SOME MAL'CEV CONDITIONS FOR VARIETIES OF ALGEBRAS

by

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Dedicated to the memory of
my late brother
Ernest

PREFACE

The work described in this thesis was carried out under the supervision of Professor Teo Sturm, Department of Mathematics and Applied Mathematics, University of Natal, Durban, from March 1989 to December 1991.

The thesis represents original work by the author and has not been submitted in any form to another University. Where use was made of the work of others it has been duly acknowledged in the text.

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ABSTRACT

This dissertation deals with the classification of varieties according to their Mal'cev properties. In general the so called Mal'cev-type theorems illustrate an interplay between first order properties of a given class of algebras and the lattice properties of the congruence lattices of algebras of the considered class.

CHAPTER 1. A survey of some notational conventions, relevant definitions and auxiliary results is presented. Several examples of less frequently used algebras are given together with the important properties of some of them. The term algebra $\mathbf{T}(X)$ and useful results concerning 'term' operations are established. A $\mathcal{K}$-reflection is defined and a connection between a $\mathcal{K}$-reflection of an algebra and whether a class $\mathcal{K}$ satisfies an identity of the algebra is established.

CHAPTER 2. The Mal'cev-type theorems are presented in complete detail for varieties which are congruence permutable, congruence distributive, arithmetical, congruence modular and congruence regular. Several examples of varieties which exhibit these properties are presented together with the necessary verifications.

CHAPTER 3. A general scheme of algorithmic character for some Mal'cev conditions is presented. R. Wille (1970) and A. F. Pixley (1972) provided algorithms for the classification of varieties which exhibit strong Mal'cev properties. This chapter is largely devoted to a modification of

the Wille-Pixley schemes. It must be noted that this modification is quite different from all such published schemes. The results are the same as in Wille's scheme but slightly less general than in Pixley's. The text presented here, however is much simpler. As an example, the scheme is used to confirm Mal'cev's original theorem on congruence permutable varieties. Finally, the so-called *Chinese variety* is defined and Mal'cev conditions are established for such a variety of algebras.

CHAPTER 4. A comprehensive survey of literature concerning Mal'cev conditions is given in this chapter.

CONTENTS

INTRODUCTION

The Chinese Remainder Theorem, dating back to about the first century, is probably the first known theorem concerning congruences. It involves the simultaneous solution of a system of congruences $x \equiv b_i (mod\ a_i)$ where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are given integers. The theorem, apparently, was used to predict the simultaneous occurrence of periodically recurring astronomical events. Today, however, congruences play a vital part in almost every branch of algebra.

In the early part of the nineteenth century Galois introduced normal subgroups which play a fundamental role in defining quotient groups and in the so-called homomorphism and isomorphism theorems. In the second half of the nineteenth century Dedekind introduced ideals which play an analogous role in the theory of rings. It was inevitable that a concept be introduced in the theory of universal algebra to unify the notions of normal subgroups and ideals. It turned out that congruences on an algebra filled this important gap. In addition to providing the means for defining quotient algebras and related notions, several mathematicians engaged their attention in the relatively new theme of classifying varieties according to the behaviour of congruences in their algebras.

In 1954 A.I. Mal'cev proved that the algebras in a variety $\mathcal{K}$ have permuting congruences if and only if there exists a ternary term t of the type of $\mathcal{K}$ such that the equations $t(x,y,y) = x$ and $t(x,x,y) = y$ hold in $\mathcal{K}$. This theorem has had a profound influence on the subsequent development of universal algebra. In recognition of Mal'cev's contribution, congruence permutable varieties are referred to as Mal'cev varieties and the term t such that $t(x,y,y) = x$ and $t(x,x,y) = y$ is called a Mal'cev term.

In the years that followed, much work on the classification of varieties was done by G. Grätzer, B. Jónsson, A.F. Pixley, W. Taylor, R. Wille and others. It was indeed Grätzer who coined the phrase "Mal'cev-type condition". Pixley, in 1963, established Mal'cev-type conditions for varieties whose algebras have permutable and distributive congruences. Jónsson [1967] determined Mal'cev type conditions for varieties with distributive congruences. A. Day, in 1968 established Mal'cev conditions for congruence modular varieties. Mal'cev conditions for congruence regular varieties were established by B. Czákany, G. Grätzer and R. Wille in 1970.

From the seventies attention was given to the development of schemes for classifying varieties according to their Mal'cev properties. In 1970, R. Wille (in [33]) and in 1972, A. F. Pixley (in [27]) provided two such schemes. In 1973, W. Taylor (in [34]) presented his generalized characterization of Mal'cev conditions. The third chapter of this thesis is devoted to a modification of the Wille-Pixley schemes.

CHAPTER 1

A GENERAL SURVEY

In this chapter we present some notational conventions, definitions and general results which will be used throughout this dissertation.

§1. SOME NOTATIONAL CONVENTIONS

We shall commence this section with a casual review of the well known notions of operations, types and algebras.

Suppose $n \in \omega$ where ω is the set of non-negative integers and S is a set. An *n-ary operation* on S is a mapping from S^n to S. We shall frequently use the terms nullary, unary, binary, ternary and quaternary operations to mean 0-ary, 1-ary, 2-ary, 3-ary and 4-ary operations respectively. A nullary operation o on the set S is a mapping from $S^0 = \{\emptyset\}$ to S. Since the mapping $o : \{\emptyset\} \to S$ is determined by the element $o(\emptyset)$ in S, nullary operations are referred to as constants.

A *type* $\mathcal{T}$ is an ordered pair $\langle \mathbf{O}, ar \rangle$ where $\mathbf{O}$ is a set of objects referred to as operation symbols, and $ar : \mathbf{O} \to \omega$ is a mapping called the arity function.

A *universal algebra* $\mathbf{A}$ of type $\mathcal{T}$ is an ordered pair

$$\mathbf{A} = \langle A, F \rangle$$

where A is a nonempty set, the so-called underlying set of $\mathbf{A}$, and $F = \langle o^{\mathbf{A}}; o \in \mathbf{O} \rangle$ is a family of finitary operations on $\mathbf{A}$, where $o^{\mathbf{A}}$ is an $ar(o)$-ary operation on $\mathbf{A}$. In the absence of any ambiguity the superscript $\mathbf{A}$ in $o^{\mathbf{A}}$ will be omitted. If $o \in \mathbf{O}$ is an operation symbol of $\mathcal{T}$ then $o^{\mathbf{A}}$ refers to the fundamental operation of $\mathbf{A}$ indexed by o; we also say that $o^{\mathbf{A}}$ is the interpretation of o in $\mathbf{A}$. For brevity we shall use uppercase boldface letters to represent algebras and the corresponding uppercase letters to represent their universes. Further, all our considerations shall be related to a general but fixed type $\mathcal{T}$. Modifications to this do arise in chapter 3 and appropriate attention will be given to the changes involved.

1.1.1 Some Symbols. Throughout this thesis, we shall use the language and notation of set and model theory. We make mention of the following symbols:

$\subset$: proper inclusion.

$\hookrightarrow$: an inclusion mapping.

id_M : the identity function on set M.

ω : the set of non-negative integers.

X^n : the set $\{\langle x_0, \ldots, x_{n-1}\rangle;\ x_0, \ldots, x_{n-1} \in X\}$ of ordered n-tuples on X, $n \in \omega$.

Y^X : the set of all functions from set X to set Y.

$ker(f)$: the kernel of a mapping $f : X \to Y$. It is defined as

$$ker(f) = \{\langle x_0, x_1\rangle \in X^2;\ f(x_0) = f(x_1)\}.$$

$f|X$: the restriction of a mapping f to a subset X of its domain.

1.1.2 The Set of Variables V. We shall assume that V is an infinite countable set of variables with a given bijection $\omega \to V$. That is,

$$V = \{x_n;\ n \in \omega\}$$

where $x_i \neq x_j$ for different indices $i, j \in \omega$. Whenever we refer to a set of variables X we shall assume that $X \subseteq V$. Also, we will frequently use the metavariables x, y, z, u, v, w, etc., which range over V.

1.1.3 The Relational Product $(r, s)^n$. Suppose r and s are binary relations on a set. As usual $r \circ s$ will denote the relational product of r and s. If n is a positive integer, we define the relational product $(r, s)^n$ inductively as follows:

$$\begin{aligned} (r,s)^1 &= r \\ (r,s)^{2n} &= (r,s)^{2n-1} \circ s \\ (r,s)^{2n+1} &= (r,s)^{2n} \circ r. \end{aligned}$$

Since the relational product is associative, the parenthesis and the symbol $\circ$ shall be frequently omitted. For example, we will write

$$(r,s)^2 = rs, \ (r,s)^3 = rsr, \ (r,s)^4 = rsrs, \text{ etc.}$$

§2. CONGRUENCES ON AN ALGEBRA

1.2.1 The Supremum of Congruences. Let **A** be a $\mathcal{T}$-algebra. The set of all congruences on **A** will be denoted by Con **A**. It is known that **Con A** is a complete sublattice of the complete lattice $\mathbf{E}(A)$ of all equivalence relations on the set A, where the lattice order on $E(A)$ is the inclusion.

Suppose $\sigma, \tau \in \operatorname{Con}\mathbf{A}$ and that n is a positive integer. Then the supremum of σ and τ in Con **A** is given by

$$\sigma \vee \tau = \bigcup_{n=1}^{\infty} (\sigma, \tau)^n$$

where $(\sigma, \tau)^n$ has been defined in Section 1.1.3.

1.2.2 The Congruence $\Theta_{\mathbf{A}}(r)$. For a $\mathcal{T}$-algebra **A** and $r \subseteq A \times A$ we define

$$\Theta_{\mathbf{A}}(r) = \bigcap \{\rho \in \operatorname{Con}\mathbf{A};\ r \subseteq \rho\}.$$

Indeed $\Theta_{\mathbf{A}}(r)$ is the least congruence on **A** which contains r as a subset. In the absence of any ambiguity we shall omit the subscript **A** and simply refer to $\Theta(r)$.

1.2.3 The Congruence $\Theta(M)$. If M is a subset of A, **A** a $\mathcal{T}$-algebra, then

$$\Theta(M) = \bigcap \{\rho \in \operatorname{Con}\mathbf{A};\ M^2 \subseteq \rho\}.$$

In particular, if $a, b \in A$, $a \neq b$, then

$$\Theta(a,b) = \bigcap\{\rho \in \operatorname{Con} \mathbf{A};\ \{a,b\}^2 \subseteq \rho\}.$$

The congruence $\Theta(a,b)$ is called a *principal congruence* on **A**.

§3. EXAMPLES OF ALGEBRAS

A *groupoid* is an algebra with one binary operation. Other commonly encountered algebras are *groups*, *Abelian groups*, *semigroups*, *monoids*, *rings* and *lattices.* The following are more examples of algebras which we shall refer to in the course of this thesis. We shall adopt the usual convention of denoting the product resulting from the binary operation $\cdot$ by juxtaposition.

1.3.1 Quasigroups. According to [13], p.355, a *quasigroup* is an algebra $\mathbf{Q} = \langle Q; \cdot, /, \backslash \rangle$ with three binary operations satisfying the identities:

1. $(x/y)y = x$
2. $(xy)/y = x$
3. $x(x\backslash y) = y$
4. $x\backslash(xy) = y$.

□

1.3.1.1 Lemma. *Let $\langle Q; \cdot, /, \backslash \rangle$ be a quasigroup. Then for all $a, b, c \in Q$ we have*

(i) $a/b = c$ *iff* $a = cb$,

(ii) $a\backslash b = c$ *iff* $b = ac$,

(iii) $ab = ac$ *iff* $b = c$ *and*

(iv) $ba = ca$ *iff* $b = c$.

PROOF.

(i) Assume $a/b = c$. Then $cb = (a/b)b = a$ by Axiom 1. Conversely, let $cb = a$. Then $a/b = (cb)/b = c$ by Axiom 2.

(ii) Assume $a\backslash b = c$. Then $ac = a(a\backslash b) = b$ by Axiom 3. Conversely, let $ac = b$. Then $a\backslash b = a\backslash(ac) = c$ by Axiom 4.

(iii) Let $ab = ac$. Then by Axiom 4 we have $b = a\backslash(ab) = a\backslash(ac) = c$. The converse implication is trivial.

(iv) Let $ba = ca$. Then by Axiom 2 we have $b = (ba)/a = (ca)/a = c$. The converse implication is trivial. □

The above lemma enables another description of quasigroups which could be helpful to the reader:

1.3.1.2 Corollary. (i) *Let* $\langle Q; \cdot \rangle$ *be a groupoid which enjoys the following property: For all* $a, b \in Q$, *each of the equations* $ax = b$ *and* $ya = b$ *has a unique solution in* Q. *If we define* $a\backslash b \overset{def}{=} x$ *and* $b/a \overset{def}{=} y$, *then* $\langle Q; \cdot, /, \backslash \rangle$ *is a quasigroup.*

(ii) *Conversely, let* $\langle Q; \cdot, /, \backslash \rangle$ *be a quasigroup. Then for all* $a, b \in Q$, *each of the equations* $ax = b$ *and* $ya = b$ *has a unique solution in* Q, *namely,*

$$x = a\backslash b \quad \textit{and} \quad y = b/a.$$

PROOF.

(i) By the definition of $a\backslash b$ and b/a we have

$$\begin{aligned} (b/a)a &= ya = b \quad \text{and} \\ a(a\backslash b) &= ax = b. \end{aligned}$$

Thus $\langle Q; \cdot \rangle$ satisfies Axioms 1 and 3 respectively. Now for Axiom 2, we note that $u \overset{def}{=} (ab)/b$ is the unique solution of $ub = ab$ (by our definition of the operation). Because a is also a solution of that equation, we have that $a = u = (ab)/b$ by the uniqueness of the solution. Axiom 4 can be verified in a similar fashion.

(ii) We have that $ax = a(a\backslash b) = b$ by Axiom 3. The uniqueness of the solution follows from Lemma 1.3.1.1 (iii). That $y = b/a$ is the unique solution of $ya = b$ in Q can be proved in a similar manner. □

1.3.2 Right-complemented Semigroups. According to [13], p.356, a *right-complemented semigroup* is an algebra $\mathbf{S} = \langle S; \cdot, * \rangle$ with two binary operations satisfying the identities:

1. $x(x * y) = y(y * x)$
2. $(xy) * z = y * (x * z)$
3. $x(y * y) = x$.

1.3.3 Boolean Algebras. A *Boolean algebra* is an algebra $\mathbf{B} = \langle B; \vee, \wedge, ', 0, 1 \rangle$ with two binary, one unary and two nullary operations satisfying the following conditions:

1. $\langle B; \vee, \wedge \rangle$ is a distributive lattice
2. $x \wedge 0 = 0$ and $x \vee 1 = 1$
3. $x \wedge x' = 0$ and $x \vee x' = 1$.

1.3.4 Heyting Algebras. An *Heyting algebra* is defined in [2], p.26, as an algebra $\mathbf{H} = \langle H; \vee, \wedge, \rightarrow, 0, 1 \rangle$ with three binary operations and two nullary operations satisfying the following conditions:

1. $\langle H, \vee, \wedge, 0, 1 \rangle$ is a distributive lattice with greatest element 1 and least element 0.
2. $x \rightarrow x = 1$
3. $(x \rightarrow y) \wedge y = y$
4. $x \wedge (x \rightarrow y) = x \wedge y$
5. $1 \rightarrow x = x$
6. $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
7. $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

Clearly Axiom (5) can be deduced from Axiom (4) as follows:

$$\begin{aligned} x &= 1 \wedge x \\ &= 1 \wedge (1 \to x) \\ &= 1 \to x. \end{aligned}$$

Let $\leq$ denote the associated lattice order on H induced by the lattice structure of **H**. We shall establish several more properties of Heyting algebras. On account of their usefulness in Section 2.3.4, we embody these properties in the following lemma:

1.3.4.1 Lemma. *Let* **H** *be an Heyting algebra. Then for all* $x, y, z \in H$ *we have*

(i) $y \leq x \to y$

(ii) $x \leq y$ *iff* $x \to y = 1$

(iii) $x \wedge z \leq y$ *iff* $z \leq x \to y$

(iv) $x \to (y \to z) = (x \wedge y) \to z$.

PROOF.

(i) In consequence of our lattice order, this is just an equivalent formulation of Axiom (3).

(ii) $\Rightarrow$: Since $x \leq y$ we have that $x = x \wedge y$. Then

$$\begin{aligned} 1 &= x \to x \\ &= x \to (x \wedge y) \\ &= (x \to x) \wedge (x \to y) \qquad \text{[by Axiom (6)]} \\ &= 1 \wedge (x \to y) \\ &= x \to y. \end{aligned}$$

$\Leftarrow$: Since $\leq$ is the lattice order it will suffice to show that $x = x \wedge y$: Assuming that $x \to y = 1$, we have

$$x = x \wedge 1$$

$$
\begin{aligned}
&= x \wedge (x \to y) \\
&= x \wedge y \quad \text{[by Axiom (4)]}.
\end{aligned}
$$

(iii) $\Rightarrow$: Assume that $x \wedge z \leq y$. Then

$$
\begin{aligned}
z &\leq x \to z && \text{[by (i)]} \\
&= (x \to z) \wedge (x \to x) && \text{[by Axiom (2)]} \\
&= x \to (x \wedge z) && \text{[by Axiom (6)]} \\
&= x \to (x \wedge z \wedge y) && \text{[by Assumption]} \\
&= (x \to (x \wedge z)) \wedge (x \to y) && \text{[by Axiom (6)]} \\
&\leq x \to y.
\end{aligned}
$$

$\Leftarrow$: If $z \leq x \to y$, then

$$
\begin{aligned}
x \wedge z &\leq x \wedge (x \to y) \\
&= x \wedge y \quad \text{[by Axiom (4)]} \\
&\leq y.
\end{aligned}
$$

We observe here that $x \to y$ is an upper bound of the set $\{z \in H;\ x \wedge z \leq y\}$. By Axiom (4) it is in fact the greatest element of that set.

(iv) We first show that $x \to (y \to z) \leq (x \wedge y) \to z$:

$$
\begin{aligned}
(x \wedge y) \wedge (x \to (y \to z)) &= y \wedge (x \wedge (x \to (y \to z))) \\
&= y \wedge (x \wedge (y \to z)) && \text{[by Axiom (4)]} \\
&\leq y \wedge (y \to z) \\
&= y \wedge z && \text{[by Axiom (4)]} \\
&\leq z.
\end{aligned}
$$

Since $(x \wedge y) \wedge (x \to (y \to z)) \leq z$, it follows from (iii) that

$$x \to (y \to z) \leq (x \wedge y) \to z.$$

To prove the converse inequality, we note that

$$
\begin{aligned}
(y \wedge x) \wedge ((y \wedge x) \to z) &= (y \wedge x) \wedge z \quad \text{[by Axiom (4)]} \\
&\leq z.
\end{aligned}
$$

Hence by (iii), $x \wedge ((x \wedge y) \to z) \leq y \to z$. Using (iii) again we conclude that $(x \wedge y) \to z \leq x \to (y \to z)$. □

1.3.5 R-modules. For a given ring $\mathbf{R}$, an $\mathbf{R}$- *module*, according to [2], p.25, is an algebra $\mathbf{M} = \langle M; +, -, \langle f_r, r \in R \rangle, 0 \rangle$ where the operation $+$ is binary, $-$ is unary, each f_r is unary and 0 is nullary such that the following conditions hold:

1. $\langle M; +, -, 0 \rangle$ is an Abelian group
2. $f_r(x + y) = f_r(x) + f_r(y)$, for $r \in R$
3. $f_{r+s}(x) = f_r(x) + f_s(x)$, for $r, s \in R$
4. $f_r(f_s(x)) = f_{rs}(x)$, for $r, s \in R$.

1.3.6 BCK Algebras. A *BCK algebra*, (see [7], p.102 and [17], p.2), is an algebra $\mathbf{A} = \langle A; \cdot, 0 \rangle$ with one binary operation and one nullary operation satisfying the following identities:

1. $((xy)(xz))(zy) = 0$
2. $(x(xy))y = 0$
3. $xx = 0$
4. $0x = 0$
5. $xy = 0 = yx$ implies that $x = y$.

□

With a BCK-algebra $\mathbf{A}$ we can associate a partially ordered set $\langle A, \leq \rangle$ where we define for all $x, y \in A$,

$$x \leq y \quad \text{iff} \quad xy = 0.$$

We also note that the above axioms can be used to prove that

6. $x0 = x$ and
7. $(xy)z = (xz)y$.

These proofs which are due to K. Iséki and S. Tanaka are presented in [17] as theorem 2 (9) and theorem 1 respectively.

We note that the binary operation $\cdot$ for BCK-algebras is not associative. Therefore we have to use parentheses for BCK-terms. We simplify our notation by the following convention: Let $a, b_0, b_1, \ldots$ be elements of a BCK-algebra $\mathbf{A}$. Then for any $n \in \omega$, we define

$$ab_0 \ldots b_{n+1} \overset{def}{=} (ab_0 \ldots b_n)b_{n+1}.$$

In consequence of the above mentioned BCK-identity $x0 = x$, we now have that $a = a0 \ldots 0$ for any number of 0's on the right hand side.

1.3.7 Implication Algebras. An *implication algebra* is defined in [13], p.356, as a groupoid $\mathbf{G} = \langle G; \rightarrow \rangle$ with one binary operation which satisfies the following three identities for all x, y, z in G:

1. $(x \rightarrow y) \rightarrow x = x$
2. $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$
3. $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

□

We shall use the above three identities to establish some useful facts about implication algebras. Thereafter we shall show that an implication algebra is just a special case of a BCK-algebra. For brevity of notation, we shall replace the operation symbol $\rightarrow$ by $\cdot$ and write (as it is usual in the theory of BCK-algebras)

$$xy \text{ instead of } \quad y \rightarrow x.$$

We shall hereafter use the following equivalent formulation of the above three axioms:

I $x(yx) = x$

II $x(xy) = y(yx)$

III $(xy)z = (xz)y$.

In view of the above we shall now refer to an implication algebra as a groupoid $\langle G; \cdot \rangle$ which satisfies Axioms I, II and III.

1.3.7.1 Lemma. *If* $\mathbf{G} = \langle G; \cdot \rangle$ *is an implication algebra, then for all* $x, y \in G$, *we have that* $x = y$ *iff* $xy = yx$.

PROOF.

$\Rightarrow$: This is trivial.

$\Leftarrow$: Assume $xy = yx$. Then

$$\begin{aligned} x &= x(yx) && [\text{by Axiom (I)}] \\ &= x(xy) && [\text{by Assumption}] \\ &= y(yx) && [\text{by Axiom (II)}] \\ &= y(xy) && [\text{by Assumption}] \\ &= y && [\text{by Axiom (I)}] \end{aligned}$$

□

1.3.7.2 Lemma. *For an implication algebra* $\mathbf{G} = \langle G; \cdot \rangle$, *we have that for all* $x, y \in G$,

$$xx = yy.$$

PROOF.

As a result of Axiom (I) we have that $x = x\{[x(yy)]x\}$. Therefore

$$\begin{aligned} xx &= (x\{[x(yy)]x\})x && \\ &= (x[(xx)(yy)])x && [\text{by Axiom (III)}] \\ &= (xx)[(xx)(yy)] && [\text{by Axiom (III)}] \\ &= (yy)[(yy)(xx)] && [\text{by Axiom (II)}] \\ &= (y[(yy)(xx)])y && [\text{by Axiom (III)}] \\ &= (y\{[y(xx)]y\})y && [\text{by Axiom (III)}] \\ &= yy && [\text{by Axiom (I)}]. \end{aligned}$$

□

1.3.7.3 Convention. In consequence of Lemma 1.3.7.2 we have established the existence of a distinguished element 0 such that

$$0 \overset{def}{=} aa$$

for all $a \in G$. We observe that 0 does not depend on our choice of a; in fact 0^{G} is a derived constant. □

1.3.7.4 Lemma. *Let* G *be an implication algebra. Then for all $x \in G$, we have that*

(i) $x0 = x$ *and*

(ii) $0x = 0$.

PROOF.

(i): By Section 1.3.7.3 above, and by Axiom (I) respectively, we have that $x0 = x(xx) = x$.

(ii): Again by Section 1.3.7.3, we have that $xx = 0$ for all $x \in G$. Therefore

$$\begin{aligned} 0x &= (xx)x \\ &= (xx)[x(xx)] && \text{[by Axiom (I)]} \\ &= \{x[x(xx)]\}x && \text{[by Axiom (III)]} \\ &= \{(xx)[(xx)x]\}x && \text{[by Axiom (II)]} \\ &= [(xx)x][(xx)x] && \text{[by Axiom (III)]} \\ &= 0. \end{aligned}$$

□

1.3.7.5 Corollary. *If $\langle B; \cdot \rangle$ is an implication algebra, then $\langle B; \cdot, 0 \rangle$ is a BCK-algebra, where 0 is the distinguished element of B such that $0 = bb$ for all $b \in B$.*

PROOF.

Axioms (3), (4) and (5) of BCK-algebras have been established in Lemmas 1.3.7.2, 1.3.7.4 (ii) and 1.3.7.1 respectively. We only need to show that

(i) $((xy)(xz))(zy) = 0$ and

(ii) $(x(xy))y = 0$.

(i): Observe that

$$\begin{aligned}
((xy)(xz))(zy) &= ((x(xz))y)(zy) && \text{[by Axiom (III)]}\\
&= ((z(zx))y)(zy) && \text{[by Axiom (II)]}\\
&= ((z(zy))y)(zx) && \text{[by Axiom (III)]}\\
&= ((y(yz))y)(zx) && \text{[by Axiom (II)]}\\
&= ((yy)(yz))(zx) && \text{[by Axiom (III)]}\\
&= (0(yz))(zx) &&\\
&= 0 && \text{[by Lemma 1.3.7.4 (ii)].}
\end{aligned}$$

(ii): Using Axiom (III) and Section 1.3.7.3 respectively we have

$$(x(xy))y = (xy)(xy) = 0.$$

□

1.3.7.6 Definition. A BCK-algebra $\langle B; \cdot, 0\rangle$ is said to be *implicative* if it satisfies the identity $x(yx) = x$. □

1.3.7.7 Theorem. *Let $\langle B; \cdot\rangle$ be a groupoid, and let 0 be an element of B. Then the following conditions are equivalent:*

(i) *$\langle B; \cdot, 0\rangle$ is an implicative BCK- algebra.*

(ii) *$\langle B; \cdot\rangle$ satisfies the following identities for all $x, y, z \in B$.*

(a) $x(yx) = x$

(b) $x(xy) = y(yx)$

(c) $(xy)z = (xz)y$

PROOF.

(i)$\Rightarrow$ (ii): $\langle B; \cdot \rangle$ satisfies (a) by Definition 1.3.7.6 and (c) by (7) of Section 1.3.6. We shall now show that the identity (b) is also satisfied in $\langle B; \cdot \rangle$. We first make the following simple observations about BCK- algebras:

O1. $u \leq v \Rightarrow uw \leq vw$.

O2. $y(y(yx)) = yx$.

Recall that the partial order $\leq$ was defined after the definition in Section 1.3.6. Now

$$\begin{aligned} (x(xy))(y(yx)) &\leq y(y(yx)) \quad \text{[Ax. 2, Sec. 1.3.6 and O1]} \\ &= yx \quad \text{[by O2]} \end{aligned}$$

Therefore we have that in any BCK-algebra,

O3. $(x(xy))(y(yx)) \leq yx$.

Moreover,

$$\begin{aligned} ((x(xy))(yx))(y(yx)) &= ((x(xy))(y(yx)))(yx) \quad \text{[(7) of Sec. 1.3.6]} \\ &\leq (yx)(yx) \quad \text{[by O1 and O3]} \\ &= 0 \quad \text{[Axiom 3, Sec. 1.3.6]} \end{aligned}$$

Therefore $(x(xy))(yx) \leq y(yx)$. Now since

$$\begin{aligned} x(xy) &= (x(yx))(xy) \quad \text{[Axiom I]} \\ &= (x(xy))(yx) \quad \text{[(7) of Sec. 1.3.6]} \end{aligned}$$

we have that $x(xy) \leq y(yx)$. By interchanging x and y we have that $y(yx) \leq x(xy)$ also, and hence (b) is satisfied in $\langle B; \cdot \rangle$.

(ii)$\Rightarrow$ (i): This follows from Corollary 1.3.7.5. □

1.3.7.8 Remark. Let us note firstly that a BCK-variety is a variety of algebras $\langle A; \cdot, 0 \rangle$ (where $\cdot$ is binary and 0 is nullary) all of whose members are BCK-algebras.

The class of implication algebras, interpreted as BCK-algebras in the sense given in Corollary 1.3.7.5, forms a variety of BCK- algebras, the so-called variety of *implicative* BCK- algebras. An equational base of this BCK-variety is given for instance by

$$\begin{aligned} (xy)z &= (xz)y, \\ x(xy) &= y(yx), \\ x(yx) &= x, \text{ and} \\ xx &= 0. \end{aligned}$$

In fact, it is the least nontrivial variety of BCK-algebras; see Cornish's survey [7], p.106, for details. Further, this BCK- variety is the class intersection of the variety of the so-called *commutative* BCK-algebras [that is, BCK-algebras satisfying $x(xy) = y(yx)$] and the variety of the so-called *positive implicative* BCK-algebras [that is, BCK-algebras which satisfy $xy = (xy)y$].

In view of the last mentioned observation we present the following proposition.

1.3.7.9 Proposition. *A BCK-algebra $\langle B; \cdot, 0\rangle$ is implicative if and only if it is commutative and positive implicative.*

PROOF.

$\Rightarrow$: We need only show that $xy = (xy)y$: Using Axiom I two times, we have

$$xy = (xy)(y(xy)) = (xy)y.$$

$\Leftarrow$: In any BCK-algebra we have that $x(yx) \leq x$. We shall now establish the converse inequality.

$$\begin{aligned} x(x(yx)) &= (yx)((yx)x) \quad \text{[Def. commutative]} \\ &= (yx)(yx) \quad \text{[Def. positive implicative]} \\ &= 0. \end{aligned}$$

Therefore $x \leq x(yx)$ and $\langle B; \cdot, 0\rangle$ is implicative. □

§4. FREE ALGEBRAS AND TERM ALGEBRAS

1.4.1 Definition. Let $\mathcal{K}$ be a class of $\mathcal{T}$- algebras. A $\mathcal{T}$-algebra $\mathbf{A}$ is said to be *free over* $\mathcal{K}$ if $\mathbf{A} \in \mathcal{K}$ and there is a set $X \subseteq A$ such that for every $\mathbf{B} \in \mathcal{K}$ and for each mapping $f : X \to B$, there exists a unique homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$ which extends f. Every such X is said to be a free generating set of $\mathbf{A}$, and we say that X freely generates $\mathbf{A}$.

A $\mathcal{T}$-algebra $\mathbf{A}$ is called an *absolutely free* algebra if it is free over the class of all $\mathcal{T}$-algebras. □

With respect to the above definition we make the following important observations which follow directly from the above definition:

1.4.1.1 Let $\mathbf{B} \in \mathcal{K}$ be generated by a $Y \subseteq B$ (written as $\mathbf{B} = \langle Y \rangle$). Then Y is a $\mathcal{K}$-free generating set of $\mathbf{B}$ if and only if for every $\mathbf{C} \in \mathcal{K}$ and each mapping $g : Y \to C$ there is at least one homomorphism $\psi : \mathbf{B} \to \mathbf{C}$ which extends g.

Conversely, if $\mathbf{A}$ is free over $\mathcal{K}$ with free generating set X, and if $\mathcal{K}$ is closed under the formation of subalgebras, then $\mathbf{A} = \langle X \rangle$.

1.4.1.2 A free algebra over a class $\mathcal{K}$ is determined, up to isomorphism, by the cardinality of any of its free generating sets. That is , if $\mathbf{A}_1$ and $\mathbf{A}_2$ are two free algebras over $\mathcal{K}$, freely generated by X_1 and X_2 respectively, then $\mathbf{A}_1$ is isomorphic to $\mathbf{A}_2$ whenever X_1 and X_2 have the same cardinality.

1.4.2 The Set of Terms $T(X)$. Let $\mathcal{T}$ be a type of algebras and let $\mathcal{T}_0$ be the set of nullary operation symbols. Suppose $X \subseteq V$ such that $X \cap \mathcal{T}_0 = \emptyset$. Then the set $T(X)$ is the least set Y such that

(i) $X \cup \mathcal{T}_0 \subseteq Y$.

(ii) If $t_0, \ldots, t_{m-1} \in Y$ and o is an m-ary operation symbol of $\mathcal{T}$, then

$$o(t_0, \ldots, t_{m-1}) \in Y.$$

The elements of $T(X)$ are referred to as *terms* or polynomial symbols of type $\mathcal{T}$ over X.

A term t is said to be n-ary, where $n \in \omega$, if

$$t \in T(\{x_i;\ i \in \omega \text{ and } 0 \leq i < n\}).$$

If t is an n-ary term, we also write $t(x_0, \ldots, x_{n-1})$ instead of t. □

Since we have $T(X) \subseteq T(Y)$ whenever $X \subseteq Y \subseteq V$, every n-ary term is also m-ary for each integer $m \geq n$. Note that there exist nullary terms of type $\mathcal{T}$ if and only if there exist nullary operation symbols of $\mathcal{T}$. Of course, a constant can be defined even by a term which is not nullary (see for instance Lemma 1.3.7.2).

1.4.3 Term Interpretation in an Algebra. Let $t \in T(X)$ be a term of type $\mathcal{T}$ over a set $X \subseteq V$ and let $\mathbf{A}$ be a $\mathcal{T}$-algebra. The *interpretation* ${}^{\mathbf{A}}t : A^X \to A$ of t is defined inductively on its complexity as follows:

(i) if t is a variable, that is, $t = x$ for some $x \in X$, then we define

$$^{\mathbf{A}}t = {}^{\mathbf{A}}x \overset{def}{=} \pi_x : A^X \to A,$$

where π is the well known projection mapping.

(ii) if $t = o$ is a nullary operation symbol of $\mathbf{O}$, then ${}^{\mathbf{A}}t$ is the constant function defined by

$$^{\mathbf{A}}t(u) = o^{\mathbf{A}} \quad \text{for} \quad u \in A^X.$$

(iii) if $t = o(t_0, \ldots, t_{r-1})$ where $o \in \mathbf{O}$ is of positive arity r and $t_0, \ldots, t_{r-1} \in T(X)$ such that the interpretations ${}^{\mathbf{A}}t_0, \ldots, {}^{\mathbf{A}}t_{r-1}$ have already been defined, then

$$^{\mathbf{A}}o(t_0, \ldots, t_{r-1})(u) = o^{\mathbf{A}}({}^{\mathbf{A}}t_0(u), \ldots, {}^{\mathbf{A}}t_{r-1}(u))$$

for $u \in A^X$.

□

Suppose $\{x_0, \ldots, x_n\} \subseteq X$ and that $t(x_0, \ldots, x_n)$ is an $n+1$-ary term. It is a simple fact that ${}^{\mathbf{A}}t(f) = {}^{\mathbf{A}}t(g)$ for all $f, g \in A^X$ if $f(x_i) = g(x_i)$ for all $i = 0, \ldots, n$.

We can therefore define ${}^{\mathbf{A}}t(a_0, \ldots, a_n)$ for $\langle a_0, \ldots, a_n\rangle \in A^{n+1}$ by

$${}^{\mathbf{A}}t(a_0, \ldots, a_n) = {}^{\mathbf{A}}t(f)$$

where $f \in A^X$ with $f(x_i) = a_i$ for $i = 0, \ldots, n$. □

We observe that for every $t \in T(X)$ there is a unique interpretation

$${}^{\mathbf{A}}t : A^X \to A.$$

We refer to ${}^{\mathbf{A}}t$ as the *term function*, (or a *polynomial*), induced by t. As a convention we shall use left superscripts for term interpretations. However, in the absence of any ambiguity the superscript $\mathbf{A}$ will be omitted.

The set of terms $T(X)$ can be transformed into an algebra in a very natural way as follows.

1.4.4 The Term Algebra $\mathbf{T}(X)$. Let $\mathcal{T}$ be a type and X a set. If $T(X) \neq \emptyset$, then the term algebra $\mathbf{T}(X)$ is the algebra with universe $T(X)$ and with $\mathcal{T}$ operations defined as follows:

(i) if $o \in \mathbf{O}$ is a nullary operation symbol, then

$$o^{\mathbf{T}(X)} = o,$$

(ii) if $o \in \mathbf{O}$ is an n-ary operation symbol and if $t_0, \ldots, t_{n-1} \in T(X)$, then

$$o^{\mathbf{T}(X)}(t_0, \ldots, t_{n-1}) = o(t_0, \ldots, t_{n-1}).$$

□

We note that the $\mathcal{T}$-algebra $\mathbf{T}(X)$ is defined if and only if $X \neq \emptyset$ or there is at least one nullary $\mathcal{T}$-operation symbol. Further, every term algebra $\mathbf{T}(X)$ is absolutely free and it is freely generated by X (see [2], p.66). This fact was established by Birkhoff in 1935.

1.4.5 Definition. Suppose $s, t \in T(X)$. Then the equation $s = t$ is referred to as an *identity*. An algebra $\mathbf{A}$ is said to satisfy the identity $s = t$, written as

$$\mathbf{A} \models s = t,$$

if the interpretations ${}^{\mathbf{A}}s : A^X \to A$ and ${}^{\mathbf{A}}t : A^X \to A$ are equal. A class $\mathcal{K}$ of algebras is said to satisfy the identity $s = t$ if every algebra of $\mathcal{K}$ satisfies that identity. A class $\mathcal{K}$ is said to be a *variety* if there exists a set Σ of identities such that

$$\mathcal{K} = \{\mathbf{A}; \mathbf{A} \text{ is a } \mathcal{T}\text{-algebra and } \mathbf{A} \models \Sigma\}.$$

□

Before the next definition we shall comment briefly on quasiprimitive and primitive classes of algebras. A *quasiprimitive* class is a collection of algebras that is closed under the formation of isomorphic images, subalgebras and direct products. A *primitive* class is closed under the formation of homomorphic images, subalgebras and direct products. Trivially every primitive class is a quasiprimitive class. In 1935 Birkhoff proved that a class $\mathcal{K}$ of algebras is a variety if and only if it is a primitive class (see [2], p.75).

1.4.6 Definition. Suppose $\mathcal{K}$ is a non-trivial variety and $X \subseteq V$. Then $\mathbf{T}_{\mathcal{K}}(X)$ denotes a free algebra over $\mathcal{K}$ with free generating set X. If κ is a cardinal number, then $\mathbf{T}_{\mathcal{K}}(\kappa)$ denotes an algebra $\mathbf{T}_{\mathcal{K}}(X)$ where $|X| = \kappa$. □

Following from Section 1.4.1.2, we note that $\mathbf{T}_{\mathcal{K}}(\kappa)$ is determined only up to isomorphism. Further, $\mathbf{T}_{\mathcal{K}}(0)$ exists if and only if the type has nullary operations. If $\kappa \neq 0$, then $\mathbf{T}_{\mathcal{K}}(\kappa)$ exists if and only if $\mathcal{K}$ has nontrivial members or $\kappa = 1$.

§5. SOME USEFUL LEMMAS

We note that an n-ary term $t(x_0, \ldots, x_{n-1})$ defines an n-ary operation on the universe of an algebra $\mathbf{A}$ by

$${}^{\mathbf{A}}t : \langle a_0, \ldots, a_{n-1}\rangle \longmapsto {}^{\mathbf{A}}t(a_0, \ldots, a_{n-1})$$

for $a_0, \ldots, a_{n-1} \in A$. Such an operation is called a derived operation since it is derived from the basic $\mathcal{T}$- operations of $\mathbf{A}$. The first of the following three lemmas shows that term functions can be effectively used for the formation of subalgebras. The next two show that they behave just like fundamental operations when it comes to the preservation of homomorphisms and the compatibility of congruences.

1.5.1 Lemma. *Suppose* $\mathbf{A}$ *is a* $\mathcal{T}$*-algebra, and* $M \subseteq A$*. Then the subalgebra* $\langle M \rangle$ *of* $\mathbf{A}$ *generated by* M *is given by*

$$\langle M \rangle = \{{}^{\mathbf{A}}t(a_0, \ldots, a_n);\ n \in \omega,\ t(x_0, \ldots, x_n) \in T(V),\ a_0, \ldots, a_n \in M\}$$

where $\mathbf{T}(V)$ *is the term algebra defined in Section* 1.4.4*. Especially, if* M *is a nonempty finite set with* r *elements, then*

$$\langle M \rangle = \{{}^{\mathbf{A}}t(a_0, \ldots, a_{r-1});\ t(x_0, \ldots, x_{r-1}) \in T(X),\ a_0, \ldots, a_{r-1} \in M\}$$

where X *is any subset of* V *satisfying* $\{x_0, \ldots, x_{r-1}\} \subseteq X$*.*

PROOF.

Suppose $\mathbf{G}$ is any subalgebra of $\mathbf{A}$ such that $M \subseteq G$, and let

$$H = \{{}^{\mathbf{A}}t(a_0, \ldots, a_n);\ n \in \omega,\ t(x_0, \ldots, x_n) \in T(X),\ a_0, \ldots, a_n \in M\}.$$

We first show by induction on the complexity of t that $H \subseteq G$: Suppose $t = o$, a nullary operation symbol of $\mathcal{T}$. Then

$${}^{\mathbf{A}}t(a_0, \ldots, a_n) = {}^{\mathbf{A}}o(a_0, \ldots, a_n) = o^{\mathbf{A}}.$$

Since $\mathbf{G}$ is a subalgebra of $\mathbf{A}$, we have $o^{\mathbf{A}} \in G$. Next, assume that t is a variable x_i, $i = 0, \ldots, n$. Then ${}^{\mathbf{A}}t(a_0, \ldots, a_n) = a_i$ and $a_i \in M \subseteq G$. Finally, let $t = o(t_0, \ldots, t_{m-1})$ for o an m-ary operation symbol and $t_j \in T(X)$, for $j = 0, \ldots, m-1$, such that each $t_j(a_0, \ldots, a_n) \in G$ (in the light of our observation given in Section 1.4.2 we can assume that all the t_j are of the same arity). Then

$${}^{\mathbf{A}}t(a_0, \ldots, a_n) = o({}^{\mathbf{A}}t_0(a_0, \ldots, a_n), \ldots, {}^{\mathbf{A}}t_{m-1}(a_0, \ldots, a_n))$$

which is an element of G, $\mathbf{G}$ being a subalgebra of $\mathbf{A}$. Thus H is a subset of G.

It is a well known that

$$\langle M \rangle = \bigcap\{\mathbf{G};\ \mathbf{G} \text{ a subalgebra of } \mathbf{A}, M \subseteq G\}.$$

Hence to prove $\mathbf{H} = \langle M \rangle$, all that remains is to show that H is a subuniverse of $\mathbf{A}$ containing M: Let o be a nullary operation symbol of $\mathcal{T}$. Then

$$o^{\mathbf{A}} = {}^{\mathbf{A}}o(a_0, \ldots, a_n) \in H.$$

For $m \in \omega - \{0\}$, let o be an m-ary operation symbol of $\mathcal{T}$ and let $h_0, \ldots, h_{m-1} \in H$. Then there are terms $t_j(x_0, \ldots, x_n) \in T(X)$, such that for $j = 0, \ldots, m-1$, we have $h_j = {}^{\mathbf{A}}t_j(a_0, \ldots, a_n)$. Then

$$\begin{aligned} o^{\mathbf{A}}(h_0, \ldots, h_{m-1}) &= o^{\mathbf{A}}({}^{\mathbf{A}}t_0(a_0, \ldots, a_n), \ldots, {}^{\mathbf{A}}t_{m-1}(a_0, \ldots, a_n)) \\ &= {}^{\mathbf{A}}(o(t_0, \ldots, t_{m-1}))(a_0, \ldots, a_n) \\ &\in H. \end{aligned}$$

Hence $\mathbf{H}$ is a subalgebra of $\mathbf{A}$ and $\langle M \rangle = \mathbf{H}$. □

1.5.2 Lemma. *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $\mathcal{T}$*-algebras and that* $\varphi : \mathbf{A} \to \mathbf{B}$ *is a homomorphism. Let* $t(x_0, \ldots, x_n) \in \mathbf{T}(X)$. *Then for all* $a_0, \ldots, a_n \in A$ *we have*

$$\varphi({}^{\mathbf{A}}t(a_0, \ldots, a_n)) = {}^{\mathbf{B}}t(\varphi(a_0), \ldots, \varphi(a_n)).$$

PROOF.

We proceed by induction on the complexity of t: Let $t = o$, where o is a nullary operation symbol of $\mathcal{T}$. Then since φ is a homomorphism, we have

$$\begin{aligned} \varphi({}^{\mathbf{A}}t(a_0, \ldots, a_n)) &= \varphi(o^{\mathbf{A}}(a_0, \ldots, a_n)) \\ &= \varphi(o^{\mathbf{A}}) \\ &= o^{\mathbf{B}} \\ &= {}^{\mathbf{B}}o(\varphi(a_0), \ldots, \varphi(a_n)) \\ &= {}^{\mathbf{B}}t(\varphi(a_0), \ldots, \varphi(a_n)). \end{aligned}$$

Now let t be a variable such that $t(x_0, \ldots, x_n) = x_i$ for some $i = 0, \ldots, n$. Then

$$\varphi({}^{\mathbf{A}}t(a_0, \ldots, a_n)) = \varphi(a_i) = {}^{\mathbf{B}}t(\varphi(a_0), \ldots, \varphi(a_n)).$$

Finally, for $m \in \omega - \{0\}$ and o an m-ary operation symbol, let $t = o(t_0, \ldots, t_{m-1})$ where for each $j = 0, \ldots, m-1$ we have

$$\varphi({}^{\mathbf{A}}t_j(a_0, \ldots, a_n)) = {}^{\mathbf{B}}t_j(\varphi(a_0), \ldots, \varphi(a_n)).$$

Then,

$$\begin{aligned}
\varphi({}^{\mathbf{A}}t(a_0,\ldots,a_n)) &= \varphi(o^{\mathbf{A}}({}^{\mathbf{A}}t_0(a_0,\ldots,a_n),\ldots,\ {}^{\mathbf{A}}t_{m-1}(a_0,\ldots,a_n))) \\
&= o^{\mathbf{B}}(\varphi({}^{\mathbf{A}}t_0(a_0,\ldots,a_n),\ldots,{}^{\mathbf{A}}\,t_{m-1}(a_0,\ldots,a_n))) \\
&= o^{\mathbf{B}}({}^{\mathbf{B}}t_0(\varphi(a_0),\ldots,\varphi(a_n)),\ldots,\ {}^{\mathbf{B}}t_{m-1}(\varphi(a_0),\ldots,\varphi(a_n))) \\
&= {}^{\mathbf{B}}t(\varphi(a_0),\ldots,\varphi(a_n)).
\end{aligned}$$

□

1.5.3 Lemma. *Let* $\mathbf{A}$ *be an algebra,* $\sigma \in \operatorname{Con}\mathbf{A}$ *and* $t(x_0,\ldots,x_n) \in \mathbf{T}(X)$*. Then for all* $a_0,\ldots,a_n,b_0,\ldots,b_n \in A$ *such that* $a_i \equiv b_i\ (\sigma)$*,* $i = 0,\ldots,n$*, we have*

$$ {}^{\mathbf{A}}t(a_0,\ldots,a_n) \equiv {}^{\mathbf{A}}t(b_0,\ldots,b_n)\ (\sigma).$$

PROOF.

Again we proceed by induction on the complexity of t: Let $t = o$ where o is a nullary operation symbol. Then ${}^{\mathbf{A}}t(a_0,\ldots,a_n) \equiv {}^{\mathbf{A}}t(b_0,\ldots,b_n)\ (\sigma)$ quite trivially since

$$ {}^{\mathbf{A}}t(a_0,\ldots,a_n) = o^{\mathbf{A}} = {}^{\mathbf{A}}t(b_0,\ldots,b_n),$$

and σ is reflexive. Next, let t be a variable and assume that $t(x_0,\ldots,x_n) = x_i$ for some $i = 0,\ldots,n$. Then since $a_k \equiv b_k\ (\sigma)$ for $k = 0,\ldots,n$, we have

$$ {}^{\mathbf{A}}t(a_0,\ldots,a_n) = a_i \equiv b_i = {}^{\mathbf{A}}t(b_0,\ldots,b_n)\ (\sigma).$$

Finally, for $m \in \omega - \{0\}$ and o an m-ary operation symbol, let $t = o(t_0,\ldots,t_{m-1})$ where $t_0,\ldots,t_{m-1}$ satisfy the lemma (as usual, we assume that all the $t_0,\ldots,t_{m-1}$ are of the same arity n; see Section 1.4.2). Then for $j = 0,\ldots,m-1$, we have

$$t_j(a_0,\ldots,a_n) \equiv t_j(b_0,\ldots,b_n)\ (\sigma).$$

Since σ is a congruence on $\mathbf{A}$, we have

$$o^{\mathbf{A}}(t_0(a_0,\ldots,a_n),\ldots,t_{m-1}(a_0,\ldots,a_n)) \equiv o^{\mathbf{A}}(t_0(b_0,\ldots,b_n),\ldots,t_{m-1}(b_0,\ldots,b_n))\ (\sigma),$$

and hence $t(a_0,\ldots,a_n) \equiv t(b_0,\ldots,b_n)\ (\sigma)$ for this case as well. □

To conclude this section we present a lemma which turns out to be rather useful in the next chapter. We recall that the congruence $\Theta(r)$ was defined in Section 1.2.2.

1.5.4 Lemma. *Suppose $\mathcal{K}$ is a nontrivial variety, $\emptyset \neq X \subseteq Z \subseteq V$ and $r \subseteq Z^2$, such that $r \cap X^2 = \emptyset$, and whenever $\langle u, v\rangle$, $\langle w, x\rangle \in r$, the elements u, v, w, x are distinct. Further, let $\mathbf{T}_{\mathcal{K}}(Z)$ be the free algebra over $\mathcal{K}$ with free generating set Z. If $\mathbf{G}$ is the subalgebra of $\mathbf{T}_{\mathcal{K}}(Z)$ generated by X, then*

$$G^2 \cap \Theta(r) = id_G.$$

PROOF.

Let us denote $\mathbf{F} = \mathbf{T}_{\mathcal{K}}(Z)$. We shall first show that $\mathbf{G}$ is freely generated by X. Take a mapping $\varphi : X \to A$ for some $\mathbf{A} \in \mathcal{K}$. Since $\mathbf{F}$ is free over $\mathcal{K}$, there is a homomorphism $\psi : \mathbf{F} \to \mathbf{A}$ extending φ. Now $\mathbf{G}$ is a subalgebra of $\mathbf{F}$, generated by X, so by Lemma 1.5.1 and 1.5.2, the restriction of ψ to $\mathbf{G}$ is the unique homomorphism from $\mathbf{G}$ to $\mathbf{A}$ extending φ. Thus $\mathbf{G}$ is free over $\mathcal{K}$, freely generated by X.

We choose any element $y \in X$ and define a mapping $f : Z \to G$ as follows:

$$\begin{aligned} f(x) &= x \quad \text{for each } x \in X; \\ f(u) &= w \quad \text{if } \langle u, w\rangle \in r \text{ or } \langle w, u\rangle \in r \text{ for some } w \in X; \\ f(z) &= y \quad \text{if } z \in Z \text{ and } \langle z, v\rangle, \langle v, z\rangle \notin r \text{ for all } v \in X. \end{aligned}$$

Notice that f is well defined because of our hypotheses concerning X and r. Since $\mathbf{F}$ is free, there is a unique homomorphism $g : \mathbf{F} \to \mathbf{G}$ extending f. If we set $h = g|G$, it is clear that $h : G \to G$ with $h|X = id_X$. Since X freely generates $\mathbf{G}$ we must have $h = id_G$. We know that $\Theta(r) \subseteq ker(g)$, by the definition f. Therefore

$$\begin{aligned} G^2 \cap \Theta(r) &\subseteq G^2 \cap ker(g) \\ &= ker(h) \\ &= id_G, \end{aligned}$$

hence $G^2 \cap \Theta(r) = id_G$. □

§6. $\mathcal{K}$-REFLECTIONS

1.6.1 Reflective classes. Refer to the diagram below:

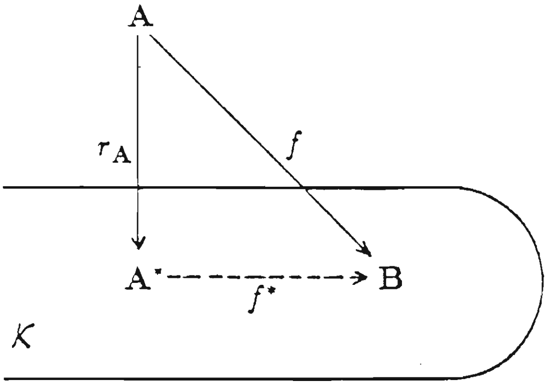

Figure 1.6.1

A subclass $\mathcal{K}$ of a class $\mathcal{L}$ of algebras of a fixed type is said to be *reflective* in $\mathcal{L}$ if for each $\mathbf{A} \in \mathcal{L}$ there exists a homomorphism $r_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^*$ for $\mathbf{A}^* \in \mathcal{K}$, with the following universal property:

> For each homomorphism $f : \mathbf{A} \to \mathbf{B}$ with the target $\mathbf{B} \in \mathcal{K}$ there is a unique homomorphism $f^* : \mathbf{A}^* \to \mathbf{B}$ such that $f = f^* \circ r_{\mathbf{A}}$.

The homomorphism $r_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^*$ is called a $\mathcal{K}$*-reflection* of $\mathbf{A}$. □

The above definition is a modification of the definition of reflective subcategories appearing in [1], page 193. We note that the $\mathcal{K}$-reflection $r_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^*$ is determined uniquely up to isomorphism in the following sense:

> The homomorphism $g : \mathbf{A} \to \mathbf{C}$ is a $\mathcal{K}$- reflection of $\mathbf{A} \in \mathcal{L}$ if and only if $\mathbf{C} \in \mathcal{K}$ and there is an isomorphism $f : \mathbf{A}^* \to \mathbf{C}$ such that $f \circ r_{\mathbf{A}} = g$.

1.6.2 Lemma. *Suppose $\mathcal{K}$ and $\mathcal{L}$ are classes of algebras such that $\mathcal{K} \subseteq \mathcal{L}$. Let $\mathbf{A}$ be a free algebra over $\mathcal{L}$ with $\mathcal{L}$-free generating set X. If $r : \mathbf{A} \to \mathbf{B}$ is a $\mathcal{K}$-reflection, then $\mathbf{B}$ is a free algebra over $\mathcal{K}$ with $r[X]$ as its $\mathcal{K}$-free generating set.*

PROOF.

Refer to the diagram in figure 1.6.2. Take a mapping $f : r[X] \to C$, $\mathbf{C} \in \mathcal{K}$. Since $f \circ (r|X)$ is a mapping of X into C, $\mathbf{C} \in \mathcal{K} \subseteq \mathcal{L}$ and $\mathbf{A}$ is free over $\mathcal{L}$, there exists a unique homomorphism $\varphi : \mathbf{A} \to \mathbf{C}$ such that

$$\varphi | X = f \circ (r|X).$$

Since $r : \mathbf{A} \to \mathbf{B}$ is a $\mathcal{K}$-reflection, there exists a unique homomorphism $\psi : \mathbf{B} \to \mathbf{C}$ such that $\varphi = \psi \circ r$. In particular we have

$$f \circ (r|X) = \varphi|X = (\psi \circ r)|X.$$

$$\begin{tikzcd}
X \arrow[r, hook, "\text{inclusion}"] \arrow[d, "r|X"'] & \mathbf{A} \arrow[d, "r"'] \arrow[rdd, dashed, "\varphi"] & \\
r[X] \arrow[r, hook, "\text{inclusion}"] \arrow[rrd, "f"'] & \mathbf{B} \arrow[rd, dashed, "\psi"'] & \\
 & & \mathbf{C}
\end{tikzcd}$$

Figure 1.6.2

Therefore $f = \psi|r[X]$. All that is left now is to establish the uniqueness of the homomorphism $\psi : \mathbf{B} \to \mathbf{C}$ which extends $f : r[X] \to \mathbf{C}$: Let $\chi : \mathbf{B} \to \mathbf{C}$ be a homomorphism satisfying

$$\psi|r[X] = f = \chi|r[X].$$

Then we have

$$(\psi \circ r)|X = (\chi \circ r)|X.$$

So $(\chi \circ r)|X = \varphi|X$. Now $\mathbf{A}$ is free over $\mathcal{L}$ with X as $\mathcal{L}$-free generating set. Therefore,

$$\chi \circ r = \varphi = \psi \circ r.$$

Also, since $r : \mathbf{A} \to \mathbf{B}$ is a $\mathcal{K}$- reflection of $\mathbf{A}$, ψ has to be determined in a unique way. Therefore $\chi = \psi$ and $\mathbf{B}$ is free over $\mathcal{K}$ with free generating set $r[X]$. $\square$

Suppose $\mathcal{K}$ is a class of algebras, and r a $\mathcal{K}$- reflection of an algebra $\mathbf{A}$. Can we use the $\mathcal{K}$- reflection r to determine whether $\mathcal{K}$ satisfies an identity of $\mathbf{A}$? The next and final lemma of this chapter addresses this question. However, before we proceed we need to first generalize the usual notion of an identity given in Definition 1.4.5. This prompts the following definition.

1.6.3 Definition. If $\mathbf{A}$ is an algebra, then each ordered pair $\langle a, b\rangle$ of elements of $\mathbf{A}$ is said to be an $\mathbf{A}$-*identity.* We shall say that an algebra $\mathbf{B}$ satisfies an $\mathbf{A}$-identity $\langle a, b\rangle$, written as

$$\mathbf{B} \models_{\mathbf{A}} \langle a, b\rangle,$$

if and only if for each homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$, we have $\varphi(a) = \varphi(b)$. A *class* $\mathcal{K}$ of algebras satisfies an $\mathbf{A}$-identity $\langle a, b\rangle$, written as

$$\mathcal{K} \models_{\mathbf{A}} \langle a, b\rangle,$$

if $\mathbf{B} \models_{\mathbf{A}} \langle a, b\rangle$ for all $\mathbf{B} \in \mathcal{K}$. □

As a consequence of the above definition, we can regard a term identity

$$t(x_0, \ldots, x_n) = s(x_0, \ldots, x_n)$$

as a $\mathbf{T}(X)$-identity $\langle t, s\rangle$. We now identify $\mathbf{B} \models t = s$ with the expression

$$\mathbf{B} \models_{\mathbf{T}(X)} \langle t, s\rangle.$$

1.6.4 Lemma. *Suppose $\mathcal{K}$ and $\mathcal{L}$ are classes of algebras such that $\mathcal{K} \subseteq \mathcal{L}$. Let $\mathbf{A}$ be a free algebra over $\mathcal{L}$ with $\mathcal{L}$-free generating set X. If $r : \mathbf{A} \to \mathbf{B}$ is a $\mathcal{K}$-reflection, then the following propositions are equivalent for each $\mathbf{A}$-identity $\langle a, b\rangle$:*

(i) $\mathcal{K} \models_{\mathbf{A}} \langle a, b\rangle$

(ii) $\mathbf{B} \models_{\mathbf{A}} \langle a, b\rangle$

(iii) $r(a) = r(b)$.

Especially, we have that $\langle a, b\rangle \in ker(r)$ if and only if $\mathcal{K} \models_{\mathbf{A}} \langle a, b\rangle$.

PROOF.

(i)$\Rightarrow$ (ii): This is trivial since $\mathbf{B} \in \mathcal{K}$.

(ii)$\Rightarrow$ (iii): Since $r : \mathbf{A} \to \mathbf{B}$ is a homomorphism, Definition 1.6.3 yields the required result, namely, that

$$r(a) = r(b).$$

(iii)$\Rightarrow$ (i): Take any $\mathbf{C} \in \mathcal{K}$ and a homomorphism

$$\varphi : \mathbf{A} \to \mathbf{C}.$$

Since $r : \mathbf{A} \to \mathbf{B}$ is a $\mathcal{K}$-reflection, there exists a homomorphism $\psi : \mathbf{B} \to \mathbf{C}$ such that $\psi \circ r = \varphi$. Therefore, using (iii) we have

$$\varphi(a) = \psi(r(a)) = \psi(r(b)) = \varphi(b).$$

Hence by Definition 1.6.3, we conclude that $\mathcal{K} \models_{\mathbf{A}} \langle a, b \rangle$. □

CHAPTER 2

SOME MAL'CEV CONDITIONS

In this chapter we shall discuss the Mal'cev-type conditions for varieties of algebras which are congruence n-permutable, congruence distributive, arithmetical, congruence modular and congruence regular.

§1. CONGRUENCE n-PERMUTABLE VARIETIES

2.1.1 Definition. Let $n \geq 2$ be an integer. An algebra $\mathbf{A}$ is said to be *congruence n-permutable* if its congruence lattice is n-permutable, that is, if

$$(\sigma, \tau)^n = (\tau, \sigma)^n \text{ for } \sigma, \tau \in \operatorname{Con} \mathbf{A},$$

where $(\sigma, \tau)^n$ has been defined in Section 1.1.3. A class $\mathcal{K}$ of algebras is said to be *congruence n- permutable* if every element of $\mathcal{K}$ is congruence n- permutable. We shall usually refer to congruence 2-permutability as congruence permutability. □

The main theorem of this section establishes Mal'cev conditions for varieties of algebras with n-permuting congruences. However, we shall first prove a lemma. If $\mathcal{K}$ is a class of algebras and s and t are terms, under what conditions will $\mathcal{K}$ satisfy the identity $s = t$? The following lemma gives some necessary and sufficient conditions under certain assumptions about $\mathcal{K}$.

2.1.2 Lemma. *Let n be a positive integer and let $X \subseteq V$. Further suppose that $s(x_0, \ldots, x_n)$, $t(x_0, \ldots, x_n) \in T(X)$, $\mathcal{K}$ is a class of algebras and $\mathbf{A}$ is a free algebra over $\mathcal{K}$ with free generating set G. Let $H = \{x_0, \ldots, x_n\}$ and let $f : H \to G$ be any mapping. Then:*

(i) ${}^{\mathbf{A}}s(f(x_0), \ldots, f(x_n)) = {}^{\mathbf{A}}t(f(x_0), \ldots, f(x_n))$ *iff for each $\mathbf{B} \in \mathcal{K}$ and for every $g : H \to B$ we have ${}^{\mathbf{B}}s(g(x_0), \ldots, g(x_n)) = {}^{\mathbf{B}}t(g(x_0), \ldots, g(x_n))$ whenever $ker(f) \subseteq ker(g)$.*

(ii) *If the mapping* $f : H \to G$ *is an injection then* ${}^{\mathbf{A}}s(f(x_0), \ldots, f(x_n)) = {}^{\mathbf{A}}t(f(x_0), \ldots, f(x_n))$ *iff* $K \models s = t$.

(iii) *If* $a_0, \ldots, a_n$ *are pairwise different elements of the set* G*, then we have that* ${}^{\mathbf{A}}s(a_0, \ldots, a_n) = {}^{\mathbf{A}}t(a_0, \ldots, a_n)$ *iff* $K \models s = t$.

PROOF.

(i) ($\Leftarrow$): This is trivial. If we let $\mathbf{B} = \mathbf{A}$, and $g = f$ then $ker(f) \subseteq ker(g)$ and ${}^{\mathbf{A}}s(f(x_0), \ldots, f(x_n)) = {}^{\mathbf{A}}t(f(x_0), \ldots, f(x_n))$.

($\Rightarrow$): Refer to the figure 2.1.1. Take $\mathbf{B} \in K$ and $g : H \to B$ such that $ker(f) \subseteq ker(g)$. We define a mapping $h_1 : G \to B$ such that $h_1 \circ f = g$ as follows: If $y \in G$ and $y = f(x)$, we define $h_1(y) = g(x)$ which is possible because $ker(f) \subseteq ker(g)$. If $y \in G \setminus f[H]$, then we choose any $b \in B$ and define $h_1(y) = b$. Now since $\mathbf{A}$ has free generating set G, there exists a unique homomorphism $h : \mathbf{A} \to \mathbf{B}$ which extends h_1. Therefore,

$$\begin{tikzcd}
H \arrow[r, "f"] \arrow[d, "g = h_1 \circ f"'] & G \arrow[r, hook] \arrow[dl, "h_1"] & \mathbf{A} = \langle G \rangle \arrow[dll, "h \text{ (homomorphism)}"] \\
\mathbf{B} & &
\end{tikzcd}$$

Figure 2.1.1

$$\begin{aligned}
{}^{\mathbf{B}}s(g(x_0), \ldots, g(x_n)) &= {}^{\mathbf{B}}s((h_1 \circ f)(x_0), \ldots, (h_1 \circ f)(x_n)) \\
&= {}^{\mathbf{B}}s((h \circ f)(x_0), \ldots, (h \circ f)(x_n)) \\
&= h({}^{\mathbf{A}}s(f(x_0), \ldots, f(x_n))) \quad \text{[by Lemma 1.5.2]}.
\end{aligned}$$

Similarly, we can show that ${}^{\mathbf{B}}t(g(x_0), \ldots, g(x_n)) = h({}^{\mathbf{A}}t(f(x_0), \ldots, f(x_n)))$. Since ${}^{\mathbf{A}}s(f(x_0), \ldots, f(x_n)) = {}^{\mathbf{A}}t(f(x_0), \ldots, f(x_n))$, we have

$${}^{\mathbf{B}}s(g(x_0), \ldots, g(x_n)) = {}^{\mathbf{B}}t(g(x_0), \ldots, g(x_n)).$$

(ii). If f is an injection then for every $g : H \to B$ we have $ker(f) \subseteq ker(g)$. The result then follows from (i).

(iii). Since $X \subseteq V$, the elements $x_0, \ldots, x_n$ of X are distinct. Hence we can define $f : H \to G$ by $f(x_i) = a_i$. Thus f is an injection and the required result follows from (ii). $\square$

2.1.3 Theorem (J. Hagemann and A. Mitschke [1973]). *Suppose $\mathcal{K}$ is a variety of algebras and $n \geq 2$ is an integer. Then the following conditions are equivalent:*

(i) *$\mathcal{K}$ is congruence n-permutable.*

(ii) *The free algebra $\mathbf{T}_{\mathcal{K}}(n+1)$ in $\mathcal{K}$ over $n+1$ free generators is congruence n-permutable.*

(iii) *There exist ternary terms $t_1, \ldots, t_{n-1}$ such that $\mathcal{K}$ satisfies the following identities:*

$$\begin{aligned} t_1(x,z,z) &= x, \\ t_{n-1}(x,x,z) &= z \text{ and} \\ t_i(x,x,z) &= t_{i+1}(x,z,z) \text{ for } i = 1, \ldots, n-2. \end{aligned}$$

PROOF.

We shall consider the case when n is even, say $n = 2m$. The case when n is odd can be treated in an analogous way.

(i)$\Rightarrow$ (ii): This is trivial since $\mathbf{T}_{\mathcal{K}}(n+1) \in \mathcal{K}$.

(ii)$\Rightarrow$ (iii): Let the free generators of $\mathbf{T}_{\mathcal{K}}(n+1)$ be $a_0, \ldots, a_n$. We shall take two n-permuting congruences, ρ and σ in $\operatorname{Con} \mathbf{T}_{\mathcal{K}}(n+1)$, and show that identities in (iii) hold in $\mathcal{K}$. Before we proceed, we recall from Section 1.2.3, that the principal congruence $\Theta(a,b)$ on an algebra $\mathbf{A}$ is defined as

$$\Theta(a,b) = \bigcap\{\tau \in \operatorname{Con} \mathbf{A}; \{a,b\}^2 \subseteq \tau\}.$$

Also, from Section 1.2.1, we have for $\rho, \sigma \in \operatorname{Con} \mathbf{A}$,

$$\rho \vee \sigma = \bigcup_{k=1}^{\infty} (\rho, \sigma)^k.$$

We now choose

$$\rho = \bigvee_{i=0}^{m-1} \Theta(a_{2i}, a_{2i+1}) \text{ and } \sigma = \bigvee_{i=0}^{m-1} \Theta(a_{2i+1}, a_{2i+2}).$$

Then

(1) $$a_0 \, \rho \, a_1 \, \sigma \, a_2 \, \rho \, a_3 \, \sigma \, \ldots \, \rho \, a_{n-1} \, \sigma \, a_n.$$

By the n-permutability of congruences on $\mathbf{T}_K(n+1)$, there exist elements b_i ($i = 0, \ldots, n$) such that

(2) $$a_0 = b_0 \, \sigma \, b_1 \, \rho \, b_2 \, \sigma \, b_3 \, \rho \, \ldots \, \sigma \, b_{n-1} \, \rho \, b_n = a_n.$$

Since $\mathbf{T}_K(n+1)$ is generated by $\{a_0, \ldots, a_n\}$, we have by Lemma 1.5.1 that each b_i is of the form

$$b_i = q_i(a_0, \ldots, a_n) \text{ for some term } q_i$$

where $q_i(a_0, \ldots, a_n)$ is the interpretation of $q_i(x_0, \ldots, x_n)$ in $\mathbf{T}_K(n+1)$. From Lemma 2.1.2 (iii) we deduce, since $a_0 = b_0$ and $a_n = b_n$, that

$$\begin{aligned} K &\models q_0(x_0, \ldots, x_n) &= x_0 \\ K &\models q_n(x_0, \ldots, x_n) &= x_n. \end{aligned}$$

We now consider two cases:

Case I: i an even number, $0 \leq i < n$.

Then from (2) we have $q_i(a_0, \ldots, a_n) \, \sigma \, q_{i+1}(a_0, \ldots, a_n)$. From the definition of σ, $a_{2j+1} \equiv a_{2j+2} \, (\sigma)$ for $j = 0, \ldots, m-1$. By Lemma 1.5.3 and the reflexivity of σ we have

(3) $$q_i(a_0, a_1, a_1, a_3, a_3, \ldots) \equiv q_{i+1}(a_0, a_1, a_1, a_3, a_3, \ldots) \, (\sigma).$$

From Lemma 1.5.4 the restriction of σ to the subalgebra generated by a_0 and the elements a_{2j+1} ($j = 1, \ldots, m-1$) is the identity on the subalgebra. Hence (3) implies that

$$q_i(a_0, a_1, a_1, a_3, a_3, \ldots) = q_{i+1}(a_0, a_1, a_1, a_3, a_3, \ldots).$$

Therefore by Lemma 2.1.2 (iii), for i even

$$\mathcal{K} \models q_i(x_0, x_1, x_1, x_3, x_3, \ldots) = q_{i+1}(x_0, x_1, x_1, x_3, x_3, \ldots).$$

Case II: i an odd number, $0 \leq i < n$.
Then from (2) we have $q_i(a_0, \ldots, a_n) \,\rho\, q_{i+1}(a_0, \ldots, a_n)$. From the definition of ρ, $a_{2j} \equiv a_{2j+1}\,(\rho)$ for $j = 1, \ldots, m-1$. By Lemma 1.5.3 and the reflexivity of ρ,

$$q_i(a_0, a_0, a_2, a_2, \ldots) \equiv q_{i+1}(a_0, a_0, a_2, a_2, \ldots)\ (\rho).$$

Again by Lemma 1.5.4 these terms are equal. Therefore, for i odd,

$$\mathcal{K} \models q_i(x_0, x_0, x_2, x_2, \ldots) = q_{i+1}(x_0, x_0, x_2, x_2, \ldots)$$

by Lemma 2.1.2. We now let

$$t_i(x, y, z) = q_i(x, \ldots, x, y, z, \ldots, z) \text{ for } i = 1, \ldots, n-1,$$

with x occurring i times and z occurring $n - i$ times. Therefore,

$$\begin{aligned} t_i(x, x, z) &= q_i(\underbrace{x, \ldots, x}_{i}, x, \underbrace{z, \ldots, z}_{n-i}) \\ &= q_{i+1}(\underbrace{x, \ldots, x, x}_{i+1}, z, \underbrace{z, \ldots, z}_{n-(i+1)}) \\ &= t_{i+1}(x, z, z), \end{aligned}$$

for $i = 1, \ldots, n-2$, while

$$\begin{aligned} t_1(x, z, z) &= q_1(x, z, z, \ldots, z) \\ &= q_0(x, z, z, \ldots, z) \\ &= x \end{aligned}$$

and

$$\begin{aligned} t_{n-1}(x, x, z) &= q_{n-1}(x, x, \ldots, x, z) \\ &= q_n(x, x, \ldots, z) \\ &= z. \end{aligned}$$

(iii)$\Rightarrow$ (i): Suppose $\mathbf{A} \in \mathcal{K}$ and let $\rho, \sigma \in \operatorname{Con}\mathbf{A}$. We show that ρ and σ permute n times. It suffices to show that

$$(\rho, \sigma)^n \subseteq (\sigma, \rho)^n.$$

Take $\langle a_0, a_n \rangle \in (\rho, \sigma)^n$. Then there exist $a_1, \ldots, a_{n-1} \in A$ such that

$$a_0 \, \rho \, a_1 \, \sigma \, a_2 \, \rho \, a_3 \, \sigma \, \ldots \, \rho \, a_{n-1} \, \sigma \, a_n.$$

Let $a_0 = b_0$ and $a_n = b_n$. We now have to find b_i $(0 < i < n)$ such that

$$a_0 = b_0 \, \sigma \, b_1 \, \rho \, b_2 \, \sigma \, b_3 \, \rho \, \ldots \, \sigma \, b_{n-1} \, \rho \, b_n = a_n.$$

Let $b_i = {}^{\mathbf{A}}t_i(a_{i-1}, a_i, a_{i+1})$ for $i \in \omega$ with $0 < i < n$, and consider the following two cases:

Case I: i even and $1 < i < n - 1$.
Then from

$$a_{i-1} \equiv a_i \, (\sigma), \quad a_i \equiv a_i \, (\sigma) \text{ and } a_{i+1} \equiv a_{i+1} \, (\sigma)$$

we have by Lemma 1.5.3, that

$${}^{\mathbf{A}}t_i(a_{i-1}, a_i, a_{i+1}) \equiv {}^{\mathbf{A}}t_i(a_i, a_i, a_{i+1}) \, (\sigma).$$

Similarly,

$${}^{\mathbf{A}}t_{i+1}(a_i, a_{i+1}, a_{i+1}) \equiv {}^{\mathbf{A}}t_{i+1}(a_i, a_{i+1}, a_{i+2}) \, (\sigma).$$

Also by (iii),

$${}^{\mathbf{A}}t_i(a_i, a_i, a_{i+1}) = {}^{\mathbf{A}}t_{i+1}(a_i, a_{i+1}, a_{i+1}).$$

Hence

$$\begin{aligned} b_i &= {}^{\mathbf{A}}t_i(a_{i-1}, a_i, a_{i+1}) \, \sigma \, {}^{\mathbf{A}}t_i(a_i, a_i, a_{i+1}) \\ &= {}^{\mathbf{A}}t_{i+1}(a_i, a_{i+1}, a_{i+1}) \, \sigma \, {}^{\mathbf{A}}t_{i+1}(a_i, a_{i+1}, a_{i+2}) \\ &= b_{i+1}. \end{aligned}$$

So σ identifies b_i with b_{i+1} for i even and $0 < i < n$.

Case II: i odd, $1 < i < n-1$.

Then for similar reasons as in case I,

$$\begin{aligned} b_i &= {}^{\mathbf{A}}t_i(a_{i-1}, a_i, a_{i+1})\ \rho\ {}^{\mathbf{A}}t_i(a_i, a_i, a_{i+1}) \\ &= {}^{\mathbf{A}}t_{i+1}(a_i, a_{i+1}, a_{i+1})\ \rho\ {}^{\mathbf{A}}t_{i+1}(a_i, a_{i+1}, a_{i+2}) \\ &= b_{i+1}. \end{aligned}$$

So ρ identifies b_i with b_{i+1} for i odd and $0 < i < n$.

Both cases together yield

$$a_0 = b_0\, \sigma\, b_1\, \rho\, b_2\, \sigma\, b_3\, \rho \ldots \sigma\, b_{n-1}\, \rho\, b_n.$$

Hence $\langle a_0, a_n \rangle \in (\sigma, \rho)^n$ and

$$(\rho, \sigma)^n \subseteq (\sigma, \rho)^n.$$

□

Clearly, setting $n = 2$ in Theorem 2.1.3, we obtain Mal'cev's original theorem characterizing congruence permutable algebras (as stated in the Introduction). In §5 of the next chapter, we shall show that without prior knowledge of this theorem, it is possible to construct, in a purely mechanical way, the equations $t(x, y, y) = x$ and $t(x, x, y) = y$ from the inequality $\rho \circ \sigma \subseteq \sigma \circ \rho$. This will serve to illustrate a more general algorithm for "finding" Mal'cev conditions.

2.1.4 Examples of Congruence n-Permutable Varieties.

The following are examples of varieties of algebras which are congruence n-permutable. Those algebras which are not commonly known have been defined in §3 of Chapter 1.

(i) The variety of *quasigroups* is congruence permutable with Mal'cev term

$$t_1(x, y, z) = (x/(y\backslash y))(y\backslash z).$$

Notice that $t_1(x, z, z) = (x/(z\backslash z))(z\backslash z) = x$ by Axiom 1 of Section 1.3.1. To show that $t_1(x, x, z) = z$, we require the identity, $x = x/(x\backslash x)$. This follows from the fact that $x = x/(x\backslash x)$ iff $x = x(x\backslash x)$ [by Lemma 1.3.1.1 (i)], and clearly the latter statement is true by Axiom 3 of Section 1.3.1. Hence

$$t_1(x, x, z) = (x/(x\backslash x))(x\backslash z) = x(x\backslash z) = z.$$

Remark. We note that the congruence permutability of groups and rings are consequences of the congruence permutability of quasigroups as follows.

1. For Groups: Let $\langle G; \cdot, ^{-1}, 1\rangle$ be a group. For $a, b \in G$ set

$$a/b \overset{def}{=} ab^{-1} \quad \text{and} \quad a\backslash b \overset{def}{=} a^{-1}b.$$

Then $\langle G; \cdot, /, \backslash\rangle$ is a quasigroup. The four axioms of Section 1.3.1 are verified below.

(i) $(x/y)y = (xy^{-1})y = x(y^{-1}y) = x1 = x$,

(ii) $(xy)/y = (xy)y^{-1} = x(yy^{-1}) = x$,

(iii) $x(x\backslash y) = x(x^{-1}y) = (xx^{-1})y = y$, and

(iv) $x\backslash(xy) = x^{-1}(xy) = (x^{-1}x)y = y$.

We can now take our Mal'cev terms t_0, t_1 and t_2 for the congruence permutability of the variety of quasigroups (noting our definition of / and \), and conclude by Theorem 2.1.3 that the variety of all groups is congruence permutable.

2. For Rings: We make the following general observation: For $i = 1, 2$, let $\mathcal{T}_i = \langle \mathbf{O}_i, ar_i\rangle$ be types such that

$$\mathbf{O}_1 \subseteq \mathbf{O}_2 \quad \text{and} \quad ar_1 = ar_2|\mathbf{O}_1.$$

Let $\mathbf{A}_i$ be a $\mathcal{T}_i$-algebra such that $A_1 = A_2$ and $o^{\mathbf{A}_1} = o^{\mathbf{A}_2}$ for all $o \in \mathbf{O}_1$. Then

$$\operatorname{Con} \mathbf{A}_2 \subseteq \operatorname{Con} \mathbf{A}_1.$$

Since both $\operatorname{Con} \mathbf{A}_2$ and $\operatorname{Con} \mathbf{A}_1$ are sublattices of the equivalence lattice $\mathbf{E}(A_1)$ which is equal to $\mathbf{E}(A_2)$, it follows that $\operatorname{Con} \mathbf{A}_2$ is a sublattice of $\operatorname{Con} \mathbf{A}_1$. Hence , if $\operatorname{Con} \mathbf{A}_1$ enjoys a property P which is expressible as a quasi-identity in the language of lattices, enriched with the additional binary operation $\circ$, then $\mathbf{Con}\,\mathbf{A}_2$ will also enjoy the same property P. The property P could be congruence n-permutability, congruence distributivity, arithmeticity, congruence modularity, or congruence regularity. (Note that for congruence regularity, the inclusion $\mathrm{Con}\,\mathbf{A}_2 \subseteq \mathrm{Con}\,\mathbf{A}_1$ is sufficient; we do not require the stronger result that $\mathbf{Con}\,\mathbf{A}_2$ is a sublattice of $\mathbf{Con}\,\mathbf{A}_1$).

Therefore, since groups are permutable, so are rings, R-modules, or for that matter any algebra which contains the group operations $\cdot$ and $^{-1}$ in its type and satisfies the group axioms.

(ii) P. M. Idziak shows in [16] that any variety of BCK- algebras is 3-permutable without employing any Mal'cev-type condition. By means of Idziak's results, H. Komori and P. M. Idziak proved the following theorem in [16], Theorem 1 and Lemma 1.

Let $\mathcal{K}$ be a BCK-variety. Then there are positive integers n, m and $\{\cdot\}$-terms $u_i = u_i(x,y)$ and $v_j = v_j(x,y)$ satisfying for every BCK-algebra, the identities,

$$u_i(x,x) = 0 \quad \textit{and} \quad v_j(x,x) = 0 \quad \textit{for } i = 1,\ldots,n \textit{ and } j = 1,\ldots,m$$

such that $\mathcal{K}$ satisfies the identity

$$xu_1(x,y)\ldots u_n(x,y) = yv_1(x,y)\ldots v_m(x,y).$$

The above mentioned theorem enables us to determine Mal'cev terms for the congruence 3-permutability of a variety of BCK-algebras, the existence of which is guaranteed by Theorem 2.1.3. We choose

$$\begin{aligned} t_1(x,y,z) &= xu_1(y,z)\ldots u_n(y,z) \text{ and} \\ t_2(x,y,z) &= zv_1(x,y)\ldots v_m(x,y). \end{aligned}$$

The existence and properties of the terms $u_1,\ldots,u_n$ and $v_1,\ldots,v_m$ are given in [30], Sections 2.6 and 2.7. The identities of Theorem 2.1.3 are verified below.

$$\begin{aligned} t_1(x,z,z) &= xu_1(z,z)\ldots u_n(z,z) = x\,\underbrace{0\ldots 0}_{n\text{-times}} = x. \\ t_1(x,x,z) &= xu_1(x,z)\ldots u_n(x,z) = zv_1(x,z)\ldots v_m(x,z) = t_2(x,z,z). \end{aligned}$$

$$t_2(x,x,z) \;=\; zv_1(x,x)\dots v_m(x,x) \;=\; z\underbrace{0\dots0}_{m-\text{times}} \;=\; z.$$

□

It is possible to deduce from the Mal'cev characterization of congruence permutability that no non trivial variety of BCK- algebras is congruence permutable. The proof is as follows. First note that if $t(x_0,\dots,x_n)$ $(n \geq 0)$ is any $\{\cdot,0\}$-term then for some $i \in \{0,\dots,n\}$, every algebra satisfies the (abbreviated) identity $t(x_0,\dots,x_n) \leq x_i$. This follows, by induction on complexity, from 4 and 6 (p. 11), the fact that $\leq$ is a partial order with least element 0 in every BCK-algebra, and the BCK-property $xy \leq x$ (since $(xy)x = ((xy)(x0))(0y) = 0$). Now if $\mathcal{K}$ is a congruence permutable variety of BCK- algebras then for some ternary $\{\cdot,0\}$-term t, $\mathcal{K}$ satisfies $t(x,x,y) = y$ and $t(x,y,y) = x$. By the above, $\mathcal{K}$ also satisfies $t(x,y,z) \leq u$ for some $u \in \{x,y,z\}$. If $u = x$ then $\mathcal{K}$ satisfies $y = t(x,x,y) \leq x$, and therefore, by symmetry of variables, $\mathcal{K}$ satisfies $y = x$, that is, $\mathcal{K}$ is trivial. Similarly, $\mathcal{K}$ is trivial if $u = y$ or $u = z$. This fact was first observed in Corollary 3.16 of W. H. Cornish: 3-permutability and quasicommutative BCK- algebras, Math. Japon., **25** (4) (1980), 477–496.

(iii) (A. Mitschke [1971], J. Hagemann and A. Mitschke [1973]). The variety of *implication algebras* is congruence 3-permutable. The Mal'cev terms are:

$$\begin{aligned} t_1(x,y,z) &= (z \to y) \to x \text{ and} \\ t_2(x,y,z) &= (x \to y) \to z. \end{aligned}$$

Using the axioms and properties of implication algebras given in Section 1.3.7 it is clear that

$$t_1(x,z,z) = (z \to z) \to x = 0 \to x = x,$$

$$t_1(x,x,z) = (z \to x) \to x \text{ and } t_2(x,z,z) = (x \to z) \to z = (z \to x) \to x,$$

$$t_2(x,x,z) = (x \to x) \to z = 0 \to z = z.$$

It is appropriate to note here that in the light of Theorem 1.3.7.7, Remark 1.3.7.8 and Proposition 1.3.7.9, the congruence 3-permutability of the variety of implication algebras follows directly from example (ii) above.

(iv) The variety of *right complemented semigroups* has congruences that are 3-permutable. As our Mal'cev terms we have

$$\begin{aligned} t_1(x,y,z) &= x(y*z) \text{ and} \\ t_2(x,y,z) &= z(y*x). \end{aligned}$$

Then,

$$t_1(x,z,z) = x(z*z) = x,$$

$$t_1(x,x,z) = x(x*z) = z(z*x) \text{ and } t_2(x,z,z) = z(z*x),$$

$$t_2(x,x,z) = z(x*x) = z.$$

It is true that for each integer $n \geq 2$, there is an algebra, called an *n-Boolean algebra* (see [18]), which is congruence $(n+1)$-permutable but not congruence n-permutable. These algebras were introduced by E. T. Schmidt (On n- permutable equational classes, Acta Sci. Math. (Szeged) **33** (1972), 29–30) solely to distinguish between degrees of permutability.

§2. CONGRUENCE DISTRIBUTIVE VARIETIES

2.2.1 Definition. An algebra **A** is said to be *congruence distributive* if its congruence lattice is distributive, that is, if for all $\rho, \sigma, \tau \in \operatorname{Con} \mathbf{A}$,

$$\rho \vee (\sigma \cap \tau) = (\rho \vee \sigma) \cap (\rho \vee \tau).$$

A class $\mathcal{K}$ of algebras is said to be *congruence distributive* if every algebra of $\mathcal{K}$ is congruence distributive. □

The following definition is necessary for the proof of the next lemma.

2.2.2 Definition. If $f : M \to N$ is a mapping, then for $x, y \in M$ we define

$$f_2(\langle x, y\rangle) = \langle f(x), f(y)\rangle.$$

□

It is therefore clear that $f_2 : M^2 \to N^2$. Hence for a binary relation r on M we have

$$f_2[r] = \{\langle f(x), f(y)\rangle; \langle x, y\rangle \in r\}$$

for all $x, y \in M$.

2.2.3 Lemma. *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are algebras of a given type and the mapping* $f : \mathbf{A} \to \mathbf{B}$ *is a surjective homomorphism. Then the mapping* $g : \operatorname{Con}\mathbf{B} \to \operatorname{Con}\mathbf{A}$ *defined by*

$$g(\sigma) = f_2^{-1}[\sigma] \quad \textit{for} \quad \sigma \in \operatorname{Con}\mathbf{B}$$

is an injective lattice homomorphism from $\operatorname{Con}\mathbf{B}$ *into* $\operatorname{Con}\mathbf{A}$.

PROOF.

The mapping $g : \operatorname{Con}\mathbf{B} \to \operatorname{Con}\mathbf{A}$ is well defined since whenever $\sigma \in \operatorname{Con}\mathbf{B}$, we have $g(\sigma) \in \operatorname{Con}\mathbf{A}$ as follows: Without loss of generality we take $o \in \mathbf{O}$ to be a binary operation. Let $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle \in g(\sigma)$. Then by the definition of g we have that $\langle f(x_1), f(y_1)\rangle$, $\langle f(x_2), f(y_2)\rangle \in \sigma$. Since σ is a congruence on $\mathbf{B}$, we have that

$$\langle o(f(x_1), f(x_2)), o(f(y_1), f(y_2))\rangle \in \sigma.$$

But f is a homomorphism. Therefore

$$\langle f(o(x_1, x_2)), f(o(y_1, y_2))\rangle \in \sigma.$$

Hence, again by the definition of g, we have that $\langle o(x_1, x_2), o(y_1, y_2)\rangle \in g(\sigma)$.

We shall now show that the mapping g is an injection: If $\sigma \subseteq \tau$ then, trivially, $g(\sigma) \subseteq g(\tau)$. We need to prove that if $\sigma \subset \tau$ then $g(\sigma) \subset g(\tau)$. Take $\langle u, v\rangle \in \tau - \sigma$ with $u = f(x)$ and $v = f(y)$. Then $\langle x, y\rangle \in g(\tau) - g(\sigma)$. Hence g is a one to one mapping.

To conclude the proof we show that $g : \operatorname{Con}\mathbf{B} \to \operatorname{Con}\mathbf{A}$ is a lattice homomorphism: For $\langle Y_i,\ i \in I\rangle$, a family of subsets of A where $I \neq \emptyset$, we have that

$$f^{-1}[\bigcap_{i\in I} Y_i] = \bigcap_{i\in I} f^{-1}[Y_i].$$

This is also true for f_2, that is, $f_2^{-1}[\sigma \cap \tau] = f_2^{-1}[\sigma] \cap f_2^{-1}[\tau]$ for $\sigma, \tau \in \mathrm{Con}\,\mathbf{B}$. Hence,

$$g(\sigma \wedge \tau) = g(\sigma \cap \tau) = g(\sigma) \cap g(\tau) = g(\sigma) \wedge g(\tau). \tag{4}$$

For the join operation, it is trivial that

$$g(\sigma) \vee g(\tau) \subseteq g(\sigma \vee \tau).$$

To establish the converse inclusion we note firstly that

$$\sigma \vee \tau = \sigma \cup \sigma\tau \cup \sigma\tau\sigma \cup \sigma\tau\sigma\tau \cup \ldots.$$

Now $\langle x, y \rangle \in g(\sigma \vee \tau)$ implies that $\langle f(x), f(y) \rangle \in \sigma \vee \tau$. Therefore there exist $u_0, \ldots, u_n \in \mathbf{B}$ such that

$$f(x) = u_0\, \sigma\, u_1\, \tau\, u_2\, \sigma\, u_3\, \tau \ldots u_n = f(y).$$

Hence there exist $z_i \in A$ for $i = 0, \ldots, n$ such that $u_i = f(z_i), z_0 = x, z_n = y$ and

$$x = z_0\, g(\sigma)\, z_1\, g(\tau)\, z_2 \ldots z_n = y.$$

This means that $\langle x, y \rangle \in g(\sigma) \vee g(\tau)$ and

$$g(\sigma \vee \tau) \subseteq g(\sigma) \vee g(\tau).$$

Both inclusions yield

$$g(\sigma \vee \tau) = g(\sigma) \vee g(\tau). \tag{5}$$

From (4) and (5) we conclude $g : \mathbf{ConB} \to \mathbf{ConA}$ is a lattice homomorphism. □

2.2.4 Theorem (B. Jónsson [1967]). *Suppose K is a non-trivial variety and $X \subseteq V$ is a set with at least three elements. Then the following conditions are equivalent:*

(i) *K is congruence distributive.*

(ii) *$\mathbf{T}_K(X)$ is congruence distributive.*

(iii) *$\mathbf{T}_K(3)$ is congruence distributive.*

(iv) *For each* $\mathbf{A} \in \mathcal{K}$ *and for all* $a, b, c \in \mathbf{A}$,

$$\langle a, c\rangle \in (\Theta(a,b) \wedge \Theta(a,c)) \vee (\Theta(b,c) \wedge \Theta(a,c)).$$

(v) *There exist a positive integer* n *and ternary terms* $t_0, \ldots, t_n$ *such that* $\mathcal{K}$ *satisfies the following identities:*

$$\begin{aligned}
t_0(x,y,z) &= x, \\
t_n(x,y,z) &= z, \\
t_i(x,y,x) &= x \text{ for } i = 1, \ldots, n-1, \\
t_{i-1}(x,x,z) &= t_i(x,x,z) \text{ for } 1 \leq i \leq n,\ i \text{ odd, and} \\
t_{i-1}(x,z,z) &= t_i(x,z,z) \text{ for } 2 \leq i \leq n,\ i \text{ even.}
\end{aligned}$$

PROOF.

(i)⇒ (ii): This is trivial since $\mathbf{T}_{\mathcal{K}}(X) \in \mathcal{K}$.

(ii)⇒ (iii): Let $Y = \{x, y, z\}$ be a three-element free generating set of $\mathbf{T}_{\mathcal{K}}(3)$. Consider a surjective mapping $f : X \rightarrow Y$. Since both $\mathbf{T}_{\mathcal{K}}(X)$ and $\mathbf{T}_{\mathcal{K}}(3)$ are free over $\mathcal{K}$, f is extendible to a surjective homomorphism $\varphi : \mathbf{T}_{\mathcal{K}}(X) \rightarrow \mathbf{T}_{\mathcal{K}}(3)$. By Lemma 2.2.3 there exists a lattice embedding

$$\psi : \mathbf{Con}\mathbf{T}_{\mathcal{K}}(3) \rightarrow \mathbf{Con}\mathbf{T}_{\mathcal{K}}(X)$$

Thus the congruence distributivity of $\mathbf{T}_{\mathcal{K}}(X)$ implies that $\mathbf{T}_{\mathcal{K}}(3)$ is congruence distributive.

(iii)⇒ (iv): Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by a three-element set $\{a, b, c\}$. Then there exists a unique surjective homomorphism $h : \mathbf{T}_{\mathcal{K}}(3) \rightarrow \mathbf{B}$ such that $h(x) = a$, $h(y) = b$ and $h(z) = c$, where x, y, z are three free generators of $\mathbf{T}_{\mathcal{K}}(3)$. By Lemma 2.2.3 there exists a lattice embedding

$$\chi : \mathbf{Con}\mathbf{B} \rightarrow \mathbf{Con}\mathbf{T}_{\mathcal{K}}(3).$$

Since $\mathbf{T}_{\mathcal{K}}(3)$ is congruence distributive, we have that $\mathbf{B}$ is congruence distributive. Therefore,

$$\langle a, c\rangle \in \Theta(a,c) \wedge (\Theta(a,b) \vee \Theta(b,c))$$

$$= (\Theta(a,b) \wedge \Theta(a,c)) \vee (\Theta(b,c) \wedge \Theta(a,c)).$$

(iv)$\Rightarrow$ (v): Let $\mathbf{T}_K(3) = \mathbf{T}_K(x,y,z)$. If we substitute x for a, y for b and z for c in (iv) we have

$$\langle x,z\rangle \in (\Theta(x,y) \wedge \Theta(x,z)) \vee (\Theta(y,z) \wedge \Theta(x,z))$$

In view of this and Lemma 1.5.1, there exist ternary terms $t_0,\ldots,t_n$ in the absolutely free algebra $\mathbf{T}(V)$ such that $t_0(x,y,z) = x$, $t_n(x,y,z) = z$ and

$$x = t_0\,(\Theta(x,y) \wedge \Theta(x,z))\,t_1\,(\Theta(y,z) \wedge \Theta(x,z))\,t_2 \ldots t_n = z,$$

where each $t_i(x,y,z)$ is considered to be the interpretation of this term in $\mathbf{T}_K(3)$. Note that the first two identities of (v) are satisfied already. For the remaining three sets of identities, we observe the following:

(I) For all $i = 0,\ldots,n$, $t_0(x,y,z) \equiv t_i(x,y,z)(\Theta(x,z))$. Also, since $\langle x,z\rangle \in \Theta(x,z)$, we have $t_i(x,y,z) \equiv t_i(x,y,x)(\Theta(x,z))$ by Lemma 1.5.3. In particular, $x \equiv t_i(x,y,x)(\Theta(x,z))$. By Lemma 1.5.4, the restriction of $\Theta(x,z)$ to the subalgebra generated by $\{x,y\}$ is the identity on the subalgebra. Hence by Lemma 2.1.2,

$$K \models x = t_i(x,y,x) \quad \text{for } i = 1,\ldots,n.$$

(II) For $1 \leq i \leq n$ and i odd, we have $t_{i-1}(x,y,z) \equiv t_i(x,y,z)(\Theta(x,y))$. But $t_{i-1}(x,x,z) \equiv t_{i-1}(x,y,z)(\Theta(x,y))$ and $t_i(x,y,z) \equiv t_i(x,x,z)(\Theta(x,y))$. Hence $t_{i-1}(x,x,z) \equiv t_i(x,x,z)(\Theta(x,y))$. Again by Lemma 1.5.4,

$$K \models t_{i-1}(x,x,z) = t_i(x,x,z) \quad \text{for } i \text{ odd}.$$

(III) For $2 \leq i \leq n$ and i even, we have $t_{i-1}(x,y,z) \equiv t_i(x,y,z)(\Theta(y,z))$. But $t_{i-1}(x,y,y) \equiv t_{i-1}(x,y,z)(\Theta(y,z))$ and $t_i(x,y,z) \equiv t_i(x,y,y)(\Theta(y,z))$. Hence $t_{i-1}(x,y,y) \equiv t_i(x,y,y)(\Theta(y,z))$. Also by Lemma 1.5.4,

$$K \models t_{i-1}(x,y,y) = t_i(x,y,y) \quad \text{for } i \text{ even}.$$

(v)$\Rightarrow$(i): Let $\mathbf{A} \in K$ and $\rho,\sigma,\tau \in \operatorname{Con}\mathbf{A}$. We need only show that

$$\rho \wedge (\sigma \vee \tau) \subseteq (\rho \wedge \sigma) \vee (\rho \wedge \tau)$$

as the converse inclusion is true for any lattice. For each positive integer m we let

$$q_m = (\sigma, \tau)^m$$

where $(\sigma, \tau)^m$ is defined in Section 1.1.3. Since $\sigma \vee \tau = \bigcup\{q_m; m \in \omega - \{0\}\}$, we shall now prove that

$$\rho \cap q_m \subseteq (\rho \cap \sigma) \vee (\rho \cap \tau) \tag{6}$$

We proceed by induction on m: For $m = 1$, the inclusion (6) is satisfied since $q_1 = (\sigma, \tau)^1 = \sigma$. We assume that (6) is true for a positive integer m and assert that (6) is true for an integer $m + 1$. To prove our assertion, let us suppose that

$$\langle a, b\rangle \in \rho \cap q_{m+1}.$$

Then there is a $c \in A$ such that $\langle a, b\rangle \in \rho$, $\langle a, c\rangle \in q_m$ and $\langle c, b\rangle \in q$ where $q \in \{\sigma, \tau\}$. For $i = 0, \ldots, n$ we set

$$d_i = {}^{\mathbf{A}}t_i(a, c, b)$$

where ${}^{\mathbf{A}}t_i(a, c, b)$ is the interpretation of $t_i(x, y, z)$ in $\mathbf{A}$. Since $\rho \in \operatorname{Con}\mathbf{A}$ and $\langle a, b\rangle \in \rho$, we have by Lemma 1.5.3 that $\langle {}^{\mathbf{A}}t_i(a, c, a), {}^{\mathbf{A}}t_i(a, c, b)\rangle \in \rho$. From (v), ${}^{\mathbf{A}}t_i(a, c, a) = a$, so that

$$\langle a, d_i\rangle \in \rho \text{ for all } i. \tag{7}$$

Since $\langle a, c\rangle \in q_m$, we have that $\langle c, a\rangle \in q_m^{-1}$. Also,

$$q_m = \sigma\,\tau\,\sigma \ldots p \qquad (m \text{ factors})$$

with $p \in \{\sigma, \tau\}$. Hence we have from the symmetry of congruences that

$$q_m^{-1} = p^{-1} \ldots \sigma^{-1}\tau^{-1}\sigma^{-1} = p \ldots \sigma\,\tau\,\sigma.$$

Therefore $\langle c, a\rangle \in q_m^{-1}$ implies that there exist $c_1, \ldots, c_{m-1} \in A$ such that

$$c\,p\,c_1 \ldots c_{m-3}\,\sigma\,c_{m-2}\,\tau\,c_{m-1}\,\sigma\,a.$$

Then by Lemma 1.5.3,

$$d_{i-1} = {}^{\mathbf{A}}t_{i-1}(a, c, b)\,p\,{}^{\mathbf{A}}t_{i-1}(a, c_1, b) \ldots {}^{\mathbf{A}}t_{i-1}(a, c_{m-1}, b)\,\sigma\,{}^{\mathbf{A}}t_{i-1}(a, a, b).$$

We therefore conclude that

$$(8) \qquad \langle d_{i-1}, {}^{\mathbf{A}}t_{i-1}(a,a,b)\rangle \in q_m^{-1}.$$

Similarly, $\langle d_i, {}^{\mathbf{A}}t_i(a,a,b)\rangle \in q_m^{-1}$ and we have

$$(9) \qquad \langle {}^{\mathbf{A}}t_i(a,a,b), d_i\rangle \in q_m.$$

Case I: Let i be an odd number. Now by Lemma 1.5.3 and the fact that $\langle a,b\rangle \in \rho$, we establish that

$$\langle {}^{\mathbf{A}}t_i(a,a,b), {}^{\mathbf{A}}t_i(a,a,a)\rangle \in \rho.$$

This means that $\langle {}^{\mathbf{A}}t_i(a,a,b), a\rangle \in \rho$. This together with (7) yields

$$(10) \qquad \langle {}^{\mathbf{A}}t_i(a,a,b), d_i\rangle \in \rho^2 = \rho.$$

In the same way, $\langle {}^{\mathbf{A}}t_{i-1}(a,a,b), d_{i-1}\rangle \in \rho$, and by the symmetry of ρ,

$$(11) \qquad \langle d_{i-1}, {}^{\mathbf{A}}t_{i-1}(a,a,b)\rangle \in \rho.$$

From (8) and (11) we have

$$\langle d_{i-1}, {}^{\mathbf{A}}t_{i-1}(a,a,b)\rangle \in \rho \cap q_m^{-1},$$

and from (9) and (10) we have

$$\langle {}^{\mathbf{A}}t_i(a,a,b), d_i\rangle \in \rho \cap q_m.$$

However, by (v), ${}^{\mathbf{A}}t_{i-1}(a,a,b) = {}^{\mathbf{A}}t_i(a,a,b)$ for i odd. Therefore

$$\langle d_{i-1}, d_i\rangle \in (\rho \cap q_m^{-1})(\rho \cap q_m) = (\rho \cap q_m)^{-1}(\rho \cap q_m)$$

by the symmetry of ρ and the fact that $\rho^{-1} \cap q_m^{-1} = (\rho \cap q_m)^{-1}$. By the induction assumption,

$$\begin{aligned}(\rho \cap q_m)^{-1}(\rho \cap q_m) &\subseteq ((\rho \cap \sigma) \vee (\rho \cap \tau))^{-1}((\rho \cap \sigma) \vee (\rho \cap \tau)) \\ &= (\rho \cap \sigma) \vee (\rho \cap \tau)\end{aligned}$$

Thus for i odd,

$$\langle d_{i-1}, d_i\rangle \in (\rho \cap \sigma) \vee (\rho \cap \tau).$$

Case II: Let i be an even number. Then by (v),

$$^{\mathbf{A}}t_{i-1}(a, c, c) = {}^{\mathbf{A}}t_i(a, c, c).$$

Again by Lemma 1.5.3 and the fact that $\langle c, b\rangle \in q \in \{\sigma, \tau\}$,

$$d_{i-1} = {}^{\mathbf{A}}t_{i-1}(a, c, b)\, q\, {}^{\mathbf{A}}t_{i-1}(a, c, c) = {}^{\mathbf{A}}t_i(a, c, c)\, q\, {}^{\mathbf{A}}t_i(a, c, b) = d_i.$$

Hence,

$$\langle d_{i-1}, d_i\rangle \in q$$

Also, since $\langle a, b\rangle \in \rho$, $d_{i-1} = {}^{\mathbf{A}}t_{i-1}(a, c, b)\, \rho\, {}^{\mathbf{A}}t_{i-1}(a, c, a) = a$. This together with (7) yields

$$\langle d_{i-1}, d_i\rangle \in \rho^2 = \rho.$$

Thus for i even also

$$\langle d_{i-1}, d_i\rangle \in \rho \cap q \subseteq (\rho \cap \sigma) \vee (\rho \cap \tau).$$

Since $d_0 = {}^{\mathbf{A}}t_0(a, c, b) = a$ and $d_n = {}^{\mathbf{A}}t_n(a, c, b) = b$, we have

$$\langle a, b\rangle \in (\rho \cap \sigma) \vee (\rho \cap \tau),$$

and finally

$$\rho \cap q_{m+1} \subseteq (\rho \cap \sigma) \vee (\rho \cap \tau).$$

Therefore K is congruence distributive. □

2.2.5 Examples of a Congruence Distributive Varieties.

(i) The variety of all lattices $\mathbf{L} = \langle L; \wedge, \vee\rangle$ is congruence distributive: We choose $n = 2$ and our Mal'cev terms are:

$$\begin{aligned} t_0(x, y, z) &= x, \\ t_1(x, y, z) &= (x \vee y) \wedge (x \vee z) \wedge (y \vee z), \quad \text{and} \\ t_2(x, y, z) &= z. \end{aligned}$$

Observe that with

$$\begin{aligned} t_0(x,y,x) &= x, \\ t_1(x,y,x) &= (x \vee y) \wedge (x \vee x) \wedge (y \vee x) = x, \\ t_1(x,x,z) &= (x \vee x) \wedge (x \vee z) \wedge (x \vee z) = x = t_0(x,x,z), \quad \text{and} \\ t_1(x,z,z) &= (x \vee z) \wedge (x \vee z) \wedge (z \vee z) = z = t_2(x,z,z), \end{aligned}$$

the identities of theorem 2.2.4 (v) are clearly satisfied.

(ii) Every variety $\mathcal{K}$ of BCK-algebras is congruence distributive: If $\mathcal{K}$ is a trivial variety, then trivially $\mathcal{K}$ is congruence distributive. So, we shall assume that $\mathcal{K}$ is nontrivial. Then by the Komori-Idziak theorem ([16], Theorem 1 and Lemma 1), there are binary $\{\cdot\}$-terms $u_1, \ldots, u_n, v_1, \ldots, v_m$ such that $u_i(x,x) = 0 = v_j(x,x)$ for every BCK-algebra ($i = 1, \ldots, n$ and $j = 1, \ldots, m$), and such that $\mathcal{K}$ satisfies the identity

(a) $xu_1(x,y) \ldots u_n(x,y) = yv_1(x,y) \ldots v_m(x,y)$.

We now choose $n = 3$ and define our Mal'cev terms as follows:

$$\begin{aligned} t_0(x,y,z) &= x, \\ t_1(x,y,z) &= x(u_1(x,y)u_1(z,y)) \ldots (u_n(x,y)u_n(z,y)) \\ t_2(x,y,z) &= z(v_1(x,z)v_1(y,z)) \ldots (v_m(x,z)v_m(y,z)), \quad \text{and} \\ t_3(x,y,z) &= z. \end{aligned}$$

All that remains is to verify the identities of Theorem 2.2.4 (v). The first two identities are satisfied by our choice of $t_0(x,y,z)$ and $t_3(x,y,z)$. For the third set of identities we have $t_0(x,y,x) = x$,

$$\begin{aligned} t_1(x,y,x) &= x(u_1(x,y)u_1(x,y)) \ldots (u_n(x,y)u_n(x,y)) \\ &= x \underbrace{0 \ldots 0}_{n\text{-times}} \\ &= x, \end{aligned}$$

and

$$t_2(x,y,x) = x(v_1(x,x)v_1(y,x)) \ldots (v_m(x,x)v_m(y,x))$$

$$\begin{aligned} &= x(0v_1(y,x))\dots(0v_m(y,x)) \\ &= x\ \underbrace{0\dots0}_{m\text{-times}} \\ &= x. \end{aligned}$$

For the fourth set of identities of Theorem 2.2.4 (v) we have

$$\begin{aligned} t_1(x,x,z) &= x(u_1(x,x)u_1(z,x))\dots(u_n(x,x)u_n(z,x)) \\ &= x(0u_1(z,x))\dots(0u_n(z,x)) \\ &= x \\ &= t_0(x,x,z). \end{aligned}$$

and

$$\begin{aligned} t_2(x,x,z) &= z(v_1(x,z)v_1(x,z))\dots(v_m(x,z)v_m(x,z)) \\ &= z\ \underbrace{0\dots0}_{m\text{-times}} \\ &= z \\ &= t_3(x,x,z). \end{aligned}$$

Finally, for the last identity (in our case) of Theorem 2.2.4 (v) we have

$$\begin{aligned} t_1(x,z,z) &= x(u_1(x,z)u_1(z,z))\dots(u_n(x,z)u_n(z,z)) \\ &= x(u_1(x,z)0)\dots(u_n(x,z)0) \\ &= xu_1(x,z)\dots u_n(x,z) \\ &= zv_1(x,z)\dots v_m(x,z) \quad [\text{by (a) with } y = z] \\ &= z(v_1(x,z)v_1(z,z))\dots(v_m(x,z)v_m(z,z)) \\ &= t_2(x,z,z). \end{aligned}$$

§3. ARITHMETICAL VARIETIES

2.3.1 Definition. An algebra is said to be *arithmetical* if it is both congruence permutable and congruence distributive. A class $\mathcal{K}$ of algebras is said to be *arithmetical* if every algebra of $\mathcal{K}$ is arithmetical. □

In §6 of chapter 3 we shall define arithmetical classes in terms of the so called Chinese classes. We shall then apply the scheme developed in chapter 3 to establish the Mal'cev type conditions for arithmetical varieties.

2.3.2 Theorem (A.F. Pixley [1963]). *Suppose $\mathcal{K}$ is a non-trivial variety and that $X \subseteq V$ is a set with at least three elements. Then the following conditions are equivalent:*

(i) *$\mathcal{K}$ is an arithmetical variety.*

(ii) *$\mathbf{T}_{\mathcal{K}}(X)$ is an arithmetical algebra.*

(iii) *There exist ternary terms $t(x,y,z)$ and $s(x,y,z)$ such that $\mathcal{K}$ satisfies the following identities:*

$$\begin{aligned} t(x,z,z) &= x, \\ t(x,x,z) &= z, \\ s(x,x,z) &= x, \\ s(x,y,x) &= x, \quad \textit{and} \\ s(x,z,z) &= z. \end{aligned}$$

PROOF.
(i)⇒ (ii): This is trivial since $\mathbf{T}_{\mathcal{K}}(X) \in \mathcal{K}$.

(ii)⇒(iii): Since $\mathbf{T}_{\mathcal{K}}(X)$ is congruence permutable, we apply Theorem 2.1.3 in the case when $n = 2$ (where we write t instead of t_1) and arrive at the result:

$$\begin{aligned} \mathcal{K} \models x &= t(x,z,z) \text{ and} \\ \mathcal{K} \models z &= t(x,x,z). \end{aligned}$$

Also, $\mathbf{T}_{\mathcal{K}}(X)$ is congruence distributive. So we can again choose $n = 2$ in Theorem 2.2.4 and write s instead of t_1 to obtain the result: (Note that the choice of $n = 2$ in this case is justified by the fact that $\rho \vee \sigma = \rho\sigma$ for $\rho, \sigma \in \operatorname{Con} \mathbf{T}_{\mathcal{K}}(X)$ since $\mathbf{T}_{\mathcal{K}}(X)$ is congruence permutable).

$$\mathcal{K} \models x = s(x,y,x),$$

$$\begin{aligned} \mathcal{K} \models x &= s(x,x,z), \text{ and} \\ \mathcal{K} \models z &= s(x,z,z). \end{aligned}$$

(iii)⇒(i): By Theorems 2.1.3 and 2.2.4 respectively, the first two identities in (iii) imply that $\mathcal{K}$ is congruence permutable, while the last three identities in (iii) imply that $\mathcal{K}$ is congruence distributive. Hence $\mathcal{K}$ is arithmetical. □

2.3.3 Remark. We note that condition (iii) of Theorem 2.3.2 can be reduced to a condition with one Mal'cev term. So we have the following two equivalent conditions for a non- trivial variety $\mathcal{K}$:

(P) There exist two ternary terms $t(x,y,z)$ and $s(x,y,z)$ such that $\mathcal{K}$ satisfies the identities:

$$\begin{aligned} x &= t(x,z,z), \\ z &= t(x,x,z), \\ x &= s(x,x,z), \\ x &= s(x,y,x), \text{ and} \\ z &= s(x,z,z). \end{aligned}$$

(Q) There exists one ternary term $p(x,y,z)$ such that $\mathcal{K}$ satisfies the identities:

$$x = p(x,z,z) = p(x,y,x) = p(z,z,x).$$

PROOF.

(P)⇒(Q): Let $p(x,y,z) = t(x,s(x,y,z),z)$.Then

$$\begin{aligned} p(x,z,z) = t(x,s(x,z,z),z) = t(x,z,z) &= x, \\ p(x,y,x) = t(x,s(x,y,x),x) = t(x,x,x) &= x, \text{ and} \\ p(z,z,x) = t(z,s(z,z,x),x) = t(z,z,x) &= x. \end{aligned}$$

(Q)⇒(P): Let $t(x,y,z) = p(x,y,z)$. Then the first two identities of (P) are satisfied as follows:

$$\begin{aligned} t(x,z,z) = p(x,z,z) &= x \text{ and} \\ t(x,x,z) = p(x,x,z) &= z. \end{aligned}$$

Now let $s(x,y,z) = p(x,p(x,y,z),z)$. Then

$$\begin{aligned} s(x,x,z) = p(x,p(x,x,z),z) &= p(x,z,z) = x \\ s(x,y,x) = p(x,p(x,y,x),x) &= p(x,x,x) = x \text{ and} \\ s(x,z,z) = p(x,p(x,z,z),z) &= p(x,x,z) = z. \end{aligned}$$

□

2.3.4 Examples of Arithmetical Varieties.

(i) The variety of all *Boolean algebras* $\mathbf{B} = \langle B; \wedge, \vee, ', 0, 1\rangle$ is arithmetical with Mal'cev term,

$$p(x,y,z) = (x \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y' \wedge z).$$

Observe the following:

$$\begin{aligned} p(x,z,z) &= (x \wedge z) \vee (x \wedge z' \wedge z') \vee (x' \wedge z' \wedge z) \\ &= (x \wedge z) \vee (x \wedge z') \vee (x' \wedge 0) \\ &= (x \wedge (z \vee z')) \vee 0 \\ &= x \wedge 1 \\ &= x, \end{aligned}$$

$$\begin{aligned} p(x,y,x) &= (x \wedge x) \vee (x \wedge y' \wedge x') \vee (x' \wedge y' \wedge x) \\ &= x \vee (0 \wedge y') \vee (0 \wedge y') \\ &= x \end{aligned}$$

and

$$\begin{aligned} p(z,z,x) &= (z \wedge x) \vee (z \wedge z' \wedge x) \vee (z' \wedge z' \wedge x) \\ &= (z \wedge x) \vee (0 \wedge x) \vee (z' \wedge x) \\ &= x \wedge (z \vee z') \\ &= x. \end{aligned}$$

(ii) The variety of *Heyting algebras* $\mathbf{H} = \langle H; \wedge, \vee, \rightarrow, 0, 1\rangle$ is arithmetical. The Mal'cev term is

$$p(x,y,z) = ((x \rightarrow y) \rightarrow z) \wedge ((z \rightarrow y) \rightarrow x) \wedge (x \vee z).$$

We will require Lemma 1.3.4.1 to verify the identities:

$$\begin{aligned} p(x,y,x) &= ((x \rightarrow y) \rightarrow x) \wedge ((x \rightarrow y) \rightarrow x) \wedge (x \vee x) \\ &= ((x \rightarrow y) \rightarrow x) \wedge x \\ &= x \quad \text{[by Axiom (3), Sec. 1.3.4].} \end{aligned}$$

For the next identity we first note that by Lemma 1.3.4.1 (iv) we have

$$\begin{aligned} x \rightarrow ((x \rightarrow z) \rightarrow z) &= (x \wedge (x \rightarrow z)) \rightarrow z \\ &= (x \wedge z) \rightarrow z \quad \text{[by Axiom (4), Sec. 1.3.4]} \\ &= 1 \quad [x \wedge z \leq z \text{ and Lemma 1.3.4.1(ii)}]. \end{aligned}$$

Therefore by (ii) of the same lemma we have

(a) $x \leq ((x \rightarrow z) \rightarrow z)$.

Now,

$$\begin{aligned} p(x,z,z) &= ((x \rightarrow z) \rightarrow z) \wedge ((z \rightarrow z) \rightarrow x) \wedge (x \vee z) \\ &= ((x \rightarrow z) \rightarrow z) \wedge (1 \rightarrow x) \wedge (x \vee z) \quad \text{[Axiom 2, Sec. 1.3.4]} \\ &= ((x \rightarrow z) \rightarrow z) \wedge (x \wedge (x \vee z)) \quad \text{[Axiom 5, Sec. 1.3.4]} \\ &= ((x \rightarrow z) \rightarrow z) \wedge x \\ &= x \quad \text{[from (a) above].} \end{aligned}$$

Finally,

$$\begin{aligned} p(z,z,x) &= ((z \rightarrow z) \rightarrow x) \wedge ((x \rightarrow z) \rightarrow z) \wedge (z \vee x) \\ &= (1 \rightarrow x) \wedge ((x \rightarrow z) \rightarrow z) \wedge (z \vee x) \\ &= x \wedge ((x \rightarrow z) \rightarrow z) \wedge (x \vee z) \\ &= (x \wedge (x \vee z)) \wedge ((x \rightarrow z) \rightarrow z) \\ &= x \wedge ((x \rightarrow z) \rightarrow z) \\ &= x \quad \text{[by (a) above].} \end{aligned}$$

§4. CONGRUENCE MODULAR VARIETIES

2.4.1 Definition. An algebra $\mathbf{A}$ is said to be *congruence modular* if its congruence lattice is modular, that is, if for all $\rho, \sigma, \tau \in \operatorname{Con}\mathbf{A}$,

$$\tau \subseteq \rho \Rightarrow \rho \wedge (\sigma \vee \tau) = (\rho \wedge \sigma) \vee \tau.$$

A class $\mathcal{K}$ of algebras is said to be *congruence modular* if every algebra of $\mathcal{K}$ is congruence modular. □

It is well known that the above lattice quasi-identity is equivalent to the following lattice identity:

$$((\rho \wedge \tau) \vee \sigma) \wedge \tau = (\rho \wedge \tau) \vee (\sigma \wedge \tau).$$

2.4.2 Theorem (A. Day [1969]). *Suppose $\mathcal{K}$ is a non-trivial variety, $X \subseteq V$ is a set with at least four elements and n is a positive integer. Then the following conditions are equivalent:*

(i) *$\mathcal{K}$ is congruence modular.*

(ii) *$\mathbf{T}_{\mathcal{K}}(X)$ is congruence modular.*

(iii) *$\mathbf{T}_{\mathcal{K}}(4)$ is congruence modular.*

(iv) *For each $\mathbf{A} \in \mathcal{K}$ and for all $a, b, c, d \in \mathbf{A}$,*

$$\langle a, d \rangle \in (\Phi \wedge \Psi) \vee (\Phi \wedge \Theta),$$

where $\Phi = \Theta(a, d) \vee \Theta(b, c)$, $\Psi = \Theta(a, b) \vee \Theta(c, d)$ and $\Theta = \Theta(b, c)$.

(v) *There exist quaternary terms $t_0, \ldots, t_n$ such that $\mathcal{K}$ satisfies the following identities:*

$$t_0(x, y, z, u) = x,$$

$$\begin{aligned} t_n(x,y,z,u) &= u, \\ t_i(x,y,y,x) &= x \textit{ for } i=0,\ldots,n-1, \\ t_i(x,y,y,u) &= t_{i+1}(x,y,y,u) \textit{ for } i \textit{ odd, } 1 \le i < n, \textit{ and} \\ t_i(x,x,u,u) &= t_{i+1}(x,x,u,u) \textit{ for } i \textit{ even, } 2 \le i < n. \end{aligned}$$

PROOF.

(i)⇒ (ii): This is trivial since $\mathbf{T}_K(X) \in K$.

(ii)⇒ (iii): Let $Y = \{x,y,z,u\}$ be a four element free generating set of $\mathbf{T}_K(4)$. Consider a surjective mapping $f : X \to Y$. Since both $\mathbf{T}_K(X)$ and $\mathbf{T}_K(4)$ are free over K, f is extendible to a surjective homomorphism $\varphi : \mathbf{T}_K(X) \to \mathbf{T}_K(4)$. By Lemma 2.2.3 there exists a lattice embedding

$$\psi : \mathbf{Con}\,\mathbf{T}_K(4) \to \mathbf{Con}\,\mathbf{T}_K(X).$$

Hence, since $\mathbf{T}_K(X)$ us congruence modular it follows that $\mathbf{T}_K(4)$ is congruence modular.

(iii)⇒ (iv): Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by $\{a,b,c,d\}$. Then there exists a unique surjective homomorphism $h : \mathbf{T}_K(4) \to \mathbf{B}$ such that $h(x) = a$, $h(y) = b$, $h(z) = c$ and $h(u) = d$. By Lemma 2.2.3 there exists a lattice embedding

$$\chi : \mathbf{Con}\,\mathbf{B} \to \mathbf{Con}\,\mathbf{T}_K(4).$$

Since $\mathbf{T}_K(4)$ is congruence modular, we have that $\mathbf{B}$ is congruence modular. Therefore, since $\langle a,d\rangle \in \Phi$ and $\langle a,d\rangle \in \Psi \vee (\Phi \wedge \Theta)$, we have

$$\begin{aligned} \langle a,d\rangle &\in \Phi \wedge (\Psi \vee (\Phi \wedge \Theta)) \\ &= (\Phi \wedge \Psi) \vee (\Phi \wedge \Theta). \end{aligned}$$

(iv)⇒(v): Let $\mathbf{T}_K(4) = \mathbf{T}_K(x,y,z,u)$. If we substitute $x = a$, $y = b$, $z = c$ and $u = d$ in (iv) we have

$$\langle x,u\rangle \in (\Phi \wedge \Psi) \vee (\Phi \wedge \Theta).$$

In view of this and Lemma 1.5.1, there exist a positive integer n and quaternary terms $t_0,\ldots,t_n$ in the absolutely free algebra $\mathbf{T}(V)$ such that $t_0(x,y,z,u) = x$, $t_n(x,y,z,u) = u$ and

$$x = t_0\,(\Phi \wedge \Psi)\,t_1\,(\Phi \wedge \Theta)\,t_2\ \ldots\ t_n = z \tag{12}$$

where each $t_i(x,y,z,u)$ is considered to be the interpretation of this term in $\mathbf{T}_{\mathcal{K}}(4)$. Clearly the first two identities of (v) are satisfied. We establish the remaining three sets of identities as follows:

(I) From (12) we see that for $i = 0,\ldots,n$, $t_0(x,y,z,u) \equiv t_i(x,y,z,u)\,(\Phi)$. Since $\langle x,u\rangle \in \Phi$ and $\langle y,z\rangle \in \Phi$, we have by the reflexivity and symmetry of Φ, and Lemma 1.5.3 that $t_i(x,y,z,u) \equiv t_i(x,y,y,x)\,(\Phi)$. By transitivity, $x = t_0(x,y,z,u) \equiv t_i(x,y,y,x)\,(\Phi)$. But by Lemma 1.5.4, the congruence Φ, restricted to the subalgebra generated by $\{x,y\}$ identifies $t_i(x,y,y,x)$ and x. Therefore, by Lemma 2.1.2,

$$\mathcal{K} \models t_i(x,y,y,x) = x \quad \text{for} \quad i = 1,\ldots,n.$$

(II) For i odd with $1 \le i < n$ we have $t_i\,(\Phi \wedge \Theta)\,t_{i+1}$. Hence $t_i \equiv t_{i+1}\,(\Theta)$. Because $\langle y,z\rangle \in \Theta$, we have $t_i(x,y,y,u) \equiv t_i(x,y,z,u)\,(\Theta)$ and $t_{i+1}(x,y,z,u) \equiv t_{i+1}(x,y,y,u)\,(\Theta)$. Thus by transitivity, $t_i(x,y,y,u) \equiv t_{i+1}(x,y,y,u)\,(\Theta)$. From Lemma 1.5.4, the restriction of Θ to the subalgebra generated by $\{x,y,u\}$ is the identity on the subalgebra. Therefore, by Lemma 2.1.2,

$$\mathcal{K} \models t_i(x,y,y,u) = t_{i+1}(x,y,y,u) \quad \text{for} \quad i \quad \text{odd}.$$

(III) For i even with $0 \le i < n$ we have $t_i\,(\Phi \wedge \Psi)\,t_{i+1}$. Hence $t_i(x,y,z,u) \equiv t_{i+1}(x,y,z,u)\,(\Psi)$. Since $\langle x,y\rangle \in \Psi$ and $\langle z,u\rangle \in \Psi$, we have by Lemma 1.5.3, that $t_i(x,y,z,u) \equiv t_i(x,x,u,u)\,(\Psi)$ and $t_{i+1}(x,y,z,u) \equiv t_{i+1}(x,x,u,u)\,(\Psi)$. Thus we have that $t_i(x,x,u,u) \equiv t_{i+1}(x,x,u,u)\,(\Psi)$. Also by Lemma 1.5.4, the restriction of Ψ to the subalgebra generated by $\{x,u\}$ is the identity on the subalgebra. Therefore, by Lemma 2.1.2,

$$\mathcal{K} \models t_i(x,x,u,u,) = t_{i+1}(x,x,u,u) \quad \text{for} \quad i \quad \text{even}.$$

(v)$\Rightarrow$(i): Take $\mathbf{A} \in \mathcal{K}$ and $\rho,\sigma,\tau \in \operatorname{Con}\mathbf{A}$ such that $\tau \subseteq \rho$. We have to show that

$$\rho \cap (\sigma \vee \tau) \subseteq (\rho \cap \sigma) \vee \tau.$$

For each positive integer m we set

$$q_m = (\sigma,\tau)^m$$

where $(\sigma,\tau)^m$ is defined in Section 1.1.3. Since $\sigma \vee \tau = \bigcup\{q_m; m \in \omega - \{0\}\}$, we now prove that

$$(13) \qquad \rho \cap q_m \subseteq (\rho \cap \sigma) \vee \tau.$$

We shall proceed by induction on m: For $m = 1$, (13) is trivially satisfied since $q_1 = \sigma$. The inclusion (13) is also true for $m = 2$ as follows: If $\langle a,b\rangle \in \rho \cap (\sigma\tau)$, there is a $c \in A$ such that $\langle a,b\rangle \in \rho$, $\langle a,c\rangle \in \sigma$, and $\langle c,b\rangle \in \tau$. Since $\tau \subseteq \rho$, we have $\langle c,b\rangle \in \rho$, and since $\langle a,b\rangle \in \rho$ we have $\langle a,c\rangle \in \rho$ as well. Thus $\langle a,c\rangle \in \rho \cap \sigma$. But $\langle c,b\rangle \in \tau$. Therefore $\langle a,b\rangle \in (\rho \cap \sigma)\tau$ and (13) is satisfied for $m = 2$. We now assume that for $m \geq 3$, (13) is satisfied for all non-negative integers less than or equal to $m - 1$.

Let m be an odd number and take $\langle a,b\rangle \in \rho \cap q_m$. Then there exist $c, d \in A$ such that $\langle a,b\rangle \in \rho$, $\langle a,c\rangle \in q_{m-2}$, $\langle c,d\rangle \in \tau$, and $\langle d,b\rangle \in \sigma$. For each $i = 0,\ldots,n$ we define

$$d_i = {}^{\mathbf{A}}t_i(a,c,d,b).$$

where all term interpretations shall be in the algebra $\mathbf{A}$. Then by the third set of identities of (v) and the application of Lemma 1.5.3, we have that for every i,

$$a = {}^{\mathbf{A}}t_i(a,d,d,a)\,\tau\,{}^{\mathbf{A}}t_i(a,c,d,a)\,\rho\,{}^{\mathbf{A}}t_i(a,c,d,b) = d_i$$

Therefore $\langle a,d_i\rangle \in \tau\rho \subseteq \rho\rho = \rho$. Since $\langle a,c\rangle \in q_{m-2} = \sigma\,\tau\,\sigma \ldots \sigma$ ($m-2$ factors), there are $c_j \in A$ for $j = 1,\ldots,m-3$ such that

$$a\,\sigma\,c_1\,\tau\,c_2\,\sigma\,c_3\,\tau\,\ldots\,\tau\,c_{m-3}\,\sigma\,c.$$

Hence, by Lemma 1.5.3 and the fact that $\langle b,d\rangle \in \sigma$, we have for $0 < i \leq n$,

$${}^{\mathbf{A}}t_{i-1}(a,a,b,b)\,\sigma\,{}^{\mathbf{A}}t_{i-1}(a,c_1,d,b)\,\tau\,{}^{\mathbf{A}}t_{i-1}(a,c_2,d,b)\ldots\sigma\,{}^{\mathbf{A}}t_{i-1}(a,c,d,b) = d_{i-1}.$$

Thus we have proved that for $0 < i \leq n$, we have

$$(14) \qquad \langle d_{i-1},\ {}^{\mathbf{A}}t_{i-1}(a,a,b,b,)\rangle \in q^{-1}_{m-2}.$$

Also,

$${}^{\mathbf{A}}t_i(a,a,b,b)\,\sigma\,{}^{\mathbf{A}}t_i(a,c_1,d,b)\,\tau\,{}^{\mathbf{A}}t_i(a,c_2,d,b)\ldots\sigma\,{}^{\mathbf{A}}t_i(a,c,d,b) = d_i,$$

and we have that

$$(15) \qquad \langle {}^{\mathbf{A}}t_i(a,a,b,b), d_i\rangle \in q_{m-2}.$$

Further, since $\langle c,d\rangle \in \tau \subseteq \rho$ and $\langle b,a\rangle \in \rho$, we can apply the third set of identities of (v) to obtain

$$\begin{aligned} d_{i-1} &= {}^{\mathbf{A}}t_{i-1}(a,c,d,b)\ \rho\ {}^{\mathbf{A}}t_{i-1}(a,d,d,a) \\ &= a \\ &= {}^{\mathbf{A}}t_{i-1}(a,a,a,a)\ \rho\ {}^{\mathbf{A}}t_{i-1}(a,a,b,b). \end{aligned}$$

Therefore by (14),

$$\langle d_{i-1},\ {}^{\mathbf{A}}t_{i-1}(a,a,b,b)\rangle \in q_{m-2}^{-1} \cap \rho^2 = q_{m-2}^{-1} \cap \rho.$$

Similarly, we have $\langle {}^{\mathbf{A}}t_i(a,a,b,b), d_i\rangle \in \rho$ and in view of (15),

$$\langle {}^{\mathbf{A}}t_i(a,a,b,b), d_i\rangle \in q_{m-2} \cap \rho.$$

Case I: For i odd with $1 \leq i < n$ we have ${}^{\mathbf{A}}t_i(a,a,b,b) = {}^{\mathbf{A}}t_{i-1}(a,a,b,b)$. Hence

$$\begin{aligned} \langle d_{i-1}, d_i\rangle &\in (\rho \cap q_{m-2}^{-1})(\rho \cap q_{m-2}) \\ &= (\rho \cap q_{m-2})^{-1}(\rho \cap q_{m-2}) \\ &\subseteq ((\rho \cap \sigma) \vee \tau)((\rho \cap \sigma) \vee \tau) \qquad \text{[Ind. Assumption]} \\ &= (\rho \cap \sigma) \vee \tau. \end{aligned}$$

Case II: For i even with $2 \leq i < n$, we note that ${}^{\mathbf{A}}t_{i-1}(a,c,c,b) = {}^{\mathbf{A}}t_i(a,c,c,b)$. Since $\langle c,d\rangle \in \tau$, we have

$$\begin{aligned} d_{i-1} &= {}^{\mathbf{A}}t_{i-1}(a,c,d,b)\ \tau\ {}^{\mathbf{A}}t_{i-1}(a,c,c,b) \\ &= {}^{\mathbf{A}}t_i(a,c,c,b)\ \tau\ {}^{\mathbf{A}}t_i(a,c,d,b) \\ &= d_i. \end{aligned}$$

Hence for all i we have

$$\langle d_{i-1}, d_i\rangle \in \tau \subseteq (\rho \cap \sigma) \vee \tau.$$

Since $d_0 = a$ and $d_n = b$, we have established that for m odd,

$$\langle a, b\rangle \in (\rho \cap \sigma) \vee \tau.$$

Finally, we consider m to be even and take $\langle a, b\rangle \in \rho \cap q_m$. Then there is a $c \in A$ such that $\langle a, b\rangle \in \rho$, $\langle a, c\rangle \in q_{m-1}$, and $\langle c, b\rangle \in \tau$. For each $i = 0, \ldots, n$ we define

$$d_i = {}^{\mathbf{A}}t_i(a, c, c, b).$$

Then by the third set of identities of (v) and Lemma 1.5.3 we have for every i, that

$$a = {}^{\mathbf{A}}t_i(a, c, c, a)\, \rho\, {}^{\mathbf{A}}t_i(a, c, c, b) = d_i.$$

Therefore $\langle a, d_i\rangle \in \rho$. Since $\langle a, c\rangle \in q_{m-1} = \sigma\,\tau\,\sigma \ldots \sigma$ $\quad (m-1$ factors), there are $c_k \in A$ for $k = 1, \ldots, m-2$ such that

$$a\,\sigma\, c_1\,\tau\, c_2\,\sigma\, c_3\,\tau\, \ldots\, \tau\, c_{m-2}\,\sigma\, c.$$

Therefore, by Lemma 1.5.3 and the fact that $\langle c, b\rangle \in \tau$,

$${}^{\mathbf{A}}t_{i-1}(a, a, b, b)\,\sigma\, {}^{\mathbf{A}}t_{i-1}(a, c_1, b, b)\,\tau\, {}^{\mathbf{A}}t_{i-1}(a, c_2, c, b) \ldots \sigma\, {}^{\mathbf{A}}t_{i-1}(a, c, c, b) = d_{i-1}.$$

Thus we have shown that

$$\langle d_{i-1},\ {}^{\mathbf{A}}t_{i-1}(a, a, b, b)\rangle \in q_{m-1}^{-1}. \tag{16}$$

Also,

$${}^{\mathbf{A}}t_i(a, a, b, b)\,\sigma\, {}^{\mathbf{A}}t_i(a, c_1, b, b)\,\tau\, {}^{\mathbf{A}}t_i(a, c_2, c, b) \ldots \sigma\, {}^{\mathbf{A}}t_i(a, c, c, b) = d_i,$$

and we have that

$$\langle {}^{\mathbf{A}}t_i(a, a, b, b), d_i\rangle \in q_{m-1}. \tag{17}$$

Further,

$$\begin{aligned} d_{i-1} &= {}^{\mathbf{A}}t_{i-1}(a, c, c, b)\, \rho\, {}^{\mathbf{A}}t_{i-1}(a, c, c, a) \\ &= a \\ &= {}^{\mathbf{A}}t_{i-1}(a, a, a, a)\, \rho\, {}^{\mathbf{A}}t_{i-1}(a, a, b, b). \end{aligned}$$

Therefore by (16),

$$\langle d_{i-1}, {}^{\mathbf{A}}t_{i-1}(a,a,b,b)\rangle \in q_{m-1}^{-1} \cap \rho^2 = q_{m-1}^{-1} \cap \rho.$$

Similarly, we have $\langle {}^{\mathbf{A}}t_i(a,a,b,b), d_i\rangle \in \rho$ and

$$\langle {}^{\mathbf{A}}t_i(a,a,b,b), d_i\rangle \in q_{m-1} \cap \rho.$$

Case I: For i odd with $1 \leq i < n$ we have ${}^{\mathbf{A}}t_i(a,a,b,b) = {}^{\mathbf{A}}t_{i-1}(a,a,b,b)$. Hence by the induction assumption

$$\begin{aligned}
\langle d_{i-1}, d_i\rangle &\in (\rho \cap q_{m-1}^{-1})(\rho \cap q_{m-1}) \\
&= (\rho \cap q_{m-1})^{-1}(\rho \cap q_{m-1}) \\
&\subseteq ((\rho \cap \sigma) \vee \tau)((\rho \cap \sigma) \vee \tau) \quad \text{[Ind. Assumption]} \\
&= (\rho \cap \sigma) \vee \tau.
\end{aligned}$$

Case II: For i even with $2 \leq i < n$, we note that ${}^{\mathbf{A}}t_{i-1}(a,c,c,b) = {}^{\mathbf{A}}t_i(a,c,c,b)$. Hence $d_{i-1} = d_i$ and

$$\langle d_{i-1}, d_i\rangle \in (\rho \cap \sigma) \vee \tau.$$

Again since $d_0 = a$ and $d_n = b$ we obtain for m even as well that

$$\langle a, b\rangle \in (\rho \cap \sigma) \vee \tau.$$

□

2.4.3 Examples of Congruence Modular Varieties.

G. Birkhoff proved that congruence permutable algebras are congruence modular. B. Jónsson proved that congruence 3-permutable algebras are congruence modular. Also, it is rather trivial that congruence distributive algebras are congruence modular.

§5. CONGRUENCE REGULAR VARIETIES

2.5.1 Definition. An algebra $\mathbf{A}$ is said to be *congruence regular* if every congruence on $\mathbf{A}$ is uniquely determined by any of its congruence classes, that is, if for all $\rho, \sigma \in \operatorname{Con}\mathbf{A}$,

$$(\mathbf{A}/\rho) \cap (\mathbf{A}/\sigma) \neq \emptyset \Rightarrow \rho = \sigma.$$

A class $\mathcal{K}$ of algebras is *congruence regular* if every element of the class is congruence regular. □

For the next lemma we recall from Section 1.2.3 that if $M \subseteq A$, $\mathbf{A}$ an algebra, then

$$\Theta(M) = \bigcap\{\tau \in \operatorname{Con}\mathbf{A}; M^2 \subseteq \tau\}.$$

2.5.2 Lemma. *An algebra $\mathbf{A}$ is congruence regular if and only if for all $a, b, c \in A$, there is a subset M of $\mathbf{A}$ such that*

$$a \equiv b(\Theta(M)) \quad \textit{and} \quad c \equiv d(\Theta(a,b)) \quad \textit{for all} \quad d \in M.$$

PROOF.

$\Rightarrow$: Let $a, b, c \in A$ and choose $M = c/\Theta(a,b)$. Then $c \equiv d(\Theta(a,b))$ for all $d \in M$. Since $M \in (\mathbf{A}/\Theta(a,b))$ we have that $\Theta(M) \subseteq \Theta(a,b)$. Hence $M \in (\mathbf{A}/\Theta(M))$ as well. Since $\mathbf{A}$ is congruence regular and $M \in (\mathbf{A}/\Theta(a,b)) \cap (\mathbf{A}/\Theta(M))$, we have $\Theta(a,b) = \Theta(M)$. Thus $a \equiv b(\Theta(M))$.

$\Leftarrow$: Let $\rho, \sigma \in \operatorname{Con}\mathbf{A}$ with $C \in (\mathbf{A}/\rho) \cap (\mathbf{A}/\sigma)$. We have to show that $\rho = \sigma$. Let $a \equiv b(\rho)$ and take $c \in C$. Then there exists $M \subseteq A$ such that $a \equiv b(\Theta(M))$ and $c \equiv d(\Theta(a,b))$ for all $d \in M$. Since $a \equiv b(\rho)$, we have that $\Theta(a,b) \subseteq \rho$. Hence $M \subseteq C$. Therefore $\langle a, b\rangle \in \Theta(M) \subseteq \Theta(C) \subseteq \sigma$, as $C \in (\mathbf{A}/\sigma)$. Thus we have the inclusion $\rho \subseteq \sigma$. The converse inclusion can be proved in a similar way. □

2.5.3 Definition. A mapping $f : A \to A$ is said to be a *translation* on an algebra $\mathbf{A}$ if for some positive integers n and i with $i \leq n$, there are elements

$a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$ and a term $t(x_1, \ldots, x_n)$ such that for all $x \in A$,

$$f(x) = {}^{\mathbf{A}}t(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n).$$

□

Before we attempt the proof of a Mal'cev type theorem for congruence regular varieties, we require two additional results. Recall from Section 1.2.2 that for a binary relation $r \subseteq A \times A$ and $\mathbf{A}$ an algebra, we define

$$\Theta(r) = \bigcap\{\tau \in \mathrm{Con}\mathbf{A}; r \subseteq \tau\}.$$

The first of the following two lemmas gives a characterisation of $\Theta(r)$ in terms of translations.

2.5.4 Lemma. *Suppose* $\mathbf{A}$ *is an algebra,* $a, b \in A$ *and* $r \subseteq A \times A$. *Then the following conditions are equivalent:*

I. $\langle a, b \rangle \in \Theta(r)$.

II. *Either*

(i) $a = b$

or

(ii) for some positive integer m, *there exist translations* $f_1, \ldots, f_m$ *on* $\mathbf{A}$ *and ordered pairs* $\langle c_1, d_1 \rangle, \ldots, \langle c_m, d_m \rangle \in r \cup r^{-1}$ *such that*

$$a = f_1(c_1), f_1(d_1) = f_2(c_2), \ldots, f_{m-1}(d_{m-1}) = f_m(c_m), f_m(d_m) = b.$$

PROOF.

We define a binary relation σ on A as follows:

$$\langle a, b \rangle \in \sigma \text{ iff } \langle a, b \rangle \text{ satisfies II.}$$

I⇒II: Evidently σ is an equivalence relation on A. We shall show that σ is a congruence on $\mathbf{A}$: For convenience let $o \in \mathbf{O}$ be a binary operation of $\mathbf{A}$. (A similar procedure to the one which follows will apply if o is n-ary for any positive integer n). Also let $a_1, a_2, b_1, b_2 \in A$ such that $a_1 \sigma b_1$ and $a_2 \sigma b_2$. We shall show that

$o^{\mathbf{A}}(a_1, a_2) \sigma o^{\mathbf{A}}(b_1, b_2)$. Suppose $a_1 \neq b_1$. Then there exists a sequence as in (ii) such that

$$a_1 = f_1(c_1),\ f_1(d_1) = f_2(c_2),\ \ldots, f_m(d_m) = b_1.$$

Since o is an operation symbol, $o(x_1, x_2)$ is a term. Consider the translation

$$f(x) = {}^{\mathbf{A}}o(x, a_2) = o^{\mathbf{A}}(x, a_2)$$

Then $f(a_1) = o^{\mathbf{A}}(a_1, a_2)$,

$$f(a_1) = f(f_1(c_1)), f(f_1(d_1)) = f(f_2(c_2)), \ldots, f(f_m(d_m)) = f(b_1),$$

and $f(b_1) = o^{\mathbf{A}}(b_1, a_2)$. Since the composition of translations is a translation, we have that

$$o^{\mathbf{A}}(a_1, a_2) \sigma o^{\mathbf{A}}(b_1, a_2). \tag{18}$$

Similarly, we have

$$o^{\mathbf{A}}(b_1, a_2) \sigma o^{\mathbf{A}}(b_1, b_2) \tag{19}$$

From (18) and (19), and the transitivity of σ we have

$$o^{\mathbf{A}}(a_1, a_2) \sigma o^{\mathbf{A}}(b_1, b_2),$$

and thus σ is a congruence on $\mathbf{A}$.

Finally we shall show that $r \subseteq \sigma$: Let $\langle a, b\rangle \in r$ and consider a term t such that t is a variable. Then ${}^{\mathbf{A}}t$, the interpretation of t in $\mathbf{A}$ is a translation. Since $a = {}^{\mathbf{A}}t(a)$ and ${}^{\mathbf{A}}t(b) = b$, we have $\langle a, b\rangle \in \sigma$ by II. Hence $\Theta(r) \subseteq \sigma$.

II$\Rightarrow$I: Suppose $\langle a, b\rangle \in \sigma$ with $a \neq b$. Then by (ii) there is a sequence

$$a = f_1(c_1),\ f_1(d_1) = f_2(c_2),\ \ldots,\ f_m(d_m) = b,$$

where the $\langle c_i, d_i\rangle \in r \cup r^{-1}$ and the f_i are translations on $\mathbf{A}$. Since $\langle c_i, d_i\rangle \in r \cup r^{-1}$, we have that $\langle c_i, d_i\rangle \in \Theta(r)$. Since a translation is defined by an n-ary operation on $\mathbf{A}$ we can apply Lemma 1.5.3 and have $\langle f_i(c_i), f_i(d_i)\rangle \in \Theta(r)$. By the transitivity of $\Theta(r)$ we have $\langle a, b\rangle \in \Theta(r)$. □

2.5.5 Lemma. *Suppose K is a non-trivial variety and $\mathbf{T}_K(X)$ is the free algebra over K with free generating set X, for $X \subseteq V$. Let $t(x_1, \ldots, x_n)$ and*

$s(x_1, \ldots, x_n)$ be terms in $\mathrm{T}(X)$. *If $\varphi : \mathrm{T}(X) \to \mathrm{T}_{\mathcal{K}}(X)$ is the unique homomorphism extending id_X then*

$$\mathcal{K} \models t = s \ \textit{ iff } \ \varphi(t) = \varphi(s).$$

PROOF.

Let us denote $\mathrm{G} = \mathrm{T}(X)$ and $\mathrm{H} = \mathrm{T}_{\mathcal{K}}(X)$.

$\Rightarrow$: Suppose $\mathcal{K} \models t = s$. Then $\mathrm{H} \models t = s$. Since $\varphi : \mathrm{G} \to \mathrm{H}$ is a homomorphism, we have by Lemma 1.5.2 that

$$\begin{aligned} \varphi(t(x_1, \ldots, x_n)) &= \varphi({}^{\mathrm{G}}t(x_1, \ldots, x_n)) = {}^{\mathrm{H}}t(\varphi(x_1), \ldots, \varphi(x_n)), \text{ and} \\ \varphi(s(x_1, \ldots, x_n)) &= \varphi({}^{\mathrm{G}}s(x_1, \ldots, x_n)) = {}^{\mathrm{H}}s(\varphi(x_1), \ldots, \varphi(x_n)). \end{aligned}$$

By the equality of t and s in H we have $\varphi(t) = \varphi(s)$.

$\Leftarrow$: Refer to figure 2.5.1. Let $\mathrm{B} \in \mathcal{K}$. We shall show that $\mathrm{B} \models t = s$. In view of Lemma 2.1.2, we need only show that ${}^{\mathrm{B}}t(u) = {}^{\mathrm{B}}s(u)$ for every mapping $u : X \to B$. Since G is freely generated by X, there exists a unique homomorphism $\alpha : \mathrm{G} \to \mathrm{B}$ extending u. Also, H is freely generated by X. Hence there exists a unique homomorphism $\beta : \mathrm{H} \to \mathrm{B}$ extending u. Therefore, for all $x \in X$, we have

$$\alpha(x) = u(x) = \beta(x) = \beta(\varphi(x)).$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] \arrow[rd, "u"] & \mathrm{G} \arrow[d, dashed, "\alpha"] \arrow[ld, "\varphi"'] \\ \mathrm{H} \arrow[r, dashed, "\beta"'] & \mathrm{B} \end{tikzcd}$$

Figure 2.5.1

Hence,

$$ {}^{\mathrm{B}}t(u) = {}^{\mathrm{B}}t(u(x_1), \ldots, u(x_n))$$

$$
\begin{aligned}
&= {}^{\mathbf{B}}t(\alpha(x_1), \ldots, \alpha(x_n)) \\
&= \alpha({}^{\mathbf{G}}t(x_1, \ldots, x_n)) \\
&= \alpha(t).
\end{aligned}
$$

Similarly,

$$ {}^{\mathbf{B}}s(u) = \beta(\varphi(s)). $$

But by our assumption $\varphi(t) = \varphi(s)$. Therefore ${}^{\mathbf{B}}t(u) = {}^{\mathbf{B}}s(u)$ and

$$ \mathcal{K} \models t = s. $$

□

2.5.6 Theorem (B. Czákany, G. Grätzer and R. Wille [1970]). *Let $\mathcal{K}$ be a non-trivial variety and let $Z \subseteq V$ be a set with at least three elements. Then the following conditions are equivalent:*

(i) *$\mathcal{K}$ is a congruence regular variety.*

(ii) *The free algebra $\mathbf{T}_{\mathcal{K}}(Z)$ is congruence regular.*

(iii) *For a positive integer n, there are terms $t_i(x,y,z)$, $u_i(x,y,z)$, and $v_i(x,y,z,u)$ for $i = 1, \ldots, n$, such that $\mathcal{K}$ satisfies the following identities:*

$$
\begin{aligned}
&(20) & z &= t_i(x,x,z) \text{ for } i = 1, \ldots, n, \\
&(21) & z &= u_i(x,x,z) \text{ for } i = 1, \ldots, n, \\
&(22) & x &= v_1(x,y,z,t_1(x,y,z)), \\
&(23) & v_{i-1}(x,y,z,u_{i-1}(x,y,z)) &= v_i(x,y,z,t_i(x,y,z)),\ i = 2, \ldots, n, \\
&(24) & y &= v_n(x,y,z,u_n(x,y,z)).
\end{aligned}
$$

PROOF.

Let us denote $\mathbf{F} = \mathbf{T}_{\mathcal{K}}(X)$.

(i)⇒ (ii): This is trivial since $\mathbf{F} \in \mathcal{K}$.

(ii)⇒ (i): Let $\mathbf{A} \in \mathcal{K}$ and let ρ and σ be two different congruences on $\mathbf{A}$ such that $(\mathbf{A}/\rho) \cap (\mathbf{A}/\sigma) \neq \emptyset$. Suppose $M \in (\mathbf{A}/\rho) \cap (\mathbf{A}/\sigma)$. Then there exists an $\langle a, b\rangle$ such that

$$ \langle a, b\rangle \in (\rho - \sigma) \cup (\sigma - \rho). $$

Let $c \in M$ and $r, s \in Z$ with $r \neq s$. Since $\mathbf{F}$ is free, there exists a unique homomorphism

$$\varphi : \mathbf{F} \to \mathbf{A}$$

such that $\varphi(r) = a$, $\varphi(s) = b$ and $\varphi(x) = c$ for all $x \in Z - \{r, s\}$. We define binary relations $\bar{\rho}$ and $\bar{\sigma}$ on F as follows: For all $u, v \in F$,

$$\begin{aligned} u \equiv v(\bar{\rho}) \text{ iff } \varphi(u) &\equiv \varphi(v)(\rho), \\ u \equiv v(\bar{\sigma}) \text{ iff } \varphi(u) &\equiv \varphi(v)(\sigma). \end{aligned}$$

We shall show that $\bar{\rho}$ and $\bar{\sigma}$ are congruences on $\mathbf{F}$: For $u \in F$, $\varphi(u) \equiv \varphi(u)(\rho)$ since ρ is a congruence on $\mathbf{A}$. Therefore $u \equiv u(\bar{\rho})$ and $\bar{\rho}$ is reflexive. For all $u, v \in F$, $u \equiv v(\bar{\rho})$ implies that $\varphi(u) \equiv \varphi(v)(\rho)$. Since ρ is symmetric, $\bar{\rho}$ is also symmetric. For $u, v, w \in F$, we have by the transitivity of ρ, that $\varphi(u) \equiv \varphi(v)(\rho)$ and $\varphi(v) \equiv \varphi(w)(\rho)$ imply that $\varphi(u) \equiv \varphi(w)(\rho)$. Hence $u \equiv v(\bar{\rho})$ and $v \equiv w(\bar{\rho})$ imply that $u \equiv w(\bar{\rho})$, making $\bar{\rho}$ transitive as well. To show that $\bar{\rho}$ is compatible with operations it suffices to consider a binary operation $o \in \mathbf{O}$. Suppose $\langle u_1, v_1\rangle, \langle u_2, v_2\rangle \in \bar{\rho}$. Then $\langle \varphi(u_1), \varphi(v_1)\rangle, \langle \varphi(u_2), \varphi(v_2)\rangle \in \rho$. But since ρ is a congruence on $\mathbf{A}$, $\langle o^{\mathbf{A}}(\varphi(u_1), \varphi(u_2)), o^{\mathbf{A}}(\varphi(v_1), \varphi(v_2))\rangle \in \rho$. By Lemma 1.5.2,

$$\langle \varphi(o^{\mathbf{F}}(u_1, u_2)), \varphi(o^{\mathbf{F}}(v_1, v_2))\rangle \in \rho.$$

Hence by the definition of $\bar{\rho}$ we have $o^{\mathbf{F}}(u_1, u_2) \equiv o^{\mathbf{F}}(v_1, v_2)(\bar{\rho})$. Thus $\bar{\rho}$ is a congruence on $\mathbf{F}$. That $\bar{\sigma}$ is a congruence on $\mathbf{F}$ can be shown in a similar manner.

Next we show $\bar{\rho} \neq \bar{\sigma}$: If $\langle a, b\rangle \in \rho$ and $\langle a, b\rangle \notin \sigma$, then $\langle \varphi(r), \varphi(s)\rangle \in \rho$ and $\langle \varphi(r), \varphi(s)\rangle \notin \sigma$. This means that $\langle r, s\rangle \in \bar{\rho}$ and $\langle r, s\rangle \notin \bar{\sigma}$, and $(\bar{\rho} - \bar{\sigma}) \neq \emptyset$. Similarly, $(\bar{\sigma} - \bar{\rho}) \neq \emptyset$ whenever $(\sigma - \rho) \neq \emptyset$. Therefore $\rho \neq \sigma$ implies that $\bar{\rho} \neq \bar{\sigma}$. Since $M \in (\mathbf{A}/\rho) \cap (\mathbf{A}/\sigma)$ and $c \in M$, we have $c/\rho = c/\sigma$. Then for $x \in Z - \{r, s\}$, $\varphi(x)/\rho = \varphi(x)/\sigma$. This implies that

$$x/\bar{\rho} = x/\bar{\sigma} \in (\mathbf{F}/\bar{\rho}) \cap (\mathbf{F}/\bar{\sigma}) \neq \emptyset.$$

But $\bar{\rho} \neq \bar{\sigma}$, thus contradicting the congruence regularity of $\mathbf{F}$. Hence $\mathcal{K}$ is congruence regular.

(i)$\Rightarrow$ (iii): We denote $\mathbf{H} = \mathbf{T}_{\mathcal{K}}(x, y, z)$ the free algebra over $\mathcal{K}$ with $\{x, y, z\}$ as its free generating set, and $\mathbf{G} = \mathbf{T}(x, y, z)$ the absolutely free algebra with free generating set $\{x, y, z\}$. We assume that x, y and z are three different variables. Then there is a unique surjective homomorphism

$$\varphi : \mathbf{G} \to \mathbf{H}$$

determined by $\varphi(x) = x$, $\varphi(y) = y$ and $\varphi(z) = z$. Since $\mathbf{H}$ is congruence regular , by Lemma 2.5.2, there is an $M \subseteq H$ such that

$$x \equiv y(\Theta(M)) \;\text{ and } z \equiv a(\Theta(x, y)) \;\text{ for all } a \in M$$

By Lemma 2.5.4, for a positive integer n, there are translations $f_1, \ldots, f_n$ on $\mathbf{H}$ and pairs $\langle a_1, b_1\rangle$, ..., $\langle a_n, b_n\rangle \in M^2$ such that

$$x = f_1(a_1),\; f_1(b_1) = f_2(a_2), \ldots,\; f_n(b_n) = y.$$

Now $\mathbf{G}$ is freely generated by $\{x, y, z\}$. Hence by Lemma 1.5.1 there are ternary terms $t_1, \ldots, t_n$ and $u_1, \ldots, u_n$ of G such that

$$\begin{aligned} a_1 &= \varphi(t_1(x, y, z)), \ldots, a_n &= \varphi(t_n(x, y, z)) \text{ and} \\ b_1 &= \varphi(u_1(x, y, z)), \ldots, b_n &= \varphi(u_n(x, y, z)). \end{aligned}$$

Let $\psi : \mathbf{H} \to \mathbf{H}$ be the unique endomorphism determined by

$$\psi(x) = \psi(y) = x \;\text{ and } \psi(z) = z$$

For all $i \in \{1, \ldots, n\}$ we have $\langle z, a_i\rangle \in \Theta(x, y)$, since $a_i \in M$. Also $\Theta(x, y) \subseteq ker(\psi)$ because $\langle x, y\rangle \in ker(\psi)$. Hence $\langle z, a_i\rangle \in ker(\psi)$ for $i = 1, \ldots, n$. Therefore we have $\varphi(z) = z = \psi(z) = \psi(a_i)$ for $i = 1, \ldots, n$, and

$$\begin{aligned} \varphi(z) &= \psi(\varphi(t_i(x, y, z))) \\ &= {}^{\mathbf{H}}t_i(\psi(\varphi(x)), \psi(\varphi(y)), \psi(\varphi(z))) \quad \text{[by Lemma 1.5.2]} \\ &= {}^{\mathbf{H}}t_i(x, x, z) \\ &= {}^{\mathbf{H}}t_i(\varphi(x), \varphi(x), \varphi(z)) \\ &= \varphi(t_i(x, x, z)). \end{aligned}$$

Consequently, by Lemma 2.5.5 we have

$$\mathcal{K} \models z = t_i(x, x, z) \quad \text{for } i = 1, \ldots, n.$$

For $i \in \{1, \ldots, n\}$, we have that $b_i \in M$ and as before, we have $\langle z, b_i \rangle \in ker(\psi)$. Hence $\varphi(z) = z = \psi(z) = \psi(b_i)$, and

$$\begin{aligned} \varphi(z) &= \psi(\varphi(u_i(x, y, z))) \\ &= {}^{\mathbf{H}}u_i(\psi(\varphi(x)), \psi(\varphi(y)), \psi(\varphi(z))) \\ &= {}^{\mathbf{H}}u_i(x, x, z) \\ &= {}^{\mathbf{H}}u_i(\varphi(x), \varphi(x), \varphi(z)) \\ &= \varphi(u_i(x, x, z)). \end{aligned}$$

Again by Lemma 2.5.5, we have

$$\mathcal{K} \models z = u_i(x, x, z) \quad \text{for } i = 1, \ldots, n.$$

We shall now derive the third identity: For every $i = 1, \ldots, n$, there is a natural number k_i, a k_i-ary term s_i and elements $a_{i,2}, \ldots, a_{i,k_i} \in H$ such that for all $a \in H$,

$$(25) \qquad f_i(a) = {}^{\mathbf{H}}s_i(a, a_{i,2}, \ldots, a_{i,k_i})$$

We note that f_i is a translation on $\mathbf{H}$. Since $\{x, y, z\}$ is a free generating set for $\mathbf{G}$, there are ternary terms $w_{i,2}, \ldots, w_{i,k_i}$ such that $a_{i,j} = \varphi({}^{\mathbf{G}}w_{i,j}(x, y, z))$ for $i = 1, \ldots, n$ and $j = 2, \ldots, k_i$. We now define quaternary terms v_i, $i = 1, \ldots, n$, by

$$(26) \qquad v_i(x, y, z, x_1) = s_i(x_1, w_{i,2}, \ldots, w_{i,k_i})$$

where $x_1 \notin \{x, y, z\}$. Then

$$\begin{aligned} \varphi({}^{\mathbf{G}}v_1(x, y, z, t_1(x, y, z))) &= \varphi({}^{\mathbf{G}}s_1(t_1(x, y, z), w_{1,2}, \ldots, w_{1,k_1})) \\ &= {}^{\mathbf{H}}s_1(\varphi(t_1), \varphi(w_{1,2}), \ldots, \varphi(w_{1,k_1})) \\ &= {}^{\mathbf{H}}s_1(a_1, a_{1,2}, \ldots, a_{1,k_1}) \\ &= f_1(a_1) \\ &= x \\ &= \varphi(x). \end{aligned}$$

By Lemma 2.5.5,

$$\mathcal{K} \models x = v_1(x, y, z, t_1(x, y, z)).$$

From (26), $v_{i-1}(x, y, z, u_{i-1}) = s_{i-1}(u_{i-1}, w_{i-1,2}, \ldots, w_{i-1,k_{i-1}})$ and

$$\begin{aligned}
\varphi({}^{\mathbf{G}}v_{i-1}(x, y, z, u_{i-1})) &= \varphi({}^{\mathbf{G}}s_{i-1}(u_{i-1}, w_{i-1,2}, \ldots, w_{i-1,k_{i-1}})) \\
&= {}^{\mathbf{H}}s_{i-1}(\varphi(u_{i-1}), \varphi(w_{i-1,2}), \ldots, \varphi(w_{i-1,k_{i-1}})) \\
&= {}^{\mathbf{H}}s_{i-1}(b_{i-1}, a_{i-1,2}, \ldots, a_{i-1,k_{i-1}}) \\
&= f_{i-1}(b_{i-1}) \\
&= f_i(a_i).
\end{aligned}$$

From the array following equation (26) we have $\varphi({}^{\mathbf{G}}v_1(x, y, z, t_1)) = f_1(a_1)$. Hence we conclude that

$$\varphi({}^{\mathbf{G}}v_i(x, y, z, t_i(x, y, z))) = f_i(a_i).$$

Applying Lemma 2.5.5 we have for $i = 2, \ldots, n$,

$$\mathcal{K} \models v_{i-1}(x, y, z, u_{i-1}) = v_i(x, y, z, t_i).$$

For the last identity we again use equation (26). Then

$$\begin{aligned}
\varphi({}^{\mathbf{G}}v_n(x, y, z, u_n)) &= \varphi({}^{\mathbf{G}}s_n(u_n, w_{n,2}, \ldots, w_{n,k_n})) \\
&= {}^{\mathbf{H}}s_n(\varphi(u_n), \varphi(w_{n,2}), \ldots, \varphi(w_{n,k_n})) \\
&= {}^{\mathbf{H}}s_n(b_n, a_{n,2}, \ldots, a_{n,k_n}) \\
&= f_n(b_n) \\
&= y \\
&= \varphi(y).
\end{aligned}$$

Therefore by Lemma 2.5.5

$$\mathcal{K} \models y = v_n(x, y, z, u_n).$$

(iii)$\Rightarrow$ (i): Let $\mathbf{A} \in \mathcal{K}$ and let $a, b, c \in A$. For $i = 1, \ldots, n$, we define

$$\begin{aligned}
a_i &= {}^{\mathbf{A}}t_i(a, b, c) \\
b_i &= {}^{\mathbf{A}}u_i(a, b, c) \\
M &= \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}.
\end{aligned}$$

If $\rho \in \mathrm{Con}\mathbf{A}$ with $\langle a, b\rangle \in \rho$, then by Lemma 1.5.3 and the identities (20) and (21),

$$\begin{aligned}\langle c, a_i\rangle &= \langle {}^{\mathbf{A}}t_i(a,a,c), {}^{\mathbf{A}}t_i(a,b,c)\rangle \in \rho \text{ and}\\ \langle c, b_i\rangle &= \langle {}^{\mathbf{A}}u_i(b,b,c), {}^{\mathbf{A}}u_i(a,b,c)\rangle \in \rho.\end{aligned}$$

This means that $c \equiv d(\rho)$ for all $d \in M$. But since $\langle a, b\rangle \in \rho$, we have that

$$c \equiv d(\Theta(a,b)) \text{ for all } d \in M. \tag{27}$$

Next, if $\sigma \in \mathrm{Con}\,\mathbf{A}$ such that $M \times M \subseteq \sigma$, then

$$\begin{aligned} a &= {}^{\mathbf{A}}v_1(a,b,c,\ {}^{\mathbf{A}}t_1(a,b,c)) \quad [\text{by } (22)]\\ &= {}^{\mathbf{A}}v_1(a,b,c,a_1)\\ &\equiv {}^{\mathbf{A}}v_1(a,b,c,b_1)\,(\sigma) \quad [\text{since } M^2 \subseteq \sigma]\\ &= {}^{\mathbf{A}}v_2(a,b,c,a_2) \quad [\text{by } (23)]\\ &\equiv {}^{\mathbf{A}}v_2(a,b,c,b_2)\,(\sigma)\\ &= \ldots \equiv \ldots = \ldots\\ &= {}^{\mathbf{A}}v_{n-1}(a,b,c,a_{n-1})\\ &\equiv {}^{\mathbf{A}}v_{n-1}(a,b,c,b_{n-1})\,(\sigma)\\ &= {}^{\mathbf{A}}v_n(a,b,c,a_n)\\ &\equiv {}^{\mathbf{A}}v_n(a,b,c,b_n)\,(\sigma)\\ &= b \quad [\text{by } (24)].\end{aligned}$$

Hence $\langle a, b\rangle \in \sigma$. However, since $\Theta(M)$ is the least congruence containing M^2, we have in particular,

$$a \equiv b(\Theta(M)). \tag{28}$$

Equipped with (27) and (28), we invoke Lemma 2.5.1 to conclude that $\mathbf{A}$ is congruence regular, and hence that $\mathcal{K}$ is congruence regular. □

2.5.7 Examples of Congruence Regular Varieties.

(i) The variety of all quasigroups is congruence regular: We take $n = 1$ and define the terms in Theorem 2.5.6 (iii) as follows.

$$\begin{aligned} t_1(x,y,z) &= y(x\backslash z), \\ u_1(x,y,z) &= z, \quad \text{and} \\ v_1(x,y,z,u) &= (y(x\backslash z))/(x\backslash u). \end{aligned}$$

We now verify the identities of part (iii) of the theorem: First we have $u_1(x,x,z) = z$ (by definition of u_1). Next,

$$t_1(x,x,z) = x(x\backslash z) = z \quad \text{[Axiom 3, Sec. 1.3.1]}.$$

Next,

$$\begin{aligned} v_1(x,y,z,t_1(x,y,z)) &= (y(x\backslash z))/(x\backslash(y(z\backslash z))) \\ &= t_1/(x\backslash t_1) \\ &= x, \end{aligned}$$

since by Lemma 1.3.1.1 (i), $t_1/(x\backslash t_1) = x$ iff $t_1 = x(x\backslash t_1)$, and clearly $t_1 = x(x\backslash t_1)$ by Axiom 3 of Section 1.3.1. Finally,

$$\begin{aligned} v_1(x,y,z,u_1(x,y,z)) &= (y(x\backslash z))/(x\backslash z) \\ &= y \quad \text{[Axiom 2, Sec 1.3.1]}. \end{aligned}$$

Therefore the variety of quasigroups is congruence regular.

In view of part 2 of the Remark in Section 2.1.4, we can conclude that groups, rings, R-modules and other such algebras (as mentioned in the Remark) are congruence regular.

(ii) The variety of Boolean algebras is congruence regular: To show that the identities in (iii) of Theorem 2.5.6 are satisfied, we take $n = 1$ and define

$$\begin{aligned} t_1(x,y,z) &= x\Delta y\Delta z, \\ u_1(x,y,z) &= z \quad \text{and} \\ v_1(x,y,z,u) &= y\Delta z\Delta u, \end{aligned}$$

where Δ is the derived Boolean operation, the so-called Boolean difference, defined by

$$x\Delta y = (x \wedge y') \vee (x' \wedge y).$$

It is a simple fact (see [22], p. 18–20), that Δ is a commutative and associative binary operation which satisfies the identities $x\Delta x = 0$ and $x\Delta 0 = x$. Then we have $u_1(x,x,z) = z$,

$$t_1(x,x,z) = x\Delta x\Delta z = 0\Delta z = z,$$

$$\begin{aligned} v_1(x,y,z,t(x,y,z)) &= (y\Delta z)\Delta(x\Delta y\Delta z) \\ &= x\Delta(y\Delta y)\Delta(z\Delta z) \\ &= x\Delta 0\Delta 0 \\ &= x, \end{aligned}$$

and

$$\begin{aligned} v_1(x,y,z,u_1(x,y,z)) &= (y\Delta z)\Delta z \\ &= y\Delta(z\Delta z) \\ &= y\Delta 0 \\ &= y. \end{aligned}$$

Therefore the variety of Boolean algebras is congruence regular. (Observe that for $n = 1$ the condition (23) of Theorem 2.5.6 (iii) is void).

It is known that every congruence regular variety is congruence modular and congruence n-permutable for some $n \geq 2$. (J. Hagemann: On regular and weakly regular congruences, Technische Hochschule Darmstadt, Preprint No. 75, June 1973).

CHAPTER 3

A GENERAL SCHEME OF SOME MAL'CEV CONDITIONS

As noted earlier, R. Wille in [33] and A. F. Pixley in [27] provided algorithms for the classification of varieties of algebras that exhibit "strong Mal'cev properties". In this chapter we present a modification of the Wille-Pixley schemes.

At the outset let us recall our convention from Section 1.1.2 that the set $V = \{x_n;\ n \in \omega\}$ is an infinite countable set of variables.

§1. THE CONGRUENCE CONDITION

3.1.1 Definition. Let X and Y be sets such that $X \subseteq Y \subseteq V$ and suppose that $X \neq \emptyset$. Then each subset of

$$Y \times Y \times exp(X \times X)$$

is said to be a *congruence condition* over $\langle X, Y\rangle$. Hence if R is a congruence condition over $\langle X, Y\rangle$, then

$$R \subseteq \{\langle u, v, r\rangle;\ \ u, v \in Y,\ \ r \subseteq X \times X\}.$$

$\square$

3.1.2 Definition. Let $\mathbf{A}$ be an algebra, R a congruence condition over $\langle X, Y\rangle$ and f a mapping such that $f : X \to A$. We say that R is *f-satisfied* in $\mathbf{A}$ if there is a mapping $g : Y \to A$ such that

(i) $g|X = f$ and

(ii) if $\langle u, v, r\rangle \in R$, then $\langle g(u), g(v)\rangle \in \Theta_{\mathbf{A}}(f_2[r])$.

where $f_2[r]$ has been defined in Section 2.2.2 and $\Theta_{\mathbf{A}}(f_2[r])$ is the congruence defined in 1.2.2. We say that R is *satisfied* in an algebra $\mathbf{A}$ if R is f-satisfied in $\mathbf{A}$ for each mapping $f : X \to A$. ☐

3.1.3 Definition. Let $E(X)$ denote the set of all equivalence relations on X and let r be a binary relation on X. Then

$$\mathcal{E}(r) = \bigcap\{e \in E(X);\ r \subseteq e\}$$

is the least equivalence on X which contains r as a subset. ☐

3.1.4 Definition. For each binary relation r on a set $X \subseteq V$, we define the mapping $\sigma_r : X \to X$ as follows. For every $x_n \in X$,

$$\sigma_r(x_n) = x_m$$

if and only if $m = min\{k \in \omega;\ \langle x_n, x_k\rangle \in \mathcal{E}(r)\}$. ☐

3.1.5 Proposition. *With respect to the mapping σ_r in Definition 3.1.4 we assert the following:*

(i) $\sigma_r \circ \sigma_r = \sigma_r$

(ii) $ker(\sigma_r) = \mathcal{E}(r)$.

PROOF.

(i) Let $\sigma_r(x_n) = x_m$. Then $\langle x_n, x_m\rangle \in \mathcal{E}(r)$. Further if $\langle x_m, x_k\rangle \in \mathcal{E}(r)$, then by transitivity $\langle x_n, x_k\rangle \in \mathcal{E}(r)$ and therefore $m \leq k$. Thus whenever $\sigma_r(x_m) = x_\ell$, then $\langle x_m, x_\ell\rangle \in \mathcal{E}(r)$ and $m \leq \ell$. For the converse inequality we note that if $\sigma_r(x_m) = x_\ell$, then $\ell \leq m$ since $\langle x_m, x_m\rangle \in \mathcal{E}(r)$ because $\mathcal{E}(r)$ is reflexive. Thus $\ell = m$ and $\sigma_r \circ \sigma_r(x_n) = \sigma_r(x_m) = x_\ell = x_m = \sigma_r(x_n)$.

(ii) Let $\langle x_n, x_m\rangle \in ker(\sigma_r)$. Then $\sigma_r(x_n) = \sigma_r(x_m)$. If $\sigma_r(x_n) = \sigma_r(x_m) = x_\ell$, then $\langle x_n, x_\ell\rangle$, $\langle x_m, x_\ell\rangle \in \mathcal{E}(r)$. By the reflexivity and transitivity of $\mathcal{E}(r)$ we have $\langle x_n, x_m\rangle \in \mathcal{E}(r)$. Hence $ker(\sigma_r) \subseteq \mathcal{E}(r)$. Conversely, let $\langle x_n, x_m\rangle \in \mathcal{E}(r)$ with $\sigma_r(x_n) = x_\ell$ and $\sigma_r(x_m) = x_p$. From $\sigma_r(x_n) = x_\ell$, we have $\langle x_n, x_\ell\rangle \in \mathcal{E}(r)$. Again since $\mathcal{E}(r)$ is reflexive and transitive, $\langle x_m, x_\ell\rangle \in \mathcal{E}(r)$. Hence $p \leq \ell$. Similarly by considering $\sigma_r(x_m) = x_p$, we have $\ell \leq p$. Thus $\ell = p$. Therefore $\sigma_r(x_n) = \sigma_r(x_m)$ and $\mathcal{E}(r) \subseteq ker(\sigma_r)$. Both inclusions yield the required result. $\square$

Before proceeding with the next theorem, we recall from Section 1.4.4 that $\mathbf{T}(Z)$ is the absolutely free algebra generated by Z. Also from Section 1.4.6, we have that $\mathbf{T}_{\mathcal{K}}(Z)$ is the free algebra over the variety $\mathcal{K}$ with Z as a free generating set. We note also that if $\mathcal{K}$ is a non- trivial variety then $Z \subseteq \mathbf{T}_{\mathcal{K}}(Z)$.

3.1.6 Theorem. *Let $\mathcal{K}$ be a non-trivial variety and R a congruence condition over $\langle X, Y\rangle$. Suppose also that X is a subset of $Z \subseteq V$. Then the following conditions are equivalent:*

(i) *R is satisfied in every algebra of $\mathcal{K}$.*

(ii) *R is satisfied in the free algebra $\mathbf{T}_{\mathcal{K}}(Z)$ with respect to the identity mapping $id_X : X \to X$.*

(iii) *There exists a mapping $p : Y \to T(Z)$ such that $id_X = p|X$ and for all $\langle u, v, r\rangle \in R$,*

$$\mathcal{K} \models_{\mathbf{T}(Z)} \langle \bar{\sigma}_r(p(u)), \bar{\sigma}_r(p(v))\rangle$$

where $\bar{\sigma}_r : \mathbf{T}(Z) \to \mathbf{T}(Z)$ is the unique homomorphism which extends the mapping $\sigma_r \cup id_{Z-X} : Z \to Z$.

PROOF.

(i)⇒ (ii): Since $\mathcal{K}$ is a non-trivial variety we note that $X \subseteq Z \subseteq T_{\mathcal{K}}(Z)$. This implication is now trivial since $\mathbf{T}_{\mathcal{K}}(Z) \in \mathcal{K}$.

(ii)⇒ (iii): By (ii) R is satisfied in the free algebra $\mathbf{T}_{\mathcal{K}}(Z)$ with respect to the mapping $id_X : X \to X \subseteq T_{\mathcal{K}}(Z)$. Hence there exists a mapping $g : Y \to T_{\mathcal{K}}(Z)$ such that $g|X = id_X$ and for all $\langle u, v, r\rangle \in R$ we have

$$\langle g(u), g(v)\rangle \in \Theta_{\mathbf{T}_{\mathcal{K}}(Z)}((id_X)_2[r]) = \Theta_{\mathbf{T}_{\mathcal{K}}(Z)}(r).$$

Let us denote $\Theta_{\mathbf{T}_{\mathcal{K}}(Z)}(r)$ by $\bar{r}$. We now show the existence of a mapping

$$p : Y \to T(Z),$$

with the special properties listed in (iii). Since $\mathbf{T}(Z)$ is an absolutely free algebra, there exists a unique homomorphism $h : \mathbf{T}(Z) \to \mathbf{T}_{\mathcal{K}}(Z)$ extending id_Z. Refer to the diagram below:

$$\begin{tikzcd}
Y \arrow[rr, "g"] \arrow[d, dashed, "p"'] & & \mathbf{T}_{\mathcal{K}}(Z) \arrow[loop right, "q"] \\
\mathbf{T}(Z) \arrow[loop left, "\bar{\sigma}_r"] \arrow[urr, "h"'] & &
\end{tikzcd}$$

Figure 3.1.1

By the definition of $\mathbf{T}_{\mathcal{K}}(Z)$, h is clearly a surjection. Hence there is a mapping $p : Y \to T(Z)$ such that $id_X \subseteq p$ and $g = h \circ p$. This is so as follows: Since $X \subseteq Z$, we have that $g|X = id_X = h|X$. So if $y \in X$ we set $p(y) = y$ to obtain $(h \circ p)(y) = h(y) = g(y)$. If $y \in Y - X$, we define $p(y) \in h^{-1}[\{g(y)\}]$ and $(h \circ p)(y) = g(y)$ in this case as well. Thus $h \circ p = g$ and $p|X = id_X$.

We now show that for all $\langle u, v, r \rangle \in R$,

$$\mathcal{K} \models_{\mathbf{T}(Z)} \langle \bar{\sigma}_r(p(u)), \bar{\sigma}_r(p(v)) \rangle.$$

where $\bar{\sigma}_r : \mathbf{T}(Z) \to \mathbf{T}(Z)$ is the unique homomorphism extending $\sigma_r \cup id_{Z-X} : Z \to Z$. Take any $\langle u, v, r \rangle \in R$. We notice from Section 1.6.1 that $h : \mathbf{T}(Z) \to \mathbf{T}_{\mathcal{K}}(Z)$ is a $\mathcal{K}$-reflection of $\mathbf{T}(Z)$. Hence by Lemma 1.6.4 we need only show that

$$h(\bar{\sigma}_r(p(u))) = h(\bar{\sigma}_r(p(v))).$$

Let $q : \mathbf{T}_{\mathcal{K}}(Z) \to \mathbf{T}_{\mathcal{K}}(Z)$ be the unique homomorphism which extends the mapping $\sigma_r \cup id_{Z-X} : Z \to Z$. Since $h \circ \bar{\sigma}_r : \mathbf{T}(Z) \to \mathbf{T}_{\mathcal{K}}(Z)$ and $q \circ h : \mathbf{T}(Z) \to \mathbf{T}_{\mathcal{K}}(Z)$ are homomorphisms such that $(h \circ \bar{\sigma}_r)|Z = (q \circ h)|Z$, we have that

$$h \circ \bar{\sigma}_r = q \circ h$$

since both are homomorphisms agreeing on a generating set. We shall now show that $\bar{r} \subseteq ker(q)$. Take $\langle x, x' \rangle \in r$ where $x, x' \in X$. Then $q(x) = \sigma_r(x)$ and

$q(x') = \sigma_r(x')$. From proposition 3.1.5 we have that $r \subseteq \mathcal{E}(r) = ker(\sigma_r)$. Thus $\langle x, x'\rangle \in ker(\sigma_r)$ which implies tha $\sigma_r(x) = \sigma_r(x')$. Hence $q(x) = q(x')$ and $\langle x, x'\rangle \in ker(q)$. Therefore we have that $r \subseteq ker(q)$. Since $ker(q)$ is a congruence on $\mathrm{T}_{\mathcal{K}}(Z)$, we have that $\bar{r} \subseteq ker(q)$ as well. We now observe that

$$h(\bar{\sigma}_r(p(u))) = q(h(p(u))) = q(g(u)) = q(g(v))$$

since $\langle g(u), g(v)\rangle \in \bar{r} \subseteq ker(q)$. Also,

$$q(g(v)) = q(h(p(v))) = h(\bar{\sigma}_r(p(v))).$$

Hence $h(\bar{\sigma}_r(p(u))) = h(\bar{\sigma}_r(p(v)))$ and $p : Y \to T(Z)$ is the required mapping.

(iii)$\Rightarrow$ (i): Refer figure 3.1.2: Take $\mathbf{A} \in \mathcal{K}$ and let $f : X \to A$ be a mapping. We have to find a mapping

$$g : Y \to A$$

such that $g|X = f$ and, for every $\langle u, v, r\rangle \in R$, $\langle g(u), g(v)\rangle \in \Theta_{\mathbf{A}}(f_2[r])$. Since $\mathbf{T}(Z)$ is absolutely free, there is a homomorphism $h : \mathbf{T}(Z) \to \mathbf{A}$ such that $f \subseteq h$.

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, "f"'] & & Y \arrow[ddll, "g"] \arrow[dd, "p"] \\
& Z \arrow[dr, hook] & \\
\mathbf{A} & & \mathrm{T}(Z) \arrow[ll, "h"'] \arrow[loop right, "\bar{\sigma}_r"]
\end{tikzcd}$$

Figure 3.1.2

We set

$$g = h \circ p,$$

and claim that g satisfies the requirements of Definition 3.1.2. That $g|X = f$ is clear since $f \subseteq h$. Take $\langle u, v, r\rangle \in R$ and denote $\Theta_{\mathbf{A}}(f_2[r])$ by Θ. We shall show that $\langle h(t), h(\bar{\sigma}_r(t))\rangle \in \Theta$ for all $t \in \mathrm{T}(Z)$: Suppose

$$S = \{t \in T(Z);\ \langle h(t), h(\bar{\sigma}_r(t))\rangle \in \Theta\}.$$

Let $o \in \mathbf{O}$ be a nullary operation symbol and let $t = o^{\mathbf{T}(Z)}$. Then since h and $\bar{\sigma}_r$ are homomorphisms, we have that

$$h(t) = h(o^{\mathbf{T}(Z)}) = o^{\mathbf{A}},$$

and

$$h(\bar{\sigma}_r(t)) = h(\bar{\sigma}_r(o^{\mathbf{T}(Z)})) = h(o^{\mathbf{T}(Z)}) = o^{\mathbf{A}}.$$

Hence $\langle h(t), h(\bar{\sigma}_r(t))\rangle \in \Theta$ for all nullary operations. We now take an n-ary operation $o \in \mathbf{O}$ for $n \geq 1$, and let $t_1, \ldots, t_n \in S$. Then

$$h(\bar{\sigma}_r(o^{\mathbf{T}(Z)}(t_1, \ldots, t_n))) = o^{\mathbf{A}}(h(\bar{\sigma}_r(t_1)), \ldots, h(\bar{\sigma}_r(t_n))),$$

and

$$h(o^{\mathbf{T}(Z)}(t_1, \ldots, t_n)) = o^{\mathbf{A}}(h(t_1), \ldots, h(t_n)).$$

Hence $\langle h(o^{\mathbf{T}(Z)}(t_1, \ldots, t_n)), h(\bar{\sigma}_r(o^{\mathbf{T}(Z)}(t_1, \ldots, t_n)))\rangle \in \Theta$ for n-ary operations as well and $\mathbf{S}$ is a subalgebra of $\mathbf{T}(Z)$. Next we show that $Z \subseteq S$: We recall that $\bar{\sigma}_r : \mathbf{T}(Z) \to \mathbf{T}(Z)$ is the unique homomorphism which extends the mapping $\sigma_r \cup id_{Z-X} : Z \to Z$ where $\sigma_r : X \to X$. If $z \in X$ then $h(z) = f(z)$, and also $h(\bar{\sigma}_r(z)) = h(\sigma_r(z)) = f(\sigma_r(z))$. But from the definition of σ_r, $\langle z, \sigma_r(z)\rangle \in \mathcal{E}(r)$. Therefore $\langle z, \sigma_r(z)\rangle \in f_2^{-1}[\Theta]$ so $f_2(\langle z, \sigma_r(z)\rangle) = \langle f(z), f(\sigma_r(z))\rangle \in \Theta$. Hence

$$\langle h(z), h(\bar{\sigma}_r(z))\rangle = \langle f(z), f(\sigma_r(z))\rangle \in \Theta.$$

If $z \in Z - X$, then $\bar{\sigma}_r(z) = id_{Z-X}(z) = z$. Thus $h(z) = h(\bar{\sigma}_r(z))$ and from the reflexivity of Θ we have

$$\langle h(z), h(\bar{\sigma}_r(z))\rangle \in \Theta.$$

Thus $Z \subseteq S$ and since Z generates $\mathbf{T}(Z)$, we conclude that $\mathbf{S} = \mathbf{T}(Z)$.

Finally for $\langle u, v, r\rangle \in R$ we have

$$\begin{aligned} g(u) &= h(p(u))\,\Theta\, h(\bar{\sigma}_r(p(u))) \\ &= h(\bar{\sigma}_r(p(v)))\,\Theta\, h(p(v)) \quad \text{[by (iii)]} \\ &= g(v). \end{aligned}$$

Hence $\langle g(u), g(v)\rangle \in \Theta_{\mathbf{A}}(f_2[r])$ and R is satisfied in every algebra of $\mathcal{K}$. □

§2. THE MAPPING $\tilde{I} : \Delta \to \operatorname{Ref} \mathbf{A}$

We shall denote the set of all reflexive relations on the universe of an algebra $\mathbf{A}$ by $\operatorname{Ref} \mathbf{A}$. If $r, s \in \operatorname{Ref} \mathbf{A}$ and if n is a positive integer we define $(r,s)^n$ as in Section 1.1.3.

3.2.1 Remark. All our considerations are related to a fixed but general type $\mathcal{T} = \langle \mathbf{O}, ar \rangle$. We shall now consider three new binary operation symbols $\wedge, \vee$ and $\circ$ which we interpret as meet, join and relation composition respectively. Then we have the auxiliary type $\mathcal{T}' = \langle \{\wedge, \vee, \circ\}, ar \rangle$, where all the considered operations are binary. We denote the absolutely free algebra of the type $\mathcal{T}'$, with $V = \{x_n;\ n \in \omega\}$ as its free generating set, by the symbol $\mathbf{\Delta}$. □

3.2.2 Definition. For each mapping $I : V \to \operatorname{Ref} \mathbf{A}$ we define a new mapping

$$\tilde{I} : \Delta \to \operatorname{Ref} \mathbf{A}$$

as follows:

(i) if $t \in V$, then $\tilde{I}(t) = I(t)$

(ii) if $t = t_1 \wedge t_2$, then $\tilde{I}(t) = \tilde{I}(t_1) \cap \tilde{I}(t_2)$

(iii) if $t = t_1 \vee t_2$, then $\tilde{I}(t) = \bigcup_{n=1}^{\infty} (\tilde{I}(t_1), \tilde{I}(t_2))^n$

(iv) if $t = t_1 \circ t_2$, then $\tilde{I}(t) = \tilde{I}(t_1) \circ \tilde{I}(t_2)$.

□

We can interpret $\operatorname{Ref} \mathbf{A}$ as a $\mathcal{T}'$-algebra with operations defined on the right hand side of the equalities in (ii) to (iv) above. Then $\tilde{I} : \mathbf{\Delta} \to \mathbf{Ref\,A}$ is the homomorphism which extends the mapping $I : V \to \operatorname{Ref} \mathbf{A}$.

3.2.3 Definition. Suppose $s, t \in \Delta$. Then we denote

$$\operatorname{Con} \mathbf{A} \models_{\mathbf{\Delta}} (t \subseteq s)$$

if $\tilde{I}(t) \subseteq \tilde{I}(s)$ for each mapping $I : V \to \operatorname{Con} \mathbf{A}$. □

Our aim is to characterise the varieties $\mathcal{K}$ of $\mathcal{T}$- algebras which, for given $s, t \in \Delta$, will satisfy $\operatorname{Con}\mathbf{A} \models_{\Delta} (t \subseteq s)$ *for all* $\mathbf{A} \in \mathcal{K}$.

3.2.4 Definition. Suppose $s, t \in \Delta$. From the terms s and t we form new terms as we did for elements of Ref $\mathbf{A}$ as follows:

$$(s,t)^1 = s, \quad (s,t)^{2n} = (s,t)^{2n-1} \circ t \ \text{ and} (s,t)^{2n+1} = (s,t)^{2n} \circ s.$$

where n is a positive integer. We also define $t^{[n]}$ as follows:

(i) if $t \in V$, then $t^{[n]} = t$

(ii) if $t = t_1 \wedge t_2$, then $t^{[n]} = (t_1 \wedge t_2)^{[n]} = t_1^{[n]} \wedge t_2^{[n]}$

(iii) if $t = t_1 \vee t_2$, then $t^{[n]} = (t_1 \vee t_2)^{[n]} = (t_1^{[n]}, t_2^{[n]})^n$

(iv) if $t = t_1 \circ t_2$, then $t^{[n]} = (t_1 \circ t_2)^{[n]} = t_1^{[n]} \circ t_2^{[n]}$.

□

Clearly $t^{[n]} \in \Delta$ for all $t \in \Delta$ and n a positive integer. Moreover, $t^{[n]}$ is a $\{\wedge, \circ\}$-term even if $\vee$ occurs in t.

3.2.5 Lemma. *Let* $\mathbf{A}$ *be a* $\mathcal{T}$*-algebra and* $I : V \to \operatorname{Ref}\mathbf{A}$ *a mapping. Then for each* $t \in \Delta$ *and for positive integers* m *and* n*, we have*

(i) $\tilde{I}(t^{[n]}) \subseteq \tilde{I}(t^{[m]})$ *if* $n \leq m$

(ii) $\tilde{I}(t) = \bigcup_{j=1}^{\infty} \tilde{I}(t^{[j]})$.

PROOF.

(i) We proceed by induction on the complexity of t: If $t \in V$, then $t^{[n]} = t = t^{[m]}$ and

$$\tilde{I}(t^{[n]}) = \tilde{I}(t) = \tilde{I}(t^{[m]}).$$

Assume that (i) holds for t_1 and t_2. If $t = t_1 \circ t_2$, then

$$\tilde{I}(t^{[n]}) = \tilde{I}((t_1 \circ t_2)^{[n]}) = \tilde{I}(t_1^{[n]} \circ t_2^{[n]}) = \tilde{I}(t_1^{[n]}) \circ \tilde{I}(t_2^{[n]}).$$

By the induction assumption, and the reflexivity of all considered relations, we have that

$$\tilde{I}(t_1^{[n]}) \circ \tilde{I}(t_2^{[n]}) \subseteq \tilde{I}(t_1^{[m]}) \circ \tilde{I}(t_2^{[m]}),$$

and

$$\tilde{I}(t_1^{[m]}) \circ \tilde{I}(t_2^{[m]}) = \tilde{I}(t_1^{[m]} \circ t_2^{[m]}) = \tilde{I}((t_1 \circ t_2)^{[m]}) = \tilde{I}(t^{[m]}).$$

If $t = t_1 \wedge t_2$, then

$$\tilde{I}(t^{[n]}) = \tilde{I}((t_1 \wedge t_2)^{[n]}) = \tilde{I}(t_1^{[n]} \wedge t_2^{[n]}) = \tilde{I}(t_1^{[n]}) \cap \tilde{I}(t_2^{[n]}).$$

By the induction assumption

$$\tilde{I}(t_1^{[n]}) \cap \tilde{I}(t_2^{[n]}) \subseteq \tilde{I}(t_1^{[m]}) \cap \tilde{I}(t_2^{[m]}),$$

and

$$\tilde{I}(t_1^{[m]}) \cap \tilde{I}(t_2^{[m]}) = \tilde{I}(t_1^{[m]} \wedge t_2^{[m]}) = \tilde{I}((t_1 \wedge t_2)^{[m]}) = \tilde{I}(t^{[m]}).$$

Finally if $t = t_1 \vee t_2$, then

$$\tilde{I}(t^{[n]}) = \tilde{I}((t_1 \vee t_2)^{[n]}) = \tilde{I}((t_1^{[n]}, t_2^{[n]})^n) = (\tilde{I}(t_1^{[n]}), \tilde{I}(t_2^{[n]}))^n.$$

Since $(\tilde{I}(t_1^{[n]}), \tilde{I}(t_2^{[n]}))^n = \tilde{I}(t_1^{[n]}) \circ \tilde{I}(t_2^{[n]}) \circ \tilde{I}(t_1^{[n]}) \circ \cdots$ to n factors, we have by the induction assumption, and the assumed reflexivity of all considered relations, that

$$(\tilde{I}(t_1^{[n]}), \tilde{I}(t_2^{[n]}))^n \subseteq (\tilde{I}(t_1^{[m]}), \tilde{I}(t_2^{[m]}))^n \subseteq (\tilde{I}(t_1^{[m]}), \tilde{I}(t_2^{[m]}))^m \quad \text{as} \quad n \leq m.$$

So we have,

$$\tilde{I}(t^{[n]}) \subseteq (\tilde{I}(t_1^{[m]}), \tilde{I}(t_2^{[m]}))^m = \tilde{I}((t_1^{[m]}, t_2^{[m]})^m) = \tilde{I}((t_1 \vee t_2)^{[m]}) = \tilde{I}(t^{[m]}).$$

(ii) Again we proceed by induction on the complexity of t: If $t \in V$, then since $t^{[j]} = t$, we have that

$$\bigcup_{j=1}^{\infty} \tilde{I}(t^{[j]}) = \bigcup_{j=1}^{\infty} \tilde{I}(t) = \tilde{I}(t).$$

Now assume that (ii) holds for t_1 and t_2. If $t = t_1 \circ t_2$ then

$$\begin{aligned}
\tilde{I}(t) &= \tilde{I}(t_1) \circ \tilde{I}(t_2) \\
&= (\bigcup_{j=1}^{\infty} \tilde{I}(t_1^{[j]})) \circ (\bigcup_{k=1}^{\infty} \tilde{I}(t_2^{[k]})) \quad \text{[by Ind. Ass.]} \\
&= \bigcup_{k,j=1}^{\infty} (\tilde{I}(t_1^{[j]}) \circ \tilde{I}(t_2^{[k]})) \quad \text{[by distr. for } \circ \text{ and } \bigcup\text{]} \\
&= \bigcup_{j=1}^{\infty} (\tilde{I}(t_1^{[j]}) \circ \tilde{I}(t_2^{[j]})) \quad \text{[by (i)]} \\
&= \bigcup_{j=1}^{\infty} \tilde{I}(t_1^{[j]} \circ t_2^{[j]}) \\
&= \bigcup_{j=1}^{\infty} \tilde{I}(t^{[j]}).
\end{aligned}$$

If $t = t_1 \wedge t_2$, then

$$\begin{aligned}
\bigcup_{j=1}^{\infty} \tilde{I}(t^{[j]}) &= \bigcup_{j=1}^{\infty} \tilde{I}((t_1 \wedge t_2)^{[j]}) \\
&= \bigcup_{j=1}^{\infty} \tilde{I}(t_1^{[j]} \wedge t_2^{[j]}) \\
&= \bigcup_{j=1}^{\infty} (\tilde{I}(t_1^{[j]}) \cap \tilde{I}(t_2^{[j]})) \\
&= \bigcup_{j=1}^{\infty} \tilde{I}(t_1^{[j]}) \cap \bigcup_{j=1}^{\infty} \tilde{I}(t_2^{[j]}),
\end{aligned}$$

by the distributive law for $\bigcup$ and $\cap$. Applying the induction assumption we have

$$\bigcup_{j=1}^{\infty} \tilde{I}(t^{[j]}) = \tilde{I}(t_1) \cap \tilde{I}(t_2) = \tilde{I}(t_1 \wedge t_2) = \tilde{I}(t).$$

If $t = t_1 \vee t_2$, then

$$\bigcup_{j=1}^{\infty} \tilde{I}(t^{[j]}) = \bigcup_{j=1}^{\infty} \tilde{I}((t_1 \vee t_2)^{[j]})$$

$$
\begin{aligned}
&= \bigcup_{j=1}^{\infty} \tilde{I}((t_1^{[j]}, t_2^{[j]})^j) \\
&= \bigcup_{j=1}^{\infty} \tilde{I}(t_1^{[j]} \circ t_2^{[j]} \circ \ldots) \\
&= \bigcup_{j=1}^{\infty} (\tilde{I}(t_1^{[j]}), \tilde{I}(t_2^{[j]}))^j \\
&\subseteq \bigcup_{j=1}^{\infty} (\tilde{I}(t_1), \tilde{I}(t_2))^j \quad \text{[by (i) and Ind. Ass.]} \\
&= \tilde{I}(t_1 \vee t_2) \\
&= \tilde{I}(t).
\end{aligned}
$$

Thus we have

$$\bigcup_{j=1}^{\infty} \tilde{I}(t^{[j]}) \subseteq \bigcup_{j=1}^{\infty} (\tilde{I}(t_1), \tilde{I}(t_2))^j.$$

For the converse inclusion we let

$$\langle a, b\rangle \in \bigcup_{j=1}^{\infty} (\tilde{I}(t_1), \tilde{I}(t_2))^j.$$

Then for some $j = 2k \geq 2$, there are $x_1, x_2, \ldots, x_{2k-1} \in \mathbf{A}$ such that

$$a\tilde{I}(t_1)x_1\tilde{I}(t_2)x_2 \ldots x_{2k-1}\tilde{I}(t_2)b.$$

By the induction assumption there are positive integers $n_1, \ldots, n_{2k}$ such that

$$a\tilde{I}(t_1^{[n_1]})x_1\tilde{I}(t_2^{[n_2]})x_2 \ldots x_{2k-1}\tilde{I}(t_2^{[n_{2k}]})b.$$

Taking $n = max\{n_1, \ldots, n_{2k}\}$, part (i) of the lemma yields

$$a\tilde{I}(t_1^{[n]})x_1\tilde{I}(t_2^{[n]})x_2 \ldots x_{2k-1}\tilde{I}(t_2^{[n]})b.$$

If we take $m = max\{2k, n\}$, then (i) and the reflexivity of $(\tilde{I}(t_1^{[n]}), \tilde{I}(t_2^{[n]}))^n$, enables us to conclude that

$$\langle a, b\rangle \in (\tilde{I}(t_1^{[m]}), \tilde{I}(t_2^{[m]}))^m.$$

Therefore

$$\langle a,b\rangle \in \bigcup_{j=1}^{\infty} (\tilde{I}(t_1^{[j]}), \tilde{I}(t_2^{[j]}))^j = \bigcup_{j=1}^{\infty} \tilde{I}(t^{[j]})$$

and hence

$$\bigcup_{j=1}^{\infty} \tilde{I}(t^{[j]}) = \bigcup_{j=1}^{\infty} (\tilde{I}(t_1), \tilde{I}(t_2))^j = \tilde{I}(t_1 \vee t_2) = \tilde{I}(t).$$

□

§3. THE SETS $\langle t,x,y\rangle_N$ AND $[t,x,y]_N$

3.3.1 Definition. As usual, $V = \{x_n;\ n \in \omega\}$. For each infinite subset N of V, we choose two infinite disjoint subsets of N and denote them by $\alpha(N)$ and $\beta(N)$. For brevity we shall denote $\alpha(V)$ by α, $\alpha(\alpha(V))$ by $\alpha\alpha$, $\alpha(\beta(V)$ by $\alpha\beta$, and so on. □

Note that we can define the sets $\alpha(N)$ and $\beta(N)$ in a more constructive way, for example, as follows: Suppose

$$N = \{x_{i_1}, x_{i_2}, x_{i_3}, \ldots\}$$

where $i_n < i_{n+1}$ for each positive integer n. Then we can set

$$\alpha(N) = \{x_{i_1}, x_{i_3}, x_{i_5}, \ldots\} \text{ and } \beta(N) = \{x_{i_2}, x_{i_4}, x_{i_6}, \ldots\}.$$

3.3.2 Notation. Recall from Section 3.2.1 the absolutely free algebra $\mathbf{\Delta}$ of type $\mathcal{T}' = \langle\{\wedge, \vee, \circ\}, ar\rangle$. We now consider another auxiliary type $\mathcal{T}'' = \langle\{\wedge, \circ\}, ar\rangle$ where all the considered operations are binary, and we denote the absolutely free algebra of type $\mathcal{T}''$, with free generating set V, by the symbol ∇. □

From Section 3.2.4, we recall that for $t \in \Delta$ and for each positive integer n we formed new $\{\wedge, \circ\}$- terms of the form $t^{[n]}$. We note that these terms are elements of the absolutely free algebra ∇ as well.

3.3.3 Definition. If $t \in \nabla$ and $x, y \in V$ then the triple $\langle t, x, y\rangle$ is an element of $\nabla \times V \times V$. Now for each infinite subset N of V, we define a set $\langle t, x, y\rangle_N \subseteq V \times V \times V$ by induction on the complexity of t as follows:

(i) if $t \in V$, then $\langle t, x, y\rangle_N = \{\langle t, x, y\rangle\}$

(ii) if $t = t_1 \wedge t_2$, then $\langle t, x, y\rangle_N = \langle t_1, x, y\rangle_{\alpha(N)} \cup \langle t_2, x, y\rangle_{\beta(N)}$

(iii) if $t = t_1 \circ t_2$, then $\langle t, x, y\rangle_N = \langle t_1, x, z\rangle_{\alpha(N)-\{y\}} \cup \langle t_2, z, y\rangle_{\beta(N)-\{x\}}$ where z is the element of $N - \{x, y\}$ with the least index.

□

3.3.4 Definition. Let N be an infinite subset of V. Then for every $t \in \nabla$ and $x, y \in V$ we define the *set* $[t, x, y]_N$ as

$$[t, x, y]_N = \{u \in V;\ (\exists w, v \in V)\ (\langle w, v, u\rangle \text{ or } \langle w, u, v\rangle \in \langle t, x, y\rangle_N)\}.$$

□

Note that by $(\langle w, v, u\rangle$ or $\langle w, u, v\rangle \in \langle t, x, y\rangle_N)$ in the above definition, we mean that

$$\langle w, v, u\rangle \in \langle t, x, y\rangle_N \quad \text{or} \quad \langle w, u, v\rangle \in \langle t, x, y\rangle_N.$$

3.3.5 Example. Consider the set V and the above mentioned constructive choice of disjoint subsets of V as

$$\begin{aligned} \alpha &= \{x_0, x_2, x_4, x_6, \ldots\} \text{ and} \\ \beta &= \{x_1, x_3, x_5, x_7, \ldots\}. \end{aligned}$$

Then

$$\begin{aligned} \alpha\beta &= \{x_1, x_5, x_9, x_{13}, \ldots\} \text{ and} \\ \beta\beta &= \{x_3, x_7, x_{11}, x_{15}, \ldots\} \end{aligned}$$

Suppose $t = (x_0 \wedge x_1) \wedge ((x_2 \circ x_3) \circ (x_4 \circ x_5))$. Then

$$\begin{aligned} \langle t, x_5, x_7\rangle_V &= \langle x_0 \wedge x_1, x_5, x_7\rangle_\alpha \cup \langle (x_2 \circ x_3) \circ (x_4 \circ x_5), x_5, x_7\rangle_\beta \\ &= \{\langle x_0, x_5, x_7\rangle, \langle x_1, x_5, x_7\rangle\} \cup B, \end{aligned}$$

where

$$\begin{aligned} B &= \langle (x_2 \circ x_3) \circ (x_4 \circ x_5), x_5, x_7 \rangle_\beta \\ &= \langle x_2 \circ x_3, x_5, x_1 \rangle_{\alpha\beta} \cup \langle x_4 \circ x_5, x_1, x_7 \rangle_{\beta\beta} \\ &= \{\langle x_2, x_5, x_9 \rangle, \langle x_3, x_9, x_1 \rangle, \langle x_4, x_1, x_3 \rangle, \langle x_5, x_3, x_7 \rangle\}. \end{aligned}$$

Also, in consequence of Definition 3.3.4, it is clear that

$$[t, x_5, x_7]_V = \{x_5, x_7, x_9, x_1, x_3\}.$$

□

3.3.6 Lemma. *Suppose* $t, t_1, t_2 \in \nabla$ *and* $x, y \in V$ *and* N *is an infinite subset of* V. *Then*

(i) $\langle t, x, y \rangle_N$ *is a finite subset of* $V \times (N \cup \{x, y\})^2$

(ii) $\{x, y\} \subseteq [t, x, y]_N$ *and* $[t, x, y]_N$ *is a finite subset of* $N \cup \{x, y\}$

(iii.a) $[t, x, y]_N = \{x, y\}$ *if* $t \in V$

(iii.b) $[t_1 \wedge t_2, x, y]_N = [t_1, x, y]_{\alpha(N)} \cup [t_2, x, y]_{\beta(N)}$ *and*
$[t_1, x, y]_{\alpha(N)} \cap [t_2, x, y]_{\beta(N)} = \{x, y\}$

(iii.c) $[t_1 \circ t_2, x, y]_N = [t_1, x, z]_{\alpha(N)-\{y\}} \cup [t_2, z, y]_{\beta(N)-\{y\}}$ *where* z *is the element of* $N - \{x, y\}$ *with the least index, and*

$$[t_1, x, z]_{\alpha(N)-\{y\}} \cap [t_2, z, y]_{\beta(N)-\{x\}} = \begin{cases} \{z\} & \text{if } x \neq y \\ \{x, z\} & \text{if } x = y. \end{cases}$$

PROOF.

(i) We proceed by induction on the complexity of t: If $t \in V$, then $\langle t, x, y \rangle_N = \{\langle t, x, y \rangle\}$ which is a finite subset of $V \times (N \cup \{x, y\})^2$. Assume that (i) holds for t_1 and t_2. If $t = t_1 \wedge t_2$, then

$$\langle t, x, y \rangle_N = \langle t_1, x, y \rangle_{\alpha(N)} \cup \langle t_2, x, y \rangle_{\beta(N)}.$$

By the induction assumption $\langle t_1, x, y\rangle_{\alpha(N)}$ and $\langle t_2, x, y\rangle_{\beta(N)}$ are finite subsets of $V \times (\alpha(N) \cup \{x,y\})^2$ and $V \times (\beta(N) \cup \{x,y\})^2$ respectively. Hence $\langle t, x, y\rangle_N$ is a finite subset of $V \times (N \cup \{x,y\})^2$. If $t = t_1 \circ t_2$, then

$$\langle t, x, y\rangle_N = \langle t_1, x, z\rangle_{\alpha(N)-\{y\}} \cup \langle t_2, z, y\rangle_{\beta(N)-\{x\}},$$

where z is the element of $N - \{x,y\}$ with the least index. By the induction assumption we have that $\langle t_1, x, z\rangle_{\alpha(N)-\{y\}}$ and $\langle t_2, z, y\rangle_{\beta(N)-\{x\}}$ are finite subsets of $V \times (\alpha(N) - \{y\} \cup \{x,y\})^2$ and $V \times (\beta(N) - \{x\} \cup \{x,y\})^2$ respectively. Hence $\langle t, x, y\rangle_N$ is a finite subset of $V \times (N \cup \{x,y\})^2$.

(ii) Again we proceed by induction on the complexity of t: By definition,

$$[t, x, y]_N = \{u \in V;\ (\exists w, v \in V)\ (\langle w, v, u\rangle \text{or } \langle w, u, v\rangle \in \langle t, x, y\rangle_N)\}.$$

If $t \in V$, then $\langle t, x, y\rangle_N = \{\langle t, x, y\rangle\}$ and

$$[t, x, y]_N = \{x, y\}$$

which is a finite subset of $N \cup \{x,y\}$. (Note that we have just proved (iii.a)). Assume that (ii) holds for t_1 and t_2. If $t = t_1 \wedge t_2$, then

$$\begin{aligned} [t, x, y]_N &= [t_1 \wedge t_2, x, y]_N \\ &= \{u \in V;\ (\exists w, v \in V)\ (\langle w, v, u\rangle \text{ or } \langle w, u, v\rangle \in \langle t_1 \wedge t_2, x, y\rangle_N)\} \\ &= [t_1, x, y]_{\alpha(N)} \cup [t_2, x, y]_{\beta(N)}, \end{aligned}$$

since $\langle t_1 \wedge t_2, x, y\rangle_N = \langle t_1, x, y\rangle_{\alpha(N)} \cup \langle t_2, x, y\rangle_{\beta(N)}$. (Note that we have now proved the first part of (iii.b)). By the induction assumption we have that $\{x,y\} \subseteq [t_1, x, y]_{\alpha(N)}$ which is a finite subset of $\alpha(N) \cup \{x,y\}$ and $\{x,y\} \subseteq [t_2, x, y]_{\beta(N)}$ which is a finite subset of $\beta(N) \cup \{x,y\}$. Thus the union $[t_1, x, y]_{\alpha(N)} \cup [t_2, x, y]_{\beta(N)}$ is a finite subset of $(\alpha(N) \cup \{x,y\}) \cup (\beta(N) \cup \{x,y\})$. Therefore $\{x,y\} \subseteq [t, x, y]_N$ which is a finite subset of $N \cup \{x,y\}$. Next, if $t = t_1 \circ t_2$, then

$$\begin{aligned} [t, x, y]_N &= [t_1 \circ t_2, x, y]_N \\ &= \{u \in V;\ (\exists w, v \in V)\ (\langle w, v, u\rangle \text{ or } \langle w, u, v\rangle \in \langle t_1 \circ t_2, x, y\rangle_N)\} \\ &= [t_1, x, z]_{\alpha(N)-\{y\}} \cup [t_2, z, y]_{\beta(N)-\{x\}}, \end{aligned}$$

where z is the element of $N - \{x, y\}$ with the least index, since

$$\langle t_1 \circ t_2, x, y\rangle_N = \langle t_1, x, z\rangle_{\alpha(N)-\{y\}} \cup \langle t_2, z, y\rangle_{\beta(N)-\{x\}}.$$

(Note that here we have just proved the first part of (iii.c)). Applying the induction assumption, $\{x, z\} \subseteq [t_1, x, z]_{\alpha(N)-\{y\}}$ which is a finite subset of $((\alpha(N) - \{y\}) \cup \{x, y\})$ and $\{z, y\} \subseteq [t_2, z, y]_{\beta(N)-\{x\}}$ which is a finite subset of $((\beta(N) - \{x\}) \cup \{x, y\})$. The union $[t_1, x, z]_{\alpha(N)-\{y\}} \cup [t_2, z, y]_{\beta(N)-\{x\}}$ is a finite subset of $((\alpha(N) - \{y\}) \cup \{x, y\}) \cup ((\beta(N) - \{x\}) \cup \{x, y\})$. Therefore $\{x, y\} \subseteq [t, x, y]_N$ and $[t, x, y]_N$ is a finite subset of $N \cup \{x, y\}$.

(iii.a) This has been established in the proof of (ii).

(iii.b) The first part has already been established in the proof of (ii). That $[t_1, x, y]_{\alpha(N)} \cap [t_2, x, y]_{\beta(N)} = \{x, y\}$ follows from (ii) since

$$\begin{aligned}\{x, y\} &\subseteq [t_1, x, y]_{\alpha(N)} \cap [t_2, x, y]_{\beta(N)}\\ &\subseteq (\alpha(N) \cup \{x, y\}) \cap (\beta(N) \cup \{x, y\})\\ &= \{x, y\}.\end{aligned}$$

(iii.c) Again, the first part was established in the proof of (ii). For the second part we consider the two conditions separately: When $x \neq y$, we have

$$\begin{aligned}\{z\} &\subseteq [t_1, x, z]_{\alpha(N)-\{y\}} \cap [t_2, z, y]_{\beta(N)-\{x\}}\\ &\subseteq ((\alpha(N) - \{y\}) \cup \{x, z\}) \cap ((\beta(N) - \{x\}) \cup \{z, y\})\\ &= \{z\}.\end{aligned}$$

When $x = y$, we have that

$$\begin{aligned}\{x, z\} &\subseteq [t_1, x, z]_{\alpha(N)-\{y\}} \cap [t_2, z, y]_{\beta(N)-\{x\}}\\ &\subseteq ((\alpha(N) - \{y\}) \cup \{x, z\}) \cap ((\beta(N) - \{x\}) \cup \{z, y\})\\ &= \{x, z\}.\end{aligned}$$

□

3.3.7 Lemma. *Suppose $t \in \nabla$, $x, y \in V$ such that $x \neq y$, and N is an infinite subset of V. Let* $\mathbf{A}$ *be a* $\mathcal{T}$*- algebra and* $I : V \to \operatorname{Ref} \mathbf{A}$ *be a mapping. If $a, b \in A$, then*

$$\langle a, b\rangle \in \tilde{I}(t)$$

if and only if there is a mapping $f : [t, x, y]_N \to A$ such that $a = f(x)$, $b = f(y)$, and if $\langle w, u, v\rangle \in \langle t, x, y\rangle_N$, then

$$\langle f(u), f(v)\rangle \in I(w).$$

PROOF.

We proceed by induction on the complexity of t: If $t \in V$, then the statement is trivially true since in this case $\langle t, x, y\rangle_N = \{\langle t, x, y\rangle\}$, $[t, x, y]_N = \{x, y\}$ and $\tilde{I}(t) = I(t)$. Assume that the lemma is satisfied for $t_1, t_2 \in \nabla$. If $t = t_1 \wedge t_2$, then

$$\langle a, b\rangle \in \tilde{I}(t_1 \wedge t_2) = \tilde{I}(t_1) \cap \tilde{I}(t_2)$$

if and only if $\langle a, b\rangle \in \tilde{I}(t_1)$ and $\langle a, b\rangle \in \tilde{I}(t_2)$. By the induction assumption the latter statement is true if and only if there are mappings

$$f : [t_1, x, y]_{\alpha(N)} \to A \quad \text{and} \quad g : [t_2, x, y]_{\beta(N)} \to A$$

which satisfy the lemma. Since $[t_1, x, y]_{\alpha(N)} \cap [t_2, x, y]_{\beta(N)} = \{x, y\}$ by the second part of Lemma 3.3.6(iii.b), the lemma will be satisfied by the mapping

$$f \cup g : [t, x, y]_N \to A.$$

Finally, if $t = t_1 \circ t_2$, then

$$\langle a, b\rangle \in \tilde{I}(t_1 \circ t_2) = \tilde{I}(t_1) \circ \tilde{I}(t_2)$$

if and only if there is a $c \in A$ such that $\langle a, c\rangle \in \tilde{I}(t_1)$ and $c, b\rangle \in \tilde{I}(t_2)$. By the induction assumption the latter statement is true if and only if there are mappings

$$f : [t_1, x, z]_{\alpha(N)-\{y\}} \to A \quad \text{and} \quad g : [t_2, z, y]_{\beta(N)-\{x\}} \to A$$

where z is the element of $N - \{x, y\}$ with the least index, such that the lemma is satisfied. This means that

$$f(x) = a, \quad f(z) = c = g(z) \quad \text{and} \quad g(y) = b,$$

and for all $\langle w, u, v\rangle \in V^3$ we have that

$$\langle w, u, v\rangle \in \langle t_1, x, z\rangle_{\alpha(N)-\{y\}} \text{ implies that } \langle f(u), f(v)\rangle \in I(w)$$

and

$$\langle w, u, v\rangle \in \langle t_2, z, y\rangle_{\beta(N)-\{x\}} \text{ implies that } \langle g(u), g(v)\rangle \in I(w).$$

Since $[t_1, x, z]_{\alpha(N)-\{y\}} \cap [t_2, z, y]_{\beta(N)-\{x\}} = \{z\}$, it is clear that the mapping

$$f \cup g : [t, x, y]_N \to A$$

is the required mapping for this case. □

§4. THE ALGORITHM

In this section we present the algorithm for finding Mal'cev conditions equivalent to a given inclusion for congruences. The algorithm is embodied in Corollary 3.4.5, Theorem 3.4.7 and Corollary 3.4.8. We begin with some definitions and two lemmas.

3.4.1 Definition. For every $t, s \in \nabla$, $x, y \in V$ and $\alpha = \alpha(V)$, let us denote

$$X = [t, x, y]_\alpha \text{ and } Y = X \cup [s, x, y]_\beta.$$

We define a congruence condition $R(t, s)$ over $\langle X, Y\rangle$ as follows:

$$\langle u, v, r\rangle \in R(t, s)$$

if and only if there exists $w \in V$ such that

$$\langle w, u, v\rangle \in \langle s, x, y\rangle_\beta \text{ and } r = \{\langle i, j\rangle;\ \langle w, i, j\rangle \in \langle t, x, y\rangle_\alpha\}.$$

□

We observe that $R(t,s)$ is indeed a congruence condition over $\langle X, Y\rangle$ as follows: If $\langle u,v,r\rangle \in R(t,s)$, then there exists an element $w \in V$ such that $\langle w,u,v\rangle \in \langle s,x,y\rangle_\beta$. Therefore

$$u,v \in [s,x,y]_\beta \subseteq Y.$$

Now $r = \{\langle i,j\rangle;\ \langle w,i,j\rangle \in \langle t,x,y\rangle_\alpha\}$. If $\langle i,j\rangle \in r$ then

$$i,j \in [t,x,y]_\alpha = X.$$

Hence $r \subseteq X^2$. So we have that if $\langle u,v,r\rangle \in R(t,s)$ then $\langle u,v,r\rangle \in Y \times Y \times exp(X^2)$. Thus $R(t,s) \subseteq Y \times Y \times exp(X^2)$.

3.4.2 Definition. Let $t,s \in \Delta$, $x,y \in V$ and let m and n be positive integers. We define

$$X_n = [t^{[n]},x,y]_\alpha \quad\text{and}\quad Y_{n,m} = X_n \cup [s^{[m]},x,y]_\beta,$$

$\square$

We remark that by the above definition $X_n \subseteq Y_{n,m}$ for all positive integers n and m.

3.4.3 Lemma. *Suppose* $\mathbf{A}$ *is a* $\mathcal{T}$*-algebra, and* $t,s \in \Delta$. *Then*

$$\mathbf{Con\,A} \models_\Delta t \subseteq s$$

if and only if for every positive integer n *and for each mapping* $f : X_n \to A$, *there is a positive integer* m *such that the congruence condition* $R(t^{[n]}, s^{[m]})$ *over* $\langle X_n, Y_{n,m}\rangle$ *is* f*-satisfied in* $\mathbf{A}$.

PROOF.

Assume $\mathbf{Con\,A} \models_\Delta t \subseteq s$ and take a mapping $f : X_n \to A$, n a positive integer. We shall show that there exists a positive integer m such that the congruence condition $R(t^{[n]}, s^{[m]})$ over $\langle X_n, Y_{n,m}\rangle$ is f-satisfied in $\mathbf{A}$: For $w \in V$ and the mapping $I : V \to \mathrm{Con}\,\mathbf{A}$ we define

$$I(w) = \bigcap\{\theta \in \mathrm{Con}\,\mathbf{A};\ (\forall \langle k,u,v\rangle \in \langle t^{[n]},x,y\rangle_\alpha)(\langle f(u), f(v)\rangle \in \theta)\}.$$

Now since $\langle f(u), f(v)\rangle \in I(w)$ for all $\langle k,u,v\rangle \in \langle t^{[n]},x,y\rangle_\alpha$, we have by Lemma 3.3.7 that $\langle f(x), f(y)\rangle \in \tilde{I}(t^{[n]})$. By Lemma 3.2.5 (ii), we have $\tilde{I}(t^{[n]}) \subseteq \tilde{I}(t)$.

Hence $\langle f(x), f(y)\rangle \in \tilde{I}(t)$, and by the assumption that $\tilde{I}(t) \subseteq \tilde{I}(s)$, we have that $\langle f(x), f(y)\rangle \in \tilde{I}(s)$. Here again by Lemma 3.2.5 (ii) we have that $\tilde{I}(s) = \bigcup_{j=1}^{\infty} \tilde{I}(s^{[j]})$. Therefore there is a positive integer m such that $\langle f(x), f(y)\rangle \in \tilde{I}(s^{[m]})$. Also by Lemma 3.3.7 there exists a mapping $g' : [s^{[m]}, x, y]_\beta \to A$ such that $f(x) = g'(x)$ and $f(y) = g'(y)$, and for all $\langle w, u, v\rangle \in \langle s^{[m]}, x, y\rangle_\beta$, we have $\langle g'(u), g'(v)\rangle \in I(w)$. Since we have that $f : X_n \to A$, we can set $g = f \cup g'$ to obtain the mapping

$$g : Y_{n,m} \to A.$$

We now show that g satisfies the requirements listed in Definition 3.1.2. It is clear that $f \subseteq g$. Now take $\langle u, v, r\rangle \in R(t^{[n]}, s^{[m]})$. By the definition of $R(t^{[n]}, s^{[m]})$ there is a $w \in V$ such that $\langle w, u, v\rangle \in \langle s^{[m]}, x, y\rangle_\beta$ and

$$r = \{\langle i, j\rangle;\ \langle w, i, j\rangle \in \langle t^{[n]}, x, y\rangle_\alpha\}.$$

We shall show that $\langle g(u), g(v)\rangle \in \Theta_{\mathbf{A}}(f_2[r])$: We have that

$$f_2[r] = \{\langle f(i), f(j)\rangle;\ \langle w, i, j\rangle \in \langle t^{[n]}, x, y\rangle_\alpha\}.$$

Therefore

$$\begin{aligned} \Theta_{\mathbf{A}}(f_2[r]) &= \Theta_{\mathbf{A}}(\{\langle f(i), f(j)\rangle;\ \langle w, i, j\rangle \in \langle t^{[n]}, x, y\rangle_\alpha\}) \\ &= \bigcap\{\theta \in \operatorname{Con}\mathbf{A};\ f_2[r] \subseteq \theta\} \\ &= I(w). \end{aligned}$$

We have, however, already shown that $\langle g'(u), g'(v)\rangle \in I(w)$. Since $g = f \cup g'$, we now have that

$$\langle g(u), g(v)\rangle \in I(w) = \Theta_{\mathbf{A}}(f_2[r]).$$

Hence the congruence condition $R(t^{[n]}, s^{[m]})$ over $\langle X_n, Y_{n,m}\rangle$ is satisfied in $\mathbf{A}$ with respect to the mapping f.

Conversely, assume that for every positive integer n and each mapping $f : X_n \to A$, there is a positive integer m such that the congruence condition $R(t^{[n]}, s^{[m]})$ over $\langle X_n, Y_{n,m}\rangle$ is f-satisfied in $\mathbf{A}$. We shall show that

$$\operatorname{Con}\mathbf{A} \models_\Delta t \subseteq s.$$

Let $I : V \to \mathrm{Con}\,\mathbf{A}$ be a mapping and $\langle a,b\rangle \in \tilde{I}(t)$. By Lemma 3.2.5 (ii) there is a positive integer n such that $\langle a,b\rangle \in \tilde{I}(t^{[n]})$. By Lemma 3.3.7 there is a mapping $f : X_n \to A$ such that $a = f(x)$, $b = f(y)$ and for every $\langle w,v,u\rangle \in \langle t^{[n]},x,y\rangle_\alpha$ we have $\langle f(u),f(v)\rangle \in I(w)$. By the assumption there is a positive integer m such that the congruence condition $R(t^{[n]},s^{[m]})$ over $\langle X_n,Y_{n,m}\rangle$ is f-satisfied in $\mathbf{A}$. Therefore f is extendible to a mapping $g : Y_{n,m} \to A$ in such a way that if $\langle u,v,r\rangle \in R(t^{[n]},s^{[m]})$ then $\langle g(u),g(v)\rangle \in \Theta_{\mathbf{A}}(f_2[r])$. We need to prove that $\langle a,b\rangle \in \tilde{I}(s)$. Take $\langle w,u,v\rangle \in \langle s^{[m]},x,y\rangle_\beta$ and denote $r = \{\langle i,j\rangle;\ \langle w,i,j\rangle \in \langle t^{[n]},x,y\rangle\}$. Since $\langle u,v,r\rangle \in R(t^{[n]},s^{[m]})$, we have

$$\langle g(u),g(v)\rangle \in \Theta_{\mathbf{A}}(f_2[r]) = \Theta_{\mathbf{A}}(\{\langle f(i),f(j)\rangle;\ \langle w,i,j\rangle \in \langle t^{[n]},x,y\rangle_\alpha\}).$$

Then by the definition of f, $\Theta_{\mathbf{A}}(f_2[r]) \subseteq I(w)$. Hence $\langle g(u),g(v)\rangle \in I(w)$ and by Lemma 3.3.7 we have $\langle g(x),g(y)\rangle \in \tilde{I}(s^{[m]})$. Hence $\langle a,b\rangle \in \tilde{I}(s^{[m]})$. By Lemma 3.2.5 (ii), $\tilde{I}(s^{[m]}) \subseteq \tilde{I}(s)$. So we now have $\langle a,b\rangle \in \tilde{I}(s)$ and thus $\tilde{I}(t) \subseteq \tilde{I}(s)$ for every $I : V \to \mathrm{Con}\,\mathbf{A}$. Therefore $\mathbf{Con}\,\mathbf{A} \models_\Delta t \subseteq s$. □

3.4.4 Lemma. *Let $t \in \nabla$. Then for each positive integer n, variables $x,y \in V$ and any infinite subset $N \subseteq V$ we have*

(i) $\langle t,x,y\rangle_N = \langle t^{[n]},x,y\rangle_N$ *and*

(ii) $[t,x,y]_N = [t^{[n]},x,y]_N$.

PROOF.

We proceed by induction on the complexity of t: If $t \in V$, then $t^{[n]} = t$ and both statements hold. Assume now that both (i) and (ii) hold for t_1 and t_2. If $t = t_1 \wedge t_2$, then

$$\begin{aligned}
\langle t,x,y\rangle_N &= \langle t_1 \wedge t_2,x,y\rangle_N \\
&= \langle t_1,x,y\rangle_{\alpha(N)} \cup \langle t_2,x,y\rangle_{\beta(N)} \\
&= \langle t_1^{[n]},x,y\rangle_{\alpha(N)} \cup \langle t_2^{[n]},x,y\rangle_{\beta(N)} \qquad \text{[induction assumption]} \\
&= \langle t_1^{[n]} \wedge t_2^{[n]},x,y\rangle_N
\end{aligned}$$

$$
\begin{aligned}
&= \langle (t_1 \wedge t_2)^{[n]}, x, y\rangle_N \\
&= \langle t^{[n]}, x, y\rangle_N .
\end{aligned}
$$

In the case of (ii) we note that by the first part of Lemma 3.3.6 (iii.b),

$$[t_1 \wedge t_2, x, y]_N = [t_1, x, y]_{\alpha(N)} \cup [t_2, x, y]_{\beta(N)}.$$

Hence $[t, x, y]_N = [t^{[n]}, x, y]_N$, in the same way as (i) was proved. Finally, if $t = t_1 \circ t_2$, then

$$
\begin{aligned}
\langle t, x, y\rangle_N &= \langle t_1 \circ t_2, x, y\rangle_N \\
&= \langle t_1, x, z\rangle_{\alpha(N)-\{y\}} \cup \langle t_2, z, y\rangle_{\beta(N)-\{x\}}
\end{aligned}
$$

where z is the element of $N - \{x, y\}$ with the least index. By the induction assumption the last expression equals $\langle t_1^{[n]}, x, z\rangle_{\alpha(N)-\{y\}} \cup \langle t_2^{[n]}, z, y\rangle_{\beta(N)-\{x\}}$ and

$$
\begin{aligned}
\langle t_1^{[n]}, x, z\rangle_{\alpha(N)-\{y\}} \cup \langle t_2^{[n]}, z, y\rangle_{\beta(N)-\{x\}} &= \langle (t_1 \circ t_2)^{[n]}, x, y\rangle_N \\
&= \langle t^{[n]}, x, y\rangle_N .
\end{aligned}
$$

The proof of (ii) follows in a similar fashion since by Lemma 3.3.6 (iii.c),

$$[t_1 \circ t_2, x, y]_N = [t_1, x, z]_{\alpha(N)-\{y\}} \cup [t_2, z, y]_{\beta(N)-\{x\}}.$$

$\square$

We recall from Definition 3.4.1, that the sets X and Y were denoted as follows:

$$X = [t, x, y]_\alpha \text{ and } Y = X \cup [s, x, y]_\beta,$$

for $t, s \in \nabla$ and $x, y \in V$. Recall also the sets X_n and $Y_{n,m}$ from Definition 3.4.2. We proceed now with the next important corollary.

3.4.5 Corollary. *Let $t, s \in \nabla$. Then*

$$\mathbf{Con\,A} \models_\Delta t \subseteq s$$

if and only if the congruence condition $R(t,s)$ over $\langle X, Y\rangle$ is satisfied in $\mathbf{A}$.

PROOF.

From Definition 3.4.1, we have that $\langle u, v, r\rangle \in R(t^{[n]}, s^{[m]})$ if and only if there is a $w \in V$ such that $\langle w, u, v\rangle \in \langle s^{[m]}, x, y\rangle_\beta$ and

$$r = \{\langle i, j\rangle;\ \langle w, i, j\rangle \in \langle t^{[n]}, x, y\rangle_\alpha\}.$$

Since $t, s \in \nabla$ we have from Lemma 3.4.4 that $\langle s^{[m]}, x, y\rangle_\beta = \langle s, x, y\rangle_\beta$ and also $\langle t^{[n]}, x, y\rangle_\alpha = \langle t, x, y\rangle_\alpha$. Therefore $R(t,s) = R(t^{[n]}, s^{[m]})$. Also from Lemma 3.4.4 we have that $[t^{[n]}, x, y]_\alpha = [t, x, y]_\alpha$ and $[s^{[m]}, x, y]_\beta = [s, x, y]_\beta$. This means that $X_n = X$ and $Y_{n,m} = Y$. The corollary is now a direct consequence of Lemma 3.4.3.

□

We note that, in consequence of the above corollary, we have for instance that an algebra $\mathbf{A}$ is congruence permutable if and only if

$$\mathbf{Con\,A} \models_\Delta x_0 \circ x_1 \subseteq x_1 \circ x_0,$$

if and only if the congruence condition $R(x_0 \circ x_1, x_1 \circ x_0)$ is satisfied in $\mathbf{A}$. We shall return to this example in §5.

We recall that Theorem 3.1.6 gave conditions under which a given congruence condition R is satisfied in a non-trivial variety $\mathcal{K}$. The following is a corollary to Theorem 3.1.6.

3.4.6 Corollary. *Suppose $t, s \in \Delta$ and $\mathcal{K}$ is a non-trivial variety. Let n be a positive integer with $X_n = [t^{[n]}, x, y]_\alpha$ and let Z be a set such that $X_n \subseteq Z \subseteq V$. If*

$$\mathbf{Con\,T}_\mathcal{K}(Z) \models_\Delta t \subseteq s$$

then there exists a positive integer m and a mapping

$$p : (X_n \cup [s^{[m]}, x, y]_\beta) \to T(X_n)$$

such that $id_{X_n} \subseteq p$. *Moreover, if there exists an element* $w \in V$ *such that* $\langle w, u, v\rangle \in \langle s^{[m]}, x, y\rangle_\beta$ *and* $r = \{\langle i, j\rangle;\ \langle w, i, j\rangle \in \langle t^{[n]}, x, y\rangle_\alpha\}$ *then for every* $\mathbf{A} \in \mathcal{K}$ *we have*

$$\mathbf{A} \models_{\mathbf{T}(X_n)} \langle \bar{\sigma}_r(p(u)); \bar{\sigma}_r(p(v))\rangle$$

where $\bar{\sigma}_r : \mathbf{T}(X_n) \to \mathbf{T}(X_n)$ *is the unique homomorphism on the absolutely free algebra* $\mathbf{T}(X_n)$ *which extends the mapping* σ_r.

PROOF.

Since $X_n \subseteq Z$ we have from $\mathbf{Con}\,\mathbf{T}_{\mathcal{K}}(Z) \models_\Delta t \subseteq s$ that $\mathbf{Con}\,\mathbf{T}_{\mathcal{K}}(X_n) \models t \subseteq s$. By Lemma 3.4.3, we have that for every positive integer n and every mapping $f : X_n \to T_{\mathcal{K}}(X_n)$ there is a positive integer m such that the congruence condition $R(t^{[n]}, s^{[m]})$ is f-satisfied in $\mathbf{T}_{\mathcal{K}}(X_n)$. In particular, we have that the congruence condition $R(t^{[n]}, s^{[m]})$ is satisfied in $\mathbf{T}_{\mathcal{K}}(X_n)$ with respect to the identity mapping $id_{X_n} : X_n \to X_n$ since $X_n \subseteq \mathbf{T}_{\mathcal{K}}(X_n)$. We now invoke the implication (ii)$\Rightarrow$ (iii) of Theorem 3.1.6: We take $Y = X_n \cup [s^{[m]}, x, y]_\beta$ and have the required mapping $p : Y \to T(X_n)$ possessing the two properties, namely, $id_{X_n} \subseteq p$, and for every $\langle u, v, r\rangle$ in $R(t^{[n]}, s^{[m]})$ and for every algebra $\mathbf{A} \in \mathcal{K}$,

$$\mathbf{A} \models_{\mathbf{T}(X_n)} \langle \bar{\sigma}_r(p(u)), \bar{\sigma}_r(p(v))\rangle$$

where $\bar{\sigma}_r : \mathbf{T}(X_n) \to \mathbf{T}(X_n)$ is the unique homomorphism extending σ_r. □

3.4.7 Theorem. *Let* $t, s \in \Delta$. *For each pair of positive integers* n *and* m, *we denote*

$$X_n = [t^{[n]}, x, y]_\alpha \quad \text{and} \quad Y_{n,m} = X_n \cup [s^{[m]}, x, y]_\beta.$$

Let $\mathcal{K}$ *be a non-trivial variety and* Z *be a set such that* $X_n \subseteq Z \subseteq V$. *Then the following conditions are equivalent:*

(i) $\mathbf{Con}\,\mathbf{A} \models_\Delta t \subseteq s$ *for all* $\mathbf{A} \in \mathcal{K}$.

(ii) $\mathbf{Con}\,\mathbf{T}_{\mathcal{K}}(Z) \models_\Delta t \subseteq s$.

(iii) *For every positive* n *there exists a positive integer* m *and a mapping* $p : Y_{n,m} \to T(X_n)$ *such that* $id_{X_n} \subseteq p$. *Moreover, if* $\langle u, v, r\rangle \in R(t^{[n]}, s^{[m]})$, *then for each*

$\mathbf{A} \in K$, *we have*

$$\mathbf{A} \models_{\mathbf{T}(X_n)} \langle \bar{\sigma}_r(p(u)), \bar{\sigma}_r(p(v)) \rangle$$

where $\bar{\sigma}_r : \mathbf{T}(X_n) \to \mathbf{T}(X_n)$ *is the unique homomorphism on the absolutely free algebra* $\mathbf{T}(X_n)$ *which extends the mapping* σ_r.

PROOF.

(i)⇒ (ii): This is trivial since $\mathbf{T}_K(Z) \in K$.

(ii)⇒ (iii): This has been proved in Corollary 3.4.6.

(iii)⇒ (i): From Theorem 3.1.6, (iii) implies that the congruence condition $R(t^{[n]}, s^{[m]})$ is satisfied in every algebra of K. As a result of Lemma 3.4.3, this in turn implies that $\mathbf{Con}\,\mathbf{A} \models_{\Delta} t \subseteq s$ for every $\mathbf{A} \in K$. □

3.4.8 Corollary. *Suppose* K *is a non-trivial variety and* $t, s \in \nabla$. *Denote*

$$X = [t, x, y]_\alpha \quad \text{and} \quad Y = X \cup [s, x, y]_\beta.$$

If Z *is a set such that* $X \subseteq Z$, *then the following conditions are equivalent:*

(i) $\mathbf{Con}\,\mathbf{A} \models_{\Delta} t \subseteq s$ *for all* $\mathbf{A} \in K$.

(ii) $\mathbf{Con}\,\mathbf{T}_K(Z) \models_{\Delta} t \subseteq s$.

(iii) *There exists a mapping* $p : Y \to T(X)$ *such that* $id_X \subseteq p$, *and for every* $\langle u, v, r \rangle \in R(t, s)$ *and for each* $\mathbf{A} \in K$ *we have*

$$\mathbf{A} \models_{\mathbf{T}(X)} \langle \bar{\sigma}_r(p(u)), \bar{\sigma}_r(p(v)) \rangle$$

where $\bar{\sigma}_r : \mathbf{T}(X) \to \mathbf{T}(X)$ *is the unique homomorphism on the absolutely free algebra* $\mathbf{T}(X)$ *which extends the mapping* σ_r.

PROOF.

In Lemma 3.4.4 we have shown that for $t \in \nabla$, $\langle t^{[n]}, x, y \rangle_N = \langle t, x, y \rangle_N$ and $[t^{[n]}, x, y]_N = [t, x, y]_N$ where $N \subseteq V$. Also in the proof of Corollary 3.4.5 we showed that $R(t^{[n]}, s^{[m]}) = R(t, s)$ for $t, s \in \nabla$. The corollary now follows directly from Theorem 3.4.7. □

§5. AN EXAMPLE

We recall Mal'cev's classical theorem for congruence permutable varieties: A non-trivial variety $\mathcal{K}$ is congruence permutable if and only if there exists a ternary term, the so called *Mal'cev term*, such that

$$\mathcal{K} \models t(x,x,y) = y \quad \text{and} \quad \mathcal{K} \models t(x,y,y) = x.$$

As noted earlier, we shall use the algorithm of §4 to give another proof of necessity of this theorem. Before we proceed, we note that in the definition of the relational product in 1.1.3, we agreed to omit the symbol $\circ$ in the interest of brevity. For the same reason we shall write xy instead of $x \circ y$.

3.5.1 Example. *Let $\mathcal{K}$ be a non-trivial variety. Then $\mathcal{K}$ is congruence permutable if and only if it has a Mal'cev term.*

PROOF OF NECESSITY.

By Definition 3.2.3, we know that $\mathcal{K}$ is congruence permutable if and only if, for every $\mathbf{A} \in \mathcal{K}$,

$$\mathbf{Con\,A} \models xy \subseteq yx.$$

In consequence of Corollary 3.4.5, $\mathbf{Con\,A} \models xy \subseteq yx$ if and only if the congruence condition $R(xy, yx)$ over $\langle X, Y\rangle$ is satisfied in $\mathbf{A}$. We set $t = xy$ and $s = yx$. From Section 3.3.1 we recall our constructive choice of α and β as disjoint subsets of V. We set

$$\alpha = \{x_0, x_2, x_4, \ldots\} \quad \text{and} \quad \beta = \{x_1, x_3, x_5, \ldots\}.$$

To determine the sets X and Y of Definition 3.4.1, we let $x = x_0$, $y = x_1$ and $z = x_2$. Then

$$\begin{aligned} \langle t, x, y\rangle_\alpha &= \langle xy, x, y\rangle_\alpha \\ &= \langle x, x, z\rangle_{\alpha\alpha-\{y\}} \cup \langle y, z, y\rangle_{\beta\alpha-\{x\}}, \end{aligned}$$

since z is the first element of $\alpha - \{x, y\}$. Hence

$$\langle t, x, y\rangle_\alpha = \{\langle x, x, z\rangle; \langle y, z, y\rangle\}.$$

Also,

$$\begin{aligned}\langle s,x,y\rangle_\beta &= \langle yx,x,y\rangle_\beta\\ &= \langle y,x,x_3\rangle_{\alpha\beta-\{y\}}\cup\langle x,x_3,y\rangle_{\beta\beta-\{x\}},\end{aligned}$$

since x_3 is the first element of $\beta-\{x,y\}$. Hence

$$\langle s,x,y\rangle_\beta=\{\langle y,x,x_3\rangle,\langle x,x_3,y\rangle\}.$$

Thus we have

$$[t,x,y]_\alpha=\{x,y,z\}\ \text{ and }\ [s,x,y]_\beta=\{x,y,x_3\}.$$

Hence

$$X=\{x,y,z\}\ \text{ and }\ Y=X\cup[s,x,y]_\beta=\{x,y,z,x_3\}.$$

We shall now determine the congruence condition $R(t,s)$ over $\langle X,Y\rangle$. By Definition 3.4.1 $\langle u,v,r\rangle\in R(t,s)$ if and only if there exists $w\in V$ such that

$$\langle w,u,v\rangle\in\langle s,x,y\rangle_\beta\ \text{ and}$$

$$r=\{\langle i,j\rangle;\ \langle w,i,j\rangle\in\langle t,x,y\rangle_\alpha\}.$$

The two possibilities for w are $w=y$ and $w=x$. If $w=y$, then $u=x$, $v=x_3$ and $r=\{\langle z,y\rangle\}$. If $w=x$, then $u=x_3$, $v=y$ and $r=\{\langle x,z\rangle\}$. Hence,

$$R(t,s)=\{\langle x,x_3,\{\langle z,y\rangle\}\rangle,\langle x_3,y,\{\langle x,z\rangle\}\rangle\}.$$

By Corollaries 3.4.5 and 3.4.8 we have that $R(t,s)$ is satisfied in $\mathcal{K}$ if and only if there exists a mapping $p:Y\to T(X)$ such that $id_X\subseteq p$ and for every $\langle u,v,r\rangle\in R(t,s)$, we have

$$\mathcal{K}\models_{\mathbf{T}(X)}\langle\bar\sigma_r(p(u)),\bar\sigma_r(p(v))\rangle$$

where $\bar\sigma_r:\mathbf{T}(X)\to\mathbf{T}(X)$ is the unique homomorphism which extends the mapping $\sigma_r:X\to X$. From the first property of p we have

$$p(x)=x,\quad p(y)=y\ \text{ and }\ p(z)=z.$$

To use the second property of p we denote, for $q(x,y,z)$ a term in $T(X)$,

$$p(x_3)=q(x,z,y),$$

where we have interchanged y and z. We now take the elements of $R(t,s)$, one at a time, and show that the identities $x = q(x,y,y)$ and $y = q(x,x,y)$ hold in $\mathcal{K}$.

(i) Take $\langle x, x_3, \{\langle z,y\rangle\}\rangle \in R(t,s)$. By the definitions of $\mathcal{E}(r)$ and σ_r in Sections 3.1.3 and 3.1.4 respectively, we have from $\mathcal{E}(r) = \{x\}^2 \cup \{z,y\}^2$ that

$$\sigma_r(x) = x \quad\text{and}\quad \sigma_r(y) = y = \sigma_r(z).$$

Since $u = x$ and $v = x_3$ we have that $\mathcal{K} \models \bar{\sigma}_r(p(x)) = \bar{\sigma}_r(p(x_3))$. But

$$\bar{\sigma}_r(p(x)) = \sigma_r(x) = x$$

and

$$\begin{aligned}\bar{\sigma}_r(p(x_3)) &= \bar{\sigma}_r(q(x,z,y))\\ &= q(\bar{\sigma}_r(x), \bar{\sigma}_r(z), \bar{\sigma}_r(y)]\\ &= q(x,y,y).\end{aligned}$$

Hence $x = q(x,y,y)$ holds in $\mathcal{K}$.

(ii) Take $\langle x_3, y, \{\langle x,z\rangle\}\rangle \in R(t,s)$. Here again, since $\mathcal{E}(r) = \{y\}^2 \cup \{x,z\}^2$, we have that

$$\sigma_r(x) = x, \quad \sigma_r(y) = y \quad\text{and}\quad \sigma_r(z) = x.$$

Since $u = x_3$ and $v = y$, we have that $\mathcal{K} \models \bar{\sigma}_r(p(x_3)) = \bar{\sigma}_r(p(y))$. But

$$\begin{aligned}\bar{\sigma}_r(p(x_3)) &= \bar{\sigma}_r(q(x,z,y))\\ &= q(\bar{\sigma}_r(x), \bar{\sigma}_r(z), \bar{\sigma}_r(y))\\ &= q(x,x,y),\end{aligned}$$

and

$$\bar{\sigma}_r(p(y)) = \bar{\sigma}_r(y) = y.$$

Hence $y = q(x,x,y)$ holds in $\mathcal{K}$. □

§6. CHINESE VARIETIES

In this section we develop Mal'cev conditions for the so-called Chinese varieties. We begin with a convention and a definition.

3.6.1 A Convention. Let $\mathbf{A}$ be an algebra. Then a set X will be referred to as a *normal subset* of A if $X \in \mathbf{A}/\sigma$ for some $\sigma \in \operatorname{Con}\mathbf{A}$. □

3.6.2 Definition. A $\mathcal{T}$-algebra $\mathbf{A}$ is said to satisfy the *Chinese Remainder Theorem* if for each nonempty, finite set N of normal subsets of A we have $\bigcap N \neq \emptyset$ whenever $X \cap Y \neq \emptyset$ for all $X, Y \in N$.

For brevity $\mathbf{A}$ will be called a *Chinese algebra* if the Chinese Remainder Theorem holds in it. A class $\mathcal{K}$ of algebras is called a *Chinese class* if all its elements are Chinese algebras. □

3.6.3 Theorem. *Let $\mathbf{A}$ be a $\mathcal{T}$-algebra. Then the following conditions are equivalent:*

(i) *$\mathbf{A}$ is a Chinese algebra.*

(ii) *If X_1, X_2, X_3 are normal subsets of A such that $X_i \cap X_j \neq \emptyset$ for $i, j \in \{1,2,3\}$, then $X_1 \cap X_2 \cap X_3 \neq \emptyset$.*

(iii) *For all $\rho, \sigma, \tau \in \operatorname{Con}\mathbf{A}$,*

$$\rho \cap (\sigma\tau) = (\rho \cap \sigma)(\rho \cap \tau).$$

PROOF.

(i)$\Rightarrow$ (ii): In consequence of Definition 3.6.2, this is trivial.

(ii)$\Rightarrow$ (i): Let N be a nonempty finite system of normal subsets of A such that $X \cap Y \neq \emptyset$ for all $X, Y \in N$. We shall show that $\bigcap N \neq \emptyset$ by induction on the size of N. If $|N| = 1$, then (i) is trivially satisfied. Let $n \geq 2$ be an integer and let $\bigcap N \neq \emptyset$ for each system N of normal subsets of A with $|N| \leq n$. If the sets $X_1, \ldots, X_{n+1}$ are normal subsets of A such that $X_i \cap X_j \neq \emptyset$ holds for $i, j \in \{1, \ldots, n+1\}$, then

$$Y_i \overset{\text{def}}{=} X_i \text{ for } i = 1, \ldots, n-1, \quad \text{and} \quad Y_n \overset{\text{def}}{=} X_n \cap X_{n+1}$$

are clearly normal in A. Also,

$$Y_i \cap Y_j = X_i \cap X_j \neq \emptyset$$

for $i, j \in \{1, \ldots, n-1\}$, and

$$Y_n \cap Y_j = X_{n+1} \cap X_n \cap X_j$$

for $j \in \{1, \ldots, n-1\}$. By (ii), the set $X_{n+1} \cap X_n \cap X_j$ is nonempty. Therefore by the induction assumption we have that

$$\emptyset \neq \bigcap_{i=1}^{n} Y_i = \bigcap_{i=1}^{n+1} X_i.$$

(ii)$\Rightarrow$ (iii): Let $\langle a, b \rangle \in \rho \cap (\sigma\tau)$. Then there is a $c \in A$ such that $\langle a, b \rangle \in \rho$, $\langle a, c \rangle \in \sigma$ and $\langle c, b \rangle \in \tau$. Let us denote

$$X_1 = a/\rho, \quad X_2 = c/\sigma \text{ and } X_3 = b/\tau.$$

Since $a \in X_1 \cap X_2$, $c \in X_2 \cap X_3$ and $b \in X_1 \cap X_3$, we have that the intersection $X_1 \cap X_2 \cap X_3$ is nonempty. Let $d \in X_1 \cap X_2 \cap X_3$. Then

$$\langle a, d \rangle \in \rho \cap \sigma \text{ and } \langle d, b \rangle \in \rho \cap \tau,$$

and therefore $\langle a, b \rangle \in (\rho \cap \sigma)(\rho \cap \tau)$. Hence $\rho \cap (\sigma\tau) \subseteq (\rho \cap \sigma)(\rho \cap \tau)$. We conclude that (iii) holds since the converse inclusion $(\rho \cap \sigma)(\rho \cap \tau) \subseteq \rho \cap (\sigma\tau)$ is always satisfied in Con **A**.

(iii)$\Rightarrow$ (ii): Suppose X_1, X_2 and X_3 are normal subsets of A and let

$$a \in X_1 \cap X_2, \quad b \in X_2 \cap X_3 \text{ and } c \in X_1 \cap X_3.$$

Then there are congruences $\rho, \sigma, \tau \in \operatorname{Con} \mathbf{A}$ such that $X_1 \in \mathbf{A}/\rho$, $X_2 \in \mathbf{A}/\sigma$ and $X_3 \in \mathbf{A}/\tau$. Now we have that

$$\langle a, c \rangle \in \rho \cap (\sigma\tau) = (\rho \cap \sigma)(\rho \cap \tau) \qquad \text{[by (iii)]}.$$

Therefore there is an element $d \in A$ such that

$$\langle a, d \rangle \in \rho \cap \sigma \text{ and } \langle c, d \rangle \in \rho \cap \tau.$$

Hence $d \in X_1 \cap X_2 \cap X_3$ and we have that $X_1 \cap X_2 \cap X_3 \neq \emptyset$. □

3.6.4 Theorem. *Suppose $\mathcal{K}$ is a variety of $\mathcal{T}$-algebras. If $\mathcal{K}$ is a Chinese variety then there exists a ternary term $q(x,y,z)$ such that $\mathcal{K}$ satisfies the following identities:*

$$\begin{aligned} q(x,x,y) &= x, \\ q(x,y,x) &= x \text{ and} \\ q(x,y,y) &= y. \end{aligned}$$

PROOF.

In consequence of Theorem 3.6.3 we have that $\mathcal{K}$ is a Chinese variety if and only if

$$\mathbf{Con\,A} \models_{\Delta} x \wedge (yz) \subseteq (x \wedge y)(x \wedge z)$$

for each $\mathbf{A} \in \mathcal{K}$. By Corollary 3.4.5, $\mathrm{Con}\,\mathbf{A} \models_{\Delta} x \wedge (yz) \subseteq (x \wedge y)(x \wedge z)$ if and only if the congruence condition $R(x \wedge (yz), (x \wedge y)(x \wedge z))$ is satisfied in $\mathcal{K}$. We set $t = x \wedge (yz)$ and $s = (x \wedge y)(x \wedge z)$. Here also we use our constructive choice of subsets α and β of V as in Definition 3.3.1. That is,

$$\alpha = \{x_0, x_2, x_4, x_6, \ldots\} \text{ and } \beta = \{x_1, x_3, x_5, x_7, \ldots\}.$$

Therefore

$$\beta\alpha = \{x_2, x_6, x_{10}, \ldots\}.$$

To determine the sets X and Y of Definition 3.4.1, we let $x = x_0$, $y = x_1$ and $z = x_2$. Then

$$\begin{aligned} \langle t, x, y\rangle_\alpha &= \langle x \wedge (yz), x, y\rangle_\alpha \\ &= \langle x, x, y\rangle_{\alpha\alpha} \cup \langle yz, x, y\rangle_{\beta\alpha} \\ &= \langle x, x, y\rangle_{\alpha\alpha} \cup \langle y, x, z\rangle_{\alpha\beta\alpha-\{y\}} \cup \langle z, z, y\rangle_{\beta\beta\alpha-\{x\}}, \end{aligned}$$

since z is the first element of $\beta\alpha - \{x, y\}$. Hence

$$\langle t, x, y\rangle_\alpha = \{\langle x, x, y\rangle, \langle y, x, z\rangle, \langle z, z, y\rangle\}.$$

Next we have

$$\begin{aligned} \langle s, x, y\rangle_\beta &= \langle (x \wedge y)(x \wedge z), x, y\rangle_\beta \\ &= \langle x \wedge y, x, x_3\rangle_{\alpha\beta-\{y\}} \cup \langle x \wedge z, x_3, y\rangle_{\beta\beta-\{x\}} \end{aligned}$$

since x_3 is the first element of $\beta - \{x,y\}$. Hence

$$\langle s,x,y\rangle_\beta = \{\langle x,x,x_3\rangle, \langle y,x,x_3\rangle, \langle x,x_3,y\rangle, \langle z,x_3,y\rangle\}$$

Therefore $X = \{x,y,z\}$ and $Y = \{x,y,z,x_3\}$. We now use Definition 3.4.1 to find the elements of the congruence condition $R(t,s)$ over $\langle X,Y\rangle$. Taking the cases $w = x$, $w = y$ and $w = z$, we find that $R(t,s)$ is the set

$$\{\langle x,x_3,\{\langle x,y\rangle\}\rangle, \langle x_3,y,\{\langle x,y\rangle\}\rangle, \langle x,x_3,\{\langle x,z\rangle\}\rangle, \langle x_3,y,\{\langle z,y\rangle\}\rangle\}.$$

Applying Theorem 3.1.6, we have that

$$p(x) = x,\ p(y) = y \ \text{and}\ p(z) = z.$$

Also let us denote

$$p(x_3) = q(x,y,z)$$

where $q(x,y,z)$ is a ternary term of $T(X)$. We shall now take the elements of $R(t,s)$, one at a time, and show that the identities in (iii) hold in $\mathcal{K}$.

From the first triple of R we have $u = x$, $v = x_3$ and $r = \{\langle x,y\rangle\}$. Therefore,

$$\bar{\sigma}_r(p(u)) = \bar{\sigma}_r(p(x)) = \bar{\sigma}_r(x) = x,$$

and

$$\begin{aligned}
\bar{\sigma}_r(p(v)) &= \bar{\sigma}_r(p(x_3)) \\
&= \bar{\sigma}_r(q(x,y,z)) \\
&= q(\bar{\sigma}_r(x), \bar{\sigma}_r(y), \bar{\sigma}_r(z)) \\
&= q(\sigma_r(x), \sigma_r(y), \sigma_r(z)) \\
&= q(x,x,z).
\end{aligned}$$

The last step follows from the definition of σ_r and the fact that $\mathcal{E}(r) = \{z\}^2 \cup \{x,y\}^2$. We note that $\bar{\sigma}_r(p(u)) = \bar{\sigma}_r(p(v))$ by the second property of p in Theorem 3.1.6. Hence if we exchange y and z, we have the identity

$$q(x,x,y) = x.$$

From the second triple of R we have $u = x_3$, $v = y$ and $r = \{\langle x, y\rangle\}$. In a similar manner as for the first triple we obtain, in this instance also, that

$$q(x, x, y) = x.$$

From the third triple of R we have $u = x$, $v = x_3$ and $r = \{\langle x, z\rangle\}$. Again, proceeding as before,

$$\begin{aligned}
\bar{\sigma}_r(p(x)) &= x \\
&= \bar{\sigma}_r(p(x_3)) \\
&= q(\sigma_r(x), \sigma_r(y), \sigma_r(z)) \\
&= q(x, y, x).
\end{aligned}$$

Hence we have the identity

$$q(x, y, x) = x.$$

From the fourth triple of R we have $u = x_3$, $v = y$ and $r = \{\langle z, y\rangle\}$. Finally, here also,

$$\begin{aligned}
\bar{\sigma}_r(p(x_3)) &= q(\bar{\sigma}_r(x), \bar{\sigma}_r(y), \bar{\sigma}_r(z)) \\
&= q(x, y, y) \\
&= \bar{\sigma}_r(p(y)) \quad [p(x_3) = p(y) \text{ by Thm. 3.1.6}] \\
&= y.
\end{aligned}$$

Hence we have the identity

$$q(x, y, y) = y.$$

□

According to Definition 2.3.1, a class $\mathcal{K}$ of algebras is said to be *arithmetical* if it is both congruence permutable and congruence distributive (A. F. Pixley, 1963). We now have the following theorem.

3.6.5 Theorem. *Suppose* **A** *is a congruence permutable algebra. Then* **A** *is arithmetical if and only if* **A** *is a Chinese algebra. A variety is arithmetical if and only if it is a Chinese variety.*

PROOF.

Since $\mathbf{A}$ is congruence permutable, we have for all $\rho, \sigma, \tau \in \operatorname{Con} \mathbf{A}$, that

$$\rho \cap (\sigma\tau) = \rho \cap (\sigma \vee \tau) \quad \text{and}$$

$$(\rho \cap \sigma)(\rho \cap \tau) = (\rho \cap \sigma) \vee (\rho \cap \tau).$$

Then we have from the equivalence of (i) and (iii) in Theorem 3.6.3 that $\mathbf{A}$ is a Chinese algebra if and only if

$$\rho \cap (\sigma \vee \tau) = (\rho \cap \sigma) \vee (\rho \cap \tau),$$

that is, if and only if $\mathbf{A}$ is congruence distributive. The first assertion now follows from the definition of an arithmetical algebra. To deduce the second assertion it suffices to note that a Chinese variety has a Mal'cev term (by Theorem 3.6.4), and is therefore congruence permutable. □

CHAPTER 4

A SURVEY OF LITERATURE

In this chapter we present a comprehensive list of available literature on Mal'cev conditions for varieties of algebras published during the last decade.

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Symbols