EXPLORING THE DISCOURSES OF PRESERVICE MATHEMATICS TEACHERS WHEN SOLVING GEOMETRY PROBLEMS

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- My family, for their unending support and motivation during the difficult times.
- I am grateful for not giving up on the work I have reported in this thesis Even though the COVID-19 pandemic caused difficulties during the data collection stage, I was persistent enough to pursue the completion of this degree in record time.
- Dedicated to my son, Luphiwe Lutsandvo Mahlaba, continue to inspire my energy.

ABSTRACT

Research on teaching and learning that aim to improve preservice mathematics teachers' (PMTs) knowledge of geometry is increasing globally. The current study explored PMTs' discourses when solving geometry problems. The amalgamation of the commognitive theory and the Van Hiele levels of geometrical thinking theory was used as the theoretical basis for this study. The study uses the difference between ritualistic and explorative discourse as explicated by Sfard (2008) in commognition together with the four Van Hiele levels of geometrical thinking to view and analyse the data. It goes deeper into the theory of commognition to use not only objectification of mathematical discourse but also the four elements of mathematical discourse to reach its conclusion. The current study aimed to answer the main question: How does preservice mathematics teachers' thinking as evident in their mathematical discourse during Euclidean geometry problem solving relate to their teaching practices in Euclidean geometry? This will be done through answering four subsidiary questions. A qualitative research approach was used to generate rich and descriptive data to answer the posed research questions. Furthermore, the qualitative approach allowed for the collection of data representing participants' geometry problem solving experiences which was the core of the current study. I purposively and conveniently sampled 6 participants in this study where they completed a task-based and face-to-face interviews. Consent was obtained from these participants prior their participation in the study. Data generated from the two instruments was thematically analysed. Findings from this study revealed that most PMTs use ritualistic discourse when communicating about their geometry problem solving actions. These findings are a consequence of them performing routines for social acceptance instead of generating endorsed narratives. Furthermore, it was observed that others used ritualistic discourse because they rely on scaffolding from others to perform their routines instead of developing their own routines. Despite the dominance of ritualistic discourse participation in the current study, there were instances where PMTs seemed to be using explorative discourse but get stuck somehow and return to ritualistic discourse. The Van Hiele theory revealed that most PMTs still operate within the lower levels of geometrical thinking. The main findings and contribution of this study is that for PMTs to advance their geometrical thinking from level 0 to level 3, they need to transform their discourse participation from ritualistic to explorative.

Keywords: Mathematical discourse; Euclidean geometry; Problem solving; Van Hiele theory; Ritualistic discourse; Explorative discourse; Commognition; Mathematical thinking.

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LIST OF ABBREVIATION AND ACRONYMS

4IR	–	Fourth Industrial Revolution
Bed	–	Bachelor of Education
C2005	–	Curriculum 2005
CAPS	–	Curriculum and Assessment Policy Statement
COVID-19	–	Coronavirus (first detected in Wuhan, China, in 2019)
DBE	–	Department of Basic Education
DHET	–	Department of Higher Education and Training
DOTS	–	difference of two squares
EEF	–	Education Endowment Foundation
FET	–	Further Education and Training
HEI	–	higher education institution
HSSREC	–	Human Social Sciences Research Ethics Committee
LoTL	–	language of teaching and learning
MKT	–	mathematical knowledge for teaching
NCS	–	National Curriculum Statement
NCTM	–	National Council for Teachers of Mathematics
OBE	–	outcomes-based education
PBL	–	problem-based learning
PGCE	–	Postgraduate Certificate in Education
PIRLS	–	Progress in International Reading Literacy Study
PMT	–	preservice mathematics teacher
RME	–	Realistic Mathematics Education
RNCS	–	Revised National Curriculum and Statement
SLANT	–	spoken language and new technology
STEM	–	Science, Technology, Engineering and Mathematics
TIMSS	–	Trends in Mathematics and Science Study
TMUF	–	Teaching Mathematics for Understanding Framework
WEF	–	World Economic Forum
ZPD	–	zone of proximal development

CHAPTER 1:
BACKGROUND, PROBLEM STATEMENT AND AIMS OF THE STUDY

1.1 INTRODUCTION

Teaching and learning are two of the most complex tasks, particularly in the current era, with drastic technological innovations affecting the education sector and South African (SA) learners' violent behaviour. South African mathematics education provides a rich context to investigate these complex tasks in order to position its teaching and learning within the global standards. Currently, global teaching and learning strategies are being refined to blend with digital transformation theories with the aim of developing twenty-first-century skills in learners. This digital shift describes a world in which people transition from digital to offline-reality interconnected technologies to make their lives easier to live and manage (Xu, David, & Kim, 2018). This was described by Schwab (2016, p. 7) as the Fourth Industrial Revolution (4IR). The implications of the 4IR for the education field cannot be ignored, since the skills that are important in the modern world have changed, which means that teaching and learning must be adapted to enable learners to compete in this digital world. Effective training of teachers who have to prepare learners in these skills is highly desirable in addressing these needs.

Recent changes in mathematics education are inspired by educational movements that are based on the tenets of the sociocultural theory. Freudenthal (1973) is encouraging in this regard, because his principles see learning mathematics as an activity, which shifts the task of learning to the learner. This means that teachers nowadays should be trained in such a way that they are able to facilitate classroom discourse effectively, where all learners are given equal opportunities to learn. Although this move to aligns mathematics education with recent research and reformed teaching and learning strategies based on the principles of the sociocultural theory that is more suited for the twenty-first century has been mandated in SA education (Department of Basic Education [DBE], 2019a), the challenge of poor learner performance persists. The lack of effectively planned and implemented professional development programmes for in-service teachers and the insufficient training of preservice teachers still hamper developments in mathematics education and improvement of learners' performance in the subject. In line with the sociocultural theory, recent research has signified teachers noticing of learners' thinking during teaching as an important strategy to advance learners' mathematical thinking and support their development in the subject (Choy & Dindyal, 2019; Jacobs, Lamb, & Philipp, 2010; Walkoe, 2015; Walkoe, Sherin, & Elby, 2019). The activity of teacher noticing, as described by Jacobs, Lamb and Phillip (2010) has been proposed because it gives teachers an opportunity to tap into their learners' thinking with the aim of spotting learner misconceptions, which will enable the teachers to design teaching methods to support learner development in the subject. My study was situated within this scope of 'noticing'. However, I aimed to notice preservice mathematics teachers' (PMTs) thinking during geometry problem solving, instead of learners' thinking. In this chapter, I provide the rationale and overview of the study by highlighting the problem of poor learner performance in SA school mathematics, geometry in particular, and some anecdotal experiences from teaching geometry. This is done not to assign blame to learners that they are performing poorly in geometry, but to situate the statement of the problem for the study on poor performance in mathematics. Through a brief review of literature, I provide a glimpse of research studies investigating geometry knowledge of PMTs in different SA contexts. I further explore the curriculum and assessment policy statement (CAPS) for Grade 12 geometry, and conclude by summing up all these discussions to provide a thick rationale for the current study.

1.2 BACKGROUND TO THE PROBLEM

In 1.2, I consider the background of the problem in this study by contextualising the study in South Africa. Specifically, I look at:

- developments in mathematics education;
- geometry knowledge of teachers and learners;
- issues of language in teaching and learning mathematics;
- how mathematics is taught;
- teacher training; and
- the recent developments in thinking about mathematics teaching and learning.

I conclude this section by summing up these ideas and giving a prelude to the contextual gap that this study addressed.

1.2.1 Developments within the South African mathematics education curriculum

Since the start of post-apartheid education, South Africa has undergone a plethora of policy and curriculum changes that have left most teachers confused about their roles in the classroom

(Chisholm, Motala, & Vally, 2003; Christie, 1990; Jansen, 1997; Parker, 2006). These extensive curricula changes were aimed at addressing the injustices brought to education by the apartheid government. Outcomes-based education (OBE) was established after 1994, and focused the process of teaching and learning on attainable pre-set outcomes (see Jansen, 1998). Learners memorised and regurgitated mathematical knowledge to show that they had achieved certain learning outcomes. However, OBE was seen as influenced by political agendas of the apartheid regime (Jansen, 1997; Maodzwa-Taruvinga & Cross, 2012). Thus, in 1997, the introduction of Curriculum 2005 (C2005) was announced with the aim of positioning learners as ‘active’ in the learning process and teachers as ‘facilitators’ of learning (Pudi, 2006). However, there were challenges with C2005 as well, and it was revised as the Revised National Curriculum Statement (RNCS). The RNCS challenged teachers in its implementation, because they did not have a full grasp of the assessment standards and they could not link the curriculum theory with classroom practice (see Pudi, 2006). The RNCS was later reviewed to form the National Curriculum Statement (NCS), which saw geometry, statistics and probability content in mathematics being deemed optional and written in an optional examination in Paper 3. These changes also had negative repercussions for mathematics education, as most students going to university or into teaching had very limited knowledge and understanding of these content areas. The review of the NCS saw the formation of the CAPS, which was introduced in 2012. In the CAPS, the content areas deemed optional in NCS were brought back to the compulsory curriculum, phased in from Grades 10 to 12. These further created problems for the mathematics education community, as most teachers were not fully trained in these topics (Luneta, 2014; Van Putten, Howie, & Stols, 2010).

The above-mentioned changes to the curriculum provided a background to and perhaps a rationale for the difficulties experienced in teaching and learning mathematics that reflected learner incompetency compared to global learners of similar grades.

1.2.2 The state of geometry knowledge of teachers and learners in South Africa

The CAPS was phased in from Grade 10 in 2012 up until Grade 12 in 2014. Since the introduction of CAPS, the Grade 12 mathematics pass percentage has remained below 60% (see DBE, 2020). Figure 1.1 shows the distribution of the pass percentage since 2016, and it is clear that, even though the percentage required to pass mathematics is 30%, a considerable number of learners (just below half) still fail to obtain the required 30% (DBE, 2016, 2017, 2018, 2019b). This performance is also manifested in the Trends in Mathematics and Science

Study (TIMSS) where SA Grade 9 learners achieved an average of 372, which put them as the second last (38 out of 39) country in mathematics achievement (see Reddy, et al., 2016). This limits many SA learners to pursue careers in science, technology, engineering and mathematics (STEM) in SA and other global universities. Consequently, South Africa becomes passive in STEM-related innovations, always borrowing ideas from other countries, which means the aim of CAPS, namely to produce a learner who will participate in innovation through the use of STEM is limited.

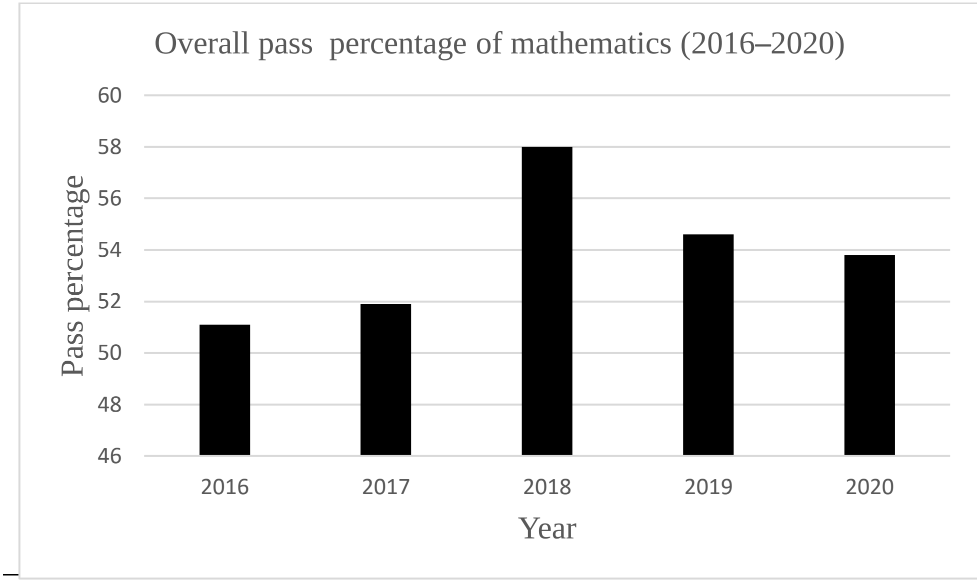
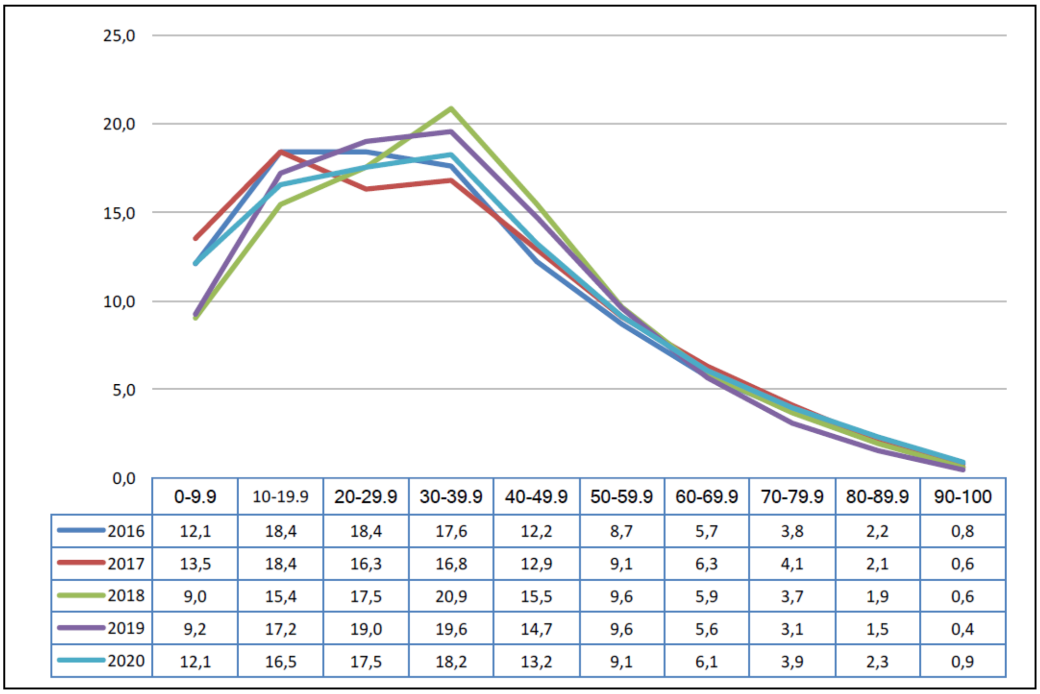


Figure 1.1: Average pass percentage for mathematics (2016–2019)

Source: DBE (2016, 2017, 2018, 2019b, 2020).

Figure 1.2 summarises SA learners' performance according to the different performance levels set out by CAPS since 2016. As seen in Figure 1.2, most learners who pass mathematics in South Africa are concentrated within the 30–39,9% threshold, while most of them fail the subject completely (0-29,9%). It is clear from Figure 1.2 that SA learners' mathematics performance reflects poor quality as most learners who pass mathematics only meet the minimum requirement. Can the problem be that SA learners are ignorant of the subject, demotivated to put effort in learning mathematics or are they inadequately taught in mathematics? Answers to these questions, though complex, can be explored through different types of research studies. The current study gravitated towards the latter part of this complex question, understanding teachers' thinking when solving geometry problems and the implications of this thinking for their practice.



	0-9.9	10-19.9	20-29.9	30-39.9	40-49.9	50-59.9	60-69.9	70-79.9	80-89.9	90-100
2016	12,1	18,4	18,4	17,6	12,2	8,7	5,7	3,8	2,2	0,8
2017	13,5	18,4	16,3	16,8	12,9	9,1	6,3	4,1	2,1	0,6
2018	9,0	15,4	17,5	20,9	15,5	9,6	5,9	3,7	1,9	0,6
2019	9,2	17,2	19,0	19,6	14,7	9,6	5,6	3,1	1,5	0,4
2020	12,1	16,5	17,5	18,2	13,2	9,1	6,1	3,9	2,3	0,9

Figure 1.2: Performance distribution curves in mathematics (percentage) (2016-2020)
Source: DBE (2020).

Focusing mainly on learner performance in geometry tasks in the 2020 NCS examinations, we can see from Figure 1.3 that on average learners obtained less than 46.3%. In his analysis of 1000 Grade 12 learners' scripts from 2008, Luneta (2015) finds that most learners were operating at Van Hiele level 2, instead of level 3, which reiterates Van Putten et al.'s (2010) conclusion that South Africa is not well prepared to include Euclidean geometry in the compulsory curriculum. This non-readiness was also observed by Alex and Mammen (2012, 2014), when they investigated Grade 10 learners' readiness to deal with Euclidean geometry content. In both studies they report that most Grade 10 learners in their sample were operating at the lowest Van Hiele level (level 0), which was of a great concern (Alex & Mammen, 2012, 2014). Recently, Naidoo and Kapofu (2020) also conclude that Grade 11 female learners possessed both positive and negative perceptions of learning geometry and these authors suggest that to improve this situation, teachers need to emancipate learners from their negative perceptions.

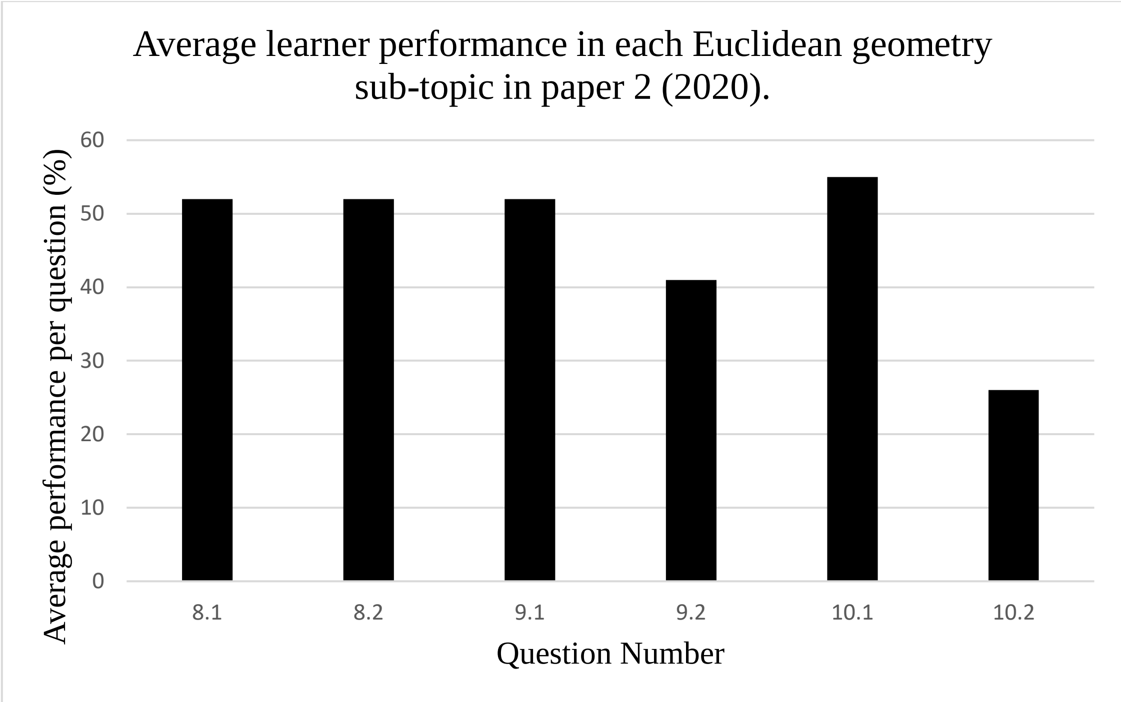


Figure 1.3: Geometry performance in the 2020 NCS examination
Source: DBE (2020).

It is because of these poor performances observed from learners, that the current study was conducted. Furthermore, there has been research evidence suggesting that SA mathematics teachers are not ready to teach Euclidean geometry. In their study, Van der Sandt and Nieuwoudt (2003) conclude that both Grade 7 in-service and preservice teachers possessed very poor geometrical knowledge to be able to teach effectively. In their later study, they find that university training does not improve teachers' geometry thinking, as teachers in their study entered university with a higher Van Hiele level of geometry thinking than when they exited (Van der Sandt & Nieuwoudt, 2005). Recently, Ubah, and Bansilal (2019) provided evidence that preservice teachers were unable to identify the similarity of triangles. Seeing that from the CAPS, similarity is a concept dealt with in Grades 7–9 and later revisited at Grade 12, these teachers were not ready to teach Euclidean geometry at these levels. In Table 1.1, I provide a glimpse of the current literature evidence from different SA contexts revealing preservice and in-service teachers' understanding and readiness to deal with Euclidean geometry content. The five studies reviewed in Table 1.1 reveal the severe nature of poor geometry content knowledge from SA in-service and preservice teachers, which is revealed in their ritualistic nature of geometry problem solving as observed in some of these studies (see Dhlamini, 2012; Ubah & Bansilal, 2019).

Table 1.1: Brief review of literature on preservice and in-service teachers understanding of geometry in South Africa

Authors	Year	Participants	Methods	Findings
Van der Sandt and Nieuwoudt	2003	In-service and preservice teachers.	Degrees of acquisition of Van Hiele level test.	They concluded that both in-service teachers and preservice teachers do not have adequate control of the Grade 7 geometry-related content knowledge.
Van Putten et al.	2010	Preservice teachers.	A case study using pre-and-post-test in a module (*intervention study*).	Even though the intervention changed preservice teachers' attitudes towards Euclidean geometry, it did not bring sufficient improvement in their acquisition of geometry content knowledge
Dhlamini	2012	In-service teachers	Written test and task-based interviews	Teachers did not possess the required Van Hiele level of geometrical thinking and subject matter knowledge of Bloom's taxonomy.
Alex and Mammen	2018	Preservice teachers	Responded to a 60-item multiple choice questionnaire.	They found that students had a fairly good understanding of basic terminology of Euclidean geometry.
Ubah and Bansilal	2019	Preservice teachers	Written responses from a questionnaire and semi-structured interviews	Preservice teachers struggled to identify similarity relationships in triangles.

The situation of poor content knowledge of SA teachers has compelled the DBE and mathematics education researchers to explore the reasons why the situation is so dire (Venkat & Spaull, 2015). By tapping into PMTs' thinking during geometry problem solving, I take a step towards understanding how PMTs' thinking, as evident in their discourse participation, can remedy the situation in geometry. Similar to numerous previous studies in mathematics education, specifically on teachers' geometry content knowledge, the question of why teachers (including PMTs) possess such poor content knowledge of geometry, is compelling and continues to drive even the current study. Perhaps, understanding the reasons why PMTs' geometry problem-solving discourse participation is ritualistic or explorative, would help explain these poor results observed in learners when it comes to Euclidean geometry-related tasks, as these have an influence in their practice. Numerous studies, such as by Taylor, Van der Berg, and Mabogoane (2013), Spaull (2013b, 2015) and Spaull and Kotze (2015), have asserted that many of the problems of todays' education are the consequences of the injustices of the apartheid government. Spaull and Kotze (2015) have also said that due to these injustices of the past, learners fall behind in their curriculum, to such a degree that catching up becomes impossible. Could this be one of the reasons for this perennial poor performance in geometry? Within the context, Spaull (2013b) further asserts that there is a strong correlation between language and context, which might explain the underperformance of SA learners, as the language of teaching and learning (LoTL) is English, which is not the home language of a majority of learners (see also Spaull & Kotze, 2015). Could this be the reason why SA teachers fail to teach geometry effectively?

1.2.3 Language and teaching mathematics in South Africa

Language is one of the most important elements of communication. Bakhtin (1981) asserts that language is principal in the lives of human beings: it transforms reality, connects human beings, shapes our experiences, and claims ideas and utterances. Middendorf (1992) asserts that language is the main means of thought. However, language does not only contain utterances (Vygotsky, 1986), it is a correspondence mechanism, consisting of a finite number of symbols and a collection of rules or grammar that controls the use of such symbols (Sfard, 2008). Hence, mathematical language includes signs and symbols, which we can use to make meaning. There is an direct relationship between thought and language (Sfard, 2008; Vygotsky, 1986; Wittgenstein, 2009),[1] which means that language can be used to articulate thinking during problem solving and to support learning (Swart, De Graaff, Onstenk, & Knèzic, 2018), and it mediates teaching and learning (Vygotsky, 1978). Humans learn language from multiple sources and cultures, so when language evolves, so does knowledge. Hence, language is considered an indispensable condition of knowing in which experience becomes knowledge (Halliday, 1993).

Mathematics is a language on its own, with its symbols and specific word usages that differ from colloquial discourses. In the formalisation of axiomatised mathematical theories, mathematicians choose proper language to refer to certain mathematical theories (Snapper, 1979). Some of the colloquial discourse expressions have different meanings and implications in mathematics, for example, table, function, integral, etc. Failure to master language (as written

[1] According to Vygotsky, language develops through social interactions. It requires not only exposure to words, but also an interdependent relationship between language and thought. He asserts that language is used in inner speech, which he describes as "thinking in pure meanings" (Vygotsky, 1986, p. 249). His idea of language is based on constructivism, a theory that recognises that learning occurs through social interactions where language is central. Hence, the development of thought and intellect depends mainly on language. For Ludwig Wittgenstein, language is not a fixed structure of signs or words imposed on the world, but a fluid structure of signs and symbols relating to our everyday practices and thought (Wittgenstein, 2009).

or spoken language or when reading language) in its colloquial and mathematical sense, may pose problems for mathematics learners. Thus, the stratification of language makes it a complex construct. Bakhtin (1981) acknowledges that language by its nature is stratified. He mentions that the language usages of a lawyer, doctor, businessman, politician, or teacher are different, not only in their vocabulary, but also in the way they manifest themselves in dialogue. This difficulty caused by the stratification of language manifests itself in the meaning of phrases and words. Tall (1989) gives the example that the word 'proof' means different things for a mathematician, judge, jury, statistician or a scientist.

However, mathematical learning starts and continues as a language, succeeds and stumbles as a consequence of language and learning is also measured in some form of language (Durkin & Shire, 1991), making language an instructional and epistemological platform for teaching and learning mathematics (Chronaki & Planas, 2018). Humans participate in dialogic discourses through language to create their meanings (Bakhtin, 1981, 1986). Bakhtin (1984),[2] asserts that dialogue is the very essence of being human, learning involves the entire individual concerning particular practices and social cultures. This supports the foundational idea in sociocultural theories that sees learning as being an active participant in the process of learning through language (Lave & Wegner, 1991; Vygotsky, 1978). Hence, dialogue is not only a form of discourse, but also a means of using language to gain consciousness.

South Africa has 11 official languages scattered across its nine provinces. However, the LoTL as mandated by the CAPS is English, which has disadvantaged numerous SA teachers and learners. Most teachers and learners fail to make meaning in mathematics, because English is not a home language to the majority of South Africans. Reflecting from the PIRLS 2011, Spaull (2013a, 2015) concludes that most SA learners struggle in later grades because of their lack of acquisition of the LoTL in the earlier grades. The PIRLS 2016 study reveals that South Africa was the lowest performing country in literacy compared to 50 other countries (Howie et al., 2017). This study found that 78% of SA Grade 4 learners performed below the international benchmarks in reading (Howie et al., 2017), which explains why they fall behind in other subjects presented in English. The teaching and learning of mathematics in the contexts of multilingual classrooms presents the education fraternity with plenty of challenges (Adler, 1997, 1999, 2002; Barwell, Chapsam, Nkambule, & Setati-Phakeng, 2016; Gorgorió & Planas,

[2] "The nature of human life itself, in dialogue a person participates wholly and throughout his whole life: with his eyes, lips, hands, soul, spirit, with his whole body" (Bakhtin, 1984, p. 293).

2001; Setati, 2005). Consequently, most SA classrooms use code-switching or learners' home languages as the LoTL, instead of English (Chikiwa & Schäfer, 2016; Grobler, 2018; Kretzer & Kaschula, 2019; Maluleke, 2019). However, as much as teachers draw from learners' home language to develop learners' conceptual understanding during code-switching, teachers experience some pedagogical difficulties due to the limited vocabulary of African languages for scientific concepts (Mavuru & Ramnarain, 2019). Thus, despite the research findings that conclude that if the learners' home language is given a prominent place in the classroom, they are in a strong position to make sense in the classroom (Probyn, 2019), issues of teaching through code-switching need to be handled carefully. Following Spaull and Kotze's (2015) argument that SA learners fall behind in their learning to such a degree that catching up becomes impossible, we expect that even SA PMTs will have the same language difficulties as Grade 4 learners. In fact, university students' demands for the inclusion of African languages as LoTL has continued to inundate higher education institutions (HEIs) (Mayaba, Ralarala, & Angu, 2018), which is evidence that most SA university students are still not comfortable with English as the LoTL. Findings from Nomlomo and Katiya (2018) show that learners need to be literate in both their home language and the LoTL in order to be successful in grasping the subject matter.

1.2.4 Mathematics teaching in South Africa

Mathematics teaching and learning in South Africa has been under siege for the past few years due to the poor performance observed from learners in both national and international tests. Because of the low pass rate in Grade 12 mathematics, teachers are under constant pressure from school principals and the DBE to produce good results with little or no adequate support. To make matters worse, novice mathematics teachers get little to no encouragement and advice from experienced teachers within their schools. Although support is provided by the DBE in different districts in the form of professional development programmes, the continuation of poor results from SA learners has shown that this is a futile exercise. The key concern is that the goals and values of CAPS vary considerably from what occurs in realistic circumstances at school and during classroom instruction. Similar to the National Council of Teachers of Mathematics (2000), one of the CAPS principles is to ensure that active and critical learning is encouraged in the classroom, instead of uncritical rote learning (DBE, 2011, 2019a). However, facilitating active and critical learning becomes a challenge for teachers who teach in overcrowded classrooms, like SA classrooms (West & Meier, 2020). Furthermore, CAPS aims to produce mathematics learners who can demonstrate global awareness as a collection of interconnected processes by realising that problem-solving environments do not occur in isolation (DBE, 2011). Hence, teaching and learning mathematics should concentrate on solving mathematical problems in various ways to show the interrelationship between content, context and problem-solving skills in maths education.

From my extensive anecdotal experiences as a mathematics teacher, and teacher trainer, I have observed that even nowadays, mathematics teaching and learning in South Africa is still traditional and by means of rote learning (see also Adler & Ronda, 2014, 2015; Barnard & Braund, 2016; Barwell, 2018; Chirinda & Barmby, 2018; Chisholm et al., 2000; Umugiraneza, Bansilal, & North, 2018). This may mean that recent global developments in mathematics teaching and learning have been concealed from SA mathematics education, or that research into these developments within SA mathematics education, is not well communicated to teachers, or even that most mathematics teachers are either unaware or unwilling to change in order to align their practice with the global developments. To rule out that global developments in mathematics teaching and learning have been concealed from South Africa, an attempt has been made by the DBE to align teacher practices to global standards by introducing the teaching for understanding framework (TMUF) (DBE, 2019a), but no change has been observed thus far (see Figure 1.4). The TMUF moves away from just learner-centric classrooms as proposed by CAPS, towards learning-centred classrooms where teachers design learning activities to help learners focus on learning (DBE, 2019a). Furthermore, the TMUF recognises that the current support given to SA mathematics teachers is not enough to remedy the situation of poor results and agrees that improvement is required in the teaching and learning of mathematics to remedy the situation. The TMUF recognises the importance of the Dutch realistic mathematics education (RME) approach, based on Freudenthal's (1973) ideas, and Singapore's problem-based learning approach and is wary of the contextual differences between these two countries and South Africa (DBE, 2019a). The TMUF was proposed as a teaching and learning framework designed to fit the context of South Africa.

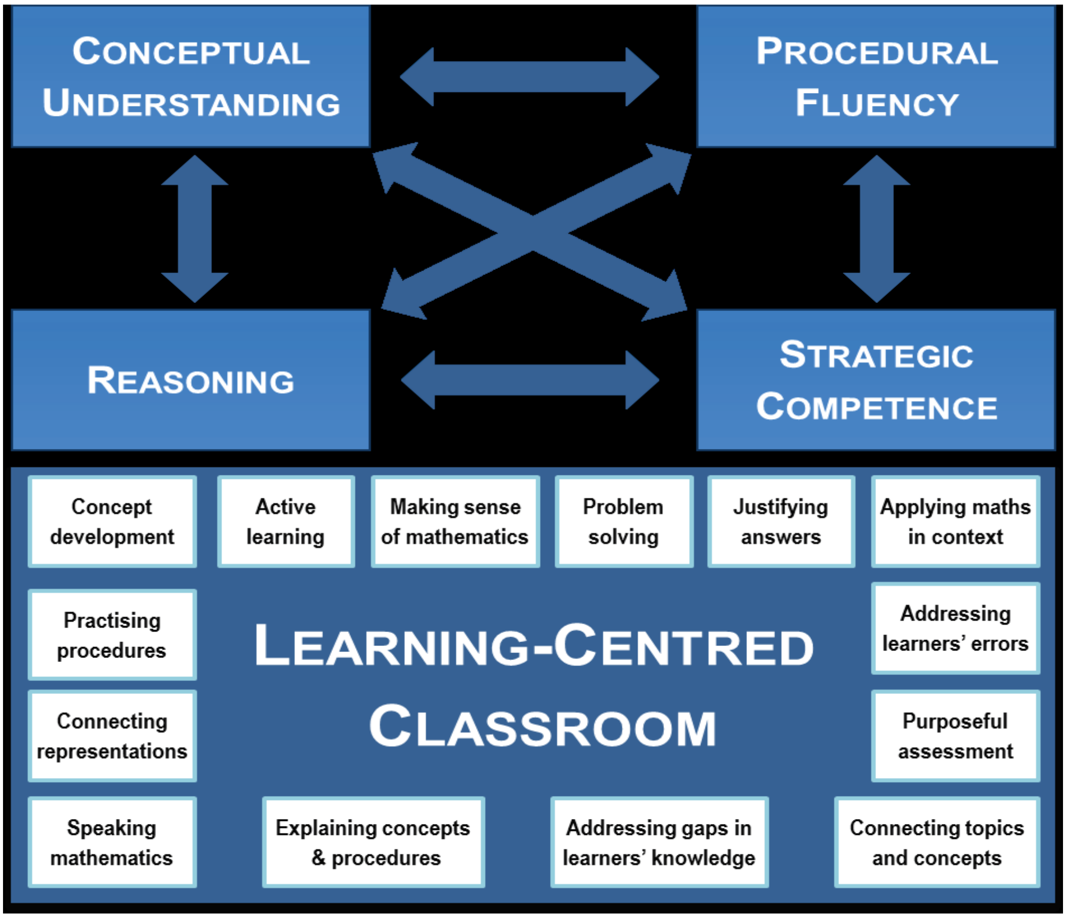


Figure 1.4: Model of teaching mathematics for understanding

Source: DBE (2019a, p. 9).

1.2.5 South African teacher's training

Although there is no doubt that teachers have a significant role to play in the provision of quality mathematics education, there is some doubt about their training to teach mathematics (Parker & Adler, 2005; Venkat & Spaull, 2015). According to the policies of the Department of Higher Education and Training, (DHET), universities, which train mathematics teachers should focus on mathematics content knowledge and classroom pedagogic practice (DHET, 2015). Shulman (1986, 1987) proposes that teachers should possess specialised knowledge for teaching a certain subject. In mathematics, this specialised knowledge for teaching includes a critical component called mathematical knowledge for teaching (MKT) (Ball & Forzani, 2009; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008), and this knowledge is linked to improved learner performance in mathematics (Hill, Rowan, & Ball, 2005). Furthermore, HEIs must give PMTs opportunities for full exposure to the classroom during their training years to obtain teaching experiences from realistic classroom environments (Diko & Feza, 2014). Successful mathematics teaching requires that teachers possess a balanced competency in mathematical content and pedagogy (Arends, Winnaar, & Mosimege, 2017). Another important skill with which mathematics teachers need to graduate, is the ability to integrate technology in their lessons, but SA teachers' ability and willingness to use the available technological artefacts to support their teaching, is controversial (Stols et al., 2015). Higher education has recently changed globally in response to the 4IR (Gleason, 2018), so complex problem solving, critical thinking, and creativity are some of the necessary skills that PMTs need to possess to teach effectively (World Economic Forum [WEF], 2018). PMTs are also expected to be self-directed lifelong learners (Knowles, 1975; WEF, 2018), able to adapt to changes such as those caused by pandemics such as COVID-19. However, it has been shown that teachers struggle to support their learners' planning for their lifelong learning (Klug, Krause, Schober, Finsterwald, & Spiel, 2014), and this can be attributed to their poor training.

Currently, in South Africa, there are two routes for becoming a mathematics teacher in any secondary school. Either one does a four-year Bachelor of Education (BEd) degree with mathematics as a major and mathematics teaching methods modules, or a Bachelor of Science degree with mathematics as a major, followed by a post-graduate certificate in education (PGCE). Both these routes have their limitations in training competent mathematics teachers for the twenty-first century. The former usually has an intake of learners who possess ritualised knowledge of mathematics, and who can perform mathematical procedures and routines without a clear understanding of their underlying principles (Adler, 2017). Furthermore, this route includes the need to balance all the content areas in the CAPS with the knowledge of teaching mathematics, and the pedagogy of mathematics, which requires more time than the allocated four years for the BEd degree. On the other hand, the latter focuses mainly on calculus modules that support the teaching of algebra. Teachers from the PGCE route graduate without being taught Euclidean geometry, but they are required to teach it. Furthermore, the PGCE route compresses the learning of mathematical pedagogy into one year, which might provide insufficient knowledge for teaching mathematics, especially Euclidean geometry. Hence, both these routes contribute to SA teachers' poor geometry knowledge for teaching, which affects learner's understanding and performance (Morrison, 2013; Pournara, Hodgen, Adler, & Pillay, 2015). Furthermore, even though most SA teacher training institutions spend much time teaching content and pedagogy (Diko & Feza, 2014), teachers still graduate with low competency in teaching mathematics effectively (Brijlal, 2014; Kaino & Moalosi, 2013), especially in teaching Euclidean geometry-related content (Adler, 2017; Luneta, 2014; Pournara et al., 2015; Van der Sandt & Nieuwoudt, 2003, 2005; Van Putten et al., 2010).

The CAPS has specific expectations for Grade 12 learners and teachers who are about to engage in the process of teaching and learning Euclidean geometry. These expectations are set based on the geometry knowledge accumulated by learners from early grades until Grade 11, and teachers' expected expertise in the subject. For example, learners in Grade 12 are expected to solve problems involving circle geometry, which is not part of the geometry curriculum for Grade 12. Though this seems like a paradox, CAPS is probably implementing one of the National Council of Teachers of Mathematics' (NCTM) properties of a curriculum, which is continuity. According to the National Council of Teachers of Mathematics (NCTM) (2000), a curriculum must be continuous. Hence, everything that learners learn in prior grades can be applied in the subsequent grades. The CAPS FET mathematics curriculum requires that Euclidean geometry teachers function at least at Van Hiele level 3, where they can perform geometrical proofs and problem-solving without any cognitive hindrance or limitation. Hence, SA mathematics PMTs are expected to graduate at Van Hiele level 3 of geometrical thinking and reasoning. However, numerous studies provide evidence that most mathematics teachers are not functioning at Van Hiele level 3, even though they are teaching mathematics at the FET phase (Van der Sandt & Nieuwoudt, 2003, 2005; Van Putten et al., 2010). Enough evidence has shown further that learners get to the FET phase with very limited knowledge and understanding of geometry (Alex & Mammen, 2014, 2016, 2018; Luneta, 2014). This evidence points to the fact that teacher-training institutions should put extra effort into preparing PMTs to teach Euclidean geometry in schools.

1.2.6 Recent developments in thinking about mathematics teaching and learning

There have been several teaching and learning theories that view the process of teaching and learning through different lenses. The behaviourists believe that mathematical learning is a result of conditioning human behaviour in a certain way (Watson, 1913). Behaviourists are against the view that human beings can create and advance their environment as they see fit (Skinner, 1971). Therefore, teaching is the training and conditioning of behaviour (Skinner, 1971), and Skinner (1974, p, 103) equates the process of thinking to "behaving weakly". However, this theory does not account for biological influences on learning, the willingness to learn and consequently does not explain all learning, hence this theory was debunked.

The cognitivists, on the other hand, focus on brain functions during learning. The cognitivists believe that learning occurs inside the human head and is not influenced by the environment (Fox, 1997). Instead of believing that behaviour influences cognitive processes, cognitivists believe that mental processes influence behaviour. Based on the acquisition metaphor (Sfard, 2008), learners *acquire*, memorise and reproduce knowledge in their process of internalisation (Fox, 1997; Greenwald, 1968). The cognitive learning theory does not acknowledge that learning occurs within the environment and peoples' interaction with the environment and with each other.

Thereafter, a more acceptable theory that encapsulates all the above ideas elegantly was introduced by Vygotsky (1978): the sociocultural theory of learning. In sociocultural theory, learning is a social process situated in society and culture (Lave & Wegner, 1991; Vygotsky, 1978). Vygotsky (1978) acknowledges the significance of support from the knowledgeable other who shortens the gap in the Zone of Proximal Development (ZPD). This process of facilitating the development of learners by the knowledgeable other, or 'scaffolding', is a critical one in their learning (Vygotsky, 1978; Wood, Bruner, & Ross, 1976). Hence, nowadays most teaching and learning strategies are influenced by the sociocultural theory, which is seen as the most relatable theory in the current educational era. A tenet of the sociocultural theory asserts that mathematics should be taught:

> By choosing engaging and natural problems suitable to their tastes, personalities, and level of experience. By giving them time to make discoveries and formulate conjectures. By helping them to refine their arguments and creating an atmosphere of healthy and vibrant mathematical criticism. By being flexible and open to sudden changes in direction to which their curiosity may lead. In short, by having an honest intellectual relationship with our students and our subject (Lockhart, 2009, p. 43).

In stressing the importance of teaching mathematics to be useful, Freudenthal (1968) asserts that mathematics applies to a rich variety of real-life contexts, hence, it should be explored in these contexts. Both these influential mathematicians agree that rich real-life mathematical problems should be integrated in the teaching and learning of mathematics, and they agree that mathematics teaching and learning should be centred on solving mathematics problems (Freudenthal, 1968, 1973, 1991; Lockhart, 2009). When the focus of the classroom is on learning, the role of the teacher is transformed into that of a tutor, or a consultant that learners look towards when clarity is needed or when faced with obstacles in their learning. The teacher then becomes the scaffold for learners to ‘step on’ in order to move to the next level of their mathematical thinking (Wood et al., 1976). Hence, I agree with Lockhart’s (2009) lament. Indeed, it is significant that teacher training colleges train PMTs to be able to solve realistic mathematics problems and to facilitate learning in their respective classrooms (Sevinc & Lesh, 2018). In South Africa, this initiative is evident in the recent mandate of the TMUF, which acknowledges that learners need to be engaged in real-life mathematics problem solving (DBE, 2019a). The focus on learning with the help of the more knowledgeable other, transforms the role of the learner into that of a *mini-mathematician*. When solving mathematical problems, learners are engaged in what mathematicians do when formulating mathematical knowledge. However, learners as *mini-mathematicians* do not necessarily have to invent new mathematical theory, but imitate the activity of mathematicians such that the knowledge they produce (or learn) appears as coming from them instead of from an outside authoritative voice. Learners mimic this activity of mathematicians with the help of a very determined teacher who is willing and committed to facilitate this process. Freudenthal (1991) calls this process ‘guided-reinvention’, where learning leads to learners recognising that the knowledge they learn is their own and they are responsible for it (see also Gravemeijer & Terwel, 2000). Hence, under the guidance of the more knowledgeable other, learners are able to rediscover mathematical knowledge for themselves (Halmos, 1994; Halmos, Moise, & George, 1975).

Current theories guided by the participationist view of learning (Sfard, 2008) recognise that people do not learn mathematics by listening to others talk about mathematics, just like they cannot learn swimming by watching people swim (Halmos et al., 1975). These theories position mathematics as a beautiful creative art (Halmos, 1968), that is actively performed by its learners. They assert that if teaching and learning is about memorising mathematical rules and procedures, being about right and wrong answers, or the transmission of knowledge, then learners are less likely to develop and see the beauty of the ’mathematical art’. However, if this art is facilitated through guided reinvention, there is much promise for effective learning (Flores & Park, 2016; Freudenthal, 1991; Larsen, 2013, 2018) and success in future problem solving (Gravemeijer, 2016; La Bestide-van Gemert, 2015). As Halmos (1970) explains, the purpose of exposing learners to mathematics, is for them to communicate about their ideas and not to memorise theorems, proofs and rules, and guided reinvention can be useful in this regard. However, in order to facilitate guided reinvention, teachers need to possess proficiency in mathematical knowledge. Hence, the current study aimed to understand the state of teachers’ content knowledge of geometry with the aim of exploring methods that can be used to support it. Furthermore, during guided reinvention, teachers need to exploit the power of technology in facilitating mathematics learning, like using geometry software (Geometer's Sketchpad, Cabrii) in exploring and teaching mathematics (Hanna, Reid, & De Villiers, 2019). With the emergence of multimodal learning (De Silva Joyce & Feez, 2018; Magana, Serrano, & Rebello, 2019) and blended learning (Borba et al., 2016; Lin, Tseng, & Chiang, 2017), it is significant that mathematics teachers are competent in teaching with technology in order to facilitate successful problem solving. The overall argument I am trying to present here, is that mathematics teaching and learning should take learners on a journey that will allow them to see the content as generated by themselves, instead of being imposed on them as facts and procedures that need to be memorised. During this activity, teachers should aim to utilise teaching methods that will allow them to notice their learners' thinking development at different steps of problem solving and support this development. This can change learners' attitudes about mathematics and increase their motivation to pursue the subject more than before. In this way, learners might become more interested in '*inventing'* more mathematics.

1.2.7 Summing up the background

So far it is clear that SA mathematical teaching and learning are not of the required global standard (Arends et al., 2017; Howie et al., 2017; Reddy et al., 2016). SA learners find themselves years behind in their mathematics learning and they are secretly being disqualified from further studying mathematics and their rehabilitation is almost unlikely to be successful (Spaull, 2015; Spaull & Kotze, 2015). This is a consequence of SA teachers who tend to employ traditional approaches of rote learning in their mathematics classrooms. Recently, a study has supported the use of instructivist teaching approaches in SA lower quintile (1–3) schools, due to the challenges that exist within these schools (Stott, 2018). The thrust of the current study is based on the current practice in SA education that sees most teachers as spending time preparing learners for examinations, instead of teaching them for understanding.

1.3 IDENTIFYING THE GAP

A plethora of research studies focusing on mathematics education, specifically the teaching and learning of geometry, have been conducted and published in peer-reviewed journals, books, book chapters and some presented at conferences and published in conference proceedings. These publications, though vast in their contextual and theoretical scope, seem to lack application within SA mathematics education. A substantial number of studies conducted on the teaching and learning of Euclidean geometry within the SA context focus on assessing learners', preservice teachers' and practising teachers' geometry knowledge based on the Van Hiele levels of geometry thinking. Some of these studies conclude that most learners come to the Further Education and Training (FET) phase not well prepared to deal with Euclidean geometry content from their previous grades (Alex & Mammen, 2014; Van Putten et al., 2010). In most cases, these studies conclude that teachers and PMTs do not have the required geometry knowledge to teach geometry effectively in high school, because they operate at the lower Van Hiele levels, while Grade 12 geometry requires that they should function at least at Van Hiele level 3 (Alex & Mammen, 2016, 2018; Luneta, 2014; Van der Sandt & Nieuwoudt, 2003, 2005). While it is argued that classroom instruction aligned with the Van Hiele levels of geometry thinking gives learners better opportunities to understand Euclidean geometry (Atebe & Schäfer, 2011), studies conducted with this theoretical position are limited in their scope to identify methods of helping PMTs to gain expertise in teaching Euclidean geometry through Van Hiele levels of geometry thinking. Thus, investigating PMTs' discourses as they solved geometry problems, opened a window to their thinking routines that could allow targeted scaffolding by teacher trainers to help the PMTs gain the necessary knowledge and skills to teach Euclidean geometry effectively up to Grade 12.

A considerable number of studies in SA mathematics education have been informed by the theoretical lens of commognition, as explicated by Sfard (2008). Nonetheless, a number of these studies concentrate on different subjects in mathematics and some concentrate on learners. For example, Heyd-Metzuyanim and Graven's (2016) study explored learners' discourse participation in developing numerical thinking, Mpofu and Pournara (2018) focus on analysing learners' discourse on the asymptotes of hyperbolics functions (see also Mudaly & Mpofu, 2019), and Roberts and Le Roux's (2019) study focused on learners' thinking about linear equations. While all these studies are pursuing discourse participation from a commognitive approach, their focus is on learners and on different mathematical topics rather than on what the current study focused. While attempting to deepen PMTs' understanding of function from a commognitive perspective, Berger and Bowie (2012) designed teaching and learning resources that aimed to give PMTs access to discourse. While Mudaly's (2015) study used the commognitive framework as an analytic lens to show that final year PMTs lack conceptual understanding of the gradient of a straight line, his focus was not on geometry problem solving that required proof and abstraction of geometry knowledge. Furthermore, Mudaly's (2015) focus was on a specific portion of geometry, gradients of straight lines, while the current study focused on the combination of these parts to solve Euclidean geometry riders, a practice useful for FET phase teachers in South Africa. Furthermore, while this topic is not part of Euclidean geometry in the CAPS, knowledge of straight lines is useful in Euclidean geometry problem solving. Van Jaarsveld (2018) explores the vocabulary of 56 third-year students training to be secondary school mathematic teachers from a commognitive approach. He focuses on these teachers solving equations and inequalities and concluded that mathematics teaching and learning should focus on developing teachers' mathematical vocabulary (Van Jaarsveld, 2018). I could not locate a research study that investigated PMTs' discourses during problem solving with the aim of diligently explaining their thinking when solving Euclidean geometry problems. While there was a study conducted by Wang (2016) that combines the theoretical tenets of commognition and Van Hiele levels of geometry thinking, the theoretical differences between Wang's (2016) study and the current study are extensively discussed in 2.9 and illustrated in Table 2.8.

1.4 THE PROBLEM STATEMENT

From the background above, I have explicated the driving force for the current study: the fact that SA learners continue to perform poorly in mathematics even after 27 years of democracy. The direness of the situation of poor performance in mathematics by SA learners has revealed itself in the results of comparison between South Africa and other countries (Reddy et al., 2016). Teachers should ensure that learners are equipped with the necessary knowledge, skills, and values to compete within the global arena and participate as critical citizens in the global economy (DBE, 2011), but seemingly this expectation is compromised. Furthermore, the education system of any country should make provisions that will enable its residents to adapt to any abrupt radical changes that may occur at any time. The training of PMTs should allow them effectively to nurture learners' skills, suitable for the 'revolution' or century in which they (will) live. This calls for teachers to be equipped with self-directed learning abilities that will allow them to develop their competency as they grow in the teaching field (Knowles, 1975). However, the current situation in South Africa, with the poor mathematical performance of learners displays a dire need to improve the results and equip learners with the knowledge, skills, and values of global mathematical standards (DBE, 2016, 2017, 2018). While there are other stakeholders involved, the task to ensure the improvement of learners' understanding and performance in mathematics, rests with teachers. Most in-service teachers have been shown to struggle with teaching Euclidean geometry (Alex & Mammen, 2018; Mayberry, 1983; Tutak

& Adams, 2015; Van der Sandt & Nieuwoudt, 2003). Furthermore, comprehensive research shows that PMTs do not possess the level of geometry knowledge required to teach geometry in schools (Tutak & Adams, 2015; Van der Sandt & Nieuwoudt, 2003, 2005). Euclidean geometry occupies a significant space in the CAPS curriculum, dominating in marks allocated in the Grade 12 paper 2 examinations. Hence, there is a need to investigate the reasons why mathematics teachers possess such poor geometry-content knowledge.

The question remains then, are HEIs training PMTs to teach geometry competently in SA secondary schools? Van der Sandt and Nieuwoudt (2005), Venkat and Spaull (2015) and Gravemeijer, Stephan, Julie, Lin, and Ohtani (2017) do not think so. Why is it that even after some millennia of research in the teaching and learning of mathematics, learners still find it extremely difficult be successful in mathematics? Even though this question seems trivial, it is one that is most difficult to answer, because of the vast number of responses possible. We have struggled to answer this question or build coherent theories to answer it, because we have ignored the essential aspects of the foundations of mathematics, one of which is altering the conception of how teachers are trained in mathematics education (Higginson, 1980), which implies that the training of PMTs needs to refocus on the foundations of mathematics. Mathematicians are abstract thinkers and if teacher trainers can understand the thinking of PMTs, the trainers are in a better position to support the PMTs' development of mathematics content and pedagogical knowledge to be better future teachers. A large body of literature in SA mathematics education addresses arithmetic concepts (functions, numbers, equations, and inequalities) and very little literature is devoted to understanding PMTs' thinking in geometry and spatial reasoning when solving geometry problems through the combination of a discursive lens and the Van Hiele theory, and how this thinking can help PMTs in their practice. Furthermore, discourse-related research in South Africa underpinned by the 'commognitive framework' tends to focus on analysing mostly the elements of mathematical discourse that reveal thinking in other mathematics fields, except geometry. These two main shortfalls in previous research studies provide a window to find the reasons for conducting this study. Hence, the current study was mainly located within the continuous, almost insurmountable problem of poor performance in geometry (see Figure 1.3) in SA education and the observed failure by teachers to remedy this situation (Adler & Pillay, 2017; Graven, 2014). The problem is located mainly in the recurring patterns occurring in PMTs' discourse on geometry problem solving. The discursive turn provided by commognition allowed for the explanation of PMTs' discourses, which are at different Van Hiele levels of geometry thinking. Being able to explain

PMTs' geometry thinking at different Van Hiele levels allows us to explain any possible reasons for the poor teaching of geometry. It further gives us a platform to look at ways in which we can improve teachers' competency in geometry problem-solving skills and their practice. In this study, I propose that quality geometry teaching can be maintained by strengthening teachers' content knowledge of geometry and their problem solving strategies. Hence, investigating PMTs' discourse participation during geometry problem solving provided a window into their thinking, which allowed me to spot certain misconceptions and ritualistic behaviours in their thinking. In addition, the semi-structured interviews broadened the lens of the findings of PMTs' discourse on geometry problem solving by explaining why their participation was the way it was.

1.5 PURPOSE OF THE STUDY

The current study sought to explore PMTs' thinking discourses through the discursive lens of commognition when solving geometry problems, better to understand the causes of poor learner performance in Euclidean geometry. As a discursive framework, commognition allows for the operationalisation of PMTs' thinking through their discourse participation during geometry problem solving. The current study recognised teacher discourse as their utterances, their writing and any form of body movements, such as gesturing or frowning, during problem solving. The study aimed to explore PMTs' thinking through their discourse participation when solving problems and to provide an explanation of how their thinking can affect their practice in the field of Euclidean geometry. The aim of this study was achieved by addressing the following objectives:

- to explore the types of discourses used by preservice teachers when solving geometry problems and why those discourses are prevalent in these PMTs;
- to explain how preservice teachers' thinking discourses affect their geometry problem solving and the implications for this in practice;
- to explore the effects of visual narratives or mediators as tools for communication used by preservice teachers when solving geometry problems; and
- to explain the implications of assessing preservice teachers' geometry thinking for their problem solving and practice in their classroom.

1.6 CRITICAL RESEARCH QUESTIONS

The current study was centred on the main question: How does preservice mathematics teachers' thinking as evident in their mathematical discourse during Euclidean geometry problem solving relate to their teaching practices in Euclidean geometry? Based on the purpose and the objectives stated above, the current study aimed to address the following subsidiary research questions:

- Which types of discourses are used by preservice mathematics teachers when solving geometry problems? Why?
- How does preservice teachers' thinking (evident in their elements of mathematical discourse) influence their geometry problem solving? What are the implications for this effect on geometry teaching?
- What effects do visual narratives used by preservice mathematics teachers have on their geometry solution paths during problem solving?

What are the implications of assessing preservice teachers' thinking for their problem solving and teaching practice?

1.7 AN OVERVIEW OF THE FOLLOWING CHAPTERS

The current study elucidated a tale of PMTs' thinking discourses during geometry problem solving through the commognitive lens. Chapter 1 provided the background to the problem by exploring research on mathematics education and highlighting from government documents the continuous poor learner performance in SA mathematics. This background was funnelled to the teaching and learning of Euclidean geometry in South Africa. It placed the problem of poor learner performance in geometry on teachers' geometry thinking and practice in the classroom. Thereafter, the aims and research questions were explained.

Chapter 2 locates the study in literature intending to reveal the literature gap, which this study fills. Here I investigate research on teacher thinking, most of which is embedded within the acquisitionist view of learning, the view that learning is the acquisition of knowledge or concepts and accommodating them in the cognitive schema, and to show understanding one needs to regurgitate this acquired knowledge. I scrutinise various methods of observing and facilitating the development of mathematical thinking in mathematics, all of which boils down to the opposite of the acquisitionist view to learning: the participationist view. Through carefully arguing the significance of the process of participation in discourse, and the explanation of meaning making in mathematics, I provide a nuanced description of mathematical thinking. Furthermore, I highlight the significance of visualisation and spatial thinking in the teaching and learning of geometry and its significance in participating in geometrical discourse. Each discussion takes a funnelling structure, starting with a broad view of the construct and slowly discussing how the construct is vested within SA literature and education. I pinpoint the gaps that this study fills, as reported towards the end of Chapter 2 and then conclude the chapter.

Chapter 3 discusses the theoretical framework used as a lens to view the findings of this study and in the discussion section to answer the research questions. It provides definitions of the important terms used by Sfard (2008) in her formulation of commognition. Furthermore, I explain the Van Hiele theory of geometry thinking. In particular, Chapter 3 explains how commognition and the Van Hiele theory of geometry thinking were used to analyse the data from this study. However, it is also made clear that the Van Hiele theory of geometry thinking is used as a supporting theory for commognition, because the study focused on discourse and geometry, which is supported by both theories.

Chapter 4 encapsulates the empirical investigation followed in this study, it highlights the paradigm, design, how the study was conducted, how ethical clearance was obtained and how trustworthiness for this study was ensured. Furthermore, it elucidates the methods of data collection, and the instruments and methods of data analysis. Chapter 5 gives the findings of the empirical investigation and Chapter 6 uses the theoretical stance explained in Chapter 3 to discuss the findings with the aim of answering the research questions. Chapter 7 gives the conclusions to answer the research questions, discusses the contribution of the study, and provides some limitations and recommendations.

CHAPTER 2:
MATHEMATICAL THINKING, DISCOURSES AND MEANING MAKING IN MATHEMATICS

2.1 INTRODUCTION

Previous research has produced compelling evidence that shows dissent from the current teaching and learning views. For example, Watson (1913) and his behaviourists believed that learning occurs through stimulus conditioning.[3] Piaget (1953, 1964, 1969) and other cognitivists believed that the brain (i.e. cognitive processing, instead of stimuli conditioning), gives rise to behaviour. However, Vygotsky's (1978) sociocultural theory of learning, from which most of the recent teaching and learning theories emanate, says that social contexts, language and social interactions are imperative for learning. The current study was conducted within the sociocultural perspective towards teaching and learning. In general, sociocultural theories situate the process of teaching and learning in a comprehensive social, historical and cultural context (Vygotsky, 1978, 1986). Active learning is at the heart of sociocultural theories of learning that accept that mathematical thought evolves as a consequence of being an active participant in the discourse of mathematics. Vygotsky's (1978) sociocultural theory has influenced a number of teaching and learning frameworks. However, the specific sociocultural-based theory underpinning the current study is Sfard's (2008) commognition. Emanating from sociocultural theory, commognition is the contemporary view that places effective mathematics learning on active participation in mathematical discourse. Learning mathematics is seen as participating in mathematical discourse, not only through discourse, but through embodiment, semiotics and the cultural-historical aspects that one brings to learning environments (Sfard, 2008).

Lockhart's (2009) *A mathematician's lament* complains about the traditional approach of mathematics instruction and suggests that mathematics classrooms should not begin with definitions or notes, but with mathematical problems worthy of the students to solve (Lockhart, 2009). In this way, the process of learning mathematics shifts to the activity of participating in mathematical discourse: mathematics is not transmitted to learners, but they reinvent it through mathematisation (Gravemeijer & Terwel, 2000; Sfard, 2008). Accordingly, Freudenthal (1973)

[3] The works of John B Watson (1913) influenced B F Skinner (1974) to such a degree that Skinner dedicated his entire life studying the effects of the different types of behaviour conditioning on learning.

complains about teaching mathematics as a finished product, the 'the anti-didactical inversion', and suggests that learners in classrooms should be taken through a 'guided reinvention' of mathematics. This cements the idea of learning as an activity where learners become active participants in knowledge construction, instead of being recipients of knowledge. Furthermore, research pioneered by Lave and Wenger (1991), Mercer (1994, 2002, 2008), Sfard (1998, 2000, 2007, 2008, 2017a), Wegerif (2008, 2011) and others, gives evidence that seeing knowledge as something to be acquired is superficial, compared to seeing knowledge as something to be constructed through active participation in discourses and becoming enculturated to a specific community. Beside the studies mentioned here, other studies provide evidence that being an active participant in mathematics discourse advances teachers' mathematical thinking and problem-solving abilities (Knuth, Zaslavsky, & Ellis, 2019; Von Renesse & Ecke, 2015; Zaslavsky, 2019). This may increase teachers' abilities to notice and support students' mathematical learning (Stockero, Rupnow, & Pascoe, 2017).

The advancement of mathematical thinking is significant in developing critical thinking skills in general, which is not only a significant skill in mathematics, but also in modern life. A good example of how important critical thinking skills are in life can be demonstrated through how one analyses news in the media. Many people fall for fake news in the media which influence decisions that such people take regarding their lives. Elder and Paul (2010) attribute this fall for fake news to the human mind. They state:

> [T]he human mind, left to its own, pursues that which is immediately easy, that which is comfortable, and that which serves its selfish interests. At the same time, it naturally resists that which is difficult, that which involves complexity, that which requires entering the thinking and predicaments of others.

Hence, many people fall for fake news because they are lazy to think critically about what is being communicated in the media (Berinsky, 2015; Pennycook & Rand, 2019). This is further proof that indeed humans are not properly developed to think critically during their early learning. As posited by Sfard (2008, 2009), a more discourse-oriented approach, where learners become active participants in the process of mathematising and teachers act as guides of the process of mathematical reinvention where they notice and support learners' mathematical thinking, has much promise in reviving the apparent deteriorating ability to think critically.

In Chapter 1, I have identified that one of the main contributors to poor performance in SA mathematics, is teachers' inabilities to support students' thinking and learning effectively and efficiently. In attempting to shed light on the gaps that exist in research about PMTs' thinking discourses when solving geometry problems and how this can affect their practice in geometry teaching, the current chapter provides a literature review focusing on the tenets that contribute to successful development of mathematical thinking for effective teaching of geometry. I begin by giving a historical background of the different mathematical minds through differentiating between intuitionism, formalism and logicism. Thereafter, I give a description of how teachers' beliefs about the nature of mathematics can affect their attempts to develop mathematical thinking in learners, by comparing three beliefs about the nature of mathematics. I then move on to situate five processes at the epicentre of the development of mathematical thinking. In this regard, I discuss the role of problem solving, teachers' noticing, metacognition, meaning making, and dialogic discourses in developing mathematical thinking. Thereafter, I discuss the significance of visualisation and spatial thinking as prerequisite skills for proficiency in geometry problem solving, teaching and learning. I then locate the literature gaps that this study will contribute to fill, before I conclude the chapter.

2.2 DIFFERENT MATHEMATICAL WORLDVIEWS

Over the past decades, mathematical thinking has been seen as an important aspect of mathematical learning globally. Although not scientifically defined succinctly, mathematical thinking is concerned with the processes involved when one is thinking about mathematics (Goos & Kaya, 2020). Tall (1997) notes that in most cases we do not always think mathematically in mathematics. He gives examples of how certain mathematical actions can be taken as a rule of thumb in our thinking, so that we end up misinterpreting these actions in different domains of mathematics (Tall, 1997). To understand the nature of mathematical thinking and how it can be developed, one can examine literature that attempts to explain this (Goos & Kaya, 2020), which is the topic of this chapter. Within the development of mathematical knowledge, we have observed different types of mathematical minds and worldviews, and research provides evidence that teachers have different beliefs about mathematics.

The most common mathematical minds are the intuitionists, the formalists and the logicist (Tall, 1991). It is important though, to note that many other different mathematical minds exist.

The intuitionists believe that mathematical concepts are derived from the laws of integers, formalists view classical mathematics as the manipulation of symbols without real-life meaning, and logicists see classical mathematics as derived from the laws of logic (Snapper, 1979; Tall, 1991). The accelerated developments in mathematics have debunked the intuitionist view based on its fallibility. Formalism was refuted by the famous Gödel's (1931) incompleteness theorem, and evidently today, mathematical knowledge is constructed based on logical deduction from axioms, postulates and formal definitions, which is the logicist way. The main aim of giving this brief discussion of the different mathematical views is to establish the existence of different thinking dispositions about mathematics, not only in mathematicians, but in mathematics teachers as well.

It is well-documented that teachers' beliefs about the nature of mathematics, conscious or not, affect their practice (Schoenfeld, 1983; Thompson, 1984). These beliefs emanate from the different mathematical worldviews teachers hold about the nature of mathematics, teaching, and learning. There is an established body of empirical studies corroborating that different teachers have different worldviews about the nature of mathematics (Chan & Wong, 2014; Chazan, 1990; Ernest, 1989). Within this body of literature, it is concluded that these teachers' different worldviews impact directly on how they view mathematics teaching and learning (Skilling, Bobis, Martin, Anderson, & Way, 2016; Spillane, Hopkins, & Sweet, 2017). So far, research has condensed teachers' beliefs about the nature of mathematical thinking into three categories: instrumental, Platonist and problem solving. A summary is given in Table 2.1 and has been discussed robustly by Ernest (1989), Beswick (2005, p. 40) and most recently, Seswono, Kohar, and Hartono (2017).

Table 2.1: Beliefs about the nature of mathematics, mathematics teaching and mathematics learning

Beliefs about the nature of mathematics	*Beliefs about mathematics teaching*	*Beliefs about mathematics learning*
Instrumentalist	Content focused with an emphasis on performance.	Skill mastery, passive reception of knowledge.
Platonist	Content focused with an emphasis on understanding.	Active construction of understanding.
Problem solving	Learner focused.	Autonomous exploration of own interests.

Source: Ernest (1989), Beswick (2005, p. 40) and Seswono, Kohar, and Hartono (2017).

Both instrumentalist and Platonist beliefs about the nature of mathematics seem to be related to the dominant 'traditional' teaching and learning methods of mathematics observed in South

Africa. However, the problem-solving belief resonates more with the participationist view of learning, where learners become ‘legitimate peripheral participants’ (Lave & Wegner, 1991) in the discourse of mathematics. Schoenfeld (1992) says that teachers’ beliefs and perceptions of mathematical ventures affect the nature of their classroom environment, and hence, their beliefs about mathematical learning can hinder or support learners’ development of mathematical thinking. The effects of teachers’ beliefs in their practice are evident in the type of mathematical tasks with which their learners engage. Breen and O’Shea (2019) say that developing tasks that can foster mathematical thinking, is a well sought-after skill for mathematics teachers. Hence, problem solving is a good identifier of mathematical thinking (Schoenfeld, 1992), which includes critical mathematising in search of a well-suited solution to a certain problem (Heyd-Metzuyanim & Graven, 2016; Van de Walle, Karp, & Bay-Williams, 2017). In another related study, Thompson (1992), drawing from the work of Lerman (1983), brings forth two mathematical views: the absolutist and the fallibilist views. Thompson (1992, p. 132) asserts:

> From an absolutist perspective, all of mathematics is based on universal, absolute foundations, and, as such, it is “the paradigm of knowledge, certain, absolute, value-free and abstract, with its connections to the real world perhaps of a platonic nature.” From a fallibilist perspective, mathematics develops through conjectures, proofs, and refutations, and uncertainty is accepted as inherent in the discipline.

Intuitionists, formalists, and logicists with their attempts to organise mathematics into an unquestionably logical structure (Ernest, 1994), subscribe to the absolutist view of mathematics. Recent developments in mathematics pedagogy negate the absolutist view’s ontology and epistemology, asserting that mathematics learning is a ‘social participative activity’ (Freudenthal, 1973; Lakatos, 1976; Sfard, 2008), which leans more towards the fallibilist view. In spite of the difficulties facing the fallibilist view of mathematics (Rowlands, Graham, & Berry, 2011), there is still a greater correlation to the sociocultural theories of today in the fallibilist view, compared to the absolutist view. Even though mathematics as a domain remains absolute,[4] its teaching and learning cannot be absolute due to the different social backgrounds and beliefs about education that exist within humans globally. Furthermore, the nature of creating, warranting and learning mathematical knowledge is dialogical (Ernest, 1994;

[4] Mathematical knowledge is developed, interpreted and endorsed from previously existing facts in a logical way. Even though mathematical knowledge changes over time, its endorsement is always based on absolute mathematical truths.

Sfard, 2008; Wegerif, 2011), which accounts for the multitudes of ideas about mathematics. Ernest (1998, p. 10) further asserts:

> [I]t is theoretically possible that any accepted knowledge including mathematical knowledge may lose its modal status as true or necessary. Such knowledge may have its justificatory warrant rejected or withdrawn (losing its status as knowledge) and be rejected as unwarranted, invalid, or even false.

Hence, the fallibilist view allows for proofs to be refuted, and evidence for this is evident throughout mathematics literature. For example, Fermats' conjecture that numbers in the form $F_n = 2^{2^n} + 1$ are prime numbers was widely accepted as true in the mathematical world, but this was disproved by Euler in 1732. In fact, a famous mathematician made it clear that in mathematics, "a proof becomes a proof after the social act of 'accepting it as a proof'. This is as true of mathematics as it is of physics, linguistics and biology" (Manin, 1977, p. 48). These discrepancies in people's views and beliefs about mathematics manifest themselves in their discourse participation when solving mathematics problems.

2.3 MATHEMATICAL THINKING FROM THE LENS OF MATHEMATICAL PROBLEM SOLVING

2.3.1 What is a problem?

Problems are tantamount to research, mathematics, and mathematics education. Halmos (1980, p. 524) says:

> I do believe that problems are the heart of mathematics, and I hope that as teachers, in the classroom, in seminars, and in the books and articles we write, we will emphasize them more and more, and that we will train our students to be better problem-posers and problem-solvers than we are.

He emphasises the significance of problem solving and problem posing as integral skills for the twenty-first century. But what is a problem in general, or even a mathematical problem? I would like to illustrate this with an example in Problem 1.

Problem 1: Solve for $\boldsymbol{x}$ *in the following equation:* $\boldsymbol{x^2 - 9 = 0}$

If we ask this question from PMTs we expect a very quick answer as a consequence of years of experience of factorising the difference of two squares (DOTS) and solving quadratic equations. So, for PMTs Problem 1 is a question that requires a recall. However, if it is asked from high school children who are just beginning to learn about factorising through DOTS and solving quadratic equations, then Problem 1 becomes an exercise for them that will provide some training of how to do DOTS and how to solve quadratic equations using DOTS. Finally, if it is asked from primary school learners who have not learned anything about DOTS, quadratic equations or how to solve an equation, this becomes a problem for them because it requires further study and an element of thought and guidance.

Polya (1962)) defines a problem as to intentionally seek out some action that will help you achieve a well-defined but not immediately accessible goal. People encounter a problem if they are presented with a situation that hinders their achievement of a certain goal(s) (Duncker, 1945; Hatch, 1988), and solving the problem requires cautious deliberation and blend of knowledge (Krulik & Rudnick, 1982). Schoenfeld (1985) argues, if a solution schema for a mathematical task is readily available to a problem solver, the activity is an exercise rather than a problem.. Van de Walle et al. (2017) assert that a problematic task in mathematics must contain mathematics content or use mathematical knowledge and methods to arrive at the solution, be accessible to learners, require justifications and motivations from the part of the learner and not just following a routine exercise and should have the ability to be explained in different ways, and should contain multiple solutions. Problems can be used as situations that initiate the process of learning (Breen & O'Shea, 2019; Lockhart, 2009; Stein, Grover, & Henningsen, 1996), for example when one is using project- or problem-based (PBL) exercises in the classroom (Dahl, 2018; Schoenfeld, 1985; Van de Walle et al., 2017). Characteristics of mathematical problems are documented in research, for example, Lappan and Phillips (1998) state the following as good characteristics of mathematics problems.

- The problem has important, useful mathematics embedded in it.
- Students can approach the problem in multiple ways using different solution strategies.
- The problem has various solutions or allows different decisions or positions to be taken and defended.
- The problem encourages student engagement and discourse.
- The problem requires higher-level thinking and problem-solving.
- The problem contributes to the conceptual development of students.

- The problem connects to other important mathematical ideas.
- The problem promotes the skilful use of mathematics.
- The problem provides an opportunity to practice important skills.
- The problem creates an opportunity for the teacher to assess what his or her students are learning and where they are experiencing difficulties.

Schoenfeld (1985) prefers not to use the word ‘problems’ in mathematics, but prefers to differentiate between routine and non-routine tasks. He takes this position, because he approaches mathematical thinking through problem-solving, and for him, routine tasks do not advance ones’ mathematical thinking compared to non-routine tasks in mathematics. This means that in order to tell whether a situation or task is a problem, we need to analyse the characteristics of the situation or task (Baumanns & Rott, 2018).

2.3.2 Some characteristics of good problem solvers

The person who encounters the problem is the problem solver and must have particular characteristics to be characterised as a good problem solver. Problem solvers must recognise that a certain situation hinders their achievement of a certain goal, otherwise no problem exists for them. They need to be motivated well enough to achieve their certain goal, which drives them to search for or use skills, knowledge, and understanding to pursue the solution to the problem (Hatch, 1988). Schoenfeld (1982, pp. 32–33) lists characteristics of good problem solvers. He states:

> To examine what accounts for expertise in problem solving, you would have to give the expert a problem for which he does not have access to a solution schema. His behavior in such circumstances is radically different from what you would see when he works on routine or familiar “non-routine” problems. On the surface his performance is no longer proficient; it may even seem clumsy. Without access to a solution schema, he has no clear indication of how to start. He may not fully understand the problem, and may simply “explore it for a while until he feels comfortable with it. He will probably try to “match” it to familiar problems, in the hope it can be transformed into a (nearly) schema-driven solution. He will bring up a variety of plausible things: related facts, related problems, tentative approaches, etc. All of these will have to be juggled and balanced. He may make an attempt solving it in a particular way, and then back off. He may try two or three things for a couple of minutes and then decide which to pursue. In the midst of pursuing one direction he may go back and say “that’s harder than it should be” and try something else.

> Or, after the comment, he may continue in the same direction. With luck, after some aborted attempts, he will solve the problem.

It is this resilience and motivation to achieve a certain goal that usually characterise good problem solvers. Successful problem solvers are further characterised by their abilities to reduce the problem into its crucial parts, reverse their own thought processes, being mindful of aspects of the problem that can help with the solution, their adjustment of position from time to time in search of the solution, and transfer procedures between different situations (Liljedahl, Santos-Trigo, Malaspina, & Bruder, 2016). When solving problems, effective problem solvers employ metacognitive strategies, while unsuccessful problem solvers usually search for a quick solution without justification (Schoenfeld, 1992, 2013). One can interview problem solvers to explain their thinking processes accurately during their solution attempts (Schoenfeld, 1982, 2013), and by means of a step-by-step decision-making process as a function of the problem-solvers' resources, orientations, and goals as significant in obtaining the solution to a certain problem (Schoenfeld, 2010). To support learners in developing their mathematical thinking during problem solving, teachers need to nurture the characteristics of being successful problem solvers.

2.3.3 What is mathematical problem solving?

It seems fuzzy to describe what problem solving is, because of the lack of a scientific definition, but it has been the subject of radical research over many years (Schoenfeld, 1992). In mathematics education and research, problem solving has been used to mean different things (Stanic & Kilpatrick, 1988). Hence, research on problem solving should be accompanied by a working definition used by the authors in that specific study (Schoenfeld, 1992). Though it seems uncertain what problem solving is, it seems concerned with the processes of attempting to obtain a feasible solution to a specific problem (Hatch, 1988). A broader conceptualisation of mathematical problem solving is presented by Hegedus' (2013, p. 89) definition. He argues regarding mathematical problem solving:

> [A]n enterprise of collaborative investigation where multiple approaches are valid. It is not just about solving a specific problem, which has a specific answer or application into the real world, but rather it is an investigation that might have multiple approaches and where students can make multiple observations.

Noticeable in this definition is the emphasis on multiple approaches during problem solving and the significance of the task to be approached from multiple perspectives. Hence, mathematical problem solving should not just give learners opportunities to repeat known procedures, but allow them to investigate solutions to non-routine or unfamiliar problems from different perspectives. This process is not only important in mathematics and its pedagogy, but also in capturing one's mathematical thinking during problem solving, usually by means of thinking aloud.

The seminal work of Polya and Schoenfeld is at the heart of mathematical problem solving (Polya, 1973; Schoenfeld, 1985). Polya (1945) gives a prominent four-step heuristic for problem solving, starting with (a) understanding the problem, (b) devising a plan, (c) carrying out the plan and (d) looking back. There is a rich body of literature on the significance of heuristics in problem solving (Karatas & Baki, 2013; Lee, 2017; Mitchell, Utgoff, & Banerji, 1983; Pearl, 1984). Polya's (1945) heuristic has been proven to be useful when learners are solving mathematics problems (Lee, 2017). However, Polya's (1945) work was developed without a full consideration of human beings as learners and how different environmental, cognitive and personal issues affect the activity of solving problems. Polya reduces problem solving to following the four steps. Hence, Schoenfeld (1985) refines Polya's (1945) work by adding environmental and personal dimensions to mathematical problem solving. He shows that problem solving is more than just developing a heuristic, it is influenced by matters of metacognition and belief systems (Schoenfeld, 1985). He posits a framework for analysing performance in mathematical problem solving, consisting of four constructs: (a) resources, (b) heuristics, (c) control and (d) belief systems (Schoenfeld, 1985).[5] This framework can be utilised to study mathematical thinking during problem solving and, in this case, it is not only practical, but important for both mathematics and its pedagogy.

2.3.4 Schoenfeld: problem solving and mathematical thinking

In his writings, Schoenfeld (1985, 2010) aims to understand what it means to think mathematically and how teachers can support learners' mathematical thinking in problem solving. He argues that learning to think mathematically does not only entail possessing more mathematical content knowledge, but includes one's flexibility, resourcefulness and efficiency in transforming or accepting the mathematical domain rules, and he describes how one can

[5] These constructs are explained in Table 2.2.

teach learners to think mathematically (Schoenfeld, 1985). He does this through a framework (see Table 2.2) that involves four elements as described above, and in doing so, he equates mathematical thinking to problem solving,[6] hence, studying mathematical thinking through one's behaviour during mathematical problem solving (Li & Schoenfeld, 2019; Schoenfeld, 1982, 1985, 1987, 2013). One of the limitations of this framework is its focus on memory and not biological traits that allow such memory. It also does not address much of the complex social factors in the social environments in which problem solving takes place (Schoenfeld, 1985), and the role of embodiment in problem solving.

Table 2.2: Theoretical framework for studying behaviours during mathematical problem solving

Resources: Mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand: this includes: - intuitions and informal knowledge regarding the domain; - facts; - algorithmic procedures; - "routine" non-algorithmic procedures; and - Understanding (propositional knowledge) about the agreed-upon rules for working in the domain.
Heuristics: Strategies and techniques for making progress on unfamiliar or nonstandard problems; rules of thumb for effective problem solving, including: - drawing figures, introducing suitable notation; - exploiting related problems; - reformulating problems, working in reverse; and - testing and verification procedures.
Control: Global decisions regarding the selection and implementation of resources and strategies. - Planning. - Monitoring and assessment. - Decision making. - Conscious metacognitive acts.
Belief systems: One's "mathematical world view", the set of (not necessarily conscious) determinants of an individual's behaviour. - About self. - About the environment. - About the topic. - About mathematics.

Source: Schoenfeld (1985, p. 15).

[6] Due to the focus of Schoenfeld (1985) on equating mathematical problem solving to the activity of mathematical thinking, this study used mathematical thinking and problem solving synonymously.

Let me attempt to elucidate some of these different elements of Schoenfeld's (1985) problem-solving framework, using Problem 2 in Table 2.3 to which I provide the solutions in Table 2.4. However, I will go into details about suggesting how one can study thinking through this model, by focusing on how this model explains mathematical thinking during problem solving.

Table 2.3: Problem 2

Problem 2: In the diagram, AEOC is a diameter of a circle with centre O. Chord CB produced meets EG at G, outside the circle, D is the midpoint of BC. OD and AB are drawn. CAEB is a cyclic quadrilateral. Prove, giving reasons, that $GE \times OA = GC \times OD$.

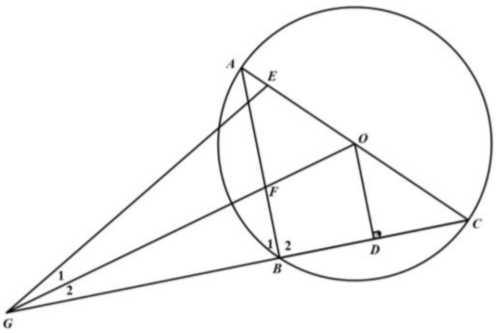


Table 2.4: Solutions to Problem 2

Solution 1	***Solution 2***
$\Delta GECIII\Delta ABC$	∠C is common
$\angle C\ is\ common$	∠DOC = ∠CGE
$\angle EGC = \angle F\ (\angle's\ in\ the\ same\ segment)$	∠ODC = ∠GEC (sum of ∠s in a Δ)
$\angle GEC = \angle B_1(sum\ of\ angles\ in\ a\ \Delta)$	∠ODC = ∠ABE (∠∠∠)
$\Rightarrow \Delta GECIII\Delta ABC\ (\angle\angle\angle)$	$\frac{CO}{CG} = \frac{OD}{EG} = \frac{ED}{EB}$ (similar Δs)
$\frac{GE}{AB} = \frac{GC}{AC}\ (similar\ \Delta s)$	CO × EG = CG × OD
$GE \times AC = AB \times GC$	but OC = OA (radii)
$But\ AC = 2 \times OA\ (diameter = 2 \times radius)$	∴ GE × AO = GC × OD
$And\ AB = 2 \times OD\ (midpoint\ theorem)$	
$GE \times 2OA = 2OD \times GC$	
$GE \times OA = OD \times GC$	

Resources are the initial toolkits brought by the problem solver to the problem, they represent all the facts, procedures and skills an individual brings to the problem(s). Resources reflect the mathematical knowledge one has experienced (mastered) prior to attempting the problem at hand, including all the tools that an individual thinks he or she can use in the current problem (Schoenfeld, 1985). Resources are important in understanding one's actions during problem solving. In solving Problem 2, one needs to draw from the knowledge of similarity, the midpoint theorem, and the triangle proportionality theorem, and the general experience of solving geometry problems (see Table 2.4). However, an individual's beliefs, knowledge, and suspects about the problems need to be considered. For example, learners can possess all these tools in the tool case, but fail to link them together with respect to the problem, because their beliefs about the problem and the solution involve a different tool case. The solution to this problem involves developmental thinking, proving that ΔGECIIIΔABC, which is mostly done in lower grades in South Africa. Furthermore, finding the solution depends on whether the problem solvers are comfortable with the algorithms and procedures for linking their tool kits to obtain the solution. A learner might realise that the problem requires the application of the triangle proportionality theorem, but fail to apply it correctly. However, Schoenfeld (1985) emphasises that the individual's resources do not necessarily need to be correct as the aim is to explain their behaviour and understand their thinking when solving mathematical problems. By understanding their behaviour through this framework, teachers are in a better position to support learners' mathematical development.

In explaining heuristics, Schoenfeld (1985) relies on Polya's (1945) four-step heuristic, and he argues that heuristics can be utilised in different ways to facilitate the solution of a problem. However, he extends to the nature of heuristics in general and discusses that some heuristics can provide quicker and neater solutions than others. If we are asked to prove that $AB \parallel OD$ in Problem 2, one would approach the solution in one of the two ways in Table 2.5. Solution 1 looks long and involves a lot of reasoning, while solution 2 is shorter and neater than solution 1.[7]

[7] Any other solution is welcome though, in this problem.

Table 2.5: Two solutions of $AB \parallel OD$ in Problem 2

Solution 1	*Solution 2*
$B_1 = 90°$ (∠subteded by diameter AC) $\angle ODC = 90°$ (line from centre to the midpoint of chord BC) $\therefore B_1 = \angle ODC = 90°$ $\Rightarrow AB \parallel OD$ (corresponding angles are equal)	$AO = OC$ (radii) $BD = DC$ (given) $\therefore AB \parallel OD$ (midpoint theorem)

The choice of heuristic, tool, algorithm or procedure towards finding the solution is an element of control. This category includes the principal decisions learners make regarding which actions to take during problem solving, which influence their success (Schoenfeld, 1985). At the control level, actions encapsulate global repercussions for the advancement towards the solution. For example, choosing solution 1 or 2 in Table 2.5 makes it a control decision because of its impact on the solution paths and time taken to reach the solution. Usually, learners do not reflect on their solutions in search of a more efficient, neater solution (Mhlolo, 2017). Hence learners do not apply metacognitive[8] knowledge during their problem solving. Control or metacognitive knowledge includes one's decision making in selecting significant, necessary and effective resources, including the time that is available to the problem solver (Schoenfeld, 1985). Hence, finding the solution to the problem in Table 2.5 through solution 1 or 2 shows the application of metacognitive knowledge. Schoenfeld (1985, p. 28–32) explains how two fully capable students fail to solve a problem due to poor decision making, a control malfunction.

From anecdotal evidence and observation of learners', and even PMTs' solutions in examinations and tests, I have noticed a trend of fearing to obtain incorrect answers when finding the roots of quadratic equations. Consequently, most learners are more comfortable to use the quadratic formulae instead of computing the factors because they believe that, if applied correctly, the quadratic formulae always give correct answers. Learners' belief systems are so engraved that they still apply the algorithm of factoring DOTS even in $x^2 + 4$. The conceptualisation of the role of 'belief systems' in mathematics problem solving seems to have spread after the work of Schoenfeld's (1985), which equated belief systems to one's mathematical worldview. Philipp (2007, p. 259) articulates belief systems in the following way.

[8] Metacognition will be discussed in 2.5.

> A metaphor for describing the manner in which one's beliefs are organized in a cluster, generally around a particular idea or object. Beliefs systems are associated with three aspects: (a) Beliefs within a beliefs system may be primary or derivative; (b) beliefs within a beliefs system may be central or peripheral; (c) beliefs are never held in isolation and might be thought of as existing in clusters.

Apparent in Philipp's (2007) conceptualisation, is the signalling relationship between different beliefs held by certain discursants, utilised to narrate their own version of reality. They are characterised by personal commitment, existence, psychological mechanisms, life span, degrees of variation, boundaries, elements, representations, evaluative and affective components, content set, and degrees of certitude (Usó-Doménech & Nescolarde-Selva, 2016). In mathematics, beliefs can be about mathematics as a subject, the discursant in mathematical discourse, or mathematics teaching and the social context where mathematical discourses occur (Philipp, 2007; Schoenfeld, 1985). Belief systems in mathematics education has been a subject of research with respect to their effects on teaching and learning, and it has been shown continuously that belief systems affect both teachers' practice and learners' learning of mathematics (Bräunling & Eichler, 2015; Cobb, 1985, 1986; McDonough & Sullivan, 2014; Philipp, 2007; Rott, Torner, Peters-Dasdemir, & Safrudiannur, 2018; Thompson, 1984). For example, a high level of religiosity might weaken learners' mathematics and science performance (Stoet & Geary, 2017). Discursants might possess two or more conflicting beliefs about mathematics learning (Schoenfeld, 1989).

In the same classroom learners might believe that knowing 'what to do' in mathematics is enough, while others seek to explore the 'what', 'how' and 'why' of mathematics, others may place more emphasis on producing correct answers, while some may believe that the teacher should make things easy for them by passionately explaining the mathematics content to them (Hoyles, 1982). Learners who believe that mathematics is an activity where the demands of the teachers need to be met, increase their dependence on their teacher, which can conceal their true abilities to construct their own mathematical knowledge (Cobb, 1986). Beliefs do not stand in isolation from other social and cognitive factors (Liebendörfer & Schukajlow, 2017; Philipp, 2007) and they have the ability to change continuously (Erens & Eichler, 2019). The social nature of mathematics teaching and learning, the classroom environment and the learners' backgrounds affect learners' belief systems (Cobb, 1986; Hoyles, 1982; Schoenfeld, 1985, 1989). Belief systems can constrain learners' thinking into a specific set of actions, influencing what they consider as a problem and an accepted solution and affecting their motivation to engage in mathematical activity (Cobb, 1985). Though it has long been shown that belief systems affect problem solving, the relationship still remains blurry and complex (Callejo & Vila, 2009).

The question remains: Can SA schools teach learners higher-order mathematical thinking skills?. This extends to geometry education, which is considered the initial enculturation to the rules of logic and mathematical rigor. The TMUF was designed with the aim of centring mathematics teaching and learning in problem solving (DBE, 2019a). This puts more significance on problem solving as a vehicle to advance learners' mathematical thinking and performance. However, it is still blurry for many SA teachers what teaching through problem solving means, and very little provision for teacher development is provided by the DBE. There is a lack of empirical studies investigating learners' problem solving in SA mathematics. The existing few studies focus more on learner performance during problem-solving tasks and not their thinking behind a certain solution path (Askew et al., 2019; Mogari & Chirove, 2017). Another study which compared the problem solving of learners of pure mathematics and mathematical literacy through a test, omits the thinking discourses of these learners (Machaba & Mwakapenda, 2016). A few studies investigate teachers' perceptions of teaching through problem solving and provide evidence that teachers face insurmountable difficulties that force them to revert to rote learning (Adler & Ronda, 2014; Chirinda & Barmby, 2018). Hence, in general, SA mathematics teachers struggle to teach through problem solving (Naroth & Luneta, 2015; Tachie & Molepo, 2019). It has been identified that overcrowding, language, over-reliance of teachers on external validation of the correctness of their answers during problem solving are some of the difficulties teachers face in terms of teaching effectively (Adu & Olaoye, 2015; Chirinda & Barmby, 2017, 2018; Venkat, 2015). Solving worthwhile, meaningful problems in mathematics may lead to the development of mathematical thinking in learners (Polya, 1945; Schoenfeld, 1985). Building from the explanation I provided under 2.2.1, if the focus of the curriculum is to develop learners' mathematical thinking, effective guidance during problem solving is important. Hence, teachers need to be developed in mathematical teaching through problem solving (Golding & Smith, 2016; Krulik & Rudnick, 1982; Livers, Harbour, & Fowler, 2019).

The work of Freudenthal (1973, 1991) portrays mathematics as an activity one participates in for enculturation into the mathematics community. For Lakatos (1976), this activity in mathematics involves proofs and refutations. Polya (1973) and Schoenfeld (1985) equate thinking to one's behaviour during problem solving, with Polya focusing on a heuristic and Schoenfeld giving a holistic view of how mathematical thinking can be assessed during problem solving. However, these two models do not reveal a complete picture of one's thinking in mathematics, because they are based on observations. What they do at least, is to provide a certain angle of studying ones' thinking in problem solving, which is based on observations. Let me extend this discourse by looking at Burton's (1984) conceptualisation of mathematical thinking and the processes involved therein.

2.3.5 Leone Burton and mathematical thinking

For Leone Burton, mathematical thinking involves not only thinking about mathematical content, but also specific dynamics, styles, operations and processes appearing to be mathematical in their nature (Burton, 1984). She also studied mathematical thinking through problem solving. Burton and Burton (1980, p. 20) refer to mathematical problems in this way.

1. A problem is frequently a problem because it is ill defined. Once the nature of the problem is identified, the method or methods for dealing with it are often clear.
2. Problems rarely have a single solution. Indeed, problems rarely have a final solution in that they are then "wrapped up" and concluded. More usually problems are open-search in the sense that the method chosen to deal with them is "best fit" rather than exact and relies upon an amalgam of objective and subjective information or even chance. What is then obtained is not a "solution", but a personal resolution of the problem, which the individual judges not as right or wrong, but as adequate.
3. Each problem resolution opens up another field of problems.
4. Problems belong to people: they are real and involve the individual. They present a challenge which the individual acknowledges.

Burton (1984) further adds that mathematical problems provoke the utilisation of mathematical knowledge, processes and operations. Mathematical thinking is invoked by problems that require mathematical knowledge and procedures to solve in any context. For Burton (1984), mathematical thinking is revealed through four steps during problem solving, which are specialising, conjecturing, generalising and convincing, as summarised in Table 2.6.

Table 2.6: The processes involved in mathematical thinking

Processes	***Explanation***
Specialising (*inductive approach*)	In problem solving, the first step is primarily based on the examination of specific examples. Each example is an opportunity for one to exploit the concrete elements of your thoughts. I may link this stage to empirical testing of a theorem, in order to achieve intuitive certainty that it is true.
Conjecturing	Conjecturing involves looking for connections. Underlying patterns are explored, expressed and substantiated.
Generalising	Once a conjecture is put forward, it becomes intuitive, one must then generalise their conjecture by proof.
Convincing (*deductive approach*)	This stage involves ensuring that the generalisation is valid globally, that it is a true mathematical statement. When everyone is convinced, this information shifts from being intuitive (personal) to being public.

Source: Burton (1984, p. 38).

The move from specialisation to conviction (Table 2.6) shows how ones' mathematical reasoning transforms from inductive reasoning to deductive thought. Table 2.6 shows that problem solvers begin by utilising an inductive approach to solving certain problems and develop towards a more deductive approach. In mathematics, specific knowledge about a certain situation needs to be persuasive and endorsed by the mathematics community. Hence, convincing requires that one produces a mathematical proof that proves the conjecture through acceptable mathematical rules, and this proof must be endorsed and accepted by the mathematics community. As a professor of mathematics, Burton (1984) believes that moving from inductive to deductive thinking is a symbolisation of how one's mathematical thinking develops. However, the development from inductive to deductive thinking is a complex one, governed by many principles. Burton (1984) believes that this development moves through a number of successive loops (Figure 2.1). The decision to move through these loops is elicited by a particular mathematical situation that compels exploration and the autonomous decision of the problem solver to solve the problem because of the inherent satisfaction and benefits of engaging in this activity (Burton, 1984; Schoenfeld, 1982, 1992).

The problem solvers grapple with the problem until they can see some sort of pattern or get an understanding of the problem. While this sense of pattern has been established, it remains implicit until it can be expressed visually, diagrammatically, or symbolically in the convincing stage of articulation. This articulation is then refined and is available for further manipulation. Therefore each loop further advances ones' mathematical thought (Burton, 1984).

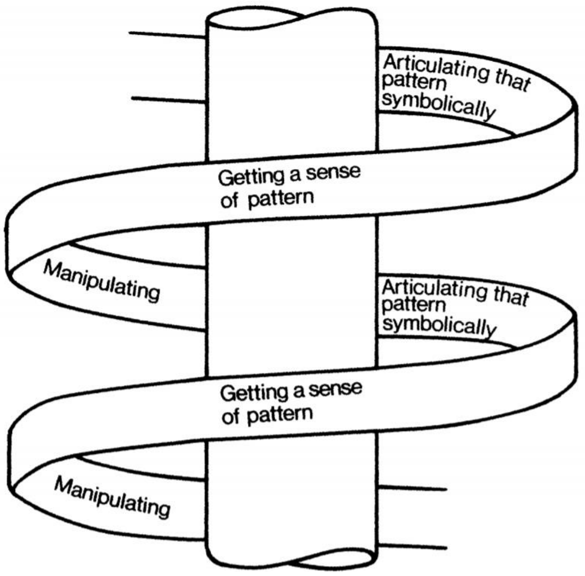


Figure 2.1: A helical model for the dynamics of mathematical thinking

Source: Burton (1984, p. 39).

According to Burton (1984), mathematical thinking passes three phases: entry, attack and review (Figure 2.2). The entry phase is characterised by the seeking of meaning through manipulating, specialising, conjecturing, and generalising. This might create a cognitive conflict that may cause the problem solver either to attack the conflict or withdraw from it.[9] These helical model of the dynamics and phases of mathematical thinking portray the development of mathematical thought as interconnected with emotion. However, this model does not provide clear guidelines for how teachers can support learners' development regarding mathematical thinking, but simply gives a framework of how mathematical thinking advances.

[9] In Schoenfeld's (1985) view, the choice of attacking the cognitive conflict or withdrawing from it, can be attributed to control or metacognition.

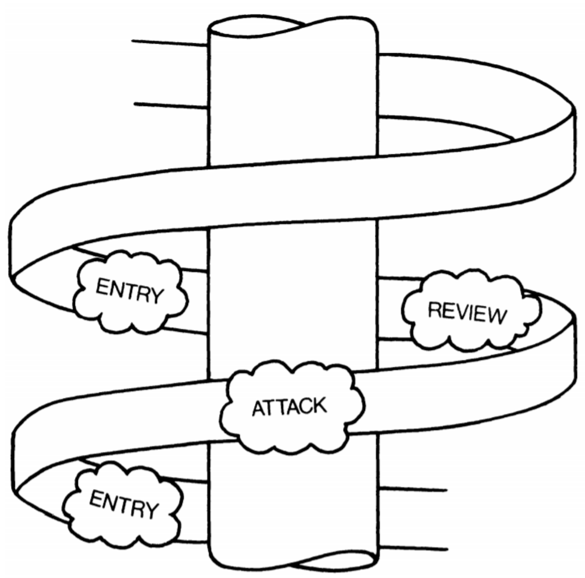


Figure 2.2: Affective phases during mathematical thinking

Source: Burton (1984, p. 42).

2.4 TEACHER'S NOTICING OF LEARNERS' MATHEMATICAL THINKING

The current study examines teacher discourses as they solve geometry problems, hoping to highlight the significance for teachers to use their learners' geometry thinking to inform the practice of these teachers. Geometry problems are divided into different classes of difficulty levels, which determine the type of actions and relational systems that can be used to solve the problems. In earlier grades, learners solve geometry problems through their experiences with the world and not through formal instruction. But, as learners proceed with their schooling, this intuitive ability seems to fade and many of their routines depend to a large degree on the formal instruction they receive in their respective classrooms (De Villiers, 2012). Hence, the teachers' roles in planning and executing lessons in mathematics classrooms become central to learners' understanding of geometry. Thinking actions about planning and executing instruction are some of the critical psychological processes most teachers face daily, and teachers' decisions in this regard are affected by their beliefs, context and the curriculum. Hence, attending to teacher thinking in mathematics education tends to grow in significance from various perspectives, because their thinking influences their practice and shapes students' learning experiences (Chapman, 2017). Teachers are confronted with a variety of learner solutions, questions and comments on a daily basis (Ball et al., 2008). It is the thought processes and beliefs of teachers that inform the type of feedback given to learners and extracted to inform practice.

In the SA context, the assessment of learners' thinking is not a primary goal of many teachers, because they focus on completing the syllabus according to the stipulated time. As observed in studies such as that of Putnam and Leinhardt (1986) and Putnam (1987), SA teachers follow the curriculum as it is, with minimal adjustments based on their learners' feedback. Aubrey, Ghent, and Kanira (2012) report that training teachers to teach their learners to think and reason in mathematics can help improve learners' problem-solving abilities and thinking. Hence, the bulk of recent research has begun to investigate teacher knowledge and beliefs about their learners' thinking, and how it informs the teachers' instruction (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Dyer & Sherin, 2016; Jacobs & Empson, 2016) and teacher noticing (Lee, 2017; Santagata & Yeh, 2016; Scheiner, 2016). Drawing from the expertise of highly skilled teachers, Jacobs and Empson (2016) propose an emergent framework for facilitating teaching actions in mathematics. These authors propose categories in the framework, instead of specific questions, because some questions might work for some situations and not for others. In a related study, Dyer and Sherin (2016) interpret two high school mathematics teachers' responsive teaching to support the significance of utilising learners' thinking during teaching and learning. Scheiner (2016) and Chapman (2017) provide commentaries along the same lines. In another study, Lee (2017) measured the pedagogical content knowledge of mathematics of 30 teachers of preschool learners. Lee found that even though these teachers were able to notice mathematical situations, they still needed support in identifying ways to enhance learners' mathematical thinking. Hence, there is still a need to devise methods to support PMTs' thinking so that they are able to scaffold learners' mathematical thinking. In ensuring that learners construct their own mathematical relational systems of knowledge, teaching and learning should be based on their existing knowledge (mathematical experiences) and teachers should be the anchor for learners' construction of their own mathematical knowledge, instead of rote memorisation. All these studies corroborate that utilising learner thinking and supporting learners in developing mathematical thinking and problem-solving abilities, is currently a promising approach. However, within the SA context, this approach has not kicked in yet, neither in basic and nor in higher mathematics education. Teaching and learning still remain traditional with a heavy focus on performance, rather than meaning making. Hopefully, the current study will ignite a move that will see SA teacher training institutions training PMTs to assess their learners thinking and use it to plan future instructional approaches.

Possessing a strong mathematical content knowledge does not suffice for successful mathematical teaching (Ball et al., 2008). An extensive knowledge of learners and how they learn effectively, is also a strong determinant of competency in teaching mathematics (Schoenfeld & Kilpatrick, 2008). Hence, there is a need to develop teachers' MKT (Chapman, 2015), specifically for them to start viewing mathematics from the learners' perspective and adapting their instruction to support learners' development of mathematical thinking. In this regard, the notion of teacher noticing has gained increased recognition in teacher training and professional development (Dyer & Sherin, 2016; Estapa et al., 2018; Jacobs et al., 2010; Mason, 1982; Santagata, Zannoni, & Stigler, 2007; Sherin & Han, 2004; Sherin & Lynn, 2019; Van Es & Sherin, 2002). The development of learners' mathematical thinking is important in mathematics education, because it affects their understanding and appreciation of, and performance in the subject. Hence, exemplary teaching in mathematics education includes giving attention to learners' thinking in the classroom (Walkoe et al., 2019), and attending to learners' thinking, and this is one of the main driving forces for the current study. The process of teachers attending to learners' thinking, with the aim of supporting the development of learners' mathematical thinking, is referred to as 'teacher noticing' and it has been making waves in preservice teacher training and in-service professional development.

Goleman (1985, p. 24) raises awareness of the significance of noticing when he states:

> [T]he range of what we think and do is limited by what we fail to notice, and because we fail to notice *that* we fail to notice there is little we can do to change until we notice how failing to notice shapes our thoughts and deeds.

The foundations of noticing are rooted within the works of Mason (1982), Goleman (1985) and Goodwin (1994), and has been advanced through the work of Sherin (2001, 2007), Sherin and Lynn (2019) and Van Es (2009, 2011, 2012). In their conceptualisation of teacher noticing, Sherin, Jacobs, and Philipp (2011, p. 5) say that teacher noticing is concerned with teachers "attending to particular events in an instructional setting" and "making sense of events in an instructional setting". It is an active process of attending to learners' thinking with the aim of supporting and improving their thinking.

Though the concept teacher noticing is a broad construct, domain-specific noticing is important to develop expertise in a certain field. Goodwin (1994, p. 606) refers to this as "professional vision". The research on teacher noticing varies from what teachers notice, how they notice, their explanation of learner thinking and actions afterwards. I want first to emphasise the significance of 'what' the teacher notices within the classroom discourse, specifically regarding what Matusov, Marjanovic-Shane, and Gradovski (2019, p. 289) refer to as "ethical discourse". Teachers are obligated to identify and respond to learners' needs in their respective classrooms (Aktaş, Yakıcı-Topbaş, & Dede, 2019), and teacher noticing allows teachers to notice the critical events of the complex mathematical classrooms (Lisa, 2016). Experiential factors, context and teacher beliefs affect what they notice (Star & Strickland, 2008; Yang, Kaiser, König, & Blömeke, 2019). Louie (2018) asserts that teacher noticing needs not only to be limited to noticing learners' thinking, but also noticing how marginalised learners participate in the classroom discourse and what spaces teachers can make for these learners to feel equal to the dominant learners. In a classroom based on participationist informed discourses, issues of respect are of outmost importance, but learners' individual mathematics abilities to participate in mathematical discourse are also critical. Hence, mathematics "teacher noticing for equity" (Baldinger, 2017, p. 232) is of significance and it can help teachers narrow learner participation gaps in the classroom (Van Es, Hand, & Mercado, 2017).

Learning to notice and support learners' mathematics is part of mathematics teachers' development of expertise and competency in teaching mathematics (Choy & Dindyal, 2019; Jacobs et al., 2010; Krupa, Huey, Lesseig, Casey, & Monson, 2017; Nickerson, Lamb, & LaRochelle, 2017; Sherin et al., 2011; Walkoe et al., 2019). What the mathematics teacher notices or does not notice, affects their future instruction, and teacher practices influence their learners' development of mathematical thinking (Didis, Erbas, Cetinkaya, Cakiroglu, & Alacaci, 2016; Lisa, 2016; Santagata & Yeh, 2016; Santagata et al., 2007). Noticing learners' actions and supporting them effectively is not just complex, but also challenging and hard for many teachers (Goleman, 1985; Lisa, 2016). However, teachers can develop abilities to notice and support their learners' mathematical thinking with support (Amador, 2016; Barnhart & Van Es, 2015; Estapa et al., 2018). Fernández, Llinares, and Rojas (2020), for example, provide evidence that teacher collaboration in an online platform enhanced their noticing of their learners' thinking and enhanced their practice (see also Johnson et al., 2019). Alternatively, Warshauer, Starkey, Herrera, and Smith (2019) use analysis of video episodes to support preservice teachers' noticing. They found that through the analysis of video episodes, preservice teachers develop a sense of noticing, because they become able to explain why students struggle with numbers and operations. Furthermore, these PMTs begin to identify teaching approaches that can support learners' development of mathematical thinking in numbers and operations (Warshauer et al., 2019). Some comparative studies show that teacher noticing varies in different contexts (Cai & Wang, 2010; Yang et al., 2019), while others show differences in the ability of novice and expert teachers noticing in mathematics (Huang & Li, 2012; Wolff, Jarodzka, Van den Bogert, & Boshuizen, 2016).

The use of video analysis of classroom practices is one dominant method used by researchers to support the development of teacher noticing in mathematics education (Mitchell & Marin, 2015; Roller, 2016; Sherin, 2007; Sherin & Han, 2004; Stockero et al., 2017; Van Es, Cashen, Barnhart, & Auger, 2017; Van Es & Sherin, 2008; Walkoe et al., 2019; Wallin & Amador, 2019). Through the use of teachers' video analysis, different frameworks have been developed. Santagata et al. (2007) developed a three-step framework for analysing lessons which includes lesson goals, the act of learner learning and alternative teaching methods to support learners' learning. Van Es and Sherin (2008), building on earlier works (Van Es & Sherin, 2002), developed their three-step framework. In their framework, they argue that learning to notice in mathematics is framed by identifying significant situations for teaching, utilising ones' knowledge of the context to support learning in a particular situation, and making connections between different events in particular situations, and the teaching and learning principles. Jacobs et al. (2010) see expertise in mathematical teaching as dependent on: –

- attending to learner strategies;
- interpreting their understanding; and
- making decisions to respond to their understanding.

Van Es (2011) proposes a four-level framework for learning to notice learners' mathematical thinking, which includes baseline noticing, mixed noticing, focused noticing, and extended noticing. In this framework, she explains these levels in terms of what teachers notice and how they notice it (Van Es, 2011). Recently, Superfine, Amador, and Bragelman (2019) used video-based discussions to facilitate prospective mathematics teachers' noticing. Particularly, they examined facilitation moves that seem to support prospective teachers' noticing of learner thinking. They found video-based discussions to be a promising method for developing mathematics teachers (Superfine et al., 2019). In another study, Walkoe et al. (2019) used annotating videos from different classes to study prospective teachers' noticing of learners' algebraic thinking (see also, Walkoe, 2015). In their study, teachers used video tagging tools to attend to their noticing of learners' thinking. They conclude that video tagging tools has benefits for both researchers and prospective teachers to identify which incidents caught teachers' attention and how they interpreted that thinking in those incidents (Walkoe et al., 2019).

However, some studies established that prospective teachers, particularly, were unsuccessful in attending to learners' thinking in a video (Ding & Domínguez, 2016; Males, 2017; Prediger & Zindel, 2017). Despite the dominance of the utilisation of video analysis in studying and supporting teacher noticing, recent studies were approaching the study of teacher noticing from different perspectives. Dreher and Kuntze (2015) focused on teacher knowledge of noticing, based on multiple representations in mathematics. They conclude that teachers had poor understanding of the role of multiple representations in mathematics classrooms and there is a need for teacher noticing of multiple representations in mathematics (Dreher & Kuntze, 2015). Another growing approach is within task analysis of learners' answers of mathematics tasks. Didis et al. (2016) utilised teacher noticing through students' work in mathematical modelling. They conclude that learner thinking is governed by describing, questioning, explaining and comparing when teachers interpret learner's thinking (Didis et al., 2016). Most recently, Sánchez-Matamoros, Fernández, and Llinares (2019) describe the relationship between attending to, interpreting and deciding how to respond to learners' tasks as characteristics of expertise in teacher noticing (see also Barnhart & Van Es, 2015; Krupa et al., 2017). Estapa et al. (2018), while recognising the significance of video use in developing noticing, say that it is not enough to understand and develop preservice teachers' noticing, and these authors suggest the usage of pedagogies of practice (Grossman et al., 2009). Pedagogies of practice are framed by representations, decompositions, and approximation of practice (Estapa et al., 2018; Grossman et al., 2009). Estapa et al. (2018) conclude that the framework of pedagogies of practice extend teacher knowledge of noticing from the representation of students' practice and what is noticed, to providing the context and authorship for prospective teachers to implement in their future lessons.

So far, I have highlighted the significance of teacher noticing and the need to develop this construct in teacher training programmes (Amador, 2016). I have explained the construct of mathematical thinking through problem solving and highlighted the significance of Polya's (1945) stage of *looking back*. The constructs of noticing and looking back in teaching and learning are framed by reflection, which forms a critical part of teachers' expertise in teaching mathematics. Now I want to draw your attention to metacognition, a construct that further signifies teacher reflection, not only in mathematical problem solving, but in the process of teaching and learning as well.

2.5 METACOGNITION – THE SIGNIFICANCE OF REFLECTION DURING MATHEMATICAL THINKING

Thinking is a complex subject for investigation, because it is difficult to study (Moseley et al., 2005). It is an innate process that sometimes escapes even the thinker. Individuals' awareness of their thinking and other better ways of thinking, is important in the development of their thinking. Dewey (1933) advocates the better way of thinking as 'reflective thinking'. He asserts that reflective thinking is "the kind of thinking that consists of turning a subject over in the mind and giving it serious and consecutive consideration" (Dewey, 1933, p. 3). The significance of reflection and reflective thought, has been studied in different fields, including the health sciences (Cahusac de Caux, Lam, Lau, Hoang, & Pretorius, 2017; Graham, 2017), higher education (Ghanizadeh, 2017), and teacher training (Beavers, Orange, & Kirkwood, 2017; Foong, Nor, & Nolan, 2018; García, Sánchez, & Escudero, 2007; Wittmann, 1981). Furthermore, other educational movements such as the Freudenthal Institutes' RME (Van den Heuvel-Panhuizen & Drijvers, 2014), the problem-solving approach (Polya, 1945, 1962; Schoenfeld, 1985), the use of technology and its effects in the teaching and learning of mathematics and geometry (Hanna et al., 2019), all stress the importance of reflection. This reflective behaviour is the one that kindles metacognition, and is referred to as metacognitive reflection. A reflective thought is useful in converting an appetitive, blind and impulsive action into an intelligent one (Dewey, 1933). Even though reflection in problem solving concerns itself with looking for memories and problem-solving strategies from past experiences (Boud, Keogh, & Walker, 1985), it does not concern itself with memorisation and recall of facts, but the purposeful act of one's attempt to understand and be aware of your thought processes during problem solving, an important metacognitive process. Similarly, the construct of teacher noticing, as explained above, is a reflective activity, which helps teachers plan better future lessons. The exercise of reflection in teaching and learning and during problem solving, kindles metacognition. This is discussed next.

The term 'metacognition' was coined by Flavell (1976), and since then researchers concerned with mathematical problem solving, became interested in metacognition during mathematical problem solving. In layman's terms, metacognition is ones' reflections about your cognitions, or thinking about your own thinking. However, Schoenfeld (1987) believes that these conceptions of metacognition are not useful in pinpointing exactly what metacognition is and why it is important to study in mathematics education. In his conceptualisation of metacognition, Flavell (1976, p. 232) proposes:

> One's knowledge concerning one's own cognitive processes and products or anything related to them, e.g. the learning-relevant properties of information or data. Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective.

Later, he describes it as "knowledge and cognitions about cognitive phenomena" (Flavell, 1979, p. 906). Schoenfeld (1985) attributes metacognition to his construct of *control* during mathematical problem-solving. However, recently the concept of metacognition has evolved in its definition, so that it is not clear what it really is. Schoenfeld (1987, 1992), Flavell, Miller, and Miller (2002) and Dinsmore, Alexander, and Loughlin (2008) raise a concern that metacognition and self-regulation have been used synonymously. This is evident even in Schoenfeld's (1987) conceptualisation of metacognition. Schoenfeld (1987, p. 190) states that metacognition is concerned with three main categories, which are:

- "your knowledge about your own thought processes";
- "control or self-regulation", which reveals this close relationship between metacognition and self-regulation; and
- "beliefs and intuitions".

Consequently, Dinsmore et al. (2008) call for research that will provide clear definitions of these terms. The significance of metacognition in mathematics education is one of the most promising approaches to enhance learners' thinking in mathematics and problem solving (Ader, 2017, 2019; Bennet, 2016; Dori, Mevarech, & Baker, 2018; Young & Worrell, 2018; Zhou & Lam, 2019).

Metacognition is divided into two domains: metacognitive knowledge and skills (Baten, Praet, & Desoete, 2017). Metacognitive knowledge concerns itself with one's learning experiences about which practices can affect the course and the results of one's cognitive enterprise, and your awareness of the mental pros and cons (Baten et al., 2017; Flavell, 1979). This is the reflective component of metacognition, which is fundamental to the development of mathematical thinking. On the other hand, metacognitive skills refer to the knowledge required for the 'control', continued engagement, and regulation of the learners' learning activities or during problem solving (Baten et al., 2017; Van der Stel & Veenman, 2014). Learners who have developed sufficient metacognitive knowledge and skills are more likely to be successful during problem solving (Hurme, Järvelä, Merenluoto, & Salonen, 2015; Izzati & Mahmudi, 2018; Kuzle, 2018; Lee, Ng, & Yeo, 2019; Mihalca, Mengelkamp, & Schnotz, 2017; Rellensmann, Schukajlow, & Leopold, 2020; Tachie, 2019; Vula, Avdyli, Berisha, Saqipi, & Elezi, 2017). Furthermore, metacognitive knowledge and skills are strong predictors of mathematical performance (Desoete et al., 2019; Jagals & Van der Walt, 2016; Marulis, Palincsar, Berhenke, & Whitebread, 2016; Shilo & Kramarski, 2019; Zhao et al., 2019), are effective in the teaching and learning of mathematics (Baten et al., 2017; Donker, De Boer, Kostons, Dignath van Ewijk, & Van der Werf, 2014; Shilo & Kramarski, 2019), and facilitates cognitive development (Nelson & Fyfe, 2019) and critical thinking (Bassett, 2016; Hogan, Dwyer, Harney, Noone, & Conway, 2015; Lemley, Ivy, Franz, & Oppenheimer, 2019; Radmehr & Drake, 2017). Teachers' metacognitive knowledge links with that of their learners' (Soodla, Jõgi, & Kikas, 2017) and teachers' knowledge of metacognition affects their practice (Wilson & Bai, 2010). Teachers require competency in metacognition in order to facilitate reflective behaviours during their teaching in the classroom. Kohen and Kramarski (2018) highlight the significance for teachers to develop pedagogical metacognition competencies in fostering learners' development of thinking. Through a balanced combination of both metacognitive knowledge and skills, one can develop into a better geometry problem solver, while developing your level of geometry thinking at the same time.

Developing metacognitive skills in the mathematics classroom requires that teachers create a mathematics culture that portrays mathematics as an important activity in learners' everyday lives, a realistic approach in which learners learn mathematics as an integral part of their everyday lives, and hence, they learn it meaningfully (Schoenfeld, 1987; Van den Heuvel-Panhuizen, 2020). Furthermore, metacognition, beliefs (including belief systems), control[10] and mathematical practices play an important role in mathematical thinking (Schoenfeld, 1992).

[10] However, in the case of control, Schoenfeld (1992, p. 364) argues that the state of literature is not well developed to include theoretical foundations of how the concept is related to mathematics education, and states:

> [T]he first issue is mechanism. We lack adequate characterization of control. That is, we do not have good theoretical models of what control is, and how it works … The second is the issue of development. We know that in some domains, children can demonstrate astonishingly subtle self-regulatory behaviours, in social situations, for example where they pick up behavioural and conversational cues regarding whether and how to pursue particular topics of conversation with their parents. How and when do children develop such skills in the social domain? How and when do they develop (or fail to develop) the analogous skills in the domain of mathematics? Are the similarities merely apparent, or do they have common base in some way? We have barely a clue regarding the answers to these questions.

Research on metacognition posits that learners require opportunities to verbalise their thinking, knowledge and skills, implying that PMTs need to be trained to facilitate classroom learning where learners are motivated to explain their thinking. In this way, learning mathematics, according to Sfard (2008), should immerse learners in participative activities that value discourse and proof as they develop their understanding of self and how to belong in the mathematics community (see also Lave & Wenger, 1991).

It has been long noticed that the main purpose of schooling is to help children to become more effective problem solvers in society (Burns, 1982), and recently this purpose is becoming more and more important. The significance of problem solving in mathematics education has been researched and discussed in the preceding subsections.[11] What is evident from these subsections, is the lack of clear guidelines on how to develop effective mathematics problem solvers from a young age, which is also absent from research on mathematical problem solving. Even though the behaviour for successful problem solvers is documented, it is still unclear what constitutes this behaviour (Schoenfeld, 1985). What seems clear, is that even though the educational changes soar in their significance, the key challenge within mathematics education (in South Africa) has always been ensuring that teachers and preservice teachers are equipped or trained to implement these new developments in their classrooms (Ader, 2019). The results of problem solving within this study might have metacognitive implications in the teaching and learning of mathematics, and geometry in particular. Even though there is not much of a gap in the research on mathematics education and metacognition, there is definitely a huge gap in geometry and metacognition, especially within the context of South Africa. In the current study, the focus was not so much on the second and third categories of metacognition as highlighted by Schoenfeld (1987), but mainly on the first one. Though PMTs' explanations of their thinking might be related to their control or beliefs and intuitions, the main concern in this study was really for PMTs to reveal their thinking, mainly because of the need to support them to develop better skills of reflecting on their thinking processes and perhaps use the same concept in their teaching.

2.6 MEANING MAKING IN MATHEMATICS

As an opening statement for this subsection, I would like to give an idea put forward by Duval and in my discussion of meaning making, I will refer continuously to this statement. He states,

[11] This is discussed in 2.2.2, 2.2.3 and 2.2.4.

"no kind of mathematical processing can be performed without using a semiotic system of representation, because mathematical processing always involves substituting some semiotic representation for another" (Duval, 2006, p. 107). One of the main aims for teaching mathematics is to provide learners with opportunities where they can construct and develop their own meaning through signs and symbols used in mathematical activities and processes. Educational research and school practice have continuously been in search of teaching and learning strategies that can alleviate learners' difficulties in mathematical problem solving for millennia. Despite having a vast body of literature in the psychology, philosophy and neuroscientific domains investigating effective methods of teaching and learning, there has not been a clear solution. Several laboratory experiments that show promise in improving the process of teaching and learning have been conducted, but still the replication in real classrooms has not been a successful one, perhaps because the transition from theory to classroom practice is a difficult one for teachers not trained in that regard. Amongst some proposed teaching and learning strategies, teaching for meaning making has been given a prominent role in recent research in mathematics education. Meaning making is a conglomerate and it is characterised by multimodality. The interaction between meaning, sense, embodiment, semiotics and discourses in mathematics mainly characterises this multimodality. In this section, I attempt to elucidate mathematical meaning making as a multimodal process requiring the connection between different modes of learning in mathematics. I will try to explain the process of meaning making as a relational process (Barwell, 2018), with meaning representing the relationship between the mind and the world (Leontiev, 2017). Blumer (1986, p. 5) asserts:

> [M]eaning is seen as social products, as creations that are formed in and through the defining activities of people as they interact … meaning of things is formed in the process of social interaction and is derived by the person from that interaction.

Blumer (1986, p. 12) further asserts "the meaning of anything and everything has to be formed, learned, and transmitted through a social process of indication – a process that is necessarily a social process". For Blumer (1986, p. 11), "the meaning of objects for a person arises fundamentally out of the way they are defined to him by others with whom he interacts". Put more succinctly, meaning does not exist in symbols, but in the activity patterns and multifaceted activity emerging from interactions about these symbols (Varela, Thompson, & Rosch, 2016). Hence, the process of making meaning is facilitated by discourse and exists in discourse. I consider Blumer's (1986) ideas about symbolic interactionism crucial in guiding my discussion of ideas about meaning making that are critical in this study, in the following paragraphs.

Much classroom culture within SA mathematics education has adopted that learning is about acquiring mathematical facts and procedures for regurgitation in the examination. Even though research and educators collaborate to improve this futile adoption of the acquisitionist perspective for a more participative one, the change seems difficult for unskilled teachers and time-consuming for those who can implement it (Voigt, 1998). Dewey (1933) says that thinking is usually confined to objects that cannot be directly perceived. In agreement with Dewey (1933), Radford, Schubring, and Seeger (2011) assert that meaning does not always refer to the existence of a concrete object, but resides in the collective and cultural activities in which these objects evolve. This is a reiteration of Vološinov's (1973, p. 28) assertion.

> After all, meaning can belong only to a sign; meaning outside a sign is a fiction. Meaning is an expression of a semiotic relationship between a particular piece of reality and another kind of reality that it stands for, represents, or depicts. Meaning is a function of the sign and is therefore inconceivable (since meaning is pure relation or function) outside the sign as some particular, independently existing thing.

Vološinov (1973, p. 23) goes on to state, "a sign is a particular material thing, but meaning is not a thing and cannot be isolated from the sign as if it were a piece of reality existing on its own part from the sign." For example, we count objects by observing them. If we could not see (say, if our eyes were closed or we were blind), we would not be able to count these object unless we can rely on experience. By touching the objects, we invoke a certain image in our brains that reminds us of the process of counting. To regress: we assign a certain number to the objects after counting them. The object is something that is concrete, but the number is not, the number exists in relation to the objects being counted. This is exactly what Vološinov (1973) refers to regarding the inextricable relationship between the sign and meaning. Evidently, meanings of concepts, mathematical or whatever, do not refer to a particular concrete object existing in the world, but is embedded in discourse in which it evolves (Radford et al., 2011). The view that meaning arises from, and is inseparable from signs, has significant implications in mathematics education, as by its nature it is considered a symbolical activity.

Meaning has to be connected to its historical, political and cultural contexts from which it emanates, but more than that, the dynamics (cultural and political) of the classroom are important considerations for meaning making. There seems to be a very close relationship between 'meaning' and 'sense', but in fact, they are different constructs that are related in a particular way. While meaning is considered to be "a historical and ideal entity", sense is "personal, *and* subjective" (Radford et al., 2011, p. 153). Thus, people make sense of situations based on their motives of the activity (Leont'ev, 1978). Seeger (2011) presents the case of Thomas trying to make sense of the number zero, where he portrays how the process of making sense becomes an intimate experience to ones' motives for learning. So far, the discussion points out that meaning does not exist in isolation from signs (symbols), and people make meaning from these signs through the social process of interaction. The situated nature of meaning so far necessitates the discussion of the role of the body in the process of making meaning.

It has long been noted, mainly by cognitive theorists, that the brain is important during learning. For example, Piaget (1964) describes how during the sensorimotor developmental stage, a child's experiential learning is used for learning at a later developmental stage. Traditional cognitive scientists claim that learning is something that only occurs in the brain and all learnt information is stored in the brain (Shapiro, 2007), without any interaction with the environment. However, Vygotsky (1978), with his sociocultural theory of learning, explains how cognition occurs through social interactions where language and culture play a significant role. From this development, Vygotsky (1978) stresses that even though the brain is important for learning, it operates and interacts with the real world (see also Clark, 1998). Mathematical thinking, in this case, is not viewed as transcending the universe, but is shaped by the human body, the neural structure of our brains and our experiences (Lakoff & Johnson, 1999). This conceptualisation means that mathematical teaching and learning must continuously involve the body, coined embodied learning (cognition). Recently, theories related to embodied cognition, are now showing that the brain is not an entity separated from the body (Macedonia, 2019), and that it is important for learning and revealing mathematical thinking. Embodied cognition is a theory within cognitive sciences, relating our cognitive processes to our body and the environment (Shapiro, 2007; Skulmowski & Rey, 2017), which has taken a loan from recent educational research, especially in mathematics education (Abrahamson & Bakker, 2016; Bohlmann, 2019; Cooper & Tisdell, 2020; Macedonia, 2019) from various methodological and theoretical positions (Burte, Gardony, Hutton, & Taylor, 2017; Edwards, 2019; Nathan & Walkington, 2017; Radford, Edwards, & Arzarello, 2009; Siposova, Tomasello, & Carpenter, 2018; Tran, Smith, & Buschkuehl, 2017). Hence, meaning making is important in the process of developing thinking in mathematics. Embodiment entails:

> (1) cognition dependent upon the kinds of experience that come from having a body with various sensorimotor capabilities; and (2) individual sensorimotor capabilities that are

> themselves embedded in a more encompassing biological and cultural context … sensory and motor process, perception and action, are fundamentally inseparable in lived cognition (Varela, 1999, pp. 11–12).

This focus on the corporeality has transformed the way we understand mathematical thinking by producing resounding evidence that the body is thoroughly involved in the process of meaning making and may reveal one's thinking (Bohlmann, 2019; Goldin-Meadow, 2017; Novack & Goldin-Meadow, 2015).

Embodied cognition acknowledges that cognitive processes are interlinked with physical actions (or experiences) within the environment (Cooper & Tisdell, 2020), and that the process of making meaning is a consequence of our awareness acquired from our sensorimotor reflexes. Embodied mathematical learning is not seen as a mere memorisation of rules not related to reality, but as a product of human participation in both historical and personal discourses (Edwards, 2019). Hence, mathematical teaching and learning should be about the (re)construction of mathematical knowledge through the integration of physical experiences and cognitive processes. Earlier, in 2.6, I gave an example of counting objects in order to assign mathematical meaning to them. People's use of their fingers, and sometimes toes, for counting (Fischer & Shaki, 2014), grounds actions to support the production of proofs (Nathan et al., 2014), which need to be coordinated through language (Nathan & Walkington, 2017). The use of paper folding and origami through learners' participation in the Th!nk3d! learning programme (Burte et al., 2017), is sufficient evidence that different mathematical aspects are embodied. Though it may seem only as something abstract, mathematical proof is an embodied phenomenon not only limited to the abstract domain of mathematics, but to historical, cultural and political domains as well (Edwards, 2019).

Teachers' use of body movements, their posture, tone, looking directly at a learner when they ask questions or make a contribution in class, assuming a formal stance by making certain continued body movements consistent to their utterances and other embodied actions, are useful in the orientation of the mathematical meaning they convey about meaning to learners (Bohlmann, 2019). For example, Goldin-Meadow, Kim, and Singer (1999) show that when speech and gestures are used in a matching manner, learners capture the mathematical concepts better than when speech does not match the gestures (see also, Alibali et al., 2013). In support of these studies, Cook, Friedman, Duggan, Cui, and Popescu (2017) further conclude that gestures can be useful in facilitating mathematical learning. In developing a grounded and embodied mathematical cognition theory, Nathan and Walkington (2017) support the use of gestures as an embodied action to support spatial thinking, visualisation and the production of geometry proofs. In an earlier article, Nathan and colleagues assert that proof production involves the use of gestures, speech, and notation which clarifies geometry as an embodied subject of mathematics (Nathan et al., 2014). It seems in this way that mathematical thinking is an embodied activity, not only requiring the mind, but sensorimotor skills as well. Despite the enormous body of literature supporting the effectiveness of embodied learning in mathematics, Yeo, Ledesma, Nathan, Alibali, and Church (2017) find that gestures were not successful in facilitating the learning of equations, showing that the combination of speech and gestures is not always successful in mathematics. They argue this for actions like 'saying 2 while pointing to 2'. However, in the same study they argue that in instruction related to graphs, gestures were of some benefit. It is through the use of body movements that learners might extract meaning from certain mathematical concepts. However, actions such as gesturing might not be as useful as embodied learning methods in the hands of a novice that will use gestures in ways that contradict their speech (Goldin-Meadow, 2010). During classroom discourse, learners utilise different gestures that portray different meanings about their learning. Learners' gestures can indicate moments of conceptual difficulties, and teachers may utilise these gestures to access learners' thinking to support their development of mathematical thinking (Novack & Goldin-Meadow, 2015). Hence, a good use of gestures in mathematics classrooms can extend from reflecting learners' thoughts to also changing them (Goldin-Meadow, 2017). The symbols and signs used by teachers in the classroom have the promise of developing learners' mathematical thinking.

So far, it is clear that signs are important in the process of meaning making in mathematics, specifically since semiotics is embedded in mathematics. As Duval (2006) says, all mathematical processes require some semiotic system of representation, failing which, mathematics processes cannot be completed. He attributes the development of semiotics to be the indispensable form of developing mathematical thinking (Duval, 2006). Mathematics teaching and learning at early stages should focus on the development of signs and symbols by learners. Proficiency in mathematical semiotics has positive effects on mathematical problem solving. In fact, successful geometry problem solvers can interchangeably utilise both visual and symbolic actions during problem solving (Ubah & Bansilal, 2019).

Semiotics is the theory and/or the study of signs (Oehler, 1987; Suhor, 1984) and it further theorises how signs signify (Radford et al., 2011), which allows us to study the role of signs in the process of meaning making in mathematics. Signs are concrete things, marks or tokens that have the ability to direct our attention to something else (Otte, 2006). In his foundational conceptualisation of semiotics, Peirce (1982) makes it clear that a sign is a placeholder for something else and contains meaning for someone (Oehler, 1987; Otte, 2011). For Peirce (1982), the sign has a triadic structure "object-representamen-interpretant" (Dörfler, 2016, p. 22). This means that the sign stands for a certain object which it represents, and the meaning of the sign makes sense only to the interpretant of that sign. Due to its focus on signs and their meaning(s), semiotics is well suited to study mathematics teaching and learning.

Mathematical thinking focuses on processes instead of content. The process involved in deriving meaning from signs depends on the interaction between the interpretant and the sign in discourse (Plowright, 2016). Hence, semiotics explains how meaning works, and how individuals make meaning through signs (Radford et al., 2011). The tenets of semiotics in mathematics education provide evidence to propel the belief that meaning is extracted from signs during social interactions (Plowright, 2016). They believe that signs are the vehicle in which the process of meaning making in mathematics rides (Kinach, 2018; Plowright, 2016; Presmeg, Radford, Roth, & Kadunz, 2018; Tomé, Purwanto, & Sa'dijah, 2019). Hence, meaning making in mathematics relies heavily on semiotics (Groth, Jones, & Knaub, 2018), and semiotics is the epicentre of mathematical activity (Duval, 2006), a vital element of geometry learning. Geometry teaching and learning are usually comprised of the utilisation of diagrams and/or visual images to help support learners' conceptual development. The evidence by Ngirishi and Bansilal (2019) that learners failed to solve geometry problems that did not have any diagrams accompanying them, and when trying to produce their own diagrams from the written statements, learners produced representations that did not correspond with the statement, which provides evidence that the study of geometry and problem solving should be accompanied by the use of diagrams for learners to draw meaning from, and visualise the problem. In fact, Peirce (1982) in his account of *diagram and diagrammatic reasoning*, considers that diagrams are objects and the means for mathematical thinking (Wille, 2019). Hence, teachers need to ensure that their learners are exposed to multiple registers of representing mathematical knowledge for better meaning making (Duval, 2006).

Considering the use of signs and symbols is not only important in understanding the process of meaning making in mathematics, but also in analysing thinking discourses during mathematical activities. If we were to analyse thinking during mathematical activity, we need to understand the relationship between different systems of representation (Duval, 2006). The focal point of this subsection is that meaning making is mediated by social interactions. Given the focus of the current study on teacher discourses, it is not just necessary but crucial to discuss the dialogical nature of mathematical thinking.

2.7 THE DIALOGICAL NATURE OF MATHEMATICAL THINKING

As opening lines to this section, I quote the words of Albert Einstein, cited in Butt (2019) and Freudenthal (1979). Albert Einstein (cited in Butt, 2019, p. 1) stated, "education is not the learning of facts, but training the mind to think", while Freudenthal (1979, p. 324) asserts, "mathematics is both a natural and social activity of man, as are speech, drawing and writing". These are important quotes, not only in the development of mathematical thinking, but in the process of developing mathematical thought. In this section I discuss the dialogical nature of meaning making and mathematical thinking. I do this by discussing the foundations of dialogism and the foundations of dialogic teaching and how these two relate to the development of mathematical thinking.

2.7.1 Monologism and dialogism: coming to know mathematics

At first glance, we think of dialogue as communication, a basic form of social interaction used by humans every day. Communication is the main means to discourse,[12] a distinct typology of communication in its utilisation of accepted actions and reactions (Sfard, 2008). In one of his seminal writings, the *Problems of Dostoevsky's poetics*, Bakhtin (1984) differentiates between monologic and dialogic discourses. He mentions that a monologically understood world is objectified, and corresponds to a strong cohesive conscience (Bakhtin, 1984) attributed to alienated discourse during objectification (Sfard, 2008). Monological discourses are impersonal and each utterance exists as if it comes from the world itself instead of being created by human beings. Hence, monological discourses are disembodied. Teachers subscribing to monological discourses in the classroom teach mathematics as finished facts which speak for themselves,

[12] Sfard (2008, p. 297) defines discourse as "a special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with reactions".

instead of their meanings being created by humans.[13] An example of a shallow monologic discourse in the classroom is the triadic dialogue (Lemke, 1990), which can be seen as a build-up of Sinclair and Coulthard's (1975) initiation-response-feedback (IRF) idea, which is characterised by the teacher posing questions for learners, who in turn answer these questions, and the teacher provides feedback to the learners (see also Matusov et al., 2019).

In many countries, mathematics teaching and learning is conducted in a monological way of sharing universal meaning with complete ignorance of the individual differences that exist among the learners in the classroom (Nesari, 2015). Instead of constructing meaning from their own experiences of the world, learners are taken through a journey of accepting universal facts without question. If learning is viewed as occurring in the same way for all learners regardless of their contexts and needs, then personal differences that exist between human beings are disregarded, and it undermines the fact that teaching in its nature can never be monologic, because of the different voices that exist within the classroom (Matusov, 2007). Furthermore, Bakhtin (1984) is clear that in a monological setting true interaction in awareness is unlikely and thus dialogue as well. Monologism is common within SA schools where the curriculum and its mandate force teachers to transmit the content for learners to reproduce in the examination. Monological discourse has been regarded as shallow and does not promote mathematical thinking and meaning making. Instead, learners should be given learning opportunities that allow them to 're-invent' mathematics under the guidance of the knowledgeable other (Freudenthal, 1973, 1991; Vygotsky, 1978).

Due to these practical inadequacies of monologic discourses in the classroom, Bakhtin (1981) proposes that classroom interaction becomes dialogic. Dialogism refers to the awareness that everything implies, and is perceived as part of a greater whole, where there is a continuous exchange of meanings, many of which may affect others (Bakhtin, 1981). It is the realisation that any utterance utilised by people in communicating with each other, is internally dialogic. Furthermore, dialogism recognises that people make meaning through interactions and that all utterances have human authors and addresses (Sfard, 2008). Here, utterances live in the discourse authentically, they are not absolute, they are considered in relation to other utterances in the discourse, and meaning exists between voices instead of being universal. Hence, narratives about the world are dependent on their human authors and they can be rejected,

[13] The monological view of discourse is not far from what Freudenthal (1973) referred to as the 'anti-didactical inversion', where mathematical activity of others is used as a starting point of mathematical instruction instead of teaching the activity itself.

modified or endorsed through interaction with others. Dialogism, unlike monologism, is a way of life that accepts our cautious, multi-voiced existence (Middendorf, 1992). In dialogism, there is always an unending space for arguments since the dialogue reflects every interlocutor's point of view, and just like the metaphor of participation (Sfard, 2015), in dialogism meaning making occurs in the activity of participating within the dialogue as part of the community of practice.[14]

In dialogic classrooms, knowledge is not passively received, but exists in humans as a consequence of socially and actively engaging in interactive discourse. However, during these interactive discourses, it is significant to note that interlocutors bring specific value to mathematical practice, justify and support their particular methods, seek and develop their capacity to make use of them (Sanchez, 2003), through participating critically in the discourse. During the dialogue, problems arise in the discourse and interlocutors (especially learners and sometimes the teacher) should be able to recognise and solve these problems as part of their mathematical activity. Hence, dialogism positions learners as participants in the process of their enculturation to a certain discourse. Though Bakhtin's (1981, 1984) dialogism appears to be a complex theory, it defines our entire existence through the use of language (Holquist, 1990). In any social interaction we become dialogic when we critique each other, listen, and respect and give each other a chance to talk, reach shared understandings, and consider everyone's point of view. Hence, issues of multiple voices (*polyphony*) and multiple languages (*heteroglossia*) are important to consider in any dialogic interaction. Unlike monologism, dialogism holds that there are always different perspectives to view the same truth. However, one might question the nature of epistemology in the theory of dialogism. How do people become aware and create their own knowledge of a certain subject matter or event around the community?

Lockhart (2009, p. 68) laments:

> [T]he student victim is first stunned and paralyzed by an onslaught of pointless definitions, propositions, and notations, and is then slowly and painstakingly weaned away from any natural curiosity or intuition about shapes and their patterns by a systematic indoctrination into the stilted language and artificial format of so-called "formal geometric proof".

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[14] Matusov et al. (2019) differentiate between ethical and discursive dialogue. They state that ethical dialogue is concerned with interlocutors treating each other with utmost respect and giving each other equal opportunities, while a discursive dialogue concerns itself with the relevant modes, habits of expression and language in classroom dialogue that characterise pedagogical activity. Though the current study sees ethical dialogism as significant in the process of teaching and learning, the main focus here was on discursive dialogism, mainly to view PMTs' discursive narratives as they solve geometry problems individually.

This is clearly not a good approach to coming to know mathematics, because it portrays mathematics as being monological. The distinctive nature of learning is that it is a process of making meaning from different situations and scenarios, instead of acquiring knowledge as finished facts. Teaching mathematics as polished facts that do not relate to society, creates a huge conceptual gap in learners' mathematical learning and may be seen as causing demotivation for learners to continue engaging in mathematics. Instead, we should come to learn and understand mathematics through being active participants in mathematical discourse. Several researchers and projects currently are positioning the process of learning as something that one actively participates in, instead of just memorising facts (Freudenthal, 1979; Sfard, 2008; Van den Heuvel-Panhuizen & Drijvers, 2014; Yuanita, Zulnaidi, & Zakaria, 2018).

Mathematics is not detached from society where all human activities occur, hence, it can be considered a human activity (Freudenthal, 1973). Mathematics is all around us and if we present it to learners as a formal closed system (like mathematicians view it), then we are eliminating the significance of relating formal mathematics to society. Freudenthal (1973) sees learning mathematics as an open activity that includes what Van den Heuvel-Panhuizen and Drijvers (2014) extend to mean students should not only mathematise reality, but also mathematics. Mathematics classrooms are very special learning environments and what happens inside them is paramount in fostering meaning making. Lunsford (1990, p. 76) describes teachers as "dialogic, multi-voiced, heteroglossic", and characterises teachers' classroom environments as "sites for dialogues and polyphonic choruses". Following Lunsford's idea, I posit that we come to know mathematics by being active participants in these sites of dialogues. We come to know mathematics by engaging in 'ontological dialogue' in which learning something new becomes an intrinsic value, viewed as an internal pleasure, and an interesting challenge (Matusov & Miyazaki, 2014). We come to know mathematics through ideologically belonging in the mathematics community, and through our interaction with others in mathematics, the community develops our own way of viewing and understanding mathematics which involves discourses about signs in our experiences (society) and history (Bakhtin, 1986; Lave & Wegner, 1991; Sfard, 2008). During the activity of coming to know mathematics as a situated subject through discourse, the role of language is important.

2.7.2 Heteroglossia and polyphony: the ubiquities of dialogism

In Chapter 1 I have explained language as the primary semiotic tool for making meaning in mathematics, and how it can reveal one's thinking in discourse. Let me now focus on how language is utilised in discourse to support the process of mathematical meaning making through explaining heteroglossia and polyphony. The terms heteroglossia and polyphony are the main foundations of Bakhtin's (1981) dialogic stance on how human beings gain consciousness of the world around them. Heteroglossia refers to the simultaneous usage of various forms of voices or signals, conflict between them and their intertwined relationships within discourse (Ivanov, 1999). Heteroglossia concerns itself with multilingualism, including intra-language, and language relating to age and profession (Bailey, 2012). Tall (1989, p. 28) articulates this multivoicedness of language through the way in which people see the word 'proof' in different professions. He says:

> The term 'proof' is just such a word. In different contexts it means very different things. To a judge and jury, it means something established by evidence 'beyond a reasonable doubt'. To a statistician it means something occurring with a probability calculated from assumptions about the likelihood of certain events happening randomly. To a scientist it means something that can be tested - the proof that water boils at 100 °C is to carry out an experiment. A mathematician wants more - simply predicting and testing is not enough - for there may be hidden assumptions (that the water boiling is always carried out at normal atmospheric pressure and not, say, on the top of Mount Everest).

Hence, heteroglossia considers the stratification of language and its functionality within different contexts and groups in society. Language is culturally diverse (Cordeiro, 2018) and there exists no unitary language (Bakhtin, 1981). The existence of a unitary language will not allow mathematical meaning making or the expression of mathematical truth from different linguistic semiotics. The existence of heteroglossia means that the process of making meaning or thinking in mathematics may not be limited to a single language, but into the pool of languages out there, even though the endorsement and usage of that meaning is mathematical. Hence, meaning making is 'heteroglossic', and incorporating heteroglossic discourse in mathematics allows learners to construct meaning from their different social histories, cultures and identities.

The simultaneous significance of each interlocutor in any dialogue brought by heteroglossia extends to the idea of polyphony. In *Problems of Dostoevsky's poetics*, Bakhtin (1984, p. 7) delineates polyphony as:

> [A] character's word about himself and his world is just as fully weighted as the author's word usually is; it is not subordinated to the character's objectified image as merely one of his characteristics, nor does it serve as a mouthpiece for the author's voice.

By this statement, Bakhtin asserts that in any dialogue every voice and utterance may represent different ideologies within the dialogue. It is argued that Bakhtin's polyphony and dialogism are synonymous (Lodge, 1990), because polyphony describes a dialogic environment where the voices of many are heard, instead of just one (monologic) voice. This is a significant stance in teaching, because a classroom dominated by a single voice usually favours rote learning while engaging different voices usually favours meaning making and develops thinking. Polyphony allows interlocutors a great deal of freedom to interact with each other, allowing the existence of different ideologies about the same phenomenon (Nesari, 2015).

Furthermore, Bakhtin (1981) asserts that the time of the language (or dialogue) and space within which it occurs is important, and he coins it the chronotope, which he describes as time-space, a unit of research to analyse texts according to the ratio and essence of the divisions of time and space represented. According to Bakhtin (1981), time and space co-exist in any dialogue and can never be separated, and indisputably, discourse occurs in both time and space. These two are continuously changing in any dialogue, and as people enter the dialogue, they make it more important than their ownership of knowledge (ideas) (Wegerif, 2016). The dialogic space is a space of mutual resonance where interlocutors either merge or clash with each other, stimulate each other to come up with new ideas. It is the space of genuine dialogue for meaningful learning (Wegerif, 2017). Hence, teachers need to create a safe dialogic space for learners to engage critically for their development of mathematical thinking (Lambirth, 2016; Teo, 2016). However, knowledge of self is also significant when one participates in dialogue. In this view, dialogue is intersubjective, and through Buber's (1970) interpretation of the 'I–Tho', it is clear that dialogue is a living thing that involves living voices and is enculturated in society (see also Wegerif & Major, 2018). The construct of dialogic space extends to dialogic teaching, a framework that does not only aim to engage learners in dialogue, but opening spaces where learners can ask about the thoughts and viewpoints of their classmates, mentors or textbooks, such that information is more discussed and built up, rather than automatically conveyed from instructor (or textbook) to student (Alexander, 2008, 2010).

2.7.3 Dialogic teaching

Children conceptualise meaning not only through the interplay between what they learn anew and their experiences, but also through interaction with others (Alexander, 2010). The conceptualisation of this meaning occurs in the zone of proximal development (ZPD) (Vygotsky, 1978) and others are critical in scaffolding children's thinking and meaning making (Wood et al., 1976). Hence, dialogic teaching believes that the development of mathematical thinking emanates from social interactions to individual analysis of learning situations (Vygotsky, 1986). Here, learners actively participate in the process of their development in mathematical thinking, while teachers continuously intervene meaningfully in learners' development of mathematical thinking through the use of dialogic talk (Alexander, 2010; Díez-Palomar & Olivé, 2015). Despite the aged body of literature on the significance of speech in learning, the Spoken Language and New Technology (SLANT) project by Mercer, Phillips, and Somekh (1991) makes a breakthrough in identifying a triadic typology of talk learners engaged in during their talk in the computer. As seen in the previous discussions, talk is critical in the process of teaching and learning, and stimulating, promoting and encouraging the growth of logical thought amongst learners through talk, is widely referred to as dialogic teaching. Here, the term dialogic teaching is used to connote the type of discourse that draws on principles of dialogism as explained by Bakhtin (1981), and exploratory talk (Mercer et al., 1991) with a heavy reliance on the work of Alexander (2020) on dialogic teaching. However, a clear definition of dialogic teaching appears in the work of Alexander (2020), who asserts that dialogic teaching is a pedagogy of the spoken word that harnesses the power of dialogue to stimulate and extend students' thinking, learning, knowing and understanding, and to enable them to discuss, reason and argue. Alexander (2020) argues that dialogic teaching unites the oral, cognitive, social, epistemic and cultural, and therefore manifests frames of mind and value as well as ways of speaking and listening (Alexander, 2020). Since I have explained the principles of dialogism above, let me now briefly explain exploratory talk as a product of the SLANT project.

Talk is ubiquitous in any classroom environment, it is the paramount 'artefact' for meaning making and it is definitely a central tool for pedagogy (Alexander, 2010, 2020). However, the type of talk that exists between learners and the teacher is a signifier of successful or unsuccessful teaching and learning. The SLANT project found types of talk evident in children's talk in the computer, disputational, cumulative and exploratory talk (Mercer, 2000).

Disputational talk is characterised by authoritarian and monologic utterances, who view their ideas as the only truth. Interlocutors are unwilling to accept the opinions of other individuals based on a constant reaffirmation of their own (Mercer, 2000). Hence, they are continuously in disagreement with one another and everyone makes their own decisions, their interaction is characterised by utterances such as, 'No! it is not that' or 'I disagree with you' (Wegerif & Mercer, 1996). The activity within such a classroom is more about competing with each other, instead of cooperating with each other (Mercer, 2000). Furthermore, interlocutors do not usually make attempts to pull out resources to support each other and do not offer constructive criticism. The text below shows an example of disputational talk.

Alpha:	*Did you know that a long time ago, human beings were apes?*
Omega:	*Then what happened?*
Alpha:	*According to Charles Darwin, they evolved.*
Omega:	*But, I did not evolve, my mother gave birth to me as a human, not an ape.*
Alpha:	*That is because your mother has already evolved to be a human, therefore, she cannot give birth to an ape. Humans give birth to humans and apes give birth to apes.*
Omega:	*No, but according to reproduction people get pregnant and give birth to humans, does that mean we are still going to evolve?*

In contrast, cumulative talk occurs when presented ideas are only accepted or rejected by interlocutors without critical evaluation. Here knowledge is grounded on and endorsed through mutual agreement between interlocutors, and knowledge development is through mutual agreement (Atwood, Turnbull, & Carpendale, 2010; Mercer, 2000). Their discourse is characterised by repetition and elaboration of each other's ideas without critical evaluation (Wegerif & Mercer, 1996) to promote group harmony and cohesion (Mercer, 2000). The short discussion below represents an example of cumulative talk.

Alpha:	*My teacher told me that an even number is a multiple of 2.*
Omega:	*I was also told the same thing, and that* $2x$ *is an even number because of the coefficient of 2.*
Beta:	*Yes, as long as we multiply any integer by 2 the result is always even, just like* $2(x^2 + 3x)$
Omega:	*What about* $2(x+1)$ *and* $2x + 1$*? Are they even?*

Alpha: *So, I am right to say that an even number is any number that is divisible by 2.*

Omega and Beta (simultaneously): *Definitely.*

Short as it is, the extract exemplifies the nature of cumulative talk. Be aware that unlike disputational talk, interlocutors are not disagreeing with each other, but they build on each other's ideas and accept each other's ideas mutually. They do not challenge each other, because that might disturb the harmony between the interlocutors. Some views within the group are ignored by everyone to display and maintain unity within the group. For them, the group identity is more important than individual identity (Mercer, 2000; Wegerif, 2011). These two types of talk lack critical appraisal and collective guarantees (Atwood et al., 2010), which are at the core of dialogic discourse. They differ in that disputational talk possesses attributes of being oppositional and defensive, while cumulative talk is optimistic and supportive (Atwood et al., 2010; Mercer, 2000). The third type of talk is dialogic, and has been coined as exploratory talk. It is closely related to dialogic talk, contains traces of Mercer et al.'s (1991) principles of talk, and resonates with Alexander's (2010) principles for dialogic teaching. Mercer (2000, p. 98) states that in exploratory talk: –

> [P]artners engage critically but constructively with each other's ideas. Relevant information is offered for joint consideration. Proposals may be challenged and counter-challenged, but if so reasons are given and alternatives are offered. Agreement is sought as a basis for joint progress. Knowledge is made publicly accountable and reasoning is visible in the talk.

Interlocutors listen carefully to one another and they ask critical questions. Everybody is seeking to gain valuable knowledge so they can exchange useful details and criticise each other's ideas objectively. Furthermore, critical and well-thought out reasons are provided for all challenges made and it is done with respect[15] and the purpose of working towards a shared understanding. The dialogue builds on what has been endorsed before, all interlocutors are encouraged to participate instead of a selected few, and the community endorses knowledge, and not individuals within the community. One seminal example of exploratory talk is the dialogue employed by Lakatos (1976) in elucidating proofs and refutations in mathematics, where the interlocutors really challenged each other's ideas and provide critical evidence to

[15] See Matusov et al.'s (2019) conceptualisation of ethical discourse and ethical dialogism already described above in footnote 14.

support their ideas. The extract below is from Hutching's (2007, pp. 6–8) handout for courses that require proof, and shows an example of exploratory talk between two interlocutors, Alpha and Beta.

Alpha: *I've just discovered a new mathematical truth!*

Beta: *Oh really? What's that?*

Alpha: *For every integer* x*, if x is even, then* x^2 *is even.*

Beta: *Hmm … are you sure that this is true?*

Alpha: *Well, isn't it obvious?*

Beta: *No, not to me.*

Alpha: *OK, I'll tell you what. You give me any integer* x*, and I'll show you that the sentence 'if* x *is even, then* x^2 *is even' is true. Challenge me.*

Beta (eyes narrowing to slits): *All right, how about.*

Alpha: *That's easy. 17 is not even, so the statement 'if 17 is even, then* 17^2 *is even' is vacuously true. Give me a harder one.*

Beta: *OK, try* $x = 62$

Alpha: *Since 62 is even, I guess I have to show you that* 62^2 *is even.*

Beta: *That's right.*

Alpha (counting furiously on her fingers):

According to my calculations, $62^2 = 3844$*, and 3844 is clearly even …*

Beta: *Hold on. It's not so clear to me that 3844 is even. The definition says that 3844 is even if there exists an integer* y *such that* $3844 = 2y$*. If you want to go around saying that 3844 is even, you have to produce an integer* y *that works.*

Alpha: *How about* $y = 1922$.

Beta: *Yes, you have a point there. So, you've shown that the sentence 'if* x *is even, then* x^2 *is even' is true when* $x = 17$ *and when* $x = 62$*. But there are billions of integers that* x *could be. How do you know you can do this for everyone?*

Alpha: *Let* x *be any integer.*

Beta: *Which integer?*

Alpha:	*Any integer at all. It doesn't matter which one. I'm going to show you, using only the fact that* x *is an integer and nothing else, that if* x *is even then* x^2 *is even.*
Beta:	*All right ... go on.*
Alpha:	*So suppose* x *is even.*
Beta:	*But what if it isn't?*
Alpha:	*If* x *isn't even, then the statement 'if* x *is even, then* x^2 *is even' is vacuously true. The only time I have anything to worry about is when* x *is even.*
Beta:	*OK, so what do you do when* x *is even?*
Alpha:	*By the definition of 'even', we know that there exist at least one integer* y *such that* $x = 2y$.
Beta:	*Only one, actually.*
Alpha:	*I think so. Anyway, let* x *be an integer such that* $x = 2y$. *Squaring both sides of this equation, we get* $x^2 = 4y^2$. *Now to prove that if* x^2 *is even, I have to exhibit an integer, twice which is* x^2.
Beta:	*Doesn't* $2y^2$ *work?*
Alpha:	*Yes, it does. So we're done.*
Beta:	*And since you haven't said anything about what* x *is, except that it's an integer, you know that this will work for any integer at all.*
Alpha:	*Right.*
Beta:	*OK, I understand now.*
Alpha:	*So here's another mathematical truth. For every integer* x, *if* x *is odd, then* x^2 *is ...*

Hence, exploratory talk is the only talk between the three options that promises good results in supporting the development of mathematical thinking and meaning making (Calcagni & Lago, 2018; Kim & Wilkinson, 2019; Rabel & Wooldridge, 2013; Wegerif, 2008).[16] This is supported by Alexander (2010), who argues that not just any talk is suitable for meaning making in dialogic teaching, but an interactive type of talk guided by the principles of dialogic talk. The focus of exploratory talk on heteroglossia and polyphony and embodied dialogue in the acculturation of others' contexts, qualifies it as an efficient talk to drive dialogic teaching

[16] Alexander (2010) describes four repertoires of talk that explains the usage of talk in various situations, talk for everyday life, talk for teaching, talk for learning, and organisational contexts.

(Hermans, 2001). Principles of dialogic teaching include collectivity and reciprocity, and it is supportive, cumulative and purposeful (Alexander, 2010, p. 38). Each characteristic is explained below.

> Collective – Teachers and learners address learning tasks together, as a group or as a class.
>
> Reciprocal – Teachers and learners listen to each other, share ideas and consider alternative viewpoints.
>
> Supportive – Learners articulate their ideas freely, without fear of embarrassment over 'wrong' answers; and they help each other to reach a common understanding.
>
> Supportive – Teachers and learners build on their own and each other's ideas and chain them into coherent lines of thinking and enquiry.
>
> Purposeful – Teachers plan and steer classroom talk with specific educational goals in mind.

Furthermore, a dialogic teaching framework is supported by four elements, which are justifications, principles, repertoires, and indicators (Alexander, 2018). Most importantly, dialogic teaching encourages learners to share their thinking within the classroom environment, which affords the teacher an opportunity to notice and support their thinking (Alexander, 2020), and is currently the most promising teaching approach (Alexander, 2019). Dialogic teaching can be used for improving the standard of classroom talk as a means of enhancing the participation, learning and achievement of learners, especially those from underprivileged backgrounds (Education Endowment Foundation [EEF], 2017). In dialogic teaching, meaning making becomes an unending activity of discourse interaction between interlocutors, focusing on 'mathematising', instead of the mathematised (Björklund, Magnusson, & Palmér, 2018). In mathematics classrooms it is then the type of teaching that utilises exploratory talk to allow learners to participate in situated mathematical discourse.[17] It encourages learners to scrutinise ideas, proposals and opinions from their teachers, peers, and other information sources in order to maximise the dialogic space and meaning making (Alexander, 2008). Children in dialogic classrooms were shown to be two to three months ahead in their mathematics, science and English learning compared to their counterparts in traditional classrooms (Alexander, 2018). It

[17] Sfard (2008, p. 128) describes the process of participating in mathematical discourse as mathematising.

has been shown to support the development of conceptual understanding in mathematics and science (Gillies, 2016; Gurbuz & Agsu, 2017), to be effective in scaffolding mathematical meaning making (Abdu, Schwarz, & Mavrikis, 2015; Kazak, Wegerif, & Fujita, 2015), can support the development of mathematical thinking and reasoning (Sedova, 2017), has the ability to improve learners' critical thinking dispositions (Hajhosseiny, 2012) and is a promising twenty-first-century teaching approach (Teo, 2019).

Given these strengths of dialogic teaching, it is critical that teachers are properly trained to use the power of talk to promote the growth of mathematical thought amongst learners (Hennessy, Dragovic, & Warwick, 2018; Van de Pol, Brindley, & Higham, 2017). Despite these merits of dialogic teaching, there are some obstacles which prevent the successful implementation of dialogic teaching, such as overcrowded classrooms (Gaillard, 2019; Marais, 2016; Matshipi, Mulaudzi, & Mashau, 2017), the preference for written work over verbal interactions (Dysthe, 1996), the lack of adequately trained teachers to facilitate classroom discourse, and the persistent belief in the transmission of mathematical facts to learners (Teo, 2019). Meaning is a dialogic relationship between genuinely interested questions and seriously provided answers (Matusov et al., 2019). Dialogic teaching and education have an epistemological belief that learning occurs as a consequence of 'actively' participating in the enculturation to a mathematics community of practice. This social practice of active participation requires that we have a clear understanding of the social world in which learning occurs (Lave & Wegner, 1991; Wegerif & Major, 2019). In the multicultural and multilingual classrooms of South Africa, the use of exploratory talk as the main means of meaning making in dialogic classrooms should cater to learners' home languages.

2.7.4 Discourse as means for developing mathematical thinking

Discourse is a word largely used in English and most recently in linguistic and educational research. According to the Oxford Dictionary, discourse is written or spoken communication or debate, a formal discussion of a topic in speech or writing, a connected series of utterances as a text or conversation. This seems to regard discourse as synonymous with a normal verbal interaction. It alludes to the activity of communicating, instead of its products (Sfard, 2012). Normally, any persons' day begins with discourse, like greeting family members, and in the case of a person living alone, thinking about taking a shower on a cold winter morning. Throughout the day, these discourses manifest themselves in different ways, for example making calls in the office, attending to students' questions during a lecture, etc. Discourse is a distinct type of interaction, characterised by its set of qualifying acts and the manner in which such acts are paired with reactions (Sfard, 2008). The meaning of discourse has proven to be dependent on semantics with a central agent of talk. Discourse-oriented researchers analyse talk to make meaning or theorise about a certain domain or phenomena. It is claimed by Cazden (2001, p. 2) that Douglas Barnes (a great reader in education from the university of Leeds) in 1974 produced a conference paper in which the following extract appeared.

> [S]peech unites the cognitive and the social. The actual (as opposed to the intended) curriculum consists in the meanings enacted or realized by a particular teacher and class. In order to learn, students must use what they already know so as to give meaning to what the teacher presents to them. Speech makes available to reflection the processes by which they relate new knowledge to old. But this possibility depends on the social relationships, the communication system, which the teacher sets up.

I believe this explains the increased need to understand classroom talk and how it influences teaching and learning. Furthermore, Descartes (1998) in *Discourse on method and mediations on first philosophy*, asserts that humans are able to arrange their talk through discourse where they make their thoughts understood. In his writings, Bakhtin (1981) makes it clear that discourse is a significant mediator of learning. In fact, Bakhtin displays in pure elegance how discourse is tantamount to thinking. What is evident in studies investigating classroom talk is that talk mediates classroom discourse and can reveal one's thinking. Even though Zack and Graves (2001, p. 265) argue, "*spoken words* do not equal *thoughts in the mind*", commognitive research shows that communicating is tantamount to thinking (Sfard, 2001, 2008).

There has been a huge increase in research aiming to understand classroom talk (Barnes, 1976, 2008, 2010; Kazak et al., 2015; Littleton & Mercer, 2013; Mercer, 2001, 2008; Mercer & Dawes, 2008; Mercer, Fernandez, Dawes, Wegerif, & Sams, 2003; Wegerif, 2011; Wegerif & Major, 2019). These studies investigate classroom talk from different perspectives, looking at the effects of talk on the outcomes of learning (Mehan, 1979), studying talk as a method of revealing one's thinking, focusing mainly on the development of discourse during learning (Kazak et al., 2015; Littleton & Mercer, 2013; Mercer, 2008; Mercer et al., 2003; Wegerif, 2011). Those that research classroom talk based on the relationship between talk in different domains, acknowledge that participation in a specific domain discourse is influenced by participating in the discourse of other domains (Bernstein, 1971). Specifically, Bernstein (1990) acknowledges the significance of understanding the rules of discourse for effective participation. Hence, the pervasive nature of talk has allowed it to be studied in different domains of life. In the linguistic domains, the most viable and dominant body of work is discourse analysis, where the researcher studies the relationship between text (spoken or written) and context (Brown & Yule, 1983). In the field of mathematical psychology, research focuses on the relationship between thinking and discourse and how these two aspects affect teaching and learning (Sfard, 2008). In education, discourse is studied as the means to mediate learning and thinking, investigating the impact of discourse on learning and the relationship between different domains of discourse. Discourse can help teachers figure out what their learners are thinking. For example, the extract below from Moschkovich (2007, p. 24) shows that learners are not content with the definition of a parallelogram, and this can serve as a starting point for further discourse in teaching and learning.

Teacher: [to the whole class]: *OK raise your hand, I want one of the groups to tell me what they do think. Is this [holding up a trapezoid] a parallelogram or not, and tell us why. I'm going to take this group right here.*

Vincent: *these two sides will never meet, but these two will.*

Teacher: *How many agree with that? So, is this a parallelogram or not?*

Students: *Half*

Teacher: *OK, if it is half, it is, or it isn't?*

Students: *Is*

Teacher: *Can we have half a parallelogram?*

Students: *Yes*

Teacher: *Yes but then could we call it a parallelogram?*

Students: *Yes*

Bakhtin (1981, 1986) asserts that classrooms are environments of dialogic polyphony and heteroglossia. In a classroom, there is a continuous exchange of words and points of view, and teachers need to ensure that everyone's voice is heard. Furthermore, classrooms are populated with learners continuously writing about what they think, and sometimes even presenting to their peers. In a mathematics classroom, analysing discourse has been made easy by the development of Sfard's (2008) commognitive framework. Since its inception, this theory has been used globally to analyse the mathematical discourse of learners (Berger, 2013; Coles & Sinclair, 2019; Heyd-Metzuyanim & Graven, 2016, 2019; Lavie, Steiner, & Sfard, 2019; Mpofu

& Pournara, 2018; Nachlieli & Michal, 2019; Roberts & Le Roux, 2019; Sfard, 2009). There have been studies that explored classroom discourse from the commognitive perspective in South Africa, focusing on calculus at university (Siyepu & Ralarala, 2014), algebra (Roberts & Le Roux, 2019), functions (Mpofu & Pournara, 2018) and teachers' discourse (Berger, 2013; Berger & Bowie, 2012; Van Jaarsveld, 2018). However, commognitive analysis of preservice teachers' discourse as they solve geometry problems in South Africa, has never been conducted before. Furthermore, this current research was a study of PMTs' thinking through their participation in discourse with the aim of understanding the significance of the coexistence of ritualistic and explorative participation in mathematics discourse, which is an avenue never explored before within the SA context.

2.8 THE SIGNIFICANCE OF DEVELOPING TEACHERS' VISUALISATION AND SPATIAL THINKING SKILLS

From a very young age children use spatial thinking to understand their surroundings. Children visualise how objects fit together when playing with assembling blocks and other toys. Concepts in geometry are generally studied with the essential aid of diagrams and shapes (Laborde, 2015). However, the SA diagnostic reports of Grade 12 mathematics learner responses since 2014 to date, have revealed that learners struggle with geometry-related questions and most of their struggles are related to their inability to utilise diagrams effectively (DBE, 2016, 2017, 2018b). Furthermore, SA learners struggle to work with geometry problems without a diagram connected to the problem (Mudaly & Rampersad, 2010; Ngirishi & Bansilal, 2019). There are continued difficulties that learners experience in their geometry learning and being behind in the curriculum (Spaull & Kotze, 2015). There was an exclusion of geometry from the compulsory curriculum in SA which might lead to SA teachers struggling to teach Euclidean geometry and not being at the required level to teach Euclidean geometry competently. All these can be related to teachers' lack of visualisation ability in geometry problem solving. Hence, the significance of visualisation in the development of competent mathematics teachers cannot be ignored.

Piaget's theory of cognitive development is monumental in the development of mental objects that become the foundation of making meaning of concepts (Piaget, 1969; Piaget & Inhelder, 1971). In Piaget's sensorimotor stage, children learn through spatio-temporal structures and in the semiotic stage, they rely on spatial skills and visualisation to learn more about their surrounding environment. However, in most cases, these spatial skills do not represent thought, but imitation (Piaget, 1969). At this stage children rely more on their visual approximations of things instead of using specific tools for certainty (Piaget, 1953). However, this is critical in their development of their thinking. Furthermore, the Van Hieles' seminal work in their development of the five levels of geometrical thinking, reveals that visualisation is the foundational level, the beginning of the development of geometrical thinking and reasoning, one that is always present in all the other levels (Usiskin, 1982; Van Hiele, 1959). Visualisation and spatial thinking play a significant role in the development of thinking in science, technology, engineering and mathematics (STEM) fields (Uttal & Cohen, 2012; Uttal, Miller, & Newcombe, 2013).

When I hear the word visualisation, I instantly think of something I can see, a visual object, a picture, a drawing or a tree. After immersing myself in the research on spatial thinking, visualisation and imagery in mathematics education, I can conclusively say that my thinking is not far off, nor was it complete. However, images can also be abstract (Johnson, 1987; Lakoff, 1987; Sfard, 1994; Tall & Vinner, 1981), which implies the existence of abstract visualisation, which most research refers to as mental imagery. In this way, images are intuitive and they represent abstract objects and events (De Leon, 2017). This is what Richardson (1969, pp. 2–3) refers to as mental imagery, which he defines:

> [A]ll those quasi-sensory or quasi-perceptual experiences of which we are self-consciously aware, and which exist for us in the absence of those stimulus conditions that are known to produce their genuine sensory or perceptual counterparts, and which may be expected to have different consequences from their sensory or perceptual counterparts.

Therefore, visualisation refers to the delicate process of forming abstract, concrete or dynamic images for decoding information during mathematical activity (Zimmermann & Cunningham, 1991). It is one's ability to create, interpret and reflect on images existing both concrete and in the brain, with the purpose of deciphering information, seeing what others might have not seen (Arcavi, 2003; Presmeg, 1986). A 2014 issue of the journal *ZDM Mathematics Education* provides ample evidence that visualisation is also a strong epistemological tool for learning mathematics (Presmeg, 2014; Rivera, Steinbring, & Arcavi, 2014). Hence, visualisation is concerned with both spatial and visual representations. Spatial representations are concerned with abstract representation existing in space, while visual representations are concrete and modality specific representations of the basic appearance of an object (Farah, Hammond, Levine, & Calvanio, 1988). Both spatial and visual representations depict information (Presmeg, 1986).

Spatial thinking and visualisation have long been recognised as important skills in mathematics and mathematics education. Einstein did most of his work driven by intuition and mental visualisation (Presmeg, 1997). Visualisation and spatial thinking are the easiest ways to learn, mainly because of their intuitive nature. Spatial thinking and visualisation have been correlated to mathematics achievement and performance (Fennema & Sherman, 1978; Geer, Quinn, & Ganley, 2019; Sherman, 1979, 1980). Other studies investigated the effects of particular interventions on performance in spatial tasks and visualisation. Burte et al. (2017) say that an embodied spatial training programme through Think3d! improves learners' spatial thinking, while Susilawati, Suryadi, and Dahlan (2017) provide empirical evidence that engaging preservice teachers in cognitive conflict improved their spatial visualisation ability, compared to their counterparts exposed to traditional teaching. Lowrie, Logan, and Hegarty (2019) show that spatial training does not only increase students' spatial thinking, reasoning and orientation, but their achievement in mathematics. Through the integration of GeoGebra, Dockendorff and Solar (2018) cement the significance of information and communications technology (ICT) in the development of preservice teachers' dynamic visualisation ability and their knowledge of teaching geometry. There have been studies focused on performance in spatial problem solving and gender differences. While some of these studies show that males perform better than females in spatial visualisation problem solving (Armstrong, 1981; Battista, 1990), other studies report no significant gender-related differences regarding spatial visualisation (Fennema & Sherman, 1978; Rodan, Gimeno, Elosua, Montoro, & Contreras, 2019; Sherman, 1980). Fennema and Tartre (1985) assert that spatial visualisation might weaken female learners' performance in mathematical problem solving more than that of male learners. However, the idea of gender differences in spatial reasoning and thinking is a myth, one that does not have compelling evidence in research (Newcombe & Stieff, 2012). Some studies assert that visualisation contributes to success in most STEM-related careers (Moe, Jansen, & Pietsch, 2018; Sorby, Veurink, & Streiner, 2018; Uttal & Cohen, 2012; Zhou & Lam, 2019) and artificial intelligence (Kunda, 2018). This puts visualisation at the epicentre of the development of competent mathematics teachers. Teachers' competency in visualisation ability does not only have the ability to improve their teaching of geometry, but their problem-solving abilities in computing multiple solutions to problems, demonstrating multiple-solution strategies to learners.

2.9 PINPOINTING THE GAPS IN LITERATURE: SITUATING THE CURRENT STUDY ON LITERATURE

In this section of the literature review, I pinpoint the gaps in literature that this study has addressed. Firstly, I discuss the literature on mathematical problem solving, then research on geometry teaching and learning with the focus on South Africa and then globally. Thereafter, I discuss commognition and how it has been used in the past to study mathematical thinking. These three main discussions are aimed at identifying the literature gaps addressed by the current study.

2.9.1 Research on mathematical problem solving

Problem solving has been on the agenda of mathematics education research for decades. Halmos (1980) argues that at the heart of mathematics exist ‘problems’ and the quest to find ‘solutions’ to these problems. Research in mathematics education has highlighted the significance of problem posing and problem solving as precursors for the development of mathematical thinking and teaching expertise. Kilpatrick (1987) asks a question about the source of good mathematical problems to provide an impetus for research to study problem posing in mathematics. Since then, generating new problems and reformulating pre-existing ones has received much attention in educational research with different foci, the main one being to display competency in teaching mathematics (Ernest, 1989; Silver, 1994). Teachers have vouched for teaching through problem posing, citing that it can be a vehicle for facilitating students’ mathematical thinking in the classroom (Li, Song, Hwang, & Cai, 2020). However, teachers’ abilities to pose mathematical problems depend on their level of mathematics training and their years of experience teaching the subject (Klein & Leikin, 2020).

Early researchers on problem solving, such as Polya (1973), argue that students become better problem solvers through imitating the actions of experienced problem solvers. However, recent research on problem solving reveals that students must solve many problems over a period of time to become good problem solvers and that teaching students problem-solving strategies and heuristics does not help them to become better problem solvers (Lester, 1994). Schoenfeld (1992) argues that the limitations of problem-solving heuristics (such as Polya’s [1973]), is that they require problem solvers to be familiar with the problem-solving strategies to implement the heuristic successfully. Recent findings further support this argument by concluding that following a step-by-step problem-solving heuristic did not improve students’ problem-solving competencies (Goulet-Lyle, Voyer, & Verschaffel, 2020). Furthermore, research has investigated the role of metacognition during mathematical activity like problem solving, and early findings (Garofalo & Lester, 1985; Lester, Garofalo, & Kroll, 1989; Schoenfeld, 1983, 1987, 1992) claim that metacognition is the driving force for mathematical problem solving, and performance with metacognitive knowledge, experience and skills influence problem-solving behaviour. Recent research on metacognition echoes the significance of metacognition in mathematical problem solving (Azevedo, 2020; Desoete & De Craene, 2019; Zhao et al., 2019). For example, metacognition has been associated with predicting problem-solving performance (Ebomoyi, 2020; Kuzle, 2018). Some researchers argue that failures in problem solving are not always brought about by the lack of metacognitive knowledge; the unawareness to activate knowledge during problem solving is also to blame (Shilo & Kramarski, 2019). In support, Erbas and Okur (2012) argue that success in mathematical problem solving cannot be judged based on metacognition only, since it depends also on problem solvers' mathematical knowledge and their repertoire of problem-solving strategies. Furthermore, metacognition is crucial in the understanding of ones' cognition, such that without fully understanding ones' metacognition, we cannot fully describe ones' thinking (Efklides, 2001). Furthermore, research on mathematics teaching and learning has signified the importance of teacher noticing and supporting students' problem-solving attempts with the aim of advancing their thinking in the classroom (Leikin, 2020).

The TMUF (DBE, 2019a), formulated around the idea of the National Research Council (2001) of mathematical proficiency to characterise successful mathematical learning, provides the impetus for implementing problem solving in learning-centred classrooms to improve mathematical thinking. However, the effects of this new reform in the teaching and learning of mathematics are still unknown. In fact, research focusing specifically on mathematical problem solving in SA education, has been stagnant for a while. Despite this, mathematical problem-solving related research in South Africa appears in literature, for example, research investigating the role of metacognition on problem solving in high school (Du Toit & Du Toit, 2013; Tachie, 2019; Tachie & Molepo, 2019) and at university (Jagals & Van der Walt, 2016). There have been fragments of research utilising Polya's (1973) heuristic to develop a model for problem solving in mathematics education (Nieuwoudt, 2015). Naroth and Luneta (2015) conclude that attempts to implement a problem-solving oriented curriculum within the SA mathematics education, has revealed that both teachers and students experienced difficulties. These findings are consistent with a recent study (Biccard, 2020), which concluded that preservice teachers have a poor understanding of problem solving and classroom norms that supported a problem-solving oriented teaching approach. In an earlier study, Chirinda and Barmby (2017) show that a professional development focusing on problem solving, improved learners' performance and prompted teachers to become facilitators of learning. Later on, Chirinda and Barmby (2018) conclude that contextual factors such as overcrowding, language difficulties, and pedagogical factors such as a lack of problem-solving strategies in students, forced teachers to retain ownership of the problem-solving activity through show-and-tell. Some studies related to teacher noticing during mathematics learning and problem solving, also appear in literature (Adler & Alshwaikh, 2019; Marais, Van der Westhuizen, & Tillema, 2013).

Thus far, research in mathematical problem solving provides evidence that problem posing is a higher order mathematical skill that depends on experience and content knowledge. Furthermore, it furnishes the mathematics education fraternity with frameworks of studying problem-solving behaviour, and through metacognition explains thinking during problem solving through observing problem-solving behaviour. Different models have been designed to support the teaching and learning of mathematics through the problem-solving approach in different global contexts. Despite the fact that this research values social interactions, mathematising and dialogic discourses in learning through problem solving, very little has been revealed about preservice mathematics teachers' discourse participation during geometry problem-solving activities. The main focus of such research has been to follow particular static frameworks or heuristics to study problem-solving behaviours and conclude about problem solvers' thinking during problem solving. Furthermore, the focus of such research is usually polarised into directions: either it is investigating the effects of a heuristic on problem solving (Nieuwoudt, 2015), or decision taking during problem solving (related to metacognition) without an attempt to balance the different foci of problem solving. However, there is a balanced distribution of problem-solving research into both learners and preservice mathematics teachers globally with more research focusing on learners in South Africa.

2.9.2 Research on geometry teaching and learning

Problem posing, conjecturing, proving and refuting are the core activities involved in the invention of mathematical knowledge (Clements, 2003). The general consensus in research is that mathematical proofs are the bearers of mathematical knowledge (Hanna & Barbeau, 2008) and Euclidean geometry is an area of school mathematics that is populated by mathematical proofs and proving. Therefore, doing well in Euclidean geometry proofs means that one possesses a good repertoire of mathematical knowledge. Proofs are at the heart of mathematical problem solving, they reincarnate various problem-solving strategies, many of which are used to solve abstract and real-life mathematical problems (Rav, 1999). The study of Euclidean geometry has the ability to enhance skills such as "visualization, critical thinking, intuition, perspective, problem-solving, conjecturing, deductive reasoning, logical argument and proof" (Jones, 2002, p. 125) and the ability to refute mathematical conjectures (De Villiers & Heideman, 2014). Despite the significance of proof and the role that Euclidean geometry plays in advancing ones' problem-solving abilities, research on the teaching and learning of Euclidean geometry globally reveals struggles in both teachers and learners (Gutiérrez & Jaime, 1999).

In 2.8 I provided an intensive review of literature to cement the general findings that in South Africa, mathematics teachers, preservice teachers and learners have poor knowledge of Euclidean geometry. While research has been successful in determining in-service teachers', preservice teachers' and learners' Van Hiele level of geometrical thinking (cf. Van Putten et al., 2010), I have not come across a study linking their Van Hiele level of geometrical thinking to their choice of problem-solving strategy, in order to understand how their levels of geometrical thinking affect their discourse participation and problem solving. Furthermore, little is known about how problem solvers come to understand Euclidean geometry problem solving and construct solutions to Euclidean geometry-related problems. Furthermore, I have not encountered a research study that focuses on SA preservice teachers' competency and proficiency in solving 'riders' in Euclidean geometry, yet this is an integral part of teaching and learning in the CAPS, especially for the FET phase (Grades 10–12). This gap will be addressed intensely by the data obtained to answer the second research question.

2.9.3 Commognition and the study of mathematical thinking

In this section I focus my discussion on the research conducted under the theoretical lens of commognition and how the gaps left by such studies are going to be addressed by the current study. I do this by first differentiating between the acquisitionist and participationist views of learning, then giving a highlight of the research done under commognition. Thereafter, I identify the gaps that these literature studies have left, and using these gaps I argue how the current study will address the gaps in literature.

2.9.3.1 Acquisitionist and participationist views of learning

For the better part of the early nineteenth century, learning was conceptualised mainly through cognitive and behaviourism theories. These theories view learning as the acquisition of bits and pieces of information that accumulates in the human brain. These conceptualisations included explaining who of different groups of people could learn the most units of information in a given time period, or who could learn these units the quickest (Säljö, 1979). In this case, knowledge is seen as an entity to be absorbed, stored in the brain and recalled whenever the need arises. In these acquisitionist schools, students are viewed as empty vessels or receptacles to be filled with bits of knowledge, and once this is done, they take ownership of this knowledge as if it were theirs from the beginning (Rogoff, 1994; Sfard, 1998). As Chisholm et al. (2000), and more recently Chirinda and Barmby (2018), argue SA education is still dominated by the acquisitionist view of learning where teaching and learning activities and processes are heavily reliant on the teacher.

However, in the early part of the twentieth century, there were developments in learning views that were strongly founded on human development as a social process (Dewey, 1938). Learning theories developed from this tenet view learning as a consequence of participating in discourse with other members of the community (Vygotsky, 1978). Lave and Wenger (1991) say in their flawless masterpiece of a sociocultural theory of learning, that learning is becoming a 'legitimate peripheral participant' in a domain-specific community of practice. This theory describes eloquently the grounds for the significance of communities of practice in the process of content learning. This view of learning contributed to the move away from cognitive views (acquisitionist) of learning which see knowledge as an entity to be acquired, towards social views (participationist) of learning that conceptualise learning as participating in discourse. Participationist views of learning differ from the acquisitionist in that they prioritise active participation with others in the process of learning. The foundational tenet of the participationist view is that "patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual" (Sfard, 2008, p. 78). In this view, all members of the community (newcomer or expert) are equally responsible for their development and the development of the community (Rogoff, 1994). The participationist view human development is a continuum starting from participating in patterned community activities to the individualisation of these activities at a personal level (Sfard, 2008). Hence, mathematics learning from the participationist view is tantamount to participating in mathematical discourse or mathematising.

2.9.3.2 Research on commognition: a highlight.

Research on commognition and commognition-based research has been booming since the introduction of the theory in mathematics education. Commognitive-based research focuses on analysing mathematical discourses with the aim of understanding how these discourses develop in learning or problem-solving contexts (Sfard, 2020). In this section I highlight the foci of educational research conducted under the lens of commognition. Prior to the formal conceptualisation of commognition in 2008, Sfard was already doing commognition-related research as a foundation to the theory (see Sfard, 1994, 1998, 2001, 2007). Sfard and colleagues analysed arithmetical discourse to argue that the sociocultural learning contexts can improve humans' skills of participating in discourse (Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005). Furthermore, Sfard (2007) argues from a commognitive standpoint for the significance of commognitive conflict in shifting mathematical discourse, and the leadership of an experienced discursant who will have a teaching-learning agreement with the students (see also Ben-Zvi & Sfard, 2007; Nardi, Ryve, Stadler, & Viirman, 2014). After Sfard's (2008) conceptualised commognition, a considerable number of mathematics education and other discourse-oriented researchers began having an interest in commognition. Research in this field has been scattered throughout all levels of learning, from elementary school to university level. Some researchers have used the commognitive standpoint to develop models of mathematics learning in schools. This includes researchers such as Jeannotte and Kieran (2017), who explain a model that can be followed to advance students' mathematical reasoning during teaching and learning. Some of the commognitively underpinned studies used the enabling power of technological artefacts to explain transformations of mathematical discourses during learning (Lu, Tao, Xu, & Stephens, 2020; Ng, 2019).

Heyd-Metzuyanim and Graven (2016) use the commognitive tenets to explain two Grade-3 learners' ('Mina' and 'Ronaldo') discourse participation in number sense. They conclude that while Mina was regarded as a good student, her discourse participation was ritualistic and aimed to please the teacher, while the troublemaker (Ronaldo) exhibited an explorative discourse participation with an interest in mathematics learning (Heyd-Metzuyanim & Graven, 2016). Furthermore, Robertson and Graven (2019), through the analysis of classroom talk on fractions and interview data, highlight episodes of the limitations caused by the usage of students' home language on their abilities to make meaning in mathematics. While these findings conflict with Robertson and Graven (2020a, 2020b), they provide an explanation for how word usage in commognition affects the process of mathematising in learning. Furthermore, Mpofu and

Pournara (2018) continue to show through commognition that SA students' discourse participation was mainly characterised by rituals of executing procedures about hyperbolic functions without the ability to explain how the answer was obtained. Similar findings were obtained by Roberts and Le Roux (2019), who saw no evidence of students transforming from ritualistic to explorative discourse participation in mathematics. In another study Mudaly and Mpofu (2019) conclude that learners exhibited fragmented knowledge about the object of asymptotes in both hyperbolic and exponential functions. Guided by the sociocultural view of the significance of the knowledgeable other during learning, Güçler (2016) argues for the significance of attending to students' meta-level rules in mathematical discourse to foster learning about functions.

In a course for in-service teachers, Berger and Bowie (2012) use the commognitive theory to explain how different resources provide access to the different elements of mathematical discourse. Later on, Berger (2013) argues that this course was able to transform some in-service teachers discourse participation from rituals to explorations, while other teachers still remained using ritualistic discourse and imitating others' routines. Park (2013) studied how students spoke about the concept of a derivative of a function and later on, Park (2015) argues for the significance of word usage and visual mediators in revealing students' difficulties with the derivative of a function. Cooper and Karsenty (2018) further use a mathematical discourse for a teaching framework informed by commognition to explain how mathematicians can contribute to the development of teaching expertise in in-service teachers (cf. Cooper, 2019). In first-year students, Ioannou (2018) concludes that objectification is crucial in the endorsement of meta-rules, and Viirman and Nardi (2019) discuss the interplay between ritualistic and explorative discourse participation of first-year biology students. Chesler (2019) investigates the word usage of pre- and in-service teachers in defining the function through the commognitive lens, while Fernández-León, Gavilán-Izquierdo, González-Regaña, Martín-Molina, and Toscano (2019) use the same lens to investigate students' routines in defining 3D geometrical objects. In a professional development course, Heyd-Metzuyanim, Smith, Bill, and Resnick (2019) investigate how teachers transformed their discourse participation, from ritualistic to explorative, using two lessons. Despite the various comprehensive studies conducted under the umbrella of commognition, there still exists some methodological and design gaps in this scholarship, some of which I highlight in the next section.

2.9.3.3 *The gaps*

In mathematics classrooms, analysing discourse has been made easy by the development of Sfard's (2008) commognitive framework. Since its inception, this theory has been used globally to analyse mathematical discourse of learners, preservice teachers and in-service teachers (Berger, 2013; Coles & Sinclair, 2019; Heyd-Metzuyanim & Graven, 2016, 2019; Lavie et al., 2019; Mpofu & Pournara, 2018; Nachlieli & Michal, 2019; Roberts & Le Roux, 2019; Sfard, 2009). There have been studies that explored classroom discourse from the commognitive perspective in South Africa, focusing on university calculus (Siyepu & Ralarala, 2014), algebra (Roberts & Le Roux, 2019), functions (Mpofu & Pournara, 2018) and teachers' discourse (Berger, 2013; Berger & Bowie, 2012b; Van Jaarsveld, 2018). However, I have not encountered a study that investigated PMTs' discourse as they solve geometry problems in South Africa through the combination of the lens of commognition and the Van Hiele levels of geometry thinking. In the United States of America, Wang (2016) combines with these two theoretical frameworks, however with a different focus compared to the current study. She describes geometry thinking through grafting the elements of mathematical discourse from commognition onto the Van Hiele theory of geometry thinking. In particular, the elements of mathematical discourse describes preservice primary teachers' geometry thinking at each level based on how these preservice teachers communicate about geometry objects (Wang, 2016). However, that study differs from the current study in design, theory and focus, as highlighted in Table 2.7.

Table 2.7: Differences between Wang's (2016) study and the current study

	Wang's (2016) study	**The current study**
Theory	This study grafts commognition onto the Van Hiele theory of geometry thinking so that participants' thinking is described through the properties of each Van Hiele level of geometry thinking, based on how they communicate about geometry objects.	The current study grafts Van Hiele theory of geometry thinking onto commognition so that participants' thinking is described through their ritualistic or explorative discourse participation.
Design	The design of this study is experimental, it includes a pre- and a post-component with a direct measurement of the participants' level of geometry thinking using a standardised questionnaire.	The design of the study is purely qualitative and focuses on how participants' thinking reveals itself during geometry problem solving.
Instruments	Participants were involved in activities that were directly related to the Van Hiele levels of geometry thinking.	Participants were involved in the activity of solving two Euclidean geometry riders.
Participants	The participants for this study were primary school preservice mathematics teachers.	The participants of this study were high school preservice mathematics teachers.

Even though Wang's (2016) study combines the Van Hiele theory and commognition, it utilises a different lens to the one adopted in the current study, and the difference will be fully explained in Chapter 3. As such, the *discourse perspectives on geometrical thought* presented by Wang (2016), is expressed through participants' development of their Van Hiele level of thinking in the pre- and post-discourse about geometry objects, while in the current study the focus is on PMTs' discourse participation during problem solving. Furthermore, with its focus on thinking viewed as communication (interpersonal and intrapersonal) during Euclidean geometry problem solving of 'riders' in South Africa, and incorporating the Van Hiele levels of geometry thinking, this study is the first of its kind in the SA context. While studies on geometry framed by commognition exists in educational research globally (Fernández-León et al., 2019; Wang, 2016), their focus is on elementary geometry, or particular geometry activities like defining, and not on geometry problem solving of 'riders'. In particular, most studies framed by commognition study the change in discourse over a certain period (Ioannou, 2018; Wang, 2016), while this study focuses on the discourse participation during a problem-solving activity.

2.10 SUMMING UP MATHEMATICAL THINKING AS PARTICIPATING IN DISCOURSE

In this literature discussion, I elucidated the significance of discourse in the development of mathematical thinking. I have argued that discourse is tantamount to the development of proficient mathematical thinking. Based on sociocultural views of Freudenthal (1973), Vygotsky (1978), and Lave and Wenger (1991), I have positioned the process of teaching and learning as a discursive one, in which the discursants (teachers and learners) become active participants. I have described the significance of providing students with opportunities to rediscover mathematical knowledge with the assistance of the teacher as a scaffolding link between what students know and what they are capable of knowing (Vygotsky, 1978). This is in line with a growing body of recent literature that views participation in discourse as tantamount to learning and the development of mathematical thinking (Gravemeijer, 2020; Lave & Wegner, 1991; Sfard, 1994, 1998, 2008, 2009, 2015). Furthermore, I have explained the construct of making meaning (mathematising) in mathematics education as one participates in discourse. As Bakhtin says, individuals participate in discourse 'wholly', using their bodies, minds and souls. I have highlighted the significance of embodied cognition in the development of mathematical thinking. Furthermore, I have shed light on the types of talk that are useful in facilitating classroom discourse in all domains of teaching and learning, with the aim of grounding exploratory talk as the most effective type of talk to facilitate classroom discourses (Robertson & Graven, 2019). Since the study is located in geometry problem solving, I have also shed some light on the pertinent issues of problem solving and research on Euclidean geometry teaching and learning. Lastly, I pinpointed the literature gaps that this study aims to fill with the aim of situating the current study in the existing literature. This means, I shed light on what is known and what is not known around the objects of the current study.

Sfard (1998, 2008) moves away from the idea of learning as acquiring mathematical knowledge, the object-level rules, towards a more comprehensive learning view, inspired by Vygotsky (1978), that learning is the conversation of humankind (Sfard, 2008) and it should be understood through the meta-rules of discourse. Lavie et al. (2019) give a compelling piece of literature highlighting the tenets of commognition, specifically that mathematical learning is all about the routinisation of one's actions in a mathematics discourse. This routinisation occurs as individuals transform their mathematical discourse participation from ritualistic to explorative. In this case, routinisation begins by imitating more knowledgeable peers (oldtimers) in the discourse and develops towards the individualisation of routines, which allows one to perform specific routines individually during problem solving (Lavie et al., 2019). Hence, each one develops his or her own routines by individualising patterned rituals into explorations (Lavie et al., 2019; Sfard, 2008). A broader view of learning mathematics as participating in mathematical discourse, is explained in the subsequent chapter, Chapter 3.

CHAPTER 3:
LEARNING MATHEMATICS AS PARTICIPATING IN MATHEMATICAL DISCOURSE

3.1 INTRODUCTION

This chapter provides the details of Sfard's (2008) theory of commognition, with a particular focus on the aspects of the theory that relates to this study. I begin with a brief explanation of the theory in general and a few key tenets of the theory. Thereafter, I explain how the theory relates to learning and thinking, and how mathematics learning is a discourse, as I locate the current study within the theory of commognition. This chapter aims to describe the keywords and the language of Sfard's (2008) commognition and how they are key in describing PMTs' discourses when solving geometry problems. Furthermore, this chapter aims to explain how commognition has enabled the study to explain PMTs' discourses and what improvements can be made to the theory in the future. Hence, this theory will be used as a lens to view and explain PMTs' discourses as they solve geometry problems.

3.2 COMMOGNITION IN A NUTSHELL

In 2008, Sfard published a book titled *Thinking as communicating: Human development, the growth of discourses, and mathematizing*, which explains a theory that can guide and be used to understand mathematical learning. Sfard (2008) uses the amalgamation of 'communication' and 'cognition' to coin the term 'commognition', which she describes as communication about thinking. She defines commognition by putting into perspective this amalgamation, stating that commognition "stresses that interpersonal communication and individual thinking are two facets of the same phenomenon" (Sfard, 2008, p. xvii). Here, Sfard asserts that thinking is correlated to communicating, stating, "thinking is defined as the *individualized version of interpersonal* communication" (Sfard, 2008, p. xvii), closely relating thinking to Bakhtin's (1986, p. 126) idea of the "superaddressee". Sfard (2008) emphasises that communication and thinking are inseparable. She describes the underpinnings of the theory from the significance of communication, objectification, and elements of mathematical discourse that are significant in mathematics classrooms. In her elucidation of commognition, Sfard (2008) differentiates between colloquial and mathematical discourse, where the former is considered to be everyday, spontaneous discourses and the latter being specifically related to mathematics. She posits four characteristics of the latter kind of discourse. A discourse becomes mathematical because of its word usage, visual mediators, narratives and routines. This will be explained in detail in this chapter. Hence, from the considerations of the tenets of commognition, I believe that it is a suitable theory to analyse the discourses of PMTs.

3.3 WHY COMMOGNITION?

Commognition is a discursive theory that is utilised here for its theoretical potential to explain PMTs' thinking. It is a theory that recognises that thinking is something that is done daily by everyone, but which is not directly available to others. Hence, commognition regards thinking as "an individualised version of *interpersonal communication*" (Sfard, 2008, p. 81). According to this view, thinking cannot just be an isolated activity, but becomes "the act of communication in itself" (Sfard, 2008, p. 82). This means that whatever utterances are made through discourse by an individual, are a consequence of that individual's thinking, and the best way to study that individual's thinking is to analyse the communication through discourse. Furthermore, school learning as it is for teacher training, should present an opportunity to extend the discourses of learners and/or PMTs (Ben-Zvi & Sfard, 2007). Commognition further recognises that, just like learning, thinking develops from a patterned collective activity. Commognition recognises that thinking can be objectified or disobjectified, but rests mainly on the significance of explaining mathematical thinking through disobjectified discourses. Therefore, thinking can be explained by analysing the discourses of PMTs. Thinking, as a patterned collective activity, happens through communicating with others and ourselves. Thinking is therefore dialogical (Bakhtin, 1981; Sfard, 2008), and thinking is modified and changed as we communicate with others.

Mathematics is considered as a difficult subject in SA schooling. Furthermore, geometry is seen as the topic where learners perform the poorest and where even teachers struggle to teach geometry effectively. Most teachers who do manage to get learners to pass geometry, use the drilling of theorems and how to prove them. Some teachers rely mainly on the possibility of questions being repeated in the standardised tests. To improve the dire situation of teachers with insufficient geometry knowledge, which leads to learners performing poorly in geometry, we need to approach this problem as a collective, ensuring that teachers are properly trained to teach geometry at their levels of teaching. We need to tap into PMTs' thinking when they solve geometry problem to see how they think and then design proper tasks and teaching strategies to enhance their level of geometry thinking to a suitable one, where they would be able to teach geometry effectively to ensure meaning making within learners. In this way, commognition offered me a window to tap into PMTs' thinking when they solve geometry problems, to understand their thinking. In commognition, thinking is voluntary, individuals engage in thinking through their continued participation in the thinking discourse. Hence, in this view, geometry learning for PMTs is a consequence of their continued participation in the community of mathematics, which mainly originates from the participationist theories of learning (Lave & Wenger, 1991). As a human activity, participation in the activity of communication has emotional implications to it, which need to be understood properly, especially if the communication is amongst competing peers (Heyd-Metzuyanim, 2013). PMTs need to move from being participants in the discourse of geometry to being individual geometry problem solvers who can teach geometry effectively. Since Vygotsky (1978), the activity of being a peripheral participant in geometry discourse for PMTs, begins with the help of a knowledgeable other. Hence, the role of the lecturer (trainer) is important as a knowledgeable other. Hence, if PMTs are learning, they become more and more independent of the lecturer and their thinking as they learn, does not require the aid of the lecturer, as they become independent thinkers.

3.4 OBJECTIFICATION IN DISCOURSE

The use of metaphors is common in all discourses, where words are partitioned into an unfamiliar discourse because of their familiarity and their readiness to be used in that discourse (Sfard, 2008). The building of mathematical knowledge from concrete objects has been long recognised (Dienes, 1960). In the current research study, it seemed significant to distinguish between PMTs' discourse about objects and how they communicated about mathematical objects. The "process in which a noun begins to be used as if it signified an extra-discursive, self-sustained entity (object), independent of human agency" (Sfard, 2008, p. 300) is known as objectification. In commognition, objectification is considered to encapsulate two inseparable discursive moves, reification and alienation. According to Sfard (2008, p. 44), "reification is the act of replacing sentences about processes and actions with propositions about states and objects". Hence, in this study reification showed PMTs transforming the talk about the process of problem solving into a talk about objects. Reification allowed PMTs to be concise about what they were communicating, which made it more flexible and applicable in mathematical discourse.

Alienation on the other hand, involves the removal of the reified discourse from the actor. Alienation refers to "using discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings" (Sfard, 2008,

p. 295). Alienation allowed PMTs to engage in the discourse of geometry problem solving in an impersonal way. These alienated geometry discourses can be thought of as theorems, axioms or postulates, etc. since they are monological (Bakhtin, 1981, 1986). In this view, a geometrical proof is the final stage of the process of objectification, where the human experiences and constructions are removed from the discourse, it is the stage of alienation itself. This is a hint that geometry teaching and learning should not begin with the process of proving, but that of investigations and explorations (De Villiers & Heideman, 2014).

Objectification has been shown to have several advantages in the process of mathematical learning. Ben-Yehuda et al. (2005) show that objectification may lead to mathematical discourses that contribute to increased levels of mathematical performance. Objectification further makes the way we communicate about mathematics more effective and provides a method of attaching objects into our mathematical discourses. Once we objectify, we create an 'object' or a 'thing' that has permanence in our mathematical discourse, which can also be an abstract entity. Through this objectified discourse, PMTs can accumulate knowledge through participating in successive mathematical discourses that increase in complexity and applicability. The reification process relates directly to the mathematical objects objectified in discourse and reification can allows PMTs to endorse the discourse as a mathematical one. Hence, objectification, in this case, underlies the patterned ways in which we think. However, objectification removes personal experiences of learning and thinking in the discourse. As Sfard (2008, p. 56) puts it, objectified "descriptions deprive a person of the sense of agency, restrict her sense of responsibility, and, in effect, exclude and disable just as much as they enable and create". This is possibly a consequence of objectification removing the PMTs' thinking and learning experiences and the way in which they might communicate in their everyday lives in the discourse. Alienation is seen as contributing to the genesis of mathematical knowledge and understanding (Morgan & Tang, 2016). Hence, once PMTs can alienate a certain mathematical discourse, they begin to understand and construct mathematical knowledge, specifically in mathematical discourse and not just colloquial discourse. The objectification of mathematical discourse means that colloquial discourse is nullified into a more specific mathematical discourse, consisting of specific word usage, routines, narratives, and some visual mediators. These, as explained in commognition, are the main characteristics of a mathematical discourse (Sfard, 2008). Hence, mathematics from this point of view, can be identified as a discourse.

3.5 MATHEMATICS AS A DISCOURSE

Sfard (2008, p. 161) describes mathematics as “an autopoietic system”, which she extends that its end products are exactly the constituents of its present discourse, as its objects. With its salient features of Lave and Wegner’s (1991) idea of ‘legitimate peripheral participation’, commognition further describes mathematical learning as participation in the mathematical discourse (Sfard, 2008). Vygotsky (1978) highlights the significance of communication in learning, and Lave and Wenger (1991) highlight the significance of participation in the process of learning. Furthermore, the idea of mathematics as a discourse has been strengthened by researchers such as Nardi et al. (2014) and Sfard (2014), saying that, to not consider mathematics as a discourse, would be ludicrous. The objects produced in a mathematical discourse are defined as abstract discursive objects containing mathematical signifiers and they are the products of objectification (Sfard, 2008). The notion of signifiers is important, also in considering mathematics as a discourse. In commognition, signifiers refer to any primary object that encapsulates its realisation procedures. Specifically, mathematical objects are defined as abstract discursive objects with distinct mathematical signifiers (Sfard, 2008). Human beings can be able to take part in different types of discourse and in some discourses, humans fall short. For example, it might be difficult for a visual artist to participate in a specialised mathematical discourse, in the same way that it would be difficult for a mathematician to participate in a discourse about project management or geography. Furthermore, Sfard (2008) differentiates between two types of discourses: colloquial and mathematical discourses. Colloquial discourses are everyday-life discourses, visually mediated by pre-existing objects (Sfard, 2008). Colloquial discourses of mathematics use everyday language that is reified and colloquial narratives can be endorsed by PMTs through their engagement in discourse with the knowledgeable other or through repetition. Mathematical discourses are, however, distinct from colloquial discourses. What makes mathematical discourses distinct, is the fact that mathematics is characterised as a domain-specific discourse that can be identified by its word usage, visual mediators, routines and narratives (Sfard, 2008). In SA mathematics education, the use of words and symbolic information is common, and it seems like an attempt to strengthen a formal mathematical discourse. This is also evident in the strong way in which symbolic language is evident in geometry proofs with very limited word usage. Mathematics is presented in an objectified way in SA high schools. The discourse does not go through the process of reification, but is alienated. This form of mathematical learning is what Freudenthal (1973) refers to as the anti-didactical view. Teachers and learners do not engage in discourse for the objective of reification, but simply aim to memorise alienated mathematical discourses. Hence, mathematics, and especially geometry, is challenging for many SA learners, as described in this chapter. Let me take a closer look at the elements of any mathematical discourse as delineated in commognition.

Word usage – the key to identifying the realm in which a discourse belongs, is the keywords used by interlocutors in the dialogue. If you hear people dialogising and the keywords, ‘mitochondria’, ‘chloroplast’ or even ‘egg cell’ occur, we know this dialogue belongs within the constraints of life sciences in the high school curriculum. Mathematical discourses in schools are similar. There is no way one can talk of ‘quadratic equations’ or ‘the Tan-Chord theorem’ in any subject other than mathematics. Even though some word usage in mathematics appears in the colloquial discourse, they form part of the formal mathematical discourse that helps in understanding mathematical concepts. The significance is highlighted by Sfard clearly when she states, “word use is an all-important matter because being tantamount to what others call “word meaning,” it is responsible for what the user can say about (and thus see in) the world” (Sfard, 2008, p. 133). A ‘table’ in colloquial discourse is different from a ‘table’ in mathematical discourse and to converse with a mathematician about a table will not be the same as that with any other citizen. Hence, in this study, the focus here was placed on PMTs’ use of mathematical keywords. Here, PMTs were observed regarding whether in their utterances they substituted a word from their colloquial discourse with a key mathematical word, and how this affected their understanding of their thinking in problem solving. This then allowed the researcher to make comments on the PMTs’ use of mathematical keywords in their practice and the use of words in mathematics learning in SA schooling.

Visual mediators – these are the visible objects that can be used by interlocutors in communicating (Sfard, 2008). These visual objects are significant, because they can act as identifiers of colloquial and mathematical discourse. In mathematical discourse, visual mediators are only developed to mediate a specific discourse that has developed or will develop at a specific time and in a specific space. These visual mediators can be used as thinking aids in mathematical discourse and they can enhance the discourse. Algebraic symbols, notations, and geometry diagrams included in this report provide clinical examples of visual mediators. PMTs engage individually with these visual mediators. Especially in geometry, diagrams are important and, in most cases, they are key to solving geometry problems. If a geometry problem is posed in words, PMTs need to come up with their own diagram to solve the problem. This visual diagram can prompt PMTs with discursive prompts, which allow PMTs to recall specific knowledge and ways of mathematical problem solving. Furthermore, visual mediators are cues for visual thinking and many prominent mathematicians have relied to a large degree on their visual thinking for success (Clements, 1981; Shepard, 1978). Commognition holds that this visual thinking and imagery is enhanced by the ability of visual mediators to induce multiple ways of thinking about a certain problem.

Narratives – in Sfard's words, narratives are "any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to *endorsement* or rejection with the help of discourse-specific substantiation procedures" (Sfard, 2008, p. 134). Narratives in discourse can be thought of as ideas that need to be discussed and endorsed mathematically, and once a certain narrative is endorsed, it is considered a theory (Sfard, 2008). The goal of mathematics is to produce endorsable narratives through the process of alienation in objectification. This means that PMTs need to produce narratives that can be derived through the use of mathematical rules and methods. In geometry, postulates, axioms, and theorems can be considered as narratives, since they can be derived using mathematical laws. These different narratives can be used in isolation or in conjunction with others to solve different mathematical problems.

Routines – routines are significant and special in mathematical practice. Routines, according to Sfard, "are repetitive patterns characteristic of the given discourse" (Sfard, 2008, p. 134). Routines are repetitive and well-articulated discursive patterns, which may include the process of mathematical generalisation or completion of certain procedures (Berger, 2013). Routines are regulated by mathematical rules, such as what counts as a definition of a square, what constitutes a mathematical proof or even how to calculate the area of a triangle. Once a narrative has been endorsed mathematically, it can be used as a routine or a reason to endorse other mathematical narratives. Different patterned ways of communicating about geometry exist and PMTs can use any method. These ways can be observed in the way PMTs use words and visual mediators to derive new or substantiate existing mathematical narratives. Furthermore, the context, the teacher, the classroom environment, the learners and other factors may influence PMTs' discourse about their thinking.

3.6 ROUTINES

In commognition, routines can be defined as “a set of metarules that describe a repetitive discursive action” (Sfard, 2008, p. 208), and these rules can be separated into two subsets:

- the *how* of a routine, which are rules that describe the course of action for a patterned activity; and
- the *when* of a routine, which are rules that describe the situations in which interlocutors deem a certain performance appropriate (Sfard, 2008).

Routines are useful determiners of one’s thinking and they allow us to examine PMTs’ discourses and how they align with the formal mathematical discourse. The metarules of a certain routine represent past actions, and the routines chosen by PMTs may be used as a basis to predict PMTs’ future actions in their geometry problem solving (Sfard, 2008). These routines can also be used to predict PMTs’ methods of objectification and assess the appropriateness of the routines they choose. Most SA learners give procedures and memorisation a prominent role in their mathematical learning, instead of the process (Adler & Pillay, 2017), and those mathematics learners suffer learning deficits at an early stage in their learning (Spaull & Kotze, 2015). In commognition, this implies that the activity of metacognition is not given any priority in SA school mathematics. Studies in South Africa show that learners usually rely on routines that they have experienced or seen before to get through a certain task or solve a mathematical problem (Berger, 2013; Nachlieli & Tabach, 2012). This implies that they usually have to remember the experiences of the classroom and extrapolate those routines to the new problem or task.

With the different mathematising outcomes of routines, commognition differentiates between these routines based on their outcomes in discourse. Sfard (2008) differentiates between three different types of routines: explorations, deeds, and rituals. *Explorations* produce narratives that are endorsable, narratives that are considered as ‘mathematical facts’. *Deeds* result in the change of objects in the environment. *Rituals*, on the other hand, do not aim to create or change narratives, but to reproduce the *how* of a routine for social reward (Sfard, 2008). According to the tenets of commognition, both deeds and rituals are the necessary predecessors of explorations. However, deeds aim to transform a certain concrete object in the environment and not to narrate a story. Hence, since geometry and problem solving are abstract constructs instead of physical objects, deeds were excluded from the current research report. It was predicted that

PMTs would communicate about the subject of geometry problem solving in an explorative or ritualised manner when they qualify to be teachers. In collecting the data, PMTs were always asked to elaborate their discursive choices, which enabled the researcher to differentiate between their discourse participation as ritualised or explorative. This distinction between ritualised and explorative geometry problem discourse participation allowed the researcher further to describe PMTs' routines of the abstract object of geometry problem solving. These descriptions can be seen as an indication of the different ways in which PMTs communicate, and hence think about the object of geometry problem solving. These routines focus on the stories PMTs tell as they engage with the object of geometry problem solving, and are connected to how they make meaning of geometry. Hence, the study examined PMTs' thinking through how they participated in the discourse of geometry problem solving, explorative or ritualised. The distinction between explorative and ritualised discourses is explained in Table 3.1.

Table 3.1: The comparison between rituals and explorations in commognition

	Ritual	Exploration
Closing condition or goal	Relationship with others (improving one's position concerning others)	Description of the world (production of endorsed narrative about the world)
By whom the routine is performed	With (scaffolded by) others	No need for scaffolding – can be performed individually
For whom the routine is performed	Others (authoritative discourse)	Others and oneself (internally persuasive discourse)
Applicability (changing the when, keeping the how constant)	Restricted, the procedure is highly situated	Broad – the procedure is applicable in a wide range of situations
Flexibility (changing the how, keeping the when constant)	Almost no degree of freedom in the course of action	The procedure is a whole class equivalence of different courses of action
Correctibility	Cannot be locally corrected – has to be reiterated in its entirety	Parts can be locally replaced with equivalent subroutines
Acceptability condition	The activity has to be shown to adhere strictly to the rules defining the routine procedure – the acceptance depends on other people	The narrative produced through the performance must be sustainable in such a way that the acceptance is independent of other people
Words' and mediators' use	Phrase driven use of keywords – as descriptions of extra-discursive mediators	Objectified use of keywords – as signifying objects in their own right

Source: Sfard (2008, p. 243).

3.6.1 Differentiating between two types of discourse participation in commognition

Mathematics education has been characterised by different beliefs about teaching and learning. A recent development was commognition, a belief that equates learning to communicating about thinking, which becomes a 'legitimate peripheral participant' in mathematical discourse (Lave & Wenger, 1991; Sfard, 2008). Rooted within sociocultural theory, commognition provides an appropriate analytic framework for analysing PMTs' discourses. Participating in the mathematical discourse during learning can be described as based on the complex dyadic relationship between discursive routines in rituals and explorations. To describe PMTs' thinking, their utterances were examined for the objectification of the object of geometry problem solving, and the substantiation of their endorsed narratives and considered explorations. In this subsection, I explain the distinction between ritualistic and explorative discourse participation in commognition.

Rituals are socially oriented actions performed to conform to society. One engages in rituals to avoid punishment, please someone or for gain certain rewards (Lavie et al., 2019; Sfard & Lavie, 2005). For example, if Grade 12 learners memorise and reproduce proofs for certain Euclidean geometry theorems because these proofs are required in the examination, the participation of the learners in classroom discourse becomes purely ritualistic.

The main driver of rituals is societal expectations. When one feels obliged to perform in a certain way to please someone, then you engage in rituals. If learners in Mrs X's classroom were directing all answers to her without giving reasons and the teacher was the one endorsing the answers as correct or incorrect through giving applause for positive evaluation, these learners' participation in this discourse was solely motivated by getting applause or positive evaluation from the teacher (Heyd-Metzuyanim & Graven, 2016). In ritualistic discourse participation, interlocutors obtain discursive cues by imitating their knowledgeable other peers or teachers or rely on previous experiences. These discursive actions are scaffolded by others, hence their level of applicability is highly restricted. Here, ritualistic discussants aim to act in harmony with no degree of flexibility in their actions, but to imitate what others are doing. As a consequence of their high reliance on performance instead of knowing, rituals tend to follow strict rules to ensure that they can be produced by others. In ritualistic discourse participation, all rituals that do not succeed, are simply repeated instead of being corrected or modified, which is a further indication of the rigidity in the applicability of rituals. This form of discourse participation does not require special substantiation of the produced narrative, because it focuses mainly on the *how* of a routine (as explained earlier). Hence, the steps for the process are listed clearly and the performer needs to follow these steps to reproduce the ritual.

As opposed to rituals, explorations are not aimed at pleasing or conforming to societal expectations, but to advance theory. Particularly, routines can be characterised as explorations if they produce narratives contributing to a mathematical theory instead of tangible objects (Sfard, 2008). Explorative participation in mathematical discourse is characterised by objectification, where words denote realistic mathematical objects, and the participant views different realisations of the same thing (such as $x^2 - 4$ and $(x-2)(x+2)$) to be interchangeable (Heyd-Metzuyanim & Graven, 2016). Rituals are mainly "process-oriented", while explorations are "outcome-oriented" (Lavie et al., 2019). Explorative activities aim at inventing, producing or discovering some truths or facts about mathematical objects. These routines are applicable in a variety of contexts because the resulting narratives are endorsed and form part of a mathematical theory. PMTs who exhibit this meta-level type of thinking, can use these endorsed narratives in a wide range of contexts while providing multiple, but acceptable, means of substantiations for their routines.

Too often learners are considered 'good' in mathematics because they conform very well to society's expectations, and those who do not conform are usually labelled as 'outcasts' and 'weak at mathematics' (Heyd-Metzuyanim & Graven, 2016). This view may affect learners' formations of their identities in the classroom community. Participation in explorative discourse requires a good understanding of some *already endorsed* mathematical knowledge, hence rituals can be thought of as significant building blocks towards explorations (Lavie et al., 2019; Nachlieli & Michal, 2019). Sfard (2008, p. 223) claims that rituals and deeds are "developmental predecessors of explorations". Hence, meta-level learning of mathematics moves from ritualistic discourse participation towards explorative participation, and the ritual develops into exploration through the process of rationalisation. At the beginning of learning, when a routine constitutes objects or metarules unfamiliar to the learner, learning becomes highly unlikely. Hence, at this stage, for beginners to be conversant with a certain mathematical discourse, they rely on imitating a knowledgeable other in that specific discourse. The beginner imitates the rules and procedures of the knowledgeable other, adapting them until he or she individualises the mathematical discourse, but at this early stage, their discourse participation is ritualised. At this stage, beginners know *how* to perform a certain routine, but have not struck the balance as to *when* to perform it. For learners to be able to transform their rituals into explorations, they need to reflect continually on their performance of the ritual, while examining the rationale for the expert performance as they participate in mathematical discourse. In Vygotsky's idea, ritualistic discourse participation occurs in the Zone of Proximal Development (ZPD), where the learner can participate in the collective patterned thinking or the performance of a routine, but is incapable of individual performance. By contrast, explorations look for the connectedness of different routines so that once learners have acquired explorative discourse participation, they hold a network of interconnected routines, instead of disconnected ones. The explorative form of objectifying particular routines allows learners to compress mathematical knowledge to allow them to solve a multitude of mathematical problems with few routines. Furthermore, commognition recognises that metalevel learning occurs "when the learner is exposed to commognitive conflict" (Sfard, 2008, p. 260). Sfard (2008, p. 296) describes the commognitive conflict as the conflict that "arises when communication occurs across incommensurable discourses". Interlocutors who are in commognitive conflict, participate in discourses that differ in their utilisation of words, mediators, and routines, which might allow them to endorse narratives that seem contradictory. Commognition warns the commognitive researcher not to confuse commognitive conflict with cognitive conflict. The difference is tabulated in Table 3.2.

Table 3.2: Comparing cognitive conflict and commognitive conflict

	Cognitive conflict	**Commognitive conflict**
The conflict is between:	The interlocutor and the world	Incommensurable discourses
Role in learning	Is an optional way for removing misconceptions	Practically indispensable for metalevel learning
How is it resolved?	By student's rational effort	By student's acceptance and rationalisation (individualisation) of the discursive ways of the expert interlocutor

Source: Sfard (2008, p. 258).

3.7 GEOMETRY THINKING IN MATHEMATICS

Commognition was used as a lens to view the findings of the current study. When studying geometry thinking and reasoning, the Van Hiele levels of geometry thinking also give an idea of one's level of geometry thinking and reasoning. In mathematics education, Euclidean geometry teaching and learning have been identified as one of the most challenging topics for both teachers and learners in South Africa (Alex & Mammen, 2014, 2016; 2018; Mogari, 2003;

Sinclair et al., 2016). Furthermore, recent studies in South Africa show that both teachers and learners are not at the required level for teaching and learning geometry (Alex & Mammen, 2018; Luneta, 2014; Van Putten et al., 2010). In South Africa, these difficulties are attributed to the erstwhile exclusion of geometry from the compulsory curriculum, as discussed in Chapter 1. In the twenty-first century, this has put much pressure on teacher training institutions to ensure that teachers graduate at an appropriate level for teaching Euclidean geometry. To measure this required level, the Van Hiele levels of geometry thinking and reasoning are usually used. The model was developed by Van Hiele-Geldof (1957) and Van Hiele (1957) towards the completion of their Doctor of Philosophy degrees at the University of Utrecht. The original model they put forward, had five levels from level 0 to level 4. However, this model was revised when Clements and Battista (1992, p. 429) identified a level that comes before level 0, which they called "pre-recognition" to start at level 1 to level 6. Within the five levels of geometry thinking, the most pertinent ones in the SA curriculum are the first four levels, on which the problems in this study focused. The levels are described as follows.

At the lowest level of geometry thinking and reasoning (level 0), geometry figures and objects are judged based on their visual appearance (form) only, without any reference to properties and any relationship that might exist thereof. At the next level (level 1), geometry figures are identified based on their properties, without considering the relationship that exists between these properties. At this level, a square is not seen as a rectangle. At the next level (level 2), learners begin to see the relationships between the properties of geometry figures. They now can relate a square and a rectangle by ordering their properties and deducing one from the other. At the next level (level 3), learners' thinking and reasoning are concerned with understanding the meaning of deduction (how to prove theorems). These are the levels that are pertinent to geometry in the Department Basic Education (DBE, 2011). Level 4 concerns itself with issues of the relationship between the systems of geometries, for example, how a line in spherical geometry compares to a line in plane geometry, which is well beyond basic education in South Africa.

There are critical issues about these levels that apply to the development of thought in geometry, especially for PMTs to use in their instruction. The language and signs used at each level are distinct, such that a relationship that is true at one level, might not be true at another (Van Hiele, 1959/1985). The second issue to be aware of, is that people who reason at different levels cannot understand each other. Hence, teachers need to attempt to reason at the level of learners, understand their routines and narratives to scaffold them to the next level. Teachers must continuously support learners to construct their deductive relational system in geometry (Van Hiele, 1985), without imposing the relational system of the teacher onto the learners. These levels are critical in the analysis of thinking and reasoning in geometry, because they reveal the characteristics of thinking for both learners and teachers. Since this study had its main focus on PMTs' thinking when solving geometry problems, it seemed useful to incorporate these levels, as they are indicators of thinking even though they will not be used as a lens to view the results of the study. Even though these levels were not assessed directly, they are pertinent in geometry education. Teachers' behaviour when participating in geometry problem solving can be related to each Van Hiele level and the elements of mathematical discourse (described above) based on its properties. This relationship is highlighted in Table 3.3.

A brief explanation of this relationship suffices and how this relationship links to the current study. Firstly, the relationship is explained based on teachers' failed behaviours during geometry problem solving instead of their successes, because this will raise awareness of the severity of the need to develop competent geometry teachers in South Africa. Teachers' behaviours described in Table 3.3, were constructed based on the properties of each Van Hiele level of geometry thinking. These behaviours can range from any example that characterises behaviour at a certain Van Hiele level of geometry thinking through the processes described by the Van Hieles. The relationship between teachers' behaviours and the Van Hiele levels is a critical one but elementary, because the teachers' behaviours were formulated based on the properties of the Van Hiele levels. Nevertheless, what is not evident here is that, since competency at each level is dependent on competency at the preceding levels, teachers' struggles at a particular level might not be because of struggling with geometry reasoning related to that level, but the preceding level(s). For example, a failure to prove a geometry theorem (level 3) might not only be hindered by a deficient competency to order objects according to the relationship that exists between their properties (level 2), but also their visualisation skills (level 0).

Table 3.3: Relationship between teacher ritualistic behaviours, Van Hiele levels of geometry thinking, and elements of mathematical discourse

Teacher's behaviour ritualistic during problem solving	Relationship to Van Hiele levels	Relationship to elements of mathematical discourse
Unable to produce a diagram from written information or depict certain geometry relationships (like theorems) in a diagram.	This can be attributed to teacher deficiency of visualisation, which is linked to Van Hiele level 0 of geometry thinking.	Teachers are struggling with their word usage, visual mediators, narratives and routines simultaneously at all Van Hiele levels.
Unable to differentiate between geometry objects (or drawings) using their properties.	Here teachers struggle with Van Hiele level 1 of geometry thinking.	
Inability to relate geometry objects (or drawings) utilising the relationship between their properties, for example, deducing how a square is related to both the rectangle and the rhombus.	These teachers have not developed Van Hiele level 2 of geometry thinking.	
Teachers are unable to use deduction to prove geometry theorems, whether it is a theorem or proving that certain lines are parallel or that certain angles are equal.	Teachers here are struggling with Van Hiele level 3 of geometry reasoning	

It was felt that, since elements of mathematical discourse are used in all mathematical discourses globally according to commognition, then at each level all, or some, of these elements of mathematical discourses might be visible (or utilised). Hence, PMTs' participation in geometry discourse at each level invokes the usage of all the elements of mathematical discourse in commognition. This link between the Van Hiele levels of geometry thinking and commognition, was useful in explaining the geometry thinking competency of PMTs during their participation in geometry problem-solving discourses. In particular, not only did it allow for the conclusion on the level of their geometry thinking, but it further also allowed one to explain why certain PMTs' geometry thinking is at a certain Van Hiele level through the analysis of their discourses. The bulk of studies on PMTs' (or teachers') Van Hiele levels of geometry thinking use statistics to arrive at conclusions about PMTs' level of geometrical thinking (Alex & Mammen, 2018), and seldom explain the reasons for the existence of these levels, besides generalising these reasons.

This relationship between commognition and the Van Hiele levels of geometry thinking provided me with the framework to fill this gap of explaining from PMTs' discourse participation why they were at a certain level, considering their usage of certain elements of mathematical discourse. Figure 3.1 attempts to show how participating in mathematical discourses can advance individuals' geometry thinking competency as they move from ritualistic to explorative discourse participation (Heyd-Metzuyanim, Smith, Bill, & Resnick, 2019; Lavie et al., 2019). Corroborating the example given in Table 3.3, Figure 3.1 shows how the elements of mathematical discourse are critical in the development from each level to the next. The underlying assumption here was a critical one in commognition: the fact that learning is viewed as participating in discourse (Lave & Wegner, 1991; Sfard, 1998, 2008), and that mathematics teaching and learning is facilitated by the different elements of mathematical discourse. Therefore, this relationship was used to explain why PMTs' geometry thinking was at a certain level, using their discourse participation.

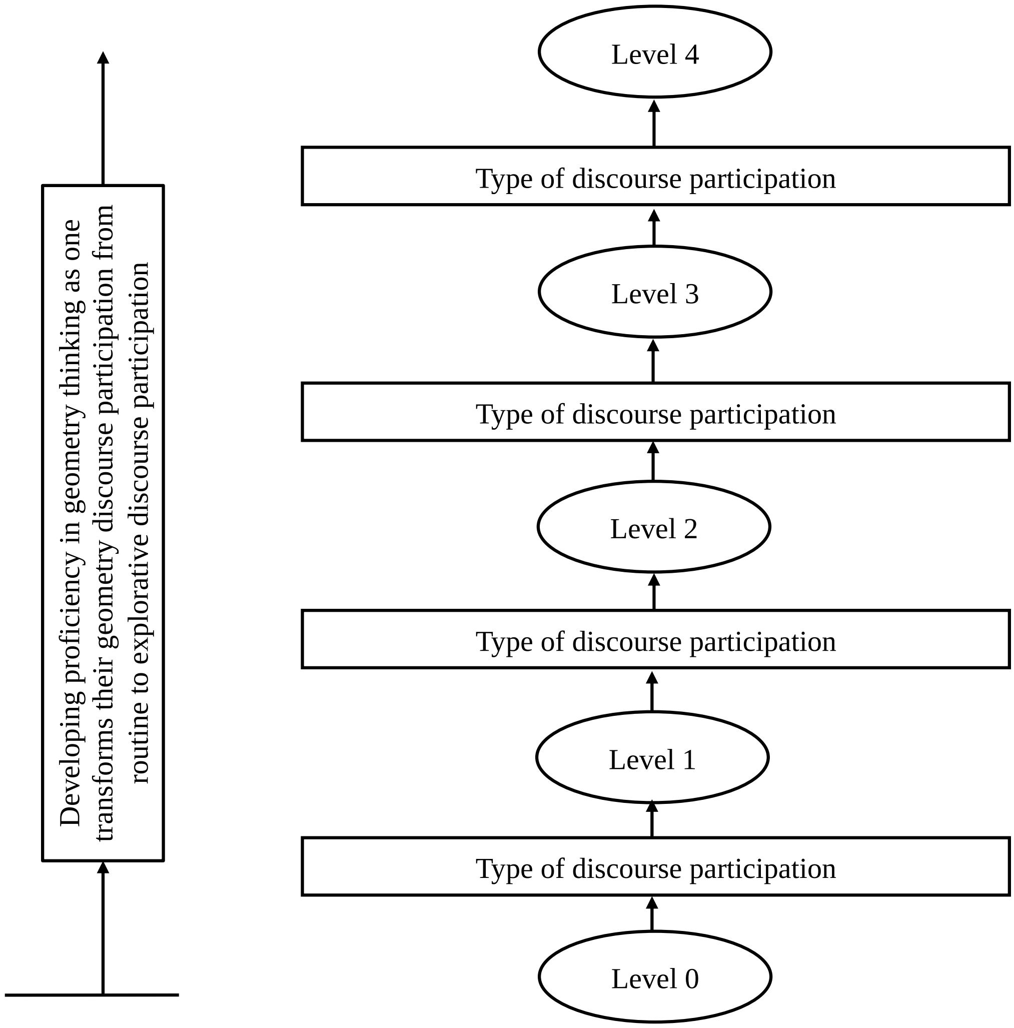


Figure 3.1: The commognitive development of geometry thinking through the Van Hiele levels

Source: Author's own compilation

3.8 SUMMARY

Commognition defines thinking as an individualised form of interpersonal communication (Sfard, 2008). Hence, it allowed the researcher to explore and tap into the thinking discourse that PMTs had with themselves when they were solving geometry problems. In particular, analysing data through this theory allowed the researcher to explore useful ways in which PMTs communicated about their thinking when solving geometry problems. This had implications for how PMTs could be supported to improve their geometry knowledge of both proofs and problem solving for better practice when they get to schools. Through the understanding of ritualistic and explorative discourses participation (Sfard, 2008), I was able to explore the types of discourses that PMTs used when solving geometry problems. How PMTs communicated about their thinking in providing reasons for why a certain approach was taken instead of any other, provided a window into their thinking and had huge implications for how they could be supported to be better geometry teachers. The elements of mathematical discourse came in handy in ensuring that teachers' discourses were regarded as mathematical or not, particularly when thinking and solving geometry problems, the difference between general and mathematical thinking was deduced based on these elements of mathematical discourse. This helped to explain PMTs' discourse and thinking when solving geometry problems. The significance of spatial thinking and visualisation is not only evident in the Van Hiele model of geometry thinking (Fuys, 1985), but also general research in mathematics education and geometry teaching and learning (Battista & Frazee, 2018; Burte et al., 2017; Mix & Battista, 2018). Hence, the elements of mathematical discourse further allowed me to explore the effects of visual mediators that these PMTs used when solving geometry problems. This theoretical framework used in this study did not aim to establish facts about thinking from the collected empirical data, but to explain a useful way of dealing with thinking and how PMTs' thinking could be advanced for their better practice in geometry.

CHAPTER 4:
RESEARCH METHODOLOGY AND DESIGN

4.1 INTRODUCTION

In this chapter, I describe the design of the study which I followed to investigate preservice mathematics teachers' (PMTs') thinking through their discourse when solving geometry problems. The study followed an interpretive paradigm to collect qualitative data that was analysed through the lens of commognition and the Van Hiele theory of geometrical thinking. The interpretive paradigm allowed the researcher to interpret PMTs' thinking using their discourse on geometry problem solving as part of their experiences with the phenomenon being investigated (Cohen, Manion & Morrison, 2018). On the other hand the qualitative approach allowed not only for the collection of qualitative data but also viewing PMT's experiences in-depth understanding of PMTs' discourse during problem solving (Denzin, 1970). Since the study was framed by a theory that believes that learning occurs through participation in discourse, I proceeded into conducting in-depth interviews with the PMTs to collect qualitative data through the utilisation of one-to-one task-based interviews. This helped in understanding the social phenomenon of PMTs' thinking from their experiences in a natural setting. Task-based interviews allowed for the collection of rich data through careful, and sometimes planned, probing. The interviews allowed me to gain knowledge about PMTs' state and developing geometry content knowledge and their problem-solving behaviours (Maher & Sigley, 2014). Task-based interviews can provide researchers with an insight into their participants' thinking, their knowledge and challenges in certain subject matter (Lewis & Fisher, 2018), which correlates task-based interviews with the foundational underpinnings of commognition. Thereafter, another discourse participation was sought from the PMTs in the form of a one-on-one, face-to-face interview where they were asked to explain their thinking when certain steps and decisions were taken during their problem solving. This gave PMTs a platform where they could clearly explain their thinking and it allowed me to ask further questions for clarity in order to gain a window into their thinking.

4.2 LOCATION OF THE STUDY

The data for this study was collected from one university located in the North West, South Africa. Both the task-based interviews and the one-on-one, face-to-face interviews took place in an enclosed space within the university premises where the participants were comfortable. The data collection process took place over a period of six months immediately after the participants completed a module in Euclidean geometry. With regard to the economic, social and political backgrounds of the participants: they all came from villages which are still ruled by kings. Hence, their economic backgrounds range from very poor to middle-class. Politics have very little effect within this province, even though these students did participate in all the political activities of the country. This province has many rural villages with schools largely populated by learners who speak Setswana as their home language, although some learners speak English or Afrikaans as their home languages. These learners usually attend the local university where the current study took place. North West, a province in South Africa, usually performs around an average of 80% in Grade 12 results since 2016, which ranks it in the upper tier of the nine provinces in South Africa. In mathematics, the North West can also be ranked in the top performing provinces, as seen in Figure 4.1. It had been within the top four performing provinces in SA mathematics education since 2017.

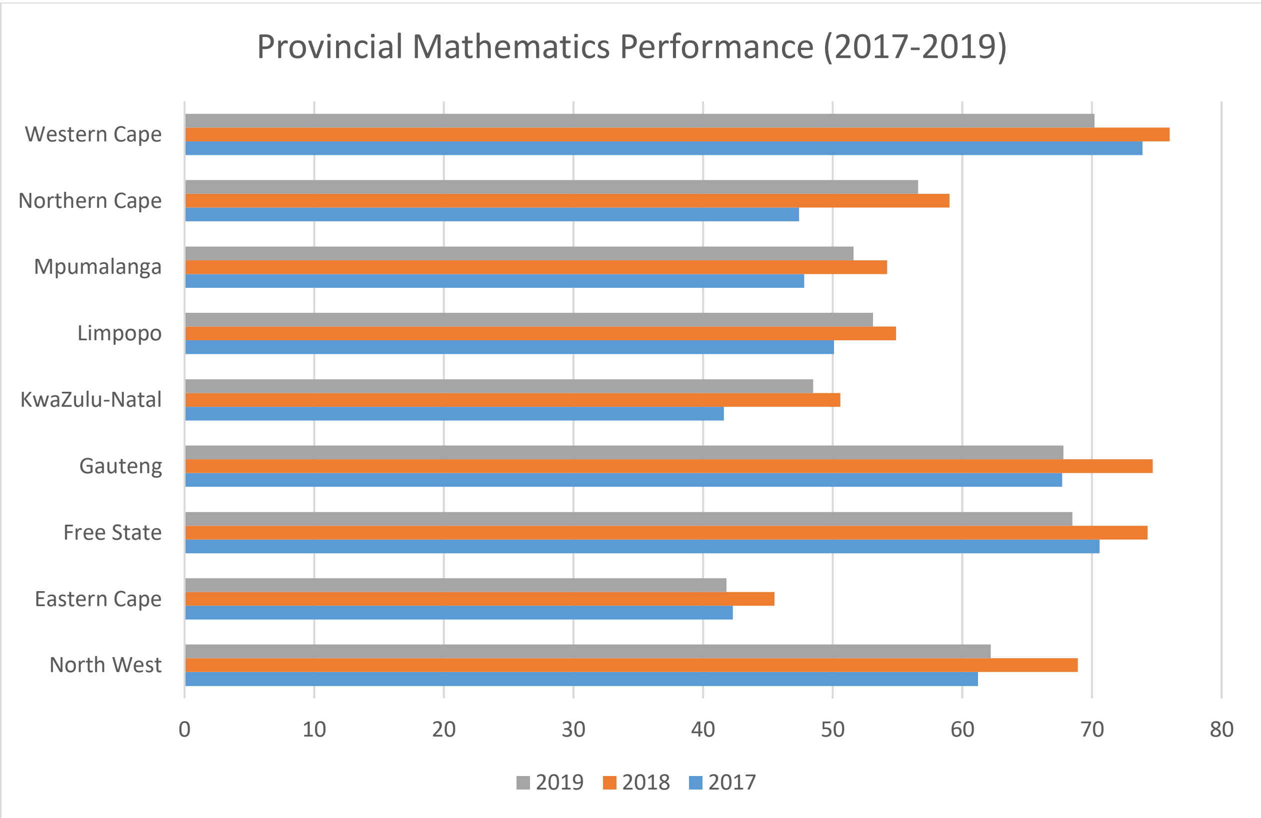


Figure 4.1: Provincial learner performance in mathematics (2017–2019)

Source: DBE (2019).

4.3 SAMPLING

In this section I discuss how the sampling was done for this study. In particular, I discuss the method of sampling used, how it was used and the targeted participants for this study.

4.3.1 The sampling method

I utilised a purposive convenient sampling method to select the participants for this study. The sampling was purposive, because the chosen participants completed the geometry module as part of their preparation to be mathematics teachers, which allowed the researcher to focus on the uniquely specific cases of their discourses when solving geometry problems, to understand their thinking during problem solving. It was convenient because the PMTs were easily accessible to the researcher for data collection purposes in terms of their location. Both purposive and convenient sampling represent small groups from a larger population, therefore, these sampling methods do not always aim to generalise about the population, but to explain those chosen cases in view of the larger population. The researcher sampled 12 participants, but only 10 were interviewed to allow space for others if one or two participants dropped out of the study. These PMTs were part of a larger group of 35 students who completed a Euclidean geometry course during the previous year (2019) and they were in their fourth year of their studies with the data collection beginning at the start of the first semester in the year 2020. The only criteria that was a requirement for participating in the study was that the PMTs should have completed a Euclidean geometry module in the second semester of the previous year (2019) from a particular university where the sample was chosen. However, when the spread of the corona virus (COVID-19) caused a nationwide lockdown, it became difficult to access participants. When the nationwide lockdown was put in place, I had already collected data from six participants and I decided that the data from these six participants was saturated and proceeded to answer the research questions.

4.3.2 The preservice mathematics teachers

The problem investigated in this study was based on the observation that high-school learners perform poorly in mathematics, especially geometry. At the beginning of this study, I proposed that supporting teachers to be aware of their learners' thinking can help teachers support their mathematics learners better. More elaboration on this note was explained in the literature review under section 2.4 and it is beginning to spread in South Africa with scattered research discussing the significance of this study. These PMTs' thinking was observed as they participated in geometry problem solving. As posited by Sfard (2008), it was expected here that PMTs who had objectified discourse participation, would be in a better position to support their learners to learn mathematics effectively. The analysis of the diagnostic report of Grade 12 since 2014 does not point to the poor performance of Grade 12 learners only, but also questions the teaching strategies utilised by their teachers. Hence, an alternative way of teaching mathematics, through tapping into learners' thinking to support them for better mathematics meaning making and meta-level learning, seems justified at this point. These 12 PMTs were in their fourth year of study. In their previous year,[18] they engaged with a Euclidean geometry module which tackled elementary and advanced geometry proof and problem solving.

In this investigation, PMTs were engaged in geometry problem-solving examples that contained diagrams, and others which required of the students to come up with their own diagrams. The participants were sampled based on the criterion that they have completed this Euclidean geometry module. These participants were all of African origins and were not sampled based on their performance in the module, nor their Grade 12 results, but their willingness and availability to participate in the study. The PMTs who participated in this study were from different social and economic backgrounds within the North West. Furthermore, they came from different cultural and linguistic backgrounds. South Africa is a multicultural and multilingual country that has eleven official languages, with other languages brought in by immigrants from other African and non-African countries. The differences in linguistic fluency of these students, from most fluent to least fluent, are shown in Table 4.1. As observed in Table 4.1, PMTs who participated in this study could speak a minimum of two languages, (Setswana and English), but were not as fluent in English[19] as they were in Setswana. However, I should state clearly that this biographical information was not used as a basis to analyse data for the study, but a means to show the diversity of the participants in terms of their fluency in the language of teaching and learning (English).

[18] Third year of study, second semester
[19] This information was gathered during the interview with each participant.

Table 4.1: Linguistic differences of PMTs who participated in this study

Pseudonym	Gender	Linguistic differences (from most fluent to least fluent)
P1	Female	Setswana, English
P2	Female	Setswana, English
P3	Male	Shona, English, Afrikaans, Setswana, IsiZulu
P4	Female	Setswana, English
P5	Male	Setswana, English, Afrikaans
P6	Male	Setswana, English

Though this study did not investigate the effects of language or culture on PMTs' geometry problem solving, it acknowledges the significance of these aspects in the process of teaching and learning, but these were outside the scope of the study. However, these aspects may have a great influence on how PMTs communicate about their thinking and their participation in mathematical discourse. From observing these PMTs' participation in classroom debates[20] and the observation that most of them listed English as their least fluent language, I decided to allow them to use code-switching a lot more than just sticking to English, since they were not as fluent in English as in another language. Despite this lecture room leniency that I gave the PMTs, they were encouraged to attempt to grasp English as the language of teaching and learning in Grades R–12 as mandated by the CAPS. Hence, during the interviews, PMTs were allowed to use code-switching whenever necessary to explain their thinking to optimise their comfort and to capture the true nature of their thinking if they could not explain it in English. These data were transcribed with the help of a translator and formed part of the data analysis to answer the research questions for this study. While the participant might not be well versed with English as the language of teaching and learning the researcher's first language is also not English (the LoTL in SA). However, the researcher has 5 years of teaching mathematics in high school and 4 years of teaching mathematics at an institution of higher learning which both requires not only competence in English but also mathematical language.

4.4 INSTRUMENT OR DATA COLLECTION METHODS

Before explaining the instrument and the methods used for data collection, I feel compelled to highlight the significance of the elements of mathematical discourse as presented by Sfard (2008). This explication of the elements enabled me to refer explicitly to these significant

[20] During the lectures I conducted with them in Euclidean geometry.

elements and I used them to explain PMTs' thinking as they solve geometry problems. Unlike other studies framed by commognition, this study aimed to explain PMTs' thinking through all the elements of mathematical discourse, instead of focusing on just one or two of these elements. As seen in Chapter 3, participating in mathematical discourse means that PMTs need to use words, narratives, visual mediators and routines as important differentiators between mathematical discourse and colloquial discourse. I decided to focus on all these elements of mathematical discourse, because they complement not only participation in mathematical discourse, but also the process of teaching and learning. Furthermore, they allowed me to have a holistic view of PMTs' thinking when they solve geometry problems to attempt to give a holistic view of the implications for teaching. Figure 4.2 represents the relationship between all these elements of mathematical discourse and how they encapsulate Sfard's (2008) theory of mathematics as communication. From this diagram, word usage, narratives and visual mediators encapsulate mathematical routines which can be ritualistic or explorative. Communication, whether in a mathematical or colloquial discourse, depends on word usage. However, it is the words that are used in the discourse that really differentiate a mathematical discourse from a colloquial one. How interlocutors refer to the objects in their discourse matters, because it is the only identifier of mathematical objects and non-mathematical objects. However, some words have homonyms and can only be identified clearly by contextual clues, which is important because mathematical objects are easily identifiable in the mathematical context, and cannot be confused with objects of the colloquial discourse. It is then important that when such homonyms are used in discourse, clarity is sought in terms of their actual context in which these words are used. In the same way, narratives and visual mediators refer to and identify mathematical objects because of their specificity in the context of mathematics. In a mathematical discourse, all these elements are significant in communicating about mathematical objects. These elements are considered to be part of mathematical routines, hence they all reveal whether a certain interlocutor thinks and learns about mathematical concepts ritualistically or in an exploratory way.

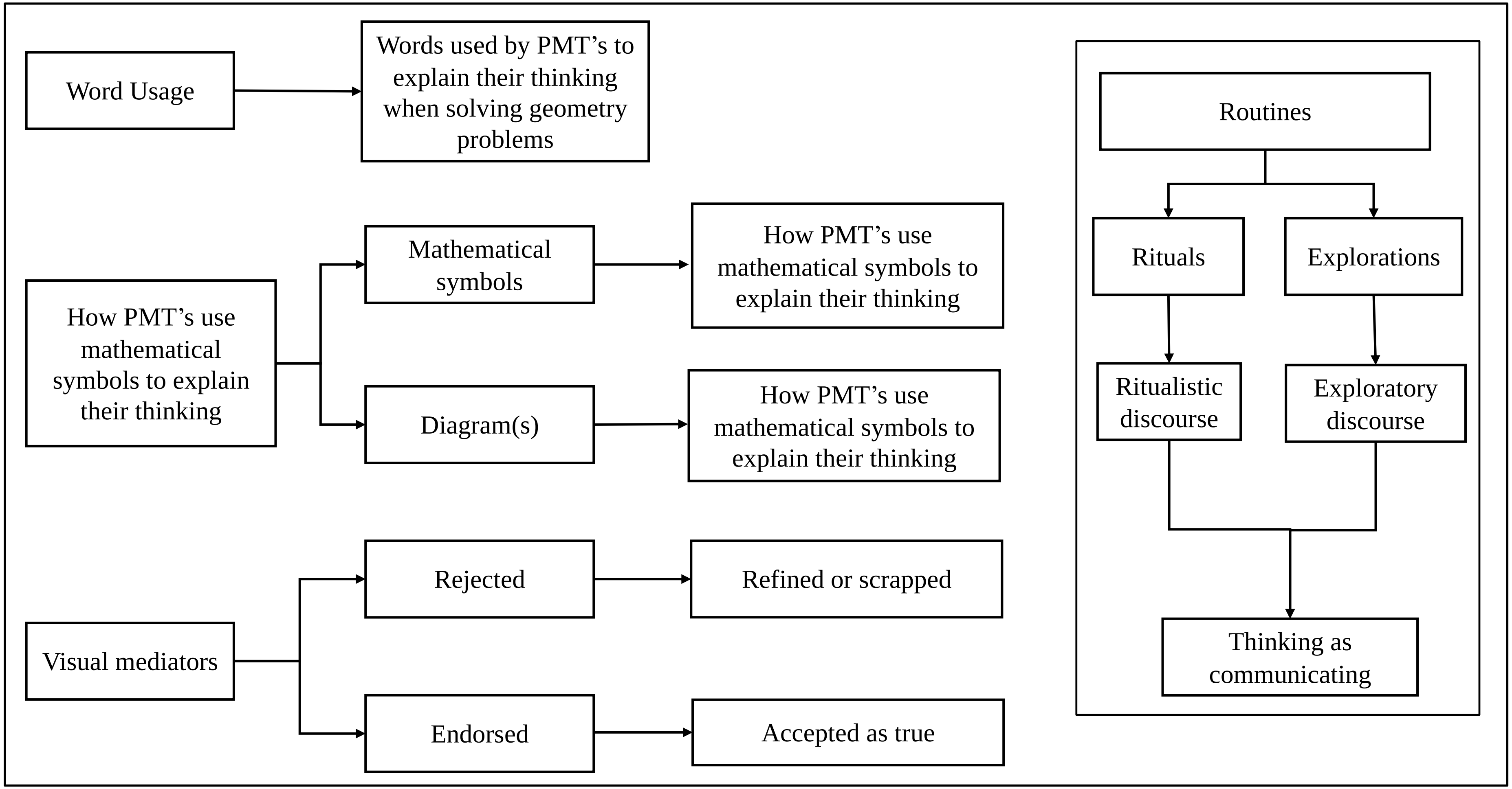


Figure 4.2: The relationship between word usage, narratives and visual mediators and how they encapsulate learners' routines in mathematical discourse

4.4.1 The task-based interview

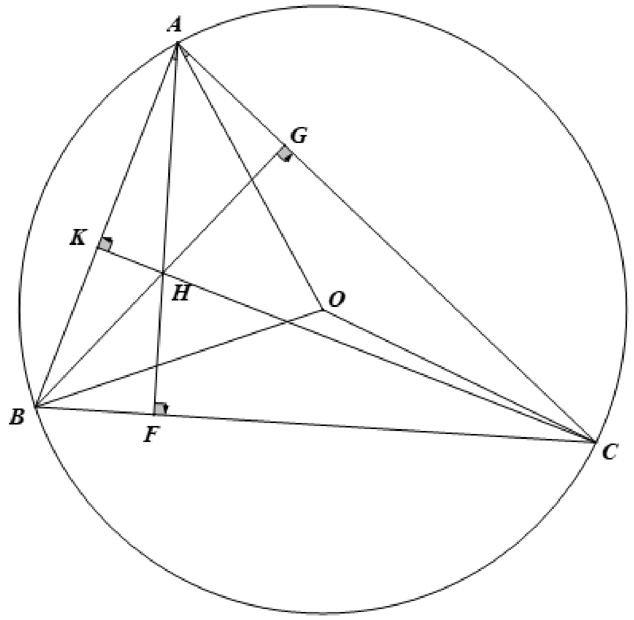
Task-based interviews are a method of interviews that has been used in mathematics education to gain knowledge about a certain individual's (or a group's) state of knowledge and their behaviour (thinking) during problem solving (Maher & Sigley, 2014). Task-based interviews are a triadic interaction between the interviewee, interviewer, and the task.[21] Task-based interviews allowed the interviewer to probe PMTs' thinking when solving geometry problems, because they are useful in observing participants thinking as they solve problems in mathematics (Goldin, 1997). Furthermore, one-on-one task-based interviews can improve teachers' expertise in understanding their learners' thinking during a certain task or problem-solving activity (Clarke, Clarke, & Roche, 2011), which can allow the teachers to support learners' mathematical learning better. In a dynamic geometry platform, task-based interviews may also allow teachers to discern their learners' visual or spatial thinking related to a certain task (Leung & Lee, 2013).

With the broad aim of understanding PMTs' thinking as they solve geometry problems, they were given two geometry problems to solve in a one-on-one task-based interview with the researcher, as seen in Figure 4.3. These problems were divided into two types. Similar to the geometry problems learners encounter in high school geometry (problem 1 in Appendix A), and what these PMTs were going to teach, the first problem contains a diagram. In the second problem (see Appendix A), however, I employed an unorthodox approach where I posed a geometry problem in words. The main aim of this was to collect data on PMTs' uses of visual mediators (diagrams) and their narratives as they solved geometry problems. I wanted to investigate if the availability (or unavailability) of a diagram enhanced PMTs' problem solving, but mainly, how did the PMTs use the given information in each diagram to guide their thinking patterns during problem solving. As explained in Chapter 2, the availability of diagrams in geometry tasks helps the process of problem solving. Hence, here I wanted to verify (or refute) this observation. From engaging with these visual mediators, PMTs produced some visual narratives and utterances which were either endorsed, refined or rejected. The interviewer requested PMTs to 'think aloud' as they solve the geometry problems, to capture data about their thinking. The visual narratives from PMTs' responses were useful in addressing the demands of the third research question, while their utterances on the use of particular visual

[21] In this case, the task is the geometry problems to be solved. In other scenarios, the task can be seen as the subject matter to be learnt.

mediators were used to support their use of a certain visual narrative. The discussions leading to the answering of this question encapsulated PMTs' (visual) thinking as they solved geometry problems. All these problems were solved in mutual, comfortable and conducive (to the participants) buildings chosen by the participants in a one-on-one setup with the researcher.

1. Let O and H denote the circumcenter and orthocenter of an acute triangle ABC, respectively. Show that $\angle BAH = \angle CAO$. (HINT: The **circumcenter** is the point where the perpendicular bisectors of a triangle intersect. The circumcenter is also the center of the triangle's circumcircle - the circle that passes through all three of the triangle's vertices. The **orthocenter** is the point where all three altitudes of the triangle intersect. An altitude is a line which passes through a vertex of the triangle and is perpendicular to the opposite side.)



2. A mirror is placed between the teacher and a student at a point C. The teacher is at point A and the student at point B. The student is standing at a point B with the distance BC = $\boldsymbol{d_s}$. The teacher moves away from point C until she can see the top of the students head in the mirror at a point A away from point C with AC =$\boldsymbol{d_t}$. The teacher has a height of $\boldsymbol{h_t}$ and she knows that the angle of incidence is equal to the angle of reflection which is equal to $\boldsymbol{\alpha}$. What is the equation that can be used to determine the height of the student, $\boldsymbol{h_s}$?

Figure 4.3: Task-based interview schedule

While PMTs were solving these problems, the researcher observed the participants and they were given access to solve the problems on their own with minimal interference from the interviewer. Furthermore, the data from these task-based interviews were further used to identify the types of discourses PMTs engaged in as they solved the geometry problems.[22] The investigation of PMTs' routines through their narratives, word usage, and visual mediators during the task-based interviews also shed light on their thinking as they solve geometry problems, which further addressed the second research question.

4.4.2 The one-on-one face-to-face interviews

Interviews are common and widely used instruments for data collection in qualitative research. They are "an interchange of views between two persons conversing about a theme of mutual interest" (Kvale, 1996, p. 2), which views knowledge as socially situated and human interaction as the main contributor to knowledge construction (Cohen et al.,, 2018; Lave & Wenger, 1991). Face-to-face interviews are relatively flexible and allow researchers to seek clarity from participants instantaneously when the need arises, which meant probing PMTs to explain their thinking of certain actions during their problem solving and their utterances during their think-aloud problem solving. After the task-based interviews, participants were engaged in semi-structured, face-to-face, one-on-one interviews[23] which lasted between 45 minutes to one hour. These semi-structured interviews were aimed at exploring preservice teachers' discourses about their thinking when solving geometry problems. During these interviews, PMTs were given a chance to explain their thinking when they were solving the geometry problems. Since the task-based interviews were recorded using a video camera, these video record(s) for each PMT was played on a big screen and each participant was asked questions probing them to explain what they were thinking while they solved the geometry problems. This allowed the researcher a chance to probe PMTs' thinking and if there was a chance, explore their alternative conceptions of solutions to the problems.

4.4.3 Data sources

The participants completed a task-sheet[24] during one-on-one, task-based interviews and that task sheet was part of the main data source for this study. During the process of the task-based interviews, the researcher observed the participants completing the task-sheet took observational notes and asked thought-provoking questions that allowed the interviewees to communicate their thinking. The task-based interviews allowed the collection of data about the lived experiences of PMTs' discourse about their thinking, which was multisensory. This

[22] Whether ritualistic or explorative discourse participation
[23] The semi-structure face-to-face one-on-one interviews can be found in Appendix D.
[24] The task-sheet contained PMTs' written solutions for the problems.

multisensory data was videotaped and audio recorded to capture both visual and vocal data as PMTs solved the geometry problems. The third data source used was the semi-structured interview transcripts that were transcribed by the researcher with the help of language specialists (Setswana and Sesotho). Since it was the interchange of views between the interviewer and the interviewee, knowledge was seen as constructed between them, instead of being measured or extracted from the participants. Hence, knowledge, in this case, was socially situated. The semi-structured interview allowed for the collection of in-depth data that revealed participants' experiences about their discourse on thinking. Out of the 6 participants who participated in this study, 6 task-sheets (written solutions), 6 interview transcriptions, and 6 observations[25] provided sufficient data to compile the thesis for this study. However, when data saturation was reached before collecting all this data from the 6 participants, then I stopped the data collection and commenced with the analysis of data.

4.5 METHODS FOR DATA ANALYSIS

In this section I discuss the methods I used to analyse the data from the current study to ensure that concise and trustworthy answers to the research questions are reached.

4.5.1 Introduction

The process of data analysis requires that one acts according to the defined terms of his chosen theoretical lens to view the data of a certain research study. The theoretical framework[26] for the current study conceptualised in Chapter 3, directly relates to the aims of the study and provides precise definitions of the terms to be used in this study. Word usage, visual mediators and narratives form part of PMTs' routines (see Figure 4.2), and these were used to categorise PMTs' discourse participation as they solved two geometry problems. With their enabling ability to identify whether a routine represents ritualistic or explorative discourse participation, each of the four elements of mathematical discourse provided a comprehensive view of PMTs' thinking. Furthermore, PMTs' explanations of their choice of certain word usages, visual mediators or narratives during their routinisation in geometry problem solving during the face-to-face interviews, opened a window that allowed for the explanation of not only their discourse participation, but why their discourse participation was either ritualistic or explorative. The

[25] An observation schedule is not included since the researcher was observing PMTs' actions as they solved geometry problems instead of specific pre-determined characteristics or phenomena.
[26] Explained in Chapter 3.

distinction between explorations and rituals[27] became the first level of data analysis from PMTs' task-based interview transcriptions which gave data to identify the difference between the two types of discourse participation. Thereafter, the second level of data analysis utilised the same data from the task-based interviews, which were coded for the Van Hiele level of geometry thinking shown by PMTs in their geometry problem solving.

4.5.2 Coding of the data from task-based interviews

In a world-class scholarly piece of writing, Saldana (2016) gives a detailed description of how qualitative researchers attach codes to transcribed verbal data in order to engage in the process of analysing data and to draw conclusions to answer the research questions. Saldana (2016) provides ample examples of showing how data can be coded and within these examples he also gives examples of coding for different purposes. A code is a word or a phrase attached to a portion of data (either a word, a sentence or a paragraph) to capture the characteristics of that portion of the data (Saldana, 2016), and captures the richness of the phenomenon being investigated (Boyatzis, 1998).

In this study I utilised qualitative data analysis through generating codes that corresponded to Table 3.1 and Table 3.2 to analyse the data from the task-based interviews. From Table 3.1, 'closing goal' represents a **category**, then "to maintain and improve ones' relationship with others" and "produce endorsed narratives about the world" are **themes,** and what follows after those are the **codes** associated with each theme. Through transcribing the data from the task-based interviews, I was able to establish an intimate relationship with PMTs' narratives, word usages, visual mediators and routines and that provided me with the introduction needed for the subsequent analysis (Saldana, 2011). The analysis began with reading through the transcribed data and identifying parts of the data that corresponded to certain themes of the theoretical framework and these parts were coded, based on PMTs' actions during problem solving. Table 4.2 represents the final coding scheme that was used to code the data from the task-based interviews.

Furthermore, the task-based interview transcripts were coded for the Van Hiele level of geometrical thinking shown by the PMTs during their geometry problem solving, using Table 4.3 to explain PMTs' level of geometrical thinking. The codes in Table 4.3 were generated using the Van Hiele theory which explains one's geometrical thinking and reasoning, using five

[27] Discussed in detail in Chapter 3 and summarised in Table 3.1.

levels which are arranged hierarchically. However, as explained in Chapter 3, the fifth level of the Van Hiele theory does not apply to high school geometry, hence, in this study, PMTs needed not to master this level but the course (the Euclidean geometry module) the fifth level was explored to ensure that the teachers' knowledge is above that of learners. Thus, for this reasons, Van Hiele level 5 was excluded in the coding process for this study. These levels show that one cannot jump levels. Each person needs to go through each level so that a person operating at Van Hiele level 2 is simultaneously operating at Van Hiele level 1 (De Villiers, 2012). Furthermore, these levels operate such that failure to master one level makes one's functionality at the next level to be obsolete and contributes to erroneous thinking or struggles in solving geometry problems posed at that level. For example, if a problem is posed at Van Hiele level 3 and a certain problem solver has not mastered either Van Hiele level 1 or 2, they will struggle to solve the problem, or their solution will be erroneous. The codes presented in Table 4.3 are divided into two categories: what teachers are able to do and not able to do during their geometry problem solving. Their inabilities are interpreted as the failure to master a certain Van Hiele level and their abilities as evidence that they have mastered a certain Van Hiele level of geometry thinking.

Table 4.2: Final codes differentiating PMTs' ritualistic and explorative discourse participation in the task-based interviews

Ritualistic discourse participation		Explorative discourse participation	
	Closing goal		
	Code		Code
To maintain, improve ones' relationship with others and perform procedures that will lead to $\angle BAH = \angle CAO$ and $h_s = \cdots$. - PMTs are worried about making mistakes errors in their solutions, - Seeking approval from others (e.g. the interviewer), - Agrees or changes narratives without questioning.	**R1**	**Produce an endorsed narrative about the world.** - Comes up with endorsed narratives about mathematical objects.	**E1**
	Who performs the routine		
With (scaffolded by) others - PMTs endorses/correct narrative after scaffolding, - PMTs require scaffolding from others (e.g. the interviewer) by asking questions, - Cannot justify procedures and processes leading to narratives, - PMTs rely on memory or previous experiences.	**R2**	**No need for scaffolding – can be performed individually** - Wants to perform the routine individually, - Comes up with approaches useful for solution, - Can justify narratives using colloquial or mathematical word.	**E2**
	Audience (for whom is the routine performed?)		
Others - PMTs view questions as coming from an outside authority or rely on outside authority, - PMTs' endorsement or discourse of narratives relies on visual appearance of diagrams, - Admits question is hard or gives up.	**R3**	**Others and oneself** - PMTs' discourses are internally persuasive and rely on themselves, geometry properties, theorems and definitions to produce endorsable narratives.	**E3**

Level of applicability of the procedure			
Restricted, the procedure is highly situated. - Follows strict procedure to come up with an endorsable narrative, - Follows the empirical approach**.**	**R4**	**Broad, applicable in a wide range of situations.** - Uses trial and error to solve problems.	**E4**
Level of flexibility in finding the solution			
Almost no degrees of freedom in the course of action. - Sceptical about certain approach, procedure or solution, - Gets stuck, and sees no other approach for solving the problem.	**R5**	**Different courses of action taken to endorse narratives.** - Sees other methods of solving the problem, - Adapts procedures accordingly to find the solution.	**E5**
Level of correctibility of the narratives			
Narrative cannot be locally corrected - PMTs cannot recognise and correct incorrect narratives.	**R6**	**Locally correct and replace narratives with equal sub-routines.** - PMTs can recognise and correct incorrect narratives.	**E6**
Acceptability and endorsement of PMTs' narratives			
Adheres to strict rules of the routine and acceptance depends on others. - PMTs' narratives are not endorsable, - Visual mediators are correct but not endorsable because of the absence reasons supporting geometrical statements, - Concludes from unendorsed narratives.	**R7**	**The acceptance of narrative is independent of other people.** - Narratives about mathematical objects are endorsable.	**E7**
Words and mediators' use			
Phrase driven use of keywords – as descriptions of extra-discursive mediators. - PMTs uses colloquial words to signify mathematical objects, - Produces or uses incorrect visual mediators.	**R8**	**Objectified use of keywords – as signifying objects in their own right**. - Narratives attached to mathematical objects, - Uses correct visual mediators.	**E8**

Table 4.3: Codes used to code task-based interviews according to the Van Hiele levels of geometrical thinking

Teacher's behaviour	Relation to Van Hiele levels	Relation to elements of mathematical discourse
Unable to produce a diagram from written information of depict certain geometrical relationships (like theorems) in a diagram.	This can be attributed to teacher deficiency of visualisation which is linked to Van Hiele level 0 of geometry thinking.	Teachers are struggling with their word usage, visual mediators, narratives and routines, simultaneously at all Van Hiele levels.
Unable to differentiate between geometry objects (or drawings) using their properties.	Here teachers struggle with Van Hiele level 1 of geometry thinking.	
Inability to relate geometry objects (or drawings) utilising the relationship between their properties, for example, deducing how a square is related to both the rectangle and the rhombus.	These teachers have not developed Van Hiele level 2 of geometry thinking.	
Teachers are unable to use deduction to prove geometry theorems, whether it is a theorem or proving that certain lines are parallel or that certain angles are equal.	Teachers here are struggling with Van Hiele level 3 of geometry reasoning	

4.5.3 Analysis of the semi-structured interview

The second phase of data collection involved face-to-face interviews where PMTs explained their reasons for choosing specific routines during their problem solving. This interview data were transcribed and each response from each participant was grouped according to each question that the participant responded to in the semi-structured interview. These interview transcripts were utilised to support explanations of why PMTs' discourse participation was either ritualistic or explorative. In particular, these PMTs' responses gave supporting evidence of how and why certain routines were performed. Due to the variation of explanations given by the participants for their performance of particular routines, I did not code the face-to-face interviews. However, I used them to support PMTs' performance of a certain routine, and to explain why the PMTs were thinking in a certain way. The task-based interviews and the supporting explanations generated from the face-to-face interviews were aligned such that the analysis became useful in answering the research questions posed in this study, contributing positively to the body of literature in geometry teaching and learning and contributing immensely to the theory of commognition. In Vignette 1, I give an example of how the data from the semi-structured interviews support PMTs' discourse participation. Please note that all examples from discourses are reproduced verbatim and unedited.

Vignette 1: Example of data analysis that combined data from the task-based and face-to-face interviews.

P3 discourse participation in the task based interview when asked to prove that $\angle BAH = \angle CAO$ in the first question P3 stated that:

"P3: Okay, what I normally do, since we are trying to prove that some given angle is equal to another angle, so I start by looking at the similarity of triangles". [Source: P3s' Task-based interview transcript]

When asked in the face-to-face about his first thoughts about question 1, P3 said:

"P3: Okay, on the first one, the first thing that I thought of was similarity, that is the first method that I thought of because in normal cases when you are trying to prove, to say angles are similar, the normal way of doing it, the most thing that I normally do is if I can prove that these two triangles are similar, then automatically the angles will be equal". [Source: P3's face-to-face interview transcript]

and when asked why these were his first thoughts P3 stated that:

"P3: In the first one, I think from my school days, from my teacher that is how I was taught. Whenever they say that prove that this angle is equal to that one, in most cases we need to locate the triangles in which those angles are in, then from there if you can prove that those triangles are similar or congruent, then we have it". [Source: P3's face-to-face interview transcript]

4.6 TRUSTWORTHINESS

This section explains the measures that were taken during this study to ensure that the findings of the study are trustworthy and credible to the readers of this thesis. Using the guidance from Guba (1981) and Guba and Lincoln (1981), I discuss how the credibility, dependability, confirmability and transferability were ensured in this study.

4.6.1 Introduction

Trustworthiness is a term that has been widely used in qualitative research to represent the measure in which the analysis of the data in a certain research project, including the conclusions of that research project, can be trusted. Hence, any study must be clear and exact in the methods used to collect and analyse data to address issues of trustworthiness and rigour. Guba and Lincoln (1981) and Guba (1981) suggest that there are four major concerns that the trustworthiness of each scientific and naturalistic research study should address: *truth value, applicability, consistency,* and *neutrality*. These four major concerns strengthen the measure of confidence of readers that the communicated research findings present the real experiences of participants (Morse, 2018). The *truth value* concerns itself with how one can obtain the ‘truth’ of the findings if the study was replicated with the same participants and context. *Applicability* seeks to find the applicability of the findings of a particular study in different contexts and participants. *Consistency* explains how the findings of a certain study can be replicated under similar conditions as the original study. *Neutrality* aims to address subjectivity and biases of a research study (Ary, Jacobs, Irvine, & Walker, 2019; Guba & Lincoln, 1981). However, approaches to ensure these criteria differ in scientific and naturalistic approaches. To achieve rigor in qualitative research, the truth value is achieved through *credibility* which gives confidence in the truth of the findings, applicability is measured by ensuring *transferability* of the research, consistency through *dependability,* and neutrality is achieved through *confirmability* (Guba, 1981). For the current study, Table 4.4 was developed to show how the issues of trustworthiness and rigor were maintained.

Table 4.4: Highlights how rigour was maintained in the current study

Credibility (truth value)	Dependability (consistency)	Transferability (generalisability)	Confirmability (neutrality)
Triangulation	Triangulation	Thick descriptions	Triangulation
Member checking	Audit trail	Purposive sampling	Audit trail
Prolonged engagement			Keep a journal (reflexivity)

4.6.2 Trustworthiness in data collection

SA mathematics education has seen incessant poor quality performance in school-leaving results of learners. Furthermore, research evidence, as discussed in Chapter 1, shows that even teachers struggle with teaching and solving geometry problems, while some research evidence shows that university preparation does not advance PMTs' geometry thinking. Despite its absence from the compulsory curriculum from 2006 to 2011, the presence of geometry in the compulsory curriculum since 2012 was enough time for the DBE to plan effective professional development programmes and for universities to design a geometry curriculum that will have an impact on advancing PMTs' geometry thinking, teaching and problem solving. The fact that this poor performance in geometry problem solving from both teachers and learners still continues even now, presented me with a quandary which prompted me to investigate PMTs' thinking through their discourse in geometry problem solving. This was done in order to find ways in which PMTs' geometry thinking can be supported to improve their geometry problem-solving skills, which will have a positive impact on their practice. This study began by selecting a convenient context (research site) where PMTs were within the sampling conditions described above. The participants were purposively sampled because they met the requirements of having completed a Euclidean geometry module from the selected research site. Thereafter, ethical clearance from the humanities and social sciences research ethics (HSSREC) and gatekeeper clearance from the research site was sought and obtained prior to collecting data with the participants. During the data collection phase, triangulation was ensured by utilising both the task-based interview schedule and the semi-structured interview schedule to collect data. Prolonged engagement with participants was ensured by keeping longer engagements with the participants in the task-based interviews and even longer follow-up with the participants through the usage of semi-structured interviews where PMTs were asked to explain the reasons for their choices of certain problem-solving strategies and actions during their geometry problem solving.

4.6.3 Trustworthiness in data analysis

Here I discuss how trustworthiness was ensured during the process of data analysis.

4.6.3.1 Credibility

To ensure credibility in this study, three elements were taken care of: triangulation, member checking and prolonged engagement. During the data analysis phase, triangulation was maintained by analysing data from different data sources, the task-based and semi-structured face-to-face interviews. Furthermore, triangulation was ensured by combining two theoretical positions, commognition and the Van Hiele level of geometry thinking, to analyse PMTs' thinking when solving geometry problems in what Denzin (1970, p. 295) refers to as "theory triangulation". For member checking, data from both the task-based and face-to-face interviews were transcribed and given to the participants to verify its authenticity and they had to sign an agreement that the transcripts represented their exact words from the interviews.

4.6.3.2 Dependability

Triangulation also influenced issues of dependability, as explained under credibility. To ensure dependability, an audit trail was kept throughout the study by explaining all the exact research steps from the design of the study, to sampling, instruments, data collection methods and data analysis methods. I gave clear and concise reasons why the specific design for this study was chosen and I gave clear guidelines for choosing the participants with a motivation for why those guidelines and particular participants were chosen. Furthermore, a clear explanation of the process of designing the instruments used in this study was given, together with how the theoretical framework in Chapter 3 was used to analyse the data for this study. This was done to ensure transparency in the way the study was conducted (Korstjens & Moser, 2018).

4.6.3.3 Transferability

Transferability was ensured by giving thick descriptions of the data and utilising clearly described purposive sampling to sample the participants of this study. Thick descriptions were given by not only describing PMTs' experiences during geometry problem solving in the task-based interviews, but also follow-up semi-structured interviews and a full description of PMTs' context was included. Furthermore, the interview transcripts were quoted verbatim to present thick descriptions of PMTs' thinking discourse during their geometry problem solving.

4.6.3.4 Confirmability

Despite triangulation (explained in 4.6.3.1) and keeping a clear audit trail (explained in 4.6.3.2), confirmability was also maintained through the usage a reflexive journal. One important conclusion from previous studies is that most researchers are not given enough guidance on how to articulate issues of reflexivity and how these issues shape their analysis of data in their research studies (Mauthner & Doucet, 2003); hence, the reflexivity issues I took care of in this study were those I was aware of as a researcher. Unlike quantitative studies that begin with the procedure and hypotheses already known, qualitative research develops through the research process such that methods, instruments and sampling are continuously being adapted by the researcher while the study is going on and findings are synthesised and constructed by the researchers from their own experience and knowledge instead of being found (Korstjens & Moser, 2018; Mauthner & Doucet, 2003). My continuous presence in the study allowed for the creation of an informed method of interpreting the data from the interviews. The data was inclusive of not only data obtained from the predesigned instruments, but also my observations of PMTs' facial expressions and their embodied actions which spoke to me differently compared to the data from the instruments. Furthermore, steps and procedures for data collection and analysis were described and motivation for each method was given.

4.7 ETHICAL CONSIDERATIONS

Ethical issues are very sensitive in qualitative research. Hence, such issues need to be addressed properly. In ensuring that all ethical standards were kept clean and clear, I began by applying for ethical clearance from the university's ethics office through the humanities and social science research ethics committee (HSSREC). The HSSREC provided me with a 'provisional clearance' pending the gatekeeper clearance from the research site. Using this provisional clearance, I applied for the gatekeeper clearance from the research site and it was provided in a space of three months (Appendix A). After getting the gatekeeper clearance from the research site, I submitted this letter to the HSSREC and full ethical clearance was provided and was valid until 03 August 2021 (Appendix B). Thereafter, I started the process of recruiting participants by sending consent forms to the students who did the Euclidean geometry module (Appendix C). The consent form explained the objectives of the study, how the participants' privacy was protected by using pseudonyms and that the research site would not be mentioned in the final write up of the thesis. Furthermore, the participants were given the transcripts of their interviews for them to go through and make changes where they feel they may have been misrepresented during the transcription process. The consent form further indicated to participants that their participation in this study was completely voluntary and they were being allowed to withdraw from the study anytime they wish to do so. All ($n = 23$) the students were willing to participate, so I purposefully selected 10 participants to participate in this study. While collecting the data from these participants, I was disturbed by the ordering of the national lockdown to curb the spread of the corona virus (COVID-19). This occurred after I collected data from six ($n = 6$) participants which was deemed saturated to answer the research questions.

CHAPTER 5:
FINDINGS OF PMTS' DISCOURSE PARTICIPATION DURING GEOMETRY PROBLEM SOLVING

5.1 INTRODUCTION

The ensuing three chapters narrate and discuss PMTs' discourse participation to reveal their thinking during geometry problem solving. Through the utilisation of commognition, which concerns itself with the individualisation of thinking and the Van Hiele theory of geometry thinking, I explain PMTs' thinking when solving two geometry problems. The current chapter focuses on findings from PMT's communication when solving these problems. To explain PMTs' individualisation of their thinking in this study, I used Sfard's (2008) notions of ritualistic and explorative discourse participation. PMTs were given a chance to solve the problems individually, but after some time elapsed, they did not produce any narrative relating to endorsing their routines and in most cases, they pleaded for scaffolding from the interviewer. Hence, the data from task-based interviews is a product of the interpersonal communication between the PMTs and the interviewer and intrapersonal communication of the PMTs. This data was coded using Table 4.2 and the Van Hiele theory of geometry thinking (Table 4.3), which provided explanations about the Van Hiele level of PMTs' geometry thinking during their geometry problem solving. PMTs' intrapersonal communication, which according to commognition, represents their thinking, was expressed as interpersonal communication with the interviewer and visual narratives in the form of diagrams and proofs.

These expressions, verbal and visual, allowed me to assess the Van Hiele level of geometry thinking for each PMT, using Table 3.3 to explain their success or failure during problem solving. Data from the face-to-face interviews were used to explain why certain routines were performed by PMTs further to cement their discourse participation as to whether it was ritualistic or explorative. These findings encapsulate the analysis that was followed to answer the critical research questions in the current study. Furthermore, these findings represent my perspective of the discourse as observing it from the inside and also from the outside, since I was both a participant and the observer during the process of data generation (Sfard, 2008, 2020). I felt the need to make it clear that what took precedence during the analysis was not just the correctness or incorrectness of the narratives produced by PMTs, but 'how' they spoke about mathematical objects during their problem solving. Unless the object of analysis in a specific theme focused on the ‘correctibility’ of narratives, then it was looked at, but also in the analysis of correctibility of the narrative, the focus was on how PMTs spoke about mathematical objects, which gave rise to either endorsed or non-endorsed narratives.

5.2 OPERATIONALISING THE ELEMENTS OF MATHEMATICAL DISCOURSE

Before the process of data analysis, which in this study relied on the elements of mathematical discourse (word usage, visual mediators, narratives and routines) and the differences between ritualistic and explorative discourse participation from commognition, I felt it necessary to explain how these terms were used in this analysis.

5.2.1 Word usage, visual mediators, and narratives

Word usage in mathematical discourse signifies mathematical objects like angles, theorems and even shapes (Sfard, 2008). On the other hand, visual mediators are visual objects used to communicate the relationship between different mathematical processes and operations during problem solving (Sfard, 2008). In a mathematical discourse, visual narratives encompass, but are not limited to, geometry proofs, geometry theorems, diagrams and any other symbolic artefacts used to communicate about the processes of mathematical objects. A narrative can be spoken or written text used to describe objects, their relationships and also the processes between objects (Sfard, 2008). Words and visual mediators are the basis for the production of narratives that are always subject to validation or refutation (Sfard, 2008). Endorsed mathematical narratives are globally accepted by the mathematics community, because they have never been successfully refuted. These narratives include mathematical proofs, theorems, definitions, axioms and postulates. Rejected narratives are those that misrepresent mathematical processes and operations which include (but are not limited to) incorrect usage of theorems, incorrect visual mediators, and incomplete mathematical proofs. In general, these narratives have been refuted by the mathematics community. As a combination of word usage and visual mediators, narratives are used in this study to categorise PMTs’ discourse. Specifically, I considered what word usage and visual mediators were used by PMTs to construct their narratives about the solutions to the two geometry problems, and I also looked at which visual mediators and word usage blocked PMTs from constructing endorsed narratives about the solutions to the geometry problems. In geometry discourse, the endorsed proof is supported by mathematical reasons that have been endorsed by the mathematics community. For example, the act of finding the missing angle x in **Error! Reference source not found.**, is not endorsable w ithout the reason “sum of ∠s in a Δ”, because without this statement, the narrative in **Error! Reference source not found.** lacks logical flow. Hence, even in PMTs’ discourse, if a statement in the proof was not supported by a valid reason (either a theorem, geometry property or axiom, etc.), it was considered as incomplete and the proof could not be endorsed. An example of this is represented in Table, where the PMTs’ visual mediators symbolising the proof of the angle sum of a triangle, did not contain supporting reasons.

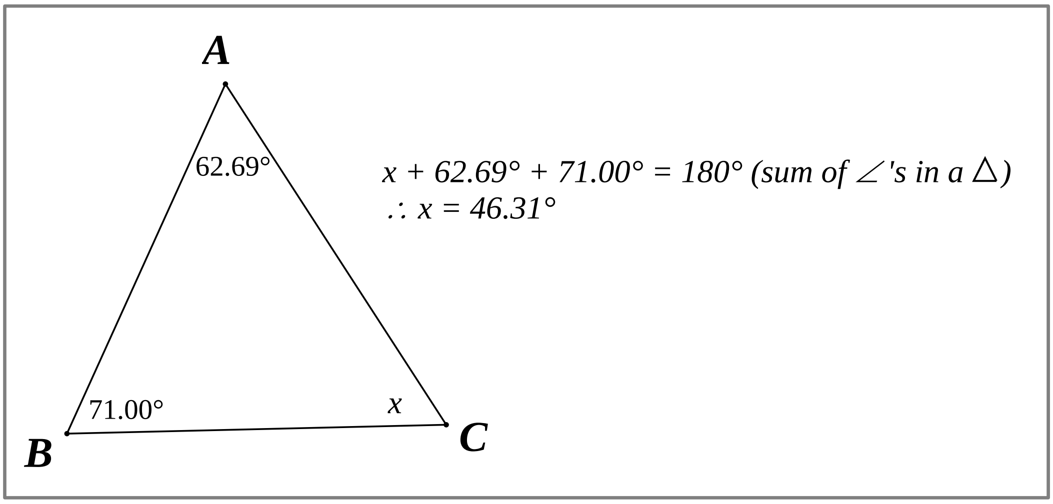


Figure 5.1: Calculating the sum of angles in a triangle

5.2.2 Routines

Sfard (2008, p. 134) defines a routine as the “repetitive patterns characteristic of the given discourse”. Routines are characterised by the regularities in word usage, visual mediators and narratives, such as finding the sum of angles in a triangle or the need to support each geometry statement during the construction of a geometry proof. As described in Chapter 3 (3.6), the how of a routine is concerned with how objects behave, and the when of a routine is concerned with PMTs’ actions during problem solving (Sfard, 2008). Given that school mathematics is dominated by discourses that describe the actions of mathematical objects rather than the actions of problem solvers and why those actions are taken (Sfard, 2008), in this study I concentrated on PMTs’ actions during problem solving and how these actions (visual or vocal) represented their thinking. Furthermore, I sought explanations as to why certain actions were taken, to obtain an understanding of PMTs’ thinking during their problem solving.

5.3 AN OVERVIEW OF THE FREQUENCY OF CODES BETWEEN PARTICIPANTS

To visualise the distribution of codes between the participants, I constructed Table 5.1, which shows the code occurrence per participant in the task-based interview. Table 5.1 gives a summary of which codes were dominant with which participant. For example, P4's discourse was more ritualistic, as most codes were classified within the ritualistic discourse participation, with only one code classified as explorative.

Table 5.1: Code occurrence per participant in the task-based interview

CODES	P1	P2	P3	P4	P5	P6	Totals
E1 [The closing goal is that the PMT produced an endorsed narrative about the world]	0	0	2	0	7	2	**11**
E2 [The PMT can perform the routine individually, no need for scaffolding]	3	0	2	0	0	0	**5**
E3 [The PMT performs the routines for others and oneself]	0	0	0	0	1	0	**1**
E4 [The PMT applies routines and procedures broadly and uses routines and procedures that are applicable in a wide range of situations]	0	0	0	0	1	0	**1**
E5 [The PMT is flexible in finding the solution and endorses narratives through different courses of action]	1	0	3	0	0	2	**6**
E6 [PMTs can locally correct and replace narratives with equal sub-routines]	0	0	0	0	0	0	**0**
E7 [The PMT accept the narrative as their own without relying on others for approval]	2	3	7	1	5	4	**22**
E8 [The PMT uses objectified keywords as signifying objects in their own right]	1	3	2	0	3	0	**9**
R1 [The PMT wants to maintain or improve their relationships with others and performs procedures that lead to the answer rigidly]	4	3	0	5	1	1	**14**
R2 [The PMTs performs routines with (scaffolded by) others]	15	8	16	11	16	9	**75**
R3 [The PMT performs the routine for others]	5	1	3	14	25	4	**52**
R4 [The PMT applies the routines and procedures in a restricted and situated manner]	1	2	0	3	5	2	**13**
R5 [The PMT shows no degree of flexibility in applying routines, procedures and endorsing narratives]	9	1	2	2	2	0	**16**
R6 [PMTs cannot locally correct their narratives and relies on scaffolding from the interviewer]	0	0	0	0	1	0	**1**
R7 [The PMT follows strict rules in endorsing routines and acceptance of the correctness of the narratives depends on others]	13	10	3	3	15	4	**48**
R8 [The PMT uses phrase driven keywords as descriptions of extra-discursive mediators]	8	7	13	4	11	13	**56**
Totals	**64**	**39**	**54**	**43**	**92**	**42**	**334**

Furthermore, it can be observed from Table 5.1 that the most dominant codes in the overall PMTs' discourse participation were R2 ($n = 75$), R3 ($n = 52$) and R8 ($n = 56$) respectively.

From this, it can be concluded that dominantly, PMTs relied mostly on other people to perform and complete their routines and they completed these routines for the satisfaction of others, instead for themselves. Furthermore, PMTs' words and mediator use were mostly ritualistic, characterised by excessive use of colloquial discourse and incorrect visual narratives.

5.4 MATHEMATICAL THINKING THROUGH PMTS' GEOMETRY DISCOURSE PARTICIPATION

This section uses the findings from the task-based interviews and the semi-structured face-to-face interviews to describe PMTs' geometry thinking as they solved geometry problems. This was done by differentiating between their ritualistic and explorative discourse participation as they solved these problems out loud. Furthermore, using the characteristics of the Van Hiele levels of geometry thinking, I also indicated the level of geometry thinking, using the utterances from PMTs' think out loud during problem solving.

5.4.1 Introduction

PMTs' discourse participation when solving the problems posed in this study, was the focal point of analysis to explain their thinking. Their utterances when solving these two geometry problems, revealed whether their discourse participation was either explorative or ritualistic. What follows below is a synthesis of the data to present a coherent narrative about PMTs' thinking during problem solving through the lens of commognition and the Van Hiele theory of geometry thinking. In Table 5.2, I give a glimpse of a vignette from PMTs' discourse participation during their problem solving and an explanation of how these utterances revealed their thinking during their problem solving. Please note that all direct quotations from participants are reproduced verbatim and unedited.

Table 5.2: An example of how PMTs' utterances during problem revealed their thinking

Speaker	What is said	What is done
P1	Okay, let me try. Can I do it alone?	
I	Yes, you can do it alone but please try to think aloud in your attempts to find the solution.	
P1	Okay, will I be wrong if I use this triangle?	Points to ΔABC
I	Okay, I will need you to solve these two problems while you are thinking aloud. Where necessary I will intervene or you can seek for clarification if the need arises.	

P3	Okay, what I normally do, since we are trying to prove that some given angle is equal to another angle, so I start by looking at the similarity of triangles.
I	Okay
P3	When I look at this, let me see, this is the first approach, it might work or might not work. Triangle, okay we can say ABH, that's the same triangle that is having our angle A that we are looking for.
P4	Keeps quiet for a while. Then redirects to problem 1.

The above extract from P1 and P3 reveals different ways of particular thinking about their approaches to problem solving. P1 firstly claims independence, trusting her abilities to solve the problem by saying, "can I do it alone", but immediately seeks approval from the interviewer about her supposed approach to solving the problem when she asks, "will I be wrong if I use this triangle?" In this regard, P1 thought that her approach during problem solving had to be verified by a more knowledgeable person (the interviewer) before implementation. According to the Van Hiele theory, the quotation from P1 shows that she was operating at a ritualistic level 0, because she only used the gesture of pointing at the visual shape "this triangle" instead of identifying the triangle using its vertices, e.g. "can I use ΔABC?". P3, on the other hand, came to the problem-solving scenario with a clear plan of how the events of his problem solving were going to unfold. He said, "what I normally do …" to signify a 'ritual' in his problem-solving experiences, which means that he thought that problems, where one is asked to show that two angles are equal, can be solved using the concept of 'similarity' only. Later on, in the face-to-face interviews, P3 further revealed that this 'ritual' emanates from his erstwhile mathematics teacher.

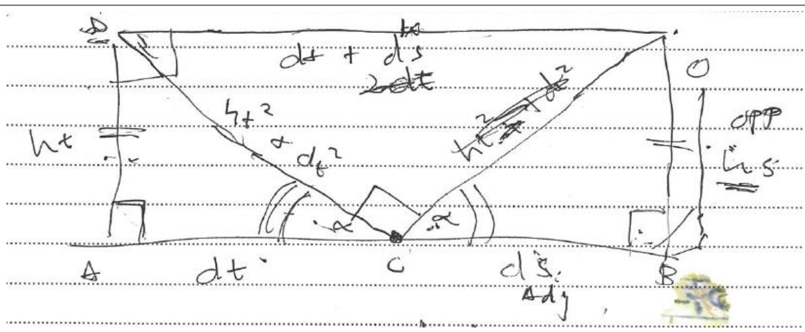
I: *Why was it so important to do that first?*

P3: *In the first one, I think from my school days, from my teacher that is how I was taught. Whenever they say that prove that this angle is equal to that one, in most cases we need to locate the triangles in which those angles are in, then from there if you can prove that those triangles are similar or congruent, then we have it.*

P4 showed retreat as she redirected from problem 2 to problem 1, due to her inability to continue with the solution. What is clear from the extracts in Table 5.2, is that PMTs' approaches to problem solving differed in many ways. Some participants relied more on interviewer approval, while others relied on their previous problem-solving experiences influenced by knowledgeable others. As seen above in the case of P3, the data from the face-to-face interviews were mainly used to support PMTs' utterances from the task-based interviews.

From Table 5.3, P1 showed that she was able to recognise the properties of the shapes from the diagram, while P2 was not able to recognise properties from the diagram. This shows that P1 at that moment was operating at a ritualistic Van Hiele level 2, because even though she was able to identify the properties and link geometry objects using their properties, her discourse was mainly colloquial and not fully alienated. P2, on the other hand, was operating at ritualistic Van Hiele level 0, because she could not link visual mediators with provided descriptions of the geometry objects, as seen in Table 5.3. P5 was able to use given descriptions of the second problem to produce a correct sketch, which is symptomatic of mastering the properties of geometry objects and being able to use them to produce visual mediators. This shows mastery of explorative Van Hiele level 2, since the diagram did not have to be constructed accurately, which would require some logical deductions about the properties of geometry objects. Another mastery of explorative Van Hiele level 2 was observed from P3, who independently related two triangles, based on the one property they shared when he stated, "I think the only thing that can connect them is the fact that they are both triangles so the sum of angles in a triangle is 180."

Table 5.3: PMTs' extracts coded according to Van Hiele level of geometry thinking

Speaker	What is said	What is done
P1	Okay, I know these are radiuses of the circle, meaning that AC is the chord of the circle, then ΔAOC is isosceles. Therefore, this angle is going to be equal to this one.	Pointing to line AO, CO and BO, indicating that they are equal. Indicating that $\angle OAC = \angle OCA$.
P2	*Reads the definitions of circumcentre, then utters,* okay then that means our circumcentre is H, yes, H is the circumcentre.	
P5		

What follows is a comprehensive explanation of PMTs' discourse participation during their problem solving in the current study, using the differences between ritualistic and explorative discourse participation and the Van Hiele theory.

5.4.2 PMTs' ritualistic discourse participation

Ritualistic discourse participation in mathematics is linked to the belief that mathematical learning requires no thought from the learner, but a mere reproduction of procedures and knowledge retrieval. The discourse participation of PMTs classified as ritualistic in this study, is characterised by their focus on producing solutions through recalling certain procedures and definitions with the aim of pleasing others (Sfard, 2008). The classification of these discourses was based on the eight themes from Table 4.2 relating to commognition.

5.4.2.1 The closing goal for PMTs' discourse

In ritualistic discourse participation, the closing goal of the participant aims to maintain or raise their relationship with others. PMTs here were hungry for acceptance, they usually sought approval from others, and they usually accepted suggestions or statements without challenging them, because these would maintain or elevate their relationships with others. There were three main characteristics of PMTs' discourse participation that had a closing goal characteristic to ritualistic discourse participation, as seen in Table 4.2. However, during the analysis, the code 'seeking approval from others' got subsumed to the code 'worried about making errors in their solutions' during the analysis.

- ***PMTs worried about making errors in their solutions***

PMTs' discourse participation contained evidence that they were worried about making errors in their problem solving. In commognition, this worry is characteristic of being worried that making errors would not serve the purpose of maintaining harmony with others. This means that PMTs might think that making errors in their problem solving would jeopardise their relationship with others. As an example, they preferred to get approval from others, before undertaking several actions during their problem solving. Through different utterances, PMTs showed that they were worried about making mistakes in their problem solving.

Table 5.4: Extracts of PMTs' discourse, worried about making mistakes in their problem solving

Speaker	What is said
I	Now that you have studied the problems, please find the solutions.
P1	Eish, my problem is that I did not do geometry in high school, so I do not know a lot of theorems about geometry.
I	Yes, we do not make assumptions like that when we prove in geometry. These angles does not have to be numbers.
P2	Okay, so it doesn't have to be a number, okay I understand. I forgot Euclidean (geometry) by the way.
I	Just a reminder, I am going to record this interview.
P4	Yoh, ay, lockdown sir.
I	What happened during lockdown?
P4	I cannot remember sir [Meaning she cannot remember the content of Euclidean geometry].

Some PMTs showed that they were worried about making mistakes, through revealing that they have forgotten Euclidean geometry as seen in the cases of P2 and P4 in Table 5.4. In particular, these two participants (P2 and P4), forgetting Euclidean geometry content, means that they would not be able to solve the problems, or they would make a lot of mistakes in their problem solving. Hence, they disclosed this information with the hope that whenever they made a mistake or struggled to solve a particular problem, their relationship with the interviewer would not be dented. Possibly, P1 finished Grade 12 during the period when Euclidean geometry was made optional in the SA mathematics curriculum. Hence, she was relying mainly on the knowledge she gained from the Euclidean geometry module that was offered by the researcher. However, P2 and P4 were clear that they have forgotten the content associated with Euclidean geometry, with P4 blaming the lockdown induced by the spread of the coronavirus (COVID-19) for her forgetting this content. P4 was the epitome of someone who wanted to maintain a relationship with others, as she blamed a realistic scenario that has caused much disturbance globally. Hence, even if she made a mistake or was not able to solve the problem, it should be understandable. While P4 required approval from the interviewer, her question in Table 5.5 was symptomatic of ritualistic Van Hiele level 0, because she could not link the visual appearance of the objects (BO and OC) with their properties.

Another way in which PMTs displayed that they were worried about making mistakes during their problem solving, was to seek approval from the interviewer for certain actions before or after the performance of a particular ritual. Instead of seeing out a narrative individually, PMTs frequently sought approval from the researcher about certain routines they were not entirely sure were endorsable. This was evident from PMTs' need for validation through utterances such as 'Am I wrong?' and 'Tell me if I am wrong', as seen in Table 5.5.

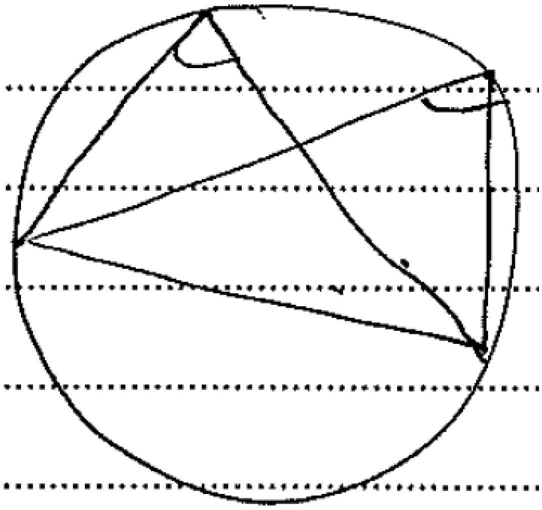
Table 5.5: Extracts of PMTs' discourse, worried about making mistakes in their problem solving

Speaker	What is said	What is done
P2	Okay if this angle is twice this, which means we are going to divide the whole angle by 3, right?	Pointing at the revolution $\angle O$
I	I see you have already came up with the diagram for the second one, can you explain to me how did you come up with this diagram?	
P2	The second one I tried it but I was lost. Here is the teacher standing at A and the mirror standing in between, and they said that the distance from the student to the mirror is d_s and from AC which is d_t, the teacher has a height of h_t and the angle of incidence is equal to the angle of reflection which is α. Tell me if I am wrong.	Pointing at A the diagram. Pointing at C in the diagram.
I	Okay, what do you think is important in this problem?	
P4	Akere, they are saying we should show that $\angle BAH$, which is this angle is equal to $\angle CAO$. Okay, ke batla o polele sir (*I need you to tell me sir*) am I wrong gore (*to say*) this BO is equal to OC?	
I	Why would you say am I wrong?	
P4	Akere (*because*) I can't remember this things because of the lockdown.	
I	Which angles are you trying to prove that are equal?	
P5	I want to prove that B is equal to A.	
I	You are asked to prove that $\angle BAH$ is equal to $\angle CAO$.	
P5	Okay, I have been trying to prove congruency in wrong triangles. Am I wrong to say BK is equal to AK?	
I	I don't know, what would the reason for those to be equal?	
P5	Keeps quiet.	

From Table 5.5, we can see that the participants required the interviewer to validate whether their narratives were correct or incorrect, which is characteristic of being worried about making mistakes during problem solving. As is evident here, PMTs attempted to perform the routine with the interviewer, instead of being independent. In their perceptions, once the interviewer endorsed their narratives, for example in the case of P2, if the interviewer said, "yes", then the narrative is mathematically correct. By seeking these validations, PMTs saw the interviewer as a knowledgeable person from whom mathematical information could be extracted. This means that P2 was going to continue with the narrative of dividing the revolution by 3 as if the angles were equal, if she did not ask the interviewer for approval with the signifier 'right?' as seen in Table 5.5. According to the Van Hiele theory, P2 lacked the understanding of the properties of angles and theorems and she made assumptions based on the visual appearance of the diagram from previous experiences of dividing angles into three equal parts. This is symptomatic of the lack of logical classification of theorems and properties, which relates to ritualistic Van Hiele level 2. Her thinking was classified as ritualistic, because her word usage was not fully objectified, characterised by the usage of signifiers such as ‘this’, instead of talking about the mathematical objects. Even in some cases where PMTs’ discourses seemed explorative, they required approval from the interviewer whether what they were doing, was indeed correct and could be endorsed. For example, P1 individually came up with endorsable sub-routines to solve the first problem (Table 5.6), but still sought the interviewer’s approval. This showed that P1 was over-reliant on the interviewer for approval to maintain the harmony between them. In the second extract, she came up with an endorsable sub-routine, but still sought the interviewer’s approval. Upon seeing that the interviewer was quiet, she remembered the theorem. In Table 5.5, where P5 asked the interviewer, “am I wrong to say BK equals AK?” also symbolised that he was reasoning at a ritualistic Van Hiele level 0, because he only used the physical appearance of the diagram for his inquiry, without any valid reason given relating to the properties of the objects.

Table 5.6: P1 seeking the interviewer’s approval during her discourse

Speaker	What is said	What is done
I	Try	Pointing to AD
P1	Let’s say I have $\angle DAC = x$ and I let $\angle CDA = y$ (Eish I’ll see what will happen). Okay and this line is passing through the centre, so it is going to be a diameter also. Yeah, and then this is going to be 90°, right?	Pointing at $\angle BOA$, looking for interviewer’s approval and the interviewer keeps quiet.
I	Can you then relate y with any other angle in this diagram?	
P1	Keeps quiet for a moment, okay from that drawing, eish what is it … ???	
I	Keeps quiet	

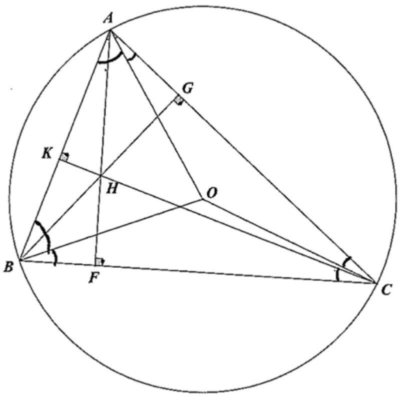
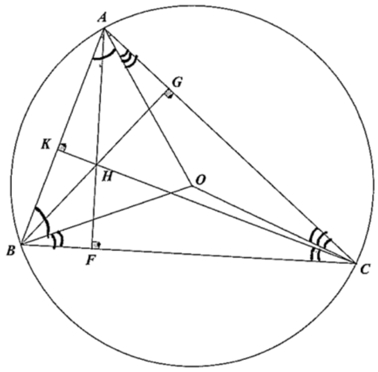
Speaker	What is said	What is done
P1	Giggling non-stop, she comes with a diagram (see below) but does not remember the theorem. Keeps quiet for a moment, then later, she remembers that it is called *angles subtended by the same arc*. 	Starts drawing

Furthermore, PMTs seemed to acquiesce whenever the interviewer suggested particular approaches, showing that they viewed the interviewer as a person of authority who was knowledgeable in problem solving in geometry.

- ***PMTs agrees without questioning the statements of suggestions***

Furthermore, to maintain harmony, PMTs became complacent and tended to agree with the interviewer's suggestions and scaffolding without question. As previously seen from participants being worried about making mistakes, seeking approval from the interviewer means that they regarded the interviewer as knowledgeable and someone who knows the solutions to the problems. Hence, questioning or disagreeing with his suggestions would cause arguments that would disturb the harmony between the interviewer and the participants, so they just followed the interviewers' suggestions and scaffolding without questioning them. P1, for example, performed the routine of solving for x in the first problem, and when the interviewer asked if this was useful, she just decided to change and solve for y instead, as seen in Table 5.7. Furthermore, the "I don't know" before choosing to change her routine, was a sign that she performed the routine of solving for x without any clear purpose of arriving at an endorsable narrative about mathematical objects. This means that the sub-routine of solving for x using the theorem of 'the sum of interior angles of a triangle' was performed, because its procedures could be remembered easily without making errors in the computation. While P2's first assertion was that the altitude CK divided the line AB into two equal parts, she changed her mind without challenging the interviewer when he suggested that CK only divided AB into two parts, but not necessarily equal.

Table 5.7: PMTs agreeing with the interviewer without questioning

<table>
<tr><th>Speaker</th><th>What is said</th><th>What is done</th></tr>
<tr><td>I</td><td>Do you think it was useful to solve for x?</td><td></td></tr>
<tr><td>P1</td><td>I don’t know, okay, let me solve for y and see.</td><td>Starts solving for y.</td></tr>
<tr><td>I</td><td>Okay, is that the only isosceles triangle or you are using this one for now?</td><td>Pointing to ΔBOC and ΔAOB.</td></tr>
<tr><td>P1</td><td>No, this one and this one.

</td><td>Indicating equal angles in the diagram.</td></tr>
<tr><td>I</td><td>So is this the indication? Are you saying all these angles are equal to each other from the different isosceles triangles?</td><td></td></tr>
<tr><td>P1</td><td>Ohhh, we are not sure of that at the moment because of the chords are different, I have to change.

</td><td>Indicating equal angles with 1, 2 and 3 arcs from different isosceles triangles.</td></tr>
<tr><td>I</td><td>Okay, how can we then use these triangles, because that’s what is important. How then are we going to use those triangles?</td><td></td></tr>
<tr><td>P2</td><td>I think we are going to use altitudes, this like from here to here which will lead to $\angle K$ giving us 90°, and this point divides this line into two equal parts.</td><td>Pointing to line CK</td></tr>
<tr><td>I</td><td>It divides them into two parts, but not equal parts.</td><td></td></tr>
<tr><td>P2</td><td>Okay, it divides them into two parts.</td><td></td></tr>
</table>

The behaviour of agreeing without questioning, or seeking further clarity from the interviewer by P2, was also observed in another incident where the interviewer mentioned that O was not the orthocentre as her first conceptualisation. P2 mentioned, "I think that we are going to use this ΔBAH, and I think H is the circumcentre and O is the orthocentre", but when the interviewer mentioned that H is the orthocentre, P2 replied by saying, "okay, it's the orthocentre". Thereafter, she changed her characterisation of centres O and H, without challenging the interviewer or asking for clarification, uttering, "that means this one is the circumcentre (pointing at O). This is a ritualistic behaviour, because it was clear that P2 did not want to be in conflict with the interviewer, but she performed her routines and produced narratives that would maintain harmony between her and the interviewer. The fact that PMTs could not correctly identify the orthocentre and the circumcentre, was also witnessed in the face-to-face interviews where they mentioned that these concepts were difficult for them. When asked what difficulties she experienced in finding the solution, P1 mentioned, "the concepts of circumcentre and the orthocentre were difficult for me, I could not relate those two concepts; they completely slipped my mind." On the other hand, P4 mentioned, "the interaction of the orthocentre and the circumcentre, how to solve problems using the concepts of orthocentre and circumcentre, was difficult". P5 further cemented the idea that the circumcentre and the orthocentre were problematic in the first problem, when he mentioned, "orthocentre and circumcentre was a problem, I did not understand what that was …"

This is also true about the behaviour of changing the visual mediators, indicating that angles are equal, from P1 in Table 5.7. Furthermore, the failure to link her endorsed narratives about the angles of the three triangles in Table 5.7 with visual mediators, is symptomatic of ritualistic Van Hiele level 2, indicating a failure to use geometry properties to produce visual narratives. However, the failure to associate a description with a visual mediator ("H is the circumcentre, and O is the orthocentre") from P2 as seen in Table 5.34, indicates that she was operating at a ritualistic Van Hiele level 1. On the same note of linking descriptions with visual mediators, P2 showed that she could identify the altitude when she mentioned, "which means the line from C to K, it is an altitude, okay". Judging from the objectivity of the discourse and the correct identification of the altitude, P2 showed evidence operating at explorative Van Hiele level 1. Traces of explorative Van Hiele level 1 were also evident from P3 who was able to substantiate individually the equality of two angles ($\angle BAO$ and $\angle ABO$), as seen in Table 5.39. However, his discourse was ritualistic, relying on signifiers of objects, instead of the objects themselves. As a result, he was classified as operating at a ritualistic Van Hiele level 1.

The complaints about forgetting Euclidean geometry-related content, seeking approval from the interviewer and agreeing with the interviewer without question, was evidence that PMTs were focused on maintaining their harmony by ending up with a final answer in the form $\angle BAH = \angle CAO$ or $h_s = \cdots$. Hence, most PMTs' answers to the second question were in the form of $h_s = \cdots$. Even though, what they considered their final narrative was not endorsable, PMTs sounded convinced that their narratives were endorsable, because they were in the form of $h_s = \cdots$. However, looking at these PMTs' extracts in Table 5.8, it was evident so far that they do not represent an endorsable narrative about the solution to the second problem. However, once they saw $h_s = \cdots$, they concluded that they have arrived at the solution, because that was their aim from the beginning. It is clear that when PMTs arrived at the narrative $h_s = \cdots$. they concluded it was the final answer, without considering the other triangle in the height of the teacher (h_t) and the distance of the teacher (d_t) in the statement. This was evident, for example from the P1's responses after she got the answer to the second question, as seen in the quotations below from the task-based interviews.

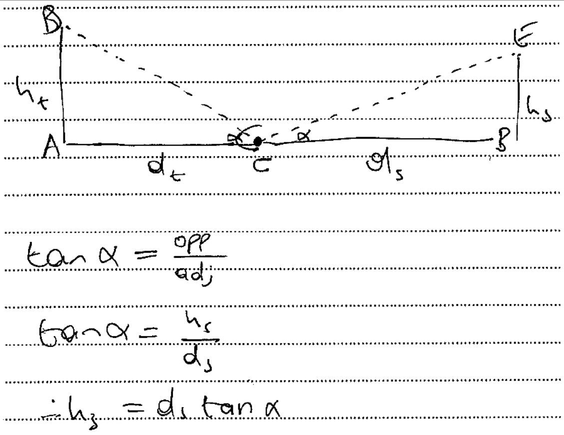
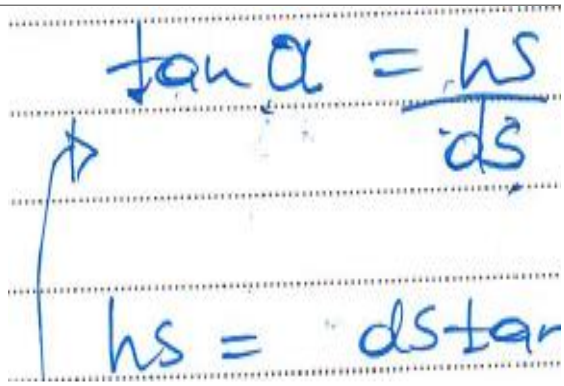
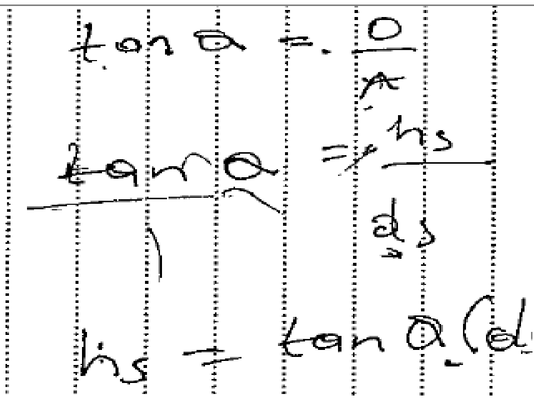
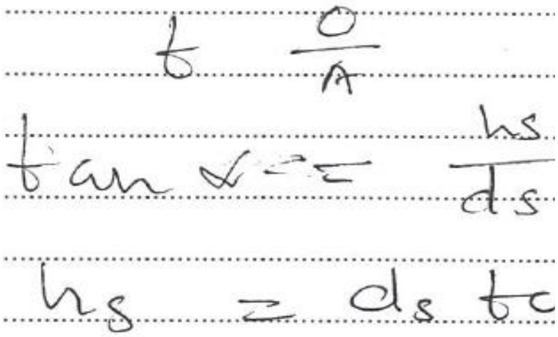
I: *Do you think this solution is efficient? What have you neglected here?*

P1*:* *Everything that has something to do with the teacher.*

I*:* *Now, do you think that is efficient, is the solution correct then?*

P1*:* *I think it is correct because I want the height of the student.*

Table 5.8: PMTs' not endorsable closing conditions about the second problem

P1	P2
	
P4	**P5**
	

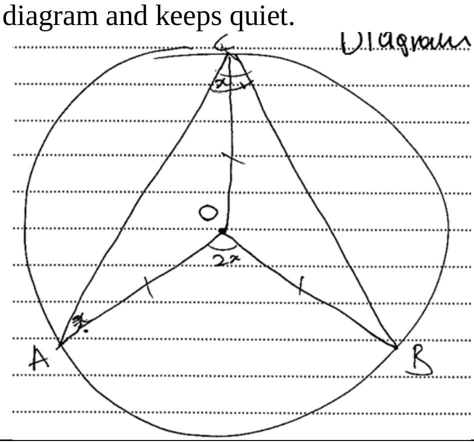
5.4.2.2 Who performs the routine?

In ritualistic discourse participation, the routine is performed with the aid of other people instead of individual efforts. PMTs relied mostly on scaffolding from the interviewer, which was achieved by asking questions or obtaining guidance to arrive at an endorsable narrative. PMTs could not justify procedures and processes leading to their narratives individually and relied mostly on their memory and past experiences. As seen from Table 5.1, R2 associated with the theme 'who performs the routine' was the most coded in the data from PMTs' problem solving. This means that PMTs mostly performed their routines with a heavy reliance on the interviewer, instead of being independent.

- ***Requiring scaffolding to move forward with a routine or to endorse a narrative***

Requiring scaffolding means that PMTs asked the interviewer some closed questions[28] or appealed with a statement that compelled the interviewer to help them in finding the solution to the problem. Table 5.9 represents P1's episode where she desperately needed some scaffolding from the interviewer to solve the first problem. By uttering the phrase "can you point me towards a certain direction please, I need to find the answer to this problem" she revealed that her performance of the routine to solve the first problem, was dependent on the interviewer giving a clue towards the solution. Hence, the arrival to the endorsed narrative of $\angle AOB$ is twice $\angle ACB$, was not going to be possible without the aid provided by the interviewer.

Table 5.9: P1's episode of requiring scaffolding when solving the first problem

Speaker	What is said	What is done
P1	Okay, the whole of $\angle BAC$, kana bae bitsang (*what do they call it*)?	Moving her fingers through line AB and AC back and forth.
I	Subtends???	
P1	Yes.	
I	What subtends what here?	
P1	Angle A is subtended by chord BC.	
I	How does that help you in finding the solution?	
P1	Eish, it does not really help me, can you point me towards a certain direction please, I need to find the answer to this problem.	
I	Can you try to break down the diagram, maybe extract the circumcentre, the radii and the triangles and see and see what you have, so try and draw it without the altitudes, try and strip it into pieces of information that you know.	
P1	Produces a diagram and keeps quiet. 	
I	Okay, now looking at this diagram, what information can you extract?	

[28] A closed question is a question that requires a direct answer.

Speaker	What is said	What is done
P1	Okay, this angle here is going to be twice this angle here.	Pointing $\angle AOB$ and $\angle ACB$. Indicating ∠AOB as 2x and ∠ACB as x.
I	What is that angle?	
P1	Ohhh, $\angle AOB$ is twice $\angle ACB$.	
I	Don't you think you need to indicate that information somewhere in the diagram where it is going to be useful?	
P1	Okay	

P2 also displayed characteristics similar to P1 in arriving at the endorsed narrative about the relationship between $\angle\mathbf{AOB}$ and $\angle\mathbf{BCA}$. To endorse the narrative fully that $\angle\mathbf{AOB}$ is twice $\angle\mathbf{BCA}$, P2 struggled with the correct angle naming and ended up using gestures (pointing) to symbolise the visual mediators she was endorsing. Hence, she relied on the interviewer for the correct angle naming, seeing that she was struggling to name the angles. Unlike P1, who was begging for the clue, P2 had the idea in her head about the relationship between these two angles, but struggled with naming them. Given the way she spoke about the theorems and the mathematical objects was not fully alienated, but with the correct linking of theorems and visual narratives, P2 could be classified as operating at Van Hiele level 2. However, her failure to use fully alienated discourse to describe mathematical objects, characterised her as using ritualistic discourse. Therefore, she was operating at a ritualistic Van Hiele level 2.

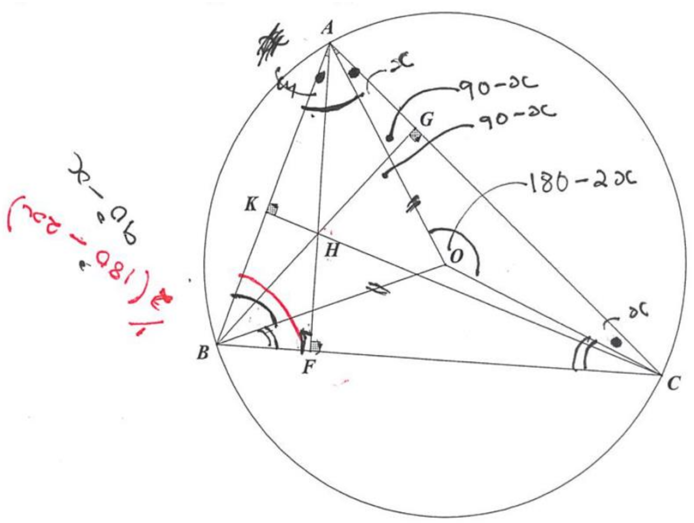
Table 5.10: P2's episode of reliance on the interviewer to endorse a narrative

Speaker	What is said	What is done
I	Yes, it's not a bisector, it's an altitude.	
P2	And we are also going to use, for finding this angle we are going to use this. This $\angle AOB$ is twice the angle CO … mmmhhhh, is it, yeah COA. Gore, $\angle AOB$ is twice $\angle B$ …, this whole angle.	Pointing at $\angle BAC$. Pointing to $\angle BCA$.
I	BCA	
P2	Yes, BCA	
I	Okay	
P2	That is what I think	

Other PMTs also required the help of the interviewer to endorse some of their narratives. For example, as seen in Table 5.11, after failing to find a reason to support why $\angle BAF$ is also equal to x, P3 withdraws and maintains that only $\angle OAC$ and $\angle OCA$ are equal to x. In this case, P3 required the scaffolding from the interviewer to realise that he could not support his claim that

$\angle BAF$ is also equal to x. Hence, the activity of endorsing this narrative was performed with the help of the interviewer.

Table 5.11: P3 changing his narrative after some scaffolding from the interviewer

Speaker	What is said	What is done
I	Okay, can you try that and see.	
P3	Okay, but we are going to have a lot of variables. Okay, let this angle be x, then this angle is x as well and that angle is x as well.	Pointing to $\angle OAC$. Pointing to $\angle OCA$ and $\angle BAF$.
I	Okay, what reason do we have for $\angle BAF$ to be also equal to x because I understand $\angle OCA$ and $\angle OAC$ will be equal because OC and OA are radii.	
P3	Ohhhh, okay no I am lying, that angle is not x. We want to prove these two. Okay, here we will have $180° - 2x$. 	Pointing to $\angle BAF$, scratches the label in the diagram. Pointing to $\angle CAO$ and $\angle BAF$.

Some cases (P3 and P5) required scaffolding from the interviewer to kick-start their performance of the procedure. P3 and P5 can be characterised as tacticians in this study, because they did a lot of questioning at the beginning of their problem solving to gather information and devised a plan to solve the problems. As seen in Table 5.12, before attempting any routine, P3 and P5 required scaffolding from the interviewer to endorse certain narratives that they thought would help find the solution.

Table 5.12: P3 and P5 requiring scaffolding in solving the first problem

Speaker	What is said	What is done
I	Okay.	
P3	So tell me, based on the orthocentre, can we say that $\angle BAH$ or $\angle BAF$ is equal to $\angle ABG$?	
I	How?	
P3	Since this is the orthocentre, are we saying these two angles are equal?	Pointing to $\angle BAH$ and $\angle ABG$
I	Why?	
P3	Keeps quiet.	
I	It was going to be easy if you were using the circumcentre because AO, BO and CO are radii. Then you will have angles opposite sides equal, but you can use you initial approach of similarity.	
P3	Okay, the whole of ΔABC is it an isosceles triangle or it is just a triangle?	
I	As stated in the given statement, it is just an acute angled triangle.	
P3	So we cannot conclude that AC is equal to BC?	
I	No, unless if you can prove it.	
P5	O, is it the centre of the circle or not?	
I	Use the definition of the circumcentre and the orthocentre to find that out.	
P5	Keeps quiet for a while. (Sighs) perpendicular bisector (laughs), the language, what is a perpendicular bisector?	
I	A line that is perpendicular to a side of a triangle and bisects the side.	
P5	Are you saying to me that OC is perpendicular to AB?	
I	No. That is something you need to think about. Can you say that OC is a perpendicular bisector of AB?	
P5	But if I extend this line, will it be perpendicular to AB?	Pointing to line OC.
I	That is what you need to think about, if you extend this line, do you think it will be perpendicular to AB?	

P3 required scaffolding from the interviewer in endorsing that angles $\angle BAH$ and $\angle ABG$ are equal to each other, so that he could use the properties of isosceles triangles to find his solution. This was evident even in his later utterances, when he attempted a new approach, he still became stuck in a loop of attempting to verify with the interviewer whether $\angle BAH$ and $\angle ABG$ were equal or not, as seen from the quotations below.

P3: *Eish, it will not work.*

I: *If you think that will not work you can move on to the next attempt.*

P3: *You know what, this will work but the only argument now that we have is that ΔABC, is it an isosceles triangle?*

I: *You are not told so and you have not proven that yet.*

P3: *So, can we say that this whole $\angle BAC$ is equal to this one, $\angle ABC$?*

I: *Not unless we can prove it.*

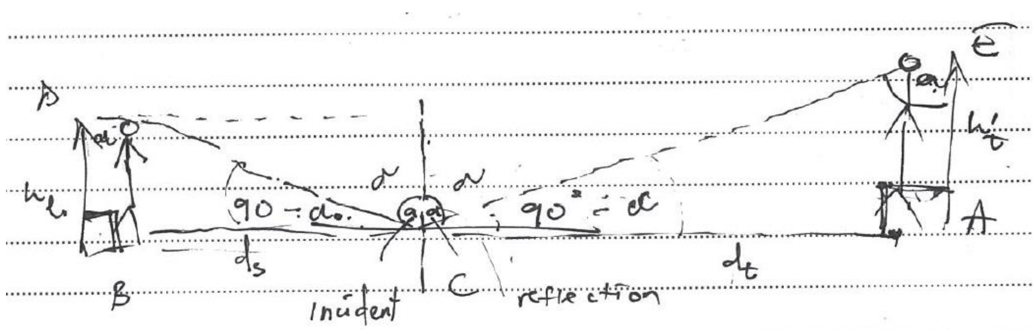
On the other hand, P5 was looking to endorse something different from P3 about the first problem. He wanted to find out if O was the centre of the circle, and also needed clarification with the definition of a perpendicular bisector. When clarity was given regarding his second worry, he fixated on the perpendicular bisectors, attempting to figure out which lines were perpendicular bisectors. He even asked, "but if I extend this line (OC), will it be perpendicular to AB?" However, in the diagram about the problem (Figure 4.3), I did not construct the perpendicular bisectors, but just joined point O (the circumcentre) to the three vertices of the triangle. Hence, his further questioning showed that he relied mostly on the visual appearance of the diagram, instead of exploiting the properties of the circumcentre to solve the problem. It seemed that P3 and P5 wanted to claim independence from the interviewer after he verified or endorsed their narratives, which landed them directly at performing the routine with the interviewer. Even in the second problem, P3's early collection of the information that would be useful to solve the problem according to his approach, was visible after he jointly endorsed the diagram useful for solving the problem with the interviewer (Table 5.13).

Table 5.13: P3's requirement of scaffolding from the interviewer in solving the second problem

Speaker	What is said
P3	Okay, what is the equation that can be used to determine the height of the student h_s. So, they did not write in terms of what?
I	It means you are open to any approach of your choice.
P3	(Laughs), Find the equation in relation to what?
I	No, remember this question is asking you in terms of the whole scenario.
P3	So, meaning to say I need to have d_s, d_t and what, what, all of them in one equation?
I	That is where your thinking comes in, what do you think?

However, there were isolated incidents where the scaffolding from the interviewer was useful for PMTs to come up with endorsable narratives. As seen in Table 5.14, P3 was able to come up with endorsable narratives after the interviewer scaffolded him about a certain theorem that was initially not visible to the participant. Furthermore, in the second part, the interviewer suggested that P1 should draw the diagram with the mirror on the ground facing upwards. Even though he was reluctant at first, he was able to produce an endorsable narrative after trying out the idea provided by the interviewer. This was evidence that the interviewer' scaffolding was useful in facilitating PMTs' production of endorsable narratives in this investigation.

Table 5.14: P3 endorsing a narrative after some scaffolding

Speaker	What is said
P3	Eish, this is killing me. Let me see. (keeps quiet for a while)
I	Okay maybe think about the theorem of angle at the centre is twice the angle at the circumference and see how it can help you.
P3	Okay, ohhh there it works. If this was drawn there for me then I was going to get it.
I	But it was already there, you just could not see it.
P3	Okay then this ($\angle AOC$) is going to be twice that ($\angle ABC$). So this one is going to be $2(180° - 2x)$, no, it's going to be $\frac{1}{2}(180° - 2x)$ (as indicated in red on diagram 2), which is simplified into $90° - x$.
P3	Okay let me try your idea and see.
I	Okay.
P3	We have the mirror here on the floor, then we have our kid, place here. There is our kid, on the ground. The kid is in point B, and then the teacher moves away from this point backwards, point A is our teacher. So the head is going to be reflected like this, then we are going to have our normal here, this is our angle alpha and alpha. Ohhh, this makes sense now, because now we have the heights now, we have the heights now, this one has some sense, and we have the distance, yeah, this is it. Then we have our reflection. [Comes up with diagram 4] 

Even though an example was only quoted from P3, other participants were, in isolated incidents, able to endorse their narratives after they received some scaffolding from the interviewer. For example, P4 was able to correct her narrative about the relationship between the triangles produced in her diagram to answer the second question after some scaffolding from the interviewer as seen in the quotations below. However, even during this time, P4 still relied on scaffolding from the interviewer to endorse some narratives.

I: *They are standing perpendicular to the ground.*

P4: *Ohhhh, Okay (giggles) perpendicular to the ground, okay* 90°.

I: *And then?*

P4: *Ohhhh, angle, angle, angle, congruency, eish sir.*

I: *Is it congruency or similarity?*

P4: *Congruency, Ohhhh, no similarity.*

P6 was also able to endorse a narrative after some scaffolding from the interviewer even though he did not use it to find the solution to the second problem. Instead of reasoning according to properties of similarity, P6's reasoning involved congruency. But after a question from the interviewer, P6 was able to change his narrative correctly as seen in the extract below.

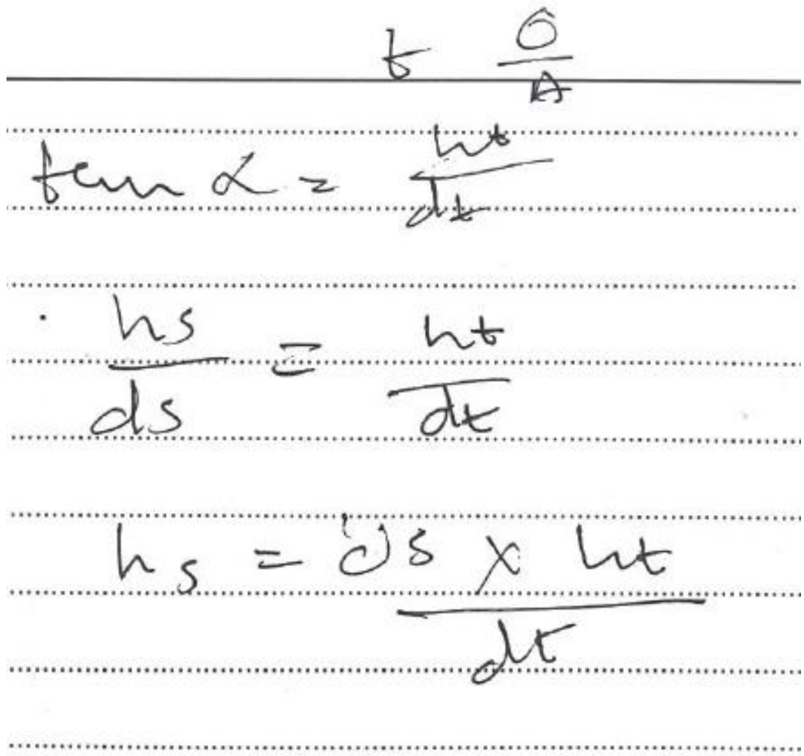
I: *Is it equal or in proportion when you are dealing with similarity?*

P6: *Oh, in proportion. So, the thing is now, how can we link this height to that* (pointing to h_t and h_s in diagram 2).

Another case of endorsing a narrative after some scaffolding, was observed from P5 when solving the second problem, as seen in the extract below.

I: *Okay, can you see that these $\tan\alpha$s are equal? And since they are equal it means their equals are also equal to each other.*

P5: *Ohh, it means that the height of the student over the distance of the student is equal to the height of the teacher over the distance of the teacher. We are looking for the height of the student, therefore the height of the student equals to the distance of the student times height of the teacher all over the distance of the teacher.*



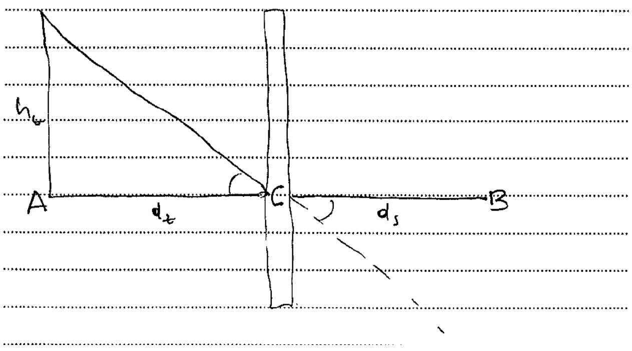
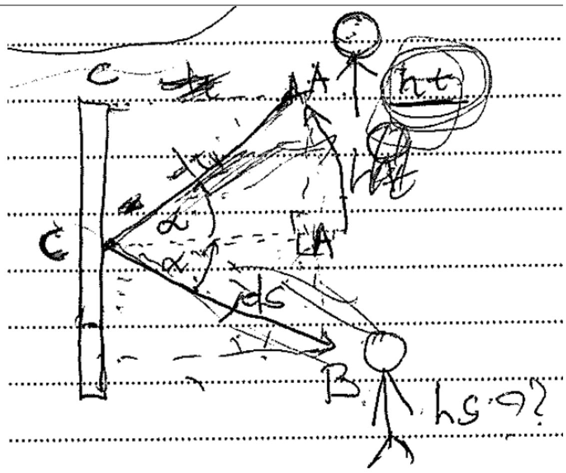
Furthermore, for the second problem, the endorsement of a visual narrative was thought of as an initial important step by the participants, with all of them stressing that the second problem could not be solved without the diagram in the face-to-face interviews (Table 5.15). However, except for P5, none of the other participants could come up with an endorsable rough sketch (visual narrative) that could help them solve the second problem without the help of the interviewer. What seems to be the main barrier, in this case, was that PMTs could not figure out the position of the mirror. There were several questions relating to the type of the mirror (this mirror, is it a both side mirror or a one-side mirror? P4) and who is the mirror facing (now the mirror is facing who?, P3). The great misconception observed from a majority of the PMTs in solving the second problem, was that they pictured the mirror as standing vertically. This can be noticed from the visual narratives produced by P1 and P6 (Table 5.16) during their discourse of solving the second problem.

Table 5.15: PMTs' comments about producing diagrams in the second problem from the face-to-face interviews

Speaker	What is said
P1	No, because without the diagram I couldn't understand what that question wanted, to me it was just words.
P2	I do not think I was going to be able to solve the problem without the help of a diagram.
P3	It wasn't helpful, it was bad. It was a bad thing not to be given a diagram because the first thing I had to have a diagram, not just a diagram but the correct diagram. I don't just need a diagram, I need a correct one.
P4	Yes it was very important because clearly you cannot solve this problem without constructing the diagram.
P5	The second one, I knew if I can draw a diagram I will understand.
P6	Question number two, it needed me to analyse and come up with the diagram because without the diagram I could not have answered that question.

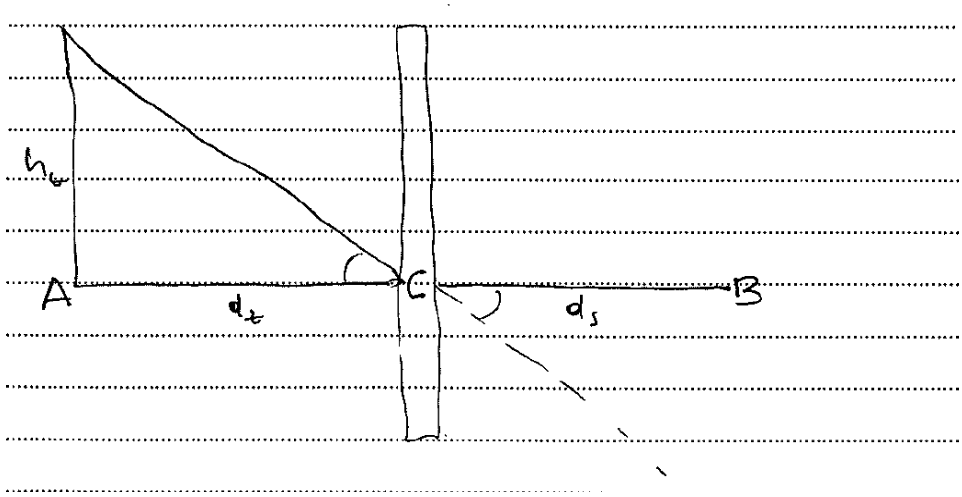
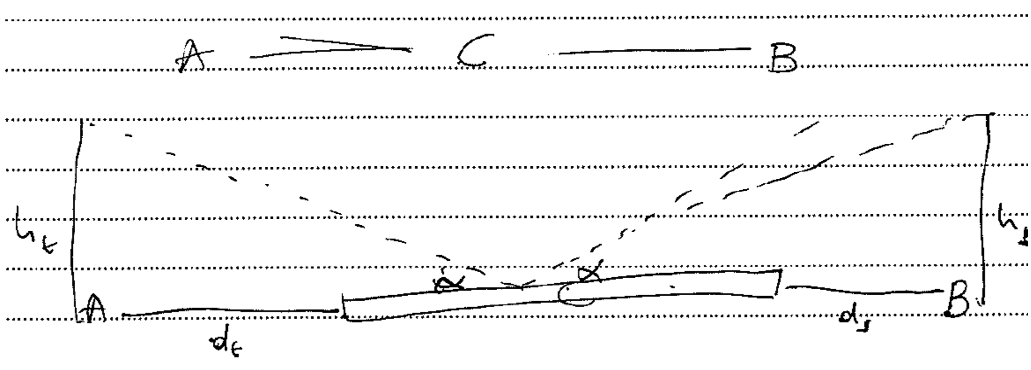
PMTs in isolated incidences were able to motivate why the production of a diagram was important to solving the problem, with P1 mentioning, "it helps me understand the diagram better … by putting it in a diagram helped to have a clear understanding of the problem". P3 mentioned that producing the diagram was useful because it helped him to visualise the problem, "so that I can visualise the problem and start solving" (P3). In fact, all the PMTs mentioned that the production of the diagram was useful for them to visualise the verbal problem that was given to them. In fact, P3 and P5 mentioned that word problems are challenging and that the most challenging part is the production of a useful visual mediator to help in solving the problem. P3 mentioned, "the general comment is that word problems in most cases, mostly, they are always challenging compared to the ones with a presented diagram", while P5 mentioned "If the diagram was there, word problems and constructing a diagram, word problems give me challenges". Furthermore, P6 also indicated, "you know also again, word problems are a very big problem for me, especially analysing them, we were not given proper foundation on solving word problems in mathematics". For all these participants, the unavailability of the diagram in the second question was a big problem, even though they did not use the availability of the diagram in the first question.

Table 5.16: Visual narratives from P1 and P6 about the second problem

P1	P6
	

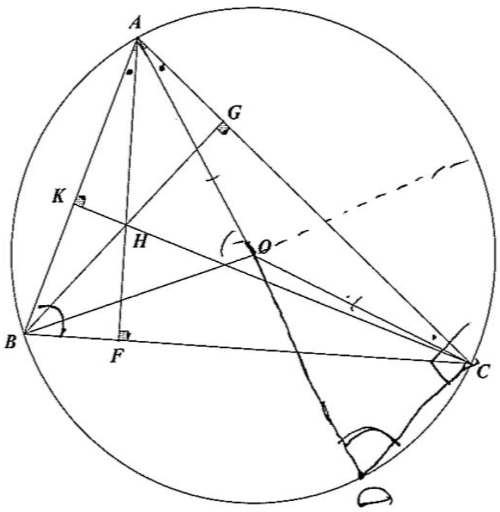
Furthermore, there were indications that PMTs wanted to go straight to problem solving, without a full understanding of the problem. For example, P6 mentioned that the availability of a diagram “gives me more options in terms of which method to use in answering the question, because I used trigonometry and Euclidean geometry and still gotten the same answer”. P3 on the other hand, also showed that he did not think of the production of the diagram in the second problem as a moment to understand the problem better. As he mentioned, “the one with the diagram, like you visualise already, you can think of a method of solving it”. In Table 5.16, I highlighted an extract of how P1 performed the activity of endorsing a visual narrative that was useful in finding the solution to the second problem with the interviewer. In most instances, as seen in Table 5.17, P1 took more time keeping quiet and coming up with no definitive answer (approach) to drawing the diagram, which symbolised that she was stuck and required some scaffolding.

Table 5.17: P1's endorsement of the visual narrative required to answer the second problem with the interviewer

Speaker	What is said and done
P1	P1 reads the statement, produces a diagram, and asks if the diagram is correct. 
I	I do not see the height of the student here. If you look at yourself in the mirror and is vertical, will you see the other side of the mirror?
P1	No, you will see you reflection.
I	How can you then position the mirror in this problem?
P1	Participant pictures the mirror sliding at a certain angle. She doesn't draw any diagram but she puts a pen tilted at a certain angle in the page and starts thinking. She realises that the image of the student will not be visible in the mirror.
I	Okay, think about another position.
	Participant keeps quiet for a while and does not find any position.
I	Try placing the mirror horizontally on the floor.
P1	I am struggling to draw this diagram if the mirror I placed horizontally on the floor. 
I	Okay then go on.

In an isolated case, P1 failed to produce an endorsable narrative about the first problem, even after some scaffolding was provided by the interviewer. Despite reaching where she could endorse the narrative from the scaffold provided by the interviewer, P1 still relied on the interviewer to guide her towards how to represent the narrative visually, using mathematical symbols. P1's discourse participation, as evident in Table 5.18, shows that she was operating at Van Hiele level 0, as she was convinced about the truth of the theorem based on the visual appearance of the diagram. Again, I must reiterate that P1's discourse in Table 5.18 was rife with keywords that are common to colloquial discourse which, according to this study, indicated that she was not operating at an explorative Van Hiele level 0, but a ritualistic level.

Table 5.18: P1 failing to endorse a narrative even after some scaffolding from the interviewer

Speaker	What is said	What is done
I	Try extending AO to the circumference at D then join D with C and see if that can help you.	
P1		Extends AO to the circumference at D then join D with C as seen in the diagram.
I	Does this diagram make it a little bit easier for you? Can you now find the solution?	
P1	Eish, it is so obvious that I don't know how to put it down, from the way it is constructed it is obvious that they will be equal but putting it down is a problem.	
I	Try it and see.	
P3	So I can relate this one and this one but how do I write it?	Pointing $\angle ADC$ and $\angle ABC$.

Furthermore, utterances such as “so I can relate this one with this, but how do I write it?” (P1), “now the mirror is facing who? Is it facing the kid or it’s facing the teacher?” (P3), “this mirror, is it a both sided mirror or a one-sided mirror” P4) and “yoh, I am stuck, now I need a clue” (P5) shows that a majority of these PMTs required some scaffolding from the interviewer. Hence, the majority of their routine performance was completed with the interviewer providing scaffolding whenever they needed it. Lavie et al. (2019) in their conceptualisation of learning as routinisation, classify such routines performed together with others as rituals.

- ***PMTs’ reliance on memory and previous experiences***

Since memory and previous experiences play a role during problem solving, the reliance on these revealed a ritualistic discourse participation when viewed from the commognitive perspective. Lavie et al. (2019) argue that learning occurs when people experience new situations and prosper in these situations by using the memories elicited by these situations. However, the application of the memory in these situations determines whether a person is acting in a ritualistic or explorative manner in dealing with the situation. In the current study, PMTs displayed some episodes where they relied on their memory and previous experiences in a ritualistic way. These PMTs’ routines were classified as ritualistic, because instead of individualising them, PMTs replicated previous experiences and even used mnemonics to perform routines. PMTs’ reliance on memory and previous experiences was characterised by the use of phrases such as “I know this theorem I just forgot what it is called” and “what I normally do”. All PMTs displayed some reliance on their memory which was characteristic of ritualistic discourse, pointing to the performance of the routines through the help of others even if they were not physically present in the problem-solving scenario. This was evident first in the case of P3 who, through the usage of the adverb “normally”, pointed to a ritual that he abided by whenever he is faced with that particular situation (Table 5.19).

Table 5.19: P3’s reliance on past experiences during problem solving

Speaker	What is said
I	Okay, I will need you to solve these two problems while you are thinking aloud. Where necessary I will intervene or you can seek for clarification if the need arises.
P3	Okay, what I normally do, since we are trying to prove that some given angle is equal to another angle, so I start by looking at similarity of triangles.

This extract suggests a ritualistic discourse performance of abiding by the rules from past experiences to solve problems in new situations. It suggests that whenever P3 was faced with a question that required of him to show that two angles were equal, he used similarity. This memory reliance might be elicited by an experience where he was successful in showing that two angles were equal through the use of similarity, or it was an activity that he observed from a highly regarded member of his mathematics community. In fact, for P3 it looked like it was a bit of both, and this can be confirmed by his utterances during the face-to-face interviews (Table 5.20), where he confirmed indeed that this activity was a ritual for him. P3 mentioned that the first thing he thought of when he saw the first problem, was similarity, because for him it was the "normal" thing to do in such situations, and also because from his school days he was taught to approach such situations in this way by his "teacher". It became clear from the face-to-face interviews that P3's past experiences evoked his approaches to solving the first problem and that even though his teacher was not physically there, he still facilitated the performance of this routine, as P3 said that this action was important, because from his school days his teacher taught him to prove that two angles were equal using similarity.

Table 5.20: P3's face-to-face interviews: justification for his reliance on past experiences (emphasis added)

Speaker	What is said
I	Tell me, when you saw the question, what were your first thoughts?
P3	Okay, on the first one, the first thing that I thought of was similarity, that is the first method that I thought of because in normal cases when you are trying to prove, to say angles are similar, the normal way of doing it, the most thing that I normally do is if I can prove that these two triangles are similar, then automatically the angles will be equal.
I	Why was it so important to do that first?
P3	In the first one, I think from my school days, from my teacher that is how I was taught. Whenever they say that prove that this angle is equal to that one, in most cases we need to locate the triangles in which those angles are in, then from there if you can prove that those triangles are similar or congruent, then we have it.
I	Okay, so it stems from your previous experiences?
P3	Yeah, when I want to prove that this angle is equal to that one, the first thing is to look for those triangles, if you can prove them to be similar, that is the first thing then you are fine, otherwise if not, then you can use other theorems. And the theorem way now it's a broad one, because now if you are not so familiar with the theorems, you are lost. But the most easy one, locate the triangles prove them to be similar or congruent, you are safe.

The practice of relying on previous experiences was later seen as a limitation by P3, because in the interviews, when he asked what else he could have done to solve the problems independently, he mentioned:

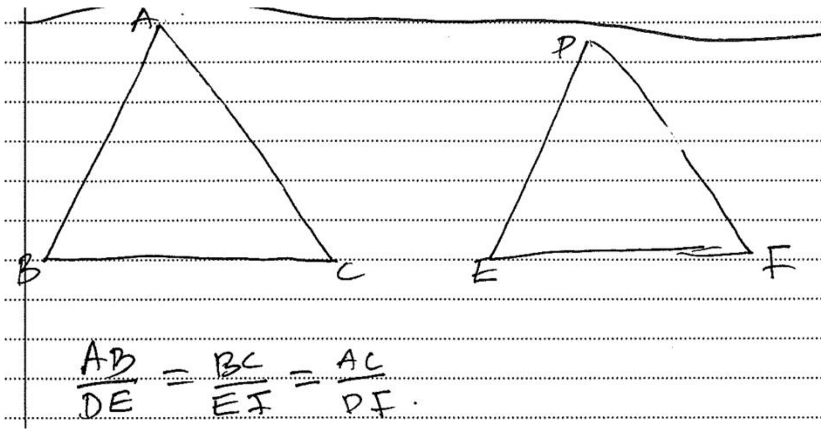
> **P3**: *one thing, I think I should not limit myself to similarities. I think that was my most approach whenever such questions* [pointing to problem 1] *comes. So I wasted most of my time on it whereas it didn't work, without trying to explore other means of doing it.*
>
> **I**: *So maybe in the future …*
>
> **P3**: *I need to open my mind to other approaches before I can say that this is the only one that can work for this problem.*

Even though later on he acknowledged that one can use Euclidean geometry theorems to show that two angles are equal, he still preferred to use similarity, because for him it is the "easy" way (Table 5.20). Furthermore, there was evidence of PMTs forgetting particular theorems and procedures. This was a sign that their routine performance was facilitated by their memory, and failure to recall certain theorems or procedures prevented them from continuing with the routine. A similar approach of attempting to reproduce past experiences was observed from P2, when producing the diagram in Figure 5.11. When asked in the interview whether there was any other approach she could have tried, the following conversation ensued between P2 and the interviewer.

> **P2**: *Yes. On number two I was thinking of reading the statement and drawing another diagram and on number one again I could have checked the sides if they are equal.*
>
> **I**: *Which sides?*
>
> **P2**: *The sides of* $\angle C$*, line CB and CA.*
>
> **I**: *So if they were equal what were you going to do?*
>
> **P2**: *I know that angle B will be equal to angle A.*

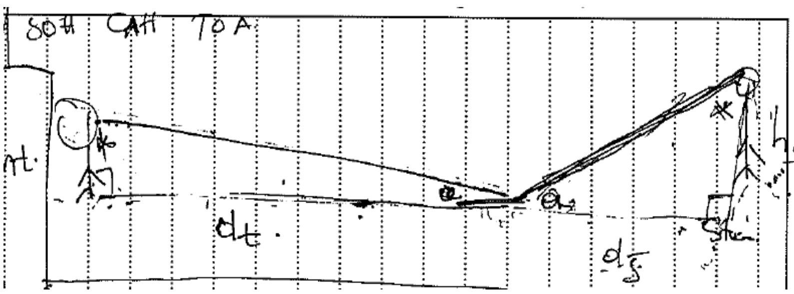
From the statement "the sides of ∠C, line CB and CA", P2 wanted to recreate the sides of an isosceles triangle, and this can be confirmed by her final statement "I know that angle B will be equal to angle A" which alludes to the fact that two angles of an isosceles triangle are equal, which is something that she "know".

Table 5.21: P1 and P4 forgetting theorems during their problem solving

Speaker	What is said	What is done
I	Why would you say, "am I wrong"?	
P4	Akere (*because*) I can't remember these things because of the lockdown.	
I	Okay, what principle of similarity can you use then now to show the formula?	
P1	We can use the sides.	
I	Okay, what principle?	
P1	In similarity, if the angle … the side that is opposite, no eish. Eish, I know this theorem I just forgot what it is called but I think it is the one we are going to use here. 	Draws a diagram and writing.

Furthermore, there was also evidence of PMTs (P4 and P5) using a mnemonic when solving the second problem. Two approaches were observed when PMTs were solving the second problem: a Euclidean geometry approach and a trigonometry approach. It was observed that when employing the trigonometry approach, P4 and P5 used a mnemonic to remind themselves about trigonometric identities. P4 utilised the all-famous "SOH CAH TOA" mnemonic to remind herself that the definition of a tangent (T) in a right-angled triangle is the side opposite (O) the angle divided by the side adjacent (A) to the angle. This was evident even from the diagram produced by P4 (Table 5.22). She even wrote the mnemonic SOH CAH TOA at the top of the diagram.

Table 5.22: P4's reliance on the mnemonic of SOH CAH TOA

Speaker	What is said	What is done
P4	Meaning this angle is equal to this. Okay, let me try the SOH CAH TOA. The opposite side and the adjacent side which is the tangent. So $\tan\theta$ is equals to (*keeps quiet for a while, then scratches* $\tan\theta$). Okay, $\tan\theta$ is equal to O over A, then $\tan\theta$ is equal to h_s over d_s, therefore, h_s is equal to tan theta times d_s. Sir, I don't know if I am right or what. Akere (*you see*), I drew this diagram the distance between the mirror and the student is d_s, akere? (*right?*). Then the height is h_t then I used tan theta which is opposite over adjacent. 	Pointing at α.

P4's reliance on the mnemonic was also evident in her visual mediators to endorse $h_s = \tan\theta\ (d_s)$, as seen in Figure 5.2 through the usage of $\tan\theta = \frac{O}{A}$ before substituting the O with h_t and the A with d_t which continued the use of the mnemonic.

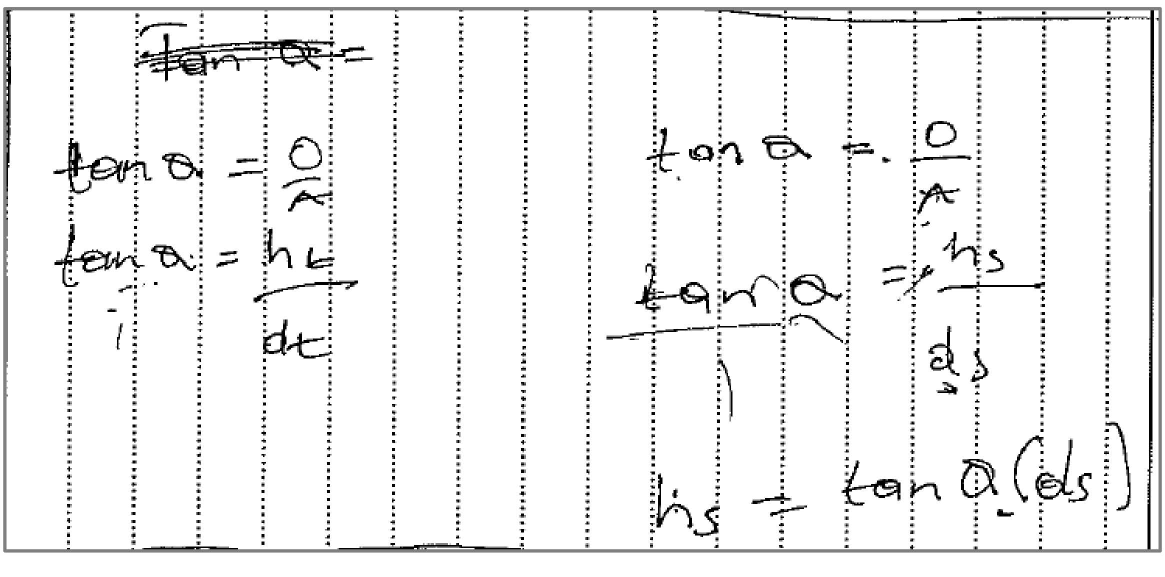
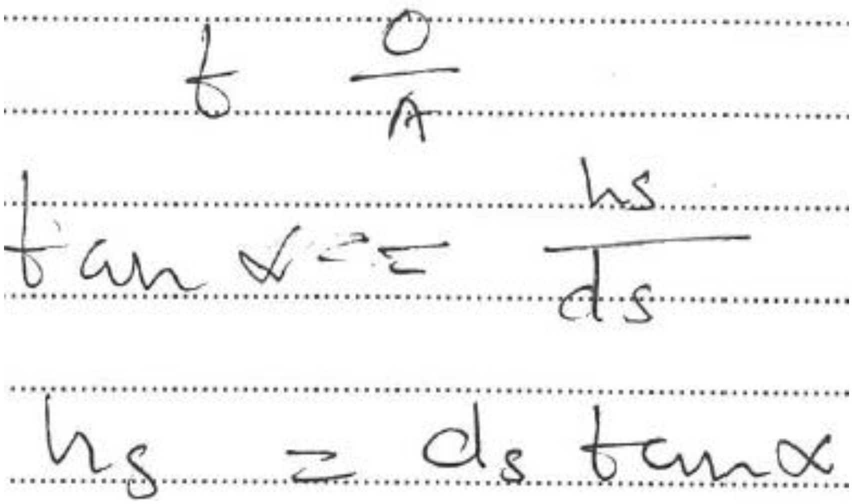
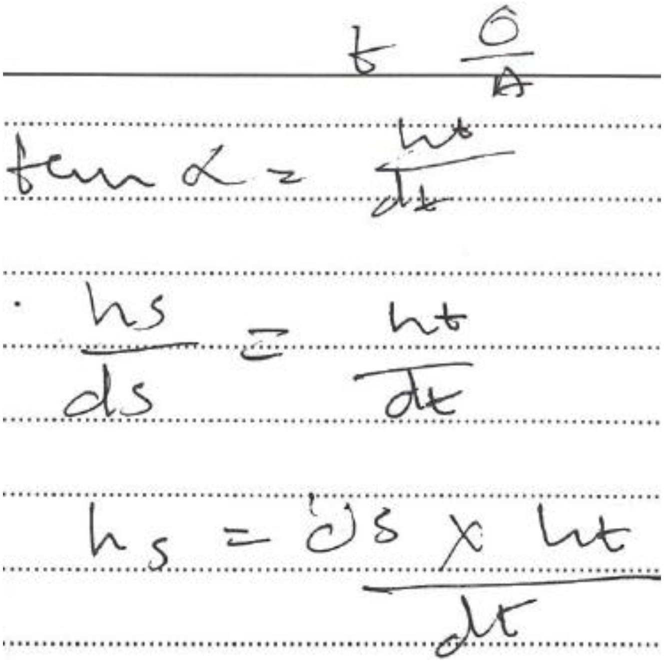


Figure 5.2: P4's trigonometry visual mediator to the second problem

On the other hand, P5 was also noticed using a mnemonic similar to that used by P4 in recalling the trigonometric identities. Though P5 did not utter the mnemonic 'SOH CAH TOA', he did utter the phrase "okay, tangent, T, O, A" and was seen writing this visual mediator down (Table 5.23). Again, this reliance on memory and mnemonic used showed PMTs' routine performance was facilitated by their past experiences, pointing to the influence of other people who were not physically present during PMTs' problem solving. For P5, this activity of using 'T, O, A' was observed twice as seen in Table 5.23.

Table 5.23: P5' trigonometry approach to answering the second question

Speaker	What is said	What is done
P5	(*Keeps quiet*) Okay this is 90 degrees this is theta, I want height, this is opposite and I also have adjacent, okay tangent, T, O, A. tan θ equals to, opposite that's the height of the student over the distance of the student, so we are looking for the height of the student, so height of the student equals distance of the student tan θ. 	*Writing mnemonic.* *Talks about tan theta but write tan alpha*.
P5	Okay, here T, O, A, tan theta equals to h_t over d_t. 	

Furthermore, there was evidence of P6 relying on his memory of similarity to prove the answer of the second question. This routine performance showed evidence of how reliance on memory facilitated P6's arrival at an endorsable routine about the second question. His episode of relying on his memory is highlighted in Table 5.24. The usage of phrases like "we did it" and "now I remember something" showed that P6's performance of the routine was reliant on his ability to recall knowledge learnt about similarity, and in particular not confuse that memory with that of congruency. As seen in his first sentence, he seemed confused between congruency and similarity.

Table 5.24: P6's reliance on memory when solving the second problem (emphasis added)

Speaker	What is said
I	Can you use another method?
P6	Yes, there is this other one, **we did it**, when the angles are like this what can I do, let's say again similarity or congruency, can we prove congruency here? No, we don't know whether the heights are the same. **Now I remember something**, I prove similarity then the corresponding parts, the two, let me check with that, gives proof of similarity.

There were also other incidents where P6 was seen to be reliant on both his memory and the interviewer to perform a routine. For example, "this one I have to remember again what the circumcentre is, Sir, what clue can you give me here". When his reliance on memory proved futile, P6 was clear that he needed a clue from the interviewer to continue with his routine performance. Hence, in this case, P6's routine performance was not only relying on his memory, but also the clue from the interviewer. This showed that indeed P6's routine performance in some episodes of his problem solving was facilitated by the presence of others, in this case, the interviewer and his memory.

PMTs' reliance on their memory, previous experiences, scaffolding and prompts from the interviewer, was a clear sign that they did not perform these routines individually, but with others. Indeed, they displayed obedience to the rules of others (the interviewer and teachers), because if the interviewer provided an incorrect scaffold, then the arrival at an endorsable narrative by these PMTs was unlikely. This analysis revealed that none of the PMTs was able to perform the routines of finding the solutions to the two problems independently. In one way or the other, they relied on metarules set out by others, either in the past (memory and previous experiences) or present (scaffolding from the interviewer). Even though some traces of individually endorsing particular narratives were visible, this only happened after being prompted and scaffolded by the interviewer or remembering particular procedures from their past experiences.

5.4.2.3 The audience of the routine

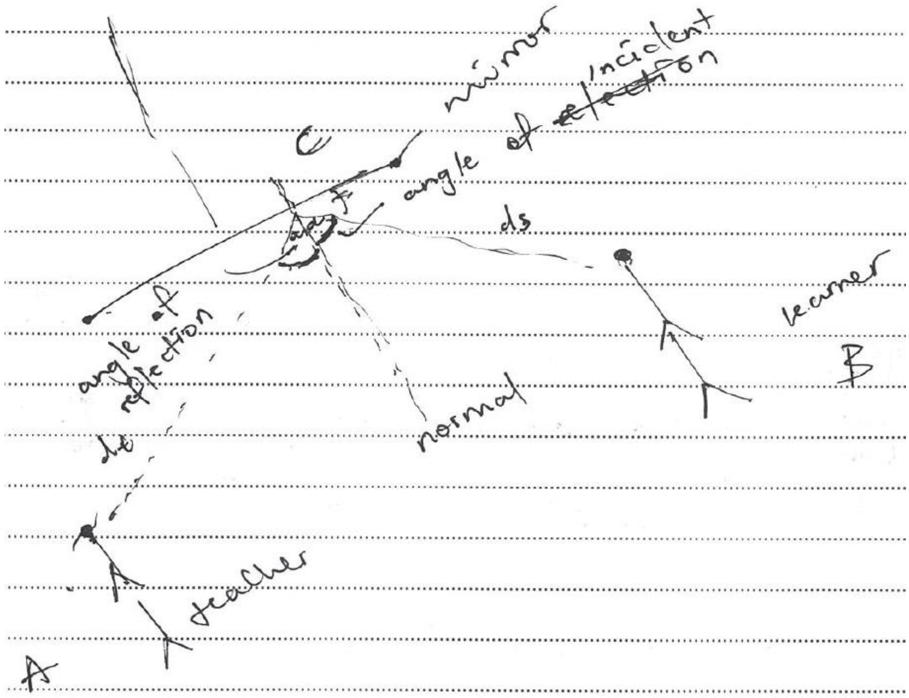
This was perhaps one of the most subjective codes during the analysis, as most of the PMTs' discourses that were characterised as authoritative, already related to data analysed earlier and in the previous sections. Hence, what became the main object of analysis here was PMTs' word usage when talking about their actions during problem solving. In particular, I looked at whether the word symbolised discourse-for-others or discourse-for-oneself (Sfard, 2008). Routines in a ritualistic discourse are specifically performed for others, and these routines are performed and endorsed based on the rules of other performers of similar routines. When a routine is performed for others, it becomes evident from how the performer talks about the mathematical objects involved in the routine, for example, using the pronoun 'they' symbolises that PMTs considered the question as coming from some outside authority and the PMTs must solve the problem to satisfy this outside authority. In this particular case, the pronoun 'they' refers to the interviewer as he is the one who formulated the problems. However, there are also traces of teachers who influenced the performance of routines. PMTs also relied on the visual appearance of the diagrams as the source of information about their narratives. Furthermore, PMTs were quoted complaining about the difficulty of the questions and some even gave up on continuing with the routine. All these themes revealed that PMTs performed their routines for the sake of pleasing the interviewer and if something was blocking them from this, they felt defeated.

The audience for routine performance in this study seemed to be other people, because PMTs continually spoke of the question as if it were coming from an outside authority. Their excessive use of the pronoun 'they' symbolises that PMTs viewed the question as imposed from an outside authority, and as such, they needed to find the solution to be in good standing with this outside authority. These PMTs engaged in this type of discourse because they knew that it made sense to the interviewer. For example, P1 asked, "they did not say anything about OC bisecting $\angle ACB$?", as if the information given in the statement were imposed on her by some authority and she did not view the statement as internally persuasive. This might induce limitation in terms of P1 finding a solution to the problem, because if 'they' did not give her the information she wanted to solve the problem, then she would be stuck. The indication of relying on an outside authority and previous experiences from P1 was also visible in her face-to-face interview where she stated, “so I thought of ways to solve it and what came first was Euclidean’s theorems, and [I] also thought of other similar problems I have done before like that one”. This means that her routine performance was dependent on her memories to produce narratives that could satisfy or be similar to those from her classroom experiences or books.

Furthermore, PMTs viewed the information given in the statements as the main source of knowledge that would help them solve the problem, instead of thinking about problem-solving strategies which could be useful in solving the problems. Hence, it seemed as if, when a certain theorem or property was not stated in the statement or the diagram, it was non-existent for them and they could not use it to solve the problem. Since it has been the goal long before, it still is even more so now, to transform this authoritative discourse to an internally persuasive one, where PMTs will construct their own perspectives about mathematical objects, instead of attempting to please others. P5 used the pronoun ‘us’ as if he were performing the routine with others. This usage of the pronoun ‘us’ also symbolised that P5 wanted to be accepted by others in his routine performance. Possibly, P5 thought he was solving the problem with the interviewer, but this was evidence that the performance of routines was for others and also thought of as happening with others. Furthermore, P3 used the adverb ‘normally’ to symbolise a routine, and in this case, the requirement of proving angles to be equal, elicited this routine in P3’s routine performance. Later in the face-to-face interviews, he mentioned, “in the first one, I think from my school days, from my teacher that is how I was taught” which symbolised that his routine performance was influenced by his erstwhile teacher, and also this teacher was his source of information in endorsing the narrative about the equality of these angles.

Table 5.25: Summary of PMTs’ utterances that regard to the questions as if it was coming from an outside authority

Speaker	What is said
P1	And **they** did not say anything about OC bisecting $\angle ACB$?
P3	Okay, what is the question that can be used to determine the height of the student h_s. So **they** did not write in terms of what?
	I think the mirror **must** be here and the student and the teacher on this side.
	Okay, what I **normally** do, since we are trying to prove that some given angle is equal to another angle, so I start by looking at the similarity of triangles.
	It **must** face down like this.

Speaker	What is said
	
P4	Yes, here **they** are saying the teacher moves away from point C … Akere (*here*) **they** are saying we **must** show that $\angle BAH$, which is this angle is equal to $\angle CAO$. Sir this question **they** are asking what is the equation that can be used to determine the height of the student akere (*right*)? But sir if that's the case then, that means there **must** be an angle in which the mirror is standing.
P5	But **they** said let O and H be the circumcentre and orthocentre. P5: Because it **must** be the same with that angle AOC, I have proven that the angle AOC which is CAO it is an isosceles. … I think when **they** say this is a perpendicular bisector. But **they** never told **us** the direction of the mirror, is it standing?
P6	… **they** say the angle of incidence is equal to the angle of reflection. Okay, **they** want the height of the student …

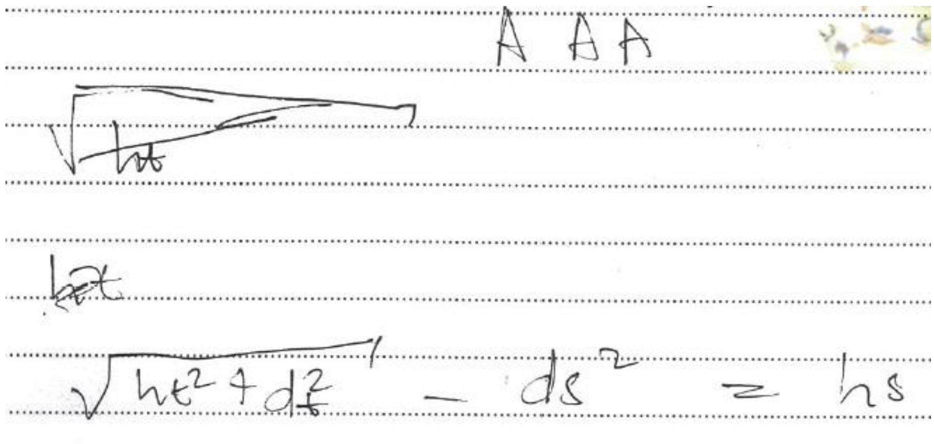
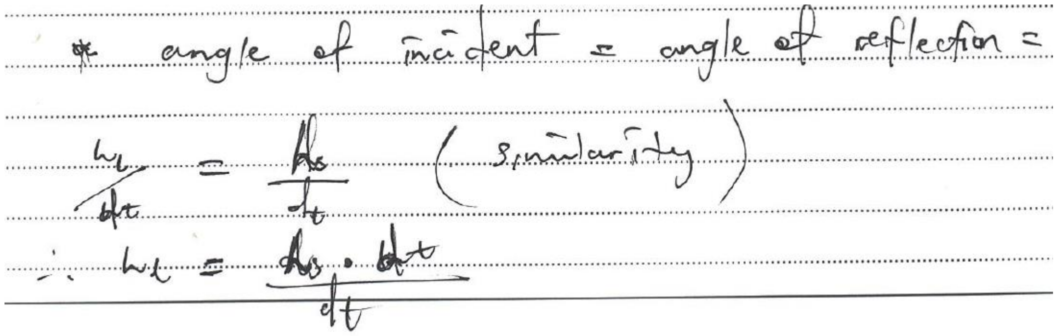
The scattered usage of the verb 'must' as seen in Table 5.25, also suggested different actions, all of which pointed to the completion of the routine for others. For example, P3 used the verb 'must' to describe the position of the mirror in relation to the student and the teacher. Even though he described the position of the mirror, as the discourse continued, P3 was not able to endorse his narrative about the position of the mirror and his declaration fell out of the discourse. On the other hand, P5 used the verb 'must' to conform to what the final answer was supposed to look like to others, which meant that he relied on the visual appearance of scenarios as the source of his actions about his narrative. The use of the verb 'must' is monologic and such discourse seemed to be imposed on others without any subject to scrutiny. However, if this monologue were disproved, it would die and fall off the discourse as if it never existed. Table 5.26 represents an example of how P5's monologic discourse was eliminated from a discourse about the first problem. What was interesting enough, in this case, was that P5 was the one who disproved his initial monologue about the equality of $\angle A$ and $\angle B$ from ΔAHB.

Table 5.26: P5's monologue about narratives in the first problem

Speaker	What is said
P5	If I go to triangle AHB, angle A will be equal to angle B.
I	Why?
P5	Because it must be the same with that angle AOC, I have proven that the angle AOC which is CAO it is an isosceles.
I	Triangle AOC is isosceles because you already know that AO and OC are equal. But can you say that triangle ABH is also isosceles?
P5	With the drawing I cannot say that but with the properties I can say that.
I	Which properties would make triangle ABH isosceles?
P5	If I can find that two angles are the same.
I	Which two angles are the same?
P5	I think A and B
I	Why?
P5	That I why I want to prove other things first so I can move towards that.
I	But you cannot use the fact that those angles are equal before you can prove that they are equal.
P5	Yes, I want to prove, that what I am trying. And then because of H, H is having these things of altitudes and orthocentre, they cannot give me information which I cannot use. That is why I said I want to prove something here, congruency first.
I	Do you think these triangles can be congruent?
P5	I have 90 degrees which is… Okay there is not enough information.

Furthermore, PMTs also relied on the visual appearance of the diagrams to endorse certain narratives without considering the geometry properties of those diagrams. For example, P5 refused to produce the proof of congruency, because for him it was 'obvious' that the two triangles were congruent, which meant he just relied on their appearance in the diagram and not on their properties (Table 5.27). The adverb 'obvious' here was used to impose on the interviewer that there was no need to write down the visual narrative, since the result was self-evident. This showed what P5 concluded about the equality of angles without giving a clear proof or substantiation using properties. So, for him, the visual appearance was enough for a conclusion which symbolised that he was reasoning at ritualistic Van Hiele level 0.

Table 5.27: P1, P3 and P5 endorsed narrative based on its visual appearance (emphasis added)

Speaker	What is said
P5	These angles are congruent.
I	Why?
P5	This is 90, this is alpha and then the remaining angles are equal, so angle, angle, angle.
I	Can you write the proof of congruency so I can see it?
P5	No, it is **obvious**. Let me continue graphically. 
P5	(*keeps quiet for a while*) I am starting to be lost now.
P3	Okay, I see now, all these angles are equal because these are 90° and then the remaining angles will also be equal from the sum of angles in a triangle. This means that these triangles are similar.
I	Can you prove that?
P3	It is proven.
I	Can you write the proof down?
P3	It is proven.
I	You need to write it down so I can see it.
P3	It's obvious
I	Okay continue.
P3	I am done, this side over this one, since they are similar, this side over this one is equal to this one over that one, done. Then I make h_l the subject of the formula. 

Speaker	What is said
P1	We can use what, similarity? Because this triangle is more similar to this one.
I	Okay, can you prove that these triangles are similar?
P1	Yes, angle, angle, angle.
I	Okay, prove it
P1	$\hat{A} = \hat{B} = 90^\circ$ ($DA \perp AC$ & $ED \perp BC$) $A\hat{C}D = E\hat{C}B = \alpha$ (Given) $A\hat{O}C = 180^\circ - 90^\circ - \alpha$ $= 90^\circ - \alpha$ also $A\hat{D}C = C\hat{E}B = 90^\circ - \alpha$ $\Delta ADC \parallel\!\mid \Delta EBC$ (A-A-A)

The same type of thinking was observed in P3, who also refused to produce a visual narrative of the proof that the two triangles were similar, even after some prompts from the interviewer to produce this visual narrative (Table 5.27). While P3 was able to come up with an endorsable narrative without the preceding visual narrative of proving similarity, P5 ended up being lost and this can be attributed to his failure to produce the proof for congruency he claimed to be true in his narratives. Furthermore, P1 showed similar characteristics to those of P3 and P5, of endorsing her narratives based on their visual appearance, even though some persistent prompts from the interviewer led to her producing an endorsable narrative to show that the two triangles were similar (Table 5.27). Furthermore, P5 struggled with Van Hiele level 2, because he endorsed the information about the two triangles based on incorrect properties. The condition of congruency he mentions (angle, angle, angle) does not belong to congruency, but similarity. The need to be right in the face of the interviewer, led P5 to committing the error of endorsing a narrative based on incorrect geometric properties. This need to be correct from P5, was also observed from an earlier narrative where he thought he had solved the problem, only to find out that he was incorrect. When solving the first problem, P5 was quoted claiming to have found the answer but later on, he acknowledged that he was incorrect and could not endorse the narrative based on his current sub-routines. This can be observed in the quotation below.

P5: *Ay I'm done, I'm done, I'm done now. I have proven the angle this one, that $\angle B$, outside here, the one of HBF it is equal to the one of GAH, that mean $\angle A$ is the same as angle B, and on this side the whole of this, up until this line. Eish, I think I celebrated too early because this includes this part (pointing to the whole of angle BAC).*

Furthermore, there were signs of PMTs giving up on solving the problems because the problems were difficult and beyond their understanding, as they claimed. In one case, P4 was seen alternating between the first and the second problem without finding the solution to either problem, and in the end, she mentioned, "I tried". In Table 5.28, I give a summary of PMTs' utterances where they surrendered to the difficulty of the problems, giving a sign that they could not continue without the help of the interviewer.

Table 5.28: Utterances of PMTs acknowledging that the questions are difficult and giving up

Speaker	What is said
P6	(keeps quiet) Eish, I don't know how to do this.
P3	Eish, this is killing me. Let me see (keeps quiet for a while)
	Okay, this is some difficult question yoh, I am struggling to place the mirror.
P4	Okay, the mirror is not at the midpoint, the mirror is placed between them and the teacher is at point B. Yoh, al let me do the other question.
	Keeps quiet for a while then redirects to the first problem.
	I managed to get to this far. This $\angle OBA$ equals $\angle OAB$, because of the isosceles triangle. From there, this $\angle OAB$ consist of $\angle BAC$, this angle and this angle, Eish from there I don't know, but I tried.
	Eish, sir, these things are difficult.
P5	Eish, this question is beyond my understanding, am I allowed to Google?
	Eish, your questions are difficult sir.

The first statement of P4 in Table 5.28, "I managed to get to this far. This ∠OBA equals ∠OAB, because of the isosceles triangle" represented that she could relate the visual appearance of the angles with their properties based on the shape as seen in her substantiation "because of the isosceles triangle". This link between the angles and the substantiation symbolised that P4 was at this instant operating at an explorative Van Hiele level 2. However, she struggled to move to level 3 to produce the proof to the problem.

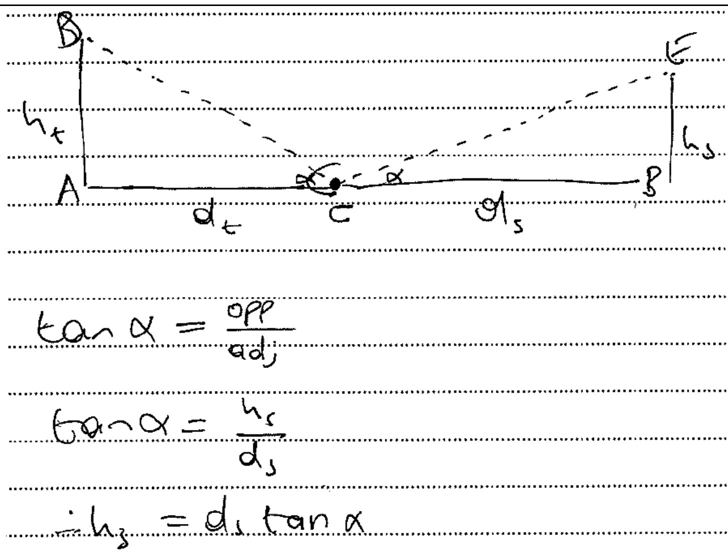
5.4.2.4 Level of applicability of the procedure

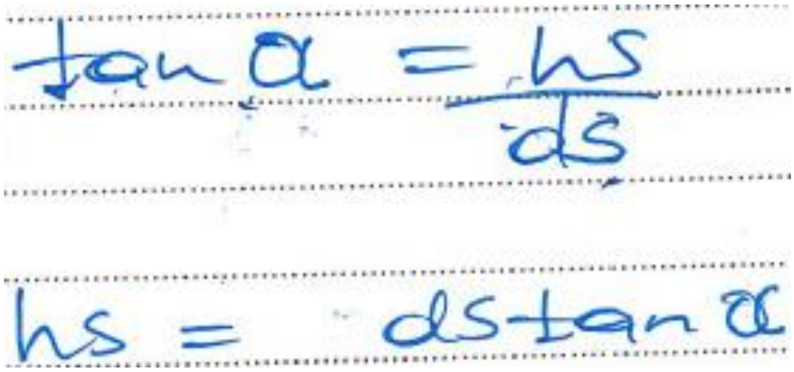
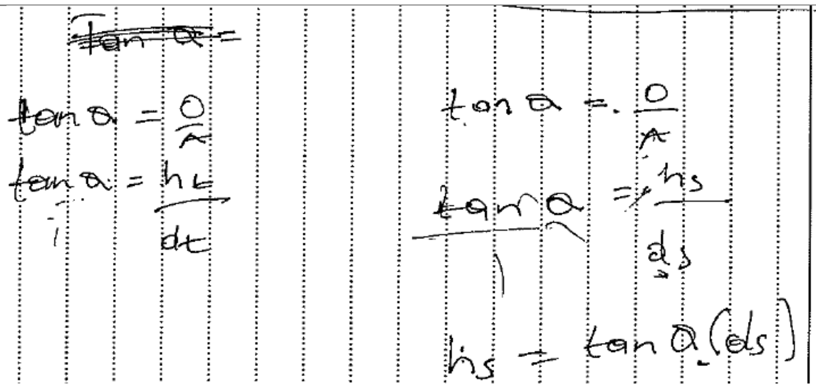
As discussed in Chapter 3, the level of applicability of rituals was both restrictive and situated. This became evident during problem solving, as when PMTs performed particular routines, they relied mainly on prompts from others, followed a strict procedure guided by the rules of endorsing that particular narrative and could not be applied in subsequent discourses, and also used empirical procedures to endorse a narrative. Therefore, the applicability of a procedure amounted to specific scenarios that could lead to the elicitation of a routine, including the physical environment or visual narratives. For example, seeing a particular visual mediator might evoke the application of previously individualised procedures. However, what was critical, was how these procedures were applied to the new scenarios. If they could lead to an endorsable narrative about the problem, then the procedure was an exploration, but if the procedure could not directly contribute to future routines or be applied in future discourses, then it was a ritual. Furthermore, the procedure was applied as if it were coming from another person or were imagined to come from another person.

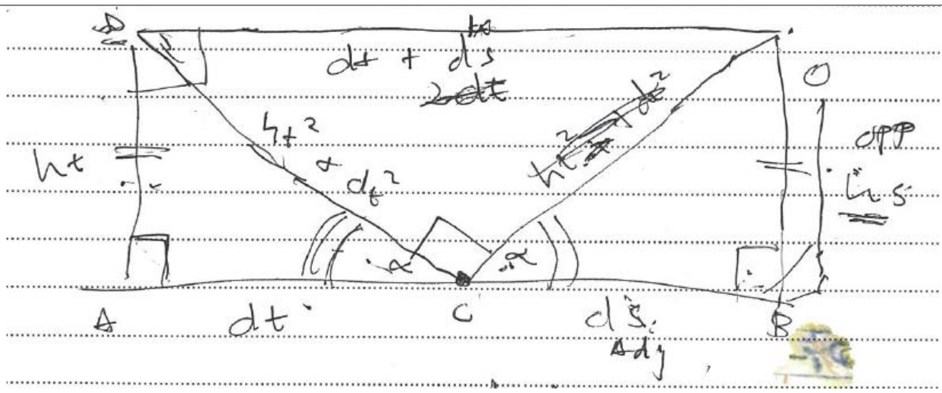
In most instances, PMTs routine application and endorsement of narratives depended on the scaffolds and prompts given by the interviewer. I have already discussed in 5.4.2.2 how PMTs relied heavily on the interviewer's scaffolding and prompts to endorse their narratives or even continue with the problem. The most dominant reliance on prompts and scaffold from the interviewer was in the second problem, where PMTs had to come up with a rough sketch (visual narrative) to help them solve the problem. Except for P5, all the other participants required the help of the interviewer to come up with an endorsable visual mediator, one that they could use in solving the second problem (Table 5.29). The production of this rough sketch seemed to evoke some past experiences from participants. Upon getting to the endorsable diagram, some PMTs were quick to apply trigonometry in solving the problem, particularly using the trigonometric definition of the tangent. This action was evoked by the two right-angled triangles that were visible in the diagram.

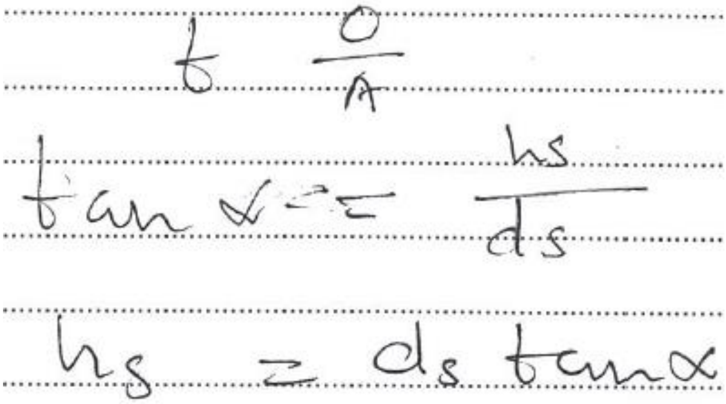
The jointly endorsed diagram for the second problem evoked the usual narratives from P5, who was quoted talking about $\tan\theta$, while writing $\tan\alpha$ as his visual mediator. Even though the angle given in the statement and the diagram was alpha, P5 spoke of theta as if it were in the statement or the diagram (Table 5.29). Furthermore, it was surprising to hear P5 speaking about theta, but writing alpha, as seen in his visual mediator in Table 5.29. The ritualised nature of discourse participation in the second problem by P1, P2, P4 and P5 was revealed when they completed their routines. The interviewer asked about the certainty of their narratives and all of them thought this was the final answer. Perhaps due to past experiences of trigonometric problem solving, these PMTs thought that the problem was solved without a full understanding of the problem. Hence, this also showed that PMTs' routines were restricted and situated on the presence of the interviewer, who through prompts and scaffolds guided the PMTs to endorse the narratives correctly. However, this was not useful for P4 who still could not find the final solution to the problem with the prompts and scaffolds from the interviewer.

Table 5.29: A summary of PMTs' ritualistic application of procedures evoked by a diagram

Speaker	What is said (done)
I	Okay, then go on.
P1	
I	Do you think that this solution is efficient? What have you neglected here?
P1	Everything that has something to do with the teacher.
I	Now, do you think that is efficient? Is the solution correct?
P1	I think it is correct because I want the height of the student.
I	Okay find the equation you can use to calculate the height of the student.

Speaker	What is said (done)
P2	Okay, this height (*referring to HS*) is opposite to this (*referring to alpha in diagram 3*) and this is adjacent (*referring to DS in diagram 3*) therefore it will be tan alpha is equals to HS over DS. 
I	How confident are you with that answer?
P2	Because it is clear that the angle is opposite the height, so using 'tan' is correct.
P4	
I	Are you done?
P4	Yes.

<table>
<tr><th>Speaker</th><th>What is said (done)</th></tr>
<tr><td>P5</td><td>

(Keeps quiet for a while) Okay this is 90 degrees this is theta, I want height neh, this is opposite and I also have adjacent, okay tangent, T, O, H (writing mnemonic). $\tan\theta$ equals to, opposite that's the height of the student over the distance of the student (talks about tan theta but write tan alpha), so we are looking for the height of the student, so height of the student equals distance of the student $\tan\theta$ (talks about tan theta but write tan alpha).

</td></tr>
<tr><td>I</td><td>Are you done?</td></tr>
<tr><td>P5</td><td>(laughs) I think so.</td></tr>
</table>

Another important feature of the applicability condition of rituals was that PMTs spoke about their narratives as relating to particular processes of their learning experiences. In particular, P6 spoke about having done particular routines through the phrases "we did this" and "we did it" to refer to his past experiences. Hence, his routine performance was facilitated by his reliance (specifically the recall) on this experience of previously 'doing' this routine. This was evident in how he followed a strict and rigid procedure when performing the routine in Figure 5.3.

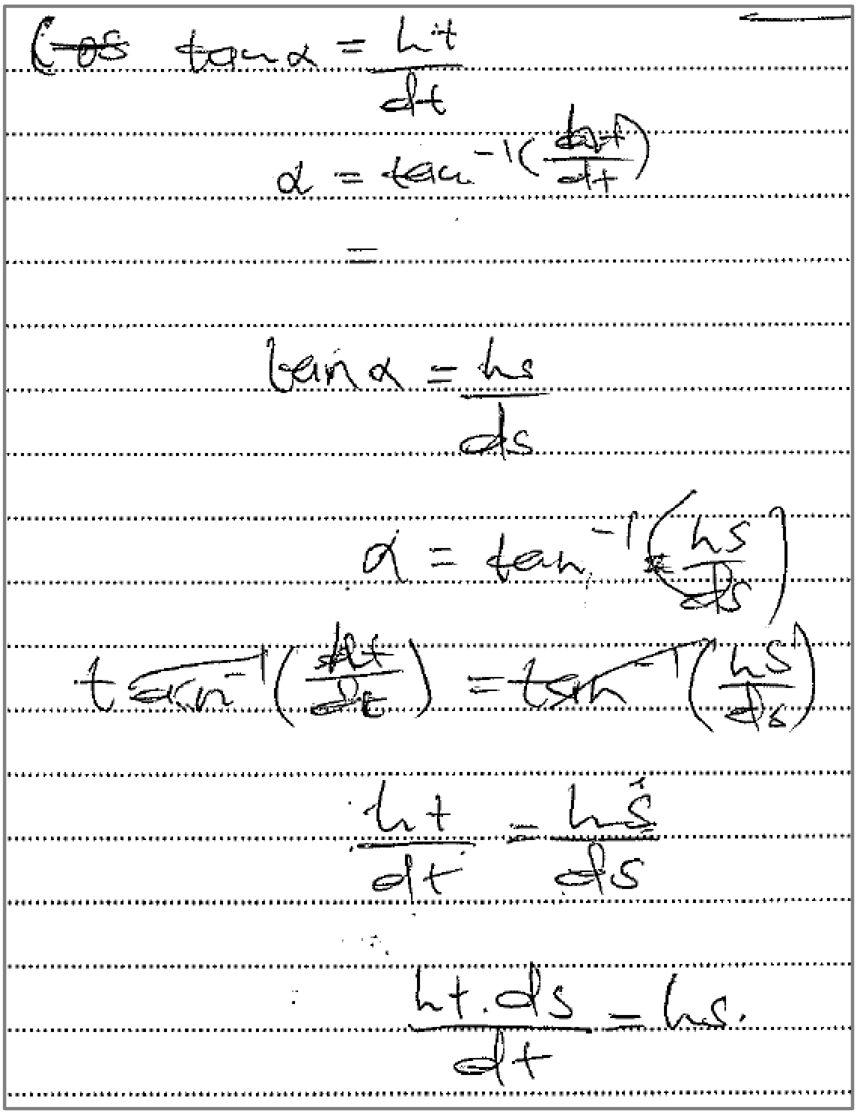


Figure 5.3: P6 Ritualistic routine performance

On the other hand, P1 spoke about not doing Euclidean geometry in high school at the beginning of her problem solving. Although this could be thought of as a hiding statement, P1 was declaring that she had little to no past experiences to facilitate her routine performance.

Table 5.30: P1 and P6 process-oriented utterances

Speaker	What is said
P6	**We did this**, this angle now, eehhh, relating these things together ... uhhmmm ...
I	Can you use another method?
P6	Yes there is this other one, **we did it**, when the angles are like this what can I do … Okay, now I **remember** something …
P1	P1 – Eish, my problem is that **I did not do** geometry in high school so I do not know a lot of theorems about geometry.

There was also evidence of PMTs utilising procedures that were restricted, like the empirical approach to endorse narratives. The empirical approach is usually used to investigate patterns for conjecturing in mathematics. However, some PMTs used this approach to endorse narratives about the first and the second problems. P2 used assumptions to assign values to angles just to get them to be equal to each other, without any clear deductive reasoning (Table 5.31). The empirical approach is restrictive, because it only applies to a special case instead of generalising about the whole routine. The empirical approach adopted by P2 and P4 during their problem solving indicated that they lacked logical deduction of properties and definitions (Van Hiele level 2) and the logical deduction of proving riders (Van Hiele level 3). This indicated that they were operating below Van Hiele level 2, which was evident from the amount of scaffolding needed by P2 in solving the two problems and the failure to solve the problems even when scaffolding was available from P4. Furthermore, phrases like "this is 30 meters and this is 10 meters" from Table 5.31 were indicative of ritualistic discourse participation, which meant that P2 and P4 in this case could be classified as operating at ritualistic Van Hiele level 1.

Table 5.31: PMTs' empirical approach to problem solving

Speaker	What is said
P2	let us assume that angle H is 60° that means angle A will be 30°.
P2	Therefore, for us to be able to say that $\angle BAH = \angle CAH$ …
P2	Therefore, 90° plus angle, 90° which is $\angle F + \angle C + \angle A$ is going to give us 180° but our $\angle A$ from this triangle it is going to be, let us say 50°, or we can simple use ΔCAO which we then going to use the orthocentre, and then from that statement we have the line which is line G to B, the altitude, therefore using the altitude, $\angle G$ it is 90°, then we have $\angle C$ and $\angle A$, and therefore that means $\angle A + \angle C + \angle O$, and therefore, from there we have $\angle O$, $\angle ACO$ we can say it is 60°, actually $\angle AOB$ is twice $\angle AOC$. Therefore, if we say AOB is 120, that means $\angle AOC$ it is 60°. Therefore, 60° plus $\angle C$ plus $\angle O$ will give us 180°. … then I did assume that $\angle KHA$ is 60° which will give us this (*pointing to* $\angle KAH$) as 30°.
P4	Okay, this is 30 metres and this is 10 metres.

5.4.2.5 Level of flexibility in finding the solution

The level of flexibility in finding the solution for each PMT was another signifier of whether their discourse participation was either ritualistic or explorative. In ritualistic discourse participation, problem solvers follow a very strict procedure in finding the solution and they perform it following strict rules that allow others to reproduce their performance (Sfard, 2008). The whole routine performance and some subroutines were considered as indicators of flexibility in finding the solution. PMTs' discourse related to ritualistic flexibility were visible from episodes where their routine performances were rigid. For example, after endorsing their narratives (or routines) following strict rules, most PMTs could not come up with an alternative approach or approaches to solving the problem. Perhaps due to the reliance on the interviewer when producing endorsable narratives, PMTs could not be flexible in their thinking and problem solving.

Furthermore, PMTs' routine performance discussed earlier, like their reliance on memory and mnemonics and admissions that they are stuck and cannot find a way to continue with the routine performance, were indications that some PMTs were not flexible in finding the solutions. Furthermore, scepticism about a certain approach, procedure or narrative showed that PMTs were not flexible in finding solutions. Even though P1 was able to produce endorsable narratives from the diagram in identifying isosceles triangles and equal angles, she was still not sure how to use this information to find the answer. This was evident from her utterances such as "I think so" and "it will help somehow", which all showed scepticism in her approach to finding the solution (Table 5.32). It was clear at this stage that P1 did not know the rules of discourse necessary in finding the solution to the first problem, nor did she have any previous experiences useful for the current problem. Her routine performance thus far seemed to be governed by endorsable subroutines which might or might not be useful in producing the solution, but she picked them up because she could endorse them. This was also confirmed in the interviews, when P1 mentioned that when she began answering the first question, she wanted to apply a Euclidean geometry theorem, which pointed to a ritualistic behaviour. Furthermore, P1 mentioned that she unpacked the diagram so that she could identify the theorems applicable to the problem, without any indication of being flexible problem-solving behaviour. All this can be seen below in her utterances during the interview.

P1: "*On the first problem which already had a diagram, I started by simplifying and analysing the question also the diagram by breaking it into pieces were by I can make sense out of it and so I could be able to applying the applicable theorems. The reason why I unpacked the diagram in the first place was so that I could be able to apply the theorems, because when it was still packed, I wasn't able to identify which theorems to use, it was just so complicated for me. For example before unpacking the diagram into other more simplified diagrams I couldn't see how angle "C" was related to angle "A", I couldn't see that I had a "cyclic quad" but after* *breaking the diagram further all those theorems were clear and I was able to answer the question".*

However, this behaviour by P1 can be related to Polya's (1973) heuristic of solving mathematical problems of breaking up a problem into smaller and simpler sub-tasks. While this observation is viable and a well-founded problem solving strategy, the extract from the interview represents a ritualistic discourse participation (from commognition) in many ways. This is because the word usage, narratives and explanation of the routine represents that P1 used a ritual in completing the solution.

Table 5.32: P1's scepticism in finding the solution to the first problem

Speaker	What is said	What is done
I	Now that you have identified your isosceles triangles and equal angles, do you think this is going to be helpful in finding the solution?	
P1	Yes, I think so.	
I	Okay, go on	Breathes heavily
P1	Because with this isosceles triangle, I want this, the A from ΔBAF and the A from ΔOAC, so with me identifying all these radiuses and the isosceles triangles, it will help somehow in me identifying how these angles are related.	

P1 further made it clear in the interviews that, besides what she did in the first problem, she did not have any other strategy of solving the problem. She mentioned, "I don't think there was something else I could have done either than what I did". Furthermore, the ability to represent the solution(s) through different approaches symbolised PMTs' lack of flexibility. After endorsing their narratives, PMTs were asked if they could have solved the problem using a different approach or approaches, and most of them could not propose a different approach. Table 5.33 provides a summary of PMTs' ritualistic flexibility during their problem-solving activities. The question relating to flexibility was asked at the end of each problem, hence, some PMTs might have not been flexible in the first problem, but could be flexible in the second problem, and vice versa. The summary given in Table 5.33 gives evidence that most PMTs were not flexible in their thinking and they felt that one solution was enough. Furthermore, this shows that PMTs relied mostly on reproducing past experiences in their solution paths and did not explore new problem-solving strategies to approach the problem from different perspectives.

Table 5.33: A summary of PMTs' flexibility during their solution paths

Speaker	What is said
P1	Ohh and then hence shown, since $\angle DAC = x$ and $\angle BAF = x$, proven. Whoah, finally.
I	Do you think there is another method we can use to solve this problem?
P1	Eish, I don't see any other way.
I	What other method can we use to get the same answer?
P2	Keeps quiet.
I	Is there any method you can think of?
P4	No sir, I cannot even find the first method, I cannot have a second method.
I	Do you think there is another method you can use to solve this problem?
P5	Yes, but I don't think we have time for that now sir.
I	You can attempt it, or at least explain it.
P5	(keeps quiet) I do not see it sir.

5.4.2.6 Level of correctibility of the narratives and the routines

Ritualistic routines and narratives cannot be corrected by PMTs and are declared true and used as they are, unless some scaffolding from a more knowledgeable other intervenes. In analysing the correctibility of narratives and routines I focused on PMTs' production of geometry statements, visual mediators, and the use of these statements in endorsing narratives. The focus was on how they spoke about mathematical objects during their problem solving and what visual mediators were used to communicate information about mathematical objects during their problem-solving activity. There were episodes where PMTs produced partially correct visual mediators (proofs) during their problem-solving activities, but these narratives were not accepted as correct because of their lack of any particular substantiation in their deductions. For example, a proof without reasons was considered partially correct, but not acceptable to describe mathematical objects as it was not structured rigorously.

During their problem-solving attempts, PMTs attempted to understand the problems through deconstructing them and extracting information from the given statement or diagram. Some PMTs were not able to use the correct definitions of mathematical terms. The definitions of the orthocentre and circumcentre were given in the statement for problem 1. However, some PMTs could not associate these definitions with the visual mediators in the diagram. P2 could not link the given definitions of the circumcentre and the orthocentre with the diagram that was given in the first problem. The same was also true for P4, who also recognised H as the circumcentre and O as the orthocentre as seen in Table 5.34. Furthermore, P2 even produced a visual mediator to symbolise her utterances about the orthocentre and the circumcentre as seen in Figure 5.4. Since she considered “H is the circumcentre” as an endorsed narrative, P2 continued to use this narrative in her problem-solving activity as seen in Table 5.34.

Table 5.34: P2, P4, and P5’s association of the definitions with visual mediators

Speaker	What is said
P2	*Reads the definition of the circumcentre and utters,* okay that means our circumcentre is H, yeah, H is our circumcentre. *Reads the definition of the orthocentre and utters,* that means our orthocentre is O. … a few moments later Okay, let me take triangle BAH and triangle CAF. Okay, the circumcentre, H, it has the line FH and CH, and KH, Okay. From the circumcentre that means, KH and, that means line BH is equal to line HG. Okay
P4	Show that $\angle BAH$ is equal to $\angle CAO$, H is the cicumcentre and O is the orthocentre.
I	Okay, you are given the definitions of both the orthocentre and the circumcentre. Did I draw the perpendicular bisectors in this diagram?
P5	I think it’s K (*pointing at point K*).
I	Don’t you think KC is the altitude?
P5	Keeps quiet.
I	What is an altitude?
P5	It is the line which passes through the vertex.

P2 also mentioned “I can use KC because from the definition of the altitude, it is perpendicular to this point (pointing at AB) and bisects this angle”. Even though an altitude is perpendicular to the side of a triangle, it does not necessarily bisect the angle of the vertex through which it passes. The inability to associate definitions with visual mediators was also visible in the case of P2, where she asserted that CK divides AB into two equal parts (Table 5.7). Furthermore, P5 was not able to define the altitude correctly, even though his definition contained correct and necessary information (passes through the vertex), it was not sufficient to characterise an altitude. This inability to define the altitude correctly symbolised that these PMTs did not master Van Hiele level 2, as their definitions of the altitude contained correct information flawed with some characteristics which, from my inside and outside observations, were influenced by the visual appearance of these discursive objects. Hence, these PMTs were still operating at ritualistic Van Hiele level 1, where they know the properties of geometry figures, but could not logically classify them to produce a correct definition and sometimes relied on the visual appearance of diagrams in their conceptualisation of definitions and properties.

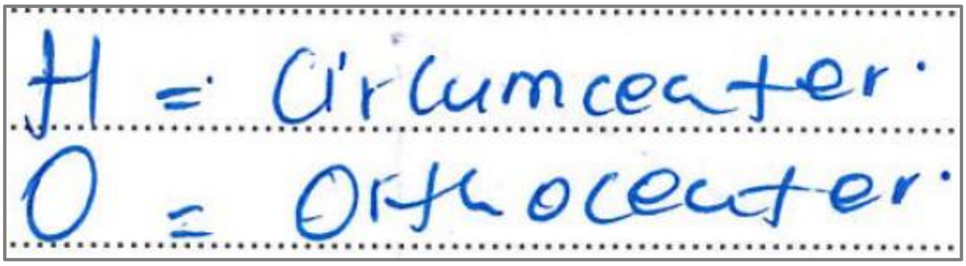


Figure 5.4: P2's visual mediator about the orthocentre and the circumcentre

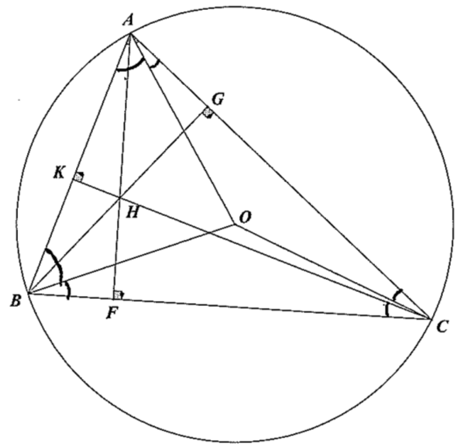
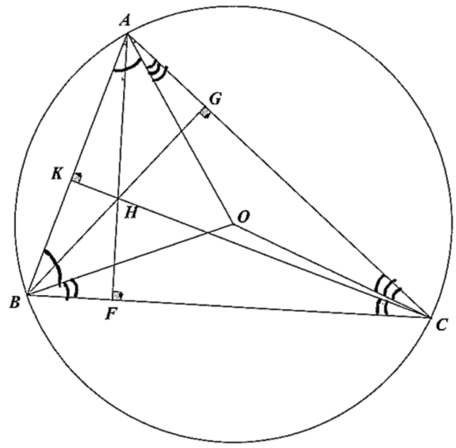
Furthermore, there was evidence of P2 endorsing her narratives based on their visual appearance. For example, she asserted, "line KH is twice line HF", without substantiating this claim with endorsed mathematical narratives. On a similar note, P6 substantiated that angles were equal in similarity, using reasons that apply to congruency. Instead of substantiating using the remaining angles of a triangle, P6 used corresponding parts of congruent triangles (CPCT) to substantiate why the remaining angles of the two triangles, each with a pair of equal angles, are equal (Table 5.35).

Table 5.35: P6 substantiating that angles are equal based on incorrect reasoning

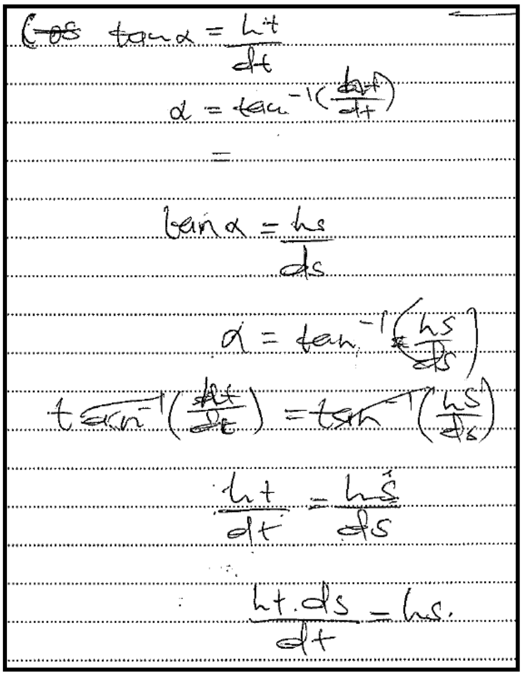
Speaker	What is said
P6	… I don't want to do this, in fact, I do not think it will allow me to use what I want to use.
I	What is that?
P6	Similarity, because the angles are the same (alpha) and these ones are 90 degrees and the remaining angles have to be the same by CPCT (meaning corresponding angles of congruent triangles). It means the corresponding parts will be equal.

In deconstructing the first problem, P1 was able to substantiate equal angles correctly in the three isosceles triangles, but produced an incorrect visual mediator to represent these angles. It was not until the questioning from the interviewer that P1 was able to see the incorrectness of her visual mediator and corrected it individually. Table 5.36 represents this situation for P1 which also presents evidence of how PMTs relied on the interviewer during their problem solving.

Table 5.36: P1's visual mediators about equal angles in isosceles triangles

Speaker	What is said	What is done
I	Okay, is that the only isosceles triangle or you are using this one for now?	
P1	No, this one and that one. 	Pointing at ΔBOC and ΔAOB. Indicates all equal angles with one arc.
I	So, with this indication are you saying that all these angles are equal to each other from the different isosceles triangles?	
P1	Ohhh, we are not sure of that at the moment because of the chords are different, I have to change. 	Indicating equal angles with 1, 2 and 3 arcs from the three different isosceles triangles.

Furthermore, some PMTs produced proofs that were endorsed based on incorrect narratives, either a step in the proof was incorrect or the substantiation given was incorrect. Consider for example, the solution produced by P6 to the second problem in Figure 5.5. This solution looks mathematically correct, but zooming into the step indicated by the arrows in Figure 5.5 reveals that it is mathematically incorrect to just cancel the arctan because in trigonometry terms, arctan is the inverse of the tangent function, which is always associated with a certain angle. Hence, cancelling the arctan only, without cancelling the angles means that the angle and the trigonometric identity are not co-joined.



What is done here is similar to the following incorrect routine.

$$\frac{\cancel{\sin} A}{\cancel{\sin} B} = \frac{A}{B}$$

Figure 5.5: P6's solution to the second problem

From this, it was clear that some PMTs endorsed their narratives based on incorrect previously endorsed narratives and sometimes their reliance on these previously endorsed narratives might have been the barrier to finding the solutions to the problems. This was evident mainly in the case of P2, who grappled with the first problem and ended up using inductive reasoning to conclude that the angles were equal because her initial endorsement of the orthocentre and circumcentre was incorrect. A geometry proof is complete if each claim is supported by true statements (properties, theorems, definitions, etc.) and is deductively correct in its form. However, there was evidence from PMTs' problem solving that some conclusions were reached based on incorrect and/or incomplete proof. For example, P3, when solving the first problem, relied on two theorems and produced his proofs based on these two theorems. The first theorem is "if an arc subtends an angle at the centre and the circumference of a circle, then the angle at the centre is twice the angle at the circumference of the circle" and he produced a complete proof which is correct because of its form and each claim is supported by valid geometry properties and theorems, as seen in Figure 5.6.

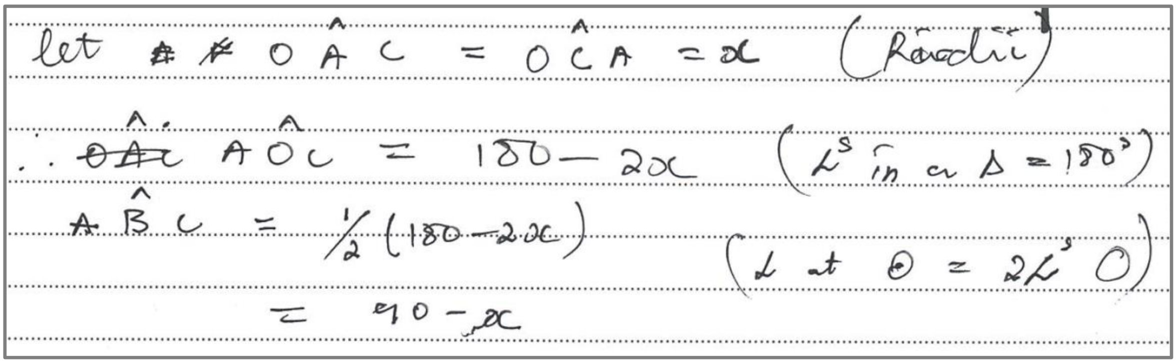


Figure 5.6: P3's proof in the first problem related to the angle at the centre theorem

Feeling that this proof was not enough to answer the question posed, P3 linked the current theorem with the theorem, which states, "the sum of the interior angles of a triangle is 180" to attempt to find the answer to the question posed. However, looking at Figure 5.7, the deduction from of the proof is true, but the proof lacks support of two statements related to the first and the second line of the proof.

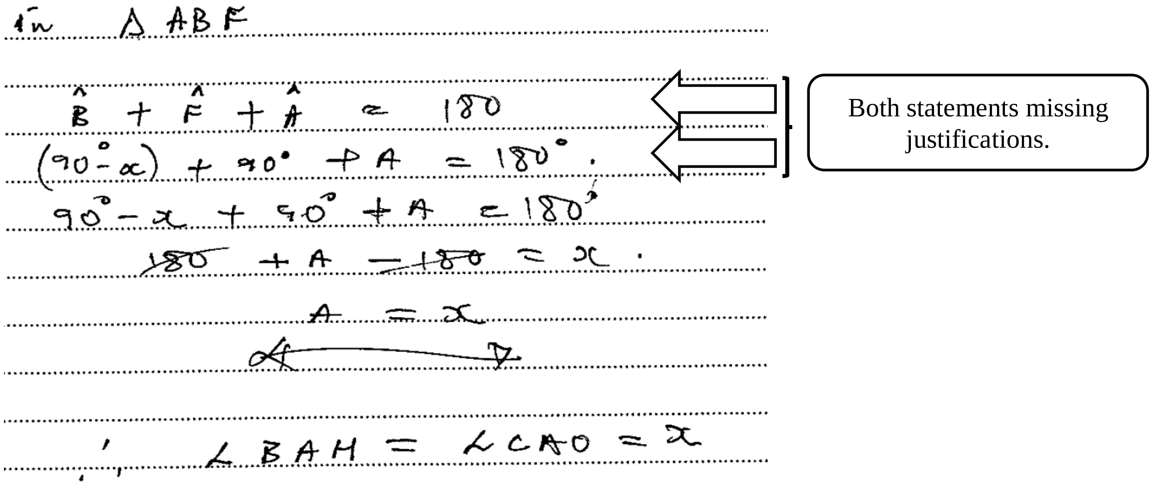


Figure 5.7: P3's linking proof related to the sum of angles of a triangle

This practice showed that P3 felt that the form of the proof was more important than its meaning, as the proof was constructed correctly, but its meaning was diminished without the justifications required in Figure 5.7, which was in line with arguments from previous research (Yang & Lin, 2008). Conclusions reached based on narratives that were not endorsable, like the one from P3 above, was evident in all the PMTs, especially when attempting to solve the first problem and this can be seen from their proofs to the sum of angles of a triangle highlighted in Table 5.37. This was the evidence that PMTs struggled with Van Hiele level 3 of logical deduction. Seeing that they produced incomplete and incorrect proofs, these PMTs were struggling to master an explorative Van Hiele level 3 of geometry thinking.

5.4.2.7 Acceptability and endorsement of PMTs' narratives

The acceptability and endorsement of PMTs' narratives focused on the reasons that PMTs found a particular narrative or routine to be acceptable and endorsable. In ritualistic discourse, the routine was accepted because it has been shown to adhere to the rules of a procedure or has been validated by others, especially more knowledgeable others. Thus far, looking at the analysis it could be broadly summarised that PMTs' proofs relied mainly on attempts to follow the rules of the procedure strictly without conceptual understanding. Looking at attempts to prove that "the sum of angles in a triangle is 180", most PMTs were mostly concerned with the structure and the symbolic manipulation that would lead them to a certain angle, instead of the substantiations necessary during geometry proofs. This was evident in Table 5.37 and Figure 5.2, to mention a few. There was a lot of worry about making mistakes presented in the first section of this data analysis (5.4.2.1), which was evidence that PMTs did not want to deviate from the procedure, and this could be seen in utterances such as "tell me if I am wrong" after a routine performance or endorsement of narratives. Furthermore, there was abundant evidence from the preceding analysis that the acceptance of various routines and narratives was predicated based on the presence of the interviewer as a discursant recognised by the PMTs as the knowledgeable other. PMTs agreeing with the suggestions from the interviewer without questioning them, and also their reliance on the interviewer's scaffolding to move forward with a routine or endorse narratives, showed that their acceptance of routines and narratives was influenced by others instead of their own knowledge that was independent of others. Instead of reiterating some of the evidence already presented above, I provided some episodes that lead to the acceptability of particular routines and narratives from different PMTs in this study, that was classified as ritualistic.

These participants, as seen in the extracts from Table 5.37, failed to produce complete geometry proofs. By 'complete', I mean a proof where each assertion made is supported by a theorem, axiom, definition, property or a postulate. They thought the production of geometry statements, for example about the angle sum of a triangle, as complete without the substantiation about the theorem. They ignored that each action in geometry, in fact mathematics, is logically deduced from existing facts that are usually used as substantiations. This, according to the Van Hiele theory of geometry thinking, means that they have not mastered level 3, while according to the current study, they operated at a ritualistic Van Hiele level 3. This means that even though they were able to follow procedures to produce endorsable narratives, these narratives could not be accepted as correct, because of the absence of the reasons substantiating each assertion or logical statement produced by these participants.

Table 5.37: PMTs' proof of the sum of angles of a triangle without substantiations

From the Δ BAH $\angle K + \angle A + \angle H = 180^\circ$ $90^\circ + \angle A + \angle H = 180^\circ$ $180^\circ - 90^\circ = \angle A + \angle H$ $\angle A + \angle H = 90^\circ$ **P2**	$A\hat{E}C + F\hat{A}C + \hat{C} = 180^\circ$ $90^\circ + F\hat{A}C + x = 180^\circ$ $F\hat{A}C = 180^\circ - 90^\circ - x$ $F\hat{A}C = 90^\circ - x$ **P1**
$180^\circ = A + 90 + \left(\frac{1}{2}(90 - A)\right)$ $180^\circ = A + 90 + 45 - \frac{1}{2}A)$ $180^\circ = A + 135^\circ - \frac{1}{2}A$ $45^\circ = \frac{1}{2}A \times 2$ $90^\circ = A$ **P5**	$2(90 - F\hat{A}B) = 180 - O\hat{A}C \pm A\hat{C}O$ $180 - 2FAB = 180 - O\hat{A}C - A\hat{C}O$ $A\hat{C}O + O\hat{A}C = 2FAB$ $OAC + OAC = 2FAB$ $2OAC = 2FAB$ $OAC = FAB$ $A\hat{C}O = OAC$ [$\angle$s opp = side] **P6**

in $\triangle ABF$

P3

$\hat{B} + \hat{F} + \hat{A} = 180$

$(90^\circ - x) + 90^\circ + A = 180^\circ$.

$90^\circ - x + 90^\circ + A = 180^\circ$

$~~180~~ + A - ~~180~~ = x$.

$A = x$

$\therefore \angle BAH = \angle CAO = x$

Considering how P1 endorsed the narrative that $\angle CAO = \angle BAF$, I noticed it was characterised by instances of either following a procedure or accepting the result based on the routine performances' adherence to the rules of the procedure. This adherence to procedures could be seen in how she calculated the sum of angles of a triangle in Figure 5.8 and how she continued to use the information from an incorrect proof in her solution, based on the adherence to the procedure of calculating the sum of angles in a triangle. Her proof was characterised by calculations which were not linked to the diagram. For example, she calculated the sum of the angles in a triangle with an assumption that it would be apparent to everyone which triangle she was referring to in the proof. This discourse participation revealed particular information about these participants' discourse participation when producing geometry proofs.

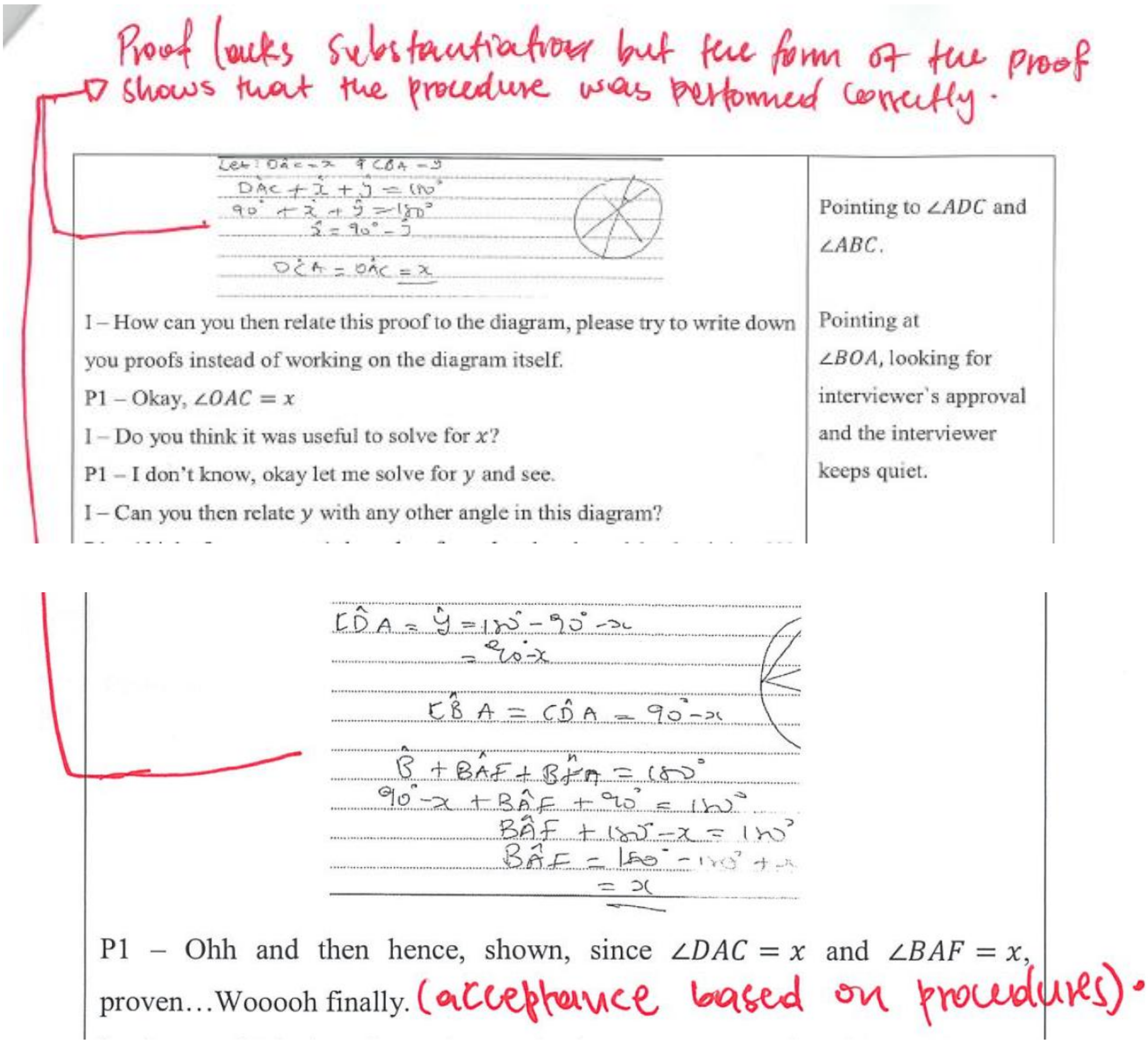
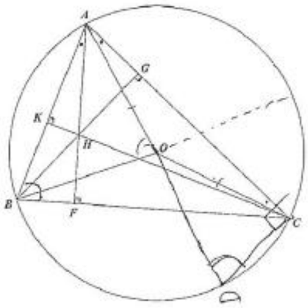


Figure 5.8: P1's adherence to the rules of procedures related to the sum of angles in a triangle theorem

Sometimes she also accepted the truth of theorems based on the visual appearance of the diagrams or based on the fact that suggestions came from the interviewer whom she considered a knowledgeable person in the subject. As seen in Figure 5.9, P1 was convinced that $\angle CAO = \angle BAF$, based on the appearance of the diagram, which was suggested by the interviewer. This showed that she was ready to accept that the answer is true without a proof, either because diagram was convincing for her or because the diagram was suggested by the interviewer. Hence, we could see that the acceptance of the narratives and routines which resulted in P1 showing that $\angle CAO = \angle BAF$, was based on the adherence to the procedures and the suggestion given by the interviewer. Her reliance on others for the acceptance of routines and narratives was evident where she utters "I can relate this one with this one, but how do I write it?". She was clear about the objects that could be related to each other, but still required the interviewer to provide guidance, perhaps a procedure to follow, when relating those mathematical objects.

To search for a different approach, the interviewer suggests a different diagram.

I – Try extending AO to the circumference at D as seen in diagram 2 and see how that can help you.



I – Does this diagram make it a little bit easier for you or not? Can you now find the solution?

P1 – Eish, it is so obvious that I don't know how to put it down, from the way it is constructed it is obvious that they will be equal but putting it down is a problem. (Acceptance based on visual appearance)

Might also be based on the fact suggestion was made by the interviewer.

I – Try it and see.

P1 - So I can relate this one with this, but how do I write it?

Figure 5.9: P1's endorsement of the solution about the first problem

Furthermore, the acceptance and endorsement of narratives and routines based on the visual appearance of diagrams, was also visible in the case of P3 (see Table 5.27), where he went on to produce a proof based on similarity without producing a formal proof about similarity. However, the case of P3 is different from P1, because he was able to give a verbal description of the proof by highlighting that all the angles were equal (Table 5.27). Even though this could count as a verbal proof, it lacked substantiation (i.e., he only mentioned that the angles are 90° without substantiating), and his verbal proof lacked one more pair of angles, which means it was incomplete. Hence, even though the final endorsed narrative was correct, it was endorsed based on narratives that were not only incorrect, but were accepted because P3 believed that they conformed to the rules of the narratives about similar triangles. The practice of accepting narratives based on visual appearances of diagrams continued to P4, who also accepted statements about mathematical objects to be true, without any justification. P4 mentioned, "angle BAH is equal to angle HAO plus angle CAO. Okay, angle BAH is equal to angle OBA minus angle BAH plus angle CAO", without justifying these statements. Furthermore, these two narratives about angles being equal could not be endorsed using the diagram given for the first problem. On the other hand, P5 accepted that two lines were equal ($AF = BG$) in the diagram provided for the first problem based on 'spirituality', which meant that he just felt that these two lines were equal and he could use this in solving the first problems (Table 5.38). Again, this reasoning based on spirituality showed that P5 was operating at a ritualistic Van Hiele level 0, because his conclusion that $AF = BG$ was not based on the properties of the geometry figures.

Table 5.38: P5's belief based on spirituality instead of mathematical reasons

Speaker	What is said
P5	I see something.
I	Okay what are you seeing?
P5	I think line AF it is equal to BG in length.
I	Why?
P5	Eish, spirituality.
P5	I want to prove that triangle ABF it is the same as ΔABG so that I can go with their properties to BAH, the one that is given, the one that I need to prove.
I	So you want to prove that triangle BGA is similar to triangle BFA?
P5	I don't think they are similar, they are congruent.
I	Congruent? For what reason.
P5	I think if I can get that information it will help me to prove that, first thing I know is that angle B and angle A, they are of equal size. The whole A because of O, because OB is radius and OA is radius.
I	What do you mean the whole A?
P5	Not the whole of A, the A that is part of BOA. And also B here, from the same triangle, they are the same. Then the second thing if I go to triangle AHB, angle A will be equal to angle B.

Furthermore, statements such as “I think if I can get that information, it will help me to prove that, first thing I know that angle B and angle A, they are of equal size. The whole A because of O, because OB is radius and OA is radius” from P4 were further proof of accepting the truth of statements without proving them and relying mainly on their visual appearance. From this statement, P5 concluded that angles A and B were equal based on incorrect reasoning and based on the way he spoke about the angles. Furthermore, P5 was quoted saying, “I don’t think they are similar, I think they are congruent”, yet when asked for a justification he could not produce one and mentioned that he was still looking for information to prove that the triangles were similar. It seemed that P5 was making his present intuitive thinking explicit about the congruency of the two triangles but the thinking was not followed by any rigorous justification or deductive formulations. This statement about the congruency of the two triangles was also an indication of ritualistic Van Hiele level 0, because he used the visual appearance of the two triangles (ΔBGA and ΔBFA) to conclude that they were congruent, instead of producing a proof. The statement “I think if I can get that information, it will help me to prove that …” was substantiated by P5 in the face-to-face interview as seen in the quotation below.

I: *So when you began to answer the question, what did you do first and why?*

P5: *The first question, the one that I thought it is Euclidean geometry, when you are given Euclidean geometry you have to find radius, diameter, then I checked if I have parallel lines, I was given perpendicular but I did not know how to use those perpendicular things. So I was stuck.*

I: *Why do you think you didn’t know how to use the perpendicular lines?*

P5: *I thought maybe theorem 1, it will help me, that the line from the centre it will bisect that one.*

I: *Radius bisects chord?*

P5: *I thought about it but it never helped me, then I thought about the theorem of angles subtended by the same chord, then G was not in the circumference, so I got stuck again. Then second question, I started by drawing the diagram which I was correct, not correct in fact (laughs), it was guiding me but it fixated my mind to san this is 45 this is 90, so that is where I got stuck.*

From this quotation, it was clear that P5’s plan was to collect as many facts (endorsed narratives) about the problem as evident in his statement “when you are given Euclidean geometry you have to find radius, diameter, then I checked if I have parallel lines, I was given perpendicular". As he mentioned even in the interview that he was looking for a particular theorem which, when he did not find the theorem useful in the problem, he was then stuck.

Furthermore, P6 endorsed a proof based on incorrect reasoning (Table 5.35) and continued to use this endorsed proof to endorse other narratives. Due to the conformity of the proof to the rules of proofs related to similarity and congruency, P6 mixed CPCT, which is used in congruent triangles, to conclude the similarity of triangles. Furthermore, he concluded, "corresponding parts will be equal" which is true in congruent triangles, yet his verbal proof was based on three pairs of angles being equal, which implied similarity of the two triangles. In conclusion, while elements from this category were dominant during the coding process, most of them linked to other themes discussed earlier, and in this section I referred back to the discussed extracts to explain how they represented a ritualistic acceptance and endorsement of routines. This was done by specifically highlighting how the narratives (word usage and visual mediators) resembled actions completed through adherence to rules, and how PMTs accepted and endorsed narratives which were based on their visual appearance or suggestions given by the interviewer. As such, most PMTs' aim was to imitate some previous experiences to solve the problems and this was evident in the way that they talked about and endorsed their routines during their problem-solving activities.

5.4.2.8 How PMTs used words and mediators in their discourse

In commognition, analysing one's think-aloud mathematical problem-solving activity, means analysing their words usage, visual symbols (proofs) and also the diagrams they produce. This section was characterised by codes that focused on how word usage and visual mediators revealed the ritualistic nature of PMTs' discourse. Hence, I divided the analysis into two subsections of using colloquial discourse to describe mathematical objects, and producing and/or using incorrect visual mediators.

- ***PMTs uses colloquial discourse, visual cues and bodily movements to describe mathematical objects***

In this subsection I included quotations that relate to how PMTs used colloquial discourse to describe mathematical objects. PMTs' discourse, that included both objectified and phrase-driven word usage, were coded as ritualistic, because they were not completely reified (Sfard, 2008). However, I only focused on the word usage that was completely phrase driven in my analysis. The first general subtheme here is PMTs' usage of pronouns such as 'this' 'them' and 'these' to refer to mathematical objects (like lines, angles, or shapes) instead of using mathematical language (see Table 5.38). Colloquial discourse that was conflicting with mathematical discourse was observed in the cased of P5, as seen in Table 5.38, where he utters, "I want to prove that ΔABF is the same as ΔABG". The use of the adjective 'same' in P5's discourse was confusing, seeing that he wanted to mean that the triangles were 'congruent' as the correct mathematical description. Even when they were asking questions from the interviewer, PMTs used colloquial discourse in the form of 'this' together with the body movement of pointing to the mathematical object they were referring to in the diagram, as seen in the cases of P3 and P5 in Table 5.39. Furthermore, I also observed similar descriptions of more than one mathematical object using colloquial discourses in the cases of P1 and P3, where they said, "this one and that one" instead of referring to the mathematical descriptions of the objects. The usage of these pronouns was so engraved in the teachers' discourse, that even though they gave objectified descriptions of mathematical objects, for example 'ΔKAH' , they still felt that qualifying pronouns such as 'this' were necessary for the description of these mathematical objects. For example, instead of saying, "I just took ΔKAH", P2 stated, "I just took this ΔKAH" (see Table 5.39). Furthermore, there were instances where PMTs started by using colloquial discourse, but then changed after some scaffolding from the interviewer. However, this scaffolded change was not permanent, since even at a later stage PMTs still continued to use colloquial discourse to describe mathematical objects (see Table 5.40).

Table 5.39: PMTs using colloquial language to describe mathematical objects

Speaker	What is said	What is done
P1	Okay, I know these are the radiuses of the circle, meaning that since AC is the chord of the circle then ΔAOC is isosceles. Therefore, this one is going to be equal to this one.	Pointing to line AO, CO and BO indicating that they are equal, indicating that $\angle OAC = \angle OCA$.
I	Okay, is that the only isosceles triangle or you are using this one for now?	
P1	No, and this one, and that one.	
P2	I think after finding this angle, and adding this angle and that angle we are going to get 180 degrees. That is where I got stuck.	Pointing to all the angles in triangle BOA.
	And on this one, I just took this ΔKAH which is 90°, then I did assume that $\angle KHA$ is 60 which will give us this as 30 which is wrong neh?	Pointing to ΔAHK. Pointing to $\angle KAH$.

Speaker	What is said	What is done
P3	So can we say that this whole angle is equal to this one?	Pointing to $\angle BAC$ and $\angle ABC$.
I	Not unless we can prove it.	
P3	But these ones are equal, we do not have to prove them because of the circumcentre.	Pointing to $\angle BAO$ and $\angle ABO$. Indicating $\angle OBC = \angle OCB$ and $\angle OCA = \angle CAO$.
I	Yes.	
P3	Okay, and this one and that one.	
P4	If they are equal it means this angle is equal to this angle akere (*right*)?	Pointing to $\angle CAO$ and $\angle OCA$, then looking at the interviewer for approval, giggling.
P5	Can I let this thing be, maybe if I say let this be x? I only have 90, I don't have this and that. Okay let me say this is x and this is y (*referring to angle ABF as x and angle BAF as y*). It will be 180 equals to 90 plus x plus y. So it will be 90 equals x plus y or if I make B the subject of the formula I have B equals $90 - A$.	Pointing to angle BAF. *Pointing to* $\angle BAF$ *and* $\angle ABF$ *in* $\angle ABF$.
P6	Okay, so this is 90 (*degrees, but did not say it*) here because they are standing, akere (*meaning right?*), there is the height, h_t, and h_s, okay let us say this, this angle and that angle are equal and this will be 90 and 90 it's fine, and then the other remaining angles will also be equal. Okay, how can I do this now, I can involve trigonometry here. Okay, I named them incorrectly earlier, now I see that the orthocentre is this one and the circumcentre is this one. Because O is also the centre of the circle it means this one is equal to that one and that one, these are radiuses. So now this angle will be equal to this angle if we say these are radiuses, so angles opposite equal sides.	Pointing to the angles labelled alpha in the diagram. Referring to the angles labelled with a 90° symbol in diagram 5. Labels them each y. Pointing to H and O. Pointing to lines AO, OC and OB. Pointing to $\angle AOC$ and $\angle OCA$.

Another observation was that when most PMTs spoke about the mathematical object of angles, most often they did not include the label 'degrees', but only the size as seen in Table 5.39 in the cases of P2, P5, and P6. This took away the mathematical meaning from their discourse and PMTs were talking about angles like they would talk about any quantity that can be quantifiable in its nature. P5 also displayed some explorative Van Hiele level 2 when he decided on his own to attach variables to the angles he wanted to use, so that he could make the work understandable for himself. Even though there was labelling for all the angles, P5 decided to use different labelling and he was able to calculate the sum of angles in a triangle using the visual mediators he attached to the angles.

Furthermore, PMTs used visual cues together with colloquial discourse to describe mathematical objects during their problem-solving activities. For example, P1 used pronouns such as ‘this’ with some gestures like pointing, such as “I want this (pointing to A)” together with visual cues such as “the A from ΔBAF and the A from ΔOAC” as seen in Table 5.41. Furthermore, P1 still used descriptions of mathematical objects related to the visual appearance of mathematical objects. As seen in Table 5.41, she used the visualisation of the height of the student and the height of the teacher to differentiate between the two triangles in the diagram. This showed that P1 was operating at a ritualistic level 2 of the Van Hiele theory, because she relied on the visual appearance to identify geometry objects, such as saying, “the A from ΔBAF” instead of saying, “I want ∠BAF”. This Van Hiele level 2 is ritualistic, because not only did P1 identify objects visually instead of using their properties, but she also used colloquial discourse such as “I want this”.

A similar discourse participation of using visuals to identify geometry objects from P1, could be observed in Table 5.9, where instead of identifying lines *BA* and *AC* she moved her fingers on top of these lines, symbolising a mathematical word, which was then suggested by the interviewer as ‘subtends’. A similar incident from P1, where she relied on the visual appearance of a diagram to remember a theorem was evident in Table 5.6, where she produced a diagram first before stating the theorem. There was another incident in the second problem when P1 was solving the problem using geometry: she used a visual appearance to remember the usage of the proportionality theorem, as seen in Table 5.21. In this particular incident, P1 did not even remember the theorem, but was able to use the procedure indicated in Table 5.21 to produce endorsable narratives about the second problem which is symptomatic of a ritualistic discourse participation. Furthermore, the way in which P1 applied the visual mnemonic in producing the endorsable narrative, revealed that she could not remember the theorem because her first assertion $\left(\frac{AD}{BE} = \frac{AC}{AB}\right)$ was not substantiated, as seen in Figure 5.13. Mathematical objects such as angles were also identified purely using visual signifiers, such as was observed from P3 in Table 5.39, when he asked about the equality of $\angle BAC$ and $\angle ABC$. Furthermore, the reliance on visual appearance of mathematical diagrams from P3, could be observed from his labelling of angles in the diagram, as seen in Table 42. These incidents in Table 5.6, Table 5.9, Table 5.21, Table 5.39 and Figure 5.13 were evidence that P1 was, at this point, operating at Van

Hiele level 0 in a ritualistic manner, because even though she knew the theorems, she relied on colloquial discourse and the visual appearance of diagrams for remembering.

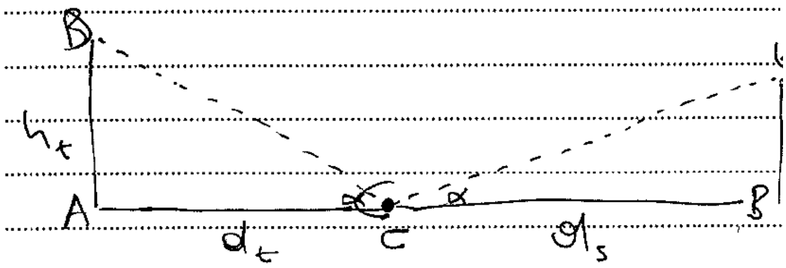
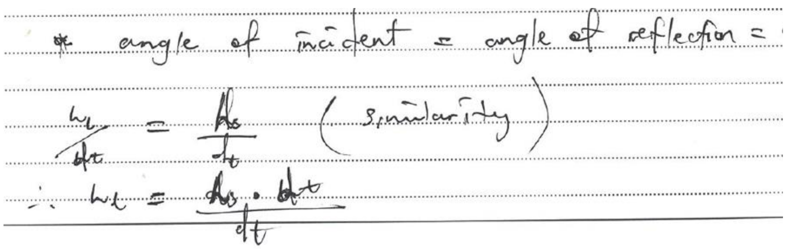
As seen in how the discourse continued in Table 5.9, the identification of geometry theorems and properties based on rituals did not help P1 to arrive at an endorsed or endorsable narrative(s), as was evident from her utterance, “eish, it does not really help me, can you point me towards a certain direction please, I need to find the answer to this problem”. Similarly, P1’s discourse in Table 5.40 showed that she spoke about a mathematical theorem (the angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference of the circle) colloquially and visually which, even though she was correct, classified her level of geometry thinking as ritualistic level 2, because of her word ‘usage’. She was not objectified in her discourse to utter, “okay, ∠AOB is twice ∠ACB”, and used signifiers to identify angles.

Table 5.40: P1 being scaffolded to using objectified discourse in her problem solving

Speaker	What is said
P1	Okay, this angle here is going to be twice this angle here.
I	What is that angle?
P1	Ohhh, $\angle AOB$ is twice $\angle ACB$.

As seen in the case of P2, the usage of visual cues for the description of mathematical objects, perhaps was due to her inability to use objectified discourse to describe angles in the diagram. Looking at Table 5.42, P2 began her talk using colloquial discourse characterised by utterances such as “if this is twice that, it means that this is also twice this whole ∠A”, but she describes an angle using two instead of three alphabetical symbols (angle CB). Furthermore, her transfer of properties from one triangle to the next, used a colloquial discourse in that it did not objectively state the triangle. Instead, she used a statement saying, “and therefore on this side” indicating that she is talking about the other triangle. Later on she revealed her struggle to name angles using the diagram where she transferred from attempting to name the angles stating, “it will tell us that ∠CAB, it is equal to angle …, mmmh B, no, akere its ∠CAH, akere ...” to using visual and embodied descriptions of the angles by pointing (“Gore this angle, ∠B, it will be equal to this angle” [pointing at ∠A]).

Table 5.41: PMTs' usage of the combination of visual cues, bodily movements and colloquial discourse to describe mathematical objects

Speaker	What is said	What is done
P1	Because with these isosceles triangle, I want this, the A from ΔBAF and the A from ΔOAC. So with me identifying all these radiuses and the isosceles triangle it will help somehow in me identifying how these angles are related. We wanted to check how this information that we have in this triangle, the one that has the height of the students is related to this one with the height of the teacher. 	*Pointing to* A
P3	I am done, this side over this one, since they are similar, this side over this one is equal to this side over that one, done. Then I make that one (h_l) subject of the formula. 	

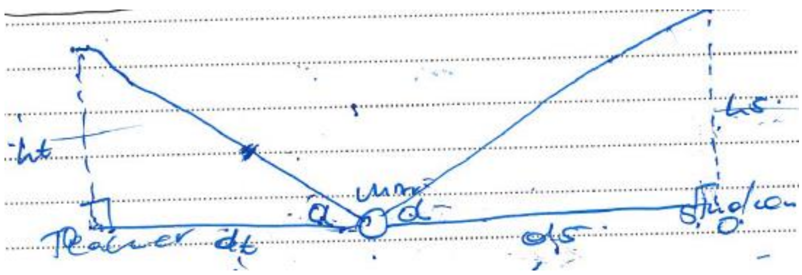
Even in her description of the definition of the tangent as a trigonometric identity, P2 relied on the visual appearance of the diagram produced to represent the second problem. In her discourse, as seen in Table 5.43, she visually identified the sides that were opposite and adjacent to the angle α before concluding that she will use the definition of the tangent in trigonometry to solve the problem. As an objectified way of saying the same thing, as P2 talked about in Table 5.43, would be to say, "HS is opposite α and DS is adjacent α, therefore, I will use the definition of the tangent because I want to solve for HS." Her discourse however is ritualistic, because of the over-reliance on pointing and the usage of pronouns. Furthermore, we can see in Table 5.43 that P3 also relied on the visual appearance to assign the variable x to the equal angles and this reliance on the visual cues led him also to think that $\angle BAF$ was equal to x.When asked later to motivate, he realised that it was not true.

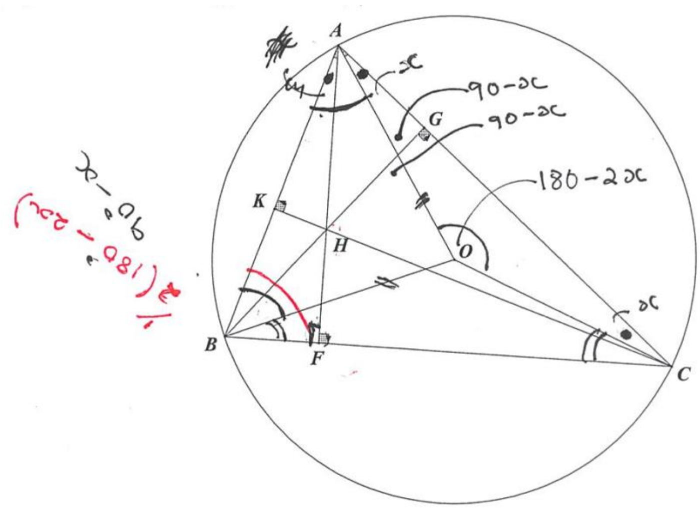
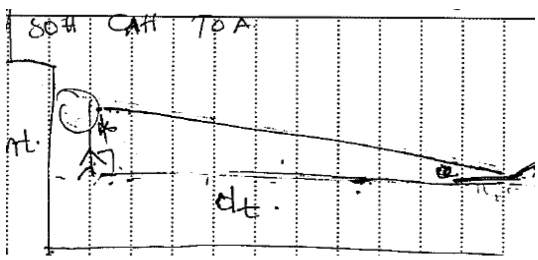
Table 5.42: P2 struggling to name angles and relying on visual cues

Speaker	What is said	What is done
P2	Sighs, okay, if this is twice as that, it means this is also twice this whole $\angle A$ which will then lead us that $\angle ABF$ or $\angle ABH$ is equal to $\angle CB$ and therefore on this side it will tell us that $\angle CAB$ it is equal to angle (keeps quiet for a while), mmmh B no, akere its $\angle CAH$, akere [*right*], gore [*like*] this angle, angle B it will be equal to this angle.	*Referring to $\angle BOC = 2 \times \angle BAC$. Pointing to $\angle A$.*
I	Okay, can you prove it?	
P2	(*Laughs…*) I cant.	

Furthermore, P3's reliance and focus on working in the diagram, labelling angles in the diagram and identifying theorems using the visual appearance of mathematical objects in a diagram, might be the influence of him talking about mathematical objects using a lot of pronouns and pointing. Even when endorsing the narrative, as seen in Table 5.43, P3 continued to use visual cues together with pointing and the use of pronouns such as 'this' when talking about mathematical objects. P2 relied on the visual appearance of the diagram to conclude that $\angle ABC$ and $\angle BAC$ were equal to one another, "gore [like] this angle, angle B it will be equal to this angle" (pointing at ∠A), without giving any substantiation or a proof for this, and when she was asked to prove her assertions, she just laughed and said, "I can't", as seen in Table 5.42. This was evidence that she was operating at ritualistic Van Hiele level 1, where properties of mathematical objects were assumed to be true based on their visual appearances.

Table 5.43: P2 and P3' reliance on visual cues to define the tangent and assign values to angles

Speaker	What is said	What is done
P2	Okay, this height is opposite to this and this is adjacent, therefore it will be tan α is equals to HS over DS. 	Pointing to HS, α and DS.
P3	Okay, but we are going to have a lot of variables. Okay, if I let this be x, then this is x as well and that is x as well.	Pointing to $\angle OAC$, $\angle OCA$, and $\angle BAF$.
I	Okay, what reason do we have that $\angle BAF$ is also equal to x, because I understand that $\angle OCA$ and $\angle OAC$ will be x because lines OC and OA are radii.	

Speaker	What is said	What is done
P3	Ohhh, Okay no I am lying, that angle is not x, We want to prove these two. Okay fine, this is $90° - x$ and this is also $90° - x$.  Eish, this is killing me. Let me see. (keeps quiet for a while)	Pointing to angle $\angle CAO$ and $\angle BAF$. Writing $90° - x$ as seen in the diagram. Pointing $\angle AOC$ and $\angle ABC$ Writes in red in the diagram.
I	Okay maybe think about the theorem of angle at the centre is twice the angle at the circumference and see how it can help you.	
P3	Okay, ohhh there it works. If this was drawn there for me then I was going to get it.	
I	But it was already there you just could not see it.	
P3	Okay then this is going to be twice that. So this one is going to be $2(180° - 2x)$, no, it's going to be $\frac{1}{2}(180° - 2x)$, which is simplified into $90° - x$.	
P4	Meaning this angle is equal to this. Okay, let me try the SOH CAH TOA. The opposite side and the adjacent side which is the tangent. 	*Pointing at the angles labelled alpha in the diagram.*

Furthermore, from Table 5.43 P4 also used visual cues to identify the method to solve the problem which was similar to that identified by P2. However, once she identified that she could use trigonometric identities, P4 relied on the mnemonic SOH CAH TOA to identify the opposite and the adjacent sides before she applied the definition of the tangent. As seen in P4's discourse, it was guided by the use of mnemonic devices instead of mathematical definitions.

Table 5.44: P5 switching between objectified and colloquial discourse when talking about mathematical objects

Speaker	What is said	What is done
P5	Okay. Sir, is it given that angle A of the triangle COA, I see you have highlighted something, is this angle equal to this?	Pointing to angle BAH and angle CAO.
I	That is what you need to prove.	
P5	Can I name H, have H1, 2 and 3?	
I	Yes, it is up to you.	
P5	Okay but I can also use that thing of naming triangles. I will say $\angle BHF$ is equal to $\angle GHA$, they are vertically opposite. Another thing, I said this angle is equal to that angle. Then because of the 90 degrees, I have angle F equals to angle G equals to 90 degrees, given. So that angle it means this angle B will be the same with the whole of H. That is something very important to me. Angle B which is HBF will be equal to GAH because it is the third angle.	Pointing to $\angle BHF$ and $\angle GHA$.

There was also evidence of P5 starting by using discourse that was objectified to refer to mathematical objects where angles were named according to the diagram given, but as his discourse developed, he started using colloquial discourse. Later on, as seen in Table 5.44, he switched between objectified and colloquial discourse to refer to mathematical objects. Again, this showed that the colloquial nature of talking about mathematical objects was engraved in these PMTs, as seen in the case of P2 in Table 5.39. Despite the fact that most of their colloquial discourse about mathematical objects was endorsable, their over-reliance on descriptions based on the visual appearance of the mathematical objects and also body movements mainly through pointing, meant that these participants' discourses about mathematical objects were mainly ritualistic.

- ***PMTs produces and/or uses incorrect visual narratives in their discourse***

Furthermore, PMTs used visual mediators to communicate about mathematical objects, specifically in the case of the second problem where they felt they had to produce a visual interpretation of the problem before they could attempt to solve it. Hence, the analysis focused on how PMTs produced and/or used visual mediators to communicate about mathematical objects, whether these were used as they were produced or they were altered before usage. I also focused on how these were altered during PMTs' discourses. In the analysis done so far, there was some evidence of PMTs producing visual mediators that were not endorsable to communicate about mathematical objects. For example, this was seen in Table 5.36 where P1 correctly identified equal angles, but used an incorrect visual mediator to communicate the information. Furthermore, most of the proofs for the sum of angles of a triangle produced by PMTs, were incomplete as they did not have the required reasoning, which also represented a case of sticking to a ritualised nature of visually communicating about mathematical objects (see Table 5.37).

I have already produced some evidence relating to the ritualistic nature of PMTs' visual mediators when communicating about mathematical objects, mainly relating to the first problem. Here I focus mainly on PMTs' evolution of the diagram related to solving the second question, as all of them mentioned in the semi-structured interviews that the second problem could not be solved without the aid of the diagram.

With the exception of P5, none the teachers could come up with their own visual interpretation that they could use to solve the second problem correctly, yet they all mentioned that a diagram was crucial in solving the second problem (Figure 5.10). P1, after reading the statement about the second problem, started to draw a sketch of the situation (Figure 5.10), but before completing the diagram, she asked the interviewer, "is this the correct sketch?" Even after some time of asking questions from the interviewer, P1 still failed to come up with a diagram she could use to solve the second problem. The interviewer suggested that she places the mirror on the ground facing upwards, but she lamented that she was struggling to draw the sketch if the mirror was in that position (Table 5.45).

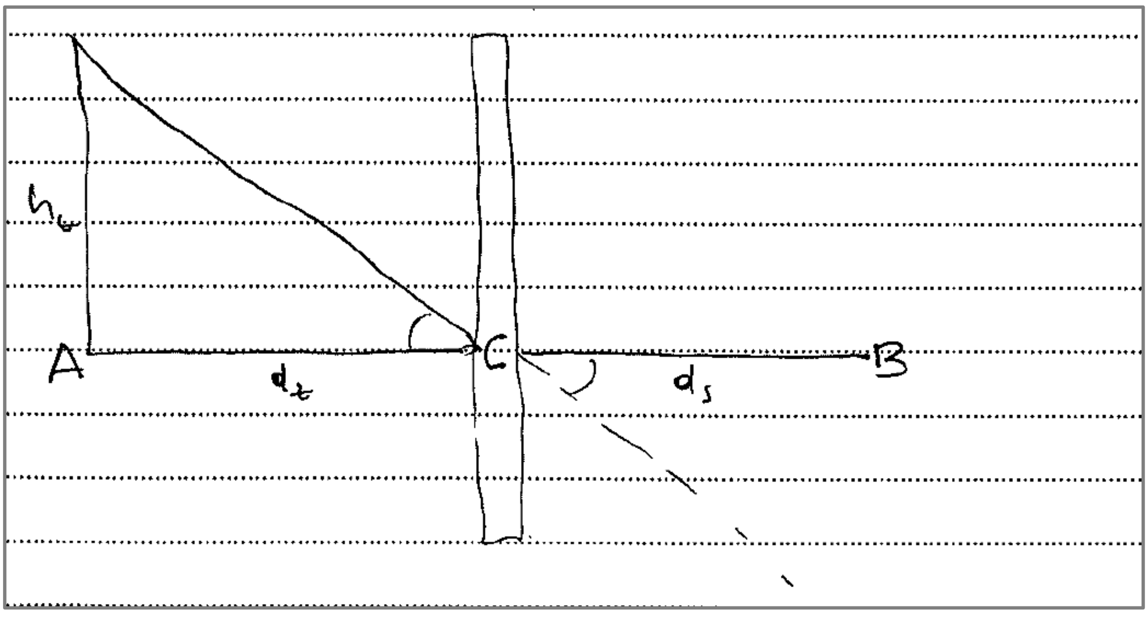
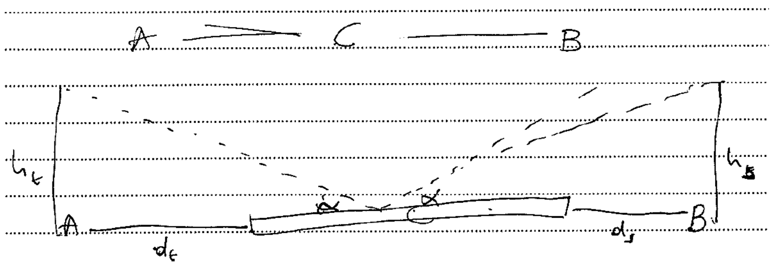


Figure 5.10: P1's attempt to produce a sketch of the second problem

However, the interviewer did not provide any further scaffolding until P1 came up with a visual sketch of what the situation could look like according to the interviewers' first suggestion. It was after the production of this visual sketch that P1 started to link the problem with some mathematical constructs, and with the help of the interviewer she was able to solve the problem. On the other hand, P2 was able to come up with a diagram individually without asking any questions from the interviewer (Figure 5.11). However, immediately after producing the diagram, she called on the interviewer to help with the first problem, to which she was not sure about the solution she had produced.

Table 5.45: P1 laments about drawing a sketch of the second problem after some scaffolding

Speaker	What is said	What is done
I	Try placing the mirror horizontally on the floor.	
P1	I am struggling to draw this diagram if the mirror I placed horizontally on the floor. She attempts to draw the diagram and comes up with the diagram below. 	

Upon clarifying the first problem, P2 admitted that when solving the second problem, she was lost, but she thought Figure 5.11 accurately presented the situation in the second problem. As with the other participants, P2 sought approval from the interviewer as to whether the visual mediator was an accurate representation of the description of the second problem. Even after producing her own visual sketch, P2 could not use this sketch to solve the second problem and required that the interviewer should help.

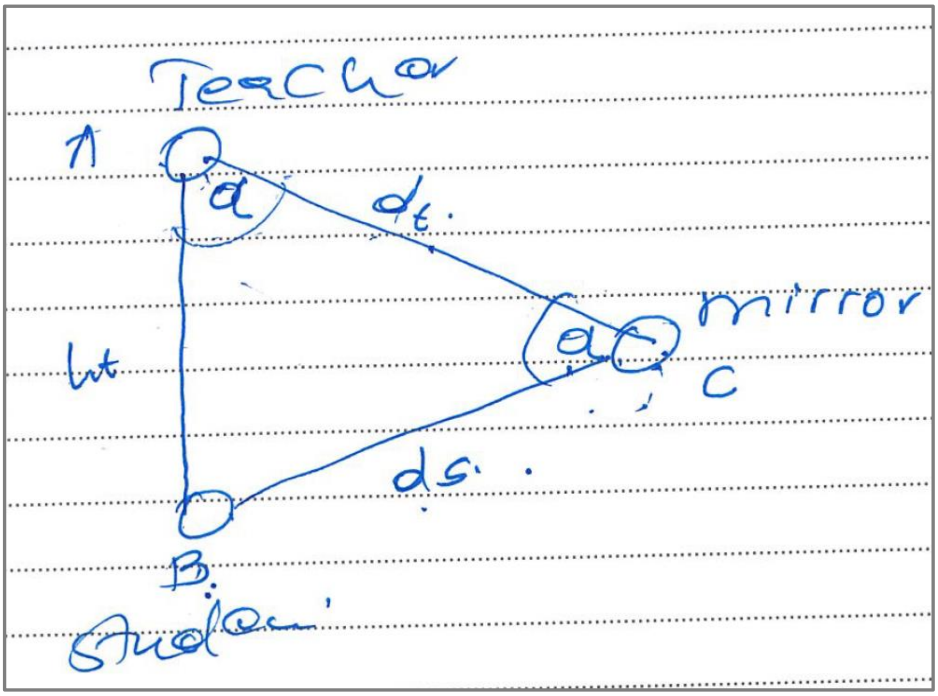


Figure 5.11: P2's individually produced sketch relating to the second problem.

P3, on the other hand, guided by the fact that the teacher should see the top of the students' head, thought that the mirror should be hanging above the teacher and the student at a certain angle as seen in Figure 5.12. Similar to P2, P3 could not use this diagram to solve the problem, but only admitted that the question is difficult and he could not figure out the position of the mirror.

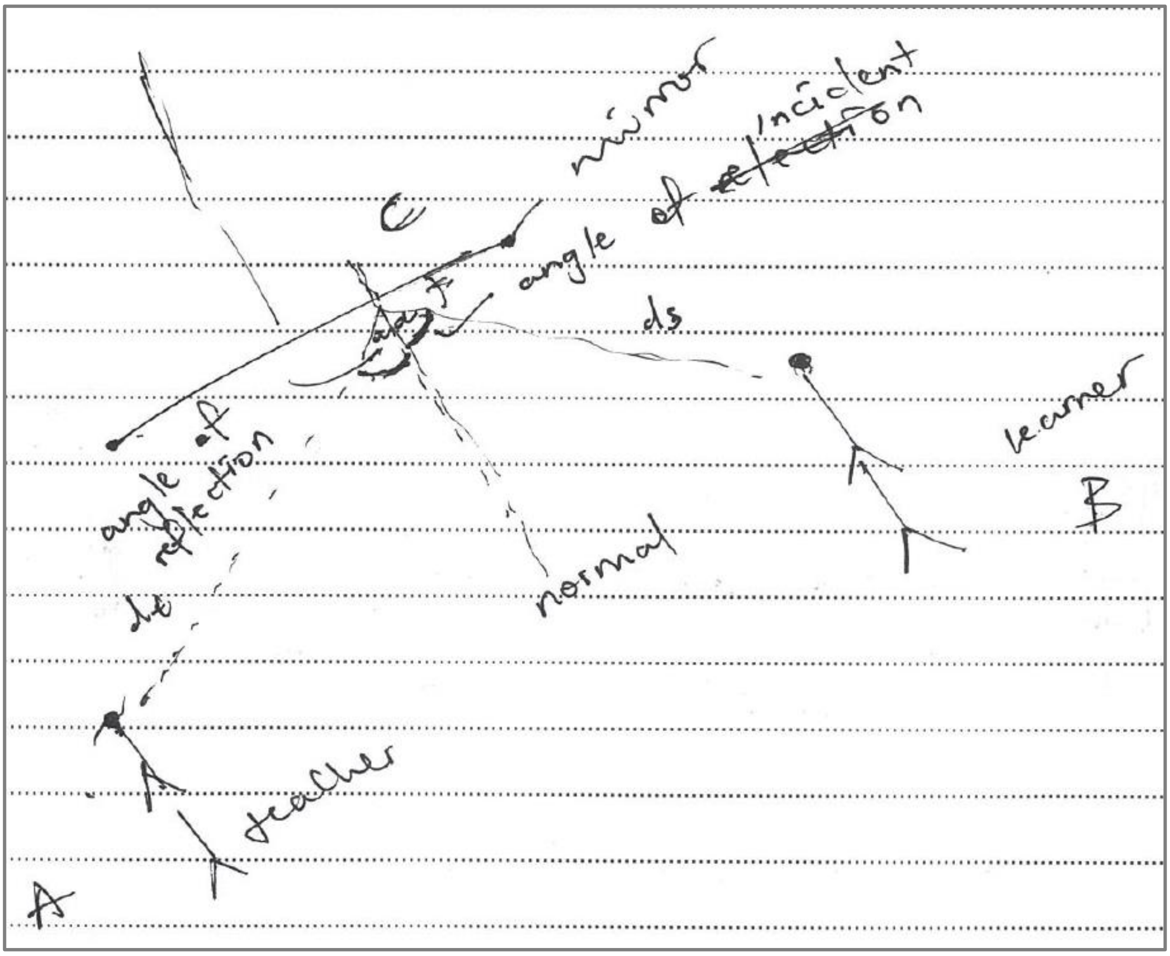


Figure 5.12: P3's representation of the situation in the second problem

P4, on the other hand, could not produce a visual mediator to represent the situation in the second problem until the scaffolding from the interviewer. Even after the scaffolding from the interviewer, P4 could not produce a complete solution of the problem, but used subroutines that needed further partitions to represent the solution to the second problem. Similar to the other participants, P6 was able only to produce the diagram related to the second problem and hence the solution, after some scaffolding from the interviewer.

Since the description of the second problem was given in words without a diagram, all the participants felt that a diagram was the first step, because without the diagram, they could not solve the problem. This was evident from their semi-structured interview responses where they all indicated that a diagram was a requirement to find the solution to the second problem, as seen in Table 5.15. This meant that they had to translate the given description of the problem into a visual representation of the problem, using mathematical properties. Given that only P5 was able to produce a visual mediator for the second problem without assistance from the interviewer, the other five participants struggled with Van Hiele level 2. Specifically, since they were not required to produce an accurate sketch, but a rough one, these participants could not translate the terminology and properties from the description into a diagram. This struggle could be classified as ritualistic, because in their utterances PMTs kept on asking questions from the interviewer before even attempting to produce the rough sketch, meaning that they required approval or scaffolds from the interviewer. A snapshot of such an inquiry can be seen in Table 5.46, where P3 asked numerous questions of the interviewer before attempting to represent the problem visually. Hence, these five teachers who struggled to translate the given description into a visual mediator useful for solving the second problem, struggled with ritualistic Van Hiele level 2.

Table 5.46: P3 asking question before producing a visual mediator for the second problem

Speaker	What is said
P3	Reads the problem twice … we need to draw a diagram. We need to present our information in a diagram.
I	Okay, you do that then.
P3	Reads the problem again. A mirror? Between?
I	Yes between the student and the teacher.
P3	Now the mirror is facing who? Is it facing the kid or it's facing the teacher?

Lastly, I noted also that P5 used a ritualistic method to identify which of the two centres in the first problems was the circumcentre. When asked which one is the circumcentre, he said, "the first one, which is O", which means he relied mainly on the statement of the problem and his identification "the first one" suggested that he did not use the characteristics of the circumcentre to identify it, but relied on the statement of the problem where the circumcentre is identified first before the orthocentre.

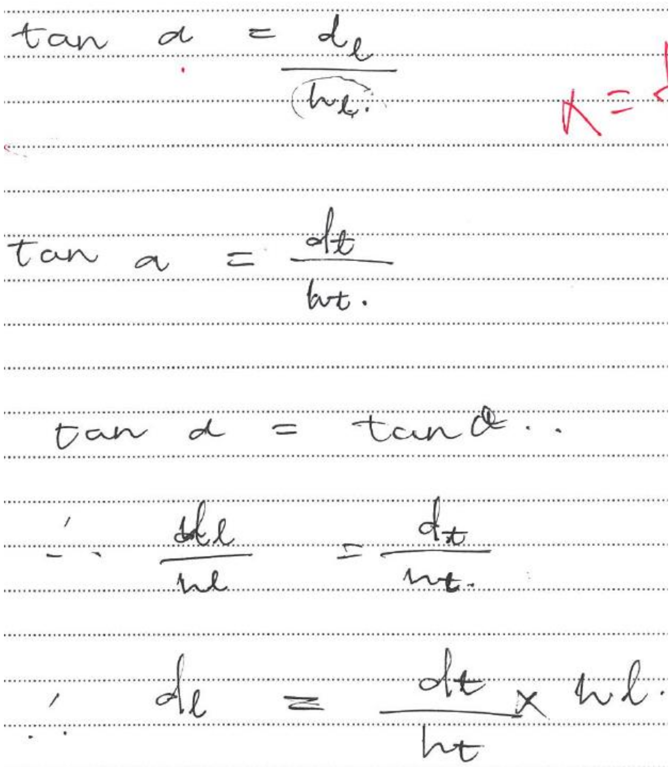
5.4.3 Explorative discourse participation

Explorative discourse is a direct opposite of ritualistic discourse in mathematics and it is linked to the belief that mathematical learning is about exercising thought in endorsing mathematical knowledge without being directly influenced by others. The goal of explorations is to produce endorsed narratives about mathematical objects that will be labelled as mathematical truths. PMTs' discourse participation classified as explorative in this study, was characterised by their focus on using acceptable rules and narratives to produce new narratives about the problem individually (Sfard, 2008). The classification of these discourses was based on the eight themes from Table 5.42 relating to commognition. Data related to explorative discourse participation was limited to PMTs' discourse participation and some justifications of explorations were only evident after some scaffolding.

5.4.3.1 The closing goal for PMTs' discourse

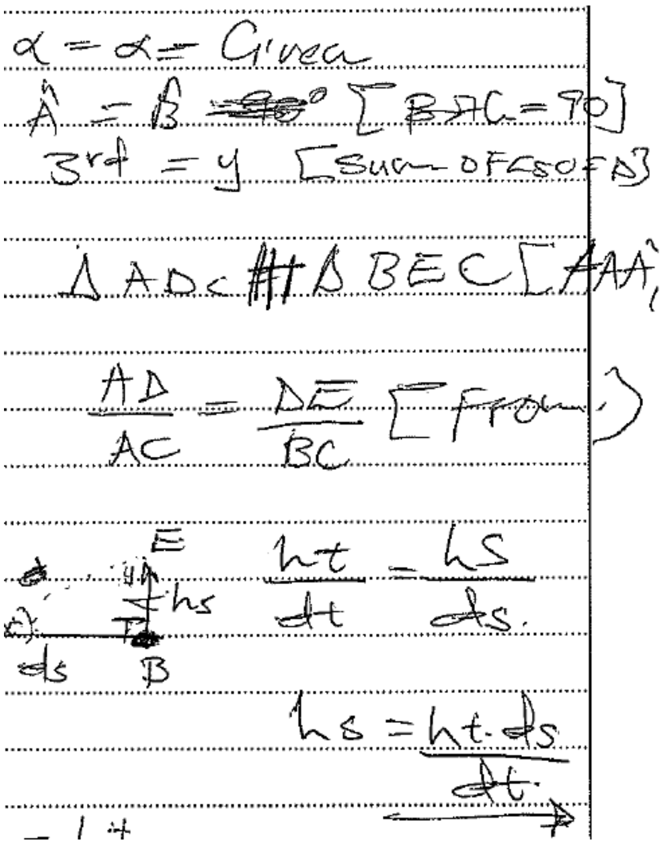
The closing condition of explorations is to produce endorsed narratives about the world, in this study the endorsed narratives about the world includes the usage of routines that contributes to the theory of mathematics (Sfard, 2008). In explorative discourse, participation routines are construction, substantiation and recall (Sfard, 2008). Construction involves the derivation of new endorsable narratives. Substantiation refers to the way in which we decide to endorse or refute constructed narratives. Recall helps problem solvers to remember narratives endorsed in the past, so they can be used in new routines (Ben-Yehuda et al., 2005). The analysis showed that none of the PMTs saw the closing condition as being to produce endorsable narratives about the two questions. Their aim was to produce realisations with the appearance, for example, $'h_s = \cdots'$ for the second problem which was evident in their solutions where they used trigonometry, saying that the solution $h_s = d_s \tan \alpha$ was correct. However, prompts and scaffolds proved to be useful in transforming some PMTs' discourse participation to more explorative ways.

Table 5.47: P3's explorative closing goal about the second problem

Speaker	What is said	What is done
I	Okay, do you think there might be any other method that you can use to solve this problem?	
P3	It might be there, using trig ratios maybe.	
I	Can you try that method and see if you get the same result?	
P3	I may try to use trig ratios.	
I	With the same diagram.	
P3	With the same diagram, yes, I can write using these two sides and those ones and then combining them. I can try to use maybe alpha, this is opposite $\tan\theta$, opposite over, if I use, ehhh, sine, this is going to be sine 90 degrees, okay let me use sine (laughs). Sine $(90° - \alpha)$, ahhh it's wrong because I need to use this one. I need to use tan, okay $\tan\alpha$ equals to d_s over h_l, and then, this side also, I can say also $\tan\alpha$ equals to distance of the teacher over height of the teacher. So now, what are we looking for, the focus is here (h_l), we make this the subject of the formulae and we substitute it there. Eish, but its not there (meaning there is no h_l in $\tan\alpha = \frac{d_t}{h_t}$). Ahhh, gwa tshwana (*meaning it's the same thing*), coming back to this, this ($\tan\alpha$) equals to that ($\tan\alpha$), that means we can equate $\frac{d_l}{h_l}$ with $\frac{d_t}{h_t}$ and then simplify.	Pointing at d_s and d_t. Pointing at h_s and h_t. Pointing at EC in ΔAEC.
		Pointing at $\tan\alpha = \frac{d_t}{h_t}$.

For example, after they found solutions to the problems, participants were asked whether there was another method of solving the problem they could think of at the time. This question prompted PMTs to reflect on their solution and find other ways of solving the problem. Even if their previous solutions were guided by rituals and scaffolding, the new solutions were produced absolutely independently of the interviewer and were substantiated to show that PMTs' main aim was to produce an endorsed narrative about the solution to the problem. P3 and P6 displayed evidence that after the request to produce a different solution, they transformed their discourse participation from ritualistic to explorative. However, in the case of P3, his desire to show the closing goal in a ritualistic way, led to him solving for d_l instead of h_l as seen in Table 5.47.

Table 5.48: P6's explorative closing condition in the second problem

Speaker	What is said
P6	Yes there is this other one, we did it, when the angles are like this what can I do, let's say again similarity or congruency, can we prove congruency here? No, we don't know whether the heights are the same. Now I remember something, I prove similarity then the corresponding parts, the two, let me check with that. 

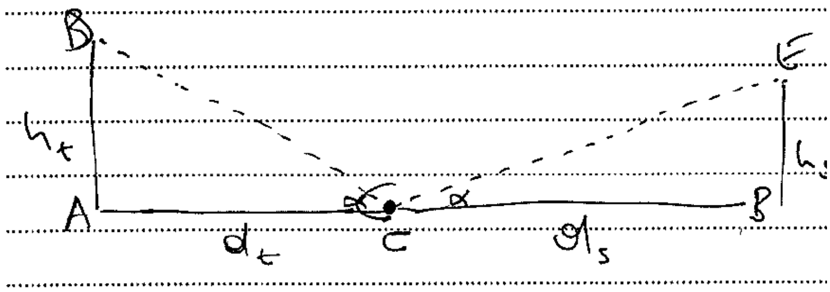
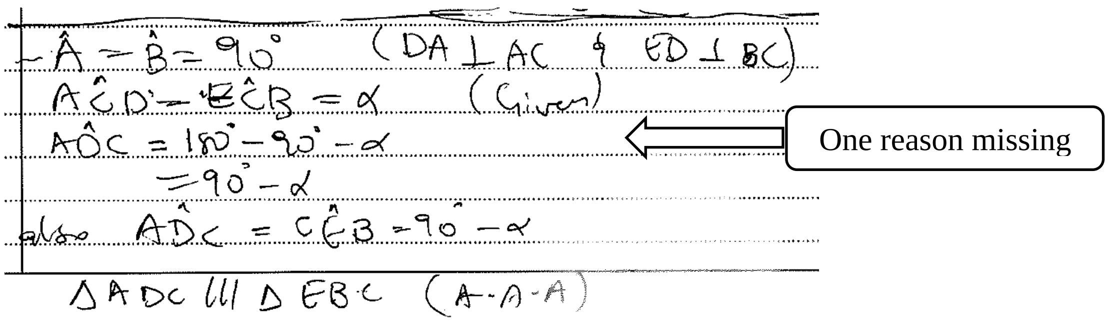
In his explanation of his alternative approach, P6 used some elements of recalling through the uttering of phrases such as “we did it”, meaning in his previous experiences he followed the approach he was proposing. Other phrases such as “now I remember something” indicated that he was recalling something from his previous experiences (Table 5.48). Furthermore, P6’s proof construction and substantiation were performed independently, which indicated that his discourse participation had transformed throughout his problem solving of the two problems from ritualistic to explorative. As argued before (Lavie et al., 2019), the prolonged independent routine performance in solving the second problem as an alternative solution approach, indicated that their routines have become fully fledged. Even though their word usage was not fully objectified in the routine performance, the process of constructing and substantiating the routine using recall, resulted in endorsable narratives about the second problem. Even though P6 could produce the proof independently, his word usage showed that he was ritualistic and had an indication of relying on past experiences and memory. Therefore, at that moment he was operating at a ritualistic Van Hiele level 3.

5.4.3.2 Who performs the routine?

In explorative discourse participation, discussants perform the routines independently without the need for scaffolding from others. The evidence produced in 5.4.3.1 indicated evidence of PMTs who were successful in performing routines completely independent of scaffolding from the interviewer. In an isolated case, P1 expressed the willingness to perform routines independently, but this proved problematic for her and she came back to rely on the scaffolding from the interviewer (Table 5.2). Furthermore, participants who were able to come up with useful approaches to endorsing the final narrative, used subroutines that fed into the overall endorsement of the narrative about the mathematical problem. For example, P1 was able to come up with a proof of similarity between the two triangles in the second problem, which allowed her to use the concept of proportionality to endorse the narrative about the second problem. Even though P1 required some scaffolding with the usage of the notion of proportionality in the second problem, the subroutine of proving similarity which fed into the usage of proportionality, was produced independently, as seen in Table 5.49. P1 was able to come up with a subroutine that fed into the endorsement of a narrative independently, and her discourse indicated that she believed that the two triangles were similar without giving any substantiation, but relied on the visual appearance of the triangles. This was evident from her utterance “we can use what, similarity? Because this triangle is more similar to this one” from Table 5.49. In this quotation, P1 did not give any substantiation until she was asked to prove the similarity, when she uttered, “yes, angle, angle, angle”. Even then, she gave the conclusion of angle, angle, angle (A.A.A) without providing a proof, meaning the visual appearance of the diagram was enough to convince her about the truth about the similarity of the two triangles. This corroborates my earlier observations that P1 in most instances was operating at Van Hiele level 0 where the visual appearance of mathematical objects is used to draw conclusions.

Furthermore, the discourse used by P1 in this case was not fully alienated and still contained the combination of word usage from colloquial discourse and the physical gestures of pointing to mathematical objects, instead of referring to them using mathematical keywords. Similar characteristics were observed from P3, who refused to prove similarity when solving the second problem, because it was obvious that the two triangles were similar. As can be observed from Table 5.27, P3 did not produce the proof, but used the truth of the proof to endorse narratives about the second problem. The failure to produce the proof of similarity in this case, because it was obvious, indicated that personally convincing visual appearance of diagrams were accepted as proof by P3. This was also the case with P5, who accepted that triangles were congruent based on visual appearance, and did not provide a proof, because, as he said, it was “obvious”. Both P3 and P5 felt no need to write proofs of similarity and congruency, perhaps because they considered the interviewer as a knowledgeable person who would see the truth of the proof without it being provided. The struggle with Van Hiele level 0 and the usage of discourse that was not fully alienated, indicated that P1, P3 and P5 were operating at ritualistic Van Hiele level 0. Furthermore, in the production of the proof, P1 missed one substantiation as indicated in Table 5.49, which required thinking at Van Hiele level 3 and this was evidence that if a student struggled in the preceding Van Hiele levels, success in the next levels was scarce, near to impossible. This was, however, an indication that P3’s usage of the truth about the similarity of the two triangles was used in an explorative way symptomatic of explorative Van Hiele level 3, as seen in Table 5.41. However, like the other participants, due to the dominance of ritualistic discourse in P3’s routine performance, I concluded that this endorsement must be classified as ritualistic Van Hiele level 3.

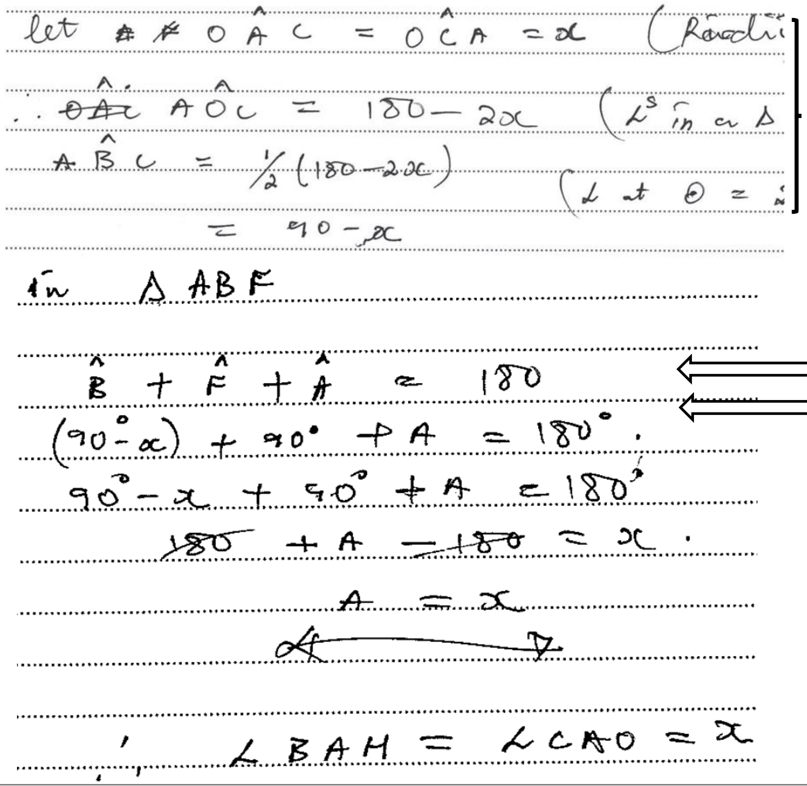
Table 5.49: P1 coming up with a subroutine that fed into the endorsement of the narrative independently

Speaker	What is said	What is done
I	Think of another method we can use.	
P1	Let's say we use the theorem of Pythagoras, we cannot link the two triangles?	
I	Since you think so, what other method can we use that can help us link the two triangles?	
P1	We can use what, similarity? Because this triangle is more similar to this one. 	Pointing at ΔABC and ΔBEC.
I	Okay, can you prove that these two triangles are similar?	
P1	Yes, angle, angle, angle	
I	Okay, prove it. 	

Even though there was an interchange between the interviewer and P1, none of the utterances from the interviewer could be thought of as having directly influenced P1's end product. She produced it completely by herself. Again, in the case of P1 as seen in Table 5.49, the construction and substantiation of the narratives were occurring individually without the need for the interviewer to scaffold, prompt or ask questions. There were select incidents where both P3, P5 and P6 were able to perform some routines independently, but could not finish the whole routine without scaffolding from the interviewer. For example, after P3 was scaffolded with which theorem to look for in solving the first problem, he was able to perform the routine independently, as seen in Table 5.50. This showed that the routine performance was not fully explorative, but the scaffolding from the interviewer elicited explorative discourse participation from P3, as seen in Table 5.50. What was of extreme importance in this performance of routines, and perhaps evidence of adhering to procedures, was that even though P3 started off his proof substantiating all his steps, in the second part of his proof P3 did not substantiate his statements, as seen in Table 5.50. This failure to substantiate his assertions while performing the routine perfectly following procedures, was indicative of operating at ritualistic Van Hiele level 3. This was because even though the routine was flawlessly performed, it seemed that P3 did not want to make mistakes in his performance so that he could obtain the answer and maintain harmony with the interviewer, while also obtaining some social acceptance and praise.

Table 5.50: P3 production of an endorsable narrative about the first problem

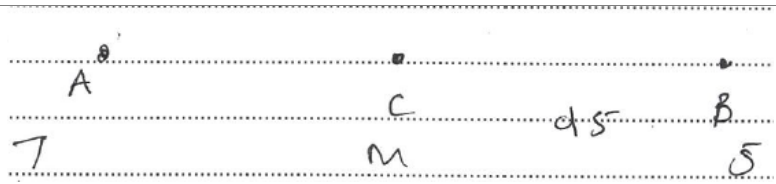
Speaker	What is said	What is done
I	Okay maybe think about the theorem of angle at the centre is twice the angle at the circumference and see how it can help you.	
P3	Okay, ohhh there it works. If this was drawn there for me then I was going to get it.	
I	But it was already there you just could not see it.	
P3	Okay then this is going to be twice that, so this one is going to be $2(180° - 2x)$, no, it's going to be $\frac{1}{2}(180° - 2x)$, which is simplified into $90° - x$.	
I	Okay.	
P3	Okay we let, huuh, I am letting angle, $\angle OAC = \angle OCA = x$, and then, but I have ran away from my approach, because that was about similarity. I left it, so now I am into using angles, naming the angles, this is another approach that I can use. So, the angle at the centre, that is $\angle ABC = \frac{1}{2}(180° - 2x)$. And then I simplified it to $90° - x$. Okay, now since this is 90° then this is x.	Pointing at $\angle AOC$ and $\angle ABC$. Pointing at $\angle AFB$ and $\angle BAH$.
I	Which triangle are you working with now?	
P3	ΔABF.	
I	Okay, continue.	
P3	Then it is going to be $\angle B + \angle F + \angle A = 180°$.	

Speaker	What is said	What is done
		All assertions substantiated correctly. Substantiations missing.

5.4.3.3 *Audience of the routine*

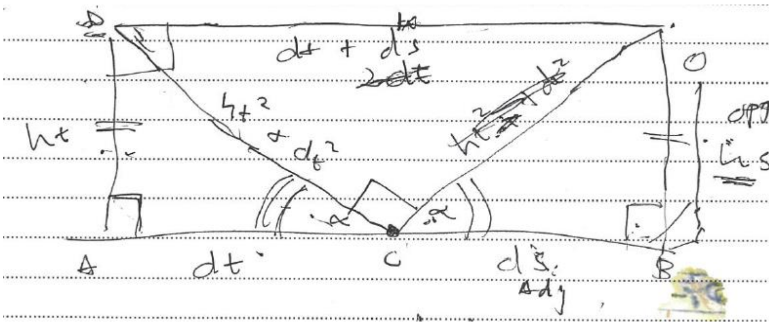
The difference between internally persuasive and authoritative discourses is not a scientific one. Where a discourse that can be internally persuasive for one person, it can be authoritative for the other person (Cooper & Selfe, 1990). Here I took the inside and outside perspective (Sfard, 2008) to analyse PMTs' intended audience of their discourses closely and critically. Upon a careful and rigorous examination of the data, very little of the PMTs' utterances or discourse participation could be linked to internal persuasions towards solving the problem. This indicated that PMTs focused mainly on obtaining the answer to the problems through the usage of metarules produced by the interviewer or those from their previous experiences. An internally persuasive discourse allows PMTs to communicate about what they are thinking during problem solving (Cooper & Selfe, 1990), and PMTs definitely did communicate what they were thinking, especially about the position of the mirror in the second problem. Their discourses were about attempting to find the perfect position for the mirror, so that the problem could make sense to them. However, it seemed that their need for individual appropriation was dependent on whether the interviewer agreed or disagreed with them. They could not produce substantiations for their thinking. Instead, when the interviewer questioned them or disagreed with their decisions, they simply changed and looked for different approaches. This showed that their discourses were not internally persuasive, but authoritative in that they required validation from the interviewer before endorsing their narratives. The only case of a discourse that showed internal persuasion was observed from P5, when producing the diagram for the second problem. After reading the statement given in the second problem, P5 began drawing a diagram of the situation, but was sceptical about some of the details given in the statement and how to express these in a diagram, as seen in Table 5.51.

Table 5.51: P5's first attempt to produce a diagram about the second problem

Speaker	What is said	What is done
P5		*Keeps quiet again, and laughs*.
I	You are trying to draw the diagram?	
P5	Yes, the teacher will move something like up, but that reflection of the mirror, it must show the head.	

After the interviewer told him to turn the problem into a mathematics problem and solve it, P5 was able to produce a correct diagram individually using recall, substantiations and construction, as seen in Table 5.52. Even though the discourse in Table 5.52 could be mistaken for an internally persuasive discourse, it might have not been possible for P5 to produce this diagram if the interviewer had not suggested that he should turn the problem into a mathematical one to solve it.

Table 5.52: P5's second attempt to produce a diagram about the second diagram

Speaker	What is said	What is done
I	You need to try and turn that into a mathematics problem and solve it.	
P5	What I know about the angle of reflection and angle of incidence it is something like this, then if they say it is equals to each other then it means the angle here is equal to the angle there, and the distance here, this is ds, AC is d_t, and the height of the teacher is h_t. What is the equation that can be used to calculate the height of the student? 	Starts to draw diagram 2. Indicating the angles by equal number of arcs.

In Matusov and Von Duyke's (2009) appropriation of internally persuasive discourses, P5's dialogue that led to the production of the endorsed narrative in Table 5.52, could be dialogically tested, but is not forever testable, because his utterance relied mainly on recalling metarules about mathematical objects that had already been endorsed by the mathematics community. As was evident, PMTs used authoritative discourse by relying on the metarules of others (especially the interviewer), their discourse was not modified to suit their individual needs and purposes as they mainly relied on the presence of some authority to validate their ideas and problem-solving attempts. Hence, PMTs' discourses was clearly meant only for others and not themselves.

5.4.3.4 Level of applicability of procedures

During their problem solving, most PMTs did not use any procedure that was applicable in different problem-solving situations, but followed situated procedures. An explorative level of applicability of procedures is characterised by the independence in the choice of applying procedures based on whether they are most likely to lead to endorsable outcomes (Sfard, 2008). Only P5 suggested the usage of "trial and error" which is a problem-solving strategy that is applicable to a variety of problems in mathematics, has been in use in mathematical discovery and invention and is likely to lead to endorsed narratives, at least in the students' approaches which symbolise internal persuasion.

Table 5.53: P5's indication of using a problem-solving strategy applicable in a variety of problems

Speaker	What is said
P5	Yes, I want to prove, that what I am trying. And then because of H, H is having these things of altitudes and orthocentre, they cannot give me information which I cannot use. That is why I said I want to prove something here, congruency first.
I	Do you think these triangles can be congruent?
P5	I have 90 degrees which is … Okay there is not enough information. But I will use trial and error, I am going to try it, if it is an error I will try another thing.

5.4.3.5 Level of flexibility in finding the solution

This feature of PMTs' explorative discourse participation was assessed based on two problem-solving behaviours: whether or not PMTs saw other methods of solving the problem besides the produced solution, and whether they could adapt their procedures accordingly to find another solution. These main themes were required to be performed individually, without the intervention of the interviewer during the performance. Given that the problems did not have any explicit requirement of producing different solutions, once PMTs produced one solution they stopped there until the interviewer asked if they could solve the problem in a different way. Hence, all PMTs were asked if they could solve each problem in a different way after finding the first solution. None of them displayed flexibility regarding the first problem, while some displayed flexibility in the second problem. When asked if they could solve the second problem differently, P3 and P6 produced solutions that involved both Euclidean geometry and trigonometry, independent of the interviewer. The different solutions produced by P3 and P6 can be seen in Table 5.47 and Table 5.48, but P3 solved for d_l instead of h_l, which might be a sign of a deeply engraved ritualistic nature in his discourse participation. P1 was able to produce an alternative narrative of the solution to the second problem, but it did not look like the first solution she produced, as seen in Figure 5.13. The solutions in Figure 5.13 were equivalent even though they were structurally different. Instead of equating the two tan alphas ($\tan\alpha = \frac{h_s}{d_s}$ and $\tan\alpha = \frac{h_t}{d_t}$) in the second solution, P1 chose to solve for α in $\tan\alpha = \frac{h_t}{d_t}$ and then substituted the result into $\tan\alpha = \frac{h_s}{d_s}$.

P1's Solution 1 **P1's solution 2**

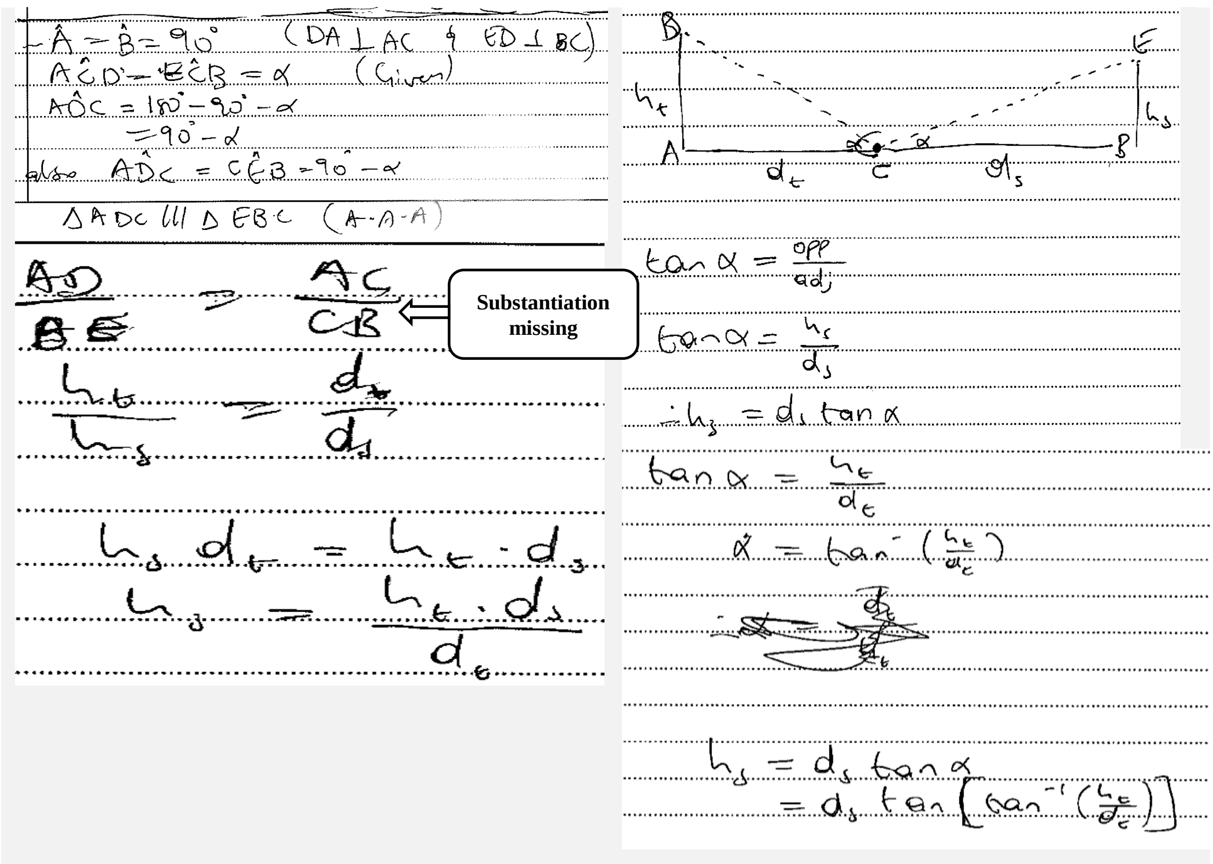


Figure 5.13: P1's two solution approaches to the second problem

Even though these solutions were structurally correct and represent glimpses of flexibility in finding solutions, the lack of reasoning in some solutions (P6), confusing the solving of h_l by solving for d_l (P3), and the application of a procedure of solving for the angles instead of equating equals (P1), they still pointed towards a ritualised nature of discourse participation by these PMTs.

5.4.3.6 Level of correctibility of the narratives and the routines

PMTs' narratives and routines in the current study did not show any explorative behaviour about how they corrected their narratives and routines, which was meaningful in terms of their discourse participation and learning. Firstly, this represented the profound nature of their ritualised discourse participation, meaning that even though PMTs were able to find the correct narratives about the two problems, their performances were governed by following strict rules and if the procedure went wrong, it was discarded and PMTs looked for other procedures that could yield their desired solution. Hence, instead of their discourse being impervious to any considerations from the interviewer, PMTs showed that it was highly influenced by others.

Secondly, while it seems that the goal of mathematics in schools is to produce an endorsed narrative about mathematical objects (Sfard, 2008), PMTs' problem-solving activities revealed that they focused on using algorithms to produce correct answers, leaning towards the mastery of the 'how' of a routine instead of the 'why'. Lastly, the fact that PMTs could not recognise errors or correct any of their narratives without the aid of the interviewer, was a sign that PMTs continually tried to look for previously learnt procedures and algorithms to apply in the new situations, without attempting to derive new narratives that could be helpful in solving the current problems.

5.4.3.7 Acceptability and endorsement of PMTs' narratives

This section examines the explorative reasons for accepting narratives and routines, that is, the acceptance of a narrative or routine can be clearly substantiated by PMTs and it is independent of outside authority or influence. It has been argued earlier in this chapter that PMTs' acceptance and endorsement of narratives was highly dependent on the interviewer's approval with utterances such as "tell me if I am wrong" (P2), symbolising that PMTs required some approval from the interviewer. The way in which PMTs endorsed their narratives was limited to following strict procedures and recalling of information, and where both routine procedures and recalling of information failed, PMTs were stuck. However, there was an explorative nature of accepting and endorsing sub-narratives that could feed into the endorsement of the entire narrative about the problems. For example, those PMTs who were able to identify the circumcentre in the first problem, used a sustainable discourse to construct and substantiate narratives about the radii of the circle. For example, P1's utterances in Table 5.3 and Table 5.39 about the radii of the circle and the characteristics of the triangle, were unsolicited by the interviewer. She immediately spotted that lines OC, OA and OB were radii and she used that to further her routine by identifying the equal angles of the isosceles ΔOAC. Even though the production of this sub-narrative did not lead to the independent endorsement of the narrative about the objects in question in the first problem, it later fed to endorsing the narrative about the objects in question, that $\angle BAF = \angle CAO$. Similar behaviour was observed from P3, who explicitly stated that $\angle BAO = \angle ABO$ and he provided the reason based on the circumcentre of the circle (see utterances by P3 in Table 5.39). P6 also showed that he was convinced about the radii and the circumcentre in his discourse about these mathematical objects. He mentioned:

> *Okay, I named them correctly earlier, now I see that the orthocentre is this one (pointing to H) and the circumcentre is this one (pointing to O). Because O is also the centre of the*

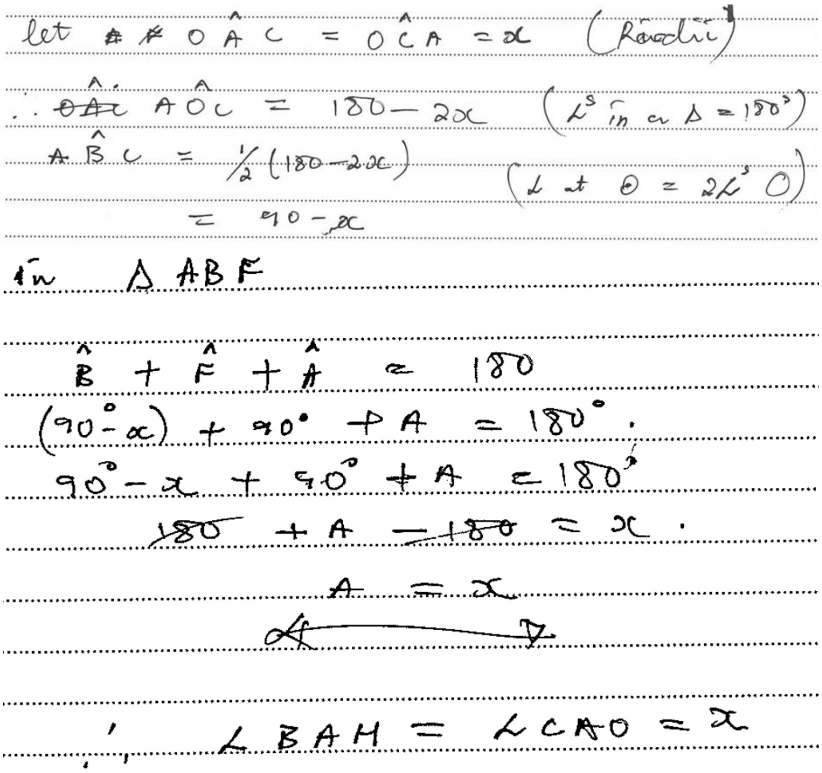
circle it means this one is equal to this one (pointing to AO and OC), these are radiuses. So now this angle will be equal to this angle (pointing to $\angle OAC$ and $\angle OCA$) if we say these are radiuses so angles opposite equal sides.

This is further proof of how explorative PMTs were in their discourse about the mathematical objects of the first problem to produce subroutines that could feed into the endorsement of the narratives.

In some cases, indirect scaffolding ended up opening windows for PMTs to remember routines and use them correctly as if they were on their own, as seen in the case of P3 in Table 5.14. Once the interviewer scaffolded P3 about the theorem mentioned in Table 5.14, he went on to produce the correct answer independently, as seen in Table 5.53. I am highlighting this move to explorative discourse participation, because of arguments that the road to exploration begins with routines (Lavie et al., 2019; Sfard, 2008). Hence, as seen in the case of P3, the provision of the earlier narrative allowed him to find the solution to the answer independently.

Table 5.54: P3 endorsing narratives independently using scaffolding given by the interviewer

Speaker	What is said	What is done
P3	Eish, this is killing me. Let me see.	Keeps quiet for a while.
I	Okay maybe think about the theorem of angle at the centre is twice the angle at the circumference and see how it can help you.	Pointing to $\angle AOC$ and $\angle ABC$ while indicating in the diagram.
P3	Okay, ohhh there it works. If this was drawn there for me then I was going to get it.	
I	But it was already there you just could not see it.	
P3	Okay then this is going to be twice that. So this one is going to be $2(180° - 2x)$, no, it's going to be $\frac{1}{2}(180° - 2x)$, which is simplified into $90° - x$.	Pointing to $\angle AFB$ and $\angle BAH$.
I	Okay	
P3	Okay we let, I am letting angle, angle OAC equal angle OCA equal x and then, but I have ran away from my approach, because that was about similarity. I left it, so now I am into using angles, naming the angles, this is another approach that I can use. So, the angle ant the centre, that is angle ABC equals $\frac{1}{2}(180° - 2x)$. And then I simplified it to $90° - x$. Okay, now since this is $90°$ then this is x.	

Speaker	What is said	What is done
I	Which triangle are you taking (working with) now?	
P3	Triangle ABF.	
	Okay, continue.	
	Then it is going to be $\angle B + \angle F + \angle A = 180°$, participant continues as indicated in the images below. 	

Even though scaffolding was provided, which directly linked to the endorsement of the narrative about the first problem, the way P3 continued with the performance of other routines was characteristic of being sustainable and independent of influence from the interviewer. P5 on the other hand, did have some moments of explorative endorsement of narratives, as seen in the second problem where he asserted, “but they never told us the direction of the mirror, is it standing? But standing there’s no way that you can see on the other side”. In this statement P5 confidently ruled out that the mirror could be standing, based on the reason that if it were standing, then the other side could not be visible. From this statement it was clear that P5 endorsed and accepted this condition without relying on any outside authority, which was symptomatic of explorative discourse participation. Similarly, P6 recognised, “the light cannot protrude through the mirror, either if they want to see each other, it has to bounce and go to the student” and that was symptomatic of accepting mathematical narratives based on sustainable reasoning.

5.4.3.8 How PMTs used words and mediators in their discourse

The explorative use of words and narrative in commognition is characterised by objectivity, that is, word usage symbolises objects that seem to exist outside the discourse, while they symbolise the discursive objects of the discourse (Sfard, 2008). PMTs excessively relied on disobjectified discourse about mathematical objects in this current study, as seen earlier in the discussion on ritualistic discourse participation. This was characterised by their usage of pronouns, such as "this one", together with body movements of pointing to the objects in the figure. In this study, objectified usage of mathematical keywords was characterised by the attachment of the words to mathematical objects and the usage of correct visual mediators. What follows is the discussion of the findings related to explorative word and visual mediator usage.

Despite the dominant usage of phrase-driven keywords and mediator use shown by the participants in this study, there were some nuances between PMTs' ritualistic and explorative word and mediator use. For example, in some cases P1 spoke about mathematical objects by naming the exact angles in the diagram instead of using determining pronouns such as "this" together with the body movement of pointing when talking about mathematical objects. The subtle distinction in P1's discourse, as seen in the quotation in Table 5.54, was her combination of "this line is passing through the centre" with the body movement of pointing, which symbolised that even though P1 was explorative in this regard, the 'usual' dominant ritualistic discourse still had an effect on how she spoke about mathematical objects. Given the scarcity of words from colloquial discourse referring to mathematical objects from P1 in Table 5.54, she used a strategy of assigning variables to particular angles so that she could use those variables to identify the angles. The approach of equating variables to mathematical objects was not only valid in general mathematical problem solving, but also represented that P1 could produce discursive mathematical objects using the properties of objects as seen in the case of the diameter in Table 5.54. A similar act was observed from P3, but in the midst of his routine performance, P3 showed a lack of logical infallibility when he also equated $\angle BAF$ to x. This ritualistic discourse dominated by colloquial discourse and the over-usage of signifiers to identify mathematical objects, indicated that P3 was operating at ritualistic Van Hiele level 1. However, P1 was operating at an explorative Van Hiele level 2 of geometry thinking, and this was explorative because of the dominantly objectified way in which she talked about the mathematical objects. The ritualistic norm of P3's discourse led to further classification of ritualistic Van Hiele level 2 of geometry thinking, even though he independently used visual mediators to extract theorems correctly, as seen in utterances from Table 5.14, Table 5.43, Table 5.50 and Table 5.53.[29] Even though P4 in Table 5.36 used colloquial discourse in describing mathematical objects, she showed understanding of the link between the visual appearance of the angles $\angle CAO$ and $\angle OCA$ and their properties as she later on supported her statement in Table 5.39 by saying, "because, O bare ke circumcentre [O is the circumcentre]".

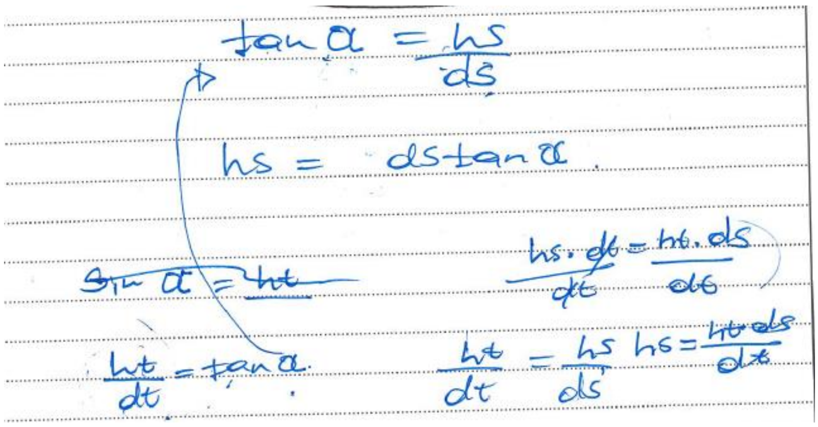
Table 5.55: P1's discourse about angles and lines in solving the first problem

Speaker	What is said	What is done
P1	Let's say I have $\angle DAC = x$ and I let $\angle CDA = y$… okay and this line is passing through the centre, so it is going to be a diameter also. Yeah, and then this is going to be 90°, right?	Pointing to AD

Similarly, in endorsing the second problem through trigonometry which most PMTs approached in a ritualistic manner, P2 seemed to talk about the mathematical objects in an explorative manner. Even though there was evidence of using the colloquial word 'over' instead of the mathematical phrase 'divided by', defining the correct relationship between the mathematical objects (h_t, d_t, d_s and h_s), her entire discourse was intertwined with some indications of explorative discourse such as '... I cross multiply' which again indicated the nuanced way in which some PMTs spoke about mathematical objects. Even though PMTs' discourses contained traces of explorative word and mediator usage, elements of phrase-driven and colloquial usage of words were still visible, as seen in the cases of P1 and P2 in Table 5.56 and Table 5.56. This was the case with all the other participants who in some cases combined both explorative and ritualistic discourse when talking about mathematical objects.

[29] In all these tables I only refer to the same quotation "Okay then this ($\angle AOC$) is going to be twice that ($\angle ABC$). So this one is going to be $2(180° - 2x)$, no, it's going to be $\frac{1}{2}(180° - 2x)$ (as indicated in red on diagram 2), which is simplified into $90° - x$" (P3).

Table 5.56: P2's talk about endorsing the second problem

Speaker	What is said	What is done
P2	Okay, I said h_t over d_t is equal to $\tan\alpha$, neh. And then I went back to my first answer which says $\tan\alpha$ is equal to h_s over d_s, and in $\tan\alpha$ I put h_t over d_t, which will then be h_t over d_t equals to h_s over d_s, and therefore, to find h_s then I cross multiply which will then be h_s times d_t equals to h_t times d_s, and therefore, I divide by d_t both sides which will give me h_s equals to h_t times d_s over d_t. 	

Even though the visual mediators may seem correct as final products, such as the one used by P2 in Table 5.56, the discourse and processes which led to them might not be entirely explorative. In fact, most of the endorsements reached in this study were ritualistic in nature. What seemed to portray an explorative usage of visual mediators was seen in P3's endorsement of the first problem from Table 5.53, specifically in his usage of letters to symbolise the angles he was referring to in his calculation. In the first part P3 symbolised his angles by using three points (letters) with the actual angle being the middle point with a OÂC and OĈA, which identified exactly the lines ($\overline{OA}$ and $\overline{AC}$) which meet at point A where the angle was located, as seen in Table 5.53. However, when he was using the same theorem in a different triangle, P3 decided to use a different notation when he began by mentioning the triangle he was working with ($in\ \Delta ABF$). This mentioning of ΔABF allowed P3 to change the notation of symbols to use a point represented by one letter where the two lines of a triangle meet (the vertex) to represent an angle as seen when he used B̂, F̂ and Â in Table 5.53. In this case P3 would alternate between two different visual mediators with endorsable narratives that referred to mathematical objects, as seen in his combination of explorative and ritualistic discourse participation in Table 5.53, indicated an objectified usage of visual mediators in this case. However, I still want to reiterate the ritualistic nature of P3's discourse that led to the explorations seen in Table 5.53. His utterance "okay, ohhh there it works. If this was drawn there for me then I was going to get it" indicated that he was looking for a way to proceed with the routine, instead of wanting to produce and endorse new narratives.

5.5 CONCLUSION

Thus far in this chapter, I have reported extensively on the data obtain from PMTs' interactions with the two problems from a task sheet and the face-to-face interviews. I have reported on this data with guidance from the tenets of commognition and the Van Hiele levels of geometry thinking. In reporting these findings, I differentiated between PMTs' ritualistic and explorative discourse participation as the main tenets of analysing mathematical thinking during problem solving, proposed by Sfard (2008) in commognition. Furthermore, I used the descriptions of the Van Hiele theory of geometry thinking levels to explain what Van Hiele level of thinking a particular PMT was operating at in a particular isolated activity during his/her problem solving. The link between these two theories is that they both describe thinking during problem solving. Commognition uses the difference between explorative and ritualistic discourse participation, while the Van Hiele theory uses the five levels of geometry thinking. Since the main theory that I used in the analysis was commognition, I grafted the Van Hiele theory onto the theory of commognition, using the descriptions in Table 4.3. This chapter gave the findings based on these theories to explain PMTs' thinking during their geometry problem solving.

CHAPTER 6:

DISCUSSION OF PMTS' THINKING EVIDENT IN THEIR DISCOURSE DURING GEOMETRY PROBLEM SOLVING

6.1 INTRODUCTION

The current section discusses the findings of the study, with the aim of answering the critical research questions posed in the current study. In this study I sought to explore PMTs' thinking discourses, through the discursive lens of commognition, when solving geometry problems. I begin this chapter by giving a synopsis of the findings based on PMTs' discourse participation of the PMTs in both ritualistic and explorative discourse participation under the eight themes of the commognitive framework in Table 3.1. Thereafter, I conceptualise the effects of PMTs' discourse on their geometry problem solving and I discuss the role of visualisation in this process of geometry problem solving. Lastly, I discuss the implications of assessing PMTs' thinking during problem solving for their practice.

6.2 A SYNOPSIS OF PMTS' DISCOURSE PARTICIPATION WHEN SOLVING GEOMETRY PROBLEMS

In the discussion in this subsection, I give a summary of PMTs' discourse participation during their problem solving, according to the main elements of commognition as presented in Table 3.1. Here I give the summary in each of the eight categories of discourse participation through comparing both ritualistic and explorative discourse participation as observed in the data from Chapter 5. This synopsis draws us closer to the nature of discourse used by PMTs to describe their thinking during problem solving and whether or not the dominant discourse could be characterised as ritualistic or explorative. The data analysis in Chapter 4 focused mainly on differentiating between explorative and ritualistic discourse participation of teachers' discourses during problem solving. This analysis, using both the tenets of commognition and the Van Hiele theory of geometry thinking, revealed elements of the data that proved to be useful in answering the research questions.

6.2.1 What is the closing condition?

The closing condition of a routine performance is related to the main reason why the problem solver engages in the process of problem solving. In ritualistic discourse participation, the closing condition for routine performance is to maintain social cohesion with others, while in explorative discourse participation it is to produce narratives that can be used to describe the world (Sfard, 2008). Similar to the findings on learners' discourse participation from Roberts and Le Roux's (2019) study, the data analysis in Chapter 5 indicated that none of the PMTs wanted to produce endorsable narratives about the world during their problem solving. Worries about making errors in their routine performance and agreeing with the interviewers' suggestions without questioning were evidence that PMTs wanted to maintain a good relationship with the interviewer. For example, the acknowledgement to the interviewer that some of the PMTs made about forgetting Euclidean geometry, indicated that even when they made a mistake, it should not dent their relationship with the interviewer, because they forgot the topic. Seeing that the goal of school mathematics is to reproduce endorsed narratives about the world, we expect from aspiring teachers to show this in their problem solving. These aspirant teachers should be able to direct learners' learning towards reproducing endorsed narratives about the world, instead of wanting to maintain social cohesion. In a way similar to ritualistic participation in the discourse, PMTs continually sought affirmation and approval from the interviewer as they solved geometry problems.

6.2.2 Who performs the routine?

Rituals are performed with the aid of others, while explorations are performed individually using some form of thoughtful imitation in their routine performance. When given the problems to solve individually, all PMTs struggled and could not solve the problems, and instead required the help of the interviewer. This means that they all performed their routines with scaffolding from the interviewer, either to kick-start their routine performance or to advance it. The role of tutoring during problem solving has been highlighted before (Wood et al., 1976), specifically as to how the role of the tutor becomes important in shortening the gap in the zone of proximal development (ZPD) (Vygotsky, 1978; Wood et al., 1976). Despite the fact that our routine performance builds on some scaffolding from the past (Sfard, 2008), PMTs' performance of their routines was overly dependent on the scaffolding provided by the interviewer. The overreliance on ritualistic means for performing routines, for these PMTs could be a consequence of their usual mathematical classroom practice, which in its nature was ritualistic, as concluded by Sfard (2017a). These problem solvers' discourse participation was usually characteristic of their own experiences during their learning. Furthermore, this could have repercussions for teachers primarily teaching by speaking about mistakes committed by students (Le Roux, 2017), so that students end up considering these mistakes as lessons, because they have no opportunities to talk about mathematical objects in their classrooms.

There was also evidence that PMTs were able to produce subroutines that were useful in endorsing the narratives about the problem. For example, they could perform the ritual of calculating the sum of angles in a triangle, but they required the help of the interviewer to link it with another theorem for the final answer. Hence, again in this case, they relied on the presence of the interviewer to endorse narratives about mathematical objects. Another observation was that PMTs relied overly much on their memories and past experiences to produce narratives. For example, P4 and P5 relied on the mnemonic "SOH CAH TOA" to calculate h_s in the second question. Even though it may seem that these PMTs could perform routines individually, their performances were overly influenced by a particular ritual, in this case a mnemonic. Furthermore, P3 made it clear that his first approach of looking for similar triangles was inspired by his high school teacher, which symbolised reliance on an outside authority even though such an authority was not present during the problem-solving situation. As a result, these PMTs did not develop their mathematical stories about the narratives of the solutions from the properties of mathematical objects, but relied on their memories and scaffolding from the interviewer (Sfard, 2017b).

As a result, there was no evidence of what Sfard (2008, p. 249) coined as "thoughtful imitation", where problem solvers monitored their decisions about what they can and cannot change in their routine performance of solving a problem. This thoughtful imitation directly relates to what Schoenfeld (1985, p. 27) refers to as "control" during problem solving, where problem solvers decide on actions to solve the problem, and these decisions about what to do during problem solving, are critical to such a degree that they determine the success and failure of finding the solution. Thus, the overreliance of PMTs on others (experience, scaffolding from the interviewer, memory, mnemonics) for which decisions to take during problem solving, indicated that they performed their routines with others instead of doing so individually. Even though scaffolding can in a long run lead to the development of problem-solving competency (Wood et al., 1976), in this study it symbolised that PMTs performed their routines with others.

6.2.3 Who is the audience for this routine performance?

The findings of this study indicated that very little of the PMTs' discourse participation contained internal persuasion, indicating that they performed their routines for others instead of for themselves. PMTs continually used the pronoun ‘they’ which indicated that the question came from some an outside authority which they had to satisfy by finding the answer to the problem. Furthermore, the pronoun ‘us’ in P5’s statement “But they never told us the direction of the mirror …” indicated a complete lack of internal persuasion as he gave this statement as if he was performing the routine with and for others. Furthermore, the utterance “The most thing that I normally do is if I can prove that these two triangles are similar, then automatically the angles will be equal” in P3’s interview, only pointed to a ritual performance, since the adverb ‘normally’ indicated that the routine was performed by imitating others. This imitation was confirmed by P3, who said “… from my teacher that is how I was taught”, which meant he was imitating not only past experiences, but also his teacher. Similar findings could be observed from Roberts and Le Roux (2019), where learners showed that they performed their routines for others instead of for internal persuasion.

Acknowledging that a problem solver’s discourse participation was located somewhere between ritualistic and explorative discourse participation (Baccaglini-Frank, 2021), there were a few instances where PMTs’ routine performances seemed to be explorative. This meant that even though PMTs’ discourse was dominantly ritualistic, there were instances of explorations in their discourse. For example, when P5 individually produced the diagram for the second problem, his discourse contained elements of internal persuasion, even though P5 did not start drawing the diagram before the interviewer guided him towards this action. This again was evidence that these PMTs were mostly reliant on scaffolding from the interviewer. Thus, from the data analysis, it was clear that the audience of the routine was dominantly others as seen from PMTs’ overreliance on authoritative discourse.

6.4.4 What is the level of applicability of the routine?

The applicability of a routine relates to the *when* of a routine, concerning when problem solvers decide to apply a certain routine during problem solving. In this study, the applicability of the routine was linked to ‘who performs the routine’ and what were the ‘closing goals’ of the routine performance. In the current study, there were instances where the closing goals were social acceptance and PMTs performed particular routines following strict procedures. This part of the *how* of a routine meant that PMTs performed the routine following strict procedures and avoided making mistakes so that they could be accepted by the interviewer. In their routine performance, they did not consciously think about the appropriateness of the routine in solving the problem.

Let us take the routine of finding the sum of the angles of a triangle as an example. All the PMTs could apply this routine in the first problem, but they could not use the result of this routine in subsequent steps to endorse other narratives about the mathematical objects in the first problem. Hence, this was proof that the level of their applicability in this case was restricted to finding the sum of angles of a triangle following the correct procedure. This meant that even though the routine was applicable in various mathematical problems, the way in which these PMTs applied this routine during their problem solving of the first problem, was narrow and restricted. This could also be observed in the second problem, where some PMTs produced a narrative $h_s = d_s \tan \alpha$ and concluded that it was the final answer, because it appeared as the answer they were looking for, since the question asked them to solve for h_s. The restrictiveness of this procedure, even though performed correctly, was that it totally disregarded the relationship between the teacher and the student that was depicted in the given statement.

PMTs' discourse participation during their problem solving focused on *what* should be done. For example, in the first problem they had to show that two angles were equal. Thereafter, they focused on how this should be done. In other words, what problem-solving strategy they could use to show this relationship between the angles. This was most evident in the case of P2, who indicated outright that he would prove that the two triangles are similar first, and then from there he could deduce that the two angles were equal. However, he could not do this when he had to write it down. This was evidence that PMTs struggled with deciding *when* a certain routine was applicable. Consequently, after producing a subroutine, the PMTs relied on scaffolding from the interviewer, which came in the form of a question or suggestion, to move to the next phase of their problem solving. In some cases, this scaffolding proved to be futile and PMTs fixated on one method, because they could not answer the scaffolding question. Furthermore, P2, P4 and P6 were observed moving between the two problems. Since they could not solve the first problem, they moved on to the second problem and when they found that the second problem was more difficult than the first (as they stated in their interviews), they moved back to the first one and started asking for clues and scaffolding from the interviewer. Thus, even though PMTs could apply procedures related to the solution of the problem, they struggled with determining when a certain procedure was applicable. Take the example of P1, who made x the subject of the formula during her problem solving, but when asked if this was the best approach, she simply said "I do not know" and changed her approach to make y the subject of the formula (see Table 5.8).

6.4.5 What is the level of flexibility in finding the solution?

The level of flexibility in finding the routines in this study was assessed from both ritualistic and explorative discourse participation. Where PMTs individually could find and substantiate different approaches to narratives about mathematical objects, they were considered flexible in finding the solutions. Furthermore, instances where PMTs could adapt their strategies to endorse narratives about the problems, they were also considered to show flexibility. In the current study, PMTs showed no flexibility in the first problem and only two PMTs (P3 and P6) showed some flexibility when asked if they could solve the second problem differently from their first solutions (see Table 5.48 and Table 5.49). On a similar note, P1 showed flexibility in the second problem with two solutions that were not the same (see Figure 5.13), and was sure that she could use the empirical method to show that these two solutions could yield the same answer. The scenario for P1 was different from that of P3 and P6. It was robust flexibility for two reasons: (a) because she was able to use trigonometry to find a solution different from the first solution, and (b) she used the routine in an unexpected way with clear justifications for the solution and she could substantiate why the two answers were the same using empirical evidence. This meant that she did not rely on the form of the answer, but derived both answers from the properties of mathematical objects instead of manipulating symbols from past experiences.[30] Thus, even though these PMTs might have produced the first solution with the aid of the interviewer, the second solutions were produced individually.

Some PMTs were not flexible in finding their solutions and this could be observed in different parts of their routine performance. For example, when asked about their endorsed narrative about the h_s in the second problem (Table 5.30), P2, P4 and P5 did not even bother to check if the answer made sense according to the problem, but instead believed that the answer was correct. Furthermore, some of these PMTs could not come up with alternative strategies to solve the problem in a different way as a sign of being explorative in their discourse participation. Hence, the overall findings were that these PMTs wanted to follow a particular procedure to get to the solution, which proved futile and then they started to become worried about making mistakes in their closing goals. This then led to them rely on the interviewer to continue solving the problems, because they were scared of making mistakes in their problem solving. They were only able to endorse the second problem after the interviewer scaffolded them from their

[30] Due to the length of the interview the student did not give a substantiation of the two solutions to check if they are equal, but I used empirical evidence to confirm this and indeed the two answers might be different in form, but they yielded the same result.

first subroutine, and as a result, they could not find any other approach to solve the problems. In conclusion, some of the PMTs showed flexibility, which pointed to explorative discourse participation, while some were rigid in their routine performance, which showed that they were ritualistic in their discourse performance.

6.4.6 What is the level of correctibility of the narratives?

Measuring the level of correctibility of PMTs' narratives, meant that I had to assess whether learners made errors in their routines or not. Furthermore, I assessed whether PMTs could recognise and correct their errors after realising the errors or not. For example, P2 read the definitions of the orthocentre and the circumcentre, which was given in the first problem, but could not link the visual mediators in the diagram and the two definitions (Figure 5.4), which was also the case with P4 in Table 5.34. This was a ritual because she could not recognise nor correct this, before the scaffolding given by the interviewer. Furthermore, there were instances where P2 and P5 defined the altitude of the triangle incorrectly and they relied on the interviewer to correct their definitions (Table 5.34). P6 drew conclusions from congruency in similarity and there were several instances from all the participants where proofs, especially regarding the sum of angles in a triangle, were produced without any substantiations. Furthermore, as highlighted in Figure 5.5, P6 drew conclusions about the correctness of the answer based on an incorrect step, and he did not recognise his mistake even after being asked if he was sure about his narrative. All these were signs of ritualistic discourse participation that was evident from the PMTs during their problem solving. There was no evidence of PMTs recognising and correcting their mistakes without scaffolding from the interviewer, which meant no PMT was explorative in the way they corrected their errors when endorsing their narratives.

6.4.7 How were narratives endorsed and accepted?

In accepting and endorsing the narratives of the PMTs in this study, I assessed whether they drew conclusions based on the visual appearance of diagrams, whether they produced narratives that were not endorsable or whether they produced visual mediators that could not be endorsed because of the lack of substantiation. As mentioned earlier in this chapter, when these PMTs were proving the sum of angles in a triangle for the first problem, they were only concerned with producing the correct form and symbolic manipulation without giving the proper substantiation. I highlighted in Figure 5.1 that the lack of the reason "sum of $\angle's$ in a Δ" after the visual mediator $x + 62.69° + 71.00° = 180°$ meant that the proof was incomplete. This can be seen in Table 5.39 where PMTs produced incomplete proofs and went on to use these proofs when endorsing the final answer to the problem. P1, for example, as seen in Figure 5.8, endorsed the narrative based on following procedures without any sustention provided. In another instance, the interviewer provided a scaffold to P1 (Figure 5.9), and she was convinced based on the visual appearance of the diagram in Figure 5.10 that $\angle CDA = \angle ABC$. Besides the visual appearance, P1 might have accepted internally that these two angles were equal, because the suggestion was given by the interviewer whom she held in high regard. Furthermore, P3 produced a proof based on similarity of two triangles without proving that the two triangles were indeed similar, because for him it was "obvious" that the two triangles were similar (Table 5.28). On the other hand, P5 accepted that two lines were equal based on "spirituality". P6 also produced a proof based on incorrect mathematical rules and went on to use it in endorsing the final narrative about the second problem. Thus, from the above summary, it can be seen that PMTs could not locally recognise nor correct their narratives, but followed procedures and ensured that the form of a proof is correct through symbolic manipulation and disregarding substantiations of statements. This was symptomatic of a ritualistic discourse participation.

However, there were glimpses of incidents where PMTs could accept narratives based on mathematical rules instead of other people. PMTs such as P1, who were able to identify the circumcentre in the first problem, were able to use the properties of isosceles triangles and radii to spot equal angles without any scaffolding from the interviewer. Similar behaviour was observed from P3 and P6 about the circumcentre. Sometimes due to scaffolding, PMTs could continue completing the routine on their own providing justifications and substantiations for their routine performance, without the aid from the interviewer. This was particularly the case with P3, who, when scaffolded about a particular theorem (Table 5.16), was able individually to produce an endorsable proof about the first problem (Table 5.53). This provided evidence of Lavie et al.'s (2019) argument that explorative discourse participation develops from some experience with ritualistic discourse participation.

6.4.8 What is the level of objectification of PMTs' discourse?

The manner (language) in which teachers describe actions about the manipulation of mathematical objects is the crux of moving from ritualistic to explorative discourse. Sfard (2017a, 2017b) emphasises this in two of her book chapters: the fact that talking about mathematical objects while using colloquial language or ritualistic mathematical language such as “transpose” instead of “subtract both sides” can be critical in the move towards explorations. In this section, I assessed the level of objectivity in PMTs’ discourse when describing mathematical objects and processes. Here, my analysis focused on how PMTs used words to describe mathematical objects and processes, and also the visual mediators they produced. The first and the most dominant finding was that PMTs used colloquial discourse, visual cues and body movements to describe mathematical objects. They used several signifiers (pronouns) for objects and gestures of pointing to describe mathematical objects. There was a common usage of the pronoun “this” together with the body movement of pointing, to refer to mathematical objects. For example, P1 said “No, this one and that one [are isosceles triangles]” together with pointing at two triangles instead of using mathematical descriptions for the triangles, such as saying “Triangle BOC and triangle AOB are isosceles triangles”. In some cases, PMTs stated theorems by using pronouns and pointing, such as P2 who stated, “if this is twice that, it means that this is also twice the whole angle A”.

Sometimes PMTs used colloquial language that was confusing to refer to mathematical processes, as was seen in the case of P5 who stated, “I want to prove that triangle ABF is the same as triangle ABG” instead of “I want to prove that triangle ABF is congruent to triangle ABG”. P2 also used disobjectified discourse to arrive at the definition of a tangent in trigonometry. What seemed to be the main signifiers of P2’s discourse was the identification of the “opposite” and “adjacent” sides she identified earlier from her mnemonic of SOH CAH TOA. Furthermore, she used pronouns and pointing in her arrival at the narrative of $\tan\alpha = \frac{h_s}{d_s}$ which showed that she participated in this discourse in a ritualistic manner. While the issue of language in SA mathematics education has been intensely debated with calls to introduce learners’ home language in mathematics teaching (Barwell et al., 2016; Planas & Setati-Phakeng, 2014; Setati, 2005), commognition believes that sticking to mathematical descriptions for properties of mathematical objects, is crucial in advancing explorative discourse participation (Sfard, 2008, 2017b). Even though PMTs’ experiences with mathematical and geometry discourses in particular, could be considered to be sufficient for them to engage in objectified discourse, this was not the case, as was evident in most of their discourse.

In the current study, PMTs’ discourse about mathematical objects was not objectified, but phrase-driven and characterised by an overreliance on colloquial discourse, accompanied by the body movement of pointing. They did not sound like people who were telling stories about mathematical objects, because they did not emphasise this in their discourse. Instead, they sounded as if they wanted to find the perfect strategy to manipulate certain marks in a paper to achieve a particular goal (Sfard, 2017b). This was perhaps a reason why the school of formalism failed to explain the foundations of mathematics, because their concern was more focused on manipulating mathematical symbols in a paper using some formal mathematical structure (Snapper, 1979), instead of finding ways to make sense of mathematical objects. As Sfard (2017b) and Le Roux (2017) acknowledge, the prolonged exposure to mixing colloquial discourse with mathematical discourse is engraved, so that as learners proceed in their grades, they consider this 'teacherese' language the only means of thinking in mathematics. In particular, most learners use this teacherese language to identify which ritualistic procedures they should apply in solving particular mathematical problems, and if this kind of language was not in the question, then they were stuck. This was evident in the participants of this study as they struggled significantly with both the problems: the first one because of the usage of the words orthocentre and circumcentre in the question, and the second one because of the unavailability of a diagram. There was no complete evidence pointing to objectified word usage from the PMTs in this study. However, their discourses were characterised by limited nuances of describing mathematical objects based on their properties and not just colloquial discourse.

6.3 CONCEPTUALISING THE EFFECTS OF PMTS' DISCOURSE PARTICIPATION ON THEIR GEOMETRY PROBLEM SOLVING AND THE IMPLICATIONS

From the discussion in 6.2 and the findings that were presented in Chapter 5, it was clear that these PMTs dominantly operated with a ritualistic discourse during their problem solving. This ritualistic discourse participation could be the main cause of these PMTs struggling in their problem solving. From the observation from PMTs' problem solving, it was clear that they could perform basic procedures like calculating the sum of angles in a triangle or using the definition of a tangent to find an unknown in a right-angled triangle. However, these PMTs became stuck afterwards because they did not know the purpose of producing those narratives, but they applied them because the narratives were self-evident and the PMTs could use their past experiences to complete these narratives. This means that these PMTs knew *how* to apply the routines, but they did not know *when* to apply it and *why* they were applying it. This speaks to Sfard's (2008) thoughtful imitation and Schoenfeld's (1985) control, and also metacognition.

These three concepts are important during problem solving because they regulate decisions about when and why particular routines are performed. Indeed, this confirms Lavie et al.'s (2019, p. 156) argument that the individualisation of routines require that students "reasoned decision making". Thus, one of the causes of PMTs struggling in their problem solving, was because they could not exercise thoughtful imitation during their problem solving. One implication of this in problem solving is that ritualistic discourse participation can limit success in geometry problem solving due to its rigidity in application and the flexibility of the routine.

Another consequence of ritualistic discourse participation that was dominant in this study, was that PMTs relied mostly on the scaffolding provided by the interviewer to kick-start and complete their routines. In some cases, the scaffolding provided by the interviewer was useful, and in some cases, not as useful, which meant that those PMTs who were able to use the interviewer's scaffolding to proceed in their problem solving, like P3 and P5, had already begun to recognise the bondedness of the steps in the routine (Lavie et al., 2019). However, those who still could not use the provided scaffolding remained stuck in their problem solving until more and more scaffolding was provided. This reliance on the scaffolding by the interviewer, which was dominant in this study, meant that the routines were performed with others, and as argued earlier, indicated the lack of thoughtful imitation from these PMTs. Despite the fact that rituals and ritualisation is a necessary component of learning mathematics (Coles & Sinclair, 2019; Lavie et al., 2019; Nachlieli & Tabach, 2019), these rituals must always be moving towards explorations, otherwise only rote learning will occur. Seeing that these PMTs mostly remained in the domain of rituals with scarce explorations in some of them, it can be concluded that their schooling experiences were not efficient to ensure complete de-ritualisation (Lavie et al., 2019). Hence, in order to emancipate PMTs from scaffolding during geometry problem solving, their future learning experience must attempt to move their ritualistic discourse participation towards an explorative one (Heyd-Metzuyanim et al., 2019). This emancipation will allow them to be flexible in solving geometry problems, instead of fixating on one method, as some of the PMTs alluded to the fact that fixating on one strategy limited them from finding the solutions to the problems. For example, P5 mentioned "the assumption … later made my mind to be fixated on thinking that the height of the student is the same as the height of the teacher", while P6 stated "… but then I was already fixed on something else to say I can answer it in this way".

Since some PMTs struggled with flexibility during their problem solving, it meant that they also struggled with endorsing correct narratives and correct incorrect ones. Consequently, they ended up endorsing narratives from incorrect realisations, because all they were concerned about was the form of the answer instead of its suitability to the question. They could not even check if their endorsed narrative was indeed suitable to their answer, because of their lack of flexibility in their problem solving. Given that PMTs kept on relying on their previous experiences to produce narratives they already knew about the mathematical objects, they did not produce any narrative that was new to them individually. Thus, according to Nachlieli and Tabach (2019), they were not explorative in their problem solving, but performed routines for social acceptance, which was evident from questions such as "Tell me if I am wrong" (P2). Flexibility is not only critical in problem solving for teachers, but also in the design or selection of mathematical tasks for learners with which to engage in their classrooms (Heyd-Metzuyanim et al., 2019). This implies that if teachers cannot show flexibility in their problem solving, it might be difficult for them to be flexible in the activities of designing or choosing mathematical tasks.

In her letter to the teacher, Sfard (2017b) gives arguments and two principles using an example from algebra regarding why teaching and learning of mathematics should be centred on explorative discourse participation. She goes on to explain the distinction between how one speaks about mathematics in ritualistic and explorative discourse participation, is not a matter of "language" of the enculturation to mathematical objects and processes (Sfard, 2017b), even though language influences thinking (Mudaly & Rampersad, 2010). PMTs' words and mediator usages were rife with phrase-driven words, colloquial discourse and the body movement of pointing to specific objects they referred to in the diagram. For example, when asked which one was the circumcentre, P5 answered "This one, the first one", and then he pointed at O in the diagram for the first problem. Another example is this utterance from P1:

> *Okay, I know <u>these are radiuses</u> of the circle, meaning that AC is the chord of the circle, then ΔAOC is isosceles. Therefore, <u>this angle is going to be equal to this one.</u>*

Clearly, the underlined statements in the above quotation represent talk about visual objects and the conclusion about the equality of angles was also made using visual descriptions. Instead of the objectified discourse "I know that AO, CO and BO are radii", P1 stated, "these are radiuses" which symbolises mathematising about visual mediators as extra-discursive entities. The level of objectification in this quotation was represented by "AC is the chord of the circle, then $\Delta\mathrm{AOC}$ is isosceles". Hence, PMTs' discourse contained nuances of objectification, but were dominated by phrase-driven and colloquial discourse with the usage of, what Radford

(2003, p. 41) refers to as “semiotic means of objectification”. The domination of ritualistic discourse with nuances of explorative discourse might be a consequence of being given very few opportunities to participate in fully-fledged discourse. However, the nuances of explorative discourse indicated that over time these PMTs’ discourse could be slowly transformed from ritualistic to explorative. As asserted elsewhere (Lavie et al., 2019), over time one’s rituals could develop into explorations such that one talks about mathematical objects instead of their concrete realisation. Once this stage is reached, a person is said to be using objectified discourse and can then derive problem-solving strategies and endorse narratives based on the properties of mathematical objects.

In conclusion, the dominance of the ritualistic discourse meant that PMTs’ routines were restricted in their applications, and they were not flexible. This could explain why PMTs struggled to solve these two problems, because they did not have any memory or past experiences that related to the problems. Furthermore, even when scaffolded, PMTs continued to make errors in their routine performances that they did not realise were incorrect and they used these incorrect steps in their procedure to find the answer. In the subroutines that they produced, PMTs followed strict procedures while talking about mathematical objects using colloquial discourse and visual cues of pointing. The main implication of this was that PMTs’ reliance on ritualistic discourse limited them from thinking of ways they could use to solve the problems instead of trying to remember past experiences related to the problems.

6.4 CONCEPTUALISING THE ROLE OF VISUALISATION ON PMTS’ PROBLEM SOLVING

Zimmermann and Cunningham (1991) define mathematical visualisation as the delicate process of forming abstract, concrete or dynamic images for decoding information during mathematical activity. This visualisation is critical in geometry problem solving as it is the first level of geometry thinking (Van Hiele, 1985/1959). There is no doubt that visualisation played a huge role in how PMTs from this study communicated about mathematical objects and how they represented mathematical information given in words. Through seeing a visual diagram (Problem 1) and imagining a visual diagram after reading a statement (Problem 2), PMTs began to engage their visualisation and visual thinking. The visualised visual mediators began to point towards a certain method of talking about the mathematical objects for PMTs in the current study. Visual mediators are critical elements of communicating about mathematical objects and eliciting visual thinking. In communicating about mathematical objects, PMTs from the current study used visual mediators as means to point at when conveying their messages. For example, they would use the pronoun 'this' together with the action of pointing to either an angle or a line, instead of using the objectified discourse of naming the angle or the line using its visual properties such as "line AO is a radius". However, this does not tell us more about PMTs' visualisation skills besides that the visual narratives in the two problems contributed to PMTs' ritualistic discourse participation.

Zooming in on the actions and utterances from PMTs after they have observed a particular image, can reveal what each PMT was visualising at the time. For example, upon identifying the radii in the circle in the first problem, P1 immediately represented the equal angles with the arc in the diagram as seen in the first diagram in Table 5.7. This means that the realisation of radii elicited the memory of isosceles triangles as she stated, but her visual mediator of the arcs in Table 5.7 was incorrect. Furthermore, P1 used visualisation to interpret theorems and related them to the diagram, which meant that the realisation of a theorem elicited a meant image of the theorem in P1 from her previous experiences. This form of visualisation transformed P1's thinking from concrete to abstract, which is critical in mathematical thinking and problem solving. This means the PMTs who acted like P1, could relate theorems with their visual representations and vice versa, which is an indication that they were operating at explorative Van Hiele level 2. However, there were instances where some PMTs could not link definitions of mathematical objects with their visual appearance, indicating that they were operating at a ritualistic Van Hiele level 0.

Furthermore, some visual narratives elicited utterances from PMTs that could not be motivated using mathematical properties. For example, P3 fixated on whether angle A and angle B in ΔABC of the first problem were equal, but could not provide any justification, and P5 who, by looking at lines AF and BG, concluded that they were equal in length and provided "spirituality" as his justification. These were cases of visual images that elicited information in PMTs' brains that they could not justify through mathematical properties or proof, but they believed this information based on the appearance of the objects in the diagram. Thus far, the concept images evoked (Tall & Vinner, 1981) by particular concepts in either the first or the second problem, did not always match the actual mathematical meaning. This meant that this particular cohort had a strong concept image associated with a weak concept definition and vice versa. It can then be concluded that visualisation also influenced PMTs' success in solving the problems in that it limited them from linking the visual images to the concepts and problem-solving strategies.

The second problem was a visual problem in the sense that all the participants began their solution by attempting to complete a diagram as a visual mediator to guide them in their solution. Furthermore, in the face-to-face interviews all the participants emphasised that without a diagram they could not have solved the problem. This could be seen in that after producing a visual mediator that they thought represented the problem, PMTs became stuck in using their visuals to produce a solution to the problem. This occurred to all the participants except for P5, who produced a correct diagram, but were stuck in using that diagram to find the solution. After scaffolding the production of the diagram, the other PMTs were then able to see the solution, with some of them approaching it from Euclidean geometry and some from trigonometry. This means that for these PMTs, the scaffolded production of the diagram enhanced their visualisation of the strategies they could use to find the solution to the problem. In conclusion, the findings from this study suggest that visualisation had a huge effect on PMTs' problem solving and their participation in the discourse of solving geometry problems. This indeed confirms that visualisation plays a critical role in learning geometry and advancing geometry problem solving (Battista, 1990).

6.5 IMPLICATIONS OF ASSESSING PMTS' THINKING DURING PROBLEM SOLVING ON THEIR PRACTICE

Assessing thinking during problem solving in mathematics has been a difficult subject in the research agenda for a long time. However, recently Sfard (2008) theorised that thinking can be assessed by analysing the discourse of a person during their problem-solving actions. Using the main tenets of this theory by Sfard (2008), I was able to shed some light onto PMTs' thinking during their problem solving. Thus far, I have shown that PMTs' thinking was predominantly ritualistic. They relied on memory, previous experiences and scaffolding from the interviewer to solve problems in such a way that the solutions were accepted by the interviewer. This implies that because the PMTs think in a ritualistic way and their problem-solving actions relate to past experiences and the reproduction of procedures, they were most likely to teach their learners in this ritualistic way. Despite the continued scaffolding given by the interviewer to these PMTs, the process of individualisation did not occur for most of these PMTs, and their problem solving was always contingent on the availability of the interviewer. This reliance on the interviewer during their problem solving, revealed that PMTs continually required support from knowledgeable others in the form of professional development courses during their practice. Furthermore, PMTs' discourses need to be scaffolded towards an explorative one so they can facilitate teaching and learning centred on explorative discourse with learners becoming participants in this discourse.

Once we have assessed and understood PMTs' thinking evident in their discourse participation during their problem solving, we can then devise interventions that can enhance this thinking as these PMTs go through their teacher training. This, of course, is possible with scaffolded instruction that aims to transform PMTs' discourse participation from ritualistic to more explorative discourse participation. In such interventions, PMTs may begin their discourse participation as outsiders where they observe more knowledgeable and experienced problem solvers solve problems. As time goes by PMTs may be introduced to the discourse as peripheral participants, and this continues with the more knowledgeable other giving lesser scaffolds and the PMTs making more individual decisions during their problem solving. This staggered and careful intervention might result in the PMTs taking control of the problem-solving process without the help from others and they might experience the discourse as discourse-for-oneself instead of discourse-for-others. In this way, the PMTs will experience learning firstly at a meta-level stage where they will be relying on scaffolding to advance their discourse and then at a later stage, once their discourse has been transformed to full-fledged explorations, they will experience learning at an object level (Sfard, 2008). At this object level, they will be able to use and apply existing meta-rules to study certain properties of the mathematical object extensively, which could result in new theorems (Ben-Zvi & Sfard, 2007).

CHAPTER 7:
CONCLUSIONS, LIMITATIONS AND RECOMMENDATIONS

7.1 INTRODUCTION

I conclude the current study, which aimed to explore PMTs' thinking when solving geometry problems and what were the implications of this thinking for their practice by referring back to the research questions posed in Chapter 1. This study was guided by the following research questions.

- What types of discourses are used by preservice mathematics teachers when solving geometry problems? Why?
- How does preservice teachers` thinking (evident in their elements of mathematical discourse) influence their geometry problem solving? What are the implications for this effect for geometry teaching?
- What effects do visual narratives used by preservice mathematics teachers have on their geometry solution paths during problem solving?
- What are the implications of assessing preservice teachers' thinking for their problem solving and teaching practice?

The first section dealt with a lay-out of the answers to these research questions, based on the findings and the themes from the discussion, using the theoretical lens of commognition supported by the van Hiele theory of geometry thinking. Thereafter, I discussed the contribution that this study has brought to the scholarship of mathematics education and the theory of commognition supported by the van Hiele theory of geometry thinking. I conclude this chapter by highlighting the limitations of this study and giving recommendations for further study.

7.2 ANSWERING THE RESEARCH QUESTIONS

This section concludes the thesis by answering the research questions that were posed in 1.6. These research questions were answered by using the themes that emerged from the data. Thereafter, I discuss the contribution that the current study has made to the teaching and learning of Euclidean geometry and the study of mathematical thinking during problem solving. Lastly, I discuss the limitations of the current study and the recommendations that arose from the current study.

7.2.1 What types of discourses are used by preservice mathematics teachers when solving geometry problems? Why?

The first question in this study dealt with the type of discourse used by PMTs during their geometry problem solving and why these PMTs were using this type of discourse. In commognition, there are two types of discourse participation: ritualistic and explorative. These types were explained in Chapter 3. In 6.2 I discussed the findings related to the type of discourse used by PMTs in their geometry problem solving. The findings indicated that PMTs relied on ritualistic discourse participation during their problem solving. This was characterised by their reliance on scaffolding from the interviewer, seeking social acceptance, and producing routines with and for others, and also by describing mathematical objects using colloquial language. Even though some sections contained traces of explorative discourse participation, the most dominant type of discourse was ritualistic. In answering the 'why' part of this question, I looked at the different reasons PMTs should be so reliant on ritualistic discourse performance. One possible reason PMTs relied on ritualistic discourse participation is similar to that observed in Baccaglini-Frank (2021), in that this discourse could get them to where they wanted to be in terms of their problem-solving performance. They wanted to obtain the right answer to ensure that they were accepted by the interviewer, so they relied on the interviewer and focused extensively on performing routines to get to where they wanted to be: the right answer. I highlight this because there is the possibility of solving mathematical problems by drawing on routines, instead of explorations and sense making (Baccaglini-Frank, 2021), and it has been highlighted before that the road to explorations passes through rituals (Lavie, et al., 2019). Thus, these PMTs tried to perform routine procedures without understanding the procedures, so they could obtain the right answer. Furthermore, Polya (1973) acknowledges that knowledge obtained from outside authority can be easily forgotten which implies that since most PMT's discourse was ritualistic, they might have experiences of obtaining knowledge from outside authority instead of creating their own knowledge.

Furthermore, the overreliance on ritualistic discourse from the participants could be related to their past learning experiences that focused on them mastering rituals and procedures instead of promoting meaning making. This could be seen, for example in the case of P3, who admitted that the first approach he took in solving the first problem was inspired by his high school teacher. He even mentioned "that is how I was taught". This could be a result of PMTs being excluded from learning opportunities that resonates with fully-fledged explorations in their classrooms (e.g. Heyd-Metzuyanim, Tabach, & Nachlieli, 2016), or be elicited by the observation from Mbhiza (2021), that most SA teachers tend to align themselves with instruction governed by ritualistic discourse participation. When rituals failed, most of the PMTs became limited in their problem solving and consequently relied on scaffolding from the interviewer. With this scaffolding, only P4 failed to solve both problems. All the other PMTs were able to solve the posed problems, based on the scaffolding given by the interviewer. Again, this showed the significance of the knowledgeable other as a *sine qua non* for success in problem solving for inexperienced problem solvers. Furthermore, it also highlights the significance of scaffolding as did Wood et al. (1976) and Vygotsky (1978) in their seminal works. However, the teaching and learning process in the classroom should develop such that students are given more and more responsibility towards their problem solving so that they require very little scaffolding as they develop in their learning.

Wood et al. (1976, p. 90) argue from a linguistic position, "comprehension of the solution must precede production" and this might be a good supporting argument as to why some PMTs (P1, P3 and P6) showed flexibility in producing alternative solutions after they have been scaffolded in the first solution. Through scaffolding, the interviewer sometimes transformed PMTs' germinal routines from rituals to exploration and PMTs were able to produce subroutines and narratives individually. This symbolises a slight movement towards explorative discourse participation. Hence, with this overreliance on scaffolding from the interviewer, it showed that PMTs' discourse participation was ritualistic.

7.2.2 How does preservice teachers` thinking (evident in their elements of mathematical discourse) influence their geometry problem solving? What are the implications for this effect for geometry teaching?

PMTs’ thinking during their problem solving depended mainly on manipulating symbols on paper without any meaning making and further explorations. PMTs focused on applying specific procedures and were not flexible when finding their solution, because of the rigidity of the procedures they followed, especially with the first problem. Furthermore, PMTs could not correct their own narratives locally and they endorsed some of their narratives using incorrect subroutines. Their strict focus on the performance of the routine to reach an envisioned goal, led to them endorsing and accepting narratives based on the visual appearances or whether the narratives would produce the answer the PMTs envisioned at the beginning of the routine performance. This overreliance on ritualistic discourse and attempting to use memories from their previous problem-solving activities caused these PMTs to struggle in their problem solving. Their struggle was exacerbated by them focusing on one strategy in their problem solving, for example P3 fixated on trying to prove similarity, but after some time he discovered that similarity could not work in solving the first problem. Hence, he was stuck after his initial strategy did not work, which meant that PMTs’ limited thinking about geometry problem-solving strategies could be a reason why they struggled to solve these two problems. Even though they were scaffolded with these, PMTs could not exercise thoughtful imitation which, according to Sfard (2008), can simplify the process of mathematical problem solving and facilitate the move from ritualistic to explorative discourse participation.

From the recent National Senior Certificate (NSC) examination Grade 12 diagnostic report, there was evidence of learners equating angles because the angles seemed to represent a particular theorem, while some of the learners provided incorrect or incomplete reasons for statements in their proofs (DBE, 2020, pp. 200–204). Since the same observations were made from these PMTs, it suffices to conclude that they will continue to perpetrate this ritualistic problem solving, even in their future teaching, producing learners who are incompetent in geometry problem solving. Furthermore, this is proof that teachers’ problem solving affects how learners solve problems, which means that if teachers can achieve competency in explorative discourse participation and geometry problem solving, it can also influence learners to solve problems through explorations. This has implications for teacher training, not only in one module, but as they grow in their journey of becoming teachers in the future.

There was also evidence that PMTs performed their routines with others (the interviewer) and for others (the interviewer) throughout their participation in the task-based interviews. These findings suggested that teachers failed to be flexible in their problem solving and they relied on limited procedures in their problem solving. This implies that even when teaching geometry, these PMTs will rely on procedures that they have learnt when solving the problem before the lesson, which means they will be teaching geometry in a ritualistic manner. Consequently, learners might experience negative learning gains if geometry is taught as a ritualistic activity of attempting to follow stringent procedures without making meaning (Machisi, 2021; Naidoo & Kapofu, 2020). Even though there were isolated incidents that symbolised explorative discourse participation, they were performed in a routine manner. For example, even though some PMTs noticed independently that they had to calculate the sum of angles in a triangle in the first problem, most of them focused on manipulating the symbols so they could get a correct answer and did not support the geometry statements made in their routine performance. It was not until the interviewer pointed that out, that they decided to support their statements.

In their problem solving, PMTs used very strict routines that were completed following strict procedures. They performed these routines without any degree of freedom. They were also not able to correct the routines without the help of the interviewer. They applied these routines for different ritualistic reasons, for example, instead of exploring the solution individually, PMTs always relied on the interviewer (with scaffolds) for social acceptance. Their endorsement of routines focused on what the answer should look like as seen in the endorsement of the second problem by some of the participants (P1, P2, P4 and P5). It is this focus on the appearance of the answer that limited PMTs' thinking about strategies to solve the problem. Instead, they focused on producing something similar to what they envisioned the answer should look like and solutions to the second problem were good examples of this. Consequently, they produced incorrect narratives or even used incorrect narratives to endorse routines in their problem solving. This means that teachers would attempt to teach geometry as a topic where, if you manipulated the symbols, you will get to the answer, instead of teaching it as an explorative topic of problem solving.

Lastly, the way PMTs spoke about mathematical problems indicated no objectification at all, because they used pronouns such as 'this' together with the body movements of pointing to communicate about mathematics objects. Instead of using mathematical language in naming angles, PMTs would utter statements such as 'this angle' accompanied by pointing towards the particular angle they are referring to in the diagram. Evidence of Grade 12 learners failing to name angles correctly, was evident in the recent NSC diagnostic test (DBE, 2020, p. 205) and this can be correlated to the disobjectified use of 'language' in the classroom. As Sfard (2017) acknowledges, the way in which teachers talk to learners in the classroom has a huge effect on the learning that goes on in the classroom. Could this disobjectified discourse used by PMTs in communicating about mathematical objects be to blame for learners' failure to name angles correctly? From the findings of the currents study, I can positively conclude that even though it might not be the main cause of this, it has a role to play. Continually using mathematical language such as saying 'angle ABC' instead of saying 'this angle', communicates to learners that angles are named according to the three points which are represented by letters. Statements such as "I want to prove that triangle ABF is the same as triangle ABG" are using colloquial discourse because the word 'same' does not mean 'congruent' in mathematics, which was what P5 wanted to prove. However, the implication for teaching geometry that can be deduced from this finding is that these PMTs are most likely to use disobjectified discourse when they are teaching geometry in the future.

7.2.3 What effects do visual mediators used by preservice mathematics teachers have on their geometry solution paths during problem solving?

As a critical geometry attribute, visualisation always complements geometry problem solving and geometry learning. The visual mediators (symbolic or diagrammatic) have a huge effect on how one solves a geometry problem. For example, P1 calculated the sum of angles in a triangle and solved for x in the process (Table 5.7), but she did not have a purpose for doing so and she started solving for y in the same calculation. In this case, the visual mediator produced by P1 in solving for x, did not give her a way forward as to how she could endorse the answer, so she changed to another variable.

Visual mediators also translate how PMTs saw and spoke about mathematical objects. Statements about visual mediators that are not mathematical in their nature have an effect on whether they will visualise the solution or the strategy to solve the problem. Visual mediators (even symbolic ones) activate ones' visual thinking and invite one's visualisation of the problem-solving scenario. Van Hiele (1957) and Van Hiele-Geldof (1957) conceptualise visualisation as the initial step of geometry thinking, and they assert that for a child to develop competence in geometry thinking, they need to develop their visualisation.

In the second problem, most PMTs produced incorrect visual mediators relating to the problem, which can be related to the lack of mastery of Van Hiele level 2 of geometry thinking. These incorrect visual mediators blocked these PMTs from making progress in their problem solving, as all of them admitted that in the second problem a diagram was the critical element of finding the answer. This means that the visual mediators they produced, did not allow them to visualise the answer or the strategy to use in answering the problem. P2, for example, who produced a diagram that looked like a triangle (Figure 5.11), admitted that she was stuck, because the visual mediator she produced was incorrect. Furthermore, P3 and P6 were relentless that once they get the correct diagram, they will be able to solve the problem. This means that visual mediators are critical in the visualisation of the strategy to solve the problem and can enhance visual thinking during problem solving. This could be observed in the case of P5, who was able to produce a correct visual mediator for the second problem and from there he was able to visualise the geometric approach to solving the problem. In the case of P5, the correct visual mediator allowed him to visualise the strategy to use when solving the problem. Thus, the effects of visual mediators, though vast, in this study included regulating the success or failure of the problem-solving attempt.

However, it should be noted that the availability of the correct visual mediator does not guarantee success in problem solving, because the first problem had a readily available diagram, but no PMT could solve the problem without the help of the interviewer. When asked about this, PMTs mentioned that they could not link the visual mediator of the centres (H and O) with the words circumcentre and orthocentre. This means that sometimes, during problem solving, the link between visual mediators and their mathematical descriptions needs to be strong, because if one is lacking, then the process of problem solving can be in jeopardy. As admitted by other PMTs like P2, that she could not solve the problem because she mistook the orthocentre for the circumcentre, the role of linking visuals with their properties is critical in Van Hiele level 2 of geometry thinking. Failure to link visual mediators with their properties (or definitions) can lead to impaired visual thinking, which can limit success in problem solving.

7.2.4 What are the implications of assessing preservice teachers' thinking for their problem solving and teaching practice?

Various implications arose from the assessment of PMTs' thinking during this study that relate both to their problem solving and their classroom practice in the future. Given the dominant ritualistic discourse participation associated with following procedures and previous memories during problem solving, we can see that even in the future, PMTs' problem solving will remain ritualistic. This will occur until they are patiently and gradually scaffolded from ritualistic to explorative discourse participation by a very patient tutor, and this will require a considerable amount of time. We saw for example, P6 making the error of dividing where he was not supposed to divide, which is a sign of a misconception that needs to be gotten rid of before the transformation from ritualistic discourse participation to an explorative one. Classroom implications for such problem-solving behaviour is that these teachers will continue to perpetrate the dominant trend of rote learning in mathematics education in South Africa. They will teach geometry problem solving as a set of procedures and routines that needs to be followed and they will refrain from exploring or teaching problems when they do not have the answer readily available to them. Furthermore, their incompetence in solving geometry problems means that they will not take time to explore solutions to problems to which they do not have the solutions, and they will not explore multiple ways of solving the same problem. Previous research studies have attributed the poor performance in mathematics to mathematics teachers' focus on transmitting procedures to learners for future regurgitation (De Villiers, 2012; Machisi, 2021).

Once PMTs' thinking during problem solving has been revealed, as this study has done, HEIs and teacher trainers are then called in to devise measures in which they can advance PMTs' geometry problem solving to influence their practice. This, however, does not mean that teacher training must focus only on advancing content without pedagogy. The two must be interwoven. However, it is advised that teachers must first know ways to solve a problem before they can think about ways of teaching it, which is why I advocate for beginning with teaching geometry problem solving before its pedagogy.

Furthermore, it was clear from PMTs' problem-solving performance that they were not monitoring and reflecting on their own thinking, hinting to no use of metacognition in their problem solving. The significance of metacognition in problems solving and how it determines the failure and success of a problem-solving activity was discussed in 2.5. This means that

PMTs could not notice their own thinking during problem solving. Thus, it will be very hard for PMTs to notice their learners' thinking in their future classrooms if they cannot notice and monitor their own thinking, nor use metacognitive processes to solve the posed geometry problems. This failure to notice learners' thinking, will have an impact on their conceptual understanding of mathematics and scaffolded mathematical learning. Teachers are going to continue to teach for the examinations and to complete the syllabi without any learners achieving an explorative discourse participation in their mathematical learning. Again, this implies that HEIs in their teacher training courses should think about ways of incorporating teaching and learning related to metacognition, and teachers noticing to improve PMTs' problem-solving competency and pedagogy.

PMTs' level of geometry thinking showed that they were dominantly using the ritualistic Van Hiele level of geometry thinking. This can be seen when they, for example, incorrectly link visual mediators with their properties or written descriptions, like in the concepts of orthocentre and circumcentre. Furthermore, they assumed that angles or lines were equal based on their visual appearances and not their properties. Consequently, teachers may in the future take their ritualistic Van Hiele level of geometry thinking to their classrooms when they start teaching. This may result in learners assuming geometry objects are equal based on their appearances or failing to attach geometry properties to visual mediators in a given diagram, as seen from the recent NSC diagnostic test (DBE, 2020). This kind of teaching may exacerbate the poor SA learner performance that is observed in both national and international tests.

7.3 CONTRIBUTION OF THE STUDY

7.3.1 Contribution in terms of literature differences

Despite the fact that there are several studies around the globe that investigate thinking during geometry problem solving, the current study was unique in the following respects.

- The study offered a thick description of teacher explanation of their behaviours during geometry problem solving, focusing on what they were thinking during certain actions of their problem solving, which is a move away from characterising geometry teachers' abilities through standardised tests.
- The study was conducted within the SA context, where commognition is gaining some popularity in studying mathematical thinking. However, from the reviewed literature I

did not find any study that focused on geometry problem solving within the SA context and the world. Furthermore, I did not find any study that combined commognition and the Van Hiele theory of geometry thinking to study thinking during geometry problem solving.

- Most of the studies using commognition in South Africa focus on school learners while this study focused on preservice mathematics teachers. Thus, it contributes towards the training of teachers for successful geometry teaching and learning in their practice. Findings from this study can be used as a guide by mathematics teacher trainers to design courses that can develop PMTs' geometry thinking and problem solving by scaffolding their discourse participation from ritualistic to explorative.
- The study explored how PMTs thought about geometry problem solving, using their discourse participation and predicted how their thinking now might influence their practice in future. The study used commognition supported by the Van Hiele theory of geometry thinking to analyse the data for this study. Even though Wang's (2016) study also used similar theories, her focus was on construction and properties of geometry objects which leaned more towards the Van Hiele theory of geometry thinking, supported by commognition. In fact, Wang (2016) grafts the elements of mathematical discourse onto each level of the Van Hiele geometry thinking. The current study is different from the Wang study, because the current study did not necessarily focus on the elements of mathematical discourse, but focused on the distinction between ritualistic and explorative discourse participation in geometry problem solving.[31] Hence, the current study provided a method of studying PMTs' geometry thinking during their problem solving, using their think-aloud discourse during problem solving.
- By identifying the difference between ritualistic and explorative Van Hiele levels that were evident in PMTs' problem solving, the study contributes immensely to the scholarship of both commognition and the Van Hiele levels of geometry thinking. This study further contributes to the very limited literature on geometry problem solving by employing realistic geometry problem solving, testing not only PMTs' geometry (deductive) thinking, but also their visio-spatial thinking.[32]

[31] Even though the elements of mathematical discourse were used in the findings and data analysis as critical descriptors of mathematical discourse.
[32] The second problem was a geometry problem posed in a realistic world context.

I am now going to be describing the main contribution of the study using Figure 7.1 describing the development of PMT's geometrical thinking through the Van Hiele levels.

7.3.2 Contribution to literature

In this section I discuss the study's main contribution to the theories of commognition and the Van Hiele theory of geometrical thinking. This allowed me to suggest how the findings from the study amalgamated commognition and van Hiele theory of geometrical thinking to contribute to new knowledge. I used Figure 7.1 to guide my discussion and this allowed me to how PMT's discourse participation can be used to enhance their development in geometrical thinking through the Van Hiele levels of geometrical thinking. The below discussion is guided by the characteristics of the Van Hiele levels and how PMTs participated in discourse in this study. Thus, I discuss this contribution based on the movement from level 0 – 1, movement from level 1 – 2 and so on ending at level 3 which is the level Grade 12 teachers should be operating. Furthermore, my discussion is guided by the type of discourse participation teachers displayed in each from the findings.

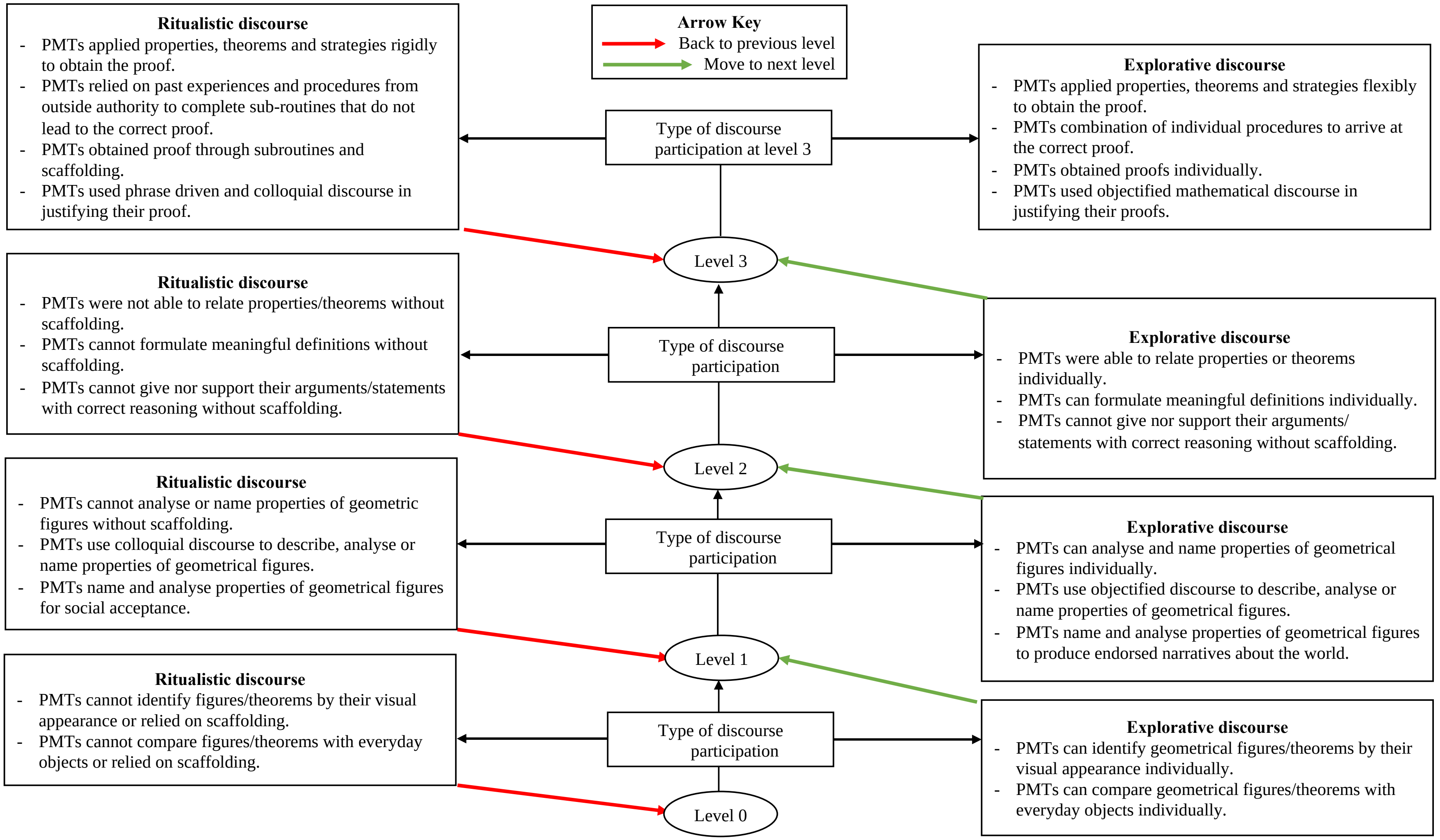


Figure 7.1: A commognitive analysis of PMTs' geometrical thinking development through Van Hiele levels of geometrical thinking

7.3.2.1 Development from Van Hiele level 0 – 1

The identification of geometrical figures by their appearances only is the critical determining skill in level 0 because only the recognition of the figure is required. PMTs who could not identify geometric figures by their appearances during discourse and those who relied on scaffolding from the interviewer lacks this critical geometrical skill. Furthermore, in advanced geometry, theorems can also be identified through the appearance of the diagrams and PMTs who could not identify particular theorems that were available in the diagram lacks visualization. As seen in the findings (P3) the lack of visualizing theorems hindered P3's problem solving but once a scaffold was given by the interviewer, he was able to complete sub-routines that involves the theorem as exploration. Thus, PMTs who lack this visualization skill cannot progress to level 1 because this visualization skill is a requirement at Van Hiele level 1. They must stay at level 0 as indicated by the red arrow in Figure 7.1. However, PMTs who could identify geometrical objects based on their appearances individually and can compare these objects with objects they encounter daily could be allowed to move to level 1 as they have mastered the skill of visualization as indicated by the green arrow in Figure 7.1. That is because the former is still reliant on ritualistic discourse participation while the latter relies on explorative discourse participation.

7.3.2.2 Development from Van Hiele level 1 – 2

The defining characteristic in Van Hiele level 1 is that PMTs should have moved from not only recognizing objects by their appearances but also by linking the objects to their properties. However, interrelating the properties of object is still not developed at level 1. It is evident from this study that some PMTs could not name properties of geometrical objects without scaffolding and some used colloquial discourse to name these properties. Some PMTs mentioned incorrect properties to gain social acceptance with the interviewer but when probed further they revealed that they did not know the properties. There is high usage of ritualistic discourse in that PMTs rely on the interviewer in most cases, they also rely on past experiences, and they wanted to conform to society in their routine performance. This shows that PMTs have not mastered the ability to attach a geometrical figure with its property and thus they should remain in level until this skill is mastered. PMTs promoted to level two are those who can individually link a particular property to a particular geometrical figure individually. They even used objectified discourse to mention and link these properties with their geometrical figures which is a characteristic of explorative discourse participation. The link of the property to it geometrical figure also allows them to produce endorsed narratives about the geometrical world. These

PMTs who shows mastery in the skill of linking geometrical figures with their properties using explorative discourse can progress to Van Hiele level 2 as they have mastered Van Hiele Level 1.

7.3.2.3 Development from Van Hiele level 2 – 3

At level 2 PMTs do not just link properties with their geometrical figures but can logically order and interrelates properties to understand the relationship between geometrical figures. Mastery of level 2 means that PMTs are getting ready for logical deduction required in proof. At this stage, PMTs should not be reliant on scaffolding to link properties and still be promoted to level 3 because that shows that they have not mastered level 2. Furthermore, the formulation of meaningful definitions is critical in level 2 and if PMTs still rely on scaffolding they are not ready for level 3. The understanding of properties and theorems is critical in proof and if one has not mastered the link between properties and the relationship between figures, then they are not ready to move to level 3. This is because PMTs will produce a whole proof without giving proper reasoning for their arguments or statements with correct reasons as evident in the current study. Only PMTs who shows logical understanding of how properties of different figures link and are able support their narrative individually using objectified discourse should be allowed to proceed to level 3 which is the last level required in the CAPS.

7.3.2.4 Discourse participation at level 3

Level 3 requires that PMTs be able to use experiences from the previous levels to understand the role of properties, theorems and the link thereof when doing geometry proof. In this level, PMTs now begin to develop longer arguments to perform geometrical proof and can successfully substantiate each argument with an endorsed mathematical narrative. PMTs who continually rely on the same procedure applied in exactly the same way instead of being flexible in their strategies are not ready for level 3 and they should be demoted to level 2. At this level, relying on procedures, sub-routines and scaffolding from others to obtain a proof does not guarantee that you will be able to prove similar problems in the future, independence is required to master level 3. Justifying your statements during proof using phrase-drive and colloquial language instead of objectified language also shows that you not mastered level 2, thus, you must be demoted back to level 2. PMTs who perform geometrical proof independently and uses explorative discourse when talking about geometrical proofs can be thought of as ready to teach geometry at Grade 12 level in the CAPS.

In conclusion, the model in Figure 7.1 gives teacher trainers an idea of monitoring the development of PMTs geometrical thinking through the Van Hiele levels using their discourse participation during geometrical problem solving. The main contribution to knowledge is that the findings of the study illuminates how the difference between ritualistic and explorative discourse can be used to improve PMT’s Euclidean geometry thinking through the van Hiele levels. While Wang (2016) clearly discusses how the Van Hiele levels relates to the elements of mathematical discourse, this study goes deeper into analysing how these elements of mathematical discourse as explained by Sfard (2008) in commognition can be used to improve PMT’s Euclidean geometry thinking through the Van Hiele levels. This is main link between the two theories and clearly identifies the study as contributing to a particular new views and knowledge on teacher training in Euclidean geometry.

7.4 LIMITATIONS OF THE STUDY

Despite the depth and compelling evidence provided in this study, there are areas which could be considered possible limitations of the current study. Firstly, the study utilised a convenient purposive sampling method to allow the researcher to gain excess to participants who exhibited characteristics that were described as the inclusion criteria. Furthermore, this was done so that the participant could also be part of the research through conducting both task-based and face-to-face interviews with the participants. Hence, a random sample was not possible given the cost implications of travelling around the country, so convenience was chosen and a motivation was given. This might have distorted the present study’s findings.

Secondly, the study focused on PMTs from a single university who finished the specific geometry module in 2019. If more PMTs from different universities were included, their proficiency levels in geometry problem solving and the way these PMTs spoke about mathematical objects could have been different. Thus, the choice of limiting the sample to one university provided results that cannot be easily generalised to a larger population. A study that could be conducted with a truly random sample with PMTs from different universities could yield findings that are different from the ones obtained in this study.

Thirdly, data was collected during the time of the COVID-19 pandemic when PMTs were involved in remote learning and they might have faced pressures that were different from face-to-face learning. This might have distorted their thinking during problem solving. One of the

PMTs even mentioned that, because of the lockdown, she had forgotten most of the geometry content she learnt in 2019.

Lastly, the position of the researcher as the participants' previous lecturers who taught them the Euclidean geometry module in 2019 might have influenced or affected the participants' responses during their problem solving and also during the interview. This might have affected the validity of the findings in the study.

7.5 RECOMMENDATIONS FOR FUTURE RESEARCH

Based on the findings from this study I would like to make the following recommendations for future research.

- This first recommendation is that a four-year-long study be conducted that will scaffold PMTs' discourse participation from ritualistic to explorative, starting from their first year of teacher training to their last year of teacher training, and then evaluate the findings afterwards. This will be an intervention study, since this current study only focused on exploring PMTs' present discourse participation and this recommendation only applies to the South African context since not all countries adhere to the four-year programme for training teachers.
- More studies can be done in the teaching and learning of Euclidean geometry in basic and higher education to investigate thinking during problem solving through combining the lenses of commognition and the Van Hiele theory of geometry thinking. This will enable teachers and teacher trainers to attend to learners' thinking that will be evident from their discourse to help them to become better geometry problem solvers.
- I also recommend that teacher trainers foster metacognition in PMTs during problem solving and also teach PMTs the art of noticing and acting upon learners' thinking with the aim of helping learners develop competency in geometry problem solving.
- In the future the researchers can be an outsider in a study to limit the effects he may have on how learners respond to questions in both the task-based interviews and the face-to-face semi-structured interviews. This can improve the quality, validity and trustworthiness of the study.

REFERENCES

Abdu, R., Schwarz, B., & Mavrikis, M. (2015). Whole-class scaffolding for learning to solve mathematics problems together in a computer-supported environment. *ZDM - Mathematics Education*, *47*(7), 1163–1178. doi:10.1007/s11858-015-0719-y

Abrahamson, D., & Bakker, A. (2016). Making sense of movement in embodied design for mathematics learning. *Cognitive Research: Principles and Implications*, *1*(1), 1–13 doi:10.1186/s41235-016-0034-3

Ader, E. (2019). What would you demand beyond mathematics? Teachers' promotion of students' self-regulated learning and metacognition. *ZDM – Mathematics Education, 51*(4), 613–624.

Adler, J. (1997). A participatory-inquiry approach and the mediation of mathematical knowledge in a multilingual classroom. *Educational Studies in Mathematics*, *33*(3), 235–258. doi:10.1023/A:1002976114883

Adler, J. (1999). The dilemma of transparency: Seeing and seeing through talk in the Mathematics classroom. *Journal for Research in Mathematics Education*, *30*(1), 47–64. doi:10.2307/749629

Adler, J. (2002). *Teaching Mathematics in multilingual classrooms*. New York, NY: Kluwer Academic.

Adler, J. (2017). Mathematics in mathematics education. *South African Journal of Science*, *113*(3/4), 1–3.

Adler, J., & Pillay, V. (2017). Mathematics education in South Africa. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming researcher's insights into improvement in mathematics teaching and learning* (pp. 9–24). Abingdon: Routledge.

Adler, J., & Ronda, E. (2014). *An analytic framework for describing teachers' mathematics discourse in instruction*. Paper presented at the Joint Meeting of PME 38 and PME-NA, Vancouver.

Adler, J., & Ronda, E. (2015). A framework for describing mathematics discourse in instruction and interpreting differences in teaching. *African Journal of Research in Mathematics,* *Science and Technology Education, 19*(3), 237–254. doi:10.1080/10288457.2015.1089677

Adu, E. O., & Olaoye, O. (2015). Language proficiency and method of instruction as determinant of Grade 9 students' academic performance in algebra. *International Journal of Educational Sciences, 8*(3), 547–555.

Aktaş, F. N., Yakıcı-Topbaş, E. S., & Dede, Y. (2019). The elementary Mathematics teachers' values underlying teacher noticing: The context of polygons. In P. Clarkson, W. T. Seah, & J. Pang (Eds.), *Values and valuing in mathematics education: Scanning and scoping the territory* (pp. 209–222). Cham: Springer International.

Alex, J. K., & Mammen, K. J. (2012). A survey of South African Grade 10 learners' geometric thinking levels in terms of the Van Hiele theory. *The Anthropologist*, *14*(2), 123–129. doi:10.1080/09720073.2012.11891229

Alex, J. K., & Mammen, K. J. (2014). An assessment of the readiness of Grade 10 learners for geometry in the context of Curriculum and Assessment Policy Statement (CAPS) expectation. *International Journal of Educational Sciences, 7*(1), 29–39. doi:10.1080/09751122.2014.11890167

Alex, J. K., & Mammen, K. J. (2016). Lessons learnt from employing Van Hiele theory based instruction in senior secondary school geometry classrooms. *Eurasia Journal of Mathematics, Science and Technology Education*, *12*(8), 2223–2236. doi:10.12973/eurasia.2016.1228a

Alex, J. K., & Mammen, K. J. (2018). Students' understanding of geometry terminology through the lens of Van Hiele theory. *Pythagoras*, *39*(1), 1–8. doi:10.4102/pythagoras.v39i1.376

Alexander, R. (2008). *Towards dialogic teaching: Rethinking classroom talk* (4th ed.). York: Dialogos.

Alexander, R. (2018). Developing dialogic teaching: Genesis, process, trial. *Research Papers in Education*, *33*(5), 561–598. doi:10.1080/02671522.2018.1481140

Alexander, R. (2019). Whose discourse? Dialogic pedagogy for a post-truth world. *Dialogic Pedagogy: An International Online Journal, 7,* 1–19. doi:10.5195/dpj.2019.268

Alexander, R. (2020). *A dialogic teaching companion*. New York, NY: Routledge.

Alibali, M. W., Nathan, M. J., Church, R. B., Wolfgram, M. S., Kim, S., & Knuth, E. J. (2013). Teachers' gestures and speech in Mathematics lessons: Forging common ground by resolving trouble spots. *ZDM*, *45*(3), 425–440. doi:10.1007/s11858-012-0476-0

Amador, J. (2016). Professional noticing practices of novice Mathematics teacher educators. *International Journal of Science and Mathematics Education, 14*(1), 217–241. doi:10.1007/s10763-014-9570-9

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*(3), 215–241. doi:10.1023/A:1024312321077

Arends, F., Winnaar, L., & Mosimege, M. (2017). Teacher classroom practices and mathematics performance in South African schools: A reflection on TIMSS 2011. *South African Journal of Education, 37*(3), 1–11.

Armstrong, J. A. (1981). Achievement and participation of women in mathematics: Results of two surveys. *Journal for Research in Mathematics Education, 12*(5), 356–372.

Ary, D., Jacobs, L. C., Irvine, C. K. S., & Walker, D. A. (2019). *Introduction to research in education* (10th ed.). Boston: Cengage Learning.

Askew, M., Venkat, H., Mathews, C., Ramsingh, V., Takane, T., & Roberts, N. (2019). Multiplicative reasoning: An intervention's impact on Foundation Phase learners' understanding. *South African Journal of Childhood Education*, *9*(1), 1–0. doi:10.4102/sajce.v9i1.622

Atebe, H. U., & Schäfer, M. (2011). The nature of geometry instruction and observed learning-outcomes opportunities in Nigerian and South African high schools. *African Journal of Research in Mathematics, Science and Technology Education*, *15*(2), 191–204. doi:10.1080/10288457.2011.10740712

Atwood, S., Turnbull, W., & Carpendale, J. I. M. (2010). The construction of knowledge in classroom talk. *The Journal of The Learning Sciences*, *19*(3), 358–402.

Aubrey, C., Ghent, K., & Kanira, E. (2012). Enhancing thinking skills in early childhood. *International Journal of Early Years Education*, *20*(4), 332–348. doi:10.1080/09669760.2012.743102

Baccaglini-Frank, A. (2021). To tell a story, you need a protagonist: How dynamic interactive mediators can fulfill this role and foster explorative participation to mathematical discourse. *Educational Studies in Mathematics*, *106*(2), 291–312.

Bailey, B. (2012). Heteroglossia. In M. Martin-Jones, A. Blackledge, & A. Creese (Eds.), *The Routledge handbook of multilingualism* (pp. 499–507). London: Routledge.

Bakhtin, M. M. (1981). *The dialogic imagination: Four essays* (C. Emerson & M. Holquist, Trans.). Austin, TX: University of Texas Press.

Bakhtin, M. M. (1984). *Problems of Dostoevsky's poetics* (C. Emerson, Trans., C. Emerson Ed.). Minneapolis, MN: University of Minnesota Press.

Bakhtin, M. M. (1986). *Speech genres and other late essays* (V. W. McGee, Trans.). Austin, TX: University of Texas Press.

Ball, D. L., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education, 60*(5), 497–511.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, *29*(1), 14–17.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. doi:10.1177/0022487108324554

Barnard, E., & Braund, M. (2016). Strategies for implementation of mathematics in Grade R: Teachers' beliefs and practices. *South African Journal of Childhood Education*, *6*(1), 1–8.

Barnes, G. (1976). *From communication to curriculum*. Portsmouth, NH: Boynton.

Barnes, G. (2008). Exploratory talk for learning. In N. Mercer & J. Hodgkinson (Eds.), *Exploratory talk in school: Inspired by the work of Douglas Barnes* (pp. 1–16). Thousand Oaks, CA: Sage.

Barnes, G. (2010). Why talk is important. *English Teaching: Practice and Critique*, *9*(2), 7–10.

Barnhart, T., & Van Es, E. (2015). Studying teacher noticing: Examining the relationship among pre-service science teachers' ability to attend, analyze and respond to student thinking. *Teaching and Teacher Education*, *45,* 83–93. doi:10.1016/j.tate.2014.09.005

Barwell, R. (2018). From language as a resource to sources of meaning in multilingual Mathematics classrooms. *Journal of Mathematical Behaviour, 50,* 155–168.

Barwell, R., Chapsam, L., Nkambule, T., & Setati-Phakeng, M. (2016). Tensions in teaching mathematics in contexts of language diversity. In R. Barwell, P. Clarkson, A. Hali, M. Kazima, J. Moschkovich, N. Planas, … M. V. Ubillus (Eds.), *Mathematics education and language diversity: The 21st ICMI study* (pp. 175–192). Dordrecht: Springer.

Bassett, M. H. (2016). Teaching critical thinking without (much) writing: Multiple-choice and metacognition. *Teaching Theology & Religion*, *19*(1), 20–40. doi:10.1111/teth.12318

Baten, E., Praet, M., & Desoete, A. (2017). The relevance and efficacy of metacognition for instructional design in the domain of mathematics. *ZDM, 49*(4), 613–623. doi:10.1007/s11858-017-0851-y

Battista, M. T. (1990). Spatial visualization and gender differences in high school Geometry. *Journal for Research in Mathematics Education*, *21*(1), 47–60. doi:10.2307/749456

Battista, M. T., & Frazee, L. M. (2018). Middle school students' use of property knowledge and spatial visualization in reasoning about 2D rotations. In P. Herbst, U. H. Cheah, P. R. Richard, & K. Jones (Eds.), *International perspectives on the teaching and learning of geometry in secondary schools* (pp. 231–263). Cham: Springer.

Baumanns, L., & Rott, B. (2018). *Is problem posing about posing "problems"? A terminological framework for researching problem posing and problem solving.* Paper presented at the 2018 Joint Conference of ProMath and the GDM Working Group on Problem Solving, Munster.

Beavers, E., Orange, A., & Kirkwood, D. (2017). Fostering critical and reflective thinking in an authentic learning situation. *Journal of Early Childhood Teacher Education, 38*(1), 3–18. doi:10.1080/10901027.2016.1274693

Bennet, M. (2016). The convergence into an ideal: Critical thinking and metacognition. *Childhood Education*, *91*(1), 75–76. doi:10.1080/00094056.2016.1134249

Ben-Yehuda, M., Lavy, I., Linchevski, L., & Sfard, A. (2005). Doing wrong with words: What bars students' access to arithmetical discourses. *Journal of Research in Mathematics Education*, *36*(3), 176–247.

Ben-Zvi, D., & Sfard, A. (2007). Ariadne's thread, daedalus' wings and the learners autonomy. *Education & Didactique*, *1*(3), 123–142.

Berger, M. (2013). Examining mathematical discourse to understand in-service teachers' mathematical activities. *Pythagoras, 34*(1), 1–10.

Berger, M., & Bowie, L. (2012). A course on functions for in-service Mathematics teachers: Changing the discourse. *Education as Change*, *16*(2), 217–229.

Berinsky, A. J. (2015). Rumors and health care reform: Experiences in political misinformation. *British Journal of Political Science*, *47*, 241–262.

Bernstein, B. (1971). *Class, codes, and control*. New York, NY: Schocken Books.

Bernstein, B. (1990). *Class, codes and control: The structuring of pedagogic discourse*. New York, NY: Routledge.

Beswick, K. (2005). The beliefs/practice connection in broadly defined contexts. *Mathematics Education Research Journal*, *17*(2), 39–68.

Björklund, C., Magnusson, M., & Palmér, H. (2018). Teachers' involvement in children's mathematizing: Beyond dichotomization between play and teaching. *European Early Childhood Education Research Journal*, *26*(4), 469–480. doi:10.1080/1350293X.2018.1487162

Blumer, H. (1986). *Symbolic interactionism: Perspective and method*. London: University of California Press.

Bohlmann, N. (2019). Unequal bodies: Corporeality and social inequality in Mathematics classrooms. *ZDM*, *51*(2), 263–275. doi:10.1007/s11858-018-1005-6

Borba, M. C., Askar, P., Engelbrecht, J., Gadanidis, G., Llinares, S., & Aguilar, M. S. (2016). Blended learning, e-learning and mobile learning in mathematics education. *ZDM*, *48*(5), 589–610. doi:10.1007/s11858-016-0798-4

Boud, D., Keogh, R., & Walker, D. A. (Eds.). (1985). *Reflection: Turning experience into learning.* London: Routledge Falmer.

Boyatzis, R. E. (1998). *Transforming qualitative information: Thematic analysis and code development.* Thousand Oaks, CA: Sage.

Bräunling, K., & Eichler, A. (2015). Teachers' beliefs systems referring to the teaching and learning of arithmetic. In C. Bernack-Schüler, R. Erens, T. Leuders, & A. Eichler (Eds.), *Views and beliefs in mathematics education: Results of the 19th MAVI Conference* (pp. 95–107). Wiesbaden: Springer Fachmedien.

Breen, S., & O'Shea, A. (2019). Designing mathematical thinking tasks. *PRIMUS*, *29*(1), 9–20. doi:10.1080/10511970.2017.1396567

Brijlal, D. (2014). Exploring the pedagogical content knowledge for teaching probability in the middle school: A South African case study. *International Journal of Educational Science*, *7*(3), 719–726.

Brown, G., & Yule, G. (1983). *Discourse analysis*. Cambridge: Cambridge University Press.

Burns, M. (1982). How to teach problem solving. *The Arithmetic Teacher*, *29*(6), 46–49. Retrieved from www.jstor.org/stable/41192015

Burte, H., Gardony, A. L., Hutton, A., & Taylor, H. A. (2017). Think3ed!: Improving mathematics learning through embodied spatial training. *Cognitive Research: Principles and Implications*, *2*(13), 1–18.

Burton, L. (1980). Problems and puzzles. *For the Learning of Mathematics*, *1*(2), 20–23.

Burton, L. (1984). Mathematical thinking: The struggle for meaning. *Journal of Research in Mathematics Education*, *15*(1), 35–49.

Cahusac de Caux, B. K. C. D., Lam, C. K. C., Lau, R., Hoang, C. H., & Pretorius, L. (2017). Reflection for learning in doctoral training: Writing groups, academic writing proficiency and reflective practice. *Reflective Practice*, *18*(4), 463–473. doi:10.1080/14623943.2017.1307725

Cai, J., & Wang, T. (2010). Conceptions of effective mathematics teaching within a cultural context: Perspectives of teachers from China and the United States. *Journal of Mathematics Teacher Education*, *13*(3), 265–287. doi:10.1007/s10857-009-9132-1

Calcagni, E., & Lago, L. (2018). The three domains for dialogue: A framework for analysing dialogic approaches to teaching and learning. *Learning, Culture and Social Interaction*, *18*, 1–12. doi:10.1016/j.lcsi.2018.03.001

Callejo, M. L., & Vila, A. (2009). Approach to mathematical problem solving and students' belief systems: Two case studies. *Educational Studies in Mathematics*, *72*(1), 111–126. doi:10.1007/s10649-009-9195-z

Carpenter, T. P., Fennema, E., Peterson, P. L., & Carey, D. A. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education*, *19*(5), 385–401. doi:10.2307/749173

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C.-P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, *26*(4), 499–531. doi:10.3102/00028312026004499

Cazden, C. B. (2001). *Classroom discourse: The language of teaching and learning* (2nd ed.). Portsmouth, NH: Heinemann.

Chan, Y.-C., & Wong, N.-Y. (2014). Worldviews, religions, and beliefs about teaching and learning: Perception of Mathematics teachers with different religious backgrounds. *Educational Studies in Mathematics*, *87*, 251–277.

Chapman, O. (2015). Understanding and supporting Mathematics teachers' knowledge for teaching. *Journal of Mathematics Teacher Education*, *18*(2), 101–103. doi:10.1007/s10857-015-9298-7

Chapman, O. (2017). Attending to Mathematics teacher thinking. *Journal of Mathematics Teacher Education*, *20*(1), 1–4. doi:10.1007/s10857-017-9363-5

Chazan, D. (1990). Quasi-empirical views of mathematics and mathematics teaching. *Interchange*, *21*(1), 14–23.

Chikiwa, C., & Schäfer, M. (2016). Teacher code switching consistency and precision in a multilingual Mathematics classroom. *African Journal of Research in Mathematics, Science and Technology Education*, *20*(3), 244–255. doi:10.1080/18117295.2016.1228823

Chirinda, B., & Barmby, P. (2017). The development of a professional development intervention for mathematical problem-solving pedagogy in a localised context. *Pythagoras, 38*(1), 1–11. doi:10.4102/pythagoras.v38i1.364

Chirinda, B., & Barmby, P. (2018). South African Grade 9 Mathematics teachers' views on the teaching of problem solving. *African Journal of Research in Mathematics, Science and Technology Education*, *22*(1), 114–124.

Chisholm, L., Motala, S., & Vally, S. (2003). *South African education policy review, 1993–2000*. Sandown: Heinemann.

Chisholm, L., Volmink, J., Ndhlovu, T., Potenza, E., Mahomed, H., Muller, J., … Mphahlele, L. (2000). *South African curriculum for the twenty first century: A report of the review committee on curriculum 2005.* Retrieved from https://www.gov.za/sites/default/files/gcis_document/201409/21cencurriculum0.pdf

Choy, B. H., & Dindyal, J. (2019). Productive teacher noticing: Implications for improving teaching. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 469–488). Singapore: Springer.

Christie, P. (1990). Reforming the racial curriculum: Curriculum change in desegregated schools in South Africa. *British Journal of Sociology of Education*, *11*(1), 37–48. doi:10.1080/0142569900110103

Chronaki, A., & Planas, N. (2018). Language diversity in mathematics education research: A move from language as representation to politics of representation. *ZDM*, *50*(6), 1101–1111. doi:10.1007/s11858-018-0942-4

Clark, A. (1998). Embodied, situated, and distributed cognition. In W. Bechtel & G. Graham (Eds.), *A companion to cognitive science* (pp. 506–517). Malden, MA: Blackwell.

Clarke, D., Clarke, B., & Roche, A. (2011). Building teachers' expertise in understanding, assessing and developing children's mathematical thinking: The power of task-based, one-to-one assessment interviews. *ZDM*, *43*(6), 901–913. doi:10.1007/s11858-011-0345-2

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York, NY: Macmillan.

Clements, K. (1981). Visual imagery and school mathematics. *For the Learning of Mathematics*, *2*(2), 2–9.

Cobb, P. (1985). Two children's anticipations, beliefs, and motivations. *Educational Studies in Mathematics*, *16*(2), 111–126.

Cobb, P. (1986). Contexts, goals, beliefs, and learning mathematics. *For the Learning of Mathematics*, *6*(2), 2–9.

Cohen, L., Manion, L., & Morrison, K. (2018). *Research methods in education* (8th ed.). London: Routledge.

Coles, A., & Sinclair, N. (2019). Ritualisation in early number work. *Educational Studies in Mathematics*, *101*(2), 177–194. doi:10.1007/s10649-018-9815-6

Cook, S. W., Friedman, H. S., Duggan, K. A., Cui, J., & Popescu, V. (2017). Hand gesture and mathematics learning: Lessons from an avatar. *Cognitive Science*, *41*(2), 518–535. doi:10.1111/cogs.12344

Cooper, A. B., & Tisdell, E. J. (2020). Embodied aspects of learning to be a surgeon. *Medical Teacher*, *42*(5) 515–522. doi:10.1080/0142159X.2019.1708289

Cooper, M. M., & Selfe, C. L. (1990). Computer conferences and learning: Authority, resistance, and internally persuasive discourse. *College English*, *52*(8), 847–869.

Cordeiro, C. M. (2018). Language as heteroglot: The bridging qualities of Swedish-English (SweE) and Singapore Colloquial English (SCE) in cross-cultural working environments. *Cross Cultural & Strategic Management*, *25*(4), 781–799. doi:10.1108/CCSM-08-2017-0105

Dahl, B. (2018). What is the problem in problem-based learning in higher education mathematics. *European Journal of Engineering Education*, *43*(1), 112–125. doi:10.1080/03043797.2017.1320354

De Leon, C. M. (2017). Mental imagery in translation process. *Journal of Language and Communication in Business*, *56*, 201–219.

De Silva Joyce, H., & Feez, S. (Eds.). (2018). *Multimodality across classrooms: Learning about and through different modalities*. New York, NY: Routledge.

De Villiers, M. (2012). *Some reflections on the Van Hiele theory*. Paper presented at the National Mathematics Congress, Swakopmund.

De Villiers, M., & Heideman, N. (2014). Conjecturing, refuting and proving within the context of dynamic geometry. *Learning and Teaching Mathematics*, (17), 20–26.

Denzin, N. K. (1970). *The research act: A theoretical introduction to sociological methods*. Piscataway, NJ: Transaction.

Department of Basic Education. (2011). *Curriculum and Assessment Policy Statement: Mathematics (Grade 10–12)*. Pretoria: Government Printer.

Department of Basic Education. (2016). *National senior certificate diagnostic report*. Pretoria: Government Printer.

Department of Basic Education. (2017). *National senior certificate diagnostic report*. Pretoria: Government Printer.

Department of Basic Education. (2018). *National senior certificate diagnostic report*. Pretoria: Government Printer.

Department of Basic Education. (2019a). *Mathematics teaching and learning framework in South Africa: Teaching mathematics for understanding*. Pretoria: Government Printer.

Department of Basic Education. (2019b). *National senior certificate diagnostic report part 1*. Pretoria: Government Printer.

Department of Basic Education. (2020). *National senior certificate diagnostic report part 1*. Pretoria: Government Printers.

Department of Higher Education and Training. (2015). Minimum requirements for teacher education qualifications. *Government Gazette*, *596*(38487). Pretoria: Government Printer.

Descartes, R. (1998). *Discourse on method and mediations on first philosophy* (D. A. Cress, Trans.). Indianapolis, IN: Hackett.

Desoete, A., Baten, E., Vercaemst, V., De Busschere, A., Baudonck, M., & Vanhaeke, J. (2019). Metacognition and motivation as predictors for mathematics performance of Belgian elementary school children. *ZDM*, *51*(4), 667–677. doi:10.1007/s11858-018-01020-w

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston, MA: D. C. Heath.

Dhlamini, S. S. (2012). *An investigation into Grade 12 teachers' understanding of geometry.* (Unpublished master's thesis). University of KwaZulu-Natal, Pinetown.

Didis, M. G., Erbas, A. K., Cetinkaya, B., Cakiroglu, E., & Alacaci, C. (2016). Exploring prospective secondary Mathematics teachers' interpretation of student thinking through analysing students' work in modelling. *Mathematics Education Research Journal*, *28*(3), 349–378. doi:10.1007/s13394-016-0170-6

Dienes, Z. P. (1960). *Building up mathematics*. London: Hutchinson.

Díez-Palomar, J., & Olivé, J. C. (2015). Using dialogic talk to teach mathematics: The case of interactive groups. *ZDM, 47*(7), 1299–1312. doi:10.1007/s11858-015-0728-x

Diko, N., & Feza, N. (2014). Training Mathematics teachers in South African university. *Mediterranean Journal of Social Sciences, 5*(23), 1456–1464.

Ding, L., & Domínguez, H. (2016). Opportunities to notice: Chinese prospective teachers noticing students' ideas in a distance formula lesson. *Journal of Mathematics Teacher Education*, *19*(4), 325–347. doi:10.1007/s10857-015-9301-3

Dinsmore, D. L., Alexander, P. A., & Loughlin, S. M. (2008). Focusing the conceptual lens on metacognition, self-regulation, and self-regulated learning. *Educational Psychology Review*, *20*(4), 391–409. doi:10.1007/s10648-008-9083-6

Dockendorff, M., & Solar, H. (2018). ICT integration in Mathematics initial teacher training and its impact on visualization: The case of GeoGebra. *International Journal of Mathematical Education in Science and Technology*, *49*(1), 66–84. doi:10.1080/0020739X.2017.1341060

Dohn, B. N. (2016). Explaining the significance of participationist approaches for understanding students' knowledge acquisition. *Educational Psychologist*, *51*(2), 188–209.

Donker, A. S., De Boer, H., Kostons, D., Dignath van Ewijk, C. C., & Van der Werf, M. P. C. (2014). Effectiveness of learning strategy instruction on academic performance: A meta-analysis. *Educational Research Review*, *11*, 1–26. doi:10.1016/j.edurev.2013.11.002

Dörfler, W. (2016). Signs and their use: Peirce and Wittgenstein. In A. Bikner-Ahsbahs, A. Vohns, R. Bruder, O. Schmitt & W. Dörfler (Eds.), *Theories in and of mathematics education: Theory strands in German speaking countries* (pp. 21–31). Cham: Springer International.

Dori, Y. J., Mevarech, Z. R., & Baker, D. R. (Eds.). (2018). *Cognition, metacognition, and culture in STEM education: Learning, teaching and assessment.* Cham: Springer.

Dreher, A., & Kuntze, S. (2015). Teachers' professional knowledge and noticing: The case of multiple representations in the Mathematics classroom. *Educational Studies in Mathematics*, *88*(1), 89–114. doi:10.1007/s10649-014-9577-8

Duncker, K. (1945). *On problem-solving* (Vol. 58). Washington, DC: The American Psychological Association.

Durkin, K., & Shire, B. (Eds.). (1991). *Language in mathematics education: Research and practice*. Milton Keynes: Oxford University Press.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1), 103–131. doi:10.1007/s10649-006-0400-z

Dyer, E. B., & Sherin, M. G. (2016). Instructional reasoning about interpretations of student thinking that supports responsive teaching in secondary mathematics. *ZDM*, *48*(1), 69–82. doi:10.1007/s11858-015-0740-1

Dysthe, O. (1996). The multivoiced classroom: Interactions of writing and classroom discourse. *Written Communication*, *13*(3), 385–425. doi:10.1177/0741088396013003004

Education Endowment Foundation. (2017). *Dialogic teaching: Evaluation report and executive summary July 2017*. Retrieved from https://educationendowmentfoundation.org.uk/public/files/Projects/Evaluation_Reports/Dialogic_Teaching_Evaluation_Report.pdf

Edwards, L. D. (2019). The body of/in proof: An embodied analysis of mathematical reasoning. In M. Danesi (Ed.), *Interdisciplinary perspectives on math cognition* (pp. 119–139). Cham: Springer International.

Elder, L., & Paul, R. (2010). *Critical thinking development: A stage theory*. The Foundation for Critical Thinking. Retrieved from https://www.criticalthinking.org/pages/critical-thinking-development-a-stage-theory/483

Erens, R., & Eichler, A. (2019). Belief changes in the transition from university studies to school practice. In M. S. Hannula, G. C. Leder, F. Morselli, M. Vollstedt, & Q. Zhang (Eds.), *Affect and mathematics education: Fresh perspectives on motivation, engagement, and identity* (pp. 345–373). Cham: Springer International.

Ernest, P. (1989). The knowledge, beliefs and attitudes of the Mathematics teacher: A model. *Journal of Education for Teaching*, *15*(1), 13–33.

Ernest, P. (1994). The dialogical nature of mathematics. In P. Ernest (Ed.), *Mathematics, education and philosophy: An international perspective* (pp. 33–48). Washington, DC: The Falmer Press.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. New York, NY: SUNY.

Estapa, A. T., Amador, J., Kosko, K. W., Weston, T., De Araujo, Z., & Aming-Attai, R. (2018). Preservice teachers' articulated noticing through pedagogies of practice. *Journal of Mathematics Teacher Education*, *21*(4), 387–415. doi:10.1007/s10857-017-9367-1

Farah, M. J., Hammond, K. M., Levine, D., & Calvanio, R. (1988). Visual and spatial mental imagery: Dissociable systems of representation. *Cognitive Psychology*, *20*, 439–462.

Fennema, E., & Sherman, J. A. (1978). Sex-related differences in mathematics education achievement and related factors: A further study. *Journal for Research in Mathematics Education*, *9*(3), 189–203.

Fennema, E., & Tartre, L. A. (1985). The use of spatial visualization in mathematics by girls and boys. *Journal for Research in Mathematics Education*, *16*(3), 184–206.

Fernández, C., Llinares, S., & Rojas, Y. (2020). Prospective Mathematics teachers' development of noticing in an online teacher education program. *ZDM – Mathematic Education,* 52(5), 959–972. doi:10.1007/s11858-020-01149-7

Fischer, M. H., & Shaki, S. (2014). Spatial associations in numerical cognition: From single digits to arithmetic. *Quarterly Journal of Experimental Psychology*, *67*(8), 1461–1483. doi:10.1080/17470218.2014.927515

Flavell, J. H. (1976). Metacognitive aspects of problem solving. In L. Resnick (Ed.), *The nature of intelligence* (pp. 231–236). Hillside, NJ: Erlbaum.

Flavell, J. H. (1979). Metacognition and cognitive monitoring: A new are of cognitive-developmental inquiry. *American Psychologist*, *34*(10), 906–911.

Flavell, J. H., Miller, P. H., & Miller, S. A. (2002). *Cognitive development* (4th ed.). Upper Saddle River, NJ: Prentice Hall.

Flores, A., & Park, J. (2016). Students' guided reinvention of definition of limit of a sequence with interactive technology. *Contemporary Issues in Technology & Teacher Education, 16*(2), 110–126.

Foong, L. Y. Y., Nor, M. B. M., & Nolan, A. (2018). The influence of practicum supervisors' facilitation styles on student teachers' reflective thinking during collective reflection. *Reflective Practice, 19*(2), 225–242. doi:10.1080/14623943.2018.1437406

Fox, S. (1997). Situated learning theory versus traditional cognitive learning theory: Why management education should not ignore management learning. *Systems Practice, 10*(6), 727–747. doi:10.1007/BF02557922

Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics, 1,* 3–8.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: D Reidel.

Freudenthal, H. (1979). New math or new education? *Prospects, 9(3), 321–331.*

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht: Kluwer.

Fuys, D. (1984). *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele*. Brooklyn, NY: National Science Foundation.

Gaillard, C. (2019). Finding the missing variable: A systematic review of mathematics improvement strategies for South African public schools. *South African Journal of Education, 39*(3), 1–9.

García, M., Sánchez, V., & Escudero, I. (2007). Learning through reflection in Mathematics teacher education. *Educational Studies in Mathematics*, *64*(1), 1–17. doi:10.1007/s10649-006-9021-9

Geer, E. A., Quinn, J. M., & Ganley, C. M. (2019). Relations between spatial skills and math performance in elementary school children: A longitudinal investigation. *Developmental Psychology*, *55*(3), 637–652.

Ghanizadeh, A. (2017). The interplay between reflective thinking, critical thinking, self-monitoring, and academic achievement in higher education. *Higher Education*, *74*(1), 101–114. doi:10.1007/s10734-016-0031-y

Gillies, R. M. (2016). Dialogic interactions in the cooperative classroom. *International Journal of Educational Research*, *76*, 178–189.

Gleason, N. W. (Ed.) (2018). *Higher education on the era of the Fourth Industrial Revolution*. Singapore: Palgrave Macmillan.

Goldin, G. A. (1997). Chapter 4: Observing mathematical problem solving through task-based interviews. *Journal for Research in Mathematics Education*, *9*, 40–177. doi:10.2307/749946

Golding, J., & Smith, C. (2016). *Beginner Mathematics teachers assessing advanced problem solving: What do they bring, what do they need, and how can the gap be bridged?* Paper presented at the British Society for Research into Learning Mathematics, Loughborough.

Goldin-Meadow, S. (2010). When gesture does and does not promote learning. *Language and Cognition*, *2*(1), 1–19. doi:10.1515/LANGCOG.2010.001

Goldin-Meadow, S. (2017). Using our hands to change our minds. *WIREs Cognitive Science*, *8*(1/2), e1368. doi:10.1002/wcs.1368

Goldin-Meadow, S., Kim, S., & Singer, M. (1999). What the teacher's hands tell the student's mind about math. *Journal of Educational Psychology*, *91*(4), 720–730.

Goleman, D. (1985). *Vital lies, simple truths: The psychology of self-direction*. New York, NY: Simon & Schuster.

Goodwin, C. (1994). Professional vision. *American Anthropologist*, *96*(3), 606–633. Retrieved from www.jstor.org/stable/682303

Goos, M., & Kaya, S. (2020). Understanding and promoting students' mathematical thinking: A review of research published in ESM. *Educational Studies in Mathematics*, *103*(1), 7–25. doi:10.1007/s10649-019-09921-7

Gorgorió, N., & Planas, N. (2001). Teaching Mathematics in multilingual classrooms. *Educational Studies in Mathematics, 47*(1), 7–33. doi:10.1023/A:1017980828943

Graham, M. J. (2017). *Reflective thinking in social work: Learning from student narratives*. London: Routledge.

Gravemeijer, K. (2016). Real, meaningful mathematics. *International Journal of Research in Undergraduate Mathematics Education*, *2*, 134–139. doi:10.1007/s40753-016-0026-1

Gravemeijer, K. (2020). A socio-constructivist elaboration of realistic mathematics education. In M. van den Heuvel-Panhuizen (Ed.), *National reflections on the Netherlands didactics of mathematics: Teaching and learning in the context of realistic mathematics education* (pp. 217–233). Cham: Springer International.

Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: A mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, *32*(6), 777–796. doi:10.1080/00220270050167170

Gravemeijer, K., Stephan, M., Julie, C., Lin, F.-L., & Ohtani, M. (2017). What mathematics education may prepare students for the society of the future? *International Journal of Science and Mathematics Education*, *15*(1), 105–123. doi:10.1007/s10763-017-9814-6

Graven, M. (2014). Poverty, inequality and mathematics performance: The case of South Africa’s post-apartheid context. *ZDM – Mathematics Education*, *46*, 1039–1049.

Greenwald, A. G. (1968). Cognitive learning, cognitive response to persuasion, and attitude change. In A. G. Greenwald, T. C. Brock, & T. M. Ostrom (Eds.), *Psychological foundations of attitudes* (pp. 147–170). New York, NY: Academic Press.

Grobler, R. (2018). Students’ perceptions of code-switching in Natural Sciences classrooms: A South African perspective. *Africa Education Review*, *15*(1), 38–51. doi:10.1080/18146627.2016.1224593

Grossman, P., Compron, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, *111*(9), 20555–22100.

Groth, R. E., Jones, M., & Knaub, M. (2018). A framework for characterizing students’ cognitive processes related to informal best fit lines. *Mathematical Thinking and Learning*, *20*(4), 251–276. doi:10.1080/10986065.2018.1509418

Guba, E. G. (1981). Criteria for assessing the trustworthiness on naturalistic inquiries. *Educational Communication and Technology, 29*(2), 75–91.

Guba, E. G., & Lincoln, Y. S. (1981). *Effective evaluation: Improving the usefulness of evaluation results through responsive and naturalistic approaches*. San Francisco, CA: Jossey-Bass.

Gurbuz, M. C., & Agsu, M. (2017). Dialogic teaching model for Ninth Class students to conceptualize inequalities. *Journal of Education and Practice*, *8*(28), 171–187.

Hajhosseiny, M. (2012). *The effect of dialogic teaching on students' critical thinking disposition*. Paper presented at the International Conference on Education and Educational Psychology, Istanbul.

Halliday, M. A. K. (1993). Towards a language-based theory of learning. *Linguistics and Education*, *5*, 93–116.

Halmos, P. R. (1968). Mathematics as a creative art. *American Scientist*, *56*(4), 375–389.

Halmos, P. R. (1970). Ten problems in Hilbert space. *Bulletin of the American Mathematical Society*, *76*, 887–933.

Halmos, P. R. (1980). The heart of mathematics. *The American Mathematical Monthly*, *87*(7), 519–524. doi:10.2307/2321415

Halmos, P. R. (1994). What is teaching. *The American Mathematical Monthly*, *101*(9), 848–854.

Halmos, P. R., Moise, E. E., & George, P. (1975). The problem of learning to teach. *The American Mathematical Monthly*, *82*(5), 466–476. doi:10.2307/2319737

Hanna, G., Reid, D. A., & De Villiers, M. (Eds.). (2019). *Proof technology in mathematics research and teaching*. Cham: Springer.

Hatch, L. (1988). Problem solving approach. In W. H. Kemp & A. E. Schwaller (Eds.), *Instructional strategies for technology education* (pp. 87–98). Mission Hills, CA: Council on Technology Teacher Education.

Hegedus, S. (2013). Young children investigation advanced mathematical concepts with haptic technologies: Future design perspectives. *The Mathematics Enthusiast*, *10*(1&2), 87–108.

Hennessy, S., Dragovic, T., & Warwick, P. (2018). A research-informed, school-based professional development workshop programme to promote dialogic teaching with interactive technologies. *Professional Development in Education*, *44*(2), 145–168. doi:10.1080/19415257.2016.1258653

Hermans, H. J. M. (2001). The dialogical self: Toward a theory of personal and cultural positioning. *Culture & Psychology*, *7*(3), 243–281. doi:10.1177/1354067x0173001

Heyd-Metzuyanim, E. (2013). The co-construction of learning difficulties in mathematics: Teacher–student interactions and their role in the development of a disabled mathematical identity. *Educational Studies in Mathematics, 83*(3), 341–368. doi:10.1007/s10649-012-9457-z

Heyd-Metzuyanim, E., & Graven, M. (2016). Between people-pleasing and mathematizing: South African learners' struggle for numeracy. *Educational Studies in Mathematics, 91*, 349–373.

Heyd-Metzuyanim, E., & Graven, M. (2019). Rituals and explorations in mathematical teaching and learning: Introduction to the special issue. *Educational Studies in Mathematics*, *101*, 141–151.

Heyd-Metzuyanim, E., Smith, M., Bill, V., & Resnick, L. B. (2019). From ritual to explorative participation in discourse-rich instructional practices: A case study of teacher learning through professional development. *Educational Studies in Mathematics*, *101*(2), 273–289. doi:10.1007/s10649-018-9849-9

Higginson, W. (1980). On the foundations of mathematics education. *For the Learning of Mathematics*, *1*(2), 3–7.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*(2), 371–406.

Hogan, M. J., Dwyer, C. P., Harney, O. M., Noone, C., & Conway, R. J. (2015). Metacognitive skill development and applied systems science: A framework of metacognitive skills, self-regulatory functions and real-world applications. In A. Peña-Ayala (Ed.), *Metacognition: Fundaments, applications, and trends: A profile of the current state-of-the-art* (pp. 75–106). Cham: Springer International.

Holquist, M. (1990). *Dialogism: Bakhtin and his world*. London: Routledge.

Howie, S., Combrinck, C., Roux, K., Tshele, M., Mokoena, G., & Palane, N. M. (2017). *Progress in international reading literacy study 2016: South African children's reading literacy achievement*. Pretoria: Centre for Evaluation and Assessment.

Hoyles, C. (1982). The pupil's view of mathematics learning. *Educational Studies in Mathematics*, *13*(4), 349–372.

Huang, R., & Li, Y. (2012). What matters most: A comparison of expert and novice teachers' noticing of Mathematics classroom events. *School Science and Mathematics*, *112*(7), 420–432. doi:10.1111/j.1949-8594.2012.00161.x

Hurme, T.-R., Järvelä, S., Merenluoto, K., & Salonen, P. (2015). What makes metacognition as socially shared in mathematical problem solving? In A. Peña-Ayala (Ed.), *Metacognition: Fundaments, applications, and trends: A profile of the current state-of-the-art* (pp. 259–276). Cham: Springer International.

Hutchings, M. (2007). *Introduction to mathematical arguments*. Washington, DC: Academy Press.

Ivanov, V. (1999). Heteroglossia. *Journal of Linguistic Anthropology*, *9*(1/2), 100–102. Retrieved from www.jstor.org/stable/43102437

Izzati, L. R., & Mahmudi, A. (2018). The influence of metacognition in mathematical problem solving. *Journal of Physics: Conference Series*, *1097*(1), [online]. doi:10.1088/1742-6596/1097/1/012107

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM*, *48*(1), 185–197. doi:10.1007/s11858-015-0717-0

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202. Retrieved from www.jstor.org/stable/20720130

Jagals, D., & Van der Walt, M. (2016). Enabling metacognitive skills for mathematics problem solving: A collective case study of metacognitive reflection and awareness. *African Journal of Research in Mathematics, Science and Technology Education*, *20*(2), 154–164. doi:10.1080/18117295.2016.1192239

Jansen, J. D. (1997). Can policy learn? Reflections on why OBE will fail. *Southern African Review of Education with Education with Production*, *3*(1), 5–10. Retrieved from https://journals.co.za/content/sare/3/1/AJA15634418_38

Jansen, J. D. (1998). Curriculum reform in South Africa: A critical analysis of outcomes-based education. *Cambridge journal of education*, *28*(3), 321-331.

Johnson, H. L., Dunlap, J. C., Verma, G., McClintock, E., DeBay, D. J., & Bourdeaux, B. (2019). Video-based teaching playgrounds: Designing online learning opportunities to foster professional noticing of teaching practices. *TechTrends*, *63*(2), 160–169. doi:10.1007/s11528-018-0286-5

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination and reason*. Chicago, IL: University of Chicago Press.

Kaino, L. M., & Moalosi, S. (2013). Teachers' mathematical pedagogical content knowledge: Some trends in search of adequate knowledge for effective teaching. *Journal of Social Science, 37*(2), 189–196.

Karatas, I., & Baki, A. (2013). The effects of learning environments based on problem solving on students' achievements of the problem solving. *International Electronic Journal of Elementary Education*, *5*(3), 249–268.

Kazak, S., Wegerif, R., & Fujita, T. (2015). Combining scaffolding for content and scaffolding for dialogue to support conceptual breakthroughs in understanding probability. *ZDM, 47*(7), 1269–1283. doi:10.1007/s11858-015-0720-5

Kim, M.-Y., & Wilkinson, I. A. G. (2019). What is dialogic teaching? Constructing, deconstructing, and reconstructing a pedagogy of classroom talk. *Learning, Culture and Social Interaction*, *21*, 70–86. doi:10.1016/j.lcsi.2019.02.003

Kinach, B. M. (2018). Progressive visualization tasks and semiotic chaining for Mathematics teacher preparation: Towards a conceptual framework. In N. Presmeg, L. Radford, W.-M. Roth, & G. Kadunz (Eds.), *Signs of signification: Semiotics in mathematics education research* (pp. 235–255). Cham: Springer International.

Klug, J., Krause, N., Schober, B., Finsterwald, M., & Spiel, C. (2014). How do teachers promote their students' lifelong learning in class? Development and first application of the LLL Interview. *Teaching and Teacher Education*, *37*, 119–129.

Knowles, M. S. (1975). *Self-directed learning: A guide for learners and teachers*. Englewood Cliffs, NJ: Cangridge Adult Education.

Knuth, E., Zaslavsky, O., & Ellis, A. (2019). The role and use of examples in learning to prove. *The Journal of Mathematical Behavior*, *53*, 256–262. doi:10.1016/j.jmathb.2017.06.002

Kohen, Z., & Kramarski, B. (2018). Promoting Mathematics teachers' pedagogical metacognition: A theoretical-practical model and case study. In Y. J. Dori, Z. R. Mevarech, & D. R. Baker (Eds.), *Cognition, metacognition, and culture in STEM education: Learning, teaching and assessment* (pp. 279–305). Cham: Springer International.

Korstjens, I., & Moser, A. (2018). Series: Practical guidance to qualitative research. Part 4: Trustworthiness and publishing. *European Journal of General Practice*, *24*(1), 120–124. doi:10.1080/13814788.2017.1375092

Kretzer, M. M., & Kaschula, R. H. (2019). (Unused) potentials of educators’ covert language policies at public schools in Limpopo, South Africa. *Current Issues in Language Planning*, *21*(3), 1–25. doi:10.1080/14664208.2019.1641349

Krulik, S., & Rudnick, J. A. (1982). Teaching problem solving to preservice teachers. *The Arithmetic Teacher*, *29*(6), 42–45. Retrieved from www.jstor.org/stable/41192013

Krupa, E. E., Huey, M., Lesseig, K., Casey, S., & Monson, D. (2017). Investigating secondary preservice teacher noticing of students’ mathematical thinking. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 49–72). Cham: Springer International.

Kunda, M. (2018). Visual mental imagery: A view from artificial intelligence. *Cortex*, *105*, 155–172.

Kuzle, A. (2018). Assessing metacognition of Grade 2 and Grade 4 students using an adaptation of multi-method interview approach during mathematics problem-solving. *Mathematics Education Research Journal*, *30*(2), 185–207. doi:10.1007/s13394-017-0227-1

Kvale, S. (1996). *InterViews: An introduction to qualitative research interviewing*. Thousand Oaks: Sage.

La Bestide-Van Gemert, S. (2015). *All positive action starts with criticism: Hans Freudenthal and the didactics of mathematics*. Dordrecht: Springer Science + Springer Media.

Laborde, C. (2015). Teaching and learning Geometry. In S. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 431–436). Springer: Cham. doi:10.1007/978-3-319-12688-3_35

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.

Lakoff, G. (1987). *Women, fire and dangerous things: What categories reveal about the mind*. Chicago, IL: University of Chicago Press.

Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh: The disembodied mind and its challenge to Western thought*. New York, NY: Basic Books.

Lambirth, A. (2016). Dialogic space theory. In T. David, K. Goouch, & S. Powell (Eds.), *The Routledge international handbook of philosophies and theories of early childhood education and care* (pp. 165–176). Abingdon: Routledge.

Lappan, G., & Phillips, E. (1998). Teaching and learning in the connected mathematics project. In L. Leutzinger (Ed.), *Mathematics in the middle* (pp. 83–92). Reston, VA: National Council of Teachers of Mathematics.

Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *Journal of Mathematical Behaviour*, *32*, 712–725.

Larsen, S. P. (2018). Didactical phenomenology: The engine that drives realistic mathematic education. *For the Learning of Mathematics, 38*(3), 25–29.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.

Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, *101*(2), 153–176. doi:10.1007/s10649-018-9817-4

Le Roux, K. (2017). Poor mathematics performance is created in multiple places. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 185–196). London: Routledge.

Lee, C. I. (2017). An appropriate prompts system based on the Polya method for mathematical problem-solving. *Eurasia Journal of Mathematics, Science & Technology Education, 13*(3), 893–910. doi:10.12973/eurasia.2017.00649a

Lee, N. H., Ng, K. E. D., & Yeo, J. B. W. (2019). Metacognition in the teaching and learning of mathematics. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 241–268). Singapore: Springer.

Lemke, J. L. (1990). *Talking science: Language, leaning, and values*. Norwood, NJ: Ablex.

Lemley, S. M., Ivy, J. T., Franz, D. P., & Oppenheimer, S. F. (2019). Metacognition and middle grade Mathematics teachers: Supporting productive struggle. *The Clearing House: A Journal of Educational Strategies, Issues and Ideas*, *92*(1/2), 15–22. doi:10.1080/00098655.2018.1547264

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. Englewood Cliffs, NY: Prentice Hall.

Leontiev, D. (2017). The divine knot: A relational view of meaning. *Journal of Constructivist Psychology*, *30*(1), 50–56. doi:10.1080/10720537.2015.1119081

Lerman, S. (1983). Problem-solving or knowledge-centred: The influence of philosophy on mathematics teaching. *International Journal of Mathematical Education in Science and Technology*, *14*(1), 59–66. doi:10.1080/0020739830140109

Leung, A., & Lee, A. M. S. (2013). Students' geometrical perception on a task-based dynamic geometry platform. *Educational Studies in Mathematics*, *82*(3), 361–377. doi:10.1007/s10649-012-9433-7

Lewis, K. E., & Fisher, M. B. (2018). Clinical interviews: Assessing and designing mathematics instruction for students with disabilities. *Intervention in School and Clinic*, *53*(5), 283–291.

Li, Y., & Schoenfeld, A. H. (2019). Problematizing teaching and learning mathematics as “given” in STEM education. *International Journal of Stem Education*, *6*(44), 1–13. doi:10.1186/s40594-019-0197-9

Liebendörfer, M., & Schukajlow, S. (2017). Interest development during the first year at university: Do mathematical beliefs predict interest in mathematics? *ZDM*, *49*(3), 355–366. doi:10.1007/s11858-016-0827-3

Liljedahl, P., Santos-Trigo, M., Malaspina, U., & Bruder, R. (2016). *Problem solving in mathematics education.* Hamburg: Springer Open.

Lin, Y.-W., Tseng, C.-L., & Chiang, P.-J. (2017). The effect of blended learning in mathematics course. *Eurasia Journal of Mathematics, Science & Technology Education*, *13*(3), 741–770. doi:10.12973/eurasia.2017.00641a

Lisa, M. J. (2016). Supporting teacher noticing of students’ mathematical strengths. *Mathematics Teacher Educator*, *4*(2), 188–199. doi:10.5951/mathteaceduc.4.2.0188

Littleton, K., & Mercer, N. (2013). *Interthinking: Putting talk into work*. Abingdon: Routledge.

Livers, S. D., Harbour, K. E., & Fowler, L. (2019). Danger! Animals in the Mathematics classroom. *Teaching Children Mathematics*, *25*(7), 406–414. Retrieved from www.jstor.org/stable/10.5951/teacchilmath.25.7.0406

Lockhart, P. (2009). *A mathematician’s lament*. New York, NY: Bellevue Literary Press.

Lodge, D. (1990). Conrad’s ‘victory’ and ‘the tempest’: An amplification. *The Morden Language Review*, *59*(2), 195–199.

Louie, N. L. (2018). Culture and ideology in Mathematics teacher noticing. *Educational Studies in Mathematics*, *97*(1), 55–69. doi:10.1007/s10649-017-9775-2

Lowrie, T., Logan, T., & Hegarty, M. (2019). The influence of spatial visualization training on students’ spatial reasoning and mathematics performance. *Journal of Cognition and Development*, *20*(5), 729–751. doi:10.1080/15248372.2019.1653298

Luneta, K. (2014). Foundation Phase teachers’ (limited) knowledge of geometry. *South African Journal of Childhood Education*, *4*(3), 71–86.

Luneta, K. (2015). Understanding students’ misconceptions: An analysis of final Grade 12 examination questions in geometry. *Pythagoras*, *36*(1), 1–11. doi:10.4102/pythagoras.v36i1.261

Lunsford, A. A. (1990). Composing ourselves: Politics, commitment, and the teaching of writing. *College Composition and Communication*, *41*(1), 71–82.

Macedonia, M. (2019). Embodied learning: Why at school the mind needs the body. *Frontiers in Psychology*, *10*(2098), 1–8. doi:10.3389/fpsyg.2019.02098

Machaba, F., & Mwakapenda, W. (2016). Learners' approaches to solving mathematical tasks: Does specialisation matter? *Africa Education Review*, *13*(3/4), 172–190. doi:10.1080/18146627.2016.1224559

Machisi, E. (2021). Grade 11 students' reflections on their Euclidean geometry learning experiences. *Eurasia Journal of Mathematics, Science and Technology Education*, *17*(2), 1–19.

Magana, A. J., Serrano, M. I., & Rebello, N. S. (2019). A sequenced multimodal learning approach to support students' development of conceptual learning. *Journal of Computer Assisted Learning*, *35*, 516–528.

Maher, C. A., & Sigley, R. (2014). Task-based interviews in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 821–824). Dordrecht: Springer.

Males, L. M. (2017). Using video of peer teaching to examine Grades 6–12 preservice teachers' noticing. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 91–109). Cham: Springer International.

Maluleke, M. J. (2019). Using code-switching as an empowerment strategy in teaching mathematics to learners with limited proficiency in English in South African schools. *South African Journal of Education*, *39*, 1–9. Retrieved from http://www.scielo.org.za/scielo.php?script=sci_arttext&pid=S0256-01002019000300006&nrm=iso

Manin, Y. I. (1977). *A course in mathematical logic*. New York, NY: Springer.

Maodzwa–Taruvinga, M., & Cross, M. (2012). Jonathan Jansen and the curriculum debate in South Africa: An essay review of Jansen's writings between 1999 and 2009. *Curriculum Inquiry*, *42*(1), 126–152. doi:10.1111/j.1467-873X.2011.00573.x

Marais, P. (2016). "We can't believe what we see": Overcrowded classrooms through the eyes of student teachers. *South African Journal of Education*, *36*(2), 1–10. doi:10.15700/saje.v36n2a1201

Marulis, L. M., Palincsar, A. S., Berhenke, A. L., & Whitebread, D. (2016). Assessing metacognitive knowledge in 3–5 year olds: The development of a metacognitive knowledge interview (McKI). *Metacognition and Learning, 11*(3), 339–368. doi:10.1007/s11409-016-9157-7

Mason, J. (1982). Attention. *For the Learning of Mathematics, 2,* 21–23.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. London: Addison-Wesley.

Matshipi, M. G., Mulaudzi, N. O., & Mashau, T. S. (2017). Causes of overcrowded classes in rural primary schools. *Journal of Social Sciences*, *51*(1–3), 109–114. doi:10.1080/09718923.2017.1305568

Matusov, E. (2007). Applying Bakhtin scholarship on discourse in education: A critical review essay. *Educational Theory*, *57*(2), 215–237.

Matusov, E., & Miyazaki, K. (2014). Dialogue on dialogic pedagogy. *Dialogic Pedagogy: An International Online Journal*, *2,* 1–47. doi:10.5195/dpj.2014.121

Matusov, E., & Von Duyke, K. (2009). *Bakhtin's notion of the internally persuasive discourse in education: Internal to what?* Paper presented at the Perspectives and limits of dialogism in Mikhail Bakhtin, Stockholm University, Stockholm.

Matusov, E., Marjanovic-Shane, A., & Gradovski, M. (Eds.). (2019). *Dialogic pedagogy and polyphonic research art: Bakhtin by and for educators*. New York, NY: Palgrave Macmillan.

Mauthner, N. S., & Doucet, A. (2003). Reflexive accounts and accounts of reflexivity in qualitative data analysis. *Sociology, 37*(3), 413–431. doi:10.1177/00380385030373002

Mavuru, L., & Ramnarain, U. D. (2019). Language affordances and pedagogical challenges in multilingual Grade 9 Natural Sciences classrooms in South Africa. *International Journal of Science Education*, 42(14), 2472–2492. doi:10.1080/09500693.2019.1655177

Mayaba, N. N., Ralarala, M. K., & Angu, P. (2018). Student voice: Perspectives on language and critical pedagogy in South African higher education. *Educational Research for Social Change*, *7*, 1–12. Retrieved from http://www.scielo.org.za/scielo.php?script=sci_arttext&pid=S2221-40702018000100002&nrm=iso

Mayberry, J. (1983). The Van Hiele levels of geometry thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education*, *14*(1), 58–69.

Mbhiza, H. (2021). Rural teachers' teaching of algebraic functions through a commognitive lens. *Interdisciplinary Journal of Rural and Community Studies*, *3*(1), 10–20.

McDonough, A., & Sullivan, P. (2014). Seeking insights into young children's beliefs about mathematics and learning. *Educational Studies in Mathematics*, *87*(3), 279–296. doi:10.1007/s10649-014-9565-z

Mehan, H. (1979). "What time is it, Denise?": Asking known information questions in classroom discourse. *Theory into Practice*, *18*(4), 285–294.

Mercer, N. (2000). *Words and minds: How we use language to think together*. London: Routledge.

Mercer, N. (2001). Why I study interthinking. *The Psychologist*, *14*(12), 648–649.

Mercer, N. (2008). Talk and the development of reasoning and understanding. *Human Development*, *51*(1), 90–100.

Mercer, N., & Dawes, L. (2008). The value of exploratory talk. In N. Mercer & J. Hodgkinson (Eds.), *Exploring talk in school: Inspired by the work of Douglas Barnes* (pp. 55–72). Thousand Oaks, CA: Sage.

Mercer, N., Fernandez, M., Dawes, L., Wegerif, R., & Sams, C. (2003). Talk about texts at the computer: Using ICT to develop children's oral and literate abilities. *Reading, Literacy and Language*, *1*(1), 81–89.

Mercer, N., Phillips, T., & Somekh, B. (1991). Spoken language and new technology (SLANT). *Journal of Computer Assisted Learning*, *7*(3), 195–202.

Mhlolo, M. K. (2017). Regular classroom teachers' recognition and support of the creative potential of mildly gifted mathematics learners. *ZDM*, *49*(1), 81–94. doi:10.1007/s11858-016-0824-6

Middendorf, M. (1992). Bakhtin and the dialogic writing class. *Journal of Basic Writing*, *11*(1), 34–46.

Mihalca, L., Mengelkamp, C., & Schnotz, W. (2017). Accuracy of metacognitive judgments as a moderator of learner control effectiveness in problem-solving tasks. *Metacognition and Learning*, *12*(3), 357–379. doi:10.1007/s11409-017-9173-2

Mitchell, R. N., & Marin, K. A. (2015). Examining the use of a structured analysis framework to support prospective teacher noticing. *Journal of Mathematics Teacher Education*, *18*(6), 551–575. doi:10.1007/s10857-014-9294-3

Mitchell, T. M., Utgoff, P. E., & Banerji, R. (1983). Learning by experimentation: Acquiring and refining problem-solving heuristics. In R. S. Michalski, J. G. Carbonell, & T. M. Mitchell (Eds.), *Machine learning: Symbolic computation* (pp. 163–190). Berlin: Springer.

Mix, K. S. S., & Battista, M. (Eds.). (2018). *Visualizing mathematics: The role of spatial reasoning in mathematics thought.* Cham: Springer.

Moe, A., Jansen, P., & Pietsch, S. (2018). Childhood preference of spatial toys. Gender differences and relationships with mental rotation in STEM and non-STEM students. *Learning and Individual Differences*, *68*, 108–115.

Mogari, D. (2003). A relationship between attitude and achievement in Euclidean geometry of Grade 10 pupils. *African Journal of Research in Mathematics, Science and Technology Education*, *7*(1), 63–72. doi:10.1080/10288457.2003.10740549

Mogari, D., & Chirove, M. (2017). Comparing Grades 10-12 Mathematics learners' non-routine problem solving. *Eurasia Journal of Mathematics Science and Technology Education*, *13*(8), 4523–4551. doi:10.12973/eurasia.2017.00946a

Morgan, C., & Tang, S. (2016). To what extent are students expected to participate in specialised mathematical discourse? Change over time in school mathematics in England. *Research in Mathematics Education*, *18*(2), 142–164. doi:10.1080/14794802.2016.1174145

Morrison, S. (2013). Exploring links between Foundation Phase teachers' content knowledge and their example spaces. *South African Journal of Childhood Education*, *3*(2), 96–111.

Morse, J. (2018). Reframing rigor in qualitative inquiry. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage handbook of qualitative research* (5th ed., pp. 1373 - 1409). Thousand Oaks, CA: Sage.

Moschkovich, J. (2007). Examining mathematical discourse practices. *For the Learning of Mathematics*, *27*(1), 24–30.

Moseley, D., Baumfield, V., Elliott, J., Higgins, S., Miller, J., Newton, D. P., & Gregson, M. (2005). *Frameworks for thinking: A handbook for teaching and learning*. Cambridge: Cambridge University Press.

Mpofu, S., & Pournara, C. (2018). Learner participation in the functions discourse: A focus on asymptotes of the hyperbola. *African Journal of Research in Mathematics, Science and Technology Education*, *22*(1), 2–13.

Mudaly, V. (2015). Pre-service teacher knowledge: Thinking about conceptual understanding. *Journal of Social Sciences*, *44*(2/3), 160–168. doi:10.1080/09718923.2015.11893475

Mudaly, V., & Mpofu, S. (2019). Learners' views on asymptotes of a hyperbola and exponential function: A commognitive approach. *Problems of Education in the 21st Century*, *77*(6), 734–744.

Mudaly, V., & Rampersad, R. (2010). The role of visualisation in learners' conceptual understanding of graphical functional relationships. *African Journal of Research in Mathematics, Science and Technology Education, 14*(1), 36–48. doi:10.1080/10288457.2010.10740671

Nachlieli, T., & Michal, T. (2019). Ritual-enabling opportunities-to-learn in Mathematics classrooms. *Educational Studies in Mathematics*, *101*(2), 253–271.

Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom: The case of function. *International Journal of Educational Research*, *51/52*, 10–27.

Nachlieli, T., & Tabach, M. (2019). Ritual-enabling opportunities-to-learn in Mathematics classrooms. *Educational Studies in Mathematics*, *101*(2), 253–271. doi:10.1007/s10649-018-9848-x

Naidoo, J., & Kapofu, W. (2020). Exploring female learners' perceptions of learning geometry in mathematics. *South African Journal of Education*, *40*, 1–11. Retrieved from http://www.scielo.org.za/scielo.php?script=sci_arttext&pid=S0256-01002020000100014&nrm=iso

Nardi, E., Ryve, A., Stadler, E., & Viirman, O. (2014). Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of calculus. *Research in Mathematics Education*, *16*(2), 182–198.

Naroth, C., & Luneta, K. (2015). Implementing the Singapore Mathematics curriculum in South Africa: Experiences of Foundation Phase teachers. *African Journal of Research in Mathematics Science and Technology Education*, *19*(3), 267–277. doi:10.1080/10288457.2015.1089675

Nathan, M. J., & Walkington, C. (2017). Grounded and embodied mathematical cognition: Promoting mathematical insight and proof using action and language. *Cognitive Research: Principles and Implications*, *2*(1), 1–20. doi:10.1186/s41235-016-0040-5

Nathan, M. J., Walkington, C., Boncoddo, R., Pier, E., Williams, C. C., & Alibali, M. W. (2014). Actions speak louder with words: The roles of action and pedagogical language for grounding mathematical proof. *Learning and Instruction*, *33*, 182–193. doi:10.1016/j.learninstruc.2014.07.001

National Council of Teachers of Mathematics. (2000). *Principles and standards for teaching school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Nelson, L. J., & Fyfe, E. R. (2019). Metacognitive monitoring and help-seeking decisions on mathematical equivalence problems. *Metacognition and Learning*, *14*(2), 167–187. doi:10.1007/s11409-019-09203-w

Nesari, A. J. (2015). Dialogism versus monologism: A Bakhtinian approach to teaching. *Procedia - Social and Behavioral Sciences, 205,* 642–647. doi:10.1016/j.sbspro.2015.09.101

Newcombe, N. S., & Stieff, M. (2012). Six myths about spatial thinking. *International Journal of Science Education*, *34*(6), 955–971. doi:10.1080/09500693.2011.588728

Ngirishi, H., & Bansilal, S. (2019). An exploration of high school learners' understanding of geometric concepts. *Problems of Education in the 21st Century*, *77*(1), 82–96.

Nickerson, S. D., Lamb, L., & LaRochelle, R. (2017). Challenges in measuring secondary Mathematics teachers' professional noticing of students' mathematical thinking. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 381–398). Cham: Springer International.

Nomlomo, V., & Katiya, M. (2018). Multilingualism and (bi)literacy development for epistemological access: Exploring students experience in the use of multilingual glossaries at a South African university. *Educational Research for Social Change*, *7,* 77–93. http://www.scielo.org.za/scielo.php?script=sci_arttext&pid=S2221-40702018000100006&nrm=iso

Novack, M., & Goldin-Meadow, S. (2015). Learning from gesture: How our hands change our minds. *Educational Psychology Review*, *27*(3), 405–412. doi:10.1007/s10648-015-9325-3

Oehler, K. (1987). An outline of Peirce's semiotics. In M. Krampen, K. Oehler, R. Posner, T. A. Sebeok, & T. von Uexküll (Eds.), *Classics of semiotics* (pp. 1–21). Boston, MA: Springer.

Otte, M. F. (2006). Mathematical epistemology from a Peircean semiotic point of view. *Educational Studies in Mathematics*, *61*(1/2), 11–38. Retrieved from www.jstor.org/stable/25472059

Otte, M. F. (2011). Evolution, learning, and semiotics from a Peircean point of view. *Educational Studies in Mathematics*, *77*(2), 313–329. doi:10.1007/s10649-011-9302-9

Parker, D. (2006). Grade 10–12 Mathematics curriculum reform in South Africa: A textual analysis of new National Curriculum Statements. *African Journal of Research in Mathematics, Science and Technology Education, 10*(2), 59–73. doi:10.1080/10288457.2006.10740605

Parker, D., & Adler, J. (2005). Constraint or catalyst? The regulation of teacher education in South Africa. *Journal of Education*, *36*(1), 59–78.

Pearl, J. (1984). *Heuristics: Intelligent search strategies for computer problem solving*. Reading: Addison-Wesley.

Pennycook, G., & Rand, D. G. (2019). Lazy, not biased: Susceptibility to partisan fake news is better explained by lack of reasoning than by motivated reasoning. *Cognition*, *188*, 39–50.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 257–315). Charlotte, NC: Information Age Publication.

Piaget, J. (1953). How children form mathematical concepts. *Scientific American*, *189*(5), 74–79.

Piaget, J. (1964). Part I: Cognitive development in children: Piaget development and learning *Journal of Research in Science Teaching*, *2*, 176–186.

Piaget, J. (1969). *The psychology of the child*. Basic Books: Basic Books.

Piaget, J., & Inhelder, B. (1971). *Mental imagery and the child*. London: Routledge & Kegan Paul.

Planas, N., & Setati-Phakeng, M. (2014). On the process of gaining language as a resource in mathematics education. *ZDM*, *46*(6), 883–893. doi:10.1007/s11858-014-0610-2

Plowright, D. (2016). *Charles Sanders Peirce: Pragmatism and education*. London: Springer.

Polya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Polya, G. (1962). *Mathematical discovery: On understanding, learning, and teaching problem solving*. New York, NY: Wiley.

Polya, G. (1973). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press.

Pournara, C., Hodgen, J., Adler, J., & Pillay, V. (2015). Can improving teachers' knowledge of mathematics lead to gains in learners' attainment in mathematics? *South African Journal of Education*, *35*(3), 1–10.

Prediger, S., & Zindel, C. (2017). Deepening prospective Mathematics teachers' diagnostic judgements: Interplay of videos, focus questions and didactic categories. *European Journal of Science and Mathematics Education*, *5*(3), 222–242.

Presmeg, N. C. (1986). Visualization in high school mathematics. *For the Learning of Mathematics*, *6*(3), 42–46.

Presmeg, N. C. (1997). Generalization using imagery in mathematics. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images* (pp. 307 – 320). New York: Lawrence Erlbaum.

Presmeg, N. C. (2014). Contemplating visualization as an epistemological learning tool in mathematics. *ZDM*, *46*(1), 151–157. doi:10.1007/s11858-013-0561-z

Presmeg, N. C., Radford, L., Roth, W.-M., & Kadunz, G. (Eds.). (2018). *Signs of significations: Semiotics in mathematics education research.* Hamburg: Springer.

Probyn, M. (2019). Pedagogical translanguaging and the construction of science knowledge in a multilingual South African classroom: Challenging monoglossic/post-colonial orthodoxies. *Classroom Discourse*, *10*(3/4), 216–236. doi:10.1080/19463014.2019.1628792

Pudi, T. (2006). 'From OBE to C2005 to RNCS': Are we still on track? *Africa Education Review*, *3*(1/2), 100–112. doi:10.1080/18146620608540445

Pudi, T. (2006). 'From OBE to C2005 to RNCS': Are we still on track?. *Africa Education Review*, *3*(1-2), 100-112.

Putnam, R. T. (1987). Structuring and adjusting content for students: A study of live and simulated tutoring of addition. *American Educational Research Journal*, *24*(1), 13–48. doi:10.3102/00028312024001013

Putnam, R. T., & Leinhardt, G. (1986). *Curriculum scripts and adjustment of content in lessons*. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA.

Rabel, S., & Wooldridge, I. (2013). Exploratory talk in mathematics: What are the benefits? *International Journal of Primary, Elementary and Early Years Education*, *41*(1), 15–22. doi:10.1080/03004279.2012.710095

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning*, *5*(1), 37–70.

Radford, L., Edwards, L., & Arzarello, F. (2009). Introduction: Beyond words. *Educational Studies in Mathematics*, *70*(2), 91–95. doi:10.1007/s10649-008-9172-y

Radford, L., Schubring, G., & Seeger, F. (2011). Signifying and meaning-making in mathematical thinking, teaching, and learning. *Educational Studies in Mathematics*, *77*(2), 149–156. doi:10.1007/s10649-011-9322-5

Radmehr, F., & Drake, M. (2017). Exploring students' mathematical performance, metacognitive experiences and skills in relation to fundamental theorem of calculus.

International Journal of Mathematical Education in Science and Technology, *48*(7), 1043–1071. doi:10.1080/0020739X.2017.1305129

Reddy, V., Visser, M., Winnaaar, L., Arends, F., Juan, A., Prinsloo, C., & Isdale, K. (2016). *TIMSS 2015: Highlights of mathematics and science achievement of Grade 9 South African learners.* Cape Town: HSRC Press.

Rellensmann, J., Schukajlow, S., & Leopold, C. (2020). Measuring and investigating strategic knowledge about drawing to solve geometry modelling problems. *ZDM*, *52*(1), 97–110. doi:10.1007/s11858-019-01085-1

Richardson, A. (1969). *Mental imagery*. New York, NY: Springer.

Rivera, F. D., Steinbring, H., & Arcavi, A. (2014). Visualization as an epistemological learning tool: An introduction. *ZDM*, *46*(1), 1–2. doi:10.1007/s11858-013-0552-0

Roberts, A., & Le Roux, K. (2019). A commognitive perspective on Grade 8 and Grade 9 learner thinking about linear equations. *Pythagoras*, *40*(1), 1–15.

Rodan, A., Gimeno, P., Elosua, M. R., Montoro, P. R., & Contreras, M. J. (2019). Boys and girls gain in spatial, but not in mathematical ability after mental rotation in primary education. *Learning and Individual Differences*, *70*, 1–11.

Roller, S. A. (2016). What they notice in video: A study of prospective secondary Mathematics teachers learning to teach. *Journal of Mathematics Teacher Education*, *19*(5), 477–498. doi:10.1007/s10857-015-9307-x

Rott, B., Torner, G., Peters-Dasdemir, J., & Safrudiannur, A. M. (Eds.). (2018). *Views and beliefs in mathematics education: The role of beliefs in the classroom*. Cham: Springer.

Rowlands, S., Graham, T., & Berry, J. (2011). Problems with fallibilism as a philosophy of mathematics education. *Science & Education*, *20*(7), 625–654. doi:10.1007/s11191-010-9234-2

Saldana, J. (2011). *Fundamentals of qualitative research*. New York, NY: Oxford University Press.

Saldana, J. (2016). *The coding manual for qualitative researchers* (3rd ed.). Los Angeles, CA: Sage.

Sanchez, J. M. C. (2003). Considering dialogue as a social instrument in the Mathematics class. *For the Learning of Mathematics*, *23*, 30–40.

Sánchez-Matamoros, G., Fernández, C., & Llinares, S. (2019). Relationships among prospective secondary Mathematics teachers' skills of attending, interpreting and responding to students' understanding. *Educational Studies in Mathematics*, *100*(1), 83–99. doi:10.1007/s10649-018-9855-y

Santagata, R., & Yeh, C. (2016). The role of perception, interpretation, and decision making in the development of beginning teachers' competence. *ZDM*, *48*(1), 153–165. doi:10.1007/s11858-015-0737-9

Santagata, R., Zannoni, C., & Stigler, J. W. (2007). The role of lesson analysis in pre-service teacher education: An empirical investigation of teacher learning from a virtual video-based field experience. *Journal of Mathematics Teacher Education*, *10*(2), 123–140. doi:10.1007/s10857-007-9029-9

Scheiner, T. (2016). Teacher noticing: Enlightening or blinding? *ZDM*, *48*(1), 227–238. doi:10.1007/s11858-016-0771-2

Schoenfeld, A. H. (1982). Some thoughts on problem-solving research and mathematics education. In F. K. Lester & J. Garofalo (Eds.), *Mathematical problem solving: Issues in research* (pp. 27–37). Philadelphia, PA: Franklin Institute Press.

Schoenfeld, A. H. (1983). Beyond the purely cognitive: Belief systems, social cognitions, and metacognitions as driving forces in intellectual performance. *Cognitive Science*, *7*(4), 329–363. doi:10.1207/s15516709cog0704_3

Schoenfeld, A. H. (1985). *Mathematical problem solving*. London: Academic Press.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189–215). Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education*, *20*(4), 338–355.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. New York, NY: Routledge.

Schoenfeld, A. H. (2013). Reflections on problem solving theory and practice. *The Mathematics Enthusiast*, *10*(1/2), 9–34.

Schoenfeld, A. H., & Kilpatrick, J. (2008). Toward a theory of proficiency in teaching Mathematics. In D. Tirosh & T. L. Wood (Eds.), *International handbook of Mathematics teacher education* (2nd ed., Vol. 2: Tools and processes in Mathematics teacher education, pp. 321–354). Brill: Sense.

Schwab, K. (2016). *The fourth industrial revolution*. Geneva, Switzerland, World Economic Forum.

Sedova, K. (2017). A case study of a transition to dialogic teaching as a process of gradual change. *Teaching and Teacher Education*, *67*, 278–290.

Seswono, T. Y. E., Kohar, A. W., & Hartono, S. (2017). Secondary teachers' mathematics-related beliefs and knowledge about mathematical problem-solving. *Journal of Physics: Conference Series*, *812*, 1–7.

Setati, M. (2005). Teaching Mathematics in a primary multilingual classroom. *Journal for Research in Mathematics Education*, *36*(5), 447–466. doi:10.2307/30034945

Sevinc, S., & Lesh, R. (2018). Training Mathematics teachers for realistic math problems: A case of modeling-based teacher education courses. *ZDM – Mathematics Education*, *50*, 301–314. doi:10.1007/s11858-017-0898-9

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, *14*(1), 44–55.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4–13.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, *46*(1/3), 13–57. Retrieved from www.jstor.org/stable/3483239

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

Sfard, A. (2009). Moving between discourses: From learning-as-acquisition to learning-as-participation. In *AIP Conference Proceedings*, *1179*(1), 55–58.

Sfard, A. (2012). Discourse. In N. M. Seel (Ed.), *Encyclopedia of the sciences of learning* (pp. 997–1001). New York, NY: Springer.

Sfard, A. (2014). University mathematics as a discourse: Why, how, and what for? *Research in Mathematics Education*, *16*(2), 199–203.

Sfard, A. (2015). Metaphors for learning. In R. Gunstone (Ed.), *Encyclopedia of science education* (pp. 634–638). Dordrecht: Springer.

Sfard, A. (2017a). Ritual for ritual, exploration for exploration or, what learners are offered is what you get from them in return. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 39–63). London: Routledge.

Sfard, A. (2017b). Teaching mathematics as an exploratory activity: A letter to the teacher. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming* *researchers' insights into improvement in mathematics teaching and learning* (pp. 135–144). London: Routledge.

Sfard, A. (2020). Commognition. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 95–101). Cham: Springer.

Sfard, A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? Early numerical thinking revisited. *Cognition and Instruction*, *23*(2), 237–309.

Shapiro, L. (2007). The embodied cognition research programme. *Philosophy Compass*, *2*(2), 338–346. doi:10.1111/j.1747-9991.2007.00064.x

Shepard, R. N. (1978). The mental image. *American Psychologist*, *33*(2), 125–137.

Sherin, M. G. (2001). Developing a professional vision of classroom events. In T. L. Wood, B. Nelson, & J. Warfield (Eds.), *Beyond classical pedagogy: Teaching mathematical school mathematics* (pp. 75–93). Mahwah, NJ: Erlbaum.

Sherin, M. G. (2007). The development of teachers' professional vision in video clubs. In R. Goldman, R. Pea, B. Barron, & S. J. Derry (Eds.), *Video research in the learning sciences* (pp. 383–395). Mahwah, NJ: Erlbaum.

Sherin, M. G., & Han, S. Y. (2004). Teacher learning in the context of a video club. *Teaching and Teacher Education*, *20*(2), 163–183. doi:10.1016/j.tate.2003.08.001

Sherin, M. G., & Lynn, J. (2019). Attending to evidence of students' thinking during instruction. *Mathematics Teaching in the Middle School*, *24*(7), 430–435. Retrieved from www.jstor.org/stable/10.5951/mathteacmiddscho.24.7.0430

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011). Situating the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teacher's eyes* (pp. 3–14). New York, NY: Routledge.

Sherman, J. A. (1979). Predicting mathematics performance in high school girls and boys. *Journal of Educational Psychology*, *71*(2), 242–249.

Sherman, J. A. (1980). Mathematics, spatial visualization, and related factors: Changes in girls and boys, grades 8–11. *Journal of Educational Psychology*, *72*(4), 476–482.

Shilo, A., & Kramarski, B. (2019). Mathematical-metacognitive discourse: How can it be developed among teachers and their students? Empirical evidence from a videotaped lesson and two case studies. *ZDM*, *51*(4), 625–640. doi:10.1007/s11858-018-01016-6

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1–22.

Sinclair, J. M., & Coulthard, M. (1975). *Towards an analysis of discourse: The English used by teachers and pupils*. London: Oxford University Press.

Sinclair, N., Bartolini Bussi, M. G., De Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: An ICME-13 survey team report. *ZDM*, *48*(5), 691–719. doi:10.1007/s11858-016-0796-6

Siposova, B., Tomasello, M., & Carpenter, M. (2018). Communicative eye contact signals a commitment to cooperate for young children. *Cognition*, *179*, 192–201. doi:10.1016/j.cognition.2018.06.010

Siyepu, S. W., & Ralarala, M. K. (2014). Making sense of mathematical discourse: Implications for success in the learning of differentiation in a university classroom. *Alternation*, *12*, 326–357.

Skilling, K., Bobis, J., Martin, A. J., Anderson, J., & Way, J. (2016). What secondary school teachers think and do about student engagement in mathematics. *Mathematics Education Research Journal*, *28*(4), 545–566.

Skinner, B. F. (1971). *Beyond freedom and dignity*. New York, NY: Penguin Books.

Skinner, B. F. (1974). *About behaviorism*. New York, NY: Vintage Books.

Skulmowski, A., & Rey, G. D. (2017). Bodily effort enhances learning and metacognition: Investigating the relation between physical effort and cognition using dual-process models of embodiment. *Advances in Cognitive Psychology*, *13*(1), 3–10.

Snapper, E. (1979). The three crises in mathematics: Logicism, intuitionism and formalism. *Mathematics Magazine*, *52*(4), 207–216. doi:10.2307/2689412

Soodla, P., Jõgi, A.-L., & Kikas, E. (2017). Relationships between teachers' metacognitive knowledge and students' metacognitive knowledge and reading achievement. *European Journal of Psychology of Education*, *32*(2), 201–218. doi:10.1007/s10212-016-0293-x

Sorby, S., Veurink, N., & Streiner, S. (2018). Does spatial skills improve STEM outcomes? The answer is yes. *Learning and Individual Differences*, *67*, 209–222.

Spaull, N. (2013a). Poverty & privilege: Primary school inequality in South Africa. *International Journal of Educational Development*, *33*(5), 436–447. doi:10.1016/j.ijedudev.2012.09.009

Spaull, N. (2013b). South Africa's education crisis: The quality of education in South Africa 1994-2011. *Johannesburg: Centre for Development and Enterprise*, *21*(1), 1–65.

Spaull, N. (2015). Schooling in South Africa: How low quality education becomes a poverty trap. *South African Child Gauge, 12*, 34–41.

Spaull, N., & Kotze, J. (2015). Starting behind and staying behind in South Africa: The case of insurmountable learning deficits in mathematics. *International Journal of Educational Development*, *41*, 13–24.

Spillane, J. P., Hopkins, M., & Sweet, T. M. (2017). School district educational infrastructure and change at scale: Teacher peer interactions and their beliefs about mathematics instruction. *American Educational Research Journal*, *55*(3), 532–571. doi:10.3102/0002831217743928

Stanic, G., & Kilpatrick, J. (1988). Historical perspectives on problem solving in the Mathematics curriculum. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 1–22). Reston, VA: National Council of Teachers of Mathematics.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice Mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education, 11*(2), 107–125. doi:10.1007/s10857-007-9063-7

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455–488. doi:10.2307/1163292

Stockero, S. L., Rupnow, R. L., & Pascoe, A. E. (2017). Learning to notice important student mathematical thinking in complex classroom interactions. *Teaching and Teacher Education*, *63*, 384–395. doi:10.1016/j.tate.2017.01.006

Stoet, G., & Geary, D. C. (2017). Students in countries with higher levels of religiosity perform lower in science and mathematics. *Intelligence, 62*, 71–78. doi:10.1016/j.intell.2017.03.001

Stols, G., Ferreira, R., Pelser, R., Olivier, W. A., Van der Merwe, A., De Villiers, C., & Venter, S. (2015). Perceptions and needs of South African Mathematics teachers concerning their use of technology for instruction. *South African Journal of Education, 35*(4), 1–13.

Stott, A. E. (2018). Are instructivist pedagogies more appropriate for learning the sciences in South African low-quintile schools than Western literature suggests? *Journal of Education*, (71), 39–57.

Suhor, C. (1984). Towards a semiotics-based curriculum. *Journal of Curriculum Studies*, *16*(3), 247–257. doi:10.1080/0022027840160304

Superfine, A. C., Amador, J., & Bragelman, J. (2019). Facilitating video-based discussions to support prospective teacher noticing. *The Journal of Mathematical Behavior*, *54*, 1–18. doi:10.1016/j.jmathb.2018.11.002

Susilawati, W., Suryadi, D., & Dahlan, J. A. (2017). The improvement of mathematical spatial visualization ability of student through cognitive conflict. *International Electronic Journal of Mathematics Education*, *12*(2), 155–166.

Swart, F., De Graaff, R., Onstenk, J., & Knèzic, D. (2018). Teacher educators' conceptualization of ongoing language development in professional learning and teaching. *Professional Development in Education*, *44*(3), 412–427. doi:10.1080/19415257.2017.1345775

Tachie, S. A. (2019). Meta-cognitive skills and strategies application: How this helps learners in mathematics problem-solving. *Eurasia Journal of Mathematics Science and Technology Education*, *15*(5), 1–12.

Tachie, S. A., & Molepo, J. M. (2019). Exploring teachers' meta-cognitive skills in Mathematics classes in selected rural primary schools in Eastern Cape, South Africa. *Africa Education Review*, *16*(2), 143–161. doi:10.1080/18146627.2017.1384700

Tall, D. (1989). The nature of mathematical proof. *Mathematics Teaching*, *127*, 28–32.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (Vol. 11, pp. 3–21). Dordrecht: Kluwer Academic.

Tall, D. (1997). *From school to university: The transition from elementary to advanced mathematical thinking*. Paper presented at the Australian Bridging Conference in Mathematics, Auckland University, Auckland.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Taylor, N., Van der Berg, S., & Mabogoane, T. (2013). Context, theory, design. In N. Taylor, S. van der Berg, & T. Mabogoane (Eds.), *Creating effective schools* (pp. 1–30). Cape Town: Pearson.

Teo, P. (2016). Exploring the dialogic space in teaching: A study of teacher talk in the pre-university classroom in Singapore. *Teaching and Teacher Education*, *56*, 47–60. doi:10.1016/j.tate.2016.01.019

Teo, P. (2019). Teaching for the 21st century: A case for dialogic pedagogy. *Learning, Culture and Social Interaction*, *21*, 170–178. doi:10.1016/j.lcsi.2019.03.009

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15*(2), 105–127. doi:10.1007/BF00305892

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics* (pp. 127–146). New York, NY: Macmillan Press.

Tomé, A. O., Purwanto, & Sa'dijah, C. (2019). Students' mathematical thinking process involving equal signs. *Journal of Physics: Conference Series*, *1227*, 1–8. doi:10.1088/1742-6596/1227/1/012012

Tran, C., Smith, B., & Buschkuehl, M. (2017). Support of mathematical thinking through embodied cognition: Nondigital and digital approaches. *Cognitive Research: Principles and Implications, 2*(16), 1–18. doi:10.1186/s41235-017-0053-8

Tutak, F. A., & Adams, T. L. (2015). A study of geometry content knowledge of elementary preservice teachers. *International Electronic Journal of Elementary Education*, *7*(3), 301–318.

Ubah, I., & Bansilal, S. (2019). The use of semiotic representations in reasoning about similar triangles in Euclidean geometry. *Pythagoras, 40*(1), 1–10. doi:10.4102/pythagoras.v40i1.480

Umugiraneza, O., Bansilal, S., & North, D. (2018). Exploring teachers' practices in teaching mathematics and statistics in KwaZulu-Natal schools. *South African Journal of Education*, *37*(2), 1–30.

Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry*. Chicago, IL: University of Chicago Press.

Usó-Doménech, J. L., & Nescolarde-Selva, J. (2016). What are belief systems? *Foundations of Science*, *21*(1), 147–152. doi:10.1007/s10699-015-9409-z

Uttal, D. H., & Cohen, C. A. (2012). Spatial thinking and STEM education: When, why and how? In B. Ross (Ed.), *Psychology of learning and motivation* (Vol. 57, pp. 148–182). Amsterdam: Academic Press.

Uttal, D. H., Miller, D. I., & Newcombe, N. S. (2013). Exploring and enhancing spatial thinking: Links to achievement in Science, Technology, Engineering, and Mathematics?

Current Directions in Psychological Science, 22(5), 367–373. doi:10.1177/0963721413484756

Van de Pol, J., Brindley, S., & Higham, R. J. E. (2017). Two secondary teachers' understanding and classroom practice of dialogic teaching: A case study. *Educational Studies, 43*(5), 497–515. doi:10.1080/03055698.2017.1293508

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2017). *Elementary and middle school Mathematics: Teaching developmentally*. New York, NY: Pearson.

Van den Heuvel-Panhuizen, M. (2020). A spotlight on mathematics education in the Netherlands and the central role of realistic mathematics education. In M. van den Heuvel-Panhuizen (Ed.), *National reflections on the Netherlands didactics of mathematics: Teaching and learning in the context of realistic mathematics education* (pp. 1–14). Cham: Springer International.

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2014). Realistic mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 521 – 525). Dordrecht: Springer.

Van der Sandt, S., & Nieuwoudt, H. D. (2003). Grade 7 teachers' prospective teachers' content knowledge of geometry. *South African Journal of Education*, *23*(3), 199–205.

Van der Sandt, S., & Nieuwoudt, H. D. (2005). Geometry content knowledge: Is pre-service training making a difference? *African Journal of Research in Mathematics, Science and Technology Education*, *9*(2), 109–120.

Van der Stel, M., & Veenman, M. V. J. (2014). Metacognitive skills and intellectual ability of young adolescents: A longitudinal study from a developmental perspective. *European Journal of Psychology of Education*, *29*(1), 117–137. doi:10.1007/s10212-013-0190-5

Van Es, E. A. (2009). Participants' roles in the context of a video club. *Journal of the Learning Sciences*, *18*(1), 100–137. doi:10.1080/10508400802581668

Van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. Jacobs, & R. Phillip (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 134–151). New York, NY: London.

Van Es, E. A. (2012). Examining the development of a teacher learning community: The case of a video club. *Teaching and Teacher Education*, *28*(2), 182–192. doi:10.1016/j.tate.2011.09.005

Van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education*, *10*(4), 571–596.

Van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education*, *24*(2), 244–276. doi:10.1016/j.tate.2006.11.005

Van Es, E. A., Cashen, M., Barnhart, T., & Auger, A. (2017). Learning to notice mathematics instruction: Using video to develop preservice teachers' vision of ambitious pedagogy. *Cognition and Instruction*, *35*(3), 165–187. doi:10.1080/07370008.2017.1317125

Van Es, E. A., Hand, V., & Mercado, J. (2017). Making visible the relationship between teachers' noticing for equity and equitable teaching practice. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 251–270). Cham: Springer International.

Van Hiele, P. M. (1985 [1959]). The child's thought and geometry. In D. Fuys, D. Geddes, & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M van Hiele* (pp. 243–252). Washington, DC: National Science Foundation.

Van Jaarsveld, P. P. (2018). Juxtaposing secondary Mathematics procedural routines and their meta-narratives: Exploring the vocabulary of student teachers. *African Journal of Research in Mathematics, Science and Technology Education*, *22*(2), 209–220.

Van Putten, S., Howie, S., & Stols, G. (2010). Making Euclidean geometry compulsory: Are we prepared? *Perspectives in Education*, *28*(4), 22–31.

Varela, F. J. (1999). *Ethical know-how: Action, wisdom, and cognition*. Palo Alto, CA: Stanford University Press.

Varela, F. J., Thompson, E., & Rosch, E. (2016). *The embodied mind: Cognitive science and human experience* (Revised ed.). Cambridge, MA: The MIT Press.

Venkat, H. (2015). Mathematical practices and mathematical modes of enquiry: Same or different? *International Journal of Stem Education*, *2*(6), 1–12. doi:10.1186/s40594-015-0018-8

Venkat, H., & Spaull, N. (2015). What do we know about primary teachers' mathematical content knowledge in South Africa? Analysis of SACMEQ 2007. *International Journal of Educational Development*, *41*, 121–130.

Voigt, J. (1998). The culture of mathematics: Negotiating the meaning of empirical phenomena. In F. Seeger, J. Voigt, & U. Wachescio (Eds.), *The culture of the Mathematics classroom* (pp. 191–220). Cambridge, MA: Campbridge University Press.

Vološinov, V. N. (1973). *Marxism and the philosophy of language*. Cambridge, MA: Harvard University Press.

Von Renesse, C., & Ecke, V. (2015). Inquiry-based learning and the art of mathematical discourse. *PRIMUS*, *25*(3), 221–237. doi:10.1080/10511970.2014.921799

Vula, E., Avdyli, R., Berisha, V., Saqipi, B., & Elezi, S. (2017). The impact of metacognitive strategies and self-regulating processes of solving math word problems. *International Electronic Journal of Elementary Education*, (1), 49–59. Retrieved from https://iejee.com/index.php/IEJEE/article/view/298

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge: Harvard University Press.

Vygotsky, L. (1986). *Thought and language*. Cambridge, MA: The MIT Press.

Walkoe, J. (2015). Exploring teacher noticing of student algebraic thinking in a video club. *Journal of Mathematics Teacher Education*, *18*(6), 523–550. doi:10.1007/s10857-014-9289-0

Walkoe, J., Sherin, M., & Elby, A. (2019). Video tagging as a window into teacher noticing. *Journal of Mathematics Teacher Education, 23,* 385:405. doi:10.1007/s10857-019-09429-0

Wallin, A. J., & Amador, J. M. (2019). Supporting secondary rural teachers' development of noticing and pedagogical design capacity through video clubs. *Journal of Mathematics Teacher Education*, *22*(5), 515–540. doi:10.1007/s10857-018-9397-3

Wang, S. (2016). *Discourse perspective of geometric thoughts*. Idaho, USA: Springer

Warshauer, H. K., Starkey, C., Herrera, C. A., & Smith, S. (2019). Developing prospective teachers' noticing and notions of productive struggle with video analysis in a mathematics content course. *Journal of Mathematics Teacher Education, 24,* 89–121. doi:10.1007/s10857-019-09451-2

Watson, J. B. (1913). Psychology as the behaviorist views it. *Psychological Review*, *20*(2), 158–177.

Wegerif, R. (2008). Dialogic or dialectic? The significance of ontological assumptions in research on educational dialogue. *British Educational Research Journal*, *34*(3), 347–361. doi:10.1080/01411920701532228

Wegerif, R. (2011). Towards a dialogic theory of how children learn to think. *Thinking Skills and Creativity*, *6*(3), 179–195.

Wegerif, R. (2016, February 5). What is 'dialogic space'? Blog posted to https://www.rupertwegerif.name/blog/what-is-dialogic-space

Wegerif, R. (2017, September 5). Dialogic space, and why we need it. Blog posted to http://www.rupertwegerif.name/blog/archives/09-2017

Wegerif, R., & Major, L. (2019). Buber, educational technology, and the expansion of dialogic space. *AI and Society*, *34*(1), 109–119.

Wegerif, R., & Mercer, N. (1996). Computers and reasoning through talk in the classroom. *Language and Education*, *10*(1), 47–64.

West, J., & Meier, C. (2020). Overcrowded classrooms: The Achilles heel of South African education? *South African Journal of Childhood Education*, *10*, 1–10. Retrieved from http://www.scielo.org.za/scielo.php?script=sci_arttext&pid=S2223-76822020000100007&nrm=iso

Wille, A. M. (2019). Activity with signs and speaking about it: Exploring students' mathematical lines of thought regarding the derivative. *International Journal of Science and Mathematics Education*, *18*(8), 1587 – 1611. doi:10.1007/s10763-019-10024-1

Wilson, N. S., & Bai, H. (2010). The relationships and impact of teachers' metacognitive knowledge and pedagogical understandings of metacognition. *Metacognition and Learning*, *5*(3), 269–288. doi:10.1007/s11409-010-9062-4

Wittgenstein, L. (1961). *Tractatus logico-philosophicus* (D. F. Pears & B. F. McGuinness, Trans.). London: Routledge & Kegan Paul.

Wittgenstein, L. (2009 [1953]). *Philosophical investigations: The German text, with an English translation* (G. E. M. Anscombe, P. M. S. Hacker, & J. Schulte, Trans., 4th ed.). Malden: MA: Blackwell.

Wittmann, E. (1981). The complementary roles of intuitive and reflective thinking in mathematics teaching. *Educational Studies in Mathematics*, *12*(3), 389–397. Retrieved from www.jstor.org/stable/3482339

Wolff, C. E., Jarodzka, H., Van den Bogert, N., & Boshuizen, H. P. A. (2016). Teacher vision: Expert and novice teachers' perception of problematic classroom management scenes. *Instructional Science*, *44*(3), 243–265. doi:10.1007/s11251-016-9367-z

Wood, D., Bruner, J. S., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry*, *17*, 89–100.

World Economic Forum. (2018). *The future of jobs report 2018*. Zurich, Switzerland: World Economic Forum.

Xu, M., David, J. M., & Kim, S. H. (2018). The Fourth Industrial Revolution: Opportunities and challenges. *International Journal of Financial Research*, *9*(2), 90–95.

Yang, K.-L., & Lin, F.-L. (2008). A model of reading comprehension of geometry proof. *Educational Studies in Mathematics*, *67*(1), 59–76. doi:10.1007/s10649-007-9080-6

Yang, X., Kaiser, G., König, J., & Blömeke, S. (2019). Professional noticing of Mathematics teachers: A comparative study between Germany and China. *International Journal of Science and Mathematics Education, 17*(5), 943–963. doi:10.1007/s10763-018-9907-x

Yeo, A., Ledesma, I., Nathan, M. J., Alibali, M. W., & Church, R. B. (2017). Teachers' gestures and students' learning: Sometimes "hands off" is better. *Cognitive Research: Principles and Implications, 2*(41), 1–11. doi:10.1186/s41235-017-0077-0

Young, A. E., & Worrell, F. C. (2018). Comparing metacognition assessments of mathematics in academically talented students. *Gifted Child Quarterly*, *62*(3), 259–275.

Yuanita, P., Zulnaidi, H., & Zakaria, E. (2018). The effectiveness of realistic mathematics education approach: The role of mathematical representation as a mediator between mathematical belief and problem solving. *PLoS One*, *13*(9), 1–20.

Zack, V., & Graves, B. (2001). Making mathematical meaning through dialogue: "Once you think of it, the Z minus three seems pretty weird". *Educational Studies in Mathematics, 46*(1/3), 229–271. Retrieved from www.jstor.org/stable/3483245

Zaslavsky, O. (2019). There is more to examples than meets the eye: Thinking with and through mathematical examples in different settings. *The Journal of Mathematical Behavior*, *53,* 245–255. doi:10.1016/j.jmathb.2017.10.001

Zhao, N., Teng, X., Li, W., Li, Y., Wang, S., Wen, H., & Yi, M. (2019). A path model for metacognition and its relation to problem-solving strategies and achievement for different tasks. *ZDM*, *51*(4), 641–653. doi:10.1007/s11858-019-01067-3

Zhou, M., & Lam, K. K. L. (2019). Metacognitive scaffolding for online information search in K-12 and higher education settings: A systematics review. *Educational Technology Research and Development*, *67*, 1353 – 1384. doi:10.1007/s11423-019-09646-

Zimmermann, W., & Cunningham, S. (1991). What is mathematical visualization? In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 1–8). Washington, DC: Mathematical Association of America.

APPENDIX A:
TASK-BASED INTERVIEW SCHEDULE

Gender: ____________________

List all languages you are fluent in from their order of fluency (from most fluent to least fluent):

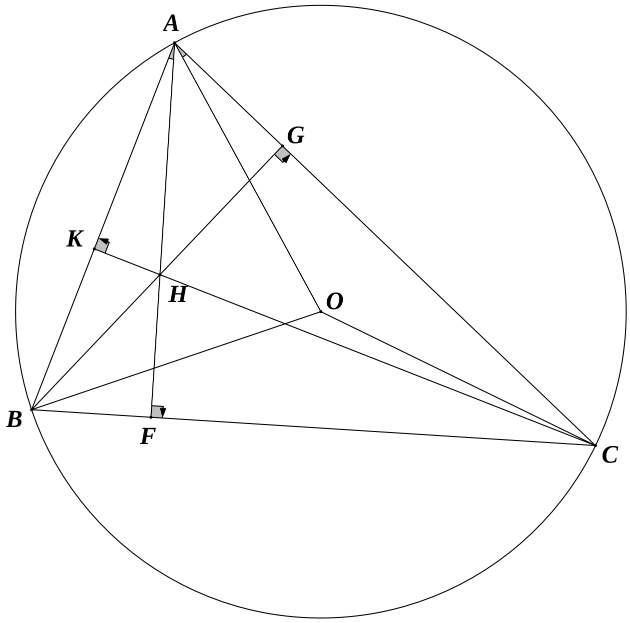
____

____

Look at both problems below carefully, read the given information with clear understanding on both of them, ask for clarity at any time you feel like you are not clear. You can choose to start with any of the two problems. Find the solution while you are thinking aloud.

Problems to be solved

1. Let O and H denote the circumcentre and orthocentre of an acute triangle ABC, respectively. Show that $\angle BAH = \angle CAO$. (HINT: The circumcentre is the point where the perpendicular bisectors of a triangle intersect. The circumcenter is also the centre of the triangle's circumcircle - the circle that passes through all three of the triangle's vertices. The orthocenter is the point where all three altitudes of the triangle intersect. An altitude is a line which passes through a vertex of the triangle and is perpendicular to the opposite side.)



2. A mirror is placed between the teacher and a student at a point C. The teacher is at point A and the student at point B. The student is standing at a point B with the distance BC = $\boldsymbol{d_s}$. The teacher moves away from point C until she can see the top of the students head in the mirror at a point A away from point C with AC =$\boldsymbol{d_t}$. The teacher has a height of $\boldsymbol{h_t}$ and she knows that the angle of incidence is equal to the angle of reflection which is equal to $\boldsymbol{\alpha}$. What is the equation that can be used to determine the height of the student, $\boldsymbol{h_s}$?

APPENDIX B:
FACE TO FACE SEMI-STRUCTURED INTERVIEWS

1. Tell me, when you saw the question, what were your first thoughts?
2. When you began to answer it, what did you do first? Why?
3. Why was it so important to do that first?
4. Do you think that there was something else you could have done?
5. During your solution path what difficulties did you experience? Why were those concepts difficult?
6. What are some other possibilities to answering that question?
7. Which problem do you think was more difficult? Why?
8. Do you think that the availability of the diagram in the first question was helpful? Why?
9. What about the second question where you were not given the diagram, do you think that was helpful? Why?
10. Do you make any reflection of your thinking at any stage? How did you do it? Why?
11. Explain how the reflection on your thinking was useful/not useful?
12. Why was it important for you construct a diagram for the second problem?
13. In the second question you produced this as your diagram that represented the problem visually for you, why this diagram?
14. What purpose did you think the diagram was going to serve in helping you to solve the problem?

APPENDIX C:

TURNITIN REPORT

EXPLORING THE DISCOURSES OF PRESERVICE MATHEMATICS TEACHERS WHEN SOLVING GEOMETRY PROBLEMS

ORIGINALITY REPORT

6%	5%	4%	1%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	mafiadoc.com Internet Source	1%
2	hdl.handle.net Internet Source	<1%
3	epdf.pub Internet Source	<1%
4	Tong Hock Tan, Rohani Ahmad Tarmizi, Aida Suraya Md. Yunus, Ahmad Fauzi Mohd. Ayub. "Understanding the Primary School Students' van Hiele Levels of Geometry Thinking in Learning Shapes and Spaces: A Q-Methodology", EURASIA Journal of Mathematics, Science and Technology Education, 2015 Publication	<1%
5	www.pmena.org Internet Source	<1%
6	hal.archives-ouvertes.fr Internet Source	<1%

APPENDIX D:

ETHICAL CLEARANCE LETTER

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

31 July 2020

Mr Sfiso Cebolenkosi Mahlaba
(210501220) School of Education
Edgewood Campus

Dear Mr Mahlaba,

Protocol reference number: HSSREC/00001173/2020
Project title: Exploring the geometry discourses of preservice teachers when solving geometry problems
Degree: PhD

Approval Notification – Expedited Application

This letter serves to notify you that your application received on 12 December 2019 in connection with the above, was reviewed by the Humanities and Social Sciences Research Ethics Committee (HSSREC) and the protocol has been granted **FULL APPROVAL**

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number. PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

This approval is valid until 03 August 2021.
To ensure uninterrupted approval of this study beyond the approval expiry date, a progress report must be submitted to the Research Office on the appropriate form 2 - 3 months before the expiry date. A close-out report to be submitted when study is finished.

All research conducted during the COVID-19 period must adhere to the national and UKZN guidelines. HSSREC is registered with the South African National Research Ethics Council (REC-040414-040).

Yours sincerely,

[redacted]

Professor Dipane Hlalele (Chair)

/ms

Humanities & Social Sciences Research Ethics Committee
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