x - 2 = x^2 + 3x + 3x + 9$. In the process of grouping like terms, all the letters are transposed to the LHS and constant to the RHS, resulting in the next line $x^2 + x^2 - 2x - 2x + 3x + 3x + 3x = 4 + 2 + 9$. In the following step, $x^2 + x^2$ on the LHS is converted to x^4. It seems that the learner acts as if there are no coefficients in front of each x^2, focusing on the exponents, where the two exponents were added and written as 4. This learner fails to link a new concept with an existing concept, that is, the prior knowledge. The exponential laws are inappropriately applied where $x^2 + x^2$ is treated as $x^2 \times x^2$

The extract below, taken from the responses of another learner's (L7) Activity Sheet, demonstrates similar inappropriate application of laws of exponents.



2.10 $(x-2)^2+3x-2=(x+3)^2$

$(x-2)(x-2)+(3x-2)=(x+3)(x+3)$

$x^2-2x-2x+4+3x-2=x^2+3x+3x-9$

$4x^4+4+x=9x^4-9$

$\frac{8x}{8}=\frac{x}{8}$

$x=\frac{1}{8}$

Figure 4.15: L7's Response showing Exponent Error

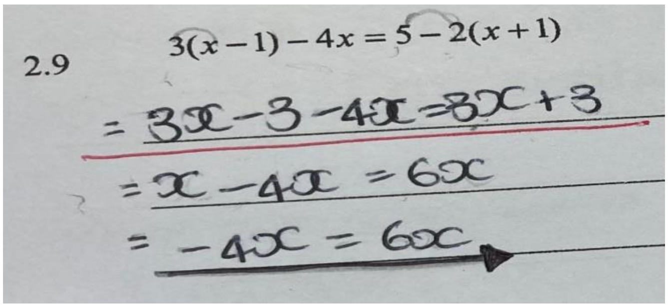
In responding to the same item, the square binomial is simplified to $x^2 - 2x - 2x + 4 + 3x - 2 = x^2 + 3x + 3x - 9$. The focus of the learner seems to be on both coefficients and exponents to the expression $x^2 - 2x - 2x$ on the LHS and the expression $x^2 + 3x + 3x$ on the RHS. However, the term x^2 in both expressions is treated as having no coefficient. The two expressions are simplified to $4x^4$ and $9x^4$ respectively, by multiplying the coefficients and adding the exponents. The equation is then simplified to $4x^4 + 4 + x = 9x^4 - 9$. The concepts of addition and subtraction of like terms in simplifying algebraic expressions appears to interfere with the previously learnt exponential laws.

This type of error was made by 46 learners, contributing 3% to the errors identified in this study.

4.2.2.2 Conjoining Errors

Pournara (2020) describes the different types of conjoining as including additive and subtractive conjoining, which incorrectly reduces two terms (like or unlike) to a single term. Examples of this are $6 + x = 6x$, $3x + x = 3x$ and $3x - x = 3x$.

The extract below, taken from the responses of one learner's (L151) Activity Sheet, illustrates additive and subtractive conjoining of like and unlike terms.



2.9 $3(x-1)-4x=5-2(x+1)$

$=3x-3-4x=3x+3$

$=x-4x=6x$

$=-4x=6x$

Figure 4.16: L151's Response showing Conjoining of Terms

L151 responds to item 2.9 by removing brackets and writing $3x + 3$ on the RHS of the equation. In the next step, $3x + 3$ is simplified to $6x$. Here, the learner ignored the variable, concentrating on the numbers 3 + 3 as if operating arithmetic, then later re-attached the variable x on the

LHS. The expression $3x - 3 - 4x$ is reduced to $x - 4x$, with the learner simplifying $3x - 3$ to x by deleting numbers and retaining the variable x. In the following step, $x - 4x$ is also simplified to $-4x$. It seems that the learner is not aware of what the coefficient of x was, so they drop it and concludes with $-4x$. This was in line with the findings of study conducted by Macgregor and Stacey (1997) to investigate the origins of students' misinterpretations of letter usage in algebra. They tried to get explanations for making the errors and also identified the possible causes of those errors. They deduced that learners can ignore letters while some of them associate them with numerical values. This observation sees learners simplifying $x - 4x$ to $-4x$ as to the appearance of x with no number means there is nothing.

In other instances, learners apply unknown rules of deleting letters or numbers in both like or unlike terms. This error involves the expression consisting of two terms that include negative or positive signs. The expression is reduced to a constant by dropping letters or reduced to a variable by deleting numbers.

The extracts below, taken from the responses of two learners' (L18 and L15) Activity Sheets, demonstrate this conjoining error. L18 (first extract, Figure 4.17) responds to item 2.8 by removing the brackets in $4(2x-1)+7=35$ and writing $8x-4+7=35$. The learner conjoins letters and numbers on the LHS and, for unknown reasons, arrives at $11-11x=35$. They then delete the numbers on the LHS, the expression $11-11x$ becoming x. The final answer is written as $x=35$.This supports Seng's (2010) observations regarding the Arithmetic–algebra gap seen in learners.

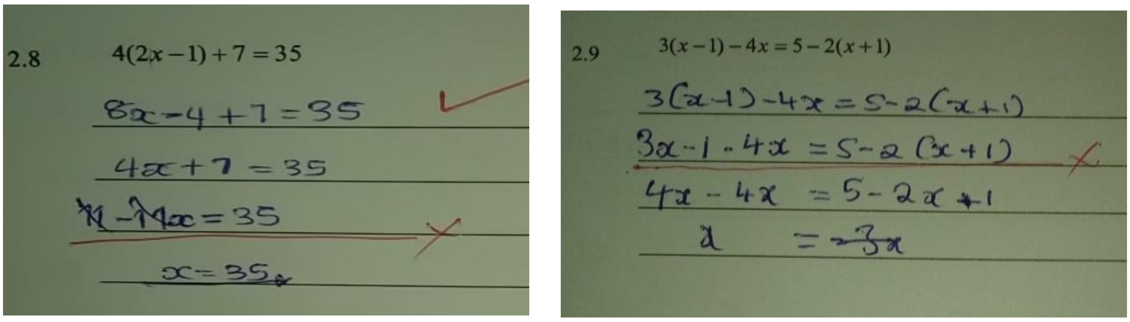


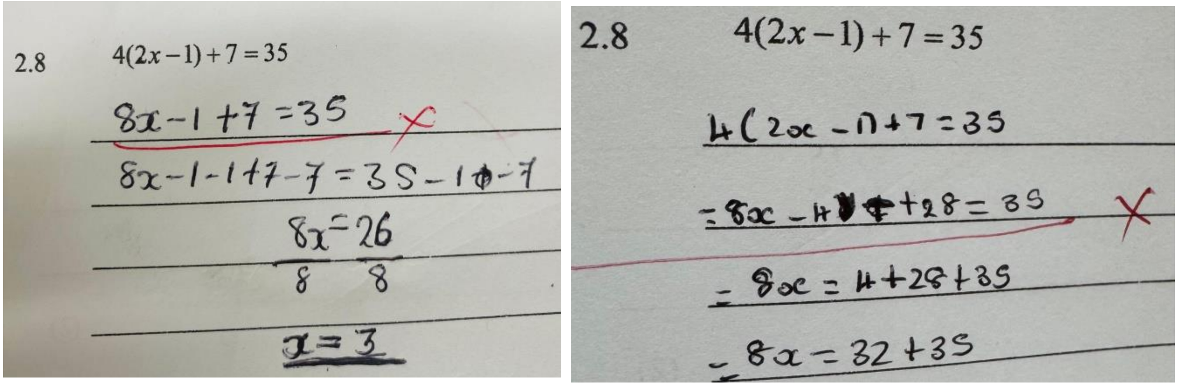
Figure 4.17: Responses of L18 and L15 showing Conjoining Error

L15 (second extract, Figure 4.17), on the other hand, responded to this item by writing $4x-4x=5-2x+1$ as the third step of the process. The learner goes on to write $x=3x$ as the final answer, appearing to have equated $4x-4x$ with 4 - 4. One *x* is taken to the next step with the RHS written as $3x$, possibly because of incorrect conjoining of like and unlike terms.

4.2.2.3 Bracket Errors

There are two types of bracket errors identified in this study, namely invalid distribution of brackets and invalid insertion of brackets.

The extracts below, taken from the responses of two learners' (L151 and L136) Activity Sheets), illustrate distribution of brackets errors.



2.8 $4(2x-1)+7=35$

$8x-1+7=35$ ✗

$8x-1-1+7-7=35-10-7$

$\frac{8x}{8}=\frac{26}{8}$

$x=3$

2.8 $4(2x-1)+7=35$

$4(2x-1)+7=35$

$=8x-4$ [illegible] $+28=35$ ✗

$=8x=4+28+35$

$=8x=32+35$

Figure 4.18: Responses L151 and L136 showing Distribution of Bracket Errors

In the above examples, the learners are unable to simplify the expression 4(2x – 1), being the multiplication of a binomial by a monomial. According to L151 (first extract, Figure 4.18), the 4 preceding the brackets in *4(2x – 1)* is connected to *2x* only, resulting in the expression being simplified as 8x – 1.

L136 (second extract, Figure 4.18) didn't understand the limits of the pre-multiplier in 4(2x – 1) + 7. The constant 7 following the brackets is also multiplied by 4, so that the learner arrives at *8x – 4 + 28.* Here, the distributive law is overgeneralised.

It is noteworthy that learners are applying known rules in appropriate situations but incorrectly adapting those known rules. Some learners, when expanding the square binomial $(a+b)^2$, mistakenly give a^2+b^2. This type of error mainly occurs in response to item 2.10, which had two such expressions, $(x-2)^2$ *and* $(x+3)^2$.

The extracts below, taken from the responses of three learners' (L122, L23 and L114) Activity Sheets, demonstrate three errors involving the incorrect expansion of a square binomial.

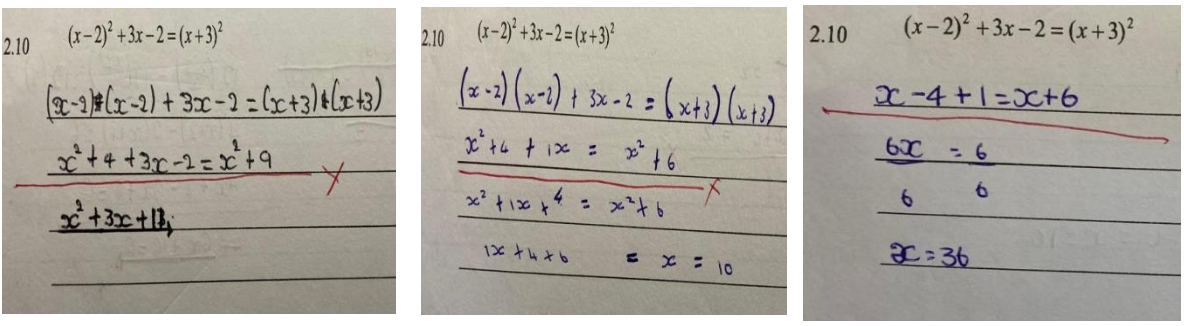


Figure 4.19: Responses of L122, L23 and L144 showing Brackets Errors involving Expansion of a Square Binomial.

In the above examples, each learner appears to follow their own idiosyncratic rule in simplifying a square binomial, which they appear to have made up.

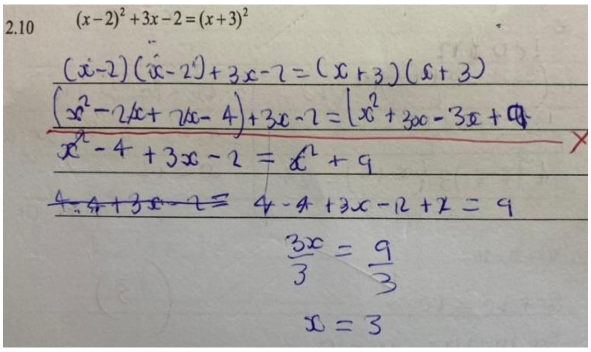
L122 (first extract, Figure 4.19) simplifies the square binomial $(x-2)^2$ as x^2+4 and $(x+3)^2$ as x^2+9· The learner seems to consistently follow a particular rule, which can be described as $(x-a)^2 = x^2+a^2$, when simplifying the square binomial.

L23 (second extract, Figure 4.19) simplifies the square binomial $(x-2)^2$ as $2x-4$ and $(x+3)^2$ as $2x+6$. Similarly, to L122, the learner simplifies the square binomial by consistently following a particular rule, described as $(x+a)^2 = 2x+2a$.

L144 (third extract, Figure 4.19) expands the square binomial $(x-2)^2$ as $x-4$ and $(x+3)^2$ as $x+6$. The learner simplifies the square binomial by following a rule, described as $(x+a)^2 = x+2a$.

These three learners fail to correctly expand the square binomials $(x-2)^2$ and $(x+3)^2$. This supports Mbewe's (2014) findings regarding misconceptions and errors in Grade 11 algebra. The learners seem to have followed rules that they had made up which were known to work for them only. Examples of such rules are $6+x=6x$, $3x+x=3x$ and $3x-x=3x$.

The extract below, taken from the responses of one learner's (L89) Activity Sheet, demonstrates a bracket error that differs from those illustrated above.



2.10 $(x-2)^2+3x-2=(x+3)^2$

$(x-2)(x-2)+3x-2=(x+3)(x+3)$

$(x^2-2x+2x-4)+3x-2=(x^2+3x-3x+9$

$x^2-4+3x-2=x^2+9$

~~$4-4+3x-2=$~~ $4-4+3x-2+2=9$

$\frac{3x}{3}=\frac{9}{3}$

$x=3$

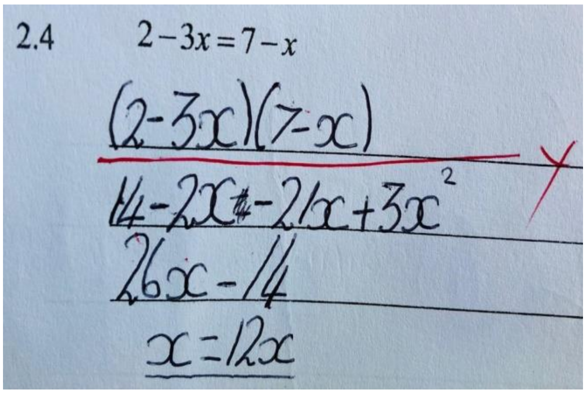
Figure 4.20: L89's Response showing Bracket Errors

In my experience, this type of error occurs more often when learners simplify square binomials (taught before) at a later stage in the syllabus, immediately after they have learnt factorising the difference of two squares. This new concept interferes with the previously learnt concept of the square binomial, which leads to rules being applied inappropriately. For example, factorising the difference of two squares $x^2 - 4$ and $x^2 - 9$ results from the multiplication of binomial by binomial $(x - 2)\ (x + 2)$ and $(x + 3)\ (x - 3)$. This new concept interferes with the previously learnt concept of the square binomial $(x - 2)^2$ and $(x + 3)^2$. L89 illustrates this error above by simplifying $(x - 2)\ (x - 2)$ as $x^2 - 2x + 2x - 4$ on the LHS and $(x + 3)\ (x + 3)$ as $x^2 + 3x - 3x + 9$ on the RHS.

Lima and Tall (2008) use the term "met-after" to describe the above situation, in which newly introduced concepts interfere with the application of rules pertaining to previously learnt concepts. For example, algebraic rules such as simplifying $5x^2 \times 4x^3$ to give $20x^5$ by multiplying the coefficients and adding powers may be misapplied in arithmetic to incorrectly simplify $5^2 \times 4^3$ as 20^5.

The second type of bracket error under consideration is the invalid insertion of brackets. This involves the introduction of brackets that do not exist, rendering the equation more cognitively demanding and leading to errors.

The extract below, taken from the responses of one learner's (L17) Activity Sheet, demonstrates invalid insertion of brackets. L7 responds item 2.4, $2 - 3x = 7 - x$, by writing it as $(2 - x)(7 - x)$. The learner ignores the equal sign and introduces brackets which are not part of the equation. At this stage, they are no longer solving for x, but rather applying the distributive property and simplifying $(2 - x)(7 - x)$ as $14 - 2x - 21x + 3x^2$. The next step conjoins unlike terms $- 2x - 21x + 3x^2$ to $26x$, and moves 14 to the end of the expression, preceded by a minus sign. The equal sign reappears at the final step with the learner arriving at their answer $x = 21x$ (x appears on both sides of the equation).



2.4 $2-3x=7-x$

$(2-3x)(7-x)$

$14-2x-21x+3x^2$

$26x-14$

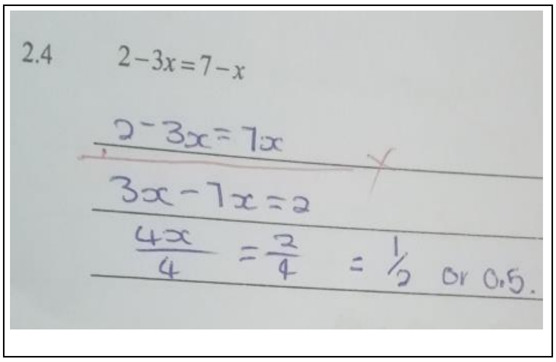
$x=12x$

Figure 4.21: L7's Response showing Bracket Error involving Invalid Insertion of Brackets.

4.2.3 Sign Errors

4.2.3.1 Minus Sign used Incorrectly (Right to Left Reasoning)

Vlassis (2004) describes right to left reasoning as applied to simplify an expression by subtracting the smaller coefficient from the larger one, working from right to left. The extracts below, taken from the responses of one learner's (L117) Activity Sheet, demonstrates right to left reasoning. The expression in the third line (in response to item 2.4), $3x - 7x$, is read from right to left as $7x - 3x$, resulting in $4x$. Similarly, in the second extract (Figure 4.22), the learner has simplified $3x$-$4x$ to x.



2.4 $2-3x=7-x$

$2-3x=7x$

$3x-7x=2$

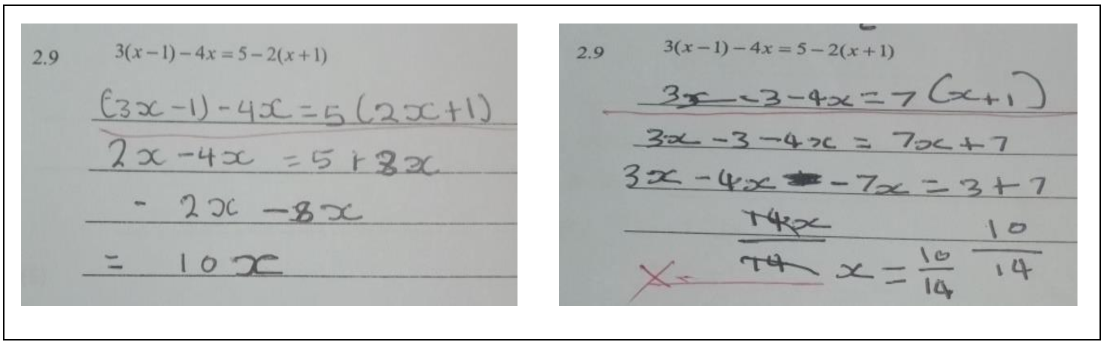
$\frac{4x}{4}=\frac{2}{4}=\frac{1}{2}$ or 0.5.

Figure 4.22: L117's Response showing Right to Left Reasoning

Ryan and Williams (2007), as cited by Sanders (2017), suggest that this error is a result of overgeneralising the commutative property of addition. For example, $a + 1 = 1 + a$ can be overgeneralised to $a – 1 = 1 – a$, which is incorrect because subtraction is not commutative.

4.2.3.2 Minus Sign Ignored

This error involves ignoring the minus sign, so that it disappears from one line of working to the next, sometimes re-appearing in the final steps of the operation. The extracts below, taken from the responses of two learners' (L16 and L121) Activity Sheets, demonstrate this sign error.



2.9 $3(x-1)-4x=5-2(x+1)$

$(3x-1)-4x=5(2x+1)$

$2x-4x=5+8x$

$-2x-8x$

$=10x$

2.9 $3(x-1)-4x=5-2(x+1)$

$3x-3-4x=7(x+1)$

$3x-3-4x=7x+7$

$3x-4x-7x=3+7$

$\frac{\cancel{14x}}{\cancel{14}}\ x=\frac{10}{14}\quad\frac{10}{14}$

Figure 4.23: Responses of L16 and L121 showing Minus Sign Ignored Error

L16 responds to item 2.9 (first extract, Figure 4.23) by simplifying the expressions on both sides of the equation $3(x – 1) – 4x = 5 – 2 (x + 1)$ to $2x – 4x$ *and* $5 + 8x$. The next steps involve inappropriate distribution of brackets and conjoining of unlike terms. The learner then ignores the equal sign and changes the equation $2x – 4x = 5 + 8x$ to an expression $– 2x – 8x$. Finally, the coefficients are added together, incorrectly simplifying the expression to $10x$.

L121 (second extract, Figure 4.23) makes a similar error in response to the same item. The minus sign disappears in the process of the operation, and the expression *3x – 4x – 7x* is incorrectly simplified to *14x.*

4.2.3.3 Minus Sign Recognised but Not Used

The extracts below, taken from the responses of two learners' (L3 and L125) Activity Sheets, demonstrating examples of learners' recognising the minus sign but failing to apply it.

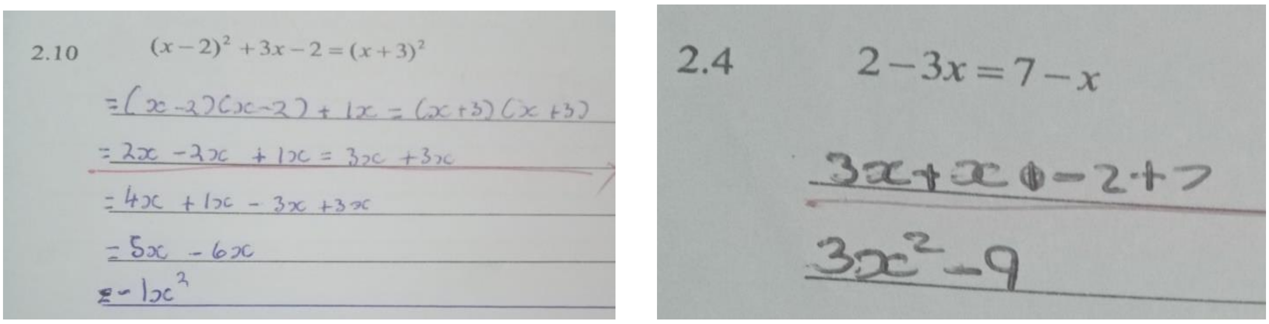


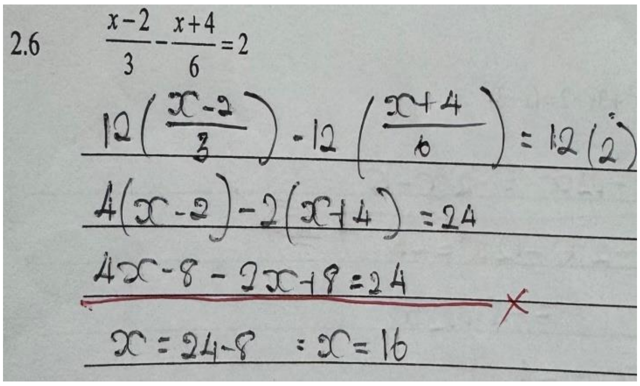
Figure 2.24: responses of L3 and L125 showing Minus Sign Recognised but Not Used Error

In response to item 2.10, L3 (first extract, Figure 2.24), incorrectly simplifies the square binomials on the LHS of the equation $(x – 2)^2 + 3x – 2= (x – 3)^2$, moving the equal sign to the beginning of the expression and writing = *4x + 1x – 3x + 3x*. In the following step, the expression is converted to *5x – 6x*. It appears that the learner adds the *4x* and *1x* to obtain *5x,* but then ignores the minus in front of the first *3x* and adds the two *3x*'s to obtain *6x*. The final line retains the negative sign, the learner arriving at an answer of *– 6x*.

Similarly, L125 responds to item 2.4 (second extract, Figure 2.24) by writing *3x + x – 2 + 7* as the first step. The like terms are added, and the next step is $3x^2 – 9$. In the process of adding like terms, the constant – 2 + 7 is converted to – 9, as if the learner is responding to – 2 – 7. Also, *3x* and *x* are added to reach $3x^2$ as if the learner is responding to $3x \times x$. Here, the learner provides logical answers to a question that is different from the one that was given.

4.2.3.4 Failure to Combine Signs and Directions

In this study, failure to combine signs of terms in front of the bracket with the operation within the bracket was more frequently observed in learners; responses to items where the minus sign appears in front of a set of brackets with two or more terms inside. The extract below, taken from the responses of one learner's (L151) Activity Sheet, illustrates this type of error.



2.6 $\frac{x-2}{3}-\frac{x+4}{6}=2$

$12\left(\frac{x-2}{3}\right)-12\left(\frac{x+4}{6}\right)=12(2)$

$4(x-2)-2(x+4)=24$

$4x-8-2x+8=24$ ✗

$x=24-8 \quad : x=16$

Figure 4.25: L151's Response showing Failure to Combine Signs and Directions

In their response to item 2.6, some learners simplified the expression $4(x-2)-2(x+4)=6$ to $4x-8-2x+8=6$. Although L151 did not struggle with the distributive property when simplifying the expression, the negative sign in front of the pre-multiplier (x + 4) is not properly used. It can be deduced from this that the learner does not consider the minus sign to have an effect on the second term inside the brackets.

A total of 131 learners made different types of sign errors with negatives. These mistakes accounted for 7% of the errors identified in this study.

4.2.4 Slips

This type of error occurs 19 times in the study, most frequently in response to item 2.10. displays the slip error. The extracts below, taken from the responses of two learners' (L33 and L38) Activity Sheets, demonstrate slip errors.

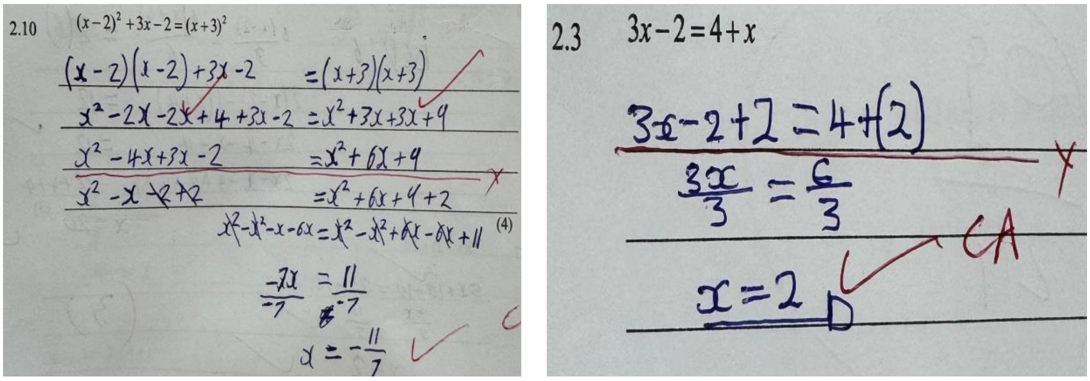
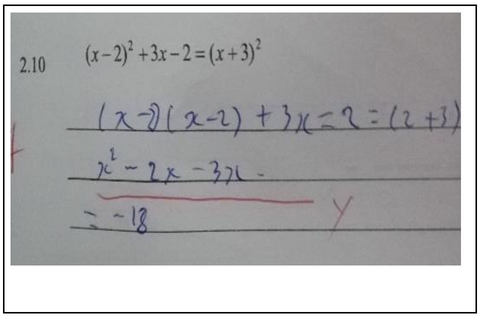


Figure 4.26: Responses of L33 and L38 showing Slip Errors

In response to item 2.10, L33 reaches $x^2 - 2x - 2x + 4 + 3x - 2 = x^2 + 3x + 3x + 9$. During the process of adding like terms on both sides, the learner then writes $x^2 - 4x + 3x - 2 = x^2 + 6x + 9$. All of the steps that followed the addition and subtraction of like terms are correct and logical but the learner has already missed the number 4 in the second step, which results in them arriving at the incorrect answer $x = -11/7$ instead of $x = -1$. Similarly, L38 responds to item 2.3, $3x - 2 = 4 + x$, by grouping like terms using transposition and writing $3x - 2 + 2 = 4 + (+2)$. The learner's intention is clear, but the letter x on the RHS is mistakenly missed, leading to the incorrect answer $x = 2$.

4.2.5 Other Errors

The 'Other' errors refer to responses that did not have any discernible patterns to explain the errors made. The extract below, taken from the responses of one learner's (L124) Activity Sheet, illustrates this type of error.



2.10 $(x-2)^2+3x-2=(x+3)^2$

$(x-2)(x-2)+3x-2=(2+3)$

$x^2-2x-3x$

$=-18$

Figure 4.27: L124's Response showing an Example of Other Errors

L124 responded to item 2.1 by first changing the square binomial, $(x-2)^2$ to $(x-2)(x-2)$ on the LHS. On the RHS, there was also a square binomial; the learner decides to substitute the x inside the brackets and write (2 + 3). The equation is then converted to an expression $x^2-2x-3x$. There is no clear understanding of the learners' intention or thought pattern here; hence, this type of error was classified as 'other'. This type of error occurred 239 times in the study, contributing 13% to the errors made.

Table 4.3 Codes of Categories and Sub-categories and the Frequency of Occurrence

Main Error Categories	Subcategories	Code	Frequency	Percentage
Correct Answers		1	526	22
No Responses		2	68	3
Equation Errors	Familiar Format Errors	3	18	1
	Division Errors	4	47	2
	Incomplete Structure Errors	5	85	4
	Inverse Errors	6	45	2
	Transposition Errors	7	47	2
	Equation changed into Expressions Errors	8	317	13

	Substitution Error	9	250	11
Variable Errors	Conjoining Errors	10	279	12
	Errors with Exponents	11	46	2
	Bracket Errors	12	250	11
Sign Errors		13	131	6
Slip Errors		14	19	4
Other Errors		15	239	10
TOTAL			2367	100

Overall, the study elicited 22% correct responses, 3% no responses (items with responses left blank on Activity Sheets) and 75% erroneous responses. Of the 75% erroneous responses, 35% were equation errors, 25% variable errors, 10% other errors, 6% sign errors and 4% slip errors.

Table 4.4 indicates the distribution of each of the errors (as coded) in terms of the individual items.

ITEM NUMBER→ ERROR CODE ↓	1.1	1.2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	2.10	TOT
Correct Response	100	100	41	82	33	18	36	5	54	38	16	3	526
No Responses (Blanks)	3	1	3	8	7	6	11	5	3	8	6	7	68
Forcing Familiar Format Errors	0	0	0	0	6	0	0	0	0	0	8	4	18
Division Errors	0	0	17	2	2	4	0	3	8	3	4	4	47
Incomplete Structure Errors	0	1	11	11	12	17	5	1	4	2	13	8	85
Inverse Errors	0	0	6	0	11	6	6	2	1	5	2	6	45
Transposition Errors	0	0	2	0	12	8	0	1	2	12	2	8	47

Equation changed to Expression Errors	0	0	28	23	33	36	15	32	32	29	30	59	317
Substitution Errors	50	42	28	1	15	21	25	16	26	16	8	2	250
Conjoining Errors	0	1	25	21	43	53	7	28	27	36	21	27	279
Errors with Exponents	0	0	0	0	8	8	1	7	0	1	3	18	46
Bracket Errors	0	0	0	0	1	36	0	45	2	2	31	14	131
Sign Errors	0	0	2	0	5	4	2	10	41	37	65	84	250
Slip Errors	0	0	3	0	6	2	0	0	0	0	0	8	19
Other Errors	3	11	23	28	24	17	44	42	7	17	11	12	239

The Activity Sheet included six items with variables on one side of the equation, which can be solved by using arithmetic methods. These items are shown in Table 4.5 below. It is noteworthy that there were a total of 377 correct responses (error code 1) to five of these six items, which indicates that correct responses were most prevalent here.

Table 4.5 Six items with variables on one side of the equation

1.1 $\frac{x}{4} = 2$	2.2 $12 = 3x$
1.2 $2x + 1 = 7$	2.7 $5(x + 2) = 20$
2.1 $3x + 7 = 35$	2.8 $4(2x - 1) + 7 = 35$

Changing equations into expressions, error code 8, was the error that occurred most in the study. In the process of solving for *x*, learners group like terms, as if they're simplifying algebraic expressions, and change equations into expressions. Items with variables on both sides are the most common source of this error, which appears 59 times in learner responses to item 2.10.

Substitution error, error code 9, was found to be the second most prevalent type of error in the study. This was observed in instances when learners use an arithmetic approach to solve equations instead of algebra. Arithmetic is used to determine *x* and substitute values of *x* that would maintain the equation's equivalence.

Sign errors, error code 13, are the most prevalent type of error in four particular items (2.7, 2.8, 2.9 and 2.10) on the Activity Sheet, appearing 227 times in learners' responses. Items 2.8, 2.9 and 2.10 included like terms presented in pairs after the step of removing brackets. According to Vlassis (2014) learners add invalid brackets around like terms which are preceded by a minus sign. For example, item 2.8 begins with $4(2x - 1) + 7 = 35$, which is simplified to $8x - 4 + 7 = 35$. To reduce $8x - 4 + 7$, some learners changed the expression to $8x - (4 + 7) = 8x - 11$.

Conjoining errors, error code 10, were found evenly distributed between Activity Sheet items, being dominant in item 2.4, appearing 53 in learners' responses. This type of error involves incorrectly reducing two terms (like or unlike) that are being added or subtracted to a single term.

The overall contribution of each error to this study is presented in Table 4.6 below. This information is also illustrated graphically in Figure 2.28 below.

Table 4.6: Most Common Errors and Frequency of Occurrence

Code	3	4	5	6	7	8	9	10	11	12	13	14	15
Freq	18	47	85	45	47	317	250	279	46	250	137	19	239
%	1	3	5	2	3	18	14	16	2	14	8	1	13

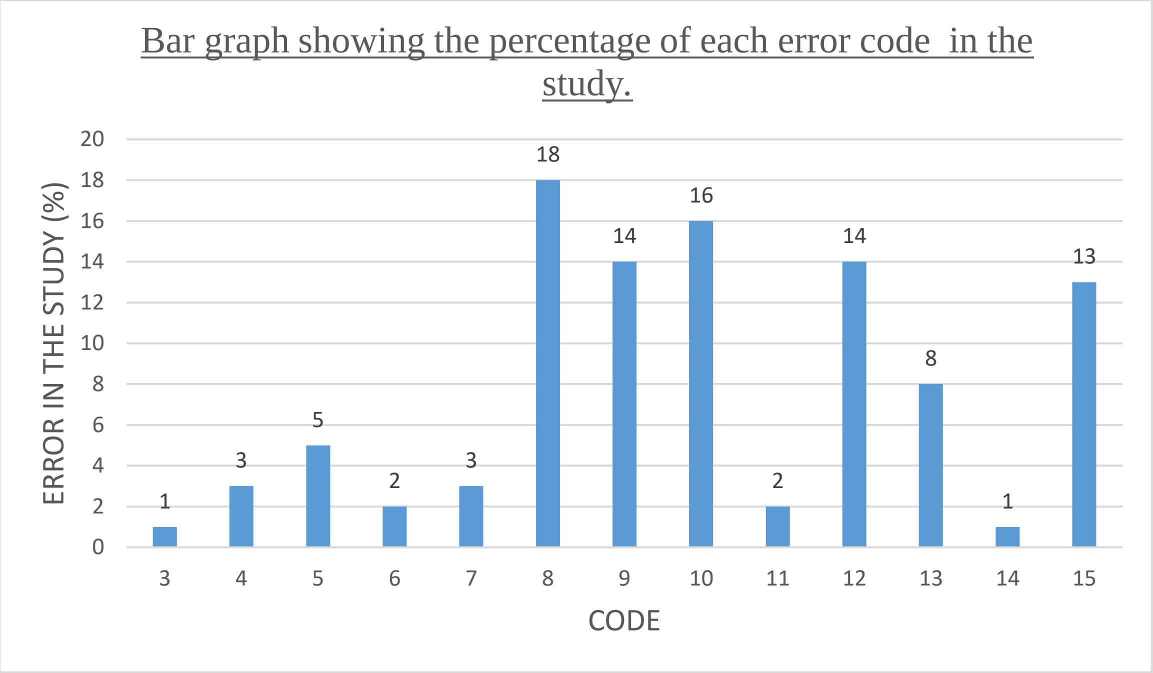


Figure 4.28: Bar Graph showing Percentage Occurrence of Error Codes

The most occurring errors involved changing equations to expression (18%), followed by conjoining errors (16%), bracket errors (16%), substitution errors (14%), other errors (13%) and sign errors (8%).

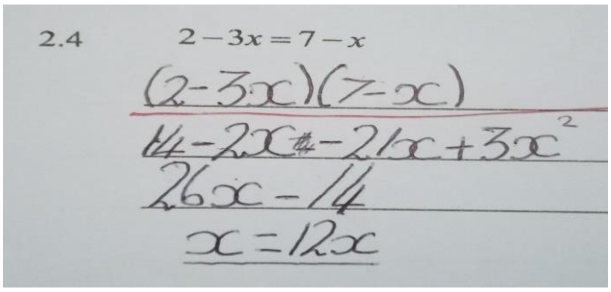
4.3 PRESENTATION OF DATA FROM THE SEMI STRUCTURED INTERVIEWS

This section discusses errors made by learners as revealed from their responses in the

semi-structured interviews. Extracts from transcriptions will be included where necessary to illustrate points made. It should be noted that although a learner's response may indicate the commission of more than one common error, focus of the discussion is on the primary error.

4.3.1 Equation changed to Expression, Conjoining and Incomplete Structure Errors

Some learners transformed equations into expressions, possibly with the intention of joining terms as if simplifying algebraic expressions. Those with limited knowledge of the general rules of algebraic expressions made further errors such as adding coefficients and constants, as well as coefficients of unlike terms. The following extract demonstrates how one learner (L7) changes an equation to an expression and then uses conjoining terms.



2.4 $2-3x=7-x$

$(2-3x)(7-x)$

$14-2x-21x+3x^2$

$26x-14$

$x=12x$

Figure 4.29: Response of L7 Illustrating change from Equation to Expression

The interview with L7 provides some insights into the struggles the learner faces in solving linear equations:

INT: I can see the equal sign is no longer there, but I see it popping up in your last step, please explain.

L7: No, I made a mistake, I forgot it.

INT: You forgot it?

L7: Yes.

INT: Ok, I see you now have 14 – 2x – 21x + 3x^2*, please explain to me.*

L7: The 14 is from 2 times 7 and I brought it here and took this sign and said minus, and I said 2 times x, I got 2x and I forgot equal sign, I went to say 3x times 7 and it gave me 21 and I put x next to it, and I multiply the two signs, negative and negative and I got positive and 3x times x and I wrote 3x exponent two.

INT: Now, below I can see the 14 is from the step above, please explain this 26x.

L7: Ay, I made a mistake here.

INT: Why do you say you made a mistake?

L7: Because I don't see where I got the 26x from.

INT: See, if you say 2 + 21 + 3, I am saying this because I'm looking at these numbers and they appear on the remaining terms after you brought down the 14.

L7: (Learner took a calculator and do some calculations) Yah, they give me 26.

INT: Meaning that is how you got 26?

L7: Yes sir, I then move to the next step.

INT: Next step, how did you get this x in x = 12x.

L7: I brought down this x from 26 because the answer is supposed to be x, so that it will be like we solving for x, then I subtracted 26 from 14 and I got 12x.

INT: So x = 12x is your final answer?

L7: Yes.

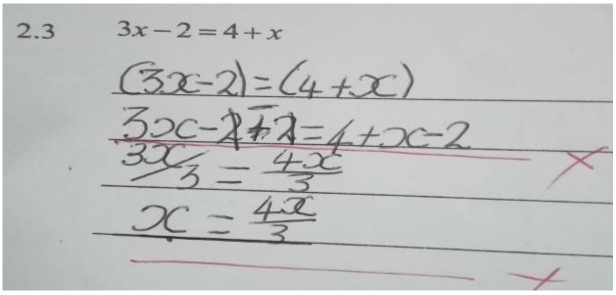
The interview extract above indicates a limited understanding of the difference between an expression and an equation. Learners in Grade 9 spend a significant amount of time on simplifying expressions. The processes of solving equations are taught at roughly the same time. This may explain why learners find it difficult to distinguish between an equation and an expression. This limited knowledge has important implications for solving linear equations (Hall, 2007).

L7 also inserts (invalid) brackets into the equation, transforming it into an expression of two binomials by ignoring the equal sign. The learner simplifies like terms to get $26x - 14$. They describe how they added terms as if applying arithmetic, for instance, 2 + 21 + 3 = 26, while temporarily ignoring the variables x and x^2. The learner then places an x back next to the numerical result. Apparently, learners who conjoin terms (like or unlike) anticipate the behaviour of algebraic expressions to be similar to that of arithmetic expressions.

The learner's responses to the interview questions and acceptance of the incomplete structure $x = 12x$ as a final answer, suggest a lack of understanding of algebraic basics. The learner is unaware of the structure of the final answer when solving for x and seems to be satisfied with having x on the LHS as a single term and any expression on the RHS.

The learner's misconception regarding the structure of the final step (which should take the form $Ax = Bx$) is entrenched, as they repeat this error in attempting to solve other items. The learner believes that the format $x = Bx$ is the final answer.

Figure 4.30 below shows the learner’s response to item 2.4, illustrating a similar error.



2.3 $3x-2=4+x$

$(3x-2)=(4+x)$

$3x-\cancel{2}+\cancel{2}=4+x-2$

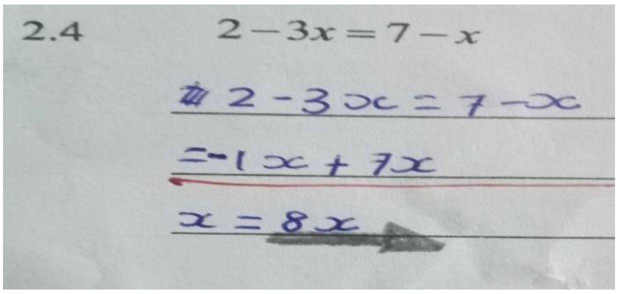
$\frac{\cancel{3}x}{\cancel{3}}=\frac{4x}{3}$

$x=\frac{4x}{3}$

Figure 4.30: L7’s Response showing Incomplete Structure Error

When the learner reaches the step $3x = 4x$, she then divides by 3 on both sides of the equation and writes her final answer as $x = \frac{4x}{3}$. The learner seems to be satisfied with isolating x from $3x$ on the LHS and leaving the RHS as $\frac{4x}{3}$. In this study, errors of this type are categorised as equation errors, falling under the subcategory of incomplete structure.

Another learner, L26, responded differently to questions about the answer left in the form $Ax = Bx$, as shown in Figure 4.31:



2.4 $2-3x=7-x$

$2-3x=7-x$

$=-1x+7x$

$x=8x$

Figure 4.31: L26’s Response showing Incomplete Structure Error

The learner's responses seem to indicate dissatisfaction with arriving at a final answer in this form.

INT: Is this your final answer? ($x = 8x$).

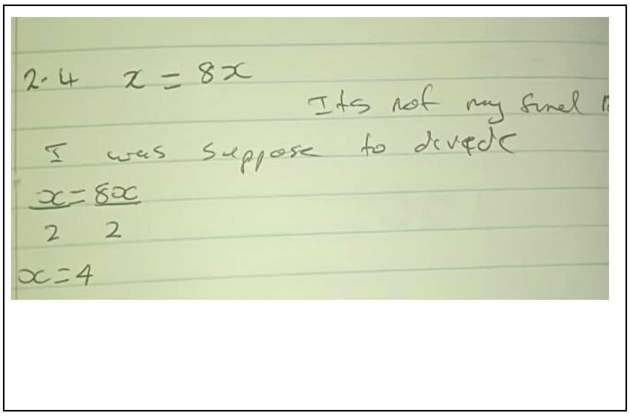
L26: No sir.

INT: But you underlined it, were you supposed to proceed?

L26: Yes I was supposed to divide.

INT: Divide, ok, how? Please show me in writing.

Extract showing learners' response:



2.4 $x = 8x$

Its not my final

I was suppose to divide

$$\frac{x}{2} = \frac{8x}{2}$$

$x = 4$

Figure 4.32: L26's Response illustrating learners' explanation of how they were supposed to divide.

INT: Oh, you divided by 2? Why dividing by 2?

L26: Because I said which number can be doubled to give us 8.

INT: What is that number?

L26: Its 4.

INT: Then where Is the 2 coming from?

L26: How many times its divide.

INT: You mean, you looked for a number that will give you 4 when dividing?

L26: Yes.

INT: And x divided by 2?

L26: It will give us 2x.

INT: I see you wrote x?

L26: Mmm.

The way that L7 and L26 respond to these questions suggests that those who left their response incomplete either did not know what to do next and ceased operating for fear of making mistakes, or they believed they had reached the final answer when the *x* is isolated on the LHS. These deductions are based on L7's and L26's responses. L7 indicates that $x = 2x$ was their final answer to the equation $2 – 3x = 7 – x$, whereas L26 says that their answer $x = 8x$ was not their final answer. Subsequently, L26 attempts to manipulate the equation but is unsuccessful in solving the equation.

L26 also seems to make similar mistakes in responding to other items, especially when they reach the incomplete structure of the form $Ax = Bx$. Figure 4.33 below shows these mistakes:

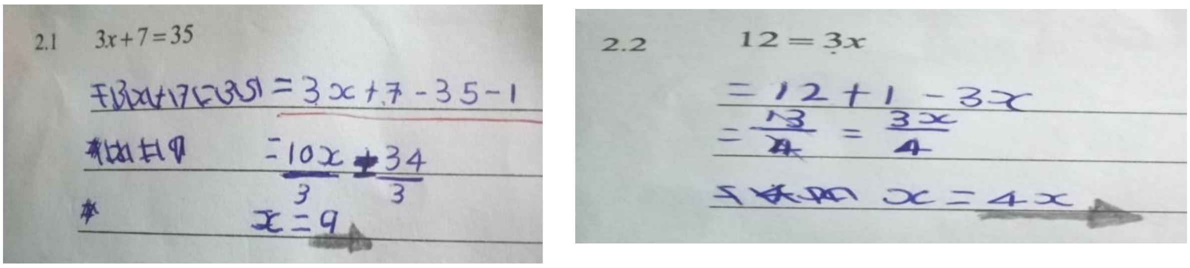


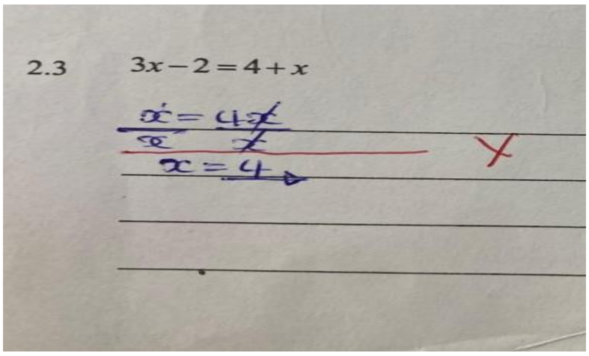
Figure 4.33: L26's Response showing Incomplete Structure Errors

L26 was unsure of the number that could be used as the dividend. In attempting to solve the problem again during the interview, the learner chose to divide by 2 when faced with $x = 8x$.

The process here was $x/2 = 8x/2$ to get $x = 4$. In responding to item 2.1, the learner divided by 3 when faced with $10x = 34$. The process here was $10x/3=34/3$ to get $x = 9$. In responding to item 2.2, the learner divided by 4 when faced with $13 = 3x$ to get $x = 4x$. Seemingly, L26 chose a random number to use as a dividend, and the quotients obtained in each case do not correspond with the numbers used. For example, $\frac{34}{3}$ is not 9, $\frac{10x}{3}$ is not x and $\frac{3x}{4}$ is not $4x$.

4.3.2 Forcing Familiar Format Errors

As mentioned, forcing familiar format and incomplete structure errors are closely related, with both errors occurring at the same step when the equation takes the form $Ax = Bx$. The learners' interview responses indicate that those who commit this error do so because they're seeking to isolate the variable on the LHS of the equation and have number only on the RHS. When they reach the step $Ax = Bx$, they divide both sides of the equation by x, however after doing so they will only cancelled the x on the RHS and leave the LHS as x. For example, Figure 4.34 below demonstrate how one learner (L18) forced the familiar format. After reaching the step $x = 4x$, L18 divides both sides of the equation by x, $\frac{x}{x} = \frac{4x}{x}$, then cancelled x on the RHS only, forcing the familiar format $x = 4$.



2.3 $3x - 2 = 4 + x$

$\frac{x}{x} = \frac{4\cancel{x}}{\cancel{x}}$

$x = 4$

Figure 4.34: L18's Response showing Forcing Familiar Format Error

L18 provides the following explanation for this response:

INT: Please explain to me, how did you went about doing this?

L18: Here teacher I don't know what happened but ………………(silent)

INT: How did you get this x? (point at x on the RHS of the equation x = 4x)

L18: Oh, I said 3 minus 2 and I got 1, we don't write 1 so I wrote down x

INT: Ok

L18: Then on this side 4 + x is 4x. I then divided by x on both sides

INT: Yes, then

L18: Then ,…………………………………………………..(silent)

INT: So x divided by x gives x?

L18: ehhhhhhhhhhh, Yes I think so

INT: You think so?

L18: Yes

INT: Why you divided by x?

L18: Because Teacher if I divided by a number with a ………. (Silent) say if I divide by 2, I was going to have 2x here (pointing to the RHS of the equation x = 4x)

INT: Yes

L: Whereas I wanted to have x on this side (LHS) and a number on this side (RHS) of the equal sign

INT: oh , ok, your intension was to have a number only on the RHS. Is that the reason the why you divided by x on both sides and chose to cancel only the RHS?

L18: yes

INT: oh, ok

INT: If it was 6 = 3x what were you going to do? were you going to divide?

L18: I was going to divide by …………………… (silent). I was going to divide by 3

INT: You were going to divide by 3

L18: …………………(Silent) Yes

INT: Ok

L18’s explanation indicates that this error results from forcing the final answer into a familiar format. In his words “ *I wanted to have x on this side (LHS) and a number on this side (RHS) of the equal sign”,* showing his intention to get the format he wanted .

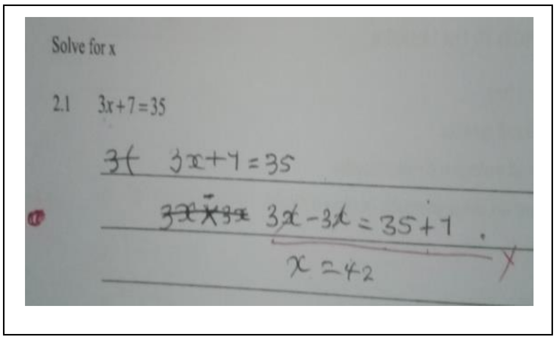
Learners attempt to structure the final answer in the form x = C by dividing both sides of the equation $Ax = Bx$ by x to isolate the variable to one side of the equation. These Learners then cancelled the variable x on one side, supposedly so that the final answer fits a familiar format, $x = C$.

When this learner reached the step $x = 4x$, it is possible that s/he looked for a value of the unknown that would satisfy this equation. S/he may have observed that the number that makes the statement $x = 4x$ true is zero, which would not have given an x on the LHS as he wanted. The learner just followed a solving procedure that h/she knows, without analysing what s/he has in hand. This learner seems to be attached to the use of procedures and relied on them too much. Freitas as cited by Lima and Tall (2008) claims that learners have difficulties with multiplication and division involving zero.

On the other hand, when L7 reach step of the form $Ax = Bx$ in item 2.3 (Figure 4.30), s/he response differs from L18’s response. L7 responds by dividing both sides of the equation $3x = 4x$ *by 3, i.e.,* $\frac{3x}{3} = \frac{4x}{3}$. The learner then writes $x = \frac{4x}{3}$ as the final answer, retaining the x on the RHS. Through this learner divided both sides by 3, s/he did not drop the variable x on the RHS to produce a familiar format structure.

4.3.3 Inverse and Conjoining Errors

Here, learners apply the additive inverse instead of the multiplicative inverse or vice versa. Figure 4.35 below demonstrates the inverse and conjoining errors made by L11.


Solve for x

2.1 $3x+7=35$

~~3(~~ $3x+7=35$

~~[illegible]~~ $3\cancel{x}-3\cancel{x}=35+7$

$x=42$

Figure 4.35: L11's Response showing Inverse and Conjoining Errors

L11 provides the following explanation for their response:

INT: Please look at this item: 3x + 7 = 35.

L11: Yes.

INT: I see you wrote 3x – 3x?

L11: Oh Sir, time was about to finish here.

INT: Time?

L11: Yes Sir, I did this one at the end

INT: You did it at the end?

L11: Yes Sir

INT: Ok, its fine. Now you wrote 3x – 3x , or let us start with 7, I see that you wrote 7 on this side (3x – 3x = 35 – 7)

L11: I wanted to … (silence), oh, I was supposed to write – 7 here (additive inverse of 7 on the LHS 3x + 7)

INT: Oh, you mean additive inverse of + 7?

L11: Yes, because we do the same thing on both sides

INT: Alright, now I can see you have written 42, please explain

L11: I added 35. I said 35 plus 7, yes.

INT: And this x (x on the left hand side)?

L11: I made a mistake and cancelled x, I was supposed to cancel these 3 (3x – 3x). After cancelling 3 I then put x at the bottom, and I said equal to 42

INT: Oh, you were supposed to cancel 3 and 3?

L11: Yes.

INT: Which means what you wrote down below is correct, but you cancelled x here (3x – 3x = 42) instead of numbers?

L11: Yes.

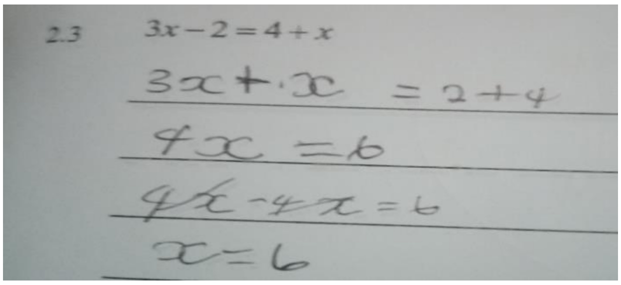
INT: So if you cancelled 3 and 3 which x was written below because I can see you had two?

L11:….(No response)

INT: So, you were left with this x?

L11: Yes.

L11 correctly transposes 7 to the RHS of the equation but is unable to do the multiplicative inverse of 3 in *3x* on the LHS. Instead, the learner decides to subtract *3x*, applying the additive inverse instead of the multiplicative inverse. The learner also makes a variable error when simplifying *3x – 3x* and, when conjoining these two like terms, the learner cancels the same coefficients, 3, bringing down *x*. This type of variable error is termed by Hall (2017) as a 'deletion error'. L11 cannot explain how they came to having *x* in the next step. Their misconception of deleting numbers seems entrenched as this error was repeated in the learner's responses to other items. The extract shown in Figure 4.36 below illustrates a similar error made by the same learner in response to a different item.



2.3 $3x-2=4+x$

$3x+x=2+4$

$4x=6$

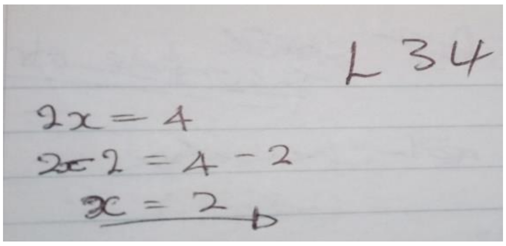
$\cancel{4}x-\cancel{4}x=6$

$x=6$

Figure 4.36: L11's Response showing Inverse Error

This illustrates the impact this type of error has on a learner's performance when they consistently make similar mistakes on different items.

Another learner (L34) responded to *2x = 4* by writing the following:



L34

$2x = 4$

$2x - 2 = 4 - 2$

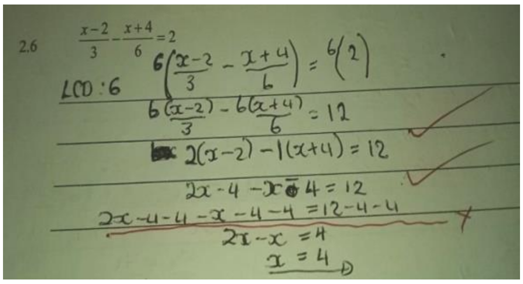
$x = 2$

Figure 4.37: L34's Response showing Inverse Error

This extract demonstrates a common challenge experienced by learners when they reach the step of the equation taking the form $Ax = B$. In the above instance, L34 subtracts 2 from both sides of the equation by applying the additive inverse of 2 instead of the multiplicative inverse. This error is similar to the inverse errors reported by Lima and Tall (2007), discussed in Chapter 2. L34 further simplifies *2x – 2*, reaching *x* by deleting 2 from both terms.

Two learners (L11 and L34) arrived at the same incorrect answer for *x* when conjoining like terms (*3x – 3x*) or unlike terms *(2x – 2)*. They gave similar explanations as to how they came up with this answer.

Additive inverse error was also evident in this study, as illustrated in L36's response to item 2.6 in Figure 4.38 below.



2.6 $\frac{x-2}{3} - \frac{x+4}{6} = 2$

LCD : 6 $6\left(\frac{x-2}{3} - \frac{x+4}{6}\right) = 6(2)$

$\frac{6(x-2)}{3} - \frac{6(x+4)}{6} = 12$ ✓

$2(x-2) - 1(x+4) = 12$ ✓

$2x - 4 - x - 4 = 12$

$2x - 4 - 4 - x - 4 - 4 = 12 - 4 - 4$ ✗

$2x - x = 4$

$x = 4$

Figure 4.38: L36's Response showing Additive Inverse Error

L36 provides the following explanation for this response:

INT: Look at the step where I put the first tick (Pointing at $2(x - 2) - 1\,(x + 4) = 12$*).*

L36: Yes.

INT: Are you done?

L36: Yes.

INT: Now explain to me how you moved to the next step.

L36: I said 2 times x and wrote 2x and drop down the negative and I said 2 times 2 and wrote 4. Then I said -1 times x is negative x. I then multiply the signs and got negative. And I said – 1 times 4 is – 4, then I wrote 12 this side.

INT: Right, next step?

L36: I drop down $2x - 4$ *and subtracted 4 and drop down* $-x - 4$ *and I subtracted to 4 again here.*

INT: You subtracted 4 here (pointing at $2x - 4 - 4$*) and you also subtracted in* $-x - 4$*, why subtract 4?*

L36: Because when you do additive inverse, you subtract

INT: So, when you do additive inverse you only do subtraction?

L36: Yes

INT: Right…

L36: I also wrote $12 - 4 - 4$ *this side (RHS).*

INT: Okay.

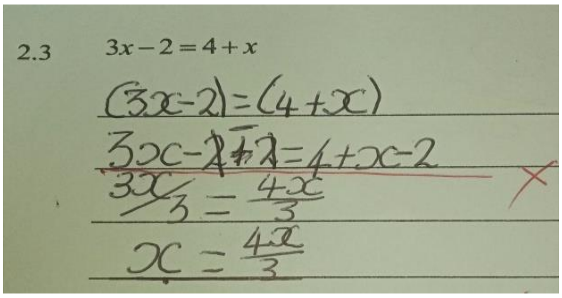
L36: *I then removed these two 4s here and the other two there and I was left with* $2x - x =$

And I said $12 - 4 - 4$ *is 4 on this side.*

L36's response shows that some learners think that the additive inverse only means subtraction. For example, given the equation $2x + 3 = 5$, where the constant on the LHS is positive, the additive inverse of + 3 is – 3. Learners convert the expression to $2x + 3 - 3 = 5 - 3$ and then reduce it to $2x = 2$. Where the constant on the LHS is negative, for example $2x - 3 = 5$, and the additive inverse of – 3 is + 3, learners continue by subtracting 3 from both sides of the equation and writing $2x - 3 - 3 = 5 - 3$. They then reduce the expression to $2x = 2$, which is the same answer reached in the equation $2x + 3 = 5$.

It is clear that adding and subtracting a constant will not yield the same results in the example provided above. Repeatedly exposing learners to similar examples with a positive constant on the RHS only could be one of the reasons for this error.

Additive inverse errors and sign errors are observed in learners' responses to other items. The following extract from L 7's response to item 2.3, demonstrates this.



2.3 $3x - 2 = 4 + x$

$(3x - 2) = (4 + x)$

$3x - \cancel{2} \overset{-}{+} \cancel{2} = 4 + x - 2$ ✗

$\frac{\cancel{3x}}{3} = \frac{4x}{3}$

$x = \frac{4x}{3}$

Figure 4.39: L7's Response showing Additive Inverse and Sign Errors

L7 responds to item 2.3 by writing $3x - 2 + 2$ on the LHS, applying the additive inverse of – 2, + 2, which is correct. However, the learner then changes the expression to 3x – 2 – 2, cancelling the plus sign. They follow this by subtracting 2 on the RHS, possibly to maintain balance. When asked about these changes, the learner gave the following explanation:

INT: I can see here …. (3x– 2) = (4 + 2) you put brackets.

L7: No, here I was doing additive inverse and I cancelled.

INT: You cancelled plus sign?

L7: Yes.

INT: Which means this is?

L7: Its 3x – 2 – 2, I then added – 2 in 4 + x like I did on that side and I wrote 4 + x – 2, I then *move down here and but 3x…*

INT: Oh, – 2 and – 2 is removed now?

L7: Yes, I cancelled them. You said what we do on this side must be done on the other side, isn’t?

INT: Yes

L7: So, I took that 2 with its minus and subtracted it on this side (pointing at 4 + x) as well then I brought down this 4 and then I divided by 3.

INT: I see it’s 4x, not 4.

L7: Eish, which means it was supposed to be 2x because I was subtracting?

INT: I see on the RHS there is 4 + x – 2, but below its now 4x?

L7: I made a mistake.

INT: You made a mistake?

L7: Yes, Sir.

INT: What were you supposed to say?

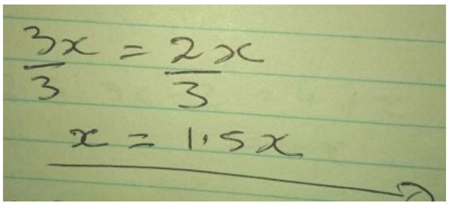
L7: Maybe I was supposed to minus 4 by 2.

INT: And this x?

L7: It was going to be here next to the 2 obtained by subtracting 2 from 4.

INT: Then?

L7: This was going to be... (writes working shown in extract below)



$$\frac{3x}{3} = \frac{2x}{3}$$

$$x = 1{,}5x$$

Figure 4.40 L7 showing Incomplete Structure Error

L7: And I divide by 3 on both sides.

INT: Then your final answer?

L7: Here (punching a calculator) … I got 1.5.

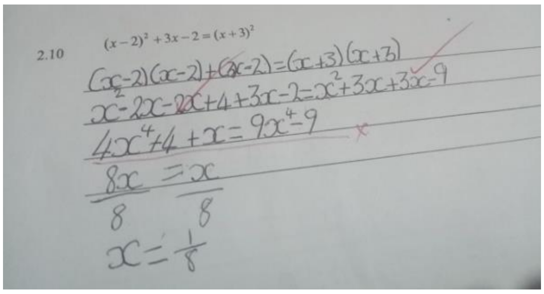
INT: And this is your final answer? (x = 1.5x)

L7: Yes.

L7's response to the questions indicates that they knew that +2 is the additive inverse of – 2 in the equation $2x - 2 = 4 + x$. However, the learner changes it from $2x - 2 + 2$ to $2x - 2 - 2$, and then proceeds to cancel – 2 and – 2, as if the expression was – 2 + 2.

4.3.4 Errors with Exponents

This type of error involves learners confusing knowledge of addition and subtraction of like terms with previously learnt knowledge of exponents. The following extract demonstrates this type of error, where L7's tries to add like terms in response to item 2.10:



2.10 $(x-2)^2 + 3x - 2 = (x+3)^2$

$(x-2)(x-2)+(3x-2)=(x+3)(x+3)$

$x^2-2x-2x+4+3x-2=x^2+3x+3x-9$

$4x^4+4+x=9x^4-9$

$\frac{8x}{8}=\frac{x}{8}$

$x=\frac{1}{8}$

Figure 4.41: L7's Response showing Error with Exponents

L7's interview responses provide some insights into their struggles with trying add like terms.

INT: I see you changed $(x - 2)^2$ *to* $x^2 - 2x - 2x + 4$ *and* $(x + 3)^2$ *to* $x^2 + 3x + 3x - 9$*. Is this – 9 or + 9?*

L7: It's minus 9.

INT: Ok. Please explain starting from this step ($x^2 - 2x - 2x + 4 + 3x - 2 = x^2 + 3x + 3x - 9$) how did you obtain $4x^4 + 4 + x$ and that $9x^4 - 9$?

L7: Let me see teacher…., oh here teacher, it's a mistake.

INT: Where is the mistake?

L7: ... (No response)

INT: So, you say from this $x^2 - 2x - 2x + 4 + 3x - 2 = x^2 + 3x + 3x - 9$ (as I write it down) there was a mistake?

L7: Yes.

INT: Where is the mistake?

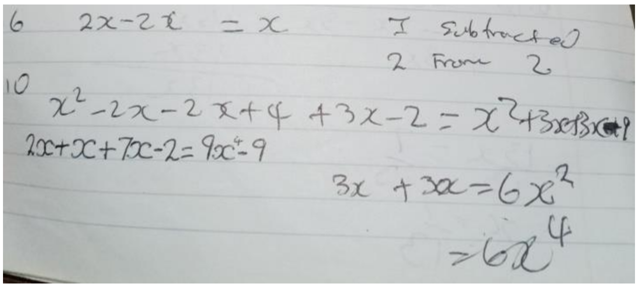
L7: This 4 (pointing at $4x^2$ in $4x^2 + 4 + x$).

INT: Please show me here (as I provide him with paper to write on) how you were supposed to write it.

L7: Must I write here at the bottom?

INT: Yes, continue from there ($x^2 - 2x -2x + 4 + 3x - 2 = x^2 + 3x + 3x - 9$).

L7: ... (writes workings shown in extract below)



6 $2x-2x = x$ I subtracted 2 From 2

10 $x^2-2x-2x+4+3x-2 = x^2+3x+3x-9$

$2x+x+7x-2=9x^4-9$

$3x+3x=6x^2$

$=6x^4$

Figure 4.42: L7's Response showing Adjustment to Item 2.10

L7: And I will do it like this after equal sign

INT: Ok. Is that $9x^4$ written over there?

L7: Yes, it is.

INT: Ok. Where does it come from?

L7: I said here teacher … (Silent), Ay it's a mistake. It was supposed to be 6x because I was going to add these two (pointing at + 3x + 3x) and it will be 6x exponent 2 because we have two x's here (pointing at 3x and 3x).

INT: So you mean 3x + 3x will give us (as I write it down next to his writing) 6x exponent 2?

L7: Oh no Sir, I made this x^4 because there were these other two x here (pointing at x^2), the ones I obtained when I was doing multiplication.

INT: Meaning it's supposed to be exponent 4?

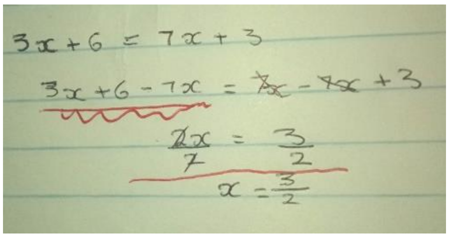
L7: Yes Sir, then I put minus 9 here.

This type of error involves learners using rules inappropriately to simplify algebraic expressions. Even though the focus was on both coefficients and exponents, learners multiply terms, such as $3x \times 3x$, instead of adding like terms. The expression $x^2 + 3x + 3x$ is simplified to $9x^4$ as if it is $x^2 \times 3x \times 3x$, while $x^2 - 2x - 2x$ is simplified to $4x^4$ as if it is $x^2 \times -2x \times -2x$. Previously learnt knowledge of simplification of exponents influenced errors made when adding and subtracting like terms.

4.3.5 Conjoining Errors

During the semi–structured interviews, learners were given an opportunity to provide an example of a linear equation and to show how they would solve it.

L13 provided the example $3x + 6 = 7x + 3$. Their response is in the extract below:



$3x + 6 = 7x + 3$

$3x + 6 - 7x = \cancel{7x} - \cancel{7x} + 3$

$\frac{\cancel{2}x}{7} = \frac{3}{2}$

$x = \frac{3}{2}$

Figure 4.43: L13's Response showing Conjoining Error

n the process of grouping like terms, L13 used the balanced method strategy. The learner applies the additive inverse of $7x$ to both sides of the equation and writes $3x + 6 – 7x = 7x – 7x + 3$. This expression is then reduced to $2x = 3$.

Seemingly, L13 here conjoins terms (like or unlike) anticipating the behaviour of algebraic expressions to be similar to that of arithmetic expressions. Constants and coefficients are added together, and the variable x is attached to a number. It appears that L13 lacks a sense that algebraic terms can be like or unlike.

L13's interview responses provide some insights into their struggles with addition and subtraction of like terms:

INT: I love your equation; it has variable x on both side of the equation. Now if you had to solve for x, how would you go about doing it?

L13: Yes Sir, I start by taking, oh! I wrote 3x and said plus 6 and then put – 7x is equal to 7x – 7x + 3.

INT: Oh! You are doing same thing (subtracting x) on both sides of the equation?

L13: Yes, sir.

INT: Ok, next step.

L13: Then I add this (3x + 6 – 7x) and say 3x + 6 is 9 and 9 minus 7 is 2x and on this side (RHS) I said (writing down 7x) subtract 7 there (7x – 7x) and left with 3. Then I moved down to this step (2x = 3) and said divided by 2, then I said x and I got, … (silent)

INT: What do you need? a calculator?

L13: Yes.

INT: Or write it as a fraction if you like?

L13: Yes.

INT: That is nice, but there is something I need to check with you again. I saw you adding 3x and 7x. its 3 and 7 gives you 10. How did you obtain 2 in 2x=3 below? Did you add all the numbers? (in 3x + 6 – 7x)

L13: I said 6 plus 7, no I said 3 plus 6 minus 7 then I got 2

INT: Oh Ok. But when I looked at 6, 6 do not have a variable x, why are we adding it to 3x and 7x?

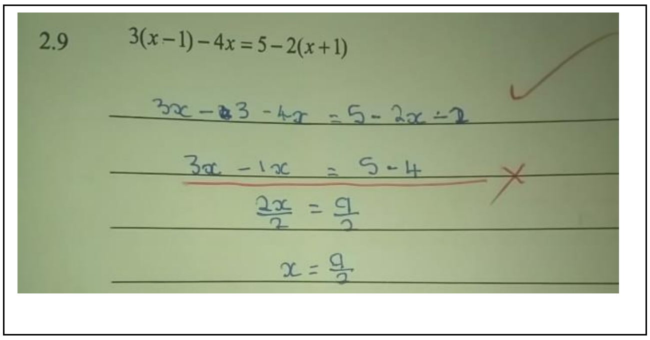
L13: Here, sir, I wanted to have one number when adding everything (3x + 6 – 7x).

INT: You adding everything?

L13: Yes.

It is apparent that L13 does not shift from operating on arithmetic. The focus was on the coefficients and constants and the intention was to have a single term on the LHS.

Another learner (L19) seems to face the same challenge of conjoining terms. The following extract demonstrates this:



2.9 $3(x-1)-4x=5-2(x+1)$

$3x-3-4x=5-2x-2$

$3x-1x=5-4$

$\frac{2x}{2}=\frac{9}{2}$

$x=\frac{9}{2}$

Figure 4.44: L19's Response showing Conjoining of Terms Error

L19 provides the following explanation for this response:

INT: You see in step 2 where there is minus x? (3x – x = 5 – 4). How did you obtain that x?

L: I took 3x as it is, and I said 3 minus 4.

INT: So, you said 3 minus 4 and you got – x?

L: Yes.

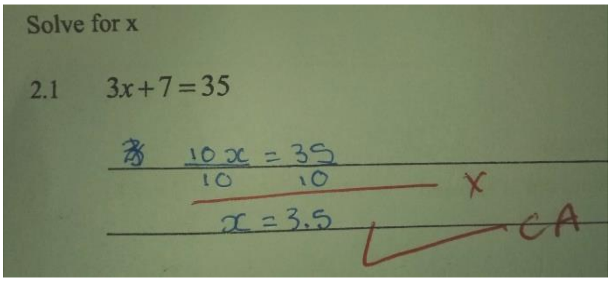
INT: On the RHS I see you wrote 5 – 4, where does the 4 comes from?

L: I took this 2 and this 2 and I add them and then I wrote 4.

INT: And where is x that was next to first 2 on the RHS?

L: I did not write it because it's supposed to be on this side only (pointing to the LHS).

The learner's misconception that the terms should be added as one would in arithmetic, is quite entrenched as this error was repeated in the learner's responses to other items.



Solve for x

2.1 $3x+7=35$

$\frac{10x}{10} = \frac{35}{10}$ ✗

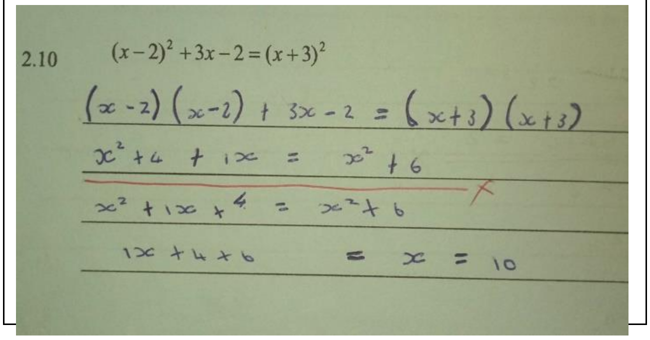
$x = 3.5$ ✓ CA

Figure 4.45: L19's Response showing Conjoining Error

L19's response demonstrates an understanding of applying the additive to isolate variable *x* in the equation *10x =35*. This is apparent from the learner's step of dividing by 10 on both sides to isolate variable *x* and maintain balance. However, incorrectly conjoining terms *3x* and 7 results in the incorrect answer here.

4.3.6 Bracket Errors

Some learners applied their own rules consistently when simplifying two square binomials. There are six different forms of responses in the study displaying invalid distribution errors. The following extract demonstrates one learner's (L23) inappropriate distribution of brackets in response to item 2.10:



2.10 $(x-2)^2+3x-2=(x+3)^2$

$(x-2)(x-2)+3x-2=(x+3)(x+3)$

$x^2+4+1x = x^2+6$ ✗

$x^2+1x+4 = x^2+6$

$1x+4+6 = x = 10$

Fig 4.46: L23's Response showing Bracket Error

Here, the square binomial $(x-2)^2$ was simplified to x^2+4, and the square binomial $(x+3)^2$ to x^2+6. L23 seems to be following a particular rule described by $(x+a)^2 = x^2 + 2a$. The learner's interview responses provide insights into their struggles in this regard:

INT: Please look at your response to item 2.10, I need you to explain to me how you went about writing it.

L23: Yes, Sir.

INT: Are you done?

L23: Yes.

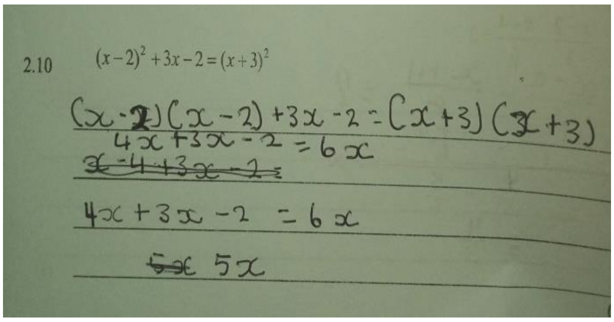
INT: Please explain your second step. How did you move from $(x-2)(x-2)+3x-2=(x+3)(x+3)$ *and wrote* $x^2+4+x=x^2+6$*?*

L23: I said x and x makes x^2 *and then I said 2 + 2 equal to 4.*

INT: And this x?

L23: I added it, I said 3x minus 2 and it gave x and then on this side I said x times x is x^2 *and then I said 3 + 3 is 6.*

By incorrectly distributing brackets, L23 is unable to simplify square binomials. As expressed by Mart (1980, p. 26) that 'learners use unknown rules in appropriate situations but incorrectly adapt the known rules'. Below is an extract from a response by another learner (L16) to the same item demonstrating another type of bracket error.



2.10 $(x-2)^2+3x-2=(x+3)^2$

$(x-2)(x-2)+3x-2=(x+3)(x+3)$

$4x+3x-2=6x$

~~$x-4+3x-2=$~~

$4x+3x-2=6x$

~~$5x$~~ $5x$

Figure 4.47: L16's Response showing Bracket Error

In the above extract, the square binomial $(x-2)^2$ is simplified to $4x$ and the square binomial $(x+3)^2$ to $6x$. The rule L16's seems to be following here, described by $(x+a)^2 = (2a)x$. The learner’s interview responses provide insights into the challenges faced by them in simplifying square binomials.

INT: Let us start here where there is 4x, ok?

L16: Yes.

INT: Look at this step (4x – 3x – 2 = 6x) and please explain it to me.

L16: Here teacher, I added like terms. These 2x which are here, [pointing at (x – 2) (x – 2)] and negative and negative is positive. I then brought down 3x and – 2.

INT: Ok, wait, you said you added like terms?

L16: Yes.

INT: Which are those like terms?

L16: It’s this (x – 2) and (x – 2).

INT: Then you got 4x?

L16: Yes, then I brought down plus 3 which was here (pointing at 3x) and I said minus 2. This side (RHS) I add like terms 3x + 3x [pointing ay (x + 3) (x + 3)] I got 6x.

INT: Oh, I see now (x – 2) and (x – 2) gives 4x?

L16: Yes.

INT: And on the RHS (x + 3) and (x + 3) gives 6x?

L16: Yes.

INT: Ok, but I see here in (x – 2) (x – 2), there is – 2 and – 2. Where is the negative?

L16: I added them, and they gave me positive.

INT: Ok, let us continue.

L16: I then brought down 4x + 3x – 2 = 6x.

INT: How did you get 5x? (next step)

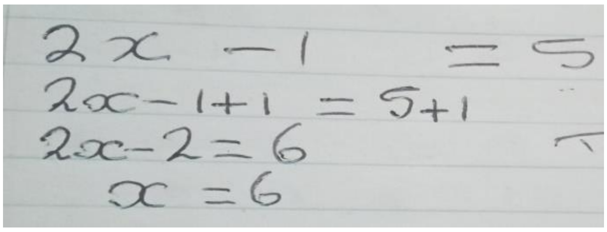
L16: Its negative here (pointing at – 2), I will start by saying 4, 5, 6, 7, (adding 4 and 3), its gives me 7 and 7 minus 2 is 5, then I wrote 5x.

INT: And this 6x, where is it?

L16: Eish, I was lost, I didn't know what to do with it.

4.3.7 Sign Errors and Conjoining Errors

Sign errors refer to errors made when dealing with the negative symbol. Various forms of sign errors involving negative symbol are described in Chapter 3. The extract below demonstrates how one learner (L7) recognises the minus sign, but does not operate with it. After providing $2x - 1 = 5$ as example of an equation, they write the following response as a solution to this equation:



$2x - 1 = 5$
$2x - 1 + 1 = 5 + 1$
$2x - 2 = 6$
$x = 6$

Figure 4.48 showing Sign Error

L23 provides the following explanation for these workings:

INT: How did you go about solving it? Please explain.

L7: Here we bring down 2x – 1 and we add 1 and we also add it on this side.

INT: Ok.

L7: Then below its 2x – 2 and its 6 here.

INT: Ok, where is the 2 coming from?

L7: This 1 and this 1 (pointing at – 1 + 1) gives…, or I am supposed to cancel them?

INT: You tell me?

L7: Ok, one plus one it's two and on this side it's 5 plus 1.

INT: Ok.

L7: Then here I subtracted these two (2x – 2) and I was left with x = 6.

INT: Oh, you subtracted 2 and 2?

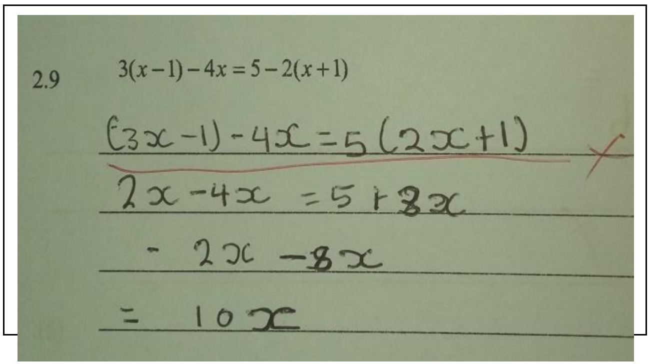
L7: Yes.

INT: Why?

L7: So that I will remain with x on this side.

Although L7 put the additive inverse of – 1 on both sides of the equation, they couldn't determine the correct answer for – 1 + 1 on the LHS of the equation. The expression 2x – 1 + 1 on the LHS is treated as 2x – (1 + 1). The minus sign is isolated from the expression and after adding 1 + 1, it is reattached. This finding aligns with Vlassis; (2004) observation that: "This is a widespread type of reasoning which puts imaginary brackets around like terms when preceded by the minus sign".

Another type of sign error in which the minus sign is ignored is identified in this study. Below is an extract from one learner (L16), demonstrating this.



2.9 $3(x-1)-4x=5-2(x+1)$

$(3x-1)-4x=5(2x+1)$

$2x-4x=5+8x$

$-2x-8x$

$=10x$

Figure 4.49: L16's Response showing Sign Error

L16 explains his response as follows:

INT: Let us start from this step where you wrote $(3x – 1) – 4x = 5(2x + 1)$*. Please explain the next step, how you went about writing it?*

L16: Here Teacher, I said 3 minus 1 is 2 and I brought down this 4, 4x and put equal and put 5 and I took 2 this side and I said 2 plus 1is 3.

INT: Like this side $(3x – 1)$*?*

L16: Yes, I then said minus 2x

INT: Oh, this minus 2x you obtained from it above $(2x – 4x)$*?*

L16: Yes, and I said $– 2x – 8x$*.*

INT: Where is the minus 8x comes from?

L16: It's from I brought down 5 from here $(5 + 3x)$ *and add it with 3.*

INT: And equal sign?

L16: ... (silent). Eish I forgot it.

INT: Ok.

L16: Then I saw that negative and negative makes positive, then I add $(– 2x – 8x)$ *and got 10x.*

INT: You said negative times negative?

L16: Yes, Sir.

INT: Ok, let us go back a bit to this step $(2x – 4x = 5 + 3x)$*. you said* $5 + 3x$ *is equal to 8x?*

L16: Yes, sir.

INT: Let us say you have $5x + 3x$ *what would be your answer?*

L16: It will be 8x.

INT: And $5 + 3x$*.*

L16: It will be… (silent). We don't have x here (pointing at 5) and here there is x (pointing at 3x), we will take this x and put it on this side (next to 5) and then we add.

INT: And the answer would be?

L16: It will be 8x as well.

INT: So answers are the same in 5x + 3x and 5 + 3x?

L16: Yes, Sir.

Vlassis (2004) discusses the overgeneralisation of the sign rule 'minus and minus gives positive', which is often initiated by learners. This overgeneralisation is evident in this study, where L16 added the coefficients of the two terms in the expression $-2x-8x$ and then said, 'negative and negative is positive', simplifying this expression to $10x$. When the learner adds like terms in step 3, they omit the equal sign. The learner is no longer solving for x, as the equation is transformed into an expression by ignoring the equal sign in step 3; when I ask him where the equal sign is, his response is that he forgot to write it. Based on this explanation, it appears that L16 wasn't solving for x here. Rather, they were simplifying an expression to obtain a single term as an answer.

4.4CHAPTER SUMMARY

This chapter presented and analysed the errors committed by learners in solving linear equation. It discussed the findings from the semi-structured interviews, which aimed to establish the possible reasons for the identified errors.

The next chapter examines these findings to answer the two questions posed in this study.

CHAPTER 5

DISCUSSIONS AND CONCLUSIONS OF RESEARCH FINDINGS

5.1 INTRODUCTION

The main focus of this study is to explore the errors made by Grade 9 learners in solving linear equations. Two research questions were posed in the introduction:

1. What errors do Grade 9 learners make in solving linear equations?
2. What are the possible reasons for making such errors in solving linear equations?

This chapter presents a discussion of the findings of the study and relates the findings to the theoretical background and literature review connected to the study. Arising from this, recommendations are made that would benefit both mathematics teachers and the mathematics research community. The chapter concludes with a discussion of my personal reflections on the study.

5.2 DISCUSSION

5.2.1 Answers to Research Question 1

The study's first objective was to identify common errors made by the learners in solving linear equations. Five common errors made by the learners were identified, including errors relating to equality, variable errors, sign errors, slip errors and other errors. The analysis shows that these errors occur in different instances while solving the linear equation. This was evident in Table 4.3, where certain errors are prevalent in learners' responses to certain items, for example, responses to items 2.7, 2.8, 2.9 and 2.10 are dominated by sign errors. These results are consistent with the findings of Mahlabela and Bansilal (2015), and Swan (2002), who suggest that errors made by learners while solving problems can be different, depending on the task itself.

5.2.1.1 Equation Errors

Equation errors contributed 46% to all errors made in this study, and within this category, changing the equation to an expression error was the most frequently occurring error, at 18% of all errors. These equation errors suggest that some learners are transforming equations to expressions by leaving out the equal sign. For example, changing the equation $3x - 2 = 4 + x$ to $10x$ (which was shown in Figure 4.9 first extract in the previous chapter) suggest that learners lack understanding of the equal sign; that the equal sign means that the two expressions appearing on each side of the equation are equal in value. The South African curriculum teaches Grade 9 algebraic expressions and equations during term 2. Hall (2002) expresses that this confusion involving the equal sign is because the procedure of simplifying expressions and that of solving equations are usually taught at the same time.

Some learners don't recognise the equal sign as a symbol expressing a relationship between expressions before and after the sign. These learners see the equal sign as a 'do something' symbol (Behr et al., 1980, p. 15), which indicates that they must calculate or compute when they see the equal sign. These learners believe that the answer' always follows the equal sign. With this level of understanding, the equal sign is nothing more than an operator symbol that connects a problem with its answer (Kieran, 1981).

In some cases, even if the equal sign was used, it does not connect the two sides of the equation as it should. The analysis suggests that learners make mistakes because they manipulate equations to obtain a familiar structure and to avoid structures that are unfamiliar to them. Data analysis indicates that learners don't only memorise algebraic rules, but also the structure of the solution to the equation without understanding the meaning of the solution. This approach is similar to the tendency of learners to mimic prototypical diagrams in geometry (Ngirishi & Bansilal, 2019). Generally, there is a tendency to shift the symbols around to achieve the desired solution. This aligns with Pournaras' (2020) observations that learners apply unknown rules, dropping variables on one side of the equation to obtain an answer in a format that is familiar to them, namely $x = k$. Some learners apply the additive inverse instead of multiplicative inverse or vice versa, in trying to isolate variable x, leading to other errors.

Breaking an equation into two parts as observed in prior research, is not seen in this study.

5.2.1.2 Variable Errors

Variable errors are the second most common errors made in the study, contributing 32% to errors identified. Accordingly, this category significantly impacts whether learners reach correct solutions to equations.

There were three instances where this error is observed. The first one involves conjoining of terms. Learners who conjoined terms anticipated the behaviour of algebraic expressions to be similar to that of arithmetic expressions. This is supported by Kieran (1992, p. 398) who suggests that learners simplify algebraic expressions by first computing according to the rules of arithmetic, and then tacking on letters.

Secondly, learners use the invalid distribution of brackets or insert unknown brackets. The learners who simplify square binomials $(x - 2)^2$ and $(x + 3)^2$ to $x^2 - 4$ and $x^2 + 9$ respectively, seem to consistently follow a particular rule which can be described by $(x - a)^2 = x^2 - a^2$ and $(x + b)^2 = x^2 + b^2$. This aligns with Mart's (1980) observations that learners use made-up rules in inappropriate situations, and incorrectly adapt the known rules. Kieran (1992) also suggests that the rules of algebra offer limited meaning to many learners, resulting in memorising algebraic rules with little or no conceptual understanding. Possibly, the learners who participated in the study confused the square binomial $(x + 3)^2$ with the application of exponential laws learnt before, for example, where $(a.b)^2$ is simplified to $a^2. b^2$.

The third case involves the overgeneralisation of exponential laws. The expression $x^2+3x+3x$ is simplified as $9x^4$. $9x^4$ is likely a result of the product of the given expression rather than the sum.

5.2.1.3 Sign Errors

Sign errors contributed 7% to errors made in this study. This error occurs when learners add constants or coefficients of like or unlike terms and in simplifying expressions that have a negative presented in front of a set of brackets. These learners appear to struggle with subtracting larger numbers from small numbers, calculating problems from right to left in order to get a positive number as an answer.

In some instances, learners detach the minus sign from the expression and disregard it. This is consistent with the findings of Ryan and Williams (2007), where the minus sign is excluded or isolated and viewed as an object on its own in isolation. The minus sign then reappears in the final answer. In expressions where the minus sign is presented in front of a set of brackets, it is only connected with the first term inside the brackets and not the other terms. As pointed out by Pournara et al., (2016), South African learners have greater difficulty in dealing with algebraic expressions which involve negatives – either as sign or as operation.

5.2.2 Answers to Research Question 2

The study's second objective was to explore possible reasons for the errors identified. Three reasons were identified, namely interference of old/new knowledge, difficulties with negatives and subtraction, and teachers' pedagogical approaches.

5.2.2.1 Interference of Old/New knowledge

Analysis shows that most errors occur during the simplification process, such as in the addition of like terms and distribution of brackets.

Mbewe (2013) notes that learners misuse previously learnt procedures and rules in situations where they are not applicable. This concurs with the findings of this study, in which some learners apply laws of exponents during the addition of like terms. One learner (L7) simplifies the expression $x^2 + 3x + 3x$ to $9x^4$ (see Figure 4.15). The explanation given for the numbers 9 and 4 in $9x^4$ was that there are four x's in the expression and 3 x 3 is 9. The learner believed that it was acceptable to add the powers because the bases (x) are the same for 3 terms. Likewise, instead of adding the coefficients, the learner multiplied them to get 9. When questioned during the semi-structured interview, L7 changed the answer to the above expression to $6x^4$. This confusion shows the learner's lack of basic conceptual knowledge in simplifying algebraic expressions.

When learners are introduced to new content, they can become confused about how this connects with their existing, 'met-before' knowledge (Lima & Tall, 2008). L7 confuses 'met-before' concepts, such as simplifying exponents, with the addition of like terms in algebra.

Learners in this study also change expressions to equations by ignoring the equal sign. They continue by conjoining terms as if they're simplifying an expression, sometimes forcing it to be one term. When two of the interviewed learners were asked where the equal sign was, they both indicated that they forgot to insert it. Some learners, when presented with equations, group and add like terms, ignoring the equal sign entirely. This kind of error indicates that learners confuse expressions and equations and don't understand the need to solve equations. Further confusion may result from the fact that the processes of simplifying expressions and solving equations are taught at roughly the same time in the curriculum (Tall, 2007).

Linking algebra with arithmetic also leads learners to make errors in solving linear equations. Learners who combine unlike terms anticipate the behaviour of algebraic expressions to be similar to that of arithmetic expressions. This error has strong roots in arithmetic and shows how the transition from arithmetic to algebra is extremely problematic. Having regard to this, it is unsurprising that errors resulting from ignoring the equal sign and conjoining terms are the most prevalent in this study.

The study also exposed difficulties faced by learners in expanding square binomials. They remove brackets inappropriately in a variety of processes (see Figures 4.18 and 4.19) and simplify square binomials using their own rules which seem to work only for them. This supports Mart's (1980) findings that learners use unknown rules in appropriate situations, but incorrectly adapt the known rules. The findings of this study also bear out Kieran's (1992) observations that the rules of algebra offer limited meaning to learners, resulting in memorising algebraic rules with little to no conceptual understanding. Additionally, Lima and Tall (2008) note that 'met-befores' can significantly impact new learning, while new knowledge and experiences can affect the way in which learners conceive existing knowledge, referred to by Tall (2007) as 'met-afters'.

5.2.2.2 Difficulties with Negatives & Subtraction

Learners encounter negative numbers for the first time in Grade 7. Prior to this, the minus symbol (–) has only one meaning: subtraction. When learners encounter negative numbers, the minus symbol takes on another meaning: it can represent a sign (negative) or an operation (subtraction). This can be confusing to learners.

The findings of the study indicate three possible reasons for errors related to negatives. Firstly, when several minus symbols (–) appear in an expression, the rule 'negative and negative gives positive' is overgeneralised. As expressed by one learner (L16) who responded to questions about this process (see Figure 4.49) saying, "Then I saw that negative and negative makes positive, then I add ($-2x-8x$) and I got $10x$."

A second possible reason, suggested by Vlassis (2004), involves learners inserting random brackets around like terms when those terms are preceded by the minus sign. They achieve this by detaching the minus sign from the expression and disregarding it, however, it's typically reattached in the final answer.

Lastly, learners' errors in simplifying algebraic expressions suggest that they may not be paying attention to signs and operations. This is most evident when an equation involves subtraction and negative, and learners focus mainly on letters and numbers. For instance, in Figure 4.24, the expression $-2(x + 4)$ is simplified to $-2x + 8$. This learner does not encounter problems with the distributive property. However, they struggle with the minus symbol in front of the pre – multiplier '2', which is erroneously associated with only the first term inside the brackets. Although some learners are able to isolate the variable x by correctly applying the inverse operation, incorrect simplification due to sign errors leads to wrong answers.

5.2.2.3 Teachers' Pedagogical Approach

Although there is no direct evidence from teachers, research suggests that teachers' pedagogical approach has an impact on how learners solve linear equations. Findings made by Musi (2023) on chapter 2 noted Teachers' lack of pedagogical knowledge and their inability to explain concepts in-depth. Teachers develop 'mal-rules' for solving equations without explaining how or why they work, and learners are expected to know and apply them. For example, rules like, 'pass the number to the other side of the equation', with the magic of 'changing signs or 'putting the number underneath the term on the other side (Lima and Tall 2008, p.8). Hall (2002) defines 'mal-rules' as rules perhaps invented by learners, which appear to work but in fact, work only under certain conditions. A study by Van Laren & Moore-Russo (2014) on teachers' beliefs about algebra noted that some teachers focus on what Skemp (1976) called 'rules without reason', indicating an emphasis on knowing what to do but not knowing why. This approach by teachers encourages learners' invention of mal-rules.

Pournara et al., (2016) also suggests that 'over-simplified and over-generalised rules' may lead to errors. This is illustrated by L16's workings in Figure 4.49 above, in which the learner over-simplifies the sign rule 'negative and negative gives positive' to simplify the expression $-2x - 8x$ to $10x$.

5.2.3 Personal Reflections

This study has provided valuable insights into several issues, including learners' errors in solving linear equations and the potential root causes of these errors. It has highlighted the importance of anticipating errors likely to be made by learners and of actively listening to learners' explanations as to how they reached their answers in order to understand where and why these errors occur. It is apparent that the errors identified crop up in different places in learners' solutions to linear equations, depending on the structure of a given equation. This is evident from the fact that learners made five different errors in simplifying the square binomials $(x - 2)^2$ and $(x + 3)^2$.

The study also demonstrates that learners not only memorise algebraic rules as suggested by research (Kieran 1992), but also the structure of the solutions to equations, without understanding the meaning of the solution. The importance of teachers' awareness of prior knowledge and experience that learners bring to the classroom cannot be overemphasised. Understanding how this knowledge is frequently misapplied, leading to errors, gives teachers the opportunity to intervene before those errors become deeply rooted.

However, it is vital for teachers to be mindful that awareness of common errors made by learners is not sufficient on its own to effect meaningful change. Teachers may operate under the assumption that teaching the same content repeatedly will assist in learners gaining a better understanding of the material. However, research suggests that this is not usually helpful because learners tend to cling to their misconceptions until they become dissatisfied with their beliefs and receptive to alternative, more plausible explanations (Hewson, 1992). Luneta and Makonye (2010) indicate further that instructors need to shift from teacher-centred methodologies and focus more on learner-centred approaches to teaching, in which instructors view themselves as being responsible for helping learners overcome the misconceptions that they've constructed.

Knowledge of learners' errors and misconceptions can also improve the quality of teaching linear equations, because teachers are able to alert learners to errors and address these while the initial teaching is taking place.

The study also indicates that when learners give wrong answers, it is not so often that the processes they apply are wrong, but rather that they are answering (what is, to their minds) a different question. This emphasised the importance of listening carefully to learners' explanations as to how they arrive at their answers. A teacher's responsibility is to find out what question they are actually answering, in order to remedy errors made. In this regard, Olivier (1989) suggests that (from a constructivist perspective), errors are considered the natural result of learner's efforts to construct their own knowledge, and these errors are intelligent constructions based on correct or incomplete (but not wrong) previous knowledge. These mistakes are deviations from what they have been taught. Research by Mutambara & Bansilal (2022) note that as learners develop their understanding of particular concepts, they develop misunderstandings and make errors. Knowledge of these could be used by instructors to improve learners' understanding.

5.2.4 Limitations

As the findings of this study are based on data obtained from Grade 9 learners from two schools only, they cannot be generalised to all Grade 9 learners in South Africa. At the outset, the intention was to analyse the learners' responses and conduct semi-structured interviews with learners, focusing on each identified error. In this instance, semi-structured interviews could only be conducted with 9 learners, which mean that the data gathered could not include all errors identified. In the circumstances, it was not possible to gain insights into learner's thinking processes on other errors made in this study. Some learner's responses did not add insight to the study because there were no real patterns in their responses even when they did not leave blank spaces.

This study did not focus on the teachers' pedagogical approaches, but rather on the learners and how they work. As such, discussion around teachers' pedagogical approaches and the extent to which these contribute to common learner errors is largely based on research previously conducted. Conducting interviews with teachers and attending their lessons would have been a useful addition to this study, but information gathering in this instance was limited to learner data.

5.3 RECOMMENDATIONS

Mamba (2012) correctly opines that any study is fruitless if the insights gained are not used for future. I believe that the insights gained from this study are going to benefit the South African mathematical landscape significantly, by making teachers, subject advisors and other stakeholders aware of common errors made by learners when solving linear equations, so that they are better equipped to address errors as they arise and avoid problems arising repeatedly as a result of those errors.

5.3.1 Recommendations for Teaching and Learning

The study's findings show primarily that learners lack a clear understanding of fundamental concepts of algebra and the meaning of equal sign. Analysis of errors reveals that learners' errors stem largely from difficulties encountered in manipulating algebraic expressions, dealing with negatives and executing the standard procedure for solving linear equations.

The lack of understanding of fundamental algebraic concepts indicates that learners do not have adequate opportunities to learn algebra content (Mtshali et. al. 2023b). In this regard, it is recommended that teachers adhere to the curriculum guidelines when delivering instruction. It is recommended that teachers should assist learners in grasping basic algebraic concepts, such as grouping and adding like terms, removing brackets, and addition and subtraction of algebraic like terms. Pournara (2020) suggests that teachers and curriculum developers need to treat equations at Grade 8 and 9 levels as an opportunity not only for their charges to learn to solve equations but also to strengthen learners' algebraic manipulation skills, particularly when working with brackets and negatives. For example, activities may be extended to include equations with more than two terms on each side, such as. $6 - 6x + 2 = 5x + 14 - x$. In this case, learners may be encouraged to simplify each side by collecting like terms, before performing inverse operations.

Another major challenge to learners in solving equations is the significant misunderstanding of the equal sign and its properties. This is clearly demonstrated in the study. To address this challenge at a beginner level, it is suggested that the following method be adopted teaching learners to solve formal equations: firstly, it should be clearly established that the equality sign is a symbol that denotes the equivalence between the LHS and RHS of an equation. This should be followed by instruction on performing the same operation on each side of the equation, and then guidance on using substitution for verification.

Additionally, it is recommended that teachers, subject advisors at the district level and all other stakeholders, should be made aware of common errors made by learners and the misconceptions they hold that led to these errors when solving linear equations. Awareness of these errors and misconceptions from the early grades may strengthen efforts to find solutions early, minimising errors and the consequences at an early stage (and subsequently, as learners learn more advanced curricula). The insights gained from the study may benefit Grade 9 teachers as custodians for learning early-stage algebra.

5.3.2 Recommendations for Further Research

It is recommended that further research focus on gathering data on the role of teachers in identifying and addressing learner errors in solving linear equations, and assisting learners to avoid errors as they progress to more advanced mathematics. By studying how teachers deliver algebraic concepts to learners, their contributions to the commitment of common errors by learners can be understood and addressed. As Gumbo (2014) observes, teachers are directly or indirectly involved in the strategies and errors that learners display in solving linear equations.

5.4 CONCLUSION

This study explores errors made by Grade 9 learners in solving linear equations. In this chapter, data analysis from the study has been used to answer the two research questions. The study's findings indicate that learners make different types of common errors in solving linear equations and that the types of errors made depend largely on the structure of the equation itself. The findings were used to explain why learners make these errors.

This chapter further provides my personal reflections on the study. It identifies the contributions insights gained from the study could make to mathematics teachers as well as mathematics research community at large, as well limitations of the study.

This chapter concludes by proposing recommendations for the teaching and learning of algebra, and suggestions for further research possibilities.

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APPENDICES

APPENDIX A: Ethical Clearance Certificate

UNIVERSITY OF KWAZULU-NATAL™
INYUVESI YAKWAZULU-NATALI

20 March 2023

S'Phamandla Robert Gasa (200202255)
School Of Education
Edgewood Campus

Dear SR Gasa,

Protocol reference number: HSSREC/00005260/2023
Project title: An analysis of errors made by learners in solving linear equations at grade 9 level
Degree: Masters

Approval Notification – Expedited Application

This letter serves to notify you that your application received on 03 March 2023 in connection with the above, was reviewed by the Humanities and Social Sciences Research Ethics Committee (HSSREC) and the protocol has been granted **FULL APPROVAL**.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number. PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

This approval is valid until 20 March 2024.
To ensure uninterrupted approval of this study beyond the approval expiry date, a progress report must be submitted to the Research Office on the appropriate form 2 - 3 months before the expiry date. A close-out report to be submitted when study is finished.

HSSREC is registered with the South African National Health Research Ethics Council (REC-040414-040).

Yours sincerely,

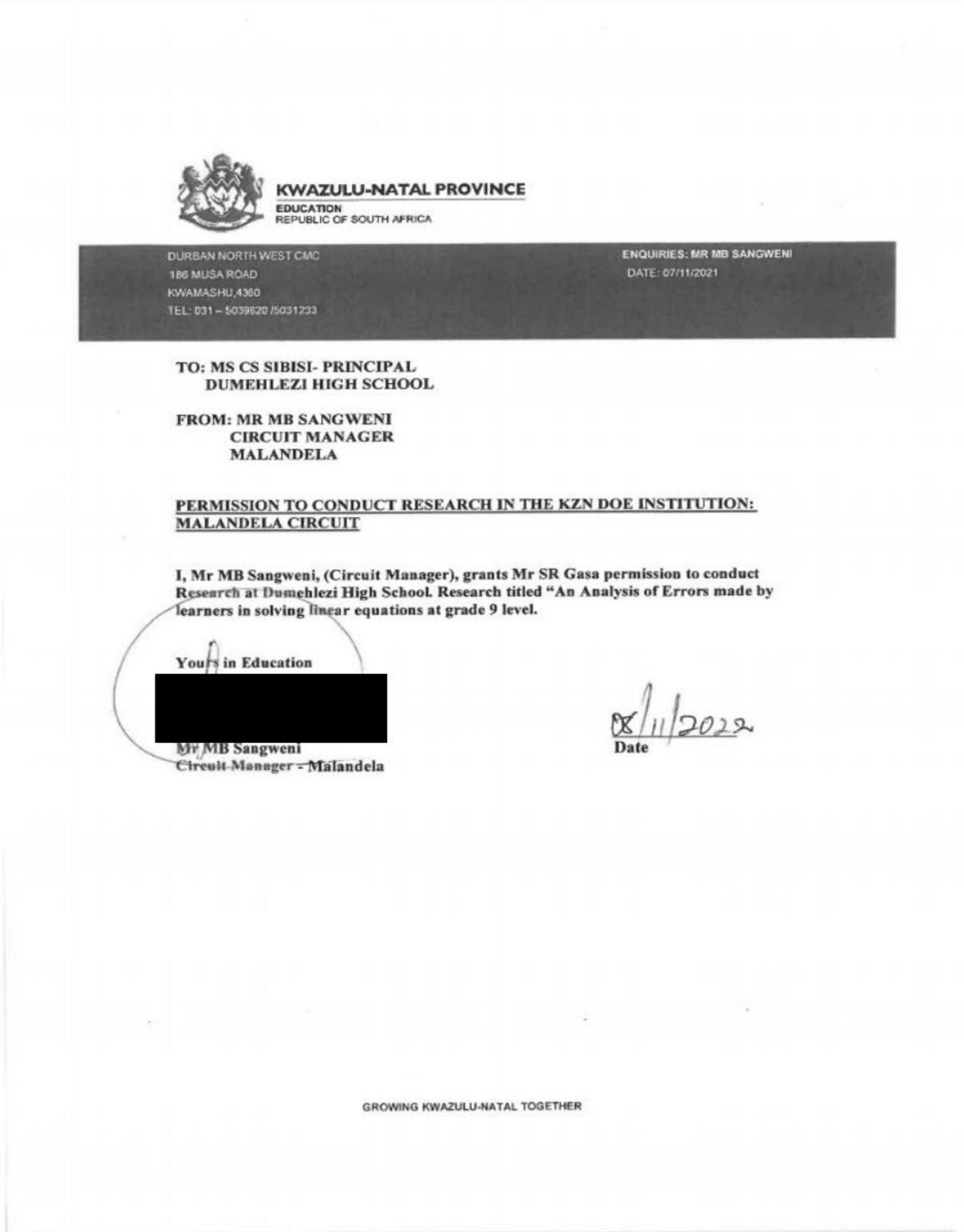
[redacted]

Professor Dipane Hlalele (Chair)

/dd

Humanities and Social Sciences Research Ethics Committee
Postal Address: Private Bag X54001, Durban, 4000, South Africa
Telephone: +27 (0)31 260 8350/4557/3587 **Email:** hssrec@ukzn.ac.za **Website:** http://research.ukzn.ac.za/Research-Ethics
Founding Campuses: Edgewood Howard College Medical School Pietermaritzburg Westville
INSPIRING GREATNESS

APPENDIX B: Gatekeeper's Letter of Permission to Conduct Research



KWAZULU-NATAL PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

DURBAN NORTH WEST CMC
186 MUSA ROAD
KWAMASHU,4360
TEL: 031 – 5039620 /5031233

ENQUIRIES: MR MB SANGWENI
DATE: 07/11/2021

TO: MS CS SIBISI- PRINCIPAL
DUMEHLEZI HIGH SCHOOL

FROM: MR MB SANGWENI
CIRCUIT MANAGER
MALANDELA

<u>PERMISSION TO CONDUCT RESEARCH IN THE KZN DOE INSTITUTION: MALANDELA CIRCUIT</u>

I, Mr MB Sangweni, (Circuit Manager), grants Mr SR Gasa permission to conduct Research at Dumehlezi High School. Research titled "An Analysis of Errors made by learners in solving linear equations at grade 9 level.

Yours in Education

[redacted]

Mr MB Sangweni
Circuit Manager - Malandela

08/11/2022
Date

GROWING KWAZULU-NATAL TOGETHER

APPENDIX C: Letter to the Department of Education

S’phamandla Robert Gasa

[redacted]

[redacted]e

3610

12 October 2022

The Circuit Manager

Department of Education

187 Musa Road

Ekuthuleni

KwaMashu

4103

Dear Sir

<u>RE: PERMISSION TO CONDUCT A STUDY AT DUMEHLEZI HIGH SCHOOL</u>

My name is Mr S.R Gasa I am a Master’s candidate studying at the University of KwaZulu-Natal, Edgewood Campus, South Africa. My study is focused on exploring learners’ errors and misconception when solving linear algebraic equations. I am interested in gathering data for research purposes at the above mentioned school. The title of my research is: AN ANALYSIS OF ERRORS MADE BY LEARNERS IN SOLVING LINEAR EQUATIONS AT GRADE 9 LEVEL. The aim is to identify learners’ errors in solving linear equations and their sources.

The intention is to come up with remedial measures which will either do away or will reduce these errors.

I intend to administer a test to grade 9 learners at the above mentioned school and then interview some of them. The participants will not be disadvantaged in any way. The right of participants to privacy, anonymity, confidentiality and respect for human dignity will be honoured during research. Participation by learners is voluntary and anyone willing to withdraw can do so without penalty. The participation of learners has no foreseeable risk.

I can be contacted at:

Email:[redacted] / 200202255@stu.ukzn.ac.za Cell: [redacted]

My supervisor is Prof S Bansilal, who is located at the School of Education, Edgewood campus, University of KwaZulu-Natal (UKZN).

Contact details:

Email: BansilalS@ukzn.ac.za Phone number: 031 260 3451

Contact Details for the Research Office: HSSREC@ukzn.ac.za

APPENDIX D: Letter to the principal

S’phamandla Robert Gasa

[redacted]

[redacted]

3610

12 October 2022

Dear Principal

RE: PERMISSION TO CONDUCT A STUDY AT YOUR SCHOOL

My name is Mr S.R Gasa I am a master’s candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. My study is focused on exploring learners’ errors and misconception when solving linear algebraic equations. I am interested in gathering data for research purposes at the above mentioned school. The title of my research is: AN ANALYSIS OF ERRORS MADE BY LEARNERS IN SOLVING LINEAR EQUATIONS AT GRADE 9 LEVEL. The aim is to identify learners’ errors in solving linear equations and their sources. The intention is to come up with remedial measures which will either do away or will reduce these errors.

I intend to administer a test to grade 9 learners at the above mentioned school and then interview some of them. The participants will not be disadvantaged in any way. The right of participants to privacy, anonymity, confidentiality and respect for human dignity will be honoured during research. Participation by learners is voluntary and anyone willing to withdraw can do so without penalty. The participation of learners has no foreseeable risk.

I can be contacted at:

Email:[redacted] / 200202255@stu.ukzn.ac.za Cell: 0[redacted]

My supervisor is Prof S Bansilal, who is located at the School of Education, Edgewood campus, University of KwaZulu-Natal (UKZN).

Contact details:

Email: BansilalS@ukzn.ac.za Phone number: 031 260 3451 Contact Details for the Research Office: HSSREC@ukzn.ac.za

APPENDIX E: Consent Form for the Principal

AN ANALYSIS OF ERRORS MADE BY LEARNERS IN SOLVING LINEAR EQUATIONS AT GRADE 9 LEVEL

I am aware that my learners will be participating in this study. The confidentiality of the school and the participants will be maintained. All research will be conducted after school hours. This study will not disrupt teaching at my school.

DECLARATION.

I ………………………………………………………………………… (Full names of principal) hereby confirm that I understand the contents of this document and the nature of the research study and I consent my learners' participation in the research study. I understand that my learners are at liberty to withdraw from the research at any time, should they so desire.

SIGNATURE OF PRINCIPAL DATE

……………………………….

…………………..

APPENDIX F: Letter to the Parent/Guardian

School of Education, College of Humanities

University of KwaZulu-Natal

Edgewood Campus

Dear parent/guardian

RE: A REQUEST FOR YOUR CHILD'S PARTICIPATION IN RESEARCH STUDY

My name is Mr S.R Gasa I am a master's candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. My study is focused on exploring learners' errors and misconception when solving linear algebraic equations. I am interested in gathering data for research purposes at the above mentioned school. The title of my research is: AN ANALYSIS OF ERRORS MADE BY LEARNERS IN SOLVING LINEAR EQUATIONS AT GRADE 9 LEVEL. The aim is to identify learners' errors in solving linear equations and their sources. The intention is to come up with remedial measures which will either do away or will reduce these errors.

I intend to administer a test to grade 9 learners at the above mentioned school and then interview some of them. The participants will not be disadvantaged in any way. The right of participants to privacy, anonymity, confidentiality and respect for human dignity will be honoured during research. Participation by learners is voluntary and anyone willing to withdraw can do so without penalty. The participation of learners has no foreseeable risk.

There is attached form at the back of this letter for you to indicate your decision to allow your child to take part in the study. May you please complete it and return it to me at your earliest convenience.

For more information concerning this request, I can be contacted at

Email:[redacted] / 200202255@stu.ukzn.ac.za Cell: [redacted]

My supervisor is Prof S Bansilal, who is located at the School of Education, Edgewood campus, University of KwaZulu-Natal (UKZN).

Email: BansilalS@ukzn.ac.za Phone number: 031 260 3451
Contact Details for the Research Office: HSSREC@ukzn.ac.za

APPENDIX G: Parent/Guardian Consent Form

Please fill in the reply slip on granting permission to your child to participate in the study.

I ………………………………………………………………………… have read and understood the conditions of the study.

My child……………………………………………………………………………… can/cannot take part in the study. (Delete the inapplicable).

PARENT/GUARDIAN’S SIGNATURE DATE

……………………………….

…………………..

APPENDIX H: Letter to the Learner

School of Education, College of Humanities

University of KwaZulu-Natal

Edgewood Campus

Dear Learner

My name is Mr S.R Gasa I am a master's candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. My study is focused on exploring learners' errors and misconception when solving linear algebraic equations. The title of my research is: AN ANALYSIS OF ERRORS MADE BY LEARNERS IN SOLVING LINEAR EQUATIONS AT GRADE 9 LEVEL. The aim is to identify learners' errors in solving linear equations and their sources. The intention is to come up with remedial measures which will either do away or will reduce these errors.

I invite you to participate in my study. You will write a short test under supervision then, depending on your responses can be interviewed on how you would have come up with your answers. Be assured that your participation will have no bearing on your grades or evaluation in the subject. The right to privacy, anonymity, confidentiality and respect for human dignity will be honoured during the research. Participation is voluntary and if you decide to withdraw, you can do so at any time without penalty. Participation in the study has no foreseeable risks.

There is an attached form for you to indicate your decision to take part in the study. Discuss your involvement in the study with your parent then complete the form and return it to me at your earliest convenience. A letter has also been sent to your parent to indicate their decision concerning your participation

For more information concerning this request, I can be contacted at

Email:[redacted]/ 200202255@stu.ukzn.ac.za Cell: 0[redacted]

My supervisor is Prof S Bansilal, who is located at the School of Education, Edgewood campus, University of KwaZulu-Natal (UKZN).

Contact details: Email: BansilalS@ukzn.ac.za Phone number: 031 260 3451

Contact Details for the Research Office: HSSREC@ukzn.ac.za

APPENDIX I: Learners' Assent form

Please fill in this form to indicate your decision to participate in the mentioned study.

I ………………………………………………………………………… have read and understood the conditions of the study. I accept/ do not accept to participate in the study. (Delete the inapplicable).

LEARNER'S SIGNATURE	DATE
……………………………….	…………………..

APPENDIX I: Editor's Letter

[redacted]
B.Soc.S. LLB (Cum Laude)

13 Susan Avenue
Ballito

Date: 4 November 2024

To Whom It May Concern

This is to certify that the thesis 'An Analysis of Errors made by Learners in Solving Linear Equations at Grade 9 Level in South Africa' by S'phamandla Robert Gasa has been edited by me for language.

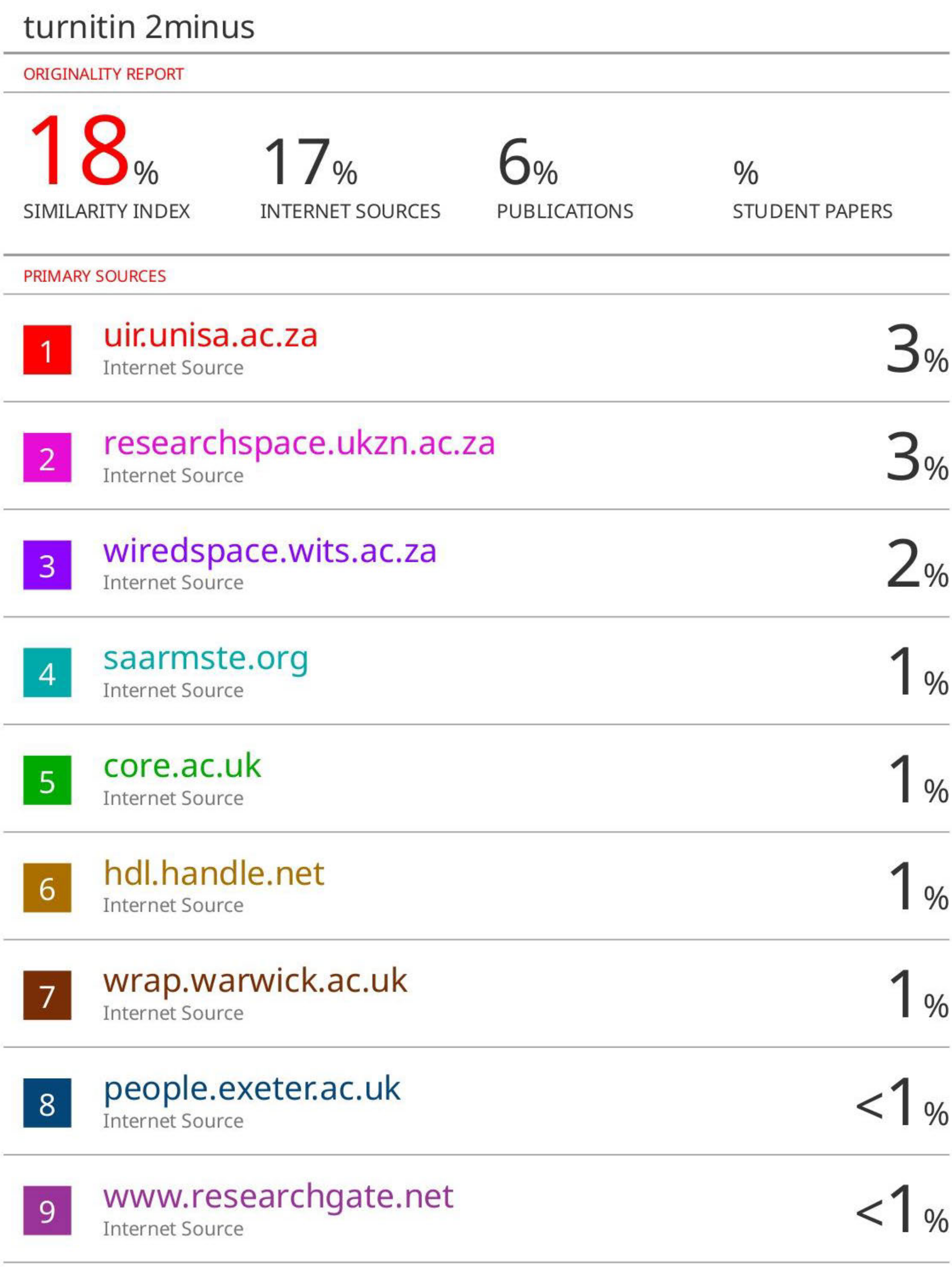
Please contact me should you require further information.

Kind regards

Lauren Frizelle
Frizelle Consulting

lauren@frizelleconsulting.co.za
[redacted]

AP



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