Global Embeddings of Pseudo-Riemannian Spaces

by

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Abstract

Motivated by various higher dimensional theories in high-energy-physics and cosmology, we consider the local and global isometric embeddings of pseudo-Riemannian manifolds into manifolds of higher dimensions. We provide the necessary background in general relativity, topology and differential geometry, and present the technique for local isometric embeddings. Since an understanding of the local results is key to the development of global embeddings, we review some local existence theorems for general pseudo-Riemannian embedding spaces. In order to gain insight we recapitulate the formalism required to embed static spherically symmetric space-times into five-dimensional Einstein spaces, and explicitly treat some special cases, obtaining local and isometric embeddings for the Reissner-Nordström space-time, as well as the null geometry of the global monopole metric. We also comment on existence theorems for Euclidean embedding spaces. In a recent result, it is claimed (Katzourakis 2005a) that any analytic n-dimensional space M may be globally embedded into an Einstein space $M \times F$ (F an analytic real-valued one-dimensional field). As a corollary, it is claimed that all product spaces are Einsteinian. We demonstrate that this construction for the embedding space is in fact limited to particular types of embedded spaces. We analyze this particular construction for global embeddings into Einstein spaces, uncovering a crucial misunderstanding with regard to the form of the local embedding. We elucidate the impact of this misapprehension on the subsequent proof, and amend the given construction so that it applies to all embedded spaces as well as to embedding spaces of arbitrary curvature. This study is presented as new theorems.

To

My mum,

for always believing in me.

Declaration

The study described in this thesis was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban. This thesis was completed under the supervision of Dr. Gareth Amery.

The research contained in this thesis represents original work by the author and has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been duly acknowledged in the text.

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Contents

Chapter 1

Introduction

Within the past century the study of higher dimensions has become a popular and interesting topic. It featured in the 1920's in the theories of Kaluza (1921) and Klein (1926) which attempt to unify general relativity and electromagnetism. More recently it has occurred in high-energy physics – c.f. string theory (Green *et al.* 1987, Vilenkin and Shellard 1994, Polchinski 1996, Schwarz 2000, Marolf 2004), Horava-Witten theory (Horava and Witten 1996) and D-brane models (Polchinski 1996, Sarangi and Tye 2002, Jones *et al.* 2003) – in the brane-world models of Arkani-Hamed-Dimopoulos-Dvali (1998, 1999), Randall-Sundrum (1999a, 1999b) and Dvali-Gabadadze-Porrati (2000); in spacetime-matter theory (Wesson and Overduin 1997, Wesson 1999), and in Gauss-Bonnet gravity (Dadhich 2004, 2006). String theory emerged as a way to combine gravity and quantum theory, and posits that fundamental particles of matter are characterized by one-dimensional strings with vibrational patterns (Green *et al.* 1987). It gave rise to five superstring theories that each specify ten dimensions for the universe. The theory of supergravity involves the problem of unifying general relativity and supersymmetry. P-branes, objects having length in p dimensions (Green *et al.* 1987), were found to be possible solutions in supergravity theory in eleven dimensions. It was then realized that the five superstring theories and supergravity are just different representations of an underlying eleven-dimensional theory called M-theory. In Horava-Witten theory, the dimensionality is reduced to five by compactifying six of the eleven dimensions. This has led to a great deal of interest in five-dimensional brane-world models such as those mentioned above. A brane-world is a particular 3-dimensional spatial or 4-dimensional space-time hypersurface embedded into a higher dimensional space, referred to as the bulk, where gravity propagates freely. Randall and Sundrum (1999a,b) considered a four-dimensional Riemannian space-time (the brane) embedded in a five-dimensional Anti-de Sitter bulk ($\mathrm{AdS}_{(5)}$) where the cosmological constant is negative. Dvali *et al.* (2000) presented a model in which 4-dimensional Newtonian gravity emerges on a 3-dimensional brane embedded in 5-dimensional Minkowski space where the extra dimension is infinite. In spacetime-matter theory (Wesson and Overduin 1997, Wesson 1999), matter is described as arising from higher dimensional effects. One of the most recent developments is Gauss-Bonnet theory (c.f. for instance, Dadhich 2004, 2006), which requires a higher-dimensional view of gravity and is based on the notion that gravity is self-interactive. It involves higher order derivatives of the metric than those used in general relativity, in order to encapsulate the next iteration of self-gravity. The new five-dimensional field equations reduce to the usual field equations of general relativity in four dimensions. All higher-dimensional theories require the understanding of how to embed one manifold into a higher dimensional manifold, and so motivate the study of embedding theory.

The subject of embeddings is historically rooted in a purely geometrical perspective: in exploring pseudo-Riemannian spaces mathematicians debated the "intrinsic" versus "extrinsic" (i.e. embedded) properties of these spaces. Here, the embedding space was usually taken to be Euclidean ($\mathbb{R}^n$). Global and local isometric embeddings into Euclidean spaces have been studied extensively in differential geometry (see, for example, Janet 1926, Cartan 1927, Friedman 1961, Nash 1954, 1956, Clarke 1970, Greene 1970, Greene and Jacobowitz 1971, Gunther 1989, 1991), with the first global result established by Nash in the 1950's. In all these cases, the embedding codimension is typically large (Stephani *et al.* 2003). The Campbell-Magaard theorem (Campbell 1926, Magaard 1963) provided the first local embedding result into a pseudo-Riemannian space that has a particular non-zero curvature tensor, and the codimension is reduced drastically to one. Other local results concerning pseudo-Riemannian embedding spaces with codimension one have been obtained in recent years (Anderson and Lidsey 2001, Dahia and Romero 2002a,b, Anderson *et al.* 2003). So, it seems that introducing curvature in the bulk helps in the reduction of the local codimension, and one might ask whether this also holds true globally. In the context of renewed interest in higher dimensional theories with non-compactified extra dimensions, this problem has become important to high-energy physics and cosmology, since the relevant natural and/or toy models are not necessarily posited in a local language (Moodley and Amery 2007). This has raised concerns as to the utility of the local results as a protective theorem for higher dimensional gravity models – see reference (Anderson 2004) where it is argued that the embedding space is not guaranteed to be well behaved with respect to physical properties such as stability and causality. We refer the reader to Wesson (2005) and Dahia and Romero (2005a,b) for responses to that paper. We also discuss this in §3.3. Global embeddings can provide insight on this issue through explicit constructions relating the local and global geometries of the bulk.

A recent theorem given by Katzourakis (2005a) claims that the Campbell-Magaard-Dahia-Romero theorem for embedding into Einstein spaces can be made global, and his subsequent papers (Katzourakis 2005b,c,d) build upon this result . However, it seems that this result holds only for Ricci-flat embedding spaces (Anderson 2004, Wesson 2005), and careful analysis of the theorem reveals that there has been a crucial misunderstanding of the local Einstein embedding result by Dahia and Romero (2002b): it is assumed that the local embedding space always has the form $M \times F$ where M is the embedded space and F is a one-dimensional analytic manifold, but this can only be true if M is Ricci-flat. Thus, as it is written, Katzourakis's result is limited since it does not apply to all embedded spaces. We focus on this problem and aim to correct and improve on Katzourakis's proof.

In order to pursue this study of embeddings we require a full understanding of the mathematics underlying it. So, we begin in Chapter 2 with a review of the necessary material in general relativity and topology. In Section 2.2 we provide a review of concepts in topology that are pertinent to this research. We discuss basic differential geometry in Section 2.3, in particular the Riemann tensor, Ricci tensor and the field equations. Curvature is an important property of a manifold and so we introduce the notions of extrinsic and intrinsic curvature in Section 2.4. In Section 2.5 we consider four-dimensional static spherically symmetric (SSS) space-times, and we present the form for the metric and calculate the components of the Ricci tensor in Section 2.5.1.

We choose to embed the Reissner-Nordström (Reissner 1916, Nordström 1918) and global monopole (Barriola and Vilenkin 1989) space-times which we discuss in Section 2.5.2 and Section 2.5.3, respectively. Both these metrics are significant in astrophysics and early universe cosmology. They occur as limiting cases for the black hole solutions in the brane-world scenario (Dadhich *et al.* 2000) and in Einstein-Gauss-Bonnet gravity (Maeda and Dadhich 2006). In Section 2.6 we provide definitions for global and local isometric embeddings of metric spaces, and in Section 2.7 we present the Cauchy-Kowalewski theorem, which is useful in proving local existence results.

Before proceeding to construct global embeddings, one needs to understand the local theory. As we just mentioned, misunderstandings of the local results can have an impact on global constructions. Since causality implies that all physics is local, local embeddings are interesting in their own right (c.f. Amery *et al.* 2007, Dadhich 2007). So, in Chapter 3, we focus on local isometric embeddings. Firstly, in Section 3.2 we outline the Gauss, Codazzi and Ricci equations that govern an embedding such that the image of the embedded space coincides with a hypersurface in the bulk, implying that co-ordinates in the embedded space are "adapted" to the embedding. We review some local existence results for pseudo-Riemannian and Euclidean embedding spaces in Section 3.3. In Section 3.4 we concentrate on particular results given by Dahia and Romero (2002a,b) for an Einstein space and a pseudo-Riemannian space with a non-degenerate Ricci tensor which is equal, up to a local analytic diffeomorphism, to the Ricci tensor of an arbitrarily given pseudo-Riemannian manifold. We summarise these proofs since they are pertinent to the construction of global embeddings. Now the embedding equations are not easily solvable even with specific choices, and a knowledge of a space-time's Killing geometry and the concept of the "rigidity" of a manifold's embedding can be useful, so we discuss these notions in Section 3.5. Then we proceed to discuss the formalism for the local isometric embedding of static spherically symmetric space-times into five-dimensional Einstein spaces in Section 3.6. Einstein spaces are common in high-energy physics – c.f. type IIA super-gravity (Howe *et al.* 1998, Lavrinenko *et al.* 1998), Wesson's "space-time-matter theory" (Wesson and Overduin 1997, Wesson 1999), the Randall-Sundrum (1999a, 1999b) brane-world scenarios, and Gauss-Bonnet gravity (Dadhich 2004, 2006). We also discuss the nature of four-dimensional space-times that may be embedded into spaces of constant curvature. As examples, we apply the procedure to the Reissner-Nordström and global monopole space-times in Section 3.7 and Section 3.8, respectively. The Reissner-Nordström embedding is quite easily obtained as is that for the global monopole's null-geometry, but the embedding equations for the full global monopole space-time remain unsolved. The work contained in this chapter will be submitted for publication (Amery *et al.* 2007). The application of the formalism for local embeddings into SSS space-times to specific examples and the subsequent analysis are the author's original contribution to this work, which extends that of Londal (2005).

In Chapter 4, we proceed to the study of global isometric embeddings. Firstly, in Section 4.2 we provide some background on global embeddings into pseudo-Euclidean spaces, and then in Section 4.3 we consider the construction given by Katzourakis (2005a) for embedding into Einstein spaces. We show that careful analysis of the theorem indicates that it applies only to Ricci-flat embedding spaces. In Section 4.4 we provide an improvement on the given theorem with comments on the stages of the proof, and in Section 4.5 we extend it to more general pseudo-Riemannian spaces. We present this work as two theorems: Theorem 1 pertaining to embeddings into Einstein spaces, and its immediate generalization, Theorem 2, pertaining to embeddings into arbitrarily specified pseudo-Riemannian spaces. We also contextualize these theorems as special cases of appropriate (and new, at least to physicists) theorems pertinent to metric spaces; and, even more generally, to paracompact manifolds. In Section 4.6 we discuss the papers (Katzourakis 2005b,c,d) that build upon the initial result, and finally, in Section 4.7 we provide further comments on embeddings. The work contained in this chapter will be submitted for publication (Moodley and Amery 2007). The analysis of Katzourakis's construction and its improvements are original research carried out by the author.

In Chapter 5, we summarize the work presented in this dissertation, comment on future prospects in the field, and note the work that we are currently pursuing.

We consistently adopt the following notational conventions: Roman lower case indices label the co-ordinates of the embedded space, Roman upper case indices label its spatial co-ordinates and Greek indices label the co-ordinates of the embedding space. We use a tilde to denote quantities pertaining to the embedded space and an overbar to denote quantities obtained from the n-dimensional component of the higher-dimensional metric. The expression C^k ($k \geq 1$) means k-times continuously differentiable and C^∞ means infinitely continuously differentiable.

Chapter 2

General Relativity and Topology

2.1 Introduction

Einstein's theory of general relativity introduces the concept that gravitation is an effect of the curvature of space-time, the four-dimensional universe that we live in. Prior to the formulation of this theory, gravity was viewed as a force in the same way that electromagnetism is a force. Einstein realized that the gravitational attraction between objects can be observed as the objects' reaction to the curvature of space-time. Since then, the theory has developed into an extensive subject and it plays a key role in the understanding of many aspects of astrophysics and cosmology. It is, so far, the most successful description of gravity. In this chapter we review the material in topology and general relativity that will be required in the construction of embeddings. The theory presented here can be found in texts on topology by Bredon (1997), Choquet-Bruhat *et al.* (1982), Munkres (1966) and Szekeres (2004), and in texts on relativity by Hawking and Ellis (1973), Hobson *et al.* (2006), Sachs and Wu (1977) and Stephani (2004). Additional references are cited as applicable. We begin in §2.2 where we review some standard concepts in topology that are relevant to the study of embeddings. In §2.3 we survey the basic structures in differential geometry. There we define a metric tensor which gives rise to the connection, the Riemann tensor and the Ricci tensor. We introduce the Einstein field equations, which are the most important mathematical expressions of general relativity. The concept of extrinsic curvature is significant in embedding theory and so we provide a detailed discussion of extrinsic and intrinsic curvature in §2.4. In §2.5 we consider static spherically symmetric space-times, as we shall investigate their embeddings later on. In §2.6 we provide definitions for global and local isometric embeddings of metric spaces, and finally, in §2.7 we present the Cauchy-Kowalewski theorem, which is useful in proving local existence results.

2.2 Topology

Definition 1. *A n-dimensional C^k (C^∞) differentiable manifold is a second countable Hausdorff space M^n together with a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ such that:*

1. *each chart is a homeomorphism $\phi_\alpha : U_\alpha \longrightarrow U'_\alpha \subset \mathbb{R}^n$ where U_α is open in M^n and U'_α is open in $\mathbb{R}^n$,*
2. *each $x \in M^n$ is in the domain of some chart i.e. the U_α cover M^n,*
3. *for any two charts $\phi : U \longrightarrow \mathbb{R}^n$ and $\psi : V \longrightarrow \mathbb{R}^n$, "the change of co-ordinates" $\phi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \phi(U \cap V)$ is C^k (C^∞), and*
4. *the collection of charts is maximal.*

Hausdorffness means that, for any two points $x \neq y$ in the manifold, there are disjoint open sets A and B such that $x \in A$ and $y \in B$. We define co-ordinates for a point $p \in U_\alpha \subseteq M^n$ by

$$x^a = u^a \circ \phi_\alpha : U_\alpha \longrightarrow \mathbb{R}, \quad a = 1, \ldots, n,$$

where $u^a : \mathbb{R}^n \longrightarrow \mathbb{R}$.

Consider a topological space M and its cover V. A cover U of M is a *refinement* of V if each element in U is a subset of some element in V. The cover U is said to be *locally finite* if each point $p \in M$ has a neighbourhood which meets, nontrivially, only a finite number of members of U. The space M is *paracompact* if any open cover of M has an open locally finite refinement. It is well known that all metric spaces are paracompact. A fundamental theorem of dimension theory states that if M is a n-dimensional manifold, then every open cover V of M has a refinement U such that no point of M lies in more than $(n+1)$ elements of U (Hurewicz and Wallman 1948, Munkres 1966). The reader is also referred to Pears (1975).

A *partition of unity* subordinate to a locally finite open cover $\{V_i\}$ of a manifold M consists of a family of differentiable functions $g_i : M \longrightarrow \mathbb{R}$ such that (Szekeres 2004)

(1) $0 \leq g_i \leq 1$ on M for all i,

(2) $g_i(p) = 0$ for all $p \notin V_i$,

(3) $\sum_i g_i(p) = 1$ for all $p \in M$.

It provides a way to "glue" together results obtained for each patch of a manifold. For every locally finite cover $\{V_i\}$ of a paracompact manifold M, there exists a partition of unity $\{g_i\}$ subordinate to this refinement. It is possible to relax the third condition: for a paracompact manifold $\mathcal{E}$ with a locally finite cover $\mathcal{W}$, there exists a "partition of something" subordinate to this cover. This is fairly standard topology (Choquet-Bruhat *et al.* 1982, Szekeres 2004). We follow the explicit construction of Katzourakis (2005a) which leaves the partition of unity unnormalized. We shall (in §4.4.2) be particularly interested in the case in which $\mathcal{W}$ is obtained from another locally finite cover $\mathcal{Q}$ by taking N copies of each patch of $\mathcal{Q}$, differentiated by means of different co-ordinates.

Statement. *There exists a family $\{f_{i_a}\}$ of C^∞ non-negative "Bell" functions on $\mathcal{E}$, with properties:*

$$\begin{aligned} f_{i_a} \;&\in\; C^\infty(\mathcal{E} \longrightarrow \mathbb{R} \cap [0, +\infty)), \\ supp(f_{i_a}) \;&\subseteq\; W_{i_a}, \;\; \forall\, [W_i] \;\in\; \mathcal{W}, \\ \sum_{\substack{i_a \in J \\ 1 \leq a \leq N}} f_{i_a}(p) \;&>\; 0, \forall\, p \in \mathcal{E}, \end{aligned}$$

and such that the f_{i_a}'s are <u>real analytic</u> within the set that they are strictly positive

$$\{p \in \mathcal{E} \mid f_{i_a}(p) > 0\} \equiv \{f_{i_a} > 0\},$$

(which is open and coincides with $int(supp(f_{i_a})) \equiv \overset{\circ}{\widehat{supp(f_{i_a})}}$):

$$f_{i_a}|_{\{f_{i_a}>0\}} = f_{i_a}|_{\overset{\circ}{\widehat{supp(f_{i_a})}}} \in C^{\infty}(\mathcal{E} \cap \{f_{i_a} > 0\} \longrightarrow \mathbb{R} \cap (0, +\infty)).$$

Proof:

Consider the C^{∞} non-negative function on $\mathbb{R}^{n+1}$:

$$f \in C^{\infty}(\mathbb{R}^{n+1} \longrightarrow [0, +\infty))$$

$$f(x) := \begin{cases} \exp\left(\frac{1}{\|x\|^2 - r^2}\right), & \text{for } \|x\| < r \\ \\ 0, & \text{for } \|x\| \geq r, \end{cases}$$

where $r > 0$ and which satisfies:

$$\{f > 0\} \equiv \{x \in \mathbb{R}^{n+1} \mid f(x) > 0\} = \mathbb{B}(0, r) \subsetneqq supp(f)$$

and

$$supp(f) = \overline{\mathbb{B}(0, r)}.$$

Here, $\mathbb{B}(0, r)$ is a "ball" of radius r. The function f is real analytic in the interior of its support as it is a composition of analytic functions, and so we have

$$f|_{\{f>0\}} = f|_{\mathbb{B}(0,r)} \in C^{\infty}(\mathbb{B}(0, r) \longrightarrow \mathbb{R}).$$

By the *Analytic Continuation Principle* (Narasimhan 1968), f can never be analytic outside of $\mathbb{B}(0, r)$, provided that $(D^{a_1+\cdots+a_s} f)(x)|_{x \to \partial\,\mathbb{B}(0,r)} = 0 \;\; \forall \;\; (a_1, \ldots, a_s) \in \mathbb{N}^s$.

Now consider the cover $\mathcal{W}$ and a refinement $\mathcal{W}_{\mathbb{B}}$ to the inverse ball located in the intersection of the images of the N co-ordinate maps for each N-element class of patches. Following the construction in $\mathbb{R}^{n+1}$ define a family of smooth real functions on $\mathcal{E}$:

$$f_{i_a} \in C^{\infty}(\mathcal{E} \longrightarrow \mathbb{R} \cap [0, +\infty)), \quad \forall\, i_a \in J, \;\; a = 1, \ldots, N$$

$$f_{i_a}(p) := \begin{cases} \exp\left(\frac{1}{\|\chi_{(i_1)}(p)\|^2 - r_i^2}\right), & \forall\, p \in \chi_{(i_1)}^{-1}(\mathbb{B}(0,r_i)) \subseteq W_{i_a} \\ \\ 0, & \forall\, p \in \mathcal{E} \setminus \chi_{(i_1)}^{-1}(\mathbb{B}(0,r_i)). \end{cases} \tag{2.2.1}$$

Here

$$\begin{aligned} \{f_{i_a} > 0\} &\equiv \{p \in \mathcal{E} \mid f_{i_a}(p) > 0\} = \chi_{(i_1)}^{-1}(\mathbb{B}(0,r_i)), \\ supp(f_{i_a}) &= \chi_{(i_1)}^{-1}(\overline{\mathbb{B}(0,r)}) \subseteq W_{i_a} \end{aligned}$$

and

$$int(supp(f_{i_a})) = \chi_{(i_1)}^{-1}(\mathbb{B}(0,r_i)),$$

which are consistent with the $\mathbb{R}^{n+1}$ scenario. Note that $\sum f_{i_a}(p) > 0$. Since f_{i_a} is defined on the real analytic manifold $\mathcal{E}$ and is a composition of analytic functions, it is itself real analytic within the interior of its support, and so we have

$$f_{i_a}|_{\{f_{i_a}>0\}} = f_{i_a}|_{\chi_{(i_1)}^{-1}(\mathbb{B}(0,r_i))} \in C^\infty(\chi_{(i_1)}^{-1}(\mathbb{B}(0,r_i)) \longrightarrow \mathbb{R}).$$

Note that the supports of the f_{i_a}'s (as well as their interior) form a locally finite cover of $\mathcal{E}$:

$$\bigcup_{\substack{i_a \in J \\ 1 \le a \le N}} supp(f_{i_a}) = \bigcup_{\substack{i_a \in J \\ 1 \le a \le N}} \{f_{i_a} > 0\} = \mathcal{E}.$$

Hence, the statement is proved.

2.3 Differential Geometry

A n-dimensional differentiable manifold M (or M^n) is essentially a topological space that locally resembles $\mathbb{R}^n$, n-dimensional Euclidean space, and on which points can be assigned the real co-ordinates $(x^1, x^2, \ldots, x^n)$. (We formally defined the concept of a differentiable manifold in §2.2.) A submanifold of M with m dimensions ($m < n$) is characterized by the parametric equations

$$x^a = x^a(u^1, u^2, \ldots, u^m) \quad a = 1, 2, \ldots, n.$$

In particular, a one-dimensional submanifold is a curve and a $(n-1)$-dimensional submanifold ($n \geq 3$) is known as a hypersurface. On a hypersurface the co-ordinates can be related by the equation

$$h(x^1, x^2, \ldots, x^n) = 0.$$

Consider two points in the manifold whose co-ordinates differ by an infinitesimal amount dx^a. We define the line element that is a measure of the infinitesimal distance ds between the neighbouring points by the relation

$$ds^2 = f(x^a, dx^a).$$

We also refer to the above relation as the metric. The distance ds^2 is invariant since it remains unchanged under a transformation of co-ordinates. In general relativity we are concerned with pseudo-Riemannian metric spaces, where the metric is expressed in the form

$$ds^2 = g_{ab}(x^c)dx^a dx^b,$$

and where g_{ab} are the components of the metric tensor $\mathbf{g}$. Here ds^2 is indefinite since it can be positive, zero or negative. If ds^2 is strictly positive, then the metric space is called Riemannian. Many results that are applicable to pseudo-Riemannian spaces hold for Riemannian spaces also, and conversely. In this dissertation, we consistently discuss only pseudo-Riemannian spaces, but note that the existence results for local isometric embeddings (and hence the global isometric embeddings) under discussion are equally applicable to a Riemannian context (Goenner 1980). The metric tensor $\mathbf{g}$ is a linear map of two vectors into their inner product:

$$\mathbf{g}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}, \quad \mathbf{u}, \mathbf{v} \in T_pM.$$

It describes the local geometry of the manifold and is a second-rank covariant tensor having the following properties:

(1) g_{ab} is symmetric i.e. $g_{ab} = g_{ba}$.

(2)

$$g_{ab}g^{bc} = \delta_a^c = \begin{cases} 1 & \text{if } c = a, \\ 0 & \text{if } c \neq a. \end{cases}$$

δ_a^c is known as the Kronecker tensor.

(3) it can be used to raise or lower indices of tensors

e.g. $u_a = g_{ab}u^b, \qquad u^a = g^{ab}u_b.$

Note that we employ Einstein's summation convention: for an index that appears as a superscript and a subscript in a term, one must sum over the index from 1 to n. For example, $\delta_r^r = 1$ when r is a fixed co-ordinate, but, in general, $\delta_a^a = \delta_1^1 + \delta_2^2 + \cdots + \delta_n^n = n$.

The functions g_{ab} can be written in matrix form as:

$$G = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}.$$

The rank of the metric is the number of independent rows of G and the signature of the metric is defined as the number of positive eigenvalues of G minus the number of negative eigenvalues of G. The space-time of general relativity is a four-dimensional manifold endowed with a metric of signature 2 (also given as $(- + ++)$). The determinant of the metric tensor is denoted by $|g_{ab}|$ and is given by the determinant of G.

A pseudo-Euclidean space ($\mathbb{R}^n$) is one which has a metric of the form

$$ds^2 = \varepsilon_1(dx^1)^2 + \cdots + \varepsilon_a(dx^a)^2 + \cdots + \varepsilon_n(dx^n)^2$$

where $\varepsilon_a = \pm 1$. If $\varepsilon_a = 1$ for all a, then the space is Euclidean. The Minkowski space-time $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ of special relativity is an example of a pseudo-Euclidean space. We sometimes denote a m-dimensional pseudo-Euclidean space, with metric having p positive eigenvalues and q negative eigenvalues, by $\mathbb{R}^m(p,q)$ where $m = p + q$.

The fundamental theorem of Riemannian geometry states that there exists a unique symmetric connection which preserves inner products under parallel transport. This connection is given by

$$\Gamma^a_{\ bc} = \frac{1}{2} g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}), \tag{2.3.1}$$

and is referred to as the Christoffel symbol of the second kind. The Christoffel symbol of the first kind is $\Gamma_{abc} = \frac{1}{2}(g_{ac,b} + g_{ba,c} - g_{bc,a})$. Both symbols are symmetric: $\Gamma^a_{\ bc} = \Gamma^a_{\ cb}$ and $\Gamma_{abc} = \Gamma_{acb}$.

The connection (2.3.1) can be used to define the covariant derivative on a manifold. The covariant derivative of a type (r, s) tensor $\mathbf{T}$ is a type $(r, s+1)$ tensor $\nabla\mathbf{T}$ and its components are:

$$\begin{aligned} T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s;d} = & T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s,d} \\ & + \Gamma^{a_1}{}_{cd} T^{ca_2 \ldots a_r}{}_{b_1 \ldots b_s} + \ldots + \Gamma^{a_r}{}_{cd} T^{a_1 \ldots c}{}_{b_1 \ldots b_s} \\ & - \Gamma^{c}{}_{b_1 d} T^{a_1 \ldots a_r}{}_{cb_2 \ldots b_s} - \ldots - \Gamma^{c}{}_{b_s d} T^{a_1 \ldots a_r}{}_{b_1 \ldots c}, \end{aligned}$$

(Here the comma represents partial differentiation with respect to one of the coordinates x^a.) Now, in general, the partial derivative of a tensor is not a tensor since taking the difference of tensors at two different points does not produce a tensor. So, in forming the covariant derivative, one "parallel transports" the tensor from one point to the other and then obtains the difference in the same way as the partial derivative. This differentiation is linear, satisfies a Leibniz rule

$$\nabla(\mathbf{U} \otimes \mathbf{V}) = (\nabla\mathbf{U}) \otimes \mathbf{V} + \mathbf{U} \otimes (\nabla\mathbf{V}),$$

and commutes with contraction. The covariant derivative of a scalar is equal to its partial derivative. It can be easily shown that the covariant derivative of a metric tensor vanishes i.e. $g_{ab;c} = 0$.

A geodesic in a manifold is a curve along which all the tangent vectors point in the same direction. It is also the curve of minimum length between two points in the manifold. Geodesics on a pseudo-Riemannian manifold can be space-like, null or time-like, which corresponds to $ds^2 > 0, ds^2 = 0$, or $ds^2 < 0$, respectively. Photons and free particles travel along null and non-null geodesics, respectively. For a metric space M with $p \in M$, the exponential map

$$exp : T_pM \longrightarrow M$$

maps the line tX, where $t \in \mathbb{R}$ and X is a tangent vector at p, into the geodesic curve through p (Choquet-Bruhat *et al.* 1982). This function is particularly useful in defining local geodesic (or normal) co-ordinates on a metric space.

The Lie derivative of a type (r, s) tensor $\mathbf{T}$ with respect to a vector field $\mathbf{X}$ is a type (r, s) tensor $\mathbf{L_X T}$ and its components are (Hawking and Ellis 1973):

$$\begin{aligned} L_{\mathbf{X}} T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} = & T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s, c} X^c \\ & - T^{c a_2 \ldots a_r}{}_{b_1 \ldots b_s} X^{a_1}{}_{,c} - \ldots - T^{a_1 \ldots c}{}_{b_1 \ldots b_s} X^{a_r}{}_{,c} \\ & + T^{a_1 \ldots a_r}{}_{c b_2 \ldots b_s} X^c{}_{,b_1} + \ldots + T^{a_1 \ldots a_r}{}_{b_1 \ldots c} X^c{}_{,b_s}. \end{aligned}$$

This derivative determines the change in a tensor from one point in the direction of the vector field $\mathbf{X}$ to a nearby point. It provides a way to transport tensors along the integral curves of $\mathbf{X}$. Again, this differentiation is linear, satisfies a Leibniz rule

$$\mathbf{L_X}(\mathbf{S} \otimes \mathbf{T}) = \mathbf{L_X S} \otimes \mathbf{T} + \mathbf{S} \otimes \mathbf{L_X T},$$

and preserves contractions. The Lie derivative is significant in the study of the symmetries of a space-time – see §3.5.

From the noncommutivity of covariant differentiation we derive a measure of curvature:

$$V^a{}_{;cd} - V^a{}_{;dc} = R^a{}_{bcd} V^b, \quad \text{and} \quad V_{b;dc} - V_{b;cd} = R^a{}_{bcd} V_a,$$

where

$$R^a{}_{bcd} = \Gamma^a{}_{bc,d} - \Gamma^a{}_{bd,c} + \Gamma^a{}_{ed} \Gamma^e{}_{bc} - \Gamma^a{}_{ec} \Gamma^e{}_{bd}, \tag{2.3.2}$$

is the curvature tensor or Riemann tensor of the second kind. The Riemann tensor of the first kind is given by

$$R_{abcd} = g_{ae}R^e{}_{bcd} = \Gamma_{abc,d} - \Gamma_{abd,c} + \Gamma_{afd}\Gamma^f{}_{bc} - \Gamma_{afc}\Gamma^f{}_{bd}.$$

We may substitute for Γ_{abc} in R_{abcd} to obtain an alternative expression

$$R_{abcd} = \frac{1}{2}(g_{ad,bc} + g_{bc,ad} - g_{ac,bd} - g_{bd,ac}) + \Gamma^e{}_{ad}\Gamma_{ebc} - \Gamma^e{}_{ac}\Gamma_{ebd}.$$

The Riemann tensor has a high degree of symmetry. R_{abcd} is antisymmetric on first and second pairs (i.e. $R_{abcd} = -R_{bacd}$ and $R_{abcd} = -R_{abdc}$), symmetric on pair exchange (i.e. $R_{abcd} = R_{cdab}$) and also satisfies the cyclic identity $R_{abcd} + R_{adbc} + R_{acdb} = 0$. Furthermore, the covariant derivative $R^a{}_{bcd;e}$ satisfies the Bianchi identity $R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0$.

The symmetric Ricci tensor R_{ab} is obtained by the contraction $R_{ab} = R^d{}_{abd}$. The full expression is

$$R_{ab} = \Gamma^d{}_{ab,d} - \Gamma^d{}_{ad,b} + \Gamma^d{}_{ed}\Gamma^e{}_{ab} - \Gamma^d{}_{eb}\Gamma^e{}_{ad}. \tag{2.3.3}$$

The Ricci scalar is given by

$$R = R^b{}_b = g^{ab}R_{ab}.$$

The connection coefficient, Riemann tensor and Ricci tensor describe the curvature of a manifold – see §2.4.

The curvature tensor $R^{ab}{}_{cd}$ can be written in the form (Stephani 2004)

$$\begin{aligned} R^{ab}{}_{cd} = C^{ab}{}_{cd} &+ \frac{1}{2}(g^a_c R^b{}_d + g^b_d R^a{}_c - g^b_c R^a{}_d - g^a_d R^b{}_c) \\ &- \frac{1}{6}R(g^a_c g^b_d - g^a_d g^b_c), \end{aligned} \tag{2.3.4}$$

where $C^{ab}{}_{cd}$ is the Weyl tensor or conformal curvature tensor. It is the conformally invariant part of the Riemann curvature tensor and is traceless i.e. $C^{ab}{}_{ad} = 0$.

A conformal transformation takes the metric $\mathbf{g}$ to the metric

$$\hat{\mathbf{g}} = \Delta \mathbf{g}, \tag{2.3.5}$$

where Δ is a function of x^i. We say that the two spaces with metrics $\mathbf{g}$ and $\hat{\mathbf{g}}$ are conformally related. This mapping induces the following change in the Ricci tensor and scalar:

$$\hat{R}_{ab} = R_{ab} - \frac{1}{2}(2\varphi_{a;b} - \varphi_a \varphi_b + g_{ab}\varphi^c \varphi_c + g_{ab}\varphi^c{}_{;c}), \tag{2.3.6}$$

$$\hat{R} = \Delta^{-1}(R - 3\varphi^c{}_{;c} - \frac{3}{2}\varphi_c \varphi^c), \tag{2.3.7}$$

where $\varphi_c = (\ln \Delta)_{,c}$. However, the Weyl tensor remains unchanged. Conformal transformations preserve the geometry of null geodesics and so can be used to study the causal structure of a space-time.

In his theory of general relativity, Einstein postulated the field equations

$$G^{ab} = R^{ab} - \frac{1}{2}Rg^{ab} = \kappa T^{ab} + \Lambda g^{ab}, \tag{2.3.8}$$

that relate space-time geometry with matter and energy. Here, T^{ab} is the energy-momentum or matter tensor, Λ is the cosmological constant and $\kappa = \frac{8\pi G}{c^4}$ is the coupling constant which is usually set to 1 for convenience. Note that c is the speed of light which we take to be one in this thesis. G^{ab} is called the Einstein tensor and has vanishing divergence, which implies energy-momentum conservation $T^{ab}{}_{;b} = 0$. For a 4-dimensional space-time with no cosmological constant, contracting (2.3.8) implies that

$$R = -\kappa T.$$

The field equations (2.3.8) are known as Einstein's field equations and may be solved to yield space-time metrics. The field equations are non-linear and cannot always be solved completely to yield exact solutions. The energy-momentum tensor is often modelled as a fluid:

$$T^{ab} = \left(\mu + \frac{p}{c^2}\right) u^a u^b + p g^{ab} + q^a u^a + q^b u^a + \pi^{ab}, \tag{2.3.9}$$

where p is the isotropic pressure, μ is the energy density, q^a is the heat flux vector ($q^a u_a = 0$) and π^{ab} is the anisotropic pressure tensor with $\pi^{ab}u_a = 0 = \pi^a_a$. The comoving fluid four-velocity $\mathbf{u}$ is unit and timelike, and so $u^a u_a = -c^2$ for a space with metric signature $(- + ++)$. For a perfect fluid, $\pi^{ab} = 0$ and $q^a = 0$ so that (2.3.9) becomes

$$T^{ab} = \left(\mu + \frac{p}{c^2}\right) u^a u^b + p g^{ab}. \tag{2.3.10}$$

In an empty space (i.e. a vacuum) $T^{ab} = 0$.

We are particularly interested in Einstein spaces and spaces of constant curvature. In a n-dimensional Einstein space, the Ricci tensor has the form

$$R_{ab} = \frac{2\Lambda}{2-n} g_{ab},$$

where Λ is the cosmological constant, and the Ricci scalar is

$$R = \frac{2n\Lambda}{2-n}.$$

The Einstein tensor is $G_{ab} = \Lambda g_{ab}$ and so Einstein spaces are empty. The case $\Lambda = 0$ corresponds to a Ricci-flat space $R_{ab} = 0 = R$. In a n-dimensional space of constant curvature, the Riemann tensor is given by

$$R_{abcd} = \frac{R}{n(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}),$$

where the Ricci scalar R is a constant. The term $\frac{R}{n(n-1)} = \varepsilon K^{-2}$ is defined as the Gaussian curvature (Stephani 2004). The Ricci and Einstein tensors become

$$R_{ab} = \frac{R}{n} g_{ab}, \quad \text{and} \quad G_{ab} = \frac{2-n}{2n} R g_{ab}.$$

Spaces of negative constant curvature are known as Anti-de Sitter spaces, and those with positive constant curvature as de Sitter spaces. This corresponds to $\Lambda < 0$ and $\Lambda > 0$, respectively.

2.4 Extrinsic and Intrinsic Curvature

The concept of curvature of a manifold is significant in general relativity and it can be perceived in two complementary ways, as intrinsic and extrinsic curvature. The intrinsic curvature of a manifold is confined to the manifold itself, whereas the extrinsic curvature of a manifold is dependent on how it is embedded in a higher dimensional space. Thus, in embedding one space into another, the extrinsic curvature will provide a description of the embedded space in relation to the embedding space. A mathematical approach to determining the curvature of a manifold is to consider the Christoffel symbol and the Riemann tensor defined in §2.3. The Riemann tensor is a measure of intrinsic curvature, and a manifold is said to be intrinsically flat if ${R^a}_{bcd}$ vanishes. Extrinsic curvature can be expressed in terms of the connection ${\Gamma^a}_{bc}$. As an example (Stephani 2004), the extrinsic curvature of a three-dimensional space in a four-dimensional space-time with metric $ds^2 = -\phi^2 dt^2 + g_{AB}dx^A dx^B$, where $\phi = \phi(t, x, y, z)$ and A and B label spatial co-ordinates, is given by

$$\Omega_{AB} \equiv -\frac{1}{2\phi}\frac{\partial g_{AB}}{\partial t} = -\phi {\Gamma^0}_{AB}.$$

Pseudo-Euclidean spaces are intrinsically and extrinsically flat. If a seemingly curved space can be transformed into a pseudo-Euclidean space globally, then the space must be intrinsically flat. To develop intuition, consider a cylinder with open ends whose surface is represented in cylindrical co-ordinates (z, ϕ) by

$$ds^2 = dz^2 + r^2 d\phi^2,$$

where r is the radius. By making the co-ordinate transformation $x = z$, $y = r\phi$, the metric can be written as the 2-dimensional Euclidean metric

$$ds^2 = dx^2 + dy^2.$$

This indicates that the surface of the cylinder is intrinsically flat, but it appears curved in 3-dimensional space. Its extrinsic curvature in the embedding space $ds^2 = dz^2 + r^2 d\phi^2 + dr^2$ has the non-zero component

$$\Omega_{\phi\phi} = -r.$$

In a more physical sense, we observe that the cylinder can be built from a flat sheet without any distortion. This cannot be done for a spherical surface, which is both extrinsically and intrinsically curved. These notions of curvature play a key role in the embedding equations with the extrinsic curvature providing a geometrical relation between the embedded and embedding spaces – see §3.2.

2.5 Static Spherically Symmetric Space-times

We consider four-dimensional space-times that are static and spherically symmetric (SSS). We use the co-ordinates (t, r, θ, ϕ) where t is time-like and (r, θ, ϕ) are spherical co-ordinates. The term "static" means that the metric components do not depend on t and that the metric remains unchanged under the transformation $t \to -t$ (Hobson *et al.* 2006). The property of being spherically symmetric implies that the space appears the same in all directions i.e. is isotropic.

2.5.1 General Space-time

The metric for a SSS space-time is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.5.1}$$

where ν and λ are functions of r. The non-zero Christoffel symbols (2.3.1) calculated from the metric are:

$$\begin{aligned}
&\Gamma^0{}_{01} = \nu', &&\Gamma^1{}_{00} = \nu' e^{2(\nu-\lambda)}, &&\Gamma^1{}_{11} = \lambda', \\
&\Gamma^1{}_{22} = -re^{-2\lambda}, &&\Gamma^1{}_{33} = -r\sin^2\theta\, e^{-2\lambda}, &&\Gamma^2{}_{12} = 1/r, \\
&\Gamma^2{}_{33} = -\sin\theta\cos\theta, &&\Gamma^3{}_{13} = 1/r, &&\Gamma^3{}_{23} = \cot\theta.
\end{aligned} \tag{2.5.2}$$

Here the prime denotes differentiation with respect to r. The non-vanishing components of the Ricci tensor (2.3.3) are given by

$$\begin{aligned}
R_{00} &= -e^{2(\nu-\lambda)}(-\nu'^2 - \nu'' + \nu'\lambda' - \frac{2}{r}\nu'), \\
R_{11} &= -\nu'^2 - \nu'' + \nu'\lambda' + \frac{2}{r}\lambda', \\
R_{22} &= 1 - e^{-2\lambda} - re^{-2\lambda}\nu' + re^{-2\lambda}\lambda', \\
R_{33} &= R_{22}\sin^2\theta,
\end{aligned} \tag{2.5.3}$$

and so the Ricci scalar is a function of r only.

2.5.2 Reissner-Nordström Space-time

The Reissner-Nordström metric (Reissner 1916, Nordström 1918) is given by

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{e^2}{r^2}} + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2.5.4}$$

which is of the form (2.5.1) with $\nu(r) = \frac{1}{2}\ln(1 - \frac{2m}{r} + \frac{e^2}{r^2})$ and $\lambda(r) = -\frac{1}{2}\ln(1 - \frac{2m}{r} + \frac{e^2}{r^2})$. It describes the exterior of a charged non-rotating black hole where m and e represent the mass and electric charge of the black hole, respectively, and $m > 0$. The metric is an asymptotically flat solution to the Einstein-Maxwell equations, and by setting the charge as zero, we recover the Schwarzschild exterior solution.

The Reissner-Nordström metric has a singularity at $r = 0$. Now, we may rewrite $(1 - \frac{2m}{r} + \frac{e^2}{r^2})$ as

$$\frac{(r-m)^2 - (m^2 - e^2)}{r^2}.$$

For $m^2 < e^2$, the above expression is always positive and so the singularity at $r = 0$ is naked. For $e^2 < m^2$, we obtain

$$\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right)=\frac{(r-m+\sqrt{m^2-e^2})(r-m-\sqrt{m^2-e^2})}{r^2}.$$

which indicates that there are two more singularities at $r_- = m - \sqrt{m^2-e^2}$ and $r_+ = m + \sqrt{m^2-e^2}$. These singularities occur as event horizons and may be removed by a co-ordinate transformation and an extension of the manifold to obtain a maximal analytic manifold (Graves and Brill 1960, Carter 1966).

Black holes with electric charge are not physically significant since neutralizing plasma in their surroundings would prevent them from admitting such charge. The embedding of black holes in higher dimensional theories can provide more physically reasonable solutions.

A metric identical to the Reissner-Nordström metric was obtained by Dadhich *et al.* (2000) as an exact solution to the Einstein field equations for a black hole localized on the brane in the Randall-Sundrum type scenario, but instead of an electric charge, there is a tidal charge represented by q. This tidal charge is interpreted as gravitational field effects in the five-dimensional bulk projected onto the brane, where matter is confined. The bulk Weyl tensor transmits these effects onto the brane. Since the Weyl tensor is assumed non-zero, the solution cannot be conformal to 5-dimensional Anti-de Sitter space-time. Note that the condition of Z_2-symmetry is imposed in the bulk since one expects no flow through the brane. For $q \geq 0$, this black hole solution has two horizons analogous to that of the Reissner-Nordström solution, and both horizons lie inside the Schwarzschild horizon. For $q < 0$, there is only one horizon that lies outside the Schwarzschild horizon. In this case the effects of the bulk tend to strengthen the gravitational field induced on the brane. This result is quite significant since we cannot have $q < 0$ for the classical Reissner-Nordström solution but we do have that property for this solution. Limits on $|q|$ have been obtained from observations, but these limits are weak since measurements deal with mostly weak field solar scales. Strong tidal effects represented by large values of $|q|$ could have implications in the strong-gravity regime (Chandrasekhar 1983) and in the formation of primordial black holes (Dadhich *et al.* 2000).

Maeda and Dadhich (2006) obtained an exact black hole solution in Einstein-Gauss-Bonnet gravity that, as $r \to \infty$, resembles the Reissner-Nordström solution in Anti-de

Sitter space-time. The 5-dimensional bulk is assumed to have the topology $M^4 \times \mathrm{AdS}_{(1)}$. Apart from the AdS background this solution is equivalent to the Reissner-Nordström type black hole on the brane (Dadhich *et al.* 2000) with Weyl charge $q < 0$.

The non-zero Christoffel symbols obtained for the Reissner-Nordström space-time (2.5.4) are

$$
\begin{aligned}
&{\Gamma^0}_{01} = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right), && {\Gamma^0}_{10} = {\Gamma^0}_{01}, \\
&{\Gamma^1}_{00} = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) \left(\frac{m}{r^2} - \frac{e^2}{r^3}\right), && {\Gamma^1}_{11} = -{\Gamma^0}_{01}, \\
&{\Gamma^1}_{22} = -r \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right), && {\Gamma^1}_{33} = -r \sin^2\theta \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right), \\
&{\Gamma^2}_{12} = {\Gamma^2}_{21} = \frac{1}{r}, && {\Gamma^2}_{33} = -\sin\theta\,\cos\theta, \\
&{\Gamma^3}_{13} = {\Gamma^3}_{31} = \frac{1}{r}, && {\Gamma^3}_{23} = {\Gamma^3}_{32} = \cot\theta.
\end{aligned}
\tag{2.5.5}
$$

Using the above symbols we compute the components of the Ricci tensor:

$$
\begin{aligned}
&R_{00} = \frac{e^2}{r^4}\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right), && R_{11} = -\frac{e^2}{r^4}\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1}, \\
&R_{22} = \frac{e^2}{r^2}, && R_{33} = \frac{e^2}{r^2}\sin^2\theta.
\end{aligned}
\tag{2.5.6}
$$

The Ricci scalar is

$$R = 0.$$

We shall use this in §3.7 to determine an embedding of the Reissner-Nordström space-time.

2.5.3 Global Monopole Space-time

In this section we refer to Barriola and Vilenkin (1989) and Vilenkin and Shellard (1994). Monopoles are topological defects formed due to spontaneous symmetry breaking phenomena in the early universe. Consider a symmetry group G breaking down to a subgroup H of G. If the vacuum manifold $M = G/H$ contains non-contractible surfaces, then local and global monopoles arise. A particular instance is when G decomposes into $K \times U(1)$, where the multiply connected group $U(1)$ is associated with electro-magnetism. The simplest global monopole occurs due to the global symmetry breaking $S0(3) \longrightarrow SO(2)$. A metric describing such a global monopole was obtained by Barriola and Vilenkin (1989) and is given by

$$ds^2 = -dt^2 + d\bar{r}^2 + (1 - 8\pi G\eta^2)\bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2),$$

which can be transformed by $\bar{r} = \sqrt{1 - 8\pi G\eta^2}\, r$ into the SSS form

$$ds^2 = -dt^2 + K^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.5.7}$$

with $\nu = 0$ and $\lambda = -\frac{1}{2}\ln K$, where $K = 1 - \kappa\eta^2$, $\kappa = 8\pi G$.

The energy-momentum tensor (2.3.9) of the defect has the non-zero components:

$$T_t^t = T_r^r = -\frac{\eta^2}{r^2},$$

where $\eta \sim 10^{16}$GeV. Since the gravitational mass density $\rho_g = T_0^0 - T_i^i = 0$, the monopole does not exert any gravitational force on the matter around it. However, the monopole can produce strong gravitational fields due to large amounts of energy in the scalar field surrounding it. The total energy or mass of the monopole is $m = \int_0^R T_t^t r^2 dr \approx 4\pi\eta^2 R$, where R is known as the cut-off radius and is usually the distance to the nearest antimonopole. Global monopoles can therefore act as gravitational lenses (Vilenkin 1984, Vilenkin and Shellard 1994).

In practice, global monopoles cannot occur in large abundances as their combined energy densities would dominate and so overclose the universe. Their (non-Gaussian) density fluctuations must be suppressed in order to maintain consistency with cosmological observations (Vilenkin and Shellard 1994). This would not be a problem if an inflationary period took place after the monopoles were formed, since then it would lead to a dilution in their number density.

Global monopoles can play a significant role in higher-dimensional theories. The global monopole space-time in an Anti-de Sitter background appears as the $r \rightarrow 0$ limit of the black hole solution in Einstein-Gauss-Bonnet gravity obtained by Maeda and Dadhich (2006). It is shown that the singularity at $r = 0$ is weakened by the presence of the Gauss-Bonnet term.

We calculate the non-zero Christoffel symbols for the global monopole space-time:

$$\begin{aligned}
&{\Gamma^1}_{22} = -Kr, && {\Gamma^1}_{33} = -Kr\sin^2\theta, \\
&{\Gamma^2}_{12} = {\Gamma^2}_{21} = \frac{1}{r}, && {\Gamma^2}_{33} = -\sin\theta\cos\theta, \\
&{\Gamma^3}_{13} = {\Gamma^3}_{31} = \frac{1}{r}, && {\Gamma^3}_{23} = {\Gamma^3}_{32} = \cot\theta,
\end{aligned} \tag{2.5.8}$$

from which we obtain the non-zero components of the Ricci tensor:

$$\begin{aligned}
R_{22} &= 1 - K, \\
R_{33} &= (1 - K)\sin^2\theta.
\end{aligned} \tag{2.5.9}$$

Hence, the Ricci scalar is

$$R = \frac{2(1-K)}{r^2}.$$

The dependence of R on r complicates the construction of a local embedding – refer to §3.8.

2.6 Embeddings

Definition 2. *Let M be a smooth manifold. For any $p \in M$, T_pM denotes the tangent space of M at p, and is the vector space of all tangent vectors to M at p. Let $f : M \longrightarrow$*

N be a smooth map between two smooth manifolds. At $p \in M$, f induces a differential map $f_* : T_pM \longrightarrow T_{f(p)}N$ given by

$$(f_*X)(\alpha) \equiv X(\alpha \circ f),$$

where $X \in T_pM$ and α is a real-valued function defined in a neighbourhood of $f(p)$ (Binney 2000).

Suppose f is a function between two manifolds M and N. f is a *homeomorphism* if

(1) f is continuous,

(2) its inverse f^{-1} is continuous, and

(3) f is bijective i.e. is $1-1$ and onto.

Furthermore, f is a C^k (C^∞) *diffeomorphism* if it is a homeomorphism with f and f^{-1} C^k (C^∞) differentiable.

Most of embedding theory involves isometric embeddings of metric spaces into metric spaces.

Definition 3. *Suppose M^n is a n-dimensional analytic manifold with metric g_{ij} and N^{n+k} is a $(n+k)$-dimensional analytic manifold with metric $\tilde{g}_{\mu\nu}$. Then $f : M^n \longrightarrow N^{n+k}$ is a global isometric embedding (Goenner 1980) if:*

(1) f is a homeomorphism onto its image,

(2) $f_* : T_pM^n \longrightarrow T_{f(p)}N^{n+k}$ is injective $(1-1)$ $\forall\ p \in M^n$, and

(3) $g_p(V,W) = \tilde{g}_{f(p)}(f_*(V), f_*(W))\ \ \forall\ V, W \in T_pM^n,\ \ \forall\ p \in M^n$.

The condition (*2*) above means that f is an immersion, and together with (*1*) it defines an embedding. The last condition further implies that the embedding is isometric at all points of M^n. The embedding is analytic (C^k, C^∞) if f is analytic (C^k, C^∞). We also have that a function $f : U \subset M^n \longrightarrow N^{n+k}$, where U is an open co-ordinated neighbourhood of a point p, is a local isometric embedding if and only if the above three conditions hold for all points in U (Goenner 1980). In co-ordinate form, f is a local isometric embedding if there exist $n+k$ differentiable functions $y^\alpha = \sigma^\alpha(x^i)$ such that the Jacobian matrix $\{\frac{\partial \sigma^\alpha}{\partial x^i}\}$ has rank n and

$$g_{ij} = \frac{\partial \sigma^\alpha}{\partial x^i}\frac{\partial \sigma^\beta}{\partial x^j}\tilde{g}_{\alpha\beta},$$

which is equivalent to the existence of solutions to the Gauss, Codazzi and Ricci equations (defined in §3.2) for local embeddings.

2.7 The Cauchy-Kowalewski Theorem

The Cauchy-Kowalewski theorem (Cauchy 1842a,b,c, Kowalewski 1875) is a local existence theorem for a system of analytic partial differential equations.

Theorem. *Consider the set of partial differential equations:*

$$\frac{\partial^2 u^A}{\partial (y^{n+1})^2} = F^A\left(y^\alpha, u^B, \frac{\partial u^B}{\partial y^\alpha}, \frac{\partial^2 u^B}{\partial y^\alpha \partial y^i}\right), \quad A, B = 1, ..., m \tag{2.7.1}$$

where $u^1, \ldots, u^m$ *are* m *unknown functions of the* $n+1$ *variables* $y^1, \ldots, y^n, y^{n+1}$*, and* $\alpha = 1, \ldots, n+1$ *and* $i = 1, \ldots, n$*. Also, let* $\xi^1, \ldots, \xi^m$*,* $\eta^1, \ldots, \eta^m$ *be analytic at* $0 \in \mathbb{R}^n$ *functions of the variables* $y^1, \ldots, y^n$*. If the functions* F^A *are analytic with respect to each of their arguments around the values evaluated at the point* $y^1 = \cdots = y^n = 0$*, then there exists a unique solution to the equations* (2.7.1) *which is analytic at* $0 \in \mathbb{R}^{n+1}$*, and that satisfies the initial conditions:*

$$\begin{aligned} u^A\left(y^1, \ldots, y^n, 0\right) &= \xi^A\left(y^1, \ldots, y^n\right), \\ \frac{\partial u^A}{\partial y^{n+1}}\left(y^1, \ldots, y^n, 0\right) &= \eta^A\left(y^1, \ldots, y^n\right), \quad A = 1, \ldots, m. \end{aligned}$$

(A proof of the theorem can be found in Petrovsky (1954).) This theorem is used extensively in local embedding theorems to ascertain the existence of solutions to the embedding equations. We shall refer to it in §3.4 where we summarize two of the local results.

Chapter 3

Local Isometric Embeddings

3.1 Introduction

In order to construct global embeddings from local embeddings into pseudo-Riemannian spaces, we require a complete understanding of the local theory. So, in this chapter we focus on local isometric embeddings. Firstly, in §3.2 we present the Gauss, Codazzi and Ricci equations that govern an embedding, and then we review some local existence results for Euclidean and pseudo-Riemannian spaces in §3.3. In §3.4 we provide a detailed summary of the theorems which are most pertinent to the formation of global embeddings. Despite results for the existence of embeddings, there are no known general solutions to the local embedding equations. Thus, one is motivated to explicitly construct local embeddings for space-times of particular interest. Once a local embedding has been determined, we can, in principle, use the relevant global existence theorems to show that a global embedding exists. Even for some specific cases, the embedding equations are not so easy to solve. The Killing geometry and rigidity of a manifold can be useful in this regard and so we discuss these concepts in §3.5. We then proceed to discuss the technique for embedding static spherically symmetric space-times in §3.6. We show that the embedding is easily obtainable for manifolds with constant Ricci scalar, however, the general case is more complicated. We comment on uniqueness of these results, and we discuss the nature of four-dimensional space-times that may be embedded into spaces of constant curvature. As examples, we consider the embeddings of the Reissner-Nordström and global monopole space-times in §3.7

and §3.8, respectively. The work contained in this chapter will shortly be submitted for publication (Amery *et al.* 2007).

3.2 The Local Embedding Equations

We want to embed an n-dimensional space V_n into an m-dimensional space V_m as a hypersurface where $m > n$. Let the metric in V_n be

$$ds^2_{(n)} = g_{ij}dx^i dx^j,$$

and the metric in V_m be

$$ds^2_{(m)} = a_{\alpha\beta}dy^\alpha dy^\beta,$$

where $y^\alpha = y^\alpha(x^i)$. We refer to V_n as the embedded space and V_m as the embedding space.

The extrinsic curvature of V_n in V_m has the components (Eisenhart 1926)

$$\Omega^{(\sigma)}_{ij} = a_{\alpha\beta}n^{\beta(\sigma)}y^\alpha{}_{,i;j},$$

where $n^{\beta(\sigma)}$ are the components of unit normal vectors orthogonal to V_n and each other, $\sigma = n, \ldots, m-1$, and the terms in brackets are labels and not indices.

A derivation by Eisenhart (1926) produces three equations required for embedding V_n into V_m as a hypersurface. Note that given V_n these equations are equivalent to solving the field equations for V_m. The equations are known as the Gauss, Codazzi and Ricci (GCR) equations and are given, respectively, by

$$R_{hijk} = \sum_\sigma e_{(\sigma)}(\Omega^{(\sigma)}{}_{hj}\Omega^{(\sigma)}{}_{ik} - \Omega^{(\sigma)}{}_{hk}\Omega^{(\sigma)}{}_{ij}) + \overline{R}_{\alpha\beta\gamma\delta}y^\alpha{}_{,h}y^\beta{}_{,i}y^\gamma{}_{,j}y^\delta{}_{,k}, \tag{3.2.1}$$

$$\Omega^{(\sigma)}{}_{ij,k} - \Omega^{(\sigma)}{}_{ik,j} = \sum_\tau e_{(\tau)}(t^{(\tau\sigma)}{}_k\Omega^{(\tau)}{}_{ij} - t^{(\tau\sigma)}{}_j\Omega^{(\tau)}{}_{ik}) + \overline{R}_{\alpha\beta\gamma\delta}y^\alpha{}_{,i}y^\gamma{}_{,j}y^\delta{}_{,k}n^{\beta(\sigma)}, \tag{3.2.2}$$

$$t^{(\tau\sigma)}{}_{j,k} - t^{(\tau\sigma)}{}_{k,j} = \sum_{\varrho} e_{(\varrho)} (t^{(\varrho\tau)}{}_j t^{(\varrho\tau)}{}_k - t^{(\varrho\tau)}{}_k t^{(\varrho\tau)}_j) + g^{lh}(\Omega^{(\tau)}{}_{lk}\Omega^{(\sigma)}{}_{hj} - \Omega^{(\tau)}{}_{lj}\Omega^{(\sigma)}{}_{hk})$$
$$- \overline{R}_{\alpha\lambda\mu\nu} y^{\mu}{}_{,j} y^{\nu}{}_{,k} n^{\lambda(\sigma)} n^{\alpha(\tau)}. \tag{3.2.3}$$

In the above, $e_{(\sigma)} = \pm 1$ and $t^{(\tau\sigma)}_{j}$ represents the twisting of the $n^{\alpha(\sigma)}$ vectors in relation to one another, where $\sigma, \tau = n, \ldots, m-1,\ \sigma \neq \tau$. The Gauss and Codazzi equations must be solved on the hypersurface V_n and the Ricci equation must be solved off the hypersurface. For embeddings with codimension one, the Ricci equation is void and the space-space components for the Ricci tensor for V_m are typically used as a propagation equation (Dahia and Romero 2002b). There does not exist any known general solution to these equations, so we must consider particular embedding spaces.

3.3 Local Existence Results

The problem of embedding a n-dimensional Riemannian manifold locally into an Euclidean manifold was first discussed by Schläfli in 1873. It was suggested that the dimension of the embedding space should be $\frac{n(n+1)}{2}$. In the 1920's, Janet (1926) and Cartan (1927) proved this true in their local existence theorem.

The Janet-Cartan theorem :
A n-dimensional Riemannian manifold with analytic positive definite metric can be locally, analytically and isometrically embedded into an Euclidean manifold $\mathbb{R}^m$ with dimension $m = \frac{n(n+1)}{2}$ (Janet 1926, Cartan 1927).

The indefinite case was treated much later by Friedman (1961). Embedding locally into Euclidean spaces has been useful as a way of investigating various properties of general relativistic space-times, and serves to classify solutions to Einstein's field equations and to obtain new exact solutions (Stephani 1967, 1968). However, there is no physical reason for preferring Euclidean embedding spaces, and other Riemannian manifolds, such as spaces of constant curvature (Rund 1972), have been utilized (Campbell 1926, Magaard 1963, Goenner 1980). In particular, the Campbell-Magaard theorem, stated by Campbell (1926) and proved by Magaard (1963), is a local existence theorem for embeddings into Ricci-flat pseudo-Riemannian spaces. It is interesting that now with a curved embedding space, the codimension of the embedding reduces to one.

The Campbell-Magaard Theorem :

A n-dimensional Riemannian manifold with analytic metric can be locally, analytically and isometrically embedded into a $(n+1)$-dimensional Ricci-flat ($\widetilde{R}_{\alpha\beta} = 0$) manifold (Campbell 1926, Magaard 1963).

This theorem, which appeals to the Cauchy-Kowalewski theorem, led to several generalizations (Anderson and Lidsey 2001, Anderson *et al.* 2003, Dahia and Romero 2002a,b). Anderson and Lidsey (2001) presented constructions embedding Einstein spaces into Einstein spaces and for the embedding of plane-wave backgrounds and Ricci-flat space-times into 5-dimensional space-times sourced by massless scalar fields. It was further shown that Einstein and Ricci-flat space-times may be embedded into space-times sourced by self-interacting scalar fields (Anderson *et al.* 2003). Dahia and Romero (2002a,b) extended the Campbell-Magaard theorem to Einstein embedding spaces, and later to a given pseudo-Riemannian manifold.

The Dahia-Romero Theorems :

- A n-dimensional pseudo-Riemannian manifold can be locally, analytically and isometrically embedded in a $(n+1)$-dimensional Einstein manifold (Dahia and Romero 2002b).

- A n-dimensional pseudo-Riemannian manifold can be locally, analytically and isometrically embedded in a $(n+1)$-dimensional pseudo-Riemannian manifold with a non-degenerate Ricci tensor which is equal, up to a local analytic diffeomorphism, to the Ricci tensor of an arbitrarily given pseudo-Riemannian manifold (Dahia and Romero 2002a).

We provide a summary of these theorems in the next section. A crucial observation drawn from these results is that the local embedding space has the metric

$$\text{diag}\,[\bar{g}_{ik}, \varepsilon\phi^2],$$

where both $\bar{g}_{ik}$ and ϕ may have functional dependence on y and x^i. This is significant for the global situation. Note that the second Dahia-Romero theorem does not guarantee, for example, the existence of embeddings into constant curvature spaces.

Embedding spaces having singular energy-momentum tensors have been considered (Dahia and Romero 2004), and the existence of harmonic (volume minimizing) locally analytic and isometric embeddings into Ricci-flat and Einstein spaces has also been established (Chervon *et al.* 2004). Note that local embedding results obtained for strictly Riemannian manifolds can be quite simply extended to the indefinite case (Goenner 1980).

Since the above results are based on the Cauchy-Kowalewski theorem, concerns have been raised regarding their relevance to physics (Anderson 2004). There is no reason to expect that the embedding space will be well-behaved with respect to physical properties such as causality and stability. In response to this, it is emphasized (Wesson 1999) that the Campbell-Magaard theorem is a local result and does not claim to guarantee a well-posed initial value problem. Furthermore, an alternative approach (Dahia and Romero 2005a,b) to the embedding problem has been considered, using the theory of local Sobolev spaces. It asserts that, for any 4-dimensional space-time, there exist initial data sets whose Cauchy development for the Einstein vacuum equations is a 5-dimensional vacuum space into which this space-time may be locally, analytically and isometrically embedded. This guarantees that we have causality and stability within particular domains for both the embedded and embedding spaces. In the same papers (Dahia and Romero 2005a,b), it is shown that perturbations outside the (local) initial hypersurface do not affect the future domain of dependence, and so causality is not violated. These results can be extended to embedding spaces with cosmological constants, but the singular energy-momentum of the brane-world scenario is more problematic (Dahia and Romero 2005a).

3.4 Summary of the Dahia-Romero Theorems

Dahia and Romero (2002b) first proved that any n-dimensional pseudo-Riemannian manifold can be locally, analytically and isometrically embedded in a $(n+1)$-dimensional

Einstein manifold. They also considered a pseudo-Riemannian embedding space with a non-degenerate Ricci tensor which is equal, up to a local analytic diffeomorphism, to the Ricci tensor of an arbitrarily given pseudo-Riemannian manifold (Dahia and Romero 2002a). The proofs for these results are essentially the same. Nevertheless, we summarize both their proofs here.

Suppose M^n is a n-dimensional analytic manifold with metric g_{ij} and N^{n+1} is a $(n+1)$-dimensional manifold with metric $\tilde{g}_{\mu\nu}$. The following theorem indicates the conditions for the existence of a local isometric embedding of M^n into N^{n+1} and is an extension of the Riemannian case proven by Magaard (1963).

Theorem. *There exists a local isometric analytic embedding of a pseudo-Riemannian manifold (M^n, g) at $p \in U \subset M^n$ into a pseudo-Riemannian manifold $(N^{n+1}, \tilde{g})$ if and only if there exist analytic functions $\bar{g}_{ik}(x^1, \dots, x^n, y)$ and $\bar{\phi}\,(x^1, \dots, x^n, y)$ in a neighbourhood of $(x_p^1, \dots, x_p^n, 0)$ with $\bar{\phi} \neq 0$, $\bar{g}_{ik} = \bar{g}_{ki}$, $|\bar{g}_{ik}| \neq 0$ and $\bar{g}_{ik}(x^1, \dots, x^n, 0) = g_{ik}(x^1, \dots, x^n)$, and such that the metric for some $V \subseteq N^{n+1}$ is*

$$\begin{aligned} ds^2 &= \tilde{g}_{\alpha\beta} dy^\alpha dy^\beta \\ &= \bar{g}_{ik} dx^i dx^k + \varepsilon \bar{\phi}^2 (dy)^2, \end{aligned} \tag{3.4.1}$$

where $\varepsilon^2 = 1$.

Note that here the metric is expressed in Gaussian-normal form. We may also, without loss of generality, set $\phi = 1$. While this is not mandated by the definition of a local isometric embedding, we may always rewrite an arbitrary metric in this form, and so this imposes no restrictions on the subsequent proof. It is sufficient to show that the embedding functions exist for an embedding of M^n into N^{n+1} given in this form. In order to prove the result, one only needs to show that the conditions of the above theorem hold.

Now, at each point in N^{n+1} there exists a co-ordinate neighbourhood in which the metric is given by (3.4.1). Consider the inclusion map

$$\iota\,(x^1, \dots, x^n) = (x^1, \dots, x^n, 0),$$

that represents an embedding of an open set of the hypersurface Σ_0, defined by $y = 0$, in N^{n+1}. The metric for Σ_0 is given by

$$g_{ik}(x^1, \ldots, x^n) = \bar{g}_{ik}(x^1, \ldots, x^n, 0),$$

where g_{ik} is unspecified for now, and the extrinsic curvature of Σ_0 is

$$\bar{\Omega}_{ik} = -\frac{1}{2\bar{\phi}}\frac{\partial \bar{g}_{ik}}{\partial y}.$$

The Gauss and Codazzi equations relate the intrinsic curvature of Σ_0 and N^{n+1} at Σ_0. The following forms of the Gauss and Codazzi equations and $\widetilde{R}_{ik}$ are obtained from the metric for N^{n+1}:

$$\widetilde{R}_{ik} = \bar{R}_{ik} + \epsilon \bar{g}^{jm}(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - 2\bar{\Omega}_{jk}\bar{\Omega}_{im}) - \frac{\epsilon}{\bar{\phi}}\frac{\partial \bar{\Omega}_{ik}}{\partial y} + \frac{1}{\bar{\phi}}\bar{\nabla}_i \bar{\nabla}_k \bar{\phi}\,, \tag{3.4.2}$$

$$\widetilde{R}_{i(n+1)} = \bar{\phi}\bar{g}^{jk}(\bar{\nabla}_j \bar{\Omega}_{ik} - \bar{\nabla}_i \bar{\Omega}_{jk}), \tag{3.4.3}$$

$$\widetilde{G}^{(n+1)}{}_{(n+1)} = -\frac{1}{2}\bar{g}^{ik}\bar{g}^{jm}(\bar{R}_{ijkm} + \epsilon(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - \bar{\Omega}_{jk}\bar{\Omega}_{im})). \tag{3.4.4}$$

Equation (3.4.4) is the Gauss equation and (3.4.3) is the Codazzi equation. Since the codimension has reduced to one and there can be no twisting in only one extra dimension, the Ricci equation falls away. It is replaced by (3.4.2), which are the space-space components of the Einstein field equations, and which we refer to as the propagation equation because it is used to propagate off the hypersurface to specify the rest of the bulk.

Einstein embedding space

Now take N^{n+1} to be an Einstein space where

$$\widetilde{R}_{\alpha\beta} = \frac{2\Lambda}{1-n}\tilde{g}_{\alpha\beta} \quad \text{and} \quad \widetilde{G}_{\alpha\beta} = \Lambda \tilde{g}_{\alpha\beta}.$$

Then the equations (3.4.2) – (3.4.4) become

$$\bar{R}_{ik} + \epsilon \bar{g}^{jm}(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - 2\bar{\Omega}_{jk}\bar{\Omega}_{im}) - \frac{\epsilon}{\bar{\phi}}\frac{\partial \bar{\Omega}_{ik}}{\partial y} + \frac{1}{\bar{\phi}}\bar{\nabla}_i\bar{\nabla}_k\bar{\phi} = \frac{2\Lambda}{1-n}\bar{g}_{ik}, \tag{3.4.5}$$

$$\bar{\phi}\bar{g}^{jk}(\bar{\nabla}_j\bar{\Omega}_{ik} - \bar{\nabla}_i\bar{\Omega}_{jk}) = 0, \tag{3.4.6}$$

$$-\frac{1}{2}\bar{g}^{ik}\bar{g}^{jm}(\bar{R}_{ijkm} + \epsilon(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - \bar{\Omega}_{jk}\bar{\Omega}_{im})) = \Lambda. \tag{3.4.7}$$

Using the fact that the quantity $\widetilde{G}_{\alpha\beta} - \Lambda\tilde{g}_{\alpha\beta}$ has vanishing divergence, one can prove that if the functions $\bar{g}_{ik}$ and $\bar{\phi}$ are analytic at $\mathbf{0}$, with $\bar{\phi} \neq 0$, $\bar{g}_{ik} = \bar{g}_{ki}$ and $|\bar{g}_{ik}| \neq 0$, and if they satisfy (3.4.5) in a neighbourhood of $\mathbf{0}$ and (3.4.6) and (3.4.7) on the hypersurface $y = 0$, then they also satisfy (3.4.6) and (3.4.7) in a neighbourhood of $\mathbf{0}$.

Take $\bar{\phi}$ to be any non-zero function that is analytic at $\mathbf{0}$. Noting that $\bar{g}_{ik} = \bar{g}_{ki}$ and using the expression for the extrinsic curvature $\bar{\Omega}_{ik}$, the propagation equation (3.4.5) can be rewritten as a set of partial differential equations for the unknown functions $\bar{g}_{ik}$ $(i \leq k)$. These functions must satisfy the initial conditions

$$\begin{aligned} \bar{g}_{ik}(x^1,\ldots,x^n,0) &= g_{ik}(x^1,\ldots,x^n), \\ \frac{\partial \bar{g}_{ik}}{\partial y}(x^1,\ldots,x^n,0) &= -2\,\bar{\phi}\,(x^1,\ldots,x^n,0)\,\Omega_{ik}(x^1,\ldots,x^n), \end{aligned}$$

where g_{ik} and Ω_{ik} are arbitrary functions that are analytic at $\mathbf{0}$ with

$$g_{ik} = g_{ki}, \quad |g_{ik}| \neq 0 \quad \text{and} \quad \Omega_{ik} = \Omega_{ki},$$

in a neighbourhood of $\mathbf{0} \in \mathbb{R}^n$. Note that $|\bar{g}_{ik}| \neq 0$ in a neighbourhood of $\mathbf{0}$. Now the Cauchy-Kowalewski theorem can be applied to ensure that there exists a unique and analytic at $\mathbf{0}$ solution $\bar{g}_{ik}$ to the set of equations. If this $\bar{g}_{ik}$ and $\bar{\phi}$ also satisfy the Gauss and Codazzi equations on $y = 0$, then they satisfy the equations in a neighbourhood of $\mathbf{0}$.

Now, take the image of embedding M^n to coincide with the hypersurface $y = 0$ in N^{n+1} so that g_{ik} is the analytic metric for M^n, and let $p \in M^n$ have the co-ordinates

$$x_p^1 = \cdots = x_p^n = 0.$$

Then M^n has a local isometric embedding (at p) in N^{n+1} if and only if there exist analytic at $\mathbf{0}$ functions Ω_{ik} satisfying

$$\Omega_{ik} = \Omega_{ki},$$

$$g^{jk}(\nabla_j \Omega_{ik} - \nabla_i \Omega_{jk}) = 0, \tag{3.4.8}$$

$$g^{ik} g^{jm} (R_{ijkm} + \varepsilon(\Omega_{ik}\Omega_{jm} - \Omega_{jk}\Omega_{im})) = -2\Lambda. \tag{3.4.9}$$

To proceed, note that since $|g_{ik}| \neq 0$ there exists an index r' obeying $g^{r'n} \neq 0$. From the $\frac{n(n+1)}{2}$ independent functions Ω_{ik} $(i \leq k)$, choose the n functions $\Omega_{r'n}$ and Ω_{1k} $(k > 1)$ as unknown and take the remaining functions (except Ω_{11} which can be determined in terms of the other Ω_{ik}) to be arbitrarily specified. Then, keeping (3.4.9) as a constraint equation, the equation (3.4.8) can be rewritten to obtain a set of n partial differential equations that must be solved subject to certain initial conditions. The Cauchy-Kowalewski theorem is applied to ensure that there exist unique and analytic solutions to the p.d.e. system. Thus, there exist analytic at $\mathbf{0}$ symmetric functions Ω_{ik} satisfying (3.4.8) and (3.4.9), and so the local isometric embedding of M^n in an Einstein space N^{n+1} is guaranteed.

Now the proof leads to three conditions that must be satisfied for the metric of embedding space to be unique:

(1) the $\frac{n(n-1)}{2} - 1$ arbitrarily specified functions Ω_{ik} for $i \leq k, i > 1$, and $(i,k) \neq (r', n)$ must be analytic at $\mathbf{0} \in \mathbb{R}^n$,

(2) the n functions $\Omega_{1k}(0, x^2, \ldots, x^n) = f_k(x^2, \ldots, x^n)$ $(k > 1)$ and $\Omega_{r'n}(0, x^2, \ldots, x^n) = f_1(x^2, \ldots, x^n)$ must be analytic at $\mathbf{0} \in \mathbb{R}^{n-1}$, and $\underset{r,s>1}{g^{rs}} \; (\underset{r \leq s}{\Omega_{rs}} + \underset{s<r}{\Omega_{sr}}) |_{\mathbf{0}} \neq 0$,

(3) a non-zero function $\bar{\phi}$ that is analytic at $\mathbf{0} \in \mathbb{R}^{n+1}$ must be chosen.

pseudo-Riemannian embedding space

One may follow the same approach to prove the existence of the local isometric embedding of M^n into a pseudo-Riemannian space N^{n+1} with a non-degenerate Ricci tensor

$\widetilde{R}_{\alpha\beta}$ which is equal, up to a local analytic diffeomorphism $\bar{f}$, to the Ricci tensor $S_{\alpha\beta}$ of an arbitrarily given pseudo-Riemannian space. So

$$\widetilde{R}_{\alpha\beta}(x^\gamma) = \frac{\partial \bar{f}^\mu}{\partial x^\alpha}\frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu}(x'^\kappa),$$

in a neighbourhood of $\mathbf{0} \in \mathbb{R}^{n+1}$ where $x'^\kappa = \bar{f}^\kappa(x^\gamma)$ and $\det(\frac{\partial \bar{f}^\mu}{\partial x^\alpha})|_0 \neq 0$. With this expression for $\widetilde{R}_{\alpha\beta}$, the Gauss, Codazzi and propagation equations become

$$\bar{R}_{ik} + \epsilon \bar{g}^{jm}(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - 2\bar{\Omega}_{jk}\bar{\Omega}_{im}) - \frac{\epsilon}{\bar{\phi}}\frac{\partial \bar{\Omega}_{ik}}{\partial y} + \frac{1}{\bar{\phi}}\bar{\nabla}_i\bar{\nabla}_k\bar{\phi} = \frac{\partial \bar{f}^\mu}{\partial x^i}\frac{\partial \bar{f}^\nu}{\partial x^k} S_{\mu\nu}(\bar{f}^\alpha), \tag{3.4.10}$$

$$\bar{\phi}\bar{g}^{jk}(\bar{\nabla}_j\bar{\Omega}_{ik} - \bar{\nabla}_i\bar{\Omega}_{jk}) = \frac{\partial \bar{f}^\mu}{\partial y}\frac{\partial \bar{f}^\nu}{\partial x^i} S_{\mu\nu}(\bar{f}^\alpha), \tag{3.4.11}$$

$$-\frac{1}{2}\bar{g}^{ik}\bar{g}^{jm}(\bar{R}_{ijkm} + \epsilon(\bar{\Omega}_{ik}\bar{\Omega}_{jm} - \bar{\Omega}_{jk}\bar{\Omega}_{im})) = \frac{\varepsilon}{2\bar{\phi}^2}\frac{\partial \bar{f}^\mu}{\partial y}\frac{\partial \bar{f}^\nu}{\partial y} S_{\mu\nu}(\bar{f}^\alpha) - \frac{\bar{g}^{jm}}{2}\frac{\partial \bar{f}^\mu}{\partial x^j}\frac{\partial \bar{f}^\nu}{\partial x^m} S_{\mu\nu}(\bar{f}^\alpha). \tag{3.4.12}$$

Define

$$\widetilde{F}^\alpha_\beta = \widetilde{G}^\alpha_\beta - \left(\tilde{g}^{\alpha\gamma}\frac{\partial \bar{f}^\mu}{\partial x^\gamma}\frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu} - \frac{1}{2}\delta^\alpha_\beta \tilde{g}^{\gamma\lambda}\frac{\partial \bar{f}^\mu}{\partial x^\gamma}\frac{\partial \bar{f}^\nu}{\partial x^\lambda} S_{\mu\nu}\right),$$

and impose that the $\bar{f}^\alpha$ satisfy

$$\widetilde{\nabla}_\alpha \left(\tilde{g}^{\alpha\gamma}\frac{\partial \bar{f}^\mu}{\partial x^\gamma}\frac{\partial \bar{f}^\nu}{\partial x^\beta} S_{\mu\nu} - \frac{1}{2}\delta^\alpha_\beta \tilde{g}^{\gamma\lambda}\frac{\partial \bar{f}^\mu}{\partial x^\gamma}\frac{\partial \bar{f}^\nu}{\partial x^\lambda} S_{\mu\nu}\right) = 0, \tag{3.4.13}$$

so that $\widetilde{F}^\alpha_\beta$ has vanishing divergence. Using this fact, it can be shown that if $\bar{g}_{ik}$, $\bar{\phi}$ and $\bar{f}^\alpha$ are analytic at $\mathbf{0}$, with $\bar{\phi} \neq 0$, $\bar{g}_{ik} = \bar{g}_{ki}$ and $|\bar{g}_{ik}| \neq 0$, and if they satisfy (3.4.10) and (3.4.13) in a neighbourhood of $\mathbf{0}$, and (3.4.11) and (3.4.12) on the hypersurface $y = 0$, then they also satisfy the equations (3.4.11) and (3.4.12) in a neighbourhood of $\mathbf{0}$.

Take $\bar{\phi} \neq 0$ to be any analytic at $\mathbf{0}$ function. Now using the fact that $|\bar{g}_{ik}| \neq 0$, $\bar{g}_{ik} = \bar{g}_{ki}$ and the expression for the extrinsic curvature $\bar{\Omega}_{ik}$, one may rewrite equations (3.4.10) and (3.4.13) as a set of partial differential equations for the unknown functions $\bar{g}_{ik}$ $(i \leq k)$ and $\bar{f}^\alpha$ that must satisfy the initial conditions

$$\begin{aligned}\bar{g}_{ik}(x^1,\ldots,x^n,0) &= g_{ik}(x^1,\ldots,x^n),\\ \frac{\partial \bar{g}_{ik}}{\partial y}(x^1,\ldots,x^n,0) &= -2\,\bar{\phi}\,(x^1,\ldots,x^n,0)\,\Omega_{ik}(x^1,\ldots,x^n),\\ \bar{f}^\alpha(x^1,\ldots,x^n,0) &= \xi^\alpha(x^1,\ldots,x^n),\\ \frac{\partial \bar{f}^\alpha}{\partial y}(x^1,\ldots,x^n,0) &= \eta^\alpha(x^1,\ldots,x^n).\end{aligned}$$

where g_{ik}, Ω_{ik}, ξ^α and η^α are analytic at $\mathbf{0} \in \mathbb{R}^n$ and $\det(\frac{\partial \bar{f}^\mu}{\partial x^\alpha})|_0 \neq 0$. The Cauchy-Kowalewski theorem is applied to ensure that there exist unique and analytic at $\mathbf{0}$ solutions $\bar{g}_{ik}$ and $\bar{f}^\alpha$ to the set of equations. Now one only needs to show that if $\bar{g}_{ik}$, $\bar{f}^\alpha$ and $\bar{\phi}$ also satisfy the Gauss and Codazzi equations on $y = 0$, then they satisfy the equations in a neighbourhood of $\mathbf{0}$.

Now specify g_{ik} to be the analytic metric for M^n so that M^n is embedded as the hypersurface $y = 0$ in N^{n+1} and set

$$\xi^i = x^i, \quad \xi^{n+1} = 0, \quad \eta^i = 0 \ \text{ and } \ \eta^{n+1} = 1,$$

so that $\det(\frac{\partial \bar{f}^\mu}{\partial x^\alpha})|_0 \neq 0$. Then the embedding exists if and only if there exist analytic at $\mathbf{0}$ functions Ω_{ik} that are symmetric and satisfy the Gauss and Codazzi equations on $y = 0$. By appealing again to the Cauchy-Kowalewski theorem, one can prove that there do exist such solutions Ω_{ik} and hence, the local isometric embedding of M^n in N^{n+1} exists.

Note that a similar construction may be carried out at any analytic point p ,with co-ordinates $(x^1_p, \ldots, x^n_p)$, throughout the manifold M^n. While we usually take the image of the embedding to coincide with the hypersurface $y = 0$, we may take it to coincide with any hypersurface $y = y_c$.

3.5 Killing Geometry and Rigidity

A conformal Killing vector $\mathbf{X}$ of a metric space is defined by the action of the Lie derivative on the metric tensor (Stephani *et al.* 2003):

$$L_{\mathbf{X}} g_{ab} = 2\,\psi(x^i)\,g_{ab}, \tag{3.5.1}$$

where ψ is the conformal factor. If $\psi = 0$, then $\mathbf{X}$ is a proper Killing vector, and (3.5.1) gives

$$X_{a;b} - X_{b;a} = 0,$$

which may be solved to yield the Killing vectors. Killing vectors are useful in simplifying the field equations and in the classification of space-times. Killing vectors generate conservation laws by Noether's theorem and play a role in the analysis of the causal structure of space-times (Hawking and Ellis 1973). Conformal Killing vectors can be applied to perturbation theory (Katz *et al.* 1997), and singularity theorems (Hawking and Ellis 1973, Joshi 1993). Note that conformal motions leave the Weyl tensor $C^{ab}{}_{cd}$ unchanged.

Now suppose that the extrinsic curvature Ω_{ab} of a manifold is generated from intrinsic quantities and their derivatives only. If these quantities are uniquely determined, then the embedded space is said to be intrinsically rigid (Goenner 1980). For a Killing vector $\mathbf{X}$, we have

$$L_{\mathbf{X}}\Gamma^{a}{}_{bc} = \frac{1}{2}\left(L_{\mathbf{X}}g^{ad}\right) + \frac{1}{2}g^{ad}\left(L_{\mathbf{X}}\frac{\partial g_{dc}}{\partial x^{b}} + L_{\mathbf{X}}\frac{\partial g_{db}}{\partial x^{c}} - L_{\mathbf{X}}\frac{\partial g_{bc}}{\partial x^{d}}\right) = 0,$$

since $L_{\mathbf{X}}g_{ab} = 0$ and $L_{\mathbf{X}}$ and $\frac{\partial}{\partial x^{i}}$ commute. It follows that $L_{\mathbf{X}}\Omega_{ab}$ must vanish for a space-time since $\Omega_{ab} = \varepsilon\phi\Gamma^{4}{}_{ab}$. This concept can be used to determine the variables upon which the extrinsic curvature depends, and so allows one to make simple assumptions for the extrinsic curvature which help in the construction of explicit embeddings. We also note that, if the extrinsic curvature depends on the Ricci tensor and the metric of the hypersurface only, then the space is said to be energetically rigid (Goenner 1980).

The Killing vectors for a static spherically symmetric space-time are (Maartens *et al.* 1995, 1996):

$$\begin{aligned}
A^{i} &= (1, 0, 0, 0),\\
B^{i} &= (0, 0, 0, 1),\\
C^{i} &= (0, 0, \sin\phi, -\cot\theta\cos\phi),\\
D^{i} &= (0, 0, -\cos\phi, \cot\theta\sin\phi).
\end{aligned}$$

By considering the vanishing Lie derivative of Ω_{ab} with respect to each of these vectors, it can be shown that the extrinsic curvature of the embedded space-time depends on r only. Furthermore, insisting that the 5-dimensional metric possess the Killing vectors

$$B^\mu = A^i \delta^\mu_i,$$

implies that $\bar{g}_{ab}$ must have a time-independent diagonal form with spherical symmetry (Amery *et al.* 2007). These assumptions shall be used to simplify the embedding of a SSS space-time.

3.6 Embedding SSS Space-times

Motivated by astrophysical considerations, we aim to embed static spherically symmetric (SSS) space-times M, with metric of the form (2.5.1), into five-dimensional Einstein spaces N. The formalism presented here is drawn from Londal (2005) and Amery *et al.* (2007).

3.6.1 The Equations

The equations (3.4.5) – (3.4.7) for an Einstein embedding space can be written as

$$\frac{\partial^2 \bar{g}_{ik}}{\partial^2 y} = -\frac{4\epsilon\Lambda\bar{g}_{ik}}{3} - 2\bar{g}^{jm}(\overline{\Omega}_{ik}\overline{\Omega}_{jm} - 2\overline{\Omega}_{im}\overline{\Omega}_{jk}) - 2\epsilon\overline{R}_{ik}, \tag{3.6.1}$$

$$0 = g^{jk}(\nabla_j\Omega_{ik} - \nabla_i\Omega_{jk}), \tag{3.6.2}$$

$$-2\Lambda = R + g^{ik}g^{jm}\epsilon(\Omega_{ik}\Omega_{jm} - \Omega_{jk}\Omega_{im})), \tag{3.6.3}$$

$$\bar{g}_{ik}(x^0, \ldots, x^3, 0) = g_{ik}(x^0, \ldots, x^3), \tag{3.6.4}$$

$$\frac{\partial\bar{g}_{ik}(x^0, \ldots, x^3, 0)}{\partial y} = -2\,\Omega_{ik}(x^0, \ldots, x^3). \tag{3.6.5}$$

Note that we have, without loss of generality, set $\phi = 1$ in the line element for N. Recall that the overbars denote quantities obtained from $\bar{g}_{ik}$. The propagation equation (3.6.1) can also be written as

$$\frac{\partial^2 \bar{g}_{ik}}{\partial^2 y} = -\frac{4\epsilon\Lambda\bar{g}_{ik}}{3} - \frac{\bar{g}^{jm}}{2}\left(\frac{\partial\bar{g}_{ik}}{\partial y}\frac{\partial\bar{g}_{jm}}{\partial y} - 2\frac{\partial\bar{g}_{im}}{\partial y}\frac{\partial\bar{g}_{jk}}{\partial y}\right) - 2\epsilon\overline{R}_{ik}. \tag{3.6.6}$$

The procedure to determine the embedding is as follows: we make an ansatz for Ω_{ik} in M in order to solve (3.6.2) and (3.6.3) on the hypersurface $y = 0$, and then by making a related ansatz for $\overline{\Omega}_{ik}$ in the bulk, we attempt to solve (3.6.1) in N subject to the conditions (3.6.4) and (3.6.5). We take dots and primes to denote differentiation with respect to y and r, respectively.

3.6.2 The General Case

In §3.5 we explained that the extrinsic curvature for a static spherically symmetric space-time depends only on r. This idea, and also the fact that a SSS space-time has $R = R(r)$, motivates the ansatz

$$\Omega_{ik} = c(r)u_i u_k + d(r)g_{ik}, \tag{3.6.7}$$

for the extrinsic curvature in M. Since $u_0 u_0 = g_{00}u_0 u^0 = -g_{00}$ and $u_i = 0$ for $i = 1, 2, 3$, the above assumption is equivalent to

$$\begin{aligned} \Omega_{ik} &= \mathrm{diag}[(d(r) - c(r))g_{00}, d(r)g_{AB}], \\ &= \mathrm{diag}[a(r)g_{00}, b(r)g_{AB}]. \end{aligned} \tag{3.6.8}$$

If the matter content is well modelled by a perfect fluid, then (3.6.7) means that we are actually assuming energetic rigidity for M (Amery *et al.* 2007). Following Londal (2005), we substitute the ansatz (3.6.7) in (3.6.2) and (3.6.3) to solve for Ω_{ik} on the hypersurface $y = 0$.

Firstly, since the covariant derivative of the metric vanishes, we can rewrite (3.6.2) as

$$0 = g^{jk}(\nabla_j \Omega_{ik}) - \nabla_i(g^{jk}\Omega_{jk}).$$

Then, applying the ansatz we obtain

$$-\nabla_i(g^{jk}\Omega_{jk}) = c^{'}\delta_i^1 - 4d^{'}\delta_i^1, \quad \text{and} \quad g^{jk}(\nabla_j\Omega_{ik}) = c\nu^{'}\delta_i^1 + d^{'}\delta_i^1,$$

so that the Codazzi equation becomes

$$c^{'} - 3d^{'} + c\nu^{'} = 0. \tag{3.6.9}$$

The Gauss equation (3.6.3) becomes

$$c = 2d + \frac{2\Lambda + R}{6\epsilon d}. \tag{3.6.10}$$

Now we may differentiate (3.6.10), substitute for $c^{'}$ using (3.6.9), and simplify to obtain

$$-d^{'}[6\epsilon d^2 + (2\Lambda + R)] + [R^{'} + (2\Lambda + R)\nu^{'}]d + 12\epsilon d^3\nu^{'} = 0.$$

Multiplying the above equation by $\frac{e^{\nu}}{d^2}$, rewriting and integrating by parts we obtain

$$I = \frac{2\Lambda + R}{d}e^{\nu} + 12\epsilon d e^{\nu} - 18\epsilon\int e^{\nu},$$

where I is a constant. Multiply the above equation by $\frac{d}{12\epsilon e^{\nu}}$ and rewrite it as

$$d^2 + \left(\frac{-18\epsilon\int e^{\nu} - I}{12\epsilon e^{\nu}}\right)d + \frac{2\Lambda + R}{12\epsilon} = 0,$$

which is an algebraic equation that can be solved for d. Hence, we have

$$d = \frac{18\epsilon\int e^{\nu} + I}{24\epsilon e^{\nu}} \pm \sqrt{\left(\frac{(18\epsilon\int e^{\nu} + I)}{24\epsilon e^{\nu}}\right)^2 - \frac{2\Lambda + R}{12\epsilon}}. \tag{3.6.11}$$

So, the extrinsic curvature on the hypersurface $y = 0$ is expressed by (3.6.8) with $c(r)$ and $d(r)$ given by (3.6.10) and (3.6.11), respectively.

Now, to propagate into the bulk, we make the ansatz

$$\overline{\Omega}_{ik}(y, r) = \text{diag}[a(y, r)g_{00}, b(y, r)g_{CD}], \tag{3.6.12}$$

where $a(y, r)$ and $b(y, r)$ must obey the initial conditions $a(0, r) = d(r) - c(r)$ and $b(0, r) = d(r)$. The definition

$$\overline{\Omega}_{ik} = -\frac{1}{2}\frac{\partial \bar{g}_{ik}}{\partial y},$$

implies that

$$\bar{g}_{ik}(y,r) = \mathrm{diag}[A(y,r)g_{00}, B(y,r)g_{CD}], \tag{3.6.13}$$

where $A(y,r) = -2\int a(y,r)dy$ and $B(y,r) = -2\int b(y,r)dy$. From the condition (3.6.4) we obtain

$$A(0,r) = 1, \quad B(0,r) = 1, \tag{3.6.14}$$

and the condition (3.6.5) implies that

$$\dot{A}(0,r) = -2a(r) \quad \text{and} \quad \dot{B}(0,r) = -2b(r). \tag{3.6.15}$$

With the assumption (3.6.13) the propagation equation (3.6.6) leads to the time-time and space-space equations:

$$\ddot{A} + \frac{\dot{A}}{2}\left(\frac{3\dot{B}}{B} - \frac{\dot{A}}{A}\right) + \frac{4\epsilon\Lambda}{3}A = -2\epsilon\overline{R}_{00}g^{00}, \tag{3.6.16}$$

$$\ddot{B} + \frac{\dot{B}}{2}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{4\epsilon\Lambda}{3}B = -\frac{2\epsilon}{3}\overline{R}_{CD}g^{CD}. \tag{3.6.17}$$

To determine the expressions $\overline{R}_{00}g^{00}$ and $\overline{R}_{AB}g^{AB}$, we first calculate the non-zero connections $\overline{\Gamma}^k{}_{ij}$ from $\bar{g}_{ij}$. We obtain:

$$\begin{aligned}
\overline{\Gamma}^0{}_{ij} &= \Gamma^0{}_{ij} + \frac{A'}{2A}\delta^0_j\delta^1_i + \frac{A'}{2A}\delta^0_i\delta^1_j, \\
\overline{\Gamma}^1{}_{ij} &= \Gamma^1{}_{ij} - \frac{A'}{2B}g^{11}g_{00}\delta^0_i\delta^0_j + \frac{1}{2}\left(1 - \frac{A}{B}\right)g^{11}g_{00,1}\delta^0_i\delta^0_j \\
&\quad + \frac{B'}{2B}\delta^1_i\delta^1_j - \frac{B'}{2B}g^{11}(g_{22}\delta^2_i\delta^2_j + g_{33}\delta^3_i\delta^3_j), \qquad (3.6.18) \\
\overline{\Gamma}^2{}_{ij} &= \Gamma^2{}_{ij} + \frac{B'}{2B}\delta^2_j\delta^1_i + \frac{B'}{2B}\delta^2_i\delta^1_j,
\end{aligned}$$

$$\overline{\Gamma}^3{}_{ij} = \Gamma^3{}_{ij} + \frac{B'}{2B}\delta^3_j\delta^1_i + \frac{B'}{2B}\delta^3_i\delta^1_j,$$

where the Γ^k_{ij} are given by (2.5.2). Using the above connections we find

$$\overline{R}_{00}g^{00} = e^{-2\lambda}\left[\frac{A}{B}(-\nu'' - \nu'^2 + \nu'\lambda' - \frac{2}{r}\nu') - \frac{A'}{B}\nu' - \frac{AB'}{2B^2}\nu'\right.$$

$$\left. + \frac{A'}{2B}\lambda' - \frac{A''}{2B} - \frac{A'}{rB} - \frac{A'B'}{4B^2} + \frac{A'^2}{4AB}\right], \quad (3.6.19)$$

$$= \frac{A}{B}R_{00}g^{00}$$
$$+ e^{-2\lambda}\left[-\frac{A'}{B}\nu' - \frac{AB'}{2B^2}\nu' + \frac{A'}{2B}\lambda' - \frac{A''}{2B} - \frac{A'}{rB} - \frac{A'B'}{4B^2} + \frac{A'^2}{4AB}\right],$$

$$\overline{R}_{AB}g^{AB} = \frac{2}{r^2} + e^{-2\lambda}\left[-\nu'' - \nu'^2 + \nu'\lambda' - \frac{2}{r^2} + \frac{4\lambda'}{r} - \frac{2\nu'}{r}\right.$$

$$- \frac{A''}{2A} + \frac{A'^2}{4A^2} + \frac{A'}{2A}\lambda' - \frac{A'}{A}\nu' - \frac{A'}{rA} - \frac{A'B'}{4AB}$$

$$\left. - \frac{2B''}{B} + \frac{3B'^2}{2B^2} - \frac{B'}{2B}\nu' + \frac{2B'}{B}\lambda' - \frac{4B'}{rB}\right] \quad (3.6.20)$$

$$= R_{AB}g^{AB} + e^{-2\lambda}\left[-\frac{A''}{2A} + \frac{A'^2}{4A^2} + \frac{A'}{2A}\lambda' - \frac{A'}{A}\nu' - \frac{A'}{rA}\right.$$

$$\left. - \frac{A'B'}{4AB} - \frac{2B''}{B} + \frac{3B'^2}{2B^2} - \frac{B'}{2B}\nu' + \frac{2B'}{B}\lambda' - \frac{4B'}{rB}\right].$$

The equations (3.6.16) and (3.6.17) must be solved subject to the conditions (3.6.14) and (3.6.15) in order to specify the embedding completely. However, solutions to these equations are not immediately clear, even though strong simplifying assumptions were made.

3.6.3 The Case $R =$ constant

For SSS space-times with constant Ricci scalar, the calculation can be simplified by setting $c(r) = 0$ and replacing $d(r)$ by the constant $\frac{-f}{2}$, so that the extrinsic curvature on the hypersurface is

$$\Omega_{ik} = \frac{-f}{2} g_{ik}.$$

The Codazzi equation (3.6.9) is satisfied and the Gauss equation (3.6.10) gives

$$f = \sqrt{-\frac{2\epsilon\Lambda}{3} - \frac{\epsilon R}{3}}.$$

Now we make the assumption

$$\overline{\Omega}_{ik} = -\frac{f(y) g_{ik}}{2},$$

where $f(0) = f$. This implies that

$$\bar{g}_{ik} = F(y) g_{ik}, \tag{3.6.21}$$

where $F(y) = \int f(y)dy$, and with the initial conditions $F(0) = 1$ and $\dot{F}(0) = f$. The propagation equation simplifies to

$$\ddot{F} + \frac{\dot{F}^2}{F} + \frac{4\epsilon\Lambda}{3} F = -\frac{\epsilon}{2} \overline{R}_{ik} g^{ik}. \tag{3.6.22}$$

With the assumption (3.6.21) we have $\overline{R}_{ik} = R_{ik}$, and so (3.6.22) becomes

$$\ddot{F} + \frac{\dot{F}^2}{F} + \frac{4\epsilon\Lambda}{3} F = -\frac{\epsilon R}{2}. \tag{3.6.23}$$

This equation can be solved (Londal 2005) subject to the above initial conditions to give the five-dimensional metric

$$ds^2 = \left[\cosh\left(\frac{1}{2}\sqrt{-\frac{2}{3}\epsilon\Lambda y}\right) + \left(1 + \frac{R}{8\Lambda A}\right)\sinh\left(\frac{1}{2}\sqrt{-\frac{2}{3}\epsilon\Lambda y}\right)\right]^2 g_{ik} dx^i dx^k + \epsilon dy^2, \tag{3.6.24}$$

where A must satisfy the consistency equation

$$\left(1+\frac{R}{8\Lambda A}\right)\sqrt{-\frac{2\epsilon}{3\Lambda}}=\sqrt{-\frac{2\epsilon}{3\Lambda}-\frac{\epsilon R}{3}}.$$

3.6.4 The Case $R = 0$

For SSS space-times with Ricci scalar $R = 0$, the propagation equation (3.6.23) becomes

$$\frac{1}{2}(F^2)^{\cdot\cdot}+\frac{4\epsilon\Lambda}{3}F^2=0, \tag{3.6.25}$$

with the initial conditions $F(0) = 1$ and $\dot{F}(0) = f = \sqrt{-\frac{2\epsilon\Lambda}{3}}$. By making the substitution $w = F^2$ this equation reduces to a second order linear equation which is easily and uniquely solvable. Thus, the metric for the 5-dimensional local embedding space is

$$ds^2=\exp\left(\sqrt{\frac{-2\epsilon\Lambda}{3}}y\right)g_{ik}dx^i dx^k+\epsilon dy^2. \tag{3.6.26}$$

3.6.5 Comments

The uniqueness of embeddings is an interesting question. Dahia and Romero (2002b) assert three conditions that must hold for the embedding metric to be unique – see §3.4. We explicitly check the uniqueness of the R = constant solution presented in §3.6.3. Note that we only consider the embedding about the points in M for which the metric components are analytic. Choose $r' = 3$. Then we have:

(1) $\Omega_{11} = \frac{-f}{2}g_{11}, \Omega_{22} = \frac{-f}{2}g_{22}, \Omega_{12} = \Omega_{13} = \Omega_{23} = 0$ are all analytic at $\mathbf{0} \in \mathbb{R}^4$ since f is a constant,

(2) at $t = 0$, $\Omega_{33} = \frac{-f}{2}g_{33}, \Omega_{01} = \Omega_{02} = \Omega_{03} = 0$ are all analytic at $\mathbf{0} \in \mathbb{R}^3$, and $(g^{11}\Omega_{11} + g^{22}\Omega_{22} + g^{33}\Omega_{33})|_{t=0} = \frac{-3f}{2}$ is non-zero since f is assumed non-zero,

(3) we have chosen $\phi = 1$ which is analytic at $\mathbf{0} \in \mathbb{R}^4$.

Note that Ω_{00} is also analytic. Thus, the embedding metric (3.6.24) is unique.

We can also check the uniqueness of solutions by noting the linearity of the embedding equations. Observe that for the case of the hypersurface having zero Ricci scalar, the propagation equation (3.6.25) can be reduced to a linear equation which has a unique solution. For the general case of static spherical symmetry on the hypersurface, the solutions (3.6.10) and (3.6.11) to the first order linear equation (3.6.9) and the algebraic equation (3.6.10) are unique. However, the propagation equations (3.6.16) and (3.6.17) do not appear linear and so one cannot say whether they admit unique solutions.

We note that the Schwarzschild space-time

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

is Ricci-flat and hence, can be locally embedded into the Einstein space (3.6.26) as an unique solution (Dahia and Romero 2002b). However, this result does not apply to the singularity at $r = 0$ since the metric is not analytic at that point.

Now we want to investigate the embedding of SSS space-times into 5-dimensional spaces of constant curvature which obey

$$\tilde{R}_{abcd} = \frac{\tilde{R}}{20}(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{bc}).$$

Explicitly calculating the Riemann tensor for (3.6.26), we find

$$\widetilde{R}_{ijkl} = R_{ijkl} - \frac{\Lambda}{6}\left(\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}\right).$$

Thus, it is of constant curvature only if the embedded space is Riemann flat. As a special case, a Ricci-flat space-time of constant curvature can be embedded into a $\Lambda = 0$ constant curvature space (also known as Minkowski space). The general question of the embeddability of space-times into 5-dimensional constant curvature remains open. The study of this issue is underway (Amery and Moodley 2008).

3.7 Embedding the Reissner-Nordström Space-time

The Reissner-Nordström space-time (see §2.5.2) has Ricci scalar $R = 0$, and so the five-dimensional Einstein embedding space has the metric

$$ds^2 = \exp\left(\sqrt{\frac{-2\epsilon\Lambda}{3}}y\right) g_{ik}^{(RN)} dx^i dx^k + \epsilon dy^2 \tag{3.7.1}$$

where $g_{ik}^{(RN)}$ represents the Reissner-Nordström metric. Here, Λ cannot be zero, or else it would yield the bulk $M^{(RN)} \times \text{AdS}_{(1)}$ ($M^{(RN)}$ representing the Reissner-Nordström space-time), and this is not an Einstein embedding since $M^{(RN)}$ is not Ricci-flat.

Letting

$$d\hat{y}^2 = \exp\left(-\sqrt{\frac{-2\epsilon\Lambda}{3}}y\right)dy^2,$$

we find that

$$\hat{y} = -\sqrt{\frac{-6\epsilon}{\Lambda}}\exp\left(-\frac{1}{2}\sqrt{\frac{-2\epsilon\Lambda}{3}}y\right),$$

which transforms the metric (3.7.1) into the form

$$ds^2 = \Delta(\hat{y})[g_{ik}dx^i dx^k + \epsilon d\hat{y}^2],$$

where $\Delta(\hat{y}) = \frac{-6\epsilon}{\Lambda\hat{y}^2}$. Thus, the embedding space is conformal to $M^{(RN)} \times \text{AdS}_{(1)}$.

3.8 Embedding the Global Monopole Space-time

The global monopole space-time (see §2.5.3) has Ricci scalar $R = \frac{2(1-K)}{r^2}$ that depends on r, and has $\nu = 0$ and $\lambda = -\frac{1}{2}\ln K$ in the SSS form of the metric. The Codazzi equation (3.6.9) can be integrated to obtain

$$c = 3d + 2I,$$

where I is an integration constant. Substituting this expression for c in (3.6.10) gives

$$d(r) = -I \pm \sqrt{I^2 + \frac{\epsilon\Lambda}{3} + \frac{\epsilon(1-K)}{3r^2}}.$$

So the extrinsic curvature on the hypersurface $y = 0$ is given by (3.6.8) with

$$a(r) = -2b(r) - 2I, \qquad b(r) = -I \pm \sqrt{I^2 + \frac{\epsilon\Lambda}{3} + \frac{\epsilon(1-K)}{3r^2}}. \tag{3.8.1}$$

Now with $\nu' = 0 = \lambda'$ the propagation equations (3.6.16) and (3.6.17) become

$$\ddot{A}+\frac{\dot{A}}{2}\left[\frac{3\dot{B}}{B}-\frac{\dot{A}}{A}\right]+\frac{4\epsilon\Lambda}{3}A=-2\epsilon K\left[-\frac{A''}{2B}-\frac{A'}{rB}-\frac{A'B'}{4B^2}+\frac{A'^2}{4AB}\right], \tag{3.8.2}$$

$$\ddot{B}+\frac{\dot{B}}{2}\left[\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right]+\frac{4\epsilon\Lambda}{3}B=\frac{4\epsilon(K-1)}{3r^2}-\frac{2\epsilon}{3}K\left[-\frac{A''}{2A}+\frac{A'^2}{4A^2}-\frac{A'}{rA}\right.$$
$$\left.-\frac{A'B'}{4AB}-\frac{2B''}{B}+\frac{3B'^2}{2B^2}-\frac{4B'}{rB}\right]. \tag{3.8.3}$$

These equations must be solved subject to the initial conditions (3.6.14) and (3.6.15) to obtain the five-dimensional Einstein embedding space. By local existence results (Dahia and Romero 2002b), we know that there does exist a solution for the embedding, however actually solving these equations is certainly nontrivial and we have not yet obtained any solutions.

We can make a negative comment on the conformal geometry of the bulk. Let us suppose that there exists a conformal Killing vector in the y-direction. This means that we are specifying isometry on null geodesics only, which is sufficient here since our bulk is empty of matter. In this case, we have $A(y,r)=kB(y,r)$ for some constant k (Amery *et al.* 2007), and so $a(y,r)=kb(y,r)$. The relations for a and b on the hypersurface imply that k be -2. However, then the condition (3.6.14) cannot be satisfied and so the assumption fails. It is impossible for the Einstein embedding space to have static spherical symmetry and a conformal Killing vector in the y-direction. Refer also to Amery *et al.* (2007).

We can obtain an explicit embedding of a space-time that is conformal to the global monopole space-time and that has zero Ricci scalar. The conformal transformation is given by

$$\hat{g}_{ik}=\Delta(r)g_{ik}.$$

where g_{ik} represents the metric of the global monopole. Note that we take the conformal factor Δ as a function of r only, motivated by the dependence of the global monopole's Ricci scalar on r. We seek $\hat{R}=0$. The expression (2.3.7) gives

$$\varphi' + \frac{1}{2}\varphi^2 + \frac{2}{r}\varphi + \frac{2(K-1)}{3Kr^2} = 0,$$

where $\varphi = (\ln \Delta)' = \frac{\Delta'}{\Delta}$. This equation has solutions

$$\varphi = \frac{P}{r}, \quad P = -1 \pm \sqrt{1 - 4\frac{(K-1)}{3K}},$$

where P is real since $0 < K < 1$. Solving $\frac{\Delta'}{\Delta} = \frac{P}{r}$ we obtain

$$\Delta = r^P, \text{ and so } \hat{g}_{ik} = r^P g_{ik}.$$

Note that this metric is not Ricci-flat. Since $\hat{R} = 0$, there exists an embedding of $\hat{g}_{ik}$ into the 5-dimensional Einstein space with metric

$$ds^2 = \exp\left(\sqrt{\frac{-2\epsilon\Lambda}{3}}y\right) r^P g_{ik} dx^i dx^k + \epsilon dy^2. \tag{3.8.4}$$

Note that this metric is not an isometric embedding of the global monopole space-time itself, since at $y = 0$ it reduces to $\hat{g}_{ik}$ and not g_{ik}, unless $P = 1$. However, $P = 1$ implies that $K = 1$, and so we regain the 4-dimensional Minkowski space-time. The composition of an embedding with a conformal transformation is not necessarily an embedding – see also §4.7. Since the conformal transformation preserves the null geometry of the global monopole space-time, this embedding of its conformally related metric can provide insight into the causal structure of the defect.

Chapter 4

Global Isometric Embeddings

4.1 Introduction

The problem of the nature of the possible global embedding manifolds for some arbitrary embedded manifold has a long history as a purely geometrical problem. Early work in the field concentrated on the embedding of Riemannian manifolds into Euclidean spaces. This was a natural historical consequence of the geometric analysis of (pseudo)-Riemannian spaces. It was found that the codimension of the embeddings are typically quite large. In §4.2 we provide a background of existence results for Euclidean embedding spaces. In §4.3 we consider a recent theorem by Katzourakis (2005a) in which it is claimed that the Campbell-Magaard-Dahia-Romero theorem for embedding into Einstein spaces can be made global. We show that careful analysis of the theorem indicates that it works only for Ricci-flat embedding spaces. There seems to have been a crucial misunderstanding of the local Einstein embedding result (Dahia and Romero 2002b): it is assumed that the local embedding space has the form $M \times F$ where M is the embedded space and F is a one-dimensional analytic manifold, but this is really only valid for Ricci-flat manifolds M. So, as written, the result (Katzourakis 2005a) is limited and in need of extension. Following Katzourakis's methodology to a large extent, we attempt to correct the given theorem in §4.4, and in §4.5 we extend its scope. In §4.4.1 we first provide an overview of the proof, before proceeding to the detailed proof in §4.4.2 which is separated into six steps. We also comment on the steps carried out in §4.4.3. These results are presented as two theorems: Theorem 1 pertains to embeddings into Einstein spaces, and Theorem 2 pertains to embeddings into arbitrarily specified pseudo-Riemannian spaces. We also provide even more general theorems pertinent to metric spaces and paracompact manifolds. In §4.6 we discuss the papers (Katzourakis 2005b,c,d) that build upon the initial result, and finally, in §4.7 we provide further comments on embeddings. The work contained in this chapter will be submitted for publication (Moodley and Amery 2007).

4.2 Euclidean Embedding Spaces

The first global existence result for isometric embeddings was given by Nash (1954), and states that

- any closed n-d Riemannian manifold has a C^1 isometric embedding in $\mathbb{R}^{2n}$, and
- any n-d Riemannian manifold has a C^1 isometric embedding in $\mathbb{R}^{2n+1}$.

The proof begins with a "short" embedding of a Riemannian manifold M in some $\mathbb{R}^k$ where the induced metric h_{ij} for M is smaller than its actual metric g_{ij}, and the embedding undergoes a series of perturbations until it is isometric. The existence of the initial embedding is guaranteed by results of Whitney (1936). More generally it is shown that

- if a closed n-d Riemannian manifold has a C^∞ embedding in $\mathbb{R}^m$ with $m \geq n+2$, then it also has an isometric embedding in $\mathbb{R}^m$, and
- if an open n-d Riemannian manifold has a "short" C^∞ embedding in $\mathbb{R}^m$ with $m \geq n+2$ and which does not coincide with its limit set (if any), then it also has an isometric embedding in $\mathbb{R}^m$.

These results were improved by Kuiper (1955) for $m \geq n+1$. The limit set of an embedding refers to the set of points in the embedding space with the property that for each point in the set, there is a divergent sequence in the embedded space whose image converges to that point (Friedman 1965).

Nash (1956) further established that every n-dimensional Riemannian manifold M is embeddable in $\mathbb{R}^m$ with

- $m = \frac{n}{2}(3n + 11)$ for M compact, or
- $m = \frac{n}{2}(n + 1)(3n + 11)$ for M non-compact,

and the embeddings are C^k isometric where $k \geq 3$. The result indicates that a greater number of dimensions are required for a smoother embedding.

Extensions by Clarke (1970) and Greene (1970) to the indefinite case followed later. Clarke (1970) showed that any n-dimensional C^∞ pseudo-Riemannian space M with C^k ($k \geq 3$) metric of rank r and signature s has a global C^k isometric embedding into $\mathbb{R}^m(p, q)$ where

- $p \geq n - \frac{1}{2}(r + s) + 1$, and
- $q \geq \frac{n}{2}(3n + 11)$, for compact M^n, or
- $q \geq \frac{n}{6}(2n^2 + 15n + 37) + 1$, for non-compact M^n.

Note that $m = p+q$. By applying this result to a strictly Riemannian and non-compact manifold, it was found that the dimension of the Euclidean embedding space is lower than that given by Nash, and so the result can be regarded as an alternative proof of Nash's theorem. Unlike the local theory, Clarke's extension of Nash's result to the indefinite case is not so trivial. Greene (1970) demonstrated that the embedding can be made C^∞ isometric with $p = q$ and $m = n(n + 5)$ for the compact case, or $m = 4(2n+1)(n+3)$ for the noncompact case. Furthermore, Gunther (1989, 1991) provided a much simplified approach to proving the existence of smooth isometric embeddings in $\mathbb{R}^m$ (see also (Yang 1998)). The case of analytic embeddings has also been considered (Greene and Jacobowitz 1971, Gromov 1970). Besides these general results, explicit embeddings of particular spaces have also been obtained: for example, Blansula (1955) showed that a n-dimensional hyperbolic space has a global C^∞ isometric embedding in $\mathbb{R}^m$ with $m = 6n - 5$ if $n > 2$, or $m = 6$ if $n = 2$.

Global embedding theory is useful as a way to find new solutions in general relativity (Stephani 1967, 1968). Classical relativistic applications of global embedding theory also include the maximal analytic extensions of the Schwarzschild solution given by Fronsdal (1959), and of the Reissner-Nordström and Kerr space-times by Plazowski (1973), as well as results by Friedman (1965) and Penrose (1965). Global embeddings provide insight into global features of a manifold, such as causality. Indeed, causal properties are related to the existence of particular embeddings (Clarke 1970). For example, a manifold M that has a smooth embedding in a normally hyperbolic pseudo-Euclidean space E cannot contain any closed time-like curves since this is impossible for such E (Clarke 1970). We note that the systematic analysis of global embeddings of exact solutions has not yet been carried out (Stephani *et al.* 2003).

4.3 Einstein Embedding Spaces

The problem of embedding a Riemannian space globally into an Einstein space was considered by Katzourakis (2004, 2005a). Initially, he treated the case of Ricci-flat embedding spaces by using the theory of fibre bundles. It is shown that a $(n + k)$-dimensional Ricci-flat bulk space-time can be constructed as a bundle structure over the embedded submanifold, which is taken to be the base space. In his later versions, Katzourakis (2005a) claims to have provided the first global generalization of the Campbell-Magaard-Dahia-Romero theorem by proving that there exists a global isometric embedding of an arbitrary n-dimensional pseudo-Riemannian space M into a $(n + 1)$-dimensional Einstein space $\mathcal{E} := M \times Y$, where Y is a 1-dimensional analytic manifold. Repeated application of the theorem would show that M can also be embedded into a space with any codimension greater than one. As a corollary to his theorem, Katzourakis (2005a) further claims that any analytic product manifold of the form $\mathcal{E}^{(n+d)} \cong M^{(n)} \times Y^{(d)}$, $d \geq 1$ admits an Einstein metric, and so is an Einstein space.

Through a careful analysis of the proof for the Katzourakis theorem, it appears to rest on the assumption that the local Einstein embedding has the form $M \times Y$ for any embedded space M. Now, Dahia and Romero (2002b) showed that any n-dimensional pseudo-Riemannian space M can be locally embedded into a $(n + 1)$-dimensional Einstein space equipped with metric diag$[\bar{g}_{ik}(x^i, y), \varepsilon\phi^2(x^i, y)]$ where $\bar{g}_{ik}$, in general, depends on the $(n+1)$th co-ordinate y, and only reduces to the metric for M along the hypersurface $y = c$. So, it is not true that the form of the local embedding is $M \times Y$ for any M. Thus, there seems to be a misunderstanding about the local embedding theorem, which is the crucial limitation of Katzourakis's result (Amery *et al.* 2007).

We illustrate this problem further with the following counter-example to his theorem. Consider a static spherically symmetric space-time (2.5.1), where the Ricci scalar is a function of r only, and set $\bar{g}_{ik} = g_{ik}$ and $\phi = \phi(y)$ in the embedding equations (3.4.5) – (3.4.7) for an Einstein embedding space. Since $\bar{g}_{ik}$ has no functional dependence on y, the extrinsic curvature $\bar{\Omega}_{ik}$ vanishes and so the Codazzi equation is trivially satisfied. The Gauss and propagation equations become

$$R = -2\Lambda, \tag{4.3.1}$$
$$R_{ik} = \frac{2\Lambda}{1-n} g_{ik}, \tag{4.3.2}$$

respectively, where Λ represents the cosmological constant. Immediately we see that Katzourakis's construction fails because (4.3.1) is not always true since R is generally not constant.

Substitute (4.3.1) in (4.3.2) so that

$$\begin{aligned} R_{ik} &= \frac{R g_{ik}}{n-1}, \qquad (4.3.3) \\ \Rightarrow \quad R^e_k &= \frac{R\delta^e_k}{n-1}, \\ \Rightarrow \quad R &= \frac{Rn}{n-1}, \end{aligned}$$

which implies that $R = 0$, and hence $R_{ik} = 0$ by (4.3.3). Thus, there does not exist a local Einstein embedding $M \times Y$ for any non-Ricci-flat space M, and so the global Einstein embedding $M \times Y$ fails.

Furthermore, since $\Lambda = 0$, we have that the global embedding space $M \times Y$ must be Ricci-flat. So the Katzourakis theorem is actually a partial global generalization (for Ricci-flat embedded spaces) of the Campbell-Magaard theorem. Thus, as it is given, this result by Katzourakis (2005a) is certainly limited. (See also Amery *et al.* (2007)).

4.4 Theorem: Global Isometric Embedding into an Einstein Space

Next, we extend the given result so that it applies to any pseudo-Riemannian embedded space. We provide a similar detailed proof to that of the Katzourakis (2005a) theorem, and while we largely follow the methodology of his proof, we also comment on the deviations from his approach.

Theorem 1. *Any n-dimensional real analytic pseudo-Riemannian manifold (M, g_M) has a global isometric analytic embedding into a $(n+1)$-dimensional Einstein manifold $(\mathcal{E}, \tilde{g}_{\mathcal{E}})$ where*

$$\widetilde{R}_{\mu\nu} = \frac{2\Lambda}{1-n}\tilde{g}_{\mu\nu} \quad (\Lambda \in \mathbb{R}).$$

4.4.1 Overview of the Proof

Before we proceed to the full development of the proof, we outline the methodology used to prove the result. First assume a global embedding space $\bar{\mathcal{E}}$ of similar topology (Einsteinian metric structure) to that of the specified local embedding space, which contains the embedded space as a hypersurface. For non-Ricci-scalar-constant spacetimes this is necessarily more subtle than a product topology: one has to manually insert the embedded manifold into the global embedding manifold. Note that it must be paracompact, and so one may consider a "partition of unity", though in an unnormalized fashion, to construct particular "Bell" functions that are essential in specifying the global analytic metric. Paracompactness further implies the existence of a locally finite cover from which one may construct several more locally finite covers. Ultimately one constructs two types of covers, both cases having as domain precisely those subsets of the original patches on which the "Bell" functions are strictly positive, but the one case having N "copies" of each domain, being distinguished by different co-ordinate systems. The arbitrariness of N allows one to "sew" together the patches (and metrics) by means of a finite number of (finite) linear systems of equations in a large (unspecified) number of arbitrary functions $\psi^{(i_a)}_{\alpha\beta}$. By choosing the number of these functions to be sufficiently large, the existence of solutions to this (meta-)system is guaranteed. We thus have a construction which ensures that the global embedding space is everywhere locally a space possessing the specified Ricci-curvature. Note that as the field equations are typically expressed in co-ordinate form, a complete local specification is not only sufficient, but also convenient.

4.4.2 Detailed Proof

The proof is separated into 6 steps, loosely matching those of the original proof (Katzourakis 2005a): Step 1 involves a different construction; Step 2 is essentially unchanged with slightly different notation and some commentary; Steps 3, 4 and 5 corresponding to Steps 4, 5 and 6, respectively, in the original proof contain some changes due to the different initial construction; and Step 5 is a little more technical than its corresponding step in the original. Katzourakis's step 3, which pertains to a standard topological proof, is presented in Section 2.2.

Step 1: The construction of the bulk $\mathcal{E}$ containing M.
We assume that there exists an $\bar{\mathcal{E}}$ which is an arbitrary $(n+1)$-dimensional real analytic pseudo-Riemannian space. Recall that $\bar{\mathcal{E}}$ is paracompact and Hausdorff since it is a metric space. We shall further insist that $\bar{\mathcal{E}}$ is an Einstein space globally, and hence locally. We note that $\bar{\mathcal{E}}$ is a manifold and so it has an open cover $\mathcal{U} \equiv \{U_i \mid i \in I\}$ where I may be infinite. Since $\bar{\mathcal{E}}$ is paracompact, there exists a locally finite refinement of this cover, given by

$$\bar{\mathcal{Q}} := \{\bar{Q}_j \mid j \in \bar{J}\},$$

where $\bar{J} \subseteq I$, and such that no point of $\bar{\mathcal{E}}$ lies in more than $(n+1)+1 = n+2$ of its elements (by dimension theory). Now we construct a set $\mathcal{Q}'$ from $\bar{\mathcal{Q}}$ by excising all points on the hypersurface Σ_c defined by $y = c$. Any patch of $\bar{\mathcal{Q}}$ that includes points on the the $y = c$ plane is split into two open patches not containing those points with $y = c$. Thus, we have that $\mathcal{Q}'$ is a cover for $\bar{\mathcal{E}} \setminus \Sigma_c$, the complement of the hypersurface Σ_c in $\bar{\mathcal{E}}$. Any point in $\bar{\mathcal{E}} \setminus \Sigma_c$ that is covered by the maximum of $n+2$ elements of $\bar{\mathcal{Q}}$ will still be covered by $n+2$ elements of $\mathcal{Q}'$. We then specify another (locally) Einstein space $\mathcal{E}$ via its cover

$$\widetilde{\mathcal{Q}} := \{\widetilde{Q}_j \mid j \in J\},$$

as the union of $\mathcal{Q}'$ and the $(n+1)$-dimensional patches generated through the application of the local (Einsteinian) embedding theorem Dahia and Romero (2002b) to a locally finite cover for M. We denote these additional patches by M' which covers all points on the $y = c$ plane. Locally this procedure yields at most $n+1$ additional patches since M is a n-dimensional manifold. Thus, each element of $\mathcal{E}$ lies in at most $d = (n+1)+(n+2) = 2n+3$ elements of the cover $\widetilde{\mathcal{Q}}$. Note that the cover $\widetilde{\mathcal{Q}}$ may be refined further, subject to the fact that we would like to retain the $(n+1)$-dimensional patches generated by the local embeddings. However, the finitude of d is sufficient to proceed with our proof. Note also that, on every patch in $\widetilde{\mathcal{Q}}$, the above construction guarantees that there exists a (local) Einstein metric.

Step 2: The covers $\mathcal{W}$, $\mathcal{W}_{\mathbb{B}}$ and $\mathcal{Q}$ of $\mathcal{E}$.

At each $p \in \mathcal{E}$, we have $p \in \widetilde{Q}_j$ for some $j \in J$. For every $\widetilde{Q}_j$ we construct N additional distinct neighbourhoods $(W_{i_a}, \chi_{(i_a)})$, $1 \leq a \leq N$, such that the W_{i_a} cover the same domain $\widetilde{Q}_j$ in $\mathcal{E}$, but are distinguished by their different co-ordinate functions $\chi_{(i_a)} : W_{i_a} \longrightarrow \mathbb{R}^{n+1}$, $1 \leq a \leq N$. Here, $N \in \mathbb{N}$ is large but finite and unspecified for now, and each $\chi_{(i_a)} = (x^1_{(i_a)}, \ldots, x^\alpha_{(i_a)}, \ldots, x^{n+1}_{(i_a)})$, where $x^\alpha_{(i_a)} := \tau^\alpha \circ \chi_{(i_a)} : W_{i_a} \longrightarrow \mathbb{R}$, $\tau^\alpha : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$. Since $\mathcal{E}$ is a pseudo-Riemannian manifold, these neighbourhoods on $\mathcal{E}$ can be chosen distinct with geodesic (or normal) co-ordinates, and an arbitrarily large but finite number of such patches do exist (Eisenhart 1926, Hawking and Ellis 1973, Sachs and Wu 1977, Campbell 1926): choose co-ordinates $\{x^\alpha_{(i_a)}\}$ of W_{i_a} and a basis $\{\mathbf{E}_\alpha\}$ of the tangent space $T_pW_{i_a}$ such that the curve with initial point p and passing through the point $r = exp(x^\alpha \mathbf{E}_\alpha)$ in W_{i_a} is a geodesic. In these co-ordinates, the connection at p vanishes and the initial direction of the curve at p depends on the choice of the basis $\{\mathbf{E}_\alpha\}$ at p. By choosing different initial directions at p, one obtains different geodesics and so different co-ordinate functions.

We form a N-element class of patches at each $p \in \mathcal{E}$:

$$[W_j] := \{(W_{i_a}, \chi_{(i_a)}) \mid a = 1, \ldots, N \mid \mathrm{dom}(W_{i_a}) = \mathrm{dom}(\widetilde{Q}_j)\}.$$

Thus, we obtain the locally finite cover:

$$
\begin{aligned}
\mathcal{W} :&= \{[W_j] \ \mid \ j \in J\}, \\
&= \{(W_{i_a}, \chi_{(i_a)}) \ \mid \ a = 1, \ldots, N \ \mid \ \mathrm{dom}(W_{i_a}) = \mathrm{dom}(\widetilde{Q}_j) \ \mid \ j \in J\},
\end{aligned} \tag{4.4.1}
$$

where the last equality provides a more verbose description of the abbreviated notation in the preceding line. Here, $\mathrm{dom}(\widetilde{Q}_j)$ denotes the domain of the patch $\widetilde{Q}_j$. Now, using a Euclidean transfer, we can identify each of the N distinct points $\chi_{(i_1)}(p), \ldots, \chi_{(i_N)}(p)$ as the origin $0 \in \mathbb{R}^{n+1}$. Consider the intersection $\bigcap_{a=1}^{N} \chi_{(i_a)}(W_{i_a})$ which is an open set in $\mathbb{R}^{n+1}$. Within this set lies an open $(n+1)$-dimensional ball $\mathbb{B}(0, R_i)$ of maximum radius $R_i > 0$. Choose any $r_i < R_i$. Then,

$$
\mathbb{B}(0, r_i) \ \subseteq \ \bigcap_{a=1}^{N} \chi_{(i_a)}(W_{i_a}).
$$

Invert $\mathbb{B}(0, r_i)$ via 1 of the coordinates, say the 1st, $\chi_{(i_1)}|_{\mathbb{B}(0,r_i)}$. So each W_{i_a} contains an analytically diffeomorphic copy of the ball $\mathbb{B}(0, r_i)$:

$$
\chi^{-1}_{(i_1)}(\mathbb{B}(0, r_i)) \ \subseteq \ W_{i_a} \ , \quad a = 1, \ldots, N.
$$

We denote $\chi^{-1}_{(i_1)}(\mathbb{B}(0, r_i))$ by $W_{\mathbb{B}i_a}$. At each $p \in \mathcal{E}$, form a class of N-elements of inversed balls, each with different induced coordinates:

$$
[\mathbb{B}_j] \ := \ \{(W_{\mathbb{B}i_a}, \ \chi_{(i_a)}|_{W_{\mathbb{B}i_a}}) \ \mid \ a = 1, \ldots, N \ \mid \ \mathrm{dom}(W_{\mathbb{B}i_a}) = \mathrm{dom}(\widetilde{Q}_j)|_{W_{\mathbb{B}i_a}}\}.
$$

Thus, we obtain the locally finite cover:

$$
\begin{aligned}
\mathcal{W}_{\mathbb{B}} :&= \{[\mathbb{B}_j] \ \mid \ j \in J\}, \\
&= \{(W_{\mathbb{B}i_a}, \ \chi_{(i_a)}|_{W_{\mathbb{B}i_a}}) \ \mid \ a = 1, \ldots, N \ \mid \ \mathrm{dom}(W_{\mathbb{B}i_a}) = \mathrm{dom}(\widetilde{Q}_j)|_{W_{\mathbb{B}i_a}} \ \mid \ j \in J\}.
\end{aligned} \tag{4.4.2}
$$

Finally, we restrict each element of $\widetilde{\mathcal{Q}}$ on its corresponding inversed ball to obtain the locally finite cover:

$$
\mathcal{Q} \ := \ \{(Q_j, \{y^A_{(j)}\}_{1 \le A \le n+1}) \ \mid \ Q_j = \widetilde{Q}_j|_{\chi^{-1}_{(i_1)}(\mathbb{B}(0,r_i))} \ \mid \ j \in J\}, \tag{4.4.3}
$$

where $y^A_{(j)} : Q_j \longrightarrow \mathbb{R}$. Note that $Q_j \subseteq \widetilde{Q}_j$, and hence is locally Einsteinian by the construction in Step 1.

While all three covers are locally finite, $\mathcal{W}$ and $\mathcal{W}_{\mathbb{B}}$ each contain more elements than $\mathcal{Q}$, and while any $p \in \mathcal{E}$ is covered by a maximum of d elements of $\mathcal{Q}$, it is also covered by a maximum of Nd elements of $\mathcal{W}$ (respectively, $\mathcal{W}_{\mathbb{B}}$). We also observe that, even though $W_{\mathbb{B}i_a} \subseteq W_{i_a} \ \forall\, i_a$, and a maximum of Nd elements of each cover contain any given point p, these bounds need not be realized, and dim $(\mathcal{W}_{\mathbb{B}}) \leq$ dim $(\mathcal{W})$. On the other hand, there exist at least N W_{i_a}'s ($W_{\mathbb{B}i_a}$'s) containing any given point p, because p lies in some $Q_j \subseteq \widetilde{Q}_j$ and hence lies in N W_{i_a}'s ($W_{\mathbb{B}i_a}$'s). This observation shall be used later in our counting arguments – see Step 5. We need to "sew" together the patches of $\mathcal{W}$ to obtain the global embedding space with metric $g_{\mathcal{E}}$, and such that it matches the local embedding patches where required. Every patch in $\mathcal{Q}$ is Einsteinian, and so provided these patches are sewn together appropriately, then $\mathcal{E}$ will be a global Einstein space. So, we require the cover $\mathcal{W}$ to specify $g_{\mathcal{E}}$ globally, and the cover $\mathcal{Q}$ to evaluate it locally. This dual perspective gives rise to a system of equations that must be satisfied to ensure the existence of the global metric – see Step 5. The cover $\mathcal{W}_{\mathbb{B}}$ of inversed balls will be used in defining the "Bell" functions (in Step 3), which will give the metric $g_{\mathcal{E}}$ the nature of being real analytic on every patch.

Step 3: The global smooth and locally analytic metric $g_{\mathcal{E}}$ on $\mathcal{E}$.

We have assumed $\mathcal{E}$ to consist of real analytic Einstein patches. It remains to specify the global metric appropriately. Consider the cover $\mathcal{W}$ and the idea of "sewing" together the W_{i_a} patches to obtain the global metric. For each i_a let $\psi^{(i_a)}_{\alpha\beta} \in C^\infty(W_{i_a} \longrightarrow \mathbb{R})$ be $\frac{1}{2}(n+1)(n+2)$ symmetric analytic functions on W_{i_a} for $\alpha, \beta \in \{1, \dots n+1\}$, and consider the co-ordinate functions $x^\alpha_{(i_a)} : W_{i_a} \longrightarrow \mathbb{R}$ where $\alpha = 1, \dots, n+1$. Define

$$g_{\mathcal{E}}(U,V) \ := \ \left(\sum_{\alpha,\beta \in \{1,\dots,n+1\}} \ \sum_{\substack{i_a \in J \\ 1 \leq a \leq N}} f_{i_a} \, \psi^{(i_a)}_{\alpha\beta} \, dx^\alpha_{(i_a)} \otimes dx^\beta_{(i_a)} \right) (U,V),$$

where $U, V \in T\mathcal{E}$ and the f_{i_a} are "Bell" functions (refer to §2.2) defined on $\mathcal{W}$, and strictly positive on $\mathcal{W}_{\mathbb{B}}$. The composition of the analytic functions f_{i_a} and $\psi^{(i_a)}_{\alpha\beta}$ is analytic which implies that $g_{\mathcal{E}}$ is real analytic. Set

$$\breve{\psi}^{(i_a)}_{\alpha\beta} \;\equiv\; f_{i_a}\, \psi^{(i_a)}_{\alpha\beta} \;\in\; C^\infty(\mathcal{E} \longrightarrow \mathbb{R}).$$

Then we have

$$g_{\mathcal{E}}(U,V) \;:=\; \left(\sum_{\alpha,\beta\in\{1,\ldots,n+1\}} \; \sum_{\substack{i_a\in J\\ 1\le a\le N}} \breve{\psi}^{(i_a)}_{\alpha\beta}\; dx^{\alpha}_{(i_a)} \otimes dx^{\beta}_{(i_a)} \right) (U,V). \tag{4.4.4}$$

Now we employ the cover $\mathcal{Q}$ to evaluate the metric locally. Taking any $Q_j \in \mathcal{Q}$ with co-ordinates $y^{\sigma}_{(j)}$, $1 \le \sigma \le n+1$, we have:

$$g_{\mathcal{E}}|_{Q_j} \;=\; \sum_{\tau,\sigma} \left(\sum_{\alpha,\beta\in\{1,\ldots,n+1\}} \; \sum_{\substack{i_a\in J\\ 1\le a\le N}} \breve{\psi}^{(i_a)}_{\alpha\beta}\, \frac{\partial x^{\alpha}_{(i_a)}}{\partial y^{\tau}_{(j)}} \frac{\partial x^{\beta}_{(i_a)}}{\partial y^{\sigma}_{(j)}} \right) |_{Q_j} \quad dy^{\tau}_{(j)} \otimes dy^{\sigma}_{(j)},$$

having used that

$$dx^{\alpha}_{(i_a)} = \frac{\partial x^{\alpha}_{(i_a)}}{\partial y^{\tau}_{(j)}}\; dy^{\tau}_{(j)}.$$

Thus, the components of $g_{\mathcal{E}}|_{Q_j}$ are

$$[g^{(j)}_{\mathcal{E}}]_{\tau\sigma} \;=\; \left(\sum_{\alpha,\beta\in\{1,\ldots,n+1\}} \; \sum_{\substack{i_a\in J\\ 1\le a\le N}} \breve{\psi}^{(i_a)}_{\alpha\beta}\, \frac{\partial x^{\alpha}_{(i_a)}}{\partial y^{\tau}_{(j)}} \frac{\partial x^{\beta}_{(i_a)}}{\partial y^{\sigma}_{(j)}} \right) |_{Q_j} \;\;,\;\; 1 \le \tau, \sigma \le n+1, \tag{4.4.5}$$

which are all analytic functions on Q_j.

Consider the metric evaluated at a point $p \in W_{i_a}$. Note that the f_{i_a} are only non-zero on the $W_{\mathbb{B}i_a} \subseteq W_{i_a}$, which implies that the metric has zero contribution from these W_{i_a} containing p outside of the corresponding $W_{\mathbb{B}i_a}$. Note also that the sums over $i_a \in J$, $1 \le a \le N$, have at least N terms for each α, β – c.f. Step 2. The evaluation of the global metric in the locally Einsteinian cover $\mathcal{Q}$ shall be employed to generate (finite) systems of equations that must be solved in order for the global metric to have the desired local properties.

Step 4: The specification of the local nature of $\mathcal{E}$.

This follows by construction: since the local embedding result stipulates that the local Einstein metric has the form $\mathrm{diag}(\bar{g}_{ab}, 1)$, for any patch Q_j in M', we must have

$$[g_{\mathcal{E}}^{(j)}]_{ab} = \bar{g}_{ab} \ , \quad 1 \leq a, b \leq n, \tag{4.4.6}$$

$$[g_{\mathcal{E}}^{(j)}]_{a(n+1)} = 0 \ , \quad 1 \leq a \leq n, \tag{4.4.7}$$

$$[g_{\mathcal{E}}^{(j)}]_{(n+1)(n+1)} = 1. \tag{4.4.8}$$

where $[g_{\mathcal{E}}^{(j)}]_{\tau\sigma}$ is given by (4.4.5). These relations ensure that the global metric $g_{\mathcal{E}}$ coincides with the local Einstein metric on any local embedding patch. Any other patch of $\mathcal{E}$ has a metric given by the local representation of the metric for the arbitrarily specified global Einstein space $\bar{\mathcal{E}}$. Thus, $[g_{\mathcal{E}}^{(j)}]_{\tau\sigma}$ is known for all patches Q_j. This local specification of $\mathcal{E}$ is utilized in the next step, where we show the existence of the unspecified functions $\psi_{\alpha\beta}^{(i_a)}$.

Step 5: The existence of functions $\psi_{\alpha\beta}^{(i_a)}$ on $\mathcal{E}$.

As yet we have not completely specified the global embedding since we need to ensure that where patches overlap, their metrics coincide, and that the functions $\psi_{\alpha\beta}^{(i_a)}$ do indeed exist on $\mathcal{E}$. Since the $f_{i_a}\psi_{\alpha\beta}^{(i_a)}$ are defined on W_{i_a} but are non-zero only inside $W_{\mathbb{B}i_a}$, and since $\mathrm{dom}(W_{\mathbb{B}i_a}) = \mathrm{dom}(Q_j)$, the components (4.4.5) of $g_{\mathcal{E}}|_{Q_j}$ can be rewritten as:

$$[g_{\mathcal{E}}^{(j)}]_{\tau\sigma} \ = \sum_{\alpha,\beta\in\{1,\ldots,n+1\}} \sum_{\Upsilon} \left\{ f_{i_a} \frac{\partial x_{(i_a)}^{\alpha}}{\partial y_{(j)}^{\tau}} \frac{\partial x_{(i_a)}^{\beta}}{\partial y_{(j)}^{\sigma}} \right\} \Big|_{Q_j} \ \psi_{\alpha\beta}^{(i_a)} \ , \tag{4.4.9}$$

where $1 \leq \tau, \sigma \leq n+1$, and $\Upsilon = \{i_a \in J \ | \ 1 \leq a \leq N \ | \ \mathrm{dom}(W_{i_a}) \cap \mathrm{dom}(Q_j) \neq \emptyset\}$. Note that the $\psi_{\alpha\beta}^{(i_a)}$ are the only unknown functions in the above relation since $[g_{\mathcal{E}}^{(j)}]_{\tau\sigma}$ is specified. The components of (4.4.9) yield $\frac{1}{2}(n+1)(n+2)$ equations on every Q_j. Now fix $\tau, \sigma \in \{1, \ldots, n+1\}$, and let

$$[g_{\mathcal{E}}^{(j)}]_{\tau\sigma} \ = \ \Phi_{(j)}, \quad \text{and} \quad \left\{ f_{i_a} \frac{\partial x_{(i_a)}^{\alpha}}{\partial y_{(j)}^{\tau}} \frac{\partial x_{(i_a)}^{\beta}}{\partial y_{(j)}^{\sigma}} \right\} \Big|_{Q_j} \ = \ \Theta_{(i_a)(j)}^{\alpha\beta} \tag{4.4.10}$$

Then (4.4.9) becomes a linear functional equation with analytic coefficients, where the $\psi_{\alpha\beta}^{(i_a)}$ are linearly independent since the patches are distinguished by different coordinate functions:

$$\Phi_{(j)} \;=\; \sum_{\alpha,\beta\in\{1,\ldots,n+1\}} \; \sum_{\Upsilon} \{\Theta^{\alpha\beta}_{(i_a)(j)} \; \psi^{(i_a)}_{\alpha\beta}\}. \tag{4.4.11}$$

From (4.4.10), the positivity of $f_{i_a}|_{Q_j}$, and the fact that at least one of the $\frac{\partial x^{\alpha}_{(i_a)}}{\partial y^{\tau}_{(j)}}$ is non-zero for any given τ, we may conclude that (4.4.11) evaluated at a point will have at least N variables $\psi^{(i_a)}_{\alpha\beta}$ with non-zero coefficients. (See also the comment to Step 2.)

Recall that any point in $\mathcal{E}$ lies in a maximum of d elements of the cover $\mathcal{Q}$. So consider the domain in $\mathcal{E}$ that consists of the maximum d overlaps of the Q_j's:

$$Q_{j_1} \bigcap \cdots \bigcap Q_{j_d} \;=\; \bigcap_{r=1}^{d} Q_{j_r}.$$

Now we want to show that there exist solutions $\psi^{(i_a)}_{\alpha\beta}$ on this intersection. Thus, (4.4.11) must be solved on $Q_{j_1}, \ldots, Q_{j_d}$ simultaneously. So on $\cap_{r=1}^{d} Q_{j_r}$ we have one system consisting of d equations:

$$(\Sigma 1) \quad \begin{cases} \Phi_{(j_r)} \;=\; \sum\limits_{\alpha,\beta} \sum\limits_{i_a=1}^{d} \sum\limits_{a=1}^{N} \{\Theta^{\alpha\beta}_{(i_a)(j_r)} \; \psi^{(i_a)}_{\alpha\beta}\}, \\ 1 \le r \le d \end{cases}$$

that must be solved. The number of independent variables in $(\Sigma 1)$ is at least N, by the above argument. Now we follow a stepwise procedure to extend the $\psi^{(i_a)}_{\alpha\beta}$ solutions on

$$Q_{j_1} \bigcap \cdots \bigcap \widehat{Q}_{j_s} \bigcap \cdots \bigcap Q_{j_d},$$

then

$$Q_{j_1} \bigcap \cdots \bigcap \widehat{Q}_{j_s} \bigcap \cdots \bigcap \widehat{Q}_{j_k} \bigcap \cdots \bigcap Q_{j_d},$$

and so on until we have solutions on each $Q_{j_1}, \ldots, Q_{j_d}$, and so on $\cup_{r=1}^{d} Q_{j_r}$. Note that the $\widehat{\ }$ means that the expression underneath must be omitted.

Now on $Q_{j_1} \bigcap \cdots \bigcap \widehat{Q}_{j_s} \bigcap \cdots \bigcap Q_{j_d}$ we have a system of $d-1$ equations:

$$(\Sigma 2) \quad \begin{cases} \Phi_{(j_r)} \;=\; \sum\limits_{\alpha,\beta} \sum\limits_{\substack{i_a=1 \\ [\mathbb{B}_i] \not\subseteq Q_{j_s}}}^{d} \sum\limits_{a=1}^{N} \{\Theta^{\alpha\beta}_{(i_a)(j_r)} \; \psi^{(i_a)}_{\alpha\beta}\}, \\ 1 \le r \le d\,, \quad r \neq s. \end{cases}$$

and there are $\binom{d}{d-1}$ such systems. The number of independent variables here is again at least N. On $Q_{j_1} \bigcap \cdots \bigcap \widehat{Q}_{j_s} \bigcap \cdots \bigcap \widehat{Q}_{j_k} \bigcap \cdots \bigcap Q_{j_d}$ we have a system of $d-2$ equations:

$$(\Sigma 3) \quad \left\{ \begin{array}{l} \Phi_{(j_r)} \; = \; \sum\limits_{\alpha,\beta} \; \sum\limits_{\substack{i_a=1 \\ [\mathbb{B}_i] \not\subseteq Q_{j_s} \cup Q_{j_k}}}^{d} \; \sum\limits_{a=1}^{N} \{\Theta^{\alpha\beta}_{(i_a)(j_r)} \; \psi^{(i_a)}_{\alpha\beta}\}, \\ 1 \leq r \leq d \,, \quad r \neq s, k \end{array} \right.$$

and there are $\binom{d}{d-2}$ such systems, each having at least N independent variables. We continue in this way until on Q_{j_r} we have a system consisting of 1 equation:

$$(\Sigma(d)) \quad \left\{ \begin{array}{l} \Phi_{(j_r)} \; = \; \sum\limits_{\alpha,\beta} \; \sum\limits_{\substack{i_a=1 \\ [\mathbb{B}_i] \not\subseteq Q_{j_1} \cup \cdots \cup \widehat{Q_{j_r}} \cup \cdots \cup Q_{j_d}}}^{d} \; \sum\limits_{a=1}^{N} \{\Theta^{\alpha\beta}_{(i_a)(j_r)} \; \psi^{(i_a)}_{\alpha\beta}\}, \end{array} \right.$$

and there are $\binom{d}{1} = d$ such systems, each having at least N independent variables. So, for each choice of τ and σ, the total number of systems is

$$\binom{d}{d} + \binom{d}{d-1} + \binom{d}{d-2} + \cdots + \binom{d}{1} \; = \; \sum_{m=1}^{d} \binom{d}{m},$$

and the total number of equations that must be solved is

$$M \; = \; (d) \cdot \binom{d}{d} + (d-1) \cdot \binom{d}{d-1} + (d-2) \cdot \binom{d}{d-2} + \cdots + \binom{d}{1} \; = \; \sum_{t=1}^{d} t \cdot \binom{d}{t}$$

Now this procedure must be carried out $\frac{1}{2}(n+1)(n+2)$ times to solve $\psi^{(i_a)}_{\alpha\beta}$ for all $\tau, \sigma \in \{1, \ldots, n+1\}$. If there are more variables than equations, then there will exist solutions $\psi^{(i_a)}_{\alpha\beta}$ on $\cup_{r=1}^{d} Q_{j_r}$. So set

$$N \; = \; \frac{1}{2}(n+1)(n+2)M + 1.$$

This value is acceptable since N is finite but arbitrarily large. Thus, we have shown that the metric $g_{\mathcal{E}}|_{\cup Q_j}$ exists.

Now, we want to extend the $\psi^{(i_a)}_{\alpha\beta}$ on the whole of $\mathcal{E}$. For any $p \in \mathcal{E}$ there exists a maximum of d patches Q_j covering p. Choose any one of these patches, say Q_{j_0}, and take any other point q in Q_{j_0}. Now q may lie in a maximum of d patches $Q_{j'}$

including Q_{j_0}. So we have 2 unions of patches $\bigcup Q_j$ and $\bigcup Q_{j'}$ that are overlapping on Q_{j_0}. Their corresponding metrics $g_{\mathcal{E}}|_{\cup Q_j}$, $g_{\mathcal{E}}|_{\cup Q_{j'}}$ must coincide on the intersection $(\bigcup Q_{j'}) \bigcap (\bigcup Q_{j'}) \subseteq Q_{j_0}$, provided that the equation

$$\Phi_{(j_0)} = \sum_{\alpha,\beta \in \{1,\ldots,n+1\}} \sum_{\substack{i_a=1 \\ [\mathbb{B}_i] \subseteq Q_{j_0}}}^{d} \sum_{a=1}^{N} \{\Theta^{\alpha\beta}_{(i_a)(j_0)} \, \psi^{(i_a)}_{\alpha\beta}\},$$

holds on both the systems solved on $\bigcup Q_j$ and $\bigcup Q_{j'}$, which it does, by construction. This implies that there exist solutions $\psi^{(i_a)}_{(\alpha\beta)}$ on $\bigcup Q_j \bigcup \bigcup Q_{j'}$. Thus, by considering all such overlapping unions, we have shown that the $\psi^{(i_a)}_{(\alpha\beta)}$ exist on the whole of $\mathcal{E}$. This implies that the patches of the global space are appropriately "sewn" together, and the global metric $g_{\mathcal{E}}$ is fully specified.

Step 6: The isometry condition.

We recall that an embedding is globally isometric if it is isometric at all points of the embedded space. Consider any $p \in M$ and note that this point is mapped to a point $f(p)$ on the hypersurface $y = c$ in $\mathcal{E}$. Now $f(p)$ lies in some $Q_j \in M'$ and $y^{(n+1)}_{(j)} = c$ at $f(p)$. The metric $g_{\mathcal{E}}$ at $f(p)$ coincides with the metric g_M at p:

$$\begin{aligned}
g_{\mathcal{E}}|_{f(p)} &= \left([g^{(j)}_{\mathcal{E}}]_{\tau\sigma} \;\; dy^{\tau}_{(j)} \otimes dy^{\sigma}_{(j)} \right) \Big|_{y^{(n+1)}_{(j)} = c} \\
&= [g^{(j)}_{\mathcal{E}}]_{ab} \;|_{y^{(n+1)}_{(j)} = c} \;\; dy^{a}_{(j)} \otimes dy^{b}_{(j)} \\
&= \bar{g}_{ab} \;|_{y^{(n+1)}_{(j)} = c} \;\; dy^{a}_{(j)} \otimes dy^{b}_{(j)} \\
&= g_{ab} \;\; dy^{a}_{(j)} \otimes dy^{b}_{(j)} \\
&= g_M|_p.
\end{aligned}$$

This can be done for all $p \in M$. Thus, $f : M \longrightarrow \mathcal{E}$ is a global isometric embedding.

By construction, the global embedding is analytic. Hence, there exists a global isometric analytic embedding of M into $\mathcal{E}$.

4.4.3 Comments on the Proof and Discussion

Now we provide several comments on the above theorem.

The principal differences between Katzourakis's proof and the one presented here lie in Steps 1 and 5: the specification of the bulk cover and the counting arguments demonstrating the existence of the global metric.

Katzourakis begins by taking Y to be a 1-dimensional analytic pseudo-Riemannian manifold with co-ordinate y, and specifying $\mathcal{E}$ to have the topology $M \times Y$ that resembles the topology of the local embedding space for M, and which is a trivial fibre bundle with projections $\pi := \mathcal{E} \longrightarrow M$ and $pr_Y := \mathcal{E} \longrightarrow Y$. Due to its product structure, $\mathcal{E}$ inherits the properties of being real analytic, Hausdorff and paracompact from M and Y via their property of being metric spaces. Crucially for Step 4, the product structure also ensures that every neighbourhood of $\mathcal{E}$ is of precisely the form of the $(n+1)$-dimensional neighbourhoods induced by the local embedding theorems (the Cauchy-Kowalewski theorem applied to the n-dimensional initial data), and hence, are Einsteinian.

The product structure of $\mathcal{E}$ yields a natural cover consisting of the product charts of M and Y, from which one may form the locally finite refinements $\mathcal{W}$, $\mathcal{W}_{\mathbb{B}}$ and $\mathcal{Q}$, that are required to fully specify the global metric $g_{\mathcal{E}}$. We have not assumed as much for the topology of $\mathcal{E}$, so we must proceed through a more elaborate construction first in Step 1.

Katzourakis takes $g_{\mathcal{E}}$ as the product metric on $M \times Y$ after assuming that topology for the embedding space, and uses an (unspecified) analytic function $\psi^{(i_a)} : W_{i_a} \longrightarrow \mathbb{R}$ to represent part of the $(n+1)$-th component of the global metric. We necessarily require more functions, but the central idea is the same though: the specification of a global metric on $\mathcal{W}$ via the introduction of analytic maps from patches in $\mathcal{W}$ to $\mathbb{R}$.

Katzourakis specifies the global Einsteinian nature of $\mathcal{E}$ through its (assumed) local Einsteinian nature on every patch of the cover $\mathcal{Q}$, whereas we construct a bulk that is globally Einstein, and therefore locally Einstein. His misinterpretation of the local embedding result (Dahia and Romero 2002b) is evident here, as he claims that the bulk $\mathcal{E}$ with topology $M \times Y$, where M is any given analytic pseudo-Riemannian manifold, is locally Einstein. However, this fails if M is not Ricci-flat. If it is, Katzourakis's construction of $\mathcal{E}$ as a product manifold ensures that every patch will be a local embedding space. Since we do not pose any restrictions on M, our local specification of $\mathcal{E}$ is different from his: we have to manually embed M in $\mathcal{E}$ to generate the local embedding patches – see Step 1.

Our construction necessarily requires $\frac{1}{2}(n+1)(n+2)$ sets of systems, rather than the one set as is the case for Ricci-flat embedding spaces. Note that the required N may be made smaller by first placing the global metric in Gaussian normal form. By construction, the local isometry induces global isometry.

4.5 Generalized Results

The above result may be extended to arbitrarily given pseudo-Riemannian embedding spaces.

Theorem 2. *Any n-dimensional real analytic pseudo-Riemannian manifold (M, g_M) has a global isometric analytic embedding into an arbitrarily specified $(n+1)$-dimensional pseudo-Riemannian manifold $(\mathcal{E}, \tilde{g}_{\mathcal{E}})$.*

Overview of the proof:
We do not provide a detailed proof since the methodology is essentially the same as for Theorem 1, but with a few modifications. In Step 1, we begin with a more general assumption by taking $\bar{\mathcal{E}}$ to be the arbitrarily specified pseudo-Riemannian space (with Ricci tensor $S_{\alpha\beta}$) of the second Dahia-Romero Theorem (Dahia and Romero 2002a). We then construct another pseudo-Riemannian space $\mathcal{E}$ having a cover formed from some cover for $\bar{\mathcal{E}}$ and the $(n+1)$-dimensional patches generated by the local embedding theorem (Dahia and Romero 2002a) applied to the n-dimensional manifold M. We denote the Ricci tensor of these patches by $\widetilde{R}_{\mu\nu}$ which is equivalent, up to a local analytic diffeomorphism, to $S_{\alpha\beta}$. M is embedded as the hypersurface Σ_c, defined by $y = c$, in $\mathcal{E}$. The global metric must now include additional factors that take the $y = c$ patches in $\mathcal{E}$ to the Ricci equivalent (diffeomorphic) patches on which the local embeddings are guaranteed. So we define

$$g_{\mathcal{E}}(U,V) \;:=\; \left(\sum_{\mu\nu} \sum_{\alpha,\beta} \sum_{\substack{i_a \in J \\ 1 \le a \le N}} f_{i_a}\, \psi^{(i_a)}_{\alpha\beta}\, \frac{\partial g^{\alpha}_{(i_a)}}{\partial x^{\mu}_{(i_a)}}\, \frac{\partial g^{\beta}_{(i_a)}}{\partial x^{\nu}_{(i_a)}}\, dx^{\mu}_{(i_a)} \otimes dx^{\nu}_{(i_a)} \right) (U,V),$$

where

$$g^{\alpha}_{(i_a)}(x^{\mu}_{(i_a)}) = x'^{\,\alpha}_{(i_a)} = \begin{cases} x^{\alpha}_{(i_a)} & \text{if } Q_i \notin M', \\ \bar{g}^{\alpha}_{(i_a)}(x^{\mu}_{(i_a)}) & \text{if } Q_i \in M', \end{cases}$$

and the $\bar{g}^{\alpha}_{(i_a)}$ satisfy

$$\widetilde{R}_{\mu\nu}(x^{\gamma}_{(i_a)}) = \frac{\partial \bar{g}^{\alpha}_{(i_a)}}{\partial x^{\mu}_{(i_a)}}\, \frac{\partial \bar{g}^{\beta}_{(i_a)}}{\partial x^{\nu}_{(i_a)}}\, S_{\alpha\beta}(x'^{\,\kappa}_{(i_a)}).$$

Note that the above expression specifies the local analytic diffeomorphism.

In fact, we may present the results yet more generally as:

Theorem 3. *If any n-dimensional real analytic metric space has a local isometric analytic embedding into some specified m-dimensional metric space ($m \ge n+1$), then there exists a global isometric analytic embedding into that space.*

or even, dropping the isometry requirement,

Theorem 4. *If any n-dimensional real analytic paracompact space has a local analytic embedding into some specified m-dimensional paracompact space ($m \ge n+1$), then there exists a global analytic embedding into that space.*

The proof of Theorem 4 is essentially the "sewing" argument presented above. Theorem 3 is at a metrical level so that we may speak of isometry and rests on our constructions for the bulk and the global metric.

In light of the above, we may consider Theorem 2 and Theorem 1 to be corollaries, in which the conditional statement in Theorem 3 is guaranteed by the Dahia-Romero results.

4.6 Singularities and Multiple Brane Scenarios

Katzourakis (2005b,c,d) generalizes his embedding theorem for Einstein spaces to include situations in which the bulk contains differential-topological singularities, or in which several branes are globally and analytically embedded into the bulk, or where there is a combination of both. The proofs for all three scenarios rely on the (limited) main result for Einstein spaces, which indicates that these results may also be limited.

First, we investigate the situation (Katzourakis 2005b) in which several, say m, branes are embedded into the bulk. The branes are specified as pseudo-Riemannian manifolds $M_1^{(n_1)}, \ldots, M_k^{(n_k)}, \ldots, M_m^{(n_m)}$ with dimension $n_1, \ldots, n_k, \ldots, n_m$, respectively. Set

$$M \equiv M_1^{(n_1)} \times \cdots \times M_k^{(n_k)} \times \cdots \times M_m^{(n_m)}.$$

By the property of its product structure, M is a pseudo-Riemannian manifold with dimension $n = \sum_{k=1}^{m} n_k$. Now, Katzourakis uses the same methodology as that of his Einstein embedding theorem applied to this M. It is claimed that $\mathcal{E} = M \times Y$, where Y is a 1-dimensional analytic manifold, is an Einstein space globally. In this way, each brane has a global isometric embedding into $\mathcal{E}$ and all the branes are disjoint submanifolds in the bulk. However, we know that this can only work provided that M is Ricci-flat, which implies that each brane must be Ricci-flat. So, this generalized result by Katzourakis is certainly limited. An analysis of the application of our construction to this situation is underway (Amery and Moodley 2008), and further work is also motivated by the physical interest in cases with singular brane energy-momentum.

Next, we consider the situation in which a n-dimensional analytic space-time M is embedded into a $(n+d)$-dimensional Einstein space $\mathcal{E}$ with unsmoothable singularities in $\mathcal{E}\setminus M$, the complement of M in $\mathcal{E}$. If the codimension $d > 1$, there can exist countable many singularities of dimension $(d-k)$, $0 \leq k \leq d$, but if $d = 1$, there can exist only one singularity that is either pointlike (0-d) or linelike (1-d). Katzourakis defines a space $\breve{\mathcal{E}} := M \times F$ (where F is an analytic manifold of dimension $d \geq 1$) and chooses a set of distinct points (or spatial "anomalies") in F to be the singularities. This set is denoted by $\mathbb{F}^{\Sigma}$ and it is assumed that $F \setminus \mathbb{F}^{\Sigma}$ remains connected. The product of $\mathbb{F}^{\Sigma}$ with points in M gives rise to a set of points in $\breve{\mathcal{E}}$. The complement of these points in

$\breve{\mathcal{E}}$ is taken as the analytic manifold $\mathcal{E}$, which Katzourakis claims is a global Einstein embedding of M. It seems that this proof also rests on misapprehensions about the form of the local (and hence global) embeddings similar to those in the original result. This issue will be the subject of future study.

4.7 Further Discussion

Now we consider the composition of an embedding with a diffeomorphism (and vice versa). The resultant embedding is not guaranteed to be isometric. We demonstrate this with the following example. Let (M, g_M) be a Ricci-flat manifold and (N, g_N) be any other Riemannian manifold that satisfy the conformal mapping σ (Stephani *et al.* 2003)

$$g_M = e^{2u(x^i)} g_N.$$

It is well-known that such transformations are diffeomorphisms (Choquet-Bruhat *et al.* 1982). Since M is Ricci-flat, it has a global isometric embedding ε into $M \times Y$ with metric (g_M, ϕ). Composing this embedding with the above diffeomorphism yields the metric $(e^{2u} g_N, \phi)$. For this composition $\varepsilon \circ \sigma$ to be an isometric embedding of N into $M \times Y$, we require

$$e^{2u} g_N = g_N \quad \text{at } y = 0,$$

which implies that $u = 0$ at $y = 0$. However, since u is a function of x^i only, this gives $u = 0$ for all y. Thus, we have $g_m = g_N$, which is a contradiction since M and N are different manifolds. Hence, $\varepsilon \circ \sigma$ is not an isometric embedding. Similarly, we can show that the composition of a conformal mapping with an embedding fails the isometric condition. This suggests that it is not sufficient to consider only embeddings for representative members of classes of diffeomorphic space-times (as is the case for the homotopy analysis carried out by Katzourakis (2005a)).

We observe that our theorems demonstrate that the work in reducing the codimension is done locally, and at a metrical level. The latter point indicates that the above construction of the global embedding is not necessarily unique from a topological view: the metrical formulation in general relativity constrains the global topology, but does not completely specify it (Lachieze-Rey and Luminet 1995, Reboucas 2004). This is evident, in the global construction, as the freedom to specify the functions $\psi_{\alpha\beta}^{(i_a)}$ in many different ways since we have requested only that the number of functions be more than the number of equations. We have shown that there do exist such functions, but we have not actually solved them. Moreover, the global embedding is not even unique at a metrical level as it appeals to the local results, which do not necessarily guarantee uniqueness. In a sense, these considerations avoid concerns that the Campbell-Magaard theorem and its extensions do not ensure a well-posed initial value problem or the non-occurrence of singularities, since while such (global) properties may be present in other constructions, here we deal only with analytic manifolds embedded, via one particular construction, into analytic manifolds. Note that the preceding caveats do not compromise the existence results: any embedding has the same existential level as any solution to the field equations.

Chapter 5

Closing Remarks

The study of higher dimensions is a popular and fertile field. The consequences of higher dimensional cosmological models in astrophysics allows one to check the consistency of these models with current observations, which may lead to new ideas of possible tests. Embedding into Euclidean spaces has been useful in the classification of space-times and in obtaining new solutions to the field equations.

The main task in this thesis was to construct global from local embeddings into pseudo-Riemannian spaces. In order to carry out this research, we required a good knowledge of concepts in general relativity and topology, which we have presented in Chapter 2. In Chapter 3, we considered the Gauss, Codazzi and Ricci equations that enable an embedding of one manifold into a higher-dimensional one. We focussed our attention on static spherically symmetric space-times which have significance in astrophysics, and discussed the formalism for embedding these space-times into five-dimensional Einstein spaces. General solutions to the resulting sets of equations are not yet known, and so it motivates one to investigate explicit embeddings of particular space-times. Embeddings of astrophysical objects into higher dimensions can provide insight into the properties of such objects. In this thesis, we have chosen to embed the Reissner-Nordström and global monopole space-times because of their relevance to astrophysics and early universe cosmology. The embedding for the Reissner-Nordström space-time is quite easily obtained, but the equations governing the embedding of the global monopole space-time are extremely complicated and remain to be solved. We then proceeded to consider global embeddings in Chapter 4. We analyzed a result by

Katzourakis (2005a) for Einstein embedding spaces and found that it applies only to the Ricci-flat case. Moreover, there seemed to be a crucial misunderstanding that the local Einstein embedding has the form $M \times F$ for any embedded space M, but this is only true for Ricci-flat manifolds M. We followed most of the construction given by Katzourakis and extended the proof to Einstein embedding spaces, and even more generally to pseudo-Riemannian embedding spaces satisfying a local analytic diffeomorphism. We also presented two further results pertaining to metric and paracompact manifolds.

As future work, one should consider space-times that possess singularities i.e. points at which the metric diverges. An example is the Schwarzschild interior space-time which is not analytic at $r = 0$. This is not such a problem for local embeddings since one can simply remove the singularity. This idea is consistent with the local notions of general relativity (Lachieze-Rey and Luminet 1995). However, the global situation is more problematic as we need to take the singularity into account when embedding M in $\mathcal{E}$. In the case of stacking a manifold M, with a point singularity, along the extra $(n + 1)$th co-ordinate, one obtains a global embedding space that now has a line singularity. Of course, one could say that singularities are not physical in reality. In this case, the existence of analytic global isometric embeddings for (say) the Reissner-Nordström or Schwarzschild exterior space-times are clearly beneficial to the study of the astrophysical effects of higher dimensions, hitherto only studied numerically – c.f. Wiseman (2002).

Other future directions include, at a global level:

- Relating the topological invariants for the global embedding space to those for the local embedding space and the embedded space,

- A revisitation of Katzourakis's homotopy analysis, using our construction,

- A similar treatment of Katzourakis's extensions to singular and multiple brane scenarios using our arguments.

A study of these questions is underway (Amery and Moodley 2008). At a local level, the following issues seem particularly interesting and are the subject of current investigation:

- Solutions to other specific embedding scenarios. This is essentially the task of solving the 5-dimensional field equations exactly, for given initial/boundary data.
- An analysis of the conditions for which a given 4-dimensional space-time is embeddable into a 5-dimensional constant curvature space. This shall involve revisiting, in detail, the original Dahia and Romero (2002a,b) proofs.

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