On zero-dimensionality of remainders of some compactifications

by

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Abstract

A compactification of a topological space is a dense embedding of the space into a compact topological space. We study different methods of compactifying a topological space with the focus on zero-dimensionality of the remainder. Freudenthal compactification is known as a maximal compactification with a zero-dimensional remainder and is guaranteed to exist for rim-compact spaces. It is shown that this compactification can be characterized using proximities. In fact, there is a one-to-one correspondence between compactifications and proximities and, in particular, between compactifications with zero-dimensional remainder and zero-dimensional proximity. Almost rim-compact spaces are spaces that are larger than the rim-compact spaces and they are shown to also have a compactification with a zero-dimensional remainder. But these do not exhaust spaces that have a compactification with a zero-dimensional remainder, for example, recently [18] it was found that spaces that lie between the locally compact part and its Freudenthal compactification also have a zero-dimensional remainder. It is known that the Freudenthal compactification is also perfect, we study the relationship between maximum compactifications with a zero-dimensional remainder and the perfectness of these compactifications.

Declaration

The study described in this dissertation was carried out in the School of Mathematics, Statistics and Computer Science at University of KwaZulu Natal in Pietermaritzburg.
This dissertation was completed under the supervision of Dr Simo Mthethwa, from March 2021 to January 2022. The research contained in this dissertation represents original work by the author and has not been submitted in any form to another University. Where use was made of the work of others, it was duly acknowledged in the text.

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As the candidate's supervisor, I have approved this dissertation for submission.

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Contents

Glossary

$\mathbb{R}$	Set of real numbers.
$\emptyset$	Empty set.
X, Y	Topological spaces.
τ	Topology defined on a space.
$\tau_{\vert Y}$	Subspace topology defined on a subset of a space.
$x \in X$	x is an element of X.
$X - A$	Complement of the set A in X.
$A \subseteq X$	A is a subset of X.
$\mathcal{P}(X)$	Collection of all subsets of X.
$Int(U)$	Interior of U.
$Cl_X U$	Closure of the set U in the space X.
$bd_X U$	Boundary of the set U in the space X.
f, g, k, h	Mappings.
$C(X)$	Ring of continuous real valued functions.
$C^*(X)$	Ring of bounded continuous real valued functions.
$\alpha X, \gamma X, ...$	Compactification of the space X
ωX	Alexandroff one-point compactification.
βX	Stone-Čech compactification.
δ	Proximity relation on $\mathcal{P}(X)$.
δ_Y	Subspace proximity defined on a subspace Y of a space X.
$\tau(\delta)$	Topology induced by the proximity δ.
$A \ll B$	A is a δ-neighborhood of B.
$E(A)$	Collection of ends that contain the set A.

$\mathcal{B}, \mathcal{M}$	Bases for a topological space.
$O(U)$	Extension of the open set U in the compactification: the largest subset of the compactification of X whose intersection with X is U.
$\mathcal{K}(X)$	Lattice of all compactifications of a space X.
$\mathcal{K}_0(X)$	Lattice of all compactifications of a space X with a zero-dimensional remainder.
$\Sigma(X)$	Collection of proximities that induce a topology on X.

Chapter 1

Introduction

In general topology, compactness is a property that generalizes the notion of a subset of the Euclidean space being closed. The theory of compact spaces is rich in its own right, however most topological spaces are not compact. Thus one of the topics of interest in the realm of topological spaces is the possibility of densely embedding a given space inside a compact Hausdorff space. This process of embedding a topological space into a compact Hausdorff space is called the compactifying such a space.

The first method of extending a non-compact space into a compact space was given by Alexandroff in 1924, in which one point is added to a locally compact space to make it compact. The Stone-Čech compactification was defined by Marshall Stone and Eduard Čech in 1937: this is the largest Hausdorff space that a completely regular topological space can be embedded in. Other methods of compactifying a space include: Wallman compactification (1938), Fan-Gottesman compactification (1952) and Magill's N-point compactification (1965), to mention a few. In Chapter 3, we look at the details of the construction of these compactifications.

Proximities provide an alternative way of looking at a topological space and the concept was axiomatized by V. A Efremovič in a paper published in 1934. Smirnov, in 1952, provided a characterization of the lattice structure of compactifications of a completely regular X by means of proximities on X that induce the topology on X; the Smirnov Theorem. Then, Skjarenko used this in 1966 to show that rim-compact spaces have a compactification with zero-dimensional remainder and also defined the Freudenthal compactification using proximities. We study the relationship between proximities and compactifications in Chapter 4.

Thinking of X as a subset of a compact Haudorff space αX, the remainder of a compactification is the complement of X in αX, denoted by $\alpha X - X$. So it is natural to endeavour to understand the nature of the points that get added to a space to make it compact. If the remainder has a basis of clopen sets then it is said to be zero-dimensional. The Freudenthal compactification is known as the maximum compactification which has a zero-dimensional remainder and it is guaranteed to exist when a space is rim-compact.

Looking at the internal structure of spaces that have a compactification with zero-dimensional remainder, B. Diamond developed a theory for the class of spaces intermediate between rim-compact spaces and spaces that have a compactification with a zero-dimensional remainder in 1987, which she called almost rim-compact spaces. In 1995, J. Hatzenbuhler and D. A Mattson gave a characterization of spaces which lie between the locally compact part and the Freudenthal compactification of the locally compact part, such spaces necessarily possess a compactification with a zero-dimensional remainder. In Chapter 5, we look at the characterization of spaces which have a compactification with zero-dimensional remainder.

In Chapter 6, we look at the maximum zero-dimensional compactification and perfectness of zero-dimensional compactifications. It turns out that a compactification with a zero-dimensional remainder is maximal if and only if it is perfect. Furthermore the maximum compactification with a zero-dimensional remainder is the minimum perfect compactification. These results were proved by McCartney in 1970.

In Chapter 7, we conclude this dessertation by looking at some recent results in zero-dimensional compactifications (or compactifications with zero-dimensional remainder) in pointfree topology and state some of the problems we would like to persue further.

Chapter 2

Preliminaries

The aim of this chapter is to provide definitions and standard results that will be used throughout this dissertation. For more details, we refer the reader to texts in general topology, for example [21] and [32].

2.1 Topological spaces

Definition 2.1.1 *A set X together with a collection τ of its subsets satisfying:*

1. $\bigcup_{i \in I} U_i \in \tau$ *for any index set I where each $U_i \in \tau$,*
2. $\bigcap_{i=1}^{n} U_i \in \tau$ *where each $U_i \in \tau$,*
3. $\emptyset, X \in \tau$.

is called a ***topological space****. The members of τ are called the* ***open sets*** *of X and τ is called a* ***topology*** *on X.*

Remark: We often omit specific mention of τ if no confusion will arise and refer to X as a topological space.

Definition 2.1.2 *Let X be a set and $d : X \times X \longrightarrow [0, +\infty)$ be a mapping. If d satisfies the following three conditions, then d is called a* ***metric*** *on X, and (X, d) is called a* ***metric space****:*

1. $d(x, y) = 0$ *if and only if* $x = y$,

2. *$d(x, y) = d(y, x)$, for every $x, y \in X$.*

3. *$d(x, z) \leqslant d(x, y) + d(y, z)$, for every $x, y, z \in X$.*

Remark: For any $\epsilon > 0$ and $x \in X$, let $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. If $U \subseteq X$ and for any $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$, we say that U is **open** in X. If (X, d) is a metric space, let τ_d consist of all open sets of X, then (X, τ_d) is a topological space. If a topological space has a topology arising from a metric, it is called a **metrizable space**. For example, let $x, y \in \mathbb{R}$ and we define $d(x, y) = |x - y|$ then $(\mathbb{R}, d)$ is a metric space, so $\mathbb{R}$ can be viewed as a topological space $(\mathbb{R}, \tau_d)$.

Definition 2.1.3 *Let X be a topological space. A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X such that*

1. *For any $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.*

2. *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Remark: If a collection of subsets $\mathcal{B}$ of X satisfies these two conditions, then we define the **topology τ generated by $\mathcal{B}$** as follows: $U \subseteq X$ is said to be **open** in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. Note that, in this case each basis element is an element of τ.

Another equivalent definition of a basis is as follows;

Definition 2.1.4 *Let X be a set and let $\mathcal{B}$ be a basis for a topology τ on X. Then τ equals the collection of all unions of elements of $\mathcal{B}$.*

Definition 2.1.5 *Let X be a topological space and $N \subseteq X$. If there exists an open set U with $x \in U \subseteq N$, then N is called a **neighborhood** of x.*

2.2 Subspaces and products

New topological spaces can be formed from the old ones by considering subsets and products:

Definition 2.2.1 *Let X be a topological space and $Y \subseteq X$. Then **the topology on Y induced by the topology τ of X** is the set $\tau_{|_Y} = \{U \cap Y : U \in \tau\}$. We say that $(Y, \tau_{|_Y})$ is a **subspace** of X.*

Theorem 2.2.2 *If A is a subspace of a topological space X, and $\mathcal{B}$ is a base for X, then $\{B \cap A | B \in \mathcal{B}\}$ is a base for A.*

Proof: See [32, Theorem 6.3]. $\square$

Remark: If a topology is not specified on a subset of a topological space, it is assumed to be the subspace topology.

Definition 2.2.3 *Let A be an index set and X_α be a set for each $\alpha \in A$. The **Cartesian product** of the sets X_α is the set*

$$\prod_{\alpha \in A} X_\alpha = \Big\{ x : A \longrightarrow \bigcup_{\alpha \in A} X_\alpha | x(\alpha) \in X_\alpha, \textit{for each } \alpha \in A \Big\}$$

*Thus $\prod_{\alpha \in A} X_\alpha$ is the set of functions defined on the indexing set. The value of $x \in \prod_{\alpha \in A} X_\alpha$ at α is denoted by x_α and is referred to as the α^{th} **coordinate of** x.*

Remark: If the index set A is the natural numbers, it is customary to denote the function x in $\prod_{k=1}^{\infty} X_\alpha$ by listing its values as an ordered $n - tuple$, i.e. $x = (x_1, x_2, ...)$. Thus $\prod_{k=1}^{\infty} X_\alpha = \Big\{ (x_1, x_2, ...) | x_k \in X_k, k = 1, 2, ... \Big\}$.

The Cartesian product can be endowed with a topology as follows;

Definition 2.2.4 *The **product topology** on $\prod_{\alpha \in A} X_\alpha$ is obtained by taking as a base for open sets, sets of the form $\prod_{\alpha \in A} U_\alpha$ where*

1. *U_α is open in X_α, for each $\alpha \in A$ and*
2. *For all but finitely many coordinates, $U_\alpha = X_\alpha$.*

Remark: Notice that the set $\prod_{\alpha \in A} U_\alpha$, where $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, ..., \alpha_n$, can be written $\prod_{\alpha \in A} U_\alpha = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap ... \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$ where π_{α_i} is the projection map of $\prod_{\alpha \in A} X_\alpha$ on X_{α_i} defined by $\pi_{\alpha_i}(x) = x_{\alpha_i}$.

2.3 Closed sets and limit points

Definition 2.3.1 *Let X be a topological space and $V \subseteq X$. If $X - V$ is open in X, we say that V is a* ***closed set*** *of X.*

Remark: Let X be a topological space. The following conditions hold:

1. $\emptyset$ and X are both open and closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

Definition 2.3.2 *A subset of X is called* ***clopen*** *if it is both open and closed.*

Definition 2.3.3 *A topological space is* ***zero-dimensional*** *if it has a basis of clopen subsets.*

Theorem 2.3.4 *Let Y be a subspace of X. A set A is closed in Y if and only if $A = C \cap Y$ for some closed set C in X.*

Proof: See [32, Theorem 6.3]. $\square$

Definition 2.3.5 *Let X be a topological space and $A \subseteq X$. The smallest closed set containing A is called the* ***closure*** *of A and is denoted by $Cl_X(A)$, that is,*

$$Cl_X(A) = \bigcap_{F\, closed, A \subseteq F} F$$

Each point $x \in Cl_X(A)$ is called a ***limit point*** *of A.*

Proposition 2.3.6 *A is closed in X if and only if $Cl_X(A) = A$*

Proof: See [32, Theorem 3.7]. $\square$

Theorem 2.3.7 *Let Y be a subspace of X and A be a subset of Y. Then $Cl_Y(A) = Cl_X(A) \cap Y$.*

Proof: See [32, Theorem 6.3]. $\square$

The next theorem gives a way of determining whether or not an element is in the closure using open sets.

Theorem 2.3.8 *Let $x \in X$. Then $x \in Cl_X(A)$ if and only if for every open set U containing x, $A \cap U \neq \emptyset$.*

Proof: See [32, Theorem 4.7]. $\square$

Definition 2.3.9 *If X is a topological space and $E \subseteq X$, the **boundary** of E is the set; $bd_X(E) = Cl_X(E) \cap Cl_X(X - E)$.*

Remark: Note that $bd_X(E) = Cl_X(E) - E$ if E is an open set of X.

Definition 2.3.10 *If $A \subseteq X$ is such that $Cl_X(A) = X$, A is said to be **dense** in X.*

2.4 Continuous functions

Definition 2.4.1 *Let X, Y be topological spaces and $f : X \longrightarrow Y$ be a mapping. If $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in X for every open set U in Y, then f is called a **continuous** mapping, or a **continuous** function.*

Continuity of a function depends not only on the function f itself, but also on the topologies specified for the domain and range.

Theorem 2.4.2 *Let X and Y be topological spaces, let $f : X \longrightarrow Y$. Then the following are equivalent:*

1. *f is continuous.*
2. *If $A \subseteq X$, then $f(Cl_X(A)) \subseteq Cl_Y(f(A))$.*
3. *If B is closed in Y, then $f^{-1}(B)$ is closed in X.*
4. *For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.*

Proof: See [32, Theorem 7.2]. $\square$

An onto map between a topological space and a set induces a topology on the set that makes the function continuous;

Definition 2.4.3 *If X is a topological space, Y is a set and $g : X \longrightarrow Y$ is an onto mapping, then the collection τ_g of subsets of Y defined by*

$$\tau_g = \{G \subseteq Y : g^{-1}(G) \text{ is open in } X\}$$

*is a topology on Y, called the **quotient topology** induced on Y by g. When Y is given such quotient topology, it is called the **quotient space** of X, and the inducing map g is called a **quotient map**.*

Definition 2.4.4 *If X and Y are topological spaces, a continuous function $f : X \longrightarrow Y$ is a **homeomorphism** if and only if f is one-to-one, onto and f^{-1} is also continuous. If f is continuous and one-to-one, we call it an **embedding** of X into Y, and say that X **is embedded in** Y **by** f.*

Remark: Thus, X is embedded in Y by f if and only if f is a homeomorphism between X and some subspace of Y. Any property of X that is entirely expressed in terms of the topology of X yields, via homeomorphism f, the corresponding property for the space Y.

Definition 2.4.5 *Let $f : X \longrightarrow Y$ be a mapping between two topological spaces. If for every open set $U \subseteq X$, $f(U)$ is an open set then f is called an **open mapping**.*

Definition 2.4.6 *Let $f : X \longrightarrow Y$ be a mapping between two topological spaces. If for every closed set $F \subseteq X$, $f(F)$ is a closed set then f is called an **closed mapping**.*

Definition 2.4.7 *If $f : X \longrightarrow Y$ is a mapping and $A \subseteq X$, the **restriction of** f to A, denoted by $f|A$, is the map from A to Y such that $(f|A)(a) = f(a)$ for each $a \in A$.*

Definition 2.4.8 *Let X be a set. Then*
$C(X) = \{f : X \longrightarrow \mathbb{R} : f \text{ is continuous}\}$ *and*
$C^*(X) = \{f \in C(X) : |f(x)| \leqslant M \text{ for some } M \in \mathbb{R} \text{ and every } x \in X\}$.

Definition 2.4.9 *$A \subseteq X$ is called a **zero-set** if there is a continuous $f : X \longrightarrow \mathbb{R}$ such that $A = \{x \in X : f(x) = 0\}$, this is usually denoted by $Z(f)$.*

2.5 Connectedness and compactness

Definition 2.5.1 *X is said to be* ***connected*** *if whenever $X = Y \cup Z$, where Y and Z are open in X, and $Y \cap Z = \emptyset$, then $X = Y$ or $X = Z$. A subset of a topological space is said to be connected if it is connected under its subspace topology*

Remark: A space is connected if and only if the only subsets of X which are clopen are the empty set and X. For; if A is a non-empty proper subset of X that is both open and closed in X, then the sets $U = A$ and $V = X - A$ are open, disjoint, non-empty and their union is X. Conversely, if $X = U \cup V$ and $U \cap V = \emptyset$ where U and V are open, then $U = X - V$ which is closed. Then U is a clopen subset of X.

Definition 2.5.2 *X is said to be* ***disconnected*** *if $X = Y \cup Z$ where $Y \cap Z = \emptyset$ for some open sets Y and Z.*

Definition 2.5.3 *A topological space X is* ***totally disconnected*** *if and only if for any $x \neq y$ in X, there exists a clopen set U of X such that $x \in U$ and $y \notin U$.*

Remark: Note that a topological space is totally disconnected if it is maximally disconnected, in the sense that it has no non-trivial connected subsets. In every topological space, the singletons are connected; in a totally disconnected space, these are the only connected subsets.

Definition 2.5.4 *Let X be a topological space and I be an index sets. A collection $\{U_i\}_{i \in I}$ of sets is a* ***cover*** *(or a* ***covering****) for X if $X \subseteq \bigcup_{i \in I} U_i$. If all U_i are open, then we call the collection $\{U_i\}_{i \in I}$ an* ***open cover****.*

Definition 2.5.5 *Let $\{U_i\}_{i \in I}$ be an open cover for X, if there exists $\{U_i\}_{i=1}^{n} \subseteq \{U_i\}_{i \in I}$ such that $X \subseteq \bigcup_{i=1}^{n} U_i$ then the collection $\{U_i\}_{i=1}^{n}$ is called a* ***finite subcover*** *of X.*

Definition 2.5.6 *A space X is* ***compact*** *if each open cover of X has a finite subcover.*

Theorem 2.5.7 *(**Tychonoff**). A non-empty product space $\prod_{\alpha \in I} X_\alpha$ is compact if and only if each X_α is compact for each $\alpha \in I$.*

Proof: See [32, Theorem 17.8]. $\square$

Definition 2.5.8 *A topological space X is said to be* ***locally compact*** *at $x \in X$ if for each neighborhood U of x there exists a compact subset K of X such that $x \in U \subseteq K$. If this holds for any $x \in X$, we say that X is* ***locally compact****.*

2.6 Separation axioms

We require that the topology on X should ideally contain enough open sets to distinguish between the points of X, in some way. We summarize separation axioms below.

Definition 2.6.1 *Let X be a topological space.*

1. *X is a T_0**-space** if: whenever x and y are distinct points in X, there is an open set containing one and not the other.*

2. *X is a T_1**-space** if: whenever x and y are distinct points in X, and U is a neighborhood of x and V is a neighborhood of y, then $y \notin U$ and $x \notin V$.*

3. *X is a **Hausdorff space (T_2-space)** if: whenever x and y are distinct points of X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.*

4. *X is a **regular** space if: whenever A is closed in X and $x \notin A$, then there are disjoint open sets U and V with $x \in U$ and $A \subseteq V$.*

5. *X is **completely regular** if: whenever A is a closed subset of X and $x \notin A$, there is a continous function $f : X \longrightarrow [0, 1]$ such that $f(x) = 0$ and $f \mid A = \{1\}$ (i.e. $f(a) = 1$ for every $a \in A$).*

6. *X is a **Tychonoff space** if: it is a completely regular and a Hausdorff space.*

7. *X is **normal** whenever A and B are disjoint closed sets in X, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.*

8. *X is called a T_4**-space** if: it is a normal and T_1-space.*

Definition 2.6.2 *A **Stone space** is a topological space that is zero-dimensional, T_0 and compact. Equivalently, a Stone space is a totally disconnected compact space.*

We capture some important results regarding compactness and separation axioms below.

Theorem 2.6.3 *The following holds true:*

1. *Every closed subspace of a compact space is compact.*
2. *If X is a Hausdorff space, then a sequence of points of X converge to at most one point of X.*
3. *Every one-point set in a Hausdorff space X is closed.*
4. *A compact subset of a Hausdorff space is closed.*
5. *A compact Hausdorff space X is a T_4-space.*
6. *A topological space X is a Tychonoff space if and only if it is homeomorphic to some subspace of the product space of closed and bounded intervals.*
7. *(**Urysohn's Lemma**). X is normal if and only if any two disjoint closed subsets can be separated by a continuous function; $A \subseteq X$ and $B \subseteq X$ are said to be separated by a function if there exists a continuous function $f : X \longrightarrow [0,1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for $b \in B$.*

Proof: For (1) see [32, Theorem 17.5], for (2) see [32, Theorem 13.7], for (3) see [32, Theorem 13.4], for (4) see [32, Theorem 17.5], for (5) see [32, Theorem 17.10], for (6) see [32, 14.3] and for (7) see [32, Lemma 15.6]. □

2.7 Boolean algebras

Definition 2.7.1 *A **partially-ordered** set, is a set A together with a relation, $\leqslant$, satisfying:*

1. reflexive*: for all $a \in A$, $a \leqslant a$,*

2. transitive*: for all* $a, b, c \in A$*, if* $a \leqslant b$ *and* $b \leqslant c$*, then* $a \leqslant c$*, and*

3. antisymmetric*: for all* $a, b, c \in A$*, if* $a \leqslant b$ *and* $b \leqslant a$ *then* $a = b$*.*

Definition 2.7.2 *An **order isomorphism** between two partially ordered sets* $(S, \leqslant_S)$ *and* $(T, \leqslant_T)$*, is a bijective mapping* $f : S \longrightarrow T$ *such that for every* $x, y \in S$*,* $x \leqslant_S y$ *if and only if* $f(x) \leqslant_T f(y)$*.*

Definition 2.7.3 *Let* A *be a subset of the partially ordered set* $(S, \leqslant)$*:*

1. *A **lower bound** of* A *is an element* $a \in S$ *such that* $a \leqslant x$ *for all* $x \in A$*. A lower bound* a *of* A *is called an **infimum** (or **greatest lower bound**) of* A *(denoted by* $\bigwedge A$*) if for all lower bounds* y *of* A *in* S*, we have that* $y \leqslant a$*.*

2. *An **upper bound** of* A *is an element* $b \in S$ *such that* $b \geqslant x$ *for all* $x \in A$*. An upper bound* b *of* A *is called **supremum** (or **least upper bound**) of* A *(denoted by* $\bigvee A$*) if for all upper bounds* z *of* A *in* S*, we have that* $z \geqslant b$*.*

Definition 2.7.4 *Let* $\{a, b\}$ *be a two element set of a partially ordered set* $(S, \leqslant)$*, then* $a \vee b$ *is the least upper bound of* $\{a, b\}$ *with* $a \wedge b$ *being the greatest lower bound.*

Definition 2.7.5 *A partially ordered* $(S, \leqslant)$ *set is a **lattice** if for each two element set* $\{a, b\}$ *in* S*, we have that* $a \vee b, a \wedge b \in S$*.*

Definition 2.7.6

1. *A lattice* $(S, \leqslant)$ *is **distributive** if* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *for any elements* $a, b, c \in S$*.*

2. *If there is* $b \in S$ *such that* $a \vee b = 1$ *and* $a \wedge b = 0$ *(where 0 is the least element and 1 the greatest element) then such an element* b *is called the **complement** of* a*, and it is denoted by* $\neg a$*.*

Definition 2.7.7 *A **Boolean algebra** is a distributive lattice in which every element has a complement.*

Remark: For a set X, let $\mathcal{P}(X) = \{B : B \subseteq X\}$, the family of all subsets of X. Then $(\mathcal{P}(X), \subseteq)$ is a Boolean algebra. Take $0 = \emptyset$ and $1 = X$. Define the meets ($\wedge$) to be intersection ($\cap$), joins ($\vee$) to be set unions ($\cup$), and complementation ($\neg$) to be set complementation of subsets of X.

Definition 2.7.8 *A **filter** on a set X is a family $\mathcal{F}$ of subsets of X such that, for all $A, B \subseteq X$, we have*

1. *$X \in \mathcal{F}$, but $\emptyset \notin \mathcal{F}$.*
2. *If $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.*
3. *If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.*

Definition 2.7.9 *The **principal filter** at an element $x \in X$ is the family $\mathcal{F}_x = \{A \subseteq X : x \in A\}$.*

Definition 2.7.10 *Let X be a topological space. The set $\mathcal{N}(x)$ of all neighbourhoods of $x \in X$ is called the **neighbourhood filter** at x*

Definition 2.7.11 *An **ultrafilter** is the maximal filter on X.*

Proposition 2.7.12 *A filter $\mathcal{F}$ on a set X is an **ultrafilter** if for $A \subseteq X$, we have that $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.*

Proof: See [32, Theorem 12.11]. □

Remark: For each $x \in X$, the principal filter $\mathcal{F}_x$ is an ultrafilter.

Theorem 2.7.13 *Let $\mathcal{F}$ be an ultrafilter on X. If $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

Proof: See [8, Part I, Section 6.4, Proposition 5]. □

Chapter 3

Some compactifications of topological spaces

One of the classical problems in topology is to embed a non-compact space into a compact one. In this chapter we define the compactification concept and look at some of the well-known methods of compactifying a space.

Definition 3.0.1 *A* ***compactification*** *of a space X is an ordered pair $(\alpha X, \alpha)$ where αX is a compact Hausdorff space and α is an embedding of X as a dense subset of αX. Thus, a* ***compactification*** *of X is a compact space αX together with a function $\alpha : X \longrightarrow \alpha X$ such that $Cl_{\alpha X}[\alpha(X)] = \alpha X$.*

Our interest is to have a compatification which is a Hausdorff space, it turns out that Tychonoff spaces are exactly the spaces that possess a Hausdorff compactification as the next theorem states. Tychonoff spaces are named after Andrey Nikolayevich Tychonoff who introduced them in 1930. For example, every metric space is Tychonoff.

Theorem 3.0.2 *A toplogical space X has a Hausdorff compactification if and only if X is a Tychonoff space.*

Proof outline : The theorem is proved by noting that a compact Hausdorff space is Tychonoff; which is proved by showing that a compact Hausdorff space is regular, then a regular compact space is a normal space which is in turn a Tychonoff space, by Urysohn's Lemma. A subset of a Tychonoff space is a Tychonoff space; since a subspace of a T_1 –space is also a T_1-space and a subspace of a completely regular space is completely regular. Therefore,

since X is embedded in a Tychonoff space, it can be thought of as a subspace of a Tychonoff space, and hence it is Tychonoff.
For the converse, let $f \in C(X)$ be a real-valued function that separates points in X (since X is assumed to be Tychonoff), which has a range contained in some closed bounded interval I_f in $\mathbb{R}$. The product space $\prod I_f$ of such functions is compact by Tychonoff theorem which states that a non-empty product space is compact if and only if each factor space is compact. We have that $\prod I_f$ is a Hausdorff space, since each I_f is a Hausdorff space. Then, the evaluation map $e : X \longrightarrow \prod I_f$ defined by $[e(x)]_f = f(x)$ is an embedding by [32, Theorem 8.12]. Thus, if X is a Tychonoff space, then it can be embedded in to the compact Hausdorff space $\prod I_f$, where $I_f \subseteq \mathbb{R}$.

Example: The space $(0,1)$ is not compact as a subset of $\mathbb{R}$ with the usual topology. $(0,1)$ is a subset of a Tychonoff space, and hence itself is Tychonoff. There are many compact Hausdorff spaces that $(0,1)$ can be embedded in;

1. $(0,1)$ can be embedded in $[0,1]$ by $h : (0,1) \longrightarrow [0,1]$ where $h(x) = x$ for $x \in (0,1)$. Then $(0,1)$ is homeomorphic to a subset of $[0,1]$ via the inclusion map and $[0,1]$ is compact since it is closed and bounded interval.

2. $(0,1)$ can be embedded in S^1 by $h : (0,1) \longrightarrow S^1$ where $h(x) = (\cos(2\pi x), \sin(2\pi x))$. Now, $(0,1)$ is homeomorphic to $S^1 - \{(1,0)\}$ which is a subspace of S^1, and S^1 is compact in $\mathbb{R}^2$ since it is a closed subset of a Hausdorff space.

3. $(0,1)$ can be embedded in $[0,1]^{\mathbb{N}}$ by $h : (0,1) \longrightarrow [0,1]^{\mathbb{N}}$ where $h(x) = (x, 1, 1, 1, ...)$. Then $(0,1)$ can be embedded in $[0,1]^{\mathbb{N}}$ (a cube) which is compact since each $[0,1]$ is compact.

In general, there are a number of ways of embedding a space into a compact Hausdorff space, we look at some of these techniques;

3.1 N-point compactifications

Definition 3.1.1 *Let n be a positive integer. A compactification αX is called an N-**point compactification** if $\alpha X - X$ consist of n points.*

Remark: A compactification with n points on the remainder has a zero-dimensional remainder since the remainder is finite.

The first general method of extending a non-compact space to compact one was the *one-point compactification* by Alexandroff (1924). The one-point compactification, denoted by ωX, is a way to extend a non-compact topological space by adjoining a single point (denoted by ∞) in such a way that the resulting space is compact and Hausdorff.
Firstly we define a topology on ωX, then prove that ωX is a compact space with the defined topology and finally construct a map that embedds X into ωX which will be our one-point compactification, as outlined in [32].

Proposition 3.1.2 *If (X, τ) is a topological space, then $(\omega X, \tau^\infty)$ is a toplogical space, with τ^∞ defined by*

$$U \in \tau^\infty \textit{ if and only if } U \in \tau \textit{ or } U = \{\infty\} \cup (X - K)$$

where K is some compact subset of X and $\infty \notin X$.

Proof: Since $\emptyset \in \tau$ then $\emptyset \in \tau^\infty$. We have $\omega X = \{\infty\} \cup (X - \emptyset)$, and since the empty set is compact in X then $\omega X \in \tau^\infty$. Let $\{U_1, U_2, ...\}$ be a collection of elements in τ^∞. If all the elements of the collection are such that they are also in τ then their union is in τ^∞ since (X, τ) is a topological space. If $\infty \in \bigcup U_j$, then there is a $U = \{\infty\} \cup (X - K)$ in $\bigcup U_j$. If U_i is open in X then there is a closed set G_i such that $U_i = X - G_i$. Hence, $\bigcup U_j = (X - K) \cup \{\infty\} \cup (X - G_1) \cup (X - G_2) \cup ... = \{\infty\} \cup [X - (K \cap G_1 \cap G_2 \cap ...)]$. Since the intersection of any closed sets is closed, and a closed subset of a compact space is compact, we have that $\bigcup U_j \in \tau^\infty$. Let $\{U_1, U_2, ...U_n\}$ be a finite collection of elements in τ^∞. If all the elements of the collection are such that they are also in τ then their intersection is in τ^∞ since (X, τ) is a topological space. If $\infty \in \bigcap_{j=1}^n U_j$ then all the elements of the collection are of the form $U_j = \{\infty\} \cup (X - K_j)$, then

$$\bigcap_{j=1}^n U_j = \bigcap_{j=1}^n [\{\infty\} \cup (X - K_j)] = \{\infty\} \cup (X - (\bigcup_{j=1}^n K_j)) = \{\infty\} \cup (X - K^*)$$

where $K^* = \bigcup_{j=1}^n K_j$, which is a compact subset of X. Therefore, the finite intersection of elements of τ^∞ is in τ^∞. $\square$

Proposition 3.1.3 *If (X, τ) is a locally compact Hausdorff space then $(\omega X, \tau^{\infty})$ is a compact Hausdorff space.*

Proof: Since X is a Hausdorff space, elements of X can be separated by disjoint open sets, hence $\omega X - \{\infty\}$ is a Hausdorff space. For $x \in X$, there is a compact neighborhood, say K, of x, since X is locally compact. That compact neighborhood has a corresponding neighborhood for $\infty \in \omega X$ given by $\{\infty\} \cup (X - K)$, which is disjoint from $U \subseteq K$ where U is an open neighborhood of x. Since the x was arbitrary, then every $x \in X$ has a neighborhood that separates it from ∞. Hence ωX is a Hausdorff space.
Let $\{U_1, U_2, ...\}$ be an open cover of ωX, then ∞ is contained in some open set $U_i = U$ from the collection. Therefore $U = \{\infty\} \cup (X - K)$ for some compact subset of X, K. Therefore $X - U = K$ which means that it is compact, hence it has a finite subcover; $X - U \subseteq \bigcup_{j=1}^{n} U_j$ implies that $\omega X = [(X - U) \cup U] \subseteq \bigcup_{j=1}^{n} U_j \cup U$, taking a finite subcover and adding on one more open set, leaves the subcover still finite, hence ωX is compact. □

Proposition 3.1.4 *If (X, τ) is non-compact Hausdorff space, the the inclusion map*

$$c : (X, \tau) \longrightarrow (\omega X, \tau^{\infty})$$

defined by:

$$c(x) = x \qquad \forall x \in X$$

is continuous, open and one-to-one, and $Cl_{\omega X}[c(X)] = \omega X$.

Proof: Let U be an open set in ωX, then it is of the form $\{\infty\} \cup (X - K)$ where K is compact or it is equal to U if it contains only elements of X by construction. Therefore $c^{-1}(\{\infty\} \cup (X - K)) = [\{\infty\} \cup (X - K)] \cap X = X - K$ is open in X, since K is a compact subset of a Hausdorff space and hence closed, or $c^{-1}(U) = U \cap X = U$ which is open in X. Hence the mapping c is continuous. Since open sets of X are mapped into open sets in ωX by definition of the map c, then it is by default an open mapping. Similarly, since an element of X is mapped to itself, the map is one-to-one. Since $Cl_{\omega X}(X)$ is the smallest closed set containing X, then $Cl_{\omega X}(X) \subseteq \omega X$. Also, if one point is added, then every open set around that point contains an element of X by the defined topology, hence the point is the limit point (hence X is not compact), which means the point must be in the closure of X, then also $\omega X \subseteq Cl_{\omega X}(X)$ which means that $Cl_{\omega X}(X) = \omega X$. □

Definition 3.1.5 *Let X be locally compact, and $\infty \notin X$. Let $\omega X = X \cup \{\infty\}$, and define τ^{∞}*

$$U \in \tau^{\infty} \text{ if and only if } U \in \tau \text{ or } U = \{\infty\} \cup (X - K)$$

*where K is some compact subset of X. Then ωX is the **one-point compactification** via the inclusion map.*

Theorem 3.1.6 *A topological space X has a one-point compactification if and only if X is locally compact and Hausdorff.*

Proof: ($\Longrightarrow$) Suppose that X has a one-point compactification:
Let $\omega X = \{\infty\} \cup X$, where $\infty \notin X$, be the one-point compactification of X. Suppose that $x \in U$ and U is an open subset of X, we want to show that there is a compact set K such that $x \in K \subseteq U$. Because ωX is a Hausdorff space, we can find disjoint open sets U and W in ωX such that $x \in V$ and $\infty \in W$. Since $\infty \notin V$, we have $V \subseteq X$ and therefore $V \cap X = V$, which means that V is also open in X. Since $x \in U \cap V$ (which is an open subset of X), we can use the regularity of X to choose an open set G in X for which $x \in G \subseteq Cl_X G \subseteq U \cap V \subseteq U$. We have $G \subseteq V \subseteq \omega X - W$, so $Cl_{\omega X} G \subseteq \omega X - W \subseteq X$ because $\omega X - W$ is closed in ωX. Therefore $Cl_X G = X \cap Cl_{\omega X} G = Cl_{\omega X} G$, meaning that $Cl_X G$ is closed in ωX and thus compact. Then $K = Cl_X G$ is the desired compact set.
($\Longleftarrow$) Suppose that X is locally compact and Hausdorff:
Choose $p \notin X$ and let $\alpha X = \{p\} \cup X$. Put a topology on αX as defined in Proposition 3.1.2 to make αX a topological space. Then by Proposition 3.1.3, αX is a compact Hausdorff space. Also X is densely embedded in αX by Proposition 3.1.4. Hence αX is a compactification of X, and since $\alpha X - X = \{p\}$ which is one point then $\alpha X = \omega X$. This means that X has a one-point compactification. $\square$

In 2013, [2] gave the neccessary and sufficient conditions for a one-point compactification to be a Stone space, which implies that the whole compactification is zero-dimensional. We record this below;

Theorem 3.1.7 *Let X be a topological space. Then the one-point compactification ωX of X is a Stone space if and only if X is a Hausdorff space and the collection of compact clopen sets is a base of X.*

Proof: ($\Longrightarrow$) Since ωX is a Stone space that is Hausdorff, then it is a compact Hausdorff space and hence X is a Hausdorff space by Theorem 3.0.2. ωX has a basis consisting of clopen sets since it is zero-dimensional and hence it has a basis consisting of compact clopen sets since it is compact. Let U be an open set of X, then U is also an open set of ωX. Therefore there exists a collection $\mathcal{U}$ of compact clopen sets of ωX such that $U = \bigcup\{O : O \in \mathcal{U}\}$. But for each $O \in \mathcal{U}$, $O \subseteq X$; so O is a compact clopen set of X. Thus the collection of compact clopen sets is a base of X.
($\Longleftarrow$) We prove that ωX is a Stone space by proving that it is totally disconnected. Let $x \neq y$ be in ωX.
Case 1. $x, y \in X$. Since X is Hausdorff and compact clopen sets form a base for X, there exists a compact clopen set U of X such that $x \in U$ and $y \notin U$. Since U is clopen in X, then it is clopen in ωX.
Case 2. $x \in X$ and $y = \infty$. Since the collection of compact clopen sets is a base of X, there exists a compact clopen set U of X such that $x \in U$. Then ωX is a Stone space, since U is a compact clopen set of ωX. $\square$

We now look at the method of compactification by n points, which was constructed by Magill in 1965. We are going to call this method of compactifying a non-compact locally compact topological space the *Magill compactification.* In [22], Magill gives a characterization of those spaces which have a compactification with n points in the remainder.

Theorem 3.1.8 *Suppose* (X, τ) *is locally compact and contains* n *non-empty open subsets* G_i *for* $i = 1, ..., n$ *which are mutually disjoint such that*

1. $K = X - \bigcup_{i=1}^{n} G_i$ *is compact and*
2. $X - \bigcup_{j \neq i} G_j$ *is not compact for each* $i = 1, ..., n$.

Let $\{\infty_i\}_{i=1}^{n}$ *be points not in* X. *Then the collection*

$$\mathcal{B}^{\rho} = \tau \cup \{H \cup \{\infty_i\} : H \in \tau \text{ and } (K \cup G_i) \cap (X - H) \text{ is compact in } X, i \in \{1, ..., n\}\}$$

is a base for a topology on a compact Hausdorff space ρX *such that* $Cl_{\rho X}(X) = \rho X$. *This* ρX *is what we call* ***Magill compactification****.*

Proof: We show that $\mathcal{B}^{\rho}$ is a base for a topology in ρX. Let $H^i = H \cup \{\infty_i\}$ and let $H_1^i, H_2^i \in \mathcal{B}^{\rho}$, then

$$(K \cup G_i) \cap [X - (H_1 \cap H_2)] = [(K \cup G_i) \cap (X - H_1)] \cup [(K \cup G_i) \cap (X - H_2)].$$

Since the union of compact sets is compact we have that $H_1^i \cap H_2^i \in \mathcal{B}^\rho$, and the other cases follow from the fact that τ is a topological space.
Since $(K \cup G_i) \cap (X - G_i) = K$ for each $i \in \{1, ..., n\}$, then we have that G_i^i and G_j^j are disjoint open sets of ρX containing ∞_i and ∞_j respectively. Now let $x \in X$ and some ∞_i be given. Since X is locally compact, there is an open subset $G^{'}$ and a compact subset $K^{'}$ of X such that $x \in G^{'} \subseteq K^{'}$. Then $(K \cup G_i) \cap K^{'}$, being a closed subset of $K^{'}$, is compact. Therefore $(X - K^{'})^i = (X - K^{'}) \cup \{\infty_i\}$ is an open set containing ∞_i which is disjoint from $G^{'}$. The remaining case follows from the fact that X is Hausdorff, hence ρX is a Hausdorff space.
Let $\mathcal{U} = \{U_1, U_2, U_3, ...\}$ be an open cover for ρX. Then n of these sets $U_1, U_2, ..., U_n$ contain, respectively, sets of the form $H_1^1, H_2^2, ..., H_n^n$ where for each i, H_i is an open set of X and $(K \cup G_i) \cap (X - H_i)$ is compact. Since $G_i \cap G_j = \emptyset$ for $i \neq j$, then we have

$$\bigcap_{i=1}^{n}[G_1 \cup ... \cup G_{i-1} \cup H_i \cup G_{i+1} \cup ...G_n] \subseteq [H_1 \cup H_2 ... \cup H_n].$$

This implies that

$$X - (H_1 \cup H_2 \cup ... \cup H_n) \subseteq \bigcup_{i=1}^{n}[X - (G_1 \cup ... \cup G_{i-1} \cup H_i \cup G_{i+1} \cup ...G_n)]$$

and since

$$H_1^1 \cup H_2^2 \cup ... \cup H_n^n \subseteq U_1 \cup U_2 \cup ... \cup U_n$$

we get

$$\begin{aligned} \rho X - (U_1 \cup U_2 \cup ... \cup U_n) &\subseteq \rho X - (H_1^1 \cup H_2^2 \cup ... \cup H_n^n) \\ &= X - (H_1 \cup H_2 \cup ... \cup H_n) \\ &\subseteq \bigcup_{i=1}^{n}[X - (G_1 \cup ... \cup G_{i-1} \cup H_i \cup G_{i+1} \cup ...G_n)] \\ &= \bigcup_{i=1}^{n}[(K \cup G_i) \cap (X - H_i)] \end{aligned}$$

which is compact. Thus $\rho X - (U_1 \cup U_2 \cup ... \cup U_n)$ is a compact subset of ρX and is therefore covered by a finite subfamily of $\mathcal{U}$. Hence ρX is compact.
It remains to show that X is a dense subset of ρX. Let H^i be a basic set

containing ∞_i. Since $(K \cup G_i) \cap (X - H)$ is compact and $K \cup G_i$ is not compact by assumption, we conclude that $H \neq \emptyset$ (i.e. $X - H \neq X$). Thus $H^i \cap X = H \neq \emptyset$. $\square$

We show that if a space has a compactification with n points in the remainder then it is locally compact and possesses a family of sets with a special property; Magill calls the family of such open sets an N-star of X.

Theorem 3.1.9 *If X has an a compactification with n points in the remainder then X is locally compact and contains n mutually disjoint, open subsets $\{G_i\}_{i=1}^n$ such that*

1. $X - \bigcup_{i=1}^n G_i$ *is compact and*
2. $X - \bigcup_{j\neq i} G_j$ *is not compact for each* $i = 1, ..., n$.

Proof: Let αX be a compactification of X with n points on the remainder. Denote the points in $\alpha X - X$ by $\infty_1, \infty_2, ..., \infty_n$. Since αX is Haudorff, there exists a mutually disjoint family $\{G'\}_{i=1}^n$ of open subsets of αX such that $\infty_i \in G_i^{'}$ for each i. Let $G_i = G_i^{'} - \{\infty_i\}$. Since X is dense in αX then each G_i is a non-empty open subset of X. Let

$$K = X - (G_1 \cup G_2 \cup ... \cup G_n) = \alpha X - (G_1^{'} \cup G_2^{'} \cup ... \cup G_n^{'})$$

then K is a closed subset of αX and is therefore compact. Now

$$\begin{aligned} K \cup G_i =& X - (G_1 \cup ... \cup G_{i-1} \cup G_{i+1} \cup ... \cup G_n) \\ =& [\alpha X - (G_1^{'} \cup ... \cup G_{i-1}^{'} \cup G_{i+1}^{'} \cup ... \cup G_n^{'})] \cap [\alpha X - \{\infty_i\}] \end{aligned}$$

If we assume that $K \cup G_i$ is compact, that implies that it is a closed subset of αX and hence

$$\{\infty_i\} \cup (G_1^{'} \cup ... \cup G_{i-1}^{'} \cup G_{i+1}^{'} \cup ... \cup G_n^{'})$$

is an open subset of αX and the intersection of this set with $G_i^{'}$ is $\{\infty_i\}$ which contradicts the fact that ∞_i is a limit point. Therefore, $K \cup G_i$ is not compact (for each i). Then the sets $\{G_i\}_{i=1}^n$ have the desired property.
To show that X is locally compact, we note that it is an open subset of a compact Hausdorff space, i.e. $X = \alpha X - \{\infty\}_{i=1}^n$ is open in αX. $\square$

3.2 Wallman compactification

The Wallman compactification is a compactification of a T_1-space that was constructed by Wallman in 1938. In [16] O. Frink generalises the compactification precedure of Wallman to obtain Hausdorff compactifications of a T_1-spaces. Frink uses what we call a Wallman base (we follow the terminogy in [32, Problem 19L]) to construct the compactifications.

Definition 3.2.1 *Let $\mathcal{A}$ be any base for the closed sets of X satisfying the following conditions:*

1. *For each closed set F and $x \notin F$, there is some $A \in \mathcal{A}$ such that $x \in A$ and $A \cap F = \emptyset$,*
2. *Finite unions and finite intersections of elements of $\mathcal{A}$ belong to $\mathcal{A}$,*
3. *If $A, B \in \mathcal{A}$ are disjoint, then for some $C, D \in \mathcal{A}$, $A \subseteq X - C$, $B \subseteq X - D$ and $(X - C) \cap (X - D) = \emptyset$.*

*Then $\mathcal{A}$ is called a **Wallman base** for X.*

Definition 3.2.2 *Let X be a topological space and $\mathcal{A}$ be a Wallman base for X. Let $\omega_{\mathcal{A}}X$ be the set of all $\mathcal{A}$-ultrafilters on X and for each $A \in \mathcal{A}$, let*

$$A^* = \{\mathcal{F} \in \omega_{\mathcal{A}}X : A \in \mathcal{F}\}$$

*The sets A^* will be the base for closed sets of $\omega_{\mathcal{A}}X$. We will call $\omega_{\mathcal{A}}X$ with this topology a **Wallman space**.*

Lemma 3.2.3 *The Wallman space $\omega_{\mathcal{A}}X$ has the following properties*

1. *If $A \in \mathcal{A}$, then $\omega_{\mathcal{A}}X - A^* = (X - A)^*$.*
2. *$A^* \cup B^* = (A \cup B)^*$.*
3. *$A^* \cap B^* = (A \cap B)^*$.*

Proof: We use the properties of filters to prove the following;
(1) :

$$\begin{aligned} \mathcal{F} \in \omega_{\mathcal{A}}X - A^* &\iff \mathcal{F} \notin A^* \\ &\iff A \notin \mathcal{F} \\ &\iff X - A \in \mathcal{F} \\ &\iff \mathcal{F} \in (X - A)^* \end{aligned}$$

(2) :

$$\begin{aligned}\mathcal{F} \in (A \cup B)^* &\iff A \cup B \in \mathcal{F}\\ &\iff A \in \mathcal{F} \text{ or } B \in \mathcal{F}\\ &\iff \mathcal{F} \in A^* \text{ or } \mathcal{F} \in B^*\\ &\iff \mathcal{F} \in A^* \cup B^*\end{aligned}$$

(3) :

$$\begin{aligned}\mathcal{F} \in A^* \cap B^* &\iff \mathcal{F} \in A^* \text{ and } \mathcal{F} \in B^*\\ &\iff A \in \mathcal{F} \text{ and } B \in \mathcal{F}\\ &\iff A \cap B \in \mathcal{F}\\ &\iff \mathcal{F} \in (A \cap B)^*.\end{aligned}$$

□

Theorem 3.2.4 *$\omega_{\mathcal{A}}X$ is a compact Hausdorff space.*

Proof: We prove that $\omega_{\mathcal{A}}X$ is compact by showing that any family of closed subsets of $\omega_{\mathcal{A}}X$ with the finite intersection property has a non-empty intersection. Let $\vartheta = \{F^* : i \in I\}$ have a finite intersection property. Let $\mathcal{F} \in \bigcap_{i=1}^{n} F_i^*$ for any finite collection in ϑ, then $F_1, F_2, ..., F_n \in \mathcal{F}$ hence $\bigcap_{i=1}^{n} F_i \neq \emptyset$. That implies that $\{F_i : i \in I\}$ has a finite intersection property. Then there is $\mathcal{G} \in \omega_{\mathcal{A}}X$ such that $F_i \in \mathcal{G}$ for each $i \in I$. Hence we have that $\mathcal{G} \in \bigcap_{i \in I} F_i^*$ as required.
We now show that $\omega_{\mathcal{A}}X$ is a Hausdorff space. Let $\mathcal{F}, \mathcal{G} \in \omega_{\mathcal{A}}X$ be distinct. Then there is a $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that $A \cap B = \emptyset$. By Definition 3.2.1, there are sets $C, D \in \mathcal{A}$ such that $A \subseteq X - C$, $B \subseteq X - D$ and $(X - C) \cap (X - D) = \emptyset$. Hence A and B are contained in disjoint open sets of X. By Lemma 3.2.3, we have $\mathcal{F} \in A^* \subseteq (X - C)^* = \omega_{\mathcal{A}}X - C^*$ and $\mathcal{G} \in B^* \subseteq (X - D)^* = \omega_{\mathcal{A}}X - D^*$. Hence $\omega_{\mathcal{A}}X - C^*$ and $\omega_{\mathcal{A}}X - D^*$ are disjoint open sets of $\omega_{\mathcal{A}}X$ that contain $\mathcal{F}$ and $\mathcal{G}$. □

Theorem 3.2.5 *The mapping $\omega_{\mathcal{A}} : X \longrightarrow \omega_{\mathcal{A}}X$, defined by*

$$\omega_{\mathcal{A}}(x) = \{F \in \mathcal{A} : x \in F\}$$

for every $x \in X$, is an embedding of X densely into $\omega_{\mathcal{A}}X$.

Proof: If $x \neq y$ in X, since X is a T_1-space, there is an open set $U \in X$ such that $x \in U$ and $y \notin U$. Let $F = X - U$, then $F \in \mathcal{A}$ such that $y \in F$ and $x \notin F$. Thus $\omega_{\mathcal{A}}(x) \neq \omega_{\mathcal{A}}(y)$.
We show that $\omega_{\mathcal{A}}$ is continuous by proving that $\omega_{\mathcal{A}}^{-1}(A^*) = A$. Now, $\omega_{\mathcal{A}}(x) \in A^*$ if and only if $A \in \omega_{\mathcal{A}}(x)$ if and only if $x \in A$ as desired. In order that $\omega_{\mathcal{A}}$ be continuous we must have that $\omega_{\mathcal{A}}(S)$ is closed for all closed subsets S of X. Since $\{A^* : A \in \mathcal{A}\}$ is a base for closed sets in $\omega_{\mathcal{A}}X$, $\omega_{\mathcal{A}}(S)$ is closed if and only if $\omega_{\mathcal{A}}(S) = \bigcap_{i \in I} A_i^*$ is closed, or if and only if $S = \bigcap_{i \in I} \omega_{\mathcal{A}}^{-1}(A_i^*) = \bigcap_{i \in I} A_i$. Since $A_i \in \mathcal{A}$, then it is closed and hence S being the intersection of closed sets is itself closed.
To prove that $\omega_{\mathcal{A}}(X)$ is dense in $\omega_{\mathcal{A}}X$, we show that every open set of $\omega_{\mathcal{A}}X$ intersects $\omega_{\mathcal{A}}(X)$. Let $\mathcal{F} \in \omega_{\mathcal{A}}X$ and N be the open set that contains $\mathcal{F}$. Since $\{A^* : A \in \mathcal{A}\}$ is a base for closed sets in $\omega_{\mathcal{A}}X$, there exists $A \in \mathcal{A}$ such that $\mathcal{F} \in \omega_{\mathcal{A}}X - A^* \subseteq N$. Taking $x \in X - A$, we have $A \notin \omega_{\mathcal{A}}(x)$, hence $\omega_{\mathcal{A}}(x) \in \omega_{\mathcal{A}}X - A^*$. This implies that $N \cap \omega_{\mathcal{A}}(X) \neq \emptyset$. □

Remark: Different choices of the Wallman base produce different compactifications. This procedure of contructing a compactification is called Wallman's method of compactifying a space X and the space $\omega_{\mathcal{A}}X$ is called the Wallman compactification of X.

We now prove that every T_1-space that is completely regular has a Wallman base.

Theorem 3.2.6 *A T_1-space is completely regular if and only if it has a Wallman base.*

Proof: ($\Longrightarrow$) We prove that the family of zero-sets is a Wallman base for a completely regular space X, that is, the sets $Z(f) = \{x \in X : f(x) = 0\}$ where f is a continuous real-valued function defined on X. Let F be a closed set such that $x \notin F$, then by definition of a completely regular space there exist a continuous function f such that $f(x) = 0$ and $f(F) = 1$. Therefore, there exist $Z(f)$ such that $x \in Z(f)$ and $F \cap Z(f) = \emptyset$. Hence Definition 3.2.1 (1) is satisfied. Since $Z(f) \cup Z(g) = Z(f \circ g)$ and $Z(f) \cap Z(g) = Z(f^2 + g^2)$, then Definition 3.2.1 (2) is also satisfied. From [17, page 17], we have that if $Z(f) \cap Z(g) = \emptyset$ then $Z(f) \subseteq X - Z(f_1) \subseteq Z(g_1) \subseteq X - Z(g)$. Therefore we have $Z(f) \subseteq X - Z(f_1)$, $Z(g) \subseteq X - Z(g_1)$ and $X - Z(f_1) \cap X - Z(g_1) = \emptyset$ as required for Definition 3.2.1 (3).
($\Longleftarrow$) If X has a Wallman base $\mathcal{A}$, then we can construct the compactifi-

cation $\omega_{\mathcal{A}}X$ by Theorem 3.2.5 which is Hausdorff by Theorem 3.2.4. Then Theorem 3.0.2 says that X must be completely regular. $\square$

In 1967 E.F. Steiner, in [31], gave conditions under which a compactification is a Wallman compactification and also defined a *regular Wallman compactification* which is neccessarily a Wallman compactification. In [4], the concept of regular Wallman compactification is used to show under which circumstances does a compactification with a zero-dimensional remainder, of a locally compact space, become a Wallman compactification. We give details of these discussions.

Definition 3.2.7 *By a* ***ring of sets*** *we will mean a family closed under finite union and finite intersection.*

Definition 3.2.8 *We will call a family $\mathcal{A}$ of closed subsets of X* ***separating*** *if whenever S is a closed subset of X and $x \notin S$ there exist sets $F, G \in \mathcal{A}$ such that $x \in F$, $S \subseteq G$ and $F \cap G = \emptyset$.*

Definition 3.2.9 *Let αX be a compactification of X and $\hat{\mathcal{A}}$ be a family of closed subsets of αX. Then $\hat{\mathcal{A}}$ has the* ***trace property with respect to*** X *if for a finite collection $F_i \in \hat{\mathcal{A}}$ we have that $F = \bigcap_{i=1}^{n} F_i \neq \emptyset$ implies that $F \cap X \neq \emptyset$.*

Theorem 3.2.10 *If the Hausdorff compactification αX possesses a separating ring $\hat{\mathcal{A}}$ with the trace property with respect to a dense subspace X, then αX is a Wallman compactification of X. In fact, αX is homeomorphic to $\omega_{\hat{\mathcal{A}} \cap X}X$.*

Proof: Let αX be a compact Hausdorff space and $\hat{\mathcal{A}}$ be a separating ring of closed sets with the trace property with respect to a dense subspace X. We have that $\hat{\mathcal{A}} \cap X = \{F \cap X : F \in \hat{\mathcal{A}}\}$. Since a compact Hausdorff space is normal and hence $\hat{\mathcal{A}}$ is a separating ring αX, then $\hat{\mathcal{A}} \cap X$ is the Wallman base for X. We want show that the natural mapping $\varphi = \omega_{\hat{\mathcal{A}} \cap X} : X \longrightarrow \omega_{\hat{\mathcal{A}} \cap X}X$ can be extended to a homeomorphism $\hat{\varphi}$ between αX and $\omega_{\hat{\mathcal{A}} \cap X}X$. If $x \in \alpha X - X$, define $\hat{\varphi}$ by

$$\hat{\varphi}(x) = \{F \cap X : x \in F \in \hat{\mathcal{A}}\}$$

We show that $\hat{\varphi}(x) \in \omega_{\hat{\mathcal{A}}}X$ by showing that $\hat{\varphi}(x)$ is maximal with respect to the finite intersection property. Let $F_i \cap X \in \hat{\varphi}(x)$ for $i = 1, 2, ..., n$,

then $x \in \bigcap_{i=1}^{n} F_i$ and since $\hat{\mathcal{A}}$ has the trace property with respect to X, we have $\bigcap_{i=1}^{n} F_i \cap X \neq \emptyset$. Thus $\hat{\varphi}(x)$ has the finite intersection property. Let $H \in \hat{\mathcal{A}}$, suppose that $H \notin \hat{\varphi}(x)$, then $x \notin H$. Since $\hat{\mathcal{A}}$ is a separating ring, we have that there exist F that contains x such that $F \cap H = \emptyset$, thus $(F \cap X) \cap (H \cap X) = \emptyset$. Hence $\hat{\varphi}(x)$ is maximal.
Let $x \neq y$ in αX. Since αX is a T_1-space, $\{y\}$ is closed and $x \notin \{y\}$. Since $\hat{\mathcal{A}}$ is a separating ring, that means that there exists $F \in \hat{\mathcal{A}}$ such that $x \in F$ and $y \notin F$. Therefore there is an F such that $F \in \hat{\varphi}(x)$ and $F \notin \hat{\varphi}(y)$. Hence $\hat{\varphi}(x) \neq \hat{\varphi}(y)$, $\hat{\varphi}$ is one-to-one.
To show that $\hat{\varphi}$ is onto, let $\mathcal{F} \in \omega_{\hat{\mathcal{A}}} X$. Consider $\{F \in \hat{\mathcal{A}} : F \cap X \in \mathcal{F}\}$. This family has the finite intersection property because $\mathcal{F}$ does. Since αX is compact, that means that there exists $x \in \bigcap\{F \in \hat{\mathcal{A}} : F \cap X \in \mathcal{F}\}$. This implies that $\mathcal{F} \subseteq \hat{\varphi}(x)$. Since $\mathcal{F}$ is maximal, $\mathcal{F} = \hat{\varphi}(x)$.
Since $\hat{\varphi}(F) = \{\hat{\varphi}(y) : F \cap X \in \hat{\varphi}(y)\} = (F \cap X)^*$ for all $F \in \hat{\mathcal{A}}$, therefore $\hat{\varphi}(F)$ is closed if and only if F is closed. Hence $\hat{\varphi}$ is a homeomorphism. □

Now we consider the question: which families of sets have the trace property?

Definition 3.2.11 *Let $\mathcal{A}$ be a family of sets in X, then we say that $\mathcal{A}$ is a family of* ***regular closed*** *sets if $A = Cl_X[Int_X(A)]$ for each $A \in \mathcal{A}$.*

Theorem 3.2.12 *If αX possesses a separating ring of regular closed sets $\hat{\mathcal{A}}$, then αX is a Wallman compactification of each of its dense subspace.*

Proof: In view of Theorem 3.2.10, it suffices to show that $\hat{\mathcal{A}}$ has the trace property with respect to any dense subspace. Let X be a dense subspace of αX. Suppose $F_1, F_2 \in \hat{\mathcal{A}}$ and $F_1 \cap F_2 \neq \emptyset$. Since $\hat{\mathcal{A}}$ is a ring of regular closed sets, $F_1 \cap F_2 = Cl_{\alpha X}[Int_{\alpha X}(F_1 \cap F_2)]$. Since X is dense in αX, then $Int_{\alpha X}(F_1 \cap F_2) \cap X \neq \emptyset$, therefore $(F_1 \cap X) \cap (F_2 \cap X) = (F_1 \cap F_2) \cap X = Cl_{\alpha X}[Int_{\alpha X}(F_1 \cap F_2)] \cap X \neq \emptyset$. This shows that $\hat{\mathcal{A}}$ has the trace property with respect to X. □

Definition 3.2.13 *If αX has a separating ring of regular closed sets, we will say that αX is a* ***regular Wallman compactification*** *of each of its dense subspaces.*

Proposition 3.2.14 *Any open subspace X of a regular Wallman space αX possesses a ring of regular closed sets.*

Proof: Let $\hat{\mathcal{A}}$ be a ring of regular closed sets in the compact space αX and X be a open subset of αX. Define $\mathcal{A} = \{F \cap X : F \in \hat{\mathcal{A}}\}$. For $F \in \hat{\mathcal{A}}$, we have

$$\begin{aligned} Cl_X[Int_X(F \cap X)] &= Cl_X[Int_X(F) \cap X] \\ &= Cl_X[Int_X(F)] \\ &= Cl_{\alpha X}[Int_X(F)] \cap X \\ &\subseteq Cl_{\alpha X}[Int_{\alpha X}(F)] \cap X \\ &= F \cap X \end{aligned}$$

For the reverse inclusion, we have

$$\begin{aligned} F \cap X &= Cl_{\alpha X}[Int_{\alpha X}(F)] \cap X \\ &= Cl_{\alpha X}[Int_{\alpha X}(F) \cap X] \cap X \\ &= Cl_X[Int_{\alpha X}(F) \cap X] \\ &\subseteq Cl_X[Int_X(F)] \\ &= Cl_X[Int_X(F) \cap X] \\ &= Cl_X[Int_X(F \cap X)] \end{aligned}$$

Hence, $Cl_X[Int_X(F \cap X)] = F \cap X$. Note that $\mathcal{A}$ is a separating ring since $\hat{\mathcal{A}}$ is a separating ring. Therefore $\mathcal{A}$ is a separating ring of regular closed sets. □

We want to prove the main theorem: if a locally compact space X possesses a separating ring of closed sets then any compactification of X with a zero-dimensional remainder is a regular Wallman compactification. We need new concepts and auxiliary results in place.
We say that a subspace is **relatively compact** if its closure is compact. Let $\mathcal{A}$ be a separating ring of regular closed sets in X and X be locally compact. Define $\mathcal{A}_*$ by

$$\mathcal{A}_* = \{F \in \mathcal{A} : F \text{ is compact or } (X - F) \text{ is relatively compact}\}$$

Then $\mathcal{A}_*$ is a separating ring of regular closed sets in X since $\mathcal{A}$ is a separating ring of regular closed sets. Let ϵX be a compactification, we define a collection $\epsilon\mathcal{A}$ of ϵX in the following manner:
$S \in \epsilon\mathcal{A}$ if and only if there are; $F \in \mathcal{A}_*$, compact $K \subseteq X$ and open subsets

V_1, V_2 of ϵX such that
(1) $F \cap K = \emptyset$,
(2) $\epsilon X - K = V_1 \cup V_2$ and $S = F \cap V_1$ where $V_1 \cap V_2 = \emptyset$.

Lemma 3.2.15 *Let X be a locally compact space, ϵX a compactification of X, and $\mathcal{A}$ a separating ring of regular closed sets in X. Then $\epsilon\mathcal{A}$ is a separating family of sets in X that is closed under finite intersection.*

Proof: We first show that $\epsilon\mathcal{A}$ is closed under finite intersection. Let $S_0, S_1 \in \epsilon\mathcal{A}$. Then by the definition of $\epsilon\mathcal{A}$; for $i = 1, 2$ there exist $F_i \in \mathcal{A}_*$, compact $K_i \subseteq X$ and open $U_i, V_i \subseteq \epsilon X$ such that $\epsilon X - K_i = U_i \cup V_i$, $F_i \cap K_i = \emptyset$, $S_i = F_i \cap U_i$. Then $S_0 \cap S_1 = (F_0 \cap F_1) \cap (U_0 \cap U_1) = F \cap U$. We have that $K = K_0 \cup K_1$ is compact, $F \cap K = \emptyset$ and

$$\begin{aligned} \epsilon X - K =& (\epsilon X - K_0) \cap (\epsilon X - K_1) \\ =& (U_0 \cup V_0) \cap (U_1 \cup V_1) \\ =& (U_0 \cap U_1) \cup (U_0 \cap V_1) \cup (V_0 \cap U_1) \cup (V_0 \cap V_1) \\ =& U \cup V \end{aligned}$$

with $V = \{(U_0 \cap V_1) \cup (V_0 \cap U_1) \cup (V_0 \cap V_1)\}$. Therefore $S_0 \cap S_1 \in \epsilon\mathcal{A}$.
We have that $\epsilon\mathcal{A} \subseteq \mathcal{A}_*$, hence $\epsilon\mathcal{A}$ is separating if $\mathcal{A}_*$ is separating. To prove that $\mathcal{A}_*$ is separating, let $x \in X$ and let G be a closed set in X such that $x \notin G$. Since G is closed and X is locally compact, we can find an open U in X such that $x \in U \subseteq X$, $Cl_X(U) \cap G = \emptyset$ and $Cl_X(U)$ is compact. Now, $\mathcal{A}$ is separating and therefore there exist $F_0, F_1 \in \mathcal{A}$ such that $x \in F_0$, $X - U \subseteq F_1$ and $F_0 \cap F_1 = \emptyset$. We have that $Cl_X(X - F_1) \subseteq Cl_X(U)$ and thus the closure of $X - F_1$ is compact, which implies that $F_1 \in \mathcal{A}_*$. Also, $F_0 \in \mathcal{A}_*$ since it is closed with a compact neighborhood. Since $G \subseteq X - Cl_X(U) \subseteq X - U \subseteq F_1$ and $x \in F_0$, then $\mathcal{A}_*$ is separating. $\square$

Lemma 3.2.16 *Let ϵX be a compactification of X and F be a regular closed subset of X. If $X - F$ is relatively compact, then*

$$Cl_{\epsilon X}(F) = F \cup (\epsilon X - X)$$

Proof:

$$\begin{aligned} Cl_{\epsilon X}(F) =& [Cl_{\epsilon X}(F) \cap X] \cup [Cl_{\epsilon X}(F) \cap (\epsilon X - X)] \\ =& Cl_X(F) \cup [Cl_{\epsilon X}(F) \cap (\epsilon X - X)] \\ =& F \cup [Cl_{\epsilon X}(F) \cap (\epsilon X - X)] \\ \subseteq& F \cup (\epsilon X - X) \end{aligned}$$

To prove the reverse inclusion, let $y \in F \cup (\epsilon X - X)$. If $y \in F$, then $y \in Cl_{\epsilon X}(F)$. Suppose that $y \in \epsilon X - X$, since $X - F$ is relatively compact then $Cl_X(X - F) = Cl_{\epsilon X}(X - F)$, hence we have

$$\begin{aligned}
&\Longrightarrow y \notin X \\
&\Longrightarrow y \notin Cl_X(X - F) \\
&\Longrightarrow y \notin Cl_{\epsilon X}(X - F) \\
&\Longrightarrow y \in \epsilon X - Cl_{\epsilon X}(X - F) \\
&\Longrightarrow y \in Cl_{\epsilon X}[\epsilon X - Cl_{\epsilon X}(X - F)]
\end{aligned}$$

And since $[\epsilon X - Cl_{\epsilon X}(X - F)] \cap X = Int_X(F)$, then

$$Cl_{\epsilon X}[\epsilon X - Cl_{\epsilon X}(X - F)] = Cl_{\epsilon X}[Int_X(F)] \subseteq Cl_{\epsilon X}(F).$$

Therefore $y \in Cl_{\epsilon X}(F)$ as required. $\square$

We are now ready to prove the main result:

Theorem 3.2.17 *Let X be a locally compact space. Then X possesses a separating ring of regular closed sets if and only if any compactification ϵX of X with a zero-dimensional remainder is a regular Wallman compactification.*

Proof: ($\Longleftarrow$) Suppose that ϵX is a regular Wallman compactification, then by Definition 3.2.13 ϵX possesses a separating ring of regular closed sets. Then, since X is locally compact and therefore X is open in ϵX, we have that X possesses a separating ring of regular closed sets by Proposition 3.2.14.
($\Longrightarrow$) Let $\mathcal{A}$ be a separating ring in X and let $\mathcal{L} = \{Cl_{\epsilon X}(S) : S \in \epsilon\mathcal{A}\}$. We first show that $\mathcal{L}$ is closed under intersection.
Claim : For $S \in \epsilon\mathcal{A}$, we have

$$Cl_{\epsilon X}(S) = S \quad \text{or} \quad Cl_{\epsilon X}(S) = S \cup [V \cap (\epsilon X - X)].$$

Let $F \in \mathcal{A}_*$ and let K be a compact subset of X such that $\epsilon X - K = V \cup U$, $F \cap K = \emptyset$ and $S = F \cap V$. If F is compact, then also S is compact, therefore $Cl_{\epsilon X}(S) = S$. Suppose that $X - F$ is relatively compact, then we have that $Cl_{\epsilon X}(F) = F \cup (\epsilon X - X)$ by Lemma 3.2.16.

$$\begin{aligned}
Cl_{\epsilon X}(S) &= Cl_{\epsilon X}(F \cap V) \\
&\subseteq [F \cup (\epsilon X - X)] \cap Cl_{\epsilon X}(V) \\
&\subseteq [F \cup (\epsilon X - X)] \cap (V \cup K) \\
&= (F \cap V) \cup [(\epsilon X - X) \cap V] \\
&= S \cup [(\epsilon X - X) \cap V]
\end{aligned}$$

Also, we have

$$\begin{aligned} S \cup [(\epsilon X - X) \cap V] =&(F \cap V) \cup [(\epsilon X - X) \cap V] \\ =&V \cap [F \cup (\epsilon X - X)] \\ =&V \cap Cl_{\epsilon X}(F) \\ \subseteq&Cl_{\epsilon X}(V \cap F) \\ =&Cl_{\epsilon X}(S) \end{aligned}$$

Therefore $Cl_{\epsilon X}(S) = S \cup [(\epsilon X - X) \cap V]$ which completes the proof for our claim.
We prove that $\mathcal{L}$ is closed under intersection by showing that; for all $S_0, S_1 \in \epsilon\mathcal{A}$ we have $Cl_{\epsilon X}(S_0) \cap Cl_{\epsilon X}(S_1) = Cl_{\epsilon X}(S_0 \cap S_1)$ and the result follows by Lemma 3.2.15 since $\epsilon\mathcal{A}$ is closed under intersection. For $i \in \{0, 1\}$, let K_i be a compact subset of X, $F_i \in \mathcal{A}_*$ and U_i, V_i be open subsets of ϵX such that $S_i = F_i \cap V_i$, while $\epsilon X - K_i = V_i \cup U_i$ and $F_i \cap K_i = \emptyset$. If S_0 is compact and S_1 is not compact then

$$\begin{aligned} Cl_{\epsilon X}(S_0) \cap Cl_{\epsilon X}(S_1) =&S_0 \cap \{S_1 \cup [(\epsilon X - X) \cap V_1]\} \\ =&(S_0 \cap S_1) \cup [S_0 \cap (\epsilon X - X) \cap V_1] \\ =&(S_0 \cap S_1) \\ =&Cl_{\epsilon X}(S_0 \cap S_1) \end{aligned}$$

since $S_0 \cap S_1$ is compact. Now suppose that neither is compact, then we have

$$\begin{aligned} Cl_{\epsilon X}(S_0) \cap Cl_{\epsilon X}(S_1) =&\{S_0 \cup [V_0 \cap (\epsilon X - X)]\} \cap \{S_1 \cup [V_1 \cap (\epsilon X - X)]\} \\ =&(S_0 \cap S_1) \cup [(\epsilon X - X) \cap V_0 \cap V_1] \end{aligned}$$

and since $S_0 \cap S_1 = (F_0 \cap F_1) \cap (U_0 \cap U_1)$ where $X - (F_0 \cap F_1)$ is relatively compact, we get that $Cl_{\epsilon X}(S_0) \cap Cl_{\epsilon X}(S_1) = Cl_{\epsilon X}(S_0) \cap Cl_{\epsilon X}(S_1)$.
Supplimenting $\mathcal{L}$ with all finite unions of its elements we get the ring of regular closed sets $\mathcal{T}$; since the finite union of finitely many regular closed sets is again regular closed sets.
Cliam: $\mathcal{T}$ is separating.
Let $x \in \epsilon X$ and let G be a closed set of ϵX such that $x \notin G$.
If $x \in X$, then since X is regular (a completely regular space is regular) and locally compact, there exist $T_0, T_1 \in \mathcal{T}$ such that $x \in T_0 = Cl_{\epsilon X}(T_0)$ and $G \subseteq Cl_{\epsilon X}(T_1)$ with $Cl_{\epsilon X}(T_0) \cap Cl_{\epsilon X}(T_1) = \emptyset$. So we may assume that $x \in \epsilon X - X$. Since $\epsilon X - X$ is zero-dimensional, it possesses a base of clopen

sets. Let C be a clopen subset of $\epsilon X - X$ such that $x \in C$ and $C \cap G = \emptyset$, of which we can find since G is closed. Define $C_0 = (\epsilon X - X) - C$. Then C and C_0 are disjoint closed subsets in ϵX such that $C_0 \cup C = \epsilon X - X$. Therefore $C_0 \cup G$ and C are disjoint closed sets also. As ϵX is normal, there exist open $U_0, U_1 \subseteq \epsilon X$ such that $C_0 \cup G \subseteq U_0$, $C \subseteq U_1$ and $U_0 \cap U_1 = \emptyset$. Then $K = \epsilon X - (U_0 \cup U_1)$ is a compact subset of X such that $K \cap G = \emptyset$. Choose a relatively compact, open O in X such that $K \subseteq O \subseteq Cl_X(O)$ and $Cl_X(O) \cap (G \cap X) = \emptyset$. Since $\mathcal{A}_*$ is separating, we have $X - O = \bigcap\{F \in \mathcal{A}_* : X - O \subseteq F\}$. Then, by compactness of K, there exist an $F \in \mathcal{A}_*$ such that $X - O \subseteq F$ and $F \cap K = \emptyset$. Define $S_0 = F \cap U_0$ and $S_1 = F \cap U_1$. Then it follows that $x \in Cl_{\epsilon X}(S_1)$ and $G \subseteq Cl_{\epsilon X}(S_0)$, while $Cl_{\epsilon X}(S_0) \cap Cl_{\epsilon X}(S_1) = \emptyset$ (since $S_0 \cap S_1 = \emptyset$). $\square$

3.3 Fan-Gottesman compactification

The Fan-Gottesman compactification was constructed by Ky Fan and Noel Gottesman. In 1952, they constructed the compactification for a regular space which has a normal base. In this section we go through their construction.

Definition 3.3.1 *Let $\mathcal{B}$ be a base for the space X, $\mathcal{B}$ is called the* ***normal base*** *if it has the following properties;*

1. *$A, B \in \mathcal{B}$ implies $A \cap B \in \mathcal{B}$.*
2. *$A \in \mathcal{B}$ implies $X - Cl_X(A) \in \mathcal{B}$.*
3. *For any open set U of X and any $A \in \mathcal{B}$ such that $Cl_X(A) \subseteq U$, there exists a $B \in \mathcal{B}$ such that $Cl_X(A) \subseteq B \subseteq Cl_X(B) \subseteq U$*

Following the terminology used in [14], a **binding family** on $\mathcal{B}$ is a non-empty family of sets of $\mathcal{B}$ such that

$$\bigcap_{i=1}^{m} Cl_X(A_i) \neq \emptyset$$

for any finite number of sets A_i of the family. By Zorn's lemma, every binding family on $\mathcal{B}$ is contained in at least one **maximal binding family** on $\mathcal{B}$.

Maximal binding families on $\mathcal{B}$ is denoted by letters $x^*, y^*, ...$
By μX, we denote the set of all maximal binding families on $\mathcal{B}$.
For each $A \in \mathcal{B}$, we define

$$A^* = \{x^* \in \mu X : \text{there exists a } B \in x^* \text{ with } Cl_X(B) \subseteq A\}$$

And let $\mathcal{B}^\mu = \{A^* : A \in \mathcal{B}\}$. We show that $\mathcal{B}^\mu$ is closed under intersection, and hence is a basis for μX.

Proposition 3.3.2 *For any two sets $A, B \in \mathcal{B}$, we have $(A \cap B)^* = A^* \cap B^*$.*

Proof: Let $x^* \in (A \cap B)^*$. Then there exists a $C \in x^*$ such that $Cl_X(C) \subseteq A \cap B$, then $Cl_X(C) \subseteq A$ and $Cl_X(C) \subseteq B$ which implies $x^* \in A^*$ and $x^* \in B^*$. Now we prove that $A^* \cap B^* \subseteq (A \cap B)^*$. Let $x^* \in A^* \cap B^*$. Then we have $C \in x^*$, $D \in x^*$ such that $Cl_X(C) \subseteq A$, $Cl_X(D) \subseteq B$. By Definition 3.3.1 (3), there exist $E, F \in \mathcal{B}$ such that $Cl_X(C) \subseteq E \subseteq Cl_X(E) \subseteq A$, $Cl_X(D) \subseteq F \subseteq Cl_X(F) \subseteq B$. From $Cl_X(C) \cap Cl_X(D) \subseteq E \cap F \in \mathcal{B}$ and $C \in x^*$, $D \in x^*$, we get $E \cap F \in x^*$ by maximality of x^*. Which together with $Cl_X(E \cap F) \subseteq A \cap B$ yields $x^* \in (A \cap B)^*$. $\square$

This means that $\mathcal{B}^\mu$ can be taken as a base for a topology on μX. Now we want to show that μX is indeed the compactification for X, but we first prove some auxilary results that we need for the proof.

Lemma 3.3.3 *If an open set U of X and finite number of sets $A_i \in \mathcal{B}$ $(i \leqslant i \leqslant n)$ satisfy*

$$Cl_X(A_1) \cap Cl_X(A_2) \cap ... \cap Cl_X(A_n) \subseteq U,$$

then there exists $B \in \mathcal{B}$ such that

$$Cl_X(A_1) \cap Cl_X(A_2) \cap ... \cap Cl_X(A_n) \subseteq B \subseteq Cl_X(B) \subseteq U.$$

Proof: For $i = 1$, this is true by Definition 3.3.1 (3). We suppose that it is true for $i = n - 1$ and prove by induction. The hypothesis may be written

$$Cl_X(A_1) \cap Cl_X(A_2) \cap ... \cap Cl_X(A_{n-1}) \subseteq U \cup (X - Cl_X(A_n)).$$

By our assumption of induction, there exists $C \in \mathcal{B}$ such that

$$Cl_X(A_1) \cap Cl_X(A_2) \cap ... \cap Cl_X(A_{n-1}) \subseteq C \subseteq Cl_X(C) \subseteq U \cup (X - Cl_X(A_n)),$$

which implies that $Cl_X(A_n) \subseteq U \cup (X - Cl_X(C))$. By Definition 3.3.1 (3), there is a $D \in \mathcal{B}$ such that

$$Cl_X(A_n) \subseteq D \subseteq Cl_X(D) \subseteq U \cup (X - Cl_X(C)).$$

Since $\bigcap_{i=1}^{n} Cl_X(A_i) \subseteq C \cap Cl_X(A_n)$, then $\bigcap_{i=1}^{n} Cl_X(A_i) \subseteq C \cap D$. Which implies that $B = C \cap D$ has the required property. $\square$

Lemma 3.3.4 *For any two sets* $A, B \in \mathcal{B}$*, we have* $A^* \cap B^* = \emptyset$ *if and only if* $A \cap B = \emptyset$.

Proof: We show that $A^* = \emptyset$ if and only if $A = \emptyset$ and the result follows by Proposition 3.3.2. Since $\emptyset^* = \emptyset$, we have that $A = \emptyset$ implies $A^* = \emptyset$. Suppose that $A \neq \emptyset$, then since X is regular, there is $B \in \mathcal{B}$ such that $B \neq \emptyset$ and $Cl_X(B) \subseteq A$. Since $Cl_X(B) \cap Cl_X(A) \neq \emptyset$ then B is contained in a binding family, by Zorn's lemma there exists $x^* \in \mu X$ such that $B \in x^*$. Hence $x^* \in A^* \neq \emptyset$. $\square$

Lemma 3.3.5 *Let* $A \in \mathcal{B}$ *and* $x^* \in \mu X$*. If* $A \notin x^*$*, then* $X - Cl_X(A) \in x^*$*, and there exists a* $B \in x^*$ *such that* $Cl_X(A) \cap Cl_X(B) = \emptyset$.

Proof: If $A \notin x^*$, then there exists a finite number of sets $A_i \in x^*$ $(1 \leqslant i \leqslant n)$ such that

$$Cl_X(A) \cap Cl_X(A_1) \cap Cl_X(A_2) \cap ... \cap Cl_X(A_n) = \emptyset.$$

Hence, we have that

$$Cl_X(A_1) \cap Cl_X(A_2) \cap ... \cap Cl_X(A_n) \subseteq X - Cl_X(A).$$

By Lemma 3.3.3, there exists $B \in \mathcal{B}$ such that

$$Cl_X(A_1) \cap Cl_X(A_2) \cap ... \cap Cl_X(A_n) \subseteq B \subseteq Cl_X(B) \subseteq X - Cl_X(A)$$

By maximality of x^*, the facts that $B \in \mathcal{B}$, $A_i \in x^*$ $(1 \leqslant i \leqslant n)$ and $\bigcap_{i=1}^{n} A_i \subseteq B$ imply that $B \in x^*$. Hence the existence of a $B \in x^*$ with $Cl_X(A) \cap Cl_X(B) = \emptyset$ is proved. From $B \in x^*$ and $B \subseteq X - Cl_X(A)$, we get that $X - Cl_X(A) \in x^*$. $\square$

Theorem 3.3.6 μX *is a Hausdorff space.*

Proof: Let $x^*, y^* \in \mu X$ and $x^* \neq y^*$. Then there is an $A \in x^*$ such that $A \notin y^*$. By Lemma 3.3.5, there is a $B \in y^*$ such that $Cl_X(A) \cap Cl_X(B) = \emptyset$. Then $Cl_X(A) \subseteq X - Cl_X(B)$. By Definition 3.3.1 (3), there is a $C \in \mathcal{B}$ such that $Cl_X(A) \subseteq C \subseteq Cl_X(C) \subseteq X - Cl_X(B)$. Let $D = X - Cl_X(C)$, then $Cl_X(B) \subseteq D$ and $D \in \mathcal{B}$ by Definition 3.3.1 (2). Now $A \in x^*$ and $Cl_X(A) \subseteq C$ imply $x^* \in C^*$. Similarly $B \in y^*$ and $Cl_X(B) \subseteq D$ imply $y^* \in D^*$. Since $C \cap D = \emptyset$, then by Lemma 3.3.4 we have that $C^* \cap D^* = \emptyset$. Hence C^* and D^* are disjoint neighborhoods of x^* and y^*. $\square$

To prove that μX is compact, we need another base for μX.

Lemma 3.3.7 *For $x^* \in \mu X$ and $A \in \mathcal{B}$, we have $x^* \in Cl_{\mu X}(A^*)$ if and only if $A \in x^*$.*

Proof: We first prove that $A \in x^*$ implies $x^* \in Cl_{\mu X}(A^*)$ by proving that every neighborhood of x^* has a non-empty intersection with A^*. Since $\mathcal{B}^\mu$ is a basis for μX, let B^* be an arbitrary neighborhood of x^* with $B \in \mathcal{B}$. From $x^* \in B^*$, we get that there is $C \in x^*$ with $Cl_X(C) \in B$. Since $A \in x^*$, we have $Cl_X(A) \cap Cl_X(C) \neq \emptyset$ and therefore $Cl_X(A) \cap B \neq \emptyset$. Hence $A \cap B \neq \emptyset$ and by Lemma 3.3.4 we have that $A^* \cap B^* \neq \emptyset$ as required.
Now assume that $A \notin x^*$. By Lemma 3.3.5, this implies that there exists $B \in x^*$ such that $Cl_X(A) \cap Cl_X(B) = \emptyset$. Then $Cl_X(B) \subseteq X - Cl_X(A)$. By Definition 3.3.1 (3), there is a $C \in \mathcal{B}$ such that $Cl_X(B) \subseteq C \subseteq Cl_X(C) \subseteq X - Cl_X(A)$. Since $B \in x^*$ and $Cl_X(B) \subseteq C$, then we have $x^* \in C^*$. But, from $A \cap C = \emptyset$, we have that $A^* \cap C^* = \emptyset$ by Lemma 3.3.4. Hence $x^* \notin Cl_{\mu X}(A^*)$. $\square$

Lemma 3.3.8 *$\mathcal{C}^* = \{\mu X - Cl_{\mu X}(A^*) : A \in \mathcal{B}\}$ is a base of μX.*

Proof: Given $x^* \in \mu X$ and $B \in \mathcal{B}$ such that $x^* \in B^*$, we are to find an $A \in \mathcal{B}$ such that

$$x^* \in \mu X - Cl_{\mu X}(A^*) \subseteq B^*$$

Since $x^* \in B^*$, there is $C \in x^*$ with $Cl_X(C) \subseteq B$. By Definition 3.3.1 (3), there is $D \in \mathcal{B}$ such that $Cl_X(C) \subseteq D \subseteq Cl_X(D) \subseteq B$. Let $A = X - Cl_X(D)$. We claim that A satisfies the above condition. From the fact that $C \in x^*$ and $Cl_X(A) \cap Cl_X(C) = \emptyset$, by Lemma 3.3.5 we have that $A \notin x^*$ and by Lemma 3.3.7 we get that $x^* \in \mu X - Cl_{\mu X}(A^*)$. Now we show that $\mu X - Cl_{\mu X}(A^*) \subseteq B^*$. Consider any point $y^* \in \mu X - Cl_{\mu X}(A^*)$. By Lemma

3.3.7, we have $A \notin y^*$ and therefore, by Lemma 3.3.5, $X - Cl_X(A) \in y^*$. This together with $Cl_X[X - Cl_X(A)] \subseteq Cl_X(D) \subseteq B$ implies that $y^* \in B^*$. $\square$

Theorem 3.3.9 *μX is compact.*

Proof: Assume that μX is not compact. Suppose that $\mathcal{U}^*$ is an open covering of μX without a finite subcovering. By Lemma 3.3.8, for every $x^* \in \mu X$, we can find a set $U^*_{x^*} \in \mathcal{U}^*$ and a set $A_{x^*} \in \mathcal{B}$ such that

$$x^* \in \mu X - Cl_{\mu X}(A^*_{x^*}) \subseteq U^*_{x^*}$$

Consider now any finite number of points $x^*_1, x^*_2, ..., x^*_n$ of μX. Since $\mathcal{U}^*$ does not have a finite cover for μX, we have

$$\bigcup_{i=1}^{n} [\mu X - Cl_{\mu X}(A^*_{x^*_i})] \neq \mu X$$

or, equivalently

$$\bigcap_{i=1}^{n} Cl_{\mu X}(A^*_{x^*_i}) \neq \emptyset.$$

If y^* is a point in this intersection, then $y^* \in Cl_{\mu X}(A^*_{x^*_i})$ for $i = 1, 2, ..., n$ which, by Lemma 3.3.7, implies that $A_{x^*_i} \in y^*$ $(1 \leqslant i \leqslant n)$. Since y^* is a binding family, we have

$$\bigcap_{i=1}^{n} Cl_X(A_{x^*_i}) \neq \emptyset.$$

And this last relation holds for any finite number of points x^*_i of μX. Hence $\{A_{x^*} : x^* \in \mu X\}$ is a binding family on $\mathcal{B}$, therefore it is contained in a maximal binding family on $\mathcal{B}$. In other words, there exists a point $a^* \in \mu X$ such that $A_{x^*} \in a^*$ for every $x^* \in \mu X$. In particular, we have $A_{a^*} \in a^*$ and by Lemma 3.3.7 we get that $a^* \in Cl_{\mu X}(A^*_{a^*})$. This contradicts that $a^* \in \mu X - Cl_{\mu X}(A^*_{a^*})$. $\square$

Theorem 3.3.10 *The mapping $\rho : X \longrightarrow \mu X$ defined by*

$$\rho(x) = \{A \in \mathcal{B} : x \in Cl_X(A)\}$$

for every $x \in X$, is an embedding of X densely into μX.

Proof: Let $x, y \in X$ and $x \neq y$. Since X is regular T_1, there exist $B \in \rho(x)$ and $C \in \rho(y)$ such that $Cl_X(B) \cap Cl_X(C) = \emptyset$. This implies that $\rho(x) \neq \rho(y)$, hence ρ is one-to-one.
Now we show that $\rho^{-1}(A^*) = A$ for every $A \in \mathcal{B}$, which would imply that ρ and ρ^{-1} are continuous. Since X is regular, $x \in A$ is equivalent to the existance of a $B \in \mathcal{B}$ such that $x \in Cl_X(B) \subseteq A$, i.e. the existence of a $B \in \rho(x)$ such that $Cl_X(B) \subseteq A$. Therefore $x \in A$ if and only if $\rho(x) \in A^*$ as required.
To show that $Cl_{\mu X}[\rho(X)] = \mu X$, we need to show that the intersection of any open set of μX with $\rho(X)$ is non-empty. We use the fact that $\rho^{-1}(A^*) = A$ and we get that $U^* \cap \rho(X) = \rho(U) \cap \rho(X) = \rho(U) \neq \emptyset$ for every non-empty $U \in \mathcal{B}$. Hence we can say that $\rho(X)$, and hence X, is dense in μX. □

Remark: We can now say that μX together with the base $\mathcal{B}^\mu$ for the topology, is a compactification of X and it is called the Fan-Gottesman compactification of X.

In 2019, C.S. Elmali and T. Ugur, defined the Fan-Gottesman compactification using closed ultrafilters on X. In their paper [12] they give necessary and sufficent conditions for Fan-Gottesman compactification of a T_3 space to be a Stone space. Since a Stone space is zero-dimensional, that implies that the compactification has a remainder that is zero-dimensional. The following definition of the Fan-Gottesman compactification is taken from their paper.

Definition 3.3.11 *Let X be a T_3-space and σX be the subcollection of all closed ultrafilters on X. For each open set $U \subseteq X$, define $U^* \subseteq \sigma X$ to be the set*

$$U^* = \{\mathcal{G} \in \sigma X : Cl_X(U) \in \mathcal{G}\}.$$

Then $\{U^* : U \text{ is open subset of } X\}$ is a base for open sets of the topology on σX, and σX is called the ***Fan-Gottesman compactification*** *of X.*
On the other hand, for each closed set $D \subseteq X$, we define $D^* \subseteq \sigma X$ by

$$D^* = \{\mathcal{G} \in \sigma X : G \subseteq D \text{ for some } G \in \mathcal{G}\}$$

The natural map $\sigma : X \longrightarrow \sigma X$ is defined by

$$\sigma(x) = \mathcal{G}_x$$

where $\mathcal{G}_x$ is the closed ultrafilter converging to $x \in X$.

The following properties, which are proved in a similar manner as in Lemma 3.2.3, are useful in our proof.
(1) If $U \subseteq X$ is open, then $\sigma X - U^* = (X - U)^*$.
(2) If $D \subseteq X$ is closed, then $\sigma X - D^* = (X - D)^*$.
(3) If U_1 and U_2 are open in X, then $(U_1 \cap U_2)^* = U_1^* \cap U_2^*$.
(4) If U_1 and U_2 are open in X, then $(U_1 \cup U_2)^* = U_1^* \cup U_2^*$.
(5) If U is open in X, then $Cl_{\sigma X}(\sigma(U)) = U^*$.

Lemma 3.3.12 *If Q is a clopen set of σX, then there exists a clopen set C of X such that $Q = C^*$.*

Proof: Since Q is open and $\{U^* : U \text{ is open subset of } X\}$ is a base for the topology of σX, then $Q = \bigcup[U^* : U \in \vartheta]$ where ϑ is a collection of open sets of X. We also have that Q is closed in σX, hence Q is a compact set of σX. Therefore there exists a finite subcollection ϑ' of ϑ such that $Q = \bigcup[U^* : U \in \vartheta']$. Let $C = \bigcup[U : U \in \vartheta']$, then by property no. 4 above we have that $C^* = Q$. Since $C = Q \cap X$, then it is a clopen set of X. $\square$

Theorem 3.3.13 *Let X be a T_3-space. Then the Fan-Gottesman compactification σX is a Stone space if and only if for each disjoint closed sets D_1 and D_2 in X, there exists a clopen set C such that $D_1 \subseteq C$ and $D_2 \cap C = \emptyset$.*

Proof: $\Longrightarrow$ Let D_1 and D_2 be two disjoint closed sets of X. Then D_1^* and D_2^* are two disjoint closed sets of σX. Let $\mathcal{D}_1 \in D_1^*$ and $\mathcal{D}_2 \in D_2^*$. Since σX is a Stone space and by Lemma 3.3.12, for each $\mathcal{D}_1 \in D_1^*$ there exists a clopen set V of X such that $\mathcal{D}_1 \in V^*$ and $\mathcal{D}_2 \notin V^*$. Taking the collection of such V^*'s, we have that $D_1^* \subseteq \bigcup_{i \in I} V_i^*$ and $\mathcal{D}_2 \notin \bigcup_{i \in I} V_i^*$. Since D_1^* is closed in σX, it is a compact set. Hence there exist a finite n such that $D_1^* \subseteq \bigcup_{i=1}^n V_i^*$ and $\mathcal{D}_2 \notin \bigcup_{i=1}^n V_i^*$. Set $F_{\mathcal{D}_2} = \bigcap_{i=1}^n (\sigma X - V_i^*)$. Since σX is a topological space and we are taking a finite intersection; $F_{\mathcal{D}_2}$ is a clopen set of σX such that $\mathcal{D}_2 \in F_{\mathcal{D}_2}$ and $F_{\mathcal{D}_2} \cap D_1^* = \emptyset$. Let $\vartheta = \{F_{\mathcal{D}_2} : \mathcal{D}_2 \in D_2^*\}$. Then $D_2^* \subseteq \bigcup[F_{\mathcal{D}_2} : F_{\mathcal{D}_2} \in \vartheta]$. Since D_2^* is compact in σX, there is a finite ϑ' such that $D_2^* \subseteq \bigcup[F_{\mathcal{D}_2} : F_{\mathcal{D}_2} \in \vartheta']$. Let $Q = \bigcup[F_{\mathcal{D}_2} : F_{\mathcal{D}_2} \in \vartheta']$. Since we are taking a finite union then Q is clopen in σX and we also have $D_1^* \subseteq \sigma X - Q$, $D_2^* \subseteq Q$. $\sigma X - Q$ is hence clopen in σX, thus by Lemma 3.3.12 we have that there exists a clopen set C in X such that $C^* = \sigma X - Q$. By property no. 5 above we have that $D_1^* \subseteq C^*$ if and only if $D_1 \subseteq C$, similarly $D_2^* \subseteq \sigma X - C = (X - C)^*$ if and only if $D_2 \subseteq X - C$. Hence, we have a clopen set C of X such that $D_1 \subseteq C$ and $D_2 \cap C = \emptyset$.

($\Longleftarrow$) We will prove that σX is totally disconnected since a Stone space is a totally disconnected compact space. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be distinct elements of σX. Then there exists closed sets D_1 and D_2 of X such that $D_1 \in \mathcal{D}_1$, $D_2 \in \mathcal{D}_2$ and $D_1 \cap D_2 = \emptyset$. Since D_1 and D_2 are disjoint, by assumption, there exist a clopen C of X such that $D_1 \subseteq C$ and $D_2 \cap C = \emptyset$. Thus $D_1^* \subseteq C^*$ and $D_2^* \cap C^* = \emptyset$. Since $D_1 \in \mathcal{D}_1$ and $D_2 \in \mathcal{D}_2$, by Definition 3.3.11 we thus have $\mathcal{D}_1 \in D_1^*$ and $\mathcal{D}_2 \in D_2^*$. So C^* is a clopen set of σX such that $\mathcal{D}_1 \in C^*$ and $\mathcal{D}_2 \notin C^*$. □

3.4 Stone-Čech compactification

Stone-Čech compactification, which was defined by Marshall Stone (1937) and Eduard Čech (1937) separately, is the process of constructing a map from a topological space X to the largest, compact Hausdorff space that contains X as a dense subset. We look at the process as outlined in [32].

Definition 3.4.1 *Let $f \in C^*(X)$, with closed and bounded range $I_f \subseteq \mathbb{R}$. Then these functions separate points in X (since X is Tychonoff), and thus the evaluation map defined by*

$$e : X \longrightarrow \prod_{f \in C^*(X)} I_f$$

where

$$[e(x)]_f = f(x)$$

*is an embedding. The closure of $e(X)$ is what is called the **Stone-Čech compactification** of X, denoted by βX.*

The product space, $\prod I_f$, is a compact Hausdorff space since it is the product of closed and bounded intervals, therefore X is embedded into a compact Hausdorff space. In order to show that the Stone-Čech compactification is the largest compactification (on X), we need to have some kind of ordering system in place for the family of compactifications that X can have, hence the next definition.

Definition 3.4.2 *Let $\mathcal{K}(X)$ denote the family of compactifications of X and let $(\alpha_1 X, \alpha_1), (\alpha_2 X, \alpha_2) \in K(X)$ be arbitrary. We say that $(\alpha_1 X, \alpha_1) \leqslant$*

$(\alpha_2 X, \alpha_2)$ if and only if there exists a continuous function $F : (\alpha_2 X, \alpha_2) \longrightarrow (\alpha_1 X, \alpha_1)$ such that $F \circ \alpha_1 = \alpha_2$.

Proposition 3.4.3 *Every continuous function from X to a compact space can be extended to βX, that is to say, if αX is a compact Hausdorff space and $\alpha : X \longrightarrow \alpha X$ is continuous then there is a continuous function $F : (\beta X, e) \longrightarrow (\alpha X, \alpha)$ such that $F \circ e = \alpha$.*

Proof: Note that $(\alpha X, \alpha)$ is a compact Hausdorff space, hence a Tychonoff space, which means that it can be embedded by the evaluation map into the product space by Theorem 2.6.3 (6),

$$e^{'} : \alpha X \longrightarrow \prod_{g \in C^*(\alpha X)} I_g,$$

We can define a map

$$H : \prod_{f \in C^*(X)} I_f \longrightarrow \prod_{g \in C^*(\alpha X)} I_g$$

by

$$[H(t)]_g = (g \circ \alpha)(t) \quad \text{for all } \ t \in \prod_{f \in C^*(X)} I_f$$

$(g \circ \alpha) \in C^*(X)$ for each $g \in C^*(\alpha X)$, therefore H is a continuous function. For $t \in e(X)$, then there exists $x \in X$ such that $t = e(x)$

$$[H(t)]_g = (g \circ \alpha)(t) = g(\alpha(t)) = g(\alpha(e(x))) = g(\alpha(x)) = [e^{'}(\alpha(x))]_g$$

Therefore, since $\alpha(x) \in \alpha X$ then $H(e(X)) \subseteq e^{'}(\alpha X)...(***)$.
If F were to be defined such

$$F : \beta X \longrightarrow \alpha X$$

with $F = e^{'-1} \circ (H|\beta X)$, then it must be that $e^{'-1}(H(\beta X)) \subseteq \alpha X$; since $e(X)$ is dense in βX, then $H(e(X))$ is dense in $H(\beta X)$ because H is a continuous map, also $e^{'}(\alpha X)$ is close (hence contains the closure of its subset as the smallest closed set) since αX is compact. Hence, using these facts and $(***)$ we have

$$H(e(X)) \subseteq H(\beta X) \subseteq e^{'}(\alpha X)$$

Therefore

$$e^{'-1}(H(\beta X)) \subseteq \alpha X$$

Hence the image of F is indeed in αX. For $x \in X$

$$(F \circ e)(x) = e^{'-1}(H(e(x))) = e^{'-1}(e^{'}(\alpha(x)) = \alpha(x)$$

Hence, also $F \circ e = \alpha$. Therefore F is the function that extends α from X to a compact space αX to F from a Stone-Čech compactification to αX. $\square$

Theorem 3.4.4 *If* $(\alpha X, \alpha) \in \mathcal{K}(X)$*, the function* α *is a dense embedding of* X *into* αX*, then* $(\alpha X, \alpha) \leqslant (\beta X, e)$ *where* $(\beta X, e)$ *is the Stone-Čech compactification of* X*.*

Proof: From the previous proposition, if $(\alpha X, \alpha)$ is a compactification, then there exists a function F that extendes α to be from a Stone-Čech compactification to αX. Therefore, there is a continuous function $F : (\beta X, e) \longrightarrow (\alpha X, \alpha)$ such that $F \circ e = \alpha$. Hence $(\alpha X, \alpha) \geqslant (\beta X, e)$. Now all that is left is to prove uniqueness.
Claim: *The Stone-Čech compactification* $(\beta X, e)$*, is the only compactification of* X *of which other compactification maps can be extended to.*
Let $(\gamma X, X) \in \mathcal{K}(X)$ be arbitrary and $(\alpha X, \alpha)$ be a compactification such that there is a continuous function $F^{'} : (\alpha X, \alpha) \longrightarrow (\gamma X, \gamma)$ with $F^{'} \circ \alpha = \gamma$. Then, there is a continuous function

$$\varphi_1 : \beta X \longrightarrow \alpha X$$

defined such that $\varphi_1 \circ e = \alpha$, hence $(\alpha X, \alpha) \leqslant (\beta X, e)$.
Similarly, a map can be defined

$$\varphi_2 : \alpha X \longrightarrow \beta X$$

such that $\varphi_2 \circ \alpha = e$, hence $(\beta X, e) \leqslant (\alpha X, \alpha)$.
Therefore, $(\alpha X, \alpha)$ is equivalent to $(\beta X, e)$. Hence, $(\beta X, e)$ is characterized up to topological equivalence by the extension property, and it can also be deduced that it is the largest element of the partially ordered lattice of all compactifications of X, $(\mathcal{K}(X), \leqslant)$. $\square$

Since βX is the maximum compactification in the collection of all compactifications $(\mathcal{K}(X), \leqslant)$, then there is always a function from βX to any other compactification by the definition of the order on $\mathcal{K}(X)$.

Definition 3.4.5 *Let αX be any compactification of X, then the mapping $f : \beta X \longrightarrow \alpha X$ such that $f(x) = x$ for any $x \in X$ is called the **natural map**.*

We are now going to show that for a strongly zero-dimensional space, its Stone-Čech compactification is zero-dimensional, a fact we get from a theorem in [24, Chapter 12, Lemma 3.9].

Definition 3.4.6 *A completely regular space X is called **strongly zero-dimensional** if for every disjoint zero-sets A and B in X, there exists a clopen set U such that $A \subseteq U$ and $U \cap B = \emptyset$.*

Lemma 3.4.7 *If A and B are disjoint zero-sets of X, then $Cl_{\beta X}(A) \cap Cl_{\beta X}(B) = \emptyset$*

Proof: If two zero-sets A and B of X are disjoint, then there exists a function $f : X \longrightarrow [0, 1]$ such that $f(a) = 1$ for $a \in A$ and $f(b) = 1$ for $b \in B$. Then f is a continuous mapping from X into a compact space, so it has a continuous $\beta f : \beta X \longrightarrow [0, 1]$. We have that $\beta f(a) = 0$ for $a \in Cl_{\beta X}(A)$ since $\beta f(a) = 0$ for $a \in A$, similalry we have that $\beta f(b) = 1$ for $b \in Cl_{\beta X}(B)$. Hence $Cl_{\beta X}(A) \cap Cl_{\beta X}(B) = \emptyset$. $\square$

Theorem 3.4.8 *A space X is strongly zero-dimensional if and only if βX is zero-dimensional.*

Proof: $\Longrightarrow$ We show that βX is totally disconnected when X is strongly zero-dimensional, then a compact Hausdorff space that is totally disconnected is zero-dimensional. If p and q are distinct points of βX, then we can choose disjoint neighborhoods A and B of p and q, respectively. So $A \cap X$ and $B \cap X$ are disjoint zero-sets of X. Then there exists a clopen set U such that $A \cap X \subseteq U$ and $(B \cap X) \cap U = \emptyset$. Thus $Cl_{\beta X}(U)$ is a clopen set containing p but not q, hence βX is totally disconnected.
$\Longleftarrow$ Assume that A and B are disjoint zero-sets of X, then $Cl_{\beta X}(A) \cap Cl_{\beta X}(B) = \emptyset$ by Lemma 3.4.7. Since βX is normal and zero-dimensional, there is a clopen set U such that $Cl_{\beta X}(A) \subseteq U$ and $U \cap Cl_{\beta X}(B) = \emptyset$. Hence $U \cap X$ is a clopen set of X such that $A \subseteq U \cap X$ and $(U \cap X) \cap B = \emptyset$. Therefore X is strongly zero-dimensional. $\square$

Remark: An element of a ring $(R, +, .)$ is called **clean** if it is the sum of a unit (an element the having multiplicative inverse) and an idempotent (an element whose square is itself), and $A \subseteq R$ is called clean if every element of A is clean. In [3, Theorem 2.5] it is shown that $C(X)$ is clean if and only if X is strongly zero-dimensional, that means that $C(X)$ is clean if and only if βX is zero-dimensional by Theorem 3.4.8.

Chapter 4

Proximity spaces and compactifications

4.1 Proximity spaces

A proximity space is an axiomatization of the intuitive notion of nearness that holds set-to-set. The concept was axiomatized by V. A. Efremovič in 1934, but published in 1951 (to which he referred to as infinitesimal space). This section aims to provide a background knowledge to the theory of proximity space, more details on proximity spaces can be found in [26]. Amongst other things, we are going to show that a compact Hausdorff space has a compatible separated proximity. The following definition gives the axiomatic characterization of proximity spaces.

Definition 4.1.1 *A* ***separated proximity*** *on a space X is a binary relation δ on $\mathcal{P}(X)$ satisfying*
(P_1) $\emptyset \not\delta A$ *for any* $A \subseteq X$,
(P_2) *If* $A, B \subseteq X$, *and* $A \cap B \neq \emptyset$, *then* $A \,\delta\, B$,
(P_3) *If* $A, B \subseteq X$, *and* $A \,\delta\, B$, *then* $B \,\delta\, A$,
(P_4) *For* $A, B_1, B_2 \subseteq X$, $A \,\delta\, (B_1 \cup B_2)$ *if and only if* $A \,\delta\, B_1$ *or* $A \,\delta\, B_2$,
(P_5) *If* $A, B \subseteq X$, *and* $A \not\delta B$, *then there exists* $E \subseteq X$ *such that* $A \not\delta E$ *and* $B \not\delta X - E$,
(P_6) $x \,\delta\, y$ *implies* $x = y$ *(we write* $x \,\delta\, y$ *instead of* $\{x\} \,\delta\, \{y\}$*).*

The following properties follow directly from the list in Definition 4.1.1 and can be found in [26].

Lemma 4.1.2 *For a proximity space* (X, δ)*, we have*

1. *If* $A \ \delta \ B$*,* $A \subseteq C$ *and* $B \subseteq D$*, then* $C \ \delta \ D$*.*
2. *If there exists an* x *such that* $A \ \delta \ x$ *and* $x \ \delta \ B$*, then* $A \ \delta \ B$*.*

If (X, δ) is a proximity space, a topology $\tau(\delta)$ is naturally defined in it in the following way: a set $A \subseteq X$ is said to be *closed* if it contains all points x such that $x \ \delta \ A$. In the next theorem we verify that all the axioms for a topological space are satisfied by this topology.

Theorem 4.1.3 *Let* A *be a subset of a proximity space* (X, δ)*. Let* A *be defined to be closed if and only if* $x \ \delta \ A$ *implies* $x \in A$*, then the collection of complements of all closed sets yields a topology on* X *denoted by* $\tau(\delta)$*.*

Proof: Since $x \not\delta \ \emptyset$ for every $x \in X$, for x not in the empty set, then $\emptyset$ is closed. X is closed because for $x \ \delta \ X$ we have $x \in X$. Let $\{A_i : i \in I\}$ be an arbitrary collection of closed sets. If $x \ \delta \ \bigcap_{i \in I} A_i$ then by (P_2) of Definition 4.1.1, $x \ \delta \ A_i$ for each $i \in I$, and so $x \in A_i$ for each $i \in I$ since A_i is closed. Thus $x \in \bigcap_{i \in I} A_i$, which means $\bigcap_{i \in I} A_i$ is closed. Finally, if A_1 and A_2 are closed and $x \ \delta \ (A_1 \cup A_2)$ then by (P_4) of Definition 4.1.1, either $x \ \delta \ A_1$ or $x \ \delta \ A_2$. But A_1 and A_2 are closed, implying that $x \in A_1$ or $x \in A_2$, hence $x \in (A_1 \cup A_2)$. Thus $A_1 \cup A_2$ is closed. □

The closure of a set in a topological space is the intersection of all closed sets which contain that set. We give a description of closed sets for the topology $\tau(\delta)$ below.

Theorem 4.1.4 *Let* (X, δ) *be a proximity space and* $\tau = \tau(\delta)$*. Then the* τ*-closure* $Cl_X(A)$ *of a set* A *is given by*

$$Cl_X(A) = \{x : x \ \delta \ A\}.$$

Proof: If $Cl_X(A)$ denotes the intersection of all closed sets containing A and $A^\delta = \{x : x \ \delta \ A\}$, then we must show that $Cl_X(A) = A^\delta$. Let $x \in A^\delta$, then $x \ \delta \ A$. By Lemma 4.1.2 (1), that means that $x \ \delta \ Cl_X(A)$. Since $Cl_X(A)$ is closed in $\tau(\delta)$ and in Theorem 4.1.3, $Cl_X(A)$ is closed when $x \ \delta \ Cl_X(A)$ implies that $x \in Cl_X(A)$. Hence we have $A^\delta \subseteq Cl_X(A)$. To prove the reverse inclusion it suffices to prove that A^δ is closed, and since $Cl_X(A)$ is the intersection of closed sets then it will be contained in it. By the definition of $\tau(\delta)$, A^δ is closed if $x \ \delta \ A^\delta$ implies $x \in A^\delta$. So let $x \notin A^\delta$, then $x \not\delta A$, so that by Definition 4.1.1 (P_5) there is a set E such that $x \not\delta E$ and $(X-E) \not\delta A$. By Lemma 4.1.2 (2) that means that there is no point $y \in (X-E)$ such that $y \ \delta \ A$, therefore $A^\delta \subseteq E$. But $x \not\delta E$, by Lemma 4.1.2 (1) that implies that $x \not\delta A^\delta$ as required to be proved. $\square$

Proposition 4.1.5 *Let G, A, B be subsets of the proximity space (X, δ). Then:*

1. *$G \in \tau(\delta)$ if and only if $x \not\delta (X-G)$ for every $x \in G$.*
2. *$A \not\delta B$ implies $Cl_X(B) \subseteq (X-A)$.*

Proof: (1) $G \in \tau(\delta)$ if and only if $(X-G)$ is closed in $(X, \tau(\delta))$, that is if and only if $(X-G)$ contains all the points x such that $x \ \delta \ (X-G)$, and this means that for $x \notin X-G$ we have $x \not\delta (X-G)$.
(2) Suppose that $Cl_X(B) \subseteq A$, by (P_2) of Definition 4.1.1 we have that $Cl_X(B) \ \delta \ A$ which in turn means $B \ \delta \ A$. $\square$

Proposition 4.1.6 *For subsets A and B of a proximity space (X, δ), we have $A \ \delta \ B$ if and only if $Cl_X(A) \ \delta \ Cl_X(B)$ (where the closure is taken with respect to $\tau(\delta)$).*

Proof: If $A \ \delta \ B$, then we have $Cl_X(A) \ \delta \ Cl_X(B)$ since $A \subseteq Cl_X(A)$ and $B \subseteq Cl_X(B)$, by Lemma 4.1.2. Now suppose that $A \not\delta B$, then by (P_5) of Definition 4.1.1 there exists an E such that $A \not\delta E$ and $(X-E) \not\delta B$. By Proposition 4.1.5 (2), $Cl_X(B) \subseteq E$, and by Lemma 4.1.2 (1) $A \not\delta E$ implies $A \not\delta Cl_X(B)$. It then follows by (P_3) of definition 4.1.1 that also $Cl_X(A) \not\delta Cl_X(B)$. $\square$

We consider the concept of δ-neighborhood in a proximity space, which was also introduced V. A. Efremovič.

Definition 4.1.7 *A subset B of a proximity space (X, δ) is a δ-**neighborhood** of A, written as $A \ll B$, if and only if $A \not\delta (X-B)$.*

Lemma 4.1.8 *The following properties hold for δ-neighborhoods*

1. *If $A_i \ll B_i$ where $i = 1, ..., k$, then $\bigcup_i A_i \ll \bigcup_i B_i$ and $\bigcap_i A_i \ll \bigcap_i B_i$.*
2. *If $A \ll C$ then there exists a B such that $A \ll B \ll C$.*

Proof: We prove (1) for $k = 2$ only. Let $A_1 \ll B_1$ and $A_2 \ll B_2$. If $B = B_1 \cup B_2$ then $A_1 \ll B$ and $A_2 \ll B$. This means that $A_1 \not\delta (X - B)$ and $A_2 \not\delta (X - B)$, therefore by (P_4) of Definition 4.1.1 we have $(A_1 \cup A_2) \not\delta (X - B)$, i.e $A_1 \cup A_2 \ll B$ as required. For the intersection, take $A = A_1 \cap A_2$. Then $A \ll B_1$ and $A \ll B_2$, i.e $A \not\delta (X - B_1)$ and $A \not\delta (X - B_2)$. By Lemma 4.1.2 (1), that means we have $A \not\delta (X - (B_1 \cap B_2))$, i.e $A \ll B_1 \cap B_2$.
For the proof of (2), let $A \ll C$. Then $A \not\delta (X - C)$, means (by (P_5) of Definition 4.1.1) that there is an E such that $A \not\delta E$ and $(X - E) \not\delta C$, but that in turn means $A \ll X - E$ and $X - E \ll C$. By choosing $B = X - E$ we get the desired result. □

Lemma 4.1.9 *Let (X, δ) be a proximity space and let $Cl_X(A)$ and $Int_X(A)$ denote, respectively, the closure and interior of A in $\tau(\delta)$. Then*

1. *$A \ll B$ implies $Cl_X(A) \ll B$, and*
2. *$A \ll B$ implies $A \ll Int_X(B)$.*

Therefore $A \subseteq Int_X(B)$, showing that a δ-neighborhood is a topological neighborhood.

Proof: (1) $A \ll B$ means that $A \not\delta (X - B)$ which implies $Cl_X(A) \not\delta (X - B)$ by Proposition 4.1.6, i.e $Cl_X(A) \ll B$
(2) By noting that $Cl_X(X - B)$ is equivalent to $(X - Int_X(B))$ we have $A \not\delta (X - B)$ implies $A \not\delta Cl_X(X - B)$ by Proposition 4.1.6, which in turn is equivalent to $A \not\delta (X - Int_X(B))$, i.e. $A \ll Int_X(B)$. □

Lemma 4.1.10 (P_5) *of Definition 4.1.1 is equivalent to the following conditions:*
(P_5^*) *$A \not\delta B$ implies there exist subsets C and D such that $A \not\delta (X - C)$, $(X - D) \not\delta B$, and $C \not\delta D$.*
(P_5^{**}) *$A \not\delta B$ implies that A and B have disjoint δ-neighborhoods.*
(P_5^{***}) *$A \not\delta B$ implies that there exist sets C and D such that $C \cup D = X$, $A \not\delta C$, and $B \not\delta D$.*

Proof: To prove that (P_5^*) implies (P_5) of Definition 4.1.1, we note that if $C \not\delta D$, then $C \subseteq (X - D)$ by Proposition 4.1.5 (2). Setting $E = X - C$, we have $A \not\delta E$ and since $(X - D) \not\delta B$ then $(X - E) \not\delta B$. On the other hand, suppose that (P_5) of Definition 4.1.1 holds. Then $A \not\delta B$ implies there is a D

such that $A \not\delta D$ and $(X - D) \not\delta B$. Moreover, since $A \not\delta D$, there exists a C such that $A \not\delta (X - C)$ and $C \not\delta D$, which completes the proof.
To see that (P_5) of Definition 4.1.1 is equivalent to (P_5^{**}) we note that if $A \not\delta E$ then $A \ll (X - E)$ and if $(X - E) \not\delta B$ then $B \ll E$ where E and $(X - E)$ are disjoint.
Finally, to see that (P_5) of Definition 4.1.1 is equivalent to (P_5^{***}), we just note that $E \cup (X - E) = X$. □

Proposition 4.1.11 *If (X, δ) is a proximity space and $A \subseteq X$, then $Cl_X(A) = \bigcap_{A \ll B} B$.*

Proof: From Lemma 4.1.9 we have that $A \ll B$ implies that $Cl_X(A) \ll B$ and hence $Cl_X(A) \subseteq B$. That means that $Cl_X(A) \subseteq \bigcap_{A \ll B} B$. To show the reverse inclusion, suppose that $x \notin Cl_X(A)$. Then $x \not\delta Cl_X(A)$, by Lemma 4.1.10, $Cl_X(A)$ has a δ-neighborhood B_x not containing x. Thus we have $x \notin \bigcap_{A \ll B} B$. □

Definition 4.1.12 *If on a set X there is a topology τ and a proximity δ such that $\tau = \tau(\delta)$, then τ and δ are said to be **compatible**.*

Definition 4.1.13 *We say that subsets A and B in X are **functionally distinguishable** if and only if there is a continuous function $f : X \longrightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.*

If we take any set $Q \subseteq X$, where X is a proximity space, and all proximity relations between subsets of Q are just as they are in X, then Q becomes a proximity space. With the compactification of X, αX, we have $X \subseteq \alpha X$, and that means if we have a compatible proximity for αX then we also have a compatible proximity for X as a subspace. In the following theorem we prove that we indeed have a compatible proximity for a compact Hausdorff space.

Theorem 4.1.14 *In a compact Hausdorff space (X, τ), $A \,\delta\, B$ if and only if $Cl_X(A) \cap Cl_X(B) \neq \emptyset$, defines a separated compatible proximity.*

Proof: A compact Hausdorff space is normal, and by Urysohn's lemma, $Cl_X(A) \cap Cl_X(B) = \emptyset$ if and only if A and B are functionally distinguishable. Then it is sufficient to prove that the proximity defined by, $A \not\delta B$ if and only

if A and B are functionally distinguishable, is compatible with τ.
Firstly we show that δ is a proximity,
(P_1) : Since $f(\emptyset) = \emptyset$ for any continuous function, that means we can always find a function that is zero at the empty set and 1 on any subset of X since the intersection is empty.
(P_2) : If $A \cap B \neq \emptyset$, then there is no function from X to $[0,1]$ such that $f(A) = 0$ and $f(B) = 1$, that is to say that A and B can not be functionally distinguishable. Hence $A \;\delta\; B$
(P_3) : If $B \not\delta A$, then A and B are functionally distinguishable, hence $A \not\delta B$.
(P_4) : If $A \;\delta\; (B_1 \cup B_2)$, then there is no function $f : X \longrightarrow [0,1]$ such that $f(A) = 0$ and $f(B_1 \cup B_2) = 1$, that means that there is no function from X to $[0,1]$ such that $f(A) = 0$ and $f(B_1) = 1$. Hence $A \;\delta\; B_1$.
(P_5) : Suppose that $A \not\delta B$ and let f be a continuous function from X to $[0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Set $E = \{x \in X : 1/2 \leqslant f(x) \leqslant 1\}$. Then $A \not\delta E$ and $(X - E) \not\delta B$. We show by example that $A \not\delta E$; Let $g : [0,1] \longrightarrow [0,1]$ be defined by $g(y) = 2y$ for $0 \leqslant y \leqslant 1/2$ and $g(y) = 1$ for $1/2 \leqslant y \leqslant 1$. Then $g \circ f$ is a continuous function from X to $[0,1]$ such that $g \circ f(A) = 0$ and $g \circ f(E) = 1$.
We show that $\tau = \tau(\delta)$. Let $G \in \tau$ and $x \in G$. Then $x \notin X - G$, which is a closed set. Since X is completely regular, there exists a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f(X - G) = 1$. By the definition of the proximity, $x \not\delta (X - G)$. Hence by Proposition 4.1.5 (1), we have that $G \in \tau(\delta)$. Conversely, if $G \in \tau(\delta)$ and $x \in G$, then by Proposition 4.1.5 $x \not\delta (X - G)$. Hence, by the definition of the proximity, there exists a τ-continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f(X - G) = 1$. Then $f^{-1}([0,1/2))$ is a τ-open neighborhood of x contained in G. Therefore G is τ-open, i.e. $G \in \tau$. To see that (P_6) is satisfied when (X,τ) is Tychonoff, we note that if $x \neq y$ then $x \notin Cl_X(\{y\})$ since (X,τ) is T_1. From the definition of a completely regular space, we are assured that x and $Cl_X(\{y\}) = y$ are functionally distinguishable, implying that $x \;\delta\; y$. □

We also have the converse, which is found in [26, Theorem 3.14], which states that:

Theorem 4.1.15 *If (X,δ) is a separated proximity space, then $\tau(\delta)$ is Tychonoff.*

We now define the concept of a proximity map and show that it is the same as a mapping being continuous in its own topology. And in turn, continuous functions that have a compact domain are proximity mappings.

Definition 4.1.16 *Let* (X, δ_1) *and* (Y, δ_2) *be two proximity spaces. A function* $f : X \longrightarrow Y$ *is a* ***proximity mapping*** *if* $A\ \delta_1\ B$ *implies* $f(A)\ \delta_2\ f(B)$.

In the following propositions, we are using the fact that a function $f : X \longrightarrow Y$ between two topological spaces is continuous if and only if $f^{-1}(F)$ is closed in X whenever F is closed in Y, that is if and only if $f(Cl_X(A)) \subseteq Cl_Y(f(A))$ for any subset A of X.

Proposition 4.1.17 *A proximity mapping* $f : (X, \delta_1) \longrightarrow (Y, \delta_2)$ *is continuous with respect to* $\tau(\delta_1)$ *and* $\tau(\delta_2)$.

Proof: In $\tau(\delta_1)$ and $\tau(\delta_2)$ we have $Cl_X(A) = \{x : x\ \delta_1\ A\}$ and $Cl_Y(f(A)) = \{y : y\ \delta_2\ f(A)\}$. So, if $x \in Cl_X(A)$, i.e $f(x) \in f(Cl_X(A))$, we have $x\ \delta_1 A$ which means that $f(x)\ \delta_2\ f(A)$ since f is a proximity mapping. Therefore $f(x) \in Cl_Y(f(A))$, from which we can conclude that $f(Cl_X(A)) \subseteq Cl_Y(f(A))$ as required. □

Proposition 4.1.18 *If* (X, δ_1) *and* (Y, δ_2) *are proximity spaces and* X *is compact, then every continuous function* f *from* X *to* Y *is a proximity mapping.*

Proof: Since X is compact, if A and B are subsets of X such that $A\ \delta_1\ B$, then $Cl_X(A) \cap Cl_X(B) \neq \emptyset$ by Theorem 4.1.14. But this implies that $f(Cl_X(A)) \cap f(Cl_X(B)) \neq \emptyset$, which in turn means that

$$f(Cl_X(A))\ \delta_2\ f(Cl_X(B))$$

by (P_2) of Definition 4.1.1. Since f is continuous, $f(Cl_X(A)) \subseteq Cl_Y(f(A))$ and $f(Cl_X(B)) \subseteq Cl_Y(f(B))$, through Lemma 4.1.2 (1) this yields

$$Cl_Y(f(A))\ \delta_2\ Cl_Y(f(B)).$$

From Proposition 4.1.6 it follows that $f(A)\ \delta_2\ f(B)$, therefore f is a proximity map. □

Definition 4.1.19 *Two proximity spaces* (X, δ_1) *and* (Y, δ_2) *are called* ***proximally isomorphic*** *(or* δ***-homeomorphic****) if and only if there exists a one-to-one mapping* f *from* X *onto* Y *such that both* f *and* f^{-1} *are proximity mappings.*

A subset of a proximity inherits a proximity from a proximity space.

Proposition 4.1.20 *Let* (X, δ) *be a proximity space and let* Y *be a subset of* X*. For any sets* A, B *of* Y*, define* $A \ \delta_Y \ B$ *if and only if* $A \ \delta \ B$*. Then* δ_Y *is a proximity on* Y*.*

Proof: (P_1) : Since a subset of Y is also a subset of X, $\emptyset \ \not\delta \ A$ for any $A \subseteq X$ implies that $\emptyset \ \not\delta_Y \ A$ for any subset A of Y.
(P_2) : If $A \cap B \neq \emptyset$ in Y then $A \cap B \neq \emptyset$ in X and since δ is a proximity on X by (P_2) of Definition 4.1.1 we have $A \ \delta \ B$ and hence $A \ \delta_Y \ B$ by the definition of δ_Y.
(P_3) : Since δ is reflexive then δ_Y is also reflexive.
(P_4) : Condition (P_4) of Dediniiton 4.1.1 is satisfied by δ_Y since it is satisfied by δ.
(P_5) : Let $A \ \not\delta_Y \ B$, we are going to show that there is a set E of Y such that $A \ \not\delta_Y \ E$ and $(Y - E) \ \not\delta_Y \ B$. If $A \ \not\delta_Y \ B$, then $A \ \not\delta \ B$ by the definition of δ_Y, hence there exists a set E_1 of X such that $A \ \not\delta \ E_1$ and $(X - E_1) \ \not\delta \ B$ by (P_5) of Definition 4.1.1 since δ is a proximity on X. If the intersection of Y and E_1 is empty then Y is a subset of $X - E_1$. Since $(X - E_1) \ \not\delta \ B$ then $Y \ \not\delta \ B$ which contradicts that B is a subset of Y. Hence the intersection of Y and E_1 is not empty. Let $E = Y \cap E_1$. Since $E \subseteq E_1$ and $A \ \not\delta \ E_1$, then $A \ \not\delta \ E$, which means $A \ \not\delta_Y \ E$. Also, since $Y - E \subseteq X - E_1$ and $(X - E_1) \ \not\delta \ B$ then $(Y - E) \ \not\delta \ B$ which means that $(Y - E) \ \not\delta_Y \ B$. □

4.2 Representation of compactifications using proximities

In [29], it is shown that the partially ordered set of inequivalent compactifications of a completely regular space X is isomorphic to the partially ordered set of proximities on X that induce the topology on X. This result is known as Smirnov theorem. It provides a characterization of the structure of compactifications of a completely regular X by means of proximities on X that induce the topology on X. In this section we are going to explore the Smirnov Theorem in detail. The concept of centered δ-system and ends, to be defined, is similar to the concept of filters and ultrafilter. The ends are used to construct a compact space from a completely regular space, which is thus used to construct the Smirnov compactification of X.

Definition 4.2.1 *A collection of sets $\mathcal{F}$ from a proximity space is said to be a* ***centered δ-system*** *if and only if*

1. *$A, B \in \mathcal{F}$ implies that $A \cap B \neq \emptyset$,*
2. *$A \in \mathcal{F}$ implies the existence of a $B \in \mathcal{F}$ such that $B \ll A$.*

Supplementing the centered δ-system $\mathcal{F}$ with the intersections of all possible finite subsystems, we get a new centered δ-system $\mathcal{F}^{'}$, any finite intersection of whose elements also belong to $\mathcal{F}^{'}$. Supplementing the system $\mathcal{F}^{'}$ with all sets $A \subseteq X$ which contain any set $B \in \mathcal{F}^{'}$, we get a new centered δ-system, which we denoted by ξ (because we want to think of it as elements of a set in what follows). Let ξ_A denote the set of all δ-neighborhoods of any non-empty set A of a proximity space X, then ξ_A is an example of a centered δ-system.

Definition 4.2.2 *An* ***end*** *of a proximity space X is any centered δ-system, ξ, which is not a subsystem of any other centered δ-system.*

It follows from the preceding remark that the intersection of any finite number of elements of an end ξ also belongs to ξ and that any set $A \subseteq X$ which contains an element of an end ξ also belongs to ξ. We denote the set of all ends of a given proximity space X by uX. We now want to define a proximity on the set of all ends, but first we define the operator $E(\ )$; $E(\ )$ corresponds to each set $A \subseteq X$ the set of all ends $\xi \in uX$ which contain A as an element, that is $E(A) = \{\xi \in uX : A \in \xi\}$.

Proposition 4.2.3 *Let $E(A) = \{\xi \in uX : A \in \xi\}$, then we have*

1. *$E(A) \cap E(B) = E(A \cap B)$.*
2. *$E(A) \cup E(B) \subseteq E(A \cup B)$.*
3. *if A and B are sets of the proximity space X such that $(X - A) \not\delta (X - B)$, then $E(A) \cup E(B) = uX$.*

Proof: (1) We have that $\xi \in E(A \cap B)$ if and only if $A \cap B \in \xi$, that is $\xi \in E(A \cap B)$ if and only if $A \in \xi$ and $B \in \xi$, which is true if and only if $\xi \in E(A) \cap E(B)$.
(2) If $\xi \in E(A) \cup E(B)$ then $\xi \in E(A)$ or $\xi \in E(B)$, i.e $A \in \xi$ or $B \in \xi$, which means that $\xi \in E(A \cup B)$.
For (3) we are going to prove that every end ξ contains either A or B if

$(X - A) \not\delta (X - B)$. In the case $X = A$ we have that $A \in \xi$ for every end. Otherwise $D = X - A$ is not empty, and so the system ξ_D of all δ-neighborhoods of D is a centered δ-system by Lemma 4.1.8 (2). Let ξ be an arbitrary end in uX. If every H of ξ meets D, then the union $\xi_D \cup \xi$ of the system ξ_D and the end ξ will be a centered δ-system. Since ξ is an end, we must have $\xi_D \subseteq \xi$. But $D \not\delta (X - B)$, which means $D \ll B$, i.e $B \in \xi_D$, and consequently $B \in \xi$. In the remaining case when D is not empty and the end ξ has an element H which does not meet D, we get that $H \subseteq A$, and so $A \in \xi$. □

We are going to define a proximity on the set of all ends uX of X, using the proximity which is defined on X. The set of all ends, with its topology induced by the defined proximity, will be shown that it is a compact space and also X will be densely embedded in it, hence it is the compactification of X. We say that this compactification is induced by the proximity defined on X.

Proposition 4.2.4 *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be any two sets of ends of X. Define $\mathcal{E}_1 \not\delta^{**} \mathcal{E}_2$ if and only if there exists sets A and B such that $A \not\delta B$, $\mathcal{E}_1 \subseteq E(A)$ and $\mathcal{E}_2 \subseteq E(B)$. Then (uX, δ^{**}) is a separated proximity space.*

Proof: (P_1) : To prove (P_1) we note that $E(\emptyset) = \emptyset$, $E(X) = uX$, and $X \not\delta \emptyset$. Since $uX \subseteq E(X)$ and $\emptyset \subseteq E(\emptyset)$, by the definition of δ^{**} we must have that $uX \not\delta^{**} \emptyset$. That means that for any non-empty set $\mathcal{E}$ of uX, we have $\mathcal{E} \not\delta^{**} \emptyset$.
(P_2) : Suppose that $\mathcal{E}_1 \not\delta^{**} \mathcal{E}_2$. Then there are sets A and B such that $\mathcal{E}_1 \subseteq E(A)$, $\mathcal{E}_2 \subseteq E(B)$, and $A \not\delta B$. Since δ is a proximity, we have $A \cap B = \emptyset$ by (P_2) of Definition 4.1.1, therefore $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$ as required.
(P_3) : $\mathcal{E}_1 \not\delta^{**} \mathcal{E}_2$ if and only if $\mathcal{E}_2 \not\delta^{**} \mathcal{E}_1$ by (P_3) of Definition 4.1.1 since δ is a proximity on X.
(P_4) : Let $\mathcal{N}, \mathcal{E}_1, \mathcal{E}_2$ be sets in uX. We will show the validity of (P_4) in the equivalent form: $\mathcal{N} \not\delta^{**} (\mathcal{E}_1 \cup \mathcal{E}_2)$ if and only if $\mathcal{N} \not\delta^{**} \mathcal{E}_1$ and $\mathcal{N} \not\delta^{**} \mathcal{E}_2$. Let $\mathcal{N} \not\delta^{**} (\mathcal{E}_1 \cup \mathcal{E}_2)$, then by definition of δ^{**} we can find sets A and B in X such that $A \not\delta B$, $\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq E(A)$ and $\mathcal{N} \subseteq E(B)$. Then there are sets A, B in X such that $A \not\delta B$, $\mathcal{E}_1 \subseteq E(A)$, and $\mathcal{N} \subseteq E(B)$, then by definition of δ^{**} we have $\mathcal{N} \not\delta^{**} \mathcal{E}_1$. Similarly, we have that $\mathcal{N} \not\delta^{**} \mathcal{E}_2$. Conversely, suppose that $\mathcal{N} \not\delta^{**} \mathcal{E}_1$ and $\mathcal{N} \not\delta^{**} \mathcal{E}_2$. By the definition of δ^{**}, we can find pairs of sets A, B and A', B' in X such that $\mathcal{E}_1 \subseteq E(A)$, $\mathcal{N} \subseteq E(B)$,

$\mathcal{E}_2 \subseteq E(A^{'})$, $\mathcal{N} \subseteq E(B^{'})$, $A \not\delta B$, and $A^{'} \not\delta B^{'}$. Then we have that $A \not\delta B \cap B^{'}$ and $A^{'} \not\delta B \cap B^{'}$, by (P_4) of Definition 4.1.1 (since δ is a proximity) we get $A \cup A^{'} \not\delta B \cap B^{'}$. And by applying (1), (2) of Proposition 4.2.3 we get $\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq E(A) \cup E(A^{'}) \subseteq E(A \cup A^{'})$ and $\mathcal{N} \subseteq E(B) \cap E(B^{'}) = E(B \cap B^{'})$. Then by the definition of δ^{**}, we must have $\mathcal{N} \not\delta^{**} (\mathcal{E}_1 \cup \mathcal{E}_2)$ as required to be proved.

(P_5) : We will prove instead an equivalent (P_5^{***}) of Lemma 4.1.10. Suppose that $\mathcal{E}_1 \not\delta^{**} \mathcal{E}_2$, then there are sets A and B of X such that $A \not\delta B$, $\mathcal{E}_1 \subseteq E(A)$, and $\mathcal{E}_2 \subseteq E(B)$. $A \not\delta B$ implies that $B \ll (X - A)$, hence by Lemma 4.1.8 (2) there is a $B^{'}$ and $B^{''}$ of X such that $B \ll B^{'} \ll B^{''} \ll (X - A)$. $B^{'} \ll B^{''}$ implies that $(X - B^{'}) \not\delta X - (X - B^{''})$, therefore by Proposition 4.2.3 (3) we have $E(B^{'}) \cup E(X - B^{''}) = uX$. But $\mathcal{E}_1 \subseteq E(A)$ and $A \not\delta B^{'}$ which means that $\mathcal{E}_1 \not\delta^{**} E(B^{'})$. In the same way we get $\mathcal{E}_2 \not\delta^{**} E(X - B^{''})$.

To show that δ^{**} satisfies (P_6), we take any distinct elements of uX, say ξ_1 and ξ_2, and show that $\xi_1 \not\delta^{**} \xi_2$. Suppose that $\xi_1 \neq \xi_2$, then we can find disjoint sets $A \in \xi_1$ and $B \in \xi_2$. Since ξ_1 is a centered δ-system, there is a set $C \in \xi_1$ such that $C \ll A$, hence $C \not\delta (X - A)$, which means that we have $C \not\delta B$. But also $\xi_1 \in E(C)$ and $\xi_2 \in E(B)$, by the definition of δ^{**}, we must have $\xi_1 \not\delta^{**} \xi_2$. □

Proposition 4.2.5 *The system ξ_x of all δ-neighborhoods of each point $x \in X$ is an end.*

Proof: For this we are going to take the system ξ_x of some point $x \in X$ and will show that if there is a collection of sets ξ which contains ξ_x as a proper subsystem, satisfying Definition 4.2.1 (2) then it does not satisfy the other condition. Hence, we show that ξ is not a centered δ-system. We choose a set $A \in (\xi - \xi_x)$. Since we assumed that ξ satisfy (2) of Definition 4.2.1, we can find sets B and C in ξ such that $C \ll B \ll A$. Since A is not a δ-neighborhood of x, we have $x \notin B$, and so $x \not\delta C$. This means that $X - C$ is in ξ_x, and so in ξ. But we also have $C \in \xi$, this means that ξ does not satisfy condition (2) of Definition 4.2.1 since $(X - C) \cap C = \emptyset$. That means that ξ_x is not a subsystem of any other centered δ-system. □

Proposition 4.2.6 *The map $u : X \longrightarrow uX$, where $u(x) = \xi_x$ for each $x \in X$, is a δ-homeomorphism of X into uX, and X is dense in uX*

Proof: We will first show that u is one-to-one. Let $x \neq y$ in X. Since δ is a separated proximity, $x \ \delta \ y$. Then by Lemma 4.1.10 (P_5^{**}), there are disjoint

δ-neighborhoods A and B such that $A \in \xi_x$ and $B \in \xi_y$. But that means that $\xi_x \neq \xi_y$. Now we need the following claim,
Claim : *For any set* $A \subseteq X$ *the preimage* $u^{-1}(E(A))$ *is the interior of* A
Let $x \in u^{-1}(E(A))$, this means that $u(x) = \xi_x \in E(A)$, that is $A \in \xi_x$ and so $x \ll A$. By Lemma 4.1.9, it follows that $x \in Int(A)$. Conversely, let $x \in Int(A)$. Since $Int(A) \in \tau(\delta)$, by Proposition 4.1.5 (1) we have $x \not\delta (X - Int(A))$. Consequently, $x \ll Int(A)$, and hence $x \ll A$. That means that $A \in \xi_x$, i.e $\xi_x \in E(A)$, and hence $x \in u^{-1}(E(A))$. The claim is proved.
We must now show that u and u^{-1} are proximity maps. Firstly, let $\mathcal{E} \not\delta^{**} \mathcal{N}$ be sets in uX. There are sets A and B in X such that $A \not\delta B$, $\mathcal{E} \subseteq E(A)$, and $\mathcal{N} \subseteq E(B)$. Then we have $u^{-1}(\mathcal{E}) \subseteq A$ and $u^{-1}(\mathcal{N}) \subseteq B$, but we have $A \not\delta B$, hence $u^{-1}(\mathcal{E}) \not\delta u^{-1}(\mathcal{N})$. That means that u is a proximity map. To show that u^{-1} is a proximity map, suppose that $A \not\delta B$ in X. Then $B \ll (X - A)$, and by Lemma 4.1.8 (2) we can choose sets B' and B'' such that $B \ll B' \ll B'' \ll (X - A)$. Since by Lemma 4.1.9, $B \ll B'$ implies that $B \subseteq Int(B')$, we can then deduce that $u(B) \subseteq E(B')$. Similarly, $B'' \ll (X - A)$ implies that $B'' \not\delta A$, i.e $A \ll (X - B'')$, which in turn implies $A \subseteq Int(X - B'')$, which means that $u(A) \subseteq E(X - B'')$. But $B' \ll B''$ means that $B' \not\delta (X - B'')$. By the definition of δ^{**}, this means that $u(A) \not\delta^{**} u(B)$.
To prove that X is dense in uX, we must consider X as a subspace of uX, then $Cl(X) = \{\xi : \xi \ \delta^{**} \ X\}$. So we must show that every point $\xi \in uX$ is such that $\xi \ \delta^{**} \ X$. For this purpose, in view of our definition of δ^{**} in uX, we will prove that any sets A and B of X such that $\xi \in E(A)$ and $X \subseteq E(B)$. Then we must have $A \ \delta \ B$. In fact we have $X = B$; an $x \in X$ is represented by ξ_x when X is considered as a subspace of uX, $X \subseteq E(B)$ means $B \in \xi_x$, which means $B \cap \{x\} \neq \emptyset$, hence $x \in B$. Therefore $X \subseteq B$. So that means $A \ \delta \ B$, as required. □

Definition 4.2.7 *By a* ***δ-extension*** *of the proximity space* X *we mean any proximity space which contains* X *as a dense subspace. We will call a proximity space* ***absolutely closed*** *if it has no* δ*-extension except for itself.*

In order to call uX a compactification of X, we must show that uX is indeed compact. We start with absolute closedness of a proximity space, and then establish compactness. We need the following criterion for absolute closedness of a proximity space;

Proposition 4.2.8 *A proximity space (X, δ) is absolutely closed if and only if every centered δ-system of sets of X have a non-empty intersection.*

Proof: Suppose X is not absolutely closed. Then there is a δ-extension $X' = X \cup \{y\}$ of X consisting of all points $x \in X$ and another point y. We consider the end η of the space X' made up of all neighborhoods of y and show that $\xi = \{X \cap H : H \in \eta\}$ is a centered system with empty intersection. Firstly, y is a limit point of X, therefore if we take H_1 and H_2 that are neighborhoods of y then they contain atleast one point of X. Therefore $(X \cap H_1) \cap (X \cap H_2) \neq \emptyset$, hence condition (1) of Definition 4.2.1 is satisfied. Since for $H \in \eta$, there is an H' such that $H' \ll H$ it follows immediately that $X \cap H' \ll X \cap H$, and hence condition (2) of Definition 4.2.1 is also satisfied. This means that ξ is a centered δ-system. It remains to show that the intersection of all $A \in \xi$ is empty. We note that for $x \in X$ we can find a δ-neighborhood which is disjoint from the δ-neighborhood of y, hence we have that the intersection of all $H \in \eta$ consists of just one point y, so that $\bigcap_{A \in \xi} A = \emptyset$, which was to be proved.
Conversely, let ξ' be a centered δ-system in the proximity space X with empty intersection. We supplement ξ' to make it an end ξ. Since $\xi' \subseteq \xi$, the intersection of all $A \in \xi$ is empty. This means that X has an end ξ which is different from any proper end ξ_x for $x \in X$. Consequently, X has a δ-extension uX different from X and this means that X is not absolutely closed. □

Theorem 4.2.9 *Let (X, δ) be a proximity space, then (uX, δ^{**}) is absolutely closed.*

Proof: By Proposition 4.2.8 it is necessary for us to establish that every centered δ-system η' of sets of uX has a non-empty intersection. We supplement the arbitrarily chosen centered δ-system η' to be an end η of uX. Then $\xi' = \{X \cap \mathcal{H} : \mathcal{H} \in \eta\}$ is a centered δ-system. We supplement this system ξ' to make it an end ξ of X and show that $\xi \in \mathcal{H}$ for every $\mathcal{H} \in \eta'$. Suppose that a set $\mathcal{H} \in \eta'$ could be found which does not contain ξ. By Definition 4.2.1, there is an $\mathcal{H}' \in \eta'$ such that $\mathcal{H}' \ll \mathcal{H}$. This means that $\mathcal{H}' \not\delta^{**} (uX - \mathcal{H})$, and since ξ is not contained in $\mathcal{H}$ then we have $\mathcal{H}' \not\delta^{**} \xi$. This means, by the definition of the proximity in uX, that we can find sets A and B in X such that $A \not\delta B$, $\xi \in E(A)$ and $\mathcal{H}' \in E(B)$. This means that $A \in \xi$ and since $B \in \mathcal{H}'$ we have $A' = X \cap \mathcal{H}' \subseteq B$. Then $A' \in \xi$ and since $A \not\delta B$ we also have $A \not\delta A^{'}$. But ξ is a centered δ-system, hence we have a contradiction. That means that $\xi \in \bigcap \mathcal{H}$ for every $\mathcal{H} \in \eta^{'}$. $\square$

In the following theorem, we are going to use the following characterization of compactness: (X, τ) is compact if $\mathcal{F} \subseteq \mathcal{P}(X)$ consists of closed sets such that any intersection of finitely many elements of $\mathcal{F}$ is non-empty (to which we say that $\mathcal{F}$ has the finite intersection property), then $\bigcap \mathcal{F} \neq \emptyset$

Theorem 4.2.10 *The proximity space (X, δ) is absolutely closed if and only if it is compact.*

Proof: If X is compact as a topological space, then there is no proximity space which contains X as a dense subspace except for itself. Conversely, suppose we have an absolutely closed proximity space X. Let ϕ be arbitrary and consist of closed sets of X with the finite intersection property, then we must show that ϕ has a non-empty intersection. For every $F \in \phi$, we consider the system ξ_F of all its δ-neighbourhood. Since no $F \in \phi$ is empty, every system ξ_F will be a centered δ-system. We let $\xi = \bigcap \xi_F$, now we prove that ξ is a centered δ-system. Firstly, on condition (1) of Definition 4.2.1, if $A, B \in \xi$ then there is F_1 and F_2 such that $A \in \xi_{F_1}$ and $B \in \xi_{F_2}$. That means that $F_1 \ll A$ and $F_2 \ll B$, which in turn implies that $F_1 \cap F_2 \ll A \cap B$. But $F_1 \cap F_2 \neq \emptyset$ since ϕ has the finite intersection property. Therefore $A \cap B$ is also non-empty since it is a δ-neighborhood of a non-empty set. Then condition (2) of Definition 4.2.1 is satisfied in ξ since it is also satisfied in ϕ. So, since X is absolutely closed, the intersection of all $H \in \xi$ is non-empty. We will show now that this intersection $\bigcap_{H \in \xi} H$ coincides with the intersection of all $F \in \phi$. In fact, by Proposition 4.1.11, for every $F \in \phi$ the intersection of all $H \in \xi_F$ is F itself since it is closed, i.e $F = \bigcap_{H \in \xi_F} H$. So we can safely say that $\bigcap_{F \in \phi} F = \bigcap_{H \in \xi} H$, and since $\bigcap_{H \in \xi} H$ is non-empty, then $\bigcap_{F \in \phi} F$ is also non-empty as required. $\square$

We can now safely say that uX is a compactification of X, with the topology induced by δ^{**}, that is to say that $(uX, \tau(\delta^{**}))$ is a compactification of X. We now want this compactification to be unique, and hence the next theorem. But before we prove the theorem we need a result from the paper by Smirnov (1952) named; *Mappings of systems of open sets*, written in Russian. It states that: *in order that two compactifications of the space X be distinct, it is necessary and sufficient that there exists a pair of closed subsets of X, the* *closure of which intersect in one compactification and do not intersect in the other compactification.*

Theorem 4.2.11 *Let (X, δ) be a proximity space, then it has only one absolutely closed δ-extension, i.e there is only one proximity space (uX, δ^{**}) which is absolutely closed.*

Proof: If (uX, δ^{**}) is a proximity space which is absolutely closed then $(uX, \tau(\delta^{**}))$ is a compactification of (X, τ). Let $(vX, \tau(\delta_v^{**}))$ be another compactification which is different from uX, then there is no homeomorphism of vX onto uX which is the identity on X. But this means (by the comment on the previous paragraph) that there are sets A and B in X whose closure meet in one compactification, say vX, and are disjoint in the other compactification. This means, by Theorem 4.1.14, that $A \ \delta_v^{**} \ B$ but $A \ \not\delta^{**} \ B$. Which is impossible since X is a subspace of both vX and uX. $\square$

By Theorem 4.1.15, we have that a separated proximity induces a Tychonoff topology, hence a Hausdorff topology, we therefore have the following theorem which can be found in [26, Theorem 7.7].

Theorem 4.2.12 *Every separated proximity space (X, δ) is a dense subspace of a unique (up to δ-homeomorphism) compact Hausdorff space uX. Moreover, for $A, B \subseteq X$, $A \ \delta \ B$ if and only if $Cl_{uX}(A) \cap Cl_{uX}(B) \neq \emptyset$.*

Definition 4.2.13 *Let uX be the set of all ends of X and $\tau(\delta^{**})$ be the topology on uX induced by the proximity δ^{**}, then uX is called the* ***Smirnov compactification*** *of X.*

Definition 4.2.14 *Let $\Sigma(X)$ denote the set of all proximities on X. Define a partial order on $\Sigma(X)$ by saying that $\delta_1 \leqslant \delta_2$ if and only if $A \ \delta_2 \ B$ implies $A \ \delta_1 \ B$ for each $A, B \subseteq X$, which is also saying that $\delta_1 \leqslant \delta_2$ if and only if there is an identity proximity map $f : (X, \delta_2) \longrightarrow (X, \delta_1)$. Then $(\Sigma(X), \leqslant)$ is a partially ordered set.*

All this analysis of the proximity space has been leading up to the Smirnov Theorem which provides a characterization of the structure of compactifications of a completely regular space X by means of proximities on X that induce a topology on X, but before we state it we need the following theorem which we state without proof (the proof can be found in [26, Theorem 7.10], but the compactifications are constructed using clusters in that instance).

Theorem 4.2.15 *Every proximity mapping g of (X, δ_1) onto (Y, δ_2) has a unique extension to a continuous mapping g^* which maps the compactification of X onto the compactification of Y.*

Theorem 4.2.16 ***(Smirnov theorem)***. *Let X be a completely regular space. Then the partially ordered set $(\mathcal{K}(X), \leqslant)$ of inequivalent compactifications of X is order isomorphic to the partially ordered set $(\Sigma(X), \leqslant)$ of proximities that induce the topology on X.*

Proof: Let $\varphi : (\Sigma(X), \leqslant) \longrightarrow (\mathcal{K}(X), \leqslant)$. φ is onto since if αX is a compactification of X, then αX is a compact Hausdorff space and by Theorem 4.1.14 we can define a compitable proximity on αX, which in turn defines a proximity δ_X on X as a subspace of αX by Proposition 4.1.20. Therefore for any arbitrary choosen compactification, there exist a proximity δ_X such that $\varphi(\delta_X) = \alpha X$. Also, φ is one-to-one since every proximity space has only one δ-extension by Theorem 4.2.11 and hence one compactification generated by it. Now we show that $\delta_1 \leqslant \delta_2$ if and only if $uX \leqslant vX$ where $\varphi(\delta_1) = uX$ and $\varphi(\delta_2) = vX$. Suppose that $\delta_1 \leqslant \delta_2$, then there is a proximity map $f : (X, \delta_2) \longrightarrow (X, \delta_1)$ such that $f(x) = x$ for all $x \in X$. By Theorem 4.2.15 this map can be extended to a continuous map from vX into uX, therefore $uX \leqslant vX$. Conversely, let $uX \leqslant vX$ where $\varphi(\delta_1) = uX$ and $\varphi(\delta_2) = vX$, this means there is a continuous map $g : vX \longrightarrow uX$ such that $g(x) = x$ for $x \in X$. Since vX is compact, by Proposition 4.1.18 this g is a proximity map. Therefore the resitriction of g into X, $g|_X$, is the identity proximity map from (X, δ_2) into (X, δ_1), thus $\delta_1 \leqslant \delta_2$. ☐

4.3 Zero-dimensional proximities

There is another way of defining the Smirnov compactification using clusters. In this section we go through this construction, and also recall zero-dimensional proximities as defined in [7]. It turns out that if a space has a zero-dimensional proximity then that proximity induces a compactification which is zero-dimensional. Moreover, if a space is zero-dimensional, there is a one-to-one correspondence between its zero-dimensional proximities and its zero-dimensional compactification.

Definition 4.3.1 *A collection σ of subsets of a proximity space (X, δ) is called a* ***cluster*** *if and only if the following conditions are satisfied:*

1. *If A and B belong to σ, then $A \, \delta \, B$*

2. *If $A \, \delta \, C$ for every $C \in \sigma$, then $A \in \sigma$*

3. *If $(A \cup B) \in \sigma$, then $A \in \sigma$ or $B \in \sigma$*

Definition 4.3.2 *For each $x \in X$, the collection $\sigma_x = \{A \subseteq X : A \, \delta \, x\}$ is a cluster. We call such a cluster a* ***point cluster****.*

We have the following properties for clusters.

Proposition 4.3.3 *Let σ be a cluster of X, then*

1. *If $\{x\} \in \sigma$ for some $x \in X$, then $\sigma = \sigma_x$.*

2. *If σ is any cluster in X, then $X \in \sigma$. Hence for every subset E of X, either $E \in \sigma$ or $(X - E) \in \sigma$.*

3. *If $A \in \sigma$ and $A \subseteq B$, then $B \in \sigma$*

4. *If σ is any cluster in X, then $A \in \sigma$ if and only if $Cl_X(A) \in \sigma$*

Proof: (1) If $\{x\} \in \sigma$ then $x \, \delta \, A$ for every $A \in \sigma$, therefore $\sigma = \sigma_x$.
(2) We have $X \, \delta \, A$ for any $A \subseteq X$, therefore by Definition 4.3.1 (2) we have that $X \in \sigma$ for any cluster σ. Moreover, since $(X - E) \cup E$ then by Definition 4.3.1 (3) we have that $X - E \in \sigma$ or $E \in \sigma$.
(3) $A \in \sigma$, then $A \, \delta \, C$ for every $C \in \sigma$. Since $A \subseteq B$ then $B \, \delta \, C$ for every $C \in \sigma$, hence $B \in \sigma$ by Definition 4.3.1 (2).
(4) Since $A \, \delta \, B$ if and only if $Cl_X(A) \, \delta \, B$ then the result follows by Definition 4.3.1 (2). □

Proposition 4.3.4 *Let σ be a cluster in a proximity space (Y, δ) and let $X \in \sigma$ with $X \subseteq Y$. Then there exists a unique cluster $\sigma^{'}$ in (X, δ_X) such that $\sigma^{'} \subseteq \sigma$, namely $\sigma^{'} = \{A \subseteq X : A \in \sigma\}$.*

Proof: See [26, Theorem 5.16]. □

Proposition 4.3.5 *A proximity space is compact if and only if every cluster in the space is a point cluster.*

Proof: See [26, Theorem 5.13]. □

Definition 4.3.6 *Let uX denote the set of all clusters in X. For $\mathcal{P} \subseteq uX$, we say that a subset A of X* ***absorbs*** *$\mathcal{P}$ if and only if $A \in \sigma$ for every $\sigma \in \mathcal{P}$.*

Proposition 4.3.7 *The binary relation δ^* on the power set of uX defined by, $\mathcal{P}\ \delta^*\ \mathcal{D}$ if and only if A absorbs $\mathcal{P}$ and B absorbs $\mathcal{D}$ implies $A\ \delta\ B$, is a separated proximity on uX.*

Proof: We need to check that conditions of Definition 4.1.1 are satisfied.
(P_1) : If $\mathcal{P}\ \delta^*\ \mathcal{D}$ then there are sets A and B absorbing $\mathcal{P}$ and $\mathcal{D}$ respectively such that $A\ \delta\ B$. Since δ is a proximity, by (P_1) of Definition 4.1.1, A and B can not be empty, therefore $\mathcal{P}$ and $\mathcal{D}$ are non-empty.
(P_2) : Let $\mathcal{P} \subseteq uX$ and $\mathcal{D} \subseteq uX$. Suppose that $\mathcal{P} \cap \mathcal{D} \neq \varnothing$ and A and B absorb $\mathcal{P}$ and $\mathcal{D}$ respectively. We want to show that $A\ \delta\ B$. But we have that both A and B absorb $\mathcal{P} \cap \mathcal{D}$, so that $A\ \delta\ B$ and hence $\mathcal{P}\ \delta^*\ \mathcal{D}$.
(P_3) : The symmetry of δ^* follows from the fact that δ is symmetric.
(P_4) : We want to show $\mathcal{P}\ \delta^*\ \mathcal{R}$ or $\mathcal{D}\ \delta^*\ \mathcal{R}$ implies that $(\mathcal{P} \cup \mathcal{D})\ \delta^*\ \mathcal{R}$. Suppose that $\mathcal{D}\ \delta^*\ \mathcal{R}$, that D absorbs $(\mathcal{P} \cup \mathcal{D})$, and that C absorbs $\mathcal{R}$. Then D absorbs $\mathcal{D}$ and hence $D\ \delta\ C$ by assumption. Thus $(\mathcal{P} \cup \mathcal{D})\ \delta^*\ \mathcal{R}$. Conversely, suppose that $(\mathcal{P} \cup \mathcal{D})\ \delta^*\ \mathcal{R}$ and that $\mathcal{P}\ \not\delta^*\ \mathcal{R}$, then we need to show that $\mathcal{D}\ \delta^*\ \mathcal{R}$. Let B absorb $\mathcal{D}$ and C absorb $\mathcal{R}$, then we must show that $B\ \delta\ C$. Since $\mathcal{P}\ \not\delta^*\ \mathcal{R}$, there are sets A and D absorbing $\mathcal{P}$ and $\mathcal{R}$ respectively such that $A\ \not\delta\ D$. Since δ is a proximity, by (P_5) of Definition 4.1.1, there is an E such that $A\ \not\delta\ E$ and $(X - E)\ \not\delta\ D$. Because D absorbs $\mathcal{R}$ and $(X - E)\ \not\delta\ D$, $(X - E)$ belongs to no cluster in $\mathcal{R}$. Consequently, $(C - E)$ belongs to no cluster in $\mathcal{R}$. But $C = (C - E) \cup (C \cap E)$ absorbs $\mathcal{R}$, implying that $C \cap E$ absorbs $\mathcal{R}$. Now $(A \cup B)$ absorbs $(\mathcal{P} \cup \mathcal{D})$, which means that $(A \cup B)\ \delta\ (C \cap E)$ by assumption. Since $A\ \not\delta\ E$, we also have that $A\ \not\delta\ (C \cap E)$, and since $(A \cup B)\ \delta\ (C \cap E)$ we must have $B\ \delta\ (C \cap E)$, implying that $B\ \delta\ C$.
(P_5) : Suppose that $\mathcal{P}\ \not\delta^*\ \mathcal{D}$, then there are sets A and B absorbing $\mathcal{P}$ and $\mathcal{D}$ respectively such that $A\ \not\delta\ B$. Since δ is a proximity, by P_5 of Definition 4.1.1, there is an E such that $A\ \not\delta\ E$ and $(X - E)\ \not\delta\ B$. Since $(X - E)\ \not\delta\ B$ and B absorbs $\mathcal{D}$, $(X - E)$ belongs to no cluster of $\mathcal{D}$. Thus E absorbs $\mathcal{D}$. Now let $\mathcal{R} = \{\sigma \in uX : E \in \sigma\}$. Then we have that A absorbs $\mathcal{P}$, E absorbs $\mathcal{R}$ but $A\ \not\delta\ E$, which means that $\mathcal{P}\ \not\delta^*\ \mathcal{R}$. And since E belongs to no cluster in $(uX - \mathcal{R})$, $(X - E)$ absorbs $(uX - \mathcal{R})$. Since we have $(X - E)\ \not\delta\ B$, that implies $(uX - \mathcal{R})\ \not\delta^*\ \mathcal{D}$
We therefore have that δ^* is a proximity on uX. To prove that it is a

separated proximity, we note that A absorbs $\{\sigma\}$ if and only if $A \in \sigma$. Hence $\sigma_1\ \delta^*\ \sigma_2$ if and only if every set in σ_1 is in σ_2 and vise versa, that is if and only if σ_1 and σ_2 coincide. $\square$

Let τ^* denote the topology induced on uX by δ^*.

Lemma 4.3.8 *A absorbs $f(B)$ if and only if $B \subseteq Cl_X(A)$, where $Cl_X(A)$ is the $\tau(\delta)$ closure of A.*

Proof: Suppose that A absorbs $f(B)$. Let $b \in B$. Then $\sigma_b \in f(B)$, which implies $A \in \sigma_b$ since A absorbs $f(B)$. Therefore $b\ \delta\ A$ and hence $b \in Cl_X(A)$. Conversely, if $B \subseteq Cl_X(A)$, then for every $b \in B$, $b \in Cl_X(A)$ and hence $b\ \delta\ A$. It follows that $A \in \sigma_b$ which shows that $A \in \sigma_b$ for any $\sigma_b \in f(B)$. Therefore A absorbs $f(B)$. $\square$

Theorem 4.3.9 *Let (X, δ) be a separated proximity space and let uX denote the set of all clusters in X, and δ^* is the proximity on uX. Let f be defined such*

$$f : (X, \delta) \longrightarrow (uX, \delta^*)$$

where $f(x) = \sigma_x$ for every $x \in X$. Then (X, δ) is proximally isomorphic to $f(X)$ with the subspace proximity induced by δ^*, and $f(X)$ is dense in uX.

Proof: We first note that D absorbs $f(A)$ if and only if $A \subseteq Cl_X(D)$ by Lemma 4.3.8. Hence if $\mathcal{D}$ is any subset of uX, then $\mathcal{D}\ \delta^*\ f(A)$ if and only if C absorbs $\mathcal{D}$ and D absorbs $f(A)$ implies $C\ \delta\ D$, that is if and only if C absorbs $\mathcal{D}$ and $A \subseteq Cl_X(D)$ implies $C\ \delta\ D$, therefore if and only if C absorbs $\mathcal{D}$ implies $C\ \delta\ A$. Now if we take $\mathcal{D}$ to be the singleton $\{\sigma\}$, we obtain $\{\sigma\}\ \delta^*\ f(A)$ if and only if $C \in \sigma$ implies $C\ \delta\ A$, that is if and only if $A \in \sigma$. Since $Cl_{uX}(f(A)) = \{\sigma : \sigma\ \delta^*\ f(A)\}$, then the τ^*-closure of $f(A)$ is equal to $\{\sigma : A \in \sigma\}$. That means that $Cl_{uX}(f(X)) = \{\sigma : X \in \sigma\} = uX$ since X is contained in every cluster. Then $f(X)$ is dense in uX.
f is one-to-one since, if $x \neq y$ then $\{x\}\ \not\delta\ \{y\}$, therefore $y \notin \sigma_x$. Hence $\sigma_x \neq \sigma_y$.
Now $f(A)\ \delta^*\ f(B)$ if and only if $C\ \delta\ D$ whenever C and D absorb $f(A)$ and $f(B)$ respectively, that is if and only if $C\ \delta\ D$ whenever $A \subseteq Cl_X(C)$ and $B \subseteq Cl_X(D)$. But this last statement is equivalent to $A\ \delta\ B$, so that $f(A)\ \delta^*\ f(B)$ if and only if $A\ \delta\ B$. Thus X is proximally isomorphic to $f(X)$. $\square$

Theorem 4.3.10 *(uX, τ^*) is compact.*

Proof: By Proposition 4.3.5, it suffices to show that an arbitrary cluster μ in uX is a point cluster. Since $f(X)$ is dense in uX, we have $Cl_{uX}(f(X)) = uX \in \mu$ by Proposition 4.3.3 (2), and therefore by Proposition 4.3.3 (4) we have that $f(X) \in \mu$. By Proposition 4.3.4, there exists a unique cluster $\mu^{'}$ in $(f(X), \delta^*_{f(X)})$ such that $\mu^{'} \subseteq \mu$. By Theorem 4.3.9, (X, δ) is proximally isomorphic to $(f(X), \delta^*_{f(X)})$, hence there exists a cluster σ in X such that $\mu^{'} = \{f(A) : A \in \sigma\}$. From the proof of Theorem 4.3.9 we have that $A \in \sigma$ if and only if $\{\sigma\}\ \delta^*\ f(A)$. Hence $\{\sigma\}\ \delta^*\ C$ for every $C \in \mu^{'}$. It follows that $\{\sigma\} \in \mu^{'} \subseteq \mu$. Therefore there exists $\sigma \in uX$ such that $\{\sigma\} \in \mu$ and it follows by Proposition 4.3.3 (1) that μ is a point cluster. ☐

Lemma 4.3.11 *If $A\ \delta\ B$, then there exists a cluster σ in (X, δ) such that $A, B \in \sigma$.*

Proof: See [26, Theorem 514]. ☐

Let $A^* = \{\sigma : A \in \sigma\}$, then $A^* \subseteq uX$.

Lemma 4.3.12 *Let (X, δ) be a proximity space. For $A \subseteq X$, the following conditions are equivalent*

1. $A\ \not\delta\ (X - A)$
2. $A^* \cap (X - A)^* = \emptyset$
3. $(X - A)^* = uX - A^*$

Moreover, if one of the above conditions is satisfied, then A^* is clopen in uX

Proof: (1) implies (2). Suppose that $A\ \not\delta\ (X - A)$. Let $\sigma \in A^* \cap (X - A)^*$. Then $\sigma \in A^*$ and $\sigma \in (X - A)^*$. Therefore $A \in \sigma$ and $\sigma \in (X - A)$. Thus $A\ \delta\ (X - A)$ by Definition 4.3.1 (1) which contradicts our assumption. Hence $A^* \cap (X - A)^* = \emptyset$.
(2) implies (3). Let $\sigma \in uX - A^*$, then $\sigma \notin A^*$, therefore $A \notin \sigma$ which implies that $X - A \in \sigma$ by Proposition 4.3.3 (2). Thus $\sigma \in (X - A)^*$, hence $uX - A^* \subseteq (X - A)^*$. If $A^* \cap (X - A)^* = \emptyset$ then $(X - A)^* \subseteq uX - A^*$.
(3) implies (2). If $(X - A)^* = uX - A^*$, then $A^* \cap (X - A)^* = A^* \cap uX - A^* = \emptyset$.
(2) implies (1). Suppose that $A^* \cap (X - A)^* = \emptyset$. Let $A\ \delta\ (X - A)$. By

Lemma 4.3.11, there is a cluster σ of X such that $A, X - A \in \sigma$. Therefore $\sigma \in A^* \cap (X - A)^*$ which contradicts our assumption.
Moreover, since for $E \subseteq X$ we have that $E^* = Cl_{uX}(E)$ then E^* is closed in uX. In addition $(X - A)^* = uX - A^*$ implies that $uX - A^*$ is closed in uX. Therefore, A^* is open in uX, and so the three equivalent conditions imply that A^* is clopen in uX. $\square$

Definition 4.3.13 *Let δ be a proximity on X. We call δ a* ***zero-dimensional proximity*** *if it satisfies the following strong version of axiom (P_5):*
(SP_5) $A \not\delta B$ implies there is a $C \subseteq X$ such that $C \not\delta (X - C)$, $A \not\delta C$, and $(X - C) \not\delta B$.

We are now going to show that a zero-dimensional proximity induces a Smirnov compactification which is zero-dimensional.

Theorem 4.3.14 *If δ is a zero-dimensional proximity on X, then (uX, τ^*) is zero-dimensional.*

Proof: Let σ_1, σ_2 be two distinct points of uX. By Definition 4.3.1 (2), there exists $A \in \sigma_1$ and $B \in \sigma_2$ such that $A \not\delta B$. By (SP_5) of Definition 4.3.13, there is a $C \subseteq X$ such that $C \not\delta (X - C)$, $A \not\delta C$, and $(X - C) \not\delta B$. By Lemma 4.3.12, $C \not\delta (X - C)$ implies $(X - C)^* = uX - C^*$. By (P_2) of Definition 4.1.1, $A \not\delta C$ implies that $A \cap C = \emptyset$. Therefore $A \subseteq X - C$, so $X - C \in \sigma_1$ by Proposition 4.3.3 (3), and so $\sigma_1 \in (X - C)^* = uX - C^*$. Similarly, $(X - C) \not\delta B$ implies $(X - C) \cap B = \emptyset$. Thus, $B \subseteq C$, so $C \in \sigma_2$, and hence $\sigma_2 \in C^*$. Thus we have that $\sigma_1 \notin C^*$ and $\sigma_2 \in C^*$, and so we have found a clopen subset C^* of uX separating σ_1 and σ_2. This implies that uX is a Stone space since it is also compact by Theorem 4.3.10, hence uX is zero-dimensional. $\square$

If we have a compactification, then it follows by Smirnov theorem that there is a proximity that corresponds to it. In the following theorem we show that the corresponding proximity is a zero-dimensional proximity if the compactification of a zero-dimensional space is zero-dimensional.

Theorem 4.3.15 *Let X be a zero-dimensional Hausdorff space and let αX be a zero-dimensional compactification of X. Then the proximity that corresponds to αX via Smirnov theorem is a zero-dimensional proximity on X.*

Proof: Let δ be the proximity corresponding to αX, then by Theorem 4.2.12 we have that $A \,\delta\, B$ if and only if $Cl_{\alpha X}(A) \cap Cl_{\alpha X}(B) \neq \emptyset$. It follows from Smirnov theorem that δ is a proximity on X, so we only need to show that δ satisfies (SP_5) of Definition 4.3.13. Since αX is compact Hausdorff zero-dimensional and both $Cl_{\alpha X}(A)$ and $Cl_{\alpha X}(B)$ are closed in αX, then there is a clopen set $U \subseteq \alpha X$ such that $Cl_{\alpha X}(A) \subseteq U$ and $U \cap Cl_{\alpha X}(B) = \emptyset$. Let $C = X - \alpha^{-1}(U)$ where $\alpha : X \longrightarrow \alpha X$. Then

$$\begin{aligned} Cl_{\alpha X}(A) \cap Cl_{\alpha X}(C) &\subseteq U \cap Cl_{\alpha X}(X - \alpha^{-1}(U)) \\ &\subseteq U \cap (Y - U) = \emptyset. \end{aligned}$$

Therefore, $A \not\delta\, C$. Also, $X - C = \alpha^{-1}(U)$, and so

$$\begin{aligned} Cl_{\alpha X}(X - C) \cap Cl_{\alpha X}(B) &= Cl_{\alpha X}(\alpha^{-1}(U)) \cap Cl_{\alpha X}(B) \\ &\subseteq Cl_{\alpha X}(U) \cap Cl_{\alpha X}(B) \\ &= U \cap Cl_{\alpha X}(B) = \emptyset. \end{aligned}$$

Thus $(X - C) \not\delta\, B$. Moreover,

$$\begin{aligned} Cl_{\alpha X}(C) \cap Cl_{\alpha X}(X - C) &= Cl_{\alpha X}(X - \alpha^{-1}(U)) \cap Cl_{\alpha X}(\alpha^{-1}(U)) \\ &\subseteq (\alpha X - U) \cap U = \emptyset, \end{aligned}$$

and so $C \not\delta\, (X - C)$. Therefore, there is a $C \subseteq X$ such that $C \not\delta\, (X - C)$, $A \not\delta\, C$ and $(X - C) \not\delta\, B$. Hence δ satisfies axiom (SP_5), and so it is a zero-dimensional proximity on X. □

Theorem 4.3.16 *For a zero-dimensional Hausdorff space X, the partially ordered set $(\mathcal{K}_0(X), \leqslant)$ of inequivalent zero-dimensional compactifications of X is order isomorphic to the partially ordered set $(\Sigma_0(X), \leqslant)$ of zero-dimesional proximities on X that induce the topology on X.*

Proof: By Theorem 4.3.15, for a zero-dimensional space, if we have a zero-dimensional compactification then the corresponding proximity is a zero-dimensional proximity. Also, by Theorem 4.3.14, if we have a zero-dimensional proximity then the induced compactification is zero-dimensional. This implies that there is a bijection between $\mathcal{K}_0(X)$ and $\Sigma_0(X)$.
By using the order defined in Definition 4.2.14, it can be concluded that the mapping between $\mathcal{K}_0(X)$ and $\Sigma_0(X)$ preseves order in a simalar manner as in Theorem 4.2.16. □

Chapter 5

Spaces with a zero-dimensional remainder

In this chapter, we discuss spaces that have a compactification with zero-dimensional remainder. Rim-compact spaces, almost rim-compact spaces, and spaces between a given space and its Freudenthal compactification are examples of such spaces.

5.1 Rim-compact spaces and the Freudenthal compactification

The Freudenthal compactification was introduced by H. Freudenthal [15]. It is a compactification of a rim-compact space whose remainder is zero-dimensional. Skjarenko [28], proved Freudenthal-Morita Theorem which states that: there exists a compactification with zero-dimensional remainder for every rim-compact space. He defined a proximity on a rim-compact space that corresponds to the maximum compactification with zero-dimensional remainder, i.e Freudenthal compactification. In this section we give details of this construction.

Definition 5.1.1 *A Hausdorff space X is called* ***rim-compact*** *if there exists in this space a basis of open sets, each of which has a compact boundary.*

Definition 5.1.2 *Let $\mathcal{B}$ be a basis of the space X that consist of open sets with compact boundaries. If:*

1. $\bigcup_{i=1}^{n} B_i \in \mathcal{B}$ *whenever* $B_i \in \mathcal{B}$ *for* $i = 1, 2, ..., n$*,*

2. $\bigcap_{i=1}^{n} B_i \in \mathcal{B}$ *whenever* $B_i \in \mathcal{B}$ *for* $i = 1, 2, ..., n$ *and*

3. $X - Cl_X(B) \in \mathcal{B}$ *whenever* $B \in \mathcal{B}$*.*

We call such a basis for a rim-compact space a π***-compact bases****.*

The following Lemma arises from the fact that a rim-compact space is regular.

Lemma 5.1.3 *Let the closed set* A *of the space* X *be contained in the element* U *of the* π*-compact basis* $\mathcal{B}$*. Then there is an element* V *of the basis such that* $A \subseteq V$ *and* $Cl_X(V) \subseteq U$*.*

Proof: Let A be a closed subset of U. If $x \in bd_X U$ then $x \notin A$. Since $bd_X U$ is compact and X is a regular space, the set $bd_X U$ may be covered by a finite number of sets, $V_1, ..., V_n$ from $\mathcal{B}$ whose closure do not contain A. If $V = U - \bigcup_{i=1}^{n} Cl_X(V_i)$, then $A \subseteq V \subseteq Cl_X(V) \subseteq Cl_X(U) - \bigcup_{i=1}^{n} V_i \subseteq U$. We also have that V is an element of $\mathcal{B}$, since $\mathcal{B}$ contains the complements of the closures. □

We now define a proximity on X using the π-compact basis, then show that the proximity is compatible with the topology on X. By Smirnov theorem (Theorem 4.2.16), this would mean that there is a compactification of X associated with the proximity.

Proposition 5.1.4 *Let* $\mathcal{B}$ *be the* π*-compact basis of* X*, we define* $A \not\delta B$ *if and only if there is a neighborhood* U *in the basis* $\mathcal{B}$ *such that* $Cl_X(A) \subseteq U$ *and* $Cl_X(B) \subseteq X - Cl_X(U)$*. Then* δ *is a compatible proximity on* X*.*

Proof: Firstly we prove that δ is a proximity, by checking if the conditions of Definition 4.1.1 are met;
(P_1) : Since for $U \in \mathcal{B}$ where $Cl_X(A) \subseteq U$, we have also $Cl_X(\emptyset) = \emptyset \subseteq X - Cl_X(U)$ since the empty-set is a subset of any set. Therefore $\emptyset \not\delta A$ for any $A \subseteq X$.
(P_2) : If $A \not\delta B$ then $A \cap B = \emptyset$ by definition of the proximity.
(P_3) : If $B \not\delta A$, then there is a $U \in \mathcal{B}$ such that $Cl_X(B) \subseteq U$ and $Cl_X(A) \subseteq X - Cl_X(U)$. Let $V = X - Cl_X(U)$, then $V \in \mathcal{B}$ since $\mathcal{B}$ it contains the complement of the closures. Then we have $Cl_X(A) \subseteq V$ and $Cl_X(B) \subseteq X - Cl_X(V)$, hence $A \not\delta B$.
(P_4) : We are going to check that $A \not\delta (B_1 \cup B_2)$ if and only if $A \not\delta B_1$

and $A \not\delta B_2$. If $A \not\delta (B_1 \cup B_2)$ then there is a neighborhood $U \in \mathcal{B}$ such that $Cl_X(A) \subseteq U$ and $Cl_X(B_1 \cup B_2) \subseteq X - Cl_X(U)$. But $Cl_X(B_1 \cup B_2) = Cl_X(B_1) \cup Cl_X(B_2)$, and the result that $A \not\delta B_1$ and $A \not\delta B_2$ follows. Now suppose that $A \not\delta B_1$ and $A \not\delta B_2$, then there are neighborhoods $U_1, U_2 \in \mathcal{B}$ for which $Cl_X(B_1) \subseteq U_1$, $Cl_X(B_2) \subseteq U_2$, $Cl_X(A) \subseteq X - Cl_X(U_1)$ and $Cl_X(A) \subseteq X - Cl_X(U_2)$. Let $U = U_1 \cup U_2$, then $Cl_X(B_1 \cup B_2) = Cl_X(B_1) \cup Cl_X(B_2) \subseteq U_1 \cup U_2 = U$ and $Cl_X(A) \subseteq X - Cl_X(U_1 \cup U_2) = X - Cl_X(U)$ with $U \in \mathcal{B}$ (since $\mathcal{B}$ contains the finite union of its members). This means that $A \not\delta (B_1 \cup B_2)$.
(P_5) : Let $A \not\delta B$. There is a $U \in \mathcal{B}$ such that $Cl_X(A) \subseteq U$ and $Cl_X(B) \subseteq X - Cl_X(U)$. From Lemma 5.1.3, there is a $V \in \mathcal{B}$ for which $Cl_X(A) \subseteq V$ and $Cl_X(V) \subseteq U$, and also a set $W \in \mathcal{B}$ such that $Cl_X(B) \subseteq W$ and $Cl_X(W) \subseteq X - Cl_X(U)$. Since $Cl_X(V) \subseteq U$ then $X - U \subseteq X - Cl_X(V)$, hence $A \not\delta (X - U)$, and similarly we have $U \not\delta B$ as required.
Finally, we check the compatibility of the proximity δ with the topology of the space, we do this by checking that $Cl_X(A) = \{x : x \; \delta \; A\}$. Let $x \in Cl_X(A)$, then every neighborhood of x intersects the set A, i.e by the definition of the proximity, $x \; \delta \; A$. Conversely, suppose that $x \notin Cl_X(A)$, then there exist a neighborhood $U \in \mathcal{B}$ for which $x \in U$ and $U \cap A = \emptyset$. According to Lemma 5.1.3, there is a neighborhood $V \in \mathcal{B}$ of the point x such that $Cl_X(V) \subseteq U$. Since $Cl_X(A) \subseteq X - Cl_X(V)$, it follows that $x \not\delta A$. □

Let $\mathcal{B}$ be a π-compact basis on the space X, and δ be a proximity relation defined by means of this basis (by means of Proposition 5.1.4). We shall term the compactification δX, corresponding to this proximity relation, the **π-compactification**. We will now prove that any compactification generated through a π-compact basis has a zero-dimensional remainder, by proving that the compactifications have a basis of open sets whose boundaries are contained in X. In order to prove this, we need the lemmas below.

Definition 5.1.5 *If U is an open subset of X and $\delta X \in \mathcal{K}(X)$, then $O(U)$ is defined to be the set $\delta X - Cl_{\delta X}(X - U)$, and $O(U)$ is called the* ***extension*** *of U in δX.*

Lemma 5.1.6 *Let δX be the π-compactification of the space X associated with the π-compact basis $\mathcal{B}$. Then the collection $\{O(U) : U \in \mathcal{B}\}$ of open sets of δX is a basis for the topology on δX.*

Proof: Let ξ be any point of δX, and O_ξ be an arbitrary neighborhood of it. We select a neighborhood O'_ξ of the point ξ such that $Cl_{\delta X}(O'_\xi) \subseteq O_\xi$. The sets $Cl_{\delta X}(O_{\xi}^{'})$ and $\delta X - O_{\xi}$ do not intersect, therefore $Cl_X(O_{\xi}^{'} \cap X) \not\delta (X - O_{\xi})$ since $Cl_X(O_{\xi}^{'} \cap X)$ and $X - O_{\xi}$ do not intersect in X. Consequently, there is a neighborhood $U \in \mathcal{B}$ for which $Cl_X(O_{\xi}^{'} \cap X) \subseteq U$ and $X - O_{\xi} \subseteq X - Cl_X U$. Since $O_{\xi}^{'} \subseteq O(O_{\xi}^{'} \cap X)$, we get that $O_{\xi}^{'} \subseteq O(U)$ from $Cl_X(O_{\xi}^{'} \cap X) \subseteq U$. Therefore $\xi \in O(U)$. Also, $X - O_{\xi} \subseteq X - Cl_X U$ means that $O(U) \subseteq \delta X - (X - O_{\xi}) = O_{\xi} \cup (\delta X - X)$, and since U is a set from X then $O(U) \subseteq O_{\xi}$. So we have $\xi \in O(U) \subseteq O_{\xi}$, as required. □

Definition 5.1.7 *Let αX be the compactification of X. If for $U \subseteq X$ we have $bd_{\alpha X} O(U) = Cl_{\alpha X}(bd_X U)$ then we shall call the compactification αX* ***perfect with respect to the set*** *U. We say that αX is a* ***perfect compactification*** *if it is perfect with respect to every open subset of X.*

Lemma 5.1.8 *Let X be a completely regular space with the compactification αX. Let δ be the proximity relation on X corresponding to αX. If A and B are two open subsets of X with $A \not\delta B$ then $O(A \cup B) = O(A) \cup O(B)$.*

Proof: Suppose that $\xi \in O(U \cup V)$. Then $\xi \in Cl_{\alpha X}(U \cup V) = Cl_{\alpha X}(U) \cup Cl_{\alpha X}(V)$. Since $A \not\delta B$, we have $Cl_{\alpha X}(U) \cap Cl_{\alpha X}(V) = \emptyset$, then ξ can belong to only one of the sets $Cl_{\alpha X}(U)$, $Cl_{\alpha X}(V)$. Let $\xi \in Cl_{\alpha X}(U)$, so that $\xi \notin Cl_{\alpha X}(V)$. We show that $\xi \in O(U)$. We select some neighborhood O_{ξ} of the point ξ, contained in $O(U \cup V)$ and not intersecting V. Then $O_{\xi} \cap X \subseteq O(U \cup V) \cap X = U \cup V$. But, since $O_{\xi} \cap V = \emptyset$, it follows that $O_{\xi} \cap X \subseteq U$. And thus $\xi \in O_{\xi} \subseteq O(O_{\xi} \cap X) \subseteq O(U)$, therefore $O(U \cup V) \subseteq O(U) \cup O(V)$. Also, $O(U) \cup O(V) \subseteq O(U \cup V)$ since $O(U), O(V) \subseteq O(U \cup V)$. □

Lemma 5.1.9 *Let δ be a proximity relation on the space X corresponding to the compactification αX. The compactification αX of the space X is perfect with respect to the open set $U \subseteq X$ if and only if for every set $A \subseteq U$ we have that, $A \not\delta\, bd_X U$ implies $A \not\delta (X - U)$.*

Proof: Let αX be perfect with respect to the open set U, and let $A \subseteq U$ be such that $A \not\delta\, bd_X U$ which means that $Cl_{\alpha X}(A) \cap Cl_{\alpha X}(bd_X U) = \emptyset$ by Theorem 4.2.12. We must show that this implies that $A \not\delta (X - U)$. Suppose that $A \;\delta\; (X - U)$, then that means that $Cl_{\alpha X}(A) \cap Cl_{\alpha X}(X - U) \neq \emptyset$. Let $\xi \in Cl_{\alpha X}(A) \cap Cl_{\alpha X}(X - U)$. Since $A \subseteq U \subseteq O(U)$ then we have that $\xi \in Cl_{\alpha X}(O(U))$. Also, since $\xi \in Cl_{\alpha X}(X - U)$ and $O(U) = \alpha X - Cl_{\alpha X}(X - U)$ then $\xi \notin O(U)$. Consequently, $\xi \in bd_{\alpha X}(O(U))$.

Since $Cl_{\alpha X}(A) \cap Cl_{\alpha X}(bd_X U) = \emptyset$ and $\xi \in Cl_{\alpha X}(A)$ then we have that $\xi \notin Cl_{\alpha X}(bd_X U)$. This means that $bd_{\alpha X}(O(U)) \neq Cl_{\alpha X}(bd_X U)$ which is a contradiction to our assumption that αX is perfect with respect to U.
Let $U \subseteq X$. Suppose that for every set $A \subseteq U$ we have that, $A \not\delta\, bd_X U$ implies $A \not\delta\, (X - U)$. We want to prove that $bd_{\alpha X} O(U) = Cl_{\alpha X}(bd_X U)$, i.e that the compactification αX is perfect with respect to U. We have that $bd_X U \subseteq Cl_X(U) \subseteq Cl_{\alpha X}(O(U))$ and at the same time $bd_X U \cap O(U) = \emptyset$, therefore $bd_X U \subseteq bd_{\alpha X} O(U)$. Since the set $bd_{\alpha X} O(U)$ is closed in αX, we get that $Cl_{\alpha X}(bd_X U) \subseteq bd_{\alpha X} O(U)$. Now we want to prove that $bd_{\alpha X} O(U) \subseteq Cl_{\alpha X}(bd_X U)$. Let $\xi \notin Cl_{\alpha X}(bd_X U)$, and we shall take a neighborhood O_ξ of the point ξ in αX such that $Cl_{\alpha X}(O_\xi) \cap Cl_{\alpha X}(bd_X U) = \emptyset$. Let $A = O_\xi \cap U$ and $B = O_\xi \cap (X - Cl_X(U))$. Since we have $Cl_{\alpha X}(O_\xi) \cap Cl_{\alpha X}(bd_X U) = \emptyset$ that means we have $O_\xi \cap bd_X U = \emptyset$ which in turn means that $O_\xi \cap X = A \cup B$. So, $O_\xi \subseteq O(O_\xi \cap X) = O(A \cup B)$. Therefore $\xi \in O(A \cup B)$. Since $Cl_{\alpha X}(A) \cap Cl_{\alpha X}(bd_X U) = \emptyset$ then $A \not\delta\, bd_X U$ and by the condition of the lemma $A \not\delta\, (X - U)$ and that means $A \not\delta\, B$. Therefore on the basis of Lemma 5.1.8, either $\xi \in O(A)$ or $\xi \in O(B)$. If $\xi \in O(A)$ then, since $O(A) = O(O_\xi \cap U) \subseteq O(U)$, we have $\xi \in O(U)$. Since $O(U)$ is open in αX, we have that $\xi \notin bd_{\alpha X} O(U)$ as desired. Now, if $\xi \in O(B)$ then $\xi \in O(X - Cl_X(U))$ and since $O(U) \cap O(X - Cl_X U) = O(U \cap (X - Cl_X U)) = O(\emptyset) = \emptyset$ so $\xi \notin Cl_{\alpha X}(O(U))$ and hence $\xi \notin bd_{\alpha X}(O(U))$ as required. □

Lemma 5.1.10 *Let δX be a π-compactification of the space X, associated with the π-compact basis $\mathcal{B}$. And let γX be an arbitrary compactification of X such that $\delta X \leqslant \gamma X$. Then the compactification γX is perfect with respect to all the sets of the basis $\mathcal{B}$.*

Proof: Let δ and δ_1 be proximities corresponding to δX and γX respectively. Let $U \in \mathcal{B}$ be arbitrary, and let $A \subseteq U$ be such that $A \not\delta_1\, bd_X U$ in the sense of the proximity δ_1. The proximity δ_1 is such that $Cl_{\gamma X}(A) \cap bd_X U = \emptyset$ by Theorem 4.2.12, therefore $Cl_X(A) \cap bd_X U = \emptyset$ which means that $Cl_X(A) \subseteq U$. By Lemma 5.1.3, there is a set V in the basis $\mathcal{B}$ for which $Cl_X(A) \subseteq V$ and $Cl_X(V) \subseteq U$. But then $X - U \subseteq X - Cl_X(V)$ which means that $A \not\delta\, (X - U)$ and further $A \not\delta_1\, (X - U)$ since $\delta \leqslant \delta_1$. Then by Lemma 5.1.9, the result is proved. □

Theorem 5.1.11 *The remainder in every π-compactification is zero-dimensional.*

Proof If δX is the compactification associated with the π-compact basis $\mathcal{B}$, then it is a π-compactification. By Lemma 5.1.6, the set $\{O(U) : U \in \mathcal{B}\}$ is a basis for δX. Since $\delta X \leqslant \delta X$, Lemma 5.1.10 says that δX is perfect with respect to all elements of $\mathcal{B}$, hence we have $bd_{\delta X} O(U) = Cl_{\delta X}(bd_X U)$ for all U in $\mathcal{B}$. But $bd_X U$ is compact since $U \in \mathcal{B}$, hence $bd_{\delta X} O(U) = bd_X U$ which is contained in X. Therefore δX has a basis of open sets which have their boundaries contained in X, hence δX has a zero-dimensional remainder. □

Lemma 5.1.12 *Let X be a rim-compact space and*

$$\mathcal{M} = \{U : U \textit{ is open in } X \textit{ and } bd_X U \textit{ is compact}\}$$

Then $\mathcal{M}$ is a maximal π-compact basis of X.

Proof: Since $\mathcal{M}$ consists of all open sets with compact boundaries, we have $\mathcal{B} \subseteq \mathcal{M}$ for any π-compact basis $\mathcal{B}$. Then $\mathcal{M}$ is a basis for X consisting of open sets with compact boundaries since it is closed under intersection. Let $B = \bigcup_{j=1}^{n} B_j$ where $B_j \in \mathcal{M}$ for each $j \in J$. Then $B \subseteq \mathcal{B}$ for some π-compact basis. Since $\mathcal{B} \subseteq \mathcal{M}$ for any π-compact basis, then $B \subseteq \mathcal{M}$. Similarly, $\bigcap_{j=1}^{n} B_j \subseteq \mathcal{M}$ for $B_j \in \mathcal{M}$. We also have that $X - Cl_X(U) \subseteq \mathcal{B}$ for some π-compact basis, where $U \in \mathcal{M}$. Hence $\mathcal{M}$ contains the complement of the closures. □

Theorem 5.1.13 *The π-compactification, FX, associated with the π-compact basis $\mathcal{M}$ is the maximal compactification among all π-compactifications of a rim-compact space X.*

Proof: Let $\delta_{\mathcal{B}}$ be the proximity associated with an arbitrary π-compact basis $\mathcal{B}$. If $A \not\delta_{\mathcal{B}} B$, there is a $U \in \mathcal{B}$ such that $Cl_X(A) \subseteq U$ and $Cl_X(B) \subseteq X - Cl_X(U)$. But if U is in $\mathcal{B}$, then also $U \in \mathcal{M}$ since $\mathcal{B} \subseteq \mathcal{M}$ for any π-compact basis $\mathcal{B}$, therefore $A \not\delta_{\mathcal{M}} B$ where $\delta_{\mathcal{M}}$ is associated with $\mathcal{M}$. Which means that $\delta_{\mathcal{B}} \leqslant \delta_{\mathcal{M}}$. Since there is an order isomorphism between proximities and compactifications by Smirnov theorem, that means if αX is a compactification generated by the arbitrary chosen π-compact basis $\mathcal{B}$, then we have $\alpha X \leqslant FX$. □

Definition 5.1.14 *FX is called the Freudenthal compactification.*

We thus have a description of the Freudenthal compactification using proximities.

Theorem 5.1.15 *FX is a perfect compactification.*

Proof: Let U be an arbitrary open set of the space X, and let $A \subseteq U$ be a set such that $A \not\delta\ bd_X U$, where δ is the proximity corresponding to the maximal π-compact basis $\mathcal{M}$. By definition of the proximity δ; there is a $V \in \mathcal{M}$ such that

$$Cl_X(A) \subseteq V, \quad bd_X U \subseteq X - Cl_X(V).$$

Let $V' = U \cap V$. Since

$$Cl_X(V) \cap bd_X U = \emptyset,$$

then $bd_X V' \subseteq bd_X V$ which implies that $bd_X V'$ is compact being a closed subset of a compact set, and hence $V' \in \mathcal{M}$. We have:

$$Cl_X(A) \subseteq V', \quad Cl_X(V') \cap (X - U) = \emptyset$$

with $V' \in \mathcal{M}$. Thus $A \not\delta\ (X - U)$. By Lemma 5.1.10 we can conclude that FX is perfect with respect to the set U and since U was arbitrary then FX is perfect with respect to all open sets of X, hence FX is a perfect compactification. $\square$

By Theorem 6.2.19, that is to follow, since FX is a perfect compactification with zero-dimensional remainder then it is the maximum compactification with a zero-dimensional remainder that a rim-compact space can have. Also, by Theorem 6.2.21, FX is the minimum perfect compactification.

5.2 Almost rim-compact spaces

In 1987, Beverly E. J. Diamond [9] developed a theory for a class of spaces intermediate between rim-compact spaces and 0-spaces, which she calls *almost rim-compact spaces*. Each almost rim-compact space possesses a compactification in which each point of the remainder has a basis (in the compactification) of open sets whose boundaries do not intersect the remainder of the compactification. The approach used was to show that if a space satisfies the conditions of almost rim-compactness, then a defined quotient space of the Stone-Čech compactification of X is a compactification of X with the property that the remainder has a basis of open sets whose boundaries are disjoint with the remainder.

Definition 5.2.1 *A space X is a **0-space** if X has a compactification αX such that $\alpha X - X$ is zero-dimensional.*

Before we get to almost rim-compact spaces, we are dealing with special open sets of the space X and their related open sets in the compactification so we first prove some results about them.

Definition 5.2.2 *Let X be a topological space, and $\mathcal{K}(X)$ be a set of all compactifications of X, we have*

1. *An open set U of X is said to be π-**open** in X if $bd_X U$ is compact.*
2. *Suppose W is open in $\alpha X \in \mathcal{K}(X)$, we say W is a **small boundary** (denoted by sb) subset of αX if $bd_{\alpha X} W \subseteq X$.*

Lemma 5.2.3 *The intersection and union of finitely many π-open sets are π-open, as is the complement of π-open sets. And the union and intersection of finitely many sb subsets is an sb subset, as is the complement of the closure of an sb subset (i.e if A is an sb subset then $X - Cl_X(A)$ is an sb subset).*

We recall that if U is an open subset of X, and $\alpha X \in \mathcal{K}(X)$, the **extension** of U in αX, denoted by $O(U)$, is the set $\alpha X - [Cl_{\alpha X}(X - U)]$. $O(U)$ is the largest open subset of αX whose intersection with X is the set U.

Lemma 5.2.4 *Let $\alpha X \in \mathcal{K}(X)$, if W is an sb open subset of αX, then*

1. *$W \cap X$ is π-open in X.*
2. *$W = O(W \cap X)$.*

Proof: (1) We have $bd_X(W \cap X) \subseteq bd_{\alpha X} W \cap X = bd_{\alpha X} W$, hence the set $bd_X(W \cap X)$ is a closed subset of a compact subset of X, and is therefore compact.
(2) It is sufficient to show that $O(W \cap X) \subseteq W$, since the reverse inclusion is true for any open subset W of αX. Now $O(W \cap X) - W \subseteq Cl_{\alpha X}(W \cap X) - W = Cl_{\alpha X}(W) - W = bd_{\alpha X} W \subseteq X$. So we have $O(W \cap X) - W \subseteq X$ while $O(W \cap X) \cap X \subseteq W$, hence $O(W \cap X) \subseteq W$. $\square$

Proposition 5.2.5 *Let $\alpha X \in \mathcal{K}(X)$ and U, V be open subsets of X.*

1. *$O(U) \cap X = U$, hence $Cl_{\alpha X}(U) = Cl_{\alpha X}(O(U))$*

2. $O(U \cap V) = O(U) \cap O(V)$

3. If $Cl_{\alpha X}(U) \cap Cl_{\alpha X}(V) = \emptyset$*, then* $O(U \cup V) = O(U) \cup O(V)$

Proof: (1) $O(U) \cap X = X - Cl_X(X - U) = U$.
(2)

$$\begin{aligned} O(U) \cap O(V) =& [\alpha X - Cl_{\alpha X}(X - U)] \cap [\alpha X - Cl_{\alpha X}(X - U)] \\ =& \alpha X - [Cl_{\alpha X}(X - U) \cup Cl_{\alpha X}(X - V)] \\ =& \alpha X - Cl_{\alpha X}((X - U) \cup (X - V)) \\ =& \alpha X - Cl_{\alpha X}(X - (U \cap V)) \\ =& O(U \cap V). \end{aligned}$$

(3) Suppose that $\xi \in O(U \cup V)$. By (1) above $\xi \in Cl_{\alpha X}(U \cup V) = Cl_{\alpha X}(U) \cup Cl_{\alpha X}(V)$. Since $Cl_{\alpha X}(U) \cap Cl_{\alpha X}(V) = \emptyset$, then ξ can belong to only one of the sets $Cl_{\alpha X}(U)$, $Cl_{\alpha X}(V)$. Let $\xi \in Cl_{\alpha X}(U)$, so that $\xi \notin Cl_{\alpha X}(V)$. We show that $\xi \in O(U)$. We select some neighborhood O_ξ of the point ξ, contained in $O(U \cup V)$ and not intersecting V. Then $O_\xi \cap X \subseteq O(U \cup V) \cap X = U \cup V$. But, since $O_\xi \cap V = \emptyset$, it follows that $O_\xi \cap X \subseteq U$. And thus $\xi \in O_\xi \subseteq O(O_\xi \cap X) \subseteq O(U)$, therefore $O(U \cup V) \subseteq O(U) \cup O(V)$. Also, $O(U) \cup O(V) \subseteq O(U \cup V)$ since $O(U), O(V) \subseteq O(U \cup V)$. □

Proposition 5.2.6 *Let* αX *be a perfect compactification of* X*, and* U *a set that is* π*-open in* X*, then*

$$Cl_{\alpha X}(U) \cap (\alpha X - X) = O(U) \cap (\alpha X - X)$$

Proof: We have that $Cl_{\alpha X}(U) \cap (\alpha X - X) = Cl_{\alpha X}(O(U)) \cap (\alpha X - X)$. Since U is π-open in X and αX is a perfect compactification of X, $Cl_{\alpha X}(O(U)) - O(U) = bd_{\alpha X} O(U) = Cl_{\alpha X}(bd_X U) = bd_X U \subseteq X$ thus $(Cl_{\alpha X}(O(U)) - O(U)) \cap (\alpha X - X) = \emptyset$, hence $(Cl_{\alpha X}(U) - O(U)) \cap (\alpha X - X) = \emptyset$ which implies that $Cl_{\alpha X}(U) \cap (\alpha X - X) = O(U) \cap (\alpha X - X)$. □

Proposition 5.2.7 *Let* αX *be a perfect compactification of* X*. If* U *is* π*-open in* X *and* F *is closed in* X*, then* $F \subseteq U$ *if and only if* $Cl_{\alpha X}(F) \subseteq O(U)$*.*

Proof: If $Cl_{\alpha X}(F) \subseteq O(U)$ then $F \subseteq O(U) \cap X = U$. Also, if $F \subseteq U$, then $Cl_{\alpha X}(F) - O(U) = (Cl_{\alpha X}(F) - O(U)) \cap (\alpha X - X) \subseteq (Cl_{\alpha X}(U) - O(U)) \cap (\alpha X - X)$. By Proposition 5.2.6, we have $(Cl_{\alpha X}(U) - O(U)) \cap (\alpha X - X) = \emptyset$. Therefore $Cl_{\alpha X}(F) - O(U) = \emptyset$, which implies that $Cl_{\alpha X}(F) \subseteq O(U)$.

Theorem 5.2.8 *The Stone-Čech compactification βX, of a completely regular space X, is a perfect compactification.*

Proof: The Stone-Čech compactification is the maximum from the set of compactifications by Proposition 3.4.3, and therefore corresponds to the maximum proximity on X by Smirnov Theorem. We are going to prove that, for an arbitrary set $U \subseteq X$, the maximal proximity δ satisfies the condition of Lemma 5.1.10 for an arbitrary open set U. Let $A \subseteq U$ and $A \not\delta\, bd_X U$, i.e let there be a continuous function f on the space X such that $f(A) = 0$ and $f(bd_X U) = 1$. We define the function g in the following way: $g(x) = f(x)$ if $x \in Cl_X(U)$ and $g(x) = 1$ if $x \in X - U$. This function is continuous and we also have that $g(A) = 0$ and $g(X - U) = 1$, therefore $A \not\delta\, (X - U)$ as required. □

Other compactifications can be formed by partitioning the Stone-Čech compactification. The next result provides a characterization of such procedure.

Proposition 5.2.9 *Let X be a topological space and $\mathcal{C}$ a partition of βX. Then $\beta X/\mathcal{C}$ is a Hausdorff compactification of X under the quotient topology if and only if for every $A \in \mathcal{C}$ and for every open subset U of βX such that $A \subseteq U$, there is an open subset V of βX such that*

1. *$A \subseteq V \subseteq U$*
2. *V is a union of members of $\mathcal{C}$.*

Proof: Firstly we show that $\beta X/\mathcal{C}$ is a compact space. Let $f : \beta X \longrightarrow \beta X/\mathcal{C}$ be the mapping $x \longmapsto [x]$ and $\beta X/\mathcal{C}$ be endowed with the quotient topology. Since f is a quotient map then it is onto, that implies that $f(\beta X) = \beta X/\mathcal{C}$ and hence $\beta X/\mathcal{C}$ is compact since its a continuous image of a compact space. Suppose that $\beta X/\mathcal{C}$ is Hausdorff. Let $A \in \mathcal{C}$ and U be an open subset of βX such that $A \subseteq U$. Then $\beta X - U$ is closed and hence a compact subset of βX. Since f is continuous, $f(\beta X - U)$ is compact. Since $\beta X/\mathcal{C}$ is Hausdorff, $f(\beta X - U)$ is closed in $\beta X/\mathcal{C}$. Moreover $f(A)$ is a singleton subset of $\beta X/\mathcal{C}$ and it is contained in the open set $\beta X/\mathcal{C} - f(\beta X - U)$. Choose $V = f^{-1}(\beta X/\mathcal{C} - f(\beta X - U))$. Then V is open subset of βX since $\beta X/\mathcal{C}$ has the quotient topology. Also $A \subseteq V \subseteq U$ and $V = f^{-1}(f(V))$, hence V is an open subset of βX such that (1) and (2) are true.
Conversely, assume that for every $A \in \mathcal{C}$ and for every open subset U of βX

such that $A \subseteq U$, there is an open subset V of βX such that (1) and (2) are true. Let us fix two distinct points u_1 and u_2 in $\beta X/\mathcal{C}$. Let $A_1 = f^{-1}(u_1)$ and $A_2 = f^{-1}(u_2)$. Since $\{u_1\}$ and $\{u_2\}$ are singletons in $\beta X/\mathcal{C}$, and f is continuous, then A_1 and A_2 are closed subsets of βX. Since a compact Hausdorff space is normal, there are disjoint open subsets U_1 and U_2 in βX such that $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$. By assumption, there are disjoint open subsets V_1 and V_2 in βX such that $A_1 \subseteq V_1 \subseteq U_1$ and $A_2 \subseteq V_2 \subseteq U_2$, V_1 and V_2 are unions of members of $\mathcal{C}$ (that is to say $V_1 = f^{-1}(f(V_1))$ and $V_2 = f^{-1}(f(V_2))$). Now, since $V_1 = f^{-1}(f(V_1))$ and $V_2 = f^{-1}(f(V_2))$ are open in βX then $f(V_1)$ and $f(V_2)$ are open and also disjoint in $\beta X/\mathcal{C}$ because of the quotient topology. Therefore we have $u_1 \in f(V_1)$ and $u_2 \in f(V_2)$, this proves that $\beta X/\mathcal{C}$ is Hausdorff. $\square$

Now we have the neccessary tools to prove the main theorems of this section.

Definition 5.2.10 *If $Z \subseteq Y$, then Z is **0-dimensionally embedded** in Y if each point of Y has a base of open sets of Y whose boundaries are contained in $Y - Z$.*

The Freudenthal compactification, FX, of a rim-compact space X is a compactification such that $FX - X$ is 0-dimensionally embedded in FX, and is the maximum in the family of compactifications of X having a 0-dimensionally embedded remainder. The Freudenthal compactification possesses these properties as a consequence of X being rim-compact.

Definition 5.2.11 *If $Z \subseteq Y$, then Z is **relatively 0-dimensionally embedded** in Y if each point of Z has a base of open sets of Y whose boundaries are contained in $Y - Z$.*

If a space X has a compactification, αX, with a relatively 0-dimensional embedded remainder, then $\alpha X - X$ is zero-dimensional. We shall formulate internal conditions on a space X that will be shown to be equivalent to X having a compactification with a relatively 0-dimensional embedded remainder. We follow [9] in defining the following concepts.

Definition 5.2.12 *Let X be a topological space;*

1. *If $F_1, F_2 \subseteq X$, then F_1 and F_2 are π-**separated** in X if there is a π-open set U such that $F_1 \subseteq U$, and $Cl_X(U) \cap F_2 = \emptyset$.*

2. *If F and U are closed and open subsets of X, respectively, then F is* ***π-contained*** *in U if there is an π-open set V of X such that $F \subseteq V \subseteq Cl_X(V) \subseteq U$.*

3. *If F is closed in X, U is open in X, and $F \subseteq U$, then F is* ***nearly π-contained*** *in U if there is a compact subset K of F so that whenever $F^{'}$ is a closed subset of F, and $F^{'} \cap K = \emptyset$, $F^{'}$ is π-contained in U.*

4. *A space X is* ***nearly rim-compact*** *if whenever U is open in X, and $x \in U$, there is an open set W of X such that $x \in W$ and $Cl_X(W)$ is nearly π-contained in U.*

5. *A space X is* ***quasi-rim-compact*** *if for any $x \in X$, there is a compact set K_x of X, so that whenever F is a closed subset of X and $F \cap K_x = \emptyset$, then x and F are π-separated.*

6. *A space X is* ***almost rim-compact*** *if X is nearly rim-compact and quasi-rim-compact.*

Every rim-compact space is almost rim-compact. Neither near rim-compactness nor quasi-rim-compactness is sufficient to ensure that a space is a 0-space. We will now proceed to show that almost rim-compactness is a sufficient condition for a space to be a 0-space; quasi-rim-compactness provides a basis of *sb* open sets of βX for elements of the decomposition contained in $\beta X - X$, while near rim-compactness provides a basis of saturated open sets for each point of X.

Definition 5.2.13 *For $p \in \beta X$, let*

$$G_p = \bigcap \{Cl_{\beta X}(U) : U \text{ is a } \pi\text{-open subset of } X \text{ and } p \in O(U)\}.$$

In the following proposition we are going to use the fact that if $U \subseteq X$ is open such that $X - U$ is compact, and $\{F_i\}_{i \in I}$ is a family of closed sets such that $\bigcap_{i \in I} F_i \subseteq U$, then there is a finite subset $J \subseteq I$ such that $\bigcap_{j \in J} F_j \subseteq U$.

Proposition 5.2.14 *G_p is connected, for all $p \in \beta X$.*

Proof: Suppose that for some $p \in \beta X$, G_p is not connected. Let $G_p = G_1 \cup G_2$, where G_1 and G_2 are disjoint nonempty closed subsets of G_p. Since G_p is compact, G_1 and G_2 are disjoint compact subsets of βX; hence, there are open sets U_1 and U_2 of βX such that $G_i \subseteq U_i$ for $i = 1, 2$ and $Cl_{\beta X}(U_1) \cap$

$Cl_{\beta X}(U_2) = \emptyset$. Then $G_p \cap [\beta X - (U_1 \cup U_2)] = \emptyset$, and $\beta X - (U_1 \cup U_2)$ is compact, hence there is a finite collection $V_i, i = 1, 2, ..., n$ of π-open subsets of X such that $p \in O(V)_i$, for each i, and $\bigcap\{Cl_{\beta X}(V)_i : 1 \leqslant i \leqslant n\} \subseteq U_1 \cup U_2$. If $V = \bigcap\{V_i : 1 \leqslant i \leqslant n\}$, then V is a π-open subset of X, and by Proposition 5.2.5 (2), $p \in O(V)$.

Let $W_i = V \cap U_i$ for $i = 1, 2$. We have that $W_i \subseteq V$ and hence $bd_X W_i \subseteq bd_X V$ for $i = 1, 2$, therefore W_1 and W_2 are π-open subsets of X. As $Cl_{\beta X}(W_1) \cap Cl_{\beta X}(W_2) \subseteq Cl_{\beta X}(U_1) \cap Cl_{\beta X}(U_2) = \emptyset$, while $W_1 \cup W_2 = V \cap (U_1 \cup U_2) = V$, it follows by Proposition 5.2.5 (3) that $p \in O(W_1) \cup O(W_2)$. Assume without loss of generality that $p \in O(W_1)$, then $G_p \subseteq Cl_{\beta X}(W_1)$, therefore $G_p \cap U_2 \subseteq Cl_{\beta X}(W_1) \cap U_2 = \emptyset$, and hence $G_p \cap U_2 = \emptyset$ is a contradiction to our choice of U_i for $i = 1, 2$. □

Theorem 5.2.15 *If X is quasi-rim-compact, and $p \in \beta X - X$, then G_p is the (compact connected) quasi-component of p in $\beta X - X$. The set G_p has a basis of open sets of βX whose boundaries are a subset of X.*

Proof: Firsly, we are going to show that $G_p \subseteq \beta X - X$ for $p \in \beta X - X$ by showing that $x \notin \beta X - X \Longrightarrow x \notin G_p$.

If $p \in \beta X - X$ and $x \in X$ (that is $x \notin \beta X - X$) then there is a closed set F of X such that $F \cap K_x = \emptyset$ (by quasi-rim-compactness), and $p \in Cl_{\beta X}(F)$. Then x and F are π-separated, while $p \in Cl_{\beta X}(F)$. That is, there is a π-open set U of X such that $x \notin Cl_X(U) \subseteq Cl_{\beta X}(U)$ and $F \subseteq U$. Since $p \in Cl_{\beta X}(F)$, and we want $p \in O(U)$, that means we must have $Cl_{\beta X}(F) \subseteq O(U)$ and hence the next claim.

Claim: *If U is π-open in X, and F is closed in X, then $F \subseteq U$ implies that $Cl_{\beta X}(F) \subseteq O(U)$.*

Since βX is a perfect compactification by Theorem 5.2.8, we have $Cl_{\beta X}(bd_X U) = bd_{\beta X}(O(U))$ by definition. Then

$$\begin{aligned}(Cl_{\beta X}(U) - O(U)) \cap (\beta X - X) =& (Cl_{\beta X} O(U) - O(U)) \cap (\beta X - X)\\ =& (bd_{\beta X} O(U)) \cap (\beta X - X)\\ =& (Cl_{\beta X}(bd_X U)) \cap (\beta X - X)\end{aligned}$$

since U is π-open in X we have $(Cl_{\beta X}(bd_X U)) \cap (\beta X - X) = bd_X U \cap (\beta X - X) = \emptyset$. Therefore we have $(Cl_{\beta X}(U) - O(U)) \cap (\beta X - X) = \emptyset$. So we now that means $Cl_{\beta X}(F) - O(U) = (Cl_{\beta X}(F) - O(U)) \cap (\beta X - X) \subseteq (Cl_{\beta X}(U) - O(U)) \cap (\beta X - X) = \emptyset$ and the claim that $Cl_{\beta X}(F) \subseteq O(U)$ is

proved.
This implies that $p \in Cl_{\beta X}(F) \subseteq O(U)$, in other words there is a π-open U such that $p \in O(U)$ and $x \notin Cl_{\beta X}(U)$, that is $x \notin G_p$. Therefore $G_p \subseteq \beta X - X$. Now we prove that G_p is a quasi-component for $p \in \beta X - X$. Let $\mathcal{G}_p = \{U : U \text{ is } \pi\text{-open in } X \text{ and } p \in O(U)\}$, then $G_p = G_p \cap (\beta X - X) = \bigcap\{Cl_{\beta X}(U) \cap (\beta X - X) : U \in \mathcal{G}_p\}$. Since βX is a perfect compactification and by Proposition 5.2.6, we have that for each $U \in \mathcal{G}_p$;

$$\begin{aligned} bd_{\beta X}[Cl_{\beta X}(U) \cap (\beta X - X)] =& bd_{\beta X}[O(U) \cap (\beta X - X)] \\ \subseteq& bd_{\beta X}O(U) \cap (\beta X - X) \\ =& Cl_{\beta X}(bd_X U) \cap (\beta X - X) \\ =& bd_X U \cap (\beta X - X) \subseteq X \cap (\beta X - X) \\ =& \emptyset \end{aligned}$$

hence $Cl_{\beta X}(U) \cap (\beta X - X)$ has an empty boundary and is therefore clopen. This means that the quasi-component of p in $\beta X - X$ is contained in G_p. On the other hand, G_p is connected, therefore G_p is contained in the quasi-component of p in $\beta X - X$. Since it has been showen by Proposition 5.2.14 that G_p is compact and connected, that means that G_p is the connected compact quasi-component of p in $\beta X - X$.
To prove the last statement, we note the intersection of finitely many members of $\mathcal{G}_p$ is again a member of $\mathcal{G}_p$. Then by compactness, if T is a closed subsest of βX such that $G_p \cap T = \emptyset$, then there is $U \in \mathcal{G}_p$ such that $G_p \subseteq Cl_{\beta X}(U) \subseteq \beta X - T$. Since $G_p \subseteq \beta X - X$, we have that $G_p \subseteq Cl_{\beta X}(U) \cap (\beta X - X) = O(U) \cap (\beta X - X)$ by Proposition 5.2.6. Then the collection of sets $\{O(U) : U \in \mathcal{G}_p\}$ is a basis for G_p consisting of open sets of βX whose boundaries are contained in X, since $bd_{\beta X}O(U) = Cl_{\beta X}(bd_X U) = bd_X U \subseteq X$. □

Theorem 5.2.16 *If X is almost rim-compact, and U is an open subset of X, then*

$$U^S = \bigcup\{G_p : p \in \beta X - X \text{ and } G_p \subseteq O(U)\} \cup U$$

is a saturated open subset of βX.

Proof: U^S is saturated since it is the union of connected components. To show that U^S is open in βX, we show that if $p \in U^S$, then there is an open set W of βX such that $p \in W \subseteq U^S$.
First suppose that $p \in U^S \cap (\beta X - X)$: Then $G_p \subseteq O(U)$, and since the set

$\{O(U) : U \in \mathcal{G}_p\}$ (as defined in the proof of Theorem 5.2.15) is a basis for G_p, then there is a π-open set V of X such that $p \in G_p \subseteq O(V) \subseteq O(U)$. We are going to show that $O(V) \subseteq U^S$. Since $O(V) \cap X \subseteq O(U) \cap X = U$ by Proposition 5.2.5 (1), we have $O(V) \cap X \subseteq U^S$. If $q \in O(V) \cap (\beta X - X)$, then $G_q \subseteq O(V) \subseteq O(U)$, since $O(V) \cap (\beta X - X)$ is clopen in $\beta X - X$. In other words, $q \in U^S$. Since q is an arbitrary element of $O(V) \cap (\beta X - X)$, then $O(V) \cap (\beta X - X) \subseteq U^S$. Thus $O(V) \subseteq U^S$. Then $W = O(V)$ is an open set of βX having the desired property.

Now suppose that $p \in U^S \cap X = U$. Since X is nearly rim-compact we choose V to be the open set of X such that $p \in V$ and $Cl_X(V)$ is nearly π-contained in U. We show that $Cl_{\beta X}(V) \subseteq U^S$. Since $Cl_{\beta X}(V) \cap X \subseteq Cl_X(V) \subseteq U$, then $Cl_{\beta X}(V) \cap X \subseteq U^S$. Suppose $r \in Cl_{\beta X}(V) - X$. Then since $r \notin Cl_{\beta X}(K) = K$ for any compact subset K of X, there is a closed subset F of $Cl_X(V)$ such that $r \in Cl_{\beta X}(F)$ and $F \cap K = \emptyset$, where K is the compact subset of $Cl_X(V)$ withnessing the fact that $Cl_X(V)$ is nearly π-contained in U. Then F is π-contained in U; let V_1 be a π-open subset of X such that $F \subseteq Cl_X(V_1) \subseteq U$. Then $r \in Cl_{\beta X}(F) \subseteq Cl_{\beta X}(V_1) \subseteq O(U)$ because $Cl_{\beta X}(Cl_X(V_1)) = Cl_{\beta X}(V_1)$ and by Proposition 5.2.7. Since $r \in Cl_{\beta X}(V_1) \cap (\beta X - X) = O(V_1) \cap (\beta X - X)$ and $O(V_1) \cap (\beta X - X)$ is clopen in $\beta X - X$, it follows by an argument in the preceding paragraph that $r \in U^S$. Since $r \in Cl_{\beta X}(V) \cap (\beta X - X)$ was arbitrary chosen, therefore we have $Cl_{\beta X}(V) \cap (\beta X - X) \subseteq U^S$. Hence $Cl_{\beta X}(V) \subseteq U^S$. Since $Cl_{\beta X}(O(V)) = Cl_{\beta X}(V)$ by Proposition 5.2.5 (1), then $O(V) \subseteq Cl_{\beta X}(V)$, which means we can choose $W = O(V)$ as the desired open set of βX.

We have shown that if $p \in U^S$, then there is an open subset W of βX such that $p \in W \subseteq U^S$. Thus U^S is a saturated open subset of βX. $\square$

This means that if X is almost rim-compact, then $x \in X$ has an open basis in βX of saturated open sets of βX; Since the collection of open sets $\{O(U) : U$ is open in X, $x \in U\}$ is a basis for x in βX, the collection $\{U^S : U$ is open in X, $x \in U\}$ is a basis for x in βX consisting of saturated open sets.

We now have a partition of βX into compact subsets that is an upper semi-continuous decomposition, and hence the next theorem.

Theorem 5.2.17 *If X is an almost rim-compact space, then X has a compactification with relatively 0-dimensional embedded remainder.*

Proof: According to Proposition 5.2.14 and Theorem 5.2.15 $\mathcal{C}(\beta X) = \{\{x\} :$

$x \in X\} \cup \{G_p : p \in \beta X - X\}$ is a decomposition of βX into connected compact sets, and according to Theorem 5.2.16 $\mathcal{C}(\beta X)$ has a basis of saturated open sets, this means that $\mathcal{C}(\beta X)$ is an upper semicontinuous decomposition. Hence by Proposition 5.2.9, $\beta X/\mathcal{C}(\beta X)$ is a compactification of X. Theorem 5.2.15 also says that the elements of $\mathcal{C}(\beta X)$ contained in $\beta X - X$ have neighbourhood base in βX of open sets whose boundaries are a subset of X, therefore $\beta X - X$ is relatively 0-dimensionally embedded in βX. Thus, we have that $\beta X/\mathcal{C}(\beta X)$ is a compactification of X with relatively 0-dimensionally embedded remainder. □

5.3 Spaces between the locally compact part and the Freudenthal compactification

Let FX denote the Freudenthal compactification of a rim-compact, completely regular Hausdorff space X. In what follows, we characterize the spaces Y such that $X \subseteq Y \subseteq FX$. From this, a characterization of when X lies between its locally compact part $L(X)$ and its Freudenthal compactification $FL(X)$ follows, as shown by J. Hatzenbuhler and D.A. Mattson in 1995 in [18]. Such spaces necessarily possess a compactification αX for which $Cl_{\alpha X}(\alpha X - X)$ is zero-dimensional.

Definition 5.3.1 *We say that X is a **perfect subspace** of Y if and only if X is dense in Y and whenever U is a π-open set in X, then there is a π-open set U_1 in Y such that $U = U_1 \cap X$ and $bd_Y U_1 = bd_X U$.*

Lemma 5.3.2 *Suppose that X is a rim-compact perfect subspace of Y. If every point of $Y - X$ has a base of π-open neighbourhoods in Y with boundaries in X, then Y is rim-compact.*

Proof: We need only consider points in X since points in $Y - X$ are already assumed to have a base of π-open Y-neighbourhoods. Let O_p be any Y-open neighbourhood of a point $p \in X$. Choose a Y-open neighborhood N_p of p such that $Cl_Y(N_p) \subseteq O_p$. Now, since X is rim-compact, $N_p \cap X$ contains a π-open X-neighbourhood U_p of p. Since X is a perfect subspace of Y, there is a π-open set V_p in Y such that $U_p = V_p \cap X$ and $bd_Y V_p = bd_X U_p$. Then $V_p \subseteq Cl_Y(V_p) = Cl_Y(V_p \cap X) = Cl_Y(U_p) \subseteq Cl_Y(N_p) \subseteq O_p$. Thus, $V_p \subseteq O_p$, so that p has a base of π-open neighbourhood in Y. $\square$

Firstly, we say that a closed subset F **separates** a space X into sets G_1, G_2 if $X - F = G_1 \cup G_2$, where G_1 and G_2 are nonempty, disjoint, open subsets of X. In the following theorem, we shall use the equivalent characterization of a **perfect compactification** which states: αX is a perfect compactification of X if a closed set F separates X into H_1, H_2, then $Cl_{\alpha X}(F)$ separates αX into G_1, G_2 where $G_i \cap X = H_i$ $(i = 1, 2)$.

Theorem 5.3.3 *Let X be a rim-compact, dense subspace of Y. Then Y satisifies $X \subseteq Y \subseteq FX$ if and only if X is a perfect subspace of Y and each point of $Y - X$ has a base of π-open Y-neighborhoods with boundaries in X. Moreover, under these conditions, $FX = FY$.*

Proof: Suppose $X \subseteq Y \subseteq FX$. Since each point of FX has a base of neighborhoods with compact boundaries that are subsets of X, then all points of $Y - X$ also possess this property since $Y - X \subseteq FX$. Next we want to prove that X is a perfect subspace of Y. Let U be any π-open set in X. If $Cl_X(U) = X$, set $U_1 = Y - bd_X U$ then $U_1 \cap X = X - bd_X U = U$ and $bd_Y U_1 = bd_Y(bd_X U) = bd_X U$. Otherwise, since FX is perfect and since $X - bd_X U = U \cup (X - Cl_X(U))$, then $bd_X U$ separates FX into disjoint V and W such that $V \cap X = U$ and $W \cap X = X - Cl_X(U)$. Evidently, $bd_{FX} V = bd_X U$. Setting $U_1 = V \cap Y$ provides the required π-open subset of Y. Hence X is a perfect subspace of Y.
Conversely, assume the given conditions on Y hold. By Lemma 5.3.2, Y is rim-compact and hence FY exists. The inclusion map of X into Y has an extension f mapping βX onto βY. Let p_1 and p_2 be the natural projections of βX and βY onto FX and FY respectively. We now show that there is a continuous bijection g of FX onto FY such that $p_2 \circ f = g \circ p_1$. Firstly we show that g is well-defined. Observe that the restrictions of f, p_1 and p_2 to X are identity functions. Take $q \in FX - X$ so that $C_q = p_1^{-1}(q)$ is a component of $\beta X - X$ because FX is a perfect compactification. Since f is continuous, the connected set $f(C_q)$ lies in $\beta Y - X$ so that $C = p_2 \circ f(C_q)$ lies in $FY - X$. Since $Y - X$ and $FY - Y$ are zero-dimensional and hence totally disconnected, if C is not a singleton it must contain points $x \in Y - X$ and $y \in FY - Y$. Let N_x be an open FY-neighborhood of x for which $y \notin Cl_{FY}(N_x)$. Then, by assumption, there is a π-open neighborhood M_x of x in Y satisfying $Cl_Y(M_x) \subseteq N_x \cap Y$ and $bd_Y M_x \subseteq X$. Since $Y - bd_Y M_x = M_x \cup (Y - Cl_Y(M_x))$ then $bd_Y M_x$ separates FY into disjoint open G and H such that $G \cap Y = M_x$

and $H \cap Y = Y - Cl_Y(M_x)$ because FY is a perfect compactification. Since $bd_Y M_x \subseteq X$, this implies that $C \subseteq G \cup H$ which means G and H disconnect C, a contradiction since C is a connected component. Thus C is a single point and $p_2 \circ f$ is constant on each C_q, hence g exists. And since $g \circ p_1$ is surjective, then g is also onto. Hence g is a continuous surjection. Next we prove that g is one-to-one. Suppose that g is not injective, then there are points p and q in $FX - X$ for which $g(p) = g(q)$. Since FX is the Freudenthal compactification of X, then there is a FX-open neighborhood W of p such that $q \notin Cl_{FX}(W)$ and $bd_{FX}W \subseteq X$. By Lemma 5.2.4 (1), $U = W \cap X$ is π-open in X, and since X is a perfect subspace of Y there is a π-open set U_1 in Y for which $U = U_1 \cap X$ and $bd_Y U_1 = bd_X U$. Set $V_1 = Y - Cl_Y U_1$. Since FY is a perfect compactification of Y, it follows that $Cl_{FY}(U_1) \cap (FY - X)$ and $Cl_{FY}(V_1) \cap (FY - X)$ are disjoint. But $p \in Cl_{FX}(U)$ and for $V = X - Cl_X(U)$ we have $q \in Cl_{FX}(V)$. Thus $g(p) \in g(Cl_{FX}(U)) = Cl_{FY}(g(U)) = Cl_{FY}(U) \subseteq Cl_{FY}(U_1)$. Similarly $g(q) \in Cl_{FY}(V_1)$. But then $g(p) = g(q)$ contradicts the fact that $Cl_{FY}(U_1) \cap (FY - X)$ and $Cl_{FY}(V_1) \cap (FY - X)$ are disjoint. Hence g is one-to-one. Moreover, FX and FY are homeomorphic under g. ☐

If X is any space, let $L(X)$ be the set of all points in X which possess compact neighborhoods in X. Then the residue of X is $R(X) = X - L(X)$. It follows that for any compactification αX of X, $R(X) = X \cap Cl_{\alpha X}(\alpha X - X)$.

Lemma 5.3.4 *If X has a compactification αX with $Cl_{\alpha X}(\alpha X - X)$ zero-dimensional, then X is rim-compact.*

Proof: For $x \in L(X)$, then we have that x has a compact neighborhood, therefore it has a base of π-open sets. We only need to prove that if $x \in R(X)$, then x has a base in X of π-open sets. Suppose that T is a closed subset of X and that $x \in R(X) - T$. Then we have that $x \notin Cl_{\alpha X}(T)$. Let $S = Cl_{\alpha X}(T) \cap Cl_{\alpha X}(\alpha X - X)$. The set S is closed in $Cl_{\alpha X}(\alpha X - X)$, and $x \in [Cl_{\alpha X}(\alpha X - X)] - S$. Since $Cl_{\alpha X}(\alpha X - X)$ is compact and zero-dimensional, there is a compact clopen set W of $Cl_{\alpha X}(\alpha X - X)$ with $x \in W$ and $W \cap S = \emptyset$. Then the sets W and $[Cl_{\alpha X}(\alpha X - X) - W] \cup Cl_{\alpha X}(T)$ are disjoint compact subsets of the compact Hausdorff space αX. Then there is a continuous function $f : \alpha X \longrightarrow [0, 1]$ with $f(W) = 0$ and $f([Cl_{\alpha X}(\alpha X - X) - W] \cup Cl_{\alpha X}(T)) = 1$. We then have $bd_{\alpha X} f^{-1}([0, 1/2)) \subseteq bd_{\alpha X} f^{-1}(0) \cup bd_{\alpha X} f^{-1}((0, 1)) \subseteq X$, which implies that $f^{-1}([0, 1/2))$ is a sb open subset of αX. By Lemma 5.2.4 (1), this implies that $f^{-1}([0, 1/2)) \cap X$ is π-open subset of X which contains x, as required. ☐

Lemma 5.3.5 *Let FX be the Freudenthal compactification, then $Cl_{FX}(FX-X)$ is zero-dimensional.*

Proof: We have that $\{O(U) : bd_X U \text{ is compact}\}$ is a basis for FX. Since FX is a perfect compactification, $bd_{FX}O(U) = bd_X U$. Since $bd_X U$ is compact, we have that $bd_X U \subseteq L(X)$ which means that $bd_{FX}O(U) \subseteq L(X)$. Therefore $Cl_{FX}(FX-X) \subseteq FX$ has a basis of open sets whose boudaries are contained in $L(X)$. $\square$

Theorem 5.3.6 *For any space X, $L(X) \subseteq X \subseteq FL(X)$ if and only if $L(X)$ is a perfect subspace of X and $Cl_{\alpha X}(\alpha X - X)$ is zero-dimensional for some compactification αX of X.*

Proof: Assume $L(X) \subseteq X \subseteq FL(X)$. Then X has the compactification $\alpha X = FL(X)$ for which $Cl_{\alpha X}(\alpha X - X)$ is zero-dimensional by Lemma 5.3.5. Also, by Theorem 5.3.3, $L(X)$ is a perfect subspace of X.
Conversely, assume that $L(X)$ is a perfect subspace of X and for some compactification αX, $Cl_{\alpha X}(\alpha X - X)$ is zero-dimensional. By Lemma 5.3.4, FX exists and by Lemma 5.3.5, $Cl_{FX}(FX - X)$ is zero-dimensional. Since $X - L(X) = R(X)$, let N be any X-open neighborhood of $p \in R(X)$, then there is a FX-open neighborhood N_1 of p for which $N = N_1 \cap X$. Now $N_1 \cap Cl_{FX}(FX - X)$ is a neighborhood of p in $Cl_{FX}(FX - X)$, so there exists a clopen neighborhood N_2 of p in $Cl_{FX}(FX - X)$ such that $N_2 \subseteq N_1 \cap Cl_{FX}(FX - X)$ since $Cl_{FX}(FX - X)$ is zero-dimensional. Then N_2 and $Cl_{FX}(FX - X) - N_2$ can be separated by disjoint sb FX-open sets U and V, respectively, since FX is normal and a Freudenthal compactification. Moreover, we can assume that $U \subseteq N_1$. Set $U_1 = U \cap X$. Then by Lemma 5.2.4 (1), U_1 is π-open. Also, since FX is a perfect compactification, $bd_X U_1 = bd_{FX}O(U_1) = bd_{FX}O(U \cap X)$, and by Lemma 5.2.4 (2) we have $bd_{FX}O(U \cap X) = bd_{FX}U$. Therefore U_1 is a π-open in X with $bd_X U_1 \subseteq L(X)$ and $U_1 \subseteq N$. Now Theorem 5.3.3 provides that $L(X) \subseteq X \subseteq FL(X)$. $\square$

Chapter 6

Maximum zero-dimensional compactifications: The perfectness

We recall that a partial order, $\leqslant$, can be defined on the family of compactifications of X by: $\alpha_1 X \leqslant \alpha_2 X$ if there is a continuous function from $\alpha_2 X$ onto $\alpha_1 X$ which leaves the points of X fixed. In this chapter we are interested in the maximum element of $\mathcal{K}_0(X)$, where $\mathcal{K}_0(X)$ is the collection of compactifications of X with a zero-dimensional remainder. We are also interested in the perfectness of such compactifications.

6.1 Maximum compactifications

McCartney [23] constructs a compactification between two given compactifications, which has a zero-dimensional remainder under certain conditions. He used this compactification to prove that the maximum compactification with a zero-dimensional remainder is equivalent to the maximum compactification with a countable remainder. We give details of this derivation.

Lemma 6.1.1 *Let $\alpha_1 X, \alpha_2 X \in \mathcal{K}(X)$. If $\alpha_1 X \leqslant \alpha_2 X$, let $f : \alpha_2 X \longrightarrow \alpha_1 X$ be the continuous onto function that fixes the elements of X, then $f(\alpha_2 X - X) = \alpha_1 X - X$*

Proof: Suppose $f(\alpha_2 X - X) \nsubseteq \alpha_1 X - X$. This implies that, for some $x \in X$ and $y \in \alpha_2 X - X$, we have $f(x) = f(y)$. Since $\alpha_2 X$ is Hausdorff, pick disjoint

neighborhoods U of x and V of y. Now, $f(U \cap X)$ is a neighborhood of $f(x)$ in X, since $f_{|X}$ is a homeomorphism on X. So, $f(U \cap X) = f(U) \cap X$, where $f(U)$ is a neighborhood of $f(x)$ in $\alpha_1 X$. That means that $f(U)$ is the neighborhood of $f(y)$ in $\alpha_1 X$. But $f^{-1}(f(U)) \subseteq U$ is not a neighborhood of y in $\alpha_2 X$, which implies that f is not continuous at y, a contradiction.
Since f is onto, we also have $\alpha_1 X - X \subseteq f(\alpha_2 X - X)$. □

The following proposition implies that, if X possesses a zero-dimensional compactification, then it has a maximum zero-dimensional compactification.

Proposition 6.1.2 *Let $\mathcal{K}_0(X)$ be a family of zero-dimensional compactifications of a space X. Then there exists a compactification $\alpha_S X$ that is the least upper bound of $\mathcal{K}_0(X)$ with respect to the partial order $\leqslant$.*

Proof: Let $\mathcal{K}_0(X) = \{\alpha_i X : i \in I\}$. Let $P = \prod_{i \in I} \alpha_i X$ and consider the function $\alpha_S : X \longrightarrow P$ determined by $\alpha_S(x) = (\alpha_1(x), \alpha_2(x), ...)$ where $\alpha_i : X \longrightarrow \alpha_i X$. Then α_S is a homeomorphism from X into a subspace of P. Let $\alpha_S X = Cl_P(X)$. Choose αX to be an arbitrary zero-dimensional compactification, and let p denote the projection of P onto αX. Then the restriction of p on $\alpha_S X$ is the continuous map from $\alpha_S X$ onto αX that is an identity on X. Since αX was arbitrary, then $\alpha_i X \leqslant \alpha_S X$ for any $\alpha_i X \in \mathcal{K}_0(X)$.
Now, suppose that there is a compactification $\alpha' X$ such that there are continuous functions $g_i : \alpha' X \longrightarrow \alpha_i X$, $\alpha_i X \in \mathcal{K}_0(X)$. Then $F : \alpha' X \longrightarrow \alpha_S X$ defined by $F(x) = (g_1(x), g_2(x), ...)$ satisfies the condition $F \circ \alpha' = \alpha_S$, hence $\alpha_S X \leqslant \alpha' X$ and therefore $\alpha_S X$ is the desired least upper bound for $\mathcal{K}_0(X)$.
Finally, let p_i be the projection of P onto $\alpha_i X$, since $p_i(\alpha_S X - X) \subseteq \alpha_i X - X$ for all $i \in I$ by Lemma 6.1.1. This implies that $\alpha_S X - X \subseteq \prod\{\alpha_i X - X : i \in I\}$, and hence $\alpha_S X - X$ is a subspace of a zero-dimensional space, which is therefore zero-dimensional. □

Definition 6.1.3 *We say that αX is a* ***countable compactification*** *if $\alpha X - X$ is a countable set.*

The following lemma says that the family of countable compactifications is contained in $\mathcal{K}_0(X)$. The following lemma is similar to that of [30, Theorem 5] where it was shown that a countable compactification is a Wallman compactification.

Lemma 6.1.4 *If αX is any countable compactification of X, then αX has a base of open sets, whose boundaries are contained in X.*

Proof: Let $x \in \alpha X$ have the open neighborhood O_x. Then $\alpha X - O_x$ is a closed subset of αX. Since αX is a compact Hausdorff space, then αX is Tychonoff, which implies that there exists a continuous function $f : \alpha X \longrightarrow [0, 1]$ such that $f(\alpha X - O_x) = \{0\}$ and $f(x) = 1$. Choose r such that $0 < r < 1$ and r is not contained in the countable set $f(\alpha X - X)$. Let $V = \{y \in \alpha X : f(y) > r\}$. If $y \in V$, then $f(y) \notin f(\alpha X - O_x)$ which implies that $y \in O_x$, hence $x \in V \subseteq O_x$. Also, $bd_{\alpha X}V = \{y \in \alpha X : f(y) = r\}$ and since r is not contained in $f(\alpha X - X)$ then we have that $bd(V) \subseteq X$. □

We now construct a compactification, which is between two given compactifications and is zero-dimensional under certain conditions:
Suppose that $\alpha_1 X$ and $\alpha_3 X$ are compactifications of X with $\alpha_1 X \leqslant \alpha_3 X$ such that the natural map $g : \alpha_3 X \longrightarrow \alpha_1 X$ is not one-to-one. Let $z \in \alpha_1 X - X$ be any point such that $g^{-1}(z)$ contains more than one point. Assume that there is an open neighborhood G of z in $\alpha_1 X$ such that $g^{-1}(G)$ contains two disjoint open subsets H_1, H_2 of $\alpha_3 X$ with the following properties:

- $g^{-1}(G) \cap (\alpha_3 X - X) \subseteq H_1 \cup H_2$,
- $g^{-1}(z) \cap H_i \neq \emptyset$ for $i = 1, 2$.

Lemma 6.1.5 *Define a relation $\backsim$ on $\alpha_3 X$ by $x_1 \backsim x_2$ if and only if either*

$$x_1, x_2 \in H_i \cap g^{-1}(y) \text{ for some } y \in \alpha_1 X \text{ and } i = 1 \text{ or } 2,$$

or

$$x_1, x_2 \in g^{-1}(y) - (H_1 \cup H_2) \text{ for some } y \in \alpha_1 X.$$

Then $\backsim$ is an equivalence relation and for each $x \in X$, so that $\{x\}$ is the equivalence class of x.
Let $\alpha_M X$ denote the quotient space $\alpha_3 X/\backsim$.

Proof: It is clear that the relation is reflexive. If $x_1, x_2 \in H_i \cap g^{-1}(y)$ for some $y \in \alpha_1 X$ then $x_2, x_1 \in H_i \cap g^{-1}(y)$, similarly for $x_1, x_2 \in g^{-1}(y) - (H_1 \cup H_2)$, hence the relation is symmetric. Suppose $x_1 \backsim x_2$ and $x_2 \backsim x_3$. If $x_1, x_2 \in H_i \cap g^{-1}(y)$ for some $y \in \alpha_1 X$ then $x_2 \notin g^{-1}(y) - (H_1 \cup H_2)$ then also $x_3 \in H_i \cap g^{-1}(y)$ since $x_2 \backsim x_3$ hence $x_1 \backsim x_3$. To show that the equivalence class of $x \in X$ is $\{x\}$ we note that $g^{-1}(x) = x$ and hence x is the only element in $H_i \cap g^{-1}(x)$ or $g^{-1}(x) - (H_1 \cup H_2)$. □

Proposition 6.1.6 *$\alpha_M X$ is a compactification of X such that $\alpha_1 X \leqslant \alpha_M X \leqslant \alpha_3 X$, and the natural map $f : \alpha_M X \longrightarrow \alpha_1 X$ is not a homeomorphism. Furthermore, $f^{-1}(y)$ contains at most two elements for each $y \in \alpha_1 X - X$.*

Proof: Let $h : \alpha_3 X \longrightarrow \alpha_M X$ be the quotient function where each $y \in \alpha_3 X$ is sent into an equivalence class $\{y\} \in \alpha_M X$. Under the defined equivalence relation in Lemma 6.1.5, we have $h(x) = x$ for every $x \in X$ which implies that $\alpha_M X \leqslant \alpha_3 X$. For any $w \in \alpha_M X$, we have that $h^{-1}(w) \subseteq g^{-1}(y)$ for some $y \in \alpha_1 X$. So we have the function $f : \alpha_M X \longrightarrow \alpha_1 X$ such that $g = f \circ h$. We want to show that f is continuous. Suppose that V is an open set in $\alpha_1 X$, therefore $g^{-1}(V) = (h^{-1} \circ f^{-1})(V)$ is open in $\alpha_3 X$ since g is continuous. Since $\alpha_M X$ has the quotient topology, then $f^{-1}(V)$ is open in $\alpha_M X$, which means that f is continuous. This implies that $\alpha_1 X \leqslant \alpha_M X$. Also, we can see from the construction that $f^{-1}(y)$ contains at most two elements for each $y \in \alpha_1 X - X$. If $\alpha_M X$ is a compactification of X, then we indeed have $\alpha_1 X \leqslant \alpha_M X \leqslant \alpha_3 X$.

Since h is a quotient mapping, then it is onto, which means that $h(\alpha_3 X) = \alpha_M X$. This implies that $\alpha_M X$ is compact since it is a continuous image of a compact space.

We now show that $\alpha_M X$ is Hausdorff. Suppose that w_1, w_2 are distinct points of $\alpha_M X$. Then $h^{-1}(w_1), h^{-1}(w_2)$ are contained in $g^{-1}(y_1), g^{-1}(y_2)$ respectively for some $y_1, y_2 \in \alpha_1 X$.

If $y_1 = y_2$ in $\alpha_1 X$, then $g^{-1}(y_1) = h^{-1}(w_1) \cup h^{-1}(w_2)$. We may assume that $h^{-1}(w_i) \subseteq H_i$ $(i = 1, 2)$. There are sets $L_1, L_2 \subseteq \alpha_M X$ such that $h^{-1}(L_i) = H_i$ for $i = 1, 2$. These subsets L_1, L_2 are open subsets of $\alpha_M X$, they are also disjoint and contain w_1, w_2 respectively.

If $y_1 \neq y_2$. Since $\alpha_1 X$ is Hausdorff, we can find disjoint sets G_1, G_2 which are open in $\alpha_1 X$ with $\{y_i\} \subseteq G_i$ $(i = 1, 2)$. Then $g^{-1}(G_i) = h^{-1}(E_i)$ for some $E_i \subseteq \alpha_M X$ $(i = 1, 2)$. Since g and h are continuous, thus E_1 and E_2 are disjoint open subsets of $\alpha_M X$ containing w_1, w_2 respectively. □

In the following proposition, we are using the following lemma:

Lemma 6.1.7 *If $\varphi : X \longrightarrow Y$ is a closed function then whenever U is an open subset of X then the set $\{y \in Y : \varphi^{-1}(y) \subseteq U\}$ is also an open subset of Y.*

Proof: Let U be open in X. Then $X - U$ is closed, and since φ is a closed function then $\varphi(X - U) = Y - \{y \in Y : \varphi^{-1}(y) \subseteq U\}$ is closed. That implies that $\{y \in Y : \varphi^{-1}(y) \subseteq U\}$ is open in Y. $\square$

Proposition 6.1.8 *If $\alpha_1 X$ is a zero-dimensional compactification of X, and G is chosen such that $G \cap (\alpha_1 X - X)$ is clopen in $\alpha_1 X - X$, then the compactification $\alpha_M X$ is also zero-dimensional.*

Proof: By Lemma 6.1.1, we have that $\alpha_3 X - X = h^{-1}(\alpha_M X - X) = g^{-1}(\alpha_1 X - X)$. If $w \in \alpha_M X - X$ and L is any open neighborhood of w in $\alpha_M X$, then $h^{-1}(L)$ is an open neighborhood of $h^{-1}(w)$ in $\alpha_3 X$. Also $h^{-1}(w) \subseteq g^{-1}(y)$ for some $y \in \alpha_1 X - X$.
If $h^{-1}(w) = g^{-1}(y)$, then since g is a closed function, there is an open subset M of $\alpha_1 X$ containing y such that $g^{-1}(M) \subseteq h^{-1}(L)$. We can choose M such that $(\alpha_1 X - X) \cap M$ is clopen in $\alpha_1 X - X$, since $\alpha_1 X - X$ is zero-dimensional. Then there is a subset $E \subseteq \alpha_M X$ such that $h^{-1}(E) = g^{-1}(M)$. So, $w \in E \subseteq L$ and since g and h are continuous, E is open in $\alpha_M X$.

$$\begin{aligned} h^{-1}(E \cap (\alpha_M X - X)) =& h^{-1}(E) \cap (\alpha_3 X - X) \\ =& g^{-1}(M) \cap (\alpha_3 X - X) \\ =& g^{-1}(M \cap (\alpha_1 X - X)). \end{aligned}$$

Thus, $E \cap (\alpha_M X - X)$ is clopen in $\alpha_M X - X$.
If $h^{-1}(w)$ is properly contained in $g^{-1}(y)$, then we may assume that $h^{-1}(w) = g^{-1}(y) \cap H_1$. Then $h^{-1}(L) \cup H_2$ is an open subset of $\alpha_3 X$ containing $g^{-1}(y)$. Since g is a closed function, we can find an open set M' of $\alpha_1 X$ containing y, such that $M' \cap (\alpha_1 X - X)$ is clopen in $\alpha_1 X - X$ (since $\alpha_1 X - X$ is zero-dimensional), and $g^{-1}(M') \subseteq h^{-1}(L) \cup H_2$. Thus $g^{-1}(M') \cap H_1$ is open in $\alpha_3 X$. It contains $h^{-1}(w)$, and $g^{-1}(M') \cap H_1 \cap (\alpha_3 X - X)$ is clopen in $\alpha_3 X - X$. But there is a subset E' of $\alpha_M X$ with $h^{-1}(E') = g^{-1}(M') \cap H_1$. Then $\{w\} \subseteq E' \subseteq L$, E' is open in $\alpha_M X$, and $E' \cap (\alpha_M X - X)$ is clopen in $\alpha_M X - X$. $\square$

The construction of $\alpha_M X$ has been leading to the following theorem.

Theorem 6.1.9 *If αX is a maximum countable compactification of X, then it is also a maximum zero-dimensional compactification of X.*

Proof: Suppose that αX is the maximum countable compactification but it is not the maximum zero-dimensional compactification. Then there is a zero-dimensional compactification $\alpha^* X$ such that $\alpha X \leqslant \alpha^* X$ and the natural map

$\alpha^* X \longrightarrow \alpha X$ is not a homeomorphism. Thus, by the construction above, we have the compactification $\alpha_M X$ such that $\alpha X \leqslant \alpha_M X \leqslant \alpha^* X$. However, the construction of $\alpha_M X$ is such that $|\alpha_M X - X| \leqslant 2|\alpha X - X|$. This contradics the fact that αX is a maximum countable compactification. Hence, αX is a maximum zero-dimensional compactification. $\square$

6.2 Perfect compactifications

In this section we investigate conditions under which a compactification with a zero-dimensional remainder is perfect. J. R. McCartney [23] showed that a compactification with a zero-dimensional remainder is perfect if and only if it is the maximum compactification with a zero-dimensional remainder. As a consequence of this, the maximum compactification with a zero-dimensional remainder is the minimum perfect compactification a space can have. Firstly, we go through some results on filters which we are going to need in our proofs, most of these results can be found in [8].

Definition 6.2.1 *Let $\mathcal{F}$ be a collection of sets of X. Then $\mathcal{F}_A = \{B \cap A : B \in \mathcal{F}\}$ is called the* ***trace*** *of $\mathcal{F}$ on A.*

Lemma 6.2.2 *Let $\mathcal{F}$ be a filter on a set X and A be a subset of X. Then $\mathcal{F}_A$ is a filter if and only if each set of $\mathcal{F}$ meets A.*

Proof: We check that conditions of Definition 2.7.8 are satisfied by sets of $\mathcal{F}_A$.
(1): To have $\emptyset \notin \mathcal{F}_A$, we must have that $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$, of which we do by assumption.
(2): Let $W \subseteq V$ where $W \in \mathcal{F}_A$. If $W = F \cap A$, then $F \cap A \subseteq V \subseteq A$ implies that $V = (F \cup V) \cap A$. Since $F \subseteq F \cup V$ and $F \in \mathcal{F}$, then $F \cup V \in \mathcal{F}$ which implies $V \in \mathcal{F}_A$.
(3): Let $W_1, W_2 \in \mathcal{F}_A$. Since $W_1 \cap W_2 = (F_1 \cap A) \cap (F_2 \cap A) = (F_1 \cap F_2) \cap A$, we have that $W_1 \cap W_2 \in \mathcal{F}_A$ because $F_1 \cap F_2 \in \mathcal{F}$. $\square$

Definition 6.2.3 *Let $\mathcal{C}$ be a set of subsets of a set X. If $\mathcal{C}$ satisfies (B_I) and (B_{II}) below, we say that $\mathcal{C}$ is a* ***filter base****.*
(B_I): The intersection of two sets of $\mathcal{C}$ contains a set of $\mathcal{C}$.
(B_{II}): $\mathcal{C}$ is not empty, and the empty subset of X is not in $\mathcal{C}$.

Lemma 6.2.4 *Let $\mathcal{F}$ be a filter on X, and let f be a continuous function from X into Y, then $f(\mathcal{F})$ is a filter base on Y.*

Proof: Let $C_1, C_2 \in f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$. Assume that $C_1 = f(F_1)$ and $C_2 = f(F_2)$ for $F_1, F_2 \in \mathcal{F}$. Then $C_1 \cap C_2 = f(F_1) \cap f(F_2) \supseteq f(F_1 \cap F_2)$. Since $F_1 \cap F_2 \in \mathcal{F}$ then we have that $C_1 \cap C_2$ contains a set of $f(\mathcal{F})$ which means that condition (B_I) of Definition 6.2.3 is satisfied.
(B_{II}) : From the fact that $F \neq \emptyset$ implies $f(F) \neq \emptyset$, we have that $\emptyset \notin \mathcal{F}$ and hence $\emptyset \notin f(\mathcal{F})$ as required for condition (B_{II}) of Definition 6.2.3. $\square$

Definition 6.2.5 *Let X be a topological space and $\mathcal{F}$ be a filter on X. If $\mathcal{N}(x)$ is a neighborhood filter of $x \in X$, then x is said to be a* ***limit point*** *of $\mathcal{F}$ if $\mathcal{N}(x) \subseteq \mathcal{F}$. Also, $\mathcal{F}$ is said to* ***converge*** *to x, written as $\mathcal{F} \longrightarrow x$.*

Definition 6.2.6 *In a topological space X, a point x is a* ***cluster point*** *of a filter base $\mathcal{C}$ on X if $x \in \bigcap\{Cl_X(F) : F \in \mathcal{C}\}$, in that case we write $\mathcal{C} \longrightarrow x$. If $\mathcal{C}$ a filter base of $\mathcal{F}$ and $\mathcal{C} \longrightarrow x$ then $\mathcal{F} \longrightarrow x$.*

Lemma 6.2.7 *The set of cluster points of a filter base on a topological space X is closed in X.*

Proof: The set of cluster points of a filter base $\mathcal{C}$ on X is by definition the set $\bigcap\{Cl_X(F) : F \in \mathcal{C}\}$, which is the intersection of closed sets and hence closed. $\square$

The following is another characterization of compactness.

Lemma 6.2.8 *If a topological space is compact then it has at least one cluster point.*

Proof: See [32, Theorem 17.4]. $\square$

Lemma 6.2.9 *A filter on a compact space converges if it has a single cluster point.*

Proof: Suppose that $\mathcal{F}$ is a filter on a compact topological space X and that $\mathcal{F}$ has only one cluster point x. Suppose that $\mathcal{F}$ does not converge to x. That implies that the neighborhood filter $\mathcal{N}(x)$ is not a subset of $\mathcal{F}$, i.e. there is a neighborhood V of x such that $V \notin \mathcal{F}$. We may assume that V

is open, since $Int(V) \in \mathcal{F}$ would imply $V \in \mathcal{F}$ by properties of filters. Then the family

$$\mathcal{V} = \{F - V : F \in \mathcal{F}\}$$

is a filter for the compact set $X - V$, hence by Lemma 6.2.8, $\mathcal{V} \longrightarrow y$ for some $y \in X - V$. Then y is also a cluster point of $\mathcal{F}$. Which is a contradiction to our assumption that $\mathcal{F}$ has only one cluster point. Therefore $\mathcal{F} \longrightarrow x$. □

Definition 6.2.10 *Let X be a topological space and $\mathcal{C}$ be a filter base on X. A point $x \in X$ is said to be a* ***limit of a filter base*** *$\mathcal{C}$, and $\mathcal{C}$ is said to converge to x, if the filter whose base is $\mathcal{C}$ converges to x.*

Lemma 6.2.11 *Let X be a Hausdorff space. Then, no filter on X has more than one limit point.*

Proof: Suppose that X is Hausdorff and $\mathcal{F}$ is a filter on X with $\mathcal{F} \longrightarrow x$ and $\mathcal{F} \longrightarrow y$. Then each neighborhood U of x and each neighborhood V of y belongs to $\mathcal{F}$ by Definition 6.2.5, so $U \cap V \neq \emptyset$ since $\mathcal{F}$ is a filter. But that means that, since X is Hausdorff, we must have $x = y$. □

The following is a corollary of the proposition given in [8, Part I, Section 7.5, Proposition 9].

Lemma 6.2.12 *Let X, Y be two topological spaces, and f be a function from X into Y which is continuous at a point $a \in X$. Then, for every filter base $\mathcal{C}$ on X which converges to a, the filter base $f(\mathcal{C})$ converges to $f(a)$.*

Definition 6.2.13 *Let $f : X \longrightarrow Y$ be continuous, and let $\mathcal{F}$ be a filter on X. A point $y \in Y$ is said to be a* ***limit point of f with respect to the filter*** *$\mathcal{F}$ if y is a limit point of the filter base $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$. In this case, we will write*

$$lim_{\mathcal{F}} f = y.$$

Proposition 6.2.14 *Let $f : X \longrightarrow Y$ be continuous, and let $\mathcal{F}$ be a filter on X. A point $y \in Y$ is a limit of f with respect to the filter $\mathcal{F}$ if and only if, for each neighborhood V of y in Y, there is a set $M \in \mathcal{F}$ such that $f(M) \subseteq V$.*

Proof: See [8, Part I, Section 7.3, Proposition 7]. □

Theorem 6.2.15 *Let A be a dense subset of X, and $f : A \longrightarrow Y$ a continuous function from A into a regular space Y. For $x \in X$, let $\mathcal{F}_A(x)$ be the trace on A of the neighborhood filter of x in X. Then f extends to a continuous function $\hat{f} : X \longrightarrow Y$ if and only if for each $x \in X$ we have that*

$$lim_{\mathcal{F}_A(x)} f$$

exists in Y.

Proof: Since we assume that $\hat{f}$ is continuous and we have that x is the limit of $\mathcal{F}_A(x)$, for each $x \in X$ we have that $\hat{f}(x) = lim_{\mathcal{F}_A(x)} \hat{f}$ by Lemma 6.2.12. We note that, if $y \in Y$ is a limit of the filter base $f[\mathcal{F}_A(x)]$ then it is the limit of the filter base $\hat{f}[\mathcal{F}_A(x)]$, thus we have

$$\hat{f}(x) = lim_{\mathcal{F}_A(x)} \hat{f} = lim_{\mathcal{F}_A(x)} f$$

This means that the filter base $f[\mathcal{F}_A(x)]$ converge to a point as required. Conversely, suppose that the condition is satisfied and define

$$\hat{f}(x) = lim_{\mathcal{F}_A(x)} f$$

for each $x \in X$; $\hat{f}$ is well-defined since Y is a Hausdorff space by Lemma 6.2.11. We have to show that $\hat{f}$ is continuous at each point of $x \in X$. Let V' be a closed neighborhood of $\hat{f}(x)$ in Y. Then, since $\hat{f}(x)$ is the limit point of f with respect to $\mathcal{F}_A(x)$, there exists $M \in \mathcal{F}_A(x)$ such that $f(M) \subseteq V'$ by Proposition 6.2.14. Let V be the open neighborhood such that $M = V \cap A$. Since V is a neighborhood of each of its points, we have

$$\mathcal{F}_A(z) = \mathcal{F}_{V \cap A}(z)$$

for each $z \in V$. It follows that

$$\hat{f}(z) = lim_{\mathcal{F}_{V \cap A}(x)} f$$

for each $z \in V$. From this, it follows that $\hat{f}(z) \in Cl_Y[f(V \cap A)]$, and because we have $f(V \cap A) \subseteq V'$ and V' is closed, then $Cl_Y[f(V \cap A)] \subseteq V'$. We thus have $\hat{f}(V) \subseteq V'$, and since Y has a neighborhood base of closed sets by regularity, that implies that $\hat{f}$ is continuous. □

Definition 6.2.16 *We say that a closed subset F **separates** a space X into sets G_1, G_2 if $X - F = G_1 \cup G_2$, where G_1 and G_2 are non-empty disjoint open subsets of X.*

We give several equivalent definitions of perfect compactifications of which we are going to use here. See, for example, Dickman and McCoy [10] and Sklyarenko [28] for the proof of equivalences.

Definition 6.2.17 *A compactification αX of X is a* ***perfect compactification*** *if it satisfies any of the following equivalent conditions*

1. *If $y \in \alpha X - X$, then for any neighborhood N of y, $N \cap X$ can not be expressed as the union of two disjoint sets V_1, V_2 each open in $N \cap X$, and having $y \in Cl_{\alpha X}(V_1) \cap Cl_{\alpha X}(V_2)$.*
2. *If f is the natural map $f : \beta X \longrightarrow \alpha X$, then $f^{-1}(z)$ is connected for all $z \in \alpha X$*
3. *If a closed set F separates X into H_1, H_2, then $Cl_{\alpha X}(F)$ separates αX into G_1, G_2, where $G_i \cap X = H_i$ $(i = 1, 2)$.*

In the proof of the next theorem, we need the following lemma whose proof is extracted from the argument of the proof of [19, Lemma 1.4].

Lemma 6.2.18 *If G is an open subset of βX and $G \cap X$ is partitioned by open sets V_1, V_2 of X, then $\{G \cap Cl_{\beta X}(V_1), G \cap Cl_{\beta X}(V_1)\}$ is a partition of G in βX.*

Proof: Let $G \cap X = V_1 \cup V_2$. Since X is dense in βX, then $G \subseteq Cl_{\beta X}(G \cap X) = Cl_{\beta X}(V_1) \cup Cl_{\beta X}(V_2)$, then $G \subseteq [Cl_{\beta X}(V_1) \cap G] \cup [Cl_{\beta X}(V_2) \cap G]$. If $Cl_{\beta X}(V_1) \cap Cl_{\beta X}(V_2) = \emptyset$ then $\{G \cap Cl_{\beta X}(V_1), G \cap Cl_{\beta X}(V_1)\}$ is a partition of G. $\square$

We are now ready to prove the main theorems.

Theorem 6.2.19 *A compactification γX of X which has a zero-dimensional remainder is the maximum compactification with a zero-dimensional remainder if and only if it is perfect.*

Proof: ($\Longleftarrow$) Consider any element $y \in \gamma X - X$, and let $\mathcal{N}_X(y)$ denote the trace on X of the neighborhood filter of y in γX. Then by Lemma 6.2.2, $\mathcal{N}_X(y)$ is a filter on X. Let αX be any compactification of X with a zero-dimensional remainder, then $\mathcal{N}_X(y)$ is a filter base in αX. If the continuous function $f : X \longrightarrow \alpha X$ is such that $f[\mathcal{N}_X(y)]$ converges in αX, then since γX is regular and has X as a dense subset, by Theorem 6.2.15,

we must have that there exists a continuous extension of $\hat{f} : \gamma X \longrightarrow \alpha X$ of the function f. But f takes elements of X to elements of X, hence we have $f[\mathcal{N}_X(y)] = \mathcal{N}_X(y)$. Therefore, we need to prove that the filter base $\mathcal{N}_X(y)$ converges in αX. By Lemma 6.2.9 and Definition 6.2.10, we must show that $\mathcal{N}_X(y)$ has a single cluster point in αX, that is, we must show that $adh_\alpha \mathcal{N}_X(y) = \bigcap \{Cl_{\alpha X}(F) : F \in \mathcal{N}_X(y)\}$ contains only one point. Once we do that, that would imply that $\alpha X \leqslant \gamma X$ by using $\hat{f}$ as our natural map, and since αX was arbitrary that would mean that γX is the maximum compactification with a zero-dimensional remainder.
We shall show that $adh_\alpha \mathcal{N}_X(y)$ contains only one point for each $y \in \gamma X - X$. Suppose this were not so for some $y \in \gamma X - X$. It would follow that there are distinct points $z_1, z_2 \in adh_\alpha \mathcal{N}_X(y)$. By Lemma 6.2.7, $adh_\alpha \mathcal{N}_X(y)$ is closed and thus a compact in αX. So we can partition it by disjoint closed subsets K_1, K_2 of αX containing z_1, z_2 respectively. Since a compact Hausdorff space is normal, there are disjoint open sets L_1, L_2 of αX containing K_1, K_2 respectively. Set $H_i = X \cap L_i$ and $F = X - (H_1 \cup H_2)$. Then F separates X into H_1, H_2 and so, since γX is perfect, by Definition 6.2.17 (3) we must have $\gamma X - Cl_{\gamma X}(F) = G_1 \cup G_2$ where $G_i \cap X = H_i$ for $i = 1, 2$. Suppose that $y \in Cl_{\gamma X}(F)$, then F together with $\mathcal{N}_X(y)$ would form the base of a filter $\mathcal{F}$, for which

$$adh_\alpha \mathcal{F} = Cl_{\alpha X}(F) \cap adh_\alpha \mathcal{N}_X(y) \subseteq [\alpha X - (L_1 \cup L_2)] \cap (K_1 \cup K_2) = \emptyset$$

since $F \subseteq \alpha X - (L_1 \cup L_2)$ and $\alpha X - (L_1 \cup L_2)$ is closed in αX. But this is contradicting the compactness of αX by Lemma 6.2.8, hence $y \notin Cl_{\gamma X}(F)$. Therefore $y \in G_1 \cup G_2$. Suppose that $y \in G_1$, then G_1 is a neighborhood for y and hence $H_1 \in \mathcal{N}_X(y)$. Since we have that $L_2 \cap H_1 = \emptyset$, and we must have every neighborhood of z_2 containing an element of H_1 for it to be in $Cl_{\alpha X}(H_1)$, then $z \notin adh_\alpha \mathcal{N}_X(y)$, which contradicts our assumption. Hence $adh_\alpha \mathcal{N}_X(y)$ consists of a single point.

($\Longrightarrow$) Suppose γX is a compactification with a zero-dimensional remainder, that is not perfect. By Definition 6.2.17 (1) of perfectness of a compactification, there is a point $z \in \gamma X - X$ such that z has an open neighborhood G with $G \cap X = V_1 \cup V_2$, where $V_1 \cap V_2 = \emptyset$, $z \in Cl_{\gamma X}(V_i)$ for $i = 1, 2$, and V_1, V_2 are open subsets of X. We can choose G such that $G \cap (\gamma X - X)$ is clopen in $\gamma X - X$, since $\gamma X - X$ is zero-dimensional. Let the continuous map be $g : \beta X \longrightarrow \gamma X$, then $g^{-1}(G)$ is an open subset of βX and $X \cap g^{-1}(G) =$

$V_1 \cup V_2$. Then by Lemma 6.2.18, if $H_i = Cl_{\beta X}(V_i) \cap g^{-1}(G)$ for $i = 1, 2$, then $g^{-1}(G) = H_1 \cup H_2$ which implies that $g^{-1}(G) \cap (\beta X - X) \subseteq H_1 \cup H_2$. Also, since $z \in Cl_{\gamma X}(V_i)$, we have that $g^{-1}(z) \in Cl_{\beta X}(V_i)$, hence $g^{-1}(z) \cap H_i \neq \emptyset$ for $i = 1, 2$. Thus G, H_1, H_2 satisfy the conditions required in the previous section to construct a compactification $\alpha_M X$ of X, and by Proposition 6.1.8 $\alpha_M X$ is a compactification with a zero-dimensional remainder. Since we have $\gamma X \leqslant \alpha_M X$, then γX is not the maximum compactification with a zero-dimensional remainder. □

Lemma 6.2.20 *If γX is a maximum compactification with a zero-dimensional remainder of X, there is a one-to-one relation between the connected components of $\beta X - X$ and the elements of $\gamma X - X$.*

Proof: Suppose that γX is a maximum compactification with a zero-dimensional remainder. Then by Theorem 6.2.19, it is a perfect compactification. So, for the continuous function $f : \beta X \longrightarrow \gamma X$, we have that $f^{-1}(z)$ is connected for each $z \in \gamma X - X$ by Definition 6.2.17 (2). Thus $f^{-1}(z)$ is contained in a connected component of $\beta X - X$. Conversely, if M is a connected component of $\beta X - X$, then $f(M)$ is connected since f is continuous, and so contains only one element. Hence $f^{-1}(z)$ is a component of $\beta X - X$ for each $z \in \gamma X - X$. □

Theorem 6.2.21 *If X has a maximum zero-dimensional compactification γX, this is also the minimum perfect compactification.*

Proof: Suppose that αX is any perfect compactification of X, let h, g be the natural maps; $h : \beta X \longrightarrow \alpha X$ and $g : \beta X \longrightarrow \gamma X$. Then for any $z \in \alpha X - X$, $h^{-1}(z)$ is connected in $\beta X - X$ by Definition 6.2.17 (2). By Lemma 6.2.20, the component of $\beta X - X$ in which $h^{-1}(z)$ is contained is mapped onto a point $y \in \gamma X - X$ by g. This implies that $h^{-1}(z) \subseteq g^{-1}(y)$, thus we have the function $f : \alpha X \longrightarrow \gamma X$ such that $g = f \circ h$. We now show that f is continuous. Let K be a closed subset of γX. Since $f^{-1}(K) = h[g^{-1}(K)]$ and g is continuous, then $g^{-1}(K)$ is closed in βX, which implies that $g^{-1}(K)$ is compact since βX is compact. Since a continuous image of a compact set is compact, we have that $h[g^{-1}(K)]$ is compact and since αX is Hausdorff that implies that $h[g^{-1}(K)] = f^{-1}(K)$ is a closed subset of αX. Therefore, f is a continuous function, thus $\gamma X \leqslant \alpha X$. □

Chapter 7

Compactifications in pointfree topology

The work carried out in this dissertation focuses on the results on compactifications with a zero-dimensional remainder for a topological space. One avenue of research is the study of compactifications in pointfree topology. Pointfree topology is a relatively new subject and some of the results that hold in point-set topology are not yet proved for pointfree topology. Pointfree topology is the study of topology without mentioning the points and it uses the machinery from lattice theory and category theory. For substantial literature in pointfree topology we refer the reader to [20] and [27].

7.1 Frame theory

Definition 7.1.1 *A* ***frame*** *is a complete lattice L in which distributivity holds, that is*

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

for any $a \in L$ and $S \subseteq L$.

The top element and the bottom element of L will be denoted by e and 0 respectively.

Definition 7.1.2 *A* ***frame homomorphism*** *is a mapping $h : M \longrightarrow L$ such that*

1. *$h(\bigvee S) = \bigvee \{h(s) : s \in S\}$ for $S \subseteq L$, including $h(0_M) = 0_L$ and*

2. $h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2)$ *for every* $a_1, a_2 \in L$, *including* $h(e_M) = e_L$.

An example of a frame is the family of open sets of a topological space X (ordered by set inclusion), denoted by $\Omega(X)$ and a frame homomorphism that is determined by any continuous function $f : X \longrightarrow Y$ as $\Omega(f) : \Omega(Y) \longrightarrow \Omega(X)$ which takes a $U \in \Omega(Y)$ to $f^{-1}(U) \in \Omega(X)$. Hence, if one has a topological space then one essentially has a frame that corresponds to that topological space. A frame L is called spatial if $L \cong \Omega(X)$ for some topological space X.

Definition 7.1.3 *A frame homomorphism* $h : M \longrightarrow L$ *is said to be* ***dense*** *if* $h(x) = 0_L$ *implies* $x = 0_M$.

Definition 7.1.4 *A frame* L *is called* ***compact*** *if whenever* $e = \bigvee S$, *then there is a finite* $F \subseteq S$ *such that* $e = \bigvee F$.

Definition 7.1.5 *Let* L *be a frame, we say that* a *is* ***rather below*** b *and write* $a \prec b$ *if and only if* $a^* \vee b = e$, *where* a^* *is the* ***pseudocomplement*** *of* a *in* L, *given by*

$$a^* = \bigvee \{x \in L : x \wedge a = 0\}.$$

Definition 7.1.6 *A frame* L *is said to be* ***regular*** *if* $x = \bigvee\{y \in L : y \prec x\}$ *for each* $x \in L$.

Definition 7.1.7 *A* ***compactification*** *of a frame* L *is a dense onto frame homomorphism* $h : M \longrightarrow L$, *where* M *is a compact regular frame.*

7.2 Recent research on compactifications in pointfree topology

In this section we look at some recent research done on compactifications in pointfree topology.

7.2.1 Strong inclusions

In 1990, Banaschewski introduced the notion of a strong inclusion on a frame L, and showed that there is an isomorphism between the set $\mathcal{K}(L)$ of all compactifications of L and the set $\mathcal{S}(L)$ of all strong inclusions on L. We give a summary of the one-to-one correspondence between strong inclusions and compactifications which is from [6]:

A *strong inclusion* on a frame L is a binary relation $\lhd$ on L satisfying the following properties:
(S_1) $x \leqslant a \lhd b \leqslant y$ implies $x \lhd y$.
(S_2) $\lhd$ is a sublattice of $L \times L$.
(S_3) $a \lhd b$ implies $a \prec b$.
(S_4) $a \lhd b$ implies there is a $c \in L$ such that $a \lhd c \lhd b$.
(S_5) $a \lhd b$ implies $b^* \lhd a^*$
(S_6) For each $a \in L$, $a = \bigvee\{x \in L : x \lhd a\}$.

Given a strong inclusion $\lhd$ on L, we obtain a compactification on L as follows: Let $\mathcal{J}$ denote the collection of all *strongly regular* ideals on L (an ideal J on L is strongly regular if $x \in J$ implies there exist $y \in J$ such that $x \lhd y$). Then $\mathcal{J}$ is a compact regular frame, and the join map

$$\begin{aligned} \bigvee : \mathcal{J} &\longrightarrow L \\ J \in \mathcal{J} &\longmapsto \bigvee J \end{aligned}$$

gives a compactification on the frame L.

Conversely, given a compactification $h : M \longrightarrow L$, the associated strong inclusion is given by

$$a \lhd_h b \quad \text{if and only if} \quad r(a) \prec r(b)$$

where $r : L \longrightarrow M$ is the right adjoint of h, satisfying $h(a) \leqslant b$ if and only if $a \leqslant r(b)$.

7.2.2 Perfect and maximum compactifications

A perfect compactification for a topological space was defined in **Section 6.2**. In 2011, Baboolal introduced the concept of perfectness in frames [5]. Baboolal also stated the additional conditions that one must have on the strong inclusion in order for the corresponding compactification to be perfect. In the same paper, the Freudenthal compactification is defined for a class of rim-compact frames using strong inclusion and it is proved to be perfect.

In 2021, Mthethwa proved that the Freudenthal compactification, as defined by Baboolal, is zero-dimensional and the minimum perfect compactification [25]. We summarize what has been proved:

Definition 7.2.1 *Let $h : M \longrightarrow L$ be a compactification of L, then (M, h) is said to be a* ***perfect compactification*** *if for each element u of L we have*

$$r(u \vee u^*) = r(u) \vee r(u^*)$$

where $r : L \longrightarrow M$ is the right adjoint.

We have that $\alpha : X \longrightarrow \alpha X$ is a perfect compactification of a topological space X if and only if $\Omega(\alpha) : \Omega(\alpha X) \longrightarrow \Omega(X)$ is a perfect compactification of frame $\Omega(X)$.
Baboolal also gave a characterization of perfectness using the strong inclusion:

Theorem 7.2.2 *$h : M \longrightarrow L$ is a perfect compactification of L if and only if the associated strong inclusion $\lhd$ satisfies;*

$$\textit{For all } x, y \in L, x \leqslant y, x \lhd y \vee y^* \textit{ implies } x \lhd y.$$

Proof: See [5, Proposition 3.9]. □

We write $\uparrow a = \{x \in L : a \leqslant x\}$.

Definition 7.2.3 *A regular frame L is called a* ***rim-compact frame*** *if each $a \in L$ is a join of elements u such that $\uparrow(u \vee u^*)$ is compact.*

If X is a rim-compact topological space then $\Omega(X)$ is rim-compact as a frame.

Definition 7.2.4 *For a frame L, a* ***π-compact basis*** *$\mathcal{B}$ for L is a basis such that $a, b \in \mathcal{B}$ implies $a \wedge b, a \vee b \in \mathcal{B}$, $a \in \mathcal{B}$ implies $a^* \in \mathcal{B}$ and $\uparrow(a \vee a^*)$ is compact.*

Theorem 7.2.5 *For a rim-compact frame L, the relation defined by:*

$$a \lhd_{\mathcal{B}} b \textit{ if and only if there exists } u \in \mathcal{B} \textit{ such that } a \prec u \prec b$$

is a strong inclusion, where $\mathcal{B}$ is a π-compact basis.

Proof: See [5, Proposition 4.6]. $\square$

Let $(\gamma_{\mathcal{B}}L, \bigvee)$ be the compactification of L corresponding to a strong inclusion $\lhd_{\mathcal{B}}$. We say that a compactification (M, h) of a rim-compact frame L is a π-compactification if there is a π-compact basis $\mathcal{B}$ of L such that $(\gamma_{\mathcal{B}}L, \bigvee) \cong (M, h)$.

Definition 7.2.6 *If we define the π-compact basis as $\mathcal{M} = \{u \in L : \uparrow(u \vee u^*)$ is compact$\}$, then $\lhd_{\mathcal{M}}$ is a strong inclusion on L which corresponds to the **Freudenthal compactification**.*

Theorem 7.2.7 *The Freudenthal compactification is perfect for a class of rim-compact frames.*

Proof: See [5, Proposition 4.10]. $\square$

Mthethwa [25] introduced the concept of h-perfect elements relative to a compactification that allowed him to give the neccessary and sufficient conditions for a compactification of a frame to be a π-compactification.

Definition 7.2.8 *Let (M, h) be a compactification of a frame L. An element $a \in M$ is said to be **h-perfect** if $\uparrow(a \vee a^*) \cong \uparrow h(a \vee a^*)$, where the isomorphism is via h.*

Theorem 7.2.9 *For any compactification (M, h) of a frame L, let $\mathcal{B} = \{h(a) : a \in M, a$ is h-perfect$\}$ and $\mathcal{B}_M = \{a \in M : h(a) \in \mathcal{B}\}$. If $\mathcal{B}$ is a basis for L then it is π-compact basis, and given that $\mathcal{B}$ is a basis for L then $(M, h) \cong (\gamma_{\mathcal{B}}L, \bigvee)$ if and only if $\mathcal{B}_M$ is a basis for M.*

Proof: See [25, Lemma 3.4, Proposition 3.6]. $\square$

Mthethwa follows a similar technique to that of Baboolal, that is to prove the results of Skljarenko [28] for frames:

Definition 7.2.10 *Let $\mathcal{C}$ be any π-compact basis for a rim-compact frame L and $\widetilde{\mathcal{C}} = \{\bigvee J : J \in \gamma_{\mathcal{C}}L$ and $\uparrow(J \vee J^*) \cong \uparrow \bigvee(J \vee J^*)\}$ where the isomorphism is via the join map $\bigvee : \gamma_{\mathcal{C}}L \longrightarrow L$. We say that $\mathcal{C}$ is **full** if $\mathcal{C} = \widetilde{\mathcal{C}}$.*

Theorem 7.2.11 *There is a one-to-one correspondence between the set of all full π-compact bases and the set of all π-compactifications of a regular frame L.*

Proof: See [25, Theorem 4.11]. $\square$

Theorem 7.2.12 *The Freudenthal compactification has a full basis.*

Proof: See [25, Corollary 4.12]. $\square$

Theorem 7.2.13 *For full π-compact bases $\mathcal{C}$ and $\mathcal{D}$ of a rim-compact frame L, $\gamma_{\mathcal{C}}L \leqslant \gamma_{\mathcal{D}}L$ if and only if $\mathcal{C} \subseteq \mathcal{D}$.*

Proof: See [25, Proposition 4.10]. $\square$

Hence, since the Freudenthal compactification has a full basis which contains all the other π-compact bases, then it is the maximum π-compactification.

Theorem 7.2.14 *If a rim-compact frame L has a π-compactification then the remainder of L in its π-compactification is zero-dimensional. Specifically, the remainder of L in its Freudenthal compactification is zero-dimensional.*

Proof: See [25, Corollary 5.2]. $\square$

Theorem 7.2.15 *The Freudenthal compactification of a rim-compact frame is the minimum perfect compactification.*

Proof: See [25, Proposition 5.5]. $\square$

7.2.3 Banaschewski extension

In [11], Dimov gives necessary and sufficient conditions under which the Banaschewski extension for topological spaces is open. In 2021, Adjei and Dube [1] extended the Dimov's result to frames. Also, in [1], a characterization of continuous functions between two zero-dimensional topological space which has Banaschewski extension that is nearly open is given. This result is obtained as a corollary of a more general result in frames.

Banaschewski compactification: If for $a \in L$, we have that $a \vee a^* = 1$, then it is called a *complemented* element. The lattice of complemented elements of L is denoted by BL. Given a zero-dimensional frame L, denote ζL

the frame of ideals of BL. Then ζL is a compact zero-dimensional frame, and the map

$$\zeta_L : \zeta L \longrightarrow L \quad \text{given by} \quad \zeta_L(I) = \bigvee I$$

is a dense onto frame homomorphism. We call ζL the *Banaschewski compactification* of the frame L. The topological analogue of constructing the Banaschewski compactification is as follows: Let X be a zero-dimensional T_0 topological space with basis $\mathscr{Z}$ of clopen sets. Let ζX denote the collection of all ultrafilters on $\mathscr{Z}$, and for each $A \in \mathscr{Z}$, let $A_{\zeta X} = \{\mathcal{F} \in \zeta X : A \in \mathcal{F}\}$. It is seen that $\{A_{\zeta X} : A \in \mathscr{Z}\}$ can be taken as a basis for closed sets of a topology on ζX. The set ζX equipped with this topology is a compact Hausdorff, zero-dimensional space with the associated dense embedding given by

$$\begin{aligned} \Phi :& X \longrightarrow \zeta X \\ & x \longmapsto \{A \in \mathscr{Z} : x \in A\}. \end{aligned}$$

Definition 7.2.16 *In a category of zero-dimensional frames **ZDFrm** with frame homomorphisms, the extension of $h : L \longrightarrow M$ given by $h^\zeta : \zeta L \longrightarrow \zeta M$ is called the **Banaschewski extension**. Similarly, in the category of zero-dimensional Hausdorff spaces **ZDHaus** with continuous functions, the extension of $f : X \longrightarrow Y$ given by $f^\zeta : \zeta X \longrightarrow \zeta Y$ is called the **Banaschewski extension**.*

Dimov gave the condition on a continuous f in **ZDHaus** such that f^ζ is open (see [11, Corollary 4.9]); he proved that $f^\zeta : \zeta X \longrightarrow \zeta Y$ is open if and only f satisfies the condition that for every clopen $F \subseteq X$ then $Cl_Y[f(F)]$ is a clopen subset of Y.

Theorem 7.2.17 *For any morphism $h : L \longrightarrow M$ in **ZDFrm**, h^ζ is open if and only if $r(BM) \subseteq BL$ where r is the right adjoint of h.*

Proof: See [1, Theorem 2.2]. □

Since, they derived that a continuous function in **ZDHaus** satisfies the Dimov condition if and only if the right ajoint of $\Omega(f)$ sends complemented elements in $\Omega(X)$ to complemeted elements in $\Omega(Y)$, this motivated them to give the Dimov's condition for frame;

Definition 7.2.18 *A **ZDFrm**-morphism satisfies the **Dimov condition** if its right adjoint sends complemented elements to complemented elements.*

Adjei and Dube then extended Dimov's result to frames by proving the following theorem.

Theorem 7.2.19 *If the **ZDFrm**-morphism h satisfies the Dimov condition then h^ζ is open.*

Proof: This follows from Theorem 7.2.17 and Definition 7.2.18. $\square$

The Banaschewski extension extension may be open while the corresponding morphism in **ZDFrm** is not open, this motivated Adjei and Dube to seek condition on a **ZDFrm**-morphism h so that if it has the condition then we have that h is open if and only if h^ζ is open.

Definition 7.2.20 *A **ζ-map** as a **ZDFrm**-morphism $h : L \longrightarrow M$ such that $h^\zeta[\tau_L(a)] = \tau_M[h(a)]$ where $\tau_L : L \longrightarrow \zeta L$ and $\tau_M : M \longrightarrow \zeta M$ are the right adjoints of ζ_L and ζ_M respectively.*

Theorem 7.2.21 *A ζ-map is open if and only if its Banaschewski extension is open.*

Proof: See [1, Theorem 2.7]. $\square$

Taking their result into **ZDHaus**, they said that a continuous $f : X \longrightarrow Y$ between zero-dimensional spaces is a *ζ-map* if its the case that for every clopen subset F of X and open subset U of Y such that $F \subseteq f^{-1}(U)$, then there is a clopen subset H of Y such that $H \subseteq U$ and $F \subseteq f^{-1}(H)$. The result that a ζ-map between zero-dimensional Hausdorff spaces is open if and only if its Banaschewski extension is open then followed, since f is a ζ-map if and only if $\Omega(f)$ is a ζ-map.
It is clear that an open ζ-map satisfies Dimov condition, we also have that a dense onto ζ-map satisfies the Demov condition. But a general ζ-map need not satify the Demov condition as shown in [1]. This means that notion of ζ-map and Dimov condition are independent.

7.3 Future research

The one-to-one relationship between strong inclusion and compactifications immitates the one-to-one relation between proximities and compactifications in a topological space which was proved in [29] as done in **Section 4.2**. In

2009, in the paper [7], a zero-dimensional proximity is defined and it is shown that there is a one-to-one correspondence between zero-dimensional proximities and zero-dimensional compactifications of which we went through it in **Section 4.3**. Motivated by this paper, we would like to establish conditions that strong inclusions need to satisfy for a corresponding compactification to be zero-dimensional.

We can see from the previous section that Skljarenko's results (see **Section 5.1**) for topological spaces have been proved for frames. McCartney's result that a compactification of a topological space with a zero-dimensional remainder is the maximum compactification with a zero-dimensional remainder if and only if it is perfect (see **Section 6.2**) is not yet proved for frames, and we wish to explore the possibility of proving this going forward. Our motivation for this is the fact that it has been proved that the Freudenthal compactification for rim-compact frames has a zero-dimensional remainder and it is also the minimum perfect compactification, it is not yet proved that this is indeed the maximum compactification with a zero-dimensional remainder (in [25], it is shown that the Freudenthal compactification is the maximum π-compactification).

Also, there are other characterisations of the Freudenthal compactification for a topological space that do not involve proximities, for example see [10]. However, this compactification has not been explored enough in the point-free context, except for the work of Baboolal [5] and Mthethwa [25], who established that the Freudenthal compactification for rim-compact frames is perfect and is the minimum perfect compactification, respectively. We wish to extend the scope of knowledge and exhibit more properties and characterizations of this compactification in frames setting with the aim of inferring some information about zero-dimensionality and perfect compactifications in general.

Adjei and Dube also gave the necessary and sufficient condition for the Banaschewski extension of a morphism in **ZDFrm** to be open of which the topological result follows as a corollary. They do not consider the case where the Banaschewski extension of a morphism in **ZDFrm** is quasi-open, and Dimov characterized in **ZDHaus**. We wish to explore this going forward, as well as researching on other properties of the Banaschewski extension.

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