# Constructions and Justifications of a Generalization of Viviani's Theorem

**By**


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*Seek first the kingdom of God*

*and His righteousness, and all*

*these things shall be added to you.*

\- Mathew 6:33 -

# Declaration

I, Rajendran Govender, declare that

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## Approval by PhD Supervisor: Submission of thesis

I, Prof. Michael de Villiers, as the candidate's Supervisor, agree to the submission of this thesis by Rajendran Govender, entitled Constructions and Justifications of a Generalization of Vivianis' Theorem.

Date: 15 March 2013

Prof. Michael de Villiers (PhD Supervisor)

# Dedications

To my wife, Inderani Govender, with deep love, for without her understanding and prayer, this would not have been possible.

To my children, Trevelyan and Shannon, for their encouragement, time, patience and support.

To my parents, Munisamy Govender and Salamma Govender, for their time and investment in my school education and early tertiary education.

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# Abbreviations and Acronyms

| | |
|---|---|
| CAPS: | Curriculum and Assessement Policy Statement |
| CG: | Conjecture Generalization |
| FET: | Further Education and Training |
| FG: | Further Generalization |
| GET: | General Edcuation and Training |
| FG: | Further Generalization |
| NCS: | National Curriculum Statement |
| NCTM: | National Council of Teachers of Mathematics |
| PMT: | Pre-service Mathematics Teacher |
| R: | Researcher |
| UKZN: | University of KwaZulu-Natal |
| UWC: | University of Western Cape |
| ZPD: | Zone of Proximal Development |

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## Abstract

This qualitative study actively engaged a group of eight pre-service mathematics teachers (PMTs) in an evolutionary process of generalizing and justifying. It was conducted in a developmental context and underpinned by a strong constructivist framework. Through using a set of task based activities embedded in a dynamic geometric context, this study firstly investigated how the PMTs experienced the reconstruction of Viviani's theorem via the processes of experimentation, conjecturing, generalizing and justifying. Secondly, it was investigated how they generalized Viviani's result for equilateral triangles, further across to a sequence of higher order equilateral (convex) polygons such as the rhombus, pentagon, and eventually to 'any' convex equi-sided polygon, with appropriate forms of justifications.

This study also inquired how PMTs experienced counter-examples from a conceptual change perspective, and how they modified their conjecture generalizations and/or justifications, as a result of such experiences, particularly in instances where such modifications took place. Apart from constructivsm and conceptual change, the design of the activities and the analysis of students' justifications was underpinned by the distinction of the so-called 'explanatory' and 'discovery' functions of proof.

Analysis of data was grounded in an analytical–inductive method governed by an interpretive paradigm. Results of the study showed that in order for students to reconstruct Viviani's generalization for equilateral triangles, the following was required for all students:

- experimental exploration in a dynamic geometry context;
- experiencing cognitive conflict to their initial conjecture;
- further experimental exploration and a reformulation of their initial conjecture to finally achieve cognitive equilibrium.

Although most students still required the aforementioned experiences again as they extended the Viviani generalization for equilateral triangles to equilateral convex polygons of 4 sides (rhombi) and five sides (pentagons), the need for experimental exploration gradually subsided. All PMTs expressed a need for an explanation as to why their equilateral triangle conjecture generalization was always true, and were only able to construct a logical explanation through scaffolded guidance with the means of a worksheet.

The majority of the PMTs (i.e. six out of eight) extended the Viviani generalization to the rhombus on empirical grounds using *Sketchpad* while two did so on analogical grounds but superficially. However, as the PMTs progressed to the equilateral pentagon (convex) problem, the majority generalized on either inductive grounds or analogical grounds without the use of *Sketchpad*. Finally all of them generalized to any convex equi-sided polygon on logical grounds. In so doing it seems that all the PMTs finally cut off their ontological bonds with their earlier forms or processes of making generalizations. This conceptual growth pattern was also exhibited in the ways the PMTs justified each of their further generalizations, as they were progressively able to see the general proof through particular proofs, and hence justify their deductive generalization of Viviani's theorem.

This study has also shown that the phenomenon of *looking back* (*folding back*) at their prior explanations assisted the PMTs to extend their logical explanations to the general equi-sided polygon. This development of a logical explanation (proof) for the general case after looking back and carefully analysing the statements and reasons that make up the proof argument for the prior particular cases (i.e. specific equilateral convex polygons), namely pentagon, rhombus and equilateral triangle, emulates the 'discovery' function of proof. This suggests that the 'explanatory' function of proof compliments and feeds into the 'discovery' function of proof. Experimental exploration in a dynamic geometry context provided students with a heuristic counterexample to their initial conjectures that caused internal cognitive conflict and surprise to the extent that their cognitive equilibrium became disturbed. This paved the way for conceptual change to occur through the modification of their postulated conjecture generalizations.

Furthermore, this study has shown that there exists a close link between generalization and justification. In particular, justifications in the form of logical explanations seemed to have helped the students to understand and make sense as to why their generalizations were always true, but through considering their justifications for their earlier generalizations (equilateral triangle, rhombus and pentagon) students were able to make their generalization to any convex equi-sided polygon on deductive grounds. Thus, with 'deductive' generalization as shown by the students, especially in the final stage, justification was woven into the generalization itself.

In conclusion, from a practitioner perspective, this study has provided a descriptive analysis of a ‘guided approach’ to both the further constructions and justifications of generalizations via an evolutionary process, which mathematics teachers could use as models for their own attempts in their mathematics classrooms.

# Chapter 1: Introduction

## 1.0 Introduction

The phenomenon that was under investigation in this study was the process of generalization, with particular focus on constructions, explanations and justifications of further generalizations of the Viviani‘s result (or theorem). Viviani’s theorem holds that in an equilateral triangle, the sum of the perpendicular distances from a point within the triangle to the sides is equal to the height of the triangle (or is constant) (De Villiers, 2011, p. 1). Viviani’s theorem was chosen as it opens up the possibility to generalizations that can be turned into a set of tasks to guide students towards constructing, explaining and justifying these generalizations. The first task focussed on a particular generalization for equilateral triangles. The latter tasks were designed to provide opportunities to extend the equilateral triangle generalization and/or its associated explanatory/justification structure further to specific, but different equilateral convex polygons, such as the rhombus, pentagon, and then later more generally to any equi-sided convex polygon. The latter generalization of the Viviani result to any equi-sided convex polygon is regarded as the Generalization of Viviani’s theorem in this study.

The notion of construction in this study encompasses the development of conjecture generalizations through inductive reasoning, analogical reasoning and deductive reasoning, and the notion of justification is considered as a continuum, with empirical argument on one end and deductive argument (proof) on the other end. From a deductive perspective the focus is on a logical explanation providing insight as to why a particular conjecture generalization (or generalization) is always true.

## 1.1 Rationale for the study

> “Mathematical discoveries, conjectures, generalizations, counter-examples, refutations, proofs (empirical, generic, deductive) are all part of what it means to do mathematics. School mathematics (as well as university mathematics) should show the intuitive and creative nature of the process, and also the false starts and blind alleys, the erroneous conceptions and errors of reasoning which tend to be a part of mathematics” (Australian Education Council, 1991, p. 14; emphasis added).

This study has been motivated by my participation in teaching of mathematics at secondary school and teacher education levels over a time span of 27 years. In particular, through self-study, attending seminars, workshops and conferences I have become more and more aware as to what it means "to do mathematics" as suggested by Australian Educational Council (1991). In particular, during the period 1999-2002 whilst reading for my Master's degree in Mathematics Education, I was given an opportunity to work in a dynamic geometric environment through using *Sketchpad*. The dynamic geometry environment (see Section 2.5.1 and 2.5.2) provides a context for experimentation with mathematical objects and the making of conjectures, which can be empirically tested, validated and then generalized; refuted through provision of an empirical counter-example which could be of global form; or modified through the construction of a heuristic counter example; and/or ultimately proven to be true via deductive explanation (i.e. logical argument) (compare Hanna & De Villiers, 2008, p. 332). Furthermore, the design and nature of *Sketchpad*-based tasks that I was engaged in, enabled me to engage with many of the core mathematical processes like conjecturing, generalizing, refutation through counter-examples, justification - both empirical and deductive. This experience widened my perspective and deepened my conceptual understanding of what 'doing mathematics' really should or could mean in a classroom context.

Further to this, as a result of the aforementioned experiences I have become more aware of the range of cyclical processes of experimentation, conjecturing, testing, generalizing, refuting and justifying that mathematicians traverse, but do not report on, in order to construct the polished definitions and theorems that we find in most mathematics textbooks and curriculum documents around the world (compare De Villiers, 2004, 2010). In particular, Freudenthal (1973) talks about the way in which textbooks hide, disguise or distort the way in which real mathematics is invented, and thus argues for an approach of 're-invention'. Hence, consistent with my epistemology that knowledge can either be created or discovered, this study extends my experiences to pre-service mathematics teachers, so that they themselves can develop a deeper understanding of the processes that one may have to traverse in order to discover or create conjectures and generalizations via experimentation and proof (see De Villiers, 2003a, 2004, 2010). It is hoped that in this way they will in turn be able to engage their prospective learners in core mathematical processes in authentic mathematical learning contexts, when they go out on practice teaching sessions or when they are employed as mathematics teachers at schools.

Generalization and justification play a pivotal role in fostering both mathematical thinking and mathematical growth. For example, Mason (1996, p. 65) asserts that: "Generalization is the heartbeat of mathematics. If teachers are unaware of its presences, and not in the habit of getting students to work at expressing their own generalizations, then mathematical thinking is not taking place."

Furthermore, any form of justification invokes a particular kind of reasoning or a combination of reasoning forms, which inevitably provides the grounds on which students can naturally question, argue and conjecture, and/or construct a generalization, and/or also explain why a particular generalization is either true or false (compare Blanton & Kaput, 2002; De Villiers, 2003a; Ellis, 2007; Hanna, 2000). Studies that focused on generalization and justification have found that students exhibit difficulty in constructing and justifying generalizations (Chazan, 1993; Hoyles, 1997; Hoyles & Kuchemann, 2002; Kieran, 1992; Knuth, Slaughter, Choppin, & Sutherland, 2002; Lee & Wheeler, 1987). Specifically, many studies have shown high school students tend to rely excessively on empirical examples to justify the truth of a mathematical proposition or conjecture generalization and struggle with deductive justification (Carpenter & Levi, 1999; Chazan, 1993; Hoyles, 1997; Knuth et al., 2002; Martin & Harel, 1989 ; Weber, 2003).

Taking into consideration the need for and importance of generalization and justification as well as its centrality and immediacy to mathematical experience in terms of promoting mathematical thinking and reasoning amongst students, many curriculum frameworks and researchers have suggested a greater focus on the processes of generalizing and justifying across mathematical curricula, particularly at school level (compare Ball & Bass, 2003; Coe & Ruthven, 1994; Davidov, 1990; Department of Education, 2002a, 2002b, 2003a, 2003b, 2003c; 2011a, 2011b, 2011c, 2011d; NCTM, 2000; Yackel & Hanna, 2003). For example, with the transformation of the South African schooling curriculum in the late 1990's and the birth of Curriculum 2005, more emphasis has been placed on reasoning and on processes such as conjecturing, generalizing, justifying (or proving) in mathematics. This emphasis was explicitly expressed in the Teacher's Guide for the Development of Mathematics Learning Programmes, which was designed to provide guidance as to how teaching, learning and assessment should typically occur in an Outcomes-Based Education (OBE) classroom. It reads as follows:

> "In addition to the knowledge, skills and values explicitly listed in the Learning Outcomes and Assessment Standards of the Mathematics Learning Area Statement(s), mathematical reasoning is considered as an integral overarching skill that needs to be developed throughout the phases and across learning outcomes … The ability to reason or argue a case is not only important in Mathematics, but is also an important life skill in its own right. Mathematical reasoning teaches reasoning skills in general, but mathematical reasoning also has conventions that are peculiar to mathematics and learners need to recognize this. Some of the skills related to mathematical reasoning include: analysing; selecting; synthesizing, generalizing conjecturing, proving." (Department of Education, 2003b, p. 29).

In line with the emphasis referred to above, the development of mathematical generalizations (which directly inolves reasoning skills) through experimentation and justification has become one of the core foci of the mathematics curriculum across most grades in South African schools recently. For example, in grades 10-12, learners are expected to do the following:

- "Investigate number patterns and hence: (a) make conjectures and generalizations; (b) provide explanations and justifications and attempt to prove conjectures;
- Produce conjectures and generalizations related to triangles, quadrilaterals and other polygons, and attempt to validate, justify, explain or prove them using any logical method (Euclidean, co-ordinate and or transformation)." (Department of Education 2003a, p. 12).

More recently, in 2009 the Department of Basic Education reviewed the teaching, learning and assessment of mathematics across South African classrooms as per the National Curriculum Statements (NCS) for Mathematics (Department of Education 2002a, 2002b, 2003a, 2003b, 2003c). After considering the findings of the 2009 curriculum review, the Department of Basic Education published four Mathematics Curriculum and Assessment Policy Statements (CAPS) (Department of Education 2011a, 2011b, 2011c, 2011d) making an effort to provide a document per phase that will explicitly guide teachers about teaching, learning and assessing in our classrooms. Even across the respective CAPS for Mathematics, reasoning, conjecturing, generalizing, justifying and proving are explicitly emphasized across the content topics. For example:

- As per CAPS Mathematics Further Education and Training Phase Grades 10-12, teachers are expected to provide grade 10 leaners opportunities to investigate, make conjectures and also construct proofs when teaching the aspects related to the content topic called Euclidean Geometry, as described in the following example:

  "Define the following special quadrilaterals: the kite, parallelogram, rectangle, rhombus, square and trapezium. Investigate and make conjectures about the properties of the sides, angles, diagonals and areas of these quadrilaterals. Prove these conjectures." (p. 25).
- As per CAPS Mathematics Senior Phase Grades 7-9, teachers are expected to give learners chances at investigations that are designed within the context of the following broad guidelines:

  "Investigations promote critical and creative thinking. It can be used to discover rules or concepts and may involve inductive reasoning, identifying or testing patterns or relationships, drawing conclusions, and establishing general trends." (p. 156). These skills include:

    - "Organizing and recording ideas and discoveries using, for example, diagrams and tables.
    - Communicating ideas with appropriate explanations.
    - Calculations showing clear understanding of mathematical concepts and procedures.
    - Generalizing and drawing conclusions." (p. 156).

The aforementioned emphasis of the South African National Curriculum for schools has important pedagogical implications for classroom practitioners. It means that mathematics teachers need to have first-hand knowledge and experience of mathematical processes (like conjecturing, generalizing, and justifying) to engage our mathematics learners successfully in the core mathematical processes as articulated by both CAPS and NCS. This automatically means that mathematics teachers themselves need to be appropriately educated to enlighten and engage their learners in core mathematical processes in meaningful ways. As pre-service mathematics teachers (PMTs) would continuously feed into the pool of mathematics teachers, who are expected to implement CAPS, it is prudent that PMTs themselves experience such processes in meaningful ways supported by appropriate contexts and environments (like dynamic geometric environments). Such experience gained by pre-service teachers will hopefully influence the way they will attempt/proceed to teach their prospective learners mathematics, as argued by Molina, Hull, & Schielak (1997, p. 11):

> ‘The process of teaching mathematics and learning mathematics is iterative: the way pre-service mathematics teachers are taught influences their understanding of and beliefs about mathematics; their understanding of and beliefs about mathematics influence the way they teach; and the way they teach influences their students’ understanding and beliefs about mathematics.”

Dynamic geometry software like *Geometer’s Sketchpad* can be invaluable in promoting reasoning, thinking and development amongst prospective teachers as it allows for experimentation via the ‘drag’ feature, which allows one to drag objects and dynamically manipulate them in the comfort of full visual control (Öner, 2008). Through using the drag facility one can drag (i.e. move around) a particular element of a dynamic figure, and the figure will then adjust itself to the altered condition (Goldenberg and Cuoco, 1998), but will maintain the invariant conditions. Öner (2008, p. 348) citing Kaput (1992) says the dragging feature of *Geometer’s Sketchpad “*facilitates conjecturing and more inductive approaches to geometric knowledge, as students can reason about the generality of their hypotheses for several cases.*”* Once the students have generalized their conjecture, the facilitator can then use the opportunity to get them to deductively justify their conjecture generalization. In this respect Öner (2008, p. 350) argues: “Dynamic Geometric Software (*Geometer’s Sketchpad)* affords an environment in which students can work with hunches and gain the confidence necessary to construct proofs for their conjecture.” Structuring conjecturing, generalizing and justification activities in a connected manner within a *Sketchpad* context is likely to “give students the opportunity to experience mathematics as a process of inquiry, not a finished product that needs to be mastered” (Öner, 2008, p. 350).

The kind and sequence of generalizing-justifying *Sketchpad*- based tasks used in this study has been designed to afford the pre-service mathematics teachers an opportunity to experience the typical path (namely: conjecturing, testing, generalizing, providing informal arguments, refuting via counter-example and proving providing a logical explanation) that mathematicians normally follow in order to construct, justify and make sense of new results. This study was motivated by the drive to provide the selected group of pre-service mathematics teachers with hands-on mathematical experience in these processes so they do not look at the development of proof as purely a formal process bounded by an axiomatic deductive approach (see Schoenfeld, 1983).

Counter-examples are powerful examples in that they provide the basis upon which a conjecture generalization can be shown to be completely or partially wrong. However, many students in our classrooms, and some teachers themselves, simply see the discovery of a single counter-example as a way of immediately disproving a conjecture generalization, and not as a possible opportunity to explore further and improve or correct their conjecture generalization so that it becomes valid (compare De Villiers, 1996; Komatsu, 2010; Lakatos, 1976; Zazkis & Chernoff, 2008). The recent CAPS (Department of Educations, 2011a) seems not to be aware of the broader role of counter-examples, not only in the production of mathematics, but also in the teaching and learning where the only assertion to counter-examples is the following: "It must be explained that a single counter-example can disprove a conjecture, but numerous specific examples supporting a conjecture do not constitute a general proof" (p. 25).

Although the latter part of the aforementioned CAPS statement can appear to be congruent with views of the mathematics community as to what constitutes a mathematical proof (i.e. a logical explanation or 'proof'), the former part of the CAPS statement does not embrace the notion of a *heuristic* counter-example. In particular, if at all, when teachers and students experience counter-examples, their experience is just limited to global counter-examples, i.e. examples that totally reject the main conjecture itself (De Villiers, 1996; Lakatos, 1976), and not heuristic counter-examples. A *heuristic* counter-example (see Section 3.2) is an example that does not refute a conjecture or statement in totality, but rather challenges a condition or property within the statement, which can either be improved on, partially removed or even left out in order to make the conjecture or generalization viable or more generalizable (De Villiers, 1996; Lakatos, 1976). In order to have a broader perspective mathematics, teachers and students experience of counter-examples should not be just limited to 'global' counter-examples, i.e. examples that only reject the main conjecture itself (see Section 3.2), but they should also be exposed to heuristic counter-examples which can lead to refinement and reformulation of conjectures.

This study therefore attempts to provide a space for pre-service mathematics teachers (PMTs) to experience a heuristic counter-example through experimental exploration within a dynamic geometry context. In this way, it is envisaged that the PMTs themselves can develop a broader understanding of the nature of counter-examples and their associated roles in developing and justifying generalizations. If this happens, then it is plausible that current pre-service teachers will enter the mathematics teaching profession with more insight as to how counter-examples should be treated in terms of modifying or rejecting conjecture generalizations. It is also possible that as and when the pre-service mathematics teachers begin to collaborate with other mathematics teachers, they will be in a better position to share their experiences and knowledge of heuristic counter-examples with one another, and thus in some way contribute to improving other teachers' understanding as to how a heuristic counter-example can be used to refine/modify a conjecture generalization.

The current literature on constructivism and meaningful learning has shown that learners' prior knowledge and experiences influence their thinking, and such prior knowledge and experiences sometimes lead to the development and/or exhibition of invalid conceptions (i.e. misconceptions) (Ben-Ari, 2001; Confrey, 1991; Olivier, 1989; Smith, DiSesssa, & Roschelle, 1993; Ryan & Williams, 2000, 2007; Tsamir & Tirosh, 2003). Moreover, as students tend to learn new ideas quite often through rote memorization in an isolated way, they often fail to intuitively see that the newly experienced information contradicts their existing ideas, and hence move on innocently to use their inherent misconception to interpret new information and thus iteratively develop further misconceptions (compare Ben-Ari, 2000; Confrey, 1991; Hynd, 2003; Luque 2003; Nesher, 1987; Olivier, 2009). These inherent misconceptions could be the underlying cause of errors that we see learners and even teachers make (compare Brodie, 2010; Olivier, 1989; Smith et al., 1993; also see discussion in section 3.4 of Chapter 3).

Sections 3.1, 3.2 & 4.6 extensively discuss how students' misconceptions can be treated through inducing cognitive conflict that is consonant with Piaget's Theory of Equilibration, and hence bring about conceptual change. In the empirical part of this study a descriptive account is given of how pre-service teachers experienced a heuristic counter-example within the context of Piaget's Equilibration Theory (or Socio-Cognitive Conflict Theory) to modify a particular conjecture generalization. Pre-service teachers and other mathematics teachers could possibly use this as a guiding heuristic of how to induce conceptual change as and when students in their mathematics classrooms make conjectures, generalizations, justifications and more particularly exhibit pertinent misconceptions.

Furthermore, most students and teachers experience and see proof as a construction of an argument or chain of arguments to validate a mathematical claim or remove their doubt about the truth of a conjecture generalization or mathematical assertion (compare De Villiers, 2003a & Hanna, 2000). This study attempts to break this restrictive perception held by mathematics students, teachers and researchers, by providing a detailed exposition as to how students can see proof as ‘acts’ that serve explanatory and discovery functions and also that proof via logical explanation can enable them to discover new generalizations or justifications and vice versa. The latter is complementary with Lanin’s (2005, p. 235) assertion that “generalization cannot be separated from justification”. Emphasizing this link, (Ellis, 2007, p. 196), says:

> “…the connection between generalizing and justification is bidirectional- engaging in acts of justification may be as likely to influence students’ ability to generalize the other way around”. Learning mathematics in an environment in which providing justifications for one’s generalizations can promote the careful development of generalizations that can be sensible and can be explained….A focus on generalization may help students not only better establish conviction in generalization but also aid in the development of subsequent, more powerful generalizations.”

In addition, Lins (2001) as cited in Lanin (2005, p. 232) notes: “students’ justifications provide a window for viewing the degree to which they see the broad nature of their generalizations and their view of what they deem as socially accepted justification.” Moreover, as discussed in the literature review and theoretical considerations for this study (see Section 2.2), the predominant mode of generalization that permeates most mathematics classrooms at both school and pre-service teacher education level is of the inductive generalization type. There is very little exposition or engagement with the development of generalizations using analogy or deductive reasoning, i.e. analogical generalizations and deductive generalizations are not really persued across school and teacher education contexts.

In the same way, Blanton and Kaput (2002) assert that through engaging students in the process of justification in our classrooms they could be encouraged to conjecture and thereby construct generalizations. Otte (1994, p. 310) alludes to the link between proof and further generalization when he says: “A proof is expected to generalize, to enrich our intuition, to conquer new objects, on which our mind may subsist.” To an extent this study explores this link and also provides an exposition to alternate ways of generalizing.

Although there is some research that has reported on the development of mathematical generalizations and/or justifications within a dynamic geometric context at pre-service teacher education level, there is a general lack of an adequate descriptive analysis of a guided approach to the construction and/or justification of a generalization within a dynamic geometric context. Research at pre-service teacher education level provides little descriptive analysis of a guided approach as to how pre-service teachers can extend an established generalization from one domain to the next, and how they can justify (or prove) such generalizations as they move from one domain to the next. Hence, this study will also attempt to provide a descriptive analysis of a 'guided approach" to the development, extension and justification of a generalization across domains with the use of *Sketchpad* as and when necessary.

## 1.2 The purpose of the study

Firstly, the purpose of the study was to actively engage pre-service mathematics in an evolutionary process of generalizing and justifying, by starting with the construction and justification of the Viviani generalization for equilateral triangles, and then extending it to a sequence of equilateral convex polygons of four sides (rhombi), five sides (pentagons) and finally to general convex equilateral *n*-gons (i.e. any equi-sided convex polygon), with a goal to investigate:

(a) How pre-service mathematics teachers generalize the Viviani result for equilateral triangles to a sequence of equilateral (convex) polygons of four sides (rhombi) and five sides (pentagons), and general convex equilateral *n*-gons (i.e. any equi-sided polygon); and

(b) How pre-service mathematics teachers arrive at their justifications for each of their extended (or further) generalizations?

In particular, the study focused on ascertaining whether pre-service mathematics teachers constructed and/or justified their generalizations whether in a non-deductive manner (that is through experimentation which entailed empirical methods such as construction, measurement and dragging, visual examples, inductive or analogical reasoning, et cetera) or deductive manner (that is through the use of generic or deductive proofs). Furthermore, this study sought to explore how pre-service mathematics teachers experienced counter-examples from a conceptual change perspective, and how they modified their conjecture generalizations and/ or justifications as a result of such experiences, particularly in instances were such modifications took place.

In addition, this study sought to use the explanatory function of proof to make proof more meaningful to pre-service mathematics teachers and provide them with necessary insight as to why the Viviani result and its resultant further generalizations is always true. By exploiting the discovery function of proof and the ‘looking back’ strategy of Polya, this study sought to explore whether or not pre-service mathematics teachers can construct generalizations deductively (or otherwise) and extend a particular proof structure to explain an extended generalization, and see the general proof through particular proofs.

Last but not least, from a practitioner perspective, the purpose of this study was to arrive at a descriptive analysis of a “guided approach” to both further constructions and justifications of generalizations via an evolutionary process, which mathematics teachers could use as models for their own attempts in their mathematics classrooms.

## 1.3 Research questions

The issues and insights that I have voiced and discussed so far lead me to propose the following research questions:

- Can pre-service mathematics teachers construct a generalization, which says that the sum of the distances from a point inside an equilateral triangle to its sides is constant? If so, how do they accomplish this generalization (which is commonly referred to as Viviani’s theorem)?
- Can pre-service mathematics teachers support their equilateral triangle generalization with a justification, and if so, how do they construct (or provide) a justification for it?
- Can pre-service mathematics teachers further generalize and extend the Viviani Theorem for equilateral triangles to equilateral (convex) polygons of four sides (rhombi), five sides (pentagons), and then to equilateral convex polygons in general? If so, how do they accomplish the constructions of such further generalizations?
- Can pre-service mathematics teachers justify each of their extended generalizations to equilateral convex polygons (namely, rhombus, pentagon and general equilateral polygon generalizations)? If so, how do they accomplish the justification of each of their respective further generalizations?

## 1.4 Delimitation of the study

As outlined in the purpose of the study, this study focused on pre-service mathematics teachers' engagement in constructing and justifying a generalization of Viviani's theorem. This research study is limited to a purposive sample of eight pre-service mathematics teachers (PMTs) from the Faculty of Education at University of Western Cape, who were in their final year of study with regard to their first teacher education qualification (see Section 6.4 for further details on the sample). Each of the eight pre-service mathematics teachers participated in one-to-one task-based interviews as discussed in Section 6.5. dealing with Methodology. Furthermore, in view of this study being conducted within a dynamic geometry context and limited to eight pre-service mathematics teachers from just one University, I, as a qualitative researcher am aware that the results emanating from this study cannot be generalized across to other contexts.

## 1.5 Generalization: What is it?

> "Mathematical generalization involves a claim that some property or technique holds for a large set of mathematical objects or conditions. The scope of the claim is always larger than the set of individually verified cases; typically, it involves an infinite number of cases (e.g.,"for all integers")" (Carraher, Martinez & Schliemann, 2008, p. 3).

For example, at primary school learners might have done activities that required them to count the number of unit squares that cover the area of specific rectangles. At the end of such a series of activities they might have been able to generalize their experiences by saying $area\ of\ a\ rectangle = length \times width$ .. Such a generalization "applies even when the length and width are not integers but any real numbers" (Mason, 1999, p. 11). In analytical geometry, the rule $(x; y) \rightarrow (x\, cos\theta - y\, sin\theta; x\, sin\theta + y\, cos\theta)$ serves a general way of describing the effect of rotating any point $P(x; y)$ in the plane about the origin through any angle $\theta$ in an anti-clockwise direction (Carter, Govender, & Heany, 2007, pp. 251-252; Mason, 1999, p. 11).

Mason (1999, p. 9) says that: "Generalizing has to do with noticing patterns and properties common to several situations", and asserts that a generalization is regarded as an expression or statement that can be specialized. This means as one uses/examines particular cases to recognize a pattern or regularity to provoke the formulation of a generalization (Njisane, 1992, p. 61), one should also be able to use such a formulated generalization in turn to produce particular cases or examples which it can characteristically generalize (compare Mason, 1999, p. 10). For example, consider a point $P(2;3)$ on a circle *O*, whose centre is at the origin. One can determine the coordinates of the image of the point $P(2;3)$ after it has been rotated about the origin through each of the angles $60°$ and $330°$ by using the generalized rule: $(x;y) \rightarrow (x\,cos\theta - y\,sin\theta; x\,sin\theta + y\,cos\theta)$. The particular cases will then be:

**Particular case 1:**

$(2;3) \rightarrow (2\,cos60° - 3\,sin60°; 2\,sin60° + 3\,cos60°) \equiv (1 - \frac{3\sqrt{3}}{2}; \sqrt{3} + \frac{3}{2})$

So the coordinates of $P'$ are : $(1 - \frac{3\sqrt{3}}{2}; \sqrt{3} + \frac{3}{2})$

**Particular case 2:**

$(2;3) \rightarrow (2\,cos330° - 3\,sin330°; 2\,sin330° + 3\,cos330°) \equiv (\sqrt{3} + \frac{3}{2};\ -1 + \frac{3\sqrt{3}}{2})$

So the coordinates of $P'$ are : $(\sqrt{3} + \frac{3}{2};\ -1 + \frac{3\sqrt{3}}{2})$.

The processes of specializing and generalizing are intrinsically intertwined. In light of this, Tall (1988, p. 2) asserts that “generalization and the complementary process of specialization is common to both elementary and advanced mathematical thinking.” Picking upon this, Mason (1999, p. 33) similarly emphasizes this complementarity as follows: “Specializing can provide fodder for generalization, and generalizations must be checked to see that they do specialize back to the particular cases which spawned them”.

In most instances, it is useful for one to specialize systematically so that a pattern can emerge (see the development of the Polygon Angle Sum Measure Generalization as illustrated later in this section). On the other hand, considering extreme examples within the context of specialization could help one to: “stretch an idea to its limits, in order to see what is going on” (Mason, 1999, p. 33). In other words, specializing can enable learners to make a conjecture generalization, and also interrogate their conjecture generalization to see if it also holds true for a greater variety of cases. In this way, learners can then be prompted to accept/reject their conjecture generalization, or modify or extend it (compare Mason, 1999, p. 33). Consonant with these ideas, Dreyfus (1991, p. 35) says: “To generalize is to derive or induce from particulars, to identify commonalities, to expand domains of validity.”

In the context of expanding the domains of validity, Njisane (1992, p. 63) affirms that generalization encompasses the extension of a class. For example, grade 9 learners would have been working with the identity $(a+b)(a-b) = a^2 - b^2$ in relation to whole numbers. However, as one extends the ideas of addition and multiplication to fractions, one comes to realize that the identity can be applied to fractions as well. In other words, the operations of addition and multiplication as described by the identity are extended from whole numbers to fractions. These extensions are called generalizations (see Njisane, 1992, p. 63).

According to Dienes (1961, pp. 282; 296), "The process of generalization, instead of leading from elements to classes, leads from classes to classes … we shall regard abstraction as class formation, and generalization as class extension…". Cognizant of generalization as a process that leads from classes to classes, Dienes (1961) defines a mathematical generalization as follows:

> "A class *B* is a mathematical generalization of the class A if B includes as part an isomorphic image of *A*, in relation to all relevant properties. This means that the classes *A* and *B* consist of quite different elements, as long as there was a part of *B* which somehow has an exact image of A, mirroring the properties of *A* in all relevant aspects" (p. 282).

In consonance with the aforementioned definition, Dienes (1961, p. 282) further notes that:

> "An example of a mathematical generalization is that of passing from natural numbers to positive and negative integers. Positive integers have exactly the same properties as natural numbers, yet they form a sub-class of the class of directed numbers to which the natural numbers do not belong. The class of directed numbers is now the class B, the class of natural numbers the class A. The image of the class A in the class B is the class of positive integers."

This definition of generalization by Dienes (1961) as is Polya's (1985) definition of generalization further on is unfortunately a little restrictive. The generalized class does not necessarily always contain the original class of special case. For example, generalizing the concept of an equilateral triangle to a rhombus makes use the common property of "all sides equal", but the class of rhombi does not contain the class of equilateral traingles. However, both equilateral triangles and rhombi would be members of the class of equilateral polygons.

Polya (1985, p. 108) is of the view that: "Generalization is passing from the consideration of one object to the consideration of a set containing that object; or passing from the consideration of a restricted set to that of a more comprehensive set containing the restricted one." For example, "we generalize when we pass from the consideration of any triangle to any polygon. … In passing from triangles to polygons with $n$ sides, we replace a constant by a variable, the fixed integer 3 by the arbitrary integer $n$ (restricted only by the inequality $n \geq 3$)" (Polya, 1954a, p. 12). A construction of a generalization, namely the Convex and Concave Polygon Angle Sum Measure Generalization, which typically exemplifies the aforementioned example, can be illustrated as follows:

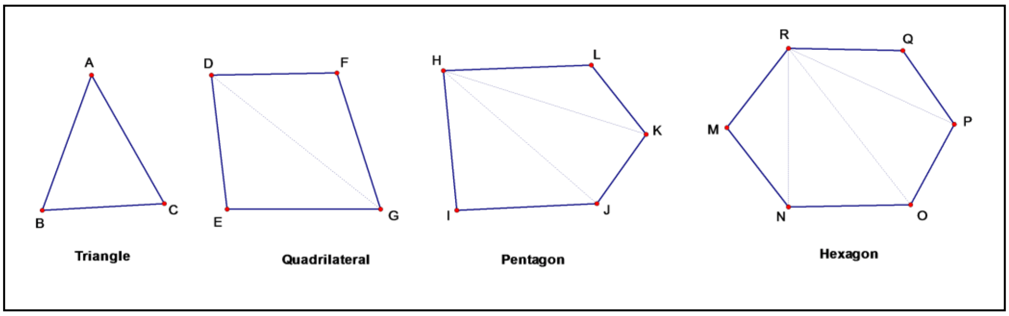



Figure 1.5.1: Convex Polygons

Figure 1.5.1 represents a set of convex polygons. Most junior high school mathematics students would have experimented with triangles, and would have discovered that the sum of the angles of a triangle is $180°$ to any triangle. They would have probably justified their generalization by providing a deductive proof. Thus, for the purposes of developing the Convex and Concave Polygon Angle Sum Measure Generalization, we could start by using the fact that sum of the angles of any triangle $ABC$ is $180°$, and proceed as follows:

(a) Arbitrary convex quadrilateral $DEFG$, having 4 sides has been divided into two triangles by drawing diagonal $DG$.

$\therefore$ the sum of the interior angles of the quadrilateral $= (4-2).180°$

$= 2.180°$

$= 360°$

(b) Arbitrary convex pentagon $HIJKL$, having 5 sides has been divided into three triangles by drawing diagonals $HJ$ and $HK$.

$\therefore$ the sum of the interior angles of the pentagon $= (5-2).180°$

$$= 3.180°$$
$$= 540°$$

(c) Arbitrary convex hexagon $MNOPQR$, having 6 sides has been divided into four triangles by drawing diagonals $RN$, $RO$ and $RP$.

$\therefore$ the sum of the interior angles of the pentagon $= (6-2).180°$

$$= 4.180°$$
$$= 720°$$

As can be seen through special cases (a), (b) and (c), the number of triangles a polygon can be divided into by drawing diagonals from one vertex to all the non-adjacent vertices is two less than the number of sides that makes up the polygon. This means that any convex polygon (i.e. any convex $n$-gon for $n \geq 3$) can be divided into $(n-2)$ triangles. Therefore, we can make the following generalization: The sum of the measure of the interior angles of any convex polygon with $n$ sides (i.e. any convex $n$-gon for $n \geq 3$) is $(n-2).180°$.

Although the generalization has been made for any convex polygon, the result also holds true for any concave polygon, but choosing 'any vertex' or drawing 'any diagonal' no longer works to divide the concave polygon into ($n$-2) triangles as some diagonals, unlike the convex case, may fall outside (see Figure 1.5.2). The above proof argument therefore needs to be adapted to state that a suitable vertex should be chosen so that all diagonals are drawn to fall inside. In arriving at this generalization, the process of abstraction played a significant role. According to Dienes (1961, p. 280),

> "…the process of abstraction is defined as the process of drawing from a number of different situations something which is common to them all. Logically speaking it is an inductive process; it consists of a search for an attribute which would describe certain elements that somehow belonged together. A class is constructed out of some elements which will then be said to belong to the class"

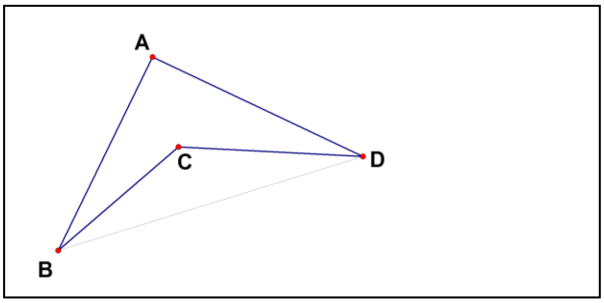



Figure 1.5.2: Concave quadrilateral with a diagonal falling outside

For example, through examining the special cases (a), (b), and (c), and then abstracting the idea that any convex or concave polygon can be divided into $(n-2)$ triangles, it is thus possible to make the generalization: The sum of the measure of the interior angles of any convex or concave polygon with $n$ sides is $(n-2).180°$.

As illustrated here, 'generalization' is characterized as a process and also a product. In particular a generalization is constructed via the process of generalization (see Yerushalmy, 1993). Similarly, Tall (1988, p. 1) says:

> "Generalization is the process of forming general conclusions from particular instances. The term also applies to the concept produced by the process, for instance "$a + b = b + a$" is considered an algebraic generalization of the arithmetic statement "$3 + 2 = 2 + 3$" and $\boldsymbol{R^n}$ is the generalization of $\boldsymbol{R^2}$"

Furthermore, looking at the phenomenon of 'generalization' through a process-product orientation, Du Toit (1992, p. 115) says: "Generalizing is the process by which an observed abstraction is generalized as valid for a larger class or a statement that is valid for a greater variety of situations." In other words, generalization is a process of formulating a statement that encompasses more possibilities and cases. For example, if we consider the Theorem of Pythagoras as re-discovered at school level, we understand that: "In any right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides" (Gonin, Archer, Slabber, & Nel, 1981, p. 394). In symbolic form, this means that for any right-angled $\Delta ABC$ having $\widehat{A} = 90°$ (see Figure 1.5.3), $a^2 = b^2 + c^2$. Clearly in this instance of the Theorem of Pythagoras, there is a condition, namely $\widehat{A}$ has to be $90°$.

Pythagoras' Theorem can be transformed into a more general statement (or generalization) that embraces a larger number of cases for which statement is true. For example, by removing the restriction that $\widehat{A} = 90°$, it is possible for us to have the following generalization of the Theorem of Pythagoras: In any triangle $\Delta ABC$, $a^2 = b^2 + c^2 - 2bc\ cosA$, where $a, b, c$ are the lengths of the sides and $\widehat{A}$ is the angle opposite side $a$ (see Mason, 1999, p. 23). In retrospect, "specialization is passing from the consideration of a given set of objects to that of smaller set, contained in the given one" Polya (1954a, p. 13). Hence in this instance, the Theorem of Pythagoras is a special case of the cosine formula $a^2 = b^2 + c^2 - 2bc\ cosA$.

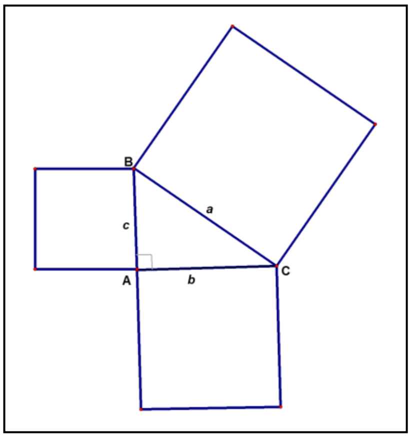



Figure 1.5.3: Right-angled triangle – Pythagoras Theorem

Alternatively, one can also generalize the theorem of Pythagoras in numerous other ways. For example, "if regular pentagons are drawn on the sides of a right-angled triangle, then the area of the pentagon on the hypotenuse is the sum of the areas of the other two" (Mason, 1999, p. 23). In fact, as long as similar figures are constructed on the respective sides of the right-angled triangle, then the area of the figure on the hypotenuse will add to the sum of the areas of similar figures on the other two sides. Moreover, one could generalize the two dimensional aspect of Pythagoras to three dimensions as follows: "The square length of the diameter of a cuboid with sides $a, b, c$ is $a^2 + b^2 + c^2$. In this instance by choosing any one of the dimensions $a, b, c$ to be zero one can return the special case, i.e. the two dimensional triangular case" (Mason, 1999, p. 24). As seen via the examples discussed in this section, that particular instances (or examples) of a generalization can be retrieved by the process of specialization. However, one should bear in mind that the retrieval or generation of such particular examples, does not necessarily tell us that the generalization is always valid (see Mason, 1999, p. 25).

Taking into account all the aforementioned definitions, views and discussions as to 'what is generalization', I believe that the following definition of a generalization as posited by Kaput (1999, p. 136) is useful for the purpose of this study:

> "Generalization involves deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and expressing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the

> patterns, procedures, structures and the relationships across and among them (which, in turn, become new, higher level objects of reasoning or communication)."

## 1.6 Types of Generalization

Generalizations can essentially be classified into three different types of generalization within mathematics. They are respectively called inductive generalization, deductive generalization, constructive (*a priori*) generalization for propositions and concepts/structures/axiom systems. I propose to discuss each of these types of generalization in sub-sections 1.6.1, 1.6.2, and 1.6.3 respectively. Further to this, I propose to focus on the kinds of reasoning such as inductive, analogical and deductive reasoning in Section 2.1.

### 1.6.1 Inductive generalizations

Inductive generalization means that a generalization is initially made on quasi-empirical grounds without necessarily any deductive thought involved, for example observing and formulating generalizations from the consideration of some particular cases (De Villiers, 1996, p. 86). Similarly, Yerushalmy (1993) citing Chi & Bassok (1989) states:

> "Induction is a well known process to reach generalizations by examination of instances or examples. An instance or a set of instances is examined, certain properties are identified. The given example is then taken as a member of a larger set and its properties are put into a larger set. Such generalization from multiple examples is developed on a similarity based approach" (p. 247).

As alluded to in the earlier paragraph, the development of an inductive generalization is facilitated by the process of inductive reasoning. Inductive reasoning (as discussed in Section 2.1) is a process of observing data, recognizing or abstracting patterns/common features/qualities across a set of objects under consideration, and then making a statement that one thinks may be true for all the given objects (cases) under consideration as well as other objects (cases) of the same type or class, though at the time of making the statement one does not really know for sure that it is generally valid (see Polya, 1954a, De Villiers, 1992). Such a statement is called a 'conjecture' and is described by Mason, Burton & Stacey (1982) in the quotation captured here:

> "A conjecture is a statement which appears reasonable, but whose truth has not been established. In other words, it has not been convincingly justified and yet it is not

known to be contradicted by any examples, nor is it known to have any consequences which are false. In other words, it has not been convincingly justified and yet is not known to be contradicted, nor is it known to have any consequences which are false" (p. 72).

Polya (1967) and Reid (2002) as cited in Candas & Castro (2005, p. 402) suggest that the process of developing an inductive generalization (i.e. a conjecture generalization) via inductive reasoning occurs in the following way:

- **Observation of particular cases**: Students experiment with particular cases of the problem posed and then try to note a pattern or observe regularity.
- **Conjecture formulation**: Students then formulate a conjecture, by making a statement about all possible cases, but with an element of doubt.
- **Conjecture validation:** At this stage students attempt to experiment with further new particular cases, but not general cases, to see if the conjecture still holds true for the new particular cases.
- **Conjecture generalization:** On seeing that a conjecture is true for some particular cases, and having experimented to see that it holds for further new particular cases (conjecture validation), students might hypothesize that the conjecture is generally true.

Similarly, as illustrated in Figure 1.6.1, James (1992, p. 160) describes the various processes, specializing, abstracting, generalizing and testing, which work in a harmonious way to build an inductive generalization. These processes can be enumerated as follows:

- **Specializing**: Students examine particular cases and become familiar with the details of each case;
- **Seeing generality:** As the students move through particular cases they begin to see some regularity across the cases. This awareness of the abstracted regularity or underlying sameness becomes more and more prominent as the students pass from one particular case to the next, and consequently boosts their sense of confidence in their observed degree of sameness across the special (or particular) cases.
- **Expressing the generality:** As soon as the student is quite confident with the underlying sameness that s/he has increasingly seen across the particular cases, then s/he begins to articulate the sameness in her/his own words and also comes to grips with concepts that underpin it.

- **Checking and Convincing (empirically)**: By empirical testing students try to see if their generalizations also hold for new particular cases as well**.**

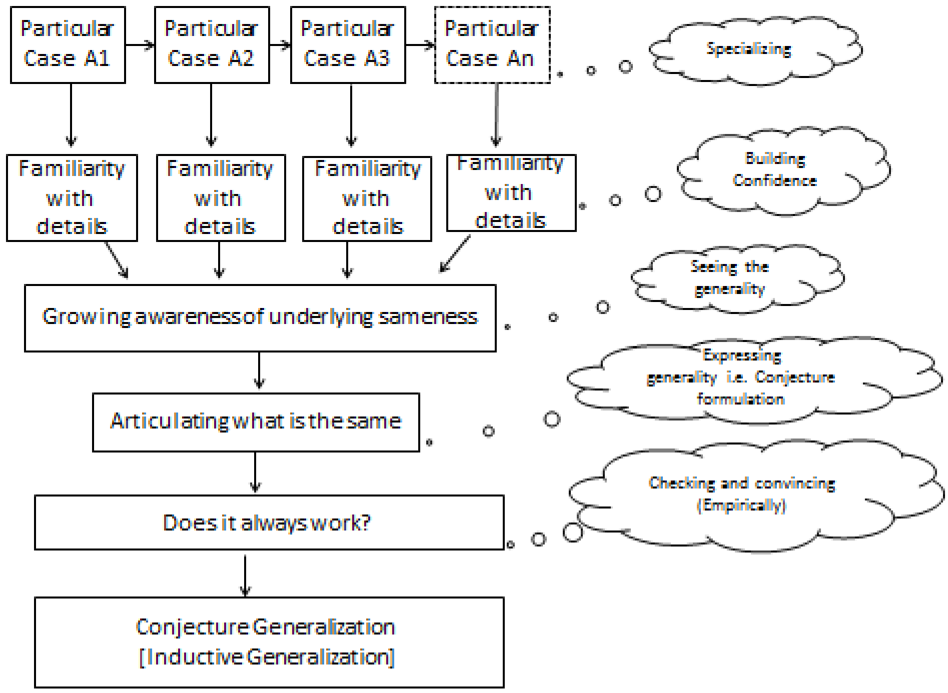



Figure 1.6.1: Development of an Inductive Generalization (see James, 1992, p. 160)

Although the process of inductive generalization serves as a way of creating a type of generalization (or conjecture), Yerushalmy (1993, p. 253) emphasizes that it is not “a process of deriving definite knowledge”. This means that such a generalization, called a ‘conjecture generalization’ in this study, needs to be justified via a logical explanation (proof) before being considered as legitimate/acceptable mathematical knowledge by the mathematics community. Despite this shortcoming, the process of inductive generalization has played a pivotal role in the birth of many mathematical ideas, propositions and theorems across the field of mathematics, and still continues to play such a role even today.

Taking into consideration that the development of inductive generalizations is underpinned by the process of inductive reasoning, Yerushalmy (1993, p. 246) says: “Generalizations are a particular kind of conjecture, created by reasoning from the specific to the general.” On the other hand, there are those that will refer to a conjecture as a conjecture, and reclassify such a conjecture as a ‘generalization’ only after it has been justified via a logical explanation (i.e. proved that conjecture is true for all cases through the use of deductive arguments). The latter notion is quite evident in most of our school and university textbooks, where for example, through inductive reasoning a conjecture is established and subsequently the status of this 'conjecture' is turned into a 'theorem' through the process of deductive justification. In such an instance the 'theorem' is then considered to be a 'generalization' and not a 'conjecture generalization'.

### 1.6.2 Deductive generalizations

Contrary to an inductive generalization, a deductive generalization is made on the basis of a logical deduction, for example by deductively analyzing the conditions of a particular theorem (or theorems) and finding from its proof that a specific condition is sufficient and others are redundant. This can enable further generalization by leaving out the redundant property (compare De Villiers, 1996, p. 86). It is fundamentally important to note that in such a case the generalization is based entirely on deductive reasoning; for example, seeing the midpoints of the sides of a kite form a rectangle because a kite has perpendicular diagonals, and then realizing the same proof would hold for *any* quadrilateral with perpendicular diagonals. (A kite has perpendicular diagonals, but there are other quadrilaterals that have perpendicular diagonals that are not kites).

### 1.6.3 Constructive (a priori) generalization

Constructive (*a priori*) generalization of concepts or structure or axiom systems takes place when a more general concept or structure or axiom system is defined by relaxing or generalizing certain conditions. For example, consider defining a 'bisecting quad' as a new concept by generalizing the concept of a parallelogram by saying that it is any quadrilateral with at least one diagonal bisecting the other. Typically, when constructive (a priori) defining takes place the resultant definition is considered true by definition as it does not involve a propositional statement. Also, this kind of generalization, is typically exemplified, "when a set of axioms (or structure) is changed through the exclusion, generalization or addition of axioms (or subset of axioms) to that set, from which totally new content is then constructed in a logical deductive way" (De Villiers, 1986, p. 4).

For example, within the process of constructive (*a priori*) generalization of concepts, we can by using analogy which is a kind of similarity (see Sections 2.1.5 & 4.4), constructively extend "the concept of a parallelogram to hexagons by defining a 'parallelo-hexagon' as a hexagon with opposite sides equal and parallel (see Figure 1.6.3)" (De Villiers, 2008, p .35) . More-over, "by relaxing one or other condition, we can generalize even further by constructively defining a parallel-hexagon as a hexagon with opposite sides parallel or an 'oppo-sided hexagon' as a hexagon with opposite sides equal" (De Villiers, 2008, p. 35).

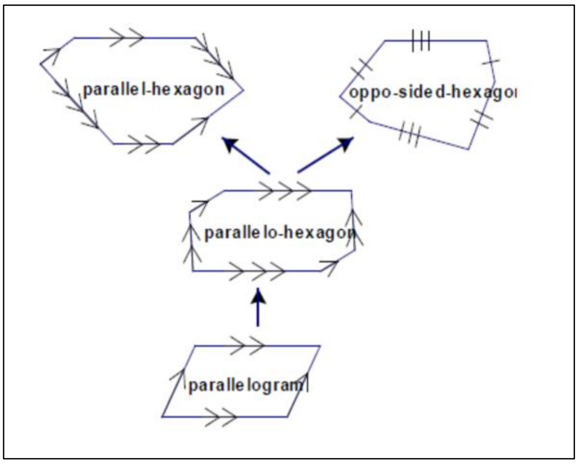



Figure 1.6.3 Generalizing a parallelogram

The same process of constructive (*a priori*) generalization, which is called condition – simplifying generalization in some literature (Holland , Holyoak, Nisbet, & Thagard, 1986), could also apply to propositions (e.g. a theorem or conjecture), whereby relaxing one or other condition of a given proposition, new conjectures or theorems are formulated. For example, generalizing the theorem of Pythagoras by asking what happens to the relationship between the sides if we relax the condition that it is a right angle. However, when it comes to generalizing propositions (theorems or conjectures) in this way, proof is needed.

## 1.7 The importance of generalization

From a mathematical perspective, generalizations are important for many reasons. For example, Harel & Martin (1988, p. 101) assert that: "Generalizations provide a broad perspective in which to observe particular phenomena, which in turn allows greater understanding of these particular phenomena." Through generalizations one can find commonality to solutions and proofs, and hence be able to solve many types of problems in a routine and efficient manner (compare Polya, 1954a, p. 17). In other words, one can extrapolate/extend generalized results/ phenomena to "new situations to solve new problems" (Harel & Martin, 1988, p. 101). In particular, generalization can afford students the opportunity to find an answer to pattern problems for any number efficiently, or extend the generalized patterns to generate new cases (Ellis, 2007, p. 461). Furthermore, the process of having students make generalizations and then testing these generalizations can impact student learning in the following ways:

- Stimulate students to think and reason mathematically (Dreyfus, 1991; Mason, 1999).
- Give students a chance to confront their misconceptions or faulty ideas (Klymchuk, 2008).
- Help students construct their own knowledge that leads to deeper understanding and reasoning abilities (Polya, 1954a).

## 1.8 Justification: What is it?

According to the Chambers Dictionary by Schwarz (1993, p. 908) justification is "the act of justifying; something which justifies." This means justification is both a process and a product. As a process, justification in the affirmative refers to the demonstration of the correctness of an assertion such as a conjecture, proposition, generalization or opinion (see Oxford Dictionary, 1995). In particular, the verb justify brings to the fore the following 'doing' words: 'vindicate, excuse, prove right, show to be just, warrant, support, validate, uphold, sanction, confirm, sustain, bear out, defend, account for, explain, make explanation for" (Shepherd, 2006, p. 465). Equivalently, Mariotti (2007, p. 288) says the role of justification is that of "explaining, arguing, corroborating, verifying a particular statement." So, in a broad sense, justification is the advancement of a reason or reasons for or against a proposition, opinion or conjecture, and this may include arguments that are verbal, data that is in numerical form, empirical examples, pictorial representations, drawings (compare Douek, 2009, p. 334) or logical arguments (i.e. deductive explanations which are commonly called proofs) (compare De Villiers, 2003a, p. 18). So in a sense, one can say that a justification is an argument or sequence of connected arguments raised in favour or against a particular claim.

In retrospect, Duval (1999, p. 3) considers an argument to be anything that is used or advanced to support or refute a proposition, and says that items such as the following could serve such a purpose: a definition; a rule; a factual statement; outcome of an experiment; an example; a belief; a contradiction. In addition, Duval (1993, p. 3) asserts that when someone uses the aforementioned items to substantiate or say "why he /she accepts a proposition" then they take on the value of justification. This in essence means that through using either plausible reasoning, such as inductive or analogical reasoning, or pure deductive reasoning

(Polya, 1954a & 1954b), one can develop an argument or a sequence of arguments that could act as a justification for a particular claim.

In particular, there are two main categories of arguments, called “arguments of plausibility”and “arguments of necessity” respectively (Cabassut, 2005, p. 391). The former are “arguments in which the warrant entitles us to draw conclusions only tentatively (qualifying it with a ‘probably’) subject to possible exceptions (‘presumably’) or conditionally (‘provided that …’)”, and the latter are “arguments in which the warrant entitles us to argue unequivocally to the conclusion” (see Cabassut, 2005, p. 391; Toulmin, 1958, p. 148).

One may therefore regard justification “as a process in which a logically connected discourse is developed” (compare Vincent, Chick, McCrea, 2005, p. 281). Furthermore, argumentation is defined as “reasoning; sequence or exchange of arguments” (Chambers Dictionary, 1993, p. 84) or “a statement advanced to justify or refute a claim in order to attain the approbation of an audience or to reach consensus on a controversial subject matter” (Oggunyi, 2007, p. 965). In light of the views voiced, it is befitting to consider argumentation and justification to be one and the same process in this study.

Although justification like argumentation is a discourse, Douek (1999, p. 127) stresses that the discourse itself does not necessarily have to be deductive in nature, and that the text generated as result of the discourse can very well be regarded as the representation of the justification. Duval (2007), Hollebrands, Conner, & Smith (2010), Krummheur (1995) as cited in Vincent et al. (2005, p. 281) all see an argument to be either part of a family of arguments posited within a complex justification or simply an outcome resulting from a justification and also asserts that justification is not just about convincing an audience, but could just very well be an ‘internal process carried out by an individual” to convince himself/herself about a particular claim. Drawing from all the aforementioned expressions about justification, one can say justification is a process of arguing on the grounds of a reason or a network of connected reason(s) such that one is enabled to build/produce a conjecture (or conjecture generalization) and/or possibly explain its validity (compare Osborne, Eduran & Simon, 2004).

Many authors in both mathematics and mathematics education view the purpose of justification as being to convince themselves or a colleague that a specific mathematical statement is correct (i.e. verify the truth of mathematical statement like a conjecture generalization) or simply remove one's doubt about a specific conjecture generalization, but nothing more (compare Hanna, 1989; Harel & Sowder, 1998, 2007; Volmink 1990, pp. 8;10). However, many often use the terms explanation, verification, and proof very loosely and interchangeably to refer to this 'convincing' or 'verification' function of justification (Marrades & Gutierrez, 2000). For example, the Department of Education (2003b, p. 8), describes justifying as "the learner is able to explain why he/she chose a particular course of action or did what he/she did." Despite this notion of justification, the Department of Education (2003a), does not elaborate on the nature of justification nor does it give any specific examples, which an educator could reflect upon or mirror in his/her classroom.

However, in other instances individuals discriminate between the general notion of justification and the deductive justification (i.e. proof). For example, Marrades and Gutierrez (2000, p. 89), in their study regarding the types of justifications secondary schools students produced whilst learning geometry in a dynamic computer environment, use "the term justification to refer to any reason given to convince people (e.g. teachers and other students) of the truth of a statement," but use "the term (formal mathematical) proof to refer to any justification which satisfies the requirements of abstraction, rigor, language, etc., demanded by professional mathematicians to accept a mathematical statement as valid within an axiomatic system."

Cabassut (2005, p. 392) uses the term 'validation' to mean a "reasoning that intends to assert, necessarily or plausibly, the truth of a statement", in order to differentiate between proof (deductive) and justification (argumentation) as follows: A proof "is a validation using only arguments of necessity," and a justification (argumentation) "is a validation using arguments of plausibility and maybe arguments of necessity." Similarly, Hanna and De Villiers (2008, p. 331), consider justification (argumentation) to be "a reasoned discourse that is not necessarily deductive but uses arguments of plausibility" and deductive proof to be "a chain of well organized deductive inferences that uses arguments of necessity."

## 1.9 Types of Justification

Bell (1976) as cited in Marrades & Gutierrez (2000, p. 89), argues that there two types of jsutifications that students use when responding to proof problems: "Empirical justification "characterized by the use of examples as element of conviction", and deductive justification, "characterized by the use of deduction to connect data with conclusions." Like Bell (1976), Balacheff (1988) after considering the results of experiments where high school students had to solve a range of proof problems, equivalently developed two distinct categories of justification, called pragmatic (empirical) justifications and conceptual (deductive) justifications. Pragmatic justifications are constructed on the grounds of examples, illustrations or actions, whereas conceptual justifications do not involve actions or showings, but instead depend on the construction of particular properties at a rather abstract level as well as possible relations between such properties. (Balacheff, 1988; also see Marrades & Gutierrez (2000, p. 89).

Amongst the types of empirical proofs suggested by Balacheff (1988, p. 219) has been the "generic example". Balacheff (1998, p. 219) argues that: "The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class." Below is a typical example cited in Balacheff (1988, p. 219):

> "The remainder on dividing a number $2\times 2$ or $5\times 5$ is the same as the remainder on dividing the number formed by the rightmost two digits by $2\times 2$ or $5\times 5$…To fix these ideas, consider the number 43 728 and the divisor $5\times 5$. The number 43 728 is equal to 43 700 + 28. However, 43 700 is divisible by 5x 5 because 43 700 is the product of 437 and 100, and as 100 is 10 x 10, or 5 x 2 x 5x 2, the factor 100 is divisible by 5x 5. The remainder on dividing 43 728 by 5x 5 or 25 is therefore the same as that on dividing 28 by 25."

Balacheff (1998, p. 219) describes 'the thought experiment' as one of the types of deductive justifications, and acknowledges that in the cases of the generic example and the thought experiment, the objective is not to show that the result is true by simply demonstrating that it works, but rather to provide the necessary reasons as to why the result is true. In particular there is a connection between the generic example and thought experiment, in the sense that the former constitutes a transitional stage which facilitates the movement from pragmatic to conceptual proofs. This is because the general character of the generic example is continually reflected upon in order to develop and construct the thought experiment, which in the final analysis practically detaches itself from the particular case under consideration. In terms of this envisaged connection, I am of the view that the generic example as a kind of justification does not belong to the set of empirical justifications as posited by Balacheff (1988, p. 219) and Marrades & Gutierrez (2000, p. 92), but rather it belongs to the continuum of deductive justifications as it allows one to see the general through the particular (see Section 5.3.1).

At this juncture it would be helpful to cite Stylianides (2008, p. 10), who asserts that a generic example "is a proof that uses a particular case as seen as representative of the general case" and hence regards a generic example as kind of proof that is used to support a mathematical claim. Similarly, Harel & Sowder (2007, p. 811) using the notion of 'proof scheme' to synonymously mean justification states: "Generic example in our taxonomy belongs to the deductive proof scheme category". Like Tall (1979), Harel & Sowder (1998, p. 271) also maintains that: "In a generic proof scheme, conjectures are interpreted in general terms but their proof is expressed in a particular context." (See Section 5.3.1).

In particular, Harel and Sowder (1996, 1998, 2007) and Sowder & Harel (1998), through working with secondary school, college and university students have differentiated between three major levels of justification, namely: external conviction, empirical and deductive justifications. Harel & Sowder (1998, 2007) note that a student exhibits an 'external conviction' type of justification, when she/he provides a justification for his/her conjecture generalization through reliance on the information provided by an authority like a teacher or mathematics textbook, the structural appearance or layout of an argument, or through the meaningless use and manipulation of symbols. For example, a student exhibiting an externally based justification could accept a proof for a given mathematical theorem in a university mathematics textbook on the grounds of seeing the structure given, required to prove, as proof, and also by seeing that the textbook has been written by a world reknowned professor in mathematics (who is an authority in the field of mathematics), without necessarily really checking if the posited argument makes any meaningful sense to himself/herself (see Harel & Sowder, 1998, 2007).

Likewise De Villiers (1992) has also reports on children's acceptance of theorems on the basis of 'authority' of the textbook or teacher or maybe parent or elder family member. In fact between 50-70% of the students sampled in this study apparently accepted the truth of the results on the basis of authority, more so than on empirical grounds, as teachers in many classrooms in South Africa do not even allow learners the opportunity to explore mathematical statements experimentally or inductively. They are either just given as 'truths' by the teacher or the teacher gives a proof: they are not even guided to rediscover the proof. Gila Hanna (2007) also mentions how even in mathematics, complicated or advanced proofs are often accepted on the authority of the mathematician rather than the ability to work through and understand the proof. For example, I accept that Andrew Weil and his partner have proven Fermat's Last Theorem, but I only have a vague understanding of how it was achieved, but little understanding of the finer details. But I accept their authority and those of the reviewers and referees that have scrutinized and examined their work. So this is far more common than people may think at first glance.

With respect to empirical justification, Harel & Sowder (1998, p. 252) asserts that: "conjectures are validated, impugned, or subverted by appeals to physical facts or sensory experiences." For example, at classroom level, quite often justifications for conjectures or conjecture generalizations are constructed solely on the basis of examples. In more explicit terms, Gutierrez, Pegg & Lawrie (2004, p. 513) asserts that: "In empirical proofs examples are the argument of conviction." In the same vein, psychologists affirm that concept formation by most individuals is naturally based on examples – and even on just a single specific example at times (Medin, 1989). Indeed in much of our daily practice, one also embraces examples in order to understand a particular point (or concept) or otherwise to check out their own sort of understanding. Consistent with the common drive by many to naturally use examples to substantiate a claim, Harel & Sowder (2007, p. 809 ) asserts that empirical justifications are: "marked by their reliance on either (a) evidence from examples of direct measurements of quantities, substitution of specific numbers in algebraic expressions, and so forth, or (b) perceptions." This means that there are typically two core kinds of empirical justifications, namely perceptual justification (i.e. visual pereception) and inductive justification (Harel & Sowder, 2007, p. 809).

Harel & Sowder (1996, p. 62) asserts that a perceptual justification is "based solely on visual or tactile perceptions." Quite often the usage of the perceptual justification, which is generally not supported by logical deduction, is prevalent amongst younger students. In a more general sense, students operate within the ambits of the perceptual justification, when they make an inference based on just one drawing in some instances or several drawings in other instances, and similarly use drawings to convince others that their inference or conclusion is true. For example, students may examine a parallelogram and infer that the opposite sides are equal just by visual examination, and may not see the causal relationship between the opposite sides being parallel and the opposite sides being equal (i.e. that opposite sides being parallel imply opposite sides are equal). In other instances, more senior students might be convinced that the line drawn from the centre of the circle to the midpoint of chord is perpendicular to the chord by just looking at several computer-generated examples, and will most likely use similar kinds of examples to convince their peers (see Harel & Sowder, 1996). Furthermore, in using the perceptual justification, it is highly probable that students might just not consider or account for the arbitrary case.

According to Harel and Sowder (1998), a student exhibits an inductive kind of justification, when he or she attempts to convince himself/herself (ascertains for themselves) or convince others (persuade others) that a specific conjecture holds true across more new specific cases. In this context, the notion of quantitative evaluations alludes to the following kinds of actions amongst others: “direct measurement of quantities, numerical computations, substitutions of specific numbers in algebraic expressions, etc.” (Harel & Sowder, 1998, p. 252). In many of our classrooms our students quite often exhibit the inductive kind of justification to remove doubt about the truth of a particular conjecture. For example, the findings from Chazan’s (1993) study revealed the predominant use of inductive proof scheme among high school students that participated in his study.

In Martin & Harel’s (1989) study, they asked their first year pre-service elementary mathematics teachers to determine the mathematical correctness of inductive and deductive verifications of either a familiar or an unfamiliar statement, and found that more than half the students accepted an inductive argument as a valid mathematical proof, and that the said acceptance was not dependent on the familiarity of the context. The use of the inductive proof scheme is also prevalent amongst students who study mathematics at advanced levels. For example, Goetting (1995, p. 43) in her study, as cited in Harel & Sowder (1998, p. 252), found that “almost 40% of her advanced undergraduates used examples as basis for judging the truth of a divisibility question.”

Like many others working in the field of mathematics and mathematics education, Harel & Sowder (1998, 2007) view deductive justification as a process that uses given data together with relevant definitions, axioms, theorems to build a logical argument that validates a mathematical claim, and hence removes one’s doubt about the truth of a mathematical claim under consideration. However, one should bear in mind that the function of deductive justification (i.e. mathematical proof) is far more than just validation, verification of a result, removing doubt or convincing oneself or others (compare De Villiers, 2003a). As discussed in Section 5.4 below, deductive justification (proof) has other functions, for example, explanation and discovery (De Villiers, 2003a; Staples, Bartlo & Thanheiser, 2012).

Simon & Blume (1996) asserts that in mathematics classrooms where justification is promoted and mathematical validation and understanding are important foci, "mathematical justification is likely to proceed from inductive toward deductive and toward greater generality" (p. 9). Through investigating and reflecting on the justifications that were articulated by the class of prospective mathematics elementary teachers, Simon & Blume (p. 17) categorize their justifications as follows:

> "Level 0 - responses identifying motivations that do not address justification
> Level 1 - Appeals to external authority
> Level 2 - Empirical demonstration
> Level 3 - Deductive justification that is expressed in terms of a particular instance (generic example)
> Level 4- Deductive justification that is independent of particular instances".

The aforementioned levels represent a progression from justifications that have no bearing on the claim under consideration, to the reliance on some form of external validation, and progressing ultimately to construction of a justification that engages with the mathematical relationships via the use of axioms, definition, theorems through logical reasoned arguments.

| **Justification Level** | **Description** |
|---|---|
| Level 0: No justification | Responses do not address justification |
| Level 1: Appeal to external authority | Reference is made to the correctness stated by some other individual or reference material. |
| Level 2: Empirical evidence | Justification is provided through the correctness of particular examples. |
| Level 3: Generic example | Deductive justification is expressed in a particular instance. |
| Level 4: Deductive Justification | Validity is given through a deductive argument that is independent of particular instances. |

Figure 1.9.1: Justification Framework (Lanin, 2005, p. 236)

Reflecting on and using Simon & Blume's (1996) categorization of justification, Lanin (2005) developed the justification framework as shown in Figure 1.9.1, to examine and analyze the kind of justifications that twenty five sixth grade students developed in order to justify the generalizations they produced as and when they were working through algebraic patterning activities using computer spreadsheets.

Distilling from the types of justifications proposed by Bell (1976a and b), Balacheff (1998), and Harel & Sowder (1998, 2007), Lanin (2005), Marrades & Gutierrez (2000), Simon & Blume (1996), it is evident that external conviction justifications; empirical justifications; generic justifications and deductive justifications constitute the core categories of justification that permeate the field of mathematics and mathematics education. These justifications can be arranged in a continuum as shown in Figure 1.10.2, with external conviction justification on the lower end and deductive justification on the upper end.

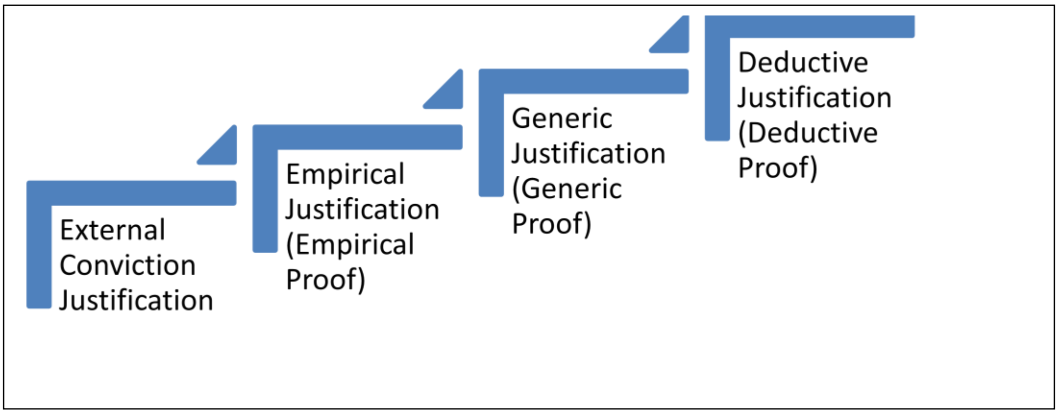



Figure 1.9.2: A continuum of justifications

This deliberate distinction of the notions of justification, suggests multiple approaches to justification. One approach is certainly justification in the form of a logical argument (proof), whilst other approaches could take the form of an empirical argument or generic proof or an external conviction. This study in its endeavour to expose pre-service mathematics teachers to alternate forms of justifications in mathematics, will consider the range of justifications as shown in Figure 1.9.2.

## 1.10 Outline of dissertation

This dissertation is organized into twelve Chapters and includes thirteen Appendices. The Chapters in the study are as follows:

**Chapter one: Introduction**

This chapter introduces the study and explores the concepts of generalization and justification as processes and products to arrive at a set of working conceptions of generalization and justifications that guides this study. In particular this Chapter discusses the different types of generalizations and justifications that emanated from the literature review done for this study, and signals the kinds of generalizations and justifications that this study attempts to engage pre-service teachers with. Moreover, Chapter one presents the rationale for the envisaged study, and in doing so the researcher describes how he came to develop an interest in the topic under study, some pertinent reasons as to why this study is worth doing, and points out some gaps in the literature with regard to generalization and justification. Furthermore, the Chapter describes the purpose of the study and the research questions. Finally the Chapter presents the delimitation aspects and the dissertation outline.

**Chapter two: Generalizations and Justifications**

This Chapter is an extension of the literature review discussed in Chapter 1. This Chapter focuses on the kinds of reasoning that underpins/defines the different kinds of generalization and justification that permeate mathematics and mathematics education. In addition this Chapter presents a literature review on research done in the context of generalizing and justifying both at school and tertiary levels. This Chapter ends with a sub-Chapter on conjecturing, generalizing and justifying within a dynamic geometric context.

**Chapter three: Misconceptions and Counter examples**

This Chapter represents a further literature review that was done on the topics misconceptions and counter-examples as result of the design of the study and the findings that emerged from the data analysis.

**Chapter four: Conceptual Framework and Theoretical Considerations**

Chapter four introduces the conceptual framework for this study from a learning theory perspective. To address the purpose of the study and answer the proposed research questions for this study, it became necessary for the researcher to engineer a conceptual framework that embraces a number of connected theoretical frameworks and topics that are compatible with the constructivist theory of learning. In view of the topics (theoretical considerations) and frameworks that have been considered to guide this study, some are discussed in this Chapter 4 and others are discussed in Chapter 5. In Chapter 4, the learning theory of constructivism in relation to cognitive and social constructivism is described and discussed. In addition, theoretical considerations pertaining to generalizations, justifications, scaffolding, discovery learning, and analogical transfer are discussed. In particular, Chapter 4 engages with the following theoretical frameworks: Piaget's Theory of Equilibration (which is also known as Piaget's Theory of Socio-cognitive Conflict); Gentner's Structure Mapping Theory and Ausubel's Theory of Meaningful Learning. The other remaining theoretical considerations are discussed in Chapter 5.

**Chapter five: Further Theoretical Considerations**

This Chapter is the continuation of the theoretical considerations and theoretical frameworks for this study as articulated in the conceptual framework guiding this study. In particular this Chapter discusses the proving process; the conception of deductive proof for this study, and the different functions of proof. It culminates with a discussion on global and heuristic counter-examples.

**Chapter six: Research Design and Methodology**

This Chapter presents the research design, methodology and research procedures that were considered for this study and provides theoretical frameworks and approaches that were used to analyze the data. Initially this Chapter starts by briefly reiterating the purpose of the study and the research questions governing this study. It then elaborates on the qualitative research approach that has been adopted for this study and provides a discussion and motivation for using the interpretive paradigm as well as a case study approach for this particular study. This Chapter also discusses the target and accessible population (sample) as well as the characteristics of the sample, and also the data collection (production) techniques and instruments that were used in this study. Moreover, this Chapter presents the inductive-deductive approach which was used to analyze the data within the context of the generalization and justification frameworks constructed for this study. Finally this Chapter culminates with a discussion on reliability and validity measures adopted for this study as well as a description of the ethical considerations governing this study.

**Chapters seven to ten: Data Analysis, Results and Discussions**

In this section of the study the researcher presents the data analysis, findings (results) and discussion in an integrated inclusive manner in relation to the equilateral triangle problem, convex rhombus problem, convex equilateral pentagon problem and ‘any’ equi-sided convex polygon problem respectively. Hence, this means that:

- **Chapter seven** presents the data analysis, findings (results) and discussion in an integrated manner in relation to the equilateral triangle problem.
- **Chapter eight** presents the data analysis, findings (results) and discussion in an integrated manner in relation to the convex rhombus problem
- **Chapter nine** presents the data analysis, findings (results) and discussion in an integrated manner in relation to the convex equilateral pentagon problem
- **Chapter ten** presents the data analysis, findings (results) and discussion in an integrated manner in relation to ‘any’ equi-sided convex polygon problem (i.e. general convex equilateral *n*-gons).

**Chapter eleven: Findings in the context of the research questions**

In line with the guiding conceptual framework of this study, this Chapter provides a consolidated discussion of the research findings in the context of the research questions, and in doing so provides the evidence to verify the research questions.

**Chapter twelve: Conclusion**

This Chapter presents the implications of the findings, limitations of this research, and recommendations for further research. Finally this Chapter presents conclusion that embraces the purpose of the study and the associated key findings of this study.

**Appendices** include the detailed task-based activities that were used during the one-to-one task based interviews as well as the interview protocols.

As part of the literature review for this study, the next Chapter (i.e Chapter 2) focuses on reasoning, generalization and justification.

# Chapter 2: Reasoning, Generalization and Justification

## 2.0 Introduction

As reasoning underpins the kind of generalizations that one could attempt and the kinds of justifications that one could propose to support and/or explain their generalizations, I wish to discuss the three types of reasoning, namely: inductive, deductive and analogical reasoning in Section 2.1. Next, I propose to use Section 2.2 to explore some of the research around generalizations that has taken place in the field of mathematics /mathematics education and also illustrate how constructed generalizations can be extended across domains. Section 2.3 is meant to explore some research related to empirical and deductive justification (proof) at school level, whilst Section 2.4 will focus on research related to justifications at pre-service teacher education level and beyond. Section 2.5 provides an in-depth discussion around conjecturing, generalizing and justifying within a dynamic geometry context.

## 2.1 Reasoning in Mathematics

Reasoning generally encompasses all thinking activities that entail making judgements and, inferences and drawing conclusions, which is quite consistent with Ball and Bass' (2003, p. 28) view that "Mathematical reasoning is no less than a basic skill." Building on this, Bjuland (2007, p. 2) says "reasoning can be defined as five interrelated processes of mathematical thinking, categorized as sense –making, conjecturing, convincing, reflection, and generalizing." Furthermore, Ross (1998, p. 254), the former President of the Mathematical Association of American (MAA) argues:

> "It should be emphasized that the foundation of mathematics is reasoning.[… ] Results may be shown to hold in a small number of cases directly, but students must recognize that all they have in that case is evidence of a conjecture until the result has been firmly established. Construction of valid arguments or proofs and criticizing arguments are integral parts of doing mathematics. If reasoning ability is not developed in the student, then mathematics simply becomes a matter of following a set of procedures and mimicking examples without thought as to why they make sense."

According to the National Council of Teachers of Mathematics 1999 year book (see Stiff & Curcio, 1999) the reasoning abilities of students can be developed through engaging them with exploratory and investigative mathematical tasks, which promote the making of conjectures that needs to be explained and justified. At a more local level, the Department of Education (2003a) also makes a similar assertion. For example, the Department of Education (2003a) has prescribed in its curriculum policy that reasoning skills should be developed by affording students activities that at least provide opportunities for analyzing, synthesizing, explaining, evaluating, justifying, convincing, proving.

Looking at the issues mentioned above in a related manner, Russell (1999, p. 1) emphasizes the following aspects about active mathematical reasoning in mathematical classrooms:

> "First, mathematical reasoning is essentially about the development, justification, and use of mathematical generalizations. In the classroom, where mathematical reasoning is the centre of the activity, the solution of an individual problem is closely linked to the generalizations behind that solution. Second, mathematical reasoning leads to an interconnected web of mathematical knowledge within a mathematical domain. Third, the development of such a web of mathematical knowledge is the foundation of what I call "mathematical memory", what we often refer to as mathematical "sense". Fourth, an emphasis on mathematical reasoning in the classroom incorporates the study of flawed or incorrect reasoning as an avenue toward deeper development of mathematical knowledge."

Since reasoning permeates the processes of conjecturing, generalizing, convincing, explaining, justifying and proving, related kinds of reasoning are discussed in this narrative. According to Polya (1967), there are essentially two broad categories of mathematical reasoning, namely demonstrative reasoning and plausible reasoning. Demonstrative reasoning, which commonly embraces deductive reasoning is akin to mathematical proof, whilst plausible reasoning refers to both inductive reasoning and analogical reasoning (Polya, 1967). Inductive reasoning, deductive reasoning and analogical reasoning will be discussed further in some detail in this Chapter.

### 2.1.1 Inductive and Deductive Reasoning

Inductive reasoning denotes the process of reasoning that occurs when, by merely looking at a set of examples, one generates a theory that one thinks will represent all the given examples as well as examples that are not present (Polya, 1967; de Villiers, 1992). Quite often this particular process is referred to as the inductive method/process or just as induction, which should not be confused with mathematical induction as a proof process. The following example can help to illustrate the use of inductive reasoning in our daily lives:

The bin truck that came on day one was black.

The bin truck that came on day two was black.

The bin truck that came on day 3 was black.

.

.

.

Therefore, the bin truck that comes daily is black.

In the above example, it was induced that the bin truck that comes daily is black, but this might not necessarily be true. It could be that on day ten a red bin truck could come. Hence, generally we may say that a conclusion reached via inductive reasoning is not necessarily valid, more so since it is usually not possible to observe every known case.

However, in other cases, inductive reasoning could yield a conclusion, which could later be shown to be valid, depending on the nature of the specific instances and conclusion. For example, if a student accurately measures the angles of seven different triangles using *Sketchpad*, and finds that the sum of the angles in each case adds up to $180^o$, then the student may inductively conclude that the sum of the angles in all triangles is $180^o$, which we know from our experience is valid, notwithstanding the fact that in normal mathematical practice one would have to show that the conclusion is valid via demonstrative reasoning.

While inductive reasoning is the process of starting with a number of specific instances and creating general statements, deductive reasoning is defined as the process which occurs when we move from some accepted generalization(s) to specific instances. The following example (see Figure 2.1.1.1), can help to illustrate the use of deductive reasoning to produce a valid conclusion:

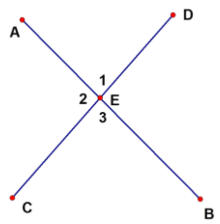



Figure 2.1.1.1: Vertically opposite angles

$\widehat{E_1} + \widehat{E_2} = 180°$ (sum of adjacent $\angle$'s on a straight line is $180^0$)……. (1)

and $\widehat{E_2} + \widehat{E_3} = 180°$ (sum of adjacent $\angle$'s on a straight line is $180^0$)…….(2)

$\therefore\ \widehat{E_1} + \widehat{E_2} = \widehat{E_2} + \widehat{E_3}$ ………………………………………………….. (3)

$\Rightarrow\ \widehat{E_1} = \widehat{E_3}$ ………………………………………………….. (4)

In the above example, statements 1 and 2 are called premises, and statement 3 a conclusion. By the common rules of deduction, if statements 1 and 2 (the premises) are true then the statement 3 (the conclusion) must be true. Hence, since statements 1 and 2 are true in the above example, we may with absolute certainty claim that statement 3 is true, and consequently declare the given argument as a valid argument, which in this case is a syllogism.

Moreover, conclusive arguments as illustrated in the vertically opposite angle's example, are called deductive, while inconclusive arguments as in the bin truck example are called inductive. It is therefore useful to distinguish between induction and deduction in terms of conclusiveness rather than just a movement between generality. In Figure 2.1.1.2 (see de Villiers, 1992, p. 46), the differences and similarities between these two reasoning processes are compared.

| | Deduction | Induction | |
|---|---|---|---|
| | True premises (General) | True Premises (Particular) | |
| Valid argument | leads to ↓ | leads to ↓ | Plausible argument |
| | True Conclusion (Particular) | Probable Conclusion (General) | |

Figure 2.1.1.2: Deduction and Induction

From Figure 2.1.1.2, it can be deduced that the relationship between premises and conclusion in a valid deduction is none other than logically leading to conclusion, whereas the relationship between premises and conclusion in induction is one only of plausible support. In other words, the conclusion arrived at via deduction is always true so long as the premises are true, where as the conclusion arrived at via the process of induction may be true or turn out to be false. For example, men who go fishing once in a while to Hout Bay may find that they only catch yellow tail fish. They then induce (or infer, or draw the conclusion) that the only fish that can be caught at Hout Bay is yellow tail fish, but this would not necessarily be true.

### 2.1.2 The complementary roles of inductive and deductive reasoning in Mathematics

In mathematics, one may observe a number of specific cases defined by specific conditions and characterisics, and find that a particular property (result) remains invariant, and hence make a conjecture. In this instance, we say inductive reasoning has been used to arrive at the conjecture. However, to provide some insight as to why the conjecture is true for all cases or some delimited sets of cases, one then proceeeds to use deductive reasoning to construct logical explanation (called proof).

For example, one might notice that every time they add the opposite angles of a cyclic quadrilateral by using *Skecthpad,* they add up to 180°. One might then conjecture that the opposite angles of cyclic quadrilateral is always 180°. These moves characterize the process of inductive reasoning (i.e. induction). Thereafter, one would normally proceed to use the axioms, definitions and theorems of geometry to prove that the the opposite angles of cyclic quadrilateral is always 180°, and when this happens we say that deduction has taken place.

Generally, inductive reasoning is seen to to be a playing a key role is the discovery and creation of new mathematics. In practice, as students examine a number of special cases, they might continuously observe a regularity (regularities), and it is this regularity that drives them to make a mathematical conjecture that they strongly feel (though not knowing with absolute certainty) is true. However, to justify (or demonstrate) with absolute certainty that the specific mathematical conjecture is true, it then beomes necessary for one to use deductive reasoning to construct a logical explanation that is mathematically sound and relevant . Polya (1954a) (as cited in de Villiers, 1992, p. 47) states that the former process is “hazardous, controversial, and provisional,” while the latter is “safe, beyond controversy, and final.”

Nonetheless, in recent years it is widely accepted that conjecturing, exploration, and the creation of new mathematical objects or results are largely underpinned by inductive and intuitive methods rather than deductive reasoning. (compare Lakatos, 1976; Polya, 1954a). Most certainly, “without inductive reasoning great mathematicians like Newton and Euler would not have made the advances they made" (de Villiers, 1992, p.47). However, without deductive reasoning we would still be doubtful of many of their results. Therefore according to de Villiers (1992, p. 47), "the one cannot do without the other. They supplement, nourish and support each other like symbiotic partners, and would stifle and suffocate on their own."

Mouly (1978) as cited in Cohen, Manion, and Morrison (2002, pp. 4-5) argues that the inductive–deductive approach is a symbiotic relationship, which consists of:

> "a back-and-forth movement in which the investigator first operates inductively from observations to hypotheses and then deductively from these hypotheses to their implications, in order to check their validity from the standpoint of compatibility with accepted knowledge. After revision, where necessary, these hypotheses are submitted to further test through the collection of data specifically designed to test their validity at the empirical level. This dual approach is the essence of the modern scientific method and marks the last stage of man's progress toward empirical science, a path that took him through folklore and mysticism, dogma, and tradition, causal observation, and finally to systematic observation."

### 2.1.3 Analogical Reasoning

#### 2.1.3.1 Reasoning via Analogy

The South African Oxford School Dictionary (2004) describes analogy as "a partial likeness between two things that are compared." In a more detailed manner, Sowa & Majumdar (2003, p. 1) lists other possible meanings of an analogy, such as: "a similarity in some respects of things that are otherwise dissimilar, a comparison that determines the degree of similarity, or an inference based on resemblance or correspondence." Polya (1954a, p. 13) also asserts that "analogy is a sort of similarity". For example, to see the similarity between an aeroplane and a bird, wherein the cockpit is matched with the head of the bird, the wings of the plane are matched with the wings of the bird, is equal to making an analogy.

Moreover, Gentner & Holyhoak (1997, p. 33), asserts that, "in analogy, the key similarities lie in the relations that hold within domains (e.g., the flow of electrons in an electric circuit is analogically similar to the flow of people in a crowded tunnel, rather than the features of the individual objects (e.g. electrons do not resemble people)." Furthermore, according to

Gentner & Holyoak (1997, p. 33), the recognition of "higher order relations", commonly referred to as relations between relations, is essential and fundamentally important for the development of analogical similarities. The following example demonstrates the notion of higher order relations: Increasing the speed of your vehicle would cause an increase in the petrol consumption, just as the faster you run would cause an increase in your oxygen consumption.

Robert Oppenheimer (1955) cited in (1989, p. 413), writes:

> "Whether or not we talk of discovery or of invention, analogies are inevitable in human thought, because we come to new things in science with what equipment we have, which is how we have learned to think, and above all how we have learned to think about the relatedness of things. We cannot, coming into something new, deal with it except on the basis of the familiar and the old fashioned. The conservation of scientific enquiry is not an arbitrary thing; it is the freight with which we operate; it is the only equipment we have. We cannot learn to be surprised or astonished at something unless we have a view of how it ought to be; and that view is almost certainly an analogy."

Thus, we may equivalently say that reasoning via analogy plays a significant role in the discovery and invention of new aspects in mathematics. In this context, one could assert that "analogical reasoning entails understanding something new by analogy with something that is known" (English, 1998, p. 126). Moreover, Gentner (1983, 1989) as cited in English & Sharry (1996 p. 138), define analogical reasoning as a process of "transfer of structural information from one system, the base (called the source) to another system (called the target) through mapping relational correspondences between the two systems." Indeed it is the very notion of "corresponding relational structures" that promotes analogy making amongst and within domains, which makes discoveries, generalizations and problem solving possible to a large extent in mathematics (see Polya, 1954a; English & Shary, 1996).

In fact, Spearman (1923) cited in Novick (1988), claims that analogical reasoning encompasses our intellectual acts. For example, when learners at school, or university students or mathematicians themselves encounter a mathematical problem (called the target problem), they quite often resort to looking at how they solved a similar (or linked) problem called the source problem, and then attempt to apply a similar kind of strategy or known procedure or if not, modify their identified strategy to solve their given problem. In particular, Novick (1988, p. 510) asserts that “retrieval of an analogous problem may enable the student (or learner or mathematician) to adapt a known procedure for use with the target problem, thus precluding the necessity of constructing a new procedure.”

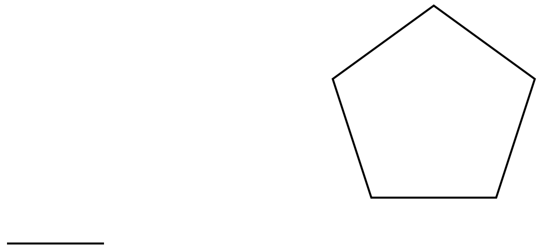
However, in ordinary everyday thought, analogy making is a very natural and spontaneous process. Quite often we use analogical reasoning to think and deliberate on issues, and also to reach conclusions related to specific designs, problems and contexts, et cetera. For example, the idea that an aeroplane should have wings in order to fly was developed from seeing a bird in flight. Similarly, the general description of the resistor situation, which constitutes Ohm’s law, was made possible by comparing the resistor situation with the water-pipe situation (Winston, 1980). Likewise, Newton was able to formulate his famous laws by simply mapping the relations between the movement of a projectile and that of the moon around the earth (de Villiers, 2009), and Archimedes developed his principle of buoyancy by reflecting on his own experience of floating in his bath (Alexander, White, & Daugherty, 1997). This most persuasively demonstrates the potential value of analogical reasoning as a generative or creative tool, and augurs well with Polya’s (1957, p. 43) assertion that “inference by analogy appears to be the most common kind of conclusion, and it is possibly the most essential kind.” However, conclusions reached via analogical reasoning should be treated with some caution, because reasoning via analogy may not necessarily contribute to the development and production of conclusions that are sound and valid, like the way deductive reasoning does.

#### 2.1.3.2 The use of analogical reasoning and analogies in mathematics

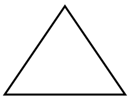
According to Gentner, Holyoak & Kokinov (2001) as cited in Sriraman (2005, p. 507), "analogy is generally defined as the ability to reason with relations, to detect patterns, identify recurrences given superficial variations and abstract from these patterns.” No doubt, such general definition is quite relevant to the area of mathematics. For example, quite often the establishment of a conjecture generalization or the construction of a “theorem” itself necessitates “the abstraction of structural relationships from a class of varying examples within which a particular pattern occurs” (Sriraman, 2005, p. 507).

Ideally in mathematics, according to Polya (1954a, p. 13), “two systems are analogous if they agree in clearly definable relations of their respective parts”. For instance, a triangle, parallelogram and circle are respectively analogous to a tetrahedron (or pyramid), prism and sphere respectively. In the plane, the triangle is the bounded by the minimum number of straight lines (i.e. 3 straight lines), but similarly in space, a tetrahedron is bounded by the minimum number of planes (i.e. 4 planes), hence, the analogy between a triangle and a tetrahedron.  We may regard a triangle and a pyramid as analogous figures, by considering the following constructions:

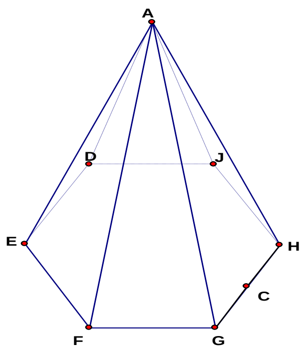
- Construct  a line segment and polygon as shown in the following figure:



- Place a point outside the line segment and connect it to the points (endpoints) of the line segment, you will then obtain a triangle.



- Place a point outside the plane of the polygon and then connect it to all the points (vertices) of the polygon, you will then obtain a pyramid.




Similarly, the analogy between a parallelogram and prism, can be easily illustrated as follows: Draw a segment parallel but opposite to the given line segment, and then by connecting the respective points, you would produce a parallelogram. Similarly, by drawing a polygon parallel but opposite to the given polygon, and connecting the respective points, you would produce a prism (see Polya, 1954a, p. 14). Moreover, “in the plane, a circle may be defined as the locus of all points equidistant from point, but similarly in space, a sphere can be defined as the surface formed by all points equidistant from a point, hence the analogy (de Villiers, 2009).

Furthermore, we could formulate theorems of solid geometry that are analogous with theorems from plane geometry. For example, in the plane, "the bisectors of three angles of a triangle meet in one point which is the centre of the circle inscribed in the triangle" but similarly in space, "the bisecting planes of the six dihedral angles of a tetrahedron meet in one point which is the centre of the sphere inscribed in the tetrahedron" (see Polya, 1954a, pp. 25 & 215).

In light of this, exploration of solid geometry can be facilitated by making the necessary analogy with plane geometry, which could possibly lead to new mathematical conjectures. For example, since the triangle is analogous to the tetrahedron, it is possible to conjecture that the method for determining the center of gravity of a tetrahedron, may be analogous to the method for determining the centre of gravity of a triangle, which normally is found by drawing the medians of the triangle and locating the point of intersection of the medians (i.e. by finding the point of concurrency of the medians, known as the centroid) (see Crystal-Alberta, 2010).

There also exist an analogy between the finite and infinite. For example, according to Polya (1954a, p. 26),

> "the infinite series and integrals are in various ways analogous to the finite sums whose limits they are; the differential calculus is analogous to the calculus of finite differences; differential equations, especially linear and homogenous differential equations, are somewhat analogous to algebraic equations, and so forth."

Furthermore, analogy can be looked upon as a similarity of relations, wherein the relations are bound by the same specific laws. For instance, the addition of numbers is analogous to the multiplication of numbers, since both operations are bound by the same rules. In particular, both addition and multiplication are commutative and associative. According to de Villiers (2009, p. 34), in a case like this, "where the analogy is determined by clearly defined rules, there exists a duality, primarily because the two operations can be interchanged," provided the commutative and associative laws are involved. In particular, de Villiers (2009, p. 34) makes the following assertion: "this duality between addition and multiplication extends to a fruitful analogy between arithmetic and geometric sequences to produce an interesting dual for the Fibonacci sequence, involving an analogous rule $T_n \times T_{n+1} = T_{n+2}$ for producing consecutive terms. Just like the Fibonacci sequence, the limit of the quotients of the logarithms of the adjacent terms of this dual sequence is also the golden ratio, i.e.: $\lim_{n \to \infty} \frac{\log T_{n+1}}{\log T_n} = \phi$."

Many authors have endorsed the "use of analogies in mathematics," which is quite succinctly nicely expressed by this limerick of Andre Weil (n.d) as cited in Sriraman (2005, p. 508):

> "As every mathematician knows,
> Nothing is more fruitful than these obscure analogies,
> These indistinct reflections of one theory into another
> These furtive caresses
> These inexplicable disagreements;
> Also nothing gives the Researcher greater pleasure…"

Furthermore, Alwyn & Dindyal (2009, p. 1) asserts that: "The exercise of analogical reasoning is not merely the memorizing of previously solved problems. It is definitely not rote learning. It involves directed, purposeful mathematical thinking in determining the structural similarities and relational properties between the source and the target problems."

For example, Archimedes' discovery of the integral calculus was made possible by constructing appropriate analogies at a relational level and also knowing how and when to use them to generate new ideas or make new findings. In actual fact, he found the area of the enclosed by a straight line and parabolic segment, and the volume of the sphere by making analogies with the equilibrium concept from mechanics (Polya, 1954a, p. 155).

Arguably, reasoning through identifying similarities from experiences is by no means low quality reasoning, but it should be regarded as powerful reasoning that can/might provide opportunities to lead to new knowledge (Crystal-Alberta, 2010; Lithner, 2003). More importantly, the ability and know-how to spot and select similarities, and thereby group or classify them is of significant importance and pre-requisite for the learning of mathematics, not forgetting that analogical reasoning serves many purposes, inter-alia explanations, discovery, problem solving as well as algebraic abstraction.

Although, the use of analogy can allow students to observe and use commonalities between different mathematical concepts, algorithms, representations, theorems, classifications, systems, et cetera, and thereby produce (or reproduce) new discoveries and conjectures, teachers should ensure that learners do not ignore or disregard differences that may exist amongst the objects under study (de Villiers, 2009). Further to this, one should refrain from using false analogies, such as referring to 'apples' and pears to prevent learners from conjoining 'unlike' terms such as $x + y$. This is a false analogy as algebraic symbols represent numbers and never represent objects, and its use can lead to serious, persistent miconceptions.

### 2.1.3.3 Some Challenges of Using Analogies

According to English (1998), quite often novice problem solvers, when trying to establish a similarity between the source and the target, erroneously focus more attention on superficial characteristics rather than the critical underlying relational structural properties, which in many instances lead to the misrepresentation of claimed analogies. This misinterpretation invariably leads to students making errors, more so in cases where students do not have the prerequisite knowledge and skills related to the specific mathematical discourse or where the analogies are weak (i.e. the levels of similarity are low when compared to the high level of dissimilarity) (see Crystal-Alberta, 2010). Furthermore, the inappropriate transfer of meaning from the source of the target, which is quite prevalent in instances of over-generalization, can also contribute to the making of significant mathematical errors (see Crystal-Alberta, 2010).

In particular, Alwyn & Dindyal (2009, p. 4), in their study of "analogical reasoning errors in mathematics at junior college level", finds students making errors dubbed misuse of the distributive property. For example, as illustrated in Figure 2.1.3.1, the error in the third line could be attributed to invalid analogical reasoning, wherein the student "erroneously applied the distributive property onto an exponential function and made the error of equating $(x^2 - 2)^{1/2\,x}$ to $(x^2)^{1/2x} - (2)^{1/2x}$". Furthermore, the error in line 5, also demonstrates another type of "analogical reasoning error", where the student "applied the procedure of differentiating a polynomial to an exponential function". Alwyn & Dindyal (2009, p. 4) with reference to Lithner (2003) makes the following assertions: "In mimicking the procedure, the student had transferred the solution strategy by identifying similarities. However it was unfortunate that the student had completely focused on the superficial likeness and had not taken the structural differences between the two forms of mathematical expressions. In using analogical reasoning, students need to purposefully direct their mathematical thinking to the structural similarities and relational properties between the source and target systems."

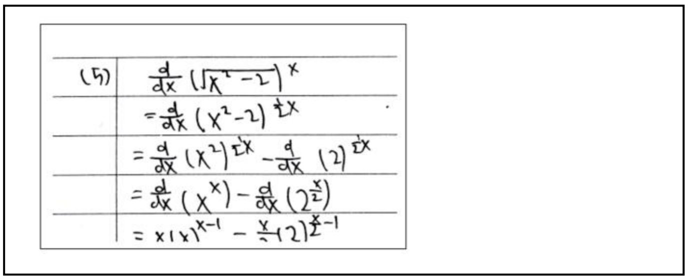


Figure 2.1.3.1: Errors in analogical reasoning (Alwyn  & Dindyal, 2009, p. 7)

## 2.2 Generalization within broader context of research and teaching

> "During generalization what occurs is, on the one hand, a search for a certain invariant in an assortment of objects and their properties, and a designation of that invariant  by a word, and, on the other hand, the use of the variant that has been singled out to identify objects in a given assortment" (Davydov, 1990, p. 5).

The conceptualization mentioned above views generalization as a process and a product that permeates the field of research on generalization to some extent across schools, teacher education contexts, and mathematics classrooms at higher education institutions.  Although generalization is considered to be both "an object and a means of thinking and communicating" (Dörfler , 1991, p. 63), a review of research on generalizations has shown that various kinds of generalizations exist within the domain of mathematics and mathematics education.  For example, Carraher, Martinez, & Schliemann (2007), Dörfler (1991) and Davydov (1990) distinguish between empirical and theoretical generalizations. For example, according to Dörfler (1991) the development of empirical generalizations entails the examination of a set of specific cases (or examples) and then distilling commonalities, regularities or qualities that exist amongst the given cases. However, Dörfler (1991) asserts that during such episodes of  empirical generalizing there is an over-reliance on particular examples, and a lack of a particular goal which could serve as a guiding light to inform one up front as to the  kind of qualities (features) that s/he should focus/concentrate upon when trying to make a generalization. As a result the chance of generalizing or extending such a generalization (i.e. an empirical generalization) further is compromised.

In keeping with the views stated earlier, Davydov (1990) points out that some of the difficulties that students experience with instructional material in the classrooms can be attributed to their largely empirical approach to constructing generalizations.  In contrast to empirical generalizations, Zakis & Liljedahl (2002, p. 379) citing Dörfler (1991) says theoretical generalization is:

> "both intentional and extentional…starts with… 'a system of action' in which essential invariants are identified and subscribed for by prototypes… is constructed through abstraction of the essential invariants …abstracted qualities are relations among objects rather than object themselves."

Carraher et al. (2007, p. 18) says: "Empirical generalizations are thought to arise from an examination of the data for underlying trends and structure. Theoretical generalization is thought to spring from the ascription of models to data." In a mathematical sense, for example, this means that the mere examination of a finite set of ordered pairs in a table form does not necessarily warrant the construction of a general formula based function for any infinite domain such as counting numbers, real numbers or complex numbers. This in essence, means that such a formula based function can only be expressed via a statement that explicitly captures its generality (Carraher et al., 2007, p. 18). Furthermore, in emphasizing the conception of theoretical generalizations, Kondakov (1954, p. 457), Strogovich (1946, p.91) and Chelpanov (1946, p. 91) as cited in Davydov (1990, p. 177) respectively say:

- "Generalization is the mental delineation of certain general properties belonging to a whole class of objects and the formulation of a conclusion that extends to every particular object in the given class"
- "Generalization is the mental transition from the attributes of particular, individual objects to attributes belonging to whole group of these objects"
- "The term generalization often designates, not just the process of singling out common properties, but its result as well, which is contained in the general concept."

Harel & Tall (1991, p. 38) sees generalization as a process that enables one to apply an established argument to a more extensive range of situations or broader context. From a cognitive perspective they posit the existence of the 3 different kinds of generalizations within a mathematical context as follows:

- "Expansive generalization occurs when the subject expands the applicability range of an existing schema without reconstructing it"
- "Reconstructive generalization occurs when the subject reconstructs an existing schema in order to widen its applicability range"

- “Disjunctive generalization occurs when, on moving from a familiar context to a new one, the subject constructs a new, disjoint, schema to deal with the new context and adds it to the array of schemas available” (Harel & Tall, 1991, p. 38).

Radford (2003) categorizes the kinds of generalizations students produced into 3 types, namely: factual, contextual and symbolic. Generalizing via numerical action typically constitutes factual generalization, and generalizing the objects that underpin such actions is characterized as contextual generalization. The understanding and use of algebraic language characterizes symbolic generalization. Furthermore, Radford (2006) differentiates between arithmetic generalizations and algebraic generalizations in the context of the elements of the generalizing process. The three elements of the generalizing process as postulated by Radford are as follows:

- The first one is: “noticing a commonality in some given particular terms”
- The second one is: “to form a general concept – a genus – by generalizing the noticed commonality to all terms of the sequence”
- The third one is to understand: “that the genus or generalized object crystallizes itself into a schema, i.e. a rule providing one with an expression of whatever term of the sequence” (p. 15).

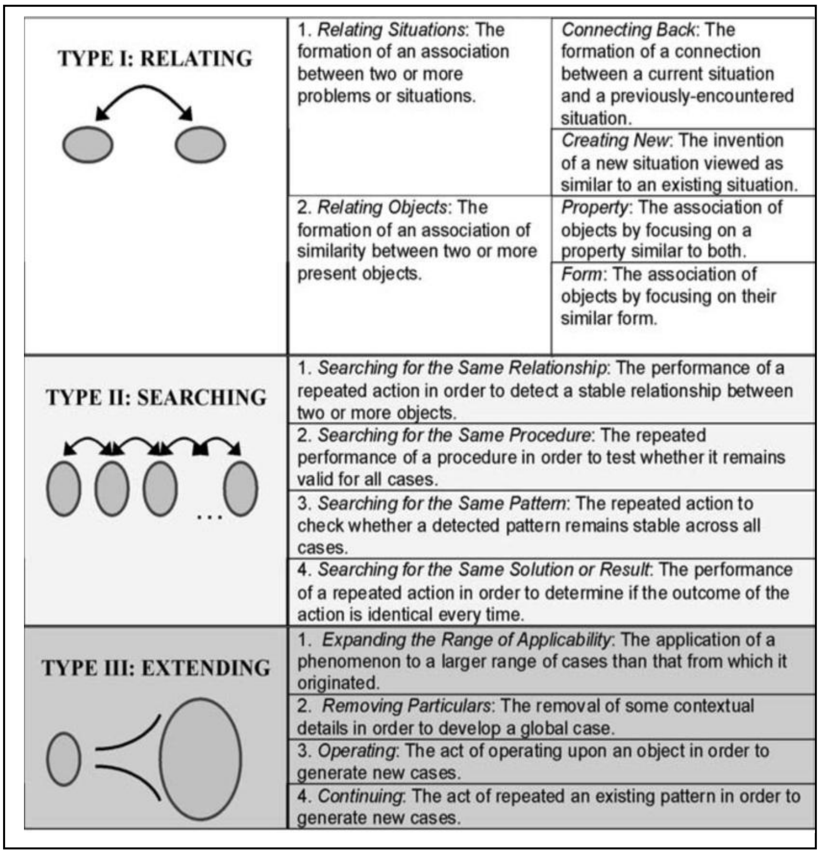
Following this line of thought, Radford (2006, p. 15) asserts that when the first two elements of the generalizing process have to be completed to perform an arithmetic generalization, and only when all three elements of the generalizing process are enacted upon and completed that an algebraic generalization can come into being.

Furthermore, through a qualitative study investigating the generalization strategies twenty-two grade nine learners exhibited whilst doing a task embracing linear patterns, Becker & Rivera (2005, p. 121) identified twenty three assortment of strategies that students used. On carefully analysing the strategies that students used, as well as their inner understanding of the use of variables, representational fluency, they found that students exhibited three types of generalizations based on similarity, namely: numerical, figural, and pragmatic. These findings were consistent with Gentners’ (1989) results, which showed that learners used different similarity strategies to construct generalizations (inductions) using everyday objects. In particular, Becker & Rivera (2005, p. 128) found students who used the numerical strategy to generalize employed trial and error as a similarity strategy but with no real sense of what the coefficients in any of the linear patterns meant. The variables were used as placeholders devoid of meaning and lacked representational fluency as and when used to generate linear sequences of numbers. Those students who used the figural generalization kind of strategy, employed perceptual similarity strategies, wherein the core focus was on the relationships amongst the set of numbers in the linear sequence under consideration. In such instances the variables were not seen as just placeholders but rather as ‘contributors’ to the development of meaningful functional relationships. On the other hand those students who characteristically employed a pragmatic generalization strategy, employed both numerical and figural kinds of strategies, with a high degree of representational fluency.

Becker & Rivera (2005) asserts that students who exhibit figural generalization do move on to eventually become pragmatic generalizers. Moreover, they found that those students (called disjunctive generalizers) who were not able to generalize, appeared to make some start by using numerical strategies, but in principle lacked the intuitive sense to try other possible approaches and see plausible connections between various representation forms and strategies for making generalizations. Likewise, Lanin (2005) finds that students who resorted to using geometric schemes to see a particular rule via visual representation were more successful in the provisioning of general arguments and developing generalizations as compared to students that employed a numerical scheme or the ‘guess and check’ kind of strategy. In retrospect, Rivera & Becker’s (2003) analysis of the induction processes that forty-two pre-service teachers invoked to construct their sets of generalizations, suggests that: “even if relationships among numerical values have had a greater contribution to similarity than did figural ones, those who induced figurally acquired a better understanding of the generalizations they constructed” (p. 63).

As evidenced in the research discussed so far, it appears that in many of our classrooms, particularly at school level, inductive generalization becomes the most common form (or if not the only type) of generalization that teachers indulge their learners in. Ellis (2007b) adds that at school level the development of generalizations is frequently pioneered through the examination of specific cases via patterning tasks. Similarly, many researchers have focused their classroom research primarily around inductive generalizations, but mainly within the context of algebra largely at school level. In such instance the generalization research focused mainly on the development of mathematical rule or property (Ellis, 2007a citing Carpenter & Franke, 2001; English & Waren, 1995; Lee, 1996).

A number of studies (compare Garcia- Criz & Marinin, 1997; Lannin, 2003; Stacey, 1989 as cited in Ellis, 2007a) have suggested that being aware of the types of generalizations that students can generate in mathematics can help to provide a much broader understanding as to the various ways in which students could “construct general rules to fit particular cases or data” (Ellis, 2007a, p. 222). Hence, via an empirical study that included teaching sessions and interviews, Ellis (2007a) proceeded to distill the different ways in which seventh graders and eighth graders constructed generalizations and also the types of mathematical generalizations they constructed. In so doing, Ellis (2007a, p. 233-234) defines ‘generalizing actions’ as the moves that students make as they generalize, and ‘reflection generalizations’ as the “final statements of generalization”. Ellis (2007a, p. 234) argues that the students generalizing actions can be classified into the three broad categories, namely: “relating, searching and extending”. Figure 2.2.1 as cited in Ellis (2007a, p. 235), provides the distinguishing attributes characterizing each of the categories of generalizing actions, which can be used to track the kind of mental activities that students engage in during the process of generalization.



| | | |
|---|---|---|
| TYPE I: RELATING | 1. *Relating Situations*: The formation of an association between two or more problems or situations. | *Connecting Back*: The formation of a connection between a current situation and a previously-encountered situation. |
| | | *Creating New*: The invention of a new situation viewed as similar to an existing situation. |
| | 2. *Relating Objects*: The formation of an association of similarity between two or more present objects. | *Property*: The association of objects by focusing on a property similar to both. |
| | | *Form*: The association of objects by focusing on their similar form. |
| TYPE II: SEARCHING | 1. *Searching for the Same Relationship*: The performance of a repeated action in order to detect a stable relationship between two or more objects. | |
| | 2. *Searching for the Same Procedure*: The repeated performance of a procedure in order to test whether it remains valid for all cases. | |
| | 3. *Searching for the Same Pattern*: The repeated action to check whether a detected pattern remains stable across all cases. | |
| | 4. *Searching for the Same Solution or Result*: The performance of a repeated action in order to determine if the outcome of the action is identical every time. | |
| TYPE III: EXTENDING | 1. *Expanding the Range of Applicability*: The application of a phenomenon to a larger range of cases than that from which it originated. | |
| | 2. *Removing Particulars*: The removal of some contextual details in order to develop a global case. | |
| | 3. *Operating*: The act of operating upon an object in order to generate new cases. | |
| | 4. *Continuing*: The act of repeated an existing pattern in order to generate new cases. | |

Figure 2.2.1: Generalizing Actions (as cited in Ellis, 2007a, p.235)

As elaborated in Figure 2.2.1, the idea of 'relating' as explained by Ellis (2007), underpins the notion of seeing some 'sort of similarity' (i.e. analogy) between two or more cases or situations as expressed by Polya (1954a), although s/he may not necessarily be able to elaborate as to how the cases are connected at that given moment. During the searching action, the student primarily repeats the same kind of procedure or action to try and see if some degree of similarity exists between the considered cases or situations (Ellis, 2007a, 2007b, 2007c). The generalizing-extending action is basically characterized by the move of a student to expand an observed pattern or a similarity relationship into a more general structure (Ellis, 2007a, 2007b, 2007c).

According to Ellis (2007a, p. 244) reflection generalizations are highly linked to the generalization action of students and represent either: "a verbal statement, a written statement, or the use of the result of a generalization." Figure 2.2.2, provides an overview of the categories of reflection generalizations that Ellis (2007a) found to be prevalent amongst the middle school learners that participated in her research study. The 'influence' category of refection generalization resonates much with Piaget's notion of adaptation (see Sections 4.2 & 4.6) and Ausubel's Theory of Meaningful learning as discussed in Section 4.5.

| | | |
|---|---|---|
| **TYPE IV: IDENTIFICATION OR STATEMENT** | 1. *Continuing Phenomenon*: The identification of a dynamic property extending beyond a specific instance. | |
| | 2. *Sameness*: Statement of commonality or similarity. | *Common Property*: The identification of the property common to objects or situations. |
| | | *Objects or Representations*: The identification of objects as similar or identical. |
| | | *Situations*: The identification of situations as similar or identical. |
| | 3. *General Principle*: A statement of a general phenomenon. | *Rule*: The description of a general formula or fact. |
| | | *Pattern*: The identification of a general pattern. |
| | | *Strategy or Procedure*: The description of a method extending beyond a specific case. |
| | | *Global Rule*: The statement of the meaning of an object or idea. |
| **TYPE V: DEFINITION** | 1. *Class of Objects*: The definition of a class of objects all satisfying a given relationship, pattern, or other phenomenon. | |
| **TYPE VI: INFLUENCE** | 1. *Prior Idea or Strategy*: The implementation of a previously-developed generalization. | |
| | 2. *Modified Idea or Strategy*: The adaptation of an existing generalization to apply to a new problem or situation. | |

Figure 2.2.2: Reflection Generalizations (as cited in Ellis, 2007a, p. 245)

As Ellis (2007a) argues that 'extending' was one of the ways that the middle school learners in her study generalized, one should bear in mind that in a mathematical context, generalization does not only accrue via inductive reasoning, but rather through other ways such as analogical reasoning and deductive reasoning (compare Yerushalmy, 1993; De Villiers, 2003b). Therefore as classroom practitioners, we should take into consideration these plausible ways (or processes) that could be used by a student to develop a generalization, and also come to understand that generalizations are not just assimilated via inductive reasoning only, but rather through looking back and reflecting on existing generalizations and asking ourselves how can we generalize the existing generalization further or across to other contexts or extend a previously established generalization further. However, in such instances one's intuitive sense and ability to reason inductively, analogically and/or deductively is crucial to the process of generalization and the construction of a further generalization (or the extension of an established generalization to other domains). For example, through looking back and reflecting on a previous generalization or result, and by analyzing the conditions governing the logical explanation for such a generalization from a deductive perspective, one can realize that a specific condition is sufficient but not necessary, and hence proceed to argue for a further generalization (de Villiers, 1996, p. 86). In addition, one could extend a particular established generalization (or theorem) further on inductive and analogical grounds as well. For example, De Villiers (1996) illustrated how the Varignon theorem can be generalized further through using a deductive approach, an analogical approach and also an inductive approach. In light of this, the discussion in the next few paragraphs focuses on the generalizing of Varignon's theorem within this context.

Firstly, Varignon's theorem as illustrated in Figure 2.2.3 states that: "If the midpoints *E, F, G* and *H* of the adjacent sides of any quadrilateral *ABCD* are consecutively connected, then *EFGH* is a parallelogram" (see De Villiers, 1996, p. 76). The deductive explanation that supported the generality of the aforementioned theorem was constructed along the following lines: In Figure 2.2.3, diagonals AC and BD were drawn. Then by using the midpoint theorem, which states that "the line joining the midpoints of two sides of a triangle is equal to half the third side and is parallel to the third side", it was shown via deductive argument that $EH//= FG$ (or $EF//= HG$) , and hence $EFGH$ is a parallelogram.

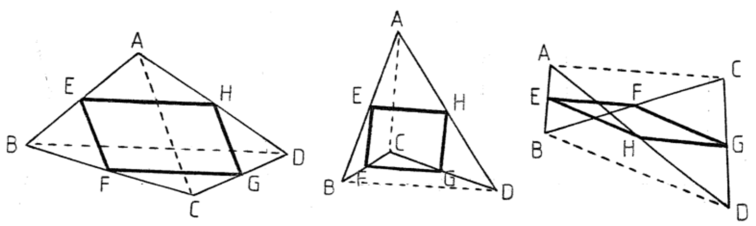



Figure 2.2.3: Varignon Theorem

In an attempt to enquire as to whether the Varignon result can analogously be extended to hexagons, De Villiers (1976, p. 78) found that when the midpoints of the adjacent sides of a regular hexagon are consecutively connected, another regular hexagon is produced as shown in Figure 2.2.4.1. In particular a hexagon with opposite sides parallel and equal is produced, i.e. a parallelo-hexagon is formed. However, De Villiers (1996, p. 77) found through experimentation in a dynamic geometry context, that when the midpoints of a non-regular hexagon are consecutively joined (see Figure 2.2.4.2), the resultant hexagon does not have opposite sides that are both parallel and equal. On the basis of the latter result, De Villiers realized that the joining of the consecutive midpoints of any hexagon does not always produce a parallelo-hexagon, i.e. it is not true in general that such a construction will produce a parallelo-hexagon. This then prompted De Villiers to ask himself the question: "under which conditions would we find a hexagon with opposite sides parallel and equal" (1996, p. 77).

Through looking back and reflecting on the logical explanation (i.e. deductive proof) that was used to provide insight as to why a parallelogram is produced when the consecutive points of any quadrilateral are joined, and also considering the characteristic property upon which the explanation depends, and analysing the properties of Figure 2.2.4.2, De Villiers (1996, p. 77) immediately fathomed out that $GH$ would be parallel and equal to the opposite side $JK$ if $AC//= DF$, and likewise realized that $BD//= EA \Rightarrow HI//= KL$ and $CE//= FB \Rightarrow IJ//= LG$. This means that for any hexagon possessing the afore-described properties, one could inscribe a parallelo-hexagon by consecutively connecting the midpoints of the sides of such an hexagon.

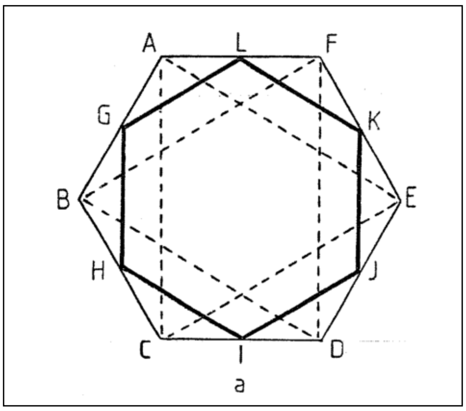


Figure 2.2.4.1: Parallelo-hexagon

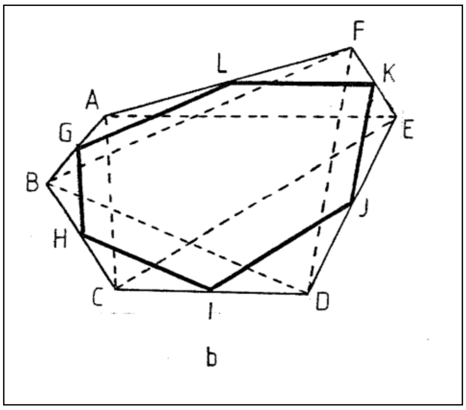


Figure 2.2.4.2: Non-parallelo-hexagon

Furthermore, De Villiers (1976, p. 77) was able to see that a hexagon $ABCDEF$ having diagonals $AC//=DF$, $BD//=EA$ and $CE//=FB$ , would indeed be parallelo-hexagon. This can be easily explained as follows: $AC//=DF \Rightarrow ACDF\ is\ a\ parm \Rightarrow AF//=CD$. Similarly, it can be shown that $BA//=ED$ and $BC//=FE$.

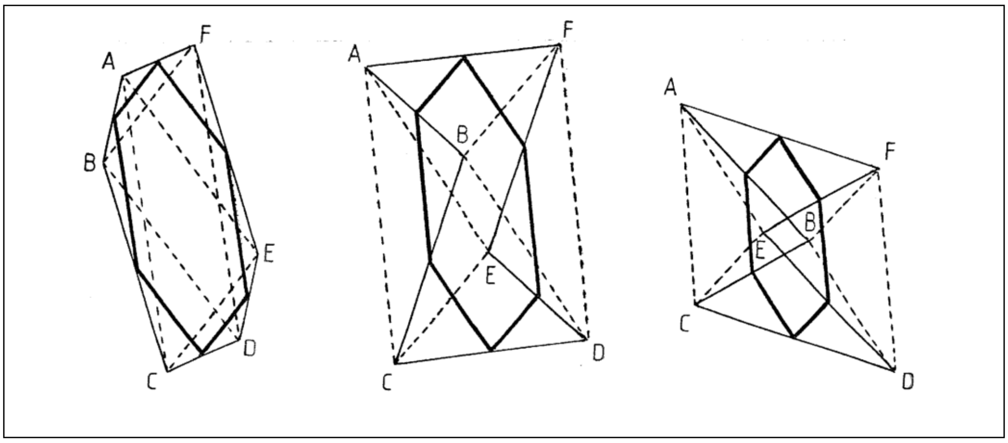


Figure 2.2.5: Examples of inscribed parallelo-hexagons

Figure 2.2.5, illustrates three examples of parallelo-hexagons wherein the midpoints have been consecutively join to also produce inscribed parallelo-hexagons. De Villiers (1996, p. 77) also pointed out that the afore-illustrated parallelo-hexagons $ABCDEF$ can alternately be drawn by constructing two congruent triangles $ACE$ and $DFB$ with $AC//DF$, $AE//DB$, and $CE//FB$, and then connecting $A, B, C, D, E, F$.

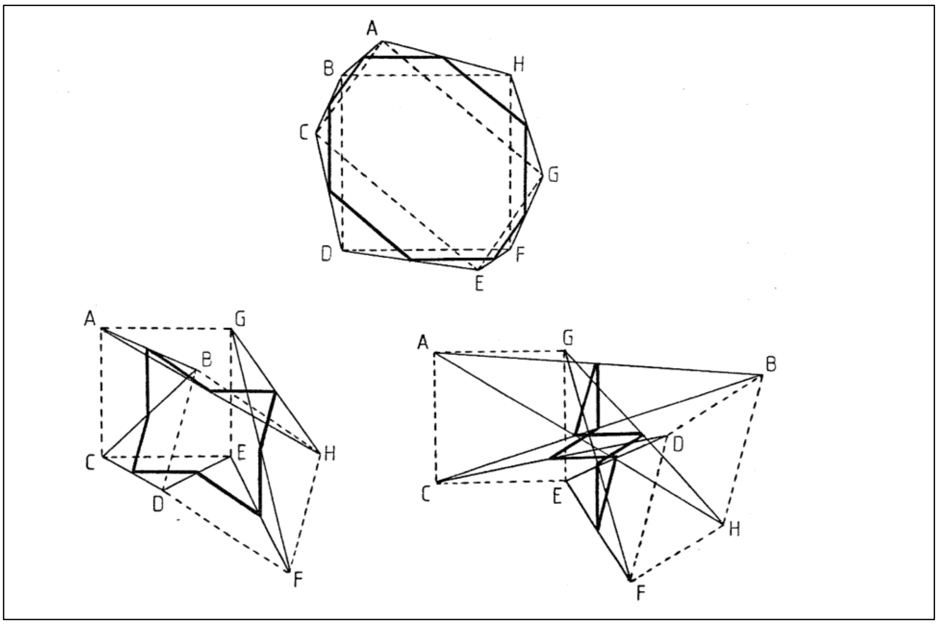


Figure 2.2.6: Inscribed parallello-octagons

Through using the same kind of reasoning postulated for the hexagon case, the result can easily be extended to an octagon $ABCDEFGH$ having $AC//=EG$, $BD//=FH$ , $CE//=GA$

and $DF//= HB$. If the midpoints of the sides of such octagons are connected then inscribed paralello-octagons are formed (see Figure 2.2.6). However, the octagons $ABCDEFGH$ themselves are not necessarily parallel-octagons. Practically such octagons can be drawn by merely constructing parallelograms $ACEG$ and $BDFH$ and then joining $A, B, C, D, E, F, G, H$.

Taking into consideration the cases as discussed, De Villiers (1996, p. 78) constructed a generalization (called Generalization 1) as shown here:

> "$If\ A_1A_2 \ldots A_{2n}(n > 1)\ is\ any$ 2*n*-*gon* $with\ A_iA_{i+2}// = A_{i+n}A_{i+n+2}\ i\ (1; 2; \ldots n)$ $and\ B_j\ are\ the\ midpoints\ A_jA_{j+2}\ (j = 1; 2; \ldots 2n), then\ B_1B_2 \ldots B_n\ is\ a$ $parallelo$-$2n$-gon."

After justifying Generalization 1 through the construction of a logical explanation,
De Villiers (1976, p. 79) argued that the result is further generalizable to $2n$-gons with the property $A_iA_{i+2}//A_{i+n}A_{i+n+2}$ or $A_iA_{i+n+2} = A_{i+n}A_{i+n+2}$ for which the respective $2n$-gons will have $B_jB_{j+1}//B_{j+n}B_{j+n+1}$ or $B_jB_{j+1} = B_{j+n}B_{j+n+1}$.

On reflecting on the original result, De Villiers (1996, p. 79) proceeded to examine as to whether it is necessary for $B_1, B_2, B_3$ and $B_4$ to be midpoints (see Figure 2.2.7).

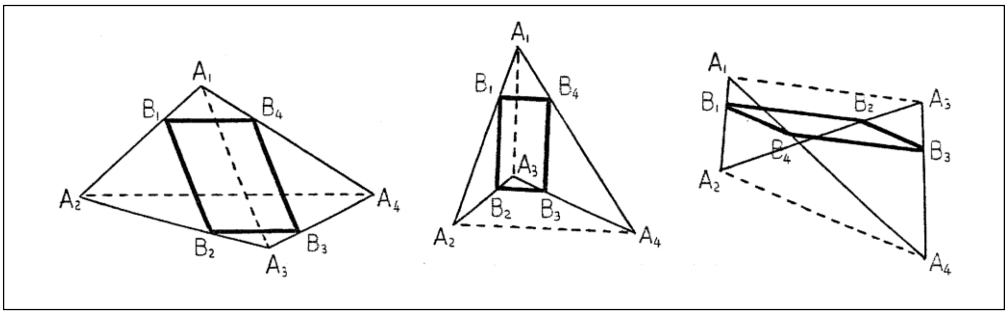


Figure 2.2.7: Second Generalization- Varignon Theorem

Thus, through taking into consideration that the result depended on $B_1B_2//A_1A_3//B_3A_4$ and $B_1B_4//A_2A_4//B_2A_3$, De Villiers (1996, p. 79) proceeded to investigate how the aforementioned relationships can be maintained if $B_1, B_2, B_3$ and $B_4$ are not the midpoints of the sides. In so doing, De Villiers extended his investigation to another particular case, namely parallel-hexagon $A_1A_2A_3A_4A_5A_6$ as shown in Figure 2.2.8, and subsequently formulated Generalization 2 as follows:

“$If\ A_1A_2 \dots A_{2n}(n > 1)$ $is\ any$ 2n-gon $with\ A_iA_{i+2}// = A_{i+n}A_{i+n+2}\ i\ (1; 2; \dots n)$ and $B_j$ $are\ the\ midpoints\ A_jA_{j+2}\ (j = 1; 2; \dots 2n), so\ that\ for$

$k = 1; 3; 5; \dots 2n - 1$: $\frac{A_kB_k}{B_kA_{k+1}} = \frac{A_{k+2}B_{k+1}}{B_{k+1}A_{k+1}} = \frac{p}{q}$,

$then\ B_1B_2 \dots B_n\ is\ a\ parallel$-$2n$-$gon\ (opposite\ sides\ parallel)$.”

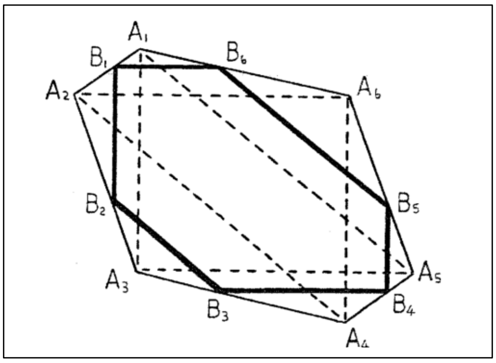



Figure 2.2.8: Parallelo-hexagon

Furthermore, through observing some particular cases as illustrated in Figure 2.2.9, de Villiers (1996, p.83) constructed the following converses to respectively Generalization 1 and 2 on inductive grounds:

**Converse to Generalization 1:**

“$If\ A_1A_2 \dots A_{2n}(n > 1)$ $is\ any$ 2n-gon $with\ A_iA_{i+2}// = A_{i+n}A_{i+n+2}\ (i\ 1; 2; \dots n)$ $and\ B_jB_{j+1}\ is\ drawn\ parallel\ to\ A_jA_{j+2}\ (j = 1; 2; \dots 2n)\ starting\ from\ the$ $midpoint\ B_1\ of\ A_jA_{j+1}$, $then\ a\ closed\ parallelo$-2n-gon $B_1B_2 \dots B_n\ is\ formed.$”

**Converse to Generalization 2:**

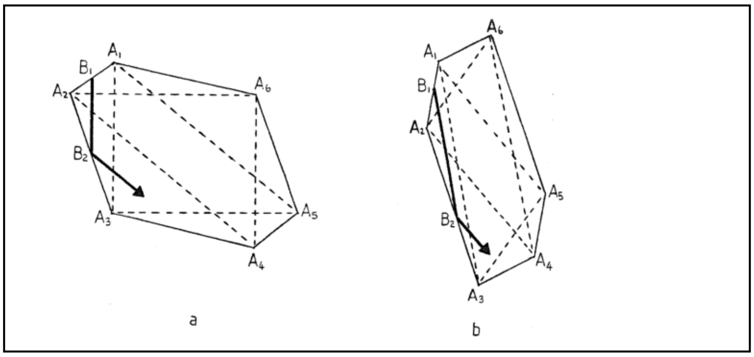



Figure 2.2.9: Hexagons – linked to converses

“$If\ A_1A_2 \dots A_{2n}(n > 1)$ $is\ any$ 2n-gon $with\ A_iA_{i+2}//\ A_{i+n}A_{i+n+2}\ (i\ 1; 2; \dots n)$

$and\ B_jB_{j+1}\ is\ drawn\ parallel\ to\ A_jA_{j+2}\ (j = 1; 2; \dots 2n)\ starting\ from\ any$ $midpoint\ B_1\ on\ a\ sided\ A_jA_{j+1}$,$then\ a\ closed\ parallel$-2n-gon $B_1B_2 \dots B_n$ $is\ formed.$"

However, De Villiers (1996, p. 94) on looking back and reflecting on the proofs that had been constructed to logically justify the converse statements (or generalizations), saw that he did not utilize the property $A_iA_{i+2}//A_{i+n}A_{i+n+2}$ at any stage in the construction of his proof. Hence, on deductive grounds he proceeded to construct the following generalization (which is called Generalization 5 in this section):

"$If\ A_1A_2 \dots A_{2n}(n > 1)\ is\ any$ 2n-gon $and\ B_jB_{j+1}\ is\ drawn\ parallel\ to$ $A_jA_{j+2}\ (j = 1; 2; \dots 2n)\ starting\ from\ any\ \ midpoint\ B_1\ on\ a\ sided\ A_jA_{j+1}$ $midpoint\ B_1\ on\ a\ sided\ A_jA_{j+1}$ ,$then\ a\ closed$ 2n-gon $B_1B_2 \dots B_n\ \ is\ formed.$"

The afore-described deductive generalization is illustrated for a hexagon as contained in Figure 2.2.10

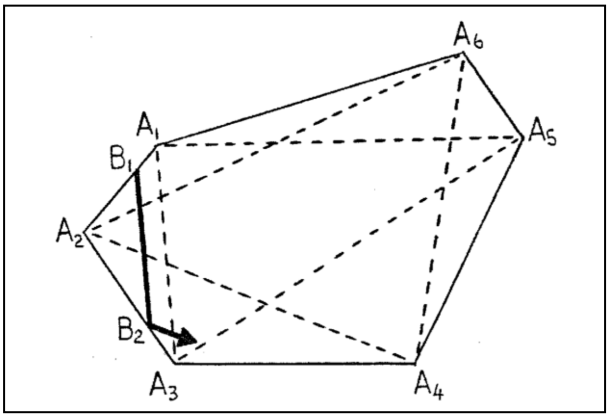



Figure 2.2.10: Hexagon – linked to Generalization 5

Moreover, through examining some specific cases like the triangle and pentagon as illustrated in Figure 2.2.11, De Villiers attempted to establish what happens when $B_jB_{j+1} = A_jA_{j+2}$. Consequently, on analogical grounds De Villiers produced the following generalization (which is called Generalization 6 in this section):

"$If\ A_1A_2 \dots A_{2n}(n > 1)\ is\ any\ \ (2n-1)$-$gon\ and\ B_jB_{j+1}\ is\ drawn\ parallel\ to$ $A_jA_{j+2}\ (j = 1; 2; \dots 4n-2)\ starting\ from\ any\ \ midpoint\ B_1\ on\ a\ sided\ A_jA_{j+2}$

$midpoint\ B_1\ on\ a\ sided\ A_jA_{j+1}$ ,$then\ a\ closed\ (4n-2)$-$gon\ B_1B_2 \ldots B_{4n-2}$ $is\ formed.$"

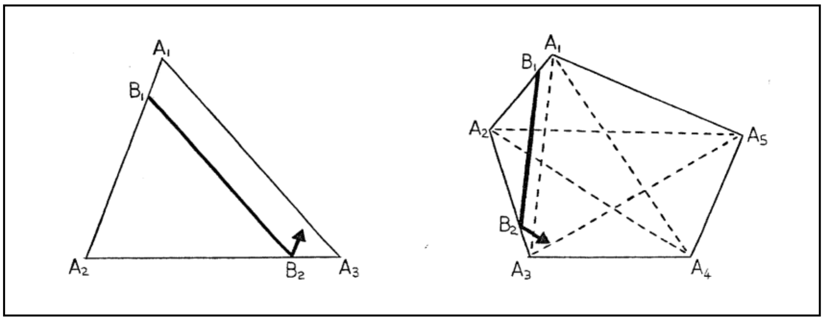



Figure 2.2.11: Triangle & pentagon cases

De Villiers (1996) has illustrated how the Varignon theorem can be generalized further in different ways, namely: deductively, analogically and inductively (also see De Villiers, 2009). However, on the contrary these ways of generalizing (particularly analogically and deductively) are not focused upon adequately in many of our pre-service mathematics teacher education classrooms. Further to this, research looking at how pre-service mathematics teachers further generalize a particular generalization or theorem across other domains seems to also be lacking as per my literature review.

## 2.3 Some research related to empirical and deductive justification (proof) at school level

In recent years, within the field of mathematics education there has been an upsurge of interest focused on developing reasoning, justifying and proving skills with a view to promoting students' understanding of mathematics. Hence, curriculum policies like C2005 (see Department of Education, 2002a, 2002b, 2003a, 2003b), the National Curriculum for England and Wales (Department for Education, 1995), and the Principles and Standards for School Mathematics (PSSM) (see NCTM, 2000), governing the teaching and learning of mathematics have been reconstructed to resonate with the aforementioned foci and views. Consequently, curriculum materials have been developed to promote and support such an endeavour. For example, within the Connected Mathematics Project (CMP) curriculum there was a pool of rich -proof tasks developed for utilization in the middle school mathematics classrooms. However, when Bieda (2010) conducted a study to investigate the enactment of students' justifying and proving competencies amongst middle school learners, results showed that the experiences of the students with the CMP tasks, which provided opportunities for encouraging the construction of conjectures, generalizations and proofs, were “insufficient for developing and understanding what constitutes valid mathematical justification” (p. 351).

In particular, Bieda’s (2010) study showed that 52 of 73 proof related problems that were implemented as per intentions outlined in the prepared CMP curriculum texts, resulted in the production of 109 conjectures (generalizations) that created the opportunity to prove. However, 59 of 109 conjectures (generalizations) were justified by the students through the use of empirical or non-proof arguments whilst the remaining 50 (46%) conjectures (generalizations) were not justified in any particular way by the students. With regard to the category of generalizations which students did not support with any justifications, the teachers of these students responded in one of the following ways:

a. Teachers gave no feedback to elicit a justification,
b. Teachers sanctioned the conjecture as valid without justification, or
c. Teachers asked other students to state whether or not they agreed with a student’s conjecture (Bieda, 2010, p. 365)

In performing an analysis of the types of justifications provided for the 59 conjectures, it was found that in 28 cases the justifications were in the form of a general argument, while in 31 cases (roughly half) the justifications were in the form of non-proof arguments. In this instance, where justifications were provided, teachers did not provide feedback to 17 justifications (10 general arguments and 7 arguments were founded solely upon confirming examples). However, the remaining 42 justifications received feedback from the teachers in one of the following forms:

a. Sanctioning of the justification (positive appraisal);
b. Questioning related to justification; or
c. Requesting of feedback from other students.

Despite the effort by the teachers to provide feedback to students’ justifications, the nature and lack of appropriate feedback did not encourage and sustain ensuing discussions pertaining to the validity of arguments posited by students, and thus resulted in students not changing their conceptions. Moreover, the findings of the research indicate that the kind of feedback provided by the teachers in a large number of instances was not equivalent in terms of its status to the kind of deductive proof that one would expect to see in a mathematics classroom. In particular, despite the teachers involved in this study having undergone special training with regard to the enactment of proof-related tasks in their classrooms according to the CPM materials, it was found that "teachers were just as likely to sanction a justification with a positive appraisal if it was a justification based on non-proof arguments as justification based on general arguments" Bieda (2010, p. 317). The aforementioned finding is not surprising, as it has likewise been previously reported in the literature (like Chazan ,1993; Hoyles and Jones (1998); Koedinger 1998, etc.) that students quite often present empirically based justifications as their proofs and not general argument as such. For example, with regard to a nation-wide study in the United Kingdom focusing on justifying and proving in school mathematics, Healy and Hoyles (1998) as cited in (Battista, 2009, p. 103), finds that students who performed extremely well in mathematics (as measured by tests and exams), did not exhibit the desired performance in terms of proof construction and in most instances resorted to the use of empirical verification as a kind of justification.

One plausible reason for the aforementioned state of affairs is the manner in which proofs are dealt with at both school and tertiary level. More often than not it is presented as a "formal confirmation of statements that pupils [students] are told are true" (Hoyles & Jones, 1998, p. 121). Such an approach with virtually no room for class discussion, debate, justification and critique causes one to see the construction of a proof as nothing more than "a standardized linear deductive presentation of an argument, in which form is often perceived as more important that content" (Hoyles & Jones, 1998, p. 121). This kind of selfish and stifling direct approach results in students experiencing conceptual difficulties as to what proof really is. For example, Hoyles & Jones (1998, p. 121) reported that students tend not to see and understand the conceptual difference between deductive and empirical arguments, and quite often prefer to use an empirical argument to justify their claim or conjecture; students see deductive proof to be just 'more' evidence that authenticates the validity of a claim; students do not use proof as part of a problem solving activity, neither do they understand its purpose in most instances, and quite often just see it as an irrelevant –add on kind of exercise/activity.

With regard to the latter point, Hershkowitz, Dreyfus, Ben-Zvi, Fiedlander, Hadas, Resnick, Tabach, & Schwarz (2002) as cited in Battista (2009, p. 103), likewise affirms that students in most instances do not really see and realize the purpose of proving. Furthermore, Battista & Clements (1995, p. 48; 2009, p. 103 ) assert that whilst mathematicians see proof as a meaningful method to establish the validity of an idea that results from their mathematical thought, students think otherwise and see it rather as a set of formal rules which have no real connections with actions and thoughts produced or experienced during a given mathematical activity.

The results of Fischbein & Kedem's (1982) study, which looked at high school students' perceptions of proof as a general argument that validates the truth of a mathematical statement, reveals that even after students constructed a valid proof for a mathematical statement or just learnt of a correct proof for a mathematical statement they were still quite adamant that surprises could be possible, and thus preferred to empirically test the conjecture further. In so doing, the students exhibited very little faith in the general argument articulated by a proof, but rather showed more faith in the generality of a posited conjecture if they could see that the conjecture could be empirically (via numerical calculations, measuring, etc.) upheld across more and more cases. All this suggests is that the students have not yet realized that a proof is in fact a general argument that asserts the truth of a conjecture for any arbitrary, and infinitely many more, cases.

The reliance on positive empirical examples by high school students to justify a conjecture pertaining to a kite-activity was also reported by Koedinger (1998, p. 327). In fact many students in the study were initially quite content by merely positing one example that they had drawn as sufficient evidence to justify their conjecture, whilst "a rare few showed any unprompted signs of thinking that further evidence was necessary or desirable" (Koedinger, 1998, p. 327). However, there was a case where a student seemed to have constructed a proof, but this was done more in respect to the adherence of traditional classroom habits rather than his/her inner personal motivated desire to validate the conjecture per se. Moreover, students were found to be in a state of confusion when requested to provide further evidence to justify their conjecture, and the majority of students were specifically told to provide a proof "'like you do in class'" prior to attempting to produce such a justification (Koedinger, 1998, p. 327). Furthermore, the study also showed that students developed false conjectures as a result of drawing over-specialized figures of a kite. For example, some students who drew a rhombus *ABCD*, a special kite, and then constructed diagonals *AC* and *BD* intersecting at *O*, made the following conjecture for all kites: $\Delta AOB \equiv \Delta AOD \equiv \Delta COD \equiv \Delta COB$.

Koedinger's (1998) study showed that students generally experienced a great deal of difficulty in setting up proofs for their formulated conjectures. Although about 10% of the students managed to work on their proofs, they struggled primarily because they did not realize that specific constructions, like the construction of diagonal *BD* to create triangles could be shown to be congruent. So most students in this category needed specific hints like: "'What methods do you know for proving congruent triangles?" and "Do you remember [Side-Side-Side]?" (Koedinger, 1998, p. 327). However, there were two students who managed to conjecture and prove that the diagonals of a kite are perpendicular with virtually very little or no prompting. Furthermore, whilst working towards a proof of the aforementioned conjecture, one of the students deduced a new conjecture, which she had not thought of previously. Although in most mathematics classrooms, "the discovery of a conjecture is usually the product of induction from examples, this student's work illustrates that proof can also serve as a discovery tool" (Koedinger, 1998, p.327). This finding is startling in the sense that our everyday mathematics educators address the production of conjectures in their classrooms through empirical investigations (induction) and reserve the use of deduction to justify the conjecture, but in this study we see that it is evidently possible to produce a conjecture through deduction.

Arising out of concerns that students treat empirical verification as deductive proof, Chazan (1993) interviewed seventeen high school students (nine urban and eight suburban), who employed empirical evidence in their geometry classes to investigate their understanding of what counts as proof. With regard to interviews pertaining to the nine urban students, the study showed that two students were of the view that by doing actual measurements across a set of examples and obtaining the desired regularity is enough to justify the validity of a conjecture for all cases, i.e., "measuring 'proves' for all members of a set with an infinite number of elements"; five did not hold the aforementioned view that the provision of "measuring examples" is a 'proof' in itself; one student was rather unclear, whilst another student seemed to have changed her view during the course of the interview to support the notion that empirical examples is not 'proof'" (Chazan 1993, p. 369). Regarding the interviews with the remaining eight suburban students, three students endorsed the notion that "measuring examples is a 'proof' for all members of a set with an infinite number of elements," with one of the three students expressing reservations as to whether the aforementioned kind of proof could rule out the possibility of finding a counter-example; three students initially disagreed with the view, but vacillated during the interview to finally endorse it; and two of the students explicitly did not support the view.

In Chazan's (1993) study, those that supported the notion that the provision of empirical examples exhibiting a desired regularity is 'proof' in itself, i.e. "that evidence is proof" (p. 369), focused on either the number of examples or the type of examples. For example, with regard to the latter focus, which pertained to the proposition related to triangles, one of the students responded as follows:

> "If I did it on a bunch of triangles, like different kinds of triangles, and then if I would find it to be all true then I'd just accept it….If you keep doing this like maybe ten more times and it just keeps on doing that, I'd just say it just would have to be that way" (Chazan, 1993, p. 369).

Furthermore, whilst there were students who believed that evidence (empirical) is proof, the study also found that there were some students who believed that "proof is just a kind of evidence" (Chazan, 1993, p. 371). The latter position held by some students can be ascribed to any of the following reasons (Chazan, 1993, p.372) :

1. Deductive proof provides no safety from counter-examples.
2. Deductive proofs are about single diagrams.
3. Deductive proofs are based on assumptions.

Notwithstanding what has been said, Battista (2009, p. 104) asserts that Chazan's findings could be attributed to either students not being fully conversant with the logic of proof or their prior experiences with incorrect proofs. Battista (2009, p. 104) also asserts that students might become "skeptical about the generality of a deductive proof … when the proof does not really lend insight into why a proposition is true," and says "in such instances, a mathematically and reasonable response is to explore the proof further by examples, perhaps reviewing the proof for some of those examples, even developing alternative proofs." Hence, the skepticism articulated by students about the generality of a proof should not always be seen from a negative perspective or be taken to imply the poor understanding of the nature of proof by students. But this should be interpreted in terms of the context in which it arises.

Healy & Hoyles's (1998) research project, Justifying and Proving in Mathematics, started in 1995 and focused on the National Curriculum for Mathematics, which was used in England and Wales. The curriculum was designed to first provide the general body students with opportunities to investigate, make conjectures, test, refine and justify their conjectures via empirical examples (reasoning or arguments), whilst the formal proofs were reserved for later stages to be attempted by high flying students only. Focusing on high performing students, the main aim of the project was to examine the impact of the aforementioned curriculum design on their views and competencies with regard to mathematical proof. Through the use of a questionnaire the researchers ascertained the competence of students in constructing proofs, as well as their views in regard to some aspects such as: generality of valid proofs, and choice of arguments to support and refute a conjecture. Results of the study showed that even the high flying students performed poorly on the questions that required the construction of a proof. In fact a large number of them did not have the necessary idea of how to begin a proof, whilst those who made an attempt could not relate their assertions via a logical argument. Moreover, Healy and Hoyles (1998, p. 2) finds that:

> "empirical verification was the most popular form of argumentation used by the students in their attempts to construct proofs, and in problems where empirical examples were not easily generated, the majority of students were unable to engage in the process of proving."

Although most students made use of empirical arguments to justify conjectures, they were quite aware that their justification would not earn the highest marks from the teachers. This suggests that students were somewhat aware that empirical justification was not adequate in itself to establish the general validity of a conjecture, but were rather handicapped in doing otherwise simply because they did not how to do otherwise (i.e. they did not have the necessary know-how to construct a deductive argument or justification). Although students experienced difficulties in evaluating arguments that appeared in the questionnaire, the majority of students tended to show appreciation for the generality that a valid proof carries with it. In other words, students seemed to be aware that "once a statement had been proved, no further work was necessary to check if it applied to a particular subset of cases within its domain of validity" Healy & Hoyles (1998, p. 3). Furthermore, whilst students were better at identifying a correct proof (valid mathematical argument) for a particular conjecture from a set of given proofs as compared to constructing a valid proof on their own, the study found that such a choice was not just limited to the correctness of the proof, but was indeed influenced by the extent to which the argument implied generality, clarified and explained the mathematics under the spotlight, as well as whether the mathematical argument was written in a formal way or not.

Further to this, Healy & Hoyles (1998) finds the kind of performance exhibited by students is shaped by the kind of conceptions of proof they held. For example, empirical arguments were posited by over one quarter of the sample of 2459 students who had "little or no sense of proof" (p. 4); those (over half of the sample of 2459 students) who appeared to have a firm understanding of the kind of generality that a proof carried with it, were much better in evaluating specific arguments and constructing proofs; and students (over one third of the sample) who were of the view that a proof should be explanatory, preferred to present their argument in narrative form instead of the formal proof form.

Pertaining to counter-examples, Galbraith's (1981) study found that many secondary students did not seem to have a firm understanding of the role of counter-examples for two reasons. Firstly, the "mechanism of refuting was not understood," (p. 17) meaning that a considerable number of students (55 out of 153) did not realize that an example could be counted as a counter-example only when it satisfies the premises (conditions) of the conjecture, but violates the conclusion of the conjecture. Secondly, "the philosophy of disproof by counter-examples was not appreciated," (p. 17) in other words some students (13 out of 73) did not realize that it takes just one and only one counter-example to refute a conjecture. Here are some typical responses that were cited in Galbraith (1981, p. 17):

> "Bill (14 years): "One example is not enough to disprove it."
> Jill (13 Years): " Need about 11."
> Michelle (15 years): "One example is enough but the more you get the more you are disproving.""

Upon reflecting on the kinds of responses about counter-examples discussed earlier, it appears the students do not seem to have a good understanding of the purpose of proof, and in particular they do not seem to know that when a valid proof is constructed to justify a conjecture it indeed justifies a conjecture for a related arbitrary case. Consequently, they find it hard to accept that it takes just one counter-example to reject a conjecture or mathematical proposition.

In summary, many students at school seem to prefer empirical arguments over deductive arguments and see an empirical argument as a valid proof. There are also pockets of students who do not seem to comprehend the generality that a valid proof articulates, and hence they either continue to empirically test posited claims after a valid proof has been constructed (or presented) to justify its existence, or look for more than one counter-example to reject a conjecture (or the conjecture proof). Moreover, it seems that the development of a conjecture is confined mostly to induction via relevant investigative activities, but virtually no opportunities are created for students to develop conjecture via deduction.

## 2.4 Some Research Related to Justifications at Pre-service teacher Education level and beyond

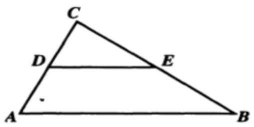
As pre-service teachers, practicing teachers, and students at both college and other tertiary institutions may have certain misconceptions in geometry (see section 3.1), they may also hold and exhibit misconceptions associated with justifications. I wish to discuss some of these misconceptions in this section. For example, Schoenfeld (1986, p. 243) engaged in a study that focused on the development of conjectures by college students within a context characterized by compass and straight edge constructions, and found that most of the students were naïve empiricists, whose approach was limited to an "empirical guess-and-test loop". With reference to the latter point, Schoenfeld (1986) found that students would go ahead to construct a figure as per the details of their conjecture, and then accept their conjecture as a statement that is generally true if their resultant construction looked sufficiently correct, with no effort to seek for a deductive justification. This move by the students of accepting their conjecture as valid based wholly on the belief their construction "looks" accurate implies that they resorted to accepting the visual appearance of their constructed figure(s) (or drawing(s)) as a form of justification that guarantees the eternal truth of their conjecture, which indeed is a misconception. For example, if a figure looked like a square, then the students would immediately conclude that it was a square without offering a justification in the form of a deductive proof. According to Schoenfeld (1986, p. 243), this misconception resulted in students, who were involved in the study, to either reject or correct solutions to posited problems simply because their resultant construction(s) (or figure(s)) were not sufficiently correct, or accept incorrect solutions to posited problems because their constructions looked reasonably good.

In the very same study (Schoenfeld, 1986), students first constructed a proof that offered an answer to a given problem. However, when requested to do the actual construction problem, a large number of students simply ignored the results (properties) that they derived initially via their proof, and hence landed up making conjectures that contradicted what they had initially proved already. According to Schoenfeld (1986, p. 243), "in some cases this was the result of impetuous behavior, since there is a strong (and incorrect) perceptual bias toward a particular hypothetical solution to the construction problem," and in other instances "students simply did not see any connection between the deductive mathematics of theorem proving and the inductive mathematics of doing constructions." Furthermore, Schoenfeld (1986, p. 243) says a large majority of the students employed an empirical standard to determine the correctness of a specific construction procedure, namely “the procedure for a construction is correct if it produces an accurate figure.” It seems that the given students relied too much on ‘visual’ representations to validate specific conjectures, with the result that they said that a particular conjecture was false if their resultant construction or presented pictorial picture did not look right.

Knuth (2002) examined the conceptions of proof that was held by a group of 16 in-service secondary school teachers who were either at van Hiele level 3 (ordering) or beyond. At van Hiele level 3, an individual is expected to have the necessary skills, knowledge and insight to classify and inter-relate figures according to their properties, and also order the properties of geometric figures logically via short chains of deductions. In contrast, at van Hiele level 4, individuals begin to construct longer sequence of statements, and start to develop understanding of the concept deduction as well as the roles of axioms, theorems, and proof. (see de Villiers, 1999 & 2003a; Gutierrez & Jaime, 1998; compare Senk, 1985 & 1989). Although most teachers in this study did not see the role of proof in mathematics as being that of explanation (i.e. proof promotes understanding and provides insight as to why a particular result is true), all the teachers did see proof as a means of verification as per their suggestions that a primary role of proof is to establish the truth of a mathematical statement. However, in doing so, 11 teachers suggested that the truth of a statement is established through the provision of a logical or deductive argument, whilst the remaining five teachers suggested that the truth of statement is established through the provision of a convincing argument. The following two responses represent the suggestions of the first group of eleven teachers:

- “I think it means to show logically that a certain statement or certain conjecture is true using theorems, logic, and going step by step;
- I see it as a logical argument that proves the conclusion. You are given a statement, and the logical argument has this statement as its conclusion” (Knuth, 2002, p. 386).

According to Knuth (2002), all the in-service teachers held the view that proof enables one to establish the truth of a mathematical statement or conclusion, but several of them demonstrated that they did not have the necessary understanding of the generality that is expressed by a proven statement. For example, in-service teachers were provided with the proof as shown in Figure 2.4.1.




Given: $\triangle ABC$ and points $D$ and $E$, which are midpoints of $AC$ and $BC$, respectively.
Prove: $AB$ is parallel to $DE$.

| | |
|---|---|
| $D$ is midpoint of $AC$ and $E$ is midpoint of $BC$ | Given. |
| $DC = (1/2)AC$ and $EC = (1/2)BC$ | Definition of a midpoint. |
| $\angle C \cong \angle C$ | Reflexive property. |
| $\triangle ABC \sim \triangle DEC$ | If two sides of a triangle are proportional to the two corresponding sides of another triangle, and the included angles are congruent, then the triangles are similar. |
| $\angle CDE \cong \angle CAB$ | Definition of similar triangles. |
| $AB$ is parallel to $DE$ | If two lines cut by a transversal ($AC$) form congruent angles with the transversal, then the lines are parallel. |

Figure 2.4.1: A focus on the generality of the proof (Knuth, 2002, p. 388)

After reading the proof, the in-service teachers acknowledged that the proof made sense to them and that they understood the proof. However, when asked whether there could be a counter-example that could refute the proof argument, some teachers drew some additional triangles before actually responding that there was no such counter-example. However, this kind of empirical view exhibited by the teachers resonates with the practice of professional mathematicians, as they often also do empirical checks after constructing a proof to check if they have not perhaps made a silly mistake in their proof or to check whether the proof is valid for all cases. In other words certainty for mathematicians is not gained exclusively by means of proof, but often by a combination of deductive proof and some empirical evidence.

Further to this, when one teacher was questioned as to why she drew additional triangles, she explained as follows: “Because proof by exhaustion. There are millions of triangles that exist, and I’ve only looked at three” (Knuth, 2002, p. 387). Even, when the facilitator drew an extremely obtuse triangle with a short base, and questioned the teachers as to whether the aforementioned conclusion as per proof in Figure 2.4.1, would hold for such an atypical triangle, five teachers first embarked on drawing sketches containing the given conditions before responding that the stated conclusion will continue to hold. A further four teachers, unconfidently (diffidently) responded that the conclusion could be true. Their responses suggested that they were not easily convinced of the truth of a conclusion by just seeing a written coherent deductive argument, but that they rather preferred to first test a conjecture via empirical examples before making any assertion as to whether a written coherent deductive argument warrants the truth of a conclusion (or conjecture) or not. The latter preference tallies with similar findings expressed by Fischbein & Kedem (1982) as well as Healy & Hoyles (1998 & 2000) in their respective studies.

In principle, the empirical moves exhibited by the teachers do not resonate with the absolutist perspective that only proof gives conviction in mathematics, but instead supports the need for empirical checks even after a proof is constructed. Hence, as facilitators in our mathematics classrooms, we need to be aware of such moves by our mathematics students and be more empathetic towards such empirical responses. Further to this, De Villiers (2004) argues that:

> "From a Lakatosian viewpoint, therefore, it is useful, by means of quasi-empirical exploration, to test not only unproved conjectures but also results already proven deductively. Such testing ought also to be encouraged, rather than suppressed, among our students, as it may give new perspectives for further research or contribute to the refinement and/or reformulation of earlier proofs, definitions, and concepts. The Lakatosian view, therefore, contrasts strongly with a traditional, rationalist view like that of Fischbein (1982) that a 'formal proof offers an absolute guarantee to a mathematical statement. Even a single practical check is superfluous' (p. 17)" (p. 407).

Furthermore, most of the in-service teachers in Knuth's (2002) study did not provide any evidence that suggested that they were conversant with the explanatory function of proof as espoused by many mathematics and mathematics educators alike, for example De Villiers (1991), Hanna (1990) and Hersh (1993). In particular, their focus on proof was more procedural in nature in that they were more interested in understanding how to proceed from the given premise to the conclusion of a proof rather than really gaining the necessary insight and understanding of the underlying mathematical relationships that a proof brought to the fore. This kind of view could be attributed to the nature of the in-service teachers' experience with proof during their earlier secondary school, college or undergraduate years of study, wherein the focus was primarily on the employment of some deductive mechanism to produce a final product called proof (see Chazan, 1993; Harel & Sowder, 1998). Schoenfeld (1994, p. 75) asserts that through practices such as the one just mentioned, we find that "proof has no personal meaning or explanatory power for students." No doubt, if we expect teachers and prospective teachers to experience proof and its explanatory power, they have to participate and experience proof like activities that inculcate an understanding of the mathematical relationships embedded in a proof which also shows why a particular theorem (or result) is indeed true.

In light of this kind of inquiry, Knuth (2002) presented the in-service teachers with five mathematical statements, wherein each statement contained justifications that ranged from proof to non-proof arguments. In all there were eight non –proof like arguments and 13 proof arguments, and the arguments were constructed via varied approaches (e.g. algebraic; proof by induction) and some arguments were more explanatory than others. Although the results showed that the given 'proof arguments' were recognized as deductive proofs by nearly all in-service teachers, many of them rated empirically based demonstrations also as deductive proofs. For example, the empirical based argument in Figure 2.4.2, was classified as a deductive proof by five teachers.

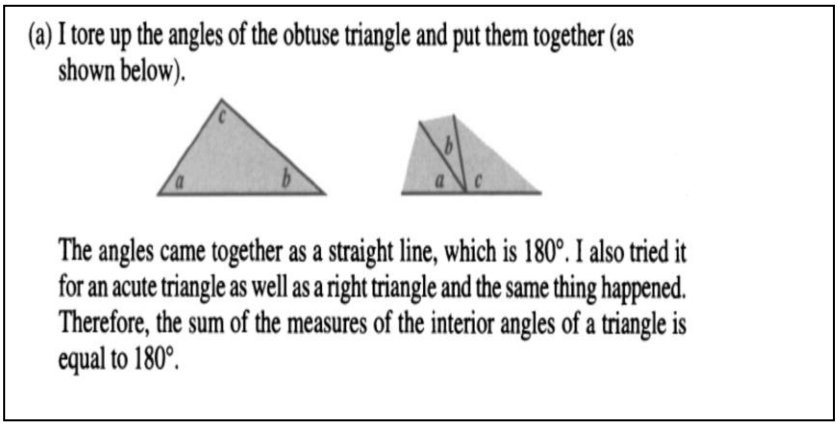



Figure 2.4.2: Empirical demonstration – Sum of the angles of a triangle is $180^{o}$ (Knuth, 2002, p. 392)

In addition, Knuth's (2002) study shows that the in-service teachers were more convinced by the form in which the argument was presented rather than the substance it contained. In much the same way, Weber (2001) also finds that college students hold the view that a proof is valid so long as it is written in the traditional two-column format. Weber's (2001) study also shows that many undergraduate students use irrelevant inferences in their failed proof attempts, and that their attempts to develop a proof are frequently characterized by the writing down of a number of rules with the hope that one of them would probably provide a lead or path to the desired proof.

Martin and Harel (1989) in their study investigated the views of proof held by a group of first year pre-service teachers that were doing a sophomore level course in mathematics. The pre-service teachers were given a task that entailed generalizations that were familiar and unfamiliar, which were accompanied by inductive examples and patterns as well as proofs of the following forms: false, particular, general, and the pre-service teachers had to judge the mathematical correctness of the posited inductive and deductive justifications. The two set sets of generalizations that the pre-service teachers had to work with are as follows:

- "**Familiar generalization**: If the sum of the digits of a whole number is divisible by 3, then the number is divisible by 3"
- **"Unfamiliar generalization:** If *a* divides *b*, and *b* divides *c*, then *a* divides *c*" (Martin & Harel, 1989, p. 43).

Results of the study showed that the particular inductive argument was accepted as a valid mathematical proof by more than half of the participating pre-service teachers, a deductive argument was accepted as valid mathematical proof by over 60% of the pre-service teachers, and just over a third of them simultaneously accepted a posited inductive argument and a correct deductive argument as being mathematically valid. Furthermore, an incorrect deductive argument for the familiar generalization was accepted by 38 % of the pre-service teachers as being mathematically correct, and 58% of them accepted an incorrect deductive argument for the unfamiliar generalization as being mathematically correct.

On reflecting on the pre-service teachers' acceptance of inductive and deductive arguments as mathematical proofs, Martin & Harel (1989, p. 49) draws the following conclusion: "the inductive frame, which is constructed at an earlier stage than the deductive frame, is not deleted from memory when students acquire the deductive frame." In particular, Martin & Harel (1989) asserts that a person's natural everyday experience of utilizing empirical evidence (or examples) to support the construction of a hypothesis or accept/ refute a posited hypothesis (or conjecture), reinforces a person's inductive frame of thinking. This natural reinforcement of the inductive frame of thinking through one's daily experiences, plausibly explains why inductive and deductive frames of thinking existed simultaneously in many of the pre-service teachers (Martin & Harel, 1989, p. 49). Furthermore, the prevalence of the simultaneous existence of both frames of thinking in many students, "suggests that the activation of both the inductive and the deductive proof frames may be required for students to believe a particular conclusion" (Martin & Harel, 1989, p. 49). The latter point corroborates to some extent with Fischbein & Kedem's (1982) finding, wherein they noted that students still wanted to empirically verify a mathematical claim despite being convinced by deductive proof. As discussed earlier, this is not uncommon even for professional mathematicians. Further to this Lakatos (1976) makes the same point, namely counter-examples to false or poorly formulated conjectures and proofs can only be found be quasi-empirical checking; so perhaps one should not be too harsh in judging these students.

Further to the point under focus, Martin & Harel (1989) finds that a false proof argument was not rejected by many of pre-service teachers who accepted a general proof argument. The authors postulate that the students belonging to this category seem to "rely on a syntactic-level deductive frame in which the verification of a statement is evaluated according to ritualistic, surface features" (Martin & Harel, 1989, p. 49). In other words, the students tend to look at the mere appearance of the argument and not the correctness of the argument, when deciding to accept or refute a posited deductive kind of argument. However, Martin & Harel (1989, p. 49) finds that there were a small number students who made their judgment about the correctness (or incorrectness) of a posited deductive argument "according to causality and purpose of argument," and hence claims that these students possessed a "conceptual-level deductive frame."

Furthermore, Martin & Harel (1989) finds that high levels of acceptance of a particular proof argument were demonstrated by those students who accepted a general proof argument, and attempted to explain the aforementioned result in the following ways. Firstly, the authors suggested that the "students may be interpreting a particular proof as an inductive argument, in which case it is seen as an instantiation of the inductive argument frame." However, the authors realized that this is quite unlikely primarily because high levels of acceptance of a particular proof also prevailed amongst students who rated inductive arguments low. Secondly, the authors postulated that "a particular proof may be viewed as an instantiation of the deductive processes used in the general proof" Martin & Harel's (1989, p. 50). In corroboration with Vinner's (1983) finding (as cited in Martin and Harel, 1989, p. 50), which noted that students in his study "preferred using a particularization of the deductive processes used to prove the statement rather than the general result." Martin and Harel (1989, p.50), suggests that the conceptual-level deductive frame is comprised of two sub-frames, namely: generalized-results and generalized deductive processes. Figure 2.4.3, represents the model of the proof frames held by students in Martin & Harel's (1989) study.

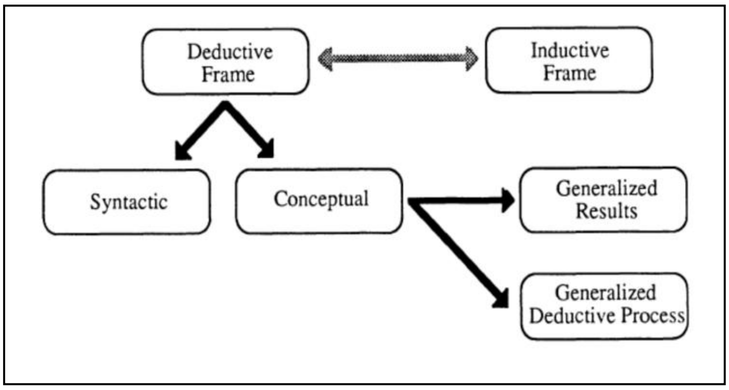



Figure 2.4.3: A schematic of students' proof scheme

## 2.5 Conjecturing, Generalizing and Justifying within a Dynamic Geometry Context

### 2.5.1 *The Geometer's Sketchpad*: What it is and what can you do with it

The development and introduction of dynamic geometry software (DGS) such as *The Geometer's Sketchpad* (*GSP*), *Cabri, Cinderella*, into the fields of Mathematics and Mathematics Education has created a classroom atmosphere wherein students' learning can go well beyond standard traditional mathematics content and practices. For example in geometry, there are now more opportunities than before for students to construct figures (or engage with readymade figures), explore, make and test conjectures, justify and refute their generalizations, pose and solve significant problems, and devise original proofs (Battista & Clements, 1995; Jackiw, 1995; Scher, 2002; Schwartz & Yerushalmy, 1986). In addition, the advent of DGS, has sparked the development of research in new fields, for example chaos theory and fractal geometry, and rejuvenated existing areas of research (De Villiers, 2007c). However, De Villiers (2007c) also cautions that there are some pitfalls associated with *DGS*, like "painless learning", "visualization makes easier" and "dynamic experimentation as sufficient verification".

Since *GSP* is the software that the participants in this study used during their one-to-one task-based activities, the ensuing discussions relate mainly to this software. *GSP* affords students opportunities to engage with already constructed dynamic figures or to actually construct dynamic figures on their own (with guidance if necessary) in terms of given mathematical relationships (or properties) using the relevant construction tools such as line segment, point, circle. The aforementioned *GSP* constructions (figures) can "always be dragged, squeezed, stretched or otherwise changed while keeping all mathematical properties interact" (Jackiw, 1991, p. 2). For example, Figure 2.5.1, called a sketch, contains a cyclic quadrilateral *ABCD*, which has been constructed through combining the following objects: points, circle and line segments.

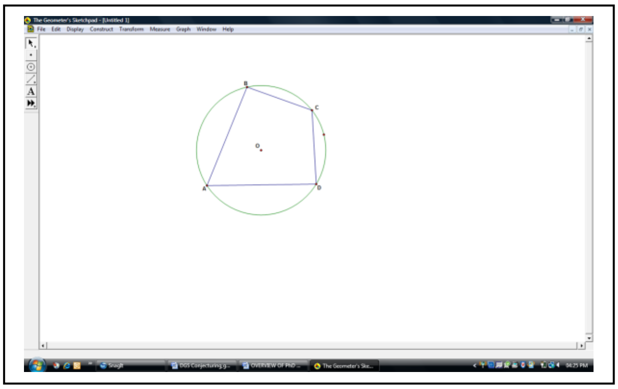


Figure 2.5.1: Dynamic Sketch

The 'drag' mode makes it possible for students to experiment with the afore-constructed cyclic quadrilateral (which is called a mathematical object) such that the relationship amongst points (*A*, *B*, *C*, *D*), line segments (*AB*, *BC*, *CD*, and *DB*) and circle *ABCD* with centre *O*, which has been defined by the construction itself, is preserved at all times either when the figure is manipulated or when one of the basic components of the construction is dragged (see Hoyles & Noss, 1994, p. 716).

For example, if point *B* is dragged in any direction, one would always obtain quadrilateral $A'B'C'D'$ such that points $A', B', C, 'D'$ are concyclic (i.e. $A'B'C'D'$ will still remain a cyclic quadrilateral). According to Driscoll, Egan, DiMatteo & Nikula (2009, p. 162), "an invariant is something about a situation that stays the same, even as parts of the situation vary". We may hence regard the vertices of quadrilateral *ABCD* being always concyclic points, as an invariant property of figure *ABCD*.

In the main one can distinguish between random dragging (i.e. wandering dragging) and directed dragging. Random dragging (wandering dragging) is done without any particular goal in mind or intent about what is being looked for and directed draging occurs when a student has an intention to move something or create some alternative experience (compare Sinclair, de Bruyn, Hanna & Harrison, 2004). Generally when students engage in random dragging, their opportunites to discover are compromised and often they become distracted from lesson activites and do not achieve the intended outcomes of the task/lesson (compare

Grayson, 2008). However, often students will do random dragging, then pause to think, and then do directed dragging when doing investigative activities using *GSP*.

In general, when a dynamic *Sketchpad* figure is dragged, the relationships defined by the construction will always be maintained, but the properties that are not defined by the construction could change or vary. So, in effect by manipulating a figure, for example through the 'drag' mode, one could explore a range of possible figures which are governed by the set of construction constraints. According to Jackiw (1991, p. 2), this feature of *GSP*, "makes it easy to distinguish between those properties that are sometimes true, and those that are always true for a given situation." In addition, *GSP* makes provision for students to record the instructional steps for their geometric constructions as scripts, thus making it possible for students to just play back a script in order to either construct or re-construct the desired figure in a sketch. The latest version of *GSP* (version 5) uses scripts in the form of a 'Create Tool' button. Being able to re-play a script provides students with the added opportunity of being able to re-trace the process (or the path that was followed) in the construction of a particular dynamic figure. In a research article, Giamati (1995, p. 456) explains: "The most useful aspect of scripting one's constructions is that students can test whether their constructions work in general or whether they have discovered a special case."

In particular the 'drag' effect, made possible by the tools embedded in *GSP*, can create opportunities for one to see many empirical examples quickly and thereby construct plausible inductive generalizations (conjectures). By dragging, students can alter the size and orientation of constructions and notice that a specific conjecture they produced always remains true, and hence become quite convinced in the truth of their conjecture such that it may result in them wanting to know why their conjecture is always true (see Mudaly, 2002). Quite often students in such circumstances feel no need for deductive justification as reported by Marioti (2001). A downside to the 'drag' effect is that students operating under an empirical proof scheme (Sowder & Harel, 1998), may mistakenly take the array of supporting empirical examples produced through 'dragging' as a means of proving (in the deductive sense) (see Izen, 1998, p. 719). With regard to the latter, De Villiers (1998), reports that in such instances he managed to encourage a deductive explanation by explicitly asking the students to determine why their results, which they have obtained through the use of interactive software, are true.

The *GSP*, regarded by many as a dynamic chalkboard, is a powerful tool for visualization. For example, Whiteley (2000, p. 2) writes:

> “The programs (*Cabri* and *Sketchpad*) expand the role of precise visual and diagrammatic reasoning in all stages of our work: posing questions, making conjectures, creating counter-examples, seeking and recognising connections. The change is dramatic and we leave a session with the program more refined information, connections and images. These refined images then run our heads as we work with problems and the solutions, both mentally and with words and pictures on paper, guided by these dynamic visual processes. These diagrams become part of our internal vocabulary and our ongoing mental processes. We also see unexpected events: an extra coincidence of lines or points in the construction; a transformation; or a mental association with some pattern in some previous geometric study.”

Furthermore, Scher’s (2002) study has shown that: “the mathematical insights that students derived from dynamic geometry software were not always those intended by the interviewers” (p. 60). In other words through using dynamic geometry software, students were able to make constructions that would not have been realizable in a paper, pencil and straight edge context, and hence managed to assimilate mathematical discoveries that were not within the expectation range of results the facilitator expected. This kind of enactment demonstrates that *GSP*, when used diligently can push the boundaries of knowledge production to an extent that new results can surprisingly be discovered with a concomitant need to logically explain why such discoveries are always true.

### 2.5.2 Research related to conjecturing, generalizing and justifying in a *GSP* context

Being able to construct dynamic figures or use ready-made sketches, which could be manipulated, particularly through dragging, has made the processes of generalizing, conjecturing, and justifying much more accessible to both students and teachers across mathematics classrooms at both school and tertiary levels. In light of this, I wish to focus on some research pertaining to the learning and/or teaching of geometry within a Sketchpad context in this section.

In a study involving university students, Giamati (1995), who viewed *GSP* as an exploratory tool that enabled students to investigate and hence uncover invariants, formulate conjectures, test and refine conjectures, presented her students, who were exploring rotations and their centers, set the following question, which was suitable for exploration within a *Sketchpad* environment:

> "If $\Delta A'B'C' = R_D^{\beta}(\Delta ABC)$, how can we locate $D$ and find $\beta$, where $D$ is the center of rotation and $\beta$ is the angle of rotation?"

To facilitate the exploration, students were given several ready- made sketches of triangles and their images under rotation about a point *D*. The students, henceforth discovered that they needed to construct the following line segments $\overline{AA'}$, $\overline{BB'}$ , and $\overline{CC'}$ , and did so accordingly. Some students proceeded further to construct the perpendicular bisectors of the aforementioned segments, with points *E*, *F* and *G* being their respective midpoints. In most instances students got into the habit of recording all their constructions as scripts of a rotation of a triangle about a given point, and this enabled students to quickly observe that the bisectors appeared to be concurrent at a point. Figure 2.5.2.1 illustrates a successful conjecture that students made, namely: "*D* is the centre of rotation for mapping $\Delta ABC$ onto $\Delta A'B'C'$" (Giamati, 1995, p. 456).

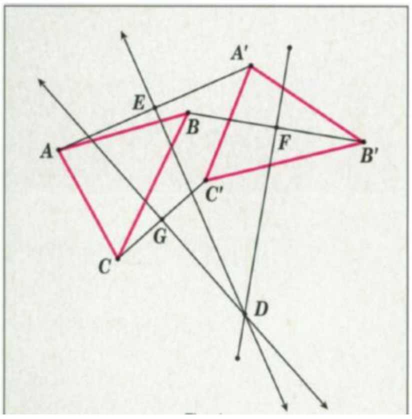



Figure 2.5.2.1: An illustration of a successful conjecture: *D* is the centre of rotation for mapping $\Delta \boldsymbol{ABC}$ onto $\Delta \boldsymbol{A'B'C'}$ . (Giamati, 1995, p. 456)

Giamati (1995) used the *GSP* environment from a facilitator perspective to support the students to develop a proof that justified the aforementioned conjecture. In the final analysis, students constructed a deductive proof that justified the existence of the following theorem: "If the two triangles are congruent and $\Delta A'B'C' = R_D^{\beta}(\Delta ABC)$, then the perpendicular bisectors of $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are concurrent at point *D*." Immediately after the construction of the aforementioned proof, Giamati (1995, p. 457) stretched the students' thinking by posing the following question: “Is this an if- and- only-if theorem?” In principle, to be considered as an if- and- only-if theorem, students were aware that they needed to show via deductive proof that its converse is also true. In the process of attempting to construct a deductive argument, students engaged in lots of explorations and required several hints. The explorations and hints, resulted in students being able to construct a counter-example that showed that the converse of theorem was not true, and thereby helped conclude that the proven theorem was indeed not of the if-and-only-if type (or their initial conjecture as per Figure 2.5.2.1, was not bi-conditional).

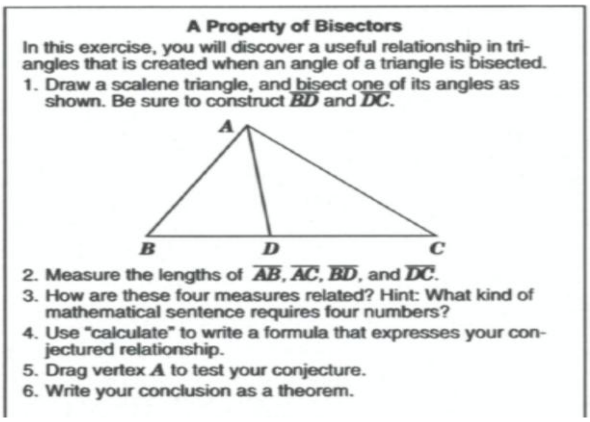
With regard to the aforementioned task being foregrounded in a *Sketchpad* environment, Giamati (1995, p. 458) maintains it was an invaluable experience for the students, and accordingly comments:

> “The students gained a deeper understanding of the problem by using their scripts to explore it and make conjectures than they would have if the results had merely been explained to them. Naturally, the exploration did not replace the proof, but it became a solid foundation on which to build the proof. The students were able to construct various examples quickly enough to recognize that certain conjectures were unreasonable…. In this example, the students found it satisfying to see the proof of their conjecture…. The power of the *Geometer’s Sketchpad* combined with the power of proof gave a complete illustration of the theorem and the aspects of ‘doing’ Mathematics.”

In a paradigm similar to that of Giamati (1995), Izen (1998) asserts that software such as GSP makes the construction of both simple and complex figures much easier for students as compared to pencil and paper; allows students to measure distances between points, lengths of segments, angles, gradients (slopes), perimeters, areas; and makes it possible for students to set up formulas that could support their conjectures. In particular, Izen (1998, p. 718), re-affirms that the move of selecting a vertex of a figure and dragging it, is equivalent to the actual construction of many figures satisfying the same given information in a conjecture. These series of constructed figures thus make it possible for students to see that their established conjecture continues to be true. Hence, in this sense Izen (1998) holds the view that inductive reasoning can be utilized to “demonstrate the likelihood that a theorem is true,” and that both the process of inductive reasoning and deductive reasoning can be used to complement each other to such an extent that students can gain full understanding of and insight into a given theorem. Furthermore, Izen (1998) firmly asserts though working in a *DGS* environment can enable one to generate compelling empirical evidence to justify the truth of a theorem, it is certainly not a mathematical proof.

Izen (1998) believes that his approach to the teaching and learning of geometry embraces the involvement of his students in dynamic geometry tasks that encourage them to discover geometric relationships; to establish the extent of the validity of their conjecture through experimentation; to construct the conclusions as theorems before getting the students to construct a justification via deductive proof for the stated theorem or established conjecture. For example, in retrospect, the construction of the following theorem, "The angle bisector of an angle of a triangle divides the opposite sides into two segments that are proportional to the other two sides" (Izen, 1998, p. 719), the activity as depicted in Figure 2.5.2.2, which had to be done using *Sketchpad*, was initially given by Izen to his students.

As a result of the explorations conducted in accordance with guiding instructions and questions described in Figure 2.5.2.2, Izen (1998, p. 719) reports that his students discovered and confirmed with a high levels of belief the following result: "If the measure of an angle of a triangle is bisected then the bisector divides the opposite sides into two segments in the same ratio to the lengths of the other two sides of the triangle." Before asking his students to prove the associated theorem, Izen gave his students exercises that involved the application of the theorem in question.


**A Property of Bisectors**

In this exercise, you will discover a useful relationship in triangles that is created when an angle of a triangle is bisected.

1. Draw a scalene triangle, and bisect one of its angles as shown. Be sure to construct $\overline{BD}$ and $\overline{DC}$.



2. Measure the lengths of $\overline{AB}$, $\overline{AC}$, $\overline{BD}$, and $\overline{DC}$.
3. How are these four measures related? Hint: What kind of mathematical sentence requires four numbers?
4. Use "calculate" to write a formula that expresses your conjectured relationship.
5. Drag vertex $A$ to test your conjecture.
6. Write your conclusion as a theorem.

Figure 2.5.2.2: Computer exercise ( Izen, 1995, p. 719)

After completing the aforementioned exercises, the students who were working in a *Sketchpad* context but provided with relevant support and guidance from the facilitator, managed to successfully construct a deductive proof that explained why the theorem is true.

The move of a student being able to explain why the given theorem (or any other theorem) is true, inculcates a sense of ownership of the developed proof in the student, and "prevents the student from feeling that the teacher is force-feeding information that makes no sense" Izen (1998, p. 719).

In a study that also bears some commonalities with Giamati's (1995) and Izen's (1998) studies in respect of the didactical approaches employed to conjecturing and proof, Mudaly (1998) conducted an experiment with the purpose of determining "learners' need for conviction and explanation within the context of dynamic geometry, and whether proof could be introduced to novices as a means of explanation" (Mudaly, 2000, p. 22). In particular a scaffolded worksheet selected from De Villiers's (1999) curriculum material, was used as a tool to facilitate the one-to one task- based interview that Mudaly (1998) conducted with a group of 17 grade nine learners. Although this group of learners was selected randomly, they had prior experience in using computers but not with *GSP*, and were not yet introduced to proof in geometry. The design of the material, which embraced a scaffolding approach, afforded learners the necessary opportunities to explore, make observations, conjecture (or discover) a plausible solution to a Viviani (see Contreras & Martinez Cruz, 2009, p. 246) linked problem, test and refine the conjectures, and ultimately explain why their discovered plausible solution is always true.

Mudaly's (1998, p. 54) research attempted to answer the following core of research questions:

- Are learners convinced about the truth of the discovered geometric conjecture and what is their level of conviction? Do they require further conviction?
- Do they exhibit a desire for an explanation for why the result is true?
- Can they construct a logical explanation for themselves with guidance?
- Do they find the guided logical explanation meaningful?

The problem, stated in the form of a Viviani problem and given to the learners to solve within a dynamic geometric context, was structured as follows:

> "Sarah, a shipwreck survivor manages to swim to a desert island. As it happens, the island closely approximates the shape of an equilateral triangle. She soon discovers that the surfing is outstanding on all three of the island's coasts and crafts a surfboard from a fallen tree and surfs every day. Where should Sarah build her house so that the

total sum of the distances from the house to all three beaches is a minimum? (She visits them with equal frequency)" (see De Villiers, 2003, p. 23)

To kick-start the inductive processes of exploration and conjecture, a ready-made dynamic sketch (see Figure 2.5.3) of an equilateral triangle containing the elements described in the Ship-wreck problem was made available to the learners in a *Sketchpad* context.

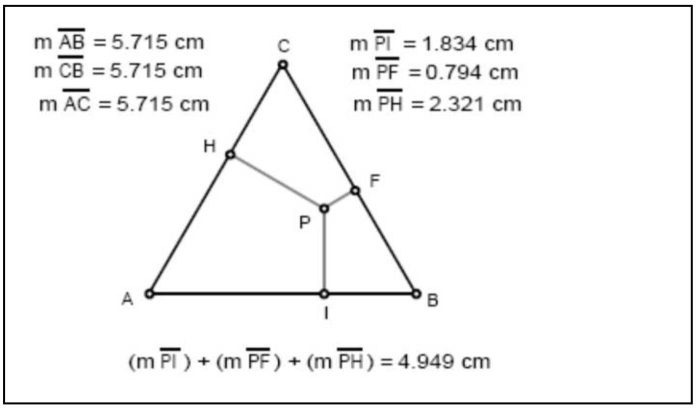



Figure 2.5.3: Dynamic Sketch of Shipwreck problem (compare Mudaly, 1998, p. 59; see Mudaly & De Villiers, 2000, p. 23)

With the *Geometer's Sketchpad* software, learners were able to drag point *P* to various positions within the confines of the equilateral triangle, and observe that the lengths of the segments *PH*, *PI* and *PF* continuously changed, but the sum of the lengths *PH*, *PI* and *PF* always remained constant. Within a few minutes after exploring the problem using the tools embedded in *GSP*, most of the students became highly convinced but surprised that Sarah could build her house at any point inside the described island.

Learners tested their conjecture via computer exploration, and thereby became more and more convinced in their conjecture, as indicated by their final levels of convictions projected below (see Mudaly & De Villiers, 2000, p. 25):

- 14 (82,3%) were 100% convinced
- 2 (11,8%) were 98% to 99% convinced
- 1(5,9%) were 55 % convinced

Despite the learners having achieved such high levels of conviction, the majority of the learners expressed a desire for an explanation as to why their conjecture is always true, independent from their need for conviction. The latter finding corroborated with De Villiers's (1991a, p. 258) finding pertaining to a non-dynamic geometry context: "Learners who have convinced themselves by quasi-empirical testing still exhibit a need for explanation, which seems to be satisfied by some sort of informal or logico-deductive argument" (as cited in Mudaly & De Villiers 2000, p. 26). In relation to the element of surprise expressed by the learners in the conjectured result, Mudaly & De Villiers (2000, p. 26) citing Hadas & Hershkowitz (1998, p. 26) makes the following assertion:"It seems that the learners' need for an explanation arose out of finding the result surprising, with the surprise causing the cognitive need to understand why it was true."

Despite the learners seeking an explanation eagerly, none of the learners were able to construct an explanation on their own when asked to do so. Thus some guidance via a scaffolded set of questions was provided to the learners, and this enabled them to build up their own logical explanations with a reasonable degree of ownership and appreciation. Therefore, it was in this sense that the learners' desire for an explanation was utilized as an opportunity to introduce the learners to proof as means of explanation rather than just verification. Despite all the learners positively acknowledging that their constructed argument satisfied their needs for explanation (or curiousity), Mudaly & De Villiers (2000, p. 28) expresses some reservations of the learners' positive responses based on the possibility that "they might have only responded positively to please the teacher." However, through reflecting on the learners' body language, facial expressions and tone of voice when presenting their positive responses, Mudaly & De Villiers (2000, p. 28) concludes that a majority of the learners seemed to have found the logical explanation satisfying.

In summary, this Chapter focussed on the kinds of reasoning in mathematics, the concept of generalization, some research done in the context of generalizaing and justifying at both school and tertiarty levels, as well as conjecturing, generalizing and justifying within a geometric context.

The next Chapter, provides a literature review on misconceptions and counter-examples.

# Chapter 3: Misconceptions and counter examples

## 3.0 Introduction

At times students have difficulty with particular aspects of mathematics because of misconceptions they possess in areas linked to the topic under consideration. Section 3.1, provides a discussion as to how misconceptions develop in students and the consequences of such misconceptions on new learning. Further to this, counter examples are discussed from both a heuristic and global perspective in Section 3.2.

## 3.1 Misconceptions

> "Misconceptions are usually an outgrowth of an already acquired system of concepts and beliefs wrongly applied to an extended domain. They should not be treated as terrible things to be uprooted since this may confuse the learner and shake his confidence in his previous knowledge. Instead, the new knowledge should be connected to the student's previous conceptual framework and put in the right perspective" (Nesher, 1987, pp 38-39).

In mathematics learners make mistakes for different reasons. For example, mistakes could be attributed to hasty reasoning, misreading information, poor concentration levels, incorrect transcription of given information or a computational error due to carelessness – and in most instances such mistakes are not regular occurrences (Almeida, 2010; Swan, n.d, p. 34). For example, in the example below a student was requested to add the following algebraic expressions: $-8xy + 4y^2 - 5x^2$; $+10y^2 - 2x^2 - 3xy$; $-6y^2 - x^2$. The student responded as follows:

$$\begin{array}{l} -5x^2 - 8xy + 4y^2 \\ -2x^2 + 3xy + 10y^2 \\ \underline{-x^2 \qquad\qquad - 6y^2} \\ \underline{-8x^2 - 5xy + 8y^2} \end{array}$$

The student's solution demonstrates that he/she understands the algorithm for the addition of algebraic expressions, but s/he did not produce the correct sum of $-8x^2 - 11xy + 8y^2$. The incorrect solution produced by the student is attributed to the incorrect transcription of the sign of second term, $-3xy$ , of the second algebraic expression in his/her write up. This is a clear example of a careless mistake, which is termed a 'slip' in the literature. Van Lehn

(1982, p. 6) defines a slip as an “unintentional, careless mistake in that little extra care apparently makes them disappear”. Equivalently, Olivier (1989) defines slips as follows: “Slips are wrong answers due to processing; they are not systematic, but are sporadically carelessly made by both experts and novices; they are easily detected and are spontaneously corrected.” (p. 196)

On the other hand, mistakes can be more serious in nature – particularly when they are committed regularly as the result of “consistent, alternative interpretations of mathematical ideas” (Swan, 2001, p. 34). These kinds of mistakes are referred specifically to as ‘errors’ in the literature. In particular, Olivier (1989, p. 197) describes errors as follows: “Errors are wrong answers due to planning, they are systematic in that they applied regularly in the same circumstances”. Furthermore, Olivier (1989, p. 197) asserts that errors “are the symptoms of the underlying conceptual structures that are the cause of errors”, and refers to the associated “underlying beliefs and principles in the cognitive structure that are the cause of systematic conceptual errors” as misconceptions. In the similar vein, Ben-Zeev (1996, 1998) as cited in Brodie (2010, p. 13), defines misconceptions as “underlying conceptual structures that explains why a learner produces a particular error or set of errors”, and asserts that misconceptions are responsible for errors.

Moreover, within the various fields of education like mathematics, science and computer science, extensive research has been done in the area of misconceptions from a constructivist perspective (see Ben-Ari, 2001; Confrey, 1991; Olivier, 1989; Smith III, diSessa, & Roschelle, 1993). According to Ben-Ari (2001, p. 58), a constructivist would see a misconception as “a logical construction based on a consistent, though nonstandard theory, held by the student” and not as a slip or (trivial) mistake. In particular Ginsburg (1977) asserts students’ mistakes (or responses) are seldom careless or capricious in nature, but are systematically grounded in their prior conceptions that they bring to the classroom, and hence from the learner’s perspective makes a lot of sense. Furthermore, research done by Smith et al. (1993, pp. 119-121) on misconceptions from a constructivist perspective, used the term misconception to “designate a student conception that produces a consistent and systematic pattern of errors across time and space” and also brought to fore the following dominant views pertaining to misconceptions:

- “Misconceptions arise from students prior learning, either in the classroom (especially for mathematics) or from their interaction with the physical and social world;”
- “Misconceptions can be stable and widespread among students;”

- “Misconceptions can be strongly held and resistant to change;”
- “Misconceptions interfere with learning;”
- “Instruction should confront misconceptions.”

The afore-mentioned research affirms that learner’s errors in mathematics emanate from rational thinking with reference to prior organized strategies, rules and generalizations, which cannot just be dismissed as ‘wrongful thinking’ but rather be construed as legitimate outputs in relation to their existing mental frameworks. For instance, in relation to earlier experiences either inside or outside the classrooms, learners develop generalizations like the following:

- “Multiplication makes numbers larger”
- “you can’t divide smaller numbers by larger ones”;
- “division always make numbers smaller”;
- “the more digits a number has, then the larger is its value”;
- “Shapes with bigger areas have bigger perimeters” (compare with Almeida, 2010; Olivier, 1989; Ryan & Williams, 2007; Swan, 2001, n.d, p. 34)

If one considers the generalization, ‘multiplication makes numbers larger’ within the context of natural numbers, then it is plausible to make such a generalization primarily because it is true for every multiplication sum involving natural numbers. Moreover, through related experience of questions, exercises, problems and classroom discussions associated strictly with the multiplication of natural numbers in early grades, the conception that multiplication makes numbers larger becomes a kind of natural belief in the minds of the young learners i.e. it becomes firmly embedded in their multiplication schema. This generalization, whilst true enough when applied to whole or natural numbers, can lend support to errors such agreeing that $\frac{3}{10} \times \frac{3}{10} > \frac{3}{10}$; $0{,}4 \text{ x } 0{,}2 = 0{,}8$; $130 \times 0.1 > 130$, that $0{,}3^2$ must be bigger than 0,3, and so on. In this instance, we explain the cause of the learner’s response by saying that s/he has applied (or extended) the generalization, ‘multiplication makes numbers larger’, which s/he has established through working with the multiplication operation in the context of whole/natural numbers, to other domains such as fractions and decimals. Hence, we can say that misconceptions result “from structures that apply appropriately in one domain being over-generalized to another” (Brodie, 2010, p. 13). Equivalently, Nesher (1987, p. 35) describes misconceptions as “a line of thinking that causes a series of errors (systematic) all resulting from an incorrect underlying premise, rather than sporadic, unconnected and non-systematic errors.”

Just like many other constructivists, Almeida (2010, p. 4) asserts that:

> "misconceptions arise frequently because a pupil is an active participant in the construction of his/her own mathematical knowledge via the reception and the interaction of new ideas within the pupils extant ideas."

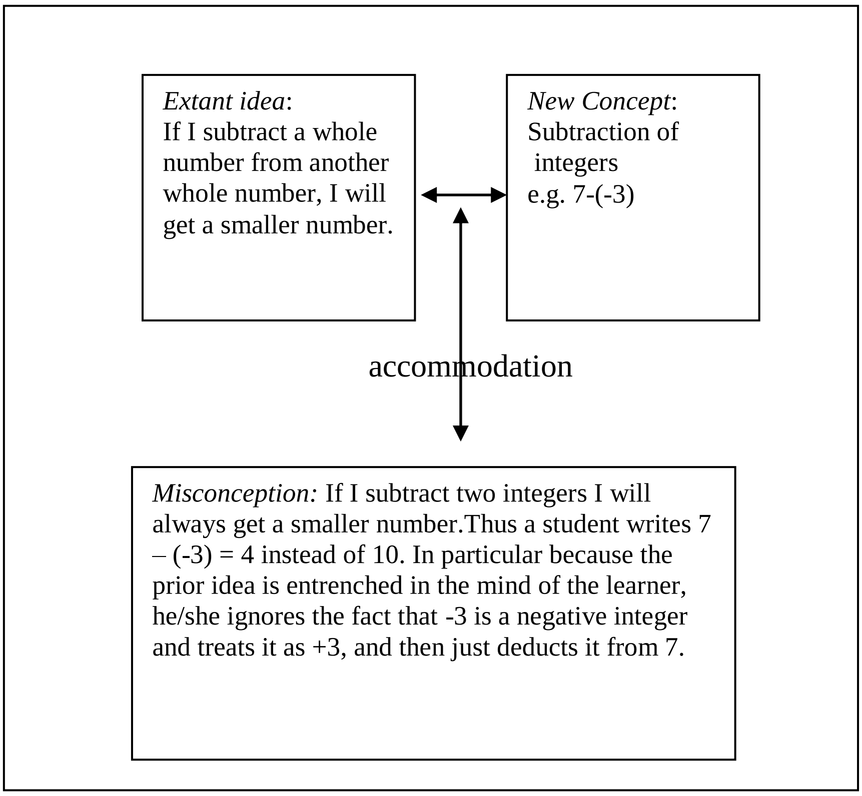



Figure 3.1.1: Development of a misconception (Adapted from Almeida, 2010, p. 4)

Nesher (1987) and Resnick et al. (1989) as cited in Smith et al. (1993, p. 120) also affirm "that misconceptions usually originate in prior instruction as students generalize prior knowledge to grapple with new tasks". For example, in early grades learners discover or experience the algebraic analogue of the distributive law of arithmetic:

$$3(x + y) = 3x + 3y \text{ or } a(b + c + d) = ab + ac + ad.$$

The aforementioned distributive law, is reinforced via many concrete examples, exercises and assignments in the classroom, where they basically work with each term independently. In fact the distribution of a given number over two or more terms becomes so entrenched in the learner's cognitive structures, that when learners are confronted with an exercise like $\sin(30° + x)$, they respond as follows: $\sin(30° + x) = \sin 30° + \sin x = \frac{1}{2} + \sin x$ .

This kind of error, which is certainly a systematic error, can be largely attributed to the over-generalization of the distributive law, and one which learners experience in arithmetic and early algebra lessons, to operations that are not distributive (Olivier, 1989; Almeida, 2010). Matz (1980) refers to errors like $\sqrt{m+n} = \sqrt{m} + \sqrt{n}$ ; $\log(a+b) = loga + logb$; $m(ny) = (mn)(my)$; $\cos(a+b) = cosa + \cos b,$ as linear extrapolation errors. In addition, Olivier (1989, p. 204) accounts for the aforementioned errors as follows:

> "these errors are an overgeneralization of the property $f(a+b) = f(a) + f(b)$, which applies only when $f$is a linear function of the form $f(a*b) = f(a)*f(b)$, where $f$ is any function and * any operation. This super-formula now acts as another deep level procedure, saying "work the parts separately", so that the indicated errors are continually being re-created, which explains its obstinate recurrence."

In light of this discussion, extrapolation errors could also be attributed the *matching rule* according to Behr & Harel (1990). The matching rule suggests that learners perceive similarity between a previous solved problem (called the base) and a newly posed problem (called the target), and hence believe that the procedure that was used to solve the base problem can also be used to solve the target problem. Naturally, in order to circumvent any conflict the student will try to apply the familiar procedure to the new (or target) problem and not apply a new or different method. Thus if students search for similarities between what is already known and the new (or the familiar and unfamiliar), and concludes that similarity exists when in fact it does not, serious errors like extrapolation errors can develop through over-generalization.

In our classrooms, there are myriad of examples, wherein generalizations that are valid in a a particular domain or context are inappropriately extended (or perhaps restricted) to other domains - in other words become 'over-generalizations' and thus result in errors. An important example of an over-generalization is characterized by the following belief that many learners hold: "The rules of invariance that apply to algebra also apply to geometrical shapes. So there must be equality in all respects when *A* becomes *B*" (Almeida, 2010, p. 35).
This aforementioned belief, which is in fact called a 'conservation' misconception, is often the course of systematic conceptual errors. For example, consider the response from a learner to an activity pertaining to area and perimeters as illustrated in Figure 3.1.2. In the learner response it is quite plausible to assume that the conservation misconception, has been responsible for the learner to make the conclusion that the perimeters are the same.

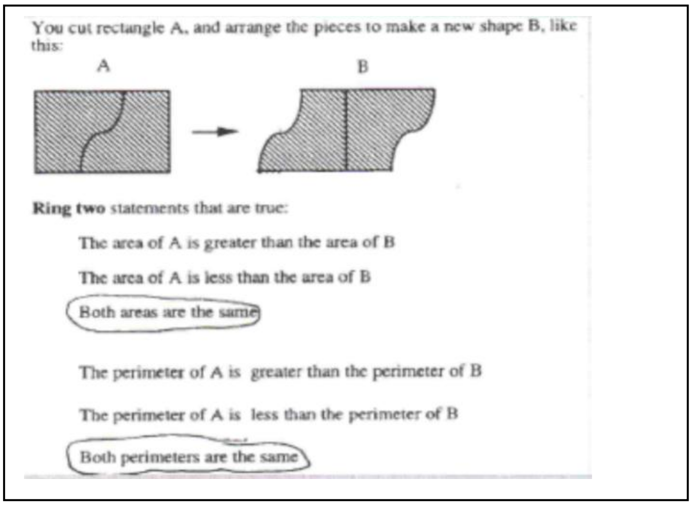



Figure 3.1.2: Development of a Misconception (Almeida 2010, p. 35)

The ‘over-generalization’ syndrome permeates all levels and aspects of mathematics. For example, at grade 10 level, learners solve quadratic equations via factorization as follows:

Case 1:

$x^2 - 2x - 24 = 0$

$\therefore (x-6)(x+4) = 0$

$\therefore (x-6) = 0 \; or \; (x+4) = 0$

$\therefore x = 6 \; or \; x = -4$

In solving, the above quadratic equation where the right hand side is zero, learners are expected to make use of the following general rule: $If a.b = 0, then\ a = 0 \; or \; b = 0$. In engaging the aforementioned rule learners are expected to proceed sequentially according to the following of the equation was given in the form $ax^2 + bx + c = 0$

- Factorize the algebraic expression on the left hand side
- Set each factor equal to zero, and then solve for $x$.

The aforementioned steps become so entrenched in the learner’s cognitive schema, that when they are given the exercise, $x^2 - 11x - 28 = 8$, some learners proceed as follows:

Case 2:

$x^2 - 11x - 28 = 8$

$\therefore (x-4)(x-7) = 8$

$\therefore (x-4) = 8 \; or \; (x-7) = 8$

$\therefore x = 12 \; or \; x = 15$

The above stated response (Case 2) shows that the learners have failed to make the right-hand side zero, before proceeding to the following steps: factorizing; setting each linear factor to zero, and solving for $x$. In effect the learners have misapplied the method to solve quadratic equations that are equal to zero (as illustrated in Case 1) to quadratic equations that are not equal to zero (as is illustrated in Case 2). In effect, the learners have not realized the pivotal role of the zero on the right hand side of the quadratic equation, but instead just treated it as any other number (Olivier, 1989; Almeida, 2010). Matz (1980) as cited in Olivier (1989, p. 201) offers an explanation for the persistent prevalence of the aforementioned error across many classrooms through the consideration of two fundamental procedures that are responsible for governing one's "cognitive functioning", namely: "surface level procedures, which are ordinary rules of arithmetic and algebra", and "deep level procedures, which create, modify, control and in general guide the surface level procedures"

With regard to the latter, "generalization over numbers" is one such kind of deep level guiding principle, which in practical terms means that "specific numbers don't matter- you could use other numbers" (see Olivier, 1989, p. 201).This notion that any numbers will work emanates from learners prior experiences with numbers in other contexts, for example when multiplying three digit numbers by two digit numbers learners come to see that the given method is not restricted to a unique set of numbers but it also works for alternate sets of numbers. Similarly, when they learn the algorithm to do long division of numbers, they do see that the algorithm works for other numbers as well. Since, this "deep level guiding principle, generalizing over numbers" (Olivier, 1989, p. 201), works extremely well in contexts such as multiplication, division, addition and subtraction of numbers, learners develop the " natural tendency to overgeneralize over numbers" (Olivier, 1989, p. 201), in other contexts. Hence, in the solving of the quadratic equation not equal to zero (Case 2), learners have over-generalized over the numbers on the right hand side of the equation i.e. they thought the algorithm will work for any number on the right hand side. Thus, the learners did not transform the given equation to the pre-requisite form $a.b = 0$, before applying the following steps:

- Factorize the algebraic expression on the left hand side
- Set each factor equal to zero, and then solve for $x$.

Reflecting on the aforementioned examples that characterize some misconceptions in Mathematics, we have seen that a 'misconception' is not necessarily wrong thinking but an intelligent construction (or belief or principle) resulting from over-generalizing of prior knowledge that was relatively correct for a particular domain to another domain where it is no longer valid (Olivier, 1989; Ryan & Williams, 2007; Swan, 2001, Smith et al.1993). Thus for example, ‘division makes smaller”, is a valid generalization in one domain (that of natural numbers) that is often misapplied to a wider domain (that of rational numbers). Likewise, when dealing with the addition of matrices (of the same order), students are able to discover and apply the following generalized rule: to get the sum of two matrices, just add the corresponding terms in each matrix. However, when multiplying two matrices, students inappropriately extend the process (i.e. over-generalize the process) pertaining to the addition of matrices to get the product of the given matrices i.e. “take each term in the two matrices and multiply them together to get the resultant corresponding term in the answer” (Bull, Jackson, & Lancaster, 2010, p. 307).

These acts of over-generalizations are endemic in mathematics and have to be circumvented through appropriate pedagogical strategies that promote discussion, justification and reasoning, and thus result in students reorganizing their own conceptions (Conn & Bauersfeld, 1995; Cobb, Yackel & McCalin, 2000, Ryan & Williams, 2000; Tsamir & Tirosh, 2003). For example when a generalization is established, an educator should pose the followings kinds of questions:

- When does the generalization not apply? Can you justify your response?
- Is the generalization true for this case? Why?
- Why does the generalization not apply for the following case?
- Can you provide other examples for which the generalization will hold? Can you justify your response in each case?

Through the kind of questions pointed out in the preceding part, it is quite possible that a learner might realize the extent to which his/her generalization can apply, and thus not over-generalize the said generalization. Through such prevention of over-generalizations, misconceptions could be prevented and invariably errors could be prevented. However, if learners experience new concepts and generalizations only via ‘nice’ functions and ‘good’ examples and not special cases (or counter-examples), then misconceptions can arise as articulated by “Tall’s generic extension principle”, which says “If an individual works in a restricted context in which all examples considered have a certain property, then in the absence of counter-examples, the mind assumes the known properties to be implicit in other contexts” (Tall, 1991 as cited in Gruenwald & Klymchuk, 2003, p. 34).

In terms of Piaget's theory of cognitive functioning, when learners are exposed to new concepts in the classrooms, they attempt to interpret and construct meaning of such new concepts within the context of already existing conceptions or schemas within their broader cognitive structure (Louw et al., 1998; Piaget, 1975, 1977, 1985). Thus, if misconceptions prevail within their existing schemas, then there is a great probability that when learners are subjected to new concepts in the classroom that their resultant conception might innocently deviate quite significantly from the intended and acceptable one, and hence perpetuate the vicious cycle of misconceptions upon misconceptions and unnecessary cognitive conflict. Worse still, the triggered cognitive conflict could become so overwhelming and frustrating for learners that it could possibly lower their morale and confidence to such an extent that they decide to disengage completely with the mathematics at hand. Hence, it is imperative that educators identify and diagnose the misconceptions learners possess and provide the necessary treatment via appropriate interventions to remedy such misconceptions. Otherwise, learners will continue to "misapply algorithms and rules in domains where they are inapplicable" (Almeida, 2010, p. 17).

With regard to the avoidance of misconceptions, Wood (1988, p. 201) argues that:

> "The only way to avoid the formation of entrenched misconceptions is through discussion and interaction. A trouble shared, in mathematical discourse, may become problem solved'

Wood's assertion, suggests that educators should create the necessary climate in the classroom for learners to debate, argue, justify and critique newly presented/discovered information, which for example could include concepts, propositions, conjectures and generalizations. Moreover, general body of research suggests that looking beyond the error itself with a view to try and understand the kind of reasoning or justification that the learner has executed in order to produce his/her response, could be more productive for educators and learners in terms of identifying and remedying the underlying misconception that causes the given error (Brodie, 2010; Chazan, 1993; Confrey, 1991; Nesher, 1987; Sasman, Linchevski, Olivier, & Liebenberg, 1998; Smith et al., 1993). Hence, with regard to errors made by our learners in our mathematics classrooms, educators should do their utmost to diagnose the related misconception (if any) by getting learners to explain or justify or provide reasons for their responses (Almeida, 2010; Smith et al. 1993). Once the reasons for the constructed error are identified, then the educator can decide what kind of example s/he could use to challenge or contrast the given misconception. For example, consider Figure 3.1.3, which contains the response of leaner as cited in Almeida (2010, p. 52).

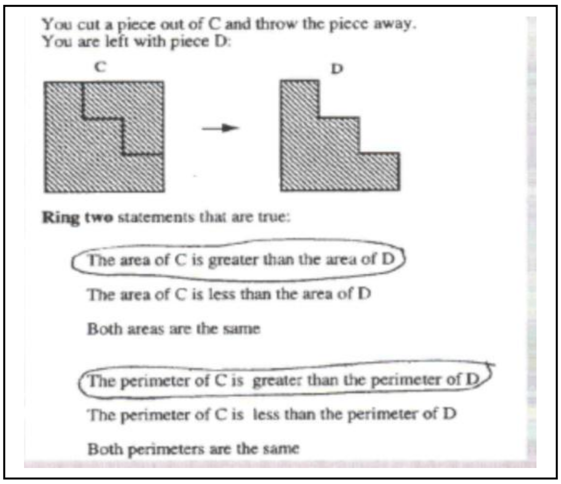



Figure 3.1.3: Development of a Misconception (Almeida 2010, p. 52)

By looking at the response of the student in Figure 3.1.3, it appears that the learner believes that a larger area implies a larger perimeter. This misconception could be based on the notion of subtraction makes smaller. According to Almieda (2010, p. 52), "one way to contrast or challenge this misconception that a larger area implies a larger perimeter is to give the pupils examples where it is not true. For example, a 1 x 2 square and a 1 x 3 rectangle. Alternatively show that any staircase *D* made out of rectangle *C* has perimeter equal to *C* by a counting exercise".

It is quite plausible that once the educator has diagnosed the underlying cause of a particular error, the educator could be in a better position to construct a special example (or a counter – example) that could contradict the learner's current ideas, and thereby create the desired cognitive conflict in the mind of a learner (Piaget, 1975, 1997). Consequently a learner's mind will be thrown into a state of non-balance, which 'alone can force a subject to go beyond his present state and to seek new equilibrium" (Piaget, 1977, p. 12). To regain such equilibrium or cognitive harmony, the learner may have to construct and reconstruct existing schemas, and subsequently also cognitive structures, to encompass (or incorporate) the new incoming (conflicting) information (or intrusions) (Piaget, 1977; Block, 1982). Regaining such equilibrium (or cognitive harmony) results in the reconstruction and transformation of misconceptions into correct conceptions, which is often a signal of intellectual growth. Thus in their classrooms, educators should be tolerant and empathetic to students that make errors or exhibit misconceptions (Olivier, 1989), make the most effort to diagnose the underlying misconceptions associated with errors, and hence create the necessary atmosphere (which includes opportunities for discussion and justifications) to challenge such misconceptions. In this regard educators ought to have the necessary teacher content knowledge (Shulman, 1986) to construct appropriate examples (counter-examples) that will induce the necessary cognitive conflict in the minds of the learner and consequently activate the processes of assimilation and accommodation to bring about the desired conceptual change.

## 3.2 Counter examples & refutations

### 3.2.1 Counter-examples

According to Bolt and Hobbs (2004) and Hummel (2000) a counter-example is a particular case (or example) which disproves a conjecture. Similarly Klymchuk (2008, p. 1) states that "a counter-example is an example that shows that a given statement (conjecture, hypothesis, proposition, rule) is false". According to Houston (2009, p. 92), "the 'counter' part of the word (counterexample) comes from the fact that we are countering, in the sense of rejecting or rebutting, the truth of a statement". For example, consider the conjecture: "$f(n) = n^2 - n + 41$ is a prime number for every natural number $n$" (Hummel, 2000). If a student attempted to establish the truth of this conjecture by cases (or a case based approach), s/he may proceed as follows as cited in Hummel (2000, p. 61):

If $n = 1$, then $n^2 - n + 41 = 41$ is a prime number
If $n = 2$, then $n^2 - n + 41 = 43$ is a prime number
If $n = 3$, then $n^2 - n + 41 = 47$ is a prime number
If $n = 4$, then $n^2 - n + 41 = 53$ is a prime number
If $n = 5$, then $n^2 - n + 41 = 61$ is a prime number
$\vdots \qquad \vdots$
If $n = 10$, then $n^2 - n + 41 = 131$ is a prime number
$\vdots \qquad \vdots$
If $n = 20$, then $n^2 - n + 41 = 421$ is a prime number
$\vdots \qquad \vdots$
If $n = 39$, then $n^2 - n + 41 = 1523$ is a prime number
If $n = 40$, then $n^2 - n + 41 = 1601$ is a prime number

Having found that the result for all 40 cases is always a prime number, the student may attempt to say that the conjecture must be true. However, not every case was considered, and the conjecture explicitly states that $n^2 - n + 41$ is a prime for *every* natural number *n*. Hence, to make such a conclusion (i.e. $n^2 - n + 41$ is a prime for *every* natural number *n*) is deemed to be fallacious in nature, particularly because not all cases have been taken into account. For example, for the case $n = 41$, we get $n^2 - n + 41 = (41)^2 - 41 + 41 = (41)^2 = 1681$, which is not a prime number. This kind of example, leads us to say that the number 41 (or case $n = 41$) is a counter-example to the statement, "If $n$ is any natural number, then $f(n) = n^2 - n + 41$ is a prime," primarily because it makes $f(41)$ a composite number rather than a prime number. In fact in terms of logic and propositional calculus, the number 41 makes the antecedent true and the consequent false, and hence we regard the number 41 as a counter-example to the given conjecture (Hummel, 2000). In equivalent terms, we say that a counter-example is "an example that satisfies all the conditions of a statement but not the conclusion" (Education Development Centre, Inc., 2002, p. 15).

However, reflecting on the student's envisaged approach, it is quite plausible to say that it is not really essential to utilize the case-based approach (or quasi –empirical testing) to ascertain the truth of the conjecture in this particular case, because it is possible that through inspection one could immediately realize that 41 is a factor of $n^2 - n + 41$ and $n^2 - n + 41 > 41$, when $n = 41$, and thereby claim that he or she has found a counter-example (see de Villiers, 2004). This single counter-example plays a significant role in illustrating to students why the given conjecture is false. Indeed a single counter-example like this is sufficient to refute a false statement (Peled & Zaslavsky, 1997). More importantly, this counter-example, may necessitate the abandonment or reformulation/refinement of the given conjecture (de Villiers, 1996). For example, the initial conjecture can be refined to read as follows:
$f(n) = n^2 - n + 41$ is a prime number for every natural number *n*, where $1 \leq n \leq 40$ .

In addition, whilst a counter-example is an example that shows a conjecture is incorrect or a case that proves the conjecture is wrong, a counter-example could also be an example which shows that a definition is inadequate (Serra, 2003, p. 136). The latter notion of a counterexample can be applied in our mathematics classrooms as follows: once a student has developed a definition of a geometric figure, s/he should test it by trying to construct or create a figure that meets the criteria of his/her definition but isn't what s/he is trying to define. In effect this means looking for a counterexample that will used as the reason that will render a given definition inadequate, notwithstanding the fact that if no counter-example can be raised against the suggested definition then there exists strong evidence that the definition of the geometric figure is a plausible one. For example, consider the following definition: a rectangle is a figure with two pairs of congruent sides. Now we can construct a counter-example, by drawing a figure with two pairs of congruent sides that is not a rectangle. Here are two counter-examples:

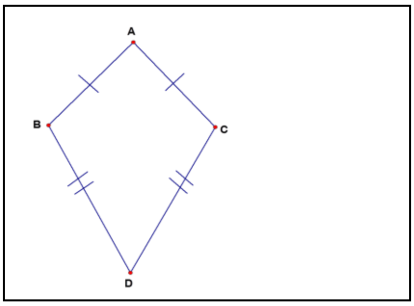



Figure 3.2.1: Kite

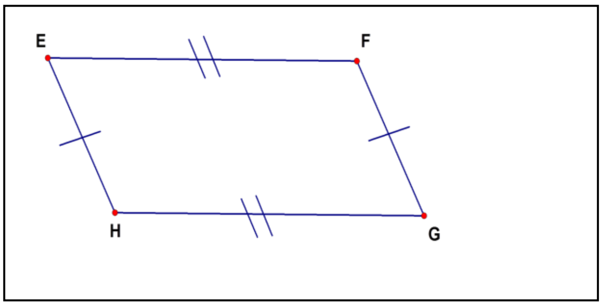



Figure 3.2.2: Parallelogram

The above counter-examples, suggest that the definition is inadequate. However, if one modifies (or refines or polish) the definition to read: a rectangle is a four- sided figure in which opposite sides are congruent and all angles measure 90°, then would no longer be in a position to construct a counterexample (Serra, 2003). Except, of course, not a 'good' definition as it is 'uneconomical' in that it contains the redundant properties of equality of sides which follow logical from the equality of angles.

The natural tendency to contest the generality of developed conjectures by mathematicians creates opportunities for the development of other conjectures and generalizations in mathematics. For example, mathematicians all over the world are still searching for a formula that generates prime numbers. In this attempt Pierre de Fermat (1601-1665) communicated to one of his fellow mathematicians, Marin Mersenne, that "the numbers $2^n + 1$ are always prime if *n* is a power of 2" (O'Connor & Robertson , 2009, p. 2). In fact Fermat tested his conjecture for the fol1owing values of *n*: 1, 2, 4, 8 and 16, and then asserted that his conjecture will definitely not hold true for cases where "*n* is not a power of 2" (O'Connor & Robertson, 2009). However, 100 years later Leonhard Euler showed that the conjecture is false for the next case $2^{2^5} + 1$ by merely showing that the number is composite as follows: $2^{2^5} + 1 = 2^{32} + 1 = 4294967297 = 641 \times 6700417$ (Klymchuk, 2008, p. 2; O'Connor & Robertson , 2009, p. 2 ).

Furthermore, also within the domain of prime numbers, Goldbach conjectured that "every odd integer $n, n > 5$, is the sum of three primes" and forwarded this to Euler in 1742 (Kahrobaei, n.d.: slide 8). However, Euler responded that "every even integer $n, n > 2$, is the sum of two primes" (Kahrobaei, n.d.: slide 8). For example, 20 = 3 +17, 24 = 11 + 13, 28 = 11 +17, and so on. The latter is called Goldbach's conjecture, and up today no one has proved that the conjecture is true nor found a counter-example that renders the conjecture to be false (Hummel, 2010; Klymchuk, 2008; Rotman, 2010). In fact as recent as 2006, the conjecture has been tested for all positive even integers up to $2.10^{17}$, and no counter-examples have been found (Kahrobaei, n.d.: slide 8).

### 3.2.2 Refutations

According to the Oxford Dictionary (2005, p. 375), refute means "prove that a person or statement, et cetra to be wrong." Reflecting on this meaning from a mathematical perspective, we find that when one has a hunch that a mathematical statement is false, then one attempts to refute such a false statement through either advancing a theoretical kind of argument based on theorems, axioms and definitions (i.e. proof argument) or providing a counter-example. Moreover within the context of mathematical discovery, counter-examples are seen as pivotal 'players' because quite often they serve as good indicators as to whether a conjecture is completely false or partially false. Hence, in this regard Komatsu (2010, p. 2) affirms Lakatos's (1976) notion that the production (or construction) of a counter-example is a "fundamental method of refutations". Lakatos (1976, pp. 7-13) differentiates between two types of refutations, namely heuristic refutation and global refutation, through the expressed notions of a "local counter-example" and "global counter-example" respectively.

#### 3.2.2.1 Global Refutation

Counter-examples are powerful examples, in that they provide the necessary basis upon which a conjecture or suggested hypothesis can be shown to be wrong. In particular, Lakatos (1976, p. 11), describes a 'global counterexample' as an example which "refutes the main conjecture itself". Furthermore de Villiers (2004, p .404) says, global refutation actually refers to "the production of a logical counter-example that meets the conditions of a statement but refutes the conclusion and thus the general validity, of the statement". The following example as cited in Houston (2009, p. 93) illustrates the notion of global refutation:

> "Let *p* and *q* be real numbers. If $p/q \in \mathbb{Q}$, then $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$. This can seem quite reasonable. However, let $p = \pi/3$ and $q = \pi/2$, then we get $p/q = 2/3$. Thus *p* and *q* are

real numbers such that $p/q \in \mathbb{Q}$ and yet both *p and q* are not in $\mathbb{Q}$. Thu*s* $p = \pi/3$ and $q = \pi/2$ provides a counterexample to the statement.*"*

Moreover, Houston (2009, p. 93) claims that a mathematical statement constructed in the form *"If …., then …"* cannot be argued to be true by the mere construction (or presentation) of just one example wherein both the premises and conclusion are true. On the other hand, as Houston (2009, p. 93) says "*just one example for which the statement is not true*" is sufficient to refute such a statement or to conclude that the statement is not a potential theorem.

However, global counter-examples are not produced by deductive reasoning only, but also by quasi-empirical testing (or experimentation) in many instances. For example, consider the following conjecture produced by a group of pre-service mathematics teachers: 'A quadrilateral with opposite sides parallel is a rhombus'. To produce a counter-example to refute this false statement, all one has to do is to use the given information to construct a quadrilateral that is not a rhombus, using *Sketchpad* or any other software, or by using the straight edge, compass pencil and paper. In this particular case, one can proceed to construct an arbitrary quadrilateral as follows using *Sketchpad*:

- Construct segment *AD*
- Construct segment *AB*
- Through *B* construct line parallel to *AD*
- Through *D* construct line parallel to *AB*
- Construct *C* as the intersection of the 2 lines.

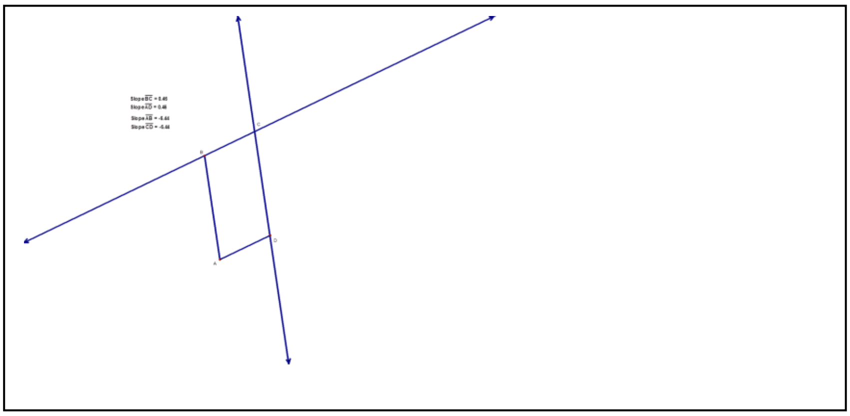


Figure 3.2.2.1: Counter example to Rhombus definition

Following, the above quasi-empirical method, it is possible to obtain the following figure, which is quadrilateral with opposite sides parallel, but is definitely not a rhombus. So essentially, we have shown quasi-empirically that 'opposite sides parallel' is not sufficient condition to produce a rhombus all the time. This Figure 3.2.2.1, is a counter-example to the conjecture: 'A quadrilateral with opposite sides parallel is a rhombus'

According to de Villiers (2004, p. 404), it is often sensible to first test unknown mathematical conjectures quasi-empirically, wherever possible, primarily because such testing serves a twofold purpose, namely:

> "The construction of a counter-example if it is false", and "the attainment of a reasonable amount of certainty (conviction), which then encourages one to start looking for a proof".

**3.2.2.2 Heuristic Refutation**

Lakatos (1976, p. 11) asserts that a 'local counterexample, is "an example which refutes a lemma (without necessarily refuting the main conjecture)", and can be regarded to be a "criticism of the proof, but not of the conjecture". In the same vein, we can say that a local or heuristic counter-example is a counter-example that does not disprove a statement in totality, but instead "challenge only one step in a logical argument or merely aspects of the domain of the validity of the proposition" (de Villiers, 2003b, p. 178). More importantly, "heuristic counter-examples are mostly not strictly logical examples, since they are after all not inconsistent with the conjecture in its intended interpretation, but are heuristic, since they spur growth of knowledge"(de Villiers, 2003b, p. 179).

This means that a heuristic example requires some modification or polishing of a concerned conjecture, theorem or its proof, but with the proviso that the core ideas of the original conjecture, theorem or its proof is upheld according to the original formulation (de Villiers, 2004 & 2010). This equivalently translates to saying: "the original conjecture (theorem) is usually still valid and true, not disproved at all, though perhaps modified, refined and much better understood" (de Villiers, 2004, p. 408).

For example, consider the following conjecture adapted from Houston (2009, p. 250): If $a < b$, then $a^2 < b^2$, $\forall a, b \in R$. However, through inspection, one can note that $a$ cannot be negative number, but that $a$ can be zero or bigger than zero. Thus we claim that $a \geq 0$. Then obviously for $a < b$, $b$ must be bigger than 0 i.e. $b > 0$. Having made such observations or conclusions, it thus becomes possible for one to revisit the original conjecture and strengthen the stated assumption by actually adding on the following set of realized conditions: $a \geq 0$ and $b>0$. This essentially means that the original conjecture, will remain valid and true, if it is merely modified to read as follows: If $a < b$, then $a^2 < b^2$, $\forall a, b \in R$ s.t. $a \geq 0$ and $b > 0$ (adapted from Houston, 2009). So, effectively in this case, one can say that we have merely altered the "domain of validity" of the conjecture.

### 3.2.3 Lakatos's theory of counter-examples in relation to proofs and refutations

Lakatos (1976), in his book Proofs and Refutations, narrates the different ways that colleagues in the mathematics community reacted to potential counter-examples that have been raised to the proof of the Descartes–Euler conjecture for all regular polyhedra, which states that $V - E + F = 2$, where $V$, $E$ and $F$ denote the number of vertices, edges and faces respectively. Associated ways in which the individuals responded to potential examples were described through the methods of "monster-barring", "exception barring", and "proofs and refutations" (Lakatos, 1976, pp. 13-33).

According to Lakatos (1976), 'monster–barring' refers to the outright rejection of a potential counter-example to a theorem. In these instances, the rejection is done without the provision of a valid reason, but by the convenient review and construction (or reconstruction) of operational definitions such that it actually excludes the identified counter-examples as part of the intended set of examples or any necessary consideration thereof. In essence, the resultant product of monster barring is "the modification or clarification of a definition" (Larsen & Zandieh , 2008, p. 208).

Whilst this kind of approach, may create an escape route from a particular counter-example, there exists a probability that the altered definition, may create a room for other counter-examples to surface, or simply exclude possible examples that may validate or support a given theorem.

'Exception barring' firstly entails citing the surfaced counter-examples, which are legitimate examples, as mere exceptions to a stated theorem, primarily to enhance the validity of the stated theorem in all other instances without any course for concern regarding the domain of validity pertaining to the stated theorem. Secondly, exception barring may involve the rephrasing or reconstruction of a conjecture by actually zoning onto the domain of validity and restricting it in such a manner as to exclude the given counter-example(s).

In general, within the context of exception barring the counter-example is considered to be legitimate, and consequently the conjecture is modified or improved upon and not the underlying definition itself as with the case of monster barring.

'Proof analysis', which is another way in which individuals can respond to potential counter-examples, is a core methodological stage in Lakatos's "more mathematically mature method of mathematical discovery, known as proof and refutations" (Swinyard & Larsen, 2010, p. 2). Larsen & Zandieh (2008, p. 207) describes the four stages of Lakatos's (1976) method of proof of *proof and refutations* as follows:

- The first stage is the construction of the initial conjecture, known as the "primitive conjecture".
- Stage two is characterized by the construction of a proof, which in Lakatos's (1976, p. 127) terms is "a rough thought experiment or argument, decomposing the primitive conjecture into sub-conjectures or lemmas".
- Stage three is characterized by the "emergence of global counter-examples". The afore-said counter-examples are "global in the sense of applying to the primitive conjecture rather than merely one of the sub-conjectures".
- Stage 4 is "the analysis of the proof to discover the lemma (perhaps hidden) to which the global example is a local counter-example. The result of this stage is an improved conjecture featuring a new proof-generated concept."

In 'proof–analysis', an individual re-examines and interrogates the constructed proof pertaining to an established primitive conjecture with a view to identify "a potentially obscured sub-conjecture for which the counter-example is problematic" (Swinyard & Larsen, 2010, p. 11). In this sense, the moves entailed in proof analysis, enables one to develop the necessary argument to motivate for the possible improvement of an initial primitive conjecture, wherein the improved conjecture will indeed include a new proof generated concept. For example, Cauchy in 1821 stated and proved the following conjecture (regarded as a primitive conjecture by Lakatos) as cited Larsen & Zandieh, 2008, p. 207: "The limit of a convergent sequence of continuous functions is continuous." However, later in 1847 Seidel (as cited in Larsen & Zandieh, 2008, p. 207) in his quest to find the hidden assumption in Cauchy's proof, "discovered the concept of uniform convergence". The discovery of the concept of uniform convergence was the catalyst responsible for the improvement of the conjecture, which ultimately contained the "new proof generated concept", namely "the limit of a uniformly convergent series of continuous functions is continuous" (Larsen & Zandieh, 2008, p. 207).

It is the proof analysis stage that makes it possible for one to distinguish the method of "proofs and refutations" from the method of "exception–barring" (Larsen & Zandieh, 2008, p. 207). In the context 'proof analysis' (with reference to Lakatos's method of 'Proof & Refutation'), the sole purpose of modifying the conjecture in question is to actually provide a definite opportunity for the proof to be realizable and acceptable to the mathematics community, but with the obvious good intention of retaining the 'counter-example' as just another ordinary example within the domain of the conjecture, rather than actually excluding it (Larsen & Zandieh, 2008)

### 3.2.4 Reframing Lakatos's ways of dealing with counter-examples for classroom practice

Larsen & Zandieh (2008) in their study focused on students responses to counter-examples and their corresponding outcomes. In their study students were engaged with the definition of a subgroup and the properties it satisfies. The Lakatosian framework for proofs and refutations, were reframed as follows for this particular study: a response from a student that signaled the rejection of a counter-example on the basis that "it is not a true instance of the relevant concept", was considered as monster barring; a response from a student that resulted in a "modification of the conjecture to exclude a counter-example without reference to a proof" was considered as exception barring; and a response which entailed a modification of the conjecture with the good intention to "make the proof work rather than simply exclude the counter-example from the domain of the conjecture", constituted the notion called proof-analysis (Larsen & Zandieh , 2008, p. 208).

The aforementioned reframed Lakatosian ways of dealing with counter-examples are described in Figure 3.2.4 with respect to the following aspects: activity type, activity focus and activity outcome. This reframed version provided a lens through which students responses to counter-examples and resultant outcomes were analyzed.

Fundamentally, the findings of Larsen & Zandieh's (2008) study, were compatible and consistent with the framework posited in Figure 3.2.4. For example, when faced with counter-examples, students displayed the elements of monster-barring by revisiting the definitions underpinning the raised the conjecture and changing them. On the other hand students showed symptoms of exception-barring by first acknowledging the legitimacy of the counter-examples, and thereafter embarking on the modification of their conjectures instead of the underlying definitions.

| **Type of activity** | **Focus of activity** | **Outcome of activity** |
|---|---|---|
| Monster-barring | Counter example & underlying definitions | Modification or clarification of an underlying definition |
| Exception barring | Counter example & conjecture | Modification of the conjecture |
| Proof-analysis | The proof, the counter example & the conjecture. | Modification of the conjecture & sometimes a definition for a new proof-generated concept. |

Figure 3.2.4: Reframing the ways of dealing with counter-examples described by Lakatos (1976) (as cited in Larsen & Zandieh, 2008, p. 9)

According to Inhleder & Caprona (1985, p. 8) as cited in Balacheff (1991, p. 89), the constructivist learning theory asserts that a student explores actively his or her environment and also that a student "participates actively in the creation of space, time and causality". Hence, in terms of the constructivist view, the student is considered as an active participant in the building of his/her own knowledge pertaining to the field of mathemetics (Balacheff, 1991). In accordance with this view, Piaget (1975) as cited in Balacheff (1991, p. 80) assert that intellectual development is initiated by statements that are contradictory in nature, which inevitably results in the equilibrium within and amongst the schemata in the cognitive structure of the learner to be disturbed; and it is through searching for explanations to overcome the experienced contradictions that new knowledge is constructed in the minds of the learners.

Indeed, Balacheff (1991) articulates that there exists a great degree of consistency between the aforementioned theory of intellectual development advanced by Piaget and Lakatos' model which describes the growth of knowledge in mathematics through the notion of 'proofs and refutations' within a broader context that incorporates a range of complex ways of responding to counter-examples. With regard to the latter, Lakatos (1976, p.5) argues that:

> "Informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of undubitably established theorems, but through the incessant

> improvement of guesses by speculation and criticism, by the logic of proof and refutations."

Moreover, within this notion of overcoming a contradiction, Balacheff (1991, p. 92) asserts: "A counter-example in the mathematics classroom is often understood as sort of catastrophe because it implies the definitive rejection of what has been refuted". On the contrary, in a mathematician's experience, this is not as such, but rather one that is fuelled at least with the following actions: careful interrogation of the counter-example; modification of the conjecture; inserting conditions into the conjecture; justifying and proving or possible rejection of the counter-example.

Drawing on the aforementioned kind of action-moves that is consonant with mathematician's kind of experience, Balacheff (1991) conducted qualitative research to ascertain how students react to counter-examples and re-look at their conjectures and proofs. Within this context, Balacheff (1991, p. 90-101), presented 13-14 year old students with an experimental activity, whereby they had to conjecture a way to calculate "the number of diagonals of a polygon once the number of its vertices is known". In keeping with Laktosian style of dealing with conjectures, the observer presented the students with counter-examples to their proposed conjectures and the students had to respond to the resultant contradictions. Results of the study showed that a range of Lakatosian ways of treating a refutation (or counter-example) was utilized by different groups of students to overcome/deal with the posited contradiction, namely: "rejection of the conjecture";"modification of the conjecture"; "exception barring"; "definition revisited"; "rejection of the counterexample" (see Balacheff, 1991, pp. 90-101).

### 3.2.5 Steering refinement of conjectures and proofs through counter examples

Interacting with counter-examples raised against a conjecture in terms of making sense of it or just constructing counter-examples against a posited conjecture, fosters the development of both reasoning and justification skills within the context of argumentation, and thus improves the opportunity for learning and successful intellectual development. Hence, to stimulate mathematical learning at school level, many mathematics education researchers (Balacheff, 1991; Larsen & Zandieh, 2008, Komatsu, 2010; Reid, 2002; Zazkis & Chernoff, 2008) strongly recommend that some of our classroom activities should mirror mathematical learning as per heuristics embedded in the process of 'proof and refutations' pioneered by Lakatos (1976).

Komatsu (2010, p. 1) embarked on a research case study “to explore how primary school students re-examine their conjectures and proofs when they confront counter-examples to the conjectures they have proved”. According to Komatsu (2010, pp. 4-7) reported on two students who proved their primitive conjecture via an action proof using manipulative objects, and who were then subsequently challenged with a counterexample. In summary, the details are as follows:

- Two students, Daiki & Takuya, after expressing a degree of surprise with regard to their calculations, “52 +25 = 77, 26 + 62 = 88 and 31 + 13 = 44”, subsequently made the following conjecture (primitive conjecture): “the tens and ones digits of the two integer sums are equal”.
- Diaka & Takuya developed a justification (action proof) for their conjecture using counters.
- The students (Diaka & Takuya) were subsequently furnished with the following counter-example by the author: “Well, can you apply the same operation to 85 +58?”
- Thereafter, through careful reflection and interrogation of their proof with aid of the colour counters (manipulatives), conceded to “85 +58” being a counter-example to their original conjecture.
- The students then attempted to improve on the truthfulness of their conjecture, by adding the following condition to their primitive conjecture: “When the sums become three digit-numbers, the digits become different”.
- Thereafter, the facilitator requested the two students re-visit the counter-example “*85 +58*” and represent the sum using counters from a horizontal viewpoint instead of the previous vertical viewpoint. Their observation from the aforementioned activity coupled with their reflection on their earlier attempted proof made it possible for them to realize how to modify their previous conjecture and produce a new conjecture that was much more comprehensive, namely: “the two- integers sums were equal with 11 multiplied by the number of pairs of the counters”.

In the main, the counter-example to the students’ primitive conjecture was the catalyst responsible for stirring the need in the students to refine their primitive conjecture and associated proof. For, example, when Komatsu (2010, p. 5) presented the students with counter-example, “Well, can you apply the same operation to 85 +58?”, they experimented and reflected upon it and after some time had passed on, they began to accept the counter-example with some degree of surprise, and responded as follows: “ This is a big problem, we must try again from the beginning” Komatsu (2010, p. 7).However, when Komatsu (2010, p.

7) posed the following question to the students, “Is the conjecture spoiled?”, one student said “no”, whilst the other said “we must overcome this problem”. According Komatsu (2010, p. 7), the latter student’s response, suggests that “this confrontation with the counter-example functioned as a driving force for their active refinement of conjecture and proofs.” Retrospectively, it was careful analysis and reflection by the students on their attempted justification of their primitive conjecture in tandem with the counter-example that made it possible for the students to realize the falseness of their primitive conjecture and also to identify the potential gaps in the primitive conjecture, and then subsequently construct a more comprehensive conjecture as they did quite successfully (Komatsu, 2010, p. 1).

Within the domain of research, Zaskis and Chernoff (2008), describe an episode of instructional interaction, which embraces a counter-example heuristic through the lens of a ‘pivotal-bridging example’’ to eradicate misconception(s) experienced by a prospective elementary school teachers. According to Zaskis and Chernoff (2008, p. 197),

> “An example is pivotal for a learner if it creates a turning point in the learner’s cognitive perception or in his or her problem solving approaches; such examples may introduce conflict or resolve it….When a pivotal example assists in conflict resolution we refer to it as pivotal-bridging example, or simply a bridging example, that is an example that serves as bridge from learner’s initial (naïve, incorrect or incomplete conceptions) towards appropriate mathematical conceptions.”

For example in Zaskis and Chernoff (2008) study, one of the prospective elementary school teachers, Selina, who was presented with a task requiring her to simplify: $\frac{13\times17}{19\times23}$, and interviewed with respect to her responses whilst attempting to reduce the fraction. Selina made the response, $\frac{13\times17}{19\times23} = \frac{221}{437}$, and then attempted to simplify by looking for factors that were common to both the numerator and denominator. However, during the continuous interview about whether the number 437 was a prime number or not, Selina replied as follows (Zazkis & Chernoff, 2008, p. 199): “Yes, it is, because it’s two prime numbers, of course it is, because two prime numbers multiplied by each other are prime, (pause).”

Thus in effect, Selina has conjectured that when two prime numbers are multiplied then the resultant product is prime. However, Selina was later requested to consider the number 15, and after some deliberation within her mind, she surprisingly realized that 15 could be expressed as a “product of two prime numbers”, namely 15 =3 x 5, even though 15 did not belong to the set of prime numbers. In this sense, the choice of the example, number 15, can be classified as a ‘pivotal example’’ for Selina, primarily because it “it introduces cognitive conflict and challenges her initial ideas” (Zazkis & Chernoff, 2008, p. 199). However, despite all the aforementioned experiences, Selina still held onto her initial conjecture, “two prime numbers multiplied by each other are prime”.

The interviewer subsequently presented the number 77 to Selina for her consideration, and only thereafter on further seeing that 77 can be expressed as the ‘product of two prime numbers’ realized that the ‘product of two prime numbers’ does not yield a prime number, and consequently abandoned her conjecture. In this sense, the number 77 served as a ‘pivotal-bridging example’ for Selina , because it helped her to see link between the numbers 15 and 437 which then made it possible for her to reconcile “her initial naïve ideas with the conventional mathematics” (Zaskis & Chernoff, 2008, p. 205). Expressed differently we can say that the pivotal example and pivotal-bridging example played a significant role in bringing about the desired conceptual change. Moreover, from the responses of Selina during the interview, Zaskis & Chernoff (2008) realized that although from a mathematical standpoint the falsity of a mathematical statement can be determined via the production/citing of just one counter-example, there exists a serious need for educators to be aware that not all counter-examples are necessarily equally effective in pedagogically assisting a learner to recognize a faulty or false conjecture. Thus from a pedagogical perspective, some students only begin to reject a conjecture after experiencing more than one counter-example. Hence, Zazkis & Chernoff (2008, p. 206) suggests that instructors (educators or facilitators) need to be aware of the convincing power of different counter-examples, so that they can appropriately choose “strategic counter-examples that may serve as pivotal examples in addressing learners misconceptions”.

Furthermore, Fujita, Jones, Kunimune, Kumakura & Matsumoto (2011), reflected on the work and results of Larsen and Zandieh’s (2008) research, which was framed within the context of the Lakatosian constructs of proof and refutation, and subsequently embarked on a teaching experiment in a Japanese lower secondary school. Apparently, at the lower secondary level, a large number of students, where sucked into notion of developing their “primitive’ conjectures with respect to 3D geometry problems by just focusing on visual images at the expense of proper geometrical reasoning. In this experiment, the researchers used the Lakatos’s constructs of monster-baring, exception- barring, and proof analysis together with practical activities and group discussions to break the cycle of reasoning via visual images and improve their levels of geometrical reasoning levels.

Results of Fujita et al.'s (2011) study, showed that appropriate management of students discussions of counter-examples in the classroom within the broader context of Lakatos' approach to counter-examples, promotes the necessary space for monster-barring and exception –barring types of responses to conjectures such that the resulting modification to the conjecture enables the proof to work as per proof-analysis heuristic. Moreover, the results show that the employment of aforesaid heuristics in the context of whole classroom discussions followed by small group discussions can promote the development of students' geometrical reasoning capability (Fujita et al., 2011 )

### 3.2.6 Refutation Schemes & generating conjectures through refutation

Lin (2005, p. 11) claims that the refuting process covers five sequential processes:

> "Entry; testing some individual examples point-wisely for sense making; testing with different kinds of examples; organizing all kinds of examples; identifying one (kind of) counterexample when realizing a falsehood".

Taking into consideration the processes encompassing the refuting process, Lin (2005) presented a group of learners with a false conjecture, and analyzed students' reasons on refuting using a coding scheme (see Lin, 2005, p. 12). The nature of the responses presented by learners suggests that there exists synergy between their thinking process and the suggested refutation process.

Furthermore, in advancing explanations for refuting the postulated false conjectures, many learners provided heuristic arguments and explicit counter-examples with supporting reasons, but some learners went to the extent of producing: "relations, known properties evidences, general rules, etc." (Lin, 2005, p. 13). These kinds of results, prompted Lin (2005) to investigate the links and relationships between the "refuting process" and conjecture production i.e. to explore possible avenues of continuity. For the envisaged investigation, two refuting-conjecture tasks (one in geometry & the other in algebra) were developed and subsequently administered grade 9 learners. Results of the investigation showed that "a high percentage of learners were able to produce correct conjectures when working on refuting –conjecture tasks" Lin (2005, p. 15) in a mathematical learning context. This affirmed the “existence of continuity between the refuting process and the production of truth statements,” which in effect meant that “refuting is an effective learning strategy for generating conjectures” Lin ( 2005, p. 15).

To provide a good opportunity for learners to construct or produce innovative conjectures, it is strongly recommended that “the content in a given false conjecture be relatively well designed” Lin (2005, p. 15). For example, Lin (2005) used the item in Figure 3.2.6, to provide space for learners to create brand new conjectures. Indeed the following is a typical kind of innovative conjecture that was produced, but in the context of a rectangle:

> “If a line cuts a rectangle along the pair of longer sides into two parts so that the cross segment is equal to the longer side, then two parts can be inverted to form a rhombus.” (Lin, 2005, p. 15)

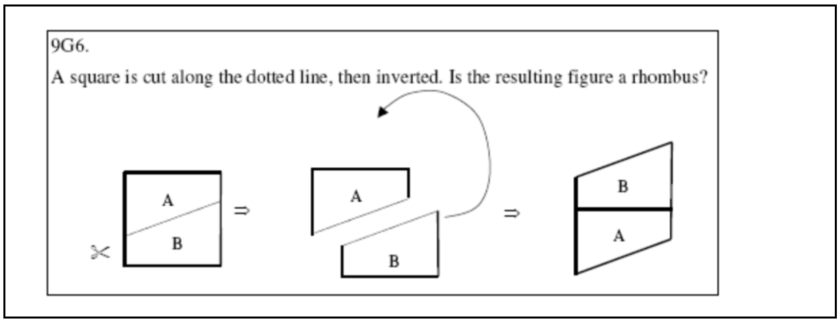



Figure 3.2.6: Geometry task investigation

### 3.2.7 Engaging with counter-examples in a teacher-education context

Counter-examples play a powerful role in terms of providing students with the necessary insight into why a particular mathematical statement is false or almost true. Hence, it is imperative that prospective mathematics teachers develop the necessary skills pertaining to the construction of counter-examples. Within this context, Ko & Knuth (2009), investigated prospective teacher’s abilities in producing counter-examples in the domain of differentiation, which is part of the calculus content in secondary schools and undergraduate mathematics courses at both universities and colleges. Results showed that the prospective secondary teachers despite having majors in mathematics struggled to identify whether a given proposition was true or false, and produce necessary counter-examples. Hence, Ko & Knuth recommended that more attention should be devoted to developing the necessary knowledge and skills of prospective mathematics educators to construct counter-examples, which they can in turn fruitfully utilize when they become fully fledged mathematics educators.

For example, the following mathematical statement was presented by Ko & Knuth to a willing sample of Taiwanese prospective secondary mathematics teachers: “Let $f$ be a function defined on a set of numbers $S$, and let $a \in S$. If $f$ is continuous at $a$, then $f$ is differentiable at $a$.” (p. 72). Results showed that about 42% could not generate a complete counter example. Two participants cited, "$f(x) = \frac{1}{x} \Rightarrow f'(x) = \ln x$", as a counter-example to refute the given mathematical statement (Ko & Nkuth, 2009, p. 72). According to Ko & Nkuth (2009, p. 72), one of the participant’s indicated the following: “$f(x)$is continuous at 0 but $f'(x)$ is not continuous at 0”, and the other showed that “$f(x)$is continuous at -1 but $f'(x)$ is not differentiable at -1”. Indeed these two participants misrepresented the first derivative of $\frac{1}{x}$ as $\ln x$, which resulted in the construction of an invalid counter-example to refute the given mathematical statement. Moreover, the first student’s response, demonstrates a misunderstanding of the notion of continuity, because $f(x) = \frac{1}{x}$ is not continuous at 0. All these excerpts, merely illustrate that these two participants did not have the necessary and sufficient understanding of continuous functions as well as differentiation. With reference to these kinds of results as the background, Ko & Knuth (2009) asserts that the conceptual understanding and strategic knowledge are pre-requisites for the determining of the truth or falsity of a proposition through either generating a proof or counter-example respectively. In the main, Ko & Knuth (2009) found that majority of the participants exhibited a lack of sound understanding of pertinent concepts in differentiation, when constructing either a proof or counter-example.

In mathematical practice, it is quite possible to raise a range of different counter-examples that can refute a particular false mathematical claim. However, from a logical perspective all such correct counter-examples share the same “logical status” and are hence considered “equally appropriate” for the refutation of a particular false claim (Peled & Zaslavsky 1997, p.49). This basically means that any single counter-example from the established range of correct counter-examples may be used to posit a mathematically sufficient argument that renders a given claim to be indeed false (Peled & Zaslavsky 1997, p. 49). On the other hand, from a pedagogical perspective, Peled & Zaslavsky (1997, p. 49) asserts that different counter-examples to a particular mathematical claim may not necessarily have the same “pedagogical status”. This translates into saying that some counter-examples to a particular claim may be much more illuminating and powerful than others with regard to the *provision of* "insight into the claim and ways to refute it" (Peled & Zaslavsky ,1997, p.49).

In a sense, just as Hanna (1990) and Hersh (1993) from a pedagogical perspective, assert that there are proofs that only prove (i.e. are convincing to a mathematician) and there are proofs that also explain (i.e. have a pedagogical power for a student), Peled & Zaslavsky (1997, p. 49) also assert that there are "counter-examples that (only) prove and there are counter-examples that (also) explain".

Furthermore, Peled & Zaslavsky (1997, p. 51) is of the view that counter-examples that have the potential to explain provide a better opportunity to facilitate learning, and in the context of disproving false claims recommends that a preferred counter-example should possess two kinds of explanatory features: "First, it should provide an explanation as to why the claim is false. Secondly, it should provide a way to see a class of additional counter-examples (if more than one exists), hence, suggest a way to generate other counter-examples." Furthermore, Peled & Zaslavsky (1997, p.51) affirms that the greater the generality of a counter-example, the greater the amount of explanatory power it possesses with respect to the explanatory features, primarily because it "provides insight into how to generate a whole family of counter-examples." Moreover, the construction or provision of a family of cases that do not satisfy a particular claim may serve as a better explanation as to why a particular claim is false as compared to the construction or provision of just one single special instance.

In particular, Peled & Zaslavsky (1997) conducted a study that focused on the explanatory nature of counter-examples, wherein the participants involved were a group of 38 in-service secondary mathematics educators who had a BSc degree in mathematics and more than 5 years of teaching experience, and a group of 45 pre-service educators in their third year of study who completed a fair number of undergraduate mathematical courses. In this study, the participants were given the following two false geometric statements (called task 1 & task 2 respectively), related to sufficient conditions for quadrilateral congruence:

1. "Two rectangles having congruent diagonals are congruent;
2. Two parallelograms having one congruent side and one congruent diagonal are congruent" Peled & Zaslavsky" (1997, p. 52).

Results showed that all of the participants, produced exactly one counter-example per given geometric statement, despite being requested to produce at least one counter-example. In terms of the correctness of the counter-examples, 37(97%) educators (teachers) and 24(53%) pre-service educators (student teachers) gave an adequate counter-example for the first geometric statement, whilst 29 (76%) educators and 19 (42%) pre-service educators gave an adequate counter-example for the second geometric statement. A Chi-square test with $p < 0{,}01$, showed that the percentage of subjects able to construct an adequate counter-example was significantly higher for educators (teachers) than pre-service educators (student teachers).

The analysis of the explanatory nature of the adequate counter-examples generated by the participants, identified the following three types of counter-examples (Peled & Zaslavsky 1997, p. 51):

1. "Specific: A counter-example which satisfies the task, as it provides a specific example contradicting the claim, yet does not give a clue to an underlying mechanism for constructing other (similar or related) counter-examples,"
2. "Semi-general: A counter-example which provides an idea about a mechanism for generating (similar or related) counter-examples for the claim, yet does not tell the "whole story" and does not cover the whole space of counter-examples,"
3. "General: A counter-example which provides a "behind the scene story", and suggests a way to generate the entire counter-example space."

In essence, specific counter-examples merely serve the purpose of refuting a claim, but are limiting in the sense that they do not contribute in any particular manner to the understanding of the general case or the development of a strategy to construct other supporting counter-examples. On the other hand, general counter-examples provide an explanation as to why the claim can be refuted and also a strategy to generate other supporting counter-examples.

Although the generation of a general counter-example to every false claim is not always feasible, the analysis in Peled & Zaslavsky (1997, p. 59) implies that "counter-examples can be used not only for satisfying tasks in the logical sense (i.e. refuting false claims) but also as didactic tools or prompts for explanations which provide insight into the meaning and essence surrounding the elements of the claim." With regard to the latter part of the implication, the findings reveal that "counter-examples are not regarded and used as didactic tools as much as they should and could be used" (Peled & Zaslavsky ,1997, p. 59).

The findings also reveal that the teachers produced more counter-examples of an explanatory nature than the student teachers. The authors attribute this disparity to teaching experience, which to an extent does contribute to raising the level of understanding of the subject-content matter as well as the level of sensitivity and appreciation of explanations. However, the authors affirm that teaching experience alone does not necessarily guarantee the production of more general counter-examples, and consequently recommends that both teachers and prospective teachers should engage in robust discussions and analysis of the pedagogical use of counter-examples over above the basic logical aspect, and also reflect continuously on their own processes of constructing counter-examples. With regard to this latter approach, Leinhardt (1993) as cited in (Peled & Zaslavsky, 1997, p. 59), argues that it will "better prepare teachers to construct powerful instructional explanations that both build on carefully chosen examples and analysis of limitations and conditions of use." So, unless teachers or prospective teachers experience such approaches & processes themselves, they will most probably not afford their learners the necessary opportunities to engage with counter-examples in manner that encourages them to explore the entire counter-example space and not just a single specific counter-example. Thus, it is necessary that teacher educators build this component regarding counter-examples into their curriculum, and continuously subject their pre-service and in-service educators to the envisaged kind experience related to counter-examples, from both a logical and pedagogical perspective (Peled & Zaslavsky, 1997).

In another study which was conducted to investigate mathematics educators "reasoning for refuting students' invalid claims" within the domain of Euclidean geometry, Potari, Zacharides and Zaslavsky (2009, p. 281-290), worked with seventy six educators that were intending to read for Master's degree in Mathematics Education.

"In a geometry lesson, in grade 10, the teacher gave the following task:
Two triangles ABT and EHZ have BT = HZ = 12 and AB = EH = 7 and the angles ATB and EZH equal to 30 degrees. Examine if the two triangles are congruent.
Two students discussed the above task and expressed the following opinions:
Student A: The two triangles have two sides and an equal angle. Therefore they are congruent.
Student B: We know from the theory that two triangles are congruent when they have two sides and a contained angle equal. Therefore the given triangles are not congruent.
If the above dialogue took place in your classroom, how would you react?"

Figure 3.2.7.1: Geometry Task

The data for their study was generated through a task that required all the educators to respond to the claim advanced by two students about the congruency of two given triangles. The details of the task are as shown in Figure 3.2.7.1. The response advanced by student *A*, stems from an over-generalization of the congruency theorem, which states: "If two triangles have two sides and the contained angle that are respectively equal then they are congruent" Potari et al. (2009, p. 281-290). The geometric construction as shown in Figure 3.2.7.2 serves as counter-example to student *A*'s explanation".

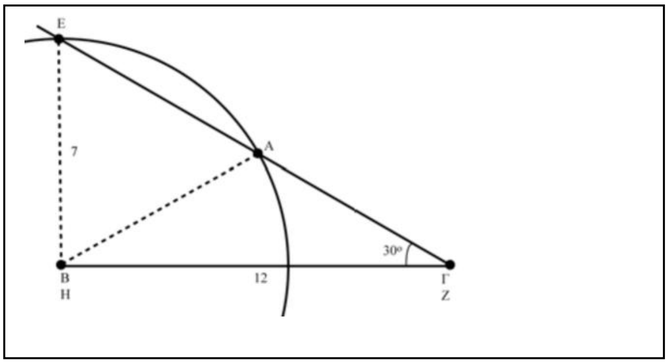



Figure 3.2.7.2: A geometric construction of a counter-example (Potari et al. , 2009, p. 283)

A reflection on the geometric construction in Figure 3.2.7.2, brings the following general geometric theorem to mind: "If the shorter of the two given sides is opposite the given angle (i.e. $b < c$, where $\beta$ is the given angle), then there will be two distinct triangles that are not congruent, except for the special case $b = cSinB$" (compare Potari et al, 2009, p. 284). Hence, a second approach to refute student's *A* claim, is to provide a deductive justification for this general theorem, and then apply the justified theorem to the specific given case (see Potari et al., 2009, p. 284). Furthermore, a third approach to refute student A's claim, could entail the use of the sine rule & cosine rule from a trigonometric perspective. For example, the application of the cosine rule for the given angle produces two possible values for the length of the third side. Furthermore, the application of the sine rule to a case where the greater of the two given sides is opposite the given angle (i.e. $b > c$ , where *B* is the given angle), yields either an acute angle or an obtuse angle opposite the larger side (see Figure 3.2.7.3). Hence, all this implies that it is possible to generate "*two (and only two) distinct non-congruent triangles*", which satisfy the given conditions of student A's claim (see Potari et al. , 2009, p. 284).

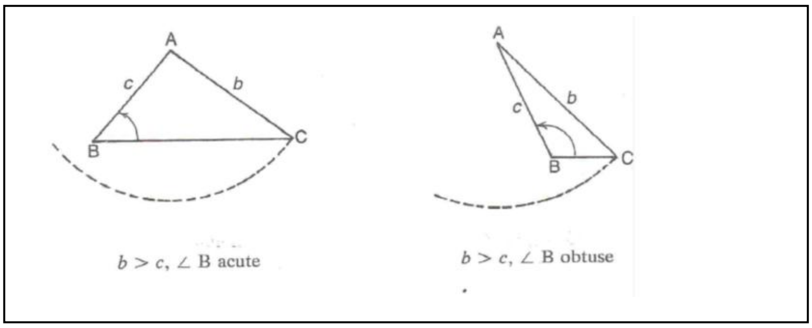



Figure 3.2.7.3 : Two distinct triangles

With regard to student *A*'s claim, results showed that no reply was obtained from three of the mathematics teachers; the triangles were regarded to be congruent by eight mathematics teachers; and the triangles were to deemed to be "not necessarily congruent" by the other sixty five mathematics teachers. In the latter case, sixty three teachers provided an explicit justification to their particular assertion or claim. Figure 3.2.7.4, gives an overview of the classification of the raised justifications into different groups. In the main, 45 out of the 63 mathematics teachers signaled that a counter-example essential to justify their claim, whilst the remaining 18 of the 63 teachers attempted to justify their claim through using known mathematical theorems relevant to the problem.

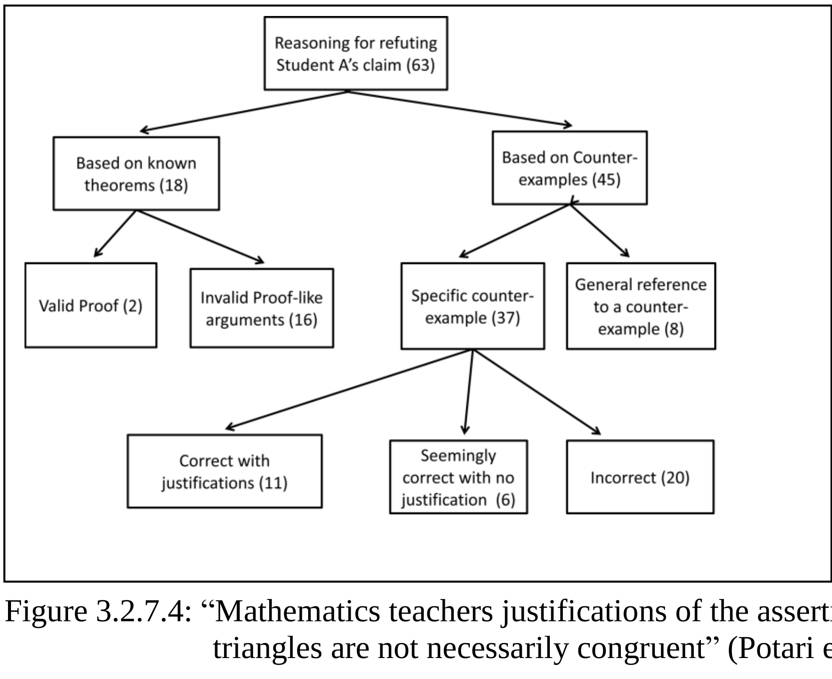



Figure 3.2.7.4: "Mathematics teachers justifications of the assertion that the two given triangles are not necessarily congruent" (Potari et al., 2009, p. 283)

As can be seen in Figure 3.2.7.4, the invalid claim raised by student A was appropriately and correctly refuted by precisely thirteen teachers. In effect 11 of them refuted correctly through constructing a counter-example whilst the other two succeeded in their refutation by using theorems. Results also showed that from a complement of 18 teachers who based their reasoning for refutation on known theorems, sixteen of the teachers purely reasoned from the perspective that none of the existing congruency theorems applies to this particular case, which is an “invalid proof-like argument” according to the authors. Potari et al. (2009, p. 284) cite the following teacher response to illustrate a typical case of an invalid proof-like argument, which teachers themselves erroneously regarded as a “valid proof for refuting student A’s claim”:

> “Student A replied without considering the known criteria for congruence triangles. I would encourage him to draw the two triangles so that to realise that these criteria cannot be applied.”

A large number (45 out of 63) of the teachers attempted to refute the student A’s claim through the notion of counter-examples. However, 8 teachers just made reference to the idea of a counter-example, but with no serious consideration of a tangible counter-example. On the other hand 37 out of the 45 teachers made an attempt to provide their own specific counter-example in order to refute student A’s claim. In their attempt, 20 of these 37 teachers actually provided incorrect counter-examples, whist six teachers provided counter-examples that was kind of correct but had no justification, and 11 teachers provided a counter-example that was correct with the necessary justification by actually constructing two triangles that satisfied the premise-conditions of student A’s claim but were not congruent (see Potari et al., 2009, p. 286)

Potari et al. (200, p. 288) are of the view that the findings of their study has some parallels with the findings of Lin’s (2005) study, in the sense that teachers either did one of the following:

i. “Confirmed the invalid claim,”
ii. “Suggested the possibility of a counter example without generating it,”
iii. “Constructed a counter-example accompanied by a mathematical proof.”

Moreover, results show that teachers reflected on their personal example spaces, when they attempted to generate counter-examples

Giannajoulias, Matorides, Potari & Zachariades (2010, p. 160) in their study investigated the nature and structure of secondary school teachers’ argumentation that was used to “convince their students about the invalidity of their mathematical claims in the context of calculus”. In particular 18 teachers of secondary school mathematics were presented with three possibilities of a student's proof to a particular mathematical statement, but the proof contained an algebraic claim that was invalid. The mathematics teachers were given the task to analyze each of the constructed proofs in order to pick up possible errors and moreover "explain how they would refute the students' invalid claims" Giannajoulias et al. (2010, p. 160). However, results of the study showed that teachers undervalued the role of counter-examples when attempting to refute invalid claims, a finding that that also surfaced across other studies pertaining to students (see for example Balacheff ,1991; Larsen & Zandieh, 2008; Lin, 2005). The study in fact showed that most teachers used theoretical arguments based on theorems to refute a claim, and by and large refrained from using counter-examples for such purpose. In particular, the study showed that a very limited number of teachers made use of counterexamples "in their argumentation", and that most teachers generally "underestimate their value as a proof method" (Giannajoulias et al., 2010, p. 160).

The rationale for such behavioral action was premised on the following kind of belief held by the teachers themselves:

> "refutation by using theorems provides stronger and more general conclusions than by using counter-examples and that counterexamples are exceptions in the sense that Lakatos (1976) discusses them" (Giannajoulias et al., 2010, p. 166).

The aforementioned study also revealed that the use of counter-examples was limited to instances where an appropriate theorem could not be used or instances where teachers wanted to just bolster some support for their theoretical arguments.

### 3.2.8 Counter-examples as a pedagogical strategy

The first experience of students with counter-examples can be challenging in the sense that they might battle to comprehend that a correct mathematical statement cannot be argued to be true by the mere provision of single example which shows that the statement is indeed true, but that just one example (known as a counter-example) which shows that a given false mathematical statement is not true is indeed enough to conclude that the given false mathematical statement is false (see Klykmuch, 2008; de Villiers, 2004). On the other hand, students may just regard a particular counter-example as a mere exception to the rule, and consequently not regard a specific false conjecture to be false. Within this context, Selden & Selden (1988) says:

> “Students quite often fail to see a single counter-example as disproving a conjecture. This can happen when a counter-example is perceived as ‘the only one that exists’, rather than being seen as generic” (As quoted in Klymchuck, 2008, p. 5).

To avoid the aforementioned kind of responses from students both at school or tertiary level, then it is incumbent upon teachers and lecturers to engage their students in counter-example activities in way that mirrors the trajectory of a mathematician’s lived experience. In this sense, Lin & Yu’s (2005) study with high school students who were involved with refuting and conjecturing, revealed that one can create an environment for the construction of counter-examples by actually deliberately posing false propositions to the student, and that by using their model of procedural refuting, learners can develop a better understanding of both the role & type of counter-examples and thereby enhance their competencies of conjecturing and indirect proving.

Similarly, Klymchuck (2008, p. 4) asserts that students should be given the necessary opportunity in the classroom to examine and comment on a “mixture of correct and incorrect statements”, which will then create a space for learners to refute an incorrect mathematical claim via counter-example(s). Also Klymchuck (2008, p. 4) suggests that counter-examples can be used teaching in the following ways:

- In your lesson or lecture make a deliberate mistake;
- Design tasks like tests, assignments, etc., that require the citing or construction of a counter-example;
- Request students to spot an error in a given text; and
- Request students to develop their own incorrect mathematical statements and counter-examples to them.

In summary, this review of several studies pertaining to counter-examples as discussed in this section has shown that the appropriate use of counter-examples can deepen students’ conceptual understanding, and also reduce or eliminate some misconceptions (see Section 3.1 for detailed discussion on misconceptions). These benefits of the use of counter-examples are also acknowledged by Klymchuck (2008, p. 2), who in addition asserts the following benefits:

- Provides a space for the development of core skills such as conjecturing, analysing, justifying, verifying refuting, proving.
- Supports a much more interactive and critical mind with respect to a learning situation or learning context.
- Promotes the development of "students' mathematical understanding beyond the merely procedural or algorithmic levels".
- Students' "example set" of particular concepts (for example functions) are developed and expanded, and in the process links and relationships among mathematical ideas are brought to the fore.

The next two Chapters (4 and 5) focuses on the theoretical considerations and frameworks related to this study.

# Chapter 4: Theoretical Considerations and Frameworks

## 4.0 Introduction

I have thus far used the previous Chapters to signpost the need to engage pre-service mathematics teachers in the processes of generalizing and justifying in a particular domain and then extending such a generalization across other domains via experimentation and justification, as well as the purpose and core research questions for this study. Having done that, I wish to use this Chapter to introduce the conceptual framework for this study from a learning theory perspective. To address the purpose of the study and answer the proposed research questions for this study, it is necessary for me to engineer a conceptual framework that embraces a number of connected theoretical frameworks and topics that are compatible with the constructivist theory of learning. In view of the extent of the topics and theoretical frameworks that have been considered to guide this study, I wish to discuss some of them in Chapter 4 and others are discussed in Chapter 5. In Chapter 4, the learning theory of constructivism in relation to cognitive and social constructivism is described and discussed. In addition theoretical considerations pertaining to generalizations, justifications, scaffolding, discovery learning, and analogical transfer are discussed. In particular, Chapter 4 engages with the following theoretical frameworks: Piaget's Theory of Equilibration (which is also known as Piaget's Theory of Socio-cognitive Conflict); Gentner's Structure Mapping Theory and Ausubel's Theory of Meaningful Learning. The other remaining theoretical considerations are discussed in Chapter 5.

## 4.1 Conceptual Framework

Maxwell (2005, p. 33) citing Miles & Huberman (1994) and Robson (2002) describes the conceptual framework of a study as "…the system of concepts, assumptions, expectations, beliefs, and theories that supports and informs your research – is a key part of your design". Similarly, Bell (2005) is of the view that a conceptual framework is the basic structure that provides the necessary grounds on which a particular research study can be built upon. Furthermore, a conceptual framework ought to facilitate sense making and contextual understanding of the findings of a research study for both practitioners and researchers. In this respect, Polit and Hungler (1995, p. 101) as cited in Al-Eissa (2009, p. 86) holds the firm view that:

“Frameworks are efficient mechanisms for drawing together and summarizing accumulated facts… The linkage of findings into a coherent structure makes the body of accumulated knowledge more accessible and, thus, more useful for both practitioners who seek to implement findings and researchers who seek to extend the knowledge base.”

Further to this, Maxwell (2005, p. 35) asserts that:

“…conceptual framework for your research study is something that is constructed, not found. It incorporates pieces borrowed from elsewhere, but the structure, the overall coherence, is something you build, not something that exists ready- made; … the most productive conceptual framework are those that integrate different approaches, lines of investigations, or theories that no one connected.”

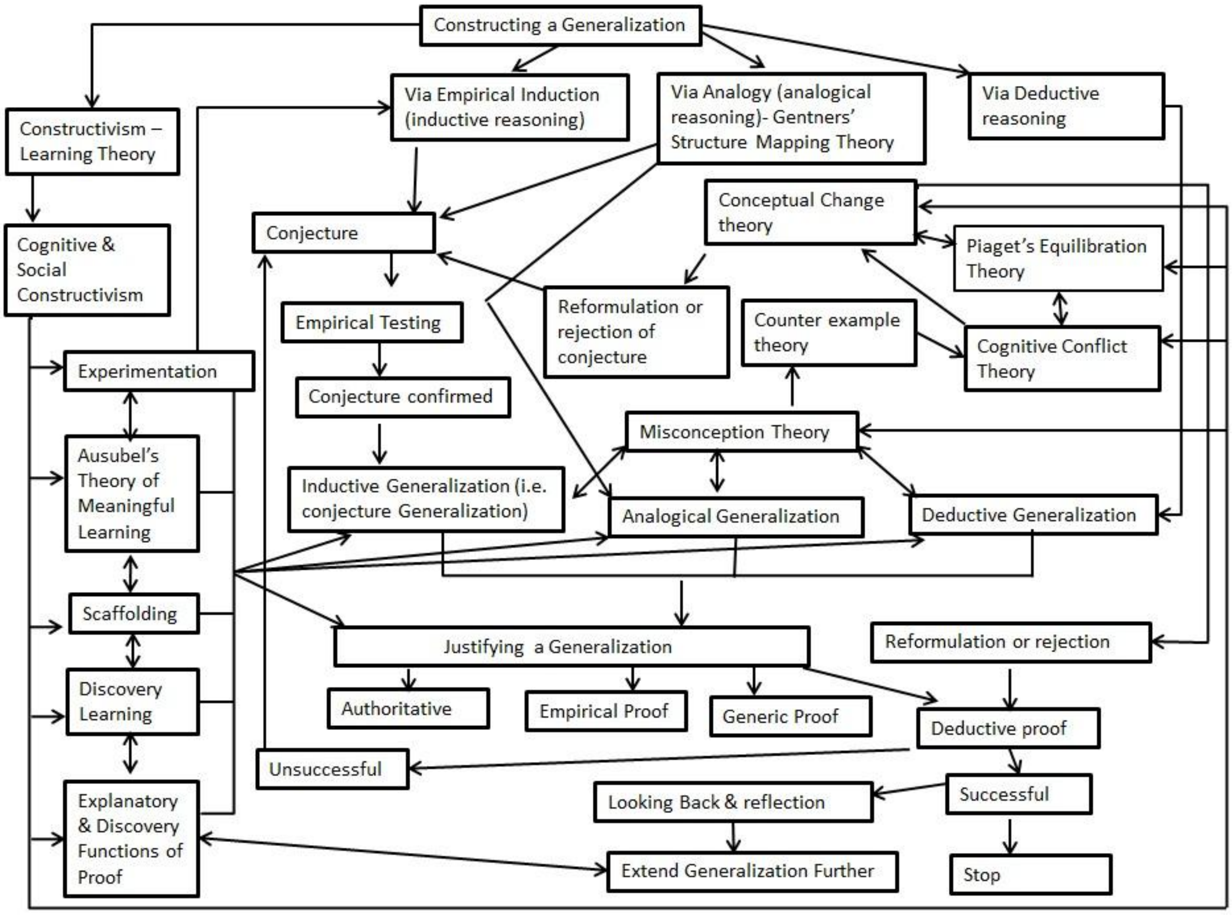



Figure 4.1: Conceptual Framework

Given this background, my conceptual framework as summarized in Figure 4.1 serves as a frame of reference for my study as it informs the rest of the research design. In other words it helps me to develop research questions that are realistic, focused and relevant, select appropriate methods and identify potential validity threats to my conclusions that can impact my study (Maxwell, 2005).

## 4.2 Constructivism: A learning theory

In this section, I wish to explore the notion of constructivism as a learning theory as well as cognitive and social constructivism. This I believe will guide the overall direction of this study as discussed further in Section 4.2.2.

### 4.2.1 The main tenets of constructivism

> **“**Learning is much more than memory. For students to really understand and be able to apply knowledge, they must work to solve problems, to discover things for themselves, to wrestle with ideas… The task of education is not to pour information into student’s heads, but to engage students’ minds with powerful and useful concepts” (Slavin, 1997, p. 269)

The above statement is consistent with constructivism, which is a learning theory that principally says “Learners construct, rather than record, knowledge” (Eggen & Kauchack, 2007, p. 234). Expanding on this principle, constructivism can be described as a view of learning that sees learning as: “A building activity in which individuals build an understanding of events, concepts and processes, based on their personal experiences and often supported by, amongst other things, activity and interaction with others” (Pritchard, 2007, p. 2). The main tenet of constructivism is the idea that students make use of their existing knowledge or information (for example, facts, concepts and procedures) to make sense of new information and new experiences, and thereby fashion a solution to a problem or modify their existing ideas or re-organize what is already known (Appleton, 1997, p. 303). This means that students continuously seek to interpret the teacher’s instructions and discussions according to their existing understandings and experiences (Cobb and Steffe, 1983; Maher, Paca, & Pancari, 1988). In other words students are active processors of information, and do not merely absorb knowledge that a teacher tries to pass onto them in class. Olivier (1992) in citing Piaget (1970) and Skemp (1979), says:

> “A constructive perspective on learning assumes that concepts are not taken directly from an experience, but that a person’s ability to learn from and what he learns from

> an experience depends on the quality of ideas he is able to bring to that experience…. Knowledge does not simply arise from experience. Rather it arises from the interaction between experience and our current knowledge structures” (1992, p. 195).

Thus, constructivism holds that when learners are able to construct a plausible solution to a problem or create new ideas or knowledge (like rules and hypothesis that explains things) through searching, using and interacting with relatable prior knowledge, meaningful learning is mostly likely to be experienced by such a cohort of learners (Ausubel, Novak, & Hanesian, 1978; Driscoll, 2005). Expressed in another way, constructivism is an epistemological view of knowledge acquisition which emphasize that meaningful learning is the active construction of knowledge by the student and not the mere transfer of objective knowledge from one student to another or from teacher to student (Applefield, Huber, & Moallem, 2000, p. 36. Olivier, 1992, p. 195; Snowman, McCowan, & Biehler , 2009, p. 490). However, in relation to the student constructing his/her own knowledge, Balacheff ( 1991, p. 89) in citing Piaget (1975) says: “The starting point of this developmental process … is the experience of a contradiction which is likely to provide a cognitive disequilibrium: It is the overcoming of such a contradiction which results in new constructions.” This then implies that as teachers and lecturers, one should constantly take cognisance of *what* one teaches and *how* we teach it (Lockhead, 1991, p. 75).

Furthermore, meaningful learning just does not happen arbitrarily or haphazardly, but it relates to information or concepts that learners already have developed (Ausubel, Novak, & Hanesian, 1978). For example, if we learn that a square is a regular figure, this information relates to our existing information about a square and about the regularity concept. In this instance the connection between ‘square’ and ‘regular’ is not arbitrary but rather a selective interplay between ‘existing’ and ‘new’ knowledge. The implication of this is that students do not enter our classrooms with blank minds, that is as ‘tabula rasa’ or passively receive information, but rather participate actively in the construction of ‘new’ knowledge by always attempting to assign new data or information meaning in relation to their individual perceptions of prior experiences (Applefield, Huber, Moallem, 2000, p. 43; Ogunnyi & Mikalsen, 2004, p. 153). It is in this sense, that Bodner (1986, p. 873) re-iterates Piaget’s constructivist perspective on learning by saying: “Knowledge is constructed in the mind of the learner.”

The afore-mentioned perspective on learning has profound implications for teaching. This view was first conjectured by Piaget who after doing research on how children acquire knowledge, provided some answers to the core question of epistemology, namely "how do we come to know what we know?" (see Bodner 1986, p. 874). It basically means that students must be given more opportunities to construct their own knowledge than is typical across many of our classrooms (Slavin, 1997, p. 270). For example, students should be given authentic tasks that allow them to experimentally explore, observe, make conjectures and construct generalizations, which they can then be encouraged to support by providing a logical explanation or explain why such a generalization is always true. This, in a sense suggests that our classrooms should become more student-centred with the teacher acting as a facilitator or guide rather than actually doing everything for the learner in a recipe mode style. As Confrey (1990, p. 100) says:

> "When I teach mathematics I am not telling students about the mathematical structures which underlie objects in the world, I am teaching them how to develop their cognition, how to see the world through a set of quantitative lenses which I believe provide a powerful way of making sense of the world."

When a classroom environment like the one described by Confrey (1990) is provided to students, it is plausible that students could make sense of new results and thereby build their own understanding of such results, i.e. they construct meaning for themselves. In keeping with Confrey's perspective of inducing meaningful learning, Von Glasersfeld (1984) as cited in Bodner (1986, p. 874), maintains: "… learners construct understanding. They do not simply mirror and reflect what they are told to do or what they read. Learners look for meaning and will try to find regularity and order in the events of the world in the absence of full or complete information".

Although learners construct their own knowledge in the form of concepts, theories, ideas, propositions, generalizations, proofs, solutions to problems, it does not mean that they are given a free reign to construct any knowledge. The knowledge they construct needs to be continually tested via their own personal experiences across similar or new contexts or though mediated learning environments (Bodner, 1986; Njisane, 1992). To Piaget the process of constructing knowledge is characterized by the learner trying to make sense of his/her new experiences in terms of the linked prior knowledge that is stored in a particular cognitive schema(s). This means that the knowledge the learner brings with him/her to a learning task has a powerful effect on the way the learner comes to see what s/he understands and subsequently build new knowledge on (Snowman, McCowan, & Biehler , 2009, p. 239, Tobin & Tippins, 1993, p. 7). Over the years practitioners in a variety of fields have embraced constructivism as a theoretical framework on which to base their activities, and in the process, a number of kinds of constructivism have emerged with particular theoretical rationale. For example, naïve constructivism, radical constructivism, cognitive constructivism, social constructivism.

Despite the various kinds of constructivism that evolved over the years, Cobb (1988, p. 89), a constructivist, asserts that:

> "A fundamental goal of mathematics instruction is or should be to help students build structures that are more complex, powerful, and abstract than those that they possess when instruction commences. The teacher's role is not merely to convey to students information about mathematics. One of the teacher's primary responsibilities is to facilitate profound cognitive restructuring and conceptual reorganization."

Through taking cognizance of Cobb's (1998) constructivist views on learning of mathematics, Clements & Battista (1988, p. 35) expresses another goal of constructivism as follows:

> "Students should become autonomous and self motivated in their mathematical activity. Such students believe that mathematics is a way of thinking about problems. They believe that they do not "get" mathematical knowledge from their teacher so much as from their own explorations, thinking, and participation in discussions. They see their responsibility in mathematics classrooms not so much as completing assigned tasks but as making sense of, and communicating about, mathematics. Such independent students have the sense of themselves as controlling and creating mathematics."

A synthesis of the views of those who have used the different brands of constructivism signals that "knowledge is personally constructed but socially mediated" (Tobin & Tippens 1993, p. 6). Thus, in essence there exists a dialectical relationship between the individuals' contribution to knowledge and the social contribution. For example, Tobins & Tippens (1993,

p. 7), citing Saxe (1992), state that "individuals construct novel understandings as they attempt to accomplish goals rooted in both their prior understandings and their socially organized activities". Furthermore, constructivists like Cobb (1990); Saxe (1992) & Wood et al. (1992), suggest that practitioners should not see the social and personal emphases of constructivism as an either/or dichotomy, particularly since they "both have important roles in thinking about knowledge, knowing, and teacher and learner roles in our classrooms" (Tobins & Tippens ,1993, p. 6). In this respect, Wood et al. (1992, p. 3) say:

> "It is useful to see mathematics as both cognitive activity constrained by social and cultural processes, and as a social and cultural phenomenon that is constituted by a community of actively cognizing individuals."

Hence, for the purpose of this study, I have adopted both cognitive and social constructivism so as to allow the emphases in constructive thought to embrace the twin processes of equilibration, namely assimilation and accommodation, as students work in a dynamic geometry environment supported by scaffolded task-based activities and myself as the facilitator and researcher. The use of this complementary paradigm, gave latitude to the researcher to design task based activities that can allow the pre-service teachers to experiment, explore, conjecture, generalize, justify, and also possibly exhibit misconception which can be corrected. By using both forms of constructivist theories of learning, I envisaged that it would be possible to account for a pre-service teacher construction of personal meaning and shared meaning, and also account for the errors and misconceptions, and episodes of conceptual change induced through experimentation and counter-examples.

### 4.2.2 Cognitive and Social constructivism

Cognitive constructivism, which is mainly an outgrowth of Piaget's ideas of cognitive development as discussed in Section 4.6, emphasizes that:

> "Knowledge is acquired as a result of a life-long constructive process in which we try to organize, structure, and restructure our experiences in the light of existing schemes (schemas) of thought, and thereby modify and expand these schemes (schemas)" (Bodner, 1986, p. 874)

Thus cognitive constructivism is a form of constructivist learning theory which emphasizes that as learners confront new information from their environment – the physical and social world around them they constantly search for meaning and understanding in relation to their existing understanding or schemas (Piaget, 1952, 1959, & 1989; Eggen & Kuachak, 2007; Pritchard, 2009) to try and maintain or re-establish their cognitive equilibrium. Schemas are large units of inter-related concepts and ideas which are stored in memory that can be retrieved and utilized to make sense of new experiences. When we are able to explain new experiences in terms of our existing schema(s), we say that the cognitive equilibrium is maintained, and when we cannot we say that our cognitive equilibrium is disturbed. Piaget (1970) says that when disequilibrium occurs, there often is a natural tendency to re-establish cognitive equilibrium, and this is when individual knowledge construction is stimulated and learners thinking advances – thus characterizing intellectual development.

As discussed in Section 4.6, equilibrium is maintained and achieved through the process of adaptation. As shown in Figure 4.2.2 in this section, adaptation consists of two reciprocal processes, namely accommodation and assimilation.

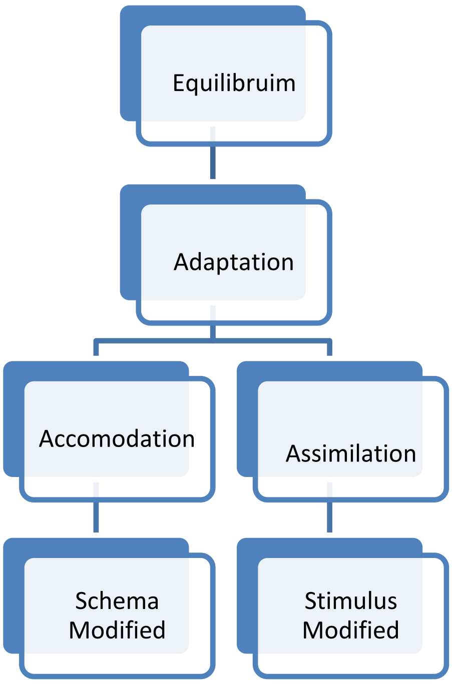



Figure 4.2.2: Maintaining equilibrium through the process of adaptation (Eggen & Kauchak, 2007, p. 37)

When new information arises, and can be fitted into a learner's existing schema with no dissonance, we say the process of assimilation has taken place. Through the assimilation of the new material or information the learner's existing schemata are enlarged, that is existing concept(s) are extended to cover a broader spectrum of ideas and examples and further distinctions through differentiation (Olivier, 1992, p. 196). However, when a learner experiences new information that is inconsistent with his current schema or conflicts with his ideas stored in his existing schema, a process of accommodation has to take place to restore the necessary equilibrium. This means that through careful and critical thinking, existing schemas have to be adjusted, modified or restructured so that the new information, ideas and procedures can be fitted in without causing any more dissonance. When this happens the desired cognitive equilibrium will be achieved, and learning would have occurred (see Louw & Louw, 1998, p. 43).

In essence cognitive constructivism focuses on how a learner builds new knowledge via innate processes like assimilation, dis-equilibration, accommodation, equilibration, which are supported by opportunities that makes it possible for the learner to interact with his teachers, peers, objects or things around him/her in a meaningful way (Snowman, McCowan, & Biehler , 2009, p. 240). For example, when learners experience a discrepant event that causes cognitive conflict, cognitive disequilibrium is induced, and they then begin to negotiate the meaning of the ideas and experiences that are not consistent with the existing schema to try and uncover the inconsistency between their current conception/knowledge and their new experience (Dalgarno, 2009). In this respect Fosnot (1996, p. 29) emphasizes that disequilibrium facilitates learning and that challenging tasks set in meaningful contexts should be given to students with the aim to allow them to explore and generate affirming and contradictory instances. However, in doing so, the errors committed by students should not be disregarded but rather be illuminated, further explored and discussed to correct the underlying misconception(s).

This is to suggest that through carefully designed individual or socially mediated discovery-oriented learning activities, the learner(s) can uncover the noted inconsistency or discrepancy, and thereby reorganize and restructure their existing conceptions or schema to accommodate the new findings or discoveries. It is in this way that new knowledge is generated and conceptual change is advanced as postulated by Piaget. Furthermore, this means that through engaging students meaningfully in investigative and problem-solving tasks that promote an environment characterized by discussions, debates, critical and reflective thinking related to their mathematical experiences inclusive of their misconceptions, students can become more actively involved in constructing progressively more complex understandings of mathematical content for themselves (see Donald, Lazarus & Lolwana 1997, p. 41). This in principle is consistent with Piaget's (1953) view that knowledge is not just transmitted 'as is' from one person to the next, but is rather actively constructed and developed to higher levels by the person encountering authentic learning experiences within their social and physical environment (see Donald, Lazarus & Lolwana, 1997, p. 41).

Building on Piaget's notion that a learner constructs his/her own knowledge through encountering particular experiences within a given environment, social constructivism places more emphasis on the building of knowledge via social interaction and language usage (Eggen & Kauchek, 2007). In fact the basic premise of social constructivism as highlighted by Vygotsky (1978) and other social constructivists like Ernest (1994) is that knowledge construction is a shared experience rather than just an individual experience, and that through the process of sharing individual perspectives learners construct understanding, which at times may not just be achievable by a learner on his/her own (Gauvain, 2001; Greeno, Collins, & Resnick, 1996). In the same vein, Snowman, McCown and Biehler (2009, p. 495) defines social constructivism as: "A form of constructivist learning theory that emphasizes how people use such cultural tools as language, mathematics, and approaches to problem solving in social settings to construct common or shared understanding of the world in which they live" (2009, p. 495). Ernest (1994, p. 304) in adopting a complementarity stance to constructivism says that a social constructivist theory to learning mathematics is: "a theory which acknowledges that both the social and individual sense making have central parts to play in the learning of mathematics." Similarly, Murray (1992) supports the view that mathematical knowledge is individual and a social construction. Furthermore, Ernest's (1991) complementarity notion of social constructivism as reported in Ernest (1994, p. 308) characterizes two key features of constructivism as follows:

> "First of all there, there is the active construction of knowledge, typically concepts and hypotheses, on the basis of experiences and previous knowledge. These provide a basis for understanding and serve the purpose of guiding future actions. Secondly there is the essential role played by experience and interaction with the physical and social worlds, in both the physical action and speech modes. This experience constitutes the intended use of the knowledge, but it provides the conflicts between

intended and perceived outcomes which lead to the restructuring of knowledge, to improve its fit with experience."

When it comes to social experiences, Piaget asserts that peer interactions are a much better catalyst of cognitive development than the sets of interaction with adult. The reason for this is that the learners find it more comfortable to engage, challenge, discuss, debate and analyse the reasons and argument (s) of another learner's view than they would be able to do similarly with an adult (Snowman, McCown & Biehler, 2009, p. 68). Further, Snowman, McCown & Biehler, (2009, p. 68) believe that:

> "It is the need to understand the ideas of a peer or playmate in order in order to formulate responses to those ideas that lead to less egocentrism and the development of new, more complex mental schemes. Put another way, a strongly felt sense of cognitive conflict impels the child to strive for a higher level of equilibrium. Formal instruction by an adult expert simply does not have the same impact regardless of how well designed it might be. That is why parents and teachers are often surprised to find children agreeing on the same issue after having rejected an adult's explanation on the very same thing."

Thus, the kinds of activities that we design for our classrooms, should allow learners to interact with their peers to a justifiable extent such that they can conjecture, discover, generalize, and justify their generalizations with arguments substantiated by appropriated reasons, warrants and backings. In this way, it is highly likely that learners could enlarge/modify their existing schemas through the processes of assimilation and accommodation as they construct and acquire new knowledge. Furthermore, this kind of learning trajectory could also be promoted by engaging learners in learning opportunities that allow them to work in socially mediated environments like *Geometer's Sketchpad* and *Geo-gebra* individually or in groups. In this sense the computer with the associated software program can serve as "an expert peer or collaborative partner to support the skills and strategies that can be internalized by the learner" (Snowman, Mcown & Biehler, 2009, p. 80).

According to Snowman, McCown & Biehler (2009, p. 78), technology inclusive of programs like *Geometer's Sketchpad*, can be used to stretch learners thinking and enhance their cognitive development in two possible ways. Firstly, technology can be used as a simulation tool or microworld to enable a learner to build new knowledge and also to overcome their misconceptions and errors in thinking. Secondly, technology can be used as a source that enables learners in a classroom to think critically, debate issues, experience ideas that do not quite fit their current conception and as a result experience cognitive conflict and disequilibrium.

From a social constructivist perspective, the kind of learning opportunities that we present to our learners either in technology based or non-technology based environments, should make provision for or take cognisance of the following:

- Learning should be facilitated preferably by a more knowledgeable person, like a teacher for example. However, in such instances the teacher should be serving as a guide (or facilitator) as the learner works collaboratively (i.e. with his/her peers or in a computer environment) to construct new conceptions or ideas, and only intervene in instances when a learner:
    - seems to be struggling to move on with the given task or parts of the given tasks; or
    - demonstrates misconceptions and errors in thinking; or
    - requests clarification or specific help; or
    - made a major breakthrough with the given problem or task, for example made a discovery or provided a logical explanation to a conjectured generalization.

- The guidance provided should enable a student(s) to link whatever knowledge and skills they are expected to acquire to their existing schemas, but in a manner that allows them to gradually internalize such new knowledge and skills so that they can in the process become increasingly self-regulated and independent (Snowman, McCown & Biehler (2009, p. 240).

- When a learner cannot complete a task when working on his/her own or collaboratively, the teacher (or facilitator) should first diagnose the problem to ascertain the conceptual and procedural difficulties that the learner has been experiencing, and in the process determine what is blocking/hindering the learner from making progress with the given task or problem or to complete the task independently on his/her own. According to his/her diagnoses the teacher or facilitator should provide the necessary treatment via scaffolding, which is the name given to:

> "the process of giving support to the learners at the appropriate time, at the appropriate level of sophistication, and in an appropriate way to meet the individual's need" (Pritchard, 2007, p. 6). In other words, scaffolding refers to the minimal assistance that is provided to a learner (or group of learners) to jump start the given task (or problem) and help the learner to complete the allocated task (or problem) as far as possible on their own (Bruner, 1985, 1986; Vygotsky, 1978). This could mean providing the learner (or group of learners) with a great deal of support at the beginning of the teacher (facilitator) intervention, and then diminishing such support as the task or problem progresses to make allowance for the learner to take increasing responsibility for his/her learning as soon as he/she is able to do so (Eggen & Kaucak, 2007, Puntambekar & Hubscher, 2005; Slavin, 1997; Snowman, McCown & Biehler 2009; Wood, Bruner, & Ross, 1976).
>
> According to Pritchard (2007), Slavin (1997) and Snowman, McCown & Biehler (2009) scaffolding can take place in various ways, for example: tailor made investigative tasks; the provision of clues and hints when a student is working on a problem, providing an example; a spontaneous question or reminder as the student is proceeding with a task; the breaking down of a problem into sub-steps; modelling; a list of reminders regarding the process/procedure to tackle a question; provision of feedback; probing via questioning; the provision of related apparatus such as *Zome* tools for 3D Geometry, tools like the scientific calculator and computer software programs like *Geometer's Sketchpad* that could help in the process of conjecturing and generalizing or completing a given problem; a writing frame to support a particular style of writing; and anything else that makes it possible for a student to grow in independence as a learner. However, in the main, when scaffolding is offered it does not mean that the teacher (or facilitator) should simplify the task, but rather that the graduated intervention of the teacher (or facilitator) should to an extent simplify the role of the learner (Greenfield, 1984).

The process of scaffolding is central to notion of the zone of proximal development (ZPD), which is described by Vygotsky (1978, p. 86) as: "the distance between the actual developmental level as determined by the leaner's independent problem solving and the level of potential development as determined through the problem solving under adult guidance or in collaboration with more capable peers". In simpler terms the difference between what a learner can successfully complete on his/her own and what he/she can accomplish with some assistance is termed ZPD. Sewell (1990) as cited in Pritchard (2007, p. 6) describes the ZPD as "a point which at which a child has partly mastered a skill but can act more effectively with the assistance of a more skilled adult or peer." Just as construction workers use scaffolding to support their building efforts, teachers can likewise use scaffolding to support the learners to make progress towards their upper limit of the ZPD and thereby achieve the desired meaning and understanding for themselves. However, the kind of support rendered should be provided according to the needs of a given learner, and not be applied blanketly to all students, i.e. it has to be differentiated. Furthermore, Pritchard (2007, p. 6) says the following about scaffolded interventions:

> "In formal situations this intervention might be planned, but often a timely and, well judged intervention depends upon circumstances that cannot necessarily be predicted, and in many situations, also upon the skill and experience of the teacher. In many cases a teacher will fulfil this role, but others are equally capable and likely to do so. In planning work for children a teacher needs to take into account the current state of the children in question, and plan accordingly and appropriately. In an ideal situation, this could mean planning for individuals, but in a more realistic situation this is not usually possible."

In general, Vygoysky (1978) asserts that when scaffolded assistance is meaningfully provided, students with a broader ZPD zone are likely to experience a greater degree of cognitive development as compared to their peers with a narrower ZPD, particularly because those students with broader ZPD have more to gain from the applied interventions or facilitation (Snowman, McCown & Biehler, 2007).

In addition to scaffolding, Snowman, McCown & Biehler (2007, p. 3) expresses the view that discovery learning is also a learning method that resonates well with the both the cognitive and social constructivist perspectives of learning. As there may be some similarities (though superficial) between discovery learning and the constructivist oriented approach to learning perspective of learning, I as the researcher am aware that epistemologically "discovery" learning and "constructivist" learning are not reconcilable. A discovery process assumes a Platonist stance and that objects exist 'a priori'; hence they are discovered. In constructivism, mental objects are not discovered, but created or re-invented. Furthermore, some constructivists like Von Glasersfeld (1991) would not see discovery learning tallying with constructivist epistemology, which assumes all knowledge is created, not discovered. However, I as the researcher, take Davis & Hersh's (1983) position that mathematics knowledge is both discovered and invented. In using a discovery approach to learning, students are often given the opportunity to experiment, explore and engage actively with objects, mathematical concepts and problems, and thereby develop their understanding of it (Slavin, 1997). It is within this realm that discovery learning feature so strongly in the design of the tasks that have been used to conduct this study. The next section 4.3 expands on the notion of discovery learning.

## 4.3 Discovery learning and exploratory environments

According to Confrey (1991, p. 112) discovery learning stresses the importance of: (1) engaging students actively in the learning process; (2) focusing more on the process of "coming to know" than just the production of correct responses or solutions; and (3) trying to unpack and make sense of the underlying concepts that are related to the given task or problem.

Thus, from a constructivist perspective on learning, teaching methods that make provision for students to work in an environment that encourages exploration, discovery of rules and conjectures, construction of generalizations and justification; establishment and modification of concepts (i.e. conceptual change) typically promotes meaningful learning (Ausubel, Novak, & Hanesain, 1978; Ausubel & Youssef, 1963; Brooks & Brooks, 2001; Bruner, 1969,1966, 1973, 1983,1986; Piaget, 1953, 1970, 1974). Resonating with this view, Confrey (1990, p. 12) is of the view that a teacher should:

> " …promote and encourage the development for each individual within his or her class of a repertoire for powerful mathematical constructions for posing, constructing, exploring, solving and justifying mathematical problems and concepts and should seek to develop in students the capacity to reflect on and evaluate the quality of their constructions."

This largely means that through using a discovery approach to learning, students should be given opportunities to learn wholly on their own by doing tasks such as investigations and problem-solving that cause them to be actively and purposefully involved in observation, reasoning, reflective abstraction, drawing of conclusions and/or the justification of such conclusions (Fostnot, 1996; Yssel & Dill, 1996; Wilcox, 1993). In most instances such discovery learning opportunities arouses students' natural curiosity, which invariably stimulates and motivates them to continue to work on their own with minimal guidance, if necessary from the teacher, until they find satisfying explanations or answers for themselves (Yssel & Dill, 1996; Slavin, 1997). In this way deeper understanding of the mathematical knowledge is promoted and the responsibility for learning is shifted to the learner (Newman, Stepich, Lehman, & Russel, 2000). Furthermore, Slavin (1997, p. 273) observe that a discovering learning approach promotes "the learning of independent problem-solving and critical-thinking skills because they must analyze and manipulate information."

Consonant with the notion of discovery learning, Bruner (1996, p. 72), a pioneer and strong promoter of discovery learning, maintains that:

> "We teach a subject not to produce living libraries of that subject, but rather to get a student to think … for himself, to consider matters as an historian does, to take part in the process of knowledge–getting. Knowing is a process not a product."

However, the process of knowledge getting as posited by Bruner does not necessarily imply that students must only discover new mathematical content such as concepts, principles, generalizations and explanations, which was not previously discovered, but rather students should be given opportunities to reinvent selected mathematical content as contained in their pre-scribed curriculum documents. This means that teachers should design tasks that allow students to experience the same kind of steps, methods and processes that the original discoverers or inventors had used or may have followed to discover or invent the selected piece of mathematical content. The latter notion of invoking students to retrace the path by which established mathematical content can be meaningfully discovered or invented, has been an underlying focus of Klein's (1924) 'bio-genetic principle' of teaching and presenting mathematical content.

In essence, Klein (1924) as cited in De Villiers (2003) argues against the sole use of the axiomatic deductive approach to present mathematical content to students, and strongly argues for the use of the 'bio-genetic principle' in teaching mathematical topics so that students can gain first-hand experience as to how mathematical content can be discovered or invented, and thereby realize that mathematics is a process and not just a 'meaningless' body of knowledge imposed upon them. In this way, mathematics students could then become empowered and motivated to experiment, explore, conjecture, wrestle with questions and controversies, justify their claims, and thereby create new mathematical content on their own or further re-discover already invented mathematics on their own (Bruner, 1967; Freudenthal, 1973; Polya, 1981).

Likewise, Human (1978, p. 20) refers to the afore-described reinvention approach to mathematical development as the 'reconstructive approach', and as cited in De Villiers (2003a, p. 13) asserts that:

> "With this term we want to indicate that content is not directly introduced to students (as finished products of mathematical activity), but that the content is newly constructed during teaching in a typical mathematical manner by the teacher and /or the students."

Consistent with the learning theory of constructivism, the use of the re-constructive approach in classrooms provides an opportunity for students to be actively engaged with genuine mathematical processes such as specializing, conjecturing, generalizing, defining, convincing, axiomatizing and justifying by which they can reconstruct or develop mathematical content that is meaningful (De Villiers, 2003a; Njisane, 1992).

One way to engage students in processes of mathematical activity that can lead up to a meaningful assimilation and/or accommodation of mathematical content is to provide environments such as *Geometer's Sketchpad*, *Geoemetric Supposer*; *Geo-gebra* and *Mathematica* that allow for discoveries and insight. For instance, students could use *Geometer's Sketchpad* to construct figures and then use the drag facility to produce numerous corresponding configurations. In this way students can observe which attributes of the figure remain invariant (i.e. constantly remain the same) when the figure is transformed in some way (i.e. when parts of the situation are varied). Hence, students could have a greater chance to visually see specific relationships playing out and/ or be encouraged make plausible conjectures, which they can later try to justify via the construction of a logical explanation (proof). In this way, students can be given an opportunity to experience a process of discovery that more closely reflects the way mathematics is often invented, and summarized by Bennett as follows: "A mathematician first visualizes and analyses a problem, making conjecture before attempting a proof" (Bennett, 2002, p. viii).

However, Balacheff (1991, p. 89) in reference to Lakatos (1976) work on proofs and refutations, is of the view that:

> “Students have to learn mathematics as social knowledge; they are not free to choose the meanings they construct. These meanings must not only be efficient in problem solving, but also be coherent with those socially recognised. This condition is necessary for the future participation of students as adults in social activities. After the first few steps, mathematics can no longer be learned by means of interactions with a physical environment, but requires the confrontation of the students’ cognitive model with that of other students or of the teacher, in the context of the given mathematical activity. Especially in dealing with refutations, the relevance of overcoming what is at stake in the confrontation of two students’ understandings of a problem and its mathematical content.”

Although discovery learning advocates that students should be given the necessary space and opportunity to work autonomously, discover as much as possible for themselves and take ‘ownership’ of their work, it is prudent that teachers do not only facilitate but track their students’ progress so that they can intervene sensitively and guide them as and when necessary (Southwood & Spanneberg, 1996, p. 70). In general, the guidance rendered should not disturb or interfere with students’ line of thinking, but should provide just enough support to enable the students to move onto a plausible problem solving path or progress with the given problem (or investigation) to the extent that they can solve the given problem or discover mathematical results largely on their own by building on their existing knowledge and experiences. The latter pedagogic move is a key practice of the guided discovery approach to learning, which is grounded in cognitive and social constructivism (Eggen & Kauchak, 2007, p. 430).

When employing the guided discovery approach in a classroom, it is customary for the teacher (or facilitator) to identify the learning outcomes for the given lesson (or task), and accordingly select the relevant target content (namely the concept, relationship, rule, generalization or proof) that s/he wants his/her students to discover or re-invent. Thereafter, the teacher (or facilitator) should design a sequential set of scaffolded questions that could lead to the target discovery or arrange the relevant information so that patterns can be found and generalizations be made. However, from the moment the students begin working on the teacher (or facilitator) designed task, the teacher (or facilitator) should continuously monitor their progress to identify their obstacles, and in the process offer the necessary guidance that will allow his/her students to overcome such obstacles and make meaningful cognitive connections that would culminate in the discovery of new results or the moment of 'Eureka' (Eggen & Kauchak, 2007; Mayer, 2004; Moreno, 2004; Snowman, McCown & Biehler, 2009). Moreover as facilitators of knowledge construction in our classrooms we need to always bear in mind that learning is not promoted only through discovery and invention but also by social discourse involving justifying, negotiation, sharing, logical explanation and evaluation (Clements & Battista, 1990; Bruner, 1986).

## 4.4 Analogical Transfer Including Gentner'' Structure Mapping Theory

### 4.4.1 Analogical Transfer

As discussed in Section 2.1.3.1, analogy is synonymous to similarity**.** Extending on this notion of similarity, Schlimm (2008, p.178) says, "an analogy is a relation of similarity between two domains, where a domain is a fixed representation of certain aspects of a phenomenon, situation, process, problem, conceptual structure, etc., that are relevant for a particular analogy". In this context, a domain constitutes of a set of elements, referred to as objects, and a number of relations that hold between them (Schlimm, 2008, p. 178). Furthermore Schlimm (2008, p.179) says: "Although inferences by analogical transfer are neither truth-preserving nor infallible, analogical reasoning plays a prominent role in advancing science and mathematics." Polya (1954a) has provided the mathematics community with an in-depth and comprehensive discussion of analogical reasoning in mathematics. According to Polya (1954a), clarifying analogies, particularly those that are vague, unclear and ambiguous, is a pivotal pre-requisite for mathematical discovery.

Thus Polya (1954a) describes three kinds of possible clarifications. Firstly, Polya (1954a, p. 28) asserts that "analogy is a similarity of relations", and affirms that the similarity is meaningful and plausible if the "relations are governed by the same laws". For example, he says "addition of numbers is analogous to multiplication of numbers", primarily because "both addition and multiplication are commutative and associative" and "both admit an inverse operation" (Polya, 1954a, p. 28).

In addition, Polya (1954, p. 28) asserts that "in general, system of objects subject to the same fundamental laws (axioms) may be considered as analogous to each other, and this kind of analogy has a completely clear meaning". This approach to analogy-making advocated by Polya, is referred to as the axiomatic approach to analogies (Schlimm, 2008).

Further to this, through a detailed discussion of the example, which illustrates addition of real numbers is analogous to the multiplication of positive numbers in another sense, Polya (1954, p. 29) demonstrates that "from any relation between original elements, we can conclude with certainty the corresponding elements of the translation , and vice versa", and goes on to further state that " a correct translation that is a one-to-one correspondence that preserves the law of certain relations, is called isomorphism in the technical language of the mathematician. Isomorphism is a fully clarified sort of analogy". In light of this, Polya (1954a, p. 29) claims that homorphism (or merohedral isomorphism) is another kind of clarified analogy, which he describes as a "systematically abridged translation", wherein subtleties might get lost whilst representing everything in the original format but ensuring that relations are preserved, despite being at a reduced level. From the clarifications, Polya has alluded to both a common set of laws or axioms, and structure–preserving mappings (which are synonymous to Gentner's (1983) Structure Mapping Theory (SMT)), as a means for making analogies precise.

Although scientists and mathematicians have successfully employed the Axiomatic Approach to analogy making (Schimm, 2008), Gentner's version of the Structure Mapping Theory (SMT) evidently seems to have received much wider use, particularly in philosophy and cognitive psychology, and has gained even more attention in the analogical reasoning literature (Gholson, Smith, Burman & Duncan, 2004).

### 4.4.2 The Structure Mapping Theory

The Structure Mapping Theory (SMT) put forth by Gentner in 1983 and reaffirmed in Gentner (1989, p. 201), holds that "an analogy is a mapping of knowledge from one domain (the base) into another (the target), which conveys that a system of relations that holds amongst the base objects also holds among the target objects." Equivalently, in mathematical language, "an analogy between two domains, is a mapping $f$ between the objects and relations of one domain (source) and another (target), such that if a relation $R$ holds for some objects $a_1$, $a_2$,... in the source domain, the corresponding relation $f(R)$ also holds for the objects $f(a_1)$, $f(a_2)$… in the target" (Schimm, 2008, p. 181).

Thus, according to Gentner (1989, p. 201), in the context of analogy making, the focus is on "relational commonalities of the objects in which those relations are embedded ". According to SMT, “in interpreting an analogy people seek to put the objects in the base in one-to-one correspondence with the objects in the target so as to obtain the maximum structural match,” but most importantly, “objects are placed in correspondence by virtue of their like roles in the common relational structure,” and not necessarily on any specific resemblance between the target objects and their corresponding base objects (Gentner, 1989, p. 201).

Moreover, within the context of analogy making, Holyoak, Gentner, & Kokinov (2001), and Gentner & Markman (1997) set forth the view that structural alignment or mapping is characterized by structural consistency, relational focus and systematicity. According to Rattermann (1997, p. 251) structural consistency means that any matching relation must subscribe to the “one-to-one mapping constraint (i.e., an element in one domain representation corresponds to at most one element in the other representation) and to the parallel connectivity constraint (i.e. if elements correspond across two representations, then the elements that are linked to them must correspond to them as well)”. In particular, Schlimm (2008, p. 181) supports Gentner & Markman’s (1997, p. 47) assertion that “parallel connectivity requires that matching relations must have matching arguments” by actually stating that “every matching relation must have matching arguments (‘parallel connectivity’). The relational focus thrust emphasizes that “analogies must involve common relations but need not involve common object descriptions” (Gentner & Markman, 1997, p. 48).

According to Holyoak, Gentner, & Kokinov (2001, p. 11), the principle of systematicity makes possible “an implicit preference for deep, interconnected systems of relations governed by higher order relations, such as causal, mathematical or functional”, while Schlimm (2008, p. 181) equivalently says “systematicity requires the mapping to include as many higher order and inter-related connections as possible.” According to Gentner & Markman (1997, p. 47), “a matching set of relations interconnected by higher order constraining relations makes a better analogical match than an equal number of matching relations that are unconnected to each other.”

### 4.4.3 Another term for the process of analogy: cognitive blending

Another term for the process of analogy is ‘cognitive blending’, which is almost certainly related to Gentner’s theory of Structural Mappings and gives helpful broad context for the central role of analogies in many forms of reasoning. In particular cognitive blending is associated with the discussion of meaning (including the technical term ‘semiotics’ the construction of meaning) and the kinds of ways we think (Fauconnier & Turner, 2002). According to Rydning (2005, p. 396), “As soon as we think of an input as a representation of another input, we have built a blend". Moreover reflecting and comprehending previous experienced ideas, theories, discourses and results enables one to bring their structures (or partial structures) into a blend characterized by a developing emergent structure that can enable one to solve a problem or explain a given result meaningfully (Rydning 2005).

### 4.4.4 Analogy in the context of similarity & explanatory structure

In order to accommodate the notion of an analogy as described in the above example, we find in the literature, the notions of relational similarity and surface similarity being clearly distinguished as follows: Surface similarity is defined as "an identity between two problem situations that play no causal role in determining the possible solution to one or the other analog" (Gentner,1989, p. 219), whereas relational similarity refers to "correspondences between objects that play parallel roles in the source and target" (Richland, Holyoak & Stigler, 2004, p. 238), and in particular, encompasses identities linked to causal relationships or "higher order relations such as cause or implied" (Gholson, Smith, Buhrman & Duncan, 1997, p. 157). In particular, Forbus, Gentner and Law (1995) and Gentner, Rattermann and Forbus (1993) as cited in Ratterman (1997, p. 250), suggest that if the representations between domains are "aligned based on common relational structures, they then exhibit relational similarity and thus "form an analogy", and "if the representations are aligned based on common objects or characteristics they form either literal similarity matches (when both objects and relations match) or mere appearance matches (when only the objects and some relations align."

According to Vosniadou (1989, p. 414) an analogy exists between any domains that "share a similar explanatory structure". In this study the broad domains are considered to be equilateral triangles, rhombi (rhombuses), pentagons, …, any equi-sided polygons. In essence, Vosniadou (1989, p. 414) postulates that analogical reasoning can occur "…between any two systems (concepts, theories, stories), which belong to fundamentally different or remote conceptual domains…", or "…to the same or at least very close conceptual domains…", as long as they share a similar explanatory structure. In particular, Vosniadou (1989, pp. 414-415) refers to the analogies as "between domain analogies" and "within domain analogies" respectively. However, "within domain analogies are regarded as literal similarities by some theories of analogy" (Vosniadou 1989, p. 415). For instance, Gentner as cited in Vosniadou (1989, p. 415) is of the firm view that "within domain comparisons are not analogies, because they involve items that are similar in many simple, descriptive, non-relational properties." Otherwise, in all other respects, Vosniadou (1989, p. 414) concurs with Gentner's Structure Mapping Theory, by actually stating (1) "the process of reasoning by analogy involves transfer of structural information from source to target system, and (2) "this transfer of knowledge is accomplished by mapping or matching processes, which consist of finding correspondences between two systems."

However, Vosniadou (1989, p. 416), argues that distinguishing between "within domain" and "between domain" comparisons, is not the driving necessity when ascertaining the "process of reasoning by analogy", but contends that reasoning by analogy can even be applied to "any two items that belong to the same fundamental category if it involves transferring an explanatory structure from one item to the other."

In more general terms the "analogy mechanism", proposed by Vosniadou (1989, p. 416), functions as follows irrespective of the nature of the given domains:

- The system retrieves a familiar source example together with an explanation of how this source example satisfies some goal (e.g., Find the dimensions of a workshop for the area to be a maximum and find its maximum area)
- The system maps the explanation derived from the source onto the target and attempts to find out if this explanation is justified by the target example.
- If the target example justifies the explanation, then it is concluded that it satisfies the goal. (Find the dimensions of a closed box for which its total surface area will be a minimum).

Below is a typical example which can illustrate the transfer of the explanatory structure from the source system to the target system. These two problems under discussion in the given example, deal with finding maxima and minima using derivatives, and thus one could say that in this case we are dealing with problems in the same fundamental category. However, the first problem relates to the concept area (or a 2-dimensional figure on plane), whilst the second problem relates to the concept volume (or a 3-dimension figure in space), and consequently one may tend to say that because these are two different conceptual domains then this equivalently means we are dealing with 'between domain analogies'.

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In the example as shown in Figure 4.4.1, it is quite evident that source (base) system can share many similar properties with the target system, but nevertheless “the reasoning process is analogical in nature because it rests on the mapping of an explanatory structure from the source system to the target system” (Vosniadou, 1989, p. 416). Moreover, the examples as shown in Figure 4.4.1, also illustrate the notion of “structural alignment” proposed by Gentner’s Structure Mapping Theory. In other words, when one reflects on the explanation structure related to the source system and target system respectively in Figure 4.4.1, we find elements of structural consistency, relational focus and systematicity upheld.

Firstly, there exists a common relation focus between both problems, primarily because both problems are associated with the application of derivatives to solve applied optimization problems (i.e. applied maximum and minimum problems using calculus methods). In this case, the student has realized that finding the dimensions of the rectangular box for which the volume will be a maximum is just the same procedure as finding the dimensions of the parking bay for which the area will be maximum, although in the target case we are dealing with a 3D-shape (or volume) and in the source target we are dealing with a 2-D shape (or area). Basically, the student has invoked or retrieved the source problem to attempt to solve the target problem. In retrieving the solution structure of the source problem, the student actually managed to retrieve or assemble the representation of the solution of the source problem on paper (or in his mind). In doing so, the student mapped the relational structure of the source problem (one that was previously solved) onto the new problem (target problem), and thus was able to solve the target problem using structure or steps of the source problem, by adapting it wherever necessary (see English, 1998; Rattermann, 1997), for example (a) in Step 2, constructing the volume formula for the given target problem, by first finding the surface area of the box in terms of the variables *x* and *h*, and then equating it to the volume of 600 $cm^3$, and thus deriving $h = \frac{600-2x^2}{6x}$; as well as in step 5, where the student had to solve a quadratic equation.

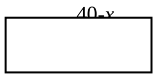
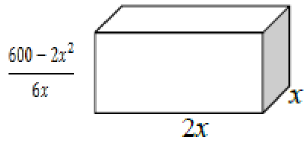
| | Source System | | Target System |
|---|---|---|---|
| Problem | A garage owner wants to build a rectangular parking bay with a perimeter of 80m so that it encloses the maximum area. Determine the dimensions of the workshop for the area to be a maximum and find this maximum area. | | A rectangular box has no lid, and is made from 600 $cm^2$ sheet metal and is $h$ cm high. The length of the container is two times as long as it is wide. Determine the dimensions of the rectangular box for which its volume will be a maximum and calculate this maximum volume. |
| Step 1 | Draw a diagram to represent the situation<br><br><br>$x$ | | Draw a diagran to represent the situation<br><br> |
| Step 2 | Insert the dimensions of the rectangle onto the diagram<br>Let length be $x$ m.<br>$\therefore$ breadth is $(40 - x)$m | | Insert the dimensions of the rectangular box onto the diagram. Let breadth be $x$ cm.<br>$\therefore$ length is $2x$ cm<br>Now the height $h$, also has to be expressed in terms of $x$:<br>Surface area of the box = 600 $cm^2$<br>$\therefore 2(2xh) + 2(xh) + 2x^2 = 600$<br>$\therefore 6xh + 2x^2 = 600$<br>$\therefore 6xh + 2x^2 = 600$<br>$\therefore\ h = \frac{600-2x^2}{6x}$ |
| Step 3 | Express the Area ($A$) of the parking bay in terms of $x$.<br>$A = x\,(40 - x)$<br>$= 40x - x^2$ | | Express the Volume of the box in terms of $x$.<br>$V = lbh = (2x)(x).\frac{600-2x^2}{6x}$<br>$= \frac{1200x^2-4x^4}{6x} = 200\,x - \frac{2x^3}{3}$ |
| Step 4 | Find $\frac{dA}{dx}$ : $\frac{dA}{dx} = 40 - 2x$ | | Find $\frac{dV}{dx}$ : $\frac{dV}{dx} = 200 - \frac{6x^2}{3}$<br>$= 200 - 2x^2$ |
| Step 5 | For maximum Area ($A$), $\frac{dA}{dx} = 0$<br>$\therefore 40 - 2x = 0$<br>$\therefore x = 20$<br>$\therefore$ for maximum area, the dimensions of the parking bay are 20m x 20m and<br>maximum area (A) = $40x - x^2$<br>$= 40(20) - (20)^2 = 400\ m^2$ | | For maximum Volume, $\frac{dV}{dx} = 0$<br>$\therefore 200 - 2x^2 = 0$<br>$\therefore x^2 = 100$<br>$\therefore x = \pm 10$<br>$\therefore x = 10$ ($x > 0$, breadth of container)<br>$\therefore$ for maximum volume, dimensions are 20 cm x 10cm x $6\frac{2}{3}$ cm, and<br>Maximum volume = $200\,(10) - \frac{2(10)^3}{3}$<br>$= 1333{,}33\ cm^3$ |

Figure 4.4.1: Explanatory structure: Structural parallelism (matching arguments)

Secondly, with regard to structural consistency, there is certainly a kind of "structural parallelism (consistent, one-to-one correspondence between the elements) that exists between the two representational structures" (Holyoak, Gentner, & Kokinov, 2001, p.11), in particular with respect to the strategy to solve the source and target problems respectively (see Figure 4.4.1). In other words, the matching relations have matching arguments (i.e. parallel connectivity is explicit between the source and target.

Thirdly, in the above example, the principle of systematicity is invoked, because in both cases, the relation is governed by the same higher order mathematical relation, namely the use of derivatives to determine the maxima or minima of a given function. In particular, the rules of differentiation were used to determine the critical points of the function $A$ in the source problem, and function $V$ in the target problem respectively, by using the fact that a critical point occurs at a point where the derivative is zero. Thus in the source case, $A'(x)$ was found and then set to be equal to zero, like $A'(x) = 0$, which was subsequently solved to find the critical value(s) that gives the maximum value of $A(x)$. Similarly, in the target case, $V'(x)$ was found, then equated to zero, and subsequently solved to determine the critical values, which will generate the maximum value of $V(x)$.

From the above discussions, it seems that Gentner's principle of structural alignment or mapping that is used to determine relational commonalities between a source and target, is also embedded in 's notion of analogical reasoning, which primarily holds the transfer of an explanatory structure as the pivotal determinant of analogical making or reasoning. However, it is not necessary to have a one-to-one mapping for every element (or step/aspects) in the source and target, but it is essential for those elements (or steps/aspects) that are connected and play a role in developing an argument or solution to the target case or problem at hand.

Further to this, from an axiomatic approach to analogies, Schlimm (2008, p.181) acknowledges that in mathematical terms, an analogy can be represented as an isomorphism, where "(i) every object of the source is mapped to a unique object of the target and (ii) every object of the target is assigned to an object of the source" , or by an embedding or partial embedding. Furthermore, according to Gentner and Markman (2005) as cited in Schlimm (2008), an analogy can also be characterized by homomorphisms, "where more than one object from the source can be mapped to a single object in the target" (Schlimm, 2008, p. 182).

Thus according to Schlimm (2008, p. 182), analogical transfer can also be described as follows: “The morphism determines a substructure that is common to the source and target domains, and those features (objects or relations) that are connected to this substructure and present in the source, but not in the target domains, are projected into the target”.

In accordance with the axiomatic approach to analogies, two systems (or domains) are analogous if the objects or elements within each system (or domain) are “subject to the same fundamental laws or axioms” (Schlimm, 2008, p. 189). Moreover, Schlimm (2008, p. 189) says that: , “the assessment of the structural alikeness of two domains in terms of laws or axioms depends on being able to find formal statements (or axioms) that, when appropriately interpreted, are true in both domains. These statements then express commonalities between the domains, that is, the positive analogies.” Schlimm (2008, p. 191) demonstrates these distinctions by using examples from group theory, concluding that “the structure- mapping explication of analogies fails in certain object rich domains (which are very common in mathematics), whereas the axiomatic approach to analogies is not subject to the same limitations and can characterize analogies in an informative way.” Despite the failure of SMT under certain conditions, it is widely used particularly in relation-rich domains, associated with philosophy, cognitive psychology, and science. However, the notion of relational similarity, which is the core entity in analogy making between domains, is embedded in both approaches, namely, the structure mapping approach and the axiomatic approach. Thus, we could regard these approaches, as approaches that support each other in analogy making, particularly in the context of mathematics.

According to Schlimm (2008, p. 192), “the process of explicating analogies by means of common axiomatization has also a number of important practical consequences, which can strongly influence further developments and prompt new discoveries.” Moreover, the axiomatic approach allows one the opportunity to determine the system of axioms that represents a specific domain, or to establish the essential commonalities amongst the domains, and thereby use it as a basis to assess whether the observed analogy can be applied to other structures in terms of finding new analogies. This essentially means that “axiomatically expressed knowledge about the structural similarity between two domains can be used directly to determine the similarity to other domains” (Schlimm, 2008, p. 194).

Regardless of whatever approach is used, the “transfer of relational knowledge across contexts is of central importance in human cognition”, more so in the context of analogical transfer leading to generalizations and new discoveries (Gentner et al., 2004, p. 586). Moreover Gick & Holyoak (1983) as cited in Gentner, Loewenstein, & Thompson (2004, p.586), asserts that, “one way to promote structural transfer is by comparing two initial examples (or cases),” primarily because the nature of the comparison itself provides a more salient and broader understanding of the common relational structure (Gentner & Markman, 1997). According to Kurtz, Miao & Gentner (2001) as cited in Gentner et al. (2004, p. 586) this process or procedure of comparing two examples or partly understood examples in order to ascertain a “common interpretation” is called “analogical encoding”. Indeed, using analogical encoding, Catrambone & Holyoak (1989) as cited in Gentner et al.(2004, p. 586), “demonstrated comparing two initial examples can facilitate deriving schema, which in turn facilitates transfer to a structurally similar cases or problem.” Indeed Gentner, Loewenstein & Thompson (2004, p. 586), further finds that “comparing two structurally similar examples (analogical encoding) not only facilitates transfer to future structurally similar cases, but also the retrieval of prior structurally similar examples or cases from memory.” Furthermore, Richland, Holyoak, & Stigler et al. (2004, p. 38) also claims that “analogical comparison can result in the formation of abstract schemas to represent the underlying structure of the source and target objects, thereby enhancing reasoners’ capacity to transfer learning across contexts.”

Based on the various discussions presented in Section 4.4, I propose to adopt Gentner’s Structure Mapping Theory ( SMT) for this study based on the following critical factors:

- Analogical reasoning involves transferring an explanatory structure between any two items (or systems or domains). In other words, analogical reasoning is not restricted to fundamentally different or remote conceptual domains (which are normally referred to as “between- domain analogies” in the literature).
- Structural parallelism is consistent, one to one correspondence between mapped elements wherein there are matching arguments (parallel connectivity).

## 4.5 Forms of Meaningful learning as viewed in Assimilation Theory

According to Ausubel's Assimilation Theory (as cited in Ausubel, Novak and Hanesian, 1978, p. 68), "new information is linked to relevant, pre-existing aspects of cognitive structure and both the newly acquired information and the pre-existing structure are modified in the process. …most meaningful learning is essentially the assimilation of new information." In other words, when new information is acquired through linking it with ones existing schemas or cognitive strcutures, then meaningful learning is said to occur (Ausubel et al,. 1978). Ausubel (as cited in Ausubel et al., 1978) proposed the following processes by which meaningful learning occurs: (i) subordinate learning which constitutes 'subsumption' (derivative subsumption and correlative subsumption), (ii) 'superordinate learning', and (iii) 'combinatorial learning'. However, "a primary process in learning is subsumption, in which new material is related to relevant ideas (or previous knowledge) in the existing cognitive structure on a non-verbatim basis (previous knowledge)" (Aziz, Razali, Hasan, & Yunos, 2009, p. 10).

### 4.5.1 Subsumption

According to Ausubel et al. (1978, p. 58), "in both concept learning and propositional learning, new information is frequently linked or anchored to relevant aspects of an individual's existing cognitive structure." This process of linking new information to pre-existing information or concepts in the individual's cognitive structure is what Ausubel has termed subsumption. However, when this new information is subsumed or incorporated into one's cognitive structure under more inclusive and general ideas, it is organized hierarchically in terms of the level of abstraction, generality, and inclusiveness of ideas, thus making it possible for the new learning to be related in a hierarchical manner to previous ideas, concepts or propositions (compare Ausubel et al., 1978; Cooper, 2009). Consequently, Ausubel et al. (1978, p. 58 ) says, "the emergence of new propositional meanings most typically reflects a subordinate relationship of the new material to the existing cognitive structure."

Ausubel (1962, p. 217) argues that when one encounters new information, the natural first process is to "subsume the new information under a relevant and more inclusive conceptual scheme." Furthermore, Ausubel (1962, p. 17) asserts that "the very fact that it is subsumable (relatable to stable elements in cognitive structure), accounts for its meaningfulness and makes possible perception of insightful relationships." However, if one encounters completely unfamiliar information or new material that cannot be subsumed, then rote learning as opposed to meaningful learning takes place (Ausubel, 1962; Cooper, 2009). Moreover, the resultant rote learning may possibly give rise to development of a new cognitive structure, which may assist with meaningful learning at some other time (see Cooper, 2009; Ausubel et al., 1978).

As alluded to earlier, there are essentially two different kinds of subsumption, namely derivative subsumption and correlative subsumption.

**4.5.1.1 Derivative subsumption**

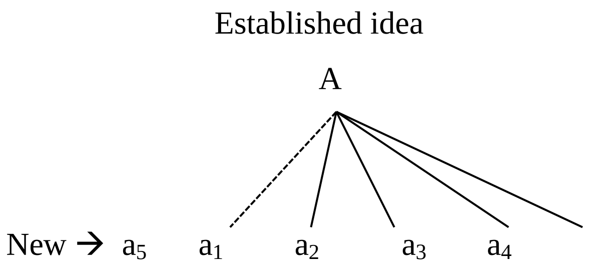
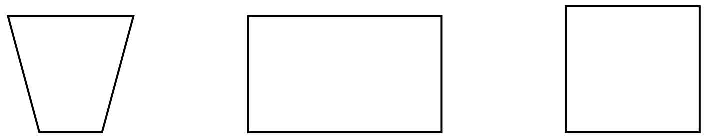



Figure 4.5.1.1: Derivative subsumption (see Ausubel et al., 1978, p .68)

According to Ausubel et al., (1978, p. 68), in derivative subsumption," new information $a_5$ is linked to the superordinate idea *A* and represents another case or extension of *A*. The critical attributes of the concept *A* are not changed, but new examples are recognized as relevant." This essentially means that the new cases or examples that learners comprehend or understand, are part and parcel or just mere examples of an established system of concepts or propositions that learners have already learned or are familiar with, or "it is just supportive or illustrative of a previously learned concept or proposition" (Ausubel et al., 1978). Furthermore, Ausubel et al., (1978, p. 58) states that "in either case the new material to be learned is directly and self-evidently derivable from or implicit in an already established and more inclusive concept or proposition in cognitive structure."

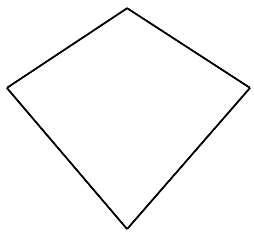
**Example 1:**

- Previous Knowledge: Let's suppose Rajen has acquired the concept of a quadrilateral in grade 5. He knows that a quadrilateral is any closed figure in the plane with four vertices and four straight sides. However, in grasping the quadrilateral concept, Rajen only experienced examples of the following shapes as quadrilaterals: trapezium, rectangle and square shapes.



Trapezium Rectangle Square

- Now in grade 6, Rajen encounters for the first time a "kite" quadrilateral, which actually fits his previous conception of a quadrilateral. In other words, the object, which in this case is the kite, reflects the characteristic property of a quadrilateral, namely, closed figure in the plane with four vertices and four straight sides.



Kite

- Consequently, the idea of a "kite" quadrilateral is linked and consolidated into his idea of a quadrilateral, without effectively changing the idea of a quadrilateral in any significant way.

#### 4.5.1.2 Correlative subsumption

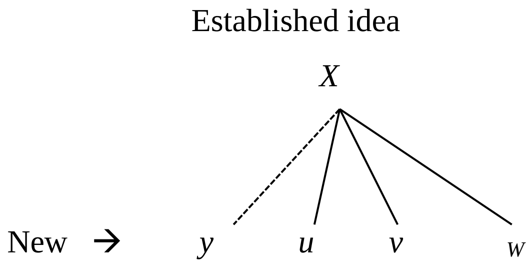



Figure 4.5.1.2**:** Correlative subsumption (see Ausubel et al., 1978, p. 68)

According to Ausubel et al., (1978, p. 68), "in correlative subsumption, new information *y* is linked to the idea *X*, but is an extension, modification, or qualification of *X*. The criterial attributes of the subsuming concept may be extended or modified with new correlative subsumption." This essentially means that an existing or previously learnt concept (or proposition) can be modified, extended or elaborated upon by subsuming the related new incoming idea into it ( Driscoll, 2000).

**Example:**

- Previous Knowledge: Let's suppose Rajen has acquired the concept of a simple closed quadrilateral in grade 5. He knows that a simple closed quadrilateral is a quadrilateral with straight sides only meeting at the vertices. Furthermore, his concept of a simple quadrilateral is limited to a convex quadrilateral, which is a simple closed quadrilateral with none of its angles reflexive.
- However, in grade 6, Rajen encounters a new kind of simple closed quadrilateral with one of its angles being reflexive, called the *concave* quadrilateral.
- Hence, Rajen has to revamp or broaden his idea of a simple closed quadrilateral, in order to accommodate the idea of concave quadrilateral. So, in other words, Rajen must change his conception of simple closed quadrilaterals from that which only contains convex quadrilateral to one that contains both convex and concave quadrilaterals. When this happens we say correlative subsumption has taken place.

In the case of correlative subsumption, one is inclined to say that this kind of learning "is more valuable learning than that of derivative subsumption", particularly because it adds value and develops a better understanding of the established concept or idea (Alexander, 2004, p. 2). Moreover, Ausubel et al. (1978, p. 58) asserts that "new subject matter is learned by correlative subsumption."

### 4.5.2. Superordinate learning:

New idea $A \rightarrow A'$

Established ideas $a_1$ $a_2$ $a_3$

Figure 4.5.2: Superordinate Learning (see Ausubel: 1978, p. 68)

"In superordinate learning, established ideas $a_{1,}$ $a_2$ and $a_3$ are recognised as more specific examples of the new idea A and become linked to A. Superordinate idea *A* is defined by a new set of criterial attributes that encompass the subordinate ideas" (Ausubel et al., 1978, p. 68). This means that in our mathematics classrooms, our learners could very well learn a new concept or inclusive proposition that could serve as an umbrella concept to subume already established ideas (Driscoll, 2000).

According to Ausubel et al., (1978, p. 59), superordinate learning is prevalent in activities that either involve inductive reasoning or synthesis of component ideas, and "occurs more commonly in conceptual than propositional learning".

**Example:**

- Rajen was familiar and well acquainted or conversant with the following quadrilaterals: rectangles, squares and isosceles trapezoids. However, he did not know that they were all examples of cyclic quadrilaterals, until he was taught that any quadrilateral with opposite angle supplementary is cyclic.
- In this case, Rajen was already aware of these quadrilaterals, but he was not aware that they were special cases of cyclic quadrilaterals, until it was explained to him (probably by his teacher) or discovered by himself perhaps by dragging a cyclic quadrilateral in a dynamic geometry environment into each of these cases. This is typically a case of superordinate learning.

### 4.5.3. Combinatorial Learning

New Idea *A* → *B* – *C*- *D* (Established Ideas)

Figure 4.5.3.1: Combinatorial Learning (see Ausubel et al., 1978, p. 68)

"In combinatorial learning New Idea *A* is seen as related to existing ideas *B, C*, and *D* but neither is more inclusive nor more specific than Ideas, *B, C*, and *D*. In this case the New Idea *A* is seen to have some critical attributes in pre-existing ideas" (Ausubel et al., 1978, p. 68). This essentially means that previous knowledge does serve as the foundation or prerequisite, upon which a new idea is developed or formulated, but the development of the new idea or proposition "bears neither a subordinate or superordinate relationship to the particular relevant ideas in the cognitive structure" (Ausubel et al., 1978, p. 59).

**Example 1:**

A carpenter wanting to design a door, may need to refer to his knowledge about the properties of a square, rectangle, parallelogram, circle, et cetera, in order to produce the final door he wants. Moreover, the carpenter must know when to use a square, rectangular or circular shaped door if he is designing a cabinet.

Furthermore**,** according to Aziz, Razali, Hasan and Yonos (2009), combinatorial learning is equivalent to analogy making or analogical thinking. According to Holyoak & Gentner (2001), the 'process of analogical thinking' is made up of connected sub-processes. For example, in a typical reasoning scenario, the following processes will be invoked as per Holyoak & Gentner's line of thinking (2001, p. 13):

- "One or more relevant analogs stored in long- term memory must be accessed"
- "A familiar analog must be mapped to the target analog to identify systematic correspondences between the two, thereby aligning the corresponding parts of each analog"
- "The resulting mapping allows analogical inferences to be made about the target analog, thus creating new knowledge to fill gaps in understanding"
- "These inferences need to be evaluated and possibly be adapted to fit the unique requirements of the target"
- "Finally in the aftermath of analogical reasoning, learning can result in the generation of new categories and schemas, the addition of new instances to memory, and new understandings of old instances and schemas that allow them to be better accessed in future."

**Example 1:**

Let's consider the factorization of a trinomial of the general form $ax^2 + bx + c$. A student for example, could easily factorize $3x^2 + 10\,xy + 8y^2$, to produce the factors $(3x + 4y)\ (x + 2y)$. However, when requested to factorize the expression, $3(a + b)^2 + 10\,(a+b)\,(a-b) + 8\,(a-b)^2$, the student observes a similarity with the example, $3x^2 + 10\,xy + 8y^2$. By using a previous idea of substitution, namely, $x = (a + b)$ and $y = (a-b)$, the student then transforms the $3(a + b)^2 + 10\,(a+b)\,(a-b) + 8\,(a-b)^2$ to read as $3x^2 + 10\,xy + 8y^2$, which s/he factorizes by referring to his/her factorization of trinomial technique. The student writes the following in his/her workbook:

$3(a + b)^2 + 10\,(a+b)\,(a-b) + 8\,(a-b)^2$

Let $x = (a + b)$ and $y = (a-b)$,

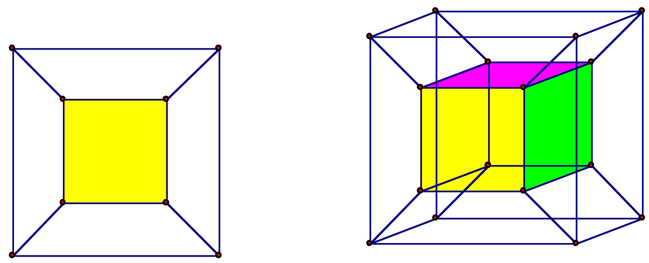
Therefore Expression = $3x^2 + 10\,xy + 8y^2$ ……… transforms trinomial

$= (3x + 4y)(x + 2y)$ ……… Factorizes the trinomial

$= \{3((a + b) + 4(a\text{-}b)\}\{(a + b) + 2(a\text{-}b)\}$ … substitutes in trinomial

$= (7a\text{-}b)(3a\text{-}b)$ ……… simplifies trinomial

**Example 2:**



a b

Figure 4.5.3.2: Analogy Making: hypercube from a 3D cube (see de Villiers, 2008, p. 35)

A hypercube (a four dimension cube) can be defined by analogy from a three-dimensional cube. According to de Villiers (2008, p. 35), “a cube when viewed directly from the front appears like a square with corresponding vertices …(i.e. a 2-D representation of a 3-D object)”. Thus, “by analogy a hypercube could be considered as a cube inside a cube (i.e. a 3-D representation of a 4D object” (see de Villiers, 2009, p. 35). This analogical process is illustrated in Figure 4.5.3.2.

Furthermore, according to Ausubel et al., (1978), students develop new generalizations , such as the hypercube from the 3D cube, even across different learning areas or fields, through combinatorial learning. For example, relationships between mass and energy, heat and volume, demand and price.

## 4.6 Piaget's Equilibration Theory, Conceptual Change & Cognitive Conflict

### 4.6.1 Piaget's Equilibration Theory

Piaget's cognitive theory suggests that cognitive development is "propelled by the human need for cognitive equilibrium – that is a state of mental balance" (Berger, 2004, p.43). In light of this, when a person is confronted with a new experience (or new information) within or outside the classroom, then he or she naturally attempts to interpret and understand such new experiences in relation to his existing understandings, knowledge and schemas. If in the event that the person's new experience or new idea 'slots' in nicely with the person's existing schemas or understandings then equilibrium may prevail upon otherwise cognitive disequilibrium. An imbalance that produces confusion, doubt, questions, dissonance, discord, , may prevail within the existing cognitive structure (schemata) (Berger, 2004; Gage & Berliner, 1992; Piaget, 1978, 1985). This imbalance then necessitates some intellectual work to restore equilibrium, which in Piaget's terms means that the person must adapt in some way to the new experience or information (Piaget, 1978, 1985).

As illustrated in Figure 4.6.1, assimilation and accommodation are two processes of adaptation that could restore equilibrium (Berger, 2004; Piaget, 1978, 1985). Assimilation "is the process of changing what is perceived so that it fits present cognitive structures" (Gage & Berliner, 1992, p. 216). This means "reinterpreting new experiences so that they fit into the old ideas" (Berger, 2004, p. 43) or already available cognitive structures (schemas). On the other hand, accommodation is "the process of changing cognitive structures so that they fit what is perceived" (Gage & Berliner, p. 216), that is "revamping old ideas so that they can fit the new experiences" (Berger, 2004, p. 43)

The journey (as illustrated in Figure 4.6.1.1) from a state of equilibrium to a state of disequilibrium as a result of some kind of conflict or contradictory statement and the subsequent return to a state of equilibrium through either the process of accommodation or process of assimilation constitutes the process of equilibration (Berger, 2004; O'Donell, Reeve & Smith, 2009; Piaget 1978, 1985).

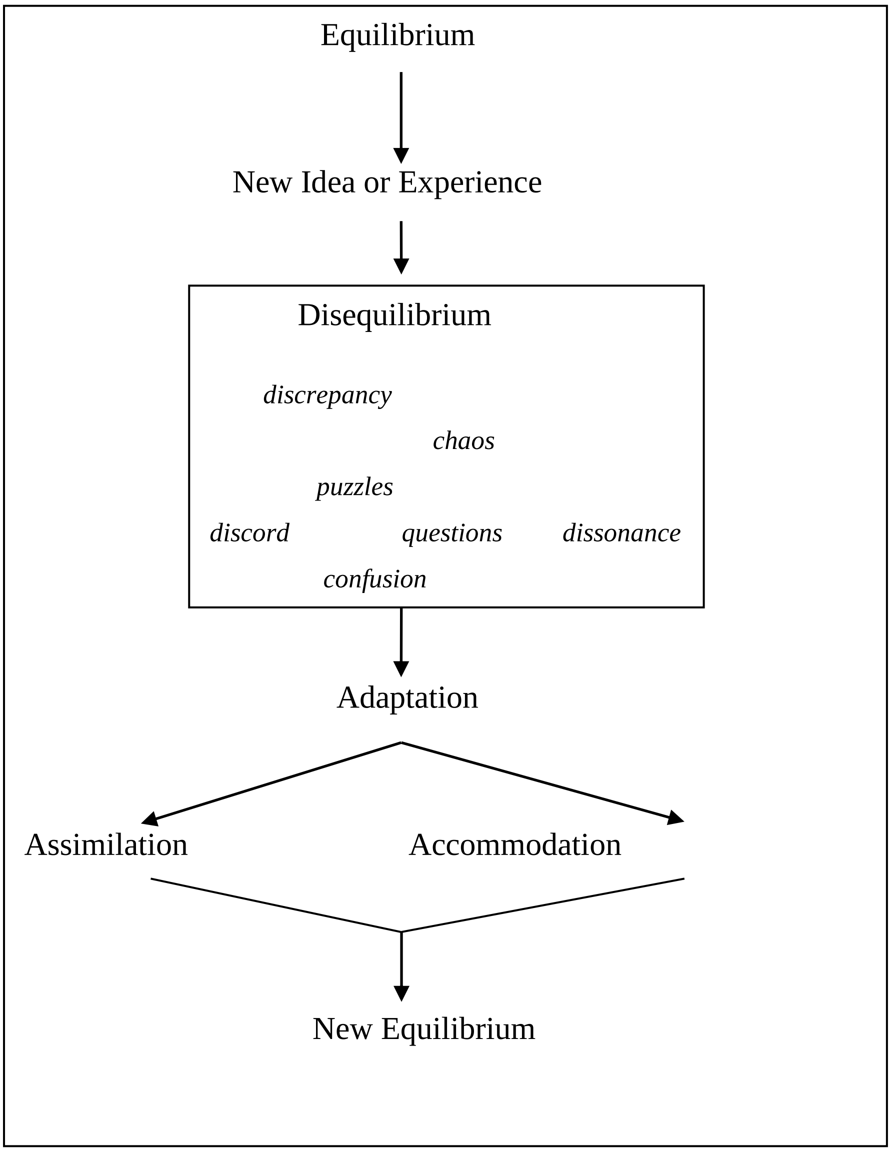



Figure 4.6.1.1: Process of Equilibration (Berger, 2004, p. 43)

Further to this, Piaget (1985, p. 6) asserts that theory of equilibration is underpinned by the following two postulates:

> "First postulate: Every assimilatory scheme tends to incorporate external elements that are compatible with it."
>
> Second postulate: Every assimilatory scheme has to be accommodated to the elements it assimilates, but the changes made to adapt it to an object's peculiarities must be effected without loss of continuity."

The first postulate reaffirms that when an individual encounters new information or challenges, the first natural attempt on the part of the individual is to map it onto existing schemata that is very similar to the new information or challenge. Furthermore, it certainly does not suggest the construction of any new structures. However, the second postulate suggests that modifications, reconstructions or refinements must be made to match existing schema to house the new information or knowledge in the mind, but the original schema that served as source for the construction of the new schema must be preserved and retained as part of the cognitive structure so that it could continue to play a role in direct assimilation of information that it was accustomed to assimilate prior to the development of the new schema. Thus, in essence the second postulate affirms that previous knowledge is never erased (see Piaget, 1978, 1985).

Below are some examples that characterize the process of equilibration. Firstly, suppose a learner's experience with quadrilaterals was limited to convex quadrilaterals only. In other words, the only quadrilaterals that the learner experienced were simple closed quadrilaterals wherein none of its angles were reflexive (see Figure 4.6.1.2)

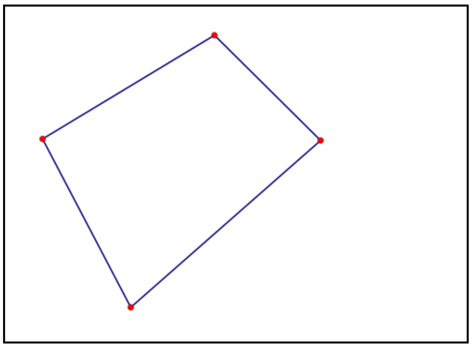


Figure 4.6.1.2: Convex Quadrilateral

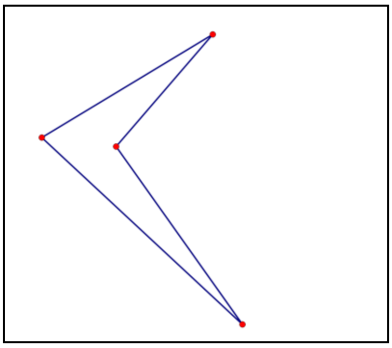


Figure 4.6.1.3: Concave Quadrilateral

Suppose that in a class tutorial, it is the first time that the learner sees a concave quadrilateral (see Figure 4.6.1.3), and the task requires the learner to describe the figure to his/her classmate. It is plausible that this task could create some perturbation in the learner, and moreover create internal cognitive conflict when attempting to reconcile the concave quadrilateral with his existing understanding or schemata. However, by initially looking at the figure with some kind of classroom support, it is plausible that the learner would realize that it is a simple closed quadrilateral by merely observing that  is a quadrilateral with sides only meeting at the vertices. Furthermore, via some facilitator or peer support the learner may look at the angles and notice that one of the angles is reflexive. Thus as a result of  invoking the existing schema of a simple closed (convex) quadrilateral schema and focusing on the measure of the angles, the learner may probably tell his classmate that it is a simple closed quadrilateral with one of its angles reflexive. In this way, the concept of simple closed quadrilaterals is expanded to include concave quadrilaterals.

This is the process of assimilation, wherein a new idea has contributed to the development of an existing schema of the learner, by: (a) expanding his/her existing concept of a simple closed quadrilateral, which was initially limited to convex quadrilaterals, to now include concave quadrilaterals as well, and (b) entrenching the aforementioned distinctions through the element of differentiation, namely reflexive angles i.e. all angles not reflexive versus one angle being reflexive (see Olivier, 1989). The aforementioned expansion invariably can help to eliminate the initial perturbation the learner has experienced and thereby contribute to the restoration of equilibrium within the cognitive structures. Moreover, since the assimilation of the concave quadrilateral into the simple closed quadrilateral schema (called the original assimilatory schema) was successful, the original assimilatory schema must be transformed into a new assimilatory schema. (i.e. modifications) must take place. This effectively means that the learner must reconstruct or reorganize his simple closed quadrilateral schema to include a new type of quadrilateral, namely a concave quadrilateral, within the parameters of Piaget's postulate 2. Once this modification (a biological process) of the simple closed quadrilateral schema has taken place, we say that accommodation of the concave quadrilateral has occurred in the cognitive structure (Melis, Ulrich & Goguadze, 2009; Piaget, 1978, 1985).

Through this process of accommodation, the mind adapts itself to the new evidence/context, and it is in this sense we say that conceptual change has occurred in the mind of the learner with regard to the concept 'simple closed quadrilateral' and that equilibrium has been established. The consequence of the resultant conceptual change and associated equilibrium is that in future when the learner sees a replica of Figure 4.6.1.3, he would most probably recognize it as a simple closed quadrilateral – free from any kind of perturbation, dissonance or discord, i.e. it will be immediately assimilated without any conflict.

Just as we have seen the interplay between the two processes, assimilation and accommodation, in bringing about conceptual change and restoring equilibrium with regard to the aforementioned concept of simple closed quadrilateral, the same kind of interplay prevails between the two processes continuously in other instances pertaining to conceptual change/development as well. However, in cases where a learner is faced with a new experience or idea that is very much incompatible with existing schema(s), cognitive disequilibrium will definitely arise. In such instances of disequilibrium the process of accommodation and not assimilation has to 'kick' in first in order to restore equilibrium i.e. schema (or schemata) must be reconstructed and re-organized to realize the desired equilibrium within the cognitive structures (Berger, 2004; Piaget 1978, 1985). The latter position is to some extent illustrated in the Hadas, Hershkowitz and Schwartz (2000) study wherein two tasks, namely the sum of the interior angles of a polygon (called Task A) and the sum of the exterior angles of a polygon (called Task B), were designed and given to about 90 grade 8 learners to do within a dynamic geometry environment pair-wise. In particular, the activity was structured as follows:

> "Task A: Measure (with the software) the sum of the interior angles in polygons as the number of sides increases. Generalize, and explain your conclusion.
> Task B: Measure (with software) the sum of the exterior angles of a quadrilateral.
> Hypothesize the sum of the exterior angles for polygons as the number of sides increases. Check your hypothesis by measuring and explain what you found" (Hadas, Hershkowitz and Schwartz, 2000, p. 132)

After experimenting with the sum of the interior angles across several simple closed polygons like the triangle, quadrilateral, pentagon, hexagon, a large number of the students (32 reports) generalized "that the sum of the interior angles increases by $180^0$ when the number of sides increases by 1"; a fair amount of students (18 reports) expressed their generalization through either one of the following general algebraic formulae (or rules), "$180°(n-2), 180°(n-4)+360°, 180°n-360°$"; and 5 pairs of students generalized in both aforementioned ways (Hadas, Hershkowitz and Schwartz, 2000, p. 132). Furthermore, the majority of the learners attempted to provide an explanation for their generalization by "generalizing their measurements, or, by adding a triangle when the number of sides increases by one." The latter kind of explanation is illustrated in Figure 4.6.1.4:

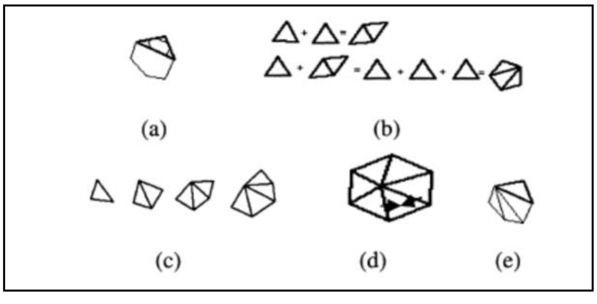



Figure 4.6.1.4: Students' drawings of various explanations in Task A (Hadas, Hershkowitz and Schwartz, 2000, p. 134).

However, it is quite plausible that the aforementioned generalization may be a new idea or experience for the student, which may not be consistent or compatible with his/her existing schema, and thus may cause some perturbation in the learner. To relieve the learner from such perturbation, it then becomes necessary that the student change or reconstruct existing schema to accept or accommodate the new information (Piaget, 1978, 1985). Through the successful accommodation of the new information, more powerful knowledge will be developed and equilibrium will thereby be restored.

Now moving onto Task B, Hadas, Hershkowitz and Schwartz (2000) reports that 37 of the 49 responses (8 individual interview responses and 41 written responses from students working in pairs) initially conjectured that the sum of the exterior angles of a simple closed polygon increases as the number of sides increases. However, the latter conjecture captured the generalization that students constructed and justified earlier in Task A. It is quite plausible that the students' experiences in Task A, wherein they constructed and explained the generalization pertaining to the interior angle sum of a polygon, and their subsequent accommodation of the idea (or generalization) within their cognitive structures, may have impacted on the construction of the students' initial conjecture in Task B. In other words, it is quite possible that the nature of the information in the hypothesis question of Task B, triggered the retrieval of the modified schema established with regard to the generalization in Task A, which acted as an assimilatory scheme for the question in task B and unfortunately resulted in an incorrect conjecture being produced in Task B.

Nevertheless, when the students attempted to validate their conjecture they made in Task B by measuring the sum of the exterior angles of particular polygons using the measuring tool within a geometric dynamic environment, they were rather surprised to find that the sum of the exterior angles were always remaining fixed at $360^0$. This contradiction appears to have challenged students' established schemata, and hence disturbed the equilibrium i.e. it brought about disequilibrium (or cognitive conflict). This experienced cognitive conflict sparked an interest in the students, causing them to want to understand "why"?. Hence, the subsequent discourse around the surprise result focused on students providing explanations for their surprising result. In total fifty different explanations spanned across the following five categories: No explanations (17 responses), inductive explanation (2 responses), partial deductive explanation (6 responses), visual-variations explanation (16 responses), and deductive explanation (9 responses) (Hadas, Hershkowitz, & Schwartz, 2000).

In such a pedagogical environment where learners are encouraged to provide explanations and come to understand why their result is true, it is plausible to conjecture that learners would be in a much better position to reconstruct and reorganize their existing schema (or schemata) to accept the new (surprising result), and thus regain cognitive harmony or equilibrium. Consequently, if such students are posed the question like 'what is the sum of the exterior angles of decagon', at some later point of their schooling career, they would probably recognize it as some kind of familiar information, and naturally invoke the 'sum of the exterior angle of a polygon" schema to respond to such a question, obviously with virtually no perturbation whatsoever, i.e. they will easily assimilate the information or question into their existing cognitive structures and respond with a fair degree of ease.

Once a new experience or idea has been accommodated, and afterwards learners come across a similar concept, idea or question in other contexts or experiences, it is plausible that may be able to assimilate it directly into the related schema easily and thereby maintain (or retain) the equilibrium status within their cognitive structures. Furthermore, we have seen via the aforementioned examples in this section that accommodation can occur first and then assimilation or vice versa, much of which depends largely on the degree of compatibility of the new idea or experience with existing schemas. In this respect, Donald, Lazarus, and Lolwana (1997, p. 45) says:

> "Progressive accommodations create more possibilities of assimilations and vice versa, in ever expanding cycles. These have to be organized and kept in a sort of dynamic balance across the child's different maps, and in relation to his 'whole' map of the world at any specific time."

In my view, the discussions presented so far underpin the process of equilibration, which inevitably is the 'engine' that drives development (Donald, Lazarus, & Lolwana,1997, p. 45) and serves as the mechanism to promote ways of thinking (Elkind, 1974).

### 4.6.2 Conceptual Change and Cognitive Conflict

Consistent with the constructive perspective on learning, learning by conceptual change is characterized by the building of new ideas in the context of old ones through the partial or major restructuring of already existing knowledge, concepts or schemata (di Sessa 2006; Biemans & Simons, 1999; Duit, 1999). It is in this sense that the emphasis is on "change" rather than acquisition. Moreover, the very shift or restructuring of existing knowledge, concepts or schemata is what distinguishes conceptual change from other types of learning, and provides students with a more fruitful conceptual framework to solve problems, explain phenomena, and function in the world (Biemans & Simons, 1999; Davis, 2011).

Prior knowledge, which is construed as all the knowledge that learners have when entering a learning environment, is regarded as the most important ingredient in the process of meaningful learning (Ausubel, 1968; Duit, 1999). In terms of the constructivist view of learning, learners are seen as active constructors of their own knowledge who naturally engage with their prior knowledge (or prior conceptions) when attempting to make sense of new information that they encounter or experience in their classrooms. Thus, when the learners are confronted with new information, it is possible that some of the prior conceptions may be compatible or coherent with the new information or experiences, and thus constitute a foundation of learning. On the other hand, new information may not be compatible with any prior linked conception(s) and thus de-rail further learning. In this instance the learners are then compelled to restructure/reorganize their existing knowledge (or conception), which then essentially means that they must face a conceptual change (Desmet, Gregore, Mussolin, 2010, p. 521).

Cognitive conflict, which is the key driver for conceptual change, as alluded to in the aforementioned set of instructional strategies, is actually a psychological state signalling a discrepancy between one's cognitive structure (schema)  and new information (or experience) or between cognitive structures (or schemas) (Lee & Kwon, 2001; Lee et al, 2003). Much of the research and development on cognitive conflict was inspired by Piaget's theory of equilibration which explains how the processes of assimilation and accommodation work in a complementary manner to restore equilibrium in the mind of the learner when it is thrown into a state of disequilibrium. The latter state of disequilibrium, which Piaget commonly referred to as cognitive conflict, can be generated by merely creating the space for a learner to experience and acknowledge some contradiction or inconsistencies in his or her ideas (Zaslavksy, 2009; Lee et al, 2003; Buchs, Butera, Mugny, Darnon, 2004). In this respect, purposively constructed counter-examples could be strategically used to introduce cognitive conflict.

Likewise, other researchers have used different kinds of terms to explain a cognitive conflict situation within the context of their research focus or epistemologies. For example, Festinger (1957) uses the term 'cognitive dissonance', which is cited and described in Zaslavsky (2005,

p. 299): "The discomfort caused by logical inconsistency or contradiction motivates the individual to modify his or her beliefs in order to bring them into closer correspondence with reality"(Zaslavsky, 2005, p. 299). Another researcher, Berlyne (1960) as cited in Zaslavsky (2005, p. 300) uses the term 'conceptual conflict' in his theory of "conceptual conflict", which affirms that cognitive conflict plays a formidable role in knowledge production/acquisition. In particular, his 'conceptual conflict theory says: "Conceptual conflict has high arousal potential, motivating the learner to attempt resolve it by seeking new information or by trying to reorganize the knowledge he or she already has" (Lee et al. , 2003 citing Berlyne 1960, 1963, 1965).

With regard to fostering and promoting conceptual change in the classroom, Nussbaum and Novick (1982) have pointed out the following instructional strategies as cited in Biemans & Simons (1999, p. 250):

- "make students aware of their own preconceptions (or the preconceptions of others) through an "exposing event",
- "create a cognitive conflict through a "discrepant event", and
- "support students' search for a solution to this conflict and encourage conceptual change."

The aforementioned "cognitive conflict" approach to conceptual change has been used quite widely in many teaching and learning contexts. For example, Tirosh and Graeber (1990) used cognitive conflict approach to probe the following misconception held by a group of pre-service elementary teachers: "In division the quotient must be less than the dividend." According to Fischbein et al., (1985) the sources of the aforementioned misconception are as follows:

- The first source of the given misconception is the partitive model - associated with the primitive thinking about division as sharing.
- The second source is associated with pace-setting at school: First division is introduced in the context of whole numbers only, and then much later is division by rational numbers, particularly rational numbers less than one, are introduced.

During the interviews, twenty-one pre-service teachers were posed the question: "In a division problem, the quotient must be less than the dividend." Results showed that a large number of the pre-service teachers (15 out of 21), argued that in division the quotient is always less than the dividend by providing the following kinds of justifications:

- "Division is sharing. When you share things, each one gets less than the whole amount. Therefore the quotient is less than the dividend (7 subjects),"
- "There are no such examples [in which the quotient is greater than the dividend] (4 subjects),"
- "Division is the inverse of multiplication. Since multiplication always make bigger, division always makes smaller,"
- "Arguments based on algorithmic procedures. For example, one of the interviewees argued that in the case of decimal divisor, "you have to change the divisor to a whole number, add zeros to the dividend, and then, ultimately the quotient is less than the dividend" (Tirosh and Graeber, 1990, p. 102).

The interviewer then posed the following question to the students: Determine $4 \div 5$. Some of the students, after using the standard algorithm, recognized their inconsistency and immediately acknowledged that quotient is not always less than the dividend. Other realized the same after a second prompting question. Moreover, the remaining students were provided with more support and directed questions in attempt to enable them to overcome the misconception. Hence, the results of the study show that as a result of the purposeful and constructive use of the conflict approach, misconceptions were alleviated and treated and thus helped pre-service teachers to develop a more accurate conception about the size relationship between the quotient and dividend, and in addition, improved their ability to construct written expressions for multiplication and division word problems (see Tirosh and Graeber (1990) for details).

Posner, Strike, Hewson, & Gertzog (1982), in their development of a model to explain conceptual change, also took cognizance of Piaget's notion of disequilibration and accommodation. Their model comprised of two major components, namely the pre-requisite conditions for accommodation to take place and a person's conceptual ecology. Conceptual ecology in this sense refers to the existing set of conceptions that a student already possesses and which inadvertently will serve as an immediate framework of reference with which the students will first engage or interact with in order to make sense of new information or experiences. In this sense, a person's conceptual ecology is said to consist of many different kinds of knowledge, such as anomalies, analogies and metaphors, examplars and images, and past experiences (Hewson & Hewson, 1984).

Posner et al (1982, p. 214) states that the following pre-requisite conditions must be fulfilled to trigger the occurrence of accommodation and thus experience conceptual change:

- The student must be dissatisfied with the existing conception, meaning the student's current existing conception must no longer be able to assist the student to comprehend or explain the new experienced concept. In other words, none of the student's existing conceptions can come to the fore to successfully resolve any anomalies or solve the envisaged problem.
- The new conception must be intelligible, coherent and internally consistent to the student.
- The student must identify the new conception as plausible. In other words the student ought to have an inner gut feeling that a newly constructed concept will be able to assist to resolve experienced anomalies or problems. Moreover, there has to be reasonable degree of consistency between the newly formulated conception and other knowledge.
- The new conception must be fruitful and make sense in many situations. Moreover, it should create new pathways of enquiry and also be able to be expanded to make sense of other experiences when the need arises.

In terms of the conditions mentioned earlier, a student upon experiencing a new phenomenon, may comprehend the new conception (i.e. find it to be intelligible), reconcile it without any necessary contradictions with previous conceptions (i.e. find the conception to be plausible) and also see the value of the new conception – for example in terms of solving a particular problem (i.e. find the conception to be fruitful). When such circumstances prevail, the student is then in a favorable position to incorporate the new conception easily into his existing cognitive structures or schema, and such a process is referred to as assimilation by Posner et al. (1982), Strike and Posner (1985), and Hewson (1989). However, if the new conception is meaningful to the students (intelligible) but contradicts existing conceptions (not plausible), then a state of conflict prevails. Consequently the acceptance of the new conception is kind of blocked by the existing set of conceptions. Thus, in order for a person to accept a new conception, it is essential that the status of the blocking conception be lowered before the new conception can begin to rise. This process is referred to as accommodation by Posner et al. (1982) (also see Hewson, 1989).

The aforementioned conceptual change theory of Posner et al.(1982), strongly suggests that necessary space and opportunities be created for learners to establish meaningful connections and relationships between prior knowledge (or conceptions) and new information to be learned, and correct their existing conceptions if necessary. However, much criticism was leveled against their theory of conceptual change (see Duit, 1999, Lederman, 1992, Pintrich et al., 1993). For example, Pintrich et al. (1993) asserted that Posner et al's theory focuses too much on the rational issues but does not consider the affective and social issues pertaining to conceptual change, i.e. it neglects the learner as a whole. In particular, it does not consider the teacher and other learners in the given learning environment, and how they influence the learner's conceptual ecology, which is an important ingredient in determining conceptual change or not. Hence, Strike and Posner (1992), after acknowledging the deficiencies in the original conceptual change theory model, revised it to accommodate the cognitive, affective, social and contextual factors, with much more emphasis on the interaction between prior conceptions and new conceptions. Moreover, in consonance with most constructivist approaches of learning and instruction, the theory supports the application of dynamic conceptual change processes as well as the provisioning of relevant "supporting" conditions to enhance conceptual change (Biemans & Simons, 1999).

Similar to the conditions posited by Posner et al (1982, p. 214), Eggen and Kauchak (2007, p. 246) assertd that the following conditions are required for students to change their thinking:

- "The existing conception must become dissatisfying; that is it must cause disequilibrium.
- An alternative conception must be understandable. The learner must be able to accommodate his or her thinking so that the alternative conception makes sense.
- The new conception must be useful in the real world. It must re-establish equilibrium, and the learner must be able to assimilate new experiences into it."

However, many other theoretical models have been developed by other academics and researchers to explain conceptual change as well. Some of them are: Carey (1985); diSessa (1988, 1993); Brewer (1987); Vosniadou (1994); and Thagard (1992). Whilst some of the models were not necessarily designed to be implemented in the classroom, a number of empirical studies were carried out in order to establish the feasibility of using them for classroom pedagogy. According to Limon (2001, p. 358), a cross analysis of the results of the various empirical studies shows that the following set of instructional strategies can best represent many of the instructional efforts made to promote conceptual change:

- "The induction of cognitive conflict through anomalous data";
- "The use of analogies to guide student's change", and
- "Cooperative and shared learning to promote collective discussion of ideas."

Moreover, in reviewing research in the field of education, Guzetti et al., (1993) stated that instructional strategies and approaches that were effective in fostering conceptual change had a common element of producing cognitive conflict. In the similar vein, Limon (2001) also reported that nearly all theoretical models that were constructed to explain conceptual change highlighted "cognitive conflict as a central condition for conceptual change" (p. 359).. However, to the contrary, cognitive conflict may not produce the necessary conceptual change, particularly if the students do not 'see' the conflict or experience difficulties in dealing with contradictory or anomalous data (Duit, 1994; Chin & Brewer, 1993; Dekkers & Thijs, 1998; Guzzeti & Glass, 1993; Strike & Posner, 1992). Worse still, the contradictory information can become a nightmare for students (and even depress them) if they do not have enough knowledge to attempt the resolution of the conflict (Lee et al., 2003). Furthermore, there exist many instances where teachers are quite successful in constructing learning episodes that make it possible for learners to experience cognitive conflict, but such opportune moments are lost in many instances because they do not have the necessary and sufficient experience (or expertise) to guide the learners to resolve the cognitive conflict (Zazkis & Chernoff, 2008, p. 196).

Despite the aforementioned shortcomings of the cognitive conflict approach, there are many studies that have demonstrated the important role that cognitive conflict plays in fostering successful conceptual change (Druyan, 1997; Hewson & Hewson, 1984; Rolka, Rosken & Liljedahl , 2007; Tirosh & Samir 2006; Thorley & Treagust, 1987; Zazkis & Chernoff, 2008; & Lieven , 2004; Watson 2007). For example in Watson's (2007) study, 58 students were posed with a number of questions pertaining to the concept of average. After considering their responses, the students were then presented with sets of alternate responses from students from other schools. Responses after the experience of cognitive conflict showed an improvement in the students' conceptual understanding of the concept of average (see Watson, 2007).

Moreover, in the application of the conflict approach in our classroom, an educator should be aware that a cognitive conflict process is triggered when a learner:

> "*a.* Recognizes an anomalous situation,
>
> b. Expresses interest or anxiety about resolving the conflict, and
>
> c. Engages in cognitive reappraisal of the situation*."* (Lee et al, 2003, p. 588).

Thus in a classroom situation, when a student realizes that his existing conceptions are not consistent with the result of an investigation, he may be surprised, become doubtful or just think of it as something strange. However, if the student becomes interested in the situation then he may display such interest by exhibiting increased levels of interest, curiosity and focused attention. On the other hand a student may become anxious about the situation, which could be gauged through notions of confusion, discomfort, frustration and probably a feeling of oppression. Thus, it is imperative that facilitators (or educators) in the classroom monitor learner responses to new information very carefully and provide the necessary support – otherwise the opportune moment for intellectual growth will be lost. If support or guidance is not provided timeously and appropriately, the student may conveniently reappraise his state and consequently decide to suspend the state. However, if appropriate constructive support is provided, the student may think longer and more critically and thus seek a more reasonable understanding, which inevitably enhances knowledge building. (see Lee et al., 2003).

Moreover, conceptual change approaches that seek to foster students thinking via discussion and argumentation related to induced cognitive conflict, have shown to be extremely powerful in developing reasoning and justification skills of students in association with their concurrent improvement of specific conceptual understandings (Mason, 2001). Quite often within such an approach a question related to the experience at hand is constructed and posed by the educator to the students, and the necessary space is created for students to provide their perspective or view on a particular kind of question. To facilitate and enhance such a learning context, the educator may use phrases such as 'give reasons', 'provide evidence', 'form an argument', 'make an assumption' in an effort to guide students in their reasoning (Anderson et al., 2001; Clark et al., 2003).

By and large cognitive conflict is regarded as an analogue to the Piagetian notion of disequilibration (Zazkis & Chernoff, 2008) and is considered as a useful pedagogical strategy to resolve or remedy misconceptions within a constructivist learning environment (Ernest, 1996). In view of the extensive incorporation of the notions of assimilation and accommodation in most models of conceptual change, this study will use Piaget's Equilibration Theory to explain identified constructs of conceptual change.

In summary, this Chapter has focused on constrcutivism as a learning theory, scaffolding, discovery learning and exploratory environments, analogical transfer including Gentner's Structure mapping theory; Ausubel's Theory of Meaningful Learning, Piaget's Equiliabration Theory, conceptual change and cognitive conflict.

The next Chapter, continues with the discussion of the following theoretical aspects that guided this study: types of generalizations (see Section 5.1); types of justifications (see Section 5.2); conception of deductive proof (see Section 5.3) and functions of proof (see Section 5.4)

# Chapter 5: Further Theoretical Considerations

## 5.0 Introduction

This chapter is a continuing narrative of issues and insights examined in Chapter 4. A classroom that provides opportunities for students to conjecture, generalize and justify their generalizations can provide the necessary intellectual space for learners to externalize their thoughts, clear their doubts, upgrade their knowledge, gain new insights, make informed decisions and even change their perceptions (Ogunniyi, 2010 notes). In this study, through the use of the *Viviani* linked activities, the necessary space is created for pre-service educators to construct their own generalizations, justify their validity through experimentation and proof, and even refine their generalizations after experiencing counter-examples. Furthermore, in this study, the set of plausible arguments resulting in the generalization and the set of valid arguments resulting in the proof of an established generalization was tracked an reported on. In particular, this Chapter, provides a discussion pf the following theoretical aspects: types of generalizations, types of justifications; conception of deductive proof, and functions of proof.

## 5.1 Generalization

As indicated in Chapter 1, this study focuses mainly on the construction of the following types of generalizations: Inductive generalization; analogical generalization and deductive generalization. Furthermore, taking into cognizance the processes of developing an inductive generalization as suggested by Canadas & Castro (2005) and James (1992), and discussed in Chapter 1, this study adopts the Canadas, Deulofu, Figuerias, Reid & Yevdokimov (2007, p. 64) model for inductive generalization, which is relevant for dynamic geometry contexts. They rececommend that the development of conjecture generalizations by empirical induction from dynamic cases could proceed through the following stages:

- “Manipulating a situation dynamically through a continuity of cases,
- Observing an invariant property in the situation,
- Formulating a conjecture that the porpoerty holds in other cases,
- Validating the conjecture,
- Generalizaing the conjecture,
- Justifing the generalization.”

## 5.2 Justification

In the context of this study, justification is defined as a rationale or argument for some mathematical generalization. As alluded to in the literature survey, justifications vary in nature depending on the kind of argument that is advanced in favour or against a specific conjecture generalization. Hence, in this study justification will be treated as a continuum (continuity spectrum) with authoritative justification on the lower end, followed by empirical justification and generic example justification respectively, with deductive justification being the ultimate kind of justification on the upper end (compare Huang, 2005). Drawing from the notions of justifications posited within research contexts by Bell (1976a & b), Balacheff (1988), De Villiers (2003a), Dörfler (1991), Hanna (2000), Marrades & Gutierrez (2000), Harel & Sowder (1996, 1998, 2007), Sowder & Harel (1998) as elaborated in Sections 1.8 and 1.9, the range of justifications that has been considered for this study includes authoritative justification, empirical proof, generic proof, and deductive proof.

### 5.2.1 Authoritative justification

It is possible that students, can construct their justifications (proof schemes) within the ambit of an authoritarian proof scheme, particularly if they believe that they can only justify a conjecture/conjecture generalization/result through the endorsement and acknowledgement of information/statement(s) in a textbook, or an educator (lecturer or professor statement), or by just obtaining the necessary confirmation from a wise and knowledgeable peer. Indeed referring to an authority for the purposes of justification, is not a misnomer in the field of mathematics, for even experts in mathematics may at times whole-heartedly accept a colleague's result, without necessarily considering or evaluating the details of the reasoning that was used to arrive at the established result (Sowder & Harel, 1998). However, classroom instruction that dwells heavily on authoritarian type of justifications, can result in an "authoritarian syndrome", whereby students will always want to convince themselves or others that a particular conjecture generalization is true, by only making reference to a statement in a textbook or citing a statement made by an educator or obtaining the 'nod' from their colleague, without fully comprehending the rationale (or reasoning) behind such "appropriated" statement(s) or confirmation(s). This kind of reliance by students on some form of authority to justify their moves or claims, can be counter-productive in the sense that such students can merely attempt to use axioms, definitions, theorems and corollaries or specific algorithms, procedures and methods to justify their conjecture generalization(s), without really making sense of them or understanding the reasons as to why they can be used.

This kind of approach invariably suppresses or curtails the opportunity to develop a deeper understanding of related topics or explain similar results successfully (Sowder & Harel, 1998), and also contributes to the development of misconceptions at times.

### 5.2.2 Empirical proof (justification)

Empirical justification, which is commonly known as *empirical proof* is a kind of justification wherein the correctness of particular examples is the argument of conviction (see Gutierrez, Pegg, Lawrie, 2004, p. 513). In other words, it is a kind of justification where one attempts to demonstrate the validity of a conjecture generalization by showing it is true in a number of randomly selected cases, but does not account for the arbitrary case (compare Balacheff, 1988; Marrades & Gutierrez, 2000; Harel & Sowder,1996, 1998, 2007; Hollebrands & Smith, 2009; Sowder & Harel 1998) This is more an informal argument commonly referred to as an empirical argument that employs inductive level of reasoning, and is regarded as an argument that "provides inconclusive evidence for the truth of a mathematical claim," i.e. it is regarded as a non-proof argument (Stylanides, 2008, p. 12). According to Canadas et al. (2007), empirical argument can be raised through either the consideration of a finite number of discrete cases or dynamic cases. Within the context of the study, pre-service educators work in a *Sketchpad* environment; they construct empirical examples that bear the invariant properties through use of the 'drag' mode; they construct conjectures (conjecture generalizations) through taking cognisance of observed regularities that prevail across one or more examples; they present their corroborating examples or observed relationships in them as the grounds (or evidence) to justify the truth of their conjecture (conjecture generalization). On the other hand, when the conjecture generalization is 'included in the statement of a problem", pre-service educators "have only to construct examples to check the conjecture generalization and justify it" (Marrades & Gutierrez, 2000, p. 91). Given the aforementioned underpinnings of empirical proof, some students working within an empirical proof scheme, more particularly in a non-Sketchpad context, tend to rely on just one type of drawing or pictorial representation to demonstrate the validity of their conjecture.

The following example (see Figure 5.2.2), wherein students demonstrate the validity of a conjecture by constructing empirical examples within the context of an interactive geometry environment, illustrates a notion of empirical justification.

**Established conjecture:** Inscribed **a**ngles in the same segment of a circle are equal

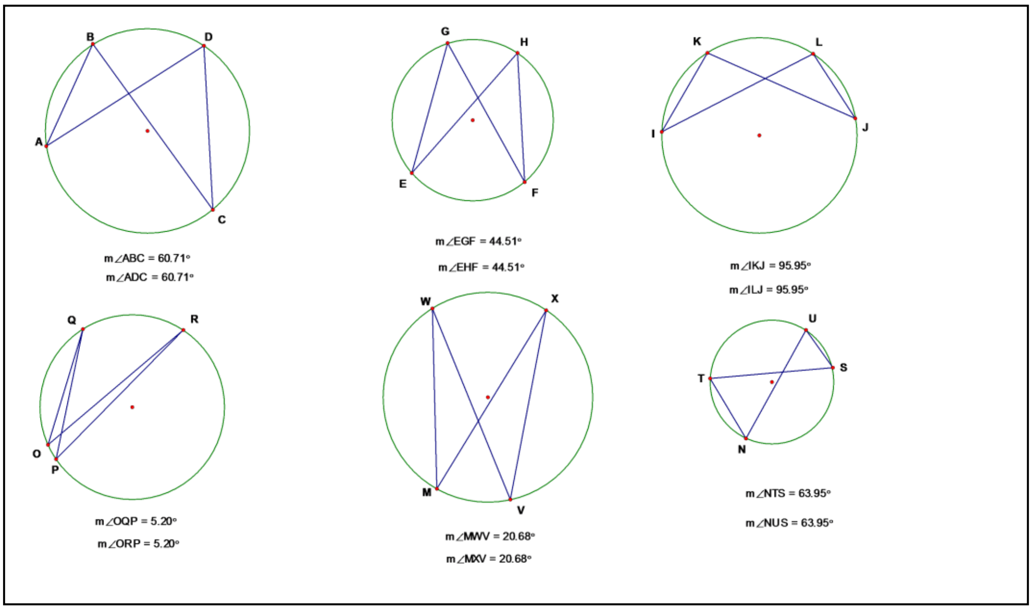



Figure 5.2.2: Empirical examples to justify inscribed angles in the same segment are equal

### 5.2.3 Generic Proof

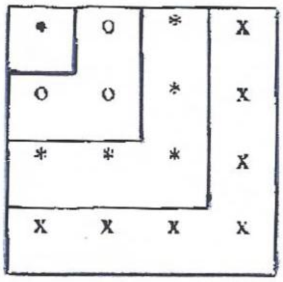
According to Leron & Zaslavsky (2009, p. 1), a generic proof is, "roughly, a proof carried out on a generic example", wherein a generic example according to Mason & Pimm (1984, p. 287), "is an actual (particular) example, but one presented in such a way as to bring out its intended role as the carrier of the general." Moreover, Stylianides (2008) with due reference to Balacheff (1988), Mason & Pimm (1984), Rowland (1998) & Harel & Sowder (1998), states that, "a generic example is a proof that uses a particular case seen as representative of the general case." Furthermore, in the main, the term "generic proving" refers to all kinds of mathematical activity revolving around a generic proof (Leron & Zalavsky, 2009).

The following elementary example, the sum of the first odd numbers, cited in Rowland (2002, p. 150) and further discussed in Rowland, Turner, Twaites, & Huckstep (2009, p. 97), can help illustrate the notion of generic proving in the context of a generic example.

Example 1: Generic Proving

**Conjecture Generalization**: The sum of 1+3 +5 + … up to any odd number is always a square number.

One possible proof for this conjecture generalization, by generic example, is as follows:



$1$

$1 + 3 = 2^2$

$1 + 3 + 5 = 3^2$

$1+ 3 +5 + 7 = 4^2$

Figure 5.2.3.1: Sum of odd numbers is a square number

The diagramatic representation in Figure 5.2.3.1, illustrates that when you start with 1 object and then add odd number of objects , namely 3, 5 and 7, consecutively, they correspondingly produce a square array of objects at each stage. Equivalently, this representation means that sum of the first odd number and each consecutive odd number 3, 5 and 7, produces a square number at each stage. In this particular case, we have merely focused on just one specific instance (i.e. 1 + 3+ 5 +7) of the general claim or conjecture generalization. This is typically "a generic example in so far as it is then clear that the addition of each subsequent odd number preserves the square array" (Rowland et al., 2009, p. 97). So effectively, we can say that a possible proof of the conjecture, the sum of 1+ 3 +5 + … up to any odd number is always a square number, has been raised through a generic example, and in this sense we may call the explanation a generic proof.

In this particular case, the generic example points to a more general truth, but does not replace the general proof itself. Furthermore, the generic proof in this instance, may be very useful for primary school children (and maybe some high school children), through which they can produce similar versions for other examples, particularly because the development of a complete formal proof may not be within their capability to do so (Leron & Zaslavsky, 2009). For example, we can use mathematical induction, which is generally beyond the reach of a primary school child and many high school students, to raise a general proof for the given conjecture generalizations as follows:

**General Proof**: Using mathematical induction

In symbolic notation, the sum of 1+3 +5 + … up to any odd number is a square can be represented as follows: $\sum_{i=1}^{n}(2i-1) = n^2$ with $n = 1, 2, 3, \ldots$.

Step 1: For $i = 1$ , $\sum_{i=1}^{1}(2i-1) = 1$ and $n^2 = 1^2$

$\therefore \sum_{i=1}^{1}(2i-1) = n^2$ for $i = 1$.

Step 2: Assume the statement is true for some $k$, i.e. , $\sum_{i=1}^{k}(2i-1) = k^2$ . Then,

$$\sum_{i=1}^{k+1}(2i-1) = \left(\sum_{i=1}^{k}(2i-1)\right) + (2(k+1)-1)$$

$= k^2 + (2k+2-1)$, by the inductive hypothesis

$= k^2 + 2k + 1$

$= (k+1)^2$

Thus if the $k$th statement is true, then so is the ($k$+1) th statement. Hence, our formula is true by the Principle of mathematical Induction.

To further illustrate the notion of generic proving, but in the context of geometry, the researcher reflects on a lesson that was observed during one of his school based practice sessions. In this lesson the student teacher was focusing on the sum of the interior angles of any polygon. The key points of the lesson are discussed in Example 2 below, with intention to illustrate the essence of generic proving.

Example 2: Generic Proving

**Generalization:** The sum of the interior angles of any (simple-closed) polygon is $(n-2)180°$

The above generalization is quite often presented to learners at junior high school level. Section 1.5 of Chapter 1, provides an outline of the generalizing process that leads to the development of this generalization. The case by case approach provides insight as to why the above-mentioned theorem holds true for the given particular instance, and not just a confirming instance of the given theorem. By reflecting on this particular case, like the quadrilateral - a generic example, learners were able to carry the “sameness” to other specific instances such as the pentagon and hexagon. The pentagon case and hexagon case can illustrate that the transparent presentation of an initial generic example (like the quadrilateral case) contributes to analogy making with other instances (such as the pentagon and hexagon cases), more easily and confidently (see Rowland, 2002), and thus contributes to ascertaining the truth. Furthermore, on reflecting on all three cases, learners can come up with the following kind of observation:

Angle sum of a polygon with 4 sides = 2 x $180^0$ = ( 4-2) x $180^0$

Angle sum of a polygon with 5 sides = 3 x $180^0$ = (5-2) x $180^0$

Angle sum of a polygon with 6 sides = 4 x $180^0$ = (6-2) x $180^0$

Through careful inspection of the above particular cases, learners can be able to see the general as follows: The sum of the angles of any (simple-closed) polygon of $n$ sides = $(n-2) \times 180°$. This essentially means, that the learners may not probably think of any "instance in which the analogy could not be achieved" (Rowland, 2002). This could be attributed to the fact that the coherent set of generic cases, provides necessary explanation and illumination of why the sum of the angles of any polygon of $n$ sides = $(n-2) \times 180°$ (see Hersh, 1993).

The notion of generic proving, permeates all levels of mathematical activities ranging from primary school to postgraduate level mathematics and beyond. Balacheff (1988, p. 219) as cited in Rowland et al. (2009, p. 97) argues that: "The generic example involves making explicit the reasons for the truth of assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class". Equivalently, Rowland (1998, p. 68) affirms that "the generic example serves not only to present a confirming instance of a proposition, which it certainly is –but to provide insight as to why the proposition holds true for that single instance". Also, Tall (1979) suggests that generic proving or the development of generic proof (s), is a potential way for students to comprehend and understand proofs that normally requires a higher level of deductive logic and thinking.

Moreover, according to Alibert & Thomas (1991, p. 216), a generic proof "works at the example level but is generic in that the examples chosen are typical of the whole class of examples and hence the proof is generalizable." However, Alibert & Thomas (1991, p. 217.), acknowledges that generic proof is different to a general proof, in that the latter "works at a more general level but consequently requires a higher level of abstraction." Furthermore, from a strict logical perspective, the necessary provision of a formal proof cannot be substituted by a generic proof, but nevertheless "the generic proof may sometimes be preferable if it results in improved understanding on the part of the students" (Alibert & Thomas, 1991, p. 217). For example, when teaching the section on arithmetic sequences, one is quite often expected to prove that the sum to $n$ terms of an arithmetic series,

$a + (a + d) + (a + 2d) + \ldots \{a + (n- 1)d\}$ is given by $S_n = \frac{n}{2}(a + l)$, where $n$ is the number of terms, $a$ is the first term and $l = a + (n - 1)d$ is the last term. To illustrate the procedure to obtain $S_n$, the sum of an arithmetic series, we take the following series as a generic example:

2 +5 + 8 +11 + 14 + 17 + 20

If $S_6$ denotes the sum of this series, then

$$S_6 = 3 + 6 + 9 + 12 + 15 + 18 + 21 \quad \ldots. (1)$$

Write the series in reverse order: $\underline{S_6 = 21 + 18 + 15 + 12 + 9 + 6 + 3} \quad \ldots.. (2)$

Adding (1) and (2): $2S_6 = 24 + 24 + 24 + 24 + 24 + 24 + 24$

6 lots of the first and last terms: $2S_6 = (3+21) + (3+21) + (3+21) + (3+21) + (3+21) + (3+21)$

$$2S_6 = 6\,(3+ 21)$$

$$\therefore\ S_6 = \frac{6}{2}(3 + 21)$$

$$= 72$$

From the above generic example, it is clear that $S_6$ = (½ × number of terms) × (sum of the first and last terms). Furthermore, the above generic proof, creates some understanding of how to develop a proof for the general case (i.e. formal proof) in a much more meaningful manner. Let us illustrate the above method in the case of the general arithmetic series,
$a + (a + d) + (a + 2d) + \ldots \{a + (n - 1)d\}$.

**General Proof:**

Consider the general arithmetic series, $a + (a + d) + (a + 2d) + \ldots \{a + (n-1)d\}$.

Let $l = \{a + (n-1)d\}$ in the general arithmetic series, for convenience.

Hence, we obtain the general arithmetic series of the form:

$a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$

Now, let us proceed using the same method as for the generic case:

$S_n = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$

Write the series in reverse order: $S_n = l + (l - d) + (l - 2d) + \ldots + (a + d) + (a + 2d) + a$

Adding (1) and (2): $2S_n = (a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l)$

$n$ lots of first and last terms $\quad 2S_n = n(a + l)$

$$\therefore S_n = \frac{n}{2}(a + l)\text{, where } l = \{a + (n-1)d\}$$

i.e. we have shown that $S_n$= (½ x number of terms) x ( sum of the first and last terms).

Given this, we can say with some degree of certainty, that through the generic example, which is a particular case in this instance, one could see the general proof of the sum to *n* terms of any arithmetic series. Moreover, this concurs with Mason and Pimm's (1984, p. 287) view that: "a generic example is an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general." In other words the example should allow one "to see the general through the particular" (Leron & Zaslavsky, 2009, p. 4). Furthermore, whilst we know the mathematical truth of any conjecture generalization, can normally be established through general proof, it is nevertheless quite evident in arithmetic series example, that the nature of the proof of the sum to *n* terms of an arithmetic series, can be realized through the development of a logical argument based on the proof-like structure that permeates the generic example (compare Rowland, 2009). Thus, essentially, we could say that generic example in this instance, provided a "window" through which the "general case" was constructed. Moreover, Movshovitz–Hadar (1988, p. 18), asserts the following:

> "The proof of generic example, should not be confused with a fully general proof. It only suggests the full proof through a generalizable concrete example. From the purely logical point of view there is no replacement for the formal proof. From a pedagogical point of view, a proof of the generic example can sometimes replace the general proof."

Furthermore, according to Rowland (2001, p. 177):

> "…the idea of a generic example is that we can 'see' beyond its particularity to what might happen in other instances. The transparent presentation of an example is intended to enable transfer of the argument to other instances. Ultimately the audience can conceive of no possible instance in which the analogy could not be achieved. "

To illustrate the notions articulated by Rowland, we will first consider a generic proof, which shows that $\sqrt{2}$ is irrational, and then illustrate how a similar argument can be used via analogy to show that $\sqrt{\frac{5}{8}}$ is also irrational.

**Generic Proof 1:** $\sqrt{2}$ is irrational (cited in Alibert & Thomas, p. 217):

"We will show that if we start with any rational $p/q$ and square it, then the result $p^2/q^2$ cannot be 2.

On squaring any integer $n$, the number of times that any prime factor appears in the factorization of n is doubled in the prime factorization of $n^2$, so each prime factor occurs an even number of times in $n^2$. In the fraction $p^2/q^2$, factorize $p^2$ and $q^2$ into primes and cancel common factors where possible. Each factor will either cancel exactly or we are left with an even number of experiences of that factor in the numerator or denominator of the fraction. The fraction $p^2/q^2$ can never be simplified to 2/1 for the latter has an odd number of 2's in the numerator. So the square of a rational $p/q$ can never be equal to 2".

**Generic Proof 2:** $\sqrt{\frac{5}{8}}$ is **irrational** (cited in Alibert & Thomas, p. 217):

"We will show that if we start with any rational $p/q$ and square it, then the result $p^2/q^2$, cannot be 5/8.

On squaring any integer $n$, the number of times that any prime factor appears in the factorization of $n$ is doubled in the prime factorization of $n^2$, so each prime factor occurs an even number of times in $n^2$. (For instance, if $n = 12 = 2^2 \times 3$, then $12^2 = 2^4 \times 3^2$). In the fraction $p^2/q^2$, factorize $p^2$ and $q^2$ into primes and cancel common factors where possible. Each factor will either cancel exactly or we are left with an even number of experiences of that factor in the numerator or denominator of the fraction. The fraction $p^2/q^2$, can never be simplified to 5/8 for the latter is $5/2^3$, which has an odd number of 5's in the numerator (and an odd number of 2's in the denominator). So the square of a rational can never be equal to 5/8".

Within the context of a generic–example assisted proof, Movshovitz–Hadar (1988, p. 17) recommends that an ideal generic example is one that is "large enough to be considered a non-specific-representative of the general case, yet small enough to serve as a concrete example." For example, Movshovitz–Hadar uses a very special 8x 8 square matrix (see Figure 5.2.3.2) to construct a generic proof which serves as a lens through which the general proof of the following theorem is seen: "For any $n \times n$ matrix, $n$ a positive integer, such that the rows form arithmetic progressions with the same common difference $d$, the sum of any n elements, no two of which are in the same row or column, are invariant" (see Albert & Thomas, 1991, pp. 217-218).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 |
| 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 |
| 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 |
| 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 |
| 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |

Figure 5.2.3.2: A representative matrix for the Theorem

Moreover, Leron & Zaslavsky (2009), asserts that the "complexity" rather than "size" of the generic example is the preferred option. For example, in relation to the theorem, a natural number which is a perfect square has an odd number of factors, the perfect square, 36, was chosen as the generic example to develop a generic proof as follows:

"All possible factorizations of 36 are; $1 \times 36$, $2\times 18$, $3 \times 12$, $4 \times 9$, $6 \times 6$.

> All the factors of 36 appear in the above list. Counting the factors, we see that all the factors appearing in all the products, except the last one come in pairs, and all are different, thus totaling to an even number. Since the last product contributes only one factor, we get a total of odd number of factors. Specifically, for the case of 36, we have $2 \times 4 + 1 = 9$ factors" (Leron & Zaslavsky, 2009, p. 1).

In particular, Leron & Zaslavsky (2009, p. 4), affirms that 36 is a good generic example for a generic proof of the 'perfect square' theorem, primarily because it has a good reasonable number of factorizations, as compared to 4, 25 or even 169 which, "would have been too special" in the sense that they "have too few factorizations".

Although, generic proofs provide an opportunity for students to see the structure of the general proof in some cases, it is also highly probable that "some of the more subtle points of a proof may not easily manifest themselves in the context of the generic proof: some steps which just happen in the example may require a special argument in the complete proof to ensure that they will always happen" (Leron & Zaslavsky, 2009, p. 4). Moreover, Rowland (2002) asserts that some scaffolding may be necessary to ensure that students are able to transfer parallel arguments from the generic proof to the more formal generalized argument (or general proof).

### 5.2.4 Deductive Proof (justification)

Deductive Justification, which is commonly known as proof (deductive) is a justification that is logically valid, based on initial assumptions, definitions and previously proved results. This is more a formal argument and is usually a deductive level of reasoning (Compare Balacheff, 1988; Harel & Sowder , 2007; Marrades & Gutierrez , 2000).

The following example illustrates the notion of deductive justification:

**Conjecture Generalization**: Inscribed angles in the same segment of a circle are equal

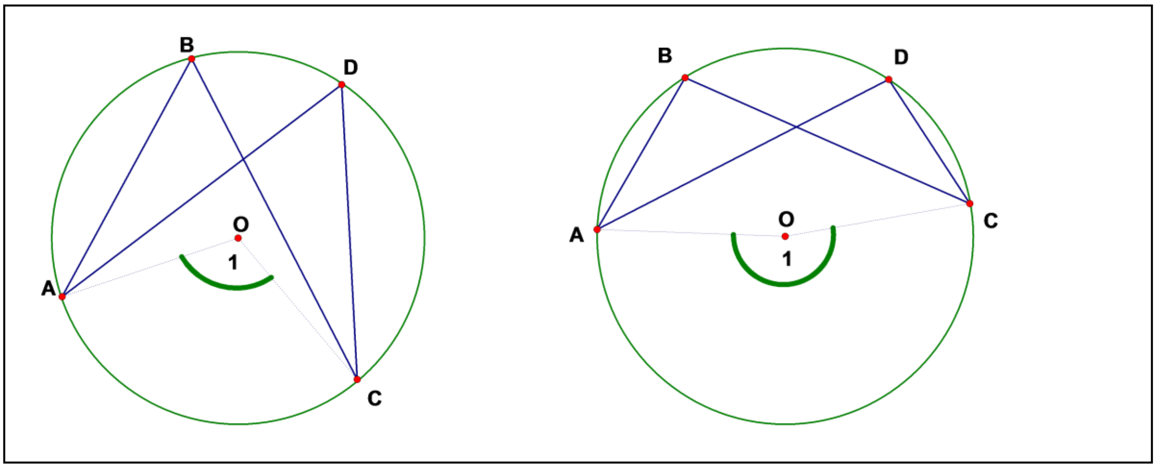



Figure 5.2.4: Inscribed angles in the same segment of a circle are equal

Given: Circle with centre $O$ and segment $AC$ subtending $A\hat{B}C$ and $A\hat{D}C$ in the same circle

Required to prove: $A\hat{B}C = A\hat{D}C$

Construction: Join $A$ and $C$ to the centre $O$ of the circle

Proof: $\hat{O}_1 = 2A\hat{B}C$ ($\angle$ at centre $=$ 2 x $\angle$ at circumference)

and $\hat{O}_1 = 2A\hat{D}C$ ($\angle$ at centre $=$ 2 x $\angle$ at circumference)

$\therefore 2A\hat{B}C = 2A\hat{D}C$

$\therefore A\hat{B}C = A\hat{D}C$

## 5.3 Conception of Deductive (Mathematical) proof adopted in this study

> "Proofs are the mathematician's way to display the mathematical machinery for solving problems and to justify a proposed solution to a problem is indeed a solution" (Rav, 1999, p. 13)

In the aforementioned conception of a proof, "*mathematical machinery*" refers to items like definitions, axioms, proven theorems, deductive reasoning, logical rules of inference such as modus ponens and modus tollens, counter examples, reasoning leading to a contradiction, etcetera, which belongs to a family of shared principles and deduction rules (mathematical theory). In a very similar way, Williams (1992, p. 42) defines a proof as "a logical, reasoned argument, presented step by step, using only definitions, axioms and proven theorems, which shows that the stated consequence or result does in fact follow" (Williams,1992, p. 342). So from a strict logical perspective, a proof in mathematics is "a logical argument that one makes to justify a claim and to convince oneself and others" (Blanton & Stylianou, 2003, p. 113).

According to the literature on proof (see Hersh, 1993 & 1997; Ramos, 2005; Martinez & Li, 2010, Douek, 1999), there are basically two main conceptions of a mathematical proof (deductive proof), namely an ordinary mathematical proof (called a practical proof/demonstration by some) and a formal mathematical proof. An ordinary mathematical proof, which is commonly used by members of the mathematical community nowadays, entails what one does to make one another believe the posited theorems or conjectured generalizations from a logical perspective; it involves the use of deductive reasoning to construct a logical argument that convinces someone like a teacher, student, prospective mathematician, or a mathematician that a given mathematical proposition is indeed true. On the other hand, a formal mathematical proof, which is embedded in mathematical logic and may involve the philosophical aspects of mathematics also, involves a more rigorous construction of a deductive argument using "a sequence of transformations of formal sentences, carried out according to the rules of predicate calculus" (Hersh, 1993, p. 391.) in order to convince the toughest skeptic/logician.

However, within the last three to four decades, the tenet that the production of formal proofs through the use of deductive reasoning is the most significant aspect of mathematics has been challenged by many mathematicians and mathematics educators. They honestly feel that there is much more to mathematics than just the construction of formal proofs, and have argued and debated a shift in focus to broaden proof to the production of arguments that foster not just the strict logical view that a proof should verify the validity of a conjecture but also that proof should promote understanding and provide an explanation as to why a conjecture is always true (de Villiers, 1990; Hanna, 2000; Hersh, 1993). Davis (1986) as cited in Hanna (1991, p. 56) describes proof as a "debating forum", wherein one could advance different kind of arguments as to why their conjecture is true or not. Furthermore, as cited in Hanna (1991, p. 56), Tymoczko (1986) sees a proof as having a "certain openness and flexibility", and Kitcher (1984) asserts that validity of a proof is a function of "correct or reasonable social practice".

As a result of views like the ones discussed in the aforementioned paragraph, there is a continuous desire amongst members of mathematics education community to conceptualize a meaning of proof that captures the salient features of a proof in terms of its structure, related processes, functions; characterizes 'proving' as a social process; and make sense to the classroom community. Moreover, Fawcett (1938) as cited in Blanton, Stylianou, & David (2003, p. 113), made the following remarks about the concept of proof:

> "The concept of proof is one which not only pervades work in mathematics but is also involved in situations where conclusions are to be reached and decisions to be made. Mathematics has a unique contribution to make in the development of this concept, and […] this concept may well serve to unify mathematical experiences of the pupil"

Within the context of the aforementioned kind of desires and remarks, Stylianides (2007, 291-292) constructed a definition of proof that encapsulates proof as an argument made up of three inter-related components, namely a "set of accepted statements", "modes of argumentation" and "modes of argument representation". According to Stylianides (2007, p. 291), proof is defined as follows:

> "Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
> It uses elements accepted by the classroom community (set of accepted statements) that are true and available without further justification;
> It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and

It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community."

Furthermore, each of the components of the aforementioned definition is exemplified via a characteristic set of examples as shown in Figure 5.3.

| **Component of an argument** | **Examples** |
|---|---|
| Set of accepted statements | Definitions, axioms, theorems, etc. |
| Modes of argumentation | Application of logical rules of inference (such as modus ponens and modus tollens), use of definitions to derive general statements, systematic enumeration of all cases to which a statement is reduced (given that their number is finite), construction of counterexamples, development of a reasoning that shows the acceptance of a statement leads to a contradiction, etc. |
| Modes of argument representation | Linguistic (e.g. oral language), physical, diagrammatic/pictorial, tabular, symbolic/algebraic, etc. |

Figure 5.3: Examples of the components of a mathematical argument (Stylianides, 2007, p. 292)

The notion of *classroom community* in Stylianide's definition refers to an audience that should determine whether an argument constructed to justify a conjecture generalization (or theorem or any other result for that matter) can be accepted as a mathematical proof or not. Furthermore, the following interpretations should be borne in mind with regard to the use of the aforementioned definition within the context of mathematics and mathematics education:

a. The notion of *classroom community* embraces all students and facilitators like prospective educators/teachers, educators, teacher educators, lecturers and also the community of professional mathematicians.

b. The terms true, valid and appropriate are used within the context of conceptions that are currently shared and agreed upon in the fields of mathematics and mathematics education. In particular the term 'true' resonates with the use of "axioms, theorems, definitions and modes of reasoning that a particular community may take as shared at a given time", and the term 'valid' is used to classify an argument wherein the assertions are connected by means of accepted rules of correct inference such as modus ponens and modus tollens (see Stylianides, 2008a,. p. 195). In addition to direct proofs, "valid arguments by counterexample, contradiction, mathematical induction, contraposition, and exhaustion are examples of proofs" (see Stylianides, 2008a, p. 196).

c. With regard to the act of using 'statements accepted by the classroom', Stylianides (2007) does not necessarily mean that each student in classroom community must understand the set of accepted statements in the same way, but rather that the statements used in the construction of a logical argument should be able stand its ground in public without further justification.

d. Similarly, with regard to 'modes of argumentation' and 'modes of argument representation' Stylianides (2007) does not necessarily mean that each student in classroom community must know or understand the respective modes in the same way, but rather suggests that the specific mode(s) of argumentation and representation a student may use to construct a logical argument could be influenced by either of the following two factors: (1) the student's prior experience with such modes of argumentation and representation; (2) the extent to which selected modes could be within the conceptual reach of a student under necessary facilitator guidance and supervision.

Taking into consideration that the proof potential of a given justication can be influenced by the form of argumentation posed (Bieda, 2010) and that the degree of comprehension of a proof by the classroom community is to a large extent a function of the form(s) of argument representation that prevails in a written justification, the definition of proof advanced by Stylianide's (2007) has been adopted for the purposes of this study, but with the following stance: A good proof should not just establish the veracity of a statement, but amongst other purposes, it should provide an explanation that sheds light as to why a particular statement claim is true, and also provide opportunities to discover new results (see de Villiers, 1990 & 2003a).

## 5.4 Functions of Proof in mathematics

From a historical perspective, proof was introduced as a device primarily to verify the correctness of a mathematical statement (Zaslavsky, Nickerson, Stylianides, Kidron, Landman, 2010). In other words proof, a deductive argument, created through the use of accepted logical rules of inference, axioms, definitions and established theorems, was seen as an argument that was constructed to enable one to remove his/her personal doubt or to remove the doubts of others about the truth of a particular mathematical conjecture or assertion (de Villiers, 1998; Harel & Rabin, 2010, p. 140). In particular, past and present mathematics educators have maintained the view that a proof enables them as well as their learners to obtain certainty, and thereby eradicate any doubt about a mathematical statement or established conjecture generalization. These views chime with the following assertions espoused given in de Villiers (1999, p. 5):

> "A proof is only meaningful when it answers the student's **doubts**, when it proves what is not obvious; - Kline (1973)
> The necessity, the functionality, of proof can only surface in situations in which students meet with **uncertainty** about the truth of mathematical propositions" - Alibert (1988).

In equivalent terms, Devlin (1988, p. 148) points out that proof is "a logically sound piece of reasoning by which one mathematician could convince another of the truth of some assertion." This notion is further echoed by Coe & Ruthven (1994, p. 42) as follows: "the most salient function of proof is that it provides grounds for belief." Similarly, Hanna (1989, p. 20) defined proof in the context of verification as follows: "A proof is an argument needed to validate a statement, an argument that may have several different forms as long as it is convincing."

Regrettably, whilst many mathematics educators are still of the view that only proof brings conviction (Bell, 1976), they have not yet realized that within actual mathematical research, conviction can be achieved by other means than that of logical proof. For example, personal conviction can be achieved through a combination of intuition, quasi-empirical verification and some form of logical (but not necessarily rigorous) proof. Indeed, an extremely high level of conviction can sometimes be realized even in the absence of a proof (see de Villiers, 2003a), with the still unproved twin pair theorem as a classic example (see Davis & Hersh, 1983, p. 369).

Furthermore, Bell (1976, p. 24) asserts that "conviction arrives most frequently as the result of mental scanning of range of items which bear on the point in question, this resulting eventually in an integration of ideas into a judgement," and proof **follows** the reaching of conviction. Effectively, this means that proof is not necessarily a pre-requisite for conviction, but rather that conviction is quite often the pre-requisite for developing a proof (de Villiers, 2003a). This position of conviction is also articulated by Polya (1954a, p. 83-84) as follows:

> "… having verified the theorem in several particular cases, we gathered strong inductive inference for it. The inductive phase overcame our initial suspicion and gave us a strong confidence in the theorem. Without such confidence we would

have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is true, you start proving it."

Moreover, Kline (1982, pp. 313-314) asserts that "…Great mathematicians know before a logical proof is ever constructed that a theorem must be true…". Although for many decades mathematicians and mathematics educators have felt that proof is just a way to verify the truth of a conjecture or generalization, several people working in the area of proof, have in recent years developed and advocated a broader perception and view of the role and function of proof in the context of mathematics. For example, Bell (1976) argues that proof serves the following core functions within the context of mathematics:

- Verification or justification: Ascertaining the truth of a proposition.
- Illumination: Develing insight as to why a proposition is true.
- Systematisation: Organising results into a deductive system of axioms, concepts and theorems.

Through reflecting on the functions of proof suggested by Bell (1976), de Villiers (1990, 1999, 2002, 2003a) developed the following model to explain the various functions of proof:

- Communication (the negotiation of meaning)
- Verification (concerned with the truth of the statement)
- Explanation (providing insight into why it is true)
- Systematization (the organization of various results into a deductive system of axioms, major concepts, and theorems)
- Discovery (the discovery or invention of new results)
- Intellectual challenge (the self-realization/fulfillment derived from constructing a proof)

Furthermore, Hanna and Jahnke (1996) in addition have suggested the following three additional functions of proof:

- Construction of an empirical theory
- Exploration of the meaning of a definition or the consequences of an assumption

- Incorporation of a well–known fact into a new framework and thus viewing it from a fresh perspective.

In addition, Hemmi (2010, p. 273) cites ‘transfer’ as another function of proof, and in this respect asserts that the kinds of thought that goes into the construction of a proof, could either provide one with techniques that one could utilize to attempt and solve other problems or improve one’s chance of understanding an item that deviates in its formulation with respect to a familiar (or original) context.

Whilst each of the functions of proof is characterized by a unique action or outcome, they do not necessarily exist in isolation from each other in the development of a given proof. For example, consider the proof of the following conjecture generalization: the bisectors of the angles of a triangle are concurrent at a point equidistant distant from the sides. The proof of this concurrency generalization in most high school textbooks, use the congruency approach, as follows:

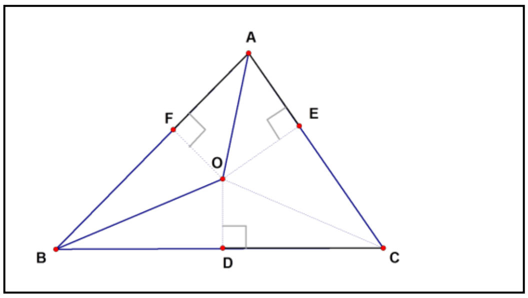



Figure 5.4.1: Concurrency - bisectors of angles of triangle

Given: $\Delta ABC$ with the bisectors of $\hat{A}$ and $\hat{B}$ meeting at $O$.

To Prove: The bisectors of the angles are concurrent at a point equidistant from the sides.

Construction: Draw $OF$, $OE$ and $OD$ perpendicular to $AB$, $AC$ and $BC$ respectively. Draw $OC$.

Proof: In $\Delta OAE$ and $\Delta OAF$

$O\hat{A}E = O\hat{A}F$ constr.

$O\hat{E}E = O\hat{F}A$ rt.$\angle$’s, constr.

$OA$ is common

$\therefore \Delta OAE \equiv \Delta OAF$ $\angle\angle$S

$\Rightarrow OE = OF$

Similarly, from $\Delta OBD$ and $\Delta OBF$ it can shown that

$OF = OD$

$\therefore \quad OD = OE$

Now, in $\Delta ODC$ and $\Delta OEC$

| | |
|---|---|
| $OD = OE$ | proved |
| $OC$ is common | |
| $O\hat{D}C = O\hat{E}C$ | rt. $\angle$'s, constr. |
| $\therefore \ \Delta ODC \equiv \Delta OEC$ | $90^{o}$HS |
| $\Rightarrow \quad O\hat{C}D = O\hat{C}E$ | |

Thus $OC$ bisects $A\hat{C}B$, and the bisectors of the angles are concurrent.

In principle, we have demonstrated the validity of the conjecture generalization. In other words we have proven the conjecture generalization, using previously established accepted statements, such as definitions and axioms. Hence, we can now call this proven conjecture generalization, a theorem. In abbreviated form, we will refer to this concurrency theorem as follows in this text: bisectors of $\angle$'s of $\Delta$ theorem.

The congruency approach to the proof bisectors of $\angle$'s of $\Delta$ theorem definitely contributes to a successful demonstration of the truth of the given theorem, through the appropriate use of conditions of congruency, in their capacity as axioms. Hence, we can say that the objective to purely establish the truth of the conjecture generalization (verification) has been realized. Also, on the other hand, the proof exposes the logical relationships between other established mathematical statements, such as other theorems, and previously accepted statements such as axioms and definitions. For instance, in the proof of the bisectors of $\angle$'s of $\Delta$ theorem, we see the building of a deductive system with the conditions of congruency assumed as some of our axioms, which is typical of the systematization function of proof. Thus, it is possible that the development of a proof of a particular mathematical statement, could underscore more than one function of proof, such as verification and systematization, as in the case of congruency proof of the bisectors of $\angle$'s of $\Delta$ theorem. Others may also claim, that the development of congruency proof for bisectors of $\angle$'s of $\Delta$ theorem, could result in some form of self realization or fulfillment, which is characteristic of the intellectual function of proof.

Hence, despite the functions of proof being characterized by particular attributes, which makes them distinguishable from each other, they often are ‘alive” and interwoven in specific proofs with varying degree of dominance, notwithstanding that in some cases certain functions may not feature at all (de Villiers, 2003a, p.10). For instance, the traditional Euclidean proof for bisectors of $\angle$'s of $\Delta$ theorem using the laborious congruency approach, does not necessarily provide a sense of explanation of why the concurrency result is true. However, one can use the symmetry approach, to establish the truth of the concurrency result, and also to provide a sense of explanation why it is true (see discussion later in Section 5.4.1).

Hence, although the explanation and discovery functions of proof are considered and discussed as part of the theoretical framework for this study, the chosen examples and their corresponding proofs may inevitably embrace or bring to the fore some of the other functions of proof as well.

### 5.4.1 Proof as a means of explanation

In cases where experimentation in mathematics is accorded the necessary pedagogical importance, we find that individuals (mathematicians, mathematics educators or university students or learners at schools) who are able to formulate a conjecture either intuitively, inductively or analogically, generally attempt to check out the authenticity of their conjecture via further quasi-empirical methods such as constructions and measurements (either using pencil and paper or software such as Geometer’s Sketchpad, Cabri or Geo-gebra, etc.), or by case–by-case numerical substitution (using perhaps a spreadsheet to create a huge table of values). For example consider the following snap-shot of a lesson, pertaining to the topic: The sum of the angles of a triangle is $180^0$:

- An educator requested learners to draw a triangle, and measure its angles and then find their sum, using Sketchpad.
- All the learners came up with the conjecture, the sum of the angles of a triangle is $180^0$.
- Thereafter, the educator requested the learners to drag the vertices of a triangle to different positions and observe the sum measure.
- In each and every case, the learners observed that the sum of the angles remained $180^0$.

- Through this process, the learners developed an extremely high level of confidence in their conjecture, making them believe that the conjecture is always true, and thus causing them to say: the sum of the angles in any triangle is always $180^0$.

However achieving such a high degree of confidence (or conviction) in their conjecture through quasi-empirical verification, does not provide the learners with necessary reasons as to "why" their conjecture is always true. In this sense, we say that all the empirical examples, merely raised the confidence 'bar', but have not provided any "satisfactory explanation why the conjecture may be true" (de Villiers, 1999). In other words, de Villiers (1999, p. 7) claims that any number of confirming empirical examples or more, will only increase one's confidence level in their conjecture, but essentially will not "provide a psychological sense of illumination or insight or understanding into how the conjecture is the consequence of other familiar results."

This is true not only in the pedagogical context of the mathematics classroom, but within the broader mathematical community. For example, on the basis of the strong "heuristic evidence" for the Riemann Hypotheses, Davis and Hersh (1983, p. 369) cited in de Villiers (1999, p. 6), concluded that the evidence is "so strong that it carries conviction even without rigorous proof." However, despite all the heuristic evidence in favour of the Riemann Hypothesis, it is inevitable that mathematicians have a burning desire for an explanation as stated by Davis and Hersh (1983, p. 368) and cited in de Villiers (1999, p. 7):

> "It is interesting to ask, in a context, such as this, why we feel the need for a proof…It seems clear that we want a proof because… if something is true and we can't deduce it in this way, this is a sign of a lack of understanding on our part. We believe, in other words, that a proof would be a way of understanding why the Riemann conjecture is true, which is something more than just knowing from convincing heuristic reasoning that it is true."

Thus, in essence, one could say that the provision of insight as to why a particular result is true and the 'thirst' for sense making of a mathematical result are the 'drivers' of 'explanation', which is also why many mathematics educationists like Hanna and Jahnke (1996) acknowledge it as an important function of proof. In a similar vein, Schoenfeld (1985, p. 172) relates to this core function of proof as follows: "'Prove it to me' comes to mean ‘explain to me why it is true’, and argumentation becomes a form of explanation, a means of conveying understanding.”

Also Hafner and Mancosu (2005, p. 218) states that “the old desire to know the reason why” motivates mathematicians to look for explanations, and suggests that the search for explanations in the context of mathematics is characterized as a search for: “the deep reasons”, “an understanding of the essence”; “a better understanding” “a satisfying reason”: “the reason why’; “the true reason”; “an account of the fact” and “the causes of”. For example, let’s revert back to the following example below, the sum of the first consecutive odd numbers is a square. A proof, by mathematical induction for this mathematical statement, has already been provided in the section on generic proving, but this proof does not provide a sense of ownership and understanding of why the sum of the first odd numbers is always a square. Thus in the context of the ‘explanatory’ function of proof, a visual-geometric approach is now adopted to try and explain, why the sum of the first odd numbers is a square.

**Example**: Show that $\sum_{i=1}^{n}(2i-1) = n^2$

We can use a visual-geometric approach to explain why $\sum_{i=1}^{n}(2i-1) = n^2$.

First notice that $(2i - 1) = (i - 1) + (i - 1) +1$. Next observe that the number of blocks in the $i$th row combined with the $i$th column of Figure 5.4.2.1 is $(i - 1) + (i - 1) + 1= 2i – 1$.

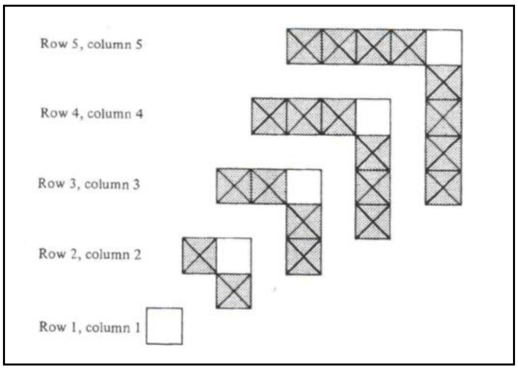



Figure 5.4.1.1: Geometric approach

For example, when $i = 3$, the number of blocks in the third row combined with third column are 2 + 2 shaded blocks plus 1 non-shaded block, for a total of 5 blocks. This situation is exemplified for the first 5 rows and columns in the figure.

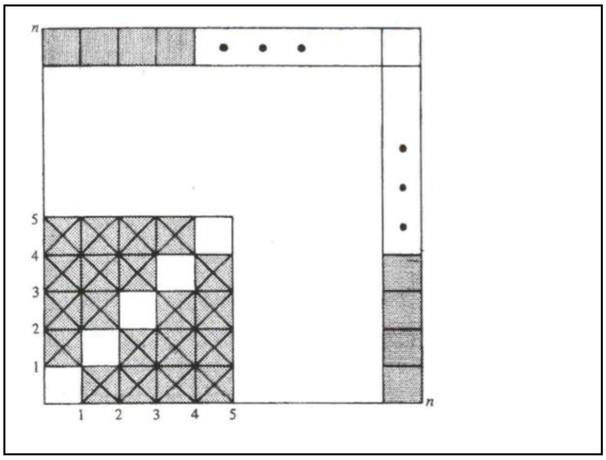



Figure 5.4.1.2: Number of Blocks

From Figure 5.4.1.2, one can further observe that the number of blocks in all rows and all columns from 1 to $n$ , is $n^2$. That is, in a square of $n$ rows and $n$ columns, there are $n^2$ elements. Such a square can be obtained by combining all the blocks of row 1, column 1, with row 2 column 2, and so on, until one finally includes the blocks of row $n$, column, $n$.

We of have previously observed that the number of blocks in row $i$, is $(i - 1) + (i - 1) + 1 = 2i - 1$. For this reason, one obtains as a result of the above discussion,

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

It is interesting to note that $\sum_{i=1}^{n}(2i-1) = 1 + 3 + 5 + 7 + \ldots + (2n - 1)$, which is the sum of the consecutive (positive) odd numbers. This result shows that the sum of $n$ consecutive odd numbers is $n^2$.

In essence, the proof by induction (see section on generic proving) can be regarded as a proof that proves, and the proof via the visual-geometric approach can be regarded as a proof that explains.

In particular, Hanna (1990, p. 9) asserts the following:

> "A proof that proves shows only that a theorem is true; it provides evidential reasons alone. It is concerned only with substantiation, with what are known as rationes cognoscendi, that is, why-we-hold-it-to-be-so reasons. A proof that explains on the other hand, also shows why a theorem is true; it provides a set of

reasons that derive from the phenomenon itself: rationes esseni, or why-it-is-so reasons."

This essentially means that both proofs that prove and proofs that explain, are considered as legitimate proofs, particularly because they both provide a coherent set of arguments to establish the validity of a statement. However, a proof that explains does more than just validating the truth of theorem or conjecture generalization, since it actually also provides an essential understanding of *why* a theorem or conjecture generalization is true. Indeed, "a proof that explains, must provide a rationale based upon the mathematical ideas involved, the mathematical properties that cause the asserted theorem to be true" (Hanna, 1990, p. 9).

Thus, we use the term, explain, in the context of proof, only when the proof highlights and invokes mathematical ideas which motivate it. However, not all proofs have explanatory power and on the other hand one can establish the validity of mathematical assertions by syntactic considerations alone, wherein the syntactic proof merely shows that a statement is true without really bringing to surface the mathematical property that makes it true (Hanna, 1990).

Hanna (1996, p. 4), also illustrates through examples, the notions of proofs that prove and explains as follows:

"Prove that the sum of the first $n$ positive integers, $S_n$, is equal to $n(n+1)/2$"

**A proof that proves**:

Proof by Mathematical Induction:

First step: For $n$ =1, we have $S_1$= 1(1+1)/2 = 1, which is true.

Second step: Assume that the theorem is true for $n = k$:

$$S_k = 1+2+3+\ldots+k = \frac{k(k+1)}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Next we must show that the formulae is true for its successor, $n = k + 1$. That is, we must show: $S_{k+1} = 1+2+3+\ldots+k+(k+1) = \; = \frac{(k+1)((k+1)+1)}{2}$ ……….. (2)

To show (2) we proceed as follows:

$$\begin{aligned} S_{k+1} &= 1+2+3+\ldots+k+(k+1) \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{k^2+k+2k+2}{2} \end{aligned}$$

$$= \frac{k^2 + 3k + 2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2}$$

We see the result, $\frac{(k+1)((k+1)+1)}{2}$ , is the expression on the right side of $S_{k+1}$ .Thus by mathematical induction, $S_n$ is true for all natural numbers $n$.

Hanna and Jahnke (1996) acknowledges that a proof by mathematical induction such as given above "demonstrates that the theorem is true, but gives the student no inkling why it is true" and moreover it "has little explanatory power". However, Hanna and Jahnke (1996, p. 904) suggests that through the use of symmetry of two representations of the given sum (which incidentally is Gauss's proof), it is possible to explain why the theorem is true as follows:

$S_n = 1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + n$

$\underline{S_n = n + (n-1) + (n-2) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + 1}$ (reversing $S_n$)

$2S_n = (n+1) + (n+1) + (n+1) + \ldots\ldots\ldots + (n+1)$ (Adding)

$= n(n+1)$ (since there are lots of $(n+1)$)

$$\therefore S_n = \frac{n(n+1)}{2}$$

In addition, Hanna and Jahnke (1996, p. 904), provides two more explanatory proofs, using triangular numbers and area respectively, which show that the sum of the first $n$ positive integers, $S_n$ , is equal to $n(n+1)/2$.

**Triangular Number proof – a proof that explains:**

In this instance, the geometric representation of the first $n$ positive integers takes the form of an isosceles right triangle of dots. In particular, the sum of the first $n$ integers can be represented as triangular numbers (see Figure 5.4.1.3).

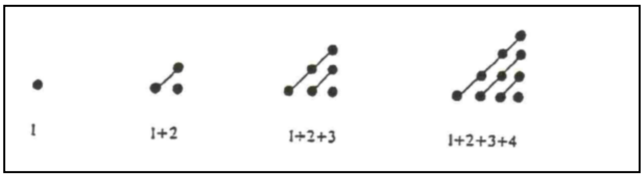



Figure 5.4.1.3: Triangular numbers

"The dots form an isosceles triangle containing $S_n = 1 + 2 + ... + n$ dots.

Two such sums, $S_n + S_n$, give a square containing $n^2$ dots and $n$ additional dots, because the diagonal of n dots is counted twice.

Therefore: $2S_n = n^2 + n$

$$S_n = \frac{n^2+n}{2}$$

$= \frac{n(n+1)}{2}$ (additional step inserted -numerator factorized)"

**Staircase-shaped area – a proof that explains**:

Hanna and Jahnke (1996, p. 905) suggests that the first $n$ integers can be represented by a staircase-shaped area as follows: "a rectangle with sides $\boldsymbol{n}$ and $\boldsymbol{n+1}$ is divided by a zigzag line (see Figure 5.4.1.4). The whole area is $n(n+1)$, and the staircase- shaped area, $1 + 2 + 3 + ... + n$ is only half, hence $\frac{n(n+1)}{2}$."

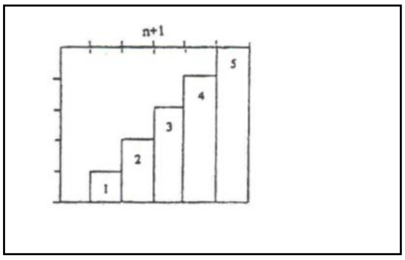



Figure 5.4.1.4: : Staircase – shape area

The alternative examples of explanatory proof signposted by Hanna and Jahnke (1996), suggest that there exists ample opportunity for mathematics educators to search (or create) and present explanatory proofs that better promote understanding. Such explanatory proofs are "much more likely to yield not only 'knowledge that', but also 'knowledge why'" (Hanna and Jahnke, 1996, p. 905). However, in reality it is not always possible to find a proof for every theorem that performs an explanatory function, but nevertheless one should make the most of any possible opportunity to engage with a proof that explains instead of one that merely proves. Hersh's (1993, pp. 396-398) is also of view that, "what proof should do for the student is to provide insight into why a theorem is true" and at the high school level "the primary role of proof is explanation", rather than just to convince because learners are any way usually already easily convinced on the basis of empirical evidence or the authority of the teacher or textbook. In particular, Hersh (1993, p. 389) asserts that, "enlightened use of proofs in the mathematics classrooms aims to stimulate the students' understanding, not to meet abstract standards of "rigor" or "honesty."

De Villiers (1992, p. 51) gives a proof that explains why the bisectors of the angles of a triangle are concurrent at a point equidistant distant from the sides (bisectors of $\angle$'s of $\Delta$), in contrast to the concurrency one that was discussed earlier in Section 5.4. Before we proceed with the actual proof that explains, we need to consider the following possible definition of the bisector of an angle (illustrated in Figure 5.4.1.5): It is the locus (path) of all the points equidistant from both arms (rays) of an angle.

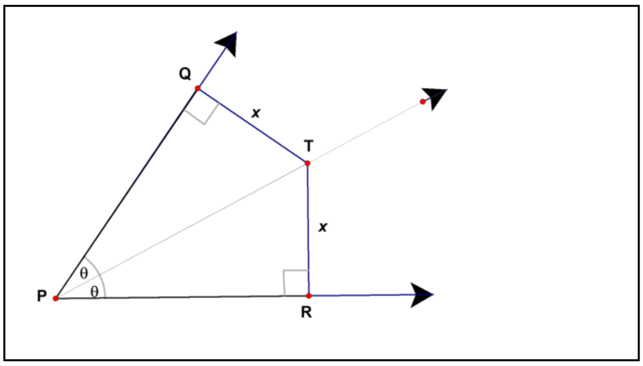



Figure 5.4.1.5: A definition of angle bisector

**Explanatory Proof : (*bisectors of* $\angle$ *'s of* $\Delta$):**

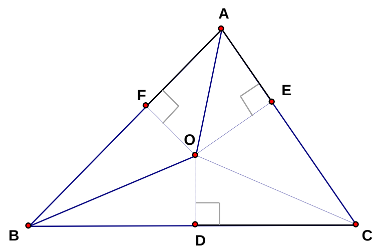



Figure 5.4.1.6: Bisectors of $\angle$'s of $\Delta$

Consider Figure 5.4.1. 6, wherein we are given $\Delta$ABC with the bisectors of $\hat{A}$ and $\hat{B}$ meeting at *O*. Using the meaning of the bisector of an angle as the locus of all the points equidistant from both arms (rays) of an angle, we develop an argument as follows using the notion of symmetry with reference to Figure 5.4.1.6:

Now, *BO* is the locus of points equidistant from *BC* and *AB*, and also *AO* is the locus of points equidistant from *AB* and *AC*. Hence, it follows that the point *O*, where *BO* and *AO* meet, must also be equidistant from *BC* and *AC* (through transitivity). Therefore the point *O* must also lie on the bisector of $\hat{C}$ (which is the locus of all the points equidistant from *BC* and *AC*), and which completes the proof of concurrency (see de Villiers, 1992, p. 51).

Explanatory proofs, such as the ones just posited, are in keeping with Hersh's (1993) humanist view of proofs, namely, "Proof is complete explanation". This essentially means that proofs are not just pure mechanistic procedures or obligatory rituals, but rather a process that entails understanding, which goes beyond just confirming that all links in a chain of deduction are correct (Hanna, 1990). This inherent drive for creating understanding through explanatory proofs, provides opportunities to increase students' understanding of concepts, methods and related applications, through some engagement with previous mathematical experiences wherever necessary. Furthermore, through well constructed explanatory proofs, there exists a greater possibility for students to obtain deeper insight into the connections of the network of mathematical ideas leading to the truth of a particular theorem or proposition. Anderson (1996, pp. 32-38) also endorses that proof in its explanatory capacity, ought to provide a kind of deeper understanding of why a particular result or conjecture generalization is always true as follows in his own words:

> "Proof should be seen as being about explaining, albeit carefully and precisely. It is where instrumental understanding gives way to relational understanding. It should be seen as the essence of mathematics and all pupils who study mathematics should meet it at some, at some level."

Furthermore, Hadas and Hershowitz (1998), in their research regarding activities that induce the need for explaining and convincing, finds that the need for explanation is raised by:

a. A surprise caused by the contradiction between the conjectures and what students get (or could not get) while working with a dynamic tool.
b. A situation where one cannot find any example for a conjecture he or she made
c. The multiple representations of the situation (geometrical, numerical and graphical) and relating these representations or resolving perceived conflicts between them

Indeed the 'surprise' element in cases where the results are convincing but surprising, stirs up the need to know why the result occurs, i.e. causes the need to understand 'why?' Moreover, for many mathematicians themselves, the explanation aspect of a proof is far more their desired goal than the mere verification aspect; "even those who have recourse to purely syntactic methods, are really more interested in the message behind the proof than its syntax, and see the mechanics of proof as a necessary but ultimately less significant aspect of mathematics" Hanna (1990, p. 12). For example, we have the proof of the famous four colour theorem, which Appel and Haken (1976) reduces to a finite (but large) number of alternatives which are resolved by the computer. Despite "the impeccable logic in the listing of the possibilities and their checking by the computer, the proof itself seems to shed no light as to why the theorem is true" (Dubinsky & Tall, 1991, p. 233).

Moreover, whilst some mathematicians may be happy with a computer checked proof, there are other mathematicians who have the dire need to have a kind of insight as to how the concept relates to other known results and not just the mere logical deduction from a proof. Unless these mathematicians have such insight, a kind of dissatisfaction or even insecurity continues to lurk in their minds because they fear that that some minor logical error (perhaps linked to errors in programming or comprehension of certain principles in the computer environment) may be found, which can then render the argument fallacious (Dubinsky & Tall, 1991). A similar kind of notion is also echoed by Otte (1994, p. 310) as follows:

> "[T]he mathematician's reaction shows quite clearly that a proof which does nothing but prove in the sense of mere verification must be very unsatisfactory. A proof is also expected to generalize, to enrich our intuition, to conquer new objects, on which our mind may subsist."

Thus, mathematics teachers and mathematics teacher educators, should strike at every opportunity to provide good mathematical explanations for established conjecture generalizations in their classrooms, with focus on the underlying mathematical ideas that contribute to the establishment of meaningful truth at all times, with the ultimate goal being to provide the much needed insight into why the conjecture generalization is always true.

### 5.4.2 Proof as a means of discovery

In the development of a mathematical proof of a conjecture generalization, we find that definitions, axioms and previously established theorems, are appropriated and used to "deduce the truth of one statement from another" (Tall, 1989, p. 30). This approach helps to establish the certainty of a particular conjecture generalization, and also provide possible "insight into how and why it works" (Tall, 1992, p. 506). However, depending on the nature of the conjecture generalization to be proved, it is quite possible that other new results or conjectures could be discovered during the ‘think-tank-time’ phase of the development of its proof, without necessarily engaging in any further kind of empirical investigations/actions. This means that new results or conjectures can be the spin-off from an attempt to prove a particular conjecture generalization or theorem. Thus we say it is possible for new results in mathematics to be discovered or invented in a purely deductive manner and not just in an intuitive, inductive or analogical manner.

For example, since 1742 numerous attempts have been made to prove the Goldbach conjecture, which states that any even number greater than 2 is the sum of two primes (Hummel, 2000). Although the Goldbach conjecture has been verified numerically (using a computer program) up to $4 \times 10^{11}$ (Saouter, 1998), no one has been able to prove it or provide a counter-example, but the search for a proof of the Goldbach conjecture has inevitably led to many other discoveries and proofs. The kind of rigour and effort that has been invested in search for a proof for the Goldbach conjecture, has earned us other good ‘dividends’, which inevitably places the inherent discovery ‘agency’ contained in particular proof making activities (such as the ones in the Goldbach case) at the helm of the construction of knew knowledge. This sentiment is also echoed as follows by Rav (1999) as cited in de Villiers (2002, p. 1.):

> “Look at the treasure which attempted proofs of the Goldbach conjecture has produced, and how much less significant by comparison its ultimate ‘truth value might be! ... Now let us suppose that one day somebody comes up with a counter-example to the Goldbach conjecture or with a proof that there exists positive even integers not representable as a sum of two primes. Would that falsify or just tarnish the magnificent theories, concepts and techniques which were developed in order to prove the now supposed incorrect conjecture? None of that. A disproof of the Goldbach conjecture would just catalyze a host of new developments, without the slightest effect on hitherto developed methods in an attempt to prove the conjecture. For we would immediately ask new questions, such as to the number of ‘non-goldbachian’even integers: finitely many? infinitely many? … New treasures would be accumulated alongside, rather than instead of the old ones, thus and so is the path of proofs in mathematics!”

Just as the search for the proof for the Goldbach conjecture catalyzed a whole new range of conjectures, lemmas and proofs, the pursuit for the proof Fermat’s Last Theorem (FLT), which states that the equation, $x^n + y^n = z^n, \quad x, y, z \neq 0$ has no integer solutions when $n$ is greater than or equal to 3, has contributed to much important mathematics (Darmon, Diamond & Taylor, 1989). In particular, the search for a proof of FLT, which was first conjectured by Pierre de Fermat in 1637 and proved by Andrew Wiles in 1995, stimulated the development of algebraic number theory in the 19th century and the proof of the modularity theorem in the 20th century (Wikipedia *b*, 2010). For example, Ernst Eduard Kummer in the 1840's as cited in Darmon et al (2007), brought in sophisticated concepts of algebraic number theory and theory of $L$ -functions to attempt to prove FLT. In this process the concepts of cyclotomic integers and integers were formulated, which undoubtedly made significant contributions towards the development of class field theory and abstract algebra (Cox, 1994). However, Kummer proved the theorem only for a large class of primes known as regular primes. Thereafter, other mathematicians, working on Kummer's proof, using sophisticated computer programs, managed to prove the conjecture for all primes less than four million (see Wikipedia, 2010*a*). The zest for the proof of FLT continued, and in the process 'Gerhard Frey's suggestion to use the modularity conjecture for elliptic curves to prove FLT was considerd(Cox, 1994). In particular, Frey worked with the Taniyama-Shimura conjecture, which states:

> *"Given an elliptic curve* $y^2 = Ax^3 + Bx^2 + Cx + D$ *over* $\mathbb{Q}$, there are nonconstant modular functions *f(z), g(z)* of the same level N such that
> $f(z)^2 = Ag(z)^3 + Bg(z)^2 + Cg(z) + D$ *."* (see Cox, 1994, p. 9)

The Taniyama- Shimura conjecture says that an elliptic curve over $\mathbb{Q}$ can be parametrized by modular functions, and that such an elliptic curve is modular. Frey tried to prove that the Taniyama-Shimura conjecture implies FLT (Cox, 1994, p. 9). However, his proof had some serious gaps, and Jeane-Pierre Serre saw that a special version of a conjecture (the so called epsilon conjecture) he made on level reduction for modular Galois representations would fill the gap. Hence, due credit goes to Frey & Serre, for showing that FLT follows from Taniyama-Shimura conjecture and the special level reduction conjecture (epsilon conjecture) made by Serre. Following this strategy, Ken Ribet in 1986, contributed to the proof of FLT by proving Serre's conjecture (the epsilon conjecture) (see Cox, 1994, p. 9). This meant that Taniyama-Shimura conjecture for semistable elliptic curves had to be proved in order to complete the proof for FLT. Inspired by this development, Andrew Wiles began to work on Taniyama-Shimura conjecture, and seven years later on 23 June 1993, he presented a proof that the conjecture is true for semistable elliptic curves (Ribet & Hayes, 1994). In essence, "the epsilon conjecture showed that any solution Fermat's equation could be used to generate a non-modular semistable eliptic curve, whereas Wiles' proof showed that such elliptic curves must be modular. This contradiction implied that there can be no solutions to Fermat's equation, thus proving Fermat's Last Theorem" (see Wikipedia, 2010*a*).

Moreover, the research mathematician Gian-Carlo Rota (1997, p. 10) as cited in de Villiers (2002) pointed out in no uncertain terms, that the value of the proof of Fermat's Last Theorem goes far beyond the verification of the result itself:

> "The actual value of what Wiles and his collaborators did is far greater than the mere proof of a whimsical conjecture. The point of the proof of Fermat's last theorem is to open up new possibilities for mathematics….The value of Wiles proof lies not in what it proves, but what it opens up, in what it makes possible."

Moreover, many people working in the field of mathematics education appear to have a general misconception that theorems or conjecture generalizations are first discovered through intuition and or empirical methods, prior to being verified by construction of logical proofs. However, there are many examples in mathematics where new results are discovered or invented just via deductive arguments. For example, suppose one was examining the properties of isosceles trapezia. Assume that we already knew that in general, as shown as shown in Figure 5.4.2.1, that an isosceles trapezium has the following properties, namely: *PQ//SR*; *PS* = *QR*; $\hat{P} = \hat{Q}$; $\hat{S} = \hat{R}$ and *PR* = *QS*.

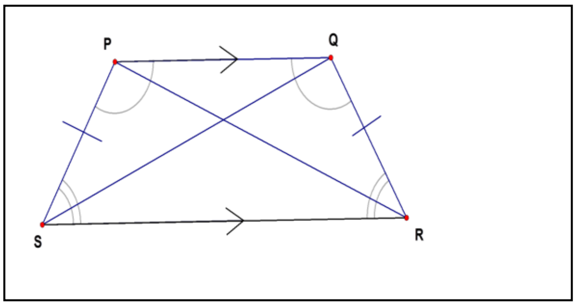



Figure 5.4.2.1 An isosceles trapezium

By considering the known properties, one could easily develop the following kind of argument by logically analysing its properties:

$$\hat{P} + \hat{S} = 180^0 \text{ (co-interior angles, PQ//SR)}$$

and $\hat{S} = \hat{R}$

$$\therefore \quad \hat{P} + \hat{R} = 180^0$$

$\Rightarrow$ the isosceles trapezium *PQRS* is a cyclic quadrilateral.

Thus, in this particular case, we have discovered new knowledge in a deductive manner and not in the conventional inductive manner by first using construction and measurement.

Moreover, a purely analytical approach whereby the properties of given objects are deductively analyzed, could in an *a priori* sense lead to the discovery of new results (see de Villiers, 2003a). For example, in Figure 5.4.2.2, we are given PX//TY//SZ and PS, PZ and XZ are transversals.

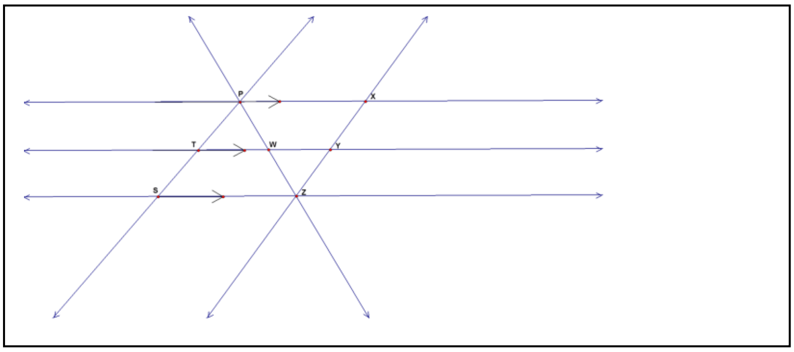



Figure 5.4.2.2: Parallel line cut by a transversal

Without actually engaging in any construction or measurement procedures, we can proceed in a deductive manner as follows:

In ΔPSZ,

*TW//SZ* (given)

$\therefore \frac{PT}{TS} = \frac{PW}{WZ}$ ... (1) (Proportional Intercept Theorem)

In ΔZPX,

*WY//PX* (given)

$\therefore \frac{PW}{WZ} = \frac{XY}{YZ}$ … (2) (Proportional Intercept Theorem)

From (1) and (2) we conclude that $\frac{PT}{TS} = \frac{XY}{YZ}$.

This basically produces the following result: three (or more) parallel lines cut all transversals in the same ratio.

Also, when after succesfully proving a particular theorem or conjecture generalization, it could be realized through careful reflection or “looking back” on the developed proof, that a particular condition or property stated in the enunciation of the theorem or conjecture generalization, is not really necessary, i.e. it may just be a superfluous property, which has no pivotal bearing on the construction of the proof. Consequently this provides an ideal opportunity to possibly generalize a result across other domains, for example from quadrilaterals to pentagons to hexagons to octagons and to a general polygon. For example, suppose learners are given the following ready- made script of a kite *ABCD* on Sketchpad, where the connected midpoints form a quadrilateral *EFGH* as shown in Figure 5.4.2.3, and are asked to experiment and then formulate a conjecture regarding the quadrilateral formed by the midpoints of its sides (see De Villiers, 2003a, p. 8).

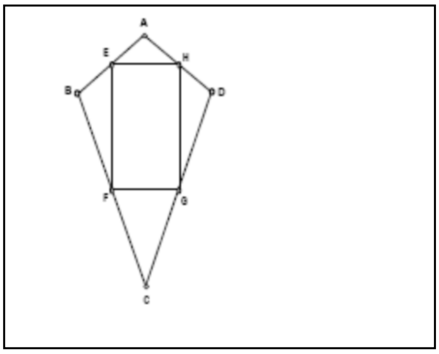



Figure 5.4.2.3: Kite *ABCD*

It so happens that the learners come up with the following conjecture generalization: The line segments consecutively connecting the midpoints of the adjacent sides of a kite form a rectangle. However, one could provide a deductive explanation of the established conjecture generalization using the following set of logical arguments:

**Deductive Explanation:**

In $\Delta ABC$,

$BE = EA$ … (given $E$ is the midpoint of $AB$)

and $BF = FC$ … (given $F$ is the midpoint of $BC$)

$\therefore$ $EF // AC$ …. (midpoint theorem)

Similarly in $\Delta ADC$, $HG // AC$

$\therefore$ $EF // HG$

Similarly, $EH // BD // FG$

Now in Quad *EFGH*:

$EF // HG$ … (proved above)

and $EH // FG$ …(proved above)

∴ Quad *EFGH* is a parallelogram …. (both pairs of opp.sides of quad *EFGH* are parallel
Since $BD \perp AC$ (property of kite) we also have $EF \perp EH$.
∴ *EFGH* is a rectangle … (*EFGH* is a parallelogram with a right angle).

On careful reflection of the aforeconstructed proof, which Polya refers to as "looking back", we observe that the property of equal adjacent sides (or an axis of symmetry through one pair of opposite angles) was not used anywhere in the development of the proof. This basically means that the result can be immediately generalized to any quadrilateral with perpendicular diagonals (a perpendicular quadrilateral) as illustrated in Figure 5.4.2.4 ( see de Villiers, 2003a).

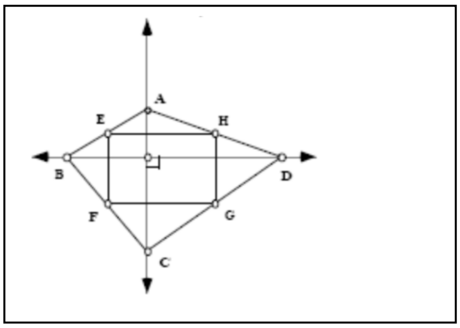



Figure 5.4.2.4: Quadrilateral with perpendicular diagonals

Furthermore this generalization is true for the concave and crossed cases as well, and demonstrates the real value of understanding why something is true. De Villiers (2003a) asserts that the general result did not emanate from any of the empirical investigations or verifications associated with the original conjecture. Furthermore De Villiers (2003a, p. 8) maintains that "even a systematic empirical investigation of various types of quadrilaterals would probably not have helped to discover the general case, since we would have probably have restricted our investigations to the familiar quadrilaterals such as parallelograms, rectangles, rhombuses, squares, and isosceles trapezoids."

Just as the previous example highlights the discovery function of proof, Hemmi & Loffwall (2011) illustrates the discovery function of proof through *looking back* at a proof constructed to justify the following statement: 'In a rectangle the midpoints are connected. Then one obtains a parallelogram'. In summary, the proof for the aforementioned statement is re-constructed using the following proportionality theorem in the argument: "If a line divides two sides of a triangle proportionally, then the line is parallel to the third side of the triangle" (see Dreyer, Dreyer, & Bissesor, 1987, p. 187).

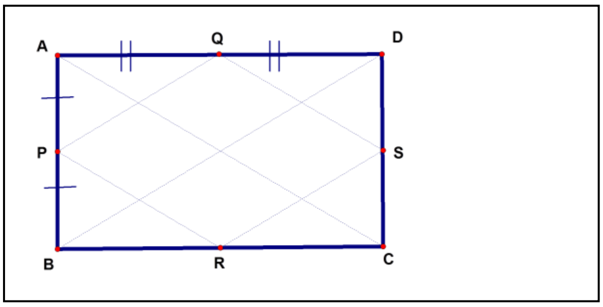


Figure 5.4.2.5: Rectangle with midpoints joined

An outline of a reconstructed proof is as follows:

Given: *P*, *Q*, *R* and *S* midpoints of sides *AB*, *AD*, *BC* and *CD* respectively

To prove: *PQSR* is a parallellogram

Construction: Draw diagonals *BD* and *AC*. Join *PQ*,*PR*, *RS* and *QS*.

Proof: Consider diagonal *BD*.

Let $AP = x\ units$ and $AQ = y\ units$,

then $PB = x\ units$ and $QD = y\ units$ (given *P* is the midpoint of *AB* & *Q* is the midpoint of *AD*)

Now,

$$\frac{AP}{PB} = \frac{x}{x} = \frac{1}{1}$$

and $$\frac{AQ}{QD} = \frac{y}{y} = \frac{1}{1}$$

$$\therefore \frac{AP}{PB} = \frac{AQ}{QD}$$

$\therefore$ PQ// BD (line divides two sides of a Δ proportionately)

Similarly, *RS//BD*

$\therefore$ *PQ//RS* ……………….. (1)

Similarly by considering the second diagonal *AC*,

*QS// PR* …………………. (2)

(1) & (2) => *PRSQ* is a parallelogram (both pairs of opp. sides of quad.*PRSQ* are parallel)

On reflecting and analyzing the afore-constructed proof, it is quite noticeable that the given information about the figure (quadrilateral) being a rectangle was not in any specific way used in the construction of the deductive justification. This then suggests that the afore-constructed deductive justification (proof) is "valid under weaker assumptions" (Hemmi & Loffwall , 2011, p .4), which in effect means that the proof produced in this instance holds for any kind of quadrilateral. Such insight enables one to realize that the initial conjecture pertaining just to the rectangle is really a special case of a more general conjecture (or generalization), and consequently makes it possible for one to produce a new generalization that is much more general in nature, for example: "Connecting the four midpoints of an arbitrary quadrilateral, yields a parallelogram" (Hemmi & Loffwall, 2011, p. 4). However, on further reflecting on the proof, we see that the proof has been constructed around a special proportion, namely $\frac{AP}{PB} = \frac{AQ}{QD} = \frac{1}{1}$, suggesting that the proof is a special case of a more general proof. In a more general sense, the proof works wells for any proportion, namely $\frac{AP}{PB} = \frac{AQ}{QD} = \frac{m}{n}$, $\forall m, n \in (0; \infty)$. Such a finding, enables one to reconstruct the original conjecture generalization into the following more general generalization: "In a quadrilateral the two sides outgoing from a corner are divided in the same proportion. The same proportion is also used to divide the two sides outgoing from the opposite corner. Connecting these four points gives a parallelogram" (Hemmi & Loffwall, 2011, p. 5). In essence, we refer to the latter generalization as a deductive generalization primarily because it was discovered (or invented) in a true deductive style upon careful reflection and analysis of the proof that was constructed to explain the validity of the initial conjecture (see de Villiers, 1990, 1999, 2003).

In much the same vein, De Villiers (2002, p. 7), discovered and proved the following interesting generalization using *Sketchpad*: "If similar triangles *PAC*, *QDC* and *RDB* are constructed on *AC*, *DC* and *DB* of any quadrilateral *ABCD* with *AD* = *BC* so that $\angle APC = \angle ASB$, where *S* is the intersection of *AD* and *BC* extended, then *P*, *Q* and *R* are collinear" (see Figure 5.4.2.6). On carefully looking back at his proof, de Villiers surprisingly realized that he had never used the property that *AD* = *BC*.

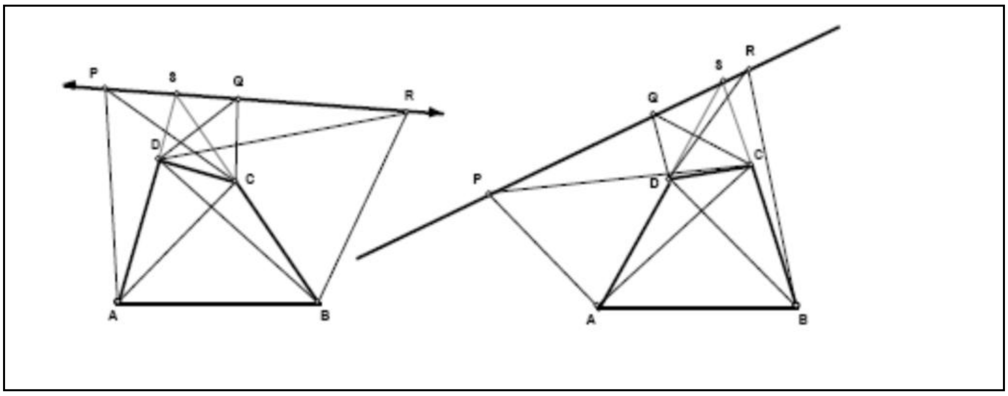



Figure 5.4.2.6: Similar triangles on any quadrilateral *ABCD*

This discovery in itself, suggested that the result could be easily generalized to *any* other quadrilateral. Indeed, this re-iterates "the value of an explanatory proof which enables one to generalize a result by the identification of fundamental properties upon which it depends" (de Villiers, 2002, p. 7).

In a similar vein, de Villiers (2007b) enlightens the discovery function of proof through reflecting on some of his experiences he had at a KwaZulu-Natal Olympiad problem solving workshop, wherein about 30 teachers were given the following problem (called Cross's theorem) to solve:
"Prove that the areas of the four shaded triangles (in Figure 5.4.2.7) are the same" (p. 2).

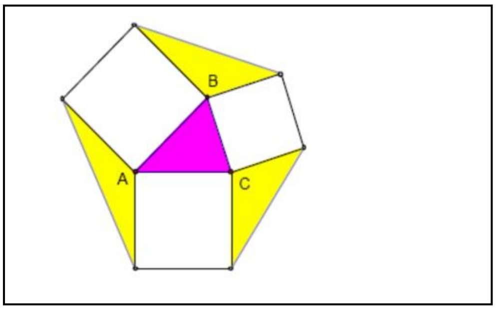



Figure 5.4.2.7: Prove Areas of the Four Shaded Triangles are the Same
(De Villiers, 2007b, p. 2)

Through reflecting and looking back on the trigonometric proof that was offered as an explanation as to why Cross's theorem is always true, and identifying the fundamental property that makes the theorem to be true, De Villiers describes how such deeper insight allows one to construct further generalizations to similar parallelograms and similar cyclic quadrilaterals, without any necessary need for experimentation. This kind of experience re-iterates that proof "*no longer just plays the role of verification, but rather that of a priori discovery*" (de Villiers, 2007b, p. 4).

As explicated in the aforementioned examples, the "discovery" function of proof means that we can develop new results from already constructed proofs, after "looking back" and carefully analyzing the statements and reasons that make up and/or do not make up the proof argument contained in the constructed proof in question (Miyazaki, 2000). The process of looking back is quite pivotal in the germination of new results, and Polya (1985, p. 15) has accordingly stressed its importance through the following words:

> "**Looking Back**: Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important instructive phase of the work. By

looking back at the completed solution, by reconsidering and reexamining the result and the path that led to it, they could consolidate their knowledge and develop their ability to solve problems. A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. There remains always something to do, with sufficient study and penetration, we could improve any solution, and, in any case, we can always improve our understanding of the solution" (Polya, 1985, p. 15).

The looking back as conceptualized by Polya has a more recent description in the process of 'folding back' on a learning trajectory in the models of Pirie and Kieran (1994, p.167). They argue that the development of understanding of any concept, topic or aspect of the mathematics learning area is really a dynamical to and fro movement through each of the eight layers as shown in Figure 5.4.2.8, and not an isolated linear process.

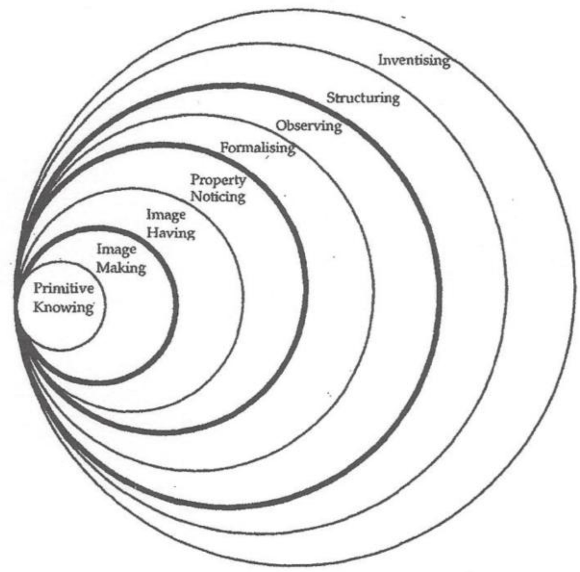



Figure 5.4.2.8: The Pirie-Kieren model for the growth of mathematical understanding
(see Pirie & Kieren, 1994, p. 167)

This means that when students encounter a problem whilst being at an outer layer, they may fold back into an inner layer with the prime purpose to reflect on their on their prior work and existing knowledge to try and build an understanding of the phenomenon or problem at hand and move forward with confidence and a deeper level of understanding that will enable him/her to solve the problem under the spotlight. In this respect, Pirie & Kieren (1994, p. 72) asserts that folding back “allows for the reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding”. Within the context of this study, folding back means shifting from a task at a higher level (Justification) to prior work with something like a *GSP* sketch exploration.

I believe that the vast array of theoretical issues and insights plus the focus on a number of relevant studies presented/discussed in the theoretical chapters can help me address the key concerns and contextual aspects of the design of my study. This is what I propose to do in the next Chapters.

In particular, the next Chapter (i.e. Chapter 6) provides a discussion on the research design and methodology that was adopted for this study.

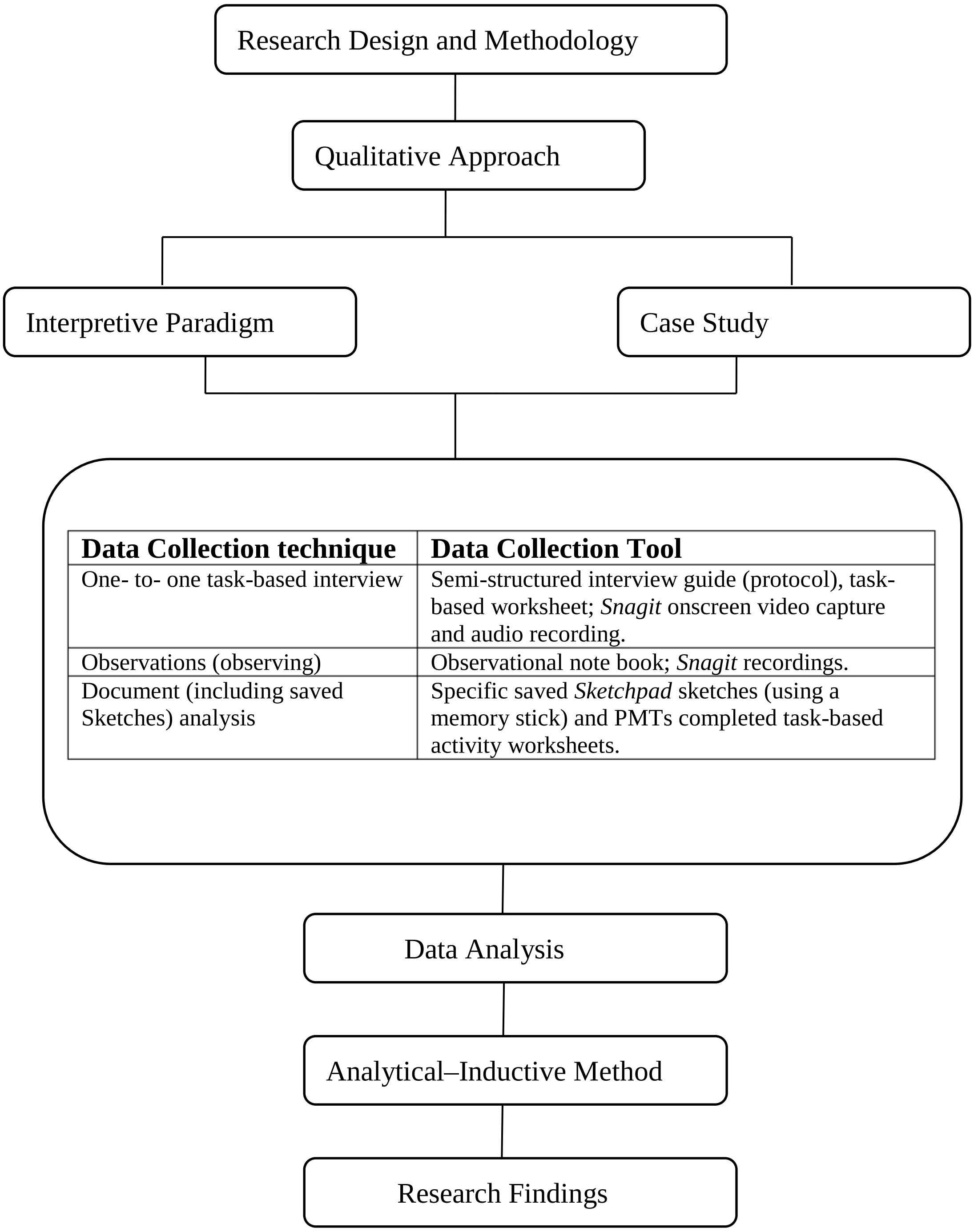
# Chapter 6: Research Design and Methodology

## 6.0 Introduction

Taking cognizance of the purpose of this study as articulated in Chapter 1, a conceptual framework located within a constructivist paradigm, which integrates a combination of theoretical considerations and existing theoretical frameworks has been constructed to provide a lens through which the following core research questions could be investigated:

- Can pre-service mathematics teachers construct a generalization, which says that the sum of the distances from a point inside an equilateral triangle to its sides is constant? If so, how do they accomplish this generalization (which is commonly referred to as Viviani's theorem)?
- Can pre-service mathematics teachers support their equilateral triangle generalization with a justification, and if so, how do they construct (or provide) a justification for it?
- Can pre-service mathematics teachers further generalize and extend the Viviani Theorem for equilateral triangles to equilateral (convex) polygons of four sides (rhombi), five sides (pentagons), and then to equilateral convex polygons in general? If so, how do they accomplish the constructions of such further generalizations?
- Can pre-service mathematics teachers justify each of their extended generalizations to equilateral convex polygons (namely, rhombus, pentagon and general equilateral polygon generalizations)? If so, how do they accomplish the justification of each of their respective further generalizations?

To augment the conceptual framework of this study in order to arrive at meaningful solutions to the posited research questions, a qualitative research methodology encompassing a case study based research strategy embedded in an interpretive paradigm has been considered. This Chapter starts with a discussion about qualitative research methodology, the interpretive paradigm and the case study research strategy as applied in this study. Next, the profiles of the participants that were involved in the research project are scrutinised together with data collection (production) techniques and tools and data analysis procedures that were adopted for this study. In addition this Chapter discusses some of the design limitations and measures taken to ensure a reasonable degree of reliability and validity in respect of the findings of this study. Lastly a brief summary of the discussions tabled in this Chapter is provided. The research design and methodology adopted for this study is shown in Figure 6.1.




| **Data Collection technique** | **Data Collection Tool** |
|---|---|
| One- to- one task-based interview | Semi-structured interview guide (protocol), task-based worksheet; *Snagit* onscreen video capture and audio recording. |
| Observations (observing) | Observational note book; *Snagit* recordings. |
| Document (including saved Sketches) analysis | Specific saved *Sketchpad* sketches (using a memory stick) and PMTs completed task-based activity worksheets. |

Figure 6.1: Research Design and Methodology

## 6.1 Qualitative Research Methodology

According to Cresswell (2003, p. 18), a qualitative approach is:

> …one in which the inquirer often makes knowledge claims based primarily on constructivist perspectives (i.e. the multiple meanings of individual experiences, meanings socially and historically constructed, with an intent of developing a theory or pattern) or …participatory perspectives (i.e.,…issue-oriented, collaborative, or change oriented. It also uses strategies inquiries such as narratives …, or case studies. The researcher collects open-ended, emerging data with the primary intent of developing themes from the data.

In essence, a qualitative research approach allows the researcher through the use of interviews, conversations, field notes, recordings and photographs to observe, interpret or make sense of participants' engagement/response/behaviour towards a phenomenon (or phenomena) under consideration in a given natural setting like a typical mathematics classroom (Denzin & Lincoln, 2005; Hitchcock & Hughes, 1995). In other words, qualitative research places individual persons or groups at the centre. It focuses on investigating, understanding, discovering meaning and explaining particular phenomena through the experiences and/or perspectives of the participants, particularly within areas of educational thought and practice (Hitchcock & Hughes, 1995; Leedy, 1993; McMillan & Schumacher, 2010). Flick (2007, p. ix) is of the view that one could interpret, make sense, explain and describe social phenomena 'from the inside' in three plausible ways, namely:

> "By analyzing experiences of individuals or groups. Experiences can be related to everyday or professional practices; they may addressed by analyzing everyday knowledge, accounts and stories. By analyzing interactions and communications in the making. This can be based on observing or recording practices of interacting and communicating and analyzing this material, by analyzing documents (texts, images, films or music) or similar traces of experiences of interactions."

In this research, the phenomenon of generalization through experimentation and justification has been explored. Through conducting one-to-one task-based interviews with pre-service mathematics teachers (PMTs) within a dynamic geometric context, the researcher has attempted to construct an holistic detailed account of the generalizing and justifying experience of PMTs. This was accentuated by recording verbal responses, collecting worksheet responses, capturing onscreen 'drag moves' within a *Sketchpad* context, and then analyzing, interpreting and making sense of the PMTs interactions, communications, worksheet responses and onscreen 'drag moves'. The goal was to capture the complexity of the world as it appeared to the subjects in the study and report to the events related to their processes of generalizing. Hence through using a qualitative approach via a case-based strategy of inquiry that is located within the interpretive paradigm, I, as the researcher, attempted to capture, account and give meaning to the PMTs' experiences with construction and justification of a generalization of Viviani's theorem, in anticipation that readers who have not experienced the evolutionary process of generalizing from one domain to the next, should then be able to make sense of it after reading the in-depth experiences of the subjects that participated in this study.

In qualitative research methodology, the language of the subjects is important. This implies that the "actual words of the subjects are thought to be critical to the process of conveying the meaning systems of the participants which eventually become the results or findings of the research" (Filstead, 1979, p. 37). In other words, whatever the subjects say (during interviews) is crucial since it allows the researcher to discover whatever is important and meaningful to the subjects in the study. The researcher's discoveries are presented as the findings of the research. These findings are then discussed, and conclusions/implications for the study are drawn.

## 6.2 Interpretive Paradigm

A feature that makes this study qualitative is its interpretative character. Carrol and Swatman (2000) argues that:

> "All researchers interpret the world through some sort of conceptual lens formed by their beliefs, previous experiences, existing knowledge, assumptions about the world and theories about knowledge and how it is accrued. The researcher's lens acts as a filter: the importance placed on the huge range of observations made in the field (choosing to record or note some observations and not others, for example) is partly determined by this filter"( pp. 118-119).

Through locating oneself within an interpretive paradigm during research activities, one can observe the experiences of participants as and when they socially engage in particular tasks or events, and then distill the meaning that such experiences have for those who experience it (Leedy, 1993, p. 141). In particular, through using an interpretive paradigm, one could attempt to: “describe, analyze, and interpret features of a specific situation, preserving its complexity and communicating the perspectives of the participants” (Hilda, Liston, & Whitcomb, 2007, p. 4). Furthermore, Denzin (1978) describes the major components of the interpretivist view in the following way:

> *“*The social world of human beings is not made up of objects that have intrinsic meaning. The meaning of objects lies in the actions that human beings take toward them…Social reality as it is sensed, known, and understood is a social production. Interacting individuals produce and define their own definition of situations and the process of defining situations is ever changing….Second, humans are … capable of shaping and guiding their own behaviour and that of others (intentionally and unintentionally), and humans learn … the definitions they attach through interactions with others*”* (p. 7).

From this perspective, meanings and actions, context and situation are inextricably linked and make no sense in isolation from each other. The ‘facts’ of human activity are social constructions; they exist only by social agreement or consensus among participants in a context and situation. For example, what counts as conjectures, generalizations, justifications, proof, counter-examples and discovery or whatever depends on the ways (and whether) these things are defined and used in human groups (compare Bredo & Feinberg, 1982, p. 16). In other words, it makes no sense for the interpretivist to do things according to preconceived principles and views without giving due attention to the factor(s) supporting the the birth of the idea (s) under consideration.

Working in an interpretivsit paradigm, requires the gathering of information that will enable the investigator to *“*make sense*”* of the world from the perspective of the participants; that is the researcher must learn how to behave appropriately in that world and how to make the world understandable to outsiders, especially in a research community. Thus, in this study, I have been involved in the activity as an insider and reflected upon it as an outsider (Cohen, Manion and Morrison, 2004). Furthermore, as it was critical in this study to capture the pre-service mathematics teachers’ voice and discourse, I as the researcher-interpretivist conducted in-depth one-to-one task-based interviews with each PMT. The one-to-one task-based interviews which were conducted in a dynamic geometry context served as the primary data source in this study, and were correspondingly augmented and triangulated with data from observations and relevant documents such as task-based worksheets. The written artifacts such as the task-based worksheets and the researcher's observational notes were constantly reviewed in association with the video tape and *Snagit* recordings. As seeking meaning in context is the key task of interpretative research, I hoped the data collected in this way to be 'rich' and more appropriate to answer the research questions.

Furthermore, consonant with methodological underpinnings of using an interpretive paradigm the data analysis in this study pursued a recursive process from the start of data collection, and the categories, patterns and themes described in this study were developed in a two- fold way. Firstly they were deduced in an analytical manner from the literature study and conceptual framework governing this study and secondly were induced from the data scanned and reviewed data (compare Hilda, Liston, & Whitcomb, 2007, p. 5). However, as the researcher working in an interpretative paradigm I am fully aware that the findings and conclusions that I have derived through analyzing my data collected in this study, could very well be different from those of other researchers analyzing the same set of data or investigating the generalizing phenomenon in the same (if not similar) context as the one considered for this study. This means that different researchers might distill different sets of conclusions (or findings) for the very same phenomenon under investigation or data collected.

Furthermore, like many other interpretivist researchers, Walsham (1993, p. 14) argues that, "the most appropriate method for conducting empirical research in the interpretive tradition is the in-depth case study." Taking this into consideration as well as the nature of this study as described by the purpose and research questions, I resolved to use the case study research strategy, as discussed in section 6.3, to gather data which could be analyzed through using the analytical – inductive method and thereby generate possible answers to the research questions.

## 6.3 Case Study

To develop a holistic understanding of the phenomenon of generalization as described in the purpose of this study informed my decision to use the case study research strategy because of its flexible form of inquiry that is most suited to study a particular phenomenon within a teacher-education context (compare Yin, 1984, 1994). According to McMillan & Schumacher (2010, p. 485) a case study is "qualitative research that examines a bounded system (i.e. a case) over time in detail, employing multiple sources of data found in the setting." Cresswell (2003, p.15) considers a case to be any one of the following: "a program, an event, an activity, a process, or one or more individuals." In this study, the case was conceptualized as a bounded system made up of eight final year pre-service mathematics teachers experiencing the constructions and justifications of a generalization (i.e. the phenomenon) of Viviani's theorem as an evolutionary process across selected domains, within a dynamic geometry context in a computer laboratory. The use of a case study-based approach enables a esearcher to get very close to the research participants via observations and interviews, and this coupled with key documents completed by participants makes it possible for a researcher to construct an intensive in-depth analysis of a case under study (Cohen et al., 2002; Cresswell, 2003; McMillan & Schumacher, 2010). Furthermore, through using various data collection procedures, such as interviews, observations, task-based worksheets it is quite likely that the researcher could get close to subjective factors such as participants' cognition, emotions and expressions (Cohen et al., 2002).

Since I was using the case study based approach supported by task-based interviews, observations, documents and video footage, I was able to explore and examine pre-service teachers' generalizing and justifying experiences associated with: (a) the reconstruction and justification of Viviani's Theorem for equilateral triangles, (b) the generalization (or extension) of Viviani's theorem to higher order polygons like the rhombus and pentagons with corresponding justifications, (c) the evolution of a generalization of Viviani's theorem for any equi-sided polygon with a logical explanation.

McMillan & Schumacher (2010, p. 345) in citing Stake (1995) argues that case studies are "intended to provide detailed, specific accounts of particular circumstances rather than offering broad generalizable findings." By using the case study approach, I was able to develop thick descriptions, use selected pieces of dialogues and quotations to provide compelling and rich representations of pre-service mathematics teachers' generalizing and justifying experiences, which ultimately led to the discovery of a generalization of Viviani's theorem and its proof. Hence, this case study can be regarded as an instrumental case study (compare McMillan & Schumacher, 2010, p. 345).

According to Hitchcock and Hughes (1995, p. 332) as cited in Cohen, Manion & Morrison (2002, p. 182), a case study has several hallmarks, namely:

> "It is concerned with a rich and vivid description of events relevant to the case; it provides a chronological narrative of events relevant to the case; it blends a

description of events with the analysis of them; it focuses on individual actors or groups of actors, and seeks to understand their perceptions of events; it highlights specific events that are relevant to the case; the researcher is integrally involved in the case; an attempt is made to portray the richness of the case in writing up the report."

Hence, through using a case study approach a researcher can capture salient features of the research activity that could help to make sense of the phenomenon under investigation or examination (Nisbet & Watt, 1984 as cited in Cohen, Manion & Morrison, 2002, p. 184 ).

Although the case study research strategy has advantages, many researcher investigators have expressed some weaknesses about it. For example, Nisbet & Watt (1984) as cited in Cohen, Manion & Morrison (2002, p. 184) makes the following claims:

> "The results may not be generalizable except where other readers/Researchers see their application; they are not easily open to cross-checking, hence they may be selective, biased, personal and subjective; they are prone to problems of observer bias, despite attempts made to address reflexivity."

Given the investigative nature of this study, the research questions, the first decisions on the focus of the research and the theoretical framework, my duty and responsibility for data collection was to:

- Develop and select materials which could be used during the one-to-one task-based interviews.
- Make observations and make video recordings of students' conversations whilst they were involved with the tasks;
- Keep discussions focused on the topics, which appeared on the worksheets;
- Take observational notes during the whole implementation process for triangulation purposes.

## 6.4 The Sample (sample)

Newman (2000, p. 198) argues that "purposeful sampling occurs when a a researcher wants to identify particular types of cases for in-depth investigation." In the same vein, Cohen et al., (2002, p. 103) says: "In purposive sampling, researchers handpick the cases to be included in the sample on the basis of their judgment of their typicality." This in essence means that the researcher takes the purpose of their study into account, and consequently builds up a sample that is a required for a specific study. As the primary purpose of this study was to actively engage pre-service mathematics teachers in an evolutionary process of generalizing and to investigate how they constructed and justified their generalizations extended from one domain to the next, this study employed the purposeful sampling strategy.

In particular, I selected the final year pre-service mathematics teachers (PMTs) who were doing Mathematics Methods modules in the School of Science and Mathematics Education (SSME) within the Faculty of Education, University of Western Cape (UWC). UWC is located approximately 30 km north of Cape Town, South Africa. As there were just five students doing the Bachelor of Education (B.Ed) Undergraduate Program, the researcher picked all five. To supplement this number of PMTs, I also included the three PMTs, who were the only students doing the Post Graduate Certificate in Education (PGCE) and specializing in mathematics at UWC. In total eight PMTs participated in this research study. These two groups had much in common, for example they all completed pure Mathematics courses up to at least $2^{nd}$ year University level, which were offered by the department of mathematics in the Faculty of Science at UWC. These modules included differential and integral calculus; linear algebra; differential equations and statistics. All eight PMTs completed their grade 12 at urban schools and obtained good passes in higher grade mathematics. Furthermore, whilst all eight PMTs were doing their mathematics methods courses, they were also participating in micro-teaching and school based teaching practice activities organized by the Faculty of Education.

Both the B.Ed and PGCE programmes are designed to provide the prospective mathematics educators with sufficient depth of knowledge and skills to enable life-long learning and systematic learning in schools from grades 7 to 12. The mathematics methods course covers a range of learning theories, teaching and learning strategies as well as assessment issues. The emphasis of the mathematics methods courses is on developing the pedagogical content knowledge of the pre-service mathematics teachers which they can use in conjunction with their mathematical content knowledge to execute effective teaching and learning in their prospective mathematics classrooms. The ultimate goal is to develop the PMTs into specialist mathematics teachers, who will be well grounded in the knowledge, skills and principles appropriate for mathematics specialization and the phase(s) they may teach. All eight pre-service mathematics teachers were computer literate as they were compelled to do the computer literacy courses offered by UWC in their first year of study. Furthermore these

PMTs were exposed to Sketchpad during their final year of study, whilst doing their mathematics method courses with me as their lecturer. Thus they were conversant with the dynamic geometric environment and the features of Sketchpad. The students needed to be conversant with the basic properties of triangles and other polygons, but this seemed to be a challenge in some instances.

## 6.5 Data collection (Data generation)

Data refers to the information collected by a researcher from which interpretations and conclusions are drawn with respect to a particular phenomenon under investigation (McMillan & Schumacher, 2010, p. 486). An important part of methodology is gathering data that is both reliable and valid, which could in this instance only be realized through the use of appropriate and purposively structured instruments that would contribute to answering the research questions of the study under consideration (Maxwell, 2005, p. 92). Some of the commonly used instruments in qualitative research encompassing a case study research strategy located with an interpretative paradigm are questionnaires, in-depth open-ended interviews, observations and documents (Cohen, Manion & Morrison, 2002, p. 184; Labuschagne 2003; McMillan & Schumacher, 2010, p. 486). In this study, data has been gathered through one-to-one task-based interviews, observations and documents (see Sections 6.5.1, 6.5.2 and 6.5.3 for detailed discussion). It is hoped that the triangulation of the various instruments and data sets would yield a more reliable and valid picture of the phenomenon of generalization as experienced by the pre-service mathematics teachers from a construction and justification perspective than other verifying strategies (compare Maxwell, 2005).

The associated tools used to collect the data as per data collection technique employed in this study are included in Figure 6.2. The one-to-one task-based interviews, observations, and documents in this study revolved largely around a set of task-based activities embedded in a *Sketchpad* context. All tasks used in this study, for example Task 1 explained in Appendix 1 and Tasks 2, 3 and 4 explained in Appendix 2, have been conceptualized within a constructivist view of learning. In so doing, the design of the tasks took into consideration the following theoretical constructs and processes: experimentation, conjecturing, generalizing, refutation via counter-examples, cognitive conflict, conceptual change, justification and proof. The curriculum material, which served as a basis for the construction of one-to-one task-based interviews and associated observations, has been based on the activities in the book, *Rethinking Proof with The Geometer's Sketchpad* (de Villiers, 2003a). Much of Task 1, which emerges from the ‘ship wreck’ problem, can be found in De Villiers (2003a, pp. 23-

26). Tasks 2-4, focus on the extension of the Viviani generalization for equilateral triangles, respectively to the rhombus, pentagon and any equi-sided polygon, and capture many of the processes and structures that are exemplified in Task 1, but with a focus on extending the generalization across domains in an evolutionary manner. Based on these results emanating from this study, it is envisaged that the material as represented in Tasks 1- 4 may have to be reviewed or re-designed. This explains the need for a classroom environment that promotes students interacting with developed curriculum materials, careful observation and thereafter reflection.

| **Data Collection technique** | **Data Collection Tool** |
|---|---|
| One- to- one task-based interview | Semi-structured interview guide (protocol), task-based worksheet; *Snag*it onscreen video capture and audio recording. |
| Observations (observing) | Observational note book; *Snagit* recordings. |
| Document (including saved Sketches) analysis | Specific saved *Sketchpad* sketches (using a memory stick) and PMTs completed task-based activity worksheets. |

Figure 6.2: Data collection techniques & tools

All the task-based activity sessions were facilitated by me, who is alternatively referred to as the researcher (facilitator) throughout the study. The task-based activity created an opportunity for me as the researcher to engage pre-service mathematics teachers in some of the core processes of mathematics ranging from generalizing through experimentation to generalizing via deductive reasoning whilst being engaged in authentic mathematical problems (as described in Tasks 1-4) in a realistic mathematical context using *Sketchpad.* Particularly the tasks aimed:

- to provide opportunities for students to experiment and create generalizations,
- to encourage students to develop appropriate justifications (arguments) for their generalizations and test their generalizations;
- to provide opportunities for students to develop proofs for their generalizations through appropriate justifications;
- to provide opportunities to explore and extend their generalizations.

Each student participated in the developed task-based activities via one-to one task-based interviews, which were held on two separate days that were almost 4 days apart from each other. As Task 1, which is described in Appendix 1, provides an opportunity for PMTs to re-construct Viviani's theorem for equilateral triangles through experimentation and proof, which can then serve as a generalization that can possibly be extended to other domains (like rhombus, pentagon, any equi-sided polygon), the researcher limited the first one-to-one task-based interview session to just the equilateral triangle problem (i.e. Task 1). Furthermore, it was anticipated that PMTs would take approximately 30 to 45 minutes to complete Task 1, and about 40 to 60 minutes to complete Tasks 2-4 as described in Appendix 2-6. As one may not necessarily concentrate adequately when a one-to-one task-based interview runs for more than an hour, as the researcher I felt it was prudent to hold the one-to-one task-based interview for each PMT over two separate sessions (referred to as the first one-to-one task-ased interview session and the second one-to-one task-based interview session in this study). Hence, the first one-to-one task-based interview session with each PMT focused on Task 1, the equilateral triangle problem. As there were eight PMTs participating in this study, and taking into account the possible time that it may take to complete the first one-to-one task-based interview through using the interview protocol as described in Appendix 3 in conjunction with Task 1, the researcher elected to hold just two sets of one-to-one task-based interviews per day. Hence, one-to-one task-based interviews for the first session spanned over 4 days. Thus, the second one-to-one task-based interview session, which spanned over four days as well, started only on the fifth day of the data collection phase. This in essence, explains why the data from each PMT was collected via one-to-one task-based interviews on two separate days. In fact, the recorded time schedules for one-to-one task-based interviews conducted in this study, show that that time taken to complete a one-to-one task based interview in the first session ranges between 25 to 50 minutes, and the time taken in the second session ranges between  45 minutes to 80 minutes.

In particular, the first session related to the development of a generalization linked to an equilateral triangle in the context of Viviani's theorem and the development of a proof for the established generalization. The second one-to-one task-based interview session focused on generalizing the Vivaini result further to and across other domains through a range of ways, namely:  (a) exploration and experimentation, (b) inductively, (c) analogically, and (d) deductively; inclusive of appropriate kinds of justifications focusing  on the development and construction of logical explanation(s) that explained why the extended generalization holds true for the given domain. All the one-to-one task-based interviews with supporting task-based activities took place in a computer laboratory on 8 consecutive school days. The one-to-one task-based interviews served as the primary data collection technique in this study, and was supported by observational notes and documents (i.e. completed task-based worksheets and saved *Sketchpad* sketches).

I conducted the one-to-one task-based interview with each PMT, and made observational notes about PMTs gestures, surprised looks, misconceptions, 'aha' moments, struggles and successes as each action played out during task- based interviews. However, in order not to disturb the flow of interactions of the PMT with the *Sketchpad*-based task and myself as the interviewer/interviewee, I made the observational notes at the end of a given task-based interview and also whenever a PMT was busy providing a written response to particular items on the task-based worksheet. The *Snagit* software (as described in Section 6.4) served a dual purpose: firstly it was used to capture the onscreen 'drag moves' of each PMT as they explored and experimented with a given problem in using *Sketchpad*; secondly it was used to audio record the task-based interview from start to beginning in tandem with 'drag moves' as they occurred during the task-based interview. In addition through using the *Snagit* software, students were encouraged to save specific strings of sketches that could support their assertions/claims.

The data from the one-to-one task-based interviews consist of direct quotations from students related to their experiences, responses, opinions, feelings and knowledge pertaining to the specific *Sketchpad* task-based activities. The data from observation consist of detailed descriptions of participants' behaviours and the full range of their interactive actions during the one-to-one task-based interviews as well as during their silent moments when they were responding in writing to some of the task-based activities. The document analysis yielded excerpts, quotations, or entire passages from the task-based worksheets as well as scripts in the form of pictures or saved sketches (compare Labuschagne, 2003). Hence, with the use of qualitative research methodology for this particular study, the researcher was able to extract rich and corroborating data concerning the development of generalization(s) and corresponding justifications thereof.

### 6.5.1 One-to-one task-based Interviews

A qualitative research interview, which is considered as one of the major approaches used to generate data, "attempts to understand the world from the subject's point of view, to unfold the meaning of people's experiences, to uncover their lived world prior to scientific explanations" (Flick, 2008, p. xvii). The purpose of an interview includes, among others, obtaining present constructions of persons, experiences, events, activities, feelings, claims, motivations, feelings, concerns, and also be used to reconstruct past experiences or predict the future of such aspects (Lincoln & Guba, 1985, pp. 268-270). However, there exist various research interview forms that are useful for different purposes, and they include forms such as the following: factual interviews, conceptual interviews, focus group interviews, narrative interviews, discursive interviews, one-to-one task-based interviews, (Davis, 1984; Flick, 2008; Golden, 2000, 2002).

In this study, I used one-to-one task-based interviews to generate data forthe research project. The basic idea associated with a task-based interview is that: "a student is seated at a table, paper and pens are provided, and the student is asked to solve a particular mathematics problem; one or more adults are present collecting data" (Davis, 1984, p .87). In addition, Davis (1984) is of the view that although paper and pens are minimum kinds of entities or equipment required, the equipment used during task-based interviews could also include rulers, compasses, hand held calculators, computers, textbooks, graph paper, geoboards, Cusenaire rods. In this study, each pre-service mathematics teacher had the opportunity to be seated behind a computer in the computer laboratory and work on series of connected problems dealing with the generalizing phenomenon by using a dynamic geometry software program called Sketchpad, and the researcher conducted the task-based interview to collect the data. Davis (1984) argues that the one-to-one task-based interview can vary from context to context, for example the level of participation by the interviewer could vary from high to low. Generally if the participation of the interviewer is curtailed (i.e. low) the problem is posed to the student, and the student is left to work on the problem with virtually no assistance from the interviewer. However, whether the interviewer participation is high or low, the student is asked beforehand and is expected to: "talk aloud, explaining in as much detail as possible what he or she is doing, why they have decided to do it, and so on" Davis (1984, p. 87). In this study, given the nature of the research questions, it was essential to engage the PMTs in four problems as contained in Tasks 1-4 (see Appendix 1 & 2), but with a moderate level of intervention from the researcher as guided by the semi-structured interview protocols (see Appendices 3-6) designed for each of Tasks 1-4 respectively

According to Davis (1984, p. 88), an interviewer may intervene or question the student's response from time to time, and such interruptions or interventions may be:

> "…in order to pose a further question, in order to provide a hint, or in order to correct an error or misunderstanding; they may also be intended to provide more motivation or perhaps some encouragement; … the interviewer may pose a new problem, after seeing what the student does with the first problem."

In this study, during the one-to-one task-based interview, I as the interviewer, intervened in a variety of ways including inter alia the following: providing a hint, correcting a misconception; posing a new problem, or just probing further to seek clarification or more insight as to how or what a particular PMT is thinking and/or generalizing, arguing and justifying his/her claims. Furthermore, Davis (1984, p. 89) asserts that during task-based interviews: "the student is expected to verbalize their thoughts as much as possible as they work at the task, or such verbalizing may take place immediately after the mathematical task has been solved… the depth of probing can vary a great deal." In this study, through appropriate probing as guided by the semi-structured interview protocols linked to each task-based activity, the PMTs were given sufficient opportunity to verbalize their thoughts and experiences and were probed to reasonable extents whenever it was deemed necessary by me as the interviewer. Through using the *Snagit* Software (which is described in Section 6.4), I was able to capture in great detail the thought processes, sense-making, reasoning, arguments and explanations of each PMT hroughout the task-based interview as they endeavored to construct a generalization of Viviani's theorem. In particular, this method allowed me as the interviewer more opportunity to observe and track how each student moved through the Viviani related tasks, as expressed by Novak and Gowin (1984, p. 12):

> "For this reason most psychologists prefer to do research in the laboratory, where variation in events can be rigidly prescribed or controlled. This approach clearly increases the chances for observing regularities in events and hence for creating new concepts."

As alluded to earlier, and reflected upon in the respective interview protocols spanning across Appendices 3, 4, 5 and 6, the interviews were semi- structured through the use of open-ended questions related to the task-based worksheets as contained in Appendices 1 & 2. Open–ended questions are considered to have advantages because they allow persons being interviewed to take whatever direction and use whatever words they want to express what they want to say (Patton, 1990). In other words, open-ended questions in the study were believed to allow flexibility, clarification, and probing of the interviewee responses. However, through using the one-to-one task-based interview, the number of external variables are reduced and the focus is narrowed, hence “giving generalizations based on findings during task-based interviews greater credibility” (Mudaly, 1998, p. 55). Davis (1984) argues that during task-based interviews, the observer needs to establish rapport with the student as this can enable the observer to probe the student further and obtain deeper insights or better data about the students dduring the process. Hence, during each of the task-based interviews for this study, I tried to maintain a good level of rapport with each PMT through providing scaffolded support to each person whenever the need arose.

Furthermore, Davis (1984, p. 87) says at the end of a task-based interview session, Researchers will have:

- an audiotape or videotape of the session;
- the paper on which the student has written;
- various written notes made by the observer(s) during the session.

This study consisted of eight task-based interviews. Consistent with Davis’ (1984) aforementioned outputs, each task-based interview was audio and video recorded using *Snagit* (see Section 6.4 for detailed discussion), supported by the collection of completed written task-based worksheets and saved Sketchpad sketches, as well as observational notes. Furthermore, a debriefing session was held with each PMT after a one-to-one task-based interview was completed.

### 6.5.2 Observations

Observations in qualitative studies supplement interviews and are often unstructured and free flowing (Daniel, 1997; Cohen, Manion & Morrison, 2002). This means that whatever one cannot grasp through an interview, one may add through observation, and thus enrich understanding of the phenomenon under study (McMillan and Schumacher 2010, p. 76). Cohen et al.(2002, p. 305) argues that “observational data affords the researcher an opportunity to gather ‘live data from ‘live’ situations. Observations in this study provided me, as the researcher, with the opportunity to look at how the processes of generalization and justification unfolded in *situ* rather than second hand, with adequate focus on the following aspects: PMTs responses, reactions, interpretations, reflections, claims, explanations of the task-based worksheet items; PMTs’ experimental exploration using Sketchpad; the kinds of gestures, facial expressions, hunches and misconceptions exhibited by PMTs; difficulties and successes experienced by PMTs (compare Elliot, 1981, p.10; Patton, 1990, pp. 203-205).

In other words, observations in this study focused primarily on gathering data on the interactional aspects as they unfolded with a PMT's interaction when he/she was: (a) working through the task-based worksheet; (b) experimenting and exploring using Sketchpad; (c) probed by me, the Researcher (Compare Morrison, 1993; Patton, 1990). The observational notes were documented in my 'observation note book' at various instances, namely: whilst PMTs were busy completing the written components of the task-based worksheets; at the end of each task-based interview session, or even after re-playing the *Snagit* audio-video recording. In particular, the *Snagit* audio-video recordings were revisited in an effort to support or validate pertinent observations related to the research questions as well as to extend the network of critical observations.

Cresswell (2003, p. 186) describes four optional roles that a researcher can assume during observations, namely: complete observer; observer as participant; participant as observer or complete participant. In this study, by assuming the role of a participant observer whilst facilitating every one-to-one task-based interview session related to the construction and justification of a generalization of Viviani's theorem, I was able to understand why the PMTs generalized and justified in the way they did and "to see things as those involved see things" (Denscombe, 1998, p. 69 ). May (2001, p. 174) acknowledges that:

> " participant observation is not an easy method to perform, or to analyse, but despite the arguments of its critics… if performed well, greatly assists in the understanding of human actions and brings with it new ways of viewing the social world."

The video recordings were looked at to support/authenticate pertinent observations related to the research questions and to extend the network of critical observations.

### 6.5.3 Documents (including saved Sketches)

The following documents supported the data gathering and analysis phase: the PMTs' completed task-based worksheets; observational notes recorded in my observational note book and PMTs' saved Sketchpad sketches. The saved Sketchpad sketches generally comprised of the sequential set of "dynamic sketches" which was constructed and used to make their generalization and also the "dynamic sketches" which were constructed to support or refute their generalization during the testing phase. In addition, each PMT saved the sketches that they used to construct a logical explanation to justify their respective generalizations as they moved from one domain to the next. The analysis of these aforementioned documents in this study served to supplement the information obtained via other methods, particularly when the reliability of evidence gathered from the one-to-one task-based interviews or observations had to be checked. Cresswell (2003; p. 187) argues that documents as a data collection type has the following advantages in research studies:

- The researcher can have easy access to the kind of language and words the participants used;
- Serves as an unobtrusive source of information, which can be visited and re-visited by a researcher at convenient times.
- Represents data that are thoughtful, in that research participants would have given due attention and consideration to compiling.
- Time and expense for transcribing are saved since the data is provided in written form.

### 6.5.4 *Snagit* : Audio and visual material

Creswell (2003, p. 188) says qualitative data can also consist of audio and visual material, and that such a category of data may take the form of photographs, videotapes, art objects, or any form of sounds. To capture all the verbal proceedings/interactions/discussions that occurred during the task-based interviews in a synchronized manner with corresponding 'drag moves' that each PMT engineered whilst experimenting, conjecturing, generalizing, refuting and justifying within a *Sketchpad* context, the researcher used the computer software program called *Snagi*t. This computer software program can capture the video display (i.e. computer onscreen visuals) and audio output (such as oral discussions and communications during task-based interviews) simultaneously. Through using the *Snagit* software I was able to audio record all the verbal communications that took place during each one-to-one task-based interview and simultaneously capture all the visual drag moves of each PMT as they happened during the one-to-one task-based interview session (*www.faculty.fairfield.edu/dgrignano/documents/SnagItFAQ.pdf*; http://download.techsmith.com/snagit/docs/onlinehelp/enu/9/default.htm?url=snagittechnicalreferenceguide.html.). Hence, I was able to produce a verbatim transcript for each one-to-one task-based interview that was in *situ* with each PMT's dynamic constructions and drag moves.

Having he audio recording synchronized with onscreen visual drag moves, made it possible for me as the researcher to replay the *Snagit* recording numerous times to reflect on what occurred during specific generalizing and justifying episodes, and hence limit the birth of premature inferences and conclusions. As *Snagit* was used to capture the interactions as they happened, it was much easier for me, as the researcher, to relive the interactive experiences at my own convenience. This became very valuable when I was analyzing the data since it was possible to rewind (backwards or forwards) the *Snagit* recording again and again to not only capture the details that I might have missed in the initial play of the *Snagit* recording, but also to do much more in-depth analysis of PMTs' verbal response(s) or interaction(s) within the context of their visual dynamic 'drag moves' (compare Ratcliff, 2006). Through using *Snagit* to capture the deliberations during the task-based interview as well as the onscreen visuals, observation note taking during the task-based interview was minimized and hence distractions were minimized (compare Gillham, 2000).

## 6.6 Data Analysis

According to Hitchcock & Hughes (1995, p. 295) "Analysis involves discovering and deriving patterns in the data, looking for general orientations in the data and, in short, trying to sort out what the data are about, why and what things might be said about them." This means that the process of data analysis involves continual reflection on collected data with a goal to develop a deeper understanding and meaning of such data in relation to the context of the study and theoretical underpinnings, and hence develop conclusion(s)/responses to the postulated research questions (Creswell, 2003, p. 190; Newman, 2000, p. 426). Furthermore, McMillan & Schumacher, 2010, p. 367) asserts that qualitative data analysis is: "a relatively systematic process of coding, categorizing, and interpreting data to provide explanations of a single phenomenon." In the main such a systematic process is facilitated via inductive analysis, which McMillan and Schumacher (2010, p. 367) describes as: "the process through which qualitative researchers synthesize and make meaning from the data, starting with specific data and ending with categories and patterns." Through using inductive analysis, one could have more general themes and conclusions emerging from data itself rather than being imposed prior to data collection.

However, the data analysis for this study was grounded in an inductive-analytical method, which is consonant with the qualitative data analysis processes employed in case studies (Bryman & Burgess, 2000, p. 4; McMillan & Schumacher, 2010, p. 367; Newman, 2000, p.

247; Spradley, 1999, p. 94). The inductive-analytical method of analysis makes provision for the development of categories or themes linked to specific codes to be developed prior to the collection of data, as well as the development of categories/themes through the use of new codes as and when the data are being examined. Codes in the context of this study are considered as “words, phrases, patterns of behavior, subjects’ ways of thinking, and events that repeat or stand out” (Bogdan & Biklen, 1998, p. 171). In this study the *apriori* categories were derived from the theoretical considerations and theories underpinning the conceptual framework, and are also motivated by the literature review conducted for the study. For example, Figure 6.6.1 represents the categories as to how pre-service mathematics teachers could possibly construct their generalizations as well as the core process involved in the construction of an inductive generalization, which the Researcher derived prior to data collection, and planned to use to code the data.

| **Type of Generalization** | **Processes** | |
|---|---|---|
| **Inductive Generalization** | Formulating a conjecture | Observe particular cases, notice patterns, make a statement about all possible cases but with element of doubt |
| | Validating the conjecture | Testing whether the conjecture is valid for new particular cases, but not in general |
| | Generalizing the conjecture | Based on a conjecture which is true for some particular cases, and having validated the conjecture for such new cases, students might hypothesize that the conjecture is true in general |
| | Refining (modifying) or Refuting the conjecture generalization through the use of counter-examples | A counter-example is a particular case which disproves a conjecture. Indeed a single counter-example is sufficient to refute a false statement. Alternatively, after careful interrogation of the counter-example, the conjecture could be modified by inserting conditions into the conjecture. |
| **Analogical generalization** | Use of some sort of similarity | |
| **Deductive generalization** | Make a conjecture generalization on logical grounds, but require visual confirmation in a GSP context. | |
| | Make a conjecture generalization immediately on logical grounds only | |

Figure 6.6.1: Analytical Framework for Generalization

Similarly, Figure 6.6.2 represents the type of justifications with qualifiers/descriptors that the Researcher envisaged to use to code the data, and hence establish how the PMTs justified their respective generalizations. In other words, Figures 6.6.1 and 6.6.2 to a large extent served as analytical frameworks for this study.

| **Justification Level** | **Description** |
|---|---|
| Level 0: No justification | Responses do not address justification |
| Level 1: Appeal to external authority | Reference is made to the correctness stated by some other individual or reference material. |
| Level 2: Empirical evidence | Justification is provided through the correctness of particular examples. |
| Level 3: Generic example | Deductive justification is expressed in a particular instance. |
| Level 4: Deductive Justification | Validity is given through a deductive argument that is independent of particular instances. |

Figure 6.6.2: Analytical framework for justification (Lannin, 2005, p. 236)

The coding process began by using the category descriptors as described in the frameworks reflected in Figures 6.6.1 and 6.6.2. However, as the researcher was scanning, interrogating the data and reflecting on his observational notes, other categories emerged from the data, and this necessitated the proposal for new codes. For example, with regard to the development of deductive generalizations, it was found that some pre-service mathematics teachers (PMTs) developed their generalizations on logical grounds but with the aid of analogical reasoning, and this had to be coded. Furthermore, with regard to the generalization to any equi-sided polygons, the data suggested the PMTs saw the general through the particular, hence were able to make their generalization to any equi-sided polygons. Also the data with regard to justifications showed the PMTs were able to see the structure of the proof for their pentagon conjecture generalization and also for the equi-sided polygon generalization through the structure of their algebraic-structured proofs for their earlier equilateral and rhombus conjecture generalization. Thus the emergent categories of generalizations and justifications like the ones described were added to the analysis.

Prior to starting the coding and categorizing processes, the researcher prepared the data by making a verbatim transcript of each of the pre-service mathematics teachers one-to-one task-based interviews as was recorded on the video tape. In doing so the time-ordered sequence of the video-taped task-based interviews was preserved for each PMT. In addition to the transcription of PMTs' and researcher's utterances as per video recorded task-based interview, the researcher also captured each PMT's gestures and facial expressions as they surfaced during their task-based interview. Furthermore, the transcriptions and associated gestures and expressions derived from the video tapes, could be supplemented with the onscreen dynamic sketches and moves that were recorded using the *Snagit* software. Thereafter episodes capturing the interaction between each PMT and the researcher were created from segments of the video and the *Snagit* onscreen-video capture. The episodes provided the Researcher with a broad contextualized perspective as to how the PMTs' constructed generalizations, refuted generalizations via counter-examples, justified their generalizations and extended their generalizations across domains. The initial analysis was managed in an orderly manner by assigning an initial code to every unit of complete conversation (or interaction) between the researcher and PMT, and a complete set of *Sketchpad* moves.

The second step proceeded to include video-transcript analysis, wherein the PMT– researcher utterances, students' gestures and dynamic sketches were supplemented with analyses pertaining to observational notes and student responses to the worksheet item completed during the task-based interviews. In constructing the aforementioned transcript analysis in combination with data supported from other sources, the researcher used a two- column table to assist with the coding and categorizing of the data. The first column was used to capture PMT responses, and the second column was used to capture the researcher's comments. Figure 6.6.3, illustrates a typical use of the two-column table.

The video transcript analysis via the use of a two-column table made it possible for the Researcher to code and analyse a range of dimensions in the data, such as PMTs' exploratory moves within a dynamics geometric context, PMTs' reasoning, generalizations and justifications, researcher's responses, the ensuing dialogue between PMT and researcher. Through using the two-column table, the researcher was able to scan and clean the data. This made it possible for the researcher to read the data and then identify data that was irrelevant, inconsistent, inaccurate or even incomplete; and also to recognize preliminary trends in the scanned data, which were then used in some or the other way to facilitate the meaningful grouping and organization of the data. In this way, the researcher was able to fine tune the data analysis, and develop the required foundation to support his claims and also extend such claims with greater degree of accountability (compare Dorit, 2000).

| PMT's responses | Researcher's comments |
|---|---|
| SHANNON:<br>The sum of the distances from a point inside the equilateral triangle is always the same…Yes. So it doesn't matter where she builds her house. | Through experimental exploration using Sketchpad, Shannon discovered that point can be located anywhere inside the equilateral triangle to yield a constant distance sum. |
| PMT: Shannon<br>…So if I've got a point *P* here, it should always … it's not always drawn from the midpoint, no. But the line here is always perpendicular. (… silence for about 3 minutes, probably trying to figure what to do). …Can you give me a hint, yes please! | To justify her equilateral triangle conjecture generalization, Shannon requested a hint from me. She was provided with a scaffolded worksheet to assist with her development of a logical explanation that justifies her conjecture generalization. |

Figure 6.6.3: Example of a two column table used to capture PMTs responses and researcher's comments

As mentioned earlier, the afore-decribed process was supplemented by the analysis of observational notes and documents. The document analysis of the data in this study encompassed reading and re-reading of the pre-service mathematics teachers' written work (completed worksheets), the viewing and re-viewing of saved *Sketchpad* scripts as well as continuous reference to the researcher's observational notes. A description of the insights gained from the read-reread and view-review process pertaining to the broad research questions was formulated and presented as a summary narrative of the sessions. Raw data obtained via observations were analyzed on the basis of the completed observation notes. A summary reporting on what took place or did not take place in the *Sketchpad* context (laboratory environment) in relation to conjecturing, generalizing, justifying, refutation via counter-examples as well as instances of confusion and surprises was completed.

During the third step of data analysis, the researcher proceeded to distill the common patterns and constructs through grouping similar and common responses into designated *apriori* categories. As categories, patterns, constructs, themes emerged, the data was re-examined and categories were refined. However, in instances where the *apriori* set of categories could not accommodate particular data or sets of data the researcher subsequently constructed new categories. The data for each PMT was then re-visited, interrogated and recoded, taking the new categories into account. Thereafter the categories were reviewed against the data sets that emanated from the one-to-one task-based interview, observational notes, *Snagit* dynamic sketches and moves as the worksheet response items, until a level of theoretical saturation prevailed. In the main, the researcher visited the respective data sets at least three times to try and ensure that categories are stable, reliable and valid. Overall the recognized patterns, themes and constructs were used by the esearcher to gauge the degree of consistency and commonality through consistent and frequent repetition across and within data sources. Finally, the patterns, constructs and themes that emerged through the process of analytical-inductive analysis are represented through tables, selected quotations and boxes in this study. This rich thick description enabled me as the researcher to convey my findings in a valid way ( compare Creswell, 2003; Maxwell, 2005).

## 6.7 Design Limitations

The study has the following limitations:

- The study was conducted with one cohort of pre-service mathematics teachers and one mathematics teacher educator who was also the researcher, and the sample was not representative of any whole mathematics teacher- educator population or mathematics pre-service teacher population.
- The activities used to facilitate the construction of a generalization of Viviani's theorem in this study was designed with the use of Sketchpad in mind, and it is not certain as to what extent a research participant would proceed toward the construction of a generalization of Viviani's theorem by using a non-dynamic context.
- The study was limited to the development of generalizations in geometry. Although some algebraic techniques have been used to build up a logical explanation to justify constructed generalizations in this study, the phenomenon of generalization does not necessarily include other areas of the mathematics curriculum such as trigonometry and analytical geometry.
- The accuracy of the data collected relies on the extent to which the participants engaged honestly with the activities and the degree of completeness provided. Since the research involves human subjects it is possible that they may have given incorrect responses or were not necessarily honest in their responses.

## 6.8 Validating the accuracy of the findings (Trustworthiness of findings)

Cresswell (2003, p. 195) argues that validity is a "strength of qualitative research" and helps to ascertain if the findings that emerge from a study "are accurate from the standpoint of the researcher, the participant or the readers". In a parallel way, the term 'trustworthiness' is used instead of 'validity' in a qualitative paradigm (Creswell & Miller, 2000; Lincoln & Guba, 1985, 2000; Trochim, 2008). In terms of trustworthiness, Sapsford & Jupp (1996, p. 1) argues that what has to be established is whether the data:

> "Do measure or characterize what the authors claim, and that the interpretations do follow them. The structure of a piece of research determines the conclusions that can be drawn from it and, most importantly, the conclusions that should not be drawn from it."

To realize the trustworthiness of the findings that emerged from my qualitative research study, I employed the following strategies from the list of strategies recommended by Cresswell (2008, p. 196): triangulation, member checking, provision of thick description(s); clarified researcher bias, and spent sufficient time with research participants. Triangulation involves the use of different data sources or collection processes to corroborate data, which in turn serves as evidence to build a coherent justification for a particular finding or set of findings. For instance, in this study in which the generalizing and justifying experiences of eight pre-service-mathematics teachers were investigated and reported on, I as a Researcher used a one-to one task-based interview strategy and recorded the discourse and interactions as they transpired using the *Snagit* recorder. The *Snagit* audio recording was transcribed and hence there was a verbatim transcript produced for each one-to-one task-based interview. Although the *Snagit* recorder captured the audio and visual data and stored it permanently, and made it possible for me as the researcher to re-visit the data as many times as possible when I needed to, the data as per *Snagit* recordings of each one-to-one task- based interview was triangulated with the following sets of data: data from the observational notes documented during and after each one-to-one task-based interview, and also the written responses of each PMT to the task-based worksheets that were used to facilitate the construction of a generalization of Viviani's theorem within a dynamic geometric context.

Through Chapters 7 to 10, which deals with data analysis, results and discussions, the researcher provided rich and thick descriptions of the discourses and discussions as they occurred during the task-based interviews as well as detailed extracts of each PMT's responses to task-based worksheet items. It is hoped that in this way the readers of this report can be transported to the research setting and context and thus give the discussion of the findings an element of shared experiences (compare Creswell, 2003, p. 196).

Furthermore, as reported in Section 6.5.1, enough time was spent with each PMT during each task- based interview. In this way, I as the researcher developed an in-depth understanding of the phenomenon of generalization under study, and also generated data that enabled me to describe in detail the generalizing and justifying experiences of each PMT. In this way credibility to the narrative accounts supporting the findings has been made possible in this study.

I am relatively known to the participants because I teach them in the Method of Mathematics Module 401. However, I was able to establish relations of trust with interviewed PMTs during the task-based interview sessions. Furthermore, as a pre-service teacher educator I hold the view that knowledge can be discovered or created and one of my areas of interest is the development and justification of generalizations within a dynamic geometric context. Thus, to minimize my bias about the phenomenon (or topic) under study, I consulted the literature extensively to distill alternative views and positions on generalizing and justifying. Furthermore, during the one-to-one task-based interviews, the Researcher tried to minimize any bias thorough not letting his personal beliefs and knowledge influence the discourse and discussions. In particular, I tried as far as possible not to allow my own assumptions, feelings and disposition direct the interview, but instead allowed for the PMT's responses to lead the way as guided by the semi-structured interview protocols (see Appendices 3-6). Moreover, through constant questioning of my practice, critical alertness and attitude toward the interpretation of the data, and through triangulation of my data, I tried to minimize the effects of any bias.

To also determine the accuracy of the findings of this study, I as the Researcher took specific descriptions of the task-based interview discourses and discussions as well as my interpretations of them (i.e. the descriptions) back to the PMTs who participated in the research. The purpose of this member checking exercise was to establish whether the given descriptions and interpretations are accurate representations of PMTs articulations, responses and interactions as expressed during the respective one-to-one task- based interviews (compare Cresswell, 2003, p. 196; Maxwell, 2005, p. 130).

## 6.9 Ethical Considerations

In view of the nature of my qualitative research being premised largely around one-to-one task-based interviews with pre-service mathematics teachers (PMTs) at UWC, I had to write to the head of Research and Development at UWC, outlining the purpose of my study and requesting permission to conduct the research using my pre-service mathematics teachers as research participants. As part of the application, I had to brief my PMTs about the details of my study so that they could give their consent to participate in my study from an informed position (McMillan and Schumacher, 2001, p. 198). In doing so, I described to the PMTs the purpose of my study, research questions, and the methodology that I planned to use in this study. Furthermore, I informed all the PMTs about their role in my study and that their anonymity would be respected and thus no participant will be referred to by his/her own name in the research report or future publications. Pseudonyms will instead be assigned to each of them. The PMTs were also advised that any information collected from them will be held in confidence. This agreement was reinforced through creating an understanding with the PMTs that the information will only be used for the purpose of this study, and not to do any harm to them or their institution. The PMTs were also informed that the participation is voluntary and they could withdraw at any stage from the research project if they wished to do so. After receiving the necessary consent from the PMTs that they were willing to participate in my study as per signed consent forms, I then completed the UWC Ethical Code of Conduct Form and submitted it to the head of Research and Development at UWC. After receiving the permission and ethical clearance from the head of Research and Development at UWC, I submitted my application for Ethical clearance and permission to the University of KwaZulu Natal (UKZN). After receiving the ethical clearance from UKZN, I had a brief meeting with all the PMTs during which I informed them about the day and time on which the one-to-one task-based interview will be conducted with each of them. At the same meeting they were briefed about the venue and set-up for the one-to-one task-based interview and also given an opportunity to ask any questions about the projected task-based interviews or seek clarification on any particular concern (De Vos, 1998, p. 26).

Furthermore, the process of interviewing, observing, transcribing and analysis of data was done transparently. Upon conclusion of this research, the researcher reported the findings of this research to PMTs that participated in this research (compare McMillan and Schumacher, 2001, p. 198). It is hoped that this study has mutually benefited the participants by equipping them with necessary skills and methodology to model the teaching of generalizations in their prospective mathematics classrooms by using *Sketchpad*. Furthermore, all transcripts, video tapes, observation notes and confidential documents have been kept in a secure place (in my office locker) during the period of the study, and will be kept for a further period of five years under lock and key as required by UKZN policy. Thereafter the *Snagit* recordings, transcripts of one-to-one task-based interviews, observational notes and diaries as well as documents like worksheets will be shredded and disposed to the waste centre. The *Snagit* recordings, which have been stored on DVDs will be incinerated and disposed to the waste centre.

In summary, this Chapter provided an overview of the methodology aspects of the research which are underpinned by a qualitative paradigm located within a constructivist framework. It also explained the various data collection techniques and accompanying tools that were used to arrive at the findings. These research tools included one-to-one task-based interviews, observations and document analysis. To try and maintain the validity of the findings of this study, I as the researcher used the following strategies: triangulation, member checking, provision of thick description(s), clarified researcher bias, and spent sufficient time with research participants. The design limitations of the study as well as the ethical considerations governing this study were also discussed.

The next Chapter, focusses on the data analysis, results and discussion with regard to the Equilateral Triangle (Viviani problem) task- based activity problem.

# Chapter 7: Equilateral Triangle (Viviani) Problem: Data Analysis, Results and Discussions

## 7.0 Introduction

In this Chapter, the data and findings related to PMTs making and justifying conjecture generalization(s) with particular reference to the Viviani task as described in Task 1 (see Appendix 1 for details) are presented in Section 7.1 and Section 7.2 respectively. In addition, Section 7.3 present the data and findings related to the generalization of the Viviani result by PMTs to other kinds triangles (which are non-equilateral).

For the purpose of convenience the Viviani problem will be repeated:

> "Sarah, a shipwreck survivor manages to swim to a desert island. As it happens, the island closely approximates the shape of an equilateral triangle. She soon discovers that the surfing is outstanding on all three of the island's coasts and crafts a surfboard from a fallen tree and surfs every day. Where should Sarah build her house so that the total sum of the distances from the house to all three beaches is a minimum? (She visits them with equal frequency)".

## 7.1 Making a conjecture generalization (inductive generalization)

In this Section, the data and findings related to PMTs making an initial conjecture in a non-*Sketchpad* context, and then a conjecture generalization by empirical induction from dynamic cases, are presented in Sections 7.1.1 and 7.1.2 respectively. This is then followed by the presentation of the data and findings in relation to:

- PMTs certainty in their conjecture generalizations in Section 7.1.3
- PMTs heuristic counter-example experience and their search for counter-examples in Section 7.1.4.
- PMTs need for an explantion as to why their conjecture generalization is always true in Section 7.1.5
- PMTs need for guidance with construction of logical explanation in Section 7.1.6.

### 7.1.1 Making an initial conjecture in a non-*Sketchpad* context

Although the focus of the study is on the development of conjecture generalization through the process of generalizing, an opportunity for PMTs to develop a conjecture without any specific restriction on the process. This means each PMT was given an opportunity to develop an initial conjecture before engaging with *Sketchpad*, either through explanation, belief, experience, deductive proof, or generalization.

Hence, initially the following question was posed to each PMT after each one read the Viviani problem:

> "Before you proceed further, locate a point in the triangle at the point where you think Sarah should build her house?"

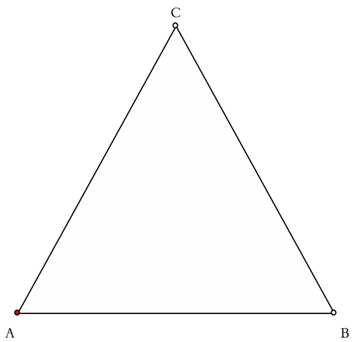



Figure 7.1.1.1 : Equilateral Triangle

All eight of the PMTs located a point in the centre of the triangle, but provided no explanation or reasoned argument for their choice, as illustrated by the responses of two of the PMTs, Renny and Victor,

**Case: Renny**

| | |
|---|---|
| RENNY: | I think it's right in the middle. |
| RESEARCHER: | Okay, you think it's right in the middle, hm? |
| RENNY: | Yes. |

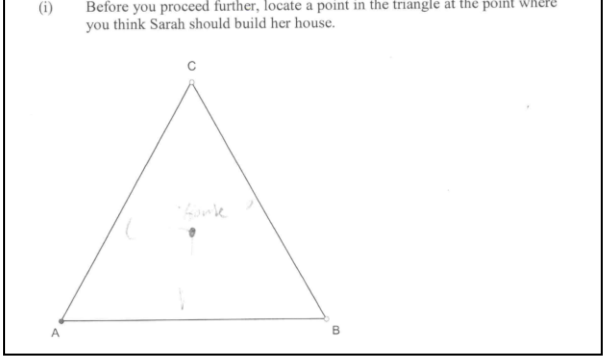



Figure 7.1.1.2 : Renny's initial response to Ship Wreck problem

**Case: Victor**

RESEARCHER: Read the question, Task 1(a)( i), and do the first part of the activity.

VICTOR: Okay, there. (*points to centre of triangle)*

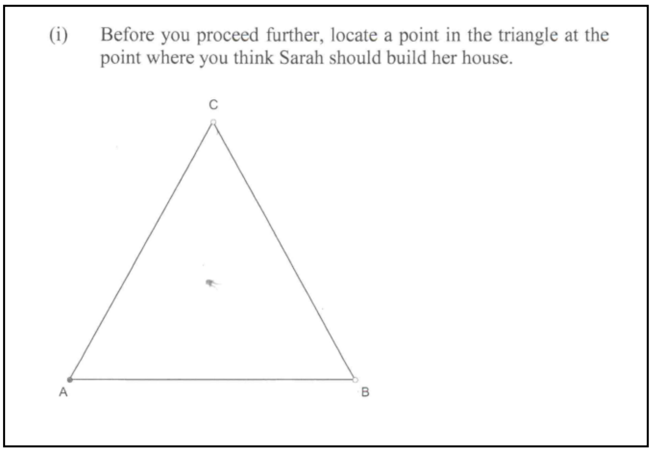



Figure 7.1.1.3: Victor's initial response to Ship Wreck problem

The PMTs' unanimous choice of the centre of the equilateral triangle as the position where Sarah should build her house, was also a unanimous choice of the grade 9 learners that participated in the same kind of activity in Mudaly's (1999) study. The PMTs choice of the centre as a point where Sarah could build her house is indeed one of the correct positions amongst many others.

After each of the PMTs located a common position, the centre, where Sarah should build her house, the following question was posed (see Task1(a) (ii)) of the Viviani worksheet in Appendix 1): Why did you choose that position? Explain or justify your choice.

Logan, Victor, and Renny seemed to come up with the incorrect argument that the distances to the sides should be equal to minimize the sum, but it is likely some of the others intuitively were thinking along similar lines. However, Victor is confusing sides with vertices – he is looking at the distances to the vertices instead of the sides. This was frequently observed also among my B.Ed Honours and PGCE students. The following three excerpts are representative of the PMTs attempted justifications that displays the misconception that the three distances to the sides have to be equal in order to minimize the sum:

**Case: Logan**

| | |
|---|---|
| LOGAN: | I think it should be there. Almost in the middle of the triangle… because she has to move equal distances to all this. If she wants to go to beach $a$ or beach $b$ or $c$, it will almost be the same. |
| RESEARCHER: | Is that your reason for choosing the middle? |
| LOGAN: | Yes, that is my reason, because they ask, where Sarah should build her house so that the total sum of the distances from the house to all the beaches is a minimum. |

**Case: Victor**

| | |
|---|---|
| RESEARCHER: | …in the middle. Okay… alright. Now why did you place it there? |
| VICTOR: | I just wanted it to be in the middle - in order for the lines from $A$ to $B$ okay, from $A$ to the point $C$, from $C$ to the point $B$ – I wanted all those lines to be equal so that this can be in the middle so that I can get the minimum distance from all the points of the triangle. |

**Case: Renny**

| | |
|---|---|
| RESEARCHER: | Alright, maybe you want to tell me why you chose that position (the middle)? |
| RENNY: | I think …okay this is an equilateral triangle…okay so if at this point the distance from here to there is equal to the distance from here to there, and this. |
| RESEARCHER: | Okay, so you're saying that the distance from the centre to each of the sides will be equal? |
| RENNY: | If she wants to reach all the coasts, but she wants to - but one distance shouldn't be further than the other, so I think this will … |

The remaining five PMTs offered no reason why they chose the centre. Two of these five remaining PMTs, Shannon and Tony, did not answer the question, just attempted to explain where the centre was (or how it could be located ) by utilizing their knowledge associated with the concurrency theorems that they learnt at high school. However, Shannon and Tony did not say WHY they thought the minimum would be at the centre. For example, Shannon responded as follows:

**Case: Shannon**

SHANNON: - but I don't need to do any calculations?

RESEARCHER: I don't know.

SHANNON: I don't think so. I think it would be more or less here.

RESEARCHER: More or less where?

SHANNON: you draw a line from here to there … from halfway between the two sides – between the sides – and where the point meets.

RESEARCHER: So are you saying that you found the midpoint of BC?

SHANNON: Yes.

RESEARCHER: And then what did you do?

SHANNON: The midpoint of *BC*, the midpoint of *AB* and the midpoint of *AC*.

RESEARCHER: And then what did you do?

SHANNON: And then construct a line towards … so that it meets angle, or point *B*. And until it meets all the angles.

RESEARCHER: So are you saying that you drew *BC* – you drew this line from the midpoint of *BC* to the vertex *A*?

SHANNON: Yes. From the midpoint of *AC* to vertex *B,* and the midpoint of *AB* to vertex C. And then where all three lines meet that would be the equal distance.

On reflecting on Shannon's response of drawing the median to locate the required centre, one can conjecture from her mentioning the 'equal distance' that it is the 'equality' (or symmetry) misconception that was at play here once more, which also surfaced more clearly later in the rhombus activity.

In general an argument from symmetry would suggest that if there is a minimum, the points holding this minimum should have the symmetries of the equilateral triangle (3-fold rotation and mirrors). So, if there is only one point, there are good general principles suggesting this is the centroid. However, none of the PMTS offered symmetry as a reason for their initial choices.

**7.1.1.1 Findings based on Section 7.1.1- Making an initial conjecture**

1. All the pre-service mathematics teachers (PMTs) located an optimal solution at the centre (or middle), which indeed is one of the correct positions amongst many others,

and this finding is similar to the finding that emerged in Mudaly's (1998) research which involved grade 9 learners.

2. Only three pre-service mathematics teachers voiced reasons why they chose the centre, and displayed a misconception that the distances have to be equal in order to minimize the sum.

3. Five pre-service mathematics teachers offered no reasons why they chose the centre, but two of these five pre-service teachers explained how it could be located using their previous knowledge about the concurrency of the medians of the triangle.

4. In the main, none of the PMTs offered symmetry as a reason for their initial choices.

### 7.1.2 Making a conjecture by empirical induction from dynamic cases in a *Geometry Sketchpad* (*GSP*) context

This Section first discusses PMTs' construction of their conjecture by empirical induction from dynamic cases, and then discusses their validation of their conjecture for new particular cases in Section 7.1.2.1, and formulations of their conjecture generalizations in Section 7.1.2.2. The findings with respect to this Section is presented in Section 7.1.2.3.

PMTs were each asked to open the sketch *Distances.gsp*, which was a ready-made sketch containing an equilateral triangle with a point *P* inside the triangle representing a possible position of the house. In addition, the sketch included a button which showed the distance sum from point *P* to the sides of the given equilateral triangle. Each student was asked to drag point *P* to experiment with their sketch (see Task 1(b) in Appendix 1 for details).

In particular, each PMT was asked to solve the following problem: Press the button to show the distance sum. Drag point *P* around the interior of the triangle. What do you notice about the sum of the distances?

By dragging point *P* around the interior of the equilateral triangle predominantly through directed dragging, each PMT essentially constructed a visual continuum of cases, but with each having a different location of point *P*. This provided an opportunity for each PMT to observe a continuity of visual cases, and identify the invariant property in the dynamic situation. The PMTs were quite surprised to find that the total sum of the distances from point *P*, which represented Sarah's house, to all three sides of the equilateral triangle, which represented all three beaches, remained constant irrespective of the position where point *P* was dragged to. This surprising result and new experience, which was made possible through experimental exploration in a dynamic geometry context, contradicted the PMTs assumption in their initial conjecture (which was made in a non-*Sketchpad* context) that the centre point (midpoint or centroid) of the equilateral triangle was the only possible point creating the minimum distance, and not that this centre point (midpoint or centroid) failed to produce the minimum distance. This in a sense means that the PMTs limited choice of the centre as the only possible position is being contradicted, and not what the student actually claimed as a conjecture.

This new dynamic result seems to have created some cognitive conflict within their cognitive structures and hence disturbed their cognitive equilibrium - their state of mental balance (Berger 2004; Piaget 1978, 1985). This in turn means that through the process of accommodation (see Section 4.6) the PMTs' existing schema could be reconstructed and reorganized to accommodate the new idea, and thereby achieve the necessary cognitive equilibrium.

Although all PMTs were surprised by their observations, namely the sum of the distances from point *P* remains constant irrespective of the position of point *P* within the triangle, they appeared to have achieved this 'cognitive equilibrium' as they were now able to impove (modify) their initial correct conjecture to embrace any point within the equilateral triangle, and be convinced about it, as shown in the following representative excerpt from the one-to-one task-based interviews with the PMTs:

**Case: Shannon**

RESEARCHER: Okay. Alright. So I want you to … uuum … now open this – so if you read the first question there, it says 'press the button to show the distance sum and drag Point P around the interior of the triangle. What do you notice about the sum of the distances?'

SHANNON: It always stays the same (with a very surprised expression on her face)

RESEARCHER: What stays the same?

SHANNON: The sum of the distances is constant from all the points.

RESEARCHER: When you say all the points, you mean from…?

SHANNON: The sum of the distances of a point inside the equilateral triangle, is always the same.

RESEARCHER: So the sum of the distances from point P to the sides will always be the same?

SHANNON: Yes. So it doesn't matter where she builds her house.

The "drag" effect, allowed the PMTs to see many empirical examples in a short space of time, and notice that the sum of the distances $h_1$ , $h_2$ and $h_3$ always remains the same, whilst the individual distances $h_1$ , $h_2$ and $h_3$ vary. Furthermore, it appears that the carefully designed empirical investigative task, which provided opportunities for dynamic visualization, as well as the necessary facilitation by the researcher, assisted the PMTs to discover that Sarah could build her house anywhere inside the equilateral triangle since the total sum of the distances from the house to all three beaches is a constant.

#### 7.1.2.1 Validating the conjecture for new particular cases

In the next task (see Task 1(b) (ii) on the worksheet in Appendix 1**),** which required the PMTs to test whether their conjecture was valid for new particular cases, each PMT was asked to do the following: "Drag a vertex of the triangle to change the triangle's size. Again, drag point P around the interior of the triangle. What do you notice?"

When the PMTs enlarged the size of the equilateral triangle, they were quick to notice that the sum of the distances from point *P* to the sides increased or decreased in correspondence with the size of the triangle. However, when they again dragged point *P* inside the already enlarged or decreased figure, they noticed with great satisfaction that the sum of distances from point *P* to the sides remained unchanged or constant. Moreover, the PMTs' observations remained consistent for a wide range of equilateral triangles created by the "drag" effect, which means they had succeeded in validating their observations for more new particular cases. The following two task-based interview excerpts are representative of the PMTs' moves and observations:

**Case: Shannon**

RESEARCHER: Now look at Question (ii). It says: drag the vertex of the triangle to change the triangle's size. Again drag point *P* around the interior of the triangle, and what do you notice?

SHANNON: I'm dragging point *P* to change the size of the triangle. It could be any size. Okay, and then the distance…

RESEARCHER: What happens to the sum there?

| | |
|---|---|
| SHANNON: | The distance sum will increase if I increase the size of the triangle. But when I drag point *P* around the interior of a triangle, the sum still stays constant. |
| RESEARCHER: | The sum of the distances from point *P* to the sides remains constant. So you increased the size of the triangle, okay. Now I want you decrease the size of the triangle …. Now what happened to the total sum? |
| SHANNON: | It's also changing. It's getting less. It's decreasing. |
| RESEARCHER: | But now investigate what happens to the sum when you drag point *P* inside this smaller size triangle? |
| SHANNON: | It still stays constant. The sum of the distances from point *P* to the sides of a triangle (referring to an equilateral triangle) still stays constant. |

**Case: Tony**

| | |
|---|---|
| RESEARCHER: | Look at Question Question (ii). It says: drag the vertex of the triangle to change the triangle's size. Again drag point *P* around the interior of the triangle, and what do you notice? |
| TONY | It also doesn't actually change. |
| RESEARCHER: | When you increased the size of the triangle, what actually happened to the sum? |
| TONY | It increased. |
| RESEARCHER: | So, once you increased the size of the triangle, the sum increased, but if you drag *P* around inside that same triangle, what happens to the sum? |
| TONY | Stays the same. |
| RESEARCHER: | So, make the triangle smaller. Drag *A* inwards. What do you notice about the sum now? |
| TONY | It shrinks. |
| RESEARCHER: | Now drag point *P* and see what happens. |
| TONY | It is still the same thing |
| RESEARCHER: | What is still the same thing? |
| TONY | The distance doesn't actually change. |
| RESEARCHER: | What distance? |

TONY … the total sum of the heights (pointing to $h_1$ , $h_2$ and $h)_3$ doesn't actually change.

### 7.1.2.2 PMTs' formulations of their conjecture generalizations

Based on their empirical observations which were true for some particular cases, and having validated them empirically for new particular cases, the PMTs were asked to write down their discoveries so far as one or more conjectures (which are considered as conjecture generalizations in this report), using complete sentences. All of the PMTs managed to capture the salient aspects in their conjecture as reflected in the following typical responses:

**Case: Shannon**

SHANNON: The discovery is the total sum of the distances from a point inside a equilateral triangle to the sides of a triangle is a constant, irrespective of where the point is in the triangle.

**Case: Trevelyan**

TREVELYAN: For any equilateral triangle, if you take a point inside that triangle, then the sum of the distance from the point, say the point is P, from P to all three sides, stays the same irrespective of where that point is located inside the triangle.

### 7.1.2.3 Finding based on Section 7.1.2 - GSP context :

1. Similar to Mudaly's (1999, p. 66) finding, all PMTs were quite surprised to find that point *P*, which represented a position of where Sarah could build her house, could be located anywhere inside the triangle to yield a constant result, namely the sum of the distances from the house to all three beaches remains constant.

2. The cognitive conflict they experienced between their initial conjecture and their new observations seemed to have served as a driving force in getting the PMTs to quickly modify their initial non-*Sketchpad* conjecture (which is correct) to capture their new experience, namely from any position of point *P* within the equilateral triangle, the sum of the distances remains constant.

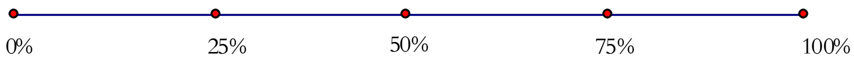
3. All PMTs made their generalizations on empirical (experimental) grounds only.

### 7.1.3 Certainty

This section fouses on the certainty PMTs expresses or had in their conjecture generalization.

Although the PMTs were quite surprised that Sarah's house could be built anywhere as a result of experimentation in a *Sketchpad* context, their expressions and use of phrases like "irrespective of where the point is in the triangle" suggested that in most cases the PMTs were reasonably confident that their conjecture was true in general. Nevertheless, to check their level of certainty, every PMT was subjected to the following question (s):

You can probably think of times when something that always appeared to be true turned out to be false sometimes. "How certain are you that you conjecture is always true? Record your level of certainty on the number line and explain or justify your choice."




Responses from the PMTs indicated that four out of eight students were 100% certain (convinced) that their discovered conjecture generalization was always true without any desire for further experimentation. Three out of the eight PMTs expressed high levels of conviction (85%; 75%; 70%) and one PMT, who expressed a medium level of conviction (50%), said he/she would move on to expressing a 100% level of certainty in his/her conjecture generalization level, only after a little further experimentation and probing.

The following response by Shannon is representative of the responses of those four PMTs who were fully convinced of their conjecture generalization without the need for any further exploration, discussion or probing.

**Case: Shannon**

RESEARCHER: …You can probably think of times when something that always appeared to be true, turned out to be false some time, right. How certain are you that your conjecture is always true. Record your level of certainty on the number line and explain or justify your choice – from your observation, of course.

SHANNON: I think it's a hundred percent.

RESEARCHER: Are you sure?
SHANNON; Yes.

Within the group of PMTs who wanted to experiment further, two PMTs, Logan & Victor, just wanted to experiment further with the given equilateral triangle, and this entailed either dragging the vertex of the given equilateral triangle and/or point *P* to further different positions within the given equilateral triangle. It seems more supporting empirical examples, with the invariant property being maintained, were necessary to fully convince these two PMTs that their CG was true for equilateral triangles. For example, the following extract from the task-based interview with Logan, represents his set of actions and responses, which moved him from an 85% conviction level to a 100 % conviction level:

**Case: Logan**

LOGAN: So I must indicate how true I think it is? So I must make a dot where I think it is. I'll say I'm 85% certain.

RESEARCHER: If you suspect your conjecture is not always true, try to supply a counter-example. Do you want to investigate more? Do you want to experiment further?

LOGAN: I just want to take another point of this… Okay. We used only point *A* to change the shape, so I thought I'd try another point to see. [Logan dragged point *B* and increased the size of the equilateral triangle, and then dragged point *P* around the interior of the equilateral triangle]

RESEARCHER: And now?

LOGAN: I'm 100% convinced.

RESEARCHER:: So what is your response to Question 2?

LOGAN: At this point I agree 100%. I don't have a counter-example.

The other two PMTs, Tony and Alan, did not initially express a 100% certainty in their conjecture generalization, because they were not completely sure as to whether their conjecture generalization was also true for non-equilateral triangle cases, namely isosceles triangles and scalene triangles. After experimenting within the context of isosceles triangles and/or scalene triangles both PMTs realized that their conjecture generalization only holds true for equilateral triangles, and consequently expressed a 100% level of certainty in their respective initial posited conjecture generalizations, which were restricted specifically to equilateral triangles. However, one should bear in mind that whether or not the result is true for other types of triangles, should logically have no effect on their certainty about the truth of the conjecture for equilateral triangles. The conjecture is about equilateral triangles, e.g. If an equilateral triangle…, and this says nothing about other triangles and certainty in that proposition ought not to be influenced about whether it is true for other triangles or not. So it seems more likely that it was not the conjecture about equilateral traingles that they were uncertain about, but about its generalization to other triangles. Of course, it could be that these two students were not making a distinction between a statement and its converse; and thought the converse also needed to be true, ie. if the sum of perpendicular distance to sides are constant, then the triangle is equilateral.

The following excerpts from the task-based interview with Tony captures the gist of the PMTs' transition to a 100% level of certainty that their conjecture generalization is always true for the case of equilateral triangles only:

**Case: Tony**

| | |
|---|---|
| RESEARCHER: | So you have an isosceles triangle, and now you have the sum of the measurements from point *P*. Drag point *P* and see what happens to the sum. |
| TONY: | It's changing … |
| RESEARCHER: | So the sum of the distances is changing. |
| TONY: | Yes. |
| RESEARCHER: | So what do you conclude then? You said you wanted to see if the result is also for an isosceles triangle, so what is your conclusion? |
| TONY: | … I think this theory that the total sum of the distances - it only applies for equilateral triangles. |

Tony's last remark in the above excerpt, "… I think this theory that the total sum of the distances – it only applies for equilateral triangles", suggests that he realized his conjecture generalization does not hold true for isosceles triangles. However, the use of the word, "think", prompted the Researcher to suspect that Tony was not 100 percent convinced of his assertion. Hence, the Researcher, asked Tony "Do you still want to investigate further?"

| | |
|---|---|
| TONY | Let's try the scalene, but I think it will still do the same thing that was done with the isosceles. The remainder of the task-based interview pertaining to the scalene triangle proceeded as follows: |

RESEARCHER: Do you still want to check?

TONY Ja.

RESEARCHER: Here is a scalene triangle … So now we have the measurements of the heights, and the sum shown. Drag point *P*.

TONY It actually changes very fast.

This experience convinced Tony that it was necessary for the triangle to be equilateral for the result to hold.

#### 7.1.3.1 Findings based on Section 7.1.3 – Certainty

Four out of eight PMTs were 100% certain about their conjecture generalization after their initial set of experimental explorations, whilst two PMTs required further experimentation within the context of equilateral triangles and two PMTs wanted further experimentation outside the context of equilateral triangles (such as isosceles and scalene triangles) before pronouncing at a 100% certainty level that their CG was true only for equilateral triangles. The latter two students were either looking at a generalization to all triangles or the converse as discussed earlier.

### 7.1.4 Heuristic counter-examples

According to a Mathematical dictionary for schools (Bolt & Hobbs, 2004), a counter-example is a particular case which disproves a conjecture or claim. In other words a *global* counter-example is an example that shows that a given statement (conjecture, hypothesis, proposition, rule) is false. Indeed a single counter-example is sufficient to refute a false statement. A *heuristic* counter-example on the other hand, in the Lakatosian sense, may just necessitate the reformulation or refinement of the given conjecture (de Villiers, 1996, 2004, 2010). Generally, though not always, counter-examples are produced largely by empirical testing rather than deductive reasoning.

With regard to counter-examples, the following two aspects are discussed in this section:

- Refinement/modification of a PMT's initial non-*Sketchpad* claim to his/her final conjecture (called conjecture generalization) in Section 7.1.4.1
- Search for counter-examples to formulated conjecture generalization in Section 7.1.4.2

#### 7.1.4.1 Refinement/modification of a PMT's initial non-*Sketchpad* conjecture

The first empirical example constructed through dragging point *P* within the interior of the equilateral triangle within *Sketchpad*, served as a heuristic counter example to the PMTs assumption that mid point (centroid or centre) of the equilateral triangle was the only point that produced the minimum distance (as per their initial conjecture that was made in a non–*Sketchpad* context). The further empirical examples that were constructed through the dragging of point *P* around the interior of the equilateral triangle, acted not as heuristic counter-examples any more, but as supporting evidence for the improvement (modification) their correct intial non-*Sketchpad* conjecture to encompass any point within the equilateral triangle to produce the minimum distance sum or generate a constant distance sum (i.e. Sarah could build her house anywhere). Actually the heuristic counter-example instilled a sense of surprise in the PMTs and with the further supporting empirical examples forced them to modify (refine) their initial conjecture that Sarah should build her house in the centre, and hence generalized to a new conjecture (or improved conjecture), which says that Sarah could build her house anywhere inside the equilateral triangle shaped island. The following two PMTs' responses are representative of the group responses:

### 7.1.5 PMTs' need for explanation (or need for understanding why the result is true)

To ascertain whether the PMTs , who all signalled a 100% certainty level in their conjecture generalization, exhibited a need for an explanation of the conjecture generalization they each constructed after experimental exploration in a dynamic geometric context, the following question was posed to each of the PMTs during the one–to-one task-based interviews:

> " If you are fully convinced of the truth of your conjecture generalization, do you still have a need for an explanation (i.e. do you want to know why it is true?) – see Task 2(c)3 of Appendix 1. In equivalent terms, the researcher, wanted to know if the PMTs had an inner desire for a deeper understanding despite being already thoroughly convinced by the supporting empirical examples that were generated through the 'drag effect' in a *Sketchpad* context.

Similar to the finding of Mudaly (1998, p. 85), all eight PMTs signaled an independent need for explanation despite having expressed a 100% certainty level in their conjecture generalization earlier on. Seven PMTs responded immediately in an eager tone that they wanted an explanation, as demonstrated in the following two typical responses:

**Case: Alan**

RESEARCHER: So with respect to that statement you made now, are you fully *convinced of the truth of your conjecture and do you still have a need for an explanation?*

ALAN: Ja, I'd like to know why it is true.

**Case: Logan**

RESEARCHER: Number 3: If you are fully convinced and you don't have a counter-example, do you still have a need for an explanation? In other words, do you want to know why it is true?

LOGAN: Ja, I would actually love to know why.

Only one of the PMTs, namely Tony, took some time to express his actual need for an explanation. This was probably due to a misconception that the series of supporting empirical examples by itself constituted an explanation as suggested by his response:

**Case: Tony**

RESEARCHER If you are fully convinced of the truth of your conjecture, do you still have a need for an explanation?'

TONY I think we explained it.

RESEARCHER: Who explained it

TONY We tried to tackle the same problem

RESEARCHER: Ja, we tackled the problem, in the sense we experimented only. You only investigated more and more, and you only confirmed how you feel about it and that your level of conviction is 100%. But does it explain to you why your result is true?

TONY Why is it actually true?

RESEARCHER: Can you explain to me why it is true?

TONY No

RESEARCHER: Do you want to know why it is true?

TONY Yes

However, after some probing, Tony realized that the empirical examples themselves did not provide the desired explanation and then expressed a desire to know why his conjecture generalization was true.

The aforementioned responses, suggest that the PMTs were intrigued by their discovery as articulated in their respective conjecture generalizations, and consequently expressed a burning desire to really 'understand' the solution to the ship wreck problem despite the overwhelming amount of empirical evidence. Similarly, but in a non-dynamic context, De Villiers (1991, p. 25) found that: "Pupils who have convinced themselves by quasi-empirical testing still exhibit a need for explanation, which seems to be satisfied by some sort of informal or formal logico-deductive argument".

**7.1.5.1 Findings based on Section 7.1.5 – PMTs need for explanation:**

Similar to Mudaly's (1998, p. 85) finding in a dynamic context and that of De Villiers' (1991, p. 258) finding in a non-dynamic context, all eight PMTs in this study appeared to have displayed some desire and need for an explanation, that is, a need to know why their conjecture generalization is always true, despite being fully convinced by quasi-empirical testing.

### 7.1.6 PMTs' need for guidance with regard to construction of a logical explanation:

Although all PMTs expressed some intrinsic desire for an explanation, none of them could produce a logical explanation when given the opportunity do so and seemed to have wanted some guidance to proceed with the construction of a logical explanation. Furtheremore, as all the points in the equilateral triangle holds the minimum distance sum to the sides of the equilateral triangle, they should also have the symmetries of the equilateral triangle. However, this notion of symmetry for the equilateral traingles was not considered as an build an explanation for their modified conjecture.

The following is a typical response of the whole group , who all indicated a need for guidance:

**Case: Shannon**

RESEARCHER: So try. Then I'll help you.

SHANNON: …So if I've got a point *P* here, it should always … it's not always drawn from the midpoint, no. But the line here is always perpendicular. (… silence for about 3 minutes, probably trying to figure what to do). …Can you give me a hint, yes please!

#### 7.1.6.1 Findings based on Section 7.1.6- PMTs' need for guidance - logical explanation

1. Similar to Mudaly's (1999, p. 101) finding, none of the PMTs were able to come up with their own informed logical explanations (proofs). They showed a definite need for guidance to develop an explanation for why their conjecture generalization is always true.

## 7.2 Justifying a Conjecture Generalization

This section firstly discusses PMTs justification via an empirical argument in Section 7.2.1, and then elaborates on how the PMTs developed a logical explanation for her/his equilateral conjecture generalization via a scaffolded guided approach in Section 7.2.2. Thereafter Section 7.2.3 focuses on the presentation of the logical explanation developed via a scaffolded approach as a coherent argument in paragraph form or two column form. Lastly, Section 7.2.4 focuses on the meaningfulness of the guided logical explanation.

### 7.2.1 Empirical evidence: Justification is provided through the correctness of particular examples

As mentioned with reference to the preceding finding (Finding 1 of Section 7.1.6.1), when PMTS were asked to support their conjecture generalization with a justification in the form of a logical explanation none could do so, but seven of the eight PMTs initially attempted to provide an empirical kind of justification (argument). The following is a typical response from several cases:

**Case: Inderani**

When Inderani was asked to explain why her conjecture was always true, she responded as follows:

INDERANI: Why I said 100% is because no matter how I rotate the angle – no matter how big or small I make it (referring to the triangle) – the distances to point *P* still add up to the same. So, in this equilateral triangle, the interior distances add up to 7,03 cm, no matter where I move point *P* to. If I change the equilateral triangle's size, these distances add up to 6,43 cm if the point was there, or whether the point was there, or there.

It was quite evident from Inderani's response, that she concluded that her conjecture generalization was true based on the fact that her claim held for some particular cases in the dynamic context. She thus provided a kind of empirical argument, which really was an argument that provided inconclusive evidence for the truth of her conjecture generalization (Stylianides, 2008). Hence, the Researcher realized that the PMTs who attempted to offer an empirical kind of argument like Inderani, should all be provided with scaffolded guidance as and when necessary to develop a logical explanation for their respective conjecture generalizations( see Section 7.2.2)

### 7.2.2 Developing a logical explanation through a scaffolded guided approach

The basic purpose of this sub-activity was to investigate whether PMTs could construct their own logical explanations through scaffolded guidance. The design of the scaffolded guidance took the form of a worksheet (see Task 1(d) of Appendix 1).

Although the levels of facilitation and intensity of probing varied as each PMT worked through each of the questions on their worksheet, all eight PMTs were able to construct a logical explanation that both justified and explained the truth of their conjecture generalization by making use of the scaffolded questions. The excerpts from the respective one-to-one task-based interviews with three PMTs, Shannon, Trevelyan and Alan, as well as the corresponding extracts from their respective worksheets capture the salient moves and responses that enabled each of them to construct their logical explanations, and equivalently represent the typical responses of the remaining five PMTs.

The case of Shannon will first be presented with relevant commentary, and thereafter the cases of Trevelyan and Alan will be presented respectively.

**Case: Shannon**

Shannon was requested to press the button to show the small triangles in their sketch, an example of which is shown in Figure 7.2.2.1. The Researcher asked Shannon to drag the vertex of the original triangle, and explain why the three different sides were all labelled *a*. Shannnon was able to explain why through more questioning as illustrated in the interview extract below:

RESEARCHER: …drag your vertices of the triangle, why are your sides all labelled '*a*', right?

SHANNON: - 'cause their lengths are all equal.

RESEARCHER: Their lengths are all equal, yes that is part of the answer, but why '$a$', and not seven or six?

SHANNON: Because 'a' represents a variable and it doesn't necessarily refer to a specific value. '$a$' can be any value and the sides will always be equal, so '$a$' just represents the (constant) same value.

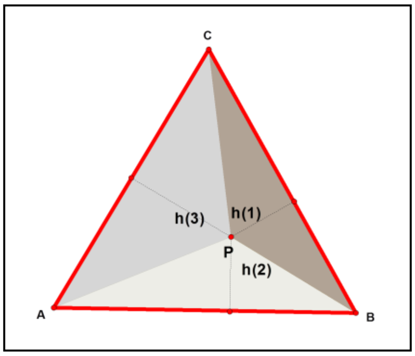



Figure 7.2.2.1 : Equilateral triangle divided into three smaller triangles

Thereafter, I, the Researcher, directed the attention of Shannon to the small triangles within the equilateral triangle, and requested her to write down the area of small $\Delta APB$, using '$a$' and the variable $h_1$.

RESEARCHER: …So can you see the small triangles in there?

SHANNON: Yes.

RESEARCHER: We have a brown one, a yellow one and a green one. So the brown triangle is … how would you label the brown triangle?

SHANNON: How will I label it? *ABP*.

RESEARCHER: Good – $\Delta ABP$, right? So this question says here: 'Write an expression for the area of the small $\Delta APB$, using '$a$' and the variable $h_1$. If you look at your worksheet…. Can you read out what you wrote?

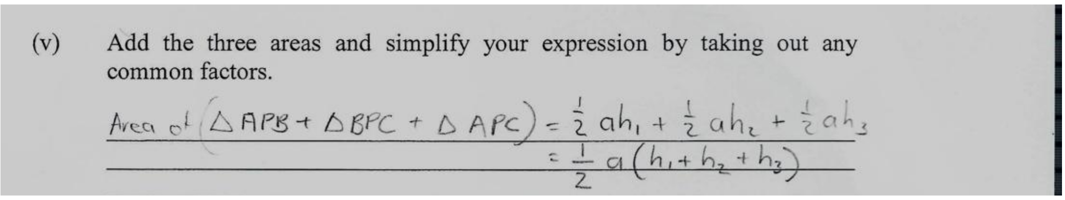
SHANNON: Okay, the area of $\Delta APB = ½\, a \times h_1$

RESEARCHER: Okay. And then, … which can also be, if you write it ½ $ah_1$, then you don't need the multiplication sign, alright?

Similarly, Shannon proceeded to express the areas of remaining triangles in terms of $a$ and their specific heights ($h_2$ or $h_3$).

| | |
|---|---|
| RESEARCHER: | …The second one? Write an expression for the area of the small $\Delta BPC$. |
| SHANNON: | $BPC$ …The area of $\Delta BPC$ is a ½ $ah_2$. |
| RESEARCHER: | Now write an expression for the area of small $\Delta APC$. |
| SHANNON: | Okay, the area of triangle $\Delta APC$ is ½ $ah_3$ |

It is quite noticeable that Shannon had a clear conceptual understanding of area of a triangle. With this confirmation, Shannon proceeded with the next question as shown in Figure 7.2.2.2 in her worksheet:



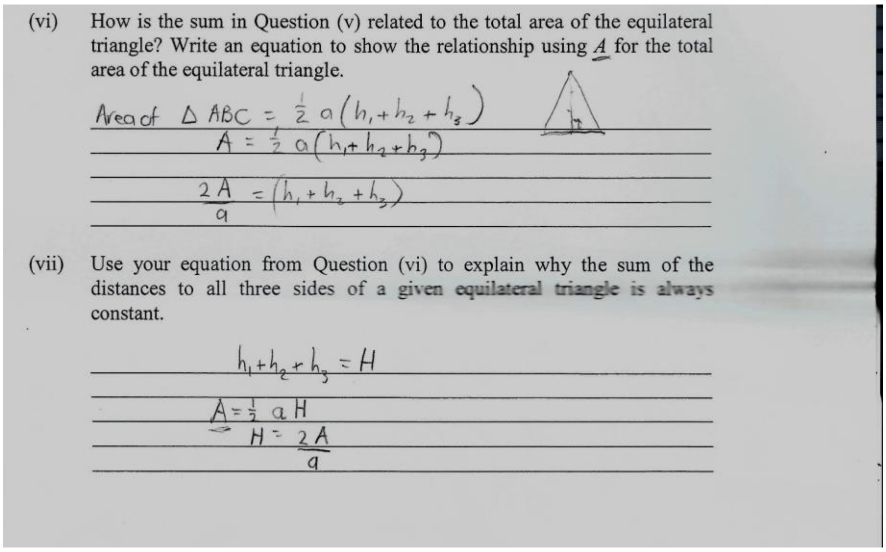
(v) Add the three areas and simplify your expression by taking out any common factors.

Area of $(\Delta APB + \Delta BPC + \Delta APC) = \frac{1}{2} ah_1 + \frac{1}{2} ah_2 + \frac{1}{2} ah_3$

$= \frac{1}{2} a(h_1 + h_2 + h_3)$

Figure 7.2.2.2: Shannon's response to Task 1(d) (v)

Immediately thereafter, Shannon proceeded with the questions (vi) and (vii) on her worksheet as shown in Figure 7.2.2.3.



(vi) How is the sum in Question (v) related to the total area of the equilateral triangle? Write an equation to show the relationship using $A$ for the total area of the equilateral triangle.

Area of $\Delta ABC = \frac{1}{2} a(h_1 + h_2 + h_3)$

$A = \frac{1}{2} a(h_1 + h_2 + h_3)$

$\frac{2A}{a} = (h_1 + h_2 + h_3)$

(vii) Use your equation from Question (vi) to explain why the sum of the distances to all three sides of a given equilateral triangle is always constant.

$h_1 + h_2 + h_3 = H$

$A = \frac{1}{2} aH$

$H = \frac{2A}{a}$

Figure 7.2.2.3: Shannon's response to Task 1(d) (vi-vii)

The following extract captures the critical/crucial moments of clarification and discussion between the Researcher and Shannon with regard to the two interrelated responses to questions (vi) and (vii).

| | |
|---|---|
| RESEARCHER: | ..Question (vi) – reads: 'How is the sum in question (v) related to the total area of the equilateral triangle? and, 'Write an equation to show the relationship, using $A$ for the total area of the equilateral triangle.' |
| SHANNON: | This is the sum in question (v) is - the total area of the equilateral triangle because the equilateral triangle consists of the three triangles. |
| RESEARCHER: | Yes, that's the first part, good. |
| SHANNON: | Okay, then we can say, we know the area of $\Delta ABC$. Is this basically what you need? |
| RESEARCHER: | Okay, so what did you write here? |
| SHANNON: | The area of $\Delta ABC = ½a\ (h_1 + h_2 + h_3)$ |
| RESEARCHER: | And what is the area of $\Delta ABC$? |
| SHANNON: | It's $A$. |
| RESEARCHER: | Okay, so? |
| SHANNON: | You just want me to say $A$ +… |
| | Because the area of a triangle will always be constant, the sum of these will always be constant. |
| RESEARCHER: | Ja, yes … I need you to explain it further. What you're saying is correct so far. You want to talk more about that? You can go ahead – do Question (vi) as well – it's fine – and then talk to me about it. |
| SHANNON: | Well, all I have concluded or derived, proved is that the area of $\Delta ABC$ is $½a\ (h_1 + h_2 + h_3)$ and that the sum of these … $(h_1 + h_2 + h_3)$ should be the length of the perpendicular line of the triangle from the base to the vertex at the top here. |
| RESEARCHER: | What are you saying about $h_1 + h_2 + h_3$ … |
| SHANNON: | It should be equal to the perpendicular height of the triangle. |
| RESEARCHER: | Of which triangle … |
| SHANNON: | … of the equilateral triangle, of the big triangle, because the area of a triangle is ½ × the base length × the perpendicular height. |
| RESEARCHER: | Okay, so let's say the perpendicular height is capital $H$ – if you want to use that, right? You can drag point $P$ to the vertex $A$… |
| RESEARCHER: | So are you talking about that length? |
| SHANNON: | Yes. |
| RESEARCHER: | Okay. Alright. Now I want … You got the expression, right? Now can you explain to me, from what you've been telling me now, why $h_1 + h_2 + h_3$ should be constant? |

SHANNON: Well, the way I see it that is because the area of a triangle – we can say that $h_1 + h_{2+} h_3$ can be capital $H$, and the area of a triangle will always stay constant, and for triangle $ABC$, the big $A$, the area, the small *'a'* which is the length of a side will always be the same, it won't change, and the $H$ also can't change because the area of the big triangle is a constant. If we change any of the three points (measurements) in $h_1 + h_{2+} h_3$, it should still add up to be $H$ …so this makes more sense to me algebraically.

Shannon seemed to argue that $h_1 + h_{2+} h_3$ is constant using two kinds of arguments:

<u>Argument 1</u>:

$\frac{2A}{a} = h_1 + h_2 + h_3$

but $A$ is constant, $a$ is constant and 2 is a constant

$\therefore \frac{2A}{a}$ is a constant

$\therefore h_1 + h_2 + h_3$ is a constant

<u>Argument 2</u>:

$h_1 + h_{2+} h_3 = H$

but $H$ is constant (because it is the height of the fixed equilateral $\Delta ABC$)

Therefore $h_1 + h_{2\,+} h_3$ is constant.

Furthermore, Shannon confirmed her appreciation and sense making of the latter algebraic explanation, by making the following remark during the interview: "*So this makes more sense to me algebraically.*"

**Case: Alan**

RESEARCHER: Let's just proceed with this task here (referring to Task 1-(d)). To explain why we have to go back to the original sketch we've been working with- It says, "Press the button to show the small triangles in your sketch. Drag the vertex of the original triangle. 'Why are the sides of the triangle all labelled *a*'?"

After Alan explained the use of the variable "*a*", the researcher drew the attention of Alan to the small triangles within the dynamic equilateral triangles, and requested him to write down the area of each small triangle, using *a* and the variables $h_1$, $h_2$ and $h_3$ respectively as indicated in the excerpts below:

RESEARCHER: Let's proceed to number (ii); it says, 'Write an expression …' Can you see the small triangles…?

ALAN: This is one, this is two, this is three.

RESEARCHER: The first one is: 'Write an expression for the area of the small $\Delta APB$, using *a* and the variable $h_1$.'

ALAN: $\Delta APB$ and the variable $h_1$ … Is $h_1$ perpendicular to *AB*?

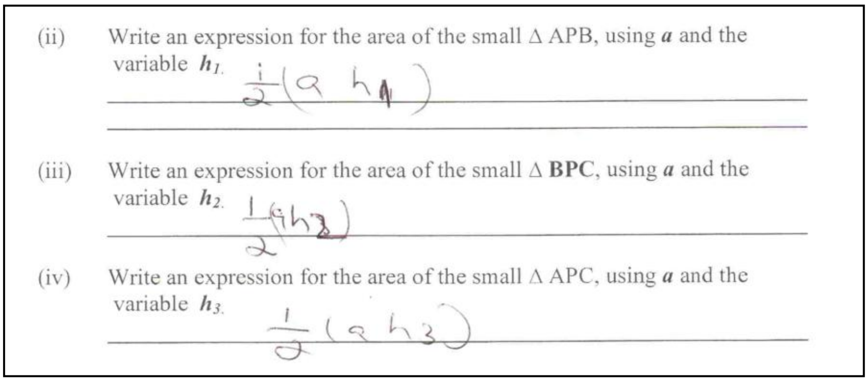
RESEARCHER: So $h_1$ in this case, is the perpendicular distance, and similarly $h_2$, and $h_3$.When we talk about distance, we talk about perpendicular distance. Does that help you now?

ALAN: So the question says here, 'Write an expression for the area of the triangle …it's ½ of base × height; and the height of the triangle is $h_1$ …it's ½ of base, which is ½ of *a*, times height, which is $h_1$.

RESEARCHER: Good. Now write an expression for the small $\Delta BPC$.
What did you write?

ALAN: ½ of $ah_2$. And the area of this triangle will be ½ of $ah_3$.

Figure 7.2.2.4 represents what Alan wrote on his worksheet:


(ii) Write an expression for the area of the small Δ APB, using *a* and the variable $h_1$.
$\frac{1}{2}(a\ h_1)$

(iii) Write an expression for the area of the small Δ **BPC**, using *a* and the variable $h_2$.
$\frac{1}{2}(ah_2)$

(iv) Write an expression for the area of the small Δ APC, using *a* and the variable $h_3$.
$\frac{1}{2}(a\ h_3)$

Figure 7.2.2.4: Alan's response to Tasks 1(d) (ii—iv)

The extract in Figure 7.2.2.4, demonstrates that Alan was able to apply the area formula correctly to each of the small triangles. This shows that if a student is given the necessary scaffolded guidance with carefully selected hints, then he/she can proceed and succeed in developing a required logical explanation.

The Researcher, then asked Alan to proceed to 1 (d) (v) on the worksheet;

RESEARCHER: And number (v)? It says, ‘Add the three areas and simplify the expression by taking out any common factors.’

ALAN: It’s *a* over 2 into ($h_1 + h_2 + h_3$).

Figure 7.2.2.5 shows what Alan wrote on his worksheet:

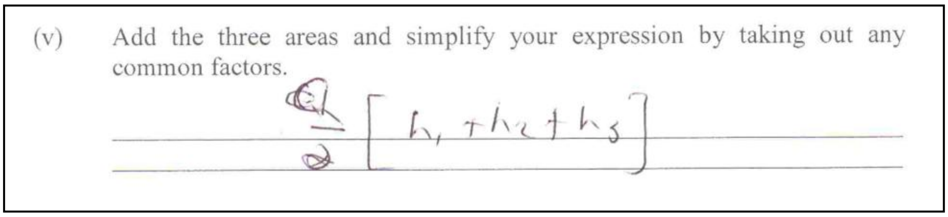



Figure 7.2.2.5 Alan’s response to Task 1(d) (v)

The researcher, then posed (vi) to Alan:

RESEARCHER: Question (vi) says: ‘How is the sum in Question (v) related to the total area of the equilateral triangle?’
Write an equation to show the relationship, using A for the total area of the equilateral triangle.

Figure 7.2.2.6 shows what Alan wrote on his worksheet.

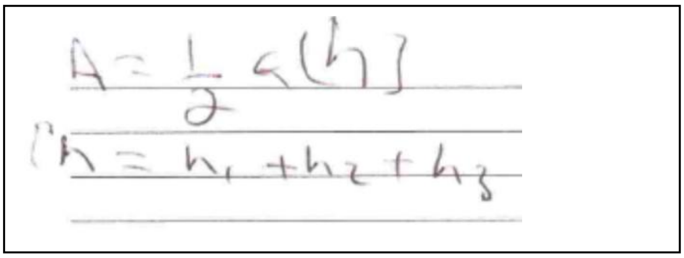



Figure 7.2.2.6 : Alan’s response to Task 1(d) (vi)

The researcher attempted to clarify the meaning of the equation that Alan wrote, with particular reference to the use of *h*, in the above equation. The excerpts below show that Alan was using *h* to represent the height of the equilateral triangle.

ALAN: From '$a$' to here, can we call that $h$ (he used $h$ to represent the height of the equilateral triangle ?

RESEARCHER: Yes.

ALAN: Using big $A$?

RESEARCHER: Yes, using big $A$ for the area of the equilateral triangle.

ALAN: If we use $A$, then $A$ will be ½ of $a$ times $h$…

RESEARCHER: What is this $h$ here?

ALAN: It's the height from any of these points.

RESEARCHER: So are you saying $h$ is the height of the equilateral triangle?

ALAN: Yes.

Immediately thereafter, the researcher, asked Alan: 'Use the equation from Question (vii), to explain why the sum of the distances to all three sides of a given triangle is always constant.' Alan then responded as follows:

ALAN: It's because the height has to remain constant, because (should have read therefore) $h$ will always be constant. So the sum of any heights (referring to $h_1 + h_2 + h_3$) inside that equilateral triangle should be equal to $h$ and $h$ is always constant.

Alan's explanation, revealed a sense of insight and understanding in that he appeared to have realized probably through dragging that the height of the equilateral triangle is equal to the sum of the heights of the 3 smaller triangles. Hence, on the basis of the fact that height of a fixed equilateral triangle is constant he rationalized that $(h_1 + h_2 + h_3)$ must also be constant. This effectively contributes to his explanation of why the sum of the distances to all three sides of a given equilateral triangle is always constant.

Thereafter, the Researcher requested Alan to write his explanation as a coherent argument.

#### 7.2.2.1 Findings based on Section 7.2.2 – Scaffolded guided approach

Whilst none of the PMTs were able to provide a logical explanation on their own when given the opportunity to do so, it was found that all eight PMTs were able to construct a logical explanation of their conjecture generalization by working through the scaffolded steps on the worksheet in association with necessary guidance instances. This finding is similar to

Mudaly's (1998, p. 101) finding, namely: "If given proper guidance, pupils were able to construct a logical explanation for the conjecture."

### 7.2.3 Presentation of explanation as an argument in paragraph form or two-column proof

After having developed the explanation through the use of a guided scaffolded approach, each PMT was given an opportunity to provide a coherent holistic explanation. The actual opportunity was phrased as follows:

> "Summarize your explanation/justification of your original conjecture. You can use questions (i)-(vii) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a Sketchpad sketch, or some other medium."

Three out of eight PMTs produced a correct and coherent logical explanation using the two-column format that is frequently used at secondary schools. In the use of the two- column format, one often spells out the following: What is given; what is required to be proved; the required constructions; and then the development of the logical argument containing statements and reasons. Two such representative cases are those of Shannon and Trevelyan:

**Case: Shannon**

In her attempt to construct her coherent logical explanation, Shannon described to the interviewer her main conceptualization of the problem and what she was expected to prove as follows:

SHANNON: Okay, so I've basically written down now what I was given, what I think I need to prove. So I was given $\Delta ABC$, with $AB$, $BC$ and $AC$ equal lengths, and I've named the lengths '$a$', or I labelled the lengths '$a$' because they are all the same, and a point $P$ in the interior of a triangle – so it can be anywhere in the interior of the triangle, with $h_1$, the line from $B$, and it's perpendicular to $BC$, $h_2$, the perpendicular on $AB$, and $h_3$, the perpendicular on $AC$. And then I must prove that $h_1 + h_2 + h_3$ are always a constant.

RESEARCHER: Yes, good.

SHANNON: Yes, so … and then I need to do the…sum so as to prove it.

Shannon then completed by writing down her logical explanation on her worksheet as indicated in Figure 7.2.3.1.

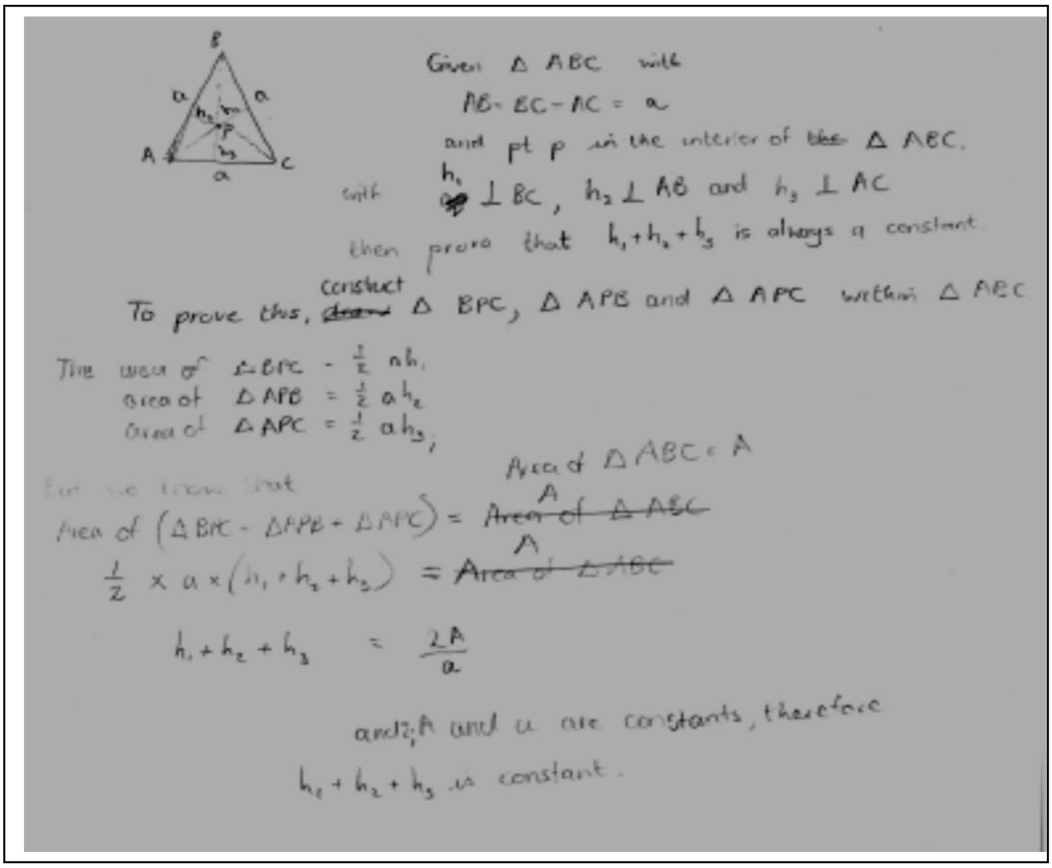



Figure 7.2.3.1: Shannon's explanation for equilateral CG

**Case: Trevelyan**

As illustrated in Figure 7.2.3.2, Trevelyan's argument and conceptualization corresponded to Shannon's explanation of her understanding of the task.

**Task 1 (e): Present your explanation/justification**
Summarize your explanation/justification of your original conjecture. You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a *Sketchpad* sketch, or some other medium.

R. T.P. the sum of the perpendicular distances from an arbitrary point p interior to the equilateral triangle is constant.

Construct:
we construct three triangles. $\triangle APC$, $\triangle BPC$, $\triangle APB$.

Proof:
we find the areas of this small triangles to be
$A\triangle APC = \frac{1}{2} a h_1$ $\quad A\triangle BPC = \frac{1}{2} a h_2$ $\quad A\triangle APB = \frac{1}{2} a h_3$
then
$A\triangle APC + A\triangle BPC + A\triangle APB = \frac{1}{2} a(h_1+h_2+h_3)$
$= \frac{1}{2} a H, \quad H = h_1+h_2+h_3$
But the sum of the areas of this small triangles equal to the area of the equilateral. therefor
$A = \frac{1}{2} a H$, A is an area of the equilateral.
Now $H = \frac{2A}{a}$ which is the constant
because $\frac{2A}{a}$ is a constant.
therefor $H = (h_1+h_2+h_3)$ is a constant □

Figure 7.2.3.2: Trevelyan's explanation for equilateral CG

On the other hand, four out of eight PMTs attempted to construct their logical explanation using the paragraph format. In the development of their respective logical explanations the PMTs indicated that $A$ and $a$ were constant, but did not explicitly give the reason, namely that the triangles was fixed. The handwritten response of Tony (see Figure 7.2.3.3) is typical of those from the other students in the latter group as well.

**Case: Tony**

**Task 1 (e): Present your explanation/justification**
Summarize your explanation/justification of your original conjecture. You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a *Sketchpad* sketch, or some other medium.

Because we added the small ~~triangle~~ areas in the equilateral triangle the total of the sum of the small ~~triag~~ triangles is equal to the ~~sum of~~ the area of the equilateral triangle therefore we deduced that

$$A = \frac{1}{2} a(h_1+h_2+h_3)$$

Where A is area of the equilateral triangle and a is a side of the equilateral triangle therefore A and a are constant
$\therefore h_1+h_2+h_3$ must also be constant

Figure 7.2.3.3: Tony's explanation for equilateral CG

#### 7.2.3.1 Findings based on Section 7.2.3: Explanation in paragraph or two column proof form

1. It seems that the PMTs' prior engagement with the scaffolded worksheet assisted seven of the eight PMTs to write down a complete coherent logical explanation for their conjecture generalization using either the two-column format or the paragraph format. Three PMTs produced their logical explanation in a two- column form in which the following aspects were clearly described: The given information; what was required to be proved; the necessary constructions; and the main body which showed the development of the logical argument with necessary and sufficient reasons. Four PMTs provided their logical explanation in paragraph form.

### 7.2.4 Meaningfulness of the guided logical explanation

When a PMT completed the guided logical explanation, piece-wise or coherently, the Researcher probed each PMT to try and establish whether they found the guided logical explanation meaningful. In particular, the Researcher wanted to ascertain if the guided logical explanation satisfied their earlier expressed need for explanation and understanding. All eight PMTs expressed a view that the guided logical explanation provided the necessary explanation and understanding as to why their conjecture generalization was true, and in fact satisfied their need for 'explanation'. The following excerpts are representative of these PMTs' expression of satisfaction:

**Case : Logan**

RESEARCHER: …do you understand this explanation?
LOGAN: Ja.
RESEARCHER: Are you satisfied?
LOGAN: Ja.

**Case: Shannon**

RESEARCHER: … So do you understand the explanation that you wrote?
SHANNON: Yes. It makes sense.
RESEARCHER: So do you now know why your result is true?
SHANNON: Yes.

**Case: Trevelyan**

RESEARCHER: So you now found out that $h_1 + h_{2+} h_3$ is a constant.

TREVELYAN: Ja, I did.

RESEARCHER: So you now explained the result.
Are you satisfied?

TREVELYAN: I am.

#### 7.2.4.1 Findings based on Section 7.2.4: Meaningfulness of the guided logical explanation.

1. Similar to Mudaly's (1998, p.103) finding, the guided logical explanation seems to have satisfied the PSTEs earlier expressed need for explanation and understanding.

### 7.3 Generalizing to other kinds of triangles

During the session that focused on the development of their conjecture generalization, two students, Alan and Tony, expressed their desire to experimentally explore other kinds of triangles such as the scalene and isosceles triangles to see if their conjecture generalization also remained true for those cases. The researcher provided the necessary scalene and isosceles triangles for the experimental exploration within the dynamic geometry context. Both students realized through the empirical evidence that their conjecture generalization cannot be applied to either scalene or isosceles triangles. The following excerpt from the task based interview with Alan captures his experiences with the isosceles triangle case:

**Case: Alan**

RESEARCHER: Here is an isosceles triangle. The measurements of $AB$ and $AC$ are given. Check that it's isosceles. And there's point $P$.

ALAN: You see it varies (referring to the sum of the distances). It doesn't vary that much, but it varies because at some point you get … [breaking off]

ALAN: I just wanted to check, that's why I said 50% because I thought we were generalizing.

RESEARCHER: So what is your statement? What do you want to say now?
Can you read out what you wrote?

ALAN: The conjecture is only true for equilateral triangles. You can't reach the same conjecture for, e.g. an isosceles triangle.

| | |
|---|---|
| RESEARCHER: | That is after the experimentation? |
| ALAN: | Yes. |

See section 7.1.3 for the excerpt representing a similar kind of response from Tony after experimental exploration. Furthermore, during the latter part of the task-based interview, each PSTE was provided the following challenge, which was constructed as follows: "In this session you may have observed, conjectured and logically explained the following result: In an equilateral triangle, the sum of the distances from a point inside the triangle to its sides is constant:

1. Do you think that the above result might be true for other kinds of triangles?

2. If not, why not? Or if so, why?"

Six out of eight PMTs appeared to have reflected on the structure of their already constructed logical explanation of their conjecture generalization for the equilateral triangle case to conclude, argue and explain why their conjecture generalization will not hold for other triangles such as scalene and isosceles. In particular, the six PMTs signaled that their conjecture generalization may not hold for other kinds of triangles like the isosceles and scalene triangles, because they may not be able to pull "a" out as part of the common factor. The following two cases, namely Logan and Shannon, illustrate the typical responses from the six PMTs.

**Case: Shannon**

The following excerpt can serve as evidence that can help explain why Shannon thinks the result for the equilateral triangle cannot apply to any other kind of triangle.

| | |
|---|---|
| RESEARCHER: | So now what you have just observed, conjectured, and logically explained is that in an equilateral triangle, the sum of the distances from the points of the triangle to its sides is constant. That's what you've just explained. |
| SHANNON: | Yes? |
| RESEARCHER: | Okay – for that equilateral triangle – am I right?<br>The question is: 'Do you think that the above result might be true for other kinds of triangles, and why? If not, why not? If so, why? Other kinds of triangles… |

SHANNON: No, I don't think so. I don't think it will be true for other kinds of triangles.

RESEARCHER: Why?

SHANNON: Because the lengths of the sides aren't going to be equal, which means that the common factor '*a*' which we took out of the equation here, won't exist. There won't be a common factor between the lengths of the heights. So your sum will basically be - if we have to redo this - is that what you want me to do – to redo it?

RESEARCHER: No.

SHANNON: No, I just want to clarify it for myself. Because you'll basically end up with say an '*a*'….

RESEARCHER: Take an isosceles triangle if you want to.

SHANNON: Because you'll basically end up with an '*a*', a '*b*' and a '*c*' here …Well you said any kind of triangle. (Note: Shannon works with a scalene triangle)

RESEARCHER: Oh, okay.

SHANNON: … where '*a*' is not equal to '*b*' is not equal to '*c*'.

RESEARCHER: What kind of triangle is that then?

SHANNON: It's a scalene.

RESEARCHER: Okay. Good.

SHANNON: And then we will basically end up with – if we had to redraw this the same way, with your triangles.

RESEARCHER: You can just explain it to me in words, it's fine.

SHANNON: No, but I want to … I don't know how to explain it in words…
You'll just end up with a ½ × …you'll end up with… because you don't have a common factor here. You can take the half out, so it will be $ah_1 + bh_2 +$… [writing] …There is nothing that I can see that will stay common. Or that will stay constant. You can't further simplify this.

RESEARCHER: Okay. Thank you very much.

Figure 7.3.1 represents a copy of what Shannon wrote in her worksheet, referred to during the above-mentioned task based interview:

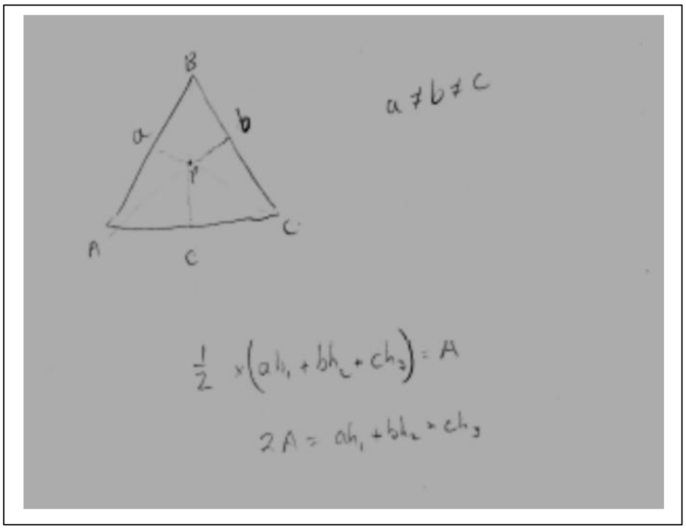



Figure 7.3.1: Shannon's Explanation – CG cannot apply to other triangles

The excerpt from the one-to-one task- based interview with Shannon and the accompanying extract from her worksheet, serve as evidence of Shannon's over-generalization of the structure of the proof, since she used the structure of the proof as a basis to argue why the result will not hold for any other kind of triangle. In particular, she was using an inverse kind of argument to justify why the result will not hold for other kinds of triangles. The inverse argument in this instance can be characterized as follows:

Equal sides ⇒ constant sum ….. established and logically explained result

Not equal sides ⇒ not constant sum …. inverse argument

Her argument, is of course logically incorrect as the original statement does not say anything about whether or not the 'constant sum' result is true for any triangle. All it says is that if it were true (which it is not incidentally) then the same proof would not apply. The same kind of inverse argument, where the proven result, namely "equal sides ⇒ constant sum", was interpreted as logically equivalent to: "not equal sides ⇒ non-constant sum" by five other PMTs as illustrated in the excerpt from the one-to-one task-based interview with Logan:

**Case: Logan**

The following excerpt brings evidence that explains why Logan thought the result for the equilateral triangle cannot apply to any other kind of triangle.

LOGAN: (You mean) like for an isosceles triangle?

RESEARCHER: Ja, is the result true for an isosceles triangle?

LOGAN: I will say no, because I assume that all the sides must be the same, which means if I take an isosceles triangle, two sides are the same, and there's one side that is not equal in size; it's different from the other two, which means that the basis will be different. In this case (referring to the equilateral triangle) we marked all three small '*a*' because we knew that the side length of all three was the same. That's why we labelled it '*a*'.

RESEARCHER: So you're saying that in an equilateral triangle, all the sides are the same, and you can have *a*. And you're saying that in the case of an isosceles triangle that's not the case.

LOGAN: Ja, because what we did here is, in this equilateral one, with all the sides being the same, we could actually take out the common factor. We could say ½ × *a* and take it out from ½ $a$ ($h_1$) + ½ $a$ ($h_2$) + ½ $a$ ($h_3$). . But in terms of the isosceles, we can't say half times *a* and take it out as the common factor because there're only two sides that are the same.

RESEARCHE R: So you're talking about this here.

LOGAN: I'm talking about this here – about that '*a*' which represents all three sides.

RESEARCHER: So you're saying that in an isosceles triangle you will not be able to pull out that *a*?

LOGAN: No, because two sides will be the same, and one side will be different.

RESEARCHER: … and in a scalene triangle?

LOGAN: Also (meaning it will not be true for a scalene triangle);
because the sides are not equal.

RESEARCHER: So are you sure?

LOGAN: I believe so.

RESEARCHER: Do you want to investigate or check your result?

LOGAN: No, I strongly believe that.

### 7.3.1 Findings based in Section 7.3- Generalizing to other kinds of triangles

1. Two PMTs, after experimentally exploring with scalene and equilateral triangles at their own request, realized that their conjecture generalization would not hold for non-equilateral triangles.

2. Six out of eight PMTs argued that their conjecture generalization, which they had justified through the construction of a logical explanation, will not hold true for non-equilateral triangles, because the sides will not all be equal and thus they would not be able to pull out the common factor "*a*". It seems that in this instance, the majority of the PMTs overgeneralized by looking back at the structure of their already developed explanation (or proof), even though in this case their incorrect logical reasoning happens to give the correct result. They seemed to be using an inverse kind of argument, namely *equal sides* ⇒ *constant sum* is logically equivalent to *not equal sides* ⇒ *non-constant sum*, which of course is incorrect, to justify why their conjecture generalization will not hold for other kind of triangles like isosceles and scalene triangles. Though not explored further in this study, it is likely that this type of reasoning would in the context of the following Rhombus activity lead them to conclude that only if a quadrilateral was a rhombus, would the sum of the distances from a point to the sides be constant. But in the case of quadrilaterals, the result is more generally true for any parallelogram, and all sides do not have to be equal for the result to hold. This might be worth investigating further in a follow-up study.

The next Chapter, focusses on the data analysis, results and discussion with regard to the Rhombus task- based activity problem.

# Chapter 8: Rhombus Problem - Data Analysis, Results and Discussion

## 8.0 Introduction

In this Chapter, we look at the generalization of the Viviani result for an equilateral triangle to the rhombus, which is a different context. The rhombus enjoys a status of being irregular, since all its sides are equal, but angles are not necessarily equal. However, the rhombus can be dragged to become a regular figure, namely a square (as all regular rhombi are squares). In particular, the consistent property across both the equilateral triangle and rhombus is that of having equal sides. The challenge during the research programme was to engage the PMTs to generalize their established result for the equilateral triangle to the rhombus. The planned questions for the rhombus task (see Appendix 2 for Task 2: Rhombus, and Appendix 3 for the Rhombus interview schedule) did not differ much from those asked in the equilateral triangle task. However, the questions were used selectively, depending on how a given student responded to a given question at a given instance.

The data and findings related to making and justifying conjecture generalization(s) with particular reference to the rhombus task are presented in this Chapter. In Section 8.1, the analyses of the data associated with making a conjecture generalization are described, and in Section 8.2 the level of certainty expressed by the PMTs towards their conjecture generalization are described. Section 8.3 describes the PMTs need for an explanation as to why their conjecture generalization is always correct, and Section 8.4 describes the kinds of justifications that the PMTs advanced in attempting to explain why their conjecture generalization is always true.

## 8.1 Making a conjecture generalization

The Rhombus Task 2(a), which was fore-grounded in a non-*Sketchpad* context, provided the following opportunities for the PMTs:

- Opportunity to construct an initial conjecture, but a general one, via analogy, which they could later confirm (or refute) through experimentation in a *GSP* context if they elect to do so, by using the guidelines provided in Task 2(b).

- Opportunity to construct an initial conjecture, but a general one, on logical grounds, which they could later confirm through experimentation in a *GSP* context if they elect to do so, using the guidelines provided in Task 2(b).

- To provide those PMTs, who did not see the similarity of the rhombus problem with the equilateral problem, an opportunity to make an initial conjecture through either their own intuition or spontaneity, which may not necessarily be correct, but could later be tested via experimentation in a Sketchpad context as outlined in Task 2(b).

Although the design of Rhombus Task 2(b), was designed for PMTs to specifically use *Sketchpad* to develop a conjecture generalization related to the rhombus problem through inductive reasoning, it also provided an opportunity for PMTs to test and validate their initial non-*Sketchpad* conjectures/conjecture generalizations within a *Sketchpad* context and experience a global counter-example in the process.

The next few subsections of this subsection 8.1 entail a discussion on how the PMTs constructed their conjecture generalizations with respect to a rhombus. Firstly, the discussion in subsection 8.1.1 focuses on the PMTs' construction and justification of an initial conjecture in a non-*Sketchpad* conjecture. The remainder of the sub-sections focuses primarily on the construction of PMTs' conjecture generalizations as observed during the one-to-one task-based interviews, and are presented in the following order:

Sub-section 8.1.2: Producing a conjecture generalization by empirical induction from dynamic cases in a *GSP* context;
Sub-section 8.1.3: Producing a conjecture generalization immediately on analogical grounds and did not require experimental confirmation with *GSP*;
Sub-section 8.1.4: Producing a conjecture on logical grounds but then requiring experimental confirmation in *GSP* context;
Sub-section 8.1.5: Producing a conjecture generalization immediately on logical grounds.

### 8.1.1 Making an initial conjecture in a non-Sketchpad context

At the start of the one-to-one task-based interview, each PMT was first presented with the rhombus worksheet, and was then requested to attempt the initial rhombus task , namely Task 2a (i-ii), as shown in Figure 8.1.1. The purpose of this initial task was to give each PMT an opportunity to develop an initial conjecture on his/her own, before engaging with Sketchpad, by possibly generalizing from the equilateral triangle case.

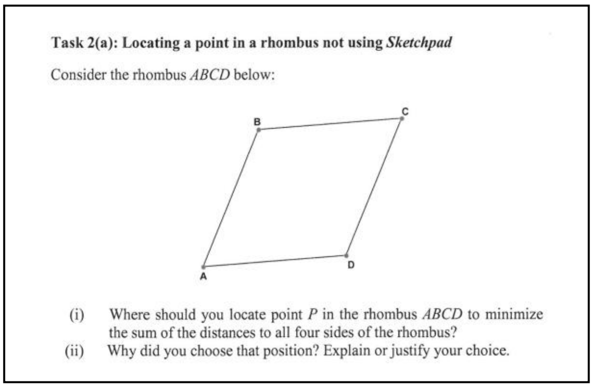



Figure 8.1.1: Rhombus Task 2(a)

With regard to Task 2(a), two PMTs, Shannon and Alan, seemed to have noticed the similarity with an equilateral triangle and conjectured that point *P* can be located anywhere inside the rhombus *ABCD* (see Section 8.1.3 for a detailed presentation of the data analysis related to the construction of their conjecture generalization via analogy and Sections 8.4.4.1 and 8.4.4.2 for their justifications). On the other hand, the remaining six PMTs, Trevelyan, Tony, Inderani, Victor, Logan and Renny, nominated just one point within the rhombus where point *P* can be located to minimize the sum of the distances to the sides of the rhombus. Alternatively, one could only assume that they have limited their search for the only point respecting both mirror symmetries of the rhombus shape. Again the expanded set of the whole interior also has the property of being invariant under both the mirrors. However, it is not clearly evident if they used the latter strategy of symmetry.
The remainder of the data analysis in this section 8.1.1, describes the latter six PMTs' initial conjecture in a non-*Sketchpad* context.

Four of the PMTs, Trevelyan, Tony, Inderani and Victor, conjectured that point *P* should be located at the centre (or middle) of rhombus. When asked to justify their choice, it seemed they exhibited the 'equality misconception' that the sum of the distances would be a minimum when the four distances to the sides would be equal to each other. The following task-based interview excerpts, which are presented case-wise, are representative of these PMTs' initial non-Sketchpad conjectures with their attempted justifications:

**Case: Trevelyan**

When the Researcher, posed Task 2a (i) to Trevelyan, which he had to do initially without using *Sketchpad*, he responded as follows: "For the start I could choose the most central point, this is the point (*pointed to centre of figure that was contained in Task 2a, which was on the hardcopy worksheet*) at which I will start – making sure that I minimize the distance…" Furthermore, when asked why he chose the centre, he replied, "I am using common sense*"*. This "common sense" relates to the following understanding stated by Trevelyan, "If you are at the centre, that means you'll travel the minimum distance to each side." Though it could be that Trevelyan misunderstood the question by thinking that only the individual distances were to be minimized, it might well be that he had the misconception that to minimize the sum of the distances $h_1 + h_2 + h_3 + h_4$ the four distances have to be equal. In other words, he was somehow thinking about or visualizing this minimum point as the centre of the rhombus.

Whatever the case may be, his responses clearly showed that he did not immediately recognise the link with the previous equilateral triangle case. The Researcher then got Trevelyan immersed into the Sketchpad Rhombus task to experimentally explore his conjecture (see section 8.1.2 for details of analysis with regard to this particular aspect)

**Case: Inderani**

The following excerpt from the one-to-one task based interview with Inderani, represents her conjecture as to where point *P* should be located and her accompanying justification thereof:

| | |
|---|---|
| RESEARCHER: | Consider the rhombus *ABCD*. Where should we locate point *P* in the rhombus *ABCD* to minimize the sum of the distances to all four sides of the rhombus? |
| INDERANI: | -Uuum…Okay, let's say in the centre. |
| RESEARCHER: | You say in the centre. Wherein the centre? Mark it… |
| INDERANI: | Let's say there (see inserted Figure 8.1.2 for plotted position) |
| RESEARCHER: | Why did you choose that position? |
| INDERANI: | - because it seems as though the distances from the point to the sides are equal. |
| RESEARCHER: | So, you're saying, put point *P* there, so it will give you the minimum distance sum. |

INDERANI: Ja.

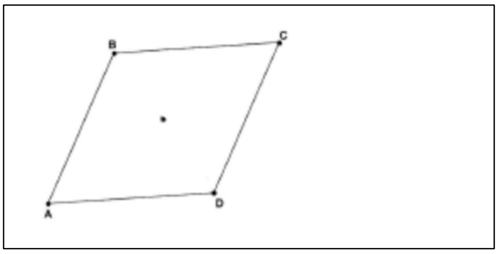



Figure 8.1.2: Inderani's response to Rhombus Task 2(a)(i)

Inderani clearly seemed to also have the 'equality' misconception that the sum of the four distances, $h_1$, $h_2$ , $h_3$ and $h_4$, would be a minimum when the four distances are equal. This showed that Inderani did not immediately recognise a link with the previous equilateral triangle case. The researcher then engaged Inderani with the *Sketchpad* Rhombus task, as illustrated in the interview excerpts and worksheet extracts in Section 8.1.2)

**Case: Victor**

The following excerpt from the one-to-one task-based interview with Victor, can illustrate his conjecture and associated justification:

RESEARCHER: Task 2(a)(i): Consider the rhombus *ABCD* below. Where should you locate point *P* in the rhombus *ABCD* to minimize the sum of the distances to the sides of the rhombus?

VICTOR: I think it can be here in the middle – at the centre.

RESEARCHER: At the centre..? Why do you choose that position?

VICTOR: Must I write it?

RESEARCHER: You can tell me.

VICTOR: - to ensure that the distance between any sides of this rhombus are equal – so that can be in the middle – to ensure that the distance from this point to the middle of *AC* is equal to the distance from this point to the middle of *AD* and the rest.

RESEARCHER: So you say that position will minimize the distance to all four sides?

VICTOR: Yes, yes.

Victor also seemed to exhibit the 'equality' misconception and thought point *P* should be located in the middle (centre) of the rhombus, and clearly saw no link or similarity to the equilateral triangle case. Further to this, Victor also erroneously thinks that the perpendicular to the side will b eat the midpoint of the side. The researcher then proceeded to engage Victor in the empirical Rhombus task-based activity using *Sketchpad* (see section 8.1.2 for details and analysis).

Note that in contrast to the cases just presented, one PMT, Logan, conjectured that point *P* should be located very close to vertex *C*, and another, Renny, conjectured that point *P* should be located very close to side *AD*. These two PMTs, Logan and Renny, seemed to have confused distances to the sides with distances to vertices in the justification of their choices. The following excerpt from the task-based interview excerpts with Logan indicative of these PMTs' conjectures and justifications:

**Case: Logan**

The sketch in Figure 8.1.3, scanned from Logan's worksheet, illustrates where Logan located his point, which was definitely not in the middle.

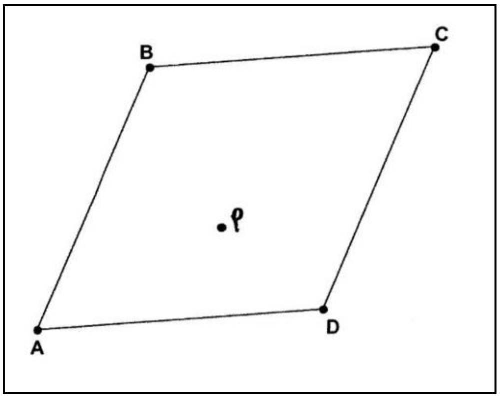



Figure 8.1.3: Logan's response to Rhombus Task 2(a)(i)

When the researcher asked Logan, "Can you describe where you located that point?", he replied: "I've located the point… not in the middle, almost nearest to the side of *AD*, but I didn't locate the point in the middle." It is not clear at all why Logan located the point there. He attempted to justify his choice as follows: "because they've asked me to locate the point so as to minimize the sum of the distances, which means the distances from point *P* are not equal for all the sides. Thus, the distance from point *P* to *D* is perhaps shorter than the distance from point *P* to *B*". However, here he seemed to be confusing distances to the sides with distances to the vertices. The Researcher then proceeded to engage Victor in the empirical rhombus task-based activity within a *Sketchpad* context (see section 8.2 for details of the investigation and analysis).

#### 8.1.1.1 Findings as per Sub-section 8.1.1: Making a conjecture generalization

1. Two PMTs, Shannon and Alan, appeared to have superficially seen the similarity with the equilateral triangle case and thereby proceeded to make the generalization that point *P* can be located anywhere inside the rhombus *P* since the sum of distances to the sides of the rhombus is constant (see section 8.1.3 for further discussion). These two PMTs, did not express any desire to confirm their conjecture generalization by means of experimental exploration with the aid of Sketchpad.
2. While limited to conjecturing in a non-*Sketchpad* context, six PMTs, Trevelyan, Tony, Inderani, Victor, Logan and Renny, were not able to generalize the result from the equilateral triangle case to the rhombus.
3. Four of the six PMTs, Trevelyan, Tony, Inderani and Victor, who were not able to generalize the result from the equilateral triangle case to the rhombus, exhibited the 'equality misconception', namely that the sum of the distances would be a minimum when the four distances are equal to each other.
4. Two of the six PMTs, Logan and Renny, who were not able to generalize the result from the equilateral triangle case to the rhombus, seemed to have been confusing distances to the sides with distances to vertices in the justification of their choice for the location of point *P*, which is also a misconception.

### 8.1.2 Producing a conjecture generalization by empirical induction from dynamic cases in a GSP context

In the engagement of the PMTs with Rhombus Task 2(b), which is shown in Figure 8.1.2.1, the conjecturing and generalizaing steps that were suggested by Canadas et al. (2007), have been taken into consideration.

The six PMTs , who did not conjecture that point *P* can be located anywhere in the rhombus to minimize the sum of the distances to the sides of the rhombus when restricted to a non-*Sketchpad* context as per Rhombus Task 2(a), were then given an opportunity to use *Sketchpad* to experimentally explore the location of point *P*. Through dragging point *P*, the PMTs constructed several empirical examples that demonstrated that point *P* need not be at the centre to minimize the sum of the distances to the sides of the rhombus, since the sum produced was always constant. As before, the initial empirical example served as a heuristic counter-example to each PMT's assumption in their initial conjecture, namely that the centre point (midpoint) was the only point creating the minimum distance. In essence, it was only the latter assumption that was contradicted and not conjecture the student actually claimed. This invariably caused each PMT to refine her/his initial non-*Sketchpad* conjecture (see De Villiers, 2004; Houston, 2009; Lakatos, 1976). However, the latter corroborating empirical examples did not act as counter-examples anymore, but instead influenced each PMT to accept the heuristic counter-example to their the assumption encompassingtheir initial non-*Sketchpad* conjecture with some degree of surprise (see Komatsu, 2010).

**Task 2(b): Using Sketchpad to develop a conjecture related to a rhombus**

Open the sketch Rhombus.gsp.

1. Press the button to show the distance sum. Drag point *P* around the interior of the rhombus. What do you know about the sum of the distances?

2. Drag vertices A, B or D of the rhombus to change the rhombus's size or shape. Again, drag point P around the interior of the rhombus. What do you notice?

3. Write your discoveries as one or more conjectures. Use complete sentences.

Figure 8.1.2.1: Rhombus Task 2(b)

The experienced contradiction appeared to have challenged the PMTs' established schemata and disturbed their cognitive equilibrium, i.e. it brought about cognitive disequilibrium, which is commonly known as cognitive conflict. In an attempt to resolve their internal cognitive conflict, the PMTs modified their initial non-*Sketchpad* conjecture and hence produced a new conjecture, which embraced point *P* as a point that could be located anywhere within the rhombus, since the sum was always constant as per their experimental observations (see Lee et al, 2003; Piaget, 1978, 1985). Hence, the heuristic counter-example functioned as a driving force for the PMTs to construct a more embracive conjecture ,which can be considered to be a new conjecture in a sense.

Through further dragging, each PMT successfully validated his/her newly constructed (or modified) *Sketchpad* conjecture. On the basis of their new *Sketchpad* conjecture, which was true for some cases, and having validated it  for new cases, each PMT went on hypothesize that the newly constructed conjecture was true in general (i.e. they made their conjecture a generalization). As each PMT was able to generalize his/her conjecture, it thus seems that through the process of accommodation each PMT adapted his/her cognitive structure, namely the ‘rhombus schemata’, to the new idea (i.e. PMTs changed their cognitive structure to fit their new *Sketchpad* experience), and thereby re-established his/her cognitive equilibrium (see Berger, 2004; Piaget, 1978, 1985).

Furthermore, the PMTs’ modification (or abandonment in a sense) of their initial non-*Sketchpad* conjecture and sub-sequent construction of a new *Sketchpad* conjecture to capture their new experiences, is consistent with constructive perspective on learning and its endorsement of cognitive conflict as a key driver for conceptual change (see Ausubel, 1968; di Sessa, 2006; Biemans & Simons, 1999; Duit, 1999; Lee et al., 2003).

The following task-based interview excerpts, which are presented case-wise, are representative of the PMTs’ aforementioned Sketchpad experiences and moves that made it possible for each of them to construct their own conjecture generalization:

**Case: Trevelyan**

The Researcher got Trevelyan immersed into the Sketchpad rhombus task 2(b), as illustrated in the following excerpts:

| | |
|---|---|
| RESEARCHER: | Okay, let’s open the sketch. Open the *Rhombus Distance.gsp.* That’s point *P* there. Click the button to show the perpendicular segments. You can see the perpendicular segments. |
| TREVELYAN: | Ja, I can. |
| RESEARCHER: | What are they? |
| TREVELYAN: | $h_1$ , $h_2$ , $h_3$, $h_4$ |
| RESEARCHER: | So they represent the … |
| TREVELYAN: | … the distance from point *P* to any side of that quad. |
| RESEARCHER: | And what is the sum at the moment, there? |
| TREVELYAN: | The sum is 13.730 cm |
| RESEARCHER: | Investigate and see where point *P* can be located to give the minimum distance sum to the sides of the rhombus. Drag point *P* around the interior of the rhombus. *(In retrospect, the Researcher should have asked the PMT here: What do you notice?, before following up with the next instruction).* Drag vertex A. |
| TREVELYAN: | - to? |
| RESEARCHER: | Outwards. Stop there. Drag point *P* and see what happens. |

You can drag vertex *B* if you want to.

After manipulating the situation dynamically through a continuity of cases, including dragging some of the vertices, made possible by the dragging feature, the researcher requested Trevleyan to refer to his observations, and then write down his discoveries as one or more conjectures. When the Researcher asked Trevelyan to read out what he wrote, he responded in a rather surprising tone as follows:

TREVELYAN: What I have is: I have noticed that if one drags one vertex of a rhombus then the sum ceases to be constant. But if you drag a point inside the rhombus, then the sum of the distances is constant.

The element of surprise contained in Trevelyan's response could possibly be attributed to him realizing that the empirical examples constructed through dragging point *P* around the interior of the rhombus, were actually contradicting his assumption governing his initial non-*Sketchpad* conjecture, namely that the point *P* should be at the centre. Although Trevelyan was faced with a set of examples that contradicted his underlying assumption that shaped his initial non-*Sketchpad* conjecture, it is quite plausible that the first contradicting example acted as a heuristic example, and the latter empirical examples supported the heursitc example to an extent that he had to accept the heuristci example. It is plausible that the encountered heuristic-counter example and corroborating empirical examples would have stirred up some cognitive conflict within his cognitive structure, namely the 'rhombus schemata', and consequently disturbed his cognitive equilibrium. This state of cognitive conflict must have been the inner force that pushed Trevelyan to refute his initial non-Sketchpad conjecture and construct his new *Sketchpad* conjecture, namely: "if you drag a point inside the rhombus, then the sum of the distances is constant".

**Case: Inderani**

The researcher engaged Inderani with the *Sketchpad* Rhombus task 2(b), as illustrated in the excerpts below.

RESEARCHER: Let us open the *Rhombus.gsp*.

Press the button to show the distance sum. Drag point *P* around the interior of the rhombus. What do you know about the sum of the distances?

INDERANI: So, I notice that the sum of the distances remains constant no matter where the point in the rhombus is. (Inderani had a surprised look on her face)

It seems that through dragging point *P* around the interior of the rhombus within a Sketchpad context, Inderani continuously witnessed that the sum of distances remained constant for different positions of point *P*, and this could have surprised her. It is quite plausible that the observation of the aforementioned regularity did not reconcile with her assumpions in her earlier non-*Sketchpad* conjecture and would have created some perturbation in the PMT and disturbed her cognitive equilibrium (see Hadas, Herskowitz and Schwartz, 2000). In retrospect, the first empirical example could have served as a heuristic counter-example, and the remaining empirical examples thereafter could merely have strengthened the case of the heuristic counter-example.

To enable Inderani to validate her observation of the invariant property, namely the sum of the distances is constant irrespective of the location of point *P* within the rhombus, the researcher requested Inderani to continue with her experimental exploration as follows: "Drag the vertices *A*, B or *D* of the rhombus to change the rhombus's size or shape. Drag point *P* around the interior of the rhombus again. What do you notice?" After experimenting as per researcher's request, Inderani convincingly responded as follows:

INDERANI: No matter what size or shape the rhombus is, the sum of the distances still remains the same from the sides to the point.

The further empirical examples constructed by dragging point *P* within each of the newly re-sized rhombi, supported Inderani's earlier observed regularity (or counter-example and corroborating examples), and this may have assisted her to convincingly accept and reaffirm her observed regularity (or counter-example status). It is quite apparent that after the latter set of experiences and further corroborating empirical examples, the PMTs would have reconstructed their existing 'rhombus schemata' to accommodate the new idea, and thereby achieved their cognitive equilibrium (see Piaget, 1978, 1985).

When the researcher requested Inderani to write down her discoveries as one as one or more conjectures, using complete sentences, she produced the following written response in her worksheet:

In a rhombus, if a point P is constructed anywhere in the interior of the rhombus, the sum of the distances from the sides to the point P will remain constant.

Figure 8.1.2.2: Inderani’s Rhombus Conjecture Generalization

Thereafter, the researcher asked Inderani as to whether her conjecture was in made with reference to any rhombus or not, and Inderani responded as follows: “that’s for any rhombus”.  Although Inderani generalized her conjecture to any rhombus, she did not refer to perpendicular distances and hence her formulation of her conjecture was inadequate. Furthermore, her generalized conjecture was written in a way that spoke about the distances from the sides to the point, which actually is the wrong order. The point is the variable and the distances are drawn from it perpendicular to the sides. Confusing the order, as Inderani did , could have been influenced by her past experience of perpendicular bisectors where the midpoint of side is the starting point, and the perpendicular is drawn to the side at the midpoint.. However, it is not the case with altitudes where perpendiculars are dropped from the vertices to the sides. It might also be that this student did not regard the order as important and believed it be be “commutative’. Thus, the source of this apparent misconception, if any, was  not clear.

**Case: Logan**

The researcher engaged Logan with the *Sketchpad* Rhombus task 2(b), as illustrated in the following excerpts:

| | |
|---|---|
| RESEARCHER: | Let’s move on to Task 2(b) which says, ‘Using *Sketchpad* to develop a conjecture related to a rhombus. So open the sketch *Rhombus.gsp*.’ Press the button to show the distance sum. What is the distance sum there? |
| LOGAN: | The distance sum there is 13.73cm. |
| RESEARCHER: | Now, drag point *P* around the interior of the rhombus. What do you notice about the sum of the distances? |
| LOGAN: | I can see that it remains constant. It doesn’t change. The distance sum remains the same – nothing changes. *(Logan sounded surprised)* |

After dragging point *P* around the interior of the rhombus, Logan had an expression of surprise on his face, more so when he announced his observation of the following invariant property: the sum of the distances from point *P* to the sides remained the same. It seemed that he was surprised to see a different result to that which he expected to see. This noted contradiction may have created some cognitive conflict in his mind, and disturbed his cognitive equilibrium.

To test whether his claim was valid for new particular cases, he changed the rhombus by dragging a vertex, and when the Researcher asked Logan, "What do you notice?" Logan replied as follows:

> "Okay, what I've noticed is that if I drag the one vertex – in this case I dragged the vertex *B*. And then the total distance sum it actually changes because it was 13,72 and now it's 14,239, but when I went back to point *P*, the point indicated inside the rhombus, the total sum – the distance sum – didn't change. Which means if you draw the vertex, the sum changes, the distance changes, but if you drag the point inside the rhombus, the distance remains the same, is constant, doesn't change even though the individual distance to sides like from point *A* to *D* might change if you drag point *B* around, but the whole distance sum remains constant."

From, the above response of Logan, it is quite evident he validated the invariant property through empirical investigation. Hence, the Researcher requested Logan to write down his discoveries as one or more conjectures, using complete sentences, which he eventually read out very confidently as illustrated in the following excerpts:

LOGAN: What I discovered is that if you locate and draw a point inside rhombus, and drag that particular point anywhere inside the rhombus, the distance sum to all the sides remains the same. However, if you drag any vertex of the rhombus, there is a change in the distance sum.

Thereafter, the researcher asked Logan to compare his latest conjecture with his initial non-*Sketchpa*d conjecture, and then state his final response with regard to his CG. Indeed, very confidently, Logan replied, "It doesn't matter where you locate that point inside the rhombus, the distance sum will remain constant – it won't change". Its seems that the empirical examples assisted Logan to reconsider and modify his initial non-*Sketchpad* conjecture about the location of the point *P* . Furthermore, the aforementioned response by Logan, suggested that through the process of accommodation he could have corrected his initial perception, namely point *P* should be located closer to side *AD*, and thereby adapted his so called 'rhombus' schemata to his newly discovered idea, and consequently achieved his cognitive equilibrium.

#### 8.1.2.1 Findings as per Section 8.1.2: Empirical induction form dynamic cases

1. Through experimental exploration within a *Sketchpad* context, the six PMTs experienced (lived through) an empirical example that served as a heuristic counter-example to their their assumptions governing the development of their initial non-*Sketchpad* conjecture. Through further dragging they experienced several corroborating empirical examples, which not only surprised them but forced them to accept the encountered heuristic counter-example. It seems that the experienced contradictions caused some internal cognitive conflict within the mind of each of the PMTs. and consequently disturbed their cognitive equilibrium. In an effort to restore their cognitive equilibrium, the PMTs were forced to modify their initial non-*Sketchpad* conjecture and hence construct a new conjecture. Each PMT's new conjecture appears to affirm that point *P* could be located anywhere inside a rhombus since it gives a constant sum of the distances to all sides of a rhombus.

2. Through further empirical testing, all six PMTs validated their conjecture for new particular cases, which were constructed by first changing the size of the rhombus by dragging vertices, *A*, *B* or *D* and then dragging point *P* within the newly sized rhombus, and consequently generalized their conjecture, i.e. made their conjecture generalization.

### 8.1.3 Producing a conjecture generalization through superficial analogical reasoning and did not require experimental confirmation with GSP.

As discussed in Sections 2.1.3 and 4.4, learning by analogy is the mapping of knowledge from one domain (i.e. the base domain) over to the target domain, where it could be applied to solve problems, develop concepts further, discover new ideas or just understand new mathematical ideas (see Alwyn & Dindyal, 2009; Gentner, 1983,1989; Polya, 1954a; Vanlehn, 1986). In the case of the Rhombus problem, two PMTs, Shannon and Alan, seemed to have produced their conjecture generalization by means of analogy from the previous equilateral triangle example, and did not require experimental exploration to confirm (or validate) their conjecture generalization.

Such development of conjecture generalizations via analogy-making has been supported and encouraged by both Polya (1954a) and Lakatos (1976) for fruitful mathematics learning to occur, and also has assisted mathematicians to discover new mathematical concepts and methods to solve problems (Lee & Sriraman, 2011). The following task-based interview excerpts, which are presented case-wise, are representative of the PMTs responses that suggest they were conjecturing through the possible use of analogical reasoning but at a superficial level. The reason for this, is that as reported later in Section 8.3, it is clear that their analogical generalization was fairly superficial as they needed an explanation and could not see the same argument as for the equilateral triangle would apply in this instance.

## Case: Shannon

The following excerpt brings evidence for how Shannon produced her conjectured generalization:

RESEARCHER: …In the previous session you observed, conjectured and logically explained that an in an equilateral triangle, the sum of the distances from a point inside the triangle to its sides is constant. You proved that. Now, I want you to look at this activity (Referring to the initial Rhombus task – see Appendix 2)

SHANNON: Well, I think, using my knowledge from the previous exercise, that you would probably still have a point *P*, …uum… anywhere within the sketch …it we draw a line from *P*… point *P* can be anywhere. And it will still minimize the sum of distances from all sides.

RESEARCHER: Are you sure?

SHANNON: Yes.

It seems that the act of reflecting on her experiences with the prior equilateral triangle activity and thereby seeing the similarity between the equilateral triangle case and the rhombus, enabled her to articulate with a good degree of certainty that point *P* can be located anywhere in the rhombus, since the sum of the distances to the sides of the rhombus is constant. In retrospect, it is plausible to say that when Shannon was confronted with the rhombus problem, she managed via analogy-making to first assimilate it into her previously accommodated idea for the equilateral triangle case, which in turn was further modified to accommodate the rhombus case. The aforementioned linking and anchoring of the new information, namely rhombus data, firstly to relevant aspects of her existing cognitive structure, namely the 'equilateral triangle' schemata, is consonant with one of the processes of meaningful learning as propounded by Ausubel's Assimilation theory, namely Subsumption (see Section 4.5).

Ssee section 8.4.4 for a continuation of the analysis of the one-to-one task-based interview responses from Shannon).

**Case: Alan**

When the researcher presented Alan with a hardcopy sketch of rhombus *ABCD* and asked him where he should locate point *P* in the rhombus *ABCD* so that the sum of the distances to all four sides of the rhombus is a minimum, he responded as follows:

ALAN: .. I can locate it anywhere inside the rhombus.… I locate point *P* inside the Rhombus (he was pointing across the interior of the rhombus).

It seems that when Alan was posed with the rhombus problem, he saw the analogical similarity with equilateral triangle, and was thus able to assimilate the rhombus case into the previously accommodated equilateral triangle idea. After that it seems that he was able to make his conjecture generalization by merely modifying his previous accommodated equilateral triangle idea to accommodate the rhombus.

In retrospect, I as the researcher should have asked both Shannon and Alan to explain why they thought that the point *P* can be located anywhere, as hearing this answer is really crucial to knowing whether they had really used analogy from the equilateral triangle case. However, the researcher did not ask tthem to explain why?, and I, as the researcher, acknowledge this as a shortcoming on my part.

#### 8.1.3.1 Finding as per subsection 8.1.3: Superficial analogical reasoning and no experimentation

1. Two PMTs were able to generalize Viviani result pertaining to the equilateral triangle to the rhombus by superficially seeing an analogical similarity with the equilateral triangle case, and did not require experimental confirmation with *GSP*.

### 8.1.4 Producing a conjecture on logical grounds and then requiring experimental confirmation in GSP context

1. None of the PMTs produced a conjecture on logical grounds and then required visual confirmation in a *GSP* context.

### 8.1.5 Producing a conjecture generalization immediately on logical grounds

1. None of the PMTs argued by using a logical explanation to produce a conjecture generalization (or generalization).

## 8.2 Level of Certainty expressed by PMTs in their conjecture generalizations

In this section, the level of certainty expressed by the group of six PMTs, who constructed their conjecture generalization through exploration are discussed as well as the two PMTs who constructed their conjecture generalization via analogy-making, are discussed in Sections 8.2.1 and 8.2.2 resepctively. Section 8.2.3 present the findings that emerged as per data analysis regarding PMTs' level of certainty in their rhombus conjecture generalization.

### 8.2.1 Level of certainty expressed by PMTs that constructed their CG by empirical induction from dynamic cases

All six PMTs, who constructed their conjecture generalization for the rhombus through experimentally exploring with several dynamic cases within a *Sketchpad* environment, expressed a hundred percent certainty in their conjecture generalization. The following two cases, namely Trevelyan and Inderani are representative of typical responses from this group of PMTs:

**Case: Trevelyan**

When the researcher asked Trevelyan "How certain are you that your conjecture is always true?" he replied "I'm sure – 100 percent sure." When the researcher probed his high level certainty in his conjecture generalization, he reaffirmed his certainty in his conjecture by responding as follows: "I'm sure that my conjecture is right. Okay, the sum of the distances in a rhombus is constant irrespective of where *P* is in the rhombus". The latter response by

Trevelyan suggests that he believes that there is no counter-example to his newly constructed (modified) conjecture generalization.

**Case: Inderani**

With regard to her conjecture generalization, Inderani also expressed a high level of certainty because she had tried many different cases and found no counter-examples. The following one-to-one task-based interview excerpt is representative of Inderani's response:

| | |
|---|---|
| RESEARCHER: | How certain are you that your conjecture is always true? Record your level of certainty on the number line and explain your choice. |
| INDERANI: | Let's say one hundred. |
| RESEARCHER: | A hundred percent? |
| INDERANI: | Hundred percent because I tried different shapes and sizes and the sum of the distances remains the same. |

### 8.2.2 Level of certainty expressed by the two PMTs that constructed their CG via analogy (superficially) with the equilateral triangle case:

The two students, Alan and Shannon, who seemed to have made their generalizations to the rhombus on analogical grounds, did not express any desire to empirically test the validity of their analogical generalizations (or conjecture generalizations), but they did express high levels of certainty in their analogical generalizations, as illustrated in the case of Alan.

**Case: Alan**

| | |
|---|---|
| ALAN: | I can locate it anywhere inside the rhombus… |
| RESEARCHER: | Are you certain about that? |
| ALAN: | Yes. |
| RESEARCHER: | How certain are you? |
| ALAN: | I'm hundred percent. |

Immediately thereafter, the researcher posed the following question to Alan, "If you suspect your conjecture is not always true, can you give me a counter-example". Alan responded very confidently (and almost arrogantly): "But I've recorded my level of certainty as a hundred percent so, there's no way that I can …".

### 8.2.3 Findings as per Section 8.2: Level of Certainty

1. All six PMTs, who constructed their conjecture generalization for the rhombus through experimentally exploring with several dynamic cases within a *Sketchpad* environment, expressed a hundred percent certainty in their conjecture generalization. It seems that the confirming visual empirical examples provided the necessary warrants that enabled all six PMTs to become absolutely convinced that there were no counter-examples to their conjecture generalization.
2. The two PMTs, who seemed to have made their generalizations to the rhombus on analogical grounds, did not express any desire to empirically test the validity of their analogical generalizations (or conjecture generalizations), but did express high levels of certainty in their analogical generalizations.

## 8.3 PMTs' need for an explanation

An explanation helps an individual to make sense of a mathematical result, which he/she could have obtained through conjecturing and generalizing. In other words, an explanation could provide a student with a psychological sense of illumination or insight as to why his/her conjecture generalization is true (see Schoenfeld, 1985, p. 172; de Villiers, 1999, p. 7). In retrospect, the experience of a contradiction by a student to one of his/her conjectures or previously known results could cause some cognitive conflict within his/her mind, and this could spark an interest in him/her and cause him/her to want to understand "why?" (see for example Hadas, Hershkowitz, & Schwartz, 2000; Movshovitz-Hadar, 1988; Zaslavsky, Nickerson, Stylanides, Idron, & Landman, 2012). In order to sharpen their understanding of "why?", students search for explanations for their observed results.. According Piaget (1975) as cited in Balacheff (1991, p. 89), this searching for explanations propels the construction of new knowledge in the minds of the learner, which is consistent with the constructivist theory of learning.

Thus, after each of the six PMTs (as reported in Findings 1 and 2 of Section 8.1.2.1 and Finding 1 of Section 8.2.3) developed their conjecture generalization for a rhombus by means of empirical induction from dynamic cases in a *GSP* context, the researcher asked each of them if they needed an explanation as to why the sum of the distances from any point *P* inside a rhombus to its sides remained constant (i.e. if they wanted know why their conjecture generalization is always true). All six PMTs expressed a desire for an explanation, and it is quite plausible that such a desire might have arisen as a result of them being surprised to find that the sum of the distances from point *P* to the sides of the rhombus were always remaining constant. This experienced contradiction could have challenged the students established ‘rhombus’ schemata, and disturbed their cognitive equilibrium, i.e. caused cognitive conflict. It is plausible, that this experienced cognitive conflict could have sparked an interest in the students to want to know “why” the sum of the distances remained constant in all dragged cases.

The following excerpts, presented case-wise, are representative of the responses of the aforementioned group of six PMTs (i.e. Group B):

**Case : Inderani:**

RESEARCHER: If you are fully convinced of the truth of your conjecture, do you still have a need for an explanation (i.e. do you want to know why it is true?)

INDERANI: Okay. Yes.

**Case: Victor**

RESEARCHER: Do you still have a need for an explanation?

Do want to know why it is true – why the result is always true?

VICTOR: Yes!

**Case: Renny**

RESEARCHER: Right. If you are fully convinced of the truth of your conjecture, do you still need an explanation?

RENNY: Ja, I want to know why it’s true.

Furthermore, the two PMTs, Shannon and Alan, who developed their conjecture generalization by means of analogy with the equilateral triangle (see Finding 1 of Section 8.1.3.1 and Finding 2 of Section 8.2.3), also expressed a desire for an explanation as to why their conjecture generalization is always true. Judging by this, it seems that their analogical generalization was rather superficial and not on logical grounds; in other words, seeing immediately that the same argument for the triangle would apply to the rhombus. The following response by Alan is representative of these two students’ need for an explanation:

**Case: Alan**

Alan expressed absolute certainty in his CG, which states that point *P* can be located anywhere inside the rhombus.

| | |
|---|---|
| RESEARCHER: | Are you sure. Okay, if you're fully convinced of the truth of your conjecture, do you still have a need for an explanation for why it is true? |
| ALAN: | Ja, I would like to know. |

#### 8.3.1 Finding as per Section 8.3 (PMTs' need for an explanation)

1. The six PMTs, who developed their conjecture generalization by means of empirical induction from dynamic cases in a GSP context, as well as the two PMTs who did so by analogy-making between the rhombus and equilateral triangle, expressed a desire and a need for an explanation as to why their conjecture generalization is always true.

## 8.4 Justifying a Conjecture Generalization

As discussed in Section 8.1 there were the two PMTs, Shannon and Alan, who were able to initially construct a conjecture generalization, namely point *P* can be located anywhere inside the rhombus to minimize the sum of the distances to the sides, whilst being restricted to work in a non-*Sketchpad* context as per Task 2(a). It appears that they could have generalized this result to the rhombus by means of analogy having seen the similarity with the equilateral triangle case (see Finding 1 of Section 8.1.3.1 and Finding 2 of Section 8.2.3). These two PMTs constructed a logical explanation for their conjecture generalization on their own accord by using the 'distance between parallel lines is constant' algebraic strategy. However, both of them were also able to provide an alternative explanation by seeing that the structure of the proof for the equilateral triangle case wherein the 'triangle-area' algebraic strategy was used could also be similarly applied to the rhombus case (see Section 8.4.4 for presentation of data and analysis).

On the other hand, there were six PMTs, Trevelyan, Tony, Inderani, Victor, Logan and Renny, who did not initially conjecture that point *P* can be located anywhere inside the rhombus whilst limited to a non-*Sketchpad* context. Hence these latter six PMTs were given an opportunity to experimentally explore their initial conjecture in a *Sketchpad* context.

Through experiencing the first empirical example, which served as a heuristic counter – example to their assumption that there was only one point creating the minimum distance as described in their respective non-*Sketchpad* initial conjectures, and also experiencing through dragging further supporting empirical examples to the heuristic counter-example, the six PMTs modified (refined) their initial conjecture and thereby produced a new conjecture, which indicated that point *P* could be located anywhere inside the rhombus since the sum of the distances to the sides was constant. Through further experimentation the PMTs validated and generalized their newly constructed conjecture. For the purpose of the presentation of the data in this section, the former two PMTs (Shannon and Alna) will be considered as belonging to group A and the latter six PMTs (Trevelyan, Tony, Inderani, Victor, Logan and Renny) to group B.

Three PMTs from group B, Tony, Inderani and Logan, were not able to develop a logical explanation on their own accord and were therefore provided with scaffolded guidance as presented in Section 8.4.1. Two PMTs from group B, Victor and Renny, managed to construct a logical explanation by using the conception that distance between the parallel sides are constant. Although these two PMTs were able to construct a logical explanation for their conjecture generalization, they were also provided an opportunity to develop an alternative explanation that employed the similar kind of 'triangle-area' algebraic strategy that was used for the equilateral triangle case, but with the aid of scaffolded guidance (see Section 8.4.2 for the presentation regarding Victor's and Renny's justifications). One PMT from group B, Trevelyan, spontaneously realized that a proof structure similar to the 'triangle- area' proof structure, which was used to construct a logical explanation for the equilateral triangle case, could be used to construct a logical explanation for the rhombus case, and hence proceeded to construct his logical explanation with no guidance (see Section 8.4.3 for discussion).

### 8.4.1 Deductive Justification for Rhombus Empirical Generalization through scaffolded guidance

This section describes the development of a logical explanation (deductive justification) through scaffolded guidance for a rhombus conjecture generalization that was produced by empirical induction from dynamic cases in *GSP* context.

One way of supporting PMTs' constructions of logical explanations, particularly when they are faced with challenges, may be through the provisioning of external scaffolding (compare Belland et al., 2008, p. 407). As discussed in Section 4.2.2, scaffolding is the support provided by an educator, lecturer, peer or other source to enable students to make progress with tasks, which they are unable to do solely on their own, i.e. independently (Wood, Bruner, & Ross, 1976). According to Puntambeka & Hubscher (2005) as cited in Belland et al. (2008, p. 408), "The goal of scaffolding is two-fold: first, to provide temporary support to students as they perform tasks that they have difficulty performing unaided, and second, to help students gain competency in the scaffolded tasks such that they can perform the tasks unaided."

With regard to the latter goal, the researcher is of the view that scaffolding should go beyond just developing competency. In particular, scaffolding should provide an opportunity for students to develop the necessary insight and understanding why a particular result is true, so that they can solve similar problems or extended problems via analogical transfer. Further to this, it seems the weakness as illustrated by the PMTs apply more to the first task involving the triangle. Here their weekness seems to be in not seeing the connection between the rhombus and the triangle, and thinking of trying the same reasoning strategy. On of Polya's (1985) heursitics is: Do I know how to solve a related problem? Clearly more than anything, they were not about the logical explanation for the equilateral triangle spontaneously, and trying to see if the same approach workd here.

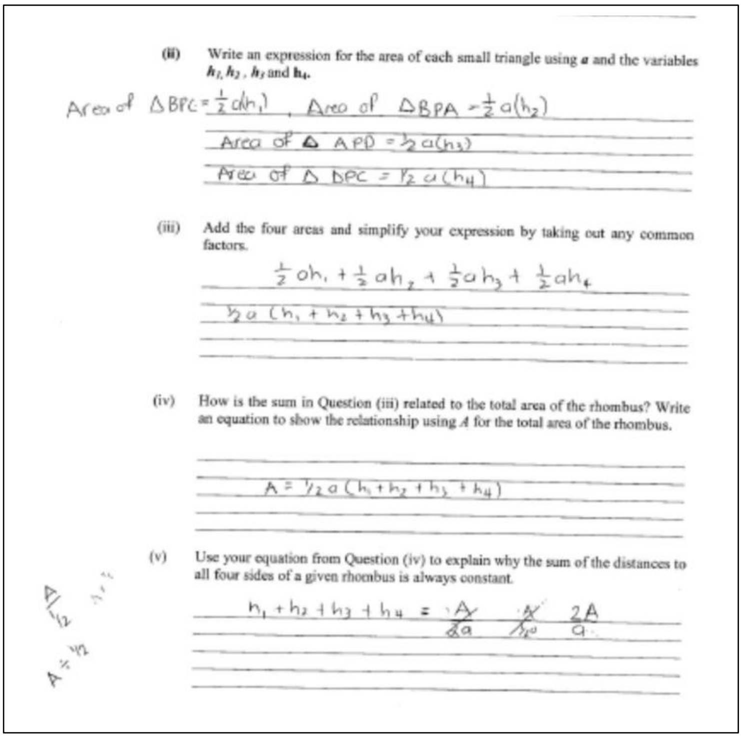
During the one-to-one task-based interviews, three PMTs, Tony, Inderani and Logan were not able to construct a logical explanation on their own to support their conjecture generalization. The nature of the scaffolding that the Researcher provided to these PMTs took the form of both soft and hard scaffolds. The hard scaffolds comprised the structured worksheet linked to the dynamic rhombus sketch within a Sketchpad context. The question prompts, which were outlined on the worksheet as shown in Figure 7.11, served as procedural guidelines and were designed to make the PMTs think about specific issues, concepts, ideas and build mini arguments (or assertions) that could finally contribute towards the development of a coherent logical explanation.  In particular, the worksheet was designed to assist each PMT to construct an explanation (justification) in a logical and sequential manner that supports their conjecture generalization.

Although PMTs had the scaffold questions, which were related to the dynamic sketch, the Researcher also provided soft scaffolding whenever he noted that a PMT was struggling to respond positively when s/he was probed during the one-to-one task-based interview. Also through differentiated probing during the one-to-one task-based interview, the Researcher was able to diagnose what additional support a specific PMT required in order to be able to accomplish a sub-task at hand, he then provided just the right amount of support at the appropriate time to enable a specific PMT to successfully move on with the development of his/her explanation.

However, an exposition of the use of scaffolding that enabled the PMTs, Inderani, Tony and Logan, to construct a coherent logical explanation is presented case-wise.

**Case: Inderani**

Inderani showed understanding as to why each of the sides of the given rhombus was labelled with the variable *a*, and proceeded to complete her worksheet as shown in Figure 8.4.1.1 with the help of limited guidance and assistance by the tesearcher.



(ii) Write an expression for the area of each small triangle using *a* and the variables $h_1, h_2, h_3$ and $h_4$.

Area of $\Delta BPC = \frac{1}{2}a(h_1)$, Area of $\Delta BPA = \frac{1}{2}a(h_2)$

Area of $\Delta APD = \frac{1}{2}a(h_3)$

Area of $\Delta DPC = \frac{1}{2}a(h_4)$

(iii) Add the four areas and simplify your expression by taking out any common factors.

$\frac{1}{2}ah_1 + \frac{1}{2}ah_2 + \frac{1}{2}ah_3 + \frac{1}{2}ah_4$

$\frac{1}{2}a(h_1 + h_2 + h_3 + h_4)$

(iv) How is the sum in Question (iii) related to the total area of the rhombus? Write an equation to show the relationship using *A* for the total area of the rhombus.

$A = \frac{1}{2}a(h_1 + h_2 + h_3 + h_4)$

(v) Use your equation from Question (iv) to explain why the sum of the distances to all four sides of a given rhombus is always constant.

$h_1 + h_2 + h_3 + h_4 = \frac{A}{\frac{1}{2}a} \quad \frac{2A}{a}$

$A/\frac{1}{2}$

$A \div \frac{1}{2}$

Figure 8.4.1.1: Inderani’s scaffolded response

Thereafter, the researcher requested Inderani, to reflect on the above scaffolded steps, and then summarize her explanation /justification of her conjecture generalization. It was encouraging to note that Inderani was able to produce a well justified coherent argument in her worksheet as illustrated in Figure 8.4.1.2

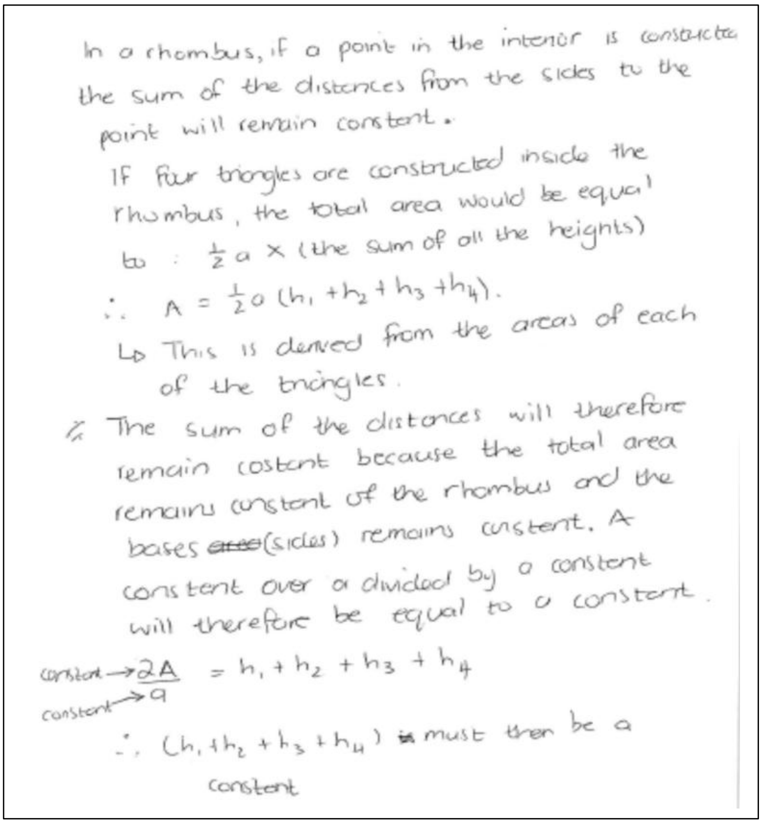

In a rhombus, if a point in the interior is constructed the sum of the distances from the sides to the point will remain constant.
If four triangles are constructed inside the rhombus, the total area would be equal to : $\frac{1}{2}a \times$ (the sum of all the heights)
$\therefore \quad A = \frac{1}{2}a(h_1 + h_2 + h_3 + h_4)$.
↳ This is derived from the areas of each of the triangles.
$\therefore$ The sum of the distances will therefore remain costant because the total area remains constant of the rhombus and the bases ~~area~~ (sides) remains constent. A constent over a divided by a constent will therefore be equal to a constent.
$\underset{\text{constent}}{2A} \,/\, \underset{\text{constent}}{a} = h_1 + h_2 + h_3 + h_4$
$\therefore (h_1 + h_2 + h_3 + h_4)$ ~~is~~ must then be a constent

Figure 8.4.1.2: Inderani’s Justification of Rhombus CG

**Case: Logan**

When the researcher asked Logan, “ Can you support your conjecture with a logical explanation or justification?”, he responded as follows:

LOGAN: The way I see it is because of the lines drawn from point *P* to the respective sides of the rhombus are perpendicular – I think this is why the distance sum remains the same. And because, the shape of… even though you drag point *P* inside the rhombus, the initial shape of the thing remains the same. There’s no change. That is why the distance sum remains the same – because the shape of the rhombus remains constant.

Logan attempted to provide an explanation, by merely describing some of his observations or some properties of the figure which in fact was not complete. Hence, the researcher, gave Logan scaffolded guidance via the use of the Rhombus worksheet as described in Appendix 2, to enable him to develop an explanation for his conjecture generalization. Figure 8.4.1.3

represents what Logan wrote on his scaffolded worksheet:

**(ii)** Write an expression for the area of each small triangle using $\boldsymbol{a}$ and the variables $h_1, h_2, h_3$ and $h_4$.

AREA = ~~$\frac{1}{2}$ a·h₁~~ AREA OF $\Delta BPC = \frac{1}{2} \cdot a \cdot h_1$

AREA OF $\Delta BPA = \frac{1}{2} \cdot a \cdot h_2$

AREA OF $\Delta APD = \frac{1}{2} \cdot a \cdot h_3$

AREA OF $\Delta CPD = \frac{1}{2} \cdot a \cdot h_4$

(iii) Add the four areas and simplify your expression by taking out any common factors.

$\frac{1}{2} \cdot a(h_1 + h_2 + h_3 + h_4)$

(iv) How is the sum in Question (iii) related to the total area of the rhombus? Write an equation to show the relationship using $A$ for the total area of the rhombus.

$A = \frac{1}{2} \cdot a(h_1 + h_2 + h_3 + h_4)$

(v) Use your equation from Question (iv) to explain why the sum of the distances to all four sides of a given rhombus is always constant.

$\frac{2A}{a} = (h_1 + h_2 + h_3 + h_4)$

LHS = RHS

Figure 8.4.1.3: Logan's scaffolded justification of Rhombus CG

When was asked to summarize his  responses as shown in Figure 8.4.1.3 into coherent logical argument., Logan responded as shown in Figure 8.4.1.4 in his worksheet.

**Task 2 (e) Present your Explanation/Justification**

Summarize your explanation/justification of your conjecture (generalization). You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a *Sketchpad* sketch, or some other medium.

IF A RHOMBUS REMAINS FIXED THE AREA OF THE RHOMBUS REMAINS CONSTANT. THE DISTANCE OF THE SIDES ALSO REMAINS CONSTANT. ~~YOU CAN A~~ BY TAKING OUT THE COMMON FACTORS OF ALL THE CALCULATED AREAS OF THE TRIANGLES INSIDE THE RHOMBUS YOU WILL ALSO BE INDICATED THAT THE ~~TOTAL AREA~~ O LHS = RHS, WHICH MEANS THE TOTAL AREA OF THE FIXED RHOMBUS REMAINS CONSTANT. THE RHS ~~RIGHT H~~ INDICATES THE SUM OF ALL THE HEIGHTS AND THE LEFT HAND SIDE INDICATES THE $\frac{1}{2} \cdot a$ WHICH IS $\frac{1}{2}$ TIMES THE DISTANCE WHICH IS THE BASE OF EACH OF THE TRIANGLES.

Figure 8.4.1.4: Logan's Justification of Rhombus CG

### 8.4.2 Deductive justifications for Rhombus Empirical Generalization: Two approaches

This Section discusses two approaches that Renny and Victor used to develop logical explanations (deductive justifications) for their Rhombus conjecture generalizations, which were produced by empirical induction from dynamic cases in *GSP* context (as reported in Section 8.1.2). Firstly Section 8.4.2.1, presents the deductive justification, wherein the idea of the distance between the parallel sides is constant, was seemingly used to logically explain their Rhombus empirical generalization on their first attempt. Then Section 8.4.2.1, presents an alternative kind of deductive explanation that each of these PMTs developed through scaffolded guidance embracing the 'triangle-area' algebraic explanatory steps as outlined in the worksheet contained in Task(2d) of Appendix 2.

#### 8.4.2.1 Deductive Justification for Rhombus Empirical CG by using the conception of the distance between parallel sides is constant

This section discusses the development of a logical explanation (deductive justification) for a conjecture generalization produced by empirical induction from dynamic cases in *GSP* context by using the conception of the distance between parallel sides is constant.

Two of the PMTs from group B, Renny and Victor, developed their logical explanation (deductive justification) for their conjecture generalization, which they produced by empirical induction from dynamic cases in a *GSP* context. They used the conception that the distance between parallel sides is constant through their own initiative, and with no assistance from the researcher during the task-based interview. Although both PMTs provided their own logical explanations, the researcher also provided them with an opportunity to construct an alternative explanation via the use of the 'area-triangle' algebraic strategy. In the latter case, the researcher provided both PMTs with the worksheet as contained in Task 2(d) of Appendix 2.The PMTs were asked to use the worksheet by making use of their onscreen dynamic rhombus sketch. The moves and deliberations that occurred during the one-to one task-based interview with and Victor are presented case-wise as follows:

**Case: Renny**

RESEARCHER: Right. If you are fully convinced of the truth of your conjecture, do you still need an explanation?

RENNY: Ja, I want to know why it's true.

RESEARCHER: Tell me why it's true.

RESEARCHER: Okay, so you're trying to support your conjecture with a logical explanation, right?

RENNY: Yes.

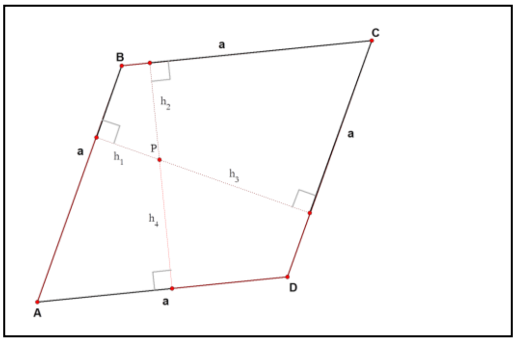
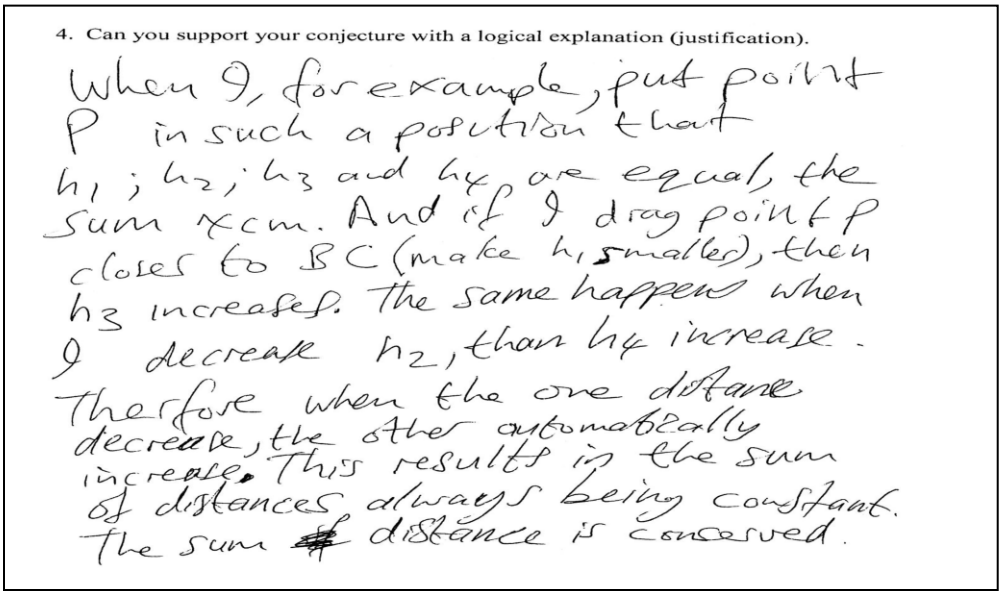



Figure 8.4.2.1 (a): A copy of Renny's onscreen dynamic sketch

With reference to a rhombus sketch like that shown in Figure 8.4.2.1(a), Renny supported his conjecture generalization with a written argument as shown in Figure 8.4.2.1 (b)



4. Can you support your conjecture with a logical explanation (justification).

When I, for example, put point P in such a position that $h_1$; $h_2$; $h_3$ and $h_4$ are equal, the sum x cm. And if I drag point P closer to BC (make $h_1$ smaller), then $h_3$ increases. The same happens when I decrease $h_2$, than $h_4$ increase. Therfore when the one distane decrease, the other automatically increase. This results in the sum of distances always being constant. The sum ~~of~~ distance is conserved.

Figure 8.4.2.1(b): Renny's Justification of Rhombus CG

Renny's explanation was correct because he was referring to distance between parallel sides being constant. With the same argument one could also prove why it is true for a parallelogram, but Renny didn't make that generalization (and the Researcher didn't pursue it further).

After Renny had logically explained why his conjecture generalization (CG) is always true, the Researcher provided him with an opportunity to construct an alternative explanation, which was similar to the 'triangle-area' algebraic explanation that was developed for the equilateral triangle case. In this instance the Researcher provided scaffolded guidance to Renny via the use of a scaffolded worksheet.

## Case: Victor

When asked to support his conjecture generalization through a logical explanation (justification), Victor constructed the argument on his worksheet as shown in Figure 8.4.2.1(c)..

Victor

Since the Rhombus is fixed the distances $h_1, h_2 h_3$ or $h_4$ are not equal but the sum of the distances would be constant because if you move closer to one side the ~~other~~ distance of the other sides increases with a same ~~amo~~ amount as the other side decreases.

Figure 8.4.2.1(c): Victor justification of Rhombus CG

It seemed that Victor had a hunch that his result could be easily explained in terms of opposite sides parallel, and proceeded to construct an explanation along those lines. However, the Researcher, should have explored further to establish what he meant, and in fact ascertaining whether he was referring to the distances between opposite parallel sides being constant, which would be a correct explanation (proof). For example, if one considers Figure 8.4.2.1 (d), then one could provide a logical explanation as described:

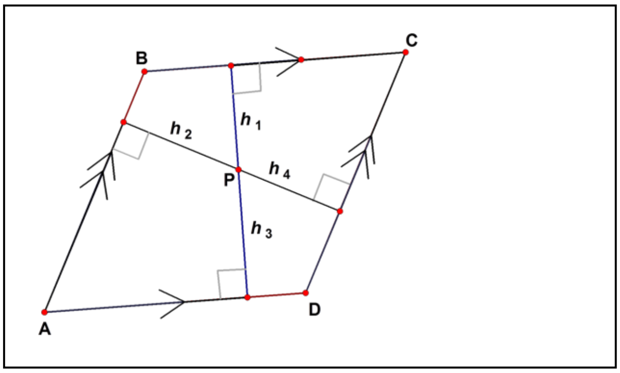



Figure 8.4.2.1(d): Rhombus- distance between parallel lines

$h_1 + h_2 =$ constant …… distance between parallel lines

$\underline{h_3 + h_4 = \text{constant}}$ …… distance between parallel lines

$\Rightarrow \sum h_i$ = constant QED.

The researcher thereafter decided to provide Victor with scaffolded guidance to develop an alternative logical explanation for his CG by using the 'triangle-area' algebraic strategy.

### 8.4.2.2 Alternative deductive justifications for Renny and Victor using the 'triangle-area' algebraic strategy through scaffolded guidance

The researcher provided Renny with scaffolded guidance to develop an alternative explanation for his conjecture generalization. The scaffolded guidance embraced the same kind of steps contained in Task 2(d) of the worksheet in Appendix 2. Renny proceeded through each of the guided steps smoothly and efficiently, and came to also understand why his rhombus conjecture generalization is true through alternatively using the 'triangle-area' algebraic strategy. Hence, when asked during the one-to-one task-based interview to reiterate his explanation as to why $h_1 + h_2 + h_3 + h_4$ were constant by referring to what he wrote on his worksheet, Renny explained as follows, by possibly drawing an analogy with the explanation of the equilateral triangle case:

RENNY: The area will not change (pointing to *A* in his equation)… They will remain constant (pointing to the sides of the rhombus on his dynamic sketch*)*. This will be constant – which is *a*, which is the side, then that will be constant (referring to *a* in his equation). So if that is constant (referring to the number 2 in his equation*)*, then these two are constant (referring to the numerator 2*A* and the denominator *a* of the expression on the right hand side of the equation) and then that means that that side will also be constant (referring to $h_1 + h_2 + h_3 + h_4$) which is on the left hand side of the equation

Thereafter, the Researcher requested Renny to proceed with Task 2(e), which required him to summarize his explanation for his conjecture generalization as an argument. Renny then produced the following argument in his worksheet as shown in Figure 8.4.2.2(a)

**Task 2 (e) Present your Explanation/Justification**

Summarize your explanation/justification of your conjecture (generalization). You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a *Sketchpad* sketch, or some other medium.

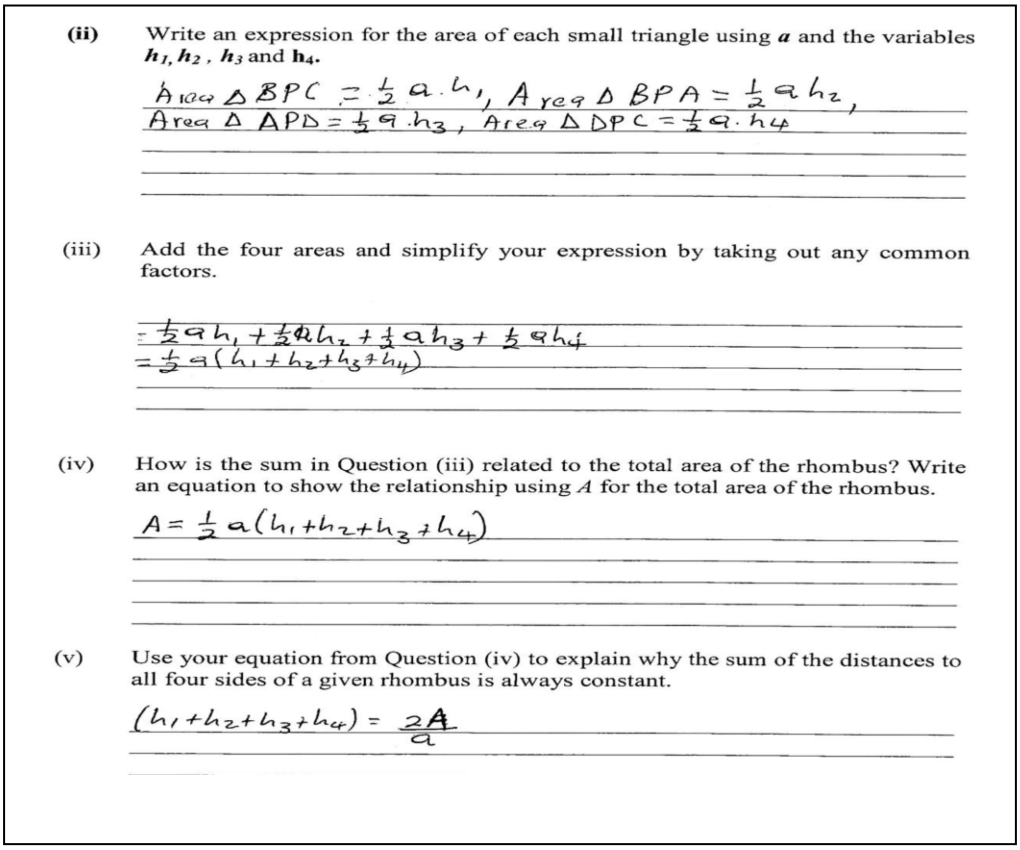
For any rhombus ABCD, with sides a, and point P inside, four triangles can be constructed: triangle BPA, $\Delta BPC$, $\Delta CPD$ and $\Delta APD$. The height of these triangles are $h_2$, $h_1$, $h_4$ and $h_3$ respectively, and the base of all is a.

The total area of the rhombus can be calculated by adding the areas of the four different triangles: $\frac{1}{2}ah_2 + \frac{1}{2}ah_1 + \frac{1}{2}ah_4 + \frac{1}{2}ah_3$. Thus $A = \frac{1}{2}a(h_1 + h_2 + h_3 + h_4)$. Making the sum of the distances the object of the equation: $(h_1 + h_2 + h_3 + h_4) = \frac{2A}{a}$. In the right side of the equation; 2 is a constant, A(area of rhombus) remains constant and "a"(sides of rhombus) remains constant. If the right-hand side of the equation is constant, the left-hand side will also not change. $(h_1 + h_2 + h_3 + h_4)$ will remain constant.

Figure 8.4.2.2(a): Renny's alternate justification of Rhombus CG

**Case: Victor**

The researcher directed Victor to the worksheet containing Task 2(d), and guided him through the worksheet. After discussing the reason for labelling each of the sides of the rhombus with the variable *a*, Victor proceeded through the scaffolded questions (ii) to (v), with a particular sense of confidence and speed.



**(ii)** Write an expression for the area of each small triangle using $a$ and the variables $h_1, h_2, h_3$ and $h_4$.

Area $\Delta BPC = \frac{1}{2} a \cdot h_1$, Area $\Delta BPA = \frac{1}{2} a h_2$,
Area $\Delta APD = \frac{1}{2} a \cdot h_3$, Area $\Delta DPC = \frac{1}{2} a \cdot h_4$

**(iii)** Add the four areas and simplify your expression by taking out any common factors.

$= \frac{1}{2} a h_1 + \frac{1}{2} a h_2 + \frac{1}{2} a h_3 + \frac{1}{2} a h_4$
$= \frac{1}{2} a (h_1 + h_2 + h_3 + h_4)$

**(iv)** How is the sum in Question (iii) related to the total area of the rhombus? Write an equation to show the relationship using $A$ for the total area of the rhombus.

$A = \frac{1}{2} a (h_1 + h_2 + h_3 + h_4)$

**(v)** Use your equation from Question (iv) to explain why the sum of the distances to all four sides of a given rhombus is always constant.

$(h_1 + h_2 + h_3 + h_4) = \frac{2A}{a}$

Figure 8.4.2.2(b) Victor's responses to scaffolded questions

By referring to his equation, $h_1 + h_2 + h_3 + h_4 = \frac{2A}{a}$, as shown in step (v) in Figure 8.4.2.2(b), Victor then went on to complete his explanation as to why $h_1 + h_2 + h_3 + h_4$ are constant as follows during the one-to-one task-based interview with the Researcher:

VICTOR: Okay… Now I have: $h_1 + h_2 + h_3 + h_4 = 2A$ which is the area of the rhombus (*referring to just A and not 2A*), over $a$ which is the side of the rhombus. $A$, which is the area of the rhombus is constant when you drag … [indistinct] …and 2 is the number that is always constant, and $a$ which is the size of the side, is also constant. Therefore, $h_1 + h_2 + h_3 + h_4$, they are also constant.

Thereafter, the researcher requested Victor, to reflect on the above scaffolded steps, and then summarize his explanation /justification of his conjecture generalization. It seems that when Victor was expected to write his explanation as coherent argument, he focused on explaining just one aspect of the envisaged argument instead, as illustrated in Figure 8.4.2.2(c).

**Task 2 (e) Present your Explanation/Justification**
Summarize your explanation/justification of your conjecture (generalization). You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a *Sketchpad* sketch, or some other medium.

In a fix Rhombus ABCD. The area of the R fixed Rambus is constant and the lenght of the sides are also constant of, wich that make the ~~distances~~ sum of the distances from point p constant which are ($h_1$, $h_2$, $h_3$ and $h_4$)

Figure 8.4.2.2(c): Victor's incomplete justification

His written response appeared to suggest that Victor was merely reiterating his response to Q(v) of the worksheet Task 2(d) only, and consequently did not consider assimilating the earlier scaffolded parts into a coherent argument.

### 8.4.3 Deductive justification of Rhombus Empirical Generalization through using analogical reasoning

This section focuses on the development of a logical explanation (deductive justification) for a conjecture generalization produced by empirical induction from dynamic cases in a *GSP* context by noticing that the 'triangle-area' proof structure for the equilateral triangle case can be applied to the rhombus case. This is analogical reasoning.

**Case: Trevelyan**

As outlined earlier (see Section 8.2.1), Trevelyan was fully convinced about the truth of his Rhombus conjecture generalization. When the researcher asked Trevelyan "Can you give me an explanation as to why it (referring to Rhombus conjecture generalization) is true?," he replied, "something like a proof!". Trevelyan's response, confirmed that he knew that there is some link between an explanation and proof, at least that of logical reasoning. For instance, when the researcher acknowledged "yes", he immediately continued that there was a connection between the explanation for the equilateral triangle case and the rhombus case, and that it was plausible to similarly use the 'triangle-area' algebraic strategy to construct an explanation for his conjecture generalization with respect to the rhombus. The following excerpt from the one-to-one task- based interview is representative of Trevelyan's thought processes:

TREVELYAN: Firstly, the sides are constant (referring to the given Sketch on the screen) – all the sides of a rhombus are the same, so that sum (referring to the distances from point P to the sides of the rhombus) should be the same, irrespective of where that *P* is. The connection to the exercise that we did the last time … the equilateral that we were working with… the sides of the equilateral are all the same. Then we observed that if we took a distance anywhere in the equilateral, then the sum would be the same. And we constructed a proof based on that.

RESEARCHER: So what are you saying about the proof for this one?

TREVELYAN: The proof for this one – ja – I think I might use/start the same way, constructing the small triangles inside such that those $h_1$, $h_{2,\ldots}$… are the perpendicular heights of each triangle that I am constructing.

Trevelyan's response indicated that he realized that he must divide the rhombus into small triangles, and use the "triangle-area" algebraic strategy to produce an explanation for his Rhombus conjecture generalization. His response suggests that he has seen the similarity between the equilateral triangle problem and the rhombus problem, and consequently felt that his explanation for his rhombus conjecture generalization could be developed by similarly using the structure of his explanation for the equilateral triangle conjecture generalization.

On proceeding with the one-to-one task-based interview, the researcher requested Trevelyan to click on the button, "Show small triangles". Then with limited probing from the Researcher, Trevelyan provided the following verbal description of his explanation with a great amount of ease and confidence, which indeed bore very close resemblance to the kind of argument that was used for the equilateral triangle case.

RESEARCHER: Okay, click on the sign: 'Show small triangles.'
So here you have your small triangles constructed.

TREVELYAN: Then I can calculate the area of each of these small triangles
and when you add all those areas, it should give us the area of the rhombus. Then if I can find that *H* is constant…

RESEARCHER: How will you explain the fact that *H* is constant?

TREVELYAN: *H* would be constant based on the fact that firstly the area of this whole thing is constant, not so? And also the sides of the rhombus are also constant. Then $H$ must be constant – of which $H$ will be the sum of these distances (referring to $h_1 + h_2 + h_3 + h_4$).

The researcher, realized that Trevelyan had an overview of his explanation (proof), and hence challenged him to write down his final proof on his worksheet.

Suppose we have a rhombus ABCD

R.T.P: we need to show that the sum of the distances from the point interior to a rhombus ABCD to each side is constant.

Construction:

We construct small triangles interior to a ~~rhomb~~ rhombus ABCD such that each distance is the perpendicular height of a small triangle.

Set the distance of each side be $x$ (sides of a rhombus are equal)

<u>Proof</u>

Area $\triangle BPC = \frac{1}{2} x h_1$, Area $\triangle BPA = \frac{1}{2} x h_2$

Area $\triangle APD = \frac{1}{2} x h_3$, Area $\triangle DPC = \frac{1}{2} x h_3$

the sum of the area of these small triangles

Area $\triangle BPC$ + Area $\triangle BPA$ + Area $\triangle APD$ + Area $\triangle DPC$

$\frac{1}{2} x h_1 + \frac{1}{2} x h_2 + \frac{1}{2} x h_3 + \frac{1}{2} x h_4$

$\frac{1}{2} x (h_1 + h_2 + h_3 + 4)$

but we know that the sum of ~~these~~ the area of these small triangles is equal to the area of the rhombus ABCD. If we set the area of rhombus ABCD to be A then

$A = \frac{1}{2} x (h_1 + h_2 + h_3 + h_4) = \frac{x}{2} H$, $H = h_1 + h_2 + h_3$

$H = \frac{2A}{x}$ then since 2, A, and $x$ are constants, thats make the right hand side a constant then ~~if they~~ therefore H is constant.

~~and~~ $H = h_1 + h_2 + h_3 + h_4$ which is the sum of the distances

Therefore the sum is constant □

Figure 8.4.3.1: Trevelyan’s deductive justification for Rhombus CG

Indeed Trevelyan, produced a coherent argument, with a clear structure as outlined in the extract contained in Figure 8.4.3.1. Trevelyan’s response showed that he was successful in producing a coherent explanation for his Rhombus conjecture generalization. In retrospect, it seemed that Trevelyan, having seen a similarity between the equilateral problem and the rhombus problem, was able to reflect on the proof for the equilateral triangle case and see the proof for the rhombus problem via analogical mapping. This cognitive experience by Trevelyan, can illustrate his experience of the discovery function of proof, which is discussed in Section 5.4.

### 8.4.4 Deductive justifications for analogical Rhombus Conjecture Generalization

This Section discusses the development of a logical explanations (deductive justification) for a conjecture generalization produced on analogical grounds. As discussed in Sections 8.1.1 and 8.1.3, two students, Shannon and Alan, seemed to have produced their conjecture generalizations on analogical grounds. Section 8.4.4.1, presents the deductive justification that each of these two PMTs constructed on their own accord by using the idea that the distance between the parallel sides is constant, whilst section 8.4.4.2 presents an alternative kind of deductive explanation that each of these PMTs developed by seeing structure of the proof from the previous equilateral triangle case.

#### 8.4.4.1 Deductive Justification for analogical Rhombus CG using using the conception of the distance between parallel sides is constant

As discussed in Sections 8.1.1, and 8.1.3, Shannon and Alan seemed to have produced their respective conjecture generalizations immediately on analogical grounds, meaning they have seen the similarity with the equilateral triangle case and proceeded to make their conjecture generalizations along the same lines. When asked to justify their conjecture generalizations, each of them proceeded to provide a correct explanation by referring to the distances between the opposite parallel sides being constant with no guidance. So the analogy they 'saw' was not between the equal sides of an equilateral triangle and that of a rhombus, but only about the conclusion (the sumof the distances will also be constant). They could have anticipated or guessed since the sum of the distances were constant for the first case, then it will also be so for the next case (the rhombus). Shannon and Alan's cases are presented with supporting excerpts from their one-to-one task-based interviews to demonstrate the development of their respective logical explanations (or deductive justifications).

**Case: Shannon**

Shannon immediately generalized her result to the rhombus, namely that point *P* can be located anywhere in the Rhombus to minimize the sum of the distances to the sides of the rhombus, probably on analogical grounds, without requiring any visual or experimental confirmation. When the researcher probed Shannon as to how certain she was about her conjecture generalization, she expressed a high level of sureness by stating "quite sure". Moreover, when the researcher posed the following question to Shannon, "Do you want to investigate, or …Can you explain to me why it's true, then?" she chose not to investigate, but instead started to explain why her generalization was true by making reference to her hardcopy (static) Rhombus sketch. Even though the Researcher presented Shannon with a

*GSP* rhombus sketch to clarify her representation of the given distances on her hard copy sketch and enable discussion, it was surprising to note that she still never dragged point *P* to empirically verify/validate her claim, but proceeded to offer her explanation. As demonstrated in the following excerpts, Shannon provided an explanation as to why her claim was true, which was quite different from the structure of the explanation that was used in equilateral triangle case. In particular she did not use the "triangle-area" algebraic strategy, but rather used the "distance between parallel lines" algebraic strategy.

RESEARCHER: Can you explain why it is true? (pause)…

SHANNON: …because these lines are parallel – this line and this line. I know how to say it, but … because this line here is parallel and this line… ow! these two lines here are parallel – *BC* and *AD* is (*sic*) parallel, and *BA* and *CD* is (*sic*) parallel, so it doesn't matter where you move point *P* …uum….$h_2 + h_4$ are always going to be the same. And $h_1 + h_3$ are going to be the same.

RESEARCHER: Ja. We know that. But why is the sum of all four constant? Why is the sum of all four constant? You're quite right. This distance will remain the same.

SHANNON: - because $h_2$ and $h_4$ (it doesn't matter where you move the point P) are always going to be constant, and $h_1 + h_3$ are always going to be constant, it doesn't matter where you move point P. It's always going to be the same.

In presenting her explanation, Shannon provided core ideas of her argument, despite the fact she did not finalize her explanation with the following kind of expected conclusion:
"Therefore, $h_1 + h_2 + h_3 + h_4$ are constant (equal to the sum of the two constant distances between the pairs of opposite sides)."
(Also see section 8.4.4 for Shannon's alternative attempt, using the 'triangle-area' algebraic strategy).

**Case: Alan**

Alan generalized his result to the rhombus probably by means of analogy having seen the similarity with the equilateral triangle, and then proceeded to give a logical explanation (proof) with Sketchpadwithout any experimental exploration. When Alan was asked, "Can you support your conjecture with a logical explanation?” he did not respond immediately *(*he probably was thinking of a strategy on how to develop a logical explanation). After about two - three minutes had passed, he drew the following sketch as shown in Figure 8.4.4.1 on his worksheet. In the development of his logical explanation Alan continuously referred to his sketch as shown in Figure 8.4.4.1, when he presented his explanation during the task-based interview.

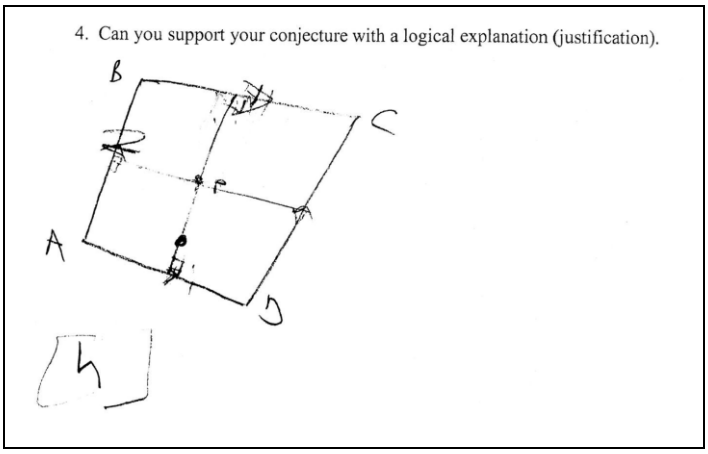



Figure 8.4.4.1: Alan’s working Figure

The excerpts from the task-based interview, illustrate that Alan seemed to be using the property that the opposite sides of the rhombus are parallel, and that the distance between each pair of parallel sides is always constant, irrespective of the position of *P*.

ALAN: …Let’s say, given a rhombus, and we know that all the sides are equal. So, if you draw a line … you have *ABCD* [indistinct] …if you draw a line perpendicular to both …sides …so that means if you have this point *P* anywhere here, this is the shortest distance from point *P* to here and from point *P* to here, even if *P* is somewhere there. This is the shortest distance from there, and then the shortest distance from something there. It’s because if you have a perpendicular line that line will be perpendicular to both two opposite sides. And on this … on this line, the length of this line will remain the same. You can put *P* anywhere on the line and it will always remain the same.

RESEARCHER: That line? (referring to the perpendicular line from P to a pair of opposite parallel sides)

ALAN: Yes. Even for this case, you can put *P* anywhere. The line will remain the same.

RESEARCHER: Yes, I know that is the same, and that is the same – constant. Those facts are correct. But why is the sum of those distances constant?

ALAN: It's because when you add these two lines, they give you *h*.

RESEARCHER: Which *h*? What is that *h*?

ALAN: The *h* is the shortest distance – the sum of the distances.

RESEARCHER: So are you saying if you add these lines (pointing to the two lines between the two pairs of opposite sides), this will give you the sum of the distances: $h_1$, $h_2$, $h_3$, $h_4$ ?

ALAN: Yes. So whenever you rotate, you drag this point. It will move with those lines. So, the sum of them will always remain the same.

RESEARCHER: Ummm… okay, that's interesting.

#### 8.4.4.2 Alternative deductive justifications for Alan and Shannon

This section presents Alan's and Shannon's alternative deductive justifications for their Rhombus conjecture generalization, which was constructed as a result of them being able to anlogically discern (i.e. see) its structure from their earlier 'triangle-area' algebraic explanation, which they constructed to logically explain their equilateral triangle conjecture generalization.

Although Shannon and Alan provided correct explanations for each of their conjecture generalizations, the researcher also provided an opportunity for them to develop an alternative explanation. In each of the cases, as soon as the researcher presented the PMTs with a dynamic sketch, which showed the division of the rhombus into four triangles and their respective heights, they immediately assimilated their explanations into their previously accommodated explanations for the equilateral triangle case, and then modified it to accommodate the rhombus case. The PMT responses, which are presented case-wise are representative of the aforementioned aspects:

**Case : Shannon**

Although Shannon provided a logical kind of explanation of why her claim was true, the Researcher also provided Shannon an opportunity to explore another type of logical explanation, which involves the "area" strategy, where the rhombus is divided into four smaller triangles in particular. The researcher presented Shannon with a *Rhombus.gsp* sketch.

As soon as Shannon clicked on the button to show small triangles and saw the small triangles with their respective heights, she immediately claimed that the proof for the equilateral triangle case can be similarly applied to produce a proof for the rhombus case. The following except from the one-to-one task-based interview represents her assertion:

RESEARCHER: … But now let's look at another way of showing or explaining the result. Can you click on 'show small triangles'?
So… can you…You want to show that $h_1 + h_2 + h_3 + h_4$ are constant, right?
So does that help you now to come up with a …(an explanation) ?

SHANNON: Yes. Similarly you can use the same proof as we did for the triangle using the area.

Thereafter, the researcher gave Shannon an opportunity to construct her proof (logical explanation), and in so doing she similarly used the 'triangle-area" strategy that was employed for the equilateral triangle case to generate her proof for the rhombus case. It seemed that Shannon was able to reflect on the structure of her proof for the equilateral triangle case and to see the proof for the rhombus case (i.e. by looking at the structure of the proof for the equilateral case she was able to mirror a similar kind of proof for the rhombus case). In terms of Gentner's Structure Mapping Theory, which is discussed in Section 4.4.2, it can be accepted that Shannon analogically mapped the structure of the 'triangle-area' proof for the equilateral triangle case onto the rhombus case. The following excerpt is representative of the construction of Shannon's logical explanation in combination with analogical reasoning:

.

RESEARCHER: Using the same proof as we did for the triangle…?
You want to write it down?

SHANNON: (writing) … So, is what are we're going to do – using the area of all the triangles? You didn't label the lengths – can I just use any variable?

RESEARCHER: You must make a statement on the side that you're doing that.

SHANNON: Yes. (continued writing) So you just want to see that this is a constant, and this is exactly the same as … so $h_1 + h_2 + h_3 + h_4 = 2A$ over $a$ (which are all constants: the big $A$ is a constant, the small $a$ is a constant) and you can make $h_1 + h_2 + h_3 + h_4$ a big $H$.

RESEARCHER: Is this like what you told me …[indistinct]

Okay, so, basically, what have you concluded in this result?

SHANNON: - that the sum of all the distances from all sides of a rhombus will be constant, it doesn't matter where point *P* … it doesn't matter where the point inside the rhombus is.

More importantly, Shannon very succinctly summarized her explanation/justification as an argument in a kind of paragraph form as indicated in the extract in Figure 8.4.4.2 which is taken from her worksheet.

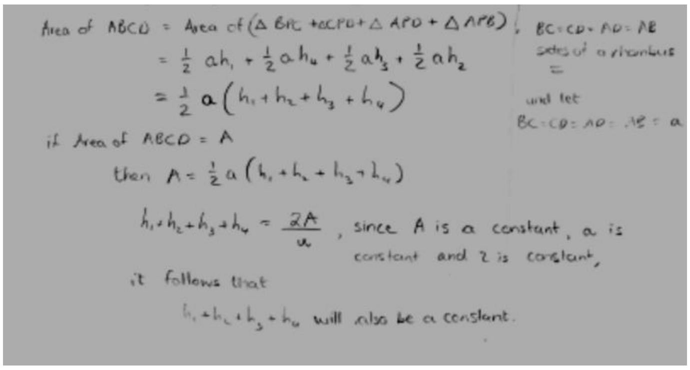



Figure 8.4.4.2: Shannon's deductive justification

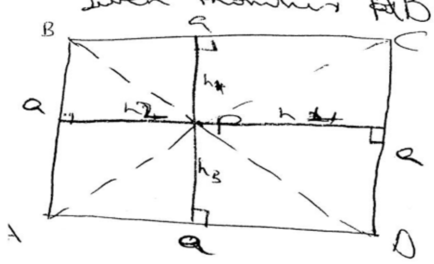
**Case: Alan**

Although Alan provided a logical kind of explanation of why his claim was true, the Researcher also provided Alan with an opportunity to explore another type of logical explanation, which involved the "triangle -area" algebraic strategy, where the rhombus is divided into four smaller triangles, in particular. Alan produced the explanation as shown in Figure 8.4.4.3, with a distinctive sense of ease, confidence and accuracy, using his own sketch, which he drew on his worksheet.

It also seemed that Alan had now seen the similarity (analogy) with the equilateral triangle case and then proceeded to construct his logical explanation by analogically mapping the structure of the 'triangle-area' explanation for the equilateral triangle case onto the rhombus case. Thus, it seemed that Alan also looked at the structure of the logical explanation (proof) for the previous equilateral case and then mirrored a similar kind of proof for the rhombus case through the process of cognitive blending (see Section 4.4.3).

In hindsight, maybe I should NOT have given them the opportunity to do an alternative explanation, but recorded what happened to them in the pentagon case next where the ‘opposite sides parallel strategy’ will not work, and to see if they then made the connection with the equilateral triangle? And switching strategy back to the ‘triangles-area’ approach?

Given rhombus ABCD. ① All sides of a rhombus are equal (given)




②Construct a perpendicular line to all sides of the rhombus from the Point P

③ Draw triangles inside the given rhombus

A → Area

$A, \triangle APB = \frac{h_2}{2} AB$

$A, \triangle APD = \frac{h_3}{2} AD$

$A, BPC = \frac{h_1}{2} BC$

$A, CPD, A = \frac{h_4}{2} CD$

Since the sides of the rhombus are all equal we can label each side with variable a

therefore: $AB = AD = BC = CD = a$

Hence we have $A_T = \frac{a}{2}[h_1 + h_2 + h_3 + h_4]$

$\Rightarrow \frac{2A_T}{a} = h_1 + h_2 + h_3 + h_4$

$A_T$ is a Constant [$A_T$ Area of the fixed rhombus]
[a the length of each side of the rhombus]

We can conclude that

$h_1 + h_2 + h_3 + h_4 = h$ will remain constant

Hence, the sum of the distances to all four sides of a given rhombus is always constant.

Figure 8.4.4.3: Alan’s deductive justification

### 8.4.5 Finding as per Section 8.4 ( Justifying a Rhombus conjecture generalization)

As discussed earlier, six PMTs (referred to as group B) needed experimental exploration to make the general conjecture, and two PMTs (referred to as group A), Shannon and Alan, seemed to have made the generalization only on analogical grounds. The findings in relation to these two groups are presented as follows:

1. Three of the six PMTs (from group B), Tony, Inderani and Logan, who required experimental exploration to make the general conjecture, needed scaffolded guidance to construct their logical explanation for their conjecture generalization.
2. Two of the six PMTs (from group B), Victor and Renny, who required experimental exploration to make the general conjecture, appeared to have explained why their result was true for a rhombus in terms of opposite sides parallel on their own accord. However, when they were exposed to an alternative explanation, namely the 'area-triangle' algebraic strategy, via a scaffolded worksheet, they proceeded through the worksheet with relative ease. This suggested that the prior experience with the scaffolded activity in the equilateral triangle case developed their competency, insight and understanding of how to use the so called 'area-triangle' algebraic strategy, i.e. analogical transfer, was enhanced.

3. One of the six PMTs (from this group), Trevelyan, who required experimental exploration to make the general conjecture, spontaneously saw a similarity between the equilateral problem and the rhombus problem, and then proceeded to use a similar kind of proof structure, namely 'triangle-area' algebraic structure, to construct a logical explanation for his conjecture generalization for a rhombus, without any assistance from the researcher. This suggested that in seeking to construct an explanation for his rhombus conjecture generalization, Trevelyan discovered, via analogical reasoning, a strategy on how to explain his rhombus conjecture generalization. This discovery in itself is an example of the discovery function of proof. In retrospect, it seemed that via analogy-making, Trevelyan first assimilated his explanation into his previously accommodated explanation for the equilateral triangle case, and then subsequently modified it to accommodate the rhombus case.

4. The two PMTs (referred to as group A), Shannon and Alan, who seemed to have made the generalization only on analogical grounds, constructed a logical explanation (deductive justification) for their conjecture generalization rather spontaneously by using the 'distance between parallel lines is constant' algebraic strategy, without receiving any assistance from the Researcher.

5. When the researcher provided both group A students, Shannon and Alan, an opportunity to develop an alternative explanation, he first asked each of them during the one-to- one task-based interview to click on the button on their dynamic Rhombus.gsp sketch to

show the small triangles. As soon as each of these PMTs saw the rhombus being broken into small triangles, they immediately suggested that they could similarly use the ‘area-triangle’ algebraic strategy, which they used for the equilateral triangle case, to construct a logical explanation for their rhombus conjecture generalization. Both students proceeded to complete their alternative explanation with both speed and accuracy and with no assistance. This suggested that their earlier scaffolded activity, which enabled these PMTs to develop a logical explanation for their conjecture generalization for a equilateral triangle, enhanced both their competency and insight on using the ‘area–triangle’ algebraic strategy in other warranted or eligible cases such as the rhombus case, for example.

The next Chapter, focusses on the data analysis, results and discussion with regard to the Pentagon task- based activity problem.

# Chapter 9: Convex Pentagon Problem: Data Analysis, Results and Discussion

## 9.0 Introduction

In this chapter, we look at whether pre-service mathematics teachers (PMTs) could generalize the following results further to a pentagon, and if so, how they accomplished this generalization.

(a) In an equilateral triangle, the sum of the distances from a point inside the triangle to its sides is constant ; and

(b) In a rhombus the sum of the distances from a point inside the rhombus to its sides is also constant.

The presentation of the data analysis and findings related to making and justifying conjecture generalization(s) with particular reference to the Pentagon task as described in Task 3 (see Appendix 2 for details), has been divided into the following four subsections in this chapter. Firstly, Section 9.1, presents the analyses of the data and findings associated with the making of a pentagon conjecture generalization, and this is followed by Section 9.2, which describes how the PMTs justified their respective pentagon conjecture generalizations. Section 9.3, describes how PMTs experience a heuristic counter-example (Mystery pentagon activity) to their initial conjecture generalization and consequently refine (or modify) their initial conjecture generalization as reported in Section 9.1, and finally Section 9.4 describes how PMTs justify their refined conjecture generalizations.

## 9.1 Pre-service mathematics teachers producing a pentagon (convex) conjecture generalization

As discussed in Sections 1.7 and 2.1, the making of a conjecture generalization can be facilitated via inductive reasoning and analogical reasoning. “Inductive reasoning is the process of reasoning from specific premises or observations to reach a general conclusion or overall rule […] It usually refers to the given instances and does therefore reach conclusions that are not necessarily valid for all possible instances” (Christou & Papageorgiou, 2007, p. 56). The kind of conjecture generalization that is developed through the process of inductive reasoning is commonly known as an inductive generalization (see section 2.1 of Chapter 2).

On the other hand, Polya (1954a) argues that a conjecture generalization can also be constructed through analogical reasoning, which is a method of processing information that compares the similarities between an already known problem (or general idea or concept or fact) with a new problem (or general idea or concept or fact) and then using the identified similarities to gain an understanding of the new problem (or idea or concept).

The analogical reasoning process begins by an individual determining the new problem or new idea (usually called the target domain) that needs to be explored, solved or explained. This new problem or idea is then compared to a known problem or idea (called the source domain) so long as some trace of similarity exists between the new problem and the known problem. On the basis of understanding that the two problems (or ideas or concepts or objects) are alike in a certain respect, one then conjectures that they alike in other respects, and thus attempts to solve a new problem by structurally mapping the solution from the source problem onto the target problem or attempts to understand a new idea (or concept or object) by drawing plausible parallels between the known idea (or concept or object ) and the new concept (or idea or object) (see Lee & Sriraman, 2011; Richland, Holyhoak & Stigler, 2004). As analogical reasoning makes it possible to understand what is likely to be true, it can be considered as a form of inductive reasoning and not deductive reasoning. However, the conjecture generalization produced through analogical reasoning is called an analogical generalization (see discussion in Section 1.7.3 and 2.1.3).

Taking cognisance of plausible ways to construct conjecture generalizations, the Pentagon Task 3(a): Generalizing a pentagon, which was fore-grounded in a non-*Sketchpad* context, provided the following opportunities for the PMTs:

- Opportunity to construct an initial conjecture generalization, but a general one, via inductive reasoning or analogy, which they could later confirm (or refute) through experimentation in a *GSP* context if they elected to do so, by using the guidelines provided in Task 3(b) of Appendix 2.

- Opportunity to construct an initial conjecture generalization, but a general one, on logical grounds, which they could later confirm through experimentation in a *GSP* context if they elected to do so, using the guidelines provided in Task 3(b) of Appendix 2.

- To provide those PMTs, who did not see the similarity of the pentagon problem with the rhombus and/or equilateral problem or did notice the pattern that 'if sides are equal the sum of distances is constant' through inductive reasoning, an opportunity to make an initial conjecture through either their own intuition or spontaneity or by just guessing, which may not necessarily be correct, but could later be tested via experimentation in a *Sketchpad* context as outlined in Task 3(b) of Appendix 2.

Although Pentagon Task 3(b), was designed for PSTEs to specifically use *Sketchpad* to develop a conjecture generalization related to the pentagon problem through inductive reasoning, it also provided an opportunity for PSTEs' to test and validate (or refute either fully or partially) their initial non-*Sketchpad* conjecture generalization within a *Sketchpad* context.

As a result of the analyses of the PMTs approach to the construction of their initial pentagon conjecture generalization via Task 3(a) and /or Task 3(b) during the one-to one task based interviews, this section has been organized into five sections to present the analyses of the data and the findings, as follows:

Section 9.1.1: Producing a conjecture generalization by empirical induction from dynamic cases in a *GSP* context.
Section 9.1.2: Producing a Regular Pentagon conjecture generalization, through analogical or inductive reasoning, and requiring some experimental confirmation or conceptual clarification in *GSP* context.
Section 9.1.3: Producing a Regular Pentagon conjecture generalization, on inductive or analogical grounds without requiring the use of *Sketchpad*.
Section 9.1.4: Producing a Regular Pentagon conjecture generalization, plausibly through either using inductive reasoning or analogical reasoning of a fairly superficial level or pure guessing, and without requiring the use of *GSP* in either of the cases.
Section 9.1.5: Producing an Equilateral Pentagon conjecture generalization which is correct analogical reasoning.
Section 9.1.6: Summarizes the findings as per discussions in Sections 91.1- 9.1.5.

### 9.1.1 PMTs producing a conjecture generalization by empirical induction from dynamic cases

1. None of the PSTEs formulated their conjecture generalization to a pentagon initially through dynamically experimenting with particular cases using *Sketchpad* .

### 9.1.2 PMTS producing a Regular Pentagon conjecture generalization through analogical or inductive reasoning

This section focuses on pre-service mathematics teachers producing a Regular Pentagon conjecture generalization through analogical or inductive reasoning and requiring some experimental or confirmation or conceptual clarification in a *GSP* context

Three students, namely Inderani, Tony and Victor, seemed to have made their conjecture generalizations on analogical or inductive grounds without initially using *Sketchpad*. When the researcher probed Inderani's level of certainty in her conjecture generalization, she expressed some uncertainty and requested to experimentally test her conjecture generalization by using *Sketchpad*. On the other hand, Victor, who expressed 75% certainty in his conjecture generalization, did not request to experimentally explore the extent of the validity of his conjecture generalization. Hence, in Victor's case the Researcher requested him to experimentally explore (test) his conjecture generalization by using *Sketchpad*.

Through dragging point *P* around the interior of the regular pentagon, and seeing that the sum of the distances to the sides of the regular pentagon was always remaining constant, the conviction level of Inderani and Victor in their respective conjecture generalizations increased to an extent that resulted in them expressing a 100% level of certainty. However, Tony's level of doubt was linked to his having not seen a pentagon with unequal sides. In this instance, Tony with the guidance of the tesearcher was requested to construct a pentagon by using *Sketchpad*. After having done so, and 'seeing' that a pentagon with unequal sides was possible, Tony moved from a 75% certainty level in his conjecture generalization to a 100% level of certainty.

The following cases of Inderani, Victor and Tony are presented with supporting excerpts from their one-to-one task based interviews and/or worksheets to demonstrate the development of their respective conjecture generalizations and corresponding levels of certainty:

**Case: Inderani**

With regard to Task 3(a) – Q 1.1, Inderani responded as follows (see Figure 9.1.2.1) in her worksheet without any reference to a dynamic *Pentagon Sketch*.

**1.** In the previous activities you may have observed, conjectured and logically explained the following results (or generalizations):

(a) In an equilateral triangle, the sum of the distances from a point inside the triangle to its sides is constant.

(b) In a rhombus the sum of the distances from a point inside the rhombus to its sides is also constant.

1.1 How would you generalize the above result(s) to a pentagon.

in a regular pentagon, the sum of the distances from a point inside the pentagon to its sides is constent

Figure 9.1.2.1: Inderani's initial pentagon generalization

It seems that Inderani just used inductive reasoning, i.e. noticed the pattern that if 'sides are equal' the sum is constant, or analogical reasoning. In reference to Inderani's written generalization, the researcher asked her "are you sure? How certain are you?", and she replied, "Maybe after looking at *Sketchpad* I'll be more confident". This response showed that Inderani wanted to confirm her claim through visual experimentation. Consequently, the researcher remarked, "So you want to test or confirm your conjecture?", and Inderani responded saying, "yes". The researcher on seeing that Inderani confined her written conjecture generalization to a regular pentagon, probed Inderani to see if she indeed really wanted to restrict her conjecture generalization to just regular pentagons, by asking her, "did you say regular pentagon here?" Inderani responded by saying, "Yes, regular pentagon". Immediately thereafter, the researcher provided Inderani with an opportunity to dynamically experiment with a regular pentagon, as illustrated in the following excerpt from the one-to-one task based interview:

| | |
|---|---|
| RESEARCHER: | Let's open a regular pentagon. Here's a regular pentagon. |
| INDERANI: | 'Show pentagon sides' … it's all constant; 'show distance sum' …okay… so let's see, if I moved around - that's constant (*she was dragging point P around numerous times*) – the sum of the distances remains constant… $h_1 + h_2 + h_3 + h_4 + h_5$ remains constant. And this is a regular pentagon. |

As Inderani manipulated the situation dynamically through a continuity of cases, by dragging point *P* around the interior of the pentagon, she continuously observed that the sum of the distances from point *P* to the sides of the regular pentagon remained constant irrespective of the location of point *P*. The latter series of observations tallied with her initial pentagon conjecture generalization (i.e. her pentagon conjecture generalization restricted to a regular pentagon), which she made prior to her *Sketchpad* investigation. It is apparent that Inderani's experimental experience boosted her level of conviction in her conjecture generalization, because when the Researcher presented Task (3c) (1) to her, she responded as follows in her worksheet (see Figure 9.1.2.2):

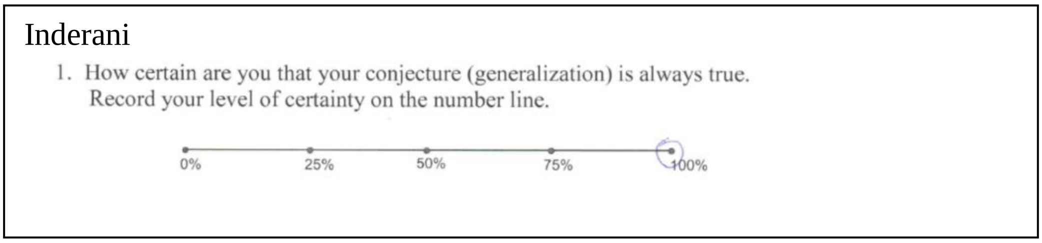
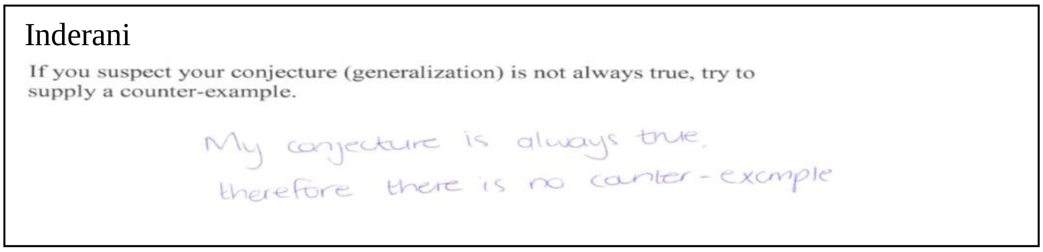



Figure 9.1.2.2: Inderani's level of certainty in her regular pentagon CG

Immediately after the aforementioned response, the Researcher asked Inderani to respond to Task (3c)(2). Inderani subsequently responded as follows in her worksheet (see Figure 9.1.2.3):


Inderani

If you suspect your conjecture (generalization) is not always true, try to supply a counter-example.

My conjecture is always true.
therefore there is no canter-exemple

Figure 9.1.2.3: Inderani asserting no counter-example to her CG

It seems that through dragging point *P* around the interior of the regular pentagon within a *Sketchpa*d context and having continuously seen that the sum of the distances remained constant for all of her different chosen positions of point *P*, Inderani came to believe that there is no counter-example to her regular pentagon conjecture generalization.

Although Inderani formulated her conjecture generalization by restricting it to the regular pentagon, the Researcher proceeded to seek clarity on Inderani's understanding of the concept of regularity, as illustrated in the following dialogue:

RESEARCHER: Are the angles equal in the pentagon? Are the angles *A*,*B*,*C*,*D* and *E* equal?

INDERANI: Yes. The angles are equal (*she made this statement without measuring the angles*) and the sides are equal.

Inderani's aforementioned response suggests that she seems to think that 'regular' is a necessary condition for equal sides.

**Case: Victor**

When the researcher asked Victor to read out his written response to Task 3(a) – Q 1.1, he responded as follows without any reference to or use of a dynamic *Pentagon Sketch*:

VICTOR: I said, in a pentagon, the sum of the distances from a point inside the pentagon to its sides is also constant.

On further probing as to the kinds of pentagons that his aforementioned conjecture generalization would hold true for, Victor responded as follows:

VICTOR: I think it can be a regular pentagon.

RESEARCHER: Do you want to test or confirm your conjecture generalization with respect to the regular pentagon?

VICTOR: …aaah… No.

It seems that Victor was able to generalize from his previous equilateral triangle and rhombus conjecture generalizations on either inductive or analogical grounds, without expressing any for the need for visual or experimental confirmation. However, when the researcher probed his level of certainty, he initially indicated 75% and later conceded to 100%. The latter response by the student, as evident in the following one-to-one task based interview excerpt, seemed more an effort to please the Researcher, and it seems doubtful the student was really 100% certain.

RESEARCHER: So are you certain that your conjecture with regard to the regular pentagon is always true? Record your level of certainty on the number line.

VICTOR: 75% (*student reads out his response from his worksheet*)

RESEARCHER: So why do you say 75%?

| | |
|---|---|
| VICTOR: | Because the sum of the distances inside the pentagon will be fixed from any position inside (he means the sum of the distances from point *P* to the sides of the regular pentagon is constant) |
| RESEARCHER: | So you're 75% sure? Why are you not 100% sure? |
| VICTOR: | Okay, I'll take 100%. |

Since the researcher was not convinced by Victor's response of "100%", he decided to give Victor some empirical investigation, although Victor himself did not request to empirically test his conjecture generalization. The following excerpt from the one-to-one task based interview with Victor, demonstrates how through experimental exploration, he did convincingly move to a 100% certainty level in his conjecture generalization:

| | |
|---|---|
| RESEARCHER: | Okay. Can you open the sketch of the pentagon?<br>Here is the regular pentagon. You say you are 75% certain – I want you look at the angle measurements and measure the pentagon's sides. By looking at the measure of the angles, are they equal? |
| VICTOR: | Yes. |
| RESEARCHER: | And are the sides all equal? |
| VICTOR: | Yes. |
| RESEARCHER: | Is it a regular pentagon that you're talking about? |
| VICTOR: | Yes. |
| RESEARCHER: | Now can you see the distances from point *P* – how many distances are there? |
| VICTOR: | Five. |
| RESEARCHER: | Now click on 'show the distance sum'. |
| VICTOR: | The distance sum is 18.77cm. |
| RESEARCHER: | Now, for that fixed pentagon, drag point *P* and see what happens. |
| VICTOR: | I see the distance sum doesn't change. It stays constant. It stays 18.777cm. It's only the distances that change, but the sum of them all doesn't change. |
| RESEARCHER: | What is your level of certainty in your conjecture generalization now? |
| VICTOR: | 100% (*says quite convincingly)* |

Victor's final observations, "I see the distance sum doesn't change. It stays constant. It stays 18.777cm. It's only the distances that change, but the sum of them all doesn't change", and his convincing expression of “100%” certainty in his conjecture generalization, made the researcher feel that Victor was absolutely sure about his conjecture generalization for any regular pentagon. Victor has limited his conjecture generalization by confining it to the regular pentagon. It is plausible that the latter confinement could be as a result of a misconception that Victor holds, namely that only regular pentagons have equal sides.

**Case: Tony**

The following excerpt illustrates how Tony extended his earlier conjecture generalizations for the equilateral triangle and rhombus cases to the pentagon case through either using analogical or inductive reasoning:

| | |
|---|---|
| TONY | If a pentagon’s sides are all equal, then the conjecture above should be the same (*referring to the conjecture generalizations for the equilateral triangle and rhombus cases*). |
| RESEARCHER: | Can you phrase that conjecture for me? |
| TONY | Therefore the sum of the distances from the point– inside the pentagon - should be the same if all sides of the pentagon are the same. |

Reflecting on his previous experiences with the equilateral triangle and the rhombus, Tony correctly identified that “sides all equal” as a sufficient condition, for the sum of the distances from a point inside the pentagon to its sides to remain constant all the time. However, when the Researcher asked Tony, “How certain are you?”, he replied, “According to these, I am actually seventy-five percent certain.” Then, as illustrated in the following excerpt, the Researcher probed Tony as to the reason for his 75% certainty level:

| | |
|---|---|
| RESEARCHER: | Do you have a counter example? |
| TONY | A counter-example – not really. |
| RESEARCHER: | So, why did you say seventy-five percent? |
| TONY | I’m not sure if the sides of a pentagon are always equal. |
| RESEARCHER: | Can you get a pentagon with sides that are not equal? |
| TONY | I haven’t seen one. |

Tony’s latter response, “I haven’t seen one”, demonstrates how limited his conceptions are of polygons beyond triangles and quadrilaterals. This suggests that in terms of the Van Hiele theory, he’s not even at Level 1 (Visualization). Because of this shortcoming, the researcher provided Tony with an opportunity to use *Sketchpad* to construct a pentagon with sides not being equal. After seeing his dynamically constructed pentagon with unequal sides, Tony reaffirmed his initial conjecture generalization with a hundred percent certainty level, as illustrated in the following excerpt:

TONY …Now, my conjecture would actually hold if the pentagon has all its sides equal. So, the distances from the point inside the pentagon to the sides would be equal if the pentagon had the same sides.

RESEARCHER: Okay. So how certain are you now?

TONY If the sides are equal, I'm a hundred percent sure.

Tony's absolute certainty in his conjecture generalization, prompted the researcher to probe Tony as to whether his conjecture generalization was limited just to the regular pentagons or not, as illustrated in the following excerpt:

RESEARCHER: Okay, that's fine. So we're looking at the regular pentagon – so you're saying your result holds for the regular pentagon only?

TONY Ja.

RESEARCHER: Do you think it will hold true for other pentagons?

TONY Because you won't have that common *a*, it won't hold true.

Tony shows the same misconception as Inderani and Victor, in that he thinks that only regular pentagons have equal sides: more specifically that equality of sides implies the equality of the angles.

### 9.1.3 PMTs producing a Regular Pentagon CG through analogical or inductive reasoning, without using GSP.

This sections focuses on re-service mathematics teachers producing a conjecture generalization through analogical or inductive reasoning, without using GSP.

In the case of the pentagon problem, three PMTs, namely Shannon, Trevelyan and Renny seemed to have produced their conjecture generalization by means of analogical or inductive reasoning from the previous cases. These three PMTs expressed a 100% level of certainty in their conjecture generalizations, without expressing a need to use *Sketchpad* to experiment and test their conjecture generalization.

The following cases of Shannon and Trevelyan, which are presented with supporting excerpts from their one-to-one task based interviews and/or worksheets, are representative of how this group of three PMTs (namely, Shannon, Trevelyan and Renny), constructed their conjecture generalizations and expressed their levels of certainty.

**Case: Shannon**

As demonstrated in the following excerpt, it seems that Shannon like Inderani and others, made her conjecture generalization via analogical or inductive grounds.

RESEARCHER: …in the previous activities we observed, conjectured and logically explained the following results:

(a) In an equilateral triangle, the sum of the distances from a point inside the triangle to its sides, is constant, right?; and also

(b) That in a rhombus, the sum of the distances from a point inside the rhombus to its sides is also constant.

How would you generalize the above results to pentagons?

SHANNON: Well, it would have to be… the result would have to be the same for a pentagon, but it would only be the same for a regular pentagon, meaning that all the sides are equal – that the sides and angles are equal so that the pentagon has got five …(indistinct mumbling). So, this is your pentagon, with five sides, all the sides have got to be equal, and all the angles have got to be equal. Then point *P* can be anywhere within the figure, and that the sum of the distances to the sides of the pentagon – the regular pentagon – will be constant.

Shannon very emphatically acknowledged that the result would have to be the same for the pentagon, provided it meets the following specific conditions: (a) all the sides have got to be equal, and (b) all the angles have got to be equal. This demonstrates that Shannon appears to have the same misconception as the other students that for a pentagon, equal sides imply equal angles, and vice versa. However, her response signals that she is probably basing her claim on the knowledge and experience she gained from her previous activities. More importantly, her claim that point *P* can be anywhere is also premised on the fact that the sum of the distances to the sides of the regular pentagon will be constant.

Shannon like others, for example Inderani, noticeably did not make an analogous connection with the previous activity, where the rhombus was not necessarily regular, i.e. whose sides were all equal but whose angles were not necessarily all equal. Shannon, just like Inderani and others, seem to think that 'regular' is a necessary condition for a pentagon to have equal sides, and this could have caused her to construct her conjecture generalization as illustrated in her responses in the afore-cited excerpt.

This misconception of the regularity concept provides an apt opportunity to show students a pentagon with equal sides that is not regular as a counter-example to refine their pentagon conjecture generalization and also remedy their alleged misconception. This kind of didactic move is typically associated with diagnostic teaching, which requires that a facilitator intervenes at critical moments in the teaching/learning sequence to correct misconceptions, false/deficit conjectures, proofs and arguments in general.

**Case: Trevelyan**

When Trevelyan was asked by the Researcher as to how he would generalize the equilateral and rhombus results (see Q1.1 of Task 3(a)), he responded as follows:

TREVELYAN: I can generalize in such a way: We observed that in an equilateral, since the sides are equal the sum was also equal (meaning the sum of the distances remains constant). We also saw that on the rhombus that the sides were equal. That means that in a pentagon, that means the sides are all the same, then it should be the same (*apparently meaning the sum of the distances remain constant*)

As discussed with reference to inductive reasoning in Sections 1.6.1 amd 2.1.1., when observations are made one tends organize them in a way that enables one to construct conjectures (or conjecture generalizations) about the behaviours of mathematical objects under the 'microscope' or about particular phenomena. One such procedural move is to look for a pattern that exists amongst already made observations (or phenomena), and thereby generalize to a conjecture or conjecture generalization (see Polya, 1967; de Villiers, 1992).

In this respect, it seems that Trevelyan reflected on his previous equilateral triangle and rhombus conjecture generalizations, and having singled out in both instances that the figures had equal sides and their distances sum were also constant, detected the following pattern: 'if sides are equal' then the sum is constant. Hence, it is plausible that when Trevelyan was asked to generalize his equilateral triangle and rhombus results (i.e. conjecture generalizations), he could have thought that he was also dealing with a pentagon with equal sides and hence inductively or analogously argued on the grounds of his detected pattern that the result would be the same for the pentagon (i.e. the distance sum for a pentagon with equal sides would also be the same as for the equilateral triangle and rhombus cases).

However, when the researcher probed Trevelyan further, as illustrated in the following excerpt, he said that he was absolutely certain that his conjecture generalization would hold only for regular pentagons.

| | |
|---|---|
| RESEARCHER: | Do you think it will only hold for regular pentagons, or will it hold for other kinds of pentagons? |
| TREVELYAN: | I think only the regular. |
| RESEARCHER: | Only the regular. …Are you certain about the regular pentagon? How certain are you? |
| TREVELYAN: | Hundred percent. |

It seems that Trevelyan like the others is thinking that 'regular' is a necessary condition for equal sides, despite having demonstrated that he knows that a regular figure has the equal sides and equal angles, as illustrated in the following task-based interview excerpt:

| | |
|---|---|
| RESEARCHER: | When you say regular, what do you mean by 'regular'? |
| TREVELYAN: | I mean the sides are all the same – are equal. |
| RESEARCHER: | Now, there are two important conditions to be met in order for a polygon to be regular: |
| TREVELYAN: | and the angles should be equal. |

It seems that Trevelyan's deficit conjecture was also influenced by limited or absent experience of encountering pentagons other than regular ones.

### 9.1.4 PMTs producing a Regular Pentagon CG using inductive reasoning or guessing or superficial analogical reasoning, without using *GSP*.

This section focuses on pre-service mathematics teachers producing a conjecture generalization, plausibly through using inductive reasoning or superficial analogical reasoning or guessing, and without the use of *Sketchpad* in either of the cases

There was just one student, namely Logan, who seemed to have constructed a Regular Pentagon conjecture generalization through either using inductive reasoning or analogical reasoning at a superficial level, or by guessing. The case of Logan is hereby presented with representative excerpts from his one-to-one task based interview with the Researcher:

**Case: Logan**

As illustrated in the following excerpt, it seems that Logan at first glance generalizes to any pentagon.

RESEARCHER: How would you generalize the above results to a pentagon?

LOGAN: I would say that, given the two foregoing statements, in a pentagon the sum of the distances from inside the pentagon to its sides is constant…

When the researcher, as illustrated in the following excerpts, further probed Logan to establish whether his conjecture generalization applied to any pentagon or not, it was realized that Logan was restricting his conjecture generalization to just 'regular' pentagons.

RESEARCHER: Okay, reading what you wrote here, you said in a pentagon the sum of the distances from a point inside the pentagon to its sides is constant. For what pentagons would that conjecture hold true? Or which pentagons are you referring to?

LOGAN: I'm talking about a fixed pentagon and a regular pentagon.

RESEARCHER: to any fixed pentagon?

LOGAN: Yes.

RESEARCHER: What do you mean by regular?

LOGAN: What I mean by regular is that the inside angles of the pentagon are all the same.

RESEARCHER: So, is that the only condition for a pentagon to be regular?

LOGAN: No. All the sides are also equal.

RESEARCHER: So the pentagon that you're referring to here, you say is a regular pentagon?

LOGAN: It's a regular pentagon, which means all the sides are the same; or all the inside angles are all equal.

RESEARCHER: You said 'or'?

LOGAN: I meant 'and'. For it to be a regular pentagon, it has to have those two characteristics.

RESEARCHER: So you're saying this result that you're talking about applies to a regular pentagon?

LOGAN: Yes, to a regular pentagon (*seems to have a misconception, namely only regular pentagons have equal sides*)

Taking cognisance of Logan's apparent misconception, namely 'only regular pentagons have equal sides', which could have probably caused him to construct his conjecture generalization, the Researcher probed Logan as to the level of certainty he had in his regular pentagon conjecture generalization. As illustrated in the following excerpt, Logan expressed absolute certainty in his conjecture generalization, and did not even want to check out the validity of the regular pentagon conjecture generalization via experimental exploration using *Sketchpad*.

RESEARCHER: How certain are you?

LOGAN: I'm a hundred percent sure.

RESEARCHER: Do you want to test or confirm the conjecture that you just spoke about?

LOGAN: No, I'm sure.

RESEARCHER: …So you don't want to investigate that result?

LOGAN: No, I don't.

### 9.1.5 Producing an Equilateral Convex Pentagon CG immediately on logical grounds Plausibly with the aid of analogical reasoning.

The analysis of the task based interview responses from the PMTs, has shown that only student, namely Alan, seems to have generalized to any equilateral pentagon on logical grounds with reasoning by analogy (i.e. cognitive blending) as a plausible aid to the development of his conjecture generalization. The case of Alan is represented with supporting excerpts from the task-based interview and task-based worksheet as follows:

**Case: Alan**

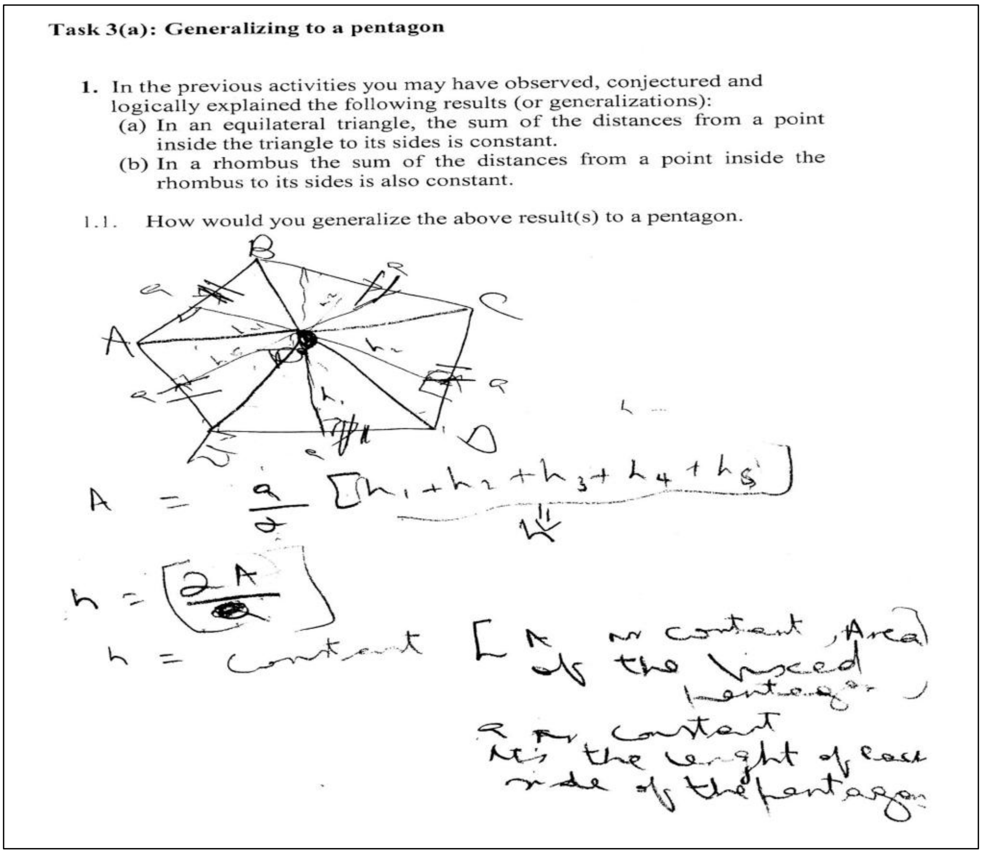



Figure 9.1.5.1: Alan reasoning deductively prior to making his CG

When Alan was presented with Task 3(a) by the Researcher, he immediately responded in the worksheet by drawing a pentagon and marking all its sides equal as shown in Figure 9.1.5.1. Without any assistance or use of *Sketchpad,* he continued to construct a written argument in the worksheet as illustrated in Figure 9.1.5.1. The act of Alan marking the sides of the pentagon as being equal, seems to suggest that he has singled out the property, namely ‘equal sides’, as the property a pentagon ought to have in order to continually extend his previous equilateral triangle and rhombus generalizations to a prospective pentagon. However, before announcing a generalization, Alan on his own chose to first construct a logical argument (see Figure 9.1.5.1) that was premised on the property, ‘equal sides’, to see if he could explain why the sum of the distances from a point inside a pentagon to its sides is always constant. In doing so, Alan similarly used the ‘triangle-area’ algebraic strategy that he had previously used in the construction of the explanations for both the equilateral triangle and rhombus conjecture generalizations, and hence reached the desired conclusion, namely the sum of the distances is constant.

When the researcher asked Alan to explain what he had written in his worksheet, he responded as follows:

ALAN: Aaaah….You have a pentagon (*referring to his pentagon with equal sides*) and you construct a point $P$ inside pentagon. Then you draw lines that are perpendicular to each side. After drawing those lines you construct a triangle – so in this case we have five triangles – and you find the area of each small triangle and they have a common factor of $a$ over 2. So the sum of all the heights of …uum..the triangles, is $h$. So you have $h$ which is going to be equal to $h_1 + h_2 + h_3 + h_4 + h_5$.. So, the total area that you are going to have is going to be the small $a$ over $2 \times h$. So when we want to prove that $h$ is always constant, we make $h$ the subject of the formula, so $h = 2$ times the area ($A$) over $a$. And we know that area ($A$) is constant because we have a fixed pentagon, and we have $a$ (small $a$) is constant as it’s the length of each side. So we have proved that $h$ ($h$ here represents the sum of the distances, i.e. $h_1 + h_2 + h_3 + h_4 + h_5$) equals a constant.

Alan’s aforementioned response suggests that he has plausibly made the conjecture generalization to the pentagon with equal sides on logical grounds with the aid of analogical reasoning. Nevertheless, the researcher probed Alan to see whether he realized that the pentagon did not need to also have equal angles (i.e. be regular) for the result to be true.

RESEARCHER: Now that is for this pentagon that you have got here where all the sides are equal (*referring to Alans’ hardcopy as shown in Figure 9.1.5.1*). Let us just go to the diagram. The pentagon that you’re talking about, which you spoke about here – are the angles at these vertices all equal or do they have different measurements in terms of what you're representing? I know you said the sides are all equal. That is fine. These angles (*Researcher pointing to angles* $BAE$; $AED$; $EDC$; $DCB$ and $CBA$) are they equal in your case or not? …So I'm asking, is this angle *BAE*, angle *ABC* and all those angles – are they equal or not?

ALAN: They are not equal.

RESEARCHER: Okay, so you're saying this result that you have here (*referring to his explained result in Figure 9.1.5.1*), is for a pentagon whose sides are equal, and whose angles are not equal?

ALAN: Ja.

Alan's responses as presented in the aforementioned excerpt, suggests clearly that he understands that his conjecture generalization will hold for equilateral pentagons that are not necessarily equiangular. To further check his understanding, the researcher asked Alan, "What happens if the angles are all equal and the sides are all equal? Will the result still hold?" He immediately responded by saying, "Ja, it will hold". Alan's latter response, suggests that he has figured out that 'equal sides' is a critical property to have in a pentagon in order to get a constant distance sum.

When the researcher asked Alan, "Will it (referring to his result deduced in Figure 9.1.5.1) hold for a pentagon whose sides are not equal?", he responded by saying "No, it won't hold". The latter response by Alan, seems to reaffirm that he has identified 'equal sides' as a critical property that should be present in a pentagon in order to make the continual generalization from the earlier cases (equilateral triangle and rhombus) to the pentagon cases. Thus he has intimated that 'equal sides' is a sufficient property a pentagon (or maybe any polygon) needs to possess to produce a constant distance sum.

After all the deliberations and discussions presented above, the Researcher asked Alan, "Can you now formulate the kind of generalization for respective pentagons?", and he verbally responded as follows:

ALAN: The $h$ (referring to the sum : $h_1 + h_2 + h_3 + h_4 + h_5$ ) is constant, if, and only if, the sides of the pentagon are all equal.

However, in expressing the required condition as an "if, and only if" condition, the student is wrong for the following reason: equal sides is just a sufficient condition not a necessary condition as the result is also true if all the angles are equal. However, in this instance it is understandable from his experience so far, particularly with the equilateral triangle explanation, and lack of experience of considering other types of pentagons.

**9.1.6 Findings as per Section 9.1 (PMTs producing a pentagon CG)**

1. Only one student, namely Alan, appeared to have made his conjecture generalization to any equilateral pentagon (not just a regular pentagon) on logical grounds, apparently with the aid of analogical reasoning (cognitive blending), and expressed full certainty in his pentagon conjecture generalization, after giving a deductive argument.

2. Three PMTs, namely Shannon , Trevelyan, and Renny, appeared to have considered their previous equilateral and rhombus conjecture generalizations and made their conjecture generalization to a regular pentagon by means of either inductive reasoning, i.e. noticed the pattern that if 'sides are equal' the sum is constant, or analogical reasoning. Furthermore, these three PMTs did not make any request to experimentally confirm their initial conjecture generalization, and expressed a 100% certainty in their pentagon conjecture generalization.

3. One student, namely Inderani, seemed to have also made her conjecture generalization to a regular pentagon by means of either inductive reasoning or analogical reasoning, but later required some experimental confirmation or clarification using *Sketchpad* to boost her level certainty in her conjecture generalization to 100%.

4. One student, namely, Victor, appeared to have also to made his conjecture generalization to a pentagon by means of either inductive reasoning or analogical reasoning, but did not on his own accord express a need to experimental confirm his conjecture generalization by using *Sketchpad*. However, the Researcher on seeing that Victor was only 75% sure about his conjecture generalization, provided an opportunity for him to explore his conjecture generalization using *Sketchpad*. Through the experimental exploration, Victor was able to see that his conjecture generalization was always true, and this made him 100% certain in his regular pentagon conjecture generalization.

5. There was one student, namely Tony, who after making his initial pentagon conjecture generalization plausibly on either analogical or inductive grounds, wanted some

clarification as to whether there existed a pentagon with unequal sides. Hence the researcher provided him with an opportunity to construct such a pentagon by using *Sketchpad* coupled with relevant guidance. Only after seeing a pentagon with unequal sides, did Tony move from a 75% level of certainty in his pentagon conjecture generalization to a 100% certainty level in his regular pentagon conjecture generalization.

6. One student, namely, Logan, seemed to have produced his initial conjecture generalization to the regular pentagon through either using inductive reasoning or guessing or analogical reasoning of a fairly superficial level, and without the use of *Sketchpad* in either of the cases. However, he did not request experimental confirmation using *Sketchpad*, but expressed 100% certainty in his conjecture generalization.

7. As described in Findings 2-6, there has been a perceptible/distinct increase in the number of students forming their initial conjecture generalizations on the basis of inductive or analogical reasoning, without any need for experimental confirmation. In fact none of the PMTs formulated their initial conjecture generalization for the regular pentagon by using *Sketchpad*, but only one student, namely Inderani, who formulated her conjecture generalization on inductive or analogical grounds, needed further experimental confirmation with *Sketchpad*.

8. All 7 PMTss (as described in Findings 2-6), namely Shannon , Trevelyan, Renny, Inderani, Victor, Tony, and Logan, limited their generalization to the pentagon by restricting it to the regular pentagon, and in so doing showed a misconception that equal sides imply equal angles. In other words, these seven PSTEs misunderstood that equal sides automatically imply equal angles, and vice versa. Having seen that the property ‘equal sides’ also prevailed in the previous rhombus and equilateral cases, they went on to claim that their conjecture generalization will hold only for regular pentagons. This misconception of thinking that ‘all sides equal” imply ‘equal angles’ or is a sufficient condition for a polygon (or pentagon) to be regular is surprising given that even for quadrilaterals it is not true that ‘equal sides’ imply regularity, for example the rhombus is a counter-example. Moreover, none of the students could themselves provide an example of a non-regular pentagon with equal sides.

## 9.2 Pre-service mathematics teachers justifying their pentagon conjecture generalization

As discussed in Section 9.1, there were seven out of the 8 PMTs, namely Inderani, Victor, Tony, Shannon, Trevelyan, Renny and Logan, who formulated their conjecture generalization for the pentagon by restricting it to the regular pentagon, apparently through having a misconception that equal sides imply equal angles. However, in the construction of their regular pentagon conjecture generalization, three PSTEs, namely Inderani, Victor and Tony, seemed to have made their generalization from the previous cases to a pentagon on inductive or analogical grounds and then required some use of *Sketchpad* to attain a hundred percent certainty level in their respective regular pentagon conjecture generalizations (see Findings 3-5 in Section 9.1.6). On the other hand 3 PMTs, namely Shannon, Trevelyan and Renny, seemed to have made their generalization from the previous cases on inductive or analogical grounds as well, but they did not require experimental confirmation with *GSP* (see Finding 2 of Section 9.1.6). Furthermore, Logan, as discussed in Section 9.1.4, seemed to have made his regular pentagon conjecture generalization through either guessing, or using inductive reasoning or using analogical reasoning at a rather superficial level. Nevertheless, there was one student, namely Alan, who seemed to have constructed his conjecture generalization to the equilateral pentagon on logical grounds plausibly with the aid of analogical reasoning, and did not limit it to just the regular pentagon (see Finding 1 of Section 9.1.6)

In contrast, Alan seemed to have spontaneously seen that the structure of the 'triangle-area' algebraic explanation, which he had previously advanced for the equilateral triangle and rhombus cases, could similarly be applied to the pentagon case. Hence, he first produced a logical kind of explanation and then announced his equilateral pentagon conjecture generalization. This cognitive experience of Alan, suggests that via analogical reasoning he first assimilated the pentagon explanation into his previously accommodated explanation(s) for cases such as the equilateral triangle and rhombus, and then subsequently modified the explanation to accommodate the pentagon case.

The remainder of this section will focus on the kinds of justifications produced by the remaining seven PMTs, namely Inderani, Victor, Tony, Shannon, Trevelyan, Renny and Logan, to explain their pentagon conjecture generalization which they confined to a regular pentagon. With regard to the latter seven PMTs, there was just one PMT, namely Logan, who was not able to develop a logical explanation on his own accord. Logan, as presented in

Section 9.2.1, was thus provided with scaffolded guidance and an opportunity to experiment in a *Sketchpad* context, and this seemed to have enabled him to develop a logical explanation for his regular pentagon conjecture generalization. The remaining six students (see Sections 9.2.2 and 9.2.3 for discussion) seemed to have been able to develop a logical explanation for their regular pentagon conjecture generalization through being able to see that that the 'triangle-area' explanatory proof structure that was used to explain their previous equilateral and rhombus conjecture generalizations, could also be applied to the regular pentagon case..

As discussed in the preceding paragraph, the justification(s) in the form of logical explanations that were developed by PMTs via guided scaffolding is (are) presented in Section 9.2.1. The justifications in the form of logical explanations that were developed by the three PMTs Inderani, Victor and Tony Victor (as decribed in Findings 3-5 of Section 9.1.6) and the three PMTs Shannon, Trevelyan and Renny (as as decribed in Finding 2 of Section 9.16) through seeing the general through the particular possibly with the aid of analogical reasoning, are presented in Sections 9.2.2 and 9.2.3. respectively.

### 9.2.1 The development of a logical explanation for a regular pentagon CG via scaffolded guidance and experimentation

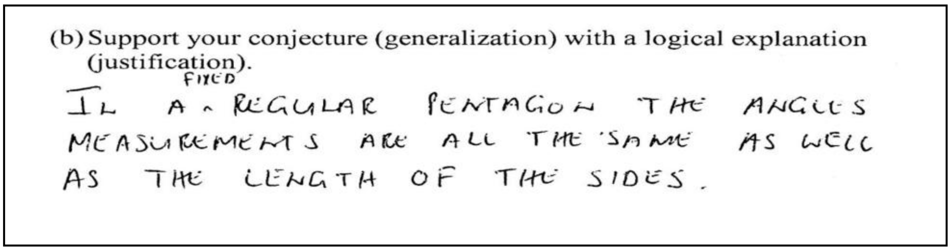
This section discusses the development of a logical explanation for a regular pentagon conjecture generalization produced by either guessing, inductive reasoning, or superficial analogical reasoning, through the use of scaffolded guidance and some experimentation within a *Sketchpad* context.

**Case: Logan**

When the researcher asked Logan to support his conjecture generalization with a justification in the form of a logical explanation, Logan replied as follows:

LOGAN: Without doing anything? …I see it's a bit difficult… in the previous exercise, I could drag the stuff and see exactly what happened. So, I would actually just like to drag ..

It seemed that Logan, probably wanted to visually/experimentally see if his result is always true. Hence, the Researcher afforded Logan an opportunity to experiment with his regular pentagon conjecture generalization and validate it. Despite this experimentatal confirmation (i.e. through dragging) in a *Sketchpad* context, he was not able to construct a logical explanation nor show any idea as to how he could possibly try (or start to try) to construct a logical explanation. In fact when, the Researcher asked Logan to support his regular pentagon conjecture generalization, he responded by saying, “this is what I wrote (*i.e. pointed to his written response in his worksheet – see Figure 9.2.1.1).*


(b) Support your conjecture (generalization) with a logical explanation (justification).

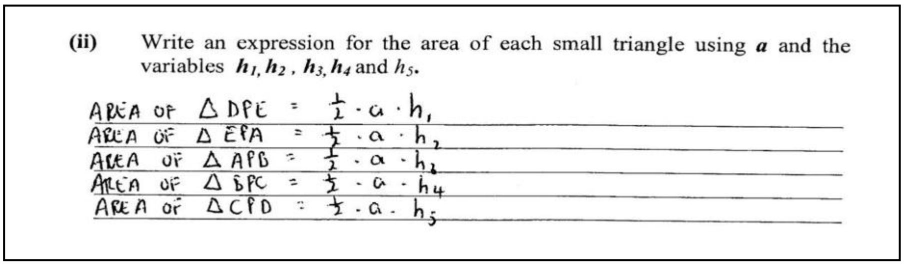
FIXED

IN A REGULAR PENTAGON THE ANGLES MEASUREMENTS ARE ALL THE SAME AS WELL AS THE LENGTH OF THE SIDES.

Figure 9.2.1.1: Logan’s failed attempt at supporting his conjecture generalization

Despite the previous two activities, the student appears to have been distracted by the properties of a regular pentagon, and have ignored the result about the sum of the distances to the sides. Hence, the Researcher had to draw the attention of Logan to the task at hand, and provide him with some scaffolded guidance, which could enable him to construct a logical explanation. The scaffolded guidance was facilitated via a worksheet task, namely Task 3(d) of the Pentagon activity (see Appendix 2).

As per Task 3(d) of Appendix 2,, Logan was first asked to “Press the button to show the small triangles in your sketch”. Immediately thereafter he was asked to proceed to do the activities in the scaffodled worksheet as contained in Task 3d. With regards to the question (ii) of the  scaffolded worksheet, and Logan then produced the following response (see Figure 9.2.1.2) in his worksheet:


**(ii)** Write an expression for the area of each small triangle using ***a*** and the variables $h_1, h_2, h_3, h_4$ and $h_5$.

AREA OF $\Delta$ DPE $= \frac{1}{2} \cdot a \cdot h_1$

AREA OF $\Delta$ EPA $= \frac{1}{2} \cdot a \cdot h_2$

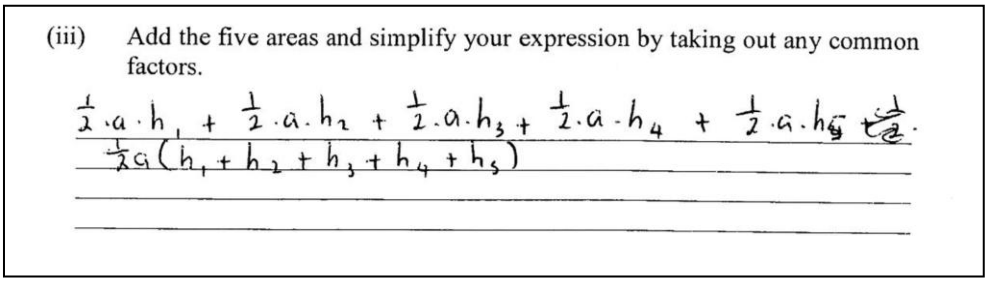
AREA OF $\Delta$ APB $= \frac{1}{2} \cdot a \cdot h_3$

AREA OF $\Delta$ BPC $= \frac{1}{2} \cdot a \cdot h_4$

AREA OF $\Delta$ CPD $= \frac{1}{2} \cdot a \cdot h_5$

Figure 9.2.1.2: Logan’s response to Task 3(d) (ii)

The researcher on seeing that Logan was able to express the area of each small triangle correctly in terms of $a$ and the variables $h_1$, $h_2$ , $h_3$, $h_4$ and  $h_5$, then requested Logan to proceed to question (iii). As shown in the excerpt from his worksheet  (see Figure 9.2.1.3), Logan on his own was able to add the five areas and simplify his expression by taking out the common factor ½ × $a$.



(iii) Add the five areas and simplify your expression by taking out any common factors.

$\frac{1}{2} \cdot a \cdot h_1 + \frac{1}{2} \cdot a \cdot h_2 + \frac{1}{2} \cdot a \cdot h_3 + \frac{1}{2} \cdot a \cdot h_4 + \frac{1}{2} \cdot a \cdot h_5$

$\frac{1}{2}a(h_1 + h_2 + h_3 + h_4 + h_5)$

Figure 9.2.1.3: Logan’s response to Task 3(d) (iii)

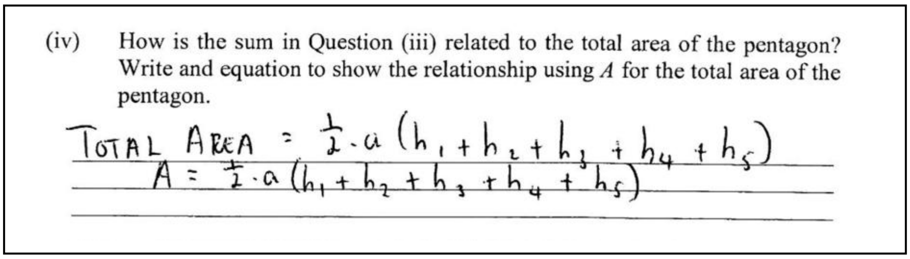
The researcher further requested him to proceed with question (iv) from the scaffolded worksheet. The following excerpt,  captures the dialogue between the researcher and Logan, that enabled Logan to finally write down the required equation as shown in Figure 9.2.1.4.

RESEARCHER: Proceed to the next question: How is the sum in Question (iii) related to the total area of the pentagon? Write an equation to show the relationship using A for the total area of the pentagon.

LOGAN: The total area of the pentagon = ½ × $a$ ($h_1$ + $h_2$ + $h_3$+ $h_4$+ $h_5$) which means the total area of the pentagon equals all the areas of the triangles, added up.

RESEARCHER: And what is the total area? What letter will you use to indicate the total area?

LOGAN: I should actually use *A*.



(iv) How is the sum in Question (iii) related to the total area of the pentagon? Write and equation to show the relationship using *A* for the total area of the pentagon.

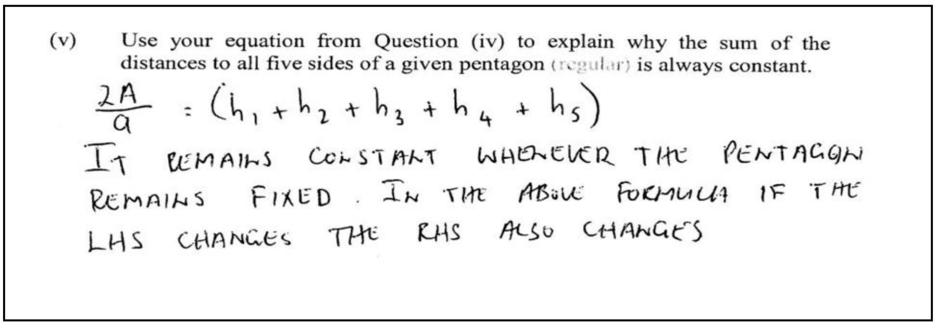
TOTAL AREA $= \frac{1}{2} \cdot a(h_1 + h_2 + h_3 + h_4 + h_5)$

$A = \frac{1}{2} \cdot a(h_1 + h_2 + h_3 + h_4 + h_5)$

Figure 9.2.1.4: Logan’s area equation

Upon observing that Logan was coping and responding well, the Researcher requested Logan to proceed to question (v) of  scaffolded worksheet, which read as follows:

“Use your equation from Question (iv) to explain why the sum of the distances to all five sides of a given pentagon is always constant”. Logan then produced the response as shown in Figure 9.2.1.5 in his worksheet:


(v) Use your equation from Question (iv) to explain why the sum of the distances to all five sides of a given pentagon (regular) is always constant.

$\frac{2A}{a} = (h_1 + h_2 + h_3 + h_4 + h_5)$

IT REMAINS CONSTANT WHENEVER THE PENTAGON REMAINS FIXED. IN THE ABOVE FORMULA IF THE LHS CHANGES THE RHS ALSO CHANGES

Figure 9.2.1.5: Logan's explanation for his conjecture generalization

In his response (as shown in Figure 9.2.1.5), Logan seems to have remembered from his two previous activities that he should first express his equation, $A = ½a\ (h_1 + h_2 + h_3 + h_4 + h_5)$, which he produced in question (iv), into the form $\frac{2A}{a} = (h_1 + h_2 + h_3 + h_4 + h_5)$. However, Logan seemed not to have quite finished off his explanation with a logical argument. Hence, the Researcher intervened and through further probing, Logan then produced the following written explanation (see Figure 9.2.1.6), using his statement $\frac{2A}{a} = (h_1 + h_2 + h_3 + h_4 + h_5)$.

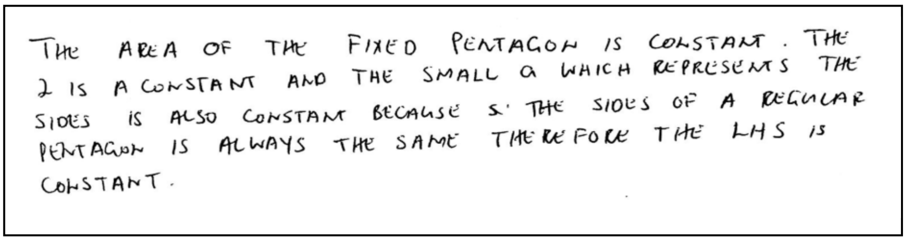

THE AREA OF THE FIXED PENTAGON IS CONSTANT. THE 2 IS A CONSTANT AND THE SMALL a WHICH REPRESENTS THE SIDES IS ALSO CONSTANT BECAUSE S' THE SIDES OF A REGULAR PENTAGON IS ALWAYS THE SAME THEREFORE THE LHS IS CONSTANT.

Figure 9.2.1.6: Logan expanding on his logical explanation

Logan's continuation of his explanation, as shown in Figure 9.2.1.6, seems to have shown a deeper understanding as to why the left hand side of the equation, namely $\frac{2A}{a}$ , is constant. Although Logan did not conclusively end his argument by stating 'therefore the right hand side, namely $(h_1 + h_2 + h_3 + h_4 + h_5)$, is constant', it is plausible that is what he meant since the task was to explain just that.

### 9.2.2 Logical explanation by Inderani, Victor and Tony for regular pentagon CG by eeing the general through the particular

This section focusses on the development of a logical explanation for a regular pentagon (convex) conjecture generalization produced by either analogical or inductive reasoning and subjected to some experimental confirmation or conceptual clarification in a *GSP* context, by analogically seeing that the 'triangle-area' proof structure for the previous equilateral triangle and rhombus cases could also be applied to the regular pentagon case:

Two students, namely Inderani and Victor, who constructed their conjecture generalizations for a pentagon and restricted it to a regular pentagon by thinking that 'regular' is a necessary condition for equal sides, used *Sketchpad* to confirm their initial regular pentagon conjecture generalization. However, Inderani on her own accord requested to use *Skecthpad* to test and confirm her conjecture generalization (see Finding 3 of Section 9.1.6), whilst this was not the case for Victor. In fact, the researcher on seeing that Victor did not express full certainty in his conjecture generalization when initially asked, gave Victor the opportunity to engage in some experimentation within a *Sketchpad* context (see see Finding 4 of Section 9.1.6) . On the other hand, Tony who also made his conjecture generalization on the grounds of the 'sides of a pentagon are equal', expressed some reservations in his conjecture generalization, primarily because he had not seen a pentagon which was not regular. Hence, in this instance, the researcher asked Tony to use *Sketchpad* and construct a pentagon by first plotting any five points and then joining them. Through, this *Sketchpad* construction Tony came to realize that not all pentagons have equal sides, and subsequently expressed 100% certainty in his initial *Sketchpad* conjecture generalization (see Finding 5 of Section 9.1.6 for details).

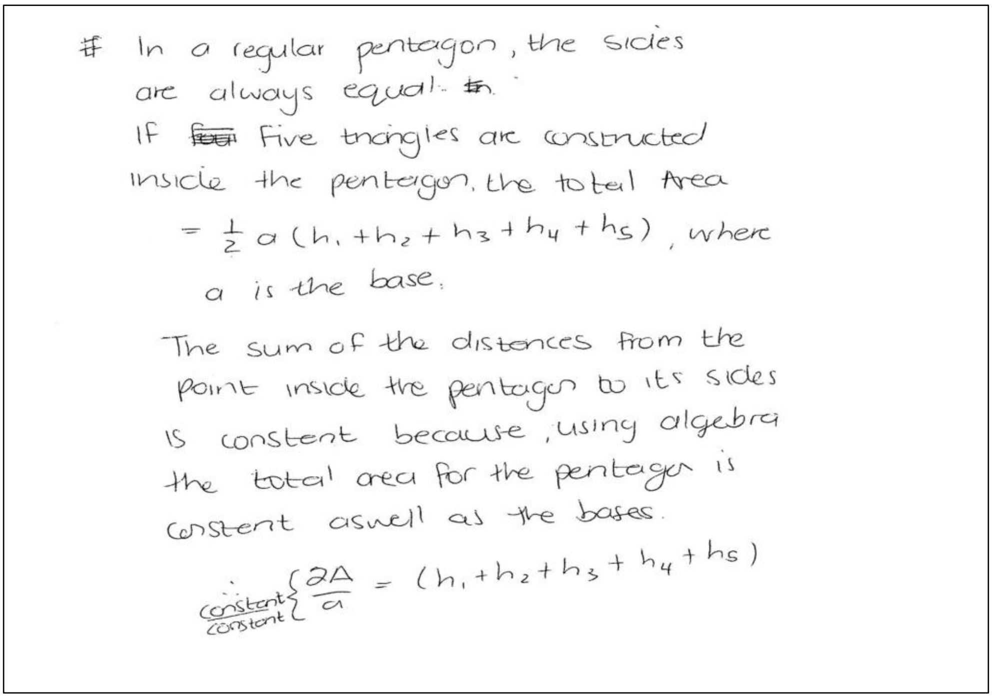
Although, the three students, namely Inderani, Victor and Tony, engaged with *Sketchpad* from different standpoints to arrive at full certainty in their regular pentagon conjecture generalizations, they seemed to have constructed their logical explanations for their regular pentagon conjecture generalizations by analogically seeing that the 'triangle-area' proof structure for the previous equilateral triangle and rhombus cases could also be applied to the regular pentagon case. The representative justifications that were constructed by Inderani, Victor and Tony during the one-to-one task based interviews are presented case-wise with accompanying dialogues and worksheet excerpts as follows.

**Case: Inderani**

When the Researcher asked Inderani to explain why her conjecture regular pentagon generalization was true, she replied as follows:

INDERANI: Using the small triangles? Because the sides of a pentagon are equal, thereforeall the bases of the small triangles inside the pentagon are all equal. So if I had to use a formula, the *a*'s are all constant there, because all the bases, all the sides of the pentagon are equal, and using the same principle as before, if you divide the total area of the pentagon by the bases, it will be a constant over a constant – this will give you a constant. And it's always true, because the bases (*referring to the sides of the regular pentagon*) of any regular pentagon are always the same – always equal.

It seems that when Inderani was constructing her logical explanation, she analogically mapped each step and idea onto the steps and ideas that was previously used to construct her logical explanations for her respective equilateral triangle and rhombus conjecture generalizations. Although her assertion that the sides of any regular bases are always equal is correct, it is plausible that she is not yet aware that one could get a pentagon with equal sides which was not regular.


In a regular pentagon, the sides are always equal.

If Five triangles are constructed inside the pentagon, the total Area

$= \frac{1}{2} a (h_1 + h_2 + h_3 + h_4 + h_5)$, where $a$ is the base.

The sum of the distences from the point inside the pentagon to its sides is constent because, using algebra the total area for the pentagon is constent aswell as the bases.

$\therefore \frac{\text{constent}}{\text{constent}} \left\{ \frac{2A}{a} = (h_1 + h_2 + h_3 + h_4 + h_5) \right.$

Figure 9.2.2.1: Inderani's logical explanation of her regular pentagon CG

When the Researcher, asked her to write down her explanation in the given worksheet, she explicitly limited her explanation to regular pentagons, and not for any other kinds of pentagons, as can be seen in the worksheet extract contained in Figure 9.2.2.1. This again suggests that she has not seen or is not aware that one could have an equilateral pentagon that is not regular (i.e. a pentagon with equal sides but unequal angles).

**Case: Victor**

When the researcher asked Victor as to whether he wanted to know why his regular pentagon conjecture generalization is true, he replied “yes”. Victor’s ‘yes’ response confirmed that he needed an explanation. When the researcher asked him “can you explain it for me?”, he immediately without any guidance produced the following insightful explanation verbally:

VICTOR: Since point *P* is inside the fixed pentagon with constant sides (meaning equal sides), if I can make the construction of triangles, I can have triangle *EPD*, triangle *DPC*, triangle *CPB*, triangle *BPA* and triangle *APE*, the sum of the areas of these triangles would – if *A* was the area of the pentagon – equal the area of triangle *EPD*, triangle *DPC*, triangle *CPB*, triangle *BPA* and triangle APE. Since all sides are equal, I can label them with *a*. Then, I can have $A = ½\, ah_1 + ½\, ah_2 + ½\, ah_3 + ½\, ah_4 + ½\, ah_5$. Then I can take out the common factor which gives me: $A = ½\, a\, (h_1 + h_2 + h_3 + h_4 + h_5)$. Then I can simplify that and get $h_1 + h_2 + h_3 + h_4 + h_5 = 2A$ over small *a*. *A* will be the area of the pentagon, which is fixed (*meaning constant*); *a* is the length of the distance of the pentagon, which is constant. Therefore $h_1 + h_2 + h_3 + h_4 + h_5$ is also constant.

Figure 9.2.2.2 represents Victor’s hand written explanation in his worksheet

Since Point P is inside the fixed pentagon
with a ~~fixed~~ Constant sides. If I can
make the construction of triangles I can
have ~~triangle~~ Δ EPD, ΔΔPC, Δ CPB, Δ BPA and Δ APE
The sum of area of these triangles would be
$A = \Delta EPD + \Delta\Delta PC + \Delta CPB + \Delta BPA + \Delta APE$
since all sides are equal I can label them
with a
then
$A = \frac{1}{2}ah_1 + \frac{1}{2}ah_2 + \frac{1}{2}ah_3 + \frac{1}{2}ah_4 + \frac{1}{2}ah_5$
$A = \frac{1}{2}a(h_1 + h_2 + h_3 + h_4 + h_5)$
$h_1 + h_2 + h_3 + h_4 + h_5 = \frac{2A}{a}$
A would be the area of the pentergon which is fixed
(a) is a lengths of the sides of the pentergon wich is
constant
$\therefore h_1 + h_2 + h_3 + h_4 + h_5$ is also constant

Figure 9.2.2.2: Victor’s logical explanation of his regular pentagon CG

Victor's explanations suggest that he was able to look at the structure of his explanations for his previous conjecture generalizations (i.e. folding back), namely an equilateral triangle and a rhombus, and mirror a similar kind of explanation for the regular pentagon case. In other words, it seems that Victor, like Inderani, first assimilated his explanation into the previously accommodated explanations for either (or both) the equilateral triangle case of rhombus case, and then subsequently modified it for the regular pentagon case.

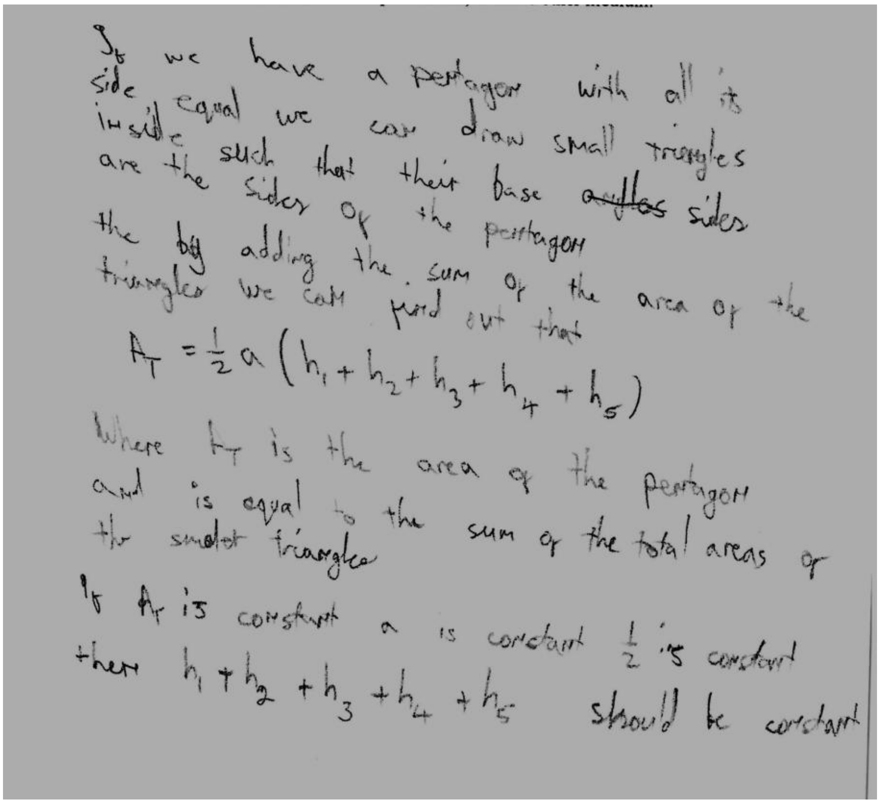
**Case: Tony**

When the researcher asked Tony to explain why he was 100 percent sure about his conjecture generalization, he spontaneously produced the following explanation:

TONY: We can make those small triangles inside but, if we make the small triangles inside, then that would mean we would have about five triangles inside with the same base and which we can call *a*. And then we can take that *a* when we find the sum of the area – meaning we find the total area of the pentagon – we can take that half of *a* out, and then we have that sum as being the constant.

The researcher could see that Tony was attempting to structure his explanation by similarly using the 'triangle-area' algebraic strategy that he had previously used to construct logical explanations for his equilateral triangle and rhombus conjecture generalizations respectively. The researcher then requested Tony to write down the main parts of his explanation in the worksheet. Figure 9.2.2.3 represents the explanation Tony produced in his worksheet.

It seems that Tony like others such as Inderani & Victor, after progressing through the previous equilateral triangle and rhombus problem, had seen the similarity between the pentagon problem, although regular in this instance, through the 'equal' sides property. Hence, it is plausible that Tony, like the others, was able to see his logical explanation for his regular pentagon conjecture generalization through the 'eyes' of his earlier 'triangle-area' logical explanations for his equilateral triangle and/ or rhombus conjecture generalizations via the connecting special property 'equal sides'. The earlier equilateral triangle conjecture generalization explanation and/ or rhombus conjecture generalization explanation semed to have served as a generic example(s) or generic proof(s), and thus enabled him to construct a logical explanation (a particular proof in the context of the general Viviani problem) for another instance, namely a proof for the regular pentagon conjecture generalization.



If we have a pentagon with all its
side equal we can draw small triangles
inside such that their base ~~angles~~ sides
are the sides of the pentagon
the by adding the sum of the area of the
triangles we can find out that

$$A_T = \frac{1}{2}a\left(h_1 + h_2 + h_3 + h_4 + h_5\right)$$

Where $A_T$ is the area of the pentagon
and is equal to the sum of the total areas of
the smaller triangles

If $A_T$ is constant a is constant $\frac{1}{2}$ is constant
then $h_1 + h_2 + h_3 + h_4 + h_5$ should be constant

Figure 9.2.2.3: Tony’s logical explanation of his regular pentagon CG

A generic example enables one to see the general through the particular as described by Mason & Pimm (1984), and in the process enables one to move from one particular proof to another particular proof by transferring the generic argument from one particular instance to another particular instance. Furthermore, as seen in the case of Tony and others like Inderani, Victor, it seems that logical explanations for equilateral triangle and/ or rhombus conjecture generalizations conforms to the notion of a generic example as described by Rowland (1998), since each provided the mentioned PMTs with insight as to why their conjecture generalization for another instance, namely a regular pentagon, held true for (see section 5.3.1 of Chapter 5).

### 9.2.3 Logical explanation by Shannon, Trevelyan and Renny for regular pentagon CG by seeing the general through the particular

This section focusses on the development of a logical explanation for a regular pentagon (convex) conjecture generalization (which they developed through either analogical or inductive reasoning and without any requirement for experimental exploration) through analogically seeing that the ‘triangle-area’ proof structure for the previous equilateral triangle and rhombus cases can also be applied to the regular pentagon case.

The three PMTs, namely Shannon, Trevelyan and Renny who produced their conjecture generalizations through either analogical or inductive reasoning, seemed to have seen analogy with the equilateral triangle and rhombus cases respectively, and hence proceeded to construct their ‘triangle-area’ algebraic explanations for the regular pentagon conjecture generalization by analogically mapping the structure of their previous triangle-area’ algebraic explanations for the equilateral triangle and rhombus case onto the regular pentagon. The following cases of Shannon and Tony, which are representative of the PMTs justifications in this group, are presented with supporting dialogue excerpts and worksheet excerpts from their one-to one task based interview sessions to illustrate the typical development of logical explanations amongst this group of PMTs (namely, Shannon, Trevelyan, and Renny).

**Case: Shannon**

When the Researcher asked Shannon to explain why her regular pentagon conjecture generalization was true, she replied as follows:

SHANNON: Because you can also divide it into triangles, and do exactly the same as you did there…uum… (Referring to the *Rhombus proof*)

However, the researcher was curious to see if Shannon could quickly articulate what she meant by “do exactly the same”. The following interview excerpts, which makes reference to both her hardcopy sketch (see Figure 9.2.3.1) and onscreen dynamic sketch, clearly demonstrate that she was able to discern with a great degree of confidence and certainty that the ‘triangle area” proof technique that she used to previously construct logical explanations for her respective equilateral triangle and rhombus conjecture generalizations, could similarly be used to construct a logical explanation for her regular pentagon conjecture generalization.

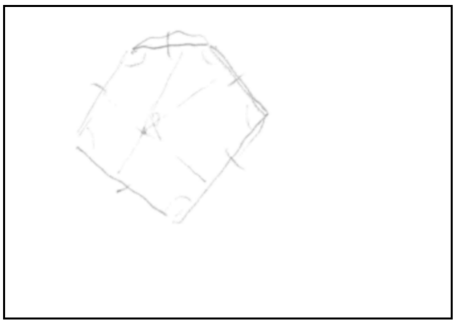


Figure 9.2.3.1.: Shannon’s sketch showing an equilateral pentagon

RESEARCHER: Okay, let's open the sketch. Let's open the regular pentagon. Enlarge the screen. Okay, you said you can do exactly the same; can you briefly tell me – just describe what you're going to do?
You can use the buttons if you want to.

SHANNON: Yes, you would draw a triangle. Can I use the small triangle button? You would use the pentagon with five triangles that the pentagon consists of, and being given the altitude of each triangle; you can prove it exactly the same as you did for the previous ones.

RESEARCHER: Ja, okay, I understand what you're saying, but can you just talk more … when you say 'the same' …? Take me through the process.

SHANNON: Well, the area of a pentagon is made up of the five triangles, as shown here (see Figure below), and then you can write down that the area of Figure *ABCDE* = the Area of each specific triangle – the five triangles you would name, respectively; write down what the formula for the area is: ½ times $a$ (and the lengths of all the sides are the same) times $h$, and you would come to the conclusion that the area of this pentagon = ½ times one of these lengths of the sides, times (in brackets) $(h_1 + h_2 + h_3 + h_4 + h_5)$, and the conclusion then is $h_1 + h_2 + h_3 + h_4 + h_5 = 2A$ (which is your area), over small $a$ which is the length of the sides. And I suppose you can carry this on for a polygon of six sides, seven sides – and all the areas would be the same.

As illustrated in the tail- end of the afore-cited excerpt, Shannon confidently suggested that the same kind of explanation would extend to other polygons like a six sided polygon and seven sided polygon. Although, the Researcher did not seize the opportunity to ask Shannon whether she was referring to a regular polygon or not, there exist a strong possibility that she could have been thinking strictly of regular polygons only. This, nevertheless illustrates her level of certainty in her regular pentagon conjecture generalization and associated explanation (or proof) thereof.

Furthermore, as described for the cases of Inderani, Victor and Tony in Section 9.2.2, it seems for Shannon (as well others described in this section 9.2.3), the logical explanations for the equilateral triangle and/ or rhombus conjecture generalizations seemed to have served as generic examples for Shannon (and others), and pointed to a more general truth. Hence, Shannon like others were able to construct a proof, although a particular proof in the context of the Viviani Problem, with the adequate insight as to why their regular pentagon conjecture generalization holds true for them. By reflecting and considering the mentioned generic examples, Shannon like the other PMTs were able to carry the 'sameness' to another specific instance, namely the construction of a logical explanation for the regular pentagon conjecture generalization (see discussion on generic proving in 5.3.1).

**Case: Renny**

After Renny affirmed that his conjecture generalization would only hold for regular pentagons, the Researcher asked him to write down a coherent explanation for his regular pentagon conjecture generalization in the provided worksheet. The worksheet extract in Figure 9.2.3.2 shows precisely what Renny wrote in his worksheet:

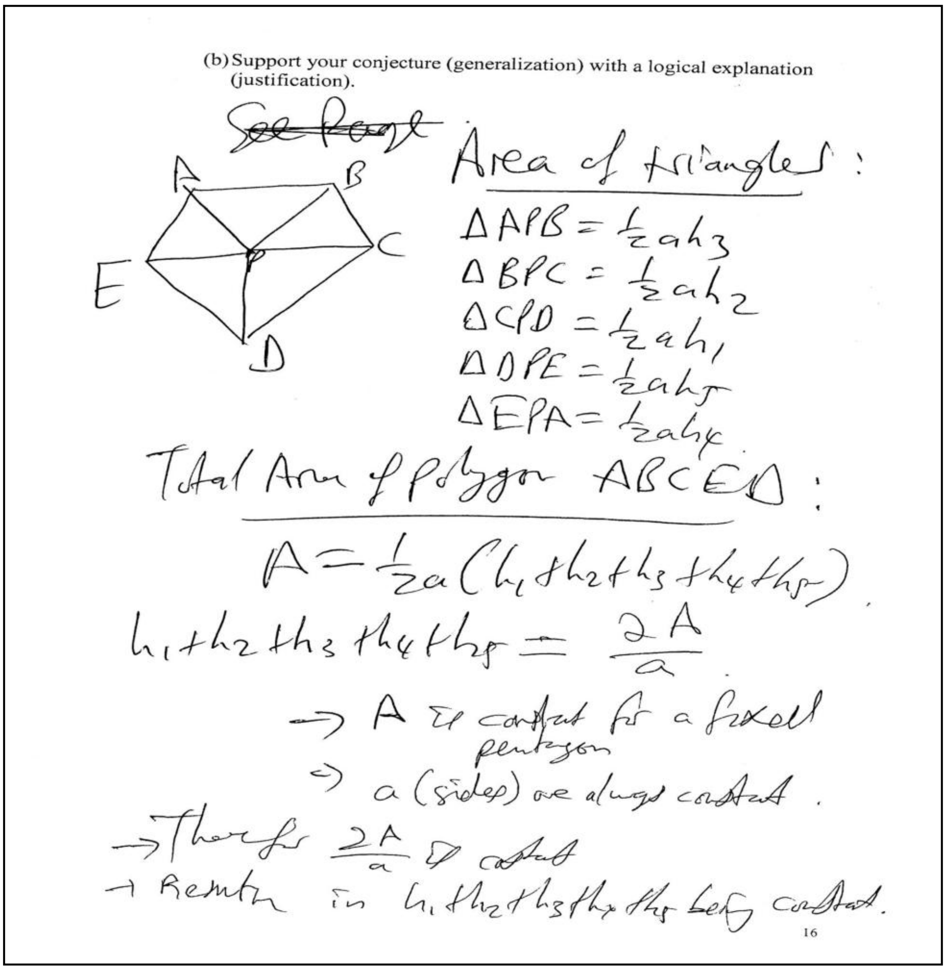



Figure 9.2.3.2: Renny's logical explanation of his regular pentagon CG

It seems that Renny, like the other 5 PMTs, was able to construct his logical explanation for his regular pentagon conjecture through a process of correlative subsumption, which is one of the primary processes of learning in which new material (or new problems) is related to previous knowledge or relevant in the existing cognitive structure (see Ausubel, 1978; Aziz et al., 2009). In particular, it seems that Renny's cognitive structure, like the other 5 PMTs, on having seen a similarity between the rhombus problem (and/or equilateral problem) and the pentagon problem (although a regular one in this instance as a result of an apparent misconception) on the grounds of seeing that 'equal sides' property manifested itself in all three problems, was triggered to correlatively subsume the regular pentagon problem (i.e. the construction of logical explanation for his regular pentagon conjecture generalization) into his earlier conceptual 'triangle-algebraic' explanatory structure for the rhombus (or equilateral triangle) conjecture generalizations. Through the process of correlative subsumption, Renny seemed to have made the cognitive link between his new problem (i.e. the construction of logical explanation for his regular pentagon conjecture generalization) and the existing 'triangle-area' algebraic explanation for his rhombus (equilateral) conjecture generalization, with aid of analogical reasoning, and then modified his 'existing 'triangle-area' algebraic explanation for his rhombus (equilateral) conjecture generalization to accommodate his 'triangle-area' algebraic explanation for his new regular pentagon conjecture generalization.

#### 9.2.4 Findings as per Section 9.2 (PMTs justifying their pentagon CG)

1. One PSTE, namely Logan, who seemed to have produced his conjecture generalization to a pentagon (convex) through either guessing, inductive reasoning or superficial analogical reasoning, but restricted it to a regular pentagon – a misconception on the grounds of thinking that equal sides imply equal angles. He struggled to put together a justification in the form of a logical explanation for his regular pentagon conjecture generalization on his own, and hence was provided with a scaffolded worksheet and necessary probing (and facilitation) by the researcher to enable him to develop a logical explanation. However, Logan was able to complete all parts of the worksheet with great ease and confidence, and it is plausible that his previous experience with a similar kind of scaffolded worksheet that was used for the construction of logical explanations for his equilateral triangle and rhombus conjecture generalizations, could have helped him to complete the worksheet with virtually no hurdles. This suggests that the earlier scaffolded activities had to an extent enhanced both his competency and insight on using the 'triangle-area' algebraic approach to explain similar conjecture generalizations.

2. a. The three students, namely, Shannon, Trevelyan and Renny, who seemed to have produced their conjecture generalization on either inductive of analogical gounds, but restricted it to a regular pentagon conjecture generalization and required no experimental confirmation, seemed to have analogically seen that the ‘triangle-area’ proof structure that they had previously used to explain their conjecture generalizations for the equilateral triangle and rhombus cases could also apply to the pentagon since it also had ‘equal sides’, and hence similarly constructed a logical explanation for their regular pentagon conjecture generalization.

   b. One PSTE, namely Inderani, who formulated her conjecture generalization on either inductive or analogical grounds, restricted it to a regular pentagon and required experimental confirmation. Like Shannon, Trevelyan and Renny, she seemed to have analogically seen that that the ‘triangle-area’ proof structure that had previously used to explain for the equilateral triangle her conjecture generalization and rhombus her conjecture generalization could also apply to the pentagon since it also had ‘equal sides’, and hence constructed a logical explanation for their regular pentagon conjecture generalization along similar lines.

   c. One PSTE, namely Victor, who formulated his conjecture generalization on either inductive or analogical grounds, restricted it to a regular pentagon, and did not request to experimentally confirm his conjecture generalization. However, the Researcher on seeing Victor showing some degree of uncertainty in his regular pentagon conjecture generalization, requested him to experimentally explore his conjecture generalization, and only after seeing that the distance sum remained constant, did Victor express full certainty in his conjecture generalization. After the aforementioned experience, Victor like Shannon, Trevelyan Renny, and Inderani, seemed to have produced his logical explanation for his regular pentagon conjecture generalization by mapping the structure of ‘triangle-algebraic’ explanation that he produced previously onto the structure of the explanation for the regular pentagon conjecture generalization.

   d. One PSTE, namely Tony, formulated his conjecture generalization on either inductive or analogical grounds, restricted it to a regular pentagon, but wanted to know if there existed a pentagon with unequal sides. The latter request by this students demonstrates how limited his conception are of polygons beyond triangles and quadrilaterals. The researcher subsequently provided an opportunity for Tony to experiment and see if such a pentagon existed, and Tony after seeing a constructed pentagon with unequal sides confessed 100 certainty in his regular pentagon conjecture generalization. However, like Shannon, Trevelyan Renny, Inderani and Victor, Tony seemed to have the 'regularity misconception', and also analogically saw that the structure of the 'triangle-area' algebraic explanation that was offered for both the equilateral and rhombus conjecture generalization could also be used to construct a logical explanation for his regular pentagon conjecture generalization, and hence went ahead to successfully produce a logical triangle-area' algebraic explanation for his regular pentagon conjecture generalization via analogical structural mapping.

3. As discussed in Findings 2 (a-d) in the preceding paragraphs, it seems that these six PMTs , namely Shannon, Trevelyan Renny, Inderani and Victor, Tony, having seen a similarity across the equilateral triangle, rhombus and pentagon problems, were then able to reflect on the 'triangle-area' explanations raised for the equilateral triangle and rhombus cases respectively, and see the logical explanation for the regular pentagon case with the aid of analogical mapping. This latter experience by six PMTs appears to demonstrates that in searching or cognizing about the logical explanation for a given conjecture generalization, it is plausible for one to discover the desired explanation via analogical mapping. Furthermore, it demonstrates that a logical explanation for a given conjecture generalization can be constructed with the aid of analogical reasoning.

   Moreover, it means that these six PMTs have 'seen' the proof for the regular pentagon conjecture generalization through a set of particular proofs (i.e. proofs for equilateral triangle conjecture generalization and rhombus conjecture generalization respectively), i.e. they have seen the proof through particular proofs, which is an inductive generalization of a proof in a sense.

4. All six PMTs as described in Findings 2 (a-d) seemed to have assimilated their explanations for the regular pentagon into their previously described explanations for the rhombus and equilateral triangle conjecture generalizations, and then modified it to accommodate the regular pentagon cases. The aforementioned assimilation – accommodation of the logical explanations, is equivalent in nature to Ausubel's theory of correlative subsumption, because it seems that in conceptualizing the logical explanation for the regular pentagon conjecture generalization a link was made to an

already existing idea, namely the ‘triangle area’ algebraic explanation for the rhombus conjecture generalization and/or equilateral triangle conjecture generalization, which was then modified to produce (or accommodate) the logical explanation for the regular pentagon case (see discussion of correlative subsumption in Section 4.5.1).

5. As described for the cases of Inderani, Victor, Tony in section 9.2.2, and Shannon and Renny in section 9.2.3, without the exclusion of Trevelyan, it seems that the logical explanations for the equilateral triangle and/ or rhombus conjecture generalization seemed to have served as generic example(s) for each PMT and pointed to a more general truth. Hence, each of the mentioned PMTs were able to construct a proof for their regular pentagon conjecture generalization, with the necessary insight as to why their regular pentagon conjecture generalization holds true for them. By reflecting on and considering the mentioned generic example(s), the stated PMTs were able to carry the ‘sameness’ to another specific instance, namely the construction of a logical explanation for the regular pentagon conjecture generalization (see discussion on generic proving in Section 5.3.1). Thus, it appears that a generic example enables the mentioned PMTs to see the general through the particular as described by Mason & Pimm (1984), and hence was able to move from one particular proof to another particular proof by transferring the generic argument from one particular instance to another particular instance

6. Since Alan produced his pentagon conjecture generalization for any pentagon immediately on logical grounds (as discussed in Section 9.1.5), he was not asked by the researcher to justify his pentagon conjecture generalization in this section, and hence there is no report on him. However, on reflecting on the argument that he used to make his pentagon conjecture generalization, it would appear that he assimilated his argument into the previous ‘triangle-algebraic’ explanations for the rhombus and equilateral triangle conjecture generalizations, and then modified it to accommodate his thought processes. I hasten to suggest that one could also construct conjecture generalizations, immediately on logical grounds with the aid of analogical reasoning as has been argued by De Villiers (2008). It seems that his earlier logical explanations for his equilateral triangle conjecture generalization and rhombus conjecture generalization acted as generic examples and thereby provided him with the necessary insight on how to construct a logical explanation for his pentagon conjecture generalization.

## 9.3 Pre-service mathematics teachers facing a Heuristic counter example (Mystery Pentagon)

This section focuses on PMTs facing a heuristic counter example (*Mystery Pentagon*) to restrictive assumption(s) embodying their regular pentagon (convex) conjecture generalization, and their subsequent modification of their pentagon (convex) CG. As discussed in Section 9.1., 7 out of 8 of the PMTs formulated their pentagon conjecture generalization by restricting it just to the regular pentagon, apparently on the grounds of having a misconception that only regular pentagons have equal sides. To try and correct this misconception, the researcher engaged all 7 PMTs in the *Mystery Pentagon* activity as shown in Figure 9.3.1, which was part of the worksheet Task 3(c) as contained in Appendix 2.

> ***Mystery Pentagon* Activity**
>
> 4(a) Do you think the result might be true for other kinds of pentagons? If not, why. If so, why?
> (Alternate Question 4a: Does your generalization hold true only for regular pentagons)
>
> (b) Do you want to test or confirm your response to Q 4 (a)?
>
> (c) Open the sketch ***Mystery Pentagon*.gsp**. Investigate whether your generalization explained in Question 3(b) holds true or not .
>
> (d) What special property must a pentagon have so that it can yield the following result: the sum of the distances from an interior point in the pentagon to its sides remain constant?
>
> 5. Consider your response to 4(d) and then write down your final generalization with respect to the sum of the distances from an interior point in a pentagon to it sides.

Figure 9.3.1 : *Mystery Pentagon* Activity (Task 3(c): 4 and 5 of Appendix 2)

Furthermore, the aim of the *Mystery pentagon* activity was to get PMTs to experience a heuristic counter-example (critical example) to their restrictive assumptions implicit in their conjecture generalization, which seemed to have caused them to restrict their CG to regular pentagons only, and consequently force them to modify their conjecture generalization to hold not only for regular pentagons, but rather any pentagon that has all its sides equal (i.e. to include an irregular pentagon that has all its sides equal, but not all angles equal). The aforementioned questions as shown in Figure 9.3.1 were not necessarily posed in the linear order as described, but rather was used as a guide to probe a PMTs response that s/he produced at a given moment during the task-based interview. For example, alternate question

4a, was posed to PMTs that restricted their conjecture generalization to regular pentagons through thinking that only regular pentagons had equal sides. In retrospect, Q4(a) and/or Q4(b), was posed to some PMTs in the previous activities as reported in Section 9.1, to try and capture the kind of immediate thoughts that a specific PMT had in his/her mind when s/he made a specific assertion or statement, rather than leaving it to be probed at some later point in the task-based interview. Hence, there was some fluidity as to the use of the questions as shown in Figure 9.3.1.

The experience of this group of seven PMTs are represented case-wise via the following four cases: Shannon, Trevelyan, Tony, Victor. Each of the accompanying case presentations contains dialogue excerpts and worksheet excerpts as it unfolded during each of the one-to-one task based-interview sessions during which each PMT was engaged with the Mystery Pentagon activity.

**Case: Shannon**

After Shannon had demonstrated that she managed to logically explain (i.e. prove) (see Section 9.2.3) her conjecture generalization, namely that the point $P$ can be positioned anywhere in a regular pentagon so that the sum of the distances to the sides remains constant, the researcher asked her: "Do you think the result might be true for other kinds of pentagons?". Shannon responded immediately and confidently by saying "Not necessarily", without doing any empirical investigations. When the researcher asked Shannon "why?", Shannon replied as illustrated in the following excerpt, by referring to the structure of the proof that she provided for the regular pentagon.

| | |
|---|---|
| RESEARCHER: | Why? |
| SHANNON: | (pause) … Oh yes, because for other pentagons, the length of the sides will not be equal – the sides aren't equal. And then you can't simplify your equation to get to a result of $h_1 + h_2 + h_3 + h_4 + h_5$ |
| RESEARCHER: | Why wouldn't you be able to simplify the other result? |
| SHANNON: | Because all the lengths of the sides $AB$, will not necessarily be equal to BC would not necessarily be equal to $CD$ would not necessarily be equal to DE would not be equal to $EA$. Because if it isn't a regular pentagon, then that property doesn't (indistinct) 'get involved'. |

Shannon clearly is set on thinking that only regular pentagons have equal sides, and hence thinking the result will not hold for irregular pentagons. Cleary she does not realize that an irregular pentagon could be one that has:

a) Equal sides but unequal angles, or

b) Unequal Sides but equal angles, or

c) Unequal sides and unequal angles.

The researcher then provided her with an irregular pentagon S*ketchpad* sketch as shown in Figure 9.3.2, but Shannon (like others) at this point in time was not informed that the given pentagon was irregular or anything of that nature. The irregular pentagon in this instance had equal sides but unequal angles. Shannon, like the other PMTs, came to realize that the mystery pentagon had equal sides and unequal angles, through clicking on the buttons on the pre-contsructed dynamic sketch, and seeing that the measurements of the respective sides of the mystery pentagon were equal whist the measure of its respective angles were not all equal. The purpose for this activity was primarily to see if the PMTs could isolate a special property that must be present in a given pentagon so that a point could be placed anywhere inside the given pentagon such that the sum of the distances from the placed point to all sides is small as possible (or remains constant).

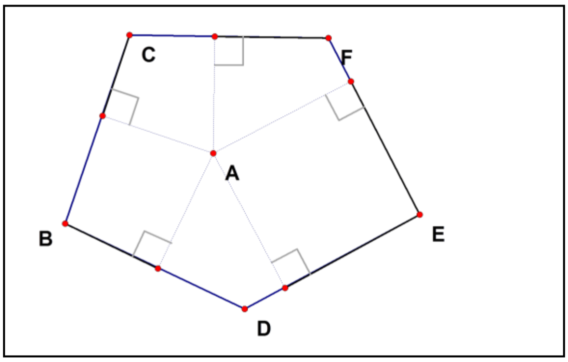



Figure 9.3.2: Irregular pentagon with equal sides but unequal angles

Thus, when the *Mystery Pentagon Sketch* activity was presented to Shannon (and also to the other PMTs) she was not informed as to whether it was regular or irregular. However, through questioning, Shannon as illustrated in the following one-to-one task based interview excerpts, was able to discover that she was dealing with a pentagon which had equal sides but unequal angles.

RESEARCHER: Okay, so you're saying your result holds only for regular pentagons. Okay. Now let's look at… just open the mystery pentagon. Here is the pentagon, there. You can click on the button to show the measure of sides. What can you tell me about the sides?

SHANNON: The sides are all equal? (*Although the measurements displayed, Shannon seemed to have made this judgement on visual grounds*)

RESEARCHER: Okay. And can you click on the button that shows 'angle measurements'?

SHANNON: The angles aren't equal.

Immediately thereafter, the researcher engaged Shannon with the empirical investigation in a *Sketchpad* context. After manipulating the situation dynamically through a continuity of cases, Shannon observed an invariant property (or result) in the situation, and she stated with an expression of surprise: "So, it (the sum of the distances to the sides) stays the same. But it's not necessarily a regular pentagon, but its sides are all equal, but its angles are different". This self discovery exhibited by Shannon, prompted the researcher to ask Shannon the following question: "Now looking at this result, what is your conjecture?' Shannon responded as follows:

SHANNON: The conjecture is we've shown the area (just to get this right in my mind … Area = ½ times the side length times ($h_1$ + $h_2$+…) but your angle was never part of the formula in the first place! So, the angle never played any role. So as long as the sides are equal – of your polygon (*in this case she was referring to a pentagon*) – the sum of the distances from the sides to a point *P* in the polygon *(in this case she was referring to a pentagon)*, would be constant.

RESEARCHER: Okay.

SHANNON: So it doesn't necessarily have to be a regular pentagon, or a regular polygon … just all the sides must be equal.

Shannon's comments as reflected in the above excerpts, appear to indicate that she has finally realized that "all sides equal" on its own is a sufficient condition to ensure that the sum of the distances from a given point to the sides of the pentagon to remain constant. She also acknowledged, that the pentagon does not have to be necessarily regular, and suggests that she has come to realize that  her conjecture generalization could be extended to irregular pentagons so long as all its sides are equal. Thus in a way, Shannon's responses as illustrated in the afore-cited excerpt, demonstrates how the appropriate  pedogogical use of a heursistic counter example to restrictive assumptions such as 'all regular penatgons are equilateral'  and 'distance is constant only for regular pentagons' can enable a student to move beyond a given assumption and hence modify (or refine) their generalization accordingly to be more encompassing.

**Case: Renny**

As discussed in section 9.1.3, Renny confirmed that his conjecture generalization would hold for regular pentagons, and subsequently as indicated in section 9.2.3 was able to provide a justification for his regular pentagon conjecture generalization in the form of a logical explanation by analogically using the 'triangle-area' algebraic strategy that was used previously to construct a logical explanation for the equilateral triangle and rhombus conjecture generalizations respectively. However, the researcher taking cognizance of Renny's earlier pentagon conjecture generalization, which he restricted to a regular pentagon seemingly on the grounds of having a misconception that 'regular is a necessary condition for equal sides', asked Renny, "…for what other kind of pentagons do you think the results will hold true?". Renny's apparent silence and then a later hesitant response, "…its …  a regular pentagon", appears to suggest that he thought that his conjecture would hold true only for regular pentagons. Hence, the Researcher used this opportune moment to engage Renny in the heuristic counter-example activity, known as the *Mystery pentagon* activity, as illustrated in the following task based interview excerpts and commentary by the Researcher.

Firstly, the researcher asked Renny to recap "what is a regular pentagon?", and Renny responded by providing the following correct definition:  "the sides are equal and the angles inside the pentagon are equal". Thereafter the researcher engaged renny with the *Mystery pentagon* as indicated in the following dialogue excerpt:

| | |
|---|---|
| RESEARCHER: | Open the Mystery  pentagon |
| RENNY: | Okay, (*working)* |
| RESEARCHER: | Okay…, show the measurement of the sides there (*the Researcher was requesting Renny to click on the button the shows the measure of the sides of the pentagon*) and show the angle measurements (*the Researcher was requesting Renny to click on the button the shows the* |

|  |  |
|---|---|
|  | *measure of the angles of the pentagon*)…. Now looking at the measurements of the sides, are the sides all equal? |
| RENNY: | Yeah! |
| RESEARCHER: | and the angles? |
| RENNY: | No… they are different, not same. (seemed surprise) |

Renny seemed to be quite surprised at having seen a pentagon with equal sides and angles not equal. The researcher then proceeded with the one-to-one task based interview session as follows:

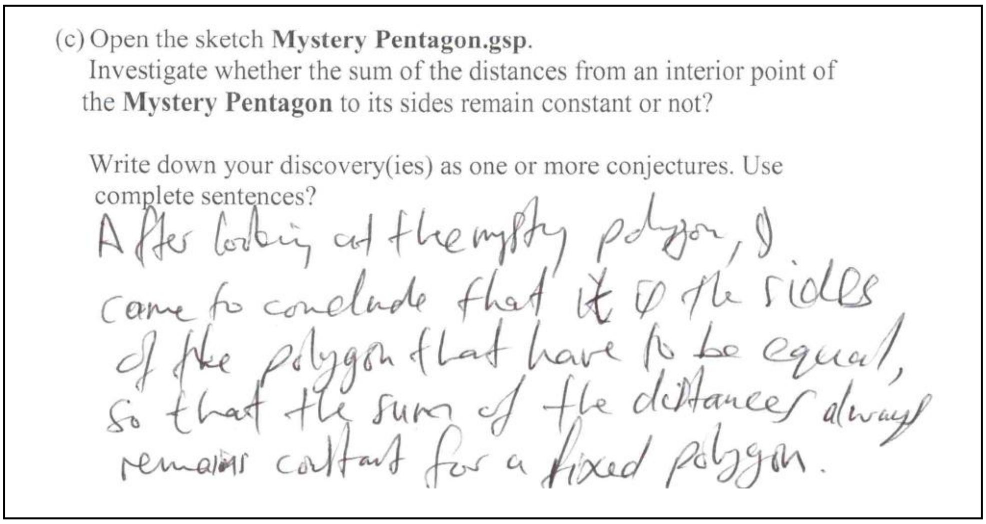
|  |  |
|---|---|
| RESEARCHER: | This pentagon that we are looking at, at the moment, its sides are equal and its angles are not equal? Let's see now what happens…, can you show the distances now? |
| RENNY: | Yes…..(*Renny clicks on the button to show distance sum*) |
| RESEARCHER: | Drag point *P*. .. What did you observe about the distance sum now? |
| RENNY: | The distances sum are the same (*was rather suprised*). |

The above excerpts, illustrate that through empirical investigation, Renny had experienced a visual heuristic counter- example to the following set of restrictive assumptions underlying his earlier CG which he though was true for only regular pentagons: 'all regular penatgons are equilateral' and 'distance is constant only for regular pentagons. Renny was rather surprised at his discovery. The researcher used this moment of surprise to enable Renny to distill a pentagon property that could be sufficient enough to allow the sum of the distances from a given point inside a pentagon to its respective sides to remain constant. The researcher facilitated this distillation process by subjecting Renny to a comparative task as illustrated in the following excerpt:

|  |  |
|---|---|
| RESEARCHER: | Now…, if you're looking at that pentagon right (pointing to the *Mystery Pentagon),* okay? And you're looking at your initial conjecture (referring to the CG for regular pentagon), you first said that it only holds true for regular pentagons? Now, this pentagon here is irregular. So, what is the important property that the pentagon must have, so that we get the result? |
| RENNY: | The sides must be equal…ag!..yeah, the sides must be equal. |

As illustrated in the aforementioned excerpt, Renny seems to have discovered that equal sides is a sufficient condition for a pentagon to produce a constant distance sum from any point

inside a pentagon to its respective sides. When the researcher asked Renny to write down his discoveries in his worksheet, he responded as in his worksheet as shown in Figure 9.3.3.


(c) Open the sketch **Mystery Pentagon.gsp**.
Investigate whether the sum of the distances from an interior point of the **Mystery Pentagon** to its sides remain constant or not?

Write down your discovery(ies) as one or more conjectures. Use complete sentences?

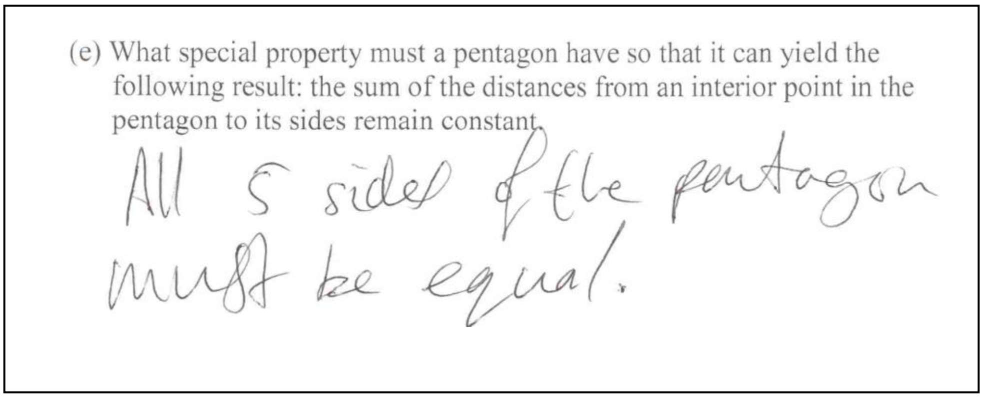
After looking at the mystery polygon, I came to conclude that it the sides of the polygon that have to be equal, So that the sum of the distances always remains constant for a fixed polygon.

Figure 9.3.3: Renny's conjecture after facing a heuristic counter-example

Despite the aforementioned written expression of Renny's discovery as shown in Figure 9..3.3, the researcher wanted to make sure about Renny's position regarding the property, "equal angles", and hence proceeded to ask Renny, "Do we need the angles to be equal?". Renny, immediately responded , "No…, the angles doesn't need to be equal, but the sides must be equal". The latter response re-affirms that Renny had finally come to realize, through experiencing a heuristic counter-example to his assumption(s), that 'equal sides' is all that a pentagon needs to have to produce a constant distance sum. Nevertheless, the researcher proceeded to probe Renny further as to a special property that a pentagon should have in order to produce a constant distance sum, as illustrated in the following excerpts:

| | |
|---|---|
| RESEARCHER: | Now, you've seen from the regular pentagon and the *Mystery pentagon* – what is important for the sum of the distances to be constant? |
| RENNY: | The five sides of the pentagon must be equal. |
| RESEARCHER: | Are you sure about that one? |
| RENNY: | Yes. |

Renny, who verbally acknowledged that 'all sides of the pentagon equal' is a special property that a pentagon should have to produce a constant distance sum, also produced the same kind of written response in his worksheet item (see Figure 9.3.4):


(e) What special property must a pentagon have so that it can yield the following result: the sum of the distances from an interior point in the pentagon to its sides remain constant.

All 5 sides of the pentagon must be equal.

Figure 9.3.4: Renny singled out the special pentagon property after facing a heuristic counter-example

The researcher on seeing that Renny was now quite sure that 'all sides equal' was a sufficient property for a pentagon to have in order for the sum of the distances from an interior point in the pentagon to its sides to remain constant, purposively provided an opportunity for Renny to refine the statement of his pentagon conjecture generalization that he had initially limited to regular pentagons only, as illustrated below:

RESEARCHER: So if you look at the first generalization you made with regard to a regular pentagon and your discovery with regard to the *Mystery pentagon*, right, I want you to see if you want to reformulate that generalization and conjecture – looking at the regular pentagon, and looking at the mystery pentagon.Can you answer that question (*Researcher pointing to Q5 of Task 3c*)?

Renny, subsequently responded as shown in Figure 9.3.5 in his worksheet. As illustrated in Figure 9.3.5, it seems that the experience of the heuristic counter-example has helped Renny to refine his earlier convex pentagon conjecture generalization, which was limited to just convex regular pentagons, to now be applicable to pentagons that just possess the following sufficient property: 'all sides equal'. Thereafter, the researcher asked Renny to provide a logical explanation for his refined conjecture.

5. Consider your response to 4(d) and if necessary edit or rephrase your initial conjecture (generalization) with respect to the sum of the distances from an interior point in a pentagon to it sides.

The sum of the distances from an interior point in a regular pentagon to its sides, will remain constant. But observing an irregular pentagon, which had all sides equal, but angles with different measures, the sum of the distance also remains constant.

If the sides of the polygon (pentagon) are equal, the sum of the distance from an interior point P to the sides of the pentagon, will not change. But the pentagon should not be changed in terms of ~~[illegible]~~ shape.

Figure 9.3.5: Renny's refined pentagon conjecture generalization

**Case: Inderani**

Although it was found, as described in Section 9.1.2., that Inderani confined her earlier pentagon conjecture generalization to a regular pentagon probably by thinking that only regular pentagons had equal sides, the researcher nevertheless asked her, "… now you spoke about the regular pentagon, do you think the result may be true for other kinds of pentagons?". Inderani responded as follows:

INDERANI: I don't think so because the sides are not equal for irregular pentagons, so therefore the bases will differ. If this was irregular and I had to draw five triangles in an irregular pentagon, you would get the bases of each triangle being different from the other one, so I don't think it would be the same.

Inderani's response, like the other PMTs, re-affirms her misconception that only regular pentagons have equal sides. Hence, the Researcher probed Inderani further about her conceptions of regular and irregular as illustrated in the following excerpts:

RESEARCHER: You spoke about an irregular pentagon – what do you mean by 'irregular pentagon'?

INDERANI: A regular pentagon means that all the sides are equal; in an irregular pentagon, all sides are not equal, and the angles are not equal. So in an irregular pentagon, the sides won't be equal, and therefore if I were to draw triangles inside that irregular pentagon, the bases will not be same, whereas for a regular pentagon all the bases are the same; in an irregular, the bases won't be the same.

The researcher continued to probe her understanding of regularity by referring her back to one of earlier dynamic sketches that she worked with, as illustrated in following excerpt:

RESEARCHER: Now, this is the pentagon (referring to Inderani's regular pentagon sketch) that you just worked with and explained before. I'm just checking before we proceed to the next sketch – here we have the measure of the sides, and we have the measure of the angles. Are the angles equal in this one?

INDERANI: The angles are all equal and the sides are all equal. So it's a regular pentagon.

RESEARCHER: So what are the properties of a regular pentagon?

INDERANI: A regular pentagon is a five-sided polygon; all five sides must be equal; and all the angles are equal.

Inderani, seems to have demonstrated correct understanding of the conception of regularity with the aid of a dynamic sketch, but it was still uncertain as to whether Inderani was aware that having an equi-sided pentagon does not necessarily mean that the pentagon is equiangular. Hence, the researcher engaged Inderani in the *Mystery pentagon* activity, as illustrated in the following series of excerpts:

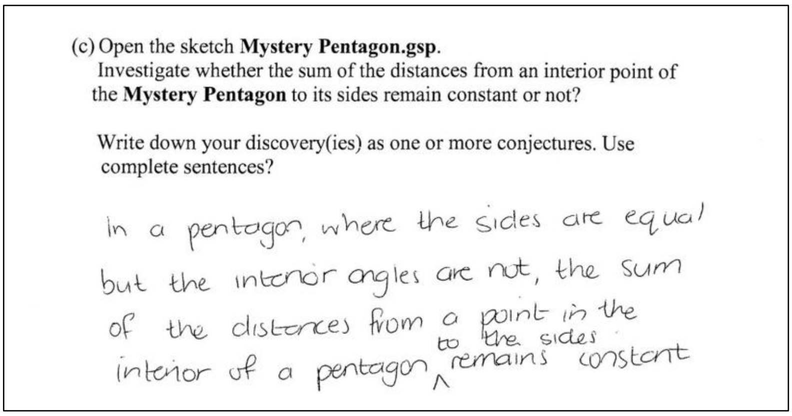
RESEARCHER: Let us investigate. You can open this polygon here – the mystery polygon –… press the button… What kind of pentagon is this

| | |
|---|---|
| | (*referring to the dynamic sketch with the measure of the angles and sides being shown).* |
| INDERANI: | This is an irregular polygon (*referring to the pentagon*), because all the angles are not equal but the sides are equal (*she sounded rather surprised*). |

Inderani appeared rather surprised to see a pentagon with equal sides but unequal angles. In the spirit of surprise, the Researcher asked Inderani to drag point *P* around the interior of the *Mystery pentagon*. After dragging point *P* around the interior of the Mystery Pentagon, the researcher asked Inderani, "…and what is happening to the sum of the distances?". Inderani immediately responded as in the following way:

| | |
|---|---|
| INDERANI: | It remains the same, the sum of the distances remains the same no matter where point *P* is positioned. But the angles are different, yet the sides will be the same. The sum of the distances remains constant. (*sounded surprised*). |

Although Inderani, did not say she was surprised, her tone and expression on her face suggested that she was surprised. It seems that she did clearly not expect to see what she saw on the *Sketchpad* screen after dragging point *P*, and this could have caused some conflict in her mind that manifested itself in an expression of surprise both in her tone and facial expression.  Although Inderani accepted the invariant property after what she saw and experienced in a *Sketchpad* context, she seemed to have already accommodated the new idea into her schemata, because when the researcher asked her to write down her discovery, she produced the following hand written response in her worksheet (see Figure 9.3.6).


(c) Open the sketch **Mystery Pentagon.gsp**.
Investigate whether the sum of the distances from an interior point of the **Mystery Pentagon** to its sides remain constant or not?

Write down your discovery(ies) as one or more conjectures. Use complete sentences?

In a pentagon, where the sides are equal but the interior angles are not, the sum of the distances from a point in the interior of a pentagon to the sides remains constant

Figure 9.3.6: Inderani's conjecture after facing a heuristic counter-example

However, when the Researcher, asked Inderani to read out what she wrote, she included the phrase, “no matter where *P* is positioned”, and this as indicated in the following excerpt, confirmed her level of confidence in her claim that articulates ‘equal sides’ as a sufficient condition for the sum of the distances to be constant.

INDERANI: In a pentagon where the sides are equal, but the interior angles are not, the sum of the distances from point *P* in the interior of the pentagon to the sides remains constant no matter where *P* is positioned.

Furthermore, it seems that the heuristic counter-example has helped Inderani to see that ‘equal sides’ is a sufficient condition for a pentagon to have a constant distance sum. The researcher also used this opportunity to get Inderani to reflect on her earlier response (or claim) to Q4 (a) of Task 3(c), and comment on its status, as illustrated in the following excerpt:

RESEARCHER: When you were asked if this result may be true for other pentagons, you said ‘no’. You said it wouldn’t be true for irregular pentagons, and now we have that case. So does your discovery here (*referring to her announced discovery after the experience of the heuristic-counter-example*), confirm or refute that response you made earlier (*referring to her earlier response to Q4(a) of Task 3(c) of the worksheet*)?

INDERANI: It refutes it.

Inderani’s response suggests that her heuristic counter-example experience prompted her to reflect on the conditions governing her original claim. In retrospect the heuristic counter-example did not refute Shannon’s conjecture generalization for a regular pentagon in its entirety but rather only refuted the condition part, meaning that the pentagon does not need to have equal angles and equal sides, but instead could just have all sides equal to produce a constant distance sum. The researcher proceeded to use Inderani’s heuristic counter-example experience to enable her to modify her initial pentagon conjecture generalization, which was restricted to just regular pentagons, to be applicable to any pentagon so long as all its sides are equal. The following responses are representative of the dialogue that occurred between the Researcher and Inderani in the build up to a modified pentagon conjecture generalization:

| | |
|---|---|
| RESEARCHER: | So now looking at these two cases here (*referring to the regular pentagon case and the irregular pentagon case*), what special property must a pentagon have so that it can yield the following result? |
| INDERANI: | The sides of the pentagon must all be equal. |
| RESEARCHER: | Considering your response to this question, can you edit or rephrase your initial conjecture? |
| INDERANI: | It says, 'In a pentagon where all the sides are equal, the sum of the distances from a point inside the pentagon to its sides remains constant.' So it's changed now to a pentagon where all the sides are equal. |

Immediately, thereafter the researcher requested Inderani to support her refined conjecture generalization with a logical explanation (deductive justification) – see Section 9.4 for Inderani's response.

**Case: Victor**

After Victor had demonstrated that he could logically explain (see Section 9.2.2) his conjecture generalization that the point *P* can be positioned anywhere in a regular pentagon so that the sum of the distances to the sides is minimum (or remains constant), the researcher asked Victor, "Do you think a pentagon other than a regular pentagon will give you the same result". Victor replied, very confidently and quickly, "No, it won't hold for other pentagons because *a* won't be the same in all other pentagons". The reason , "*a* won't be the same in all other pentagons", provided by Victor to substantiate why his CG for the regular pentagon will not hold for other pentagons, shows that he could have reflected on the structure of the proof for the regular pentagon case, and assumed that if the pentagon was not regular, then the sides would not be equal, and thus there would be no common "*a*" to pull out as a common factor. This misconception of equal sides implying regularity is the same misconception that was manifested by the other 6 PMTs.

However, when researcher asked Victor, "Would you want to test to confirm that response?", he loudly replied "No". The Researcher became curious and asked him "Why?", and Victor responded as follows:

| | |
|---|---|
| VICTOR: | Because I know that will be impossible if the sides are not equal. |

Victor's response, "because I know that will be impossible if the sides are not equal", suggest that he firmly believed that "all sides equal" is not only sufficient for his CG to hold true but a sufficient condition for a polygon to be regular.

The researcher through further probing could see that Victor was aware that a regular pentagon has both equal sides and equal angles. Hence, the Researcher proceeded to ask Victor: " Do you think a pentagon other than that (*referring to a regular pentagon*) will give you the same result (*i.e. constant distance sum*)". Victor replied with a great deal of confidence and certainty, "No, it won't give me the same result".

The researcher then gave Victor the *Mystery Pentagon* task, which considered a pentagon with equal sides but unequal angles. After Victor clicked on the buttons to show the measure of the pentagon sides and the measure of the pentagon angles at the request of the researcher, the researcher asked Victor, "What do you observe in this sketch?". On seeing the displayed measurements, Victor who seemed to very much surprised, responded as follows: " I see that the sides are equal, but the angles are not qual". This surprised look could possibly be a manifestation of a deeper inner cognitive conflict, which could have resulted from him seeing what he did not expect to see, namely a pentagon with equal sides and unequal angles. Nevertheless, it is quite plausible that the purposefully designed heuristic counter-example could have helped Victor to correct his misconception concerning the regularity concept, i.e. equal sides does not necessarily imply equal angles.

The researcher then proceeded with the one-to-one task based interview, and asked Victor to drag point *P* within the interior of the pentagon, which had equal sides but unequal angles, and to "investigate whether the sum of the distances from an interior point of the pentagon to its sides remained constant or not". After Victor experimented with his dynamic irregular pentagon, the researcher asked him, "what did you find out about the distance sum?", and Victor with a disturbed look on his face, replied " the distance sum stays constant". It seems that this observation may not have reconciled well with his expectations and previous ideas, and could have stirred up some cognitive conflict within an established schemata, and hence disturbed his cognitive equilibrium, i.e. brought about cognitive disequilibrium. It is quite plausible that the latter disequilibrium could have been the underlying cause for the perturbed look on Victor's face. The researcher then asked Victor to write down his discoveries as one or more conjectures using complete sentences in his worksheet, and Victor responded in the following way (see Figure 9.3.7):

(c) Open the sketch **Mystery Pentagon.gsp**.
Investigate whether the sum of the distances from an interior point of the **Mystery Pentagon** to its sides remain constant or not?

Write down your discovery(ies) as one or more conjectures. Use complete sentences?

The sides are equal but the angles are not equal

The sum of distances from point P is constant

any where inside the pentergon

Figure 9.3.7: Victor's conjecture after facing a heuristic counter-example

Although Victor seemed to have captured the core ideas of his discovery in his response, the researcher was not at ease with the construction of his last sentence since (as shown in the excerpt contained in Figure 9.3.7) it does not clearly articulate that the sum of the distances is from point *P* to its sides. Hence, the researcher probed Victor to clarify as to whether he meant "the sum of the distances from point *P* to the sides is constant", and Victor responded by saying "yes".

The researcher moved on with the task-based interview as shown in the following dialogue excerpt:

RESEARCHER: What special property must a pentagon have so that it can yield the following result: the sum of the distances from an interior point in the pentagon to its sides remains constant? I.e. by looking at your first result (*referring to CG for the regular pentagon*), and looking at this (*referring to observations for the mystery pentagon*), what special property must the pentagon have?

VICTOR: I think it must have equal sides.

Victor's response suggests that heuristic counter-example has helped him to see that equal sides is a sufficient condition for a pentagon to have in order to ensure that the sum of the distances from an interior point in the pentagon to its sides remains constant. Since, Victor was able to distill 'equal sides' as a sufficient pentagon property for the given problem, the Researcher proceeded with task-based interview session by asking Victor to refer to his worksheet containing Task 3(C) and respond to Q.5 (see Appendix 2 for details). Victor subsequently produced his written response as illustrated in Figure 9.3.8:

5. Consider your responses to Q4(c), Q4(d) & Q4(e) and if necessary edit or rephrase your initial conjecture (generalization) with respect to the sum of the distances from an interior point in a pentagon to it sides.

~~The~~ In a pentagon the sum of the distances from point P which is inside the pentagon is constant only if the sides are equal, eventhough the angles are not equal.

It ~~m~~ is a must that sides are equal

Figure 9.3.8: Victor's refined pentagon conjecture generalization

Immediately after Victor rephrased his CG, the researcher asked Victor "So are you saying that if any pentagon's got equal sides only, then the result will hold?", and Victor confidently responded with a resounding "Yes". On reflecting on Victor's responses, it seems that the heuristic counter-example acted as a driving force that enabled Victor to modify and refine his initial pentagon conjecture generalization, which was limited to regular pentagons because of an apparent misconception regarding regularity, to a pentagon conjecture generalization that was premised on a sufficient condition, namely 'equal sides'. Thereafter the researcher asked Victor to describe how he would go about explaining why his refined CG was always true either verbally or in writing? However, Victor responded he preferred to write ( see Section 9.4. for his written logical explanation).

In conclusion, through the enactment of the aforementioned Mystery pentagon activity, all 7 PMTS experienced a heuristic counter-example to the restrictive assumption(s) governing their regular pentagon conjecture generalization, such as: 'only regular pentagons have equal sides' or 'all equilateral pentagons are regular'; and/or 'the distance is constant only for regular pentagons'. In retrospect, the heuristic counter-example did not invalidate their conjecture generalization for a regular pentagon, but was rather a heuristic counter-example to their restrictive assumptions underlying the regular pentagon conjecture generalization and not a counter-example the regular pentagon conjecture generalization itself. This experience of a heuristic counter-example helped the PMTs to correct their misconception that only regular pentagons had equal sides, and thereby modify their previous pentagon conjecture generalizations, which was restricted to regular pentagons, to also be applicable to pentagons that just have all it sides equal only. In other words, through the heuristic counter–example experience, all the PMTs came to realize that 'all sides equal' is a sufficient condition for a pentagon to have in order to enable the sum of the distances from any point inside such a pentagon to it sides to be constant. In essence, then the examples of the mystery pentagon that was not regular ( but was still convex) opened up a conjecture generalization but these were not counter-examples to the regular pentagon conjecture generalization.

### 9.3.1 Findings as per Section 9.3 (PMTs facing a heuristic counter-example)

1. As discussed in Sections 9.1.2-9.1.4, seven students generalized to a pentagon from their previous equilateral and rhombus conjecture generalization on either inductive grounds (i.e. noticed the pattern from the previous cases that if 'the sides are equal' the sum is constant) or analogical grounds (i.e. having seen the pentagon has equal sides then they could have plausibly reasoned that their result for the pentagon should also be similar to the rhombus (or equilateral) conjecture generalization. Furthermore, seven students restricted their conjecture generalizations to the regular pentagon, showing a misconception that only regular pentagons have equal sides, and hence thinking the result will not hold for any irregular pentagons (see Finding 8 of Section 9.1.6). However, when the researcher requested each PMT to experiment with a pentagon (called a critical example or heuristic counter-example - see Section 9.3), which they initially did not know had equal sides but unequal angles, each of them were very surprised by the existence of such a pentagon.

2. It seemed that none of seven PMTs expected to see a pentagon with equal sides and unequal angles, and this may have not reconciled well with their previous idea of a pentagon with equal sides. This seemed to have caused some internal conflict in the mind of each of PMT , i.e. disturbed their cognitive equilibria, which manifested itself in the 'surprise' kind of expression on the face (or tone) of each of the seven PMTs. This heuristic counter-example in the *Mystery Pentagon* case, seemed to have helped the PMTs to correct their misconception that 'equal sides' imply equal angles (i.e. it seemed to have corrected their thinking that only regular pentagons had equal sides).

3. When each of the seven PMts experimented further with the irregular pentagon that was still convex (called the *Mystery Pentagon*), which had equal sides but unequal angles, they were surprised to also find that the sum of the distances from an interior point of the pentagon to it sides remained constant.

4. After each of the seven PMTs discovered that a constant distance also prevailed for the case of a pentagon with equal sides and unequal angles, they were questioned as to which special property a pentagon must have to yield a constant distance sum, and each of them did singled out 'equal sides' as the special property.

5. Each of the seven PMTs' recognition of 'equal sides' as the special property a pentagon (not necessarily a regular pentagon) must have to yield a constant distance sum, suggests that they each had discovered 'equal sides' as a 'sufficient' pentagon property that would enable the sum of the distances from any interior point of any pentagon, and not necessarily a regular pentagon, to its sides to be constant.

6. On seeing the heuristic counter-example (or critical example), each of the seven PMTs were able to modify (refine) the initial pentagon conjecture generalization, which was restricted to just regular pentagons, to embrace any pentagon which simply possessed the 'special' property (i.e. sufficient property), namely 'equal sides'. For example, one of PMTs produced the following refined conjecture generalization: "If the sides of the polygon (pentagon) are equal, the sum of the distance from an interior point *P* to the sides of the pentagon, will not change (*meaning the distance sum will remain constant*)". So essentially the heuristic counter-example in the *Mystery Pentagon* case served as a driving force for each PMT to waive the restrictive assumptions and hence refine her/his initial pentagon conjecture generalization. In particular, the examples of the Mystery equilateral pentagon that was not regular but still convex opened up a generalization but were not counter-examples to the regular pentagon conjecture generalization itself.

7. The heuristic counter-example appears to have caused some internal conflict in their respective minds, but they subsequently accommodated the new ideas, namely: (a) not only regular pentagons have equal sides, and (b) 'equal sides' is a sufficient condition that can be possessed by any pentagon, and not just a regular pentagon, to enable the production of a constant distance sum.

8. It seems that engagement of the PMTs in a dynamic learning environment coupled with an exploratory task (i.e. Mystery Pentagon task) enabled the PMTs to encounter a critical example, which together with necessary intervention by the facilitator and the use of appropriate questions, like Q4 (d): "what special property must a pentagon have so that it can yield the following result: the sum of the distances from an interior point in the pentagon to its sides remain constant", and Q5: "Consider your response to 4(d) and then write down your final generalization with respect to the sum of the distances from an interior point in pentagon to its sides", provided an apt opportunity for PMTs to have experienced a heuristic counter-example, which did not invalidate

their initial regular pentagon conjecture generalization, but rather made them see that there was a counter-example to the restrictive assumption that only regular pentagons had equal sides.

## 9.4. Pre-service mathematics teachers (PMTs) justifying their refined pentagon (Mystery pentagon) conjecture generalization

As discussed and illustrated in section 9.3 of this chapter, all seven PMTs (namely Inderani, Victor, Tony, Shannon, Trevelyan, Renny and Logan), who experimented with the *Mystery Pentagon* within a dynamic context, experienced a heuristic counter-example that enabled them to single out a sufficient property that a pentagon could possess to allow the distances from an interior point of such a pentagon to its sides to remain constant all the time. More importantly, the heuristic counter-example experience seemed to have helped the PMTs to modify and refine their initial pentagon conjecture generalization, which had been restricted to just regular pentagons, to a pentagon conjecture generalization that was premised on a sufficient condition, namely ‘equal sides’. By refining their conjecture generalization in the latter way, the PMTs were saying that their conjecture generalizations could apply to any pentagon with equal sides and not just regular pentagons.

When each of the seven PMTs, namely Inderani, Victor, Tony, Shannon, Trevelyan, Renny and Logan, were asked to provide a justification in the form of logical explanation (proof) for their newly refined conjecture generalization, they responded as follows:

a. Three PMTs, namely Shannon, Tony and Trevelyan, verbally acknowledged in their own tones that they would use their earlier proof they had produced to justify their regular pentagon conjecture generalization also as a proof to justify their refined conjecture generalization. Hence, the Researcher on hearing the respective responses from Shannon and Tony, did not ask them to produce a written logical explanations. However, he nevertheless asked Trevelyan to verbally describe how he would go about constructing his logical explanation (proof).
b. The four other PMTs, namely Inderani, Victor, Renny, Logan, responded by immediately producing a written logical explanation which was more or less the same as the one they had previously produced for their regular pentagon conjecture generalization.

The following two cases, namely Tony and Trevelyan, are representative of those three PMTs who verbally expressed their justifications for their refined pentagon conjecture generalization by mirroring their explanations for their earlier regular pentagon conjecture generalizations:

**Case: Tony**

The researcher asked Tony to prove his refined CG, after experiencing the Mystery Pentagon counter-example. Tony replied, with utmost confidence, “I just proved it”. In actual fact Tony was referring to his proof that he wrote for the regular pentagon case. This indeed shows that Tony could look at the structure of a given proof for a given problem and use it to generalize the proof for a similar problem.

| | |
|---|---|
| RESEARCHER: | So, to prove that for this pentagon with equal sides, how will you prove it? Just talk to me; you don’t have to write. |
| TONY | That the sum of the distances is constant! I just proved it (referring to the proof of the regular pentagon case) |
| RESEARCHER: | You say you just proved it? |
| TONY | Yes. |

**Case Trevelyan:**

When the researcher asked Trevelyan, “Briefly tell me how you’ll prove it (referring to his refined pentagon conjecture generalization”, he confidently and categorically responded by saying, “I would do the same proof”. Despite seeing the confident expression on Trevelyan’s face, the researcher asked him to briefly describe how he would actually go about to prove his conjecture generalization, as illustrated in the following excerpt:

| | |
|---|---|
| RESEARCHER: | Briefly tell me how you’ll prove it. |
| TREVELYAN: | I would do the same proof - of constructing the small triangles such that each of these triangles has a perpendicular height which is the distance from point P to the side of that small triangle, the side of the pentagon. And then from there I would do the sum of each of these triangles. Once I’d got the sum of all of those triangles, the sum of the area of all of those small triangles, then that sum should be equal to the area of the whole pentagon. And then from there I can show that the sum of those distances are constant. |

RESEARCHER: How would you show it?

TREVELYAN: I'll use the fact that each side is constant; and the area of the pentagon is also constant; and that the height of the sum should also be constant, based on that.

RESEARCHER: So are you saying to me the proof is similar to the previous proof.

TREVELYAN: It is.

Trevelyan's responses as illustrated in the afore-cited excerpt, appear to demonstrate that he had a clear plan of how to use his 'triangle-area' algebraic structure justification for his regular pentagon conjecture generalization to construct a logical explanation for his refined conjecture generalization in just the same way. One could conjecture that Trevelyan on seeing that 'equal sides' is a special property that a pentagon should possess for its distance sum to be constant and having just proved the regular pentagon conjecture generalization through the use of the 'equal sides' property, may have hence realized that the proof will be the same in both cases (i.e. for regular pentagon and irregular pentagon with equal sides).

The following three cases, namely Inderani, Victor and Renny, are representative of how the four PMTs (namely Inderani, Victor, Renny, and Logan) constructed their written logical explanations for their refined pentagon conjecture generalization by directly adopting the 'triangle-area' algebraic structure explanation that they had earlier produced to justify their regular pentagon conjecture generalization.

**Case: Inderani**

When the Researcher asked Inderani to provide a justification in the form of a logical explanation for her refined conjecture generalization, she proceeded with no guidance to support her refined conjecture generalization with the following logical explanation in her worksheet. The explanation as shown in Figure 9.4.1, was more or less the same kind of explanation that she had provided for her regular (convex) pentagon conjecture generalization (compare Figure 9.2.2.1 in Section 9.2.2).

In an equisided pentagon, if five triangles were constructed inside the pentagon, the total area $= \frac{1}{2}ah_1 + \frac{1}{2}ah_2 + \frac{1}{2}ah_3 + \frac{1}{2}ah_4 + \frac{1}{2}ah_5$

$\therefore A = \frac{1}{2}a(h_1 + h_2 + h_3 + h_4 + h_5)$

$\frac{2A}{a} = (h_1 + h_2 + h_3 + h_4 + h_5)$ (constant → $2A$; constant → $a$)

Because the total area is constant and the base (sides) is constent, the left hand side is therefore a constent.
If the left hand side is constent, the right hand side is therefore constent.

Figure 9.4.1: Inderani's justification for her refined pentagon CG

**Case: Victor**

When the Researcher asked Victor to explain why his refined CG was always true, he immediately responded as in his worksheet as shown in Figure 9.4.2, with no guidance.

It seems that Victor, like the others, supported his refined pentagon conjecture generalization by advancing more or less the same kind of 'triangle-area' logical explanation that he had constructed for his regular pentagon conjecture generalization earlier on (compare Figure 9.2.2.2 in Section 9.2.2).

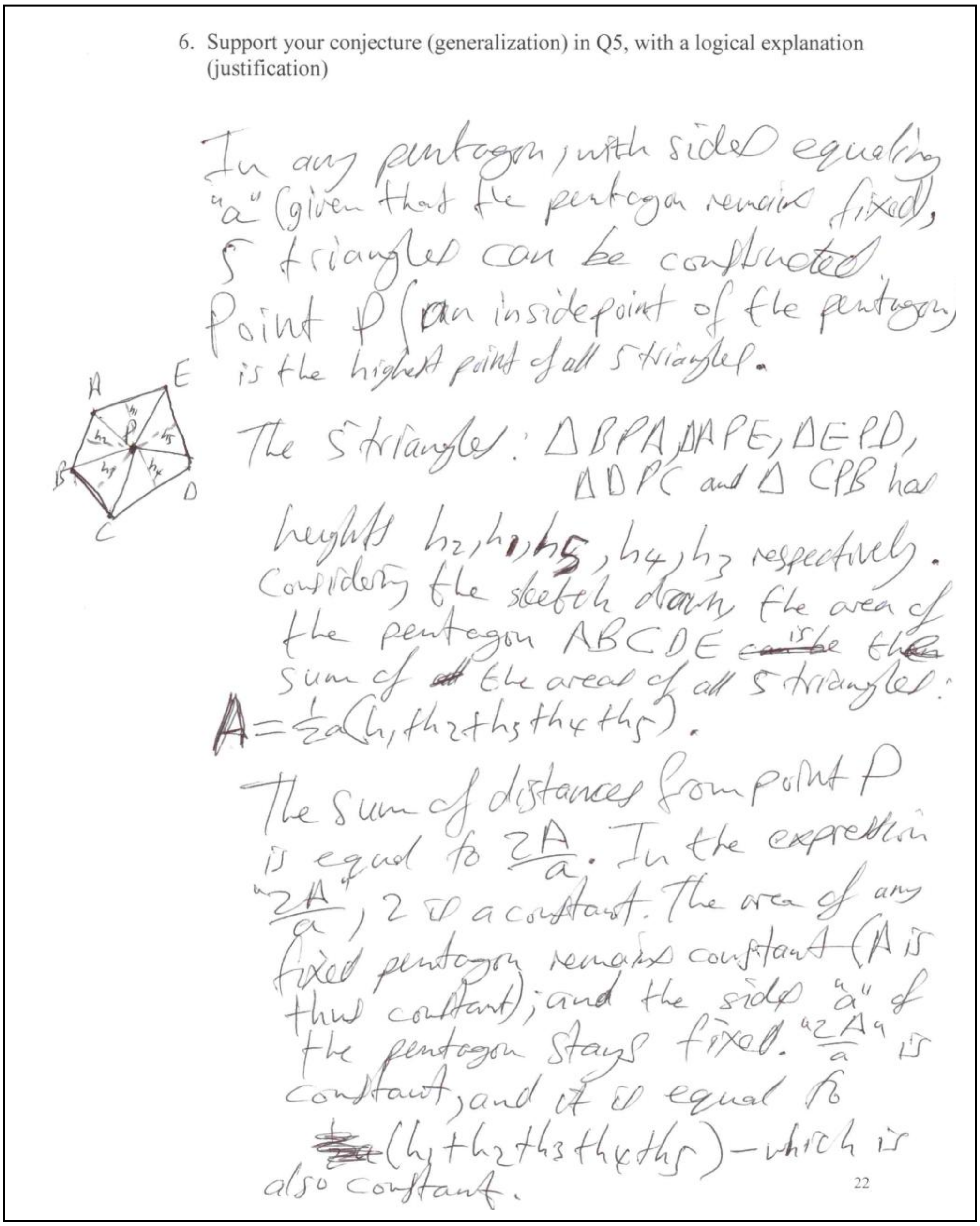
6. Support your conjecture (generalization) in Q5, with a logical explanation (justification)

Area of the fixed peontagon A
I have triangles inside the fixed pentagon which are
$\triangle APB$, $\triangle BPC$, $\triangle CPD$, $\triangle DPE$ and $\triangle EPA$
Sum of areas of the triangles is given as
$\triangle APB + \triangle BPC + \triangle CPD + \triangle DPE + \triangle EPA$
since the sides are equal I can label them by (a)
then
$A = \frac{1}{2}a.h_1 + \frac{1}{2}a\,h_2 + \frac{1}{2}a.h_3 + \frac{1}{2}a.h_4 + \frac{1}{2}a.h_5$
$A = \frac{1}{2}a(h_1 + h_2 + h_3 + h_4 + h_5)$
$\frac{2A}{a} = (h_1 + h_2 + h_3 + h_4 + h_5)$
Since A is fixed that means it's always constant and (a) is a constant $\therefore h_1 + h_2 + h_3 + h_4 + h_5$ is also constant.

Figure 9.4.2 : Victor's justification for her refined pentagon CG

**Case: Renny**

Figure 9.4.3 represents the written logical explanation Renny produced in his worksheet during the one-to-one task based interview session. It is also evident Renny, like the others, had used more or less the same logical explanation structure as for his regular pentagon conjecture generalization to construct a logical explanation for his refined pentagon conjecture generalization.



6. Support your conjecture (generalization) in Q5, with a logical explanation (justification)

In any pentagon, with sides equaling "a" (given that the pentagon remains fixed), 5 triangles can be constructed. Point P (an insidepoint of the pentagon) is the highest point of all 5 triangles.

The 5 triangles: $\Delta BPA$, $\Delta APE$, $\Delta EPD$, $\Delta DPC$ and $\Delta CPB$ has heights $h_2, h_1, h_5, h_4, h_3$ respectively. Considering the sketch drawn, the area of the pentagon ABCDE ~~can be~~ is the sum of ~~all~~ the areas of all 5 triangles:

$A = \frac{1}{2}a(h_1+h_2+h_3+h_4+h_5)$.

The sum of distances from point P is equal to $\frac{2A}{a}$. In the expression "$\frac{2A}{a}$", 2 is a constant. The area of any fixed pentagon remains constant (A is thus constant); and the side "a" of the pentagon stays fixed. "$\frac{2A}{a}$" is constant, and it is equal to $(h_1+h_2+h_3+h_4+h_5)$ — which is also constant.

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Figure 9.4.3: Renny's justification for his refined pentagon CG

Despite the different ways (i.e. verbal and written) of expressing that their logical explanations is the same (or more or less the same), it seems that all seven PMTs have looked back at their logical explanations that they developed for their earlier regular pentagon conjecture generalization (source problem).

Upon seeing that the property ‘equal sides’ was the only pentagon property that was used in construction of their logical explanation for the regular pentagon conjecture generalization (source problem) and that the very same property also defined their refined pentagon conjecture generalization (target problem), their cognitive structures seemed to have automatically subsumed the development of the logical explanation for the refined conjecture generalization under a relevant and more inclusive conceptual scheme, namely their earlier logical explanation for the regular pentagon conjecture generalization, as propounded by Ausubel’s Theory of Meaningful learning (see detailed discussion in Section 4.5). In particular, Ausubel (1962, p. 217) asserts that when one encounters new information or a problem, the natural first processes is to “subsume the new information (*or problem*) under a relevant and more inclusive conceptual scheme”, and “the very fact that a new information (*or a given problem*) is subsumable (i.e. relatable to stable element in the cognitive structure), accounts for its meaningfulness and makes possible perception of insightful relationships”.

In retrospect, on comparing the kinds of logical explanations produced by the PMTs for their earlier regular pentagon conjecture generalizations and their refined pentagon conjecture generalizations, it is plausible to conjecture that this involved a particular kind of subsumptive (subsumption) process called derivative subsumption. According to Ausubel (1978, p. 68), in derivative subsumption “new information (*or a new problem*) is linked to a superordinate idea *A* and represents another case or extension of *A*. The critical attributes of the concept *A* are not changed, but new examples are recognized as relevant”. This essentially means that the new cases or examples that learners comprehend or understand, are part and parcel or just mere examples of an established system of concepts or propositions that that learners have already learned or are familiar with, or “it is just supportive or illustrative of a previously learned concept or proposition” (Ausubel et al., 1978, p. 58). Furthermore, Ausubel et al.(1978, p. 58) asserts that “in either case the new material to be learned is directly and self-evidently derivable from or implicit in an already established and more inclusive concept or proposition in a cognitive structure.”

Hence, through having *linked* their new target problem (i.e. the construction of a logical explanation for their refined pentagon conjecture generalization) to their source problem (i.e.

logical explanation for their regular pentagon conjecture generalization) via the ‘equal sides’ property, it seems that all seven PMTs were then able to structurally map the relational structure of the ‘triangle-algebraic’ logical explanation for their source problem onto their target problem by using analogical reasoning, and thereby produced a logical explanation for their refined conjecture generalization. For example, if we look at Victor’s ‘triangle- area’ algebraic explanation of his regular pentagon conjecture generalization in Figure 9.2.2.2 and his ‘triangle- area’ algebraic explanation for his refined pentagon conjecture generalization in Figure 9.4.2, one sees a structural consistency, i.e. there is a one to one kind of correspondence between the elements that exist between the two representational structures (see Gentner Strcture Mapping Theory discussed in Section 4.4.1; also see Section 4.4.2)..

### 9.4.1 Findings as per Section 9.4 (PMTs justifying refined pentagon CG)

1. Two PMTs, namely Shannon and Tony verbally expressed that their logical explanation for their refined logical explanation would be the same as their logical explanation for their earlier conjecture generalization, and hence did not actually produce a written explanation. On account of the confidence displayed in their assertion, the Researcher did not request a written explanation .

2. One of the PMTs, namely Trevelyan, verbally expressed that his logical explanation for his refined logical explanation would be the same as his logical explanation for his earlier regular pentagon conjecture generalization. Although the Trevelyan also sounded very confident, the Researcher asked him to produce a written explanation. The written explanation Trevelyan produced for his refined conjecture generalization was more or less the same as his logical explanation for his regular pentagon conjecture generalization.

3. Four PMTs, namely Inderani, Victor, Logan and Renny, when they were asked to justify their refined pentagon conjecture generalization, immediately without any hesitation produced a written logical explanation, which was more or less the same as their logical explanation for their regular pentagon conjecture generalization.

4. It seems that the seven PMTs, who are referred to across Findings 1-3, succeeded to construct a logical explanation for their refined pentagon CG on the grounds of the following experiences and links:
   - They already worked with a pentagon though regular and constructed a logical explanation using ‘equal sides’ to explain why distance sum is constant; and
   - Plausibly saw a similarity (or link) between their regular pentagon conjecture generalization (source problem) and their refined pentagon conjecture generalization (target problem) through the ‘equal sides’ property; and
   - Hence automatically subsumed the target problem (i.e. the construction of a logical explanation for their refined pentagon conjecture generalization) under an existing but relevant explanatory structure, namely the ‘triangle-area’ algebraic explanatory structure for their regular pentagon generalization (source problem).

   Furthermore, it seems that in this instance that a particular kind of subsumption occurred, namely derivative subsumption. In the case of derivative subsumption the solution of a new problem can be derived from an already established problem solution so long as a link can be made between the solution of the source problem and target problem (see discussion of derivative subsumption in Section 4.5.1). It seems that the PMTs were able to see seen a link or relationship between the source an target problem through the ‘equal sides property, and hence succeeded in constructing an explanation for their refined pentagon conjecture generalization (target problem) by plausibly transferring the explanatory steps from the source problem onto the explanatory steps of the target via analogical structural mapping.

The next Chapter, focusses on the data analysis, results and discussion with regard to the ‘any’ equi-sided polygon task- based activity problem.

# Chapter 10: Equi-Sided Polygon (Convex) Problem: Data Analysis, Results and Discussion

## 10.0 Introduction

In Chapters 7, 8 and 9 the following generalizations were constructed and justified within the context of convex polygons:

(a) In an equilateral triangle, the sum of the distances from a point inside the triangle to its sides is constant;

(b) In a rhombus the sum of the distances from a point inside the rhombus to its sides is also constant;

(c) In any equi-sided pentagon, the sum of the distances from a point inside the pentagon to its sides is also constant.

In this Chapter, the question is whether pre-service mathematics teachers (PMTs) could generalize or further generalize these results or generalizations further to any equi-sided (i.e. equilateral) convex polygon, and if so, how they accomplished and justified this further generalization.

The data analysis and findings related to making and justifying conjecture generalization(s) with particular reference to the equi-sided convex polygon task-based activity as described in Task 4 of Appendix 2 are presented case-wise in this Chapter 10. For each case, the researcher presents the data analysis as to how a specific PMT constructed and/or justified his/her conjecture generalization in a continuous form. Hence in this regard the structure of presentation of the data analysis is slightly different from the structure of the presentation of the data analysis presented in Chapters 7, 8 and 9 wherein the production of generalizations was presented first followed later by the justifications of generalizations.

## 10.1 Pre-service mathematics teachers producing and justifying a conjecture generalization to any equi-sided convex polygon

The findings of the pentagon-task base activity, as discussed in Chapter 9, showed that one PMT, Alan, was able to construct a generalization for any equi-sided convex pentagon immediately on logical grounds with the aid of analogical reasoning, whilst the remaining 7

PMTs managed to experience a heuristic counter-example to restrictive assumption(s), which then forced them to refine their regular pentagon conjecture generalization. The latter seven PMTs, as discussed in Chapter 9, seemed to have analogically seen that the structure of their 'triangle-area' algebraic explanations that they advanced (or produced) for their earlier regular pentagon conjecture generalizations, could similarly be used to the construct a logical explanation for their equi-sided pentagon conjecture generalizations, and hence justified their refined pentagon conjecture generalization accordingly. In fact three PMTs verbally expressed that they would justify their refined pentagon conjecture generalization (i.e. equi-sided pentagon) by using the same kind of 'triangle-area' algebraic proof, which they had advanced for the regular pentagon conjecture generalization, and four PMTs actually wrote down the more or less the same kind of 'triangle- area' algebraic proof that they had advanced for the regular pentagon conjecture generalization.

Soon after each PMT demonstrated that they could produce a logical explanation for their refined pentagon conjecture generalization as discussed in Section 9.4, each of them was presented with Task 4(a) as shown in Figure 10.1.1.

**Task 4(a): Generalizing to any equi-sided polygon**

1. Below are a set of generalizations that you may have developed earlier:

G1: In any equilateral triangle, the sum of the distances from a point inside the triangle to its sides is constant, and

G2: In any rhombus the sum of the distances from a point inside the rhombus to its sides is also constant.

G3: In any equi-sided pentagon, the sum of the distances from a point inside the pentagon to its sides is also constant.

Consider the above set of generalizations. Can you generalize to polygons with a similar property?

Figure 10.1.1: Task 4(a): Generalizing to any equi-sided convex polygon

As per data analysis, which is presented case-wise, all 8 PMTs through reflecting on their earlier generalizations for specific convex polygons with equal sides (i.e. equilateral triangle, rhombus, and equilateral pentagon) were able to extend the 'constant distance sum' generalization to any equi-sided convex polygon on logical grounds, without expressing any explicit need for visual / experimental confirmation in a *Sketchpad* context (or by using *Sketchpad*). In this respect, six PMTs on seeing that they were dealing with any polygon characterised by the sufficient property, namely 'equal sides', immediately extended the same chain of ideas that prevailed for the equilateral, rhombus and equilateral pentagon generalizations across to the case of any equi-sided convex polygon.

One PMT (namely Shannon), first decided to construct a logical argument that was premised on the property, 'equal sides', to see if she could explain why the sum of the distances from point inside a equi-sided convex polygon to it sides is always constant. In doing so, Shannon similarly used the 'triangle area' algebraic strategy that she had previously used in the construction of her explanations for the equilateral triangle, rhombus and pentagon generalizations, and hence reached the desired conclusion, namely the sum of the distances is constant. Furthermore, one PMT, namely Alan, tried to use an inverse kind of argument to show that the 'constant distance sum' does not hold for polygons with unequal sides but in the process demonstrated that he realized that any polygon needs to have 'equal sides' for the distance sum to be constant.

Despite the variations in the PMTs logical approach, which governed the construction of their generalization for any equi-sided convex polygon, each PMT's 'constant distance sum' generalization for any equi-sided polygon can be regarded as a deductive generalization (see Section 1.6.2 for discussion on deductive generalizations, and the case-wise data analysis in this Chapter 10). When each PMT was asked to support their deductive generalization with a logical explanation (i.e. justification) as per question 4 of Task (4b) of Appendix 2, all 8 PMTs produced a deductive justification with no guidance. In doing so, each PMT used the similar kind of 'triangle-area' algebraic proof structure that was used to construct explanations for each of their previous particular generalizations, namely pentagon, rhombus and equilateral triangle conjecture generalizations. Hence, it seems that through reflecting on their earlier proofs for their earlier 'constant distant sum' generalizations for specific polygons that had equal sides, all PMTs saw a 'common proof structure' prevailing amongst the set of particular proofs, and without going through any stressful cognitive change (see Tall, 1991, p. 12), similarly extended the same 'common proof structure' to construct a 'general proof' for their equi-sided convex polygon generalization.

The prominence of the extension of the 'triangle-area' algebraic structure explanation from the earlier constructed explanations to the explanation for the 'equi-sided' polygon generalization, re-affirm the researcher's assertion that each of the eight PMTs have managed to generalize to any equi-sided convex polygon on logical grounds. For example, on seeing that the equi-sided convex polygon has the critical property, 'equal sides' (a sufficient condition for a constant distance sum), each PMT immediately concluded logically that the distance sum is constant.

The case-wise presentation that follows entails an analysis of how each PMT accomplished and justified their further generalization to any equi-sided polygon. Each of the accompanying case presentations contains dialogue excerpts and worksheet excerpts as it unfolded during each one-to-one task-based interview session, during which each PMT was engaged with the equi-sided convex polygon problem activity.

**Case: Shannon**

As per Task 4(a) described in Figure 10.1.1, when the researcher asked Shannon during the one-to-one task-based interview, "Can you generalize the result to any other polygon?", she immediately and spontaneously proceeded as shown in Figure 10.1.2, to first ascertain via logical argument as to whether she could also extend her earlier refined pentagon conjecture generalization (or previous equilateral triangle and rhombus generalization) in general to any equi-sided convex polygon. Only after seeing through her logical argument as shown in Figure 10.1.2, that it is possible to extend her previous refined pentagon generalization to any equi-sided convex polygon, did she confidently reply with a "Yes" to the researcher's initial question: "Can you generalize the result to any other polygon?" Although, Shannon immediately engaged with developing an algebraic proof based on area considerations on her own in her worksheet as illustrated in Figure 10.2, she did so without engaging with any dynamic sketch of an equi-sided convex polygon using *Sketchpad*.

When further probed by the researcher as illustrated in the following task-based interview excerpt, Shannon re-affirmed and verbally explained that the distance sum will be constant for all polygons with equal sides by making reference to her written logical explanation as shown in Figure.10.1.2. The kind of logical explanation that Shannon advanced for her equi-sided convex polygon conjecture generalization, was similar in structure to the 'triangle-area' algebraic explanation she gave to support her refined pentagon conjecture generalization.

SHANNON: Yes. You can generalize the area of a polygon – like I said here - is equal to ½ × (the side length) × ($h_1 + h_2 + h_3 +.. + ...$). So, the sum of the distances from the sides, depending on how many sides there are, will always be 2*A* over *a*, and will always be a constant.

RESEARCHER: So you’re saying it will hold for other polygons as well?

SHANNON: All polygons where the sides are equal.

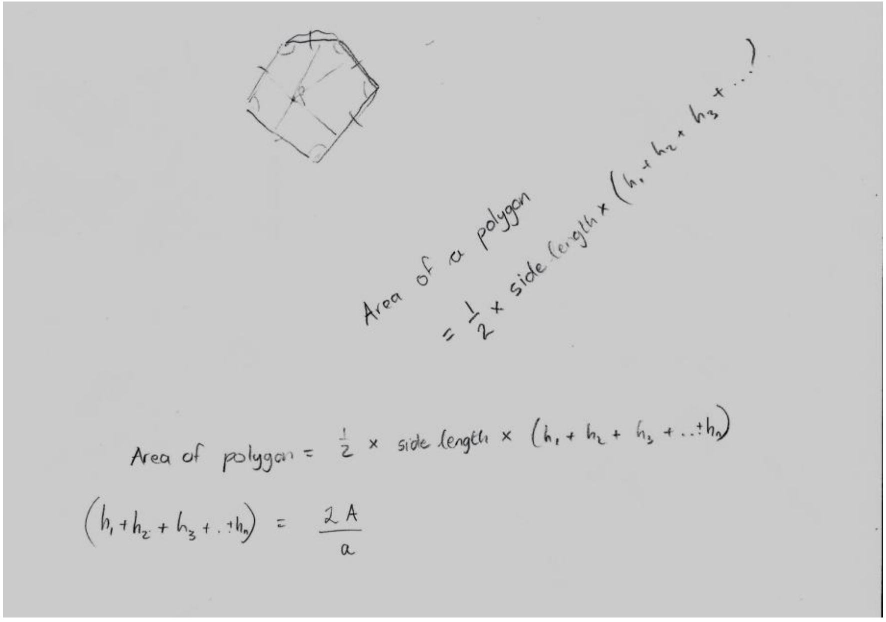



Figure 10.1.2: Shannon’s algebraic proof for her equi-sided polygon CG

Although Shannon substantiated her conjecture generalization for all equi-sided polygons with both written and verbal logical explanations as demonstrated so far, the Researcher nevertheless requested her to write down her logical explanation as a coherent argument. On this note, Shannon produced a coherent written logical explanation as shown in Figure 10.1.3, which in fact was an improvement on her earlier written logical explanation as shown in Figure 10.1.2, in that it explained in much more detail as to why the distance sum is constant.

Although, Shannon proceeded easily with the development of a coherent algebraic argument her equi-sided convex polygon conjecture generalization through triangle-area considerations, her statement “and area of polygon, side length and 2 is constants” provided the necessary warrants of why she concluded that $(h_1 + h_2 + h_3 + \ldots + h_n)$ is a constant. More importantly, Shannon’s last set of statements, “and the area of polygon, side length and 2 is constant, then that $(h_1 + h_2 + h_3 + \ldots + h_n)$ is a constant”, illustrated her understanding of the nature of propositional relationships.

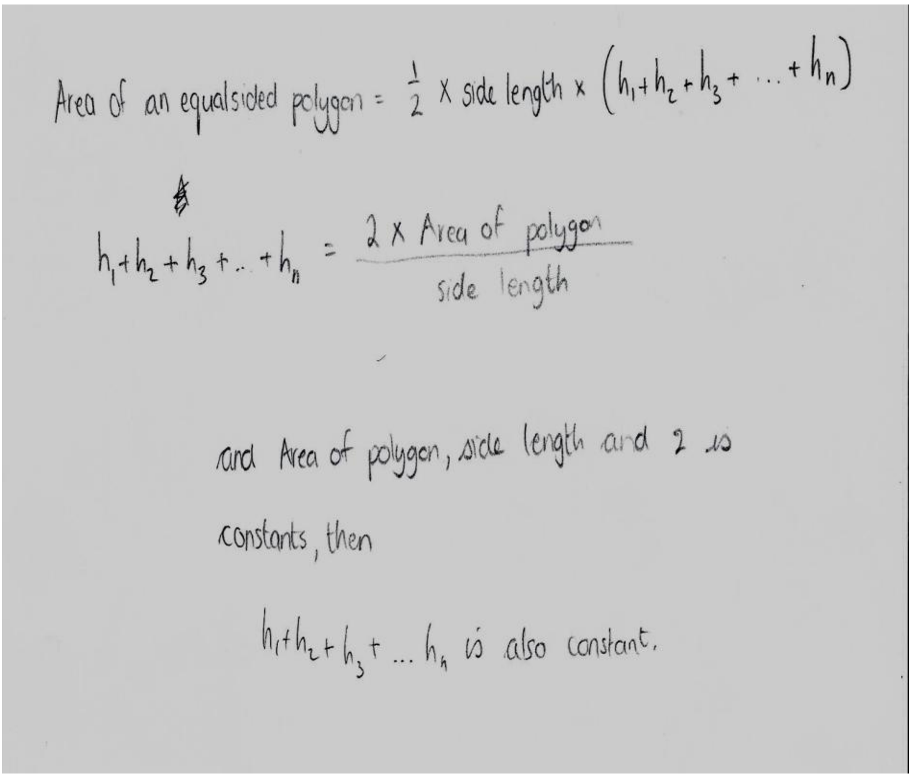



Figure 10.1.3: Shannon's improved algebraic proof for her equi-sided polygon CG

However, when the researcher attempted to probe Shannon as to whether she was satisfied with her logically constructed explanation as shown in Figure 10.1.3, she unwittingly posed the question with reference to any polygon instead of any equi-sided polygon. Nevertheless, as illustrated in the following excerpt, Shannon was quite forthright and explicit in her response, and actually stated "Not for any polygon – the polygon where all sides are equal".

| | |
|---|---|
| RESEARCHER: | So are you satisfied with the explanation of why the result is true for any polygon? |
| SHANNON: | Not for any polygon – the polygon where all the sides are equal. |
| RESEARCHER: | Yes. |
| SHANNON: | So, you can say equal-sided polygon. |

Shannon's responses as illustrated in the afore-cited excerpt, demonstrates that she has successfully accommodated the idea: 'all sides equal is a sufficient condition for any polygon to yield a constant distance sum'.

**Case: Trevelyan**

Although Trevelyan did not experiment with a polygon beyond a pentagon, he confidently stated the following conjecture when the question in Task 4(a) was asked.

TREVELYAN: My conjecture is that given any polygon, if the sides of the polygon are the same, then the sum of the distances from a point to each side of the polygon, it should be constant.

Since Trevelyan had already discovered and proved for previous particular equi-sided polygons (like the equilateral triangle, rhombus and equilateral pentagon) the sum of the distances is constant, he seemed to have now accommodated the idea of ‘equal sides’ as a sufficient property to yield a constant distance sum’ within his cognitive structure. Hence, on seeing a polygon that had all its sides equal, he logically deduced that the sum of the distances from an interior point of ‘any polygon with equal sides’ would also be constant. In particular this generalization from his earlier generalizations and its associated proofs is a good example of the ‘discovery’ function of proof described in Section 5.4.2, whereby the deductive identification of the characterising property, like ‘equal sides’ in this instance, was considered to be sufficient to enable further generalization of the ‘constant distant sum’ generalization to any polygon.

Although Trevelyan seemed to have constructed a generalization for any equi-sided convex polygon on logical grounds, the Researcher nevertheless asked him : “How will you go about proving that one (*referring to his equi-sided polygon generalization*)?”, and also referred him to a ready made dynamic *Sketch* of an equi-sided convex polygon on *Sketchpad*, thinking that he would use it to explore whether his conjecture generalization was true before proceeding to the construction of proof (or a logical explanation). However, Trevelyan did not drag point *P* nor show any intention to drag point *P* to empirically verify his conjecture generalization, but immediately replied, “I can do the same proof”, and went on further to produce a written explanation is his worksheet as shown in Figure 10.1.4.

Trevelyan’s written logical explanation as shown in Figure 10.1.4, demonstrated that he had used the layout of a two-column proof, ranging from what is given, to what is required to prove, to the requisite construction, and the development of a logical argument under the subsection called proof. As can be seen in the write up of his two column proof, Trevelyan has treated his conjecture generalization to be a conditional statement, namely: “ If a polygon has equal sides, then the sum of the distances from any interior point of the polygon to its sides is constant”. To prove his conditional statement is true, he assumed that the first part of the conditional statement is true, by writing down, “Suppose that we have a polygon with *n* sides that are equal”. Then he immediately wrote down under RTP (which means ‘required to prove’), “We need to show that the sum of the distance from any interior point to the polygon to each side of it is constant”, which is equivalent to the second part of the conditional statement. Clearly Trevelyan is aware that he must demonstrate that the antecedent implies the consequent. Furthermore, he realized that certain constructions needed to be done on his diagram, for example the construction of the smaller triangles and their respective heights.

Suppose we have a polygon with $n$ sides that are all equal.

R.T.P:

We need to show that the sum of the distance from any point interior to the polygon to each side of it is constant.

Construction:

We construct small triangles interior to the polygon such that the perpendicular hight of each triangle is one of the distances from the interior point to the side of the polygon which we can call them $h_1, h_2, \ldots, h_n$

Let each side of the polygon be $a$

Proof:

The sum of the area of these small triangles say $L$

$L = \frac{a}{2}(h_1 + h_2 + \ldots + h_n)$

Now

$L$ is equal to the area of the polygon say $A$

$A = L = \frac{a}{2}(h_1 + h_2 + \ldots + h_n) = \frac{a}{2} H, \quad H = h_1 + h_2 \ldots$

$A = \frac{a}{2} H \Rightarrow H = \frac{2A}{a}$

Then since $2$, $A$, and $a$ are constant then $H$ is constant. But $H = h_1 + h_2 + \ldots + h_n$ which is the sum of the distances from the interior point to each side of the polygon

~~therefore~~ Therefore the sum is constant $\square$

Figure 10.1 4: Trevelyan’s written logical explanation for his equi-sided polygon CG

The development of Trevelyan’s logical argument under the section he calls “Proof”, demonstrates his familiarity in the use of deductive arguments in the development of an entire logical proof or explanation.

Reflecting on Trevelyan's written logical explanation for his 'equi-sided convex polygon' generalization, it is evident that there is a connection between his earlier set of logical explanations for each of his particular generalizations (namely, equilateral triangle, rhombus and equilateral pentagon) and his logical explanation for the general case (i.e. 'equi-sided polygon' generalization). In this context, his earlier set of logical explanations for specific equi-sided convex polygons (like equilateral triangle, rhombus and equilateral pentagon) can be seen to be the underlying 'driver' that facilitated his transition from particular to general logical explanations. In other words, by reflecting (i.e. folding back) on the logical explanations of the earlier equilateral triangle, rhombus and equilateral pentagon generalizations, Trevelyan like others, saw a common proof structure (i.e. 'triangle-area' algebraic proof structure) permeating all the particular proofs (logical explanations), and used that common proof structure to develop and construct a general proof that logically explained his 'equi-sided convex polygon' generalization.

**Case: Inderani**

When, the Researcher referred Inderani to Task 4a as shown in Figure 10.1, and asked Inderani, "Can you generalize to polygons with a similar property?", she immediately wrote the following generalization as shown in Figure 10.1.5 in her worksheet, without expressing any need for visual confirmation or experimentation using *Sketchpad*:

> in a polygon that has all sides equal, the sum of the distances from a point inside the polygon to it's sides remains constent

Figure 10.1.5: Inderani's equi-sided convex polygon conjecture generalization

It appears that Inderani made her generalization to any equi-sided convex polygon on logical grounds as well. For example, from her previous generalization cases, Inderani seemed to have discerned that 'equal sides' is a sufficient property for a polygon to have in order to yield a constant distance sum, and hence on seeing that she was asked to generalize to polygons with 'equal sides', she logically concluded that the sum of the distances is also constant in such polygons. When the researcher asked Inderani to do Task 4(b)- Q.1, which probed her level of certainty in her equi-sided polygon conjecture generalization, she responded as shown in Figure 10.1.6 in her worksheet:

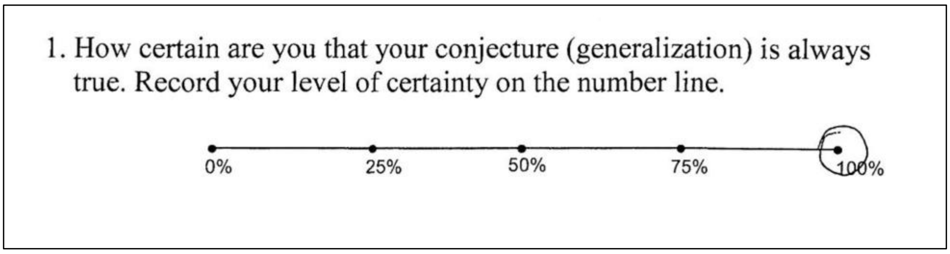
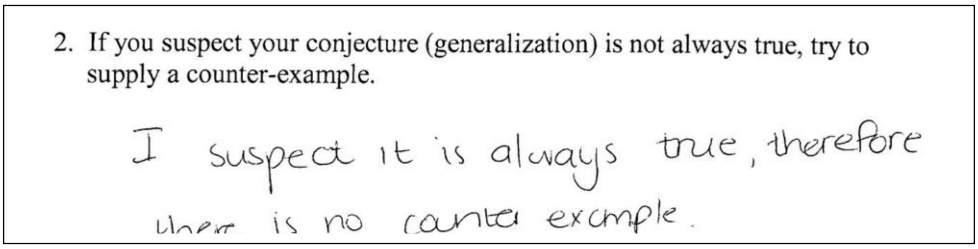

1. How certain are you that your conjecture (generalization) is always true. Record your level of certainty on the number line.



Figure 10.1.6: Inderani's level of certainty in her equi-sided polygon CG

Furthermore, when the researcher posed the following question to Inderani, "If you suspect your conjecture is not always true, try to supply a counter-example", she replied as shown in Figure 10.1.7 in her worksheet:


2. If you suspect your conjecture (generalization) is not always true, try to supply a counter-example.

I suspect it is always true, therefore there is no counter example.

Figure 10.1.7: Inderani's counter-example response

On seeing that Inderani thought there was no counter-example to her deduced equi-sided convex polygon generalization, the researcher took the opportunity to ask Inderani to justify her generalization via a written logical explanation. Immediately thereafter, Inderani produced a written explanation in her worksheet as shown in Figure 10.1.8.

Her development of a proof, seems to suggest that she has a firm grip on how to develop logical relationships and arguments. For example, she has ascertained that the area of the big equilateral polygon is equal to the area of the smaller constructed triangles, thus she was able to confidently make the following logical conclusion, "$\therefore A = ½\ a\ (h_1 + h_2 + _{\dots} + h_n)$". Furthermore, Inderani argued that the sum of the heights, $h_1 + h_2 + _{\dots} + h_n$, is constant on the basis of the following warrants: *A* is constant; and the bases are constant, which means that "*a*" is constant. Although she did not mentioned that ½ is also a constant, this is implicit in her statement. Moreover, she exhibited a strong use of a propositional relationship in her very last statement, "if the left hand side is constant, the right hand side is constant", in order to finally justify or explain why $(h_1 + h_2 + _{\dots} + h_n)$ is constant.

In an equi-sided polygon, if triangles were constructed in the interior, the total area = $\frac{1}{2}a$ (sum of the heights) of the $\Delta$'s

$\therefore A = \frac{1}{2}a(\Sigma h)$, where $a$ is the bases of each triangle and the side of the polygon

Because the total area for the polygon remains constent and the bases (sides) remains constent, the sum of the heights will therefor also remain constent

$$\frac{2A}{a} = \Sigma h$$

constent $\rightarrow 2A$

constent $\rightarrow a$

Figure 10.1.8: Inderani's written explanation for her equi-sided convex polygon CG

It seems that since Inderani was able to carry out the process of justifying each of her specific generalizations (namely equilateral triangle, rhombus and equi-lateral pentagon problems) by using the 'triangle-area' algebraic explanatory structure, she was able to generalize (expansively) to the proof for her equi-sided convex polygon generalization without much difficulty (see Tall, 1991, p.12). In fact, Inderani was able to construct her 'triangle–area' algebraic proof for her equi-sided convex polygon generalization plausibly through structural-analogical mapping, which is also referred to as cognitive blending (see Section 4.4.3). This kind of expansive approach can be regarded as generic in nature, since the PMT constructed a general proof (i.e. a proof for the general case) by referring to particular proofs (i.e. proofs for the equilateral triangle, rhombus and pentagon generalizations). According to Tall (1991, p.12), "such a generic approach (*i.e. seeing the general proof through particular proofs*) is seen both an easy method of generalization because it applies a well known process in a broader context and also a first step towards formal abstraction as it does not involve major cognitive reconstruction".

**Case: Tony**

When the researcher probed Tony, as per Task 4(a) in Figure 10.1.1, as to whether he could generalize from the set of already established generalizations to polygons with a similar property, Tony replied: "It will hold that the sum of the distances from the interior point to the sides will remain constant". On further probing by the Researcher, Tony stressed his generalization to polygons in the form of a conditional statement as follows: "if it is an equi-sided polygon then the sum of the distances (*meaning the sum of the distances from the interior point to the sides of the polygon*) will remain constant".

It seems that Tony, like Trevelyan, Inderani and others, extended his generalization to equi-sided convex polygons on logical grounds. Although the Researcher provided Tony with a dynamic sketch of an equi-sided convex hexagon, it was noted that he never showed any interest to investigate empirically whether his equi-sided convex polygon generalization is really true. It seems that Tony, like the other PMTs, was able to reflect on his 'constant distant sum' generalizations made for specific equi-sided convex polygons, such as the equilateral triangle, rhombus and equi-lateral pentagon, and thereby abstracted that the 'constant distant sum' will also prevail for any polygon so long as it is equi-sided. On the basis of his abstraction, Tony like the other PMTs, extended his 'constant distant sum generalization' to the class of polygons constituting any equi-sided polygon, without expressing any need for experimental confirmation by using *Sketchpad* . This kind of logical deduction on the part of Tony and also other PMTs, characterizes the progression of the PMTs from the making of generalizations through experimentation, inductive reasoning or analogical reasoning to the making of generalization based on logical argument. Thus Tony, like the other PMTs, progressed to make his generalization to any equi-sided polygon on logical grounds, and drew away from the use of non-deductive processes to make such a generalization. Freudenthal (1973, p. 451) refers to this change in one's thought processes, as the "cutting of the ontological bonds". This cutting of the bond with experimental reality, demonstrates that the PMTs have cognitively grown in the construction of generalizations through the processes of generalizing and abstracting (see Tall, 2002; Tall et al., 2012).

When the researcher asked Tony, "Can you give me a proof of that (*referring to his equi-sided polygon generalization*)?", Tony spontaneously and verbally indicated that the proof he would make, would be similar to the logical explanation he had constructed earlier for his pentagon conjecture generalization, as illustrated in the following one-to-one task based interview excerpt:

| | |
|---|---|
| RESEARCHER: | Can you give me a proof of that (*referring to his equi-sided polygon generalization*)? |
| TONY | The only proof that I can make is that (*referring to the pentagon proof*)… I just want to draw the small triangles inside the polygon and |

try to find the total sum of the areas of the smaller triangles inside the polygon; take out that *a* the way that we did earlier. Work it out … so it should hold for all polygons.

RESEARCHER: What have you explained here?

TONY Instead of a pentagon, I would say a polygon because we are now talking about a figure that has more sides than five.

RESEARCHER: How many sides?

TONY An infinite number of sides.

RESEARCHER: *n* sides?

TONY Yes.

The Researcher on seeing that Tony was seeing the structure of his general proof through the structure of his earlier proof for his refined pentagon conjecture generalization probed him a little further about his verbal explanation regarding the figure under consideration, as follows:

RESEARCHER: What is the thing (*referring to Tony's considered figure i.e. polygon*) you explained here?

TONY Instead of a pentagon, I would say a polygon because we are now talking about a figure that has more sides than five.

RESEARCHER: How many sides?

TONY An infinite number of sides.

RESEARCHER: *n* sides?

TONY Yes.

After hearing Tony's description of the figure under consideration, the Researcher asked him to write down his logical explanation for his conditional conjecture generalization. Tony subsequently produced in his worksheet the written explanation as shown in Figure 10.1. 9.

It seems that Tony has also been able to discern the structure of general proof for any-equi-sided convex polygon from the structure of the proof for his earlier refined convex pentagon conjecture generalization. This is quite evident in his description of the construction of the small triangles, "… we can draw small triangles inside the polygon…", and from the slip in referring to a pentagon instead of a polygon in line 4 of his explanation. The latter mentioned slip (i.e. the use of the term 'pentagon' instead of 'polygon') demonstrates that Tony may have been looking back at the proof of his refined convex pentagon conjecture generalization and simultaneously modifying each step of his refined convex pentagon conjecture generalization (source) to accommodate a logical explanation for his equi- sided convex polygon generalization (target). Hence in the process, he appears to have just ‘slipped up’ by not replacing the word ‘pentagon’ by ‘polygon in line 4.

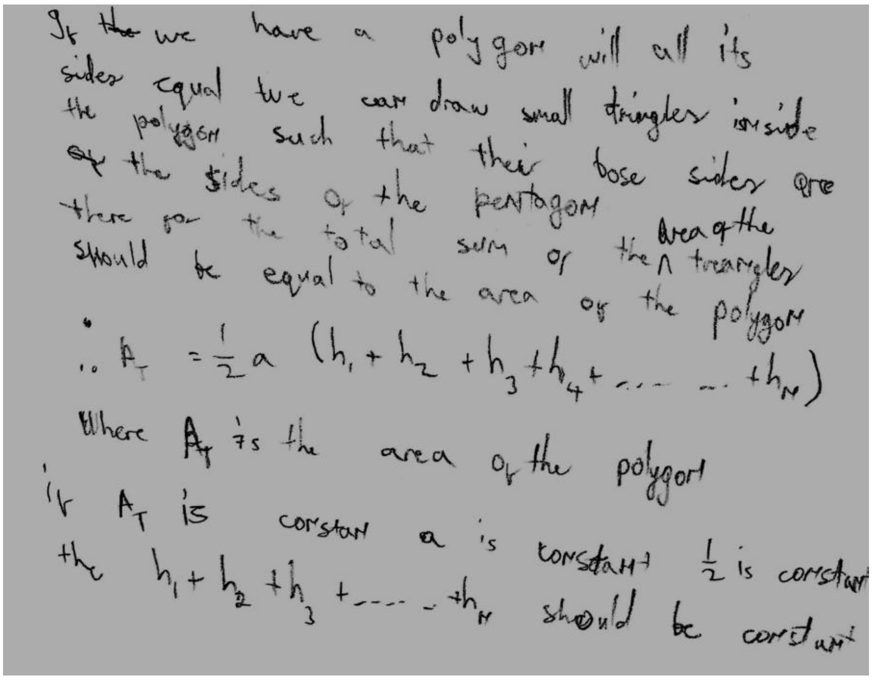

If ~~the~~ we have a polygon will all its
sides equal we can draw small tringles inside
the polygon such that their base sides are
of the sides of the pentagon
there for the total sum of the area of the n triangles
should be equal to the area of the polygon
$\therefore A_T = \frac{1}{2} a (h_1 + h_2 + h_3 + h_4 + \dots \dots + h_n)$
Where $A_T$ is the area of the polygon
if $A_T$ is constant $a$ is constant $\frac{1}{2}$ is constant
the $h_1 + h_2 + h_3 + \dots - + h_n$ should be constant

Figure 10.1.9: Tony’s written explanation for his equi-sided convex polygon CG

The aforementioned kind of parallel transfer of information from the source to the target is in consonance with Genter’s Structure Mapping Theory (SMT), as described in Section 4.4.2, which asserts that: ‘in interpreting an analogy people seek to put the objects in the base in one –to-one correspondence with the objects in the target so as to obtain the maximal structural match”, and more importantly, “ objects are placed in correspondence by virtue of their like roles in the common relational structure” (see Gentner, 1989, p. 201). This is equivalent to cognitive blending (see Section 4.4.3).

In his deductive justification, Tony considered the antecedent or the given aspect appropriately at the start of his explanation by writing, “If we have a polygon with all its sides equal, we….”. Moreover, he confidently ascertained that the area of the big equilateral triangle is equal to the area of the smaller constructed triangles, i.e.“ $\therefore A_T = ½\, a\, (h_1 + h_2 +_{\dots} + h_n)$”. Then Tony argued that the sum of the heights, $h_1 + h_2 +_{\dots} + h_n$ , is constant on the basis of the following warrants: $A$ is constant, $a$ is constant and ½ is constant.

**Case: Victor**

The following excerpt is a representation of the task-based interview between the Victor and the researcher in relation to Task 4a as shown in Figure 10.1.1.

| | |
|---|---|
| RESEARCHER: | Consider the above set of generalizations. Can you now generalize to polygons with a similar property? For example, can you generalize the result for a hexagon? Will the result hold true for a hexagon? |
| VICTOR: | Yes. |
| RESEARCHER: | And a heptagon? |
| VICTOR: | It can go through for all |
| RESEARCHER: | - an octagon? |
| VICTOR: | Yes, still. For all polygons with similar properties the results can apply. |
| RESEARCHER: | So which is the property you are talking about? |
| VICTOR: | If they all have constant sides – if all sides are equal. |

Victor's response, "If they all have constant sides – if all sides are equal", seems to suggest that through his earlier equilateral triangle, rhombus and equilateral pentagon generalizations and associated justifications thereof, he has now generalized the specific property, 'all sides equal', as a sufficient condition for the sum of the distances to remain constant in a given specific polygon, and thereby assimilated the established idea into his cognitive structure. The kind of deductive generalization exhibited by Victor, demonstrated that he had seen the general result through particular results.

The following one-to-one task based interview shows that Victor has seen that the kind of 'triangle–area' algebraic strategy which he used to provide a deductive justification for his earlier equilateral triangle, rhombus and pentagon generalizations could also be used to justify his generalization to the hexagon, heptagon, octagon and finally any equi-sided polygon

| | |
|---|---|
| RESEARCHER: | How will you prove it – that it will hold true for a hexagon or an octagon? (*Researcher is referring to equi-sided hexagons and octagons)* |
| VICTOR: | Based on my explanation there, my previous explanation (referring to his refined pentagon conjecture generalization), for any other polygon |

– if the sides are all equal – that means I can take out *a* as a common factor. But this I can prove.

RESEARCHER: So if it's six-sided…?

VICTOR: If it's six-sided that means I can have $h_1$ , $h_2$ , $h_3$ , $h_4$ ,$h_5$ and $h_6$ = to 2*A* over *a*. That can make the sum of the distances from point *P* constant.

RESEARCHER: and if it's eight-sided, how will you do it?

VICTOR: If it's eight-sided that means I can have $h_1 + h_2 + h_3 + h_4 + h_5 + h_6 + h_7 + h_8 =$ 2*A* over *a*. That can make sum of the distances from point *P* be constant.

RESEARCHER: And for an *n*-sided polygon, then?

VICTOR: Then I'm going to have $h_1$ , $h_2$ , $h_3$ up until I have $h_n$, which is equal to 2*A* over *a*. Which means all the polygons with similar properties which is those with constant sides.

RESEARCHER: So are you certain about that?

VICTOR: Yes.

Figure 10.1.10 represents the logical explanation (or proof) , which Victor eventually wrote in his worksheet:

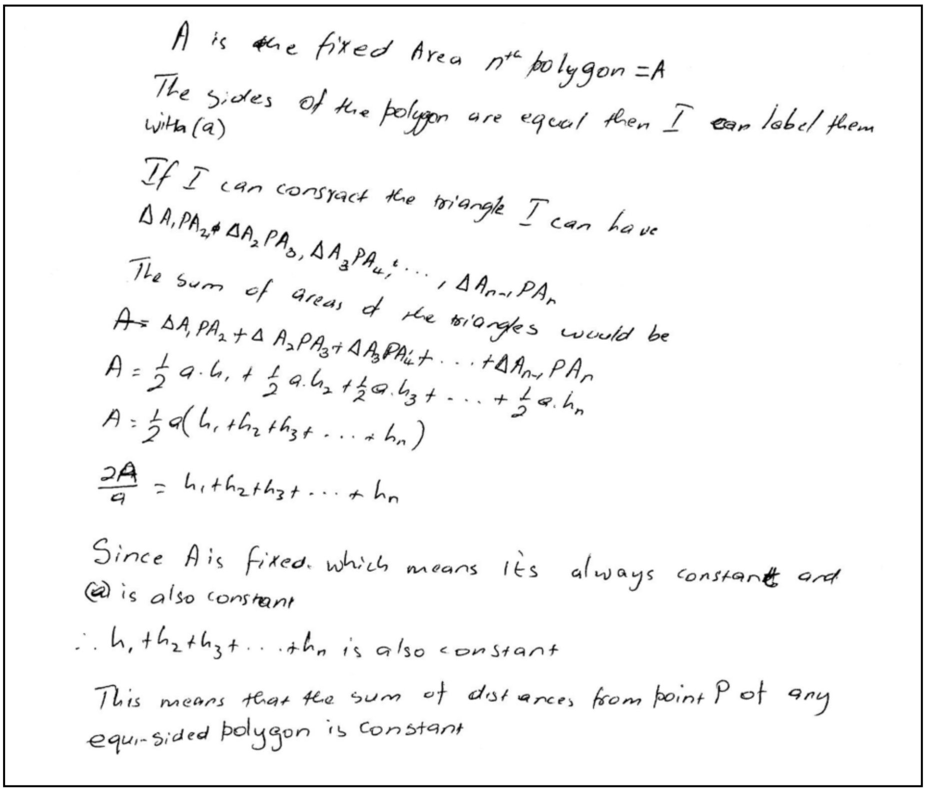

A is the fixed Area $n^{th}$ polygon $=A$
The sides of the polygon are equal then I can label them with (a)
If I can consyract the triangle I can have
$\Delta A_1PA_2, \Delta A_2PA_3, \Delta A_3PA_4, \ldots, \Delta A_{n-1}PA_n$
The sum of areas of the triangles would be
$A = \Delta A_1PA_2 + \Delta A_2PA_3 + \Delta A_3PA_4 + \ldots + \Delta A_{n-1}PA_n$
$A = \frac{1}{2}a \cdot h_1 + \frac{1}{2}a \cdot h_2 + \frac{1}{2}a \cdot h_3 + \ldots + \frac{1}{2}a \cdot h_n$
$A = \frac{1}{2}a(h_1 + h_2 + h_3 + \ldots + h_n)$
$\frac{2A}{a} = h_1 + h_2 + h_3 + \ldots + h_n$
Since A is fixed which means its always constant and (a) is also constant
$\therefore h_1 + h_2 + h_3 + \ldots + h_n$ is also constant
This means that the sum of distances from point P of any equi-sided polygon is constant

Figure 10.1.10: Victor's written explanation for his equi-sided convex polygon CG

The following one-to-one task based interview excerpt provides an in-depth description as to how Victor engaged with the development of the writing of his proof:

| | |
|---|---|
| RESEARCHER: | Can you read out what you've written there, please? |
| VICTOR: | Okay. *A* is a fixed area of a polygon – which I labelled *A*. The sides are all equal. I can label them with *a*. If I can construct the triangles, I can have triangle $A_1P\ A_2$ into triangle $A_2P\ A_3$ into triangle $A_3P\ A_4$ up until by single sequence I can have triangle $A_{n-1}\ PA_n$. Then the sum of the areas of the triangles – I can have the area of a triangle $A_1P\ A_2$ + triangle $A_2P\ A_3$ + triangle $A_3P\ A_4$ + triangle $A_4P\ A_5$ - I can add them up until I get to triangle of $A_{n-1}\ PA_n$.<br><br>When I add them together, since I said I labelled the sides by *a*, I'm going to have ½ $ah_1$ + ½ $ah_2$ + ½ $ah_3$ + ½ $ah_4$ –up until I add them up until I get to ½ $a\ h_n$. Then I take out the highest common factor which is ½ *a*. I'm going to have: $A$ = ½ *a* into $h_1 + h_2 + h_3$ + the number of *h*'s up until I get $h_n$. Then I simplify this, then I get 2*A* over *a*, which is equal to $h_1 + h_2 + h_3$ + up until I get $h_n$. Since *a* is fixed, it means it is always constant, and *a* is also a constant. Therefore $h_1 + h_2 + h_3$ + up until I get $h_n$ is also constant. This means the sum of the distances from point *P* to the sides of any equi-sided polygon is constant. |

Although Victor, did not label the sub-parts of his explanation proof, he has identified what is given, by writing down, "*A* is a fixed area of $n^{th}$ polygon = *A*, the sides of the polygon are all equal then I can label them with *a*". Furthermore, he has made the effort to describe his construction of the small triangles, "If I can construct the triangle I can have $\Delta A_1P\ A_2 + \Delta\ A_2P\ A_3 + \Delta\ A_3P\ A_4 + \ldots + \Delta\ A_{n-1}\ PA_n$". Victor, successfully added the areas of all the small triangles, and logically equated it to the area of the polygon itself, which after simplifying, resulted in the following equation:"$2A/a = h_1 + h_2 + h_3 + \ldots + h_n$". Then he logically argued that the sum of the heights, $h_1 + h_2 + _{\ldots} + h_n$, is constant on the basis of the following warrants: *A* is constant, *a* is constant, and finally concludes: "This means that the sum of distances from point *P* to the sides of any-equi-sided polygon is constant".

The steps in Victor's logical explanation, shows that there is reasonable degree of parallel connectivity between his logical explanation for his equi-sided convex polygon generalization and, for example, his logical explanation for his refined convex pentagon conjecture generalization (compare Figure 9.4.2 in Section 9.4 of Chapter 9), i.e. there is some structural consistency between the two logical explanations.

It seems that Victor, like the other 7 PMTs, was able to construct his logical explanation for his equi-sided convex polygon generalization through a process of correlative subsumption, which is one of the ways in which a new problem (or a variation of a given problem) is related to previous or relevant knowledge in the existing cognitive structure (see Ausubel, 1978; Aziz, 2009). In other words, Victor on seeing a similarity between the 'any' equi-sided convex polygon generalization and his respective generalizations for the previous three convex cases via the 'equal sides' property, was cognitively triggered to correlatively subsume the construction of his logical explanations for his 'equi-sided' convex polygon generalization into his earlier conceptual 'triangle-area' algebraic explanatory structure. Consistent with the correlative subsumptive process as posited by Ausubel (1978), with the aid of analogical reasoning, Victor modified his existing 'triangle-area' algebraic explanation for his refined pentagon generalization (or other particular generalization) to accommodate his triangle-area' algebraic explanation for his new equi-sided polygon generalization.

**Case: Alan**

The Researcher, as per the question in Task 4a (see Figure 10.1.1), asked Alan the following question, "Can you generalize the result to other polygons?". Alan replied, "You can't generalize it until you prove to those which don't have the equal sides – then the case is the same – then you can generalize it from there".

It seems clear from the afore-cited statement that Alan is aware that one cannot generalize it to 'any' polygon unless one has proven that it is also true for polygons with unequal sides. With the statement "then the case is the same – then you can generalize it from there", he seems to indicate that he has realized that the polygon needs to have equal sides. In this sense, Alan's response was insightful and relevant, more particularly because the researcher referred to just "other polygons" instead of "other polygons with a similar property" in his question.

Alan then continued by attempting to explain why the result might not hold for polygons with different sides. He in fact drew the sketch, which is shown in Figure 10.1. 11, at the back of his worksheet on his own accord:

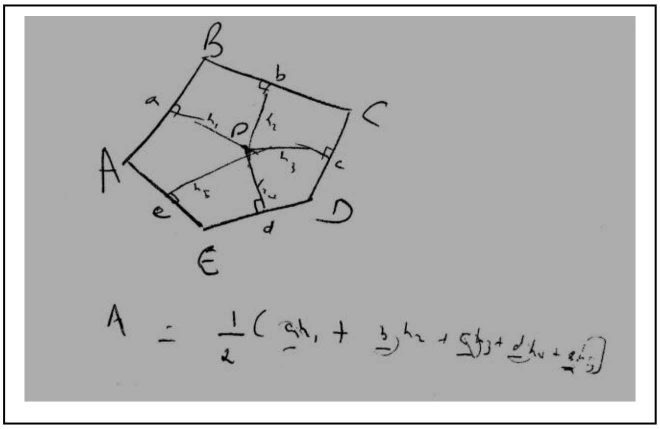



Figure 10.1.11: Alan's sketch used to explain why the result will not hold for irregular pentagons.

The following excerpts from the one-to-one task-based interview, illustrates how Alan made use of his sketch as shown in Figure 10.1.11, to logically argue why the result will not hold for irregular pentagons, which do not have equal sides:

ALAN: … we need to have, to try it in irregular polygons that don't have equal sides,… …so let's say this is a, this is b, this is c, this is d and this is e. So we have different sides. So we draw a line there, and we have a point which is going to be perpendicular to there, and then perpendicular to there, perpendicular to there and to there etc. So they all come to the same point *P*. And then if you have *a*, *b*, *c*, *d*, *e* – then for each we have $h_1$ , $h_2$ , $h_3$, $h_4$, $h_5$. So now they will have one thing in common: which is going to be ½.

RESEARCHER: What is common?

ALAN: the ½. So *A* is going to be this $\frac{1}{2}(ah_1 + b\,h_2 + c\,h_3 + d\,h_4 + e\,h_5)$
So, in this case … these vary.

RESEARCHER: Yes. So what is your point now? What are you trying to show?

ALAN: For a different-sided…uuuh… pentagon (…pause…) we can't find a constant *a*.

RESEARCHER: So what are you saying then? You cannot find a constant *a* – so what is that saying to you? What does it say about the result?

ALAN: Then *h* varies.

RESEARCHER: Ja, *a* is different here.

ALAN: *h*!

RESEARCHER: ….. does the result hold true for a pentagon whose sides are not equal? (…pause…) Will the result – the sum of the distances from a point inside the pentagon to its sides – be constant? Would it hold if the sides of the pentagon have different measurements?

ALAN: No, it won't.

The argument presented by Alan, as to why his result will not hold for irregular polygons with different (or unequal) sides, seems to rest on the fact that he would not be able to extract a common "*a*", which is the variable used to represent the equality of all sides in the polygon, as part of the common factor, ½ *a*. Indeed, it is possible that Alan looked at the general proof for the irregular pentagon with equal sides, as his reference, and thus argued that if there is no common side, *a*, then the general proof will not hold true in this case. It appears that Alan is overgeneralizing the structure of the proof of the irregular pentagon case with equal sides. Since there is no common factor ½ *a*, and he cannot get an equation as before: $A= \frac{1}{2} a( h_1 + h_2 + h_3+ h_4 + h_5 + h_6 + h_7 + ....+ h_n)$, and hence cannot explain why $( h_1 + h_2 + h_3+ h_4+ h_5+ + h_6 + h_7+ ....+ h_n)$ should be constant in such a case.

His reasoning of course, is incorrect as all his 'proof' pointedly show is that if the sides of a pentagon, or polygon, were not equal and the sum of the distances to its sides were constant, then it could not be explained (proved) in this manner. For example, we know the sum of the distances from a point to the sides of any equi-angular polygon or of a $2n$ –gon $(n \geq 2)$ with opposite sides parallel, are also constant, and these results cannot be explained (proved) in the same way as the equilateral polygon.

When the Researcher later asked the question again, but this time being more specific to polygons with a similar property, the response was as follows:

.

RESEARCHER: Will it hold for any equi-sided polygon? We don't know how many sides it has, but it has *n* sides.

ALAN: Yes, it will.

RESEARCHER: Are you sure?

ALAN: Yes

RESEARCHER: Can you just tell me why you think it will work for any equi-sided polygon . You don't have to write it down.

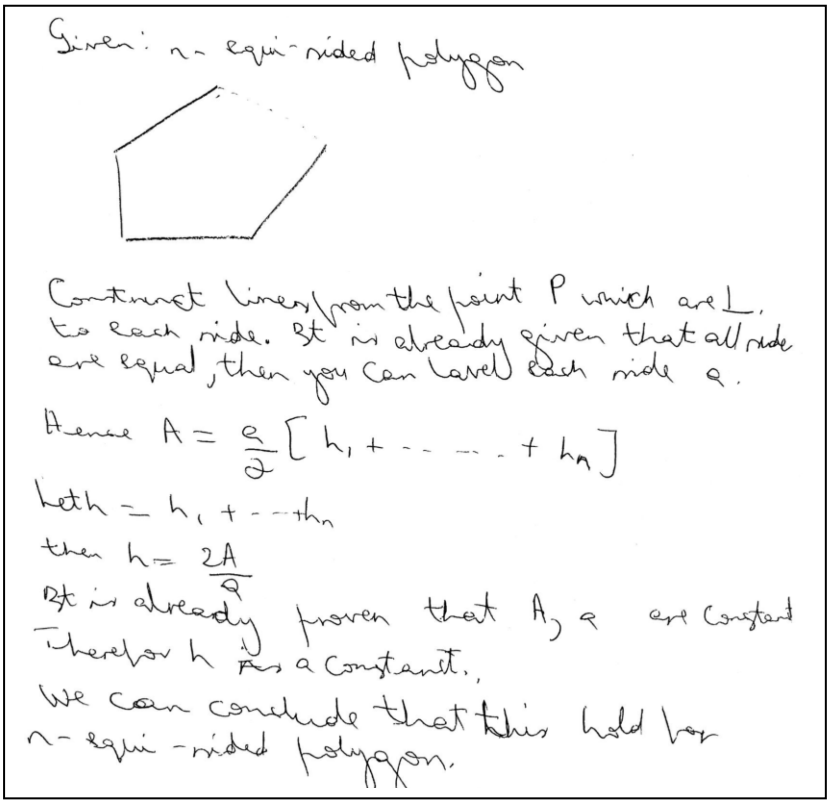
ALAN: It's a closed diagram, right?

RESEARCHER: Okay.

ALAN: So if you put a point inside it, it doesn't matter how big it is, so all the sides of that are equal, so if you put point $P$ inside and then you construct triangles with lines that are perpendicular to each side, then we can have $h$ from $h_1$ up until $h_n$. And when we collect them together, they have got a common factor of $a$ over 2. And at the end of the day you have an area which is $a$ over 2 into $h_1$ to $h_n$ (it depends on how many sides you have on that second point) and then at the end of the day you have $h$ being $2A$ over $a$, which is a constant, as we've already proved. So it doesn't matter how many sides you have.

Furthermore, when the researcher asked Alan to write down his logical explanation to justify his equi-sided convex polygon generalization, he proceeded spontaneously to produce the written logical explanation shown in Figure 10.1.12.

Alan used a kind of logical format, to write down his explanation. The sequence of his layout reflects his logical though processes. In particular, it is evident that he has identified the 'given' aspect of his conjecture or conditional statement by writing the following as given, "*n*-equi-sided polygon". Having identified the antecedent (given) part, Alan drew a diagram, which is a common requirement in most geometry proofs, and then went on to describe his set of constructions, and labelled his diagram with appropriate justifications. For example, he wrote, "it is already given that all sides are equal, then you can label each side $a$". Immediately, thereafter he logically deduced that the area of the big equilateral polygon is equal to sum of the areas of the smaller constructed triangles, and thus made the following logical deduction, "$A = a/2\ (h_1 + h_2 +_{\ldots} + h_n)$ ".He then manipulated the equation and simplified it to get $h = 2A/a$, where $h$ actually represented $h_1 + h_2 +_{\ldots} + h_n$. Then, on the basis of the data in his equation, $h = 2A/a$, Alan concluded, "therefore $h$ is a constant", on the basis of the following warrants: $A$ is constant and $a$ are constant. Using his claim, "therefore $h$ is a constant" as a premise, Alan made the following conclusion," we can conclude that this holds true for any *n*-equi-sided polygon". The overall representation and layout of Alan's explanation, suggests that he has developed the necessary skill and knowledge of how to provide/write a coherent explanation or argument, that would be easy for a reader to follow.



Given: n- equi-sided polygon

Construct lines from the point P which are $\perp$ to each side. It is already given that all side are equal, then you can label each side a.

Hence $A = \frac{a}{2}[h_1 + \ldots + h_n]$

Let $h = h_1 + \ldots + h_n$

then $h = \frac{2A}{a}$

It is already proven that A, a are constant

Therefor h is a constant.

We can conclude that this hold for n- equi-sided polygon.

Figure 10.1. 12: Alan's written explanation for his equi-sided convex polygon CG

Alan's verbal explanation and written logical argument, suggest that he has seen the structure of the general proof through the lens of the structure of the earlier particular proofs for the equilateral pentagon, rhombus and equilateral triangle cases. He then went on to complete his general proof by analogically mapping the structure of the 'triangle-area' algebraic proofs for the previous cases onto it. It appears that Alan first assimilated his explanation into the previously accommodated explanations for the equilateral pentagon, rhombus or equilateral triangle cases, which was then subsequently modified to accommodate the equi-sided convex polygon (see Sections 4.2.2 and 4.6 for a discussion on assimilation and accommodation).

**Case: Logan**

The researcher on seeing that Logan was able to provide a logical explanation for his refined pentagon generalization, asked him: "Do you think that the same result will hold for a hexagon?". Logan immediately responded as follows: "As long as the sides are all the same".

Logan's response confirms that he has identified that "all sides equal" is a sufficient condition which must exist in order for his conjecture to prevail. He further acknowledged that his earlier generalization will hold for polygons such as hexagons, octagons and nonagons, and in general for any polygon, as long as the condition, "all sides equal" is present.

RESEARCHER: - and for a heptagon?

LOGAN: I believe it will as long as the sides are all the same.

RESEARCHER: - and for an octagon?

LOGAN: - yes, as long as the sides are all the same.

RESEARCHER: What other polygons can it hold for?

LOGAN: - a nonagon.

RESEARCHER: Yes.

LOGAN: - and a heptagon and a hexagon.

RESEARCHER: How would you explain the result for a hexagon?

LOGAN: I will basically follow the same procedure.

RESEARCHER: - and for an octagon?

LOGAN: Octagon - exactly the same.

RESEARCHER: Now do you think the result will hold for an 11-sided polygon whose sides are equal?

LOGAN: Yes.

RESEARCHER: - for a twelve-sided polygon?

LOGAN: - as long as the sides are equal.

RESEARCHER: And how will you explain that?

LOGAN: As long as the sides are equal, I'll explain it in exactly the same way. …(*referring to his earlier explanations*)

RESEARCHER: And will this result hold for any polygon whose sides are equal?

LOGAN: Ja.

RESEARCHER: And how will you explain that?

LOGAN: In the same way, as long as the sides are the same…(*referring to his earlier explanations*) …

RESEARCHER: But how will you explain that result? (*probing the student*)

LOGAN: I would add all the areas of the inside triangles of the *n*-sided polygon, and take out the common factor as I did before, and then … get the heights on the one side, and because it's a fixed polygon it remains constant (*meaning its area remains constant*), and the 2 is constant, and the small *a* of the sides will be all the same (meaning the sides are constant in measure).

Although Logan did not finally conclude that the sum of the heights (which is presumably on the LHS of the equation) will be constant because 2*A*/*a* is constant, his explanation nevertheless demonstrates that he has seen the 'sameness' between his equi-sided polygon generalization and his earlier generalizations. He thus rationalized that the 'triangle-area' algebraic structure that was used to construct a logical explanation for each of his earlier generalizations could similarly be applied to construct a logical explanation that could justify his generalization for any equi-sided convex polygon (i.e. the general case). This is much more evident in line 8 of his written explanation (see Figure 10.1.13), as he used the word 'pentagon' instead of 'polygon'

Of course at the surface level one can say this was a slip or error, but at a deeper level it seems to demonstrate that he was constructing a logical explanation for his equi-sided convex polygon generalization (a target) by building an appropriate correspondence with the elements (or steps) of his earlier logical explanation for his refined convex (equilateral) pentagon generalization (base or source), and in the process he may have forgotten to replace the word 'pentagon' with the word 'polygon'. Despite, this anomaly, there appears to be a structural consistency (i.e. parallel connectivity) between the source and target (i.e. the logical explanation for the general case is relationally similar to the logical explanation for a particular case) – (see Sections 2.1.5 and 4.4 that discusses analogy and transfer in detail).

**Task 4(d): Present your explanation/justification *: n- equi-sided* polygon**

Summarize your explanation/justification of your conjecture (generalization). You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a *Sketchpad* sketch, or some other medium.

~~IF YOU ADD UP A~~

FIRSTLY THE SIDES SHOULD BE EQUAL
YOU ADD UP ALL THE AREAS OF THE INSIDE TRIANGLES
YOU TAKE OUT THE COMMON FACTOR WHICH IS $\frac{1}{2} \cdot a$ AN.
MAKE SURE YOU LEFT WITH $\frac{1}{2} \cdot a\,(h_1 + h_2 + h_3 + h_4 \ldots)$
WHEN MULTIPLYING WITH $\frac{1}{2} \cdot a$ YOU HAVE
$\frac{2A}{a} = (h_1 + h_2 + h_3 \ldots)$. BECAUSE THE $A$ IS A CONSTANT
THE 2 A CONSTANT AND THE $a$ ALSO A CONSTANT
THE $\frac{2A}{a}$ IS A CONSTANT BECAUSE THE PENTAGON
IS FIXED. THIS MEANS THE LHS IS ALSO
CONSTANT.

Figure 10.1.13: Logan's written explanation for his equi-sided convex polygon CG

Although Logan, in his last statement, wrote down LHS instead of RHS, it seems evident from his written explanation that he was able to look at the structure of the proofs (logical explanations) for the particular convex cases (equilateral pentagon, rhombus and equilateral triangle) and mirror a similar kind of proof for the general case, the equi-sided convex polygon.

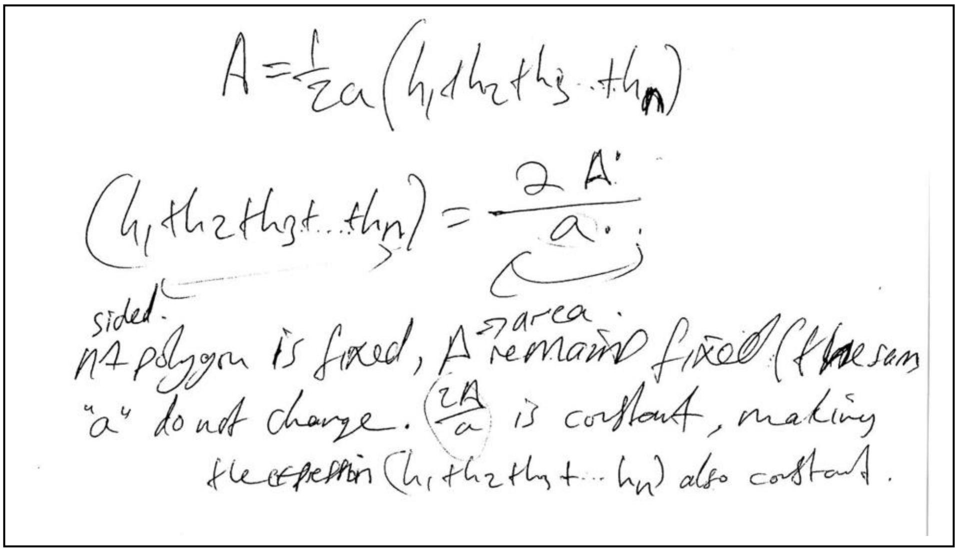
Although Logan did not label the sub-parts of his explanation proof, he has clearly identified the antecedent, which is the given part, because he wrote the following as the initial statement in his worksheet, "Firstly the sides should be equal". Although he did not draw a diagram and describe his constructions, Logan went on to logically explain that he will add up the areas of the smaller triangles and take out the common factor, which of course, was a common step in the proofs constructed for the earlier particular cases. After deriving the following equation, $2A/a = (h_1 + h_2 + h_3 \ldots)$, he went on to argue that $2A/a$ is a constant on the basis of the following warrants: $A$ is constant, 2 is a constant and , "$a$" is also a constant .

**Case: Renny**

When the Researcher, as per Task 4(a) in Figure 10.1.1, asked Renny: "Consider the above set of generalizations. Can you now generalize to polygons with a similar property?", Renny spontaneously replied: "Yes, for a polygon that has equal sides, the sum of the distances will always be constant". However, the Researcher challenged Renny, by asking "Will it be true for any polygon?", and Renny confidently replied: "yes, if all the sides are equal".

Renny' response, "yes, if all the sides are equal", seems to suggest that from his previous generalizations, he has come to realize that that 'equal sides' is a condition that a polygon must possess in order for the distance sum to be constant. In other words, Renny has come to see 'equal sides' as a sufficient condition, which could give anyone the 'go-ahead' to extend the 'constant-distance sum' generalization to any equi-sided convex polygon. This suggests that Renny has a sense of what constitutes a logical relationship, that is a propositional relationship. In other words, if the premise, which in this case is, "If all the sides are equal", prevails in a given case, then it automatically means that the conclusion, namely the sum of the distances from point $P$ to the sides is constant, will logically follow.

Despite the degree of logical deduction already displayed by Renny, the researcher asked Renny to produce a written logical explanation to justify his equi-sided polygon generalization. Figure 10.1.14 represents Renny's written logical explanation.


$A = \frac{1}{2}a(h_1 + h_2 + h_3 \ldots + h_n)$

$(h_1 + h_2 + h_3 + \ldots + h_n) = \frac{2A}{a}$

sided. area
n polygon is fixed, A remains fixed (the sum
"a" do not change. $\frac{2A}{a}$ is constant, making
the expression $(h_1 + h_2 + h_3 + \ldots h_n)$ also constant.

Figure 10.1.14: Renny's written explanation for his equi-sided convex polygon CG

Renny did not set out his explanation in terms of the following aspects such as given, diagram, construction, required to prove and proof. However, he used the same area- triangle algebraic strategy to derive the following equation in his worksheet, "$A = ½\ a\ (h_1 + h_2 + _{\ldots} + h_n)$". Thereafter Renny went on to successfully manipulate the equation and re-arranged it to get, $( h_1 + h_2 + h_3 +\ldots+ h_n) = 2A/a$. Renny then claimed that $2A/a$ is constant on the basis of the warrants, $A$ and $a$ are constant. ($A$ was considered constant the equi-sided polygon was fixed). Having logically deduced that $2A/a$ is constant, Renny then concluded by writing, ".. making the sum $(h_1 + h_2 + _{\ldots} + h_n)$ also constant".

It seems that Renny, like the other PMTs, after having reflected on his earlier 'triangle –area' algebraic structure proofs for their respective specific equi-sided convex polygon generalizations, saw the general structure of the proof that would explain his 'constant distant sum' generalization for any equi-sided convex polygon. Hence, like the other PMTs, he proceeded with spontaneity to perform the necessary transformations on one of his earlier particular proofs (like the logical explanation for their refined pentagon conjecture generalization) to produce the desired general proof.

According to Tall (2002) & Tall et al.(2012), this transition from particular proof (or generic proof) to a general proof, characterizes the PMT's cognitive growth in proof development, and simultaneously signals that he has in a sense cut off the ontological bonds (see Freudenthal, 1973, p. 451) with his earlier particular proof structures (or particular proofs).

## 10.2 Findings as per Section 10.1 (PMTs producing and justifying CG to any equi-sided polygon)

1. All PMTs were asked to consider the set of generalizations that they may have developed earlier with regard to particular convex polygons

   G1: In an equilateral triangle the sum of the distances from a point inside the triangle to its sides is constant.

G2: In a rhombus the sum of the distances from a point inside the rhombus to its sides is also constant.

G3: In any equi-sided pentagon the sum of the distances form a point inside the pentagon to its sides is also constant.

When each of the eight PMTs were asked to consider the above set of generalizations, and generalize to polygons with a similar property, they made their generalization on logical grounds, without expressing any explicit need for visual/experimental confirmation through the use of *Sketchpad* as follows:

1.1 Six of the eight PMTs extended their earlier 'constant distant sum' generalization that was made for specific equi-sided convex polygons like the equilateral pentagon, rhombus and equilateral triangle, automatically onto their 'any' equi-sided convex polygon (i.e. general equilateral convex polygon).

1.2 One PMT, namely Shannon, saw that the structure of the 'triangle-area' algebraic explanations that she had used to construct deductive justifications for each of the generalizations (i.e. G1. G2 & G3) could similarly be used to construct a logical explanation (i.e. deductive justification) for any equi-sided convex polygon, and hence made her generalization to any convex polygon with equal sides on logical grounds.

1.3 One PMT, namely Alan, demonstrated that he was aware that the general polygon needed to have equal sides to produce a constant distance sum, and tried to deductively demonstrate that the 'constant distance sum' would not prevail if the sides of a convex polygon are not all equal.

1.4 In summary, all PMTs arrived at the following generalization: The sum of the distances from a given point inside any equi-sided convex polygon to its sides will always remain constant.

2. By virtue of making their generalization to any equi-sided convex polygon on logical grounds with no need for experimental/empirical verification, it seems that all PMTs had finally cut off their ontological bonds with their earlier forms or processes of making generalizations.

3. The underlying condition, “all sides equal”, was acknowledged in different ways by all PMTs, as a sufficient condition for the sum of the distances to remain constant.

4. When asked to explain why their generalization for any equi-sided convex polygon was true, all of the PMTs equivalently mentioned: if all the sides of the convex polygon are equal, then a similar kind of procedure or explanation, i.e. the ‘triangle–area’ algebraic explanation offered for the other particular cases, like the equilateral pentagon, rhombus and equilateral triangle will hold.

5. None of the eight PMTs provided an empirical argument or made reference to *Sketchpad* to justify why their conjecture generalization for the equi-sided convex polygon was always true.

6. None of the eight PMts required any specific form of scaffolded guidance to develop a logical explanation to justify why their conjecture generalization for the equi-sided convex polygon was always true.

7. All 8 PMTs discernibly reflected on the general process and made a generic abstraction, from the earlier cases, to recognize that the ‘triangle-area’ algebraic method, can be extended, widened or extrapolated to construct a logical explanation to justify the invariance of the distance sum in any equi-sided conex polygon. Thus, in this instance the generic abstraction coupled with analogical reasoning made it possible for all PMTs to make the transition to formal abstraction, i.e. to construct and develop a general proof for any equi-sided convex polygon. This essentially means they have seen the general proof through a set of particular proofs or they have seen the general through the particular.

8. It seems that the phenomenon of ‘looking back’ (i.e. folding back) at their prior explanations assisted the PMTs to extend their logical explanations to the general equi-sided convex polygon. This development of a logical explanation (proof) for the general case after looking back and carefully analysing the statements and reasons that make up the proof argument for the prior particular cases, namely pentagon,

rhombus and equilateral triangle, emulates the discovery function of proof as discussed and elaborated in Section 5.4.2 of Chapter 5.

9. The insight and understanding gained from the logical explanations for the earlier generalizations, seemed to have created a ‘road map’ that enabled all the PMTs to discover a general proof for the equi-sided convex polygon. Thus in this sense, the explanatory function proof complimented that discovery function of proof and vice versa (see Sections 5.4.1 and 5.4.2 for discussion on the explanatory and discovery functions of proof respectively). The prior logical explanations seemed to have empowered PMTs to attempt a general argument as it increased their confidence and ability to compose it immediately on logical grounds in combination with analogical reasoning (i.e. cognitive blending).

10. All eight PMTs appeared to have assimilated their logical explanations for their ‘equi-sided convex polygon’ generalization into their previously described logical explanations for the refined pentagon generalization (or regular pentagon or rhombus or equilateral triangle generalizations), and then modified it to accommodate the regular pentagon cases. The aforementioned assimilation–accommodation of the logical explanations, is consistent with Ausubel’s theory of correlative subsumption, because it seems that in conceptualizing the logical explanation for the ‘equi-sided polygon’ generalization ( a new idea), a link was made to an already existing idea (old idea), namely the ‘triangle area’ algebraic explanation for the refined pentagon conjecture generalization (or other earlier generalizations). The new idea was assimilated into the old idea, and was subsequently modified to produce (or accommodate) the logical explanation for the regular pentagon case (see discussion of correlative subsumption in Section 4.5.1 of Chapter 4).

11. The PMTs’ justification for their equi-sided convex polygon generalization in the form of a logical explanation, showed understanding of and expressed propositional relationships explicitly, wherein the relationship between premise and conclusion was well articulated in their deductive arguments, with the necessary data, warrants and backings.

12. All PMTs provided some structure to the layout of their written explanations. For example, the antecedent of the conjecture generalization (or conditional statement)

was  appropriately considered as the ‘given’ aspect in the write up, and this is then followed by a relevant sketch, constructions, the ‘required to prove part’ and the subsequent logical development of the explanation/proof. The required to prove part, speaks to the consequent aspect of the conjecture or conditional statement.

The next Chapter, contextualizes the findings with regard to each of the four problems (Equlialteral triangle, Rhombus, Pentagon and ‘any’ Equi-sided Polygon) in relation to the research questions of this study.

# Chapter 11: Findings in Relation to Research Questions

## 11.0 Introduction

The PMTs in this study engaged with four connected task-based activities associated with a sequence of convex polygon problems, namely the equilateral triangle problem, rhombus problem, pentagon problem, and 'any' equi-sided convex polygon problem. These activities were designed with the idea of creating a learning path to support pre-service teachers in generalizing (i.e. extending) Viviani's theorem to a sequence of equilateral polygons of four sides (rhombi), five sides (pentgaons) and then ultimately to equi-sided convex polygons. The ultimate purpose of the tasks was to enable PMTs to discover a generalization of Viviani's theorem, namely: In any equi-sided convec polygon the sum of the distances from a point inside the polygon to its sides is constant.

In this Chapter the findings that emerged through PMTs engagement with each of the convex polygonal problems (equilateral triangle, rhombus, pentagon (regular & irregular) and any equi-sided polygon) are discussed and reflected upon in the context of the research questions for this study. In so doing a graphic exposition is provided to reiterate the sequence of events as the generalizing and justifications unfolded as the PMTs responded to the respective task-based activities during their one-to-one task-based interview sessions. Further to this, the researcher wishes to emphasize the effect of the changes in each of the eight PMTs as they sequentially progressed through each of the problems as well as the impact of the new discoveries on each of the PMTs. In addition reflections on the theoretical, methodological aspects of this study will be provided.

The equilateral triangle problem created space for the pre-service teachers to re-construct the Viviani result (called a particular generalization in this instance) for equilateral triangles, whilst the rhombus, pentagon and equi-sided polygon problems provided an opportunity for the PMTs to extend their established Viviani result for equilateral triangles across to a rhombus and then to a pentagon and ultimately generalize to any equi-sided convex polygon through using one or more of the following kinds of arguments/reasoning: inductive, analogical, deductive. Through progressing from the equilateral triangle problem to the rhombus problem, then to the pentagon problem and finally to the equi-sided polygon convex problem, the pre-service mathematics teachers experienced typical mathematical processes through which new content, such as generalizations, are discovered, constructed, and justified. In particular, pre-service mathematics teachers were exposed to the following kinds of processes through the designed task-based activities: intuitive guessing; conjecturing and generalizing through experimentation, inductive reasoning and/or analogical reasoning; heuristic refutation and global refutation; deductive generalizing (i.e. generalizing on logic grounds); justifying of generalizations either empirically, generically, deductively or deductively with the aid of analogical reasoning.

The engagement of the pre-service mathematics teachers in the whole process until they produced a generalization of Viviani's Theorem, was underpinned by the learning theory of constructivism. The following theoretical frameworks/theories were invoked for this study: Piaget's equilibration theory (i.e Piaget's socio-conflict theory); Ausubel's theory of meaningful learning; Gentner's Structure Mapping theory; explanatory and discovery functions of proof; counter-examples and scaffolding.

The data analysis, results and discussion Chapters 7, 8, 9 and 10 respectively reported on the pre-service teachers' constructions and justifications of their generalizations across the following convex polygons: equilateral triangle, rhombus, pentagon and general convex equi-sided (equilateral) polygon. This Chapter presents the key findings of this study in relation to each of the four research questions, which were as follows:

- Can pre-service mathematics teachers construct a generalization, which says that the sum of the distances from a point inside an equilateral triangle to its sides is constant? If so, how do they accomplish this generalization (which is commonly referred to as Viviani's theorem)?
- Can pre-service mathematics teachers support their equilateral triangle generalization with a justification, and if so, how do they construct (or provide) a justification for it?
- Can pre-service mathematics teachers further generalize and extend the Viviani Theorem for equilateral triangles to equilateral (convex) polygons of four sides (rhombi), five sides (pentagons), and then to equilateral convex polygons in general? If so, how do they accomplish the constructions of such further generalizations?
- Can pre-service mathematics teachers justify each of their extended generalizations to equilateral convex polygons (namely, rhombus, pentagon and general equilateral polygon generalizations)? If so, how do they accomplish the justification of each of their respective further generalizations?

## 11.1 Equilateral Triangle Problem

This section discusses the findings related to Research Questions 1 and 2. Research Question 1 focuses on the generalization for the equilateral triangle problem and Research Question focuses on the justification of the equilateral triangle conjecture generalization.

### 11.1.1 Research Question 1: Generalization for the Equilateral Triangle

As described in Section 7.0, the equilateral triangle problem was used as the ‘vehicle’ to answer this core question. Figure 11.1 provides an overview of the path that PMTs traversed to construct their inductive generalization for the equilateral triangle problem. I believe this can help understand the dynamism that evolved with the construction of a generalization for the equilateral triangle.

At the very start of the task based activity (see Section 7.1.1), PMTs were first given an opportunity in a non-*Sketchpad* context with an expectation that they would spontaneously use their own intuition to locate a point in the equilateral triangle (which was drawn on the hard copy worksheet) where they thought Sarah should build her house so that the total sum of the distances from the house to all three beaches is a minimum. All eight of the PMTs located a point in the centre (or middle) of the equilateral triangle (see Finding 1 of Section 7.1.1.1). The same kind of choice was also exhibited unanimously by all the grade 9 learners that participated in a similar activity in Mudaly’s (1998) study. Indeed the choice of the centre (commonly known as midpoint or centroid) is certainly one of the correct positions amongst many others that will create the minimum distance.

When the PMTs were asked to explain or justify why they chose the centre, the explanations of three PMTs, Victor, Logan and Renny, suggested they had been thinking that the distances to the sides should be equal to minimize the sum (i.e. they displayed a misconception that the distances have to be equal in order to minimize the sum). Although some of the other five PMTs may have also been intuitively thinking along the same lines, none of them provided a reason to justify their choice (see Findings 2 & 3 in Section 7.1.1.1). In particular, none of the PMTs used symmetry offered symmetry as a reason for their initial choice.

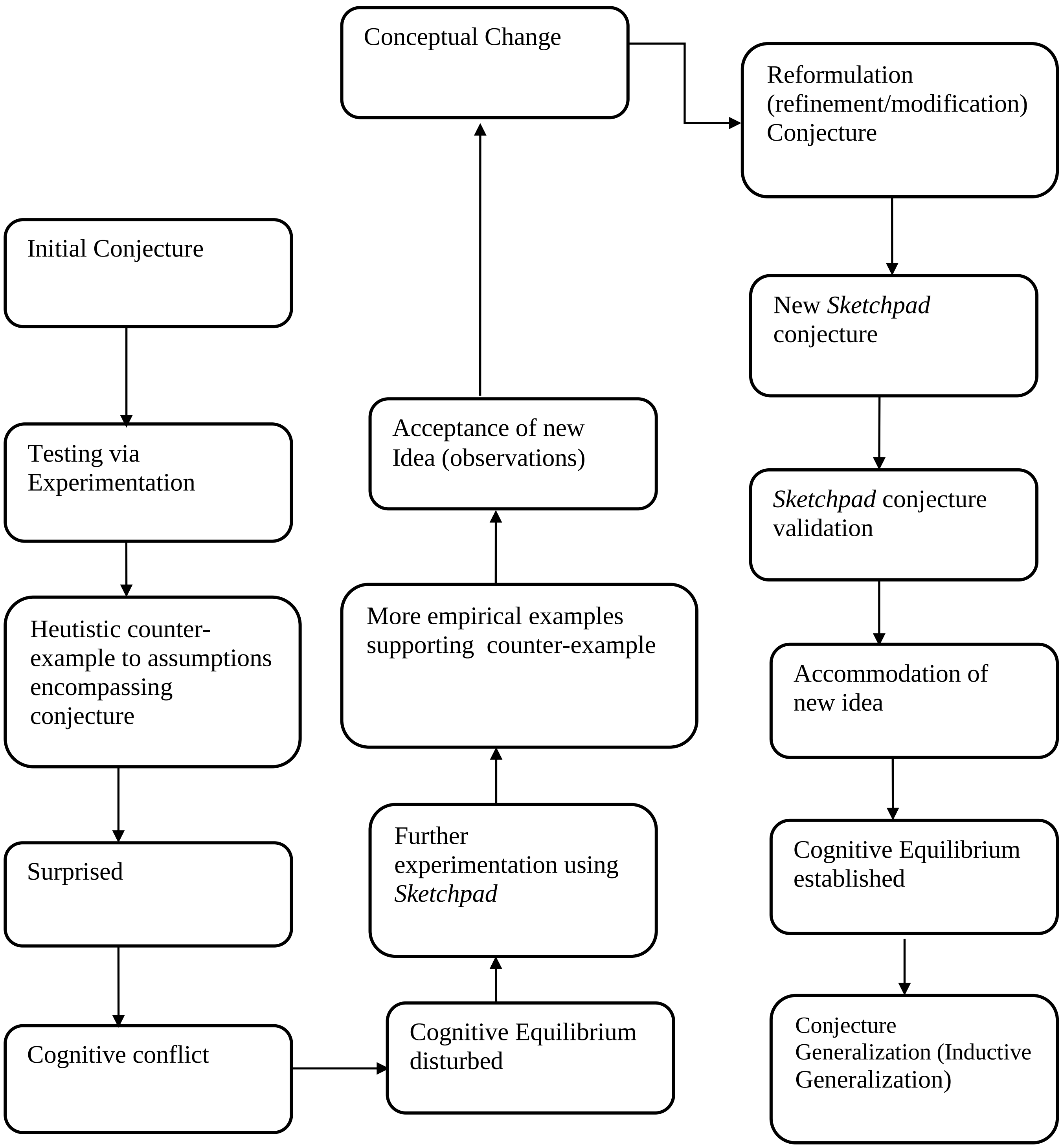



Figure 11.1: A trajectory of the PMTs Inductive Generalization

Consistent with a constructivist perspective of learning which suggests that learning should occur through the active involvement of learners as discussed in Section 4.2, the Researcher provided the PMTs an opportunity to explore and experiment with their initial conjecture using *Sketchpad*. In this instance, each PMT during their one-to-one task-based interview was given a ready-made sketch containing an equilateral triangle with a point *P* inside the triangle representing a possible position of the house. As per Task 1(b) in Appendix 1, through dragging point *P*, the PMTs constructed a visual continuum of cases with each having a different location of point *P*, and hence observed the invariance of the distance sum, namely the sum of the distances was constant. As discussed in Section 7.1.4.1, the first visual example acted as a heuristic counter example not to their initial non-*Sketchpad* conjecture but to the assumption that the centre was the only point creating the minimum distance. Although the further experienced empirical examples supported their acceptance of their heuristic counter-example. this occurred with a degree of surprise to all PMTs (see Finding 1 of Section 7.1.2.3) and prevailed across the cases as described in Section 7.1.2. This surprise can be attributed to the PMTs seeing that the displayed dynamic empirical examples, were contradicting an assumption in their initial conjecture, namely that point *P* should be at the centre. A similar element of surprise was exhibited by the grade 9 learners in Mudaly's study (1998, p. 66), when through dragging they found that the sum of the distances from point *P* to the sides of the equilateral triangle remained constant for any position of point *P* within the equilateral triangle.

As per Piaget's Equilibration Theory discussed in Section 4.6, the experienced contradiction seemed to have caused some cognitive conflict within their cognitive structures and hence disturbed their cognitive equilibrium, i.e. it brought about cognitive disequilibrium, which is commonly known as cognitive conflict. In an attempt to resolve their internal conflict, each PMT abandoned their initial conjecture and through accommodation of the new idea within their cognitive structures, reconstructed a new conjecture which embraced the invariance of the distance sum. In this way each PMT restored his/her cognitive equilibrium. Thus, within this context (see Section 7.1.4.1, it seems that the heuristic counter-example was the underlying cause of the 'cognitive conflict' that forced the PMTs to reject their earlier conjecture and generalize to a new conjecture which says that Sarah could build her house anywhere inside the equilateral triangle shaped island. The aforementioned PMTs modification (refinement) of their initial non-*Sketchpad* conjecture (see Finding 1 in Section 7.1.4.3) in favour of their new conjecture portrays cognitive conflict as a 'key driver' for conceptual change (see Ausubel, 1968; di Sessa, 2006; Biemans & Simons, 1999; Duit, 1999; Lee et al., 2003). Through further experimentation with new particular cases, and seeing that their *Sketchpad* conjecture was still true for such new cases, the PMTs began to see and believe that their *Sketchpad* conjecture was true in general. In particular, the visual empirical examples thus provided the necessary warrants for all eight PMTs to eventually be completely convinced that were no counter-examples to their conjecture generalizations (see Finding 2 in Section 7.1.4.3).

Returning to Research Question 1, the results that accrued from the Equilateral Problem Task, demonstrated that the PMTs were able to construct a conjecture generalization, namely, that the sum of the distances from a point inside an equilateral triangle to its sides is constant. This conjecture generalization was arrived at inductively, not through one single process but rather a set of complementary processes as represented by the multi-faceted trajectory depicted in Figure 11.1.

### 11.1.2 Core Research Question 2: Justification of equilateral triangle conjecture

As per finding in Section 7.1.5.1, all eight PMTs expressed some desire and need for an explanation, despite being highly convinced by quasi-empirical testing This is similar to Mudaly's (1998, p. 85) finding in a dynamic context and that of De Villiers's (1991, p. 258) finding in a non-dynamic context. As this study adopted a constructivist-oriented approach to learning (see Section 4.2), the PMTs were first given an opportunity to construct their own explanation to explain and/or justify their posited conjecture generalization. The finding that emerged in Section 7.1.6, showed that none of the eight PMTs could provide a logical explanation on their own, but instead 7 of them only attempted to justify the general truth of their conjecture generalization with reference to their dynamic geometry exploration, i.e. they provided an empirical kind of justification (see Section 7.2.1). However, as pointed by Stylianides (2008) trying to explain why a conjecture generalization is always true by merely showing that it holds true for some particular examples or cases does not provide conclusive evidence that the conjecture generalization is always true.

Like Mudaly (1998, p. 101) it was found that all eight PMTs displayed a need for guidance in constructing a logical explanation to justify why the sum of the distances always remain constant in the equilateral triangle case (see Finding 1 in Section 7.1.6.1). The researcher then provided each PMT with a scaffolded worksheet (see Task 1(c) of Appendix 1 and See Section 7.2.2). As each PMT engaged with the sub-tasks in the worksheet, they were provided with necessary guidance by the facilitator as and when it was needed by the PMTs or when it was noticed that a PMT was not able to move on from one sub-task to the next or was just struggling to respond to a given sub-task(s). The amount of intervention and guidance offered by the facilitator to enable the PMTs to accomplish the sub-tasks that he or she could otherwise not complete on their own was largely a function of the extent that PMTs could not (or struggled) to answer sub-questions in the scaffolded worksheet. To determine the kind of assistance or guidance that a particular PMT needed, the facilitator first probed a PMT about his/her response (or no response) by asking specific questions that were related to the sub- question (or sub-problem at hand).

The knowledge, skills and ideas that each PMT acquired as he/she moved through each sub-task of the scaffolded worksheet through individualized guidance whenever it was necessary, was connected to each PMT's existing schemes and were gradually internalized by each PMT. Similar to Mudaly's finding (1998, p. 101), the scaffolded worksheet coupled with some individualized guidance by the facilitator allowed each PMT to become "increasingly self-regulated and independent" as espoused by the social constructivist approach to teaching and learning (Snowman, Mcowan, and Biehler, 2009, p. 328). Through connecting and linking their respective internalized ideas as per sub-tasks, each of the eight PMTs managed to build (construct) an explanation that both explained and justified why the sum of the distances from the sides from a given point inside an equilateral triangle to its sides always remained constant.

The PMTs' progression through the various scaffolded sub-tasks in the worksheet with the aid of necessary guidance, support, probing and facilitation from the researcher, seemed to have enabled them to build on their prior knowledge, internalize new information, gain the necessary insight as to why the sum of the distances remained constant, and move onto the next level. For example, when the PMTs were afterwards asked to write down a complete coherent logical explanation as an argument in a paragraph form or two column form, such that it logically explains why their conjecture generalization is always true, seven of the PMTs were able to do so independently (see Finding 1 of Section 7.2.3.1). Three PMTs produced their logical explanation in a two- column form wherein the following aspects were clearly described: The given information; what was required to be proved; the necessary constructions; and the main body which showed the development of the logical argument with necessary and sufficient reasons. Four PMTs provided their logical explanation in paragraph form. (See Finding 1 in Section 7.2.3.1)

Returning to Research Question 2, the findings of this study show that majority (seven out of eight PMTs) initially provided an empirical justification via dynamic cases for their equilateral conjecture generalization. Subsequently through scaffolded support they were able to justify their equilateral conjecture generalization by means of a logical explanation.

### 11.2 Rhombus Problem

In this section, the findings associated with the Rhombus problem are related to Research Questions 3 and 4. Research Question 3 focuses on the extension of the Viviani generalization for equilateral triangles to the rhombus, pentagon and finally to any equi-sided convex polygon as well as how the PMTs made their generalization to the respective cases. On the other hand, Research Question 4, focuses on whether PMTs can justify each of their extended generalizations and also on 'how' they constructed their generalizations.

#### 11.2.1 Research Question 3: Rhombus Problem (Generalizations)

Figure 11.2.1, provided at this juncture of my narrative is meant to signpost my visualization of the trajectory of Rhombus Conjecture Generalizations. I believe this will augment the discursive flow of this section.

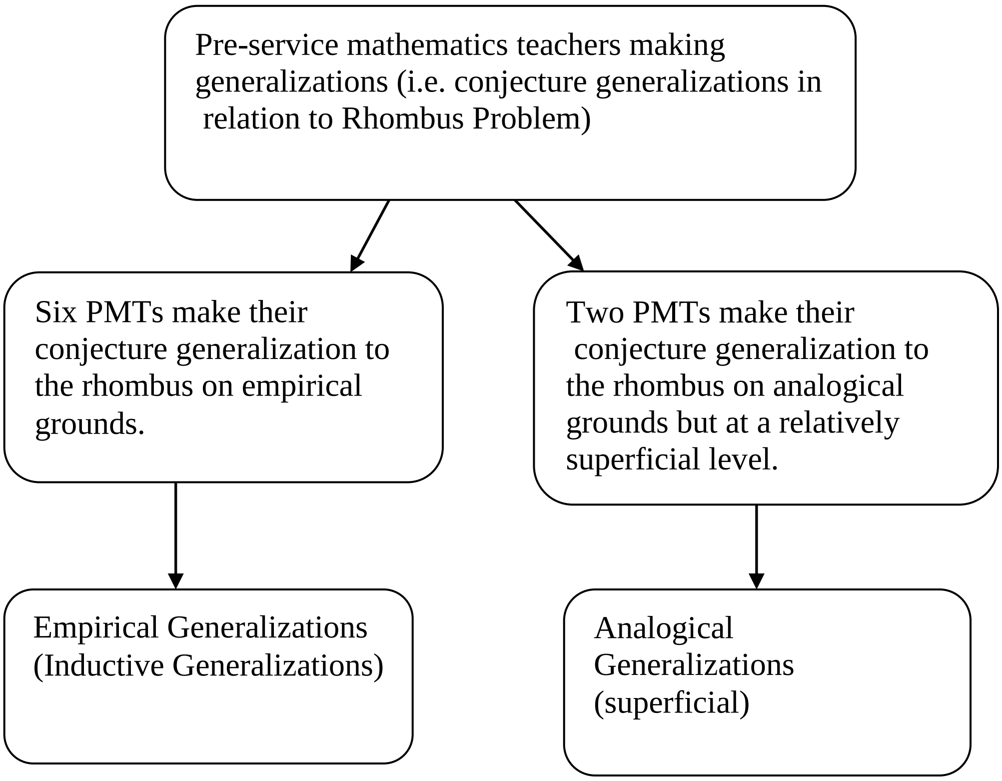



Figure 11.2.1: Trajectory of Rhombus Conjecture Generalizations

Alan and Shannon immediately generalized to a rhombus and did not display any need to experimentally explore/test their claim by using *Sketchpad* (see Finding in  Section 8.1.1.1). The remaining six PMTs, however, were not able to quickly (immediately) generalize the result from the equilateral triangle to the rhombus (see Finding 2 in Section 8.1.1.1). In fact, four of these latter  six PMTs, who all conjectured that point *P* should be located at the centre (middle) of the rhombus, again exhibited the ‘equality misconception’, namely, that the sum of the distances would be a minimum when the four distances to the sides are equal to each other (see Finding 3 in Section 8.1.1.1). In other words, they seemed to ‘subconciously’ think of this minimum point as the incentre of the rhombus. The remaining two of these six PMTs, namely Logan and Renny, seemed to confuse distances to the sides with distances to vertices in their justification of their choice for the location of point *P* at the centre (see Finding 4 in Section 8.1.1.1)

The six PMTs, who did not initially realize that point can be located anywhere with the rhombus, were given an opportunity to dynamically explore their claim using *Sketchpad* (see Section 81.2*)*. In the process, each of the PMTs surprisingly experienced a heuristic counter-example (i.e. contradiction) to the assumption that the centre or middle point is the only point that can be used to obtain the minimum distance sum as evidenced in their initial conjecture (see Komatsu, 2010). The  further empirical examples in turn supported the acceptance of the heuristic counter-example (see Finding 1 in Section 8.1.2.1). The contradiction the PMTs experienced disturbed their cognitive equilibrium. i.e. caused cognitive conflict, which triggered their surprised appearance. Hence, in their attempt to resolve their internal cognitive conflict, each PMT rejected the notion that centre as the only point and hence modified the initial conjecture to accommodate their discovery, i.e.the new idea (see Lee et al., 2003; Piaget, 1978, 1985; Sections 3.2 and 4.6). In effect, the heuristic  counter-example functioned as a driving force for conceptual change and the subsequent modification the initial non-*Sketchpad* conjecture.This effectively resulted in each  of the six PMTs constructing a new kind of conjecture (see Ausubel, 1968; di Sessa, 2006; Biemans & Simons, 1999; Duit, 1999; Lee et al., 2003; Sections 3.2.5 and 4.6.2).

As reported in Finding 2 in Section 8.1.2.1 through further exploration in a dynamic geometry context, each PMT successfully validated their new conjecture, which claims that point *P* can be located anywhere inside the rhombus to maintain the invariance of the distance sum. Consequently, each PMT generalized his/her newly established conjecture, signaling that through the process of accommodation they had adapted their ‘rhombus schemata’ to the new idea, and hence re-established their cognitive equilibrium (See Berger, 2004, Piaget, 1978, 1985, Section 4.6).

On reflecting on Research Question 3 in respect of the Rhombus problem, the findings firstly showed that a limited number (just two) of PMTs (namely Alan and Shannon) immediately extended the Viviani generalization for equilateral triangles to a rhombus on analogical grounds. However, their level of analogical reasoning turned out to be rather superficially executed (see Section 8.1.3). Further to this, these two PMTs, who expressed high levels of certainty in their rhombus conjecture generalization, did not express or exhibit any desire to confirm their newly constructed rhombus conjecture generalization through experimental exploration in a *Sketchpad* context (see Finding 2 in Section 8.2.3).

Secondly, the findings of this part of the study, showed that whilst the majority (six out of eight PMTs) were not able to initially generalize the Viviani Theorem for equilateral triangles to a rhombus whilst restricted to a non-*Sketchpad* context, they were able through experimental exploration within a *Sketchpad* context to construct a conjecture generalization for the rhombus case that was similar in construct to the Viviani generalization for equilateral triangles. However, the latter process of experimental exploration was not a linear process, but a process that was characterized by an amalgamation of experiences, actions and moves such as: encountering of a heuristic counter-example that contradicted their assumption that the centre point was the only point that could produce the required minimum distance sum; witnessing supporting examples to the heuristic example; acceptance of the heuristic example; cognitive conflict; modifying (refining) their initial non-*Sketchpad* conjecture; construction of a new *Sketchpad* conjecture for the rhombus case; achievement of cognitive equilibrium; validation of the *Sketchpad* conjecture for new cases; and then accepting that the *Sketchpad* conjecture was true in general (see Findings 1 and 2 on Section 8.1.2.1). Thirdly, the findings of this study showed that none of the PMTs produced a conjecture generalization immediately on logical grounds (see Finding 1 in Section 8.1.4 and Finding 1 in Section 8.1.5)

### 11.2.2 Research Question 3: Rhombus Problem (Justifications)

As reported in Finding 1 in Section 8.3.1, all eight PMTs expressed a desire and need for an explanation as to why their Rhombus conjecture generalization is always true. Figure 11.2.2 can assist in understanding the dynamism that evolved with reference to the trajectory of justifications for rhombus conjecture generalizations.

As reported in the Findings in Section 8.4.5, one of the six PMTs, namely Trevelyan, who made his rhombus conjecture generalization on empirical grounds, saw on his own that he could similarly use the 'triangle-area' algebraic explanation to logically explain his rhombus conjecture generalization. Further to this, three of the six PMTs, who also finally made their Rhombus conjecture generalization through empirical induction by using dynamic cases, through the use of scaffolded guidance developed a logical explanation for it (see Finding 1 of Section 8.4.5). However, two of the six PMTs, Victor & Renny, who also established their Rhombus conjecture generalization via empirical induction through using dynamic cases, used the parallel distance sum proposition to explain why the distance sum will remain constant in the rhombus case (see Finding 2 of Section 8.4.5). In addition, as discussed in Section 8.4.4.1, the two PMTs, Alan and Shannon, who presumably used analogical reasoning of superficial form to construct their Rhombus conjecture generalization, also used the parallel distance sum proposition to advance a logical argument to justify it (see Finding 4 of Section 8.4.5). This was unexpected as the researcher had assumed that having done the equilateral triangle explanation in terms of the equality of sides, they might similarly think of considering the equality of the sides of a rhombus. However, this shift in focus and approach can possibly be attributed to week-long delay between the equilateral triangle and rhombus task.

These latter four PMTs, Victor, Renny, Alan and Shannon, were each provided an opportunity to provide an alternate logical explanation for their Rhombus conjecture generalization As discussed in Section 8.4.2.2, Victor and Renny, needed scaffolded guidance. This guidance was offered to them via Task 2(d), which is task-based worksheet as outlined in Appendix 2. As reported in Finding 2 in Section 8.4.5, these two PMTs, Victor and Renny, completed each activity in the rhombus worksheet with a great amount of accuracy and ease. This ease in the analogical transfer of information from their previous scaffolded equilateral triangle-algebraic explanation across to the rhombus could have been made possible through structural parallelism, i.e. by mapping the structure of the sub-arguments encompassing the logical explanation for their equilateral triangle conjecture

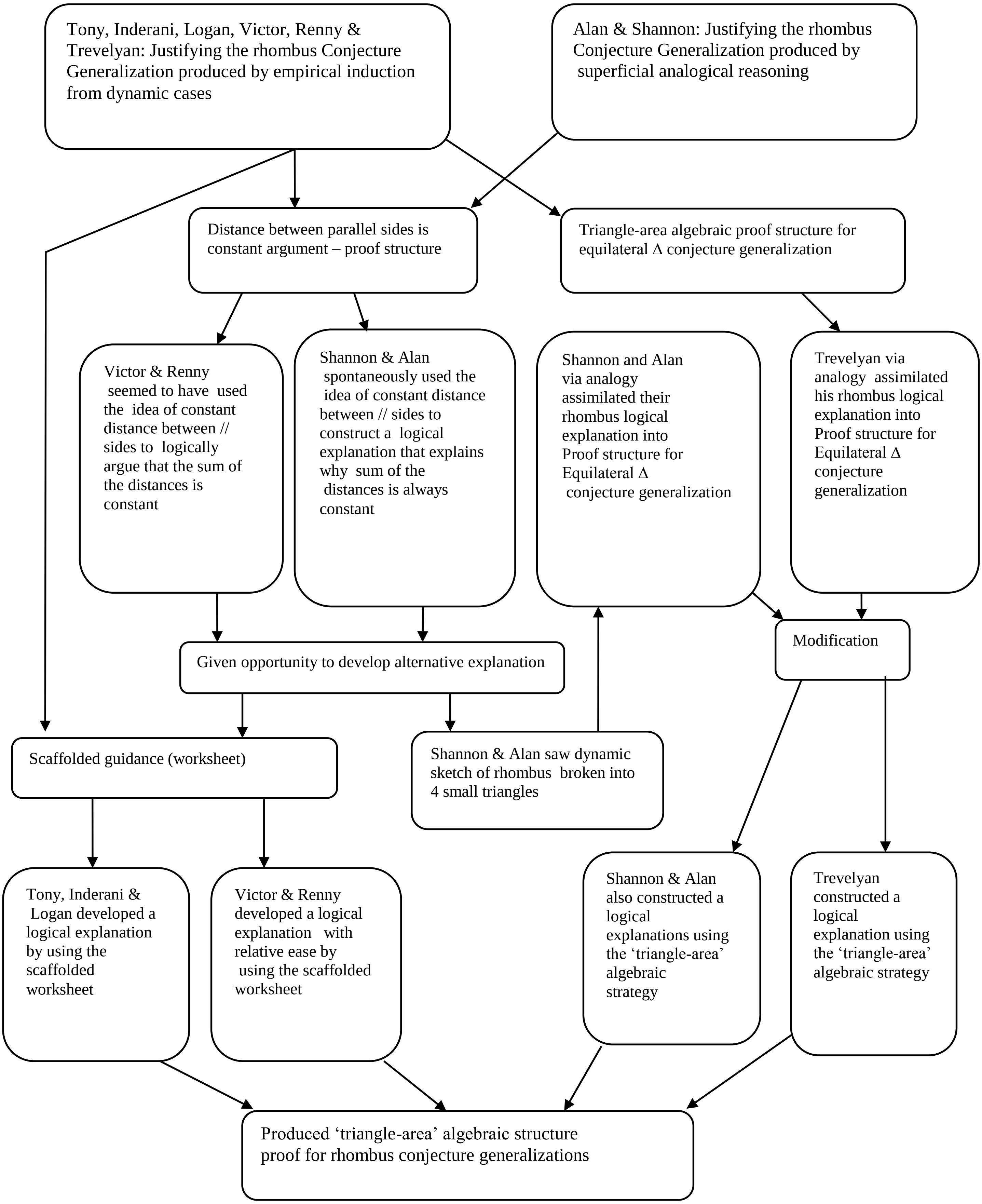



Figure 11.2.2: Trajectories of Justifications for Rhombus Conjecture Generalizations

generalization via a one- to- one correspondence onto the matching explanatory sub-arguments for the rhombus conjecture generalization (see discussion Gentner' Structure Mapping Theory in Section 4.4.2 & 4.4.3). In the main, it seems that their previous scaffolded experience enhanced their analogical transfer of information from the source problem to the target problem.

However, as reported in Finding 5 of Section 8.4.5, Alan and Shannon did not require any form of scaffolded guidance to develop a logical explanation for their conjecture generalization. As soon as they saw the dynamic rhombus, which showed the division of the rhombus into four triangles with respective heights, they quickly (spontaneously) asserted that the same kind of 'triangle-area' algebraic proof that they offered for the equilateral triangle generalization could likewise be used to construct a 'triangle-area' type of logical explanation for their rhombus conjecture generalization, and hence produced their 'triangle-area' algebraic logical explanations as shown in Figure 8.4.4.2 and Figure 8.4.4.3 respectively. This analogical-deductive move by both Alan and Shannon characteristically demonstrated that a proof in itself can through analogical structural mapping (i.e. cognitive blending) enable mathematics students to discover proof(s) of similar results or generalizations (compare Schoenfeld, 1986 and De Villiers, 2003).

The aforementioned discovery function of proof (see Section 5.4.2), was also exhibited by Trevelyan, who after making his conjecture generalization as a result of experimental exploration in a *Sketchpad* context, spontaneously saw a similarity between the equilateral problem and the rhombus problem, and thereby proceeded without any assistance from the researcher to similarly use the 'triangle-area' algebraic proof structure for the equilateral triangle case to construct a logical explanation for his rhombus empirical generalization.

In terms of Piaget's theory of assimilation and accommodation, Alan, Shannon and Trevelyan appeared to have assimilated their logical explanations for their rhombus conjecture generalization into their previously accommodated explanations for the equilateral triangle case, and then modified it to accommodate their logical explanations for the rhombus conjecture generalization (see Sections 4.2.1 & 4.6).

Returning to Research Question 4, the findings show that through the use of a scaffolded worksheet and the provision of varied guidance by the facilitator, some PMTs were able to construct a logical explanation for their rhombus conjecture generalization with relative ease.

Also through scaffolded guidance, those PMTs who initially used the proposition that the distance between parallel sides is constant to justify their Rhombus conjecture generalization, were then able to construct a logical explanation through using the ‘triangle-area’ algebraic strategy. Still focusing on the development of an alternative explanation, it was found that when PMTs were presented with a dynamic rhombus divided into triangles (i.e. a hint), they spontaneously saw a similarity between the explanatory structure for the rhombus problem and the equilateral triangle problem, and hence proceeded on their own to construct their ‘triangle-area’ algebraic explanation. However, in one case (see Trevelyan) a student on his own saw the similarity between the equilateral problem and the rhombus problem, and then through structural analogical mapping proceeded to construct a ‘triangle-algebraic’ logical explanation for his rhombus conjecture generalization. This in turn suggests that justifications in the form of logical explanations can be discovered through being able to draw parallels between explanatory structures (see discussion in Section 4.4.3).

### 11.3 Pentagon problem: Findings

In Sections 11.3.1 and 11.3.2, the findings associated with the Pentagon problem in relation to research questions 3 and 4 are reported respectively. Research question 3 focuses on the extension of the Viviani generalization for equilateral angles to a sequence of equilateral (convex polygons) of 4 sides (rhombi), five sides (pentagons) and finally to general convex equilateral *n*-gons, as well as how the PMTs made their generalization to the respective cases. On the other hand, core research question 4, focuses on whether PMTs can justify each of their extended generalizations and also on ‘how’ they constructed their generalizations.

#### 11.3.1 Research Question 3: Pentagon Problem (Generalizations)

Figure 11.3.1 can assist in clarifying how the PMTs made their respective pentagon generalizations. As illustrated in Figure 11.3.1, there was one PMT, Alan, who on his own during his first attempt extended his prior generalizations associated with equilateral triangle and rhombus problems directly to the general equilateral pentagon on logical grounds with the aid of analogical reasoning (see Finding 1 of Section 9.1.6). Hence, Sections 11.3.1.1 and 11.3.1.2 do not further report on Alan, but only on the findings associated with making of generalizations by the remaining seven PMTs in relation to regular pentagon problem and Mystery (i.e. irregular) pentagon problem respectively.

#### 11.3.1.1 Regular Pentagon Generalizations

On their first attempt seven PMTs restricted their pentagon generalization to a regular pentagon, and in doing so seemed to exhibit the following misconception: ‘equal sides always imply equal angles’, and vice versa (see Finding 8 of Section 9.1.6). In this instance all seven of them were apparently thinking (or believing) that all equilateral pentagons are regular, i.e. they appear to have been thinking that ‘all sides equal’ always imply that all angles are equal, and thus rationalized that their generalizations were limited to just regular pentagons. Despite having worked with a general rhombus in the previous Task 2d (see Appendix 2) , which had all sides equal and not all angles equal, it is rather surprising that all seven PMTs thought that ‘equal sides’ is a sufficient property to produce a regular pentagon. This can perhaps be explained by them probably never having seen any other pentagon, but a regular one. This ‘regularity’ misconception for higher polygons, which was overwhelmingly exhibited by the seven PMTs, may have already been very well embedded in a complex schema within each PMT’s mind (see Sinatra & Pintrich, 2003 as cited in Eggen & Kauchak, 2007), and hence caused the PMTs to restrict their conjecture generalization to regular pentagons only.

As reported in Findings 2 to 7 in Section 9.1.6, there had been an increase in the number of PMTs forming their conjecture generalizations via either inductive or analogical reasoning, without actually doing any experimental exploration in a dynamic geometry context prior to making their generalization. Hence, in effect none of the PMTs produced the regular pentagon conjecture generalization by empirical induction from dynamic cases in *Sketchpad* context (see Finding 1 in Section 9.1.1). Four of these seven PMTs (Shannon, Trevelyan, Renny, Logan) expressed 100% certainty in their conjecture generalization, and did not express any desire to empirically test their claim by using *Sketchpad* (see Findings 2 and 6 in Section 9.1.6).

However, there was just one PMT, Inderani, who after producing her conjecture generalization through either inductive or analogical inductive grounds, requested on her own accord to do some experimental exploration within a *Sketchpad* context to determine the extent to which her regular pentagon conjecture generalization was always true. As a result of her experimental exploration her certainty level in her regular pentagon conjecture generalization subsequently climbed to 100% (see Finding 3 of Section 9.1.6), more so because she did not encounter any visual counter-examples as she dragged the selected vertex point around. In addition, two students, Tony and Victor, seemed to have had some doubts about their regular pentagon conjecture generalization. However, through further

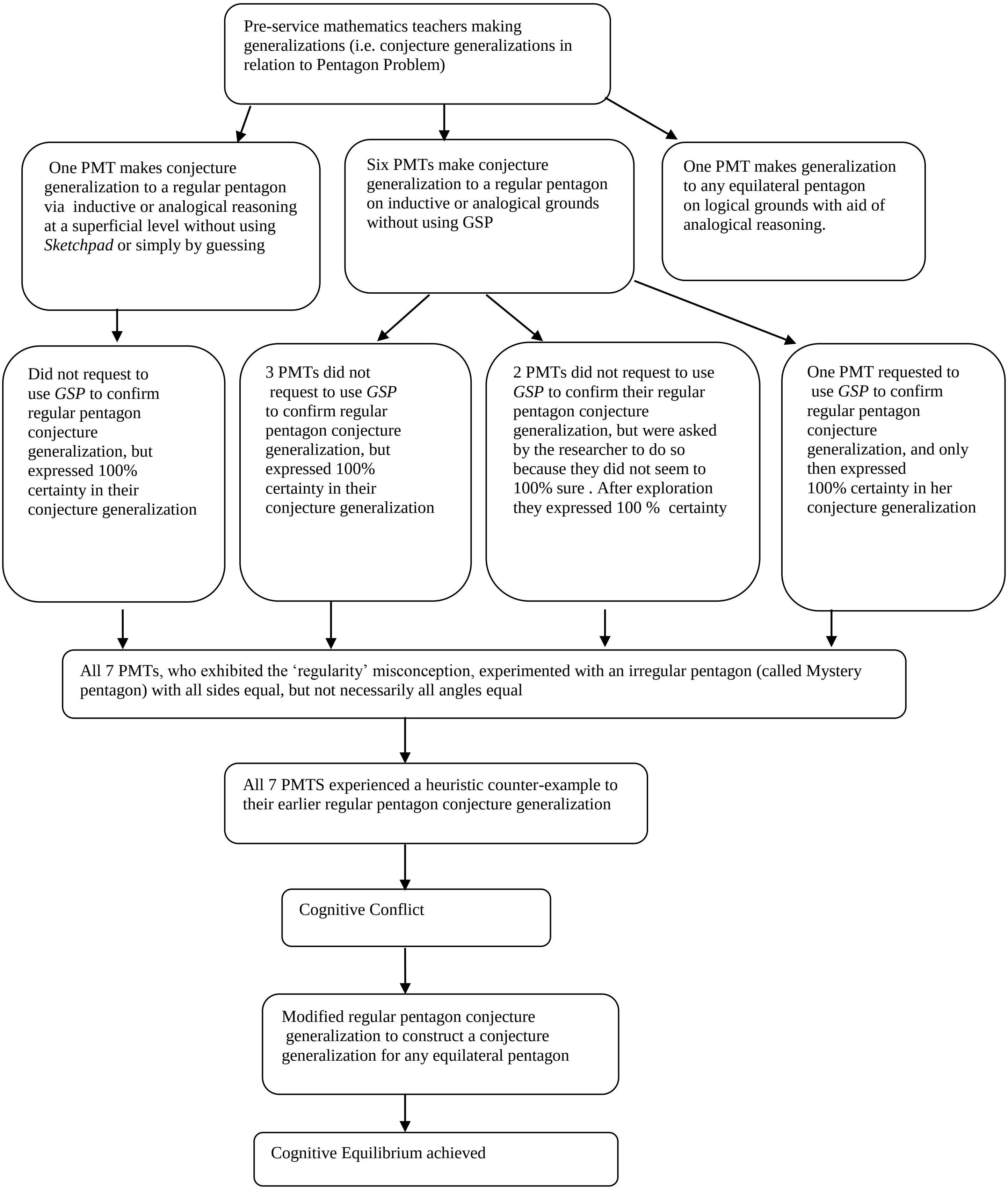



Figure 11.3.1: Trajectories of Pentagon Conjecture Generalizations

experimentation under my guidance their doubts were resolved and they then expressed 100 % certainty in their regular pentagon conjecture generalization (see Findings 4 and 5 of Section 9.1.6).

As reported in Chapter 9, the PMTs were first asked to justify their regular pentagon conjecture generalization before they were presented with the Mystery pentagon (i.e. irregular pentagon with equal sides) activity. However, to keep a holistic focus on generalizations and justifications as required of Research Questions 3 and 4 respectively, the findings associated with the refined pentagon generalizations (see Section 11.3.1.2 ) precedes the discussion of the findings associated with justifications of the regular pentagon conjecture generalization (see Section 11.3.2.1). However, the reader should read Section 11.3.2.1, before reading Section 11.3.1.2.

#### 11.3.1.2 Mystery Convex Pentagon (Irregular): Generalizations

In this section, the findings in relation to the 'Mystery Pentagon' problem are presented. In so doing it provides an exposition as to how the PMTs overcame their 'regularity' misconception through a heuristic counter-example experience, and hence refined their initial regular pentagon conjecture generalization to involve irregular pentagons with equal sides as well.

As discussed in Sections 4.6.1 and 4.6.2 (Piaget's Equilibration Theory, Conceptual Change and Cognitive Conflict), changing the PMTs 'regularity' misconception in this instance required a restructuring of their already existing 'regularity' schema through a process of conceptual change. Conceptual change is similar to Piaget's notions of disequilibrium, accommodation and assimilation as illustrated in Figure 4.6.1 (see Alparsian et al., 2004; Berger, 2004; O'Donell, Reeve and Smith, 2009; Piaget, 1978, 1985). From a conceptual change perspective, as described in Section 4.6.2 certain minimum conditions need to prevail or be created to enable students to change their thinking. Consistent with conditions for conceptual change summarized by Eggen and Kauchak (2007) (see Section 4.6.2) as well as the instructional strategies suggested by Piaget (1978, 1985), Nussbaum and Novick (1982), Lee et al., (2003), Limon (2001) (see Section 4.6.2), the researcher used the *Mystery Pentagon* activity to allow the PMTs to experiment in a dynamic geometry context and experience a pentagon that had equal sides, but angles not necessarily equal. As reported in Finding 1 in Section 9.3.1, the PMTs were rather surprised to see such a pentagon. This surprised look seemed to have been symptomatic of a deeper inner cognitive conflict (i.e. cognitive disequilibrium), that emanated from them seeing what they did not expect to see, namely a pentagon having equal sides which was not necessarily regular (see Finding 2 in Section 9.3.1). In addition, their cognitive equilibrium was further disturbed, when they found that as they dragged a point inside the irregular pentagon, which had equal sides but unequal angles, the invariance of the distance sum was maintained (see Finding 3 in Section 9.3.1). The PMTs' existing schemata (or schemes) were inadequate to explain their new unexpected experiences, and consequently resulted in the disruption of their cognitive equilibrium, which from a conceptual change perspective satisfies one of the three core conditions that are necessary to bring about change in students thinking (see Eggen & Kauchak (2007).

To re-establish their cognitive equilibrium in this instance, all PMTs adapted to their new experiences via the process of accommodation. They firstly modified their existing schemas that intimated that only regular pentagons have equal sides and created a new schema in response to their unexpected experience. The resultant new schema embraced the idea that it is possible to have a pentagon with unequal angles, but equal sides (i.e. a pentagon with equal sides). Secondly, they also modified their schema that encompassed the idea that not only in regular pentagons an invariance of the distance sum can be achieved but also in irregular pentagons (provided equality of sides were maintained). The accommodation of these new ideas suggests that the new discoveries (or results), which was made possible through the process of experimentation and appropriate questioning by the researcher, made sense to each of the seven PMTs, and thus asserts that the second core condition for conceptual change had been realized.

Thus, in a sense dynamic exploratory experience of the PMTs with the irregular pentagon (called the Mystery Pentagon in this study) brought to the fore how a purposefully designed heuristic counter-example helped the PMTs to correct their misconception that 'equal sides' imply 'equal angles' (see Finding 8 in Section 9.3.1). They came to generally realize that equal sides are not just possessed by regular polygons but could also be possessed by some irregular polygons as well. Note that this heuristic counter-example did not invalidate their conjecture generalization for a regular pentagon, but served as a counter-example to their idea (or assumption) that only regular pentagons had equal sides (see Finding 8 in Section 9.3.1). In light of this, the heuristic counter-example helped them to realize that 'all sides equal' is a sufficient pentagon property to maintain an invariance of its distance sum (see Findings 3 and 4 in Section 9.3.1). For example, one of the PMTs, Renny, stated the following refined pentagon conjecture generalization: "If the sides of the polygon (pentagon) are equal, the sum of the distance from an interior point *P* to the sides of the pentagon will not change (meaning the distance sum will remain constant)".

The heuristic counter-example in the Mystery Pentagon case therefore appears to have served as a driving force for each PMT to refine his/her initial pentagon conjecture generalization, which was limited to regular pentagons, to a pentagon conjecture generalization that encompassed any equilateral pentagon (see Finding 6 in Section 9.3.1). Such a refinement of a conjecture generalization with respect to any equilateral convex pentagon, signals the re-establishment of cognitive equilibrium within the minds of each affected PMT. From a conceptual change perspective, this re-establishment of cognitive equilibrium supplements the third core condition that is required for conceptual change (see Eggen & Kauchak, 2007). Furthermore, as demonstrated via Findings 1, 2, 7, and 10 in Section 10.2, it seems that the PMT were able to assimilate new experiences into their refined pentagon conjecture generalization schema on logical grounds.

Reverting to Research Question 3 in relation to the Pentagon problems, the findings show that PMTs were able to extend their Viviani generalization for equilateral triangles and rhombi across to any equilateral pentagon. However, this process was not a linear one but one that was fueled by the following processes and events:

- First making a pentagon conjecture generalization that was restricted to a regular pentagon through analogy or inductive reasoning.
- Experimental confirmation of the regular pentagon conjecture generalization by some PMTs.
- Experience of a heuristic counter-example within a dynamic geometry context that caused cognitive conflict and surprise.
- Further experimental exploration aggravating the cognitive conflict and cognitive disequilibrium, that resulted in the PMTs eventually accepting the counter-example, and subsequently reformulating their initial regular pentagon conjecture general to include any equilateral pentagon to achieve their cognitive equilibrium.

### 11.3.2 Research Question 4: Pentagon Problem (Justifications)

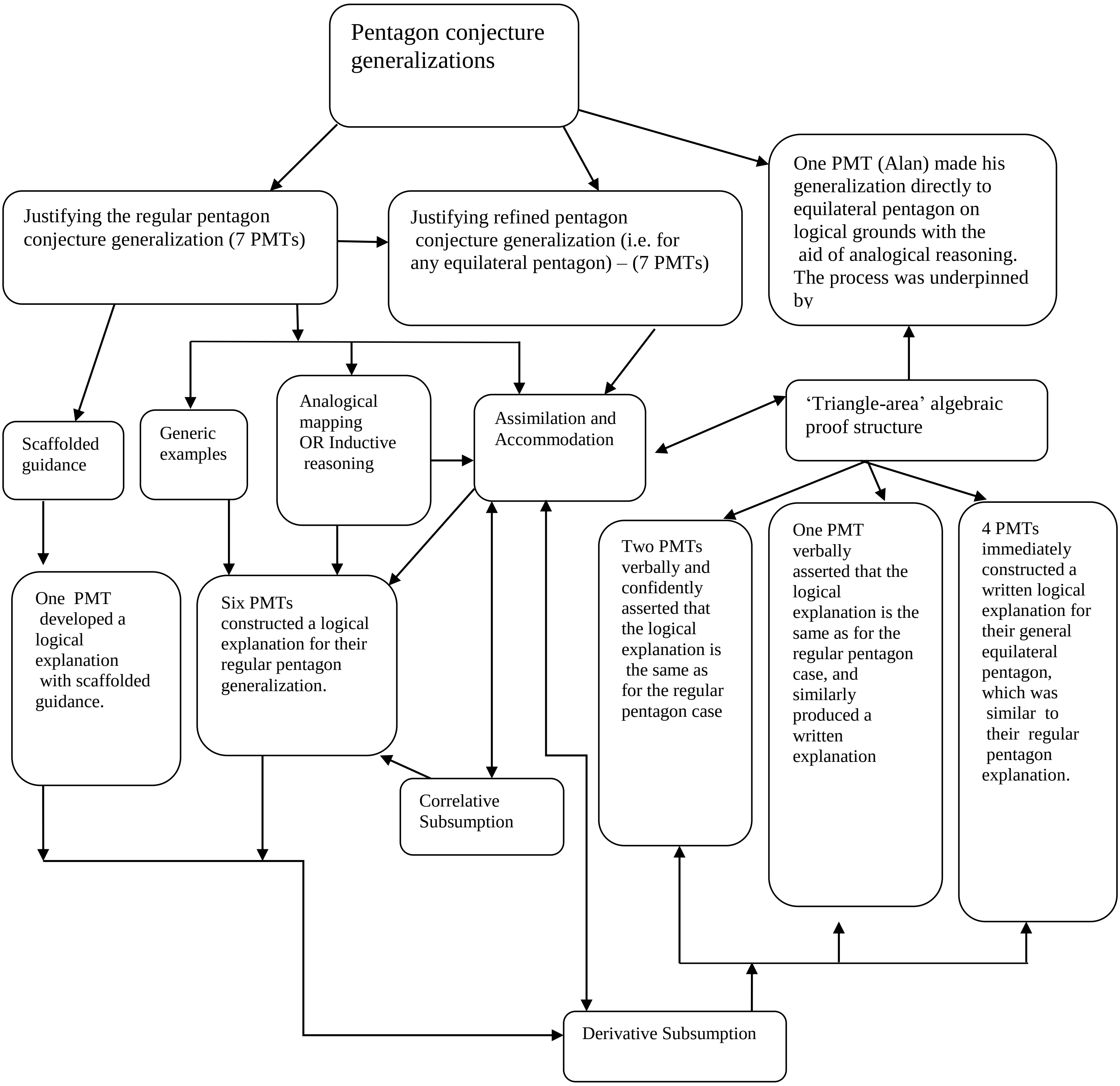



Figure 11.3.2: Trajectory of Justifications for Pentagon (Convex) Conjecture Generalizations

Figure 11.3.2 illustrates how the PMTs accomplished their respective justifications of their pentagon generalizations. As illustrated in Figure 11.3.2 and reported in Section 11.3.1, one PMT, Alan, extended his prior generalizations associated with equilateral triangle and rhombus problems directly to the general equilateral pentagon on logical grounds with the aid of analogical reasoning (see Finding 1 in Section 9.1.6) through the process of correlative subsumption. Thus, the findings connected to Alan's justification are discussed last under Section 11.3.2.3. The findings connected to the justifications of the remaining seven PMTs to their regular pentagon conjecture generalizations and refined pentagon conjecture generalizations are discussed in Sections 11.3.2.1 and 11.3.2.2 respectively.

#### 11.3.2.1 Research Question 4: Regular Pentagon Problem (Justifications)

The researcher being aware of the extent of the 'regularity' misconception exhibited by seven PMTs, nevertheless requested each of the seven PMTs to provide a justification in the form of a logical explanation for their regular pentagon conjecture generalization. As illustrated in Figure 11.3.2, the results of this showed that all seven PMTs could justify their conjecture generalizations through the construction of a logical explanation. However, the route to such a logical explanation was not necessarily the same for all PMTs. Firstly, Logan constructed his logical explanation via scaffolded guidance whilst working in a dynamic geometry context (see discussion in Section 9.2.1 and Finding 1 in Section 9.2.4). In effect, earlier scaffolded activities did enhance both his competency and insight on using the 'triangle-area' algebraic to explain similar conjecture generalizations.

Findings also showed that the other six PMTs, having seen a connection between the equilateral triangle, rhombus and pentagon problems via the 'equal sides' property, were rather quick to see that 'triangle-area' algebraic approach they had used to construct logical explanations for the previous conjecture generalizations could analogically be applied to also construct a logical explanation for their regular pentagon generalization. All 6 PMTs succeeded to explain their regular pentagon conjecture generalization in a logical manner through parallel transfer of the core steps from their previous 'triangle–area' algebraic explanations and relevant modifications thereof (see Finding 2 (a-d) and 3 in Section 9.2.4). This cognitive move in itself shows that through analogical reasoning (cognitive blending) and transfer, one may develop logical explanations for newly conjectured results, as well as discover such logical explanations (see discussion on discovery function of proofs in Section 5.4.2, and Finding 3 in Section 9.2.4)

Within this continuum of the moves exhibited by each of the six PMTs in the building up of their logical explanation for their regular pentagon conjecture generalization, all of them seemed to have used the insight and understanding (see discussion on explanatory function of proof in Section 5.4.1) gained from their previous triangle area algebraic explanations to see and build on the 'triangle area' algebraic proof for the regular pentagon conjecture generalization. Thus on seeing a connection between the regular pentagon problem and the previous problems, the PMTs through looking back on their previous triangle-area algebraic proofs were able to see the 'sameness' for the proof of the regular pentagon conjecture generalization. Hence as described by Mason and Pimm (1984), they were able to move from one particular proof to another particular proof by transferring the generic argument from one particular instance to another instance (see Finding 5 in Section 9.2.4. In this sense, we say that the PMTs were able to see the particular proof for the regular pentagon conjecture generalization through previous particular proofs, which acted as generic examples in this particular context.

According to Johnson (1975, pp. 425-426), the 'meaningfulness' of new information that learners or students come to experience in their classrooms, is seen as the "most powerful variable" that governs the learning and explanation of complex results and verbal discourses. In addition, Driscoll (2005) maintains that the basic idea underpinning any form of meaningful learning is characterized by the learner actively making some attempt to connect new ideas to already existing ones. As discussed in more detail in Section 4.5, meaningful learning is underpinned by two kinds of subsumption, namely derivative subsumption and correlative subsumption.

As reported in Finding 4 in Section 9.2.4, six PMTs constructed their logical explanations for their regular pentagon conjecture generalization through a process of correlative subsumption (see Figure 11.3.3). Consistent with the process of correlative subsumption (see Ausubel et al., 1978, & Driscoll, 2000; Section 4.5), the 'new information' in this study, namely the construction of a logical explanation for the regular pentagon CG, was assimilated into a previous idea or experience (i.e. 'established idea'), namely the triangle-area algebraic explanations for equilateral triangle and rhombus CG's (see Findings 4 and 6 in Section 9.2.4). This assimilation is likely to have been triggered by observing the presence of the same 'equal sides' property in their pentagon figure as before (general, equilateral pentagon in the case of Alan and regular pentagon for the other students). In particular, on having gained the necessary insight and experience as to how the 'equal sides' property was used to construct 'triangle-area' algebraic explanations for their earlier equilateral triangle and rhombus generalizations, the PMTs were quick to see that logical explanations for their respective pentagon conjecture generalizations could be constructed in a similar way. This means that their prior experience and insight helped them to easily assimilate their anticipated logical explanation for their new pentagon conjecture generalization into either one of their previous 'triangle-area' algebraic explanations, which were already stored in their long-term memory. Then through modifying some of the key attributes of either one of their previous logical explanations in an analogical manner, all seven PMTs appear to have managed to accommodate their 'triangle–area' algebraic explanation for their new regular pentagon conjecture generalization. In much the same manner, Alan also accomplished his 'triangle-area' algebraic explanation for his new general pentagon conjecture generalization (see Finding 6 of Section 9.2.4)

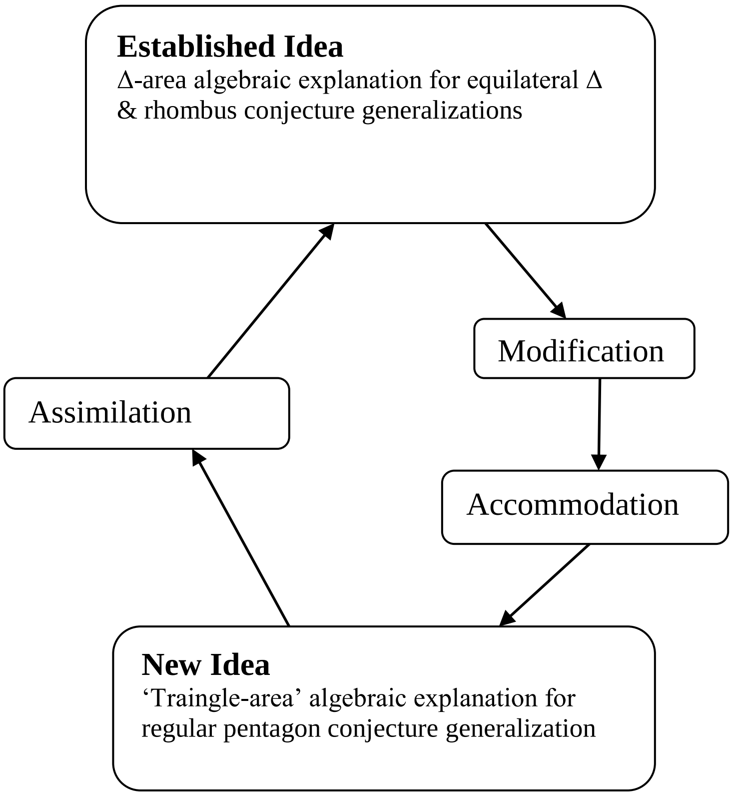



Figure 11.3.3: Correlative Subsumption

Returning to Research Question 4 in relation to justification of a regular pentagon conjecture generalization, all seven PMTs were able to justify it. The ways that were sought to accomplish their justifications could be summarized as follows:

- Scaffolded guidance through engaging with a scaffolded worksheet in a dynamic geometry context.
- Generic Justification: The general proof was seen through the previous particular proofs. The parallel transfer of information from the 'triangle-area' algebraic structure proof for the rhombus conjecture generalization to the 'triangle-area' algebraic

structure proof for the regular pentagon problem through the process of analogical reasoning. Hence, we can refer to this as an analogical-deductive generalization.

#### 11.3.2.2 Research Question 4: Justifications of the Refined Pentagon (Convex) Conjecture Generalization

As reported in Section 9.4, there were seven PMTs who after experiencing a heuristic counter-example to their assumption that 'all equilateral pentagons are regular' saw that it was possible to have an irregular pentagon with equal sides but angles not necessarily equal that still preserves the distance sum. This caused the PMTs to modify (refine) their regular pentagon conjecture generalization to include any pentagon with equal sides. Soon after each of the PMTs expressed their refined pentagon conjecture generalization, the researcher requested each of them to provide a justification in the form of a logical explanation as to why their new result is always true.

Findings show that all 7 PMTs were able to justify their refined pentagon conjecture generalization through the provision of a deductive justification (i.e. logical explanation) through similarly using the 'triangle-area' algebraic structure proof that they had previously used to justify their previous conjecture generalizations (see Findings 1-4 in Section 9.4.1). In so doing, three confidently advanced their logical explanations verbally (see Findings 1 and 2 in Section 9.4.1) and four confidently wrote down the logical explanation immediately (see Finding 3 of Section 9.4.1).

As discussed in detail in Section 9.4, these seven PMTs had already worked with the regular pentagon wherein they used the 'equal sides' property to construct a logical explanation providing insight as to why the distance sum remained constant. It appears that on seeing that the 'equal sides' property also featured as the main condition in the new refined pentagon conjecture generalization, they immediately saw the logical connection between the regular pentagon (source problem) and the equilateral pentagon (target problem). The latter is consistent with Ausubel's philosophy of meaningful learning (see Section 4.5 for a detailed discussion) which claims that when a learner encounters new ideas or phenomena he/she actively attempts to naturally connect them to already existing ideas accommodated in their cognitive structures to try and make sense and/or explain the existence of such new phenomena or ideas. In particular, the development and construction of the desired logical explanation for the refined pentagon conjecture generalization suggests that through the process of derivative subsumption (see discussion of derivative subsumption in Section 4.5.1) each of the seven PMTs automatically subsumed the target problem, namely the construction of a logical explanation for their refined pentagon conjecture generalization into an existing but relevant 'triangle-area' algebraic explanatory structure, namely the logical explanation for their regular pentagon conjecture generalization (source problem).

As indicated in Figure 11.3.2, the findings of this part of the study demonstrates how the process of derivative subsumption was facilitated by the process of analogical structural transfer (i.e. cognitive blending) of the explanatory steps from the PMT's logical explanation for their regular pentagon conjecture generalization in a parallel manner across to their logical explanation for their refined pentagon conjecture generalization (see Finding 4 in Section 9.4.1).

Returning to Research Question 4 in relation to the justification of the PMTs refined pentagon conjecture generalization, all seven PMTs were able to justify it by providing a deductive justification that was made possible through the process of derivative subsumption in consonance with analogical structural transfer.

#### 11.3.2.3 Research Question 4: Alan's Justification of his 'any' Equilateral Convex Pentagon Generalization

Although Alan produced his conjecture generalization for a general pentagon immediately on logical grounds, it seems his earlier 'triangle-area' algebraic explanations for his equilateral triangle and rhombus conjecture generalizations also acted as generic examples and thereby provided him with the necessary insight and approach to similarly construct a logical explanation for his general pentagon conjecture generalization via analogical transfer (see Finding 6 in Section 9.2.4). The latter deductive move affirms that one could very well construct conjecture generalizations immediately on logical grounds with the aid of analogical reasoning as has been argued by De Villiers (2008). In addition, it seems that in the case of Alan, his logical explanation was underpinned by the process of correlative subsumption.

## 11.4 'Any' Equi-sided convex polygon problem: Findings

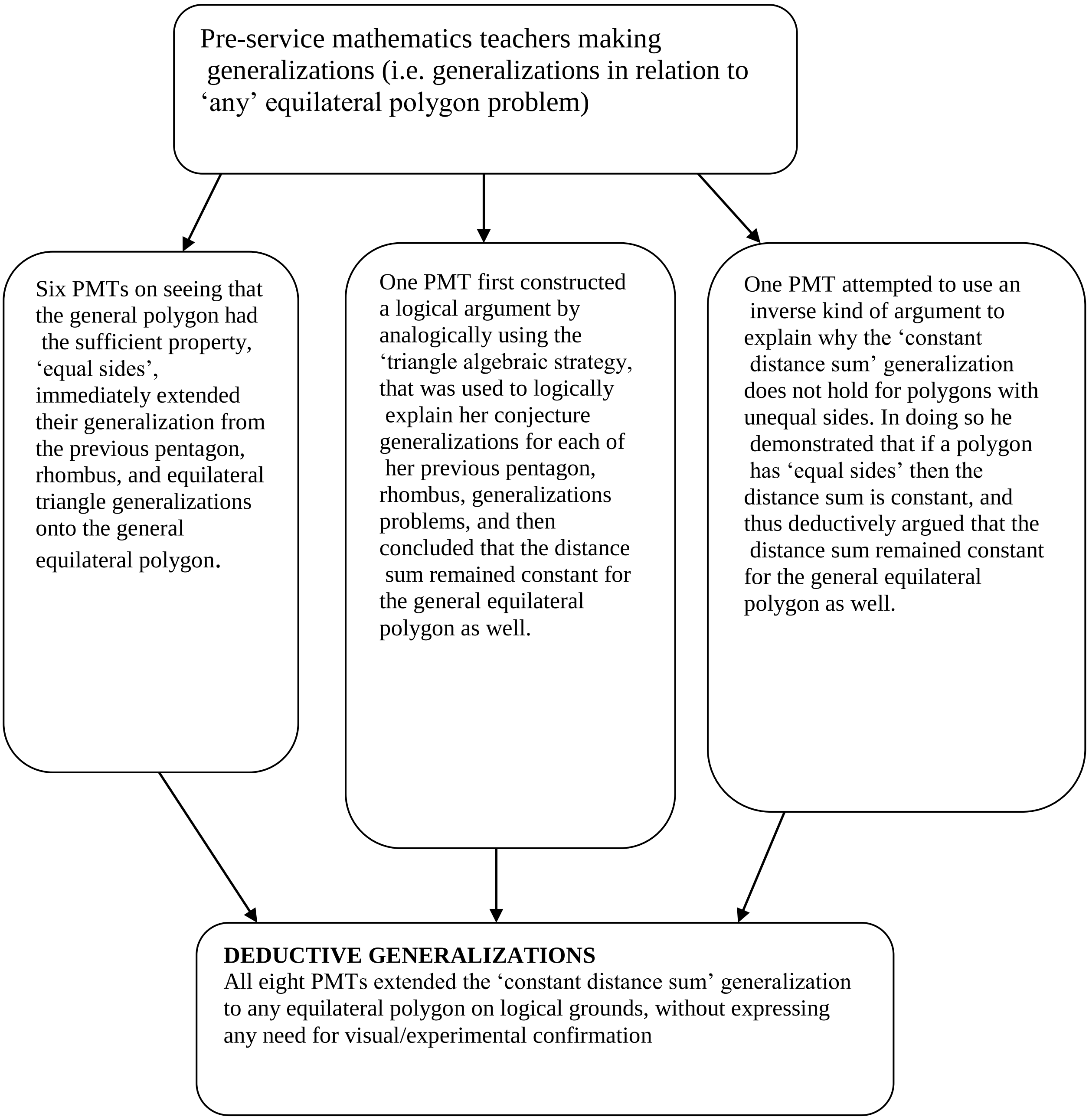



Figure 11.4: Trajectories for Generalization for 'any' equi-sided convex polygon

In this section the findings associated with 'any' equi-sided convex polygon problem in relation to Research Questions 3 and 4 are discussed. The accompanying Figure 11.4, provided at this juncture of my narrative is meant to provide an overview as to how PMTs constructed their generalization for any equi-sided (or equilateral) polygon. I believe this will augment the discursive flow of this section.

When each of the eight PMTs were asked to consider their previous generalizations, and then generalize to polygons with a similar property, they each immediately made a generalization to any equi-sided polygon on logical grounds, without expressing any need for visual/experimental confirmation in a *Sketchpad* context (See Finding 1 of Section 10.2). In particular, the results showed that six PMTs on seeing that they were dealing with any polygon with all sides being equal, readily generalized the invariance of the distance sum to an equi-sided polygon (see Finding 1.1 in Section 10.2). For example, Trevelyan without any experimental exploration confidently responded as follows: "My conjecture is that given any polygon, if the sides of the polygon are the same, then the sum of the distances from a point to each side of the polygon, it should be constant". Similarly Renny spontaneously replied: "Yes, for a polygon that has equal sides, the sum of the distances will always be constant", and went on affirm that such a generalization will hold true for any equi-sided polygon (see Section 10.1)

The responses by these six PMTS are discussed more fully in Section 10.2. These students seemed to have accommodated the idea of 'equal sides' as a sufficient property to yield a 'constant distance sum' within their respective cognitive structures. This kind of deductive generalization, which accrued from their experiences and constructions of their earlier generalizations and related logical explanations (proofs), highlights the *discovery* function of proof described in Section 5.4.2. In other words, the identification of a definitive property like 'equal sides' in this instance, was now seen to be sufficient to enable further generalization of the constant distant sum to any equi-sided polygon. The deductive generalizations produced by each of the PMTs at this stage, demonstrates that they had seen the general result through particular results.

As the PMTs progressed to make their generalization to any equi-sided convex polygon on logical grounds, they drew away from their prior use of non-deductive or empirical processes to make a generalization. Freudenthal (1973) refers to this change in one's thought processes, as the "cutting of the ontological bonds". This cutting of the bond with empirical reality, demonstrates that the PMTs had cognitively grown in the construction of generalizations through the processes of generalizing and abstracting (see Tall, 2002; Tall et al., 2012). For example, Shannon, without working with any dynamic sketch of an equi-sided convex polygon (with $n > 5$), saw that she could divide an equi-sided polygon into triangles, and thereby proceeded on her own to similarly and successfully use the 'triangle-area' algebraic strategy that she had used to justify her previous conjecture generalizations. As shown in

Figure 10.1.2, Shannon constructed a logical argument that demonstrated that the sum of the distances, namely $h_1 + h_2 + h_3 +\ldots+ h_n$ , remained constant. On the basis of her constructed logical explanation, Shannon then generalized that the 'distance sum' will remain constant for any equi-sided polygon as shown in Figure 10.1.2. When asked to produce a coherent logical application, Shannon polished up her logical argument that she had already constructed as per Figure 10.1.3, by providing the necessary warrants for her statements in her logical explanation and also demonstrated her fluency with propositional relationships (se discussion in Section 10.2).

Alan also demonstrated that he was aware that the general equilateral convex polygon needed to have equal sides to produce a constant distance sum, and produced a correct, generalized proof based on the 'triangle-area' strategy. However, upon looking back (i.e. folding back) at his proof, he then tried to deductively demonstrate that the 'constant distance sum' result will not hold true if the sides of the polygon are not equal. The argument produced by Alan to explain why 'constant distance sum' generalization will not hold for irregular polygons with unequal sides, was premised on the fact that he would not be able to extract a common "*a*", which is the variable used to represent the equality of all sides in the polygon, as part of the common factor ½*a*. In other words, he argued that if there is no common equal side in a given polygon, then it would not be possible construct a logical explanation to justify the 'constant distance sum' generalization. However, Alan's reasoning as discussed in Section 10.1 is incorrect, as all his argument shows is that if the sides of some polygon were not equal and the sum of the distances to its sides were constant, then it could not be explained (proved) in the same manner.

Like Alan and Shannon, none of the other PMTs provided an empirical argument or made reference to *Sketchpad* to justify why their conjecture generalization for the equi-sided polygon was always true (see Finding 5 in Section 10.2). Furthermore, none of the eight PMTs required any specific form of scaffolded guidance to develop a logical explanation to justify why their conjecture generalization for the equi-sided polygon was always true (see Finding 6 in Section 10.2)

Findings of this study also showed that all eight PMTs appeared to have looked back (i.e. folding back) at the logical explanations for their previous generalizations, and reflected on the general strategy (process) that was used to build a justification that logically explained each of their previous generalizations (see Finding 8 in Section 10.2). Consequently, all

PMTs made a successful generic abstraction from the earlier cases to recognize that 'triangle–area' algebraic method can be extended or extrapolated to construct a logical explanation to justify the invariance of the distance sum in any equi-sided polygon (see Finding 7 in Section 10.2). This kind of generic abstraction is consistent with the pedagogical underpinnings of the discovery function of proof as described in Chapter 4, and also suggests that the explanatory function of proof compliments and stimulates the discovery function of proof (See Finding 9 in Section 10.2; and also Sections 5.4.1 and 5.4.2). In this instance, as summarized in Figure 11.5, the generic abstraction, coupled with analogical reasoning, made it possible for all PMTs to make the transition to formal abstraction, i.e. to construct and develop a general proof for any equi-sided polygon. In short, they have seen the general proof through the earlier set of particular proofs (see Findings 7 and 9 in Section 10.2).

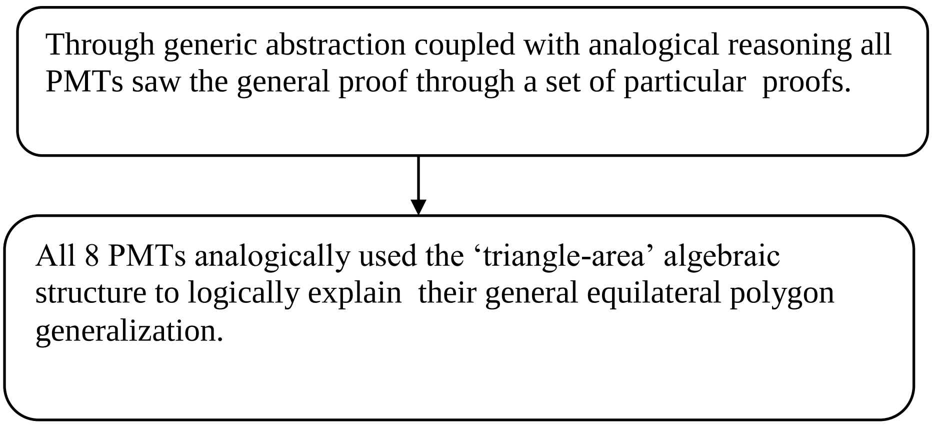



Figure 11.5: Trajectory of Justifications of Generalization for 'Any' Equilateral Convex Polygon

For example, the latter is typically illustrated in Victor's written response as shown in Figure 10.1.10 in Chapter 10. The steps in Victor's logical explanation show that there was a reasonable degree of parallel connectivity (i.e. cognitive blending) between his logical explanations for his equi-sided convex polygon generalization and his refined pentagon conjecture generalization (see Figure 9.4.2 in Section 9.4), i.e. there was some structural consistency between the two logical explanations.

From the perspective of meaningful learning as argued by Ausubel et al. (1978), it appears that all PMTs were able to construct their logical explanations for their equi-sided convex polygon generalizations through a process of correlative subsumption (see Finding 10 in Section 10.2), which is one of the ways in which a new problem (or a variation of a given problem) is related to previous or relevant knowledge in the existing cognitive structure (see

Ausubel et al.,, 1978; Aziz et al., 2009). In other words, they saw the ‘equal sides’ similarity between the equi-sided convex polygon generalization and their earlier generalizations for the equilateral convex pentagon, rhombus and equilateral triangle cases. This cognitively triggered the correlative subsumption (Ausubel, 1978) of the construction of the logical explanation for the ‘equi-sided’ polygon generalization into their earlier conceptual ‘triangle-area–algebraic’ explanatory structure. Analogical reasoning was then used to modify their existing ‘triangle-area’ algebraic explanation to accommodate the equi-sided convex polygon generalization.

Taking all the findings as discussed in this section, one can say that all eight PMTs were able to extend their previous generalizations to ‘any’ equi-sided convex polygon on logical grounds.

The next Chapter provides concluding remarks, implications of the findings, limitations of this study an some recommendations for further research.

# Chapter 12: Concluding remarks, Implications, Limitations and Recommendations

### 12.0 Introduction

This Chapter provides concluding remarks, and also discusses the implications of the findings and the limitations of this study. In addition some recommendations are suggested for future possible research studies in mathematics education

### 12.1 Concluding Remarks

In this qualitative study, which was governed by a constructivist framework, eight pre-service mathematics teachers were involved in typical mathematical processes that mathematicians normally traverse in their endeavour to construct, justify, extend and generalize generalizations. In particular, this study investigated the active engagement of eight pre-service mathematics teachers in the processes of generalizing and justifying through the use of selected activities, and their attempts to evaluate the validity of their generalizations through experimentation and justifications. In so doing, the research attempted to determine and explain how pre-service mathematics student teachers (PMTs) developed and justified their generalizations in a dynamic context. In the process, this study explored how PMTs experienced counter-examples (particularly heuristic counter-examples to assumptions embodying the development of thir initial conjectures) from a conceptual change perspective, and how they modified their conjecture generalizations and/or justifications as a result of such experiences. In addition the explanatory and discovery functions of proof were reflected upon during the constructions and justifications of further generalizations.

Analysis of data was grounded in an analytical–inductive method governed by an interpretive paradigm. Firstly, pre-service mathematics teachers were given an opportunity to reconstruct Viviani's theorem for the domain of equilateral triangles. Results of this part of the study showed that PMTs first experienced a heuristic counter-example to the assumptions defining their initial conjecture and hence resultant cognitive conflict. Subsequently through further experimental exploration and reformulation of their initial conjecture, the PMTs finally re-established their cognitive equilibrium. In summary, students reconstructed Viviani's generalization for equilateral triangles through empirical induction of dynamic cases, i.e. they constructed an inductive generalization.

Although the PMTs were highly convinced by the quasi-empirical testing, all eight of them expressed a need for an explanation. When they were given an opportunity to provide an explanation, none of them managed to provide a logical explanation on their own, but instead seven out of eight PMTs offered an empirical kind of justification. However, results of this study showed that through appropriate scaffolded guidance all eight PMTs constructed a logical explanation that justified their equilateral triangle conjecture generalization.

Results for the Rhombus task, showed that the majority (six out of eight) of PMTs were unable to initially generalize the Viviani Theorem for equilateral triangles to a rhombus, and conjectured that the minimum point would lie at the centre of the rhombus. However, through experimental exploration in a dynamic geometry context the PMTs experienced a heuristic counter-example to their assumption that the centre point would be the only point that could enable one to acquire the minimum distance to the sides of the rhombus, and caused them internal cognitive conflict. Through further experimental exploration the six PMTs accepted the heuristic counter-example to their centre point assumption as contained in their initial conjecture, and subsequently modified their initial conjecture to finally achieve the desired cognitive equilibrium. In so doing, these six PMTs constructed a rhombus conjecture generalization that was similar in construct to the Viviani generalization for equilateral triangles. Two of the PMTs seemed to have extended their equilateral triangle conjecture generalization to the rhombus on analogical grounds. However, their level of analogical reasoning was later revealed to be rather weak and superficial, as they expressed a desire and need for an explanation during their one-to-one task based interview session.

Results showed that PMTs accomplished their logical explanations (i.e. deductive justifications) for their rhombus conjecture generalization as follows:

- Three directly required  scaffolded guidance, which was connected to the with the 'triangle–area' algebraic explanation, but to a limited extent as compared to the equilateral triangle case
- Two seemed to have used and algebraic strategy encompassing the proposition: the distance between parallel lines is constant. However, later through scaffolded guidance they alternatively developed a  'triangle–area' algebraic explanation with relative ease.
- Two spontaneously used and algebraic strategy encompassing the proposition: the distance between parallel lines is constant. However, they alternatively developed a

'triangle–area' algebraic explanation without any kind of scaffolded guidance but through analogy.

- One used analogy straight away, i.e. assimilating the rhombus logical explanation into the proof structure of the equilateral triangle conjecture.

As the PMTs proceeded with the Pentagon problem, one PMT extended the 'constant- distant sum' generalization for the previous equilateral triangle and rhombus cases onto any equi-sided pentagon on logical grounds with the aid of analogical reasoning. The remaining seven PMTs proceeded as follows:

- One PMT made a conjecture generalization to a regular pentagon via inductive or analogical reasoning at a superficial level without using *Sketchpad* or simply guessing.
- Six PMTs made their conjecture generalization to a regular pentagon on inductive or analogical grounds without using *Sketchpad*.

As regards the latter seven PMTs, who restricted their pentagon conjecture generalization to a regular pentagon, results showed that one PMT developed a logical explanation encompassing the 'triangle-area' algebraic strategy via scaffolded guidance, and the remaining six PMTs did so by seeing the structure of the proof for the regular pentagon conjecture generalization through the 'triangle-area' algebraic structure proofs that were constructed respectively to logically explain their earlier rhombus and equilateral triangle conjecture generalizations. This latter move typically represents the process of correlative subsumption as explained by Ausubel et al. (1978), and is also connected to the process of analogical transfer in a parallel manner as described by Gentner's Structure Mapping Theory (Gentner, 1983, 1989) as well as cognitive blending.

In restricting their pentagon conjecture generalization to a regular pentagon, the seven PMTs exhibited an inherent misconception, namely that only regular pentagons have equal sides. However, the experience of a carefully designed heuristic counter-example functioned as a driving force for the PMTs to correct this misconception, and subsequently they refined their regular pentagon conjecture to include any equilateral convex polygon.

Each of the seven PMTs succeeded in constructing a logical explanation by similarly using the 'triangle-area' algebraic structure that was used to construct logical justifications of their earlier regular pentagon conjecture generalization. This was done by plausibly transferring the explanatory steps from the source problem onto the explanatory steps of the target problem via analogical structural mapping, i.e. parallel transfer of information from source to target (see Gentner, 1983, 1989). As the PMTs' logical explanation was linked to their previous 'triangle area' algebraic explanation for the regular pentagon, i.e. a superordinate idea, it seems that the process of derivative subsumption took place in this instance.

When each of the eight PMTs were asked to consider previous generalizations, and then generalize to any polygon with a similar property, all eight PMTs made their generalization to any equi-sided polygon on logical grounds, without expressing any explicit need for visual/experimental confirmation through the use of *Sketchpad*. This cognitive move suggests that all the PMTs had now severed the ontological bonds with their earlier forms or processes of making generalizations (see Freudenthal, 1973, p. 451), and this in turn signals cognitive growth in proof construction amongst the PMTs (see Tall, Yevdokimov, Koichu, Whitely, Kondratieva, & Cheng, 2012; Harel & Sowder, 1998, 2005). In particular, all eight PMTs reflected on the general process and made a generic abstraction from the earlier cases to recognize that the 'triangle-area' algebraic method, can be extended, widened or extrapolated to construct a logical explanation to justify the invariance distance sum in any equi-sided polygon. In the light of this, the generic abstraction coupled with analogical reasoning made it possible for all PMTs to make the transition to formal abstraction, i.e. to construct and develop a general proof for any equi-sided polygon. This essentially means they had perceived the general proof through a set of particular proofs, or that they had seen the general through the particular.

In addition, this study showed that the phenomenon of looking back (i.e. folding back) at their prior explanations assisted the PMTs to extend their logical explanations to the general equi-sided convex polygon. This development of a logical explanation (proof) for the general case after looking back and carefully analysing the statements and reasons that make up the proof argument for the prior particular cases, namely pentagon, rhombus and equilateral triangle, emulates the discovery function of proof. Furthermore, the insight and understanding gained from the logical explanations constructed for their earlier generalizations, seemed to have created a 'road map' that enabled all the PMTs to discover a general proof for the equi-sided polygon. Thus in this sense, the explanatory function of proof complimented the discovery function of proof and vice versa.

In most of our classrooms at schools and universities, the objective of proving a given mathematical statement or conjecture generalization is often just to verify the validity of such a statement or generalization. Thus students at schools, prospective mathematics teachers and qualified teachers come to view proofs as products produced through a deductive mechanism that exist in isolation from mathematical processes, and also come to see a proof as an end in itself, i.e. it has no further role to play in the development of new mathematics (see Chazan, 1993; Harel & Sowder, 1998). This limited treatment of proofs in our classrooms invariably does not provide the necessary insight to the students as to why a particular result is always true (see De Villiers, 2003a; Hanna, 1995). Furthermore it does not allow them to see how new knowledge (like proofs and further generalizations or specializations) can be constructed by looking back and reflecting on a particular proof (or set of proofs) and then analogically transferring the explanatory structure of such a proof across to new domains in a manner that enables one to build a logical explanation that could meaningfully justify an extended generalization. In a sense, the way proof is dealt with in our mathematics classrooms compromises the notion of meaningful learning as posited by Ausubel (1962) and Ausubel, Noval & Hanesian (1978) and is not used "as a vehicle to promote mathematical understanding" (Hanna, 1995, p. 42). Further to this, Schoenfeld (1994, p. 75) argues: "In most instructional contexts proof has no personal meaning or explanatory power for students".

However, this study has illustrated how proving activities can stimulate meaningful learning and can be used as a vehicle to promote mathematical understanding to an extent that pre-service mathematics teachers were able to discover a generalization of Viviani's theorem on deductive grounds on their own with/without the aid of analogical reasoning. This in a sense concurs with Piaget's notion that students have the innate capability to logically develop mathematical ideas autonomously, and that such autonomy can be stimulated through the provision of appropriate learning contexts characterized by meaningful interactions (Beth and Piaget, 1966).

As a pedagogical contribution to mathematics teaching and learning, this study provides a descriptive analysis of a 'guided approach' to both the construction and justification of generalizations via an evolutionary process, which mathematics teacher educators could use as a model for their own attempts in their mathematics classrooms. For example, it provides plausible ideas as to how students through experimental exploration can construct inductive generalizations, experience heuristic counter-examples that can cause (or force) them to modify (refine) their conjectures (or conjecture generalizations, i.e. make a conceptual change as per Piaget's model of socio-cognitive conflict. It also provides a road map of how analogical reasoning can be used to extend generalizations from one domain to the next, as well as aiding in the construction of deductive generalizations and logical explanations of particular generalizations through parallel transfer of explanatory structures from one domain across to another domain. In particular, this study brings to the fore three basic ways in which generalizations can be developed: namely inductive, analogical and deductive; and also illustrates three plausible ways of justifying a conjecture generalization: namely empirical, generic and deductive. In this respect the study provided a set of plausible trajectories for generalizations as well as justifications, which mathematics teacher educators and mathematics teachers can use in their own classroom to facilitate the construction and justification of generalizations not only from a teaching and learning perspective, but also a research perspective (see Figures 11.1;11.2.1; 11.2.2.;11.3.1,11.3.2; 11.4; 11.5).

In addition, the study showed that deductive justifications and deductive generalizations are not necessarily constructed independent of analogical reasoning; and approaching the construction of a proof from an explanatory lens enabled the PMTs to engage in the construction of proofs that made sense. As a result of this sense making they seemed to have gained the necessary insight as to why their respective generalizations were always true, and this in turn helped them to meaningfully discover new generalizations and proofs. More importantly, establishing a learning environment in our university mathematics education classrooms, wherein pre-service mathematics teachers experience the processes of generalizing and justifying as reported in this study, could very well influence how they might eventually engage their own learners in mathematical processes like conjecturing, refutation, generalizing and justification when they return to mathematics classrooms as qualified mathematics teachers. In this way, mathematics learners might come to view mathematics as a process that is within their capabilities and not just a series of 'products' that are produced, and hence learn mathematics in a more meaningful way.

This study also made some theoretical contributions to the field of mathematics education and research. On a theoretical level, this study illustrated how a multi-faceted conceptual framework, which has hardly been used before, acted as an efficient mechanism and helped integrate different approaches, processes and frameworks to work in harmony to support and inform my research. This conceptual framework as illustrated in Figure 4.1 and discussed in

Section 4.1, enabled me to interpret the findings of this study into a coherent structure, which can be easily accessed and implemented by both practitioners and researchers.

In addition, this study firstly differentiated between the following types of generalizations in the field of mathematics: Inductive generalization, deductive generalization, constructive (a priori) generalization (see Section 1.6). Further to this the study showed that analogical reasoning and deductive reasoning can work in a complimentary manner to produce an analogical-deductive kind of generalization as well as an analogical-deductive kind of justification. In other words, the study demonstrated that deductive reasoning does not necessarily happen independently from analogical reasoning. Furthermore, this study showed that the phenomenon of 'seeing the general through the particular' is a powerful didactic device not only to extend generalizations to higher order polygons but to also construct logical explanations to explain such generalizations.

This study also made some methodical contributions. From the results of study that were presented in this case study, it is clear that the method of one-to-one task-based interviews conducted within a dynamic geometry context, provided an apt way to individually track the generalizing and justifying experiences of the pre-service mathematics teachers with deeper insight and understanding. It made it possible for myself as the researcher to see the cognitive growth that took place amongst the PMTs as they proceeded through the evolutionary process of generalization. Furthermore, the one-to-one task-based interview provided an opportunity to me, the researcher, to establish rapport with the PMT and hence obtain deeper insights and better data about his/her thought processes. In particular this data collection technique made it possible for me to clarify and seek further explanations to specific expressions, statements, justifications, misconceptions and explanations that a PMT made in the process of constructing and justifying his/her generalization.

Although as a mathematics teacher and lecturer, my habitual role is to help students and see students succeed, it was a natural tendency to want to assist the students during their task based activities. However, in my role as researcher and participant observer, I managed to curtail this notion by providing just enough scaffolded support to a given PMT that the situation demanded. This was made possible through the use of appropriately designed task-based worksheets and semi-structured interview protocols that provided me as a researcher the opportunity to intervene and render support as diagnosed during a PMT's generalizing and justifying experience. As such, the use of one-to-one task-based interviews within this case study research approach enabled me to provide an in-depth account of each PMTs generalizing and justifying experience in a connected and holistic way, and hence can be accepted as methodological contribution to the research community in mathematics education.

### 12.2 Implications of the Findings

The investigation as to where a point should be located within an equilateral triangle such that the sum of the distances from the located point to its respective sides is a minimum, served as a rich context for pre-service mathematics teachers (PMTs) to engage in pivotal mathematical processes such as experimentation, conjecturing, generalizing, refuting, refining, and justifying. The equilateral triangle task provided a foundation to create rich tasks like the rhombus problem, pentagon problem (convex) and 'general' equi-sided convex polygon problem, which in different ways provided the space for PMTs to continually extend the 'constant distance sum' generalization from an equilateral triangle to a rhombus to any equi-sided convex pentagon and finally to generalize the 'constant distance sum' result to any equi-sided convex polygon. However, the ways in which PMTs extended their generalization grew mainly from an empirical approach in the rhombus case by largely using *Sketchpad* to an analogical approach in the pentagon case without initially using *Sketchpad*, and finally to a deductive approach in the general equi-sided polygon case. This suggests that mathematics teachers and mathematics educators should be prepared to initially receive generalizations made on empirical grounds when students face a problem for the first time. Further to this, they may expect their students through necessary scaffolded guidance to grow in their ways of generalizing as they progress from one polygon to higher order polygons that have at least a common characterizing property.

This evolutionary process of generalizing a particular result from one domain to another domain in an iterative manner, and in progressively different ways, enabled the pre-service mathematics teachers to construct a generalization of Viviani's theorem, namely: in any equi-sided convex polygon the sum of the distances from a point inside the polygon to its sides is constant. See Figures 11.1, 11.2.1, 11. 3.1 and 11.4 in Chapter 11, which represent the range of processes that pre-service teachers experienced and engaged with in order to help them to re-invent (or re-construct) this generalization. The developmental processes that are reflected in Figures 11.1, 11.2.1, 11.3.1, 11.4, represent typical processes mathematicians themselves engage with in their endeavor to invent or discover new knowledge. Hence, it is hoped that the experiential journey of the pre-service mathematics teachers as they progressed from the equilateral triangle problem to the general equi-sided convex polygon problem helped them to develop a broader understanding of how new mathematics is sometimes created, and hence contributed in some way to the prospective mathematics teachers' perspectives on the nature of mathematics. Providing such a generalizing and justifying experience to pre-service mathematics teachers across our teacher education institutions, can prepare, encourage and motivate them to build and create similar learning experiences for their students, during their teaching practice sessions and when they return to mathematics classrooms as fully qualified teachers. In this way they could help in some small way to break the traditional method of presenting learners with a result and proof as it is done in most textbooks and classrooms, i.e. stating a theorem or generalization and then verifying its truth by directly providing a proof. In other words, there would be greater chance for a wider learner population to experience some of the genuine mathematical processes.

Polya's (1957) notion of 'Looking Back' and Pirie and Kieran (1994) notion of 'Folding Back' embraces a much wider ground than just checking the correctness or meaningfulness of a solution. For Polya, 'Looking Back' embraces: "the consideration of alternative solutions and representations, the re-examination of the solution for a more efficient strategy, and the extension of the solution to other related problems" (Leong, Tya, Toh, Wek, & Dindyal, 2011, p. 182). These latter tenets of 'Looking Back", which promote mathematical thinking (Tall, 1991) characterizes some of the typical processes that mathematicians constantly engage with in their mathematical practices. In fact, mathematicians do not simply stop after they have found a solution to a problem, but instead look back and reflect on their solution or logical explanation and use it "as a sort of kernel to generate solutions to related problems" or construct further generalizations (Leong, Tya, Toh, Quek, & Dindyal, 2011, p. 182). In this study, the process of looking back and reflecting on each of the previous generalizations played a significant role in assisting the PMTs' in their extension of the Viviani result for equilateral triangles as well as its justification to other polygons, like the rhombus, pentagon and any equi-sided convex polygon. This suggests that mathematics educators and practicing mathematics teachers should instill in their students the habit of reflecting and looking back at their generalizations and proofs to the extent that they always seek to extend their generalizations and justifications to higher order polygons whenever the opportunity prevails.

In particular, through reflecting on their logical explanations for their respective earlier conjecture generalizations (i.e. equilateral triangle, rhombus and pentagon) all eight PMTs were able to abstract a general explanatory strategy, namely the 'triangle area algebraic method'. This method with some provided scaffolding and guidance by the researcher, especially earlier on with rhombus and pentagon problems, enabled them to eventually extend and extrapolate in order to construct both a generalization as well as a logical explanation to justify the invariance of the distance sum in an equi-sided convex polygon. This cognitive move by each of the eight PMTs, affirms that they saw a generalization of Viviani's theorem advance through their logical explanations for their earlier generalizations, and were also able to deduce the General Proof (i.e. proof of the generalization for any equi-sided polygon) from these. Furthermore, this recognition of the 'general through the particular', shows how the insight that the pre-service teachers gained via the process of constructing logical explanations to justify their earlier conjecture generalizations, enabled them to eventually 'deductively' discover a generalization of Viviani's theorem. This realization affirmed the fact that the discovery function of proof does not exist in isolation from the explanatory function of proof, but rather that these functions interact with each other in a mutualistic way to construct new knowledge. This kind of disposition to the phenomena of generalizing and justifying (or proving), which enabled the pre-service teachers to construct and justify a general generalization through self-discovery and some guidance should be built into pre-service mathematics teacher education curriculums to inculcate and promote mathematical thinking and sense making amongst them (see Leong et al., 2011; Schoenfeld, 1992; Tall, 1991).

Furthermore, this study showed that through initial and subsequent experimentation (see De Villiers, 2004) within a dynamic geometry context, pre-service mathematics teachers experienced genuine mathematical processes like abstracting, specializing, generalizing, conjecturing, global and heuristic refutation. In this study, experimentation embraced the use of non-deductive methods, and provided the space for pre-service mathematics to use intuitive, inductive or analogical reasoning to make their initial conjectures, particularly in the equilateral triangle, rhombus and pentagon problems. Through experimentation, pre-service mathematics teachers generalized their conjecture, experienced heuristic counter-examples to assumptions that played a part in the formation of their initial conjectures and finally led to re-formulation of their conjecture or conjecture generalization. As evidenced in this study the process of experimentation made it possible for students to first encounter a new idea and then generalize such an idea and also extend the resultant generalization further to other domains via inductive and/or analogical reasoning (see Polya, 1954a). Thus, teachers and teacher educators should look at opportunities within their teaching curriculum that allow students to experimentally explore specific mathematical phenomena, make a conjecture, generalize a conjecture, experience a counter-example (see Lakatos, 1976) that cause some perturbation and cause them to abandon their conjecture or modify their conjecture.

As reported in this study, through a heuristic counter-example to assumption(s) informing the development of their initial conjecture(s), PMTs experienced some perturbations (i.e. cognitive conflict) within their 'regular polygon schema' and hence cognitive disequilibrium. Through the process of conceptual change, PMTs modified their 'regular polygon schema', i.e. corrected their regularity misconception, and in so doing restored their cognitive equilibrium PMTs (see Balacheff, 1991; Komatsu, 2010; Lee et al., 2003; Piaget, 1978, 1985; Von Glasersfeld, 1989). This suggests that mathematics teacher educators can promote the conceptual growth of their pre-service mathematics teachers by identifying and creating similar opportunities that allow them to interact with mathematical tasks, reveal misconceptions and use experimental approaches to correct such misconceptions (Slavin, 1997). In other words, through appropriate pedagogical strategies mathematics teacher educators can promote experimentation, refutation, discussion, justification and reasoning, and thus help their pre-service mathematics to appropriately reorganize their own conceptions (see Clements and Battista, 1990; Cobb & Bauersfeld, 1995; Cobb, Yackel & McCalin, 2000, Ryan & Williams, 2000; Tsamir & Tirosh, 2003, Clements & Battista, 1990).

Classroom practitioners should be aware that conceptual change is synonymous with Piaget's notions of disequilibrium (i.e. cognitive conflict), accommodation and assimilation, and hence when students exhibit misconceptions in their thinking, they should first provide such students with a discrepant event that enables them to experience some degree of cognitive conflict that not only causes them to be surprised but also to become cognitively disturbed (i.e. experience cognitive disequilibrium) (see Eggen & Kauchak, 2007; Piaget, 1978, 1985; Posner et al.,1984). Whilst in this state of cognitive disequilibrium, students should be allowed to experiment further to derive convincing evidence that their existing conception is indeed either fully or partially invalid. When this heightened convincing happens, the probability of changing their existing conceptions is more likely to occur. However, to foster such a conceptual change it is essential that students accommodate their thinking so that the alternate (or new) conception becomes intelligible or makes sense (see Eggen and Kauchak, 2007; Piaget, 1978, 1985; Posner et al., 1984). Teachers can gauge such sense making through checking whether the students can successfully explain a solution to a problem using the new conception or can logically explain why the new conception itself is valid and acceptable.

Lastly, from a conceptual change perspective, the new conception must enable the students to re-establish their desired cognitive equilibrium (i.e. be fruitful), and enable students to assimilate new experiences. That means that the new conception should create new pathways of enquiry and also be expanded to make sense of other experiences when the need arises or simply explain additional examples or cases (see Berger, 2004; Gage & Berliner, 1992; Eggen & Kauchak, 2007; Hadas, Hershkowitz & Schwartz, 2000; Piaget, 1978, 1985; Posner et al. 1984; Tirosh & Graeber, 1990; Zazkis & Chernoff, 2008).

As reported in this study, scaffolding was provided to all PMTs to assist them to develop a logical explanation for their equilateral triangle conjecture generalization. Although scaffolding was provided in the subsequent task based activities associated with the rhombus and pentagon problems, the degree and extent of the scaffolding decreased as PMTs moved from the equilateral problem to the pentagon problem. As posited by Snowman, McCown & Biehler (2009), this reduction in the need for scaffolded guidance affirms that the scaffolded guidance enabled the PMTs to gradually internalize the required knowledge and skills with a fair degree of insight, and in the process became increasingly self-regulated and independent as they progressed in an evolutionary manner from one task to the next. In fact they became so independent, that through the processes of subsumption, all PMTs saw that their 'triangle-area' algebraic proof for the regular pentagon could be used analogicallyand also logically to justify their conjecture generalization for any equi-sided pentagon, and hence went on to produce a coherent 'triangle-area' logical explanation also on their own. This kind of development typically represented a trajectory of conceptual growth in each of the pre-service mathematics teachers.

Thus, for the purposes of successful teaching and learning, it is imperative that when a student cannot initially make progress with a given problem or task on his/her own, the facilitator ought to be able to diagnose the problem to ascertain the conceptual and procedural difficulties the student is experiencing or what else is hindering/blocking the student from making progress. Then according to the diagnosis by the facilitator, scaffolding should be given to provide the students with sufficient support at the beginning of the intervention. This procedure was followed to advance the development of a logical explanation for the equilateral triangle conjecture generalization in the study. Then as illustrated in this study, the facilitator may diminish such support as and when a student progresses with a similar kind task (or connected problem) to make allowance for the student to take increasing responsibility for his/her learning as soon as he/she is able to do so (Eggen & Kauchak, 2007;

Puntambekar & Hubscher, 2005; Slavin, 1997; Snowman, McCown & Biehler 2009; Wood, Bruner, & Ross, 1976).

Furthermore, this study showed that reasoning by analogy was used extensively by the pre-service mathematics teachers to extend their generalizations to higher order polygons and even to logically explain why such generalizations are generally true. This is consistent with De Villiers' (2008, p. 34) assertion that: "Analogy is often a powerful means of extending or applying mathematical results to other domains." In particular the findings of this study demonstrated pre-service mathematics teachers were able to deductively construct their generalization for the general equi-sided polygon case, with the aid of analogical reasoning, and indeed produce their coherent logical explanation through a process of parallel transfer as suggested by Gentner's theory of Structural Mapping Theory (Gentner, 1983,1989). This typically exemplifies the role that analogy can play in the discovery and invention of new mathematical knowledge (see (see De Villiers, 2008; English, 1998; Gentner 1983,1989; Lee & Sriraman, 2011; Polya, 1954a , 1981).

In a sense, the extent of the use of analogical reasoning in the processes of generalizing and justifying in this study, resonates with Polya's (1954a, p. 17) assertion: "Analogy seems to have a share in all discoveries, but in some it has the lion's share". Taking cognizance of the powerful role of analogical reasoning as illustrated in this study, it becomes prudent for mathematics teacher educators and practicing mathematics teachers to promote the use of analogy in their classrooms through appropriate and relevant activities, which are within the range of their prescribed curricula, so that their students (or learners) can likewise conjecture and justify generalizations, and then extend them across to other domains via the use of analogy on their own.

Apart from Polya (1954a), who emphasized the role of analogical reasoning in problem solving, Holyoak and Thagard (1989) also illustrated that analogical reasoning can assist students to adapt to new novel contexts. Novick (1988) and White and Mitchelmore (2010) also showed that students can transfer a representation from one context to the next via analogy-making (as described by Getner's Structure Mapping Theory). Furthermore, Benson (2007, p .4) as cited in De Villiers (2008) argues that the ability to recognize and use analogies is pivotal in attempting to solve problems primarily because it: "allows the solver to connect the familiar (a previously used method, strategy, or context) to the unfamiliar (a new problem)" (p. 38). In addition, Novick (1988, p. 510) states that "retrieval of an analogous problem may enable the student to adapt a known procedure for use with the target problem, thus precluding the necessity of constructing a new procedure. Thus it is imperative that mathematics teacher educators and practicing mathematics teachers understand analogies themselves and know how to design tasks that provide a space for their students to use analogical reasoning to discover new generalizations and/or their proofs. Pre-service mathematics teachers and practicing mathematics teachers ought to be exposed to task- based activities similar to those used in this study, so that they can develop a deeper understanding of how to use analogies (or analogical reasoning) to promote the discovery and justification of generalizations as well as problem solving amongst their learners, when they take up their positions as mathematics teachers in schools.

## 12.3 Limitations of this study

In this section, limitations associated with the sample, period of data collection, researcher role and shortage of time are presented respectively in Sections 12.3.1, 12.3.2, 12.3.3 and 12.3.4.

### 12.3.1 Sample of the study:

This study consisted of a sample of only eight final year PMTs. This sample is small and extending the sample would have given more reliable results. Therefore, it is difficult to state to what extent the findings of this study would be generalizable. Given the thick ‘descriptions’ presented in this study of the kinds of generalizations and justifications that all eight pre-service teachers constructed as well as how they arrived at their generalizations and justifications, it is plausible that pre-service mathematics teachers at other higher education institutions may move along similar ways of generalizing and justifying in mathematics with the same problem or problems of a similar kind. In this regard, the major concern will be for the reader to determine to what extent the findings of this study are applicable to a particular situation (Guba & Lincoln, 1993). However, one should bear in mind that the aim of the study was not to generalize but to provide qualitative data and analysis thereof which other teacher educators or researchers can find useful and applicable to their own particular setting.

### 12.3.2 Period of Data Collection

As this was a large in-depth study, data had to be collected for the four problems through one-to-one task-based interviews taking place within a dynamic geometry environment in a computer laboratory. There were two data collection sessions designed to collect data from each PMT. The first session was limited to the equilateral triangle problem, and the second session dealt with the rhombus, pentagon and the general equi-sided convex polygon problems because during his planning for data collection the researcher had an intuitive sense that each of these sessions would take a great deal of time. In fact, the amount of time that was consumed for the first session ranged from 25 minutes to 50 minutes, whilst the time taken for the second session ranged from 45 minutes to 80 minutes per PMT.

As this data was collected during University session times, the researcher could only engage each PMT in their respective one-to-one task based interview sessions when they were free, i.e. when they were not attending lectures. With this constraint, it was feasible for the researcher to do one task-based interview on some days and two on other days. Thus the data collection phase took about 10 days. During these 10 days, the researcher asked the pre-service mathematics teachers not to discuss any of the task-based problems or interview questions with their colleagues that were participating in this research. Despite this plea from the researcher, there is no guarantee that participating PMTs did not discuss the task-based problems and interview questions with their colleagues. If such discussions happened it is not easy to rationalize or factor in its net influence on the results of this study.

Thus, taking all this into cognizance, the researcher at first collected data via the one-to-one task-based interviews for the equilateral problem from all eight PMTs, and only thereafter proceeded with the second one-to-one task-based interview sessions to collect data associated with each of the three remaining problems. Thus the data for the equilateral triangle problem was collected during the first week, whilst the data for the remaining three problems was collected in the second week. As discussed in Chapters 7 and 10, there were five out of the eight PMTs who did not immediately see the connection between the equilateral triangle problem and rhombus problem, and it is not certain if the time delay of at least a week between sessions 1 and 2 could have contributed in some way to their not being able to immediately extend their generalization from the equilateral triangle case to the rhombus case. In fact six of them had to experimentally explore by using *Sketchpad* to arrive at the rhombus conjecture generalization.

### 12.3.3 Researcher as data collector and analyser of data

Although the data for this study came from the one-to-one task- based interviews which were audio recorded, written responses to worksheets were collected from all pre-service mathematics teachers, and their dynamic geometry moves were also recorded by using *Snagit*. However, the researcher was the only person collecting and analysing the data. Hence, my own personal philosophy about teaching and learning mathematics, developed from ten years of experience as a high school teacher and seventeen years of experience in mathematics teacher education, could have had a potential bias in the process of data collection and the subsequent analyses of the data. Taking into consideration the possibility of such bias, the researcher took the necessary precaution to minimize any bias by recording salient aspects, thoughts, clarifications, dissonances, reflections, 'aha' moments, insights and decisions as and when they occurred during the one- to- one task based interview in a note book. Furthermore, the triangulation of the data with task-based interview transcripts; worksheet responses and *Snagit* recordings affirms the findings of this study. All these data sources have been stored securely, and are available to any other researcher should they want to query any of the findings.

### 12.3.4 Exploring other possible generalizations of Viviani's Theorem

As discussed in Sections 6.5 and 12.3.2, the time taken to conduct and complete one-to-one task based interview across all the task based problems was quite extensive. Hence, there was no further time in this study to explore other generalizations to Viviani's theorem, like Generalizing to concave equilateral polygons, generalizing to any equi-angled polygon or any 2*n*-gon with opposite parallel sides.

## 12.4 Recommendations for Further Research

The following recommendations for further research are made on the basis of the results that emerged from this study:

12.4.1 The generalization of Viviani's theorem for equilateral triangles to a sequence of equilateral concave polygons of four sides (rhombi), five sides (pentagons) and general concave equilateral n-gons should be explored. If for example, one could explore a non-convex pentagon (which could be created by dragging in the mystery pentagon), one would find that the sum of the distances to the edges is a constant only for some part of the polygon, but switches to a variable quantity as the point is moved into interior regions, where the point does not 'see' along interior rays. It would be an interesting further exploration to see if the students could adapt and modify their conjectures in the face of such a different 'counter-example'.

12.4.2 A further exploration could focus on what generalizations could be made for points *p* exterior to even the equilateral triangle, or more so as to when do the sum of the areas of the triangles created = area of the polygon. More generally, what sign pattern should be used to compute the area of the polygon from areas of the triangles connected to a general point *p*.

12.4.3 This study has shown that PMTs have cognitively grown in their ways of generalizing as they progressed through the sequence of task based problems. For example, this cognitive growth (or development) is evident in the regular pentagon case as most students made their initial generalizations on analogical grounds without effectively using *Sketchpad*, whilst none of the PMTs used *Sketchpad* when they constructed their equi-sided polygon generalization of Viviani's Theorem. Research conducted on a larger scale with pre-service mathematics teachers to establish whether similar results can be obtained would be valuable. Perhaps an investigation can be carried out to see whether these results can also be obtained in a non-dynamic geometry environment.

12.4.4 This research study could be repeated as follows:

- Across other geometric content
- Across another area within mathematics, for example trigonometry, calculus, algebra or data handling.
- In a whole classroom situation where the social-constructivist dynamics of the interaction might be different and more complex.
- With learners or practicing teachers at school

12.4.5 More in-depth research be carried out to explore the extent that mathematics teacher educators and/or practicing mathematics teachers engage their students (or learners) with meaningful learning activities and pedagogical processes that promote cognitive conflict and conceptual change to help students (or learners) to not only eliminate misconceptions but refine and establish generalizations. In a much broader way, further research could encompass identifying misconceptions across mathematics at school among learners as well as teachers, and seeking ways to remediate them possibly by creating cognitive conflict.

12.4.6 In conducting further research as described in 12.4.3, the following theoretical frameworks should be considered: Piaget's equilibration theory (see discussion in Section 4.6) which embraces the processes of assimilation, disequilibrium and accommodation; and Ausubel's theory of meaningful learning that embraces the processes of subsumption, namely derivative subsumption and correlative subsumption (see discussion in Section 4.5).

12.4.7 In order to make the task of proving in our classrooms more accessible and meaningful from both an explanatory and a discovery perspective of proof, further research on a much larger scale could be conducted on the plausibility of engaging pre-service mathematics teachers, practicing mathematics teachers and/or school learners in the processes of generic proving across a broader range of mathematical content topics. This could serve as a vehicle to allow the possibility of good practices to permeate our mathematics classrooms at schools, where practicing mathematics teachers and/or learners can then also engage in development of proof from explanatory and discovery perspectives as well (see Section 5.4 for discussion on functions of proof).

12.4.8 Further research could explore the other two possible generalizations of Viviani's theorem in the plane:

- "The sum of the distances from an interior point to the sides of any equi-angular polygon constant;
- The sum of the distances from an interior point to the sides of any $2n$-gon with opposite parallel sides is constant"

(see http://frink.machighway.com/~dynamicm/viviani-general.html).

12.4.9 Research could also focus on generalizing the Viviani result to 3D. One such example is the following possibility:

"The sum of the distances from a point $P$ to the faces of a tetrahedron with faces of equal area, is constant" (see http://frink.machighway.com/~dynamicm/viviani-general.html).

12.4.10 More-over research could focus on generalizing and exploring the Viviani result in non-Euclidean geometries, i.e. hyperbolic, elliptic and taxicab geometry.

12.4.11 Other studies could focus on the processes of conjecturing and refutations, or perhaps even other more formal processes of mathematics such as defining and classifying.

12.4.12 Another study could also investigate what intuitions learners and prospective teachers have that could be utilized and built upon to produce generalizations.

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# Appendices:

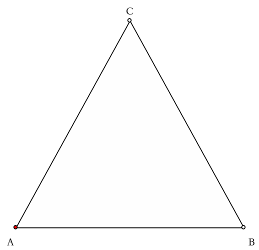
## Appendix 1: Session one task-based activity – Equilateral Triangle Problem

**Task 1:  Ship wreck problem**

Sarah, a shipwreck survivor manages to swim to a desert island. As it happens, the island closely approximates the shape of an equilateral triangle. She soon discovers that the surfing is outstanding on all three of the island's coasts and crafts a surfboard from a fallen tree and surfs everyday. Where should Sarah build her house so that the *total sum of the distances from the house to all three beaches is a minimum? (She visits them with equal frequency).*

**Task 1(a): Locating a point in the triangle *not* using *Sketchpad*.**

(i) Before you proceed further, locate a point in the triangle at the point where you think Sarah should build her house.




(ii) Why did you choose that position? Explain or justify your choice.

**Task 1- (b): Using *Sketchpad* to develop a conjecture**

Open the sketch **Distances.gsp.** Drag point *P* to experiment with your sketch.

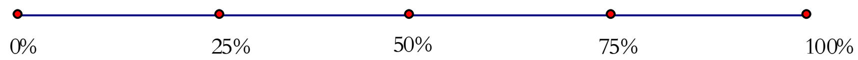
(i) Press the button to show the distance sum. Drag point *P* around the interior of the triangle. What do you notice about the sum of the distances?

(ii) Drag a vertex of the triangle to change the triangle's size. Again, drag point P around the interior of the triangle. What do you notice?

(iii) Write your discoveries so far as one or more conjectures. Use complete sentences.

**Task 1 (c): Certainty, counterexamples and logical explanations**

1. You probably can think of times when something that always appeared to be true turned out to be false sometimes. How certain are you that you conjecture is always true? Record your level of certainty on the number line and explain (or justify) your choice.




2. If you suspect your conjecture is not always true, try to supply a counter-example. If you are fully convinced of the truth of your conjecture, do you still have a need for explanation ? (i.e. do you want know why it is true?).
3. Can you support your conjecture with a logical explanation (justification)? You may use the back of this page, another sheet of paper, a *Sketchpad* sketch, or some other medium to write your explanation (or justification).

**Task 1- (d): Developing a logical explanation**

Press the button to show the small triangles in your sketch.

(i) Drag a vertex of the original triangle. Why are the three different sides all labeled *a*?

(ii) Write an expression for the area of the small $\Delta APB$ using $\boldsymbol{a}$ and the variable $\boldsymbol{h_1}$.

(iii) Write an expression for the area of the small $\Delta BPC$ using $\boldsymbol{a}$ and the variable $\boldsymbol{h_2}$.

(iv) Write an expression for the area of the small $\Delta APC$ using $\boldsymbol{a}$ and the variable $\boldsymbol{h_3}$.

(v) Add the three areas and simplify your expression by taking out any common factors.

(vi) How is the sum in Question (iii) related to the total area of the equilateral triangle? Write an equation to show the relationship using *A* for the total area of the equilateral triangle.

(vii) Use your equation from Question (iv) to explain why the sum of the distances to all three sides of a given equilateral triangle is always constant.

**[NB:** In some cases Questions (ii-iv) was asked as follows: Write an expression for the area of each small triangle using *a* and the variables $h_1, h_2, h_3$.]

**Task 1 (e): Present your explanation/justification**

Summarize your explanation/justification of your original conjecture. You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a *Sketchpad* sketch, or some other medium.

**Task 1(f): Challenge**

In this session you may have observed, conjectured and logically explained the following result: *In an equilateral triangle, the sum of the distances from a point inside the triangle to its sides is constant.*

1. Do you think that the above result might be true for other kinds of triangles?
2. If not, why not? **or** If so, why?

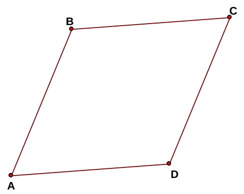
**Appendix 2: Session two task based activities - Rhombus, Pentagon and any Equi-sided Polygon Problems**

In the previous session you may have observed, conjectured and logically explained that in an equilateral triangle, the sum of the distances from a point inside the triangle to its sides is constant. In this session, we will consider the same problem for a rhombus, pentagon, and then any equi-sides polygon respectively.

**Task 2: Rhombus Task-based Activity**

**Task 2(a): Locating a point in a rhombus not using *Sketchpad***

Consider the rhombus *ABCD* below:




1. Where should you locate point *P* in the rhombus *ABCD* to minimize the sum of the distances to all four sides of the rhombus?
2. Why did you choose that position? Explain or justify your choice.

Task 2(b): Using Sketchpad to develop a conjecture related to a rhombus

Open the sketch **Rhombus.gsp**.

i) Press the button to show the distance sum. Drag point *P* around the interior of the rhombus. What do you know about the sum of the distances?

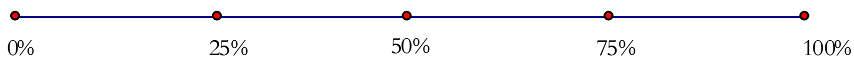
ii) Drag vertices *A*, *B* or *D* of the rhombus to change the rhombus's size or shape. Again, drag point *P* around the interior of the rhombus. What do you notice?

iii) Write your discoveries as one or more conjectures. Use complete sentences.

**Task 2 (c): Certainty, Counterexamples and Logical Explanations**

1.How certain are you that your conjecture is always true. Record your level of certainty on the number line and explain your choice

.



2. If you suspect your conjecture is not always true, try to supply a counter-example.

3. If you are fully convinced of the truth of your conjecture, do you still have a need for explanation (i.e. do you want know why it is true?)

4. Can you support your conjecture with a logical explanation (justification).

**Task 2 (d): Developing a logical explanation (proof)**

Press the button to show the small triangles in your sketch.

(ii) Drag *A*, *B* or *D* of the original rhombus. Why can you label each of the sides with the variable *a*?

(iii) Write an expression for the area of each small triangle using $\boldsymbol{a}$ and the variables $\boldsymbol{h_1}$, $\boldsymbol{h_2}$ , $\boldsymbol{h_3}$ and $\mathbf{h_4}$.

(iv) Add the four areas and simplify your expression by taking out any common factors.

(v) How is the sum in Question (iii) related to the total area of the rhombus? Write an equation to show the relationship using *A* for the total area of the rhombus.

(vi) Use your equation from Question (iv) to explain why the sum of the distances to all four sides of a given rhombus is always constant.

**Task 2 (e) Present your Explanation/Justification**

Summarize your explanation/justification of your conjecture (generalization). You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a *Sketchpad* sketch, or some other medium.

**Task 3: Pentagon Task-based activity**

**Task 3(a): Generalizing to a pentagon**

1. In the previous activities you may have observed, conjectured and logically explained the following results (or generalizations):

(a) In an equilateral triangle, the sum of the distances from a point inside the triangle to its sides is constant.

(b) In a rhombus the sum of the distances from a point inside the rhombus to its sides is also constant.

How would you generalize the above result(s) to a pentagon.

2. Do you want to test or confirm your conjecture (generalization)?

Let us experimentally investigate your conjecture using *Sketchpad:*

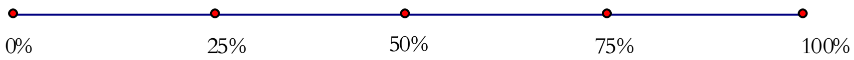
**Task 3 (b): Using Sketchpad to develop a conjecture related to a pentagon**

Open the sketch **Regular Pentagon.gsp**.

1. Press the button to show the distance sum. Drag point P around the interior of the pentagon. What do you know about the sum of the distances?
2. Drag a vertex of the pentagon to change the rhombus's size. Again, drag point P around the interior of the pentagon. What do you notice?
3. Write your discoveries as one or more conjectures. Use complete sentences.

**Task 3(c): Certainty, Counterexamples and Logical Explanations [Pentagon]**

1. How certain are you that your conjecture (generalization) is always true.
2. Record your level of certainty on the number line.




3. If you suspect your conjecture (generalization) is not always true, try to supply a counterexample.

If you are fully convinced of the truth of your conjecture (generalization), do you still have a need for explanation or do you want know why it is true?

(a) Support your conjecture (generalization) with a logical explanation (justification).

4. (a) Do you think the result might true for other kinds of pentagons? If not, why. If so, why?

(b) Would you want to test or confirm your response to Q 4 (a)?

OR

Are you already convinced. If so, why? If not, why?

(c) Open the sketch **Mystery Pentagon.gsp**.

Investigate whether your generalization explained in Question 3(b) holds true or not for the **Mystery pentagon.** You may also investigate your response to Q4(a) if necessary.

(d) What special property must a pentagon have so that it can yield the following result: the sum of the distances from an interior point in the pentagon to its sides remain constant.

5. Consider your response to 4(d) and then edit or rephrase your initial conjecture (generalization) with respect to the sum of the distances from an interior point in a pentagon to it sides.
6. Support your conjecture (generalization) in Q5, with a logical explanation (justification)

**Task 3(d): Developing a logical explanation (proof): Regular Pentagon**

Press the button to show the small triangles in your sketch.

(i) Drag a vertex of the original pentagon. Why are the five different sides all labeled *a*?

(ii) Write an expression for the area of each small triangle using *a* and the variables $h_1, h_2$, $h_3$, $h_4$ and $h_5$.

(iii) Add the five areas and simplify your expression by taking out any common factors.

(iv) How is the sum in Question (iii) related to the total area of the pentagon? Write an equation to show the relationship using *A* for the total area of the pentagon.

(v) Use your equation from Question (iv) to explain why the sum of the distances to all five sides of a given pentagon (regular) is always constant.

**Task 3(e): Present your Explanation/Justification**

Summarize your explanation/justification of your conjecture (generalization). You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, A *Sketchpad* sketch, or some other medium.

**TASK 4: Further Generalizations : To 'Any' Equi-sided Polygon**

**Task 4(a): Generalizing Further**

1. Below are a set of generalizations that you may have developed earlier:

G1: In an equilateral triangle, the sum of the distances from a point inside the triangle to its sides is constant, and

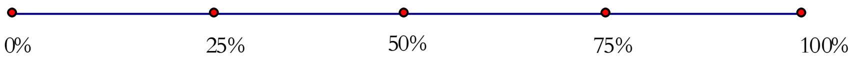
G2: In a rhombus the sum of the distances from a point inside the rhombus to its sides is also constant.

G3: In any equi-sided pentagon, the sum of the distances from a point inside the pentagon to its sides is also constant.

Consider the above set of generalizations. Can you now generalize to polygons with a similar property?

**Task 4(b): Certainty, Counterexamples and Logical Explanations**

1. How certain are you that your conjecture (generalization) is always true. Record your level of certainty on the number line.




2. If you suspect your conjecture (generalization) is not always true, try to supply a counter-example. If you are fully convinced of the truth of your conjecture (generalization), do you till have a need for explanation or do you want know why it is true?
3. Support your conjecture (generalization) with a logical explanation (justification).

**Task 4(c) : Developing a logical explanation – *n-Equi-sided polygons***

(i) Open the sketch ***n-Equi-sided Polygon.gsp*.** Press the button to show some of the *n* small triangles in your sketch.

(ii) Write an expression for the area of each small triangle using *a* and the variables $h_1, h_2, h_3, h_4 \ldots\ldots h_n$.

(iii) Add the *n* areas and simplify your expression by taking out any common factors.

(iv) How is the sum in Question (iii) related to the total area of the *n-equi sided* polygon? Write and equation to show the relationship using *A* for the total area of the *n-equi-sided* polygon.

(v) Use your equation from Question (iv) to explain why the sum of the distances to all *n* sides of a given *n-equi-sided* polygon is always constant.

**Task 4(d): Present your Explanation/Justification *: n- equi-sided* polygon**

Summarize your explanation/justification of your conjecture (generalization). You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a *Sketchpad* sketch, or some other medium.

## Appendix 3: Semi- structured interview schedule for Equilateral ∆ Task

**R**: Researcher ; **PMT**= Pre-service Matheamatics Teacher; **FG** = Further Generalization

**R** asks **PMT**:

After **PMT** has read the problem, then **R** asks **PMT** to locate a point in the triangle at the point where you think she should build her house.

↓

**R** asks **PMT**: Why did you choose that position?
Explain your choice?

Choosing 'centre' or vertex

Any point → Proceed to Task 1(b)

↓

A sheet containing Task 1(a): Using sketchpad to develop a conjecture through generalization

↓

After sufficient exploration on Sketchpad, **R** asks **PMT** to write their discoveries so far as one more conjectures, using complete sentences.

↓

**R** asks PMT: How certain are you that your conjecture is always true?

↓

**R** asks **PMT**: Record your level of certainty on the number line and explain your choice

**R** asks **PMT**: If you believe your generalization (conjecture) is always true, provide some examples to support your view

**R** asks **PMT**: If you suspect your conjecture is not always true, try to supply counterexamples.

**R asks PMT:** How would you convince your facilitator or classmate or members of your group?

**R asks PMT:** Support your conjecture with a logical explanation/justification or a convincing proof.

Unacceptable/incorrect

Acceptable/correct → Proceed to FG

**Task 1- (c)**: Developing a logical explanation (proof) through guided questions will be given to the student. **R asks PMT** : To proceed with the task.

**R asks PMT:** Summarize your explanation/justification of your original conjecture. You can use questions (i)-(vii) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, A *Sketchpad* sketch, or some other medium.

Research Questions?

## Appendix 4: Semi-structured interview schedule for Rhombus Task

**R**: Researcher ; **PMT**= Pre-service Mathematics Teacher; **FG** = Further Generalization

**R** asks **PMT**:

**R** asks **PMT** to reflect on the explanation provided for the generalization : In any equilateral triangle , the sum of the distances from a point inside the triangle to it sides is constant.

**R** asks **PMT** to consider the Rhombus ABCD.
**R** asks **PMT**: Where should you locate point *P* in the rhombus *ABCD* to minimize the sum of the distances to all four sides of the rhombus?

**R** asks **PMT**: Why did you choose that position?
Explain your choice?

Choosing ‘centre’/ vertex/ 1point

Any point → Proceed to Task 2(c)

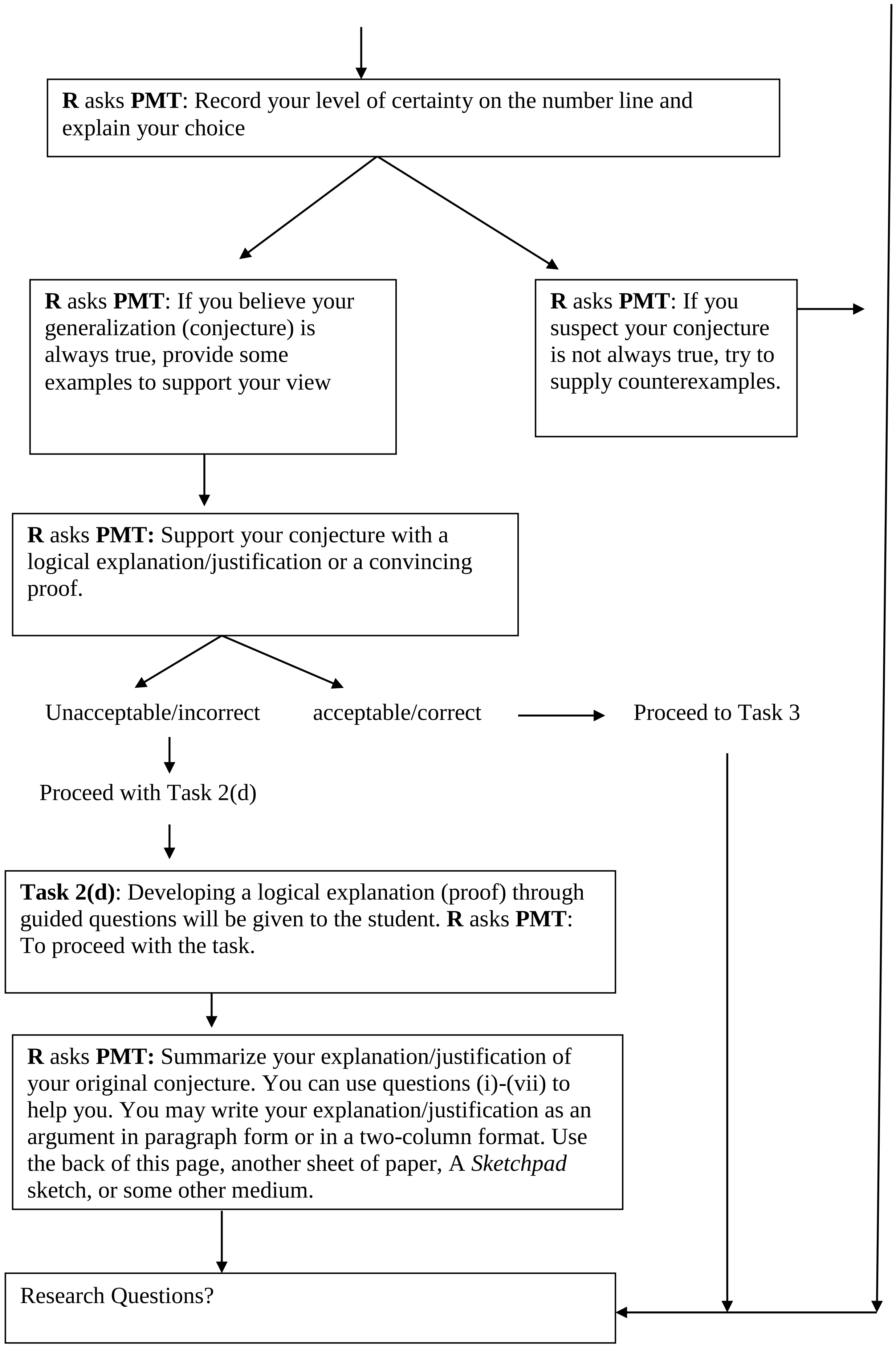
Proceeds to Task 2(b)

A sheet containing Task 2(b): Using sketchpad to develop a conjecture through generalization

After exploration on Sketchpad, **R** asks **PMT** to write their discoveries so far as one more conjectures, using complete sentences.

Proceed to Task (2c)

**R** asks **PMT**: How certain are you that your conjecture is always true?


**R** asks **PMT**: Record your level of certainty on the number line and explain your choice
**R** asks **PMT**: If you believe your generalization (conjecture) is always true, provide some examples to support your view
**R** asks **PMT**: If you suspect your conjecture is not always true, try to supply counterexamples.
**R** asks **PMT:** Support your conjecture with a logical explanation/justification or a convincing proof.
Unacceptable/incorrect
acceptable/correct
Proceed to Task 3
Proceed with Task 2(d)
**Task 2(d)**: Developing a logical explanation (proof) through guided questions will be given to the student. **R** asks **PMT**: To proceed with the task.
**R** asks **PMT:** Summarize your explanation/justification of your original conjecture. You can use questions (i)-(vii) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, A *Sketchpad* sketch, or some other medium.
Research Questions?

## Appendix 5: Semi- structured interview schedule – for Pentagon Task

**R**: Researcher ; **PMT**= Pre-service Matheamatics Teacher; **FG** = Further Generalization

**R** asks **PMT**: To proceed with task 3(a)

**R** asks **PMT**: How would you generalize the results of the equilateral triangle and rhombus to a pentagon.

For what kinds of pentagons will your conjecture (generalization) hold true for?

What properties should a pentagon have for your conjecture (generalization) to be true?

**R** asks **PMT**: Do you want to test or confirm your conjecture(s) in Q 1 above?

Yes

[Logical grounds, but needs experimental (visual)

confirmation; only experimentally (visually)]

Proceeds Task 3(b)

A sheet containing Task 3(b): Using sketchpad to develop a conjecture through generalization

After exploration on Sketchpad, **R** asks **PMT** to write their discoveries so far as one more conjectures, using complete sentences.

Proceed to Task (3c)

No. [Logical Grounds]

Proceeds Task 3(c)

**R** asks **PMT**: How certain are you that your conjecture is always true?

**R** asks **PMT**: Record your level of certainty on the number line and explain your choice

**R** asks **PMT**: If you suspect your conjecture is not always true, try to supply counterexamples.

**R** asks **PMT**: If you are fully convinced of the truth of your conjecture (generalization), do you still have a need for explanation ?(i.e. do you want to know why it is true)?

**R** asks **PMT:** Support your conjecture with a logical explanation/justification or a convincing proof.

Unacceptable/incorrect

Acceptable/correct → Proceed to Mystery Pentagon Task 3(c): Q4 :(a),4(b)

Proceed with Task 3(d)

**Task 3(d)**: Developing a logical explanation (proof) through guided questions will be given to the student. **R asks S**: To proceed with the task.

**R** asks **PMT:** Summarize your explanation/justification of your original conjecture. You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, A *Sketchpad* sketch, or some other medium.

Proceed to Mystery Pentagon: Task 3(c): Q4(a) & (4(b)

**R** asks **PMT**: Do you think the result might be true for other kinds of pentagons? (ie. For what other kinds of pentagons the above result might be true?)

No

YES

**R** asks **PMT**: Why?

Do you want to test or confirm your response to Q 4 (a) in Task 3?

Yes

No

R asks PMT: To open Mystery Pentagon.gsp and investigate whether the sum of the distances from an interior point of the mystery pentagon to its sides remain constant.

**R asks PMT:** Does this confirms or refutes your response to Q. 4(a)]

No

Yes

Investigates further

**R** asks **PMT**: Does your generalization explained in Q3(b) or made in Q1 holds true or not for the Mystery pentagon)

**R** asks **PMT**: What special property must a pentagon have so that it can yield the following result: The sum of the distances from an interior point in the pentagon to it sides remain constant?

Cannot identify property!

STOP!

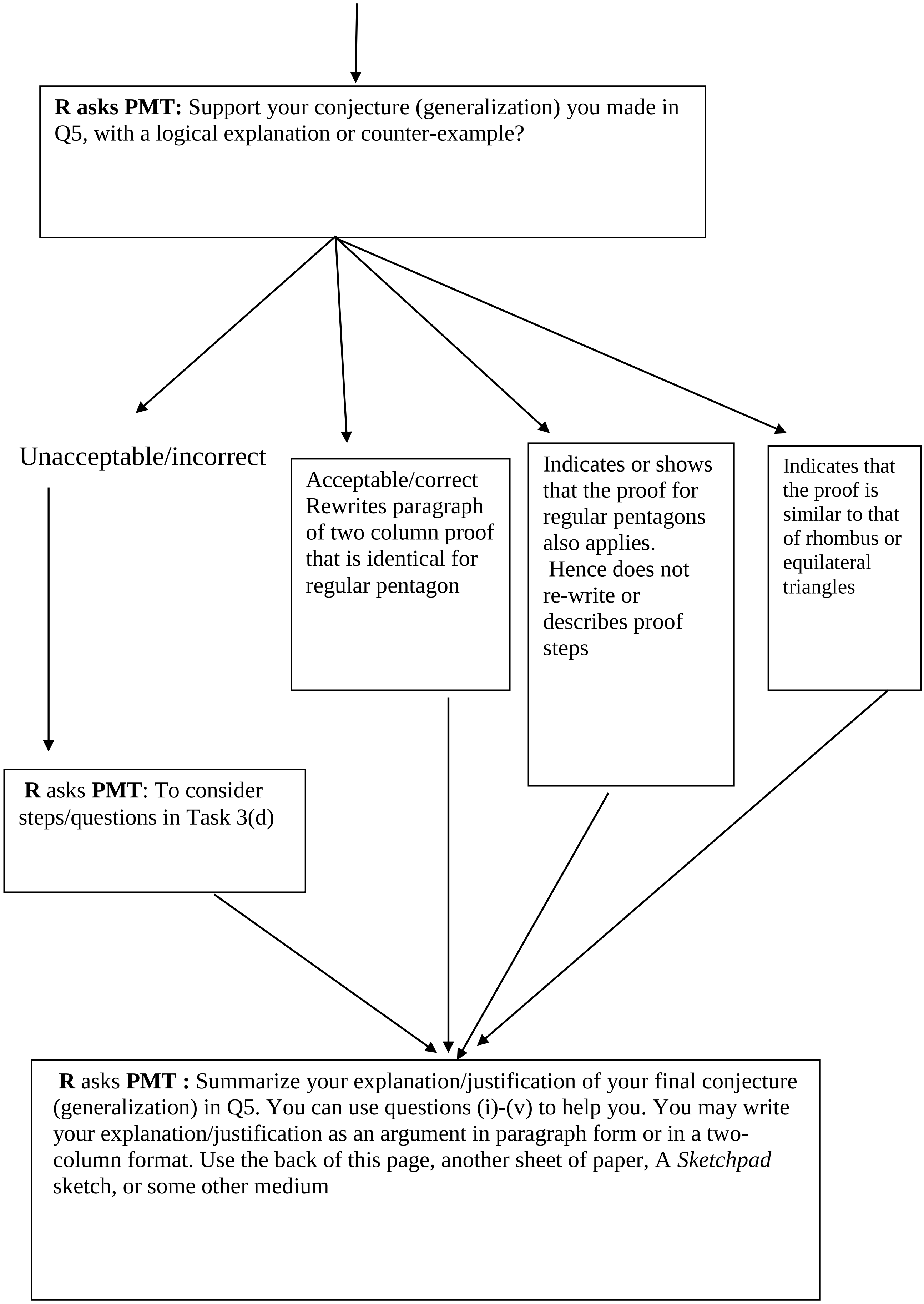
Identifies property- Proceeds

**R** asks **PMT**: Consider your response to Q4(e) and if necessary edit or modify or rephrase your initial pentagon conjecture (generalization) that you explained in Q3(b) or made earlier.

Does not modify

STOP!

Proceed if modifies?


**R asks PMT:** Support your conjecture (generalization) you made in Q5, with a logical explanation or counter-example?
Unacceptable/incorrect
Acceptable/correct Rewrites paragraph of two column proof that is identical for regular pentagon
Indicates or shows that the proof for regular pentagons also applies. Hence does not re-write or describes proof steps
Indicates that the proof is similar to that of rhombus or equilateral triangles
**R** asks **PMT**: To consider steps/questions in Task 3(d)
**R** asks **PMT :** Summarize your explanation/justification of your final conjecture (generalization) in Q5. You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, A *Sketchpad* sketch, or some other medium

## Appendix 6: Semi- structured interview schedule for $n$-Equi-sided Polygon Task

**R**: Researcher ; **PMT**= Pre-service Matheamatics Teacher; **FG** = Further Generalization

**R** asks **PMT** :

**R** asks **PMT**: How would you generalize the results of the equilateral triangle ; rhombus and equi-sided pentaggon  to polygons with a similar property

Unacceptable/Incorrect

Acceptable/ correct

R discusses the  3 specific generalizations  with PMT

Unacceptable/Incorrect

Acceptable/ correct

Proceeds Task  4(b)

Stop

**R** asks **PMT**: How certain are you that your conjecture (generalization) is always true?

**R** asks **PMT**: Record your level of certainty on the number line and explain your choice

**R** asks **PMT**: If you suspect your conjecture-generalizatiom is not always true, try to supply a- counter-example.

**R** asks **PMT**: If you are fully convinced of the truth of your conjecture-generalization, do you still have a need for explanation? (i.e. do you want to know why it is true?)

**R** asks **PMT:** Support your conjecture with a logical explanation/justification or a convincing proof.

Unacceptable/incorrect Acceptable/correct → Proceed to Task 4 (c) vi

Proceed with Task 4(c) i-v

Developing a logical explaination (proof) through guided Questions will be given to the student. R asks PMT: To Proceed with the task.

Proceed to Task 4(c) vi

R asks PMT Summarize your explanation/justification of your conjecture (generalization). You can use questions (i)-(v) to help you. You may write your explanation/justification as an argument in paragraph form or in a two-column format. Use the back of this page, another sheet of paper, a Sketchpad sketch, or some other medium.

STOP

# Appendix 7: Ethical Clearance Certificate from UKZN

UNIVERSITY OF
KWAZULU-NATAL

RESEARCH OFFICE (GOVAN MBEKI CENTRE)
WESTVILLE CAMPUS
TELEPHONE NO.: 031 – 2603587
EMAIL : ximbap@ukzn.ac.za

15 OCTOBER 2009

**MR. R GOVENDER (8217781)**
**SCIENCE, MATHEMATICS & TECHNOLOGY EDUCATION**

Dear Mr. Govender

**EXPEDITED APPLICATION**
**ETHICAL CLEARANCE APPROVAL NUMBER: HSS/0429/0**

I wish to inform you that your application for ethical clearance has been granted full approval for the following project:

> **"A dynamic interactive guided approach to the construction and validation of some geometrical generalizations"**

**PLEASE NOTE: Research data should be securely stored in the school/department for a period of 5 years**

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

........................................................................

**PROFESSOR STEVEN COLLINGS (CHAIR)**
**HUMANITIES & SOCIAL SCIENCES ETHICS COMMITTEE**

cc. Supervisor (Prof. M de Villiers)
cc. Ms. R Govender

UNIVERSITY OF
KWAZULU-NATAL

RESEARCH OFFICE (GOVAN MBEKI CENTRE)
WESTVILLE CAMPUS
TELEPHONE NO.: 031 – 2603587
EMAIL : ximbap@ukzn.ac.za

21 August 2009

MR. R GOVENDER (8217781)
SCIENCE, MATHEMATICS AND TECHNOLOGY EDUCATION

Dear Mr. Govender

**ETHICAL CLEARANCE APPROVAL**

I wish to confirm that ethical clearance has been approved for the following project:

***"A Dynamic interactive guided approach to the construction and validation of some geometrical generalization"***

**PLEASE NOTE: Research data should be securely stored in the school/department for a period of 5 years**

Yours faithfully

..................................................................................

**MS. PHUMELELE XIMBA**
**ADMINISTRATOR**
**HUMANITIES AND SOCIAL SCIENCES ETHICS COMMITTEE**

cc. Supervisor (Prof. M de Villiers)
cc. Mr. D Buchler

# Appendix 8: Approval letter from UWC, where the participants studied

FROM (TUE)APR 21 2009 9:47/ST. 9:47/No.7517367183 P 1

**OFFICE OF THE DEAN**
**DEPARTMENT OF RESEARCH DEVELOPMENT**

Private Bag X17, Bellville 7535
South Africa
Telegraph: UNIBELL
Telephone: +27 21 959-2948/2949
Fax: +27 21 959-3170
Website: www.uwc.ac.za

20 April 2009

**To Whom It May Concern**

I hereby certify that the Senate Research Committee of the University of the Western Cape has approved the methodology and the ethics of the following research project by: Mr. R Govender (Education)

| | |
|---|---|
| Research Project: | A dynamic interactive guided approach to the construction and validation of some geometrical generalizations. |
| Registration no: | 09/4/1 |

*Peter Syster*
*Research Development*
*University of the Western Cape*

**UNIVERSITY** *of the* **WESTERN CAPE**
A place of quality, a place to grow, from hope to action through knowledge

## Appendix 9: Informed consent for participants

**From: Rajendran Govender**

**Lecturer – Mathematics Education**

**School of Science and Mathematics Eduation**

**Faculty of Education**

**University of Western Cape, Modderdam Road, Bellville, 7535**

**2 April 2009**

**TO: B.Ed/PGCE Student**

**Name: ______________________**

**Student Number: _______________________**

**RE: INFORMED CONSENT TO PARTICPATE IN MY PhD RESEARCH PROJECT**

**Dear Student,**

The purpose of writing letter to you, is to seek your consent to participate in my PhD research project, which will be registered with the University of Kwa-Zulu Natal. Kindly read the information below, which I have explained to you in some detail. If you do consider to participate in my PhD research project, in terms of the information given below, then kindly complete and sign the declaration, which is found at the end of this letter.

**Below is the information about my research project and ethical aspects:**

**Project Title:** A dynamic interactive guided approach to the construction and validity of some geometric generalizations

**Purpose of Study:**

The purpose of the study is twofold:

Firstly, to investigate how to actively engage pre-service mathematics teachers in the process of generalizing through the use of selected activities and to encourage them to evaluate the validity of the generalizations through proof and experimentation. The aim is to arrive at a descriptive analysis of a "guided approach" to both the development and proof of generalizations, which teachers could use as models for their own attempts in mathematics classrooms.

Secondly, the purpose is to investigate from a conceptual point of view, the level of cognitive unity (continuity/ discontinuity) that exists between the production of a generalization and the construction of its proof, by pre-service mathematics teachers.

**The name, affiliations and contact details of the researcher with qualifications**:

Name of Researcher: Mr. Rajendran Govender

Occupation: Lecturer – Mathematics Education, School of Science and Mathematics Education.

Qualification: UDE (UDW); BSc.(UNISA); BSc. Honours – Mathematics (UDW); M.Ed (UDW).

Current PhD Study is being done at the University of Kwa- Zulu Natal, Faculty of Education.

Contact details: rgovender@uwc.ac.za; 0824513648 (cell); 021 -9592248 (office).

**Name, contact address or telephone number of an independent person whom potential subjects/participants may contact for further information:**

Project Supervisor: Prof. Michael de Villiers

Contact details: Prof Michael de Villiers
Mathematics Education (Edgewood Campus)
University of KwaZulu-Natal
Private Bag X03
3605 ASHWOOD, South Africa
Tel: 027-(0)31-2607252 (w): 027-(0)31-7083709 (h)
Fax: 027-(0)31-2603697 (w): Cell: 0836561396
http://mysite.mweb.co.za/residents/profmd/homepage.html

**You have been identified as a participant because:**

- You are a pre-service mathematics teacher
- If you are doing the B.Ed Program you will be doing Method of Mathematics 301 in your third year of your B.Ed program or Method of Mathematics 401 and

Mathematics Education 121 in your second semester of your final year ($4^{th}$ year) of study with respect to the B.Ed program. OR if you are doing the PGCE program, you will be doing Method of Mathemtics 411 & 412.

- Mathematics Education 121 – deals with the use of ICT in the teaching and learning of Mathematics; and Method of Mathematics 401 and 301 (Or 411 & 412) deals precisely design of learning materials, learning programmes and lesson plans as well as the use of the investigative approach to generalize and develop conjectures and validate them through experimentation and proof. The PGCE method modules also focusses on the use of ICT in Mathematics.

**You will participate in this research study as follows:**

You will participate in task-based activities using worksheets in an interactive dynamic geometric context, which uses a dynamic interactive guided approach to construct and validate some geometric some geometric generalizations. Throughout the task-base activities you will work on your own using the *Sketchpad* program and the worksheets. The task-based activities will take place in two separate sessions, and each session will be a different day. The first session will consist of the development of a generalization linked to the Equilateral triangle (Viviani's Theorem) and the development of proof for the established generalization. The second session will consist of the development of a further generalization to another polygon (for example a quadrilateral) and its corresponding proof. As you conduct each activity, a one-to-one task based interview will take place. The one-to-one task based interview will be video recorded by another colleague who will also serve as an observer. Observational notes will be recorded by the observer. Also, I as a researcher will record observational notes in my diary after the one-to –one interviews a well as during the time when you are working on the tasks independently (when you are not being interviewed). Another video recorder will be focused entirely on your computer screen for the entire duration of each session to record all your action moves during your generalizing phase and proof phase (session 1), further generalizing phase (session2). You will be working in a relaxed and comfortable environment with no undue stress or unfair demands.

As indicated above, you will be involved in two task-based activity sessions. Each session will be approximately and hour each per day over 2 separate days. On Day 1, you will develop a generalization and develop its proof using guided worksheets. On day 2, you will develop a further generalization and its corresponding proof.

**Potential benefits:**

Your participation in this study will provide an opportunity for you to develop the necessary pedagogical knowledge and skills to model the teaching of generalizations and their corresponding proofs in dynamic geometric context, using a dynamic interactive guided approach in a school environment. Also, you will become more competent, informed and versatile of how to use software like Sketchpad, to develop learners reasoning and proof skills.

**Payments or Reimbursements or Financial Expenses:**

You will not be paid for participation in the study.  However, you will be reimbursed for any travelling expenses to and from the University of Western Cape, for the specific days that you participate in the task-based activities one-to-one task based interviews. You will provided with lunch on the specific days that you participate in the task-based activities and one-to-one task based interviews.

**Use of any written, audio or video recordings made**

The audio and video recordings will be transcribed by the researcher. This together with the observational notes and completed worksheets (documents), will be used to identify patterns by grouping similar responses into categories, which is a way of organizing the data. Selected responses during the interviews as well from documents (like the worksheets) and associated observations will be inserted in the research report to substantiate and back specific and relevant claims that the study posits.  This data will represented as:

- tables
- selected quotations. e.g. powerful, representative or illustrative direct statements from responses to a question in an interview.
- case boxes

**How and when the gathered data will be disposed:**

The observational notes (diary) and documents (like worksheets) and video recordings and audio-tapes will be stored in my locker in my office for a period of five years. Thereafter the observational notes (diary) as well documents like worksheets will be shredded and disposed to the waste centre. The audiotapes and video recordings will be incinerated and disposed to the waste centre.

**Confidentiality or Anonymity**

Your anonymity will be respected and thus you will not be referred by your actual name in the research report or any other forms of communication. All your deliberations during the interviews as well as written documents (like worksheets) completed during the activity sessions, as well observational notes, video recordings and audio tapes, will be treated with utmost confidentially at all times.

**Participation in this study**

Furthermore, you will not be disadvantaged in anyway if you decide not to participate in this research study. Your participation in this study is absolutely voluntary and you are free to withdraw from the study at any stage and for any reason.

**DECLARATION**

[**A dynamic interactive guided approach to the construction and validity of some geometric generalizations** – PhD Study]

I………………………………………………………………………… (Full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT DATE

## Appendix 10: Approval of Change in Title of PhD Study

UNIVERSITY OF
KWAZULU-NATAL

08 November 2010

Mr R Govender
Po Box 7208
Welgemoed
9538

Dear Mr Govender

**Change in Title: Doctor of Philosophy Thesis**

The Faculty Higher Degrees committee at it meeting held on 01 November 2010 approved your request to change your title.

**From:**
*A dynamic interactive guided approach to the construction and validation of some geometrical generalizations.*

**To:**
*Constructions and justifications of a generalization of Viviani's Theorem*

Thank you,

Nomsa Ndlovu
**Postgraduate Studies & Research**

cc: Professor M de Villiers, SSMTE

**Faculty of Education**
**Deputy Dean (Postgraduate Studies and Research)**

**Postal Address:** Private Bag X03, Ashwood 3605, South Africa

**Telephone:** +27 (0)31 260 1169 **Facsimile:** +27 (0)31 260 7594 **Email:** bhanad1@ukzn.ac.za **Website:** www.ukzn.ac.za

**Founding Campuses:** Edgewood Howard College Medical School Pietermaritzburg Westville

## Appendix 11: Notice of Intention to Submit Dissertation for Examination

UNIVERSITY OF
KWAZULU-NATAL
INYUVESI
YAKWAZULU-NATALI

10 September 2012

Professor M de Villiers
School of Education
Edgewood Campus

Dear Professor M de Villiers

**INTENTION TO SUBMIT OR RESUBMIT- PhD**

At the meeting of School Research and Higher Degrees Committee on 10 September 2012 the committee noted that Rajendran Govender – 8217781 PhD served notice of intention to submit their dissertation for examination

**Title:** Constructions and Justifications of a Generalization of Viviani's Theorem

Thank you

Ntoh Mthethwa
**Postgraduate Administrator**

School of Education
Postal Address: Private Bag X03, Ashwood, 3605, South Africa
Telephone: +27 (0) 31 260 3663 Facsimile: +27 (0)31 260 3600 Email: education@ukzn.ac.za Website: www.ukzn.ac.za

1910 - 2010
100 YEARS OF ACADEMIC EXCELLENCE

Founding Campuses: Edgewood Howard College Medical School Pietermaritzburg Westville

# Appendix 12: Letters from language editors

## 12.1 Letter from First Language Editor

**TO WHOM IT MAY CONCERN**

This is to record that I have edited the dissertation by Rajendran Govender, entitled

**CONSTRUCTIONS AND JUSTIFICATIONS OF A GENERALIZATION OF VIVIANI'S THEOREM.**

M Heerden

**Monica M.A van Heerden, D.Ed.**

**HERMANUS**
**12 December 2012**

## 12.2 Letter from Second Language Editor

**University of the Western Cape**

**Private Bag X17 Bellville 7535 South Africa**
**Tel. 021-9592 442/2650**
**Fax: 021-959 3358**

***Faculty of Education***

March 11, 2013

**<u>TO WHOM IT MAY CONCERN</u>**

This is to state that I have edited the doctoral dissertation of Rajendran Govender entitled, "**Constructions and Justifications of a Generalization of Viviani's Theorem**". Furthermore, I wish to attest that all typographical errors as cited in the examiner reports have been attended to my full satisfaction.

I will be much obliged if you could take official cognizance of this letter.

Sincerely

S. Sivakumar

Dr. Sivakumar Sivasubramaniam
Professor and Head
Department of Language Education
Room E 71
Faculty of Education
University of the Western Cape
Private Bag X17
Bellville 7535
Western Cape
Republic of South Africa
Tel; 27-021-959-2449/2442
Email: ssivasubrmaniam@uwc.ac.za or sivakumar49@yahoo.com

# Appendix 13: Turnitin Certifcate



**Turnitin Originality Report**

Constructions and Justifications of a Generalization of Viviani's Theorem by Rajendran Govender

From Constructions and Justifcations of a Generalization of Viviani's Theorem (Rajen 2012)

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