An Exploration of General Education and Training (GET) teachers' mathematical knowledge and its influence on the quality of instruction in the teaching of functions

By

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Declaration

I **Hlengiwe Abigail Nguse** declare that:

(i) The research reported in this thesis, except where otherwise indicated, is my original research.

(ii) This thesis has not been submitted for any degree or examination at any other university.

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Signed ....

Dated: 24 July 2019

Dedication

I dedicate this work to my husband, Fortune. Thank you for all your continued support and encouragement when I was ready to throw in the towel. All my children, thank you guys for believing in me, I hope to have set an example for you in words and deeds that knowledge is vital.

I also dedicate this work to the two ladies who saw the God given potential in me, my aunt who gave me unconditional love and my mom who against all odds saw to it that I obtained a good education.

To the rest of the family (from both sides), I say thank you for cheering me on. Above all I thank the Lord Jesus Christ for starting this work and seeing it through to completion.

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Abstract

This thesis documents a study of the General Education and Training teachers' mathematical knowledge of functions and what this knowledge brings to the quality of instruction. The study made use of Variation Theory and Mathematical Knowledge for Teaching as theoretical frameworks. With regard to the methodology, data generation methods included, semi-structured interviews, pen and paper (written items), lesson observations, field notes and document analysis. The sample was chosen through purposive sampling. The participants were four mathematics teachers from four varying schooling contexts in KwaZulu-Natal. Data generation took place in 2016 and 2017 and a total of 28 lessons were observed.

Data generated from pen and paper items corroborated the results of the interviews and the data generated from the classroom observations. This suggested that teacher knowledge does influence the quality of classroom instruction. The findings support the literature which shows that teachers' subject matter knowledge hugely impacts on the quality of instruction. The study, however, concluded that a lack of subject matter knowledge does not stop teachers from delivering lessons of an acceptable level as required by the Curriculum and Assessment Policy Statement if they follow readily designed lesson plans and make use of prescribed curriculum materials including learner workbooks. It was concluded that when out of field teachers use these prescribed curriculum resources effectively, they are able to involve learners in worthwhile learning of mathematics similar to that made available to learners in classrooms where the teacher has a sound knowledge of the subject matter. It is equally important that textbooks and learner workbooks are checked thoroughly for errors before being printed out and distributed to schools as this can have an adverse effect on learning especially in subjects like mathematics.

It is the conclusion of this study that when teachers focus on creating a space of learning which enhances in learners the capabilities to discern which knowledge is germane, the learner and the content are placed at the centre of the process of teaching and learning which improves the quality of instruction.

Finally, the study proposes a new knowledge domain based on the model of reflective practice which aims to assist teachers with identifying individual knowledge areas of need for continued professional development.

Acronyms

CAPS	Curriculum and Assessment Policy Statement
CBMS	Conference Board of Mathematics and Science
CCK	Common Content Knowledge
CK	Content Knowledge
CPTD	Continued Professional Teacher Development
DBE	Department of Basic Education
DOE	Department of Education
FET	Further Education and Training
GET	General Education and Training
HCK	Horizon Content Knowledge
IBL	Inquiry Based Learning
IEA	International Association for the Evaluation of Educational Achievement
IEB	Independent Examination Board
IQMS	Internal Quality Management System
ISPFTEDSA	Integrated Strategic Planning Framework for Teacher Education and Development in South Africa
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
KZN	KwaZulu-Natal
MDI	Mathematical Discourse in Instruction
MIR	Macro Indicator Report
MKT	Mathematics Knowledge for Teaching
MQI	Mathematical Quality of Instruction
NEEDU	National Education Evaluation & Development Unit
NPFTED	National Policy Framework for Teacher Education and Development
PCK	Pedagogical Content Knowledge
PGCE	Post Graduate Certificate in Education
PLC	Professional Learning Community

PUFM	Profound Understanding of Fundamental Mathematics
RSA	Republic of South Africa
SA	South Africa
SACE	South African Council of (for) Educators
SACMEQ	Southern and Eastern Africa Consortium for Monitoring Educational Quality
SAKT	Self-Assessment Knowledge for Teaching
SCK	Specialised Content Knowledge
SES	Socio-Economic Status
SMK	Subject Matter Knowledge
TED	Teacher Education and Development
TIMSS	Trend in International Mathematics and Science Study
TPACK	Technological, Pedagogical, and Content Knowledge
US	United States
VSK	Value, Skills and Needs

Table of Contents

Contents

List of figures

CHAPTER 1: Introduction

The teaching of mathematics has remained a subject of great interest globally. From a pedagogical stance, teachers in mathematics have a very critical role to play in facilitating effective learning in the classroom" (Das, 2015). The quest to understand reasons behind low learner attainment in mathematics has led many researchers to look into the kind of knowledge needed in the teaching of mathematics (Ball, Hill, Blunk, Charalombous, Lewis, Phelps & Sleep, 2008; Ball & Schilling, 2008; Christiansen 2012; Hurrell, 2013; Lannin, Webb, Chval, Arbaugh, Hicks, Taylor & Bruton, 2013; Mudaly, 2015; Myers & Rivero, 2019; Llinares, 2020). Shulman (1986) prompted the discussions about the value of subject matter knowledge and the need to shift the research of the day away from a focus on learning towards that which investigates the knowledge needed in the work of teaching. Shulman (1986) introduced Pedagogical Content Knowledge (PCK) as a knowledge domain which amalgamates the knowledge of the content with the methods of teaching this content (pedagogy). Since the inception of PCK, a plethora of studies have been pursued with the aim of investigating how this knowledge domain is manifested in mathematics classrooms (Lannin et al., 2013; Adler & Venkat, 2014; Pournara, 2014; Gardee & Brodie, 2015; Sapire, Shalem, Wilson-Thompson & Paulsen, 2016; Huang, Barlow & Prince, 2016; van Staden and Motsamai, 2017, Maoto, Masha, Mokwana, 2018; Yang, Kaiser, Konig & Blomeke, 2019). This research according to Venkat & Spaull (2015) has revealed that learner attainment is more associated with PCK than content knowledge which excludes pedagogy.

Teaching and learning are two sides of the same coin and since teachers cannot pass on the knowledge they do not have (Hartley, 2010), it is important that initiatives to improve learner attainment in mathematics be linked to initiatives to developing teacher knowledge of teaching the subject. Global educational policy trends have seen the shift from focusing on increased learner attainment towards educational quality (Carnoy, 2012; Ndlovu, 2014; van Staden & Motsamai, 2017; Maoto et al., 2018; du Plessis, 2020). There is, therefore, a critical need for studies that seek to ascertain the extent to which teachers' current knowledge meets the requirements of knowledge needed to teach mathematics (Venkat & Spaull, 2015). Data that exist seem to suggest that South African teachers' conceptual knowledge generally needs attention (Kwenda, 2014; Sapire et al., 2016). McCarthy & Oliphant (2013, p.4) argue that "one of the most important factors limiting the quality of mathematics education is the poor quality of our teachers, and numeracy and mathematics teaching in particular, especially at lower grade levels". Similarly, Nel (2020) states that many South African learners fail to meet the expectations of higher education in quantitative literacy and other research has also shown that learner performance is linked to teacher subject matter knowledge (Courtney-Clarke & Wessels, 2014). According to (Chirinda & Barmby, 2017), very few professional development programme activities in South Africa are geared towards empowering teachers with pedagogical knowledge and skills needed to teach the content. Kwenda (2014) argues that the shortness of teaching practice for pre-service teachers in South Africa compared to what happens in countries like Zimbabwe, deprives these prospective teachers of action research opportunities and there is no meaningful reflection and theorising taking place during this training period.

For teachers to be effective in the classroom, their knowledge of the field of study needs to go further than the theory they learn in the institutions of higher learning. Nel & Luneta (2017) concluded that mentoring that take cognisance of teachers' pedagogical and content needs enables them to improve their ability to understand the mathematics they teach and to enhance lesson preparation and teaching skills. "Mentoring refers to a professional relationship in which an experienced person, the mentor, assists another one (the mentee) in developing specific skills and knowledge that will enhance the less experienced person's professional and personal growth" (Nel & Luneta, 2017, p.2). Research continues to show that teachers learn best through requesting advice, testing and sharing ideas with colleagues (Spangenberg, 2017; Chauraya & Brodies, 2018; Umugiraneza, Bansilal & North, 2018; Ngcoza & Southwood, 2019). It has been suggested that mentoring can act as a catalyst to improve reflection on practice (Frick, Carl & Beets, 2010).

One of this study's critical questions required teachers to offer perceptions about their own knowledge of functions. This requires the ability to reflect on one's understanding of the content while also reflecting on the ability to teach this content. It is important that teachers acquire necessary skills to reflect on their knowledge in order to identify areas of need for professional development. If teachers are not taught the skills to reflect on their practice, it will not be possible for them to know what it is that they know or do not know. Reflecting during practice also gives teachers opportunities to check for learner understanding, which is an important element of teaching. Quality instruction is therefore marked with teachers taking opportunities to reflect on their practice and having the flexibility to readjust instruction in order to enhance learning. Reflective practice is about taking moments to ponder about learner understanding prior to teaching a lesson, during the process of teaching and after the lesson (Rasmussen, 2016).

While it is clear from research that reflection has become a catchword in pre-service teacher training, researchers agree that the process of reflection is often challenging for pre-service and in-service teachers alike. For instance, Wang (2016) found that students reflect in detached ways and that a connection exists between reflection, self-understanding and self-definitions. Similarly, Costandius & Botes (2018) argue that reflection compels people to face their own biases and to realise their limitations in a manner that is challenging and often uncomfortable. Chye, Zhou, Koh and Liu (2019) further assert that pre-service teachers do not take reflective portfolios seriously because these are often not graded or assessed for marks, hence the student's objective is to simply get the portfolio work out of the way as quickly as possible. Zhou, Xu & Martinovic (2017) reported that pre-service teachers were uncomfortable with the concept of micro teaching and were reluctant to give feedback publicly after the process of micro teaching. It is clear from these findings that there is a lack of understanding of the objectives of reflection as well as lack of appreciation of the process involved. This lack of understanding of the objectives of professional learning and growth initiatives is also observed in in-service teachers. Chirinda & Barmby (2017) found that practicing teachers were reluctant to participate in mathematical problem solving pedagogy project due to fears of not finishing the syllabus on time.

One of the aims of this study is therefore to collect information about teachers' abilities to reflect on their knowledge of teaching by comparing the knowledge declared with the actual knowledge observed in the classroom.

1.1 Background of the study

It is not possible to talk about teacher knowledge in South Africa without mentioning the country's political history (Venkant & Spaull, 2015). South Africa has come a long way since the 1994 democratic elections which saw the abolishing of the former racist regime with its policies. The new dispensation brought about hope for change in areas like education, land reform and the opportunity for greater economic participation by those who had previously been excluded. Two decades later, South Africans are faced with the reality of the strongholds left by the previous system.

To those who are concerned about the country's quality of education, the paradox is that, on the one hand there is a clear need to raise educational standards and more so in the previously disadvantaged communities in order to ensure greater economic participation. On the other hand, the teaching body expected to be influential towards promoting quality education is the product of the inferior education system engineered by the apartheid regime. Teachers who were themselves educated under apartheid and those who have been taught by such teachers are now expected to teach in ways that promote quality teaching and learning. It cannot be disputed that there have been excellent and passionate teachers who despite coming from a disadvantaged educational background, have gone forth and made a noticeable impact on the learners they have taught. However, research shows that many learners from previously disadvantaged schools have failed to attain the expected results in literacy and numeracy (Taylor, 2011; Spaul, 2013).

There is consensus amongst scholars that a review of the teaching and learning strategies currently informing learning in the classroom is required in order to develop workplace-ready graduates (Modipabe & Kibirige, 2015; Mobarak, 2019; Sikhwari, Ravhuhali, Lavhelani & Pataka, 2019). According to Abdulhamid & Venkat (2014, p.138), "recent small-scale studies in South Africa have shown that primary mathematics teachers often provide limited opportunities for learners to understand mathematics in coherent ways". Similarly, Venkat & Spaull (2015) analysed the Southern and Eastern Africa Consortium for Monitoring Educational Quality (SACMEQ) study of content knowledge of Grade 6 South African teachers. The study found that 17 per cent of Grade 6 learners in South Africa were taught by mathematics teachers who had content knowledge below a Grade 4 or 5 level and 62 per cent of learners were taught by teachers with a Grade 5 or 6 level of content knowledge. The study also reported that some learners did better at answering mathematics questions than teachers who taught Grade 6 mathematics.

From 1994 to 2007, mathematics was a compulsory school subject in South Africa up to Grade 9 level. In Grade 10, learners had to select a minimum of 6 subjects with mathematics as one of the electives. The mathematics curriculum was separated into, Standard or Higher Grade mathematics. In 2008, the first cohorts of matriculants were introduced to Mathematical Literacy as an option to Core Mathematics in the National Senior Certificate (NSC) examinations. Mathematical Literacy is offered to learners in the [1]Further Education and

[1] Further Education and Training band represents grades 10 -12

Training (FET) band who are failing to obtain the minimum required pass mark. Giving these learners the opportunity to take Mathematical Literacy partly solves the problem of retaining learners in the same grade due to failing mathematics and partly ensures that the country's citizens are taught how to apply numeracy skills . According to the Department of Basic Education's (DBE) Macro Indicator Report (2013), it was shown that, a third of all South African learners at school in 2007 had repeated a grade. This report further indicates a strong link between repetition and the rate of drop-out. It is stated that the drop-out rate increases from Grade 9 upwards and that in 2011 this increase almost reached 13 per cent in Grades 10 and 11. Similarly, Spaull (2015) reported that the four most prominent reasons for dropping-out given by youth on household surveys were: lack of financing; looking for a job, failing grades; and pregnancy.

By the same token, comparative research shows that South African learners have not performed well in the previous international comparative studies in mathematics according to the Trends In Mathematics and Science Studies (TIMSS, 1995; 1999; 2003 and 2011). Reporting on the TIMSS (2011) findings, (Spaull, 2015) reveals that the average Grade 9 learner in South Africa performed worse than the average Grade 8 learner from other middle-income countries. This data also showed that these South African learners were lagging behind by between two and three Grade levels, thus making South Africa the worst of all participating countries. Similarly, reporting on the results from a large scale quantitative study that was conducted in 2013 Maniraho (2017), found that Grade 6 Rwandan learners performed better than their South African counterparts in a study that sought to compare teachers' pedagogical content knowledge (PCK) to student learning of mathematics. The most recent reports, however, paint a more promising picture and indicate that the Department of Basic Education (DBE) has been proactive about putting measures in place to try and improve the country's education system.

According to Trends in International Mathematics and Science Study (TIMMS, 2015), South Africa has shown the biggest positive change of all the participating countries, with the overall performance improvement equivalent to "two Grade levels between 2003 and 2015" (Zuze, Reddy, Visser, Winnaar & Govender, 2015, p.2). What is more encouraging is that the greatest improvement is observed at the lower end of the achievement distribution, indicating that those who were previously achieving the lowest results are showing improvement in mathematics and science scores. Although there has been improvement in the country's learner achievements in mathematics and science (Reddy, 2018), the reality, still remains, however, that South Africa is one of the lowest achieving countries in comparison to other African countries (Macro Indicator Report, 2013). Furthermore, evidence points to the need to focus attention on mathematics teaching given that only 21% of learners who wrote matric in 2017 achieved more than a 60 per cent pass in mathematics (Reddy, 2018).

In my literature review I did not find a South African research which was purely qualitative and sought to link teacher knowledge of a specific topic with the quality of instruction in varying schooling contexts. This study was hoping to address this gap in the knowledge.

1.2 The statement of the problem

Against the context highlighted above, the study sought to answer the following research question:

How does the General Education and Training (GET) teachers' mathematical knowledge of functions influence the quality of instruction in the classroom?

The study hypothesised that a teacher with a good Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) of functions will be able to demonstrate this knowledge through a high quality level of instruction. The opposite result was hypothesised for a teacher with limited knowledge of the subject and pedagogy. The following objectives were formulated:

- To investigate GET teachers' ability to reflect on their mathematical knowledge for teaching specific content.
- To explore how teachers' knowledge of specific content (functions) influences mathematical quality of their instruction;
- To explore other factors that may have an influence on the quality of instruction; and
- To make recommendations as to the measures that can be put in place in order to enhance the quality of instruction in mathematics.

To facilitate the generation and analysis of data, four research questions were posed:

1. How do teachers perceive their mathematical knowledge for the teaching of functions?
2. How does a teacher's content knowledge of functions influence their teaching?
3. What other factors influence the quality of instruction?
4. Why does content knowledge influence instruction in the way it does?

1.3 The aim of the study

The study aimed to explore how four teachers use their knowledge which includes subject matter knowledge and knowledge of pedagogy to teach functions in Grades 7 and 9 classrooms. The four participants taught in four different schooling contexts namely: Former Model C, Township, Rural and Independent schools. Making use of an existing model of Mathematical Knowledge for Teaching (MKT) developed by Ball, Hill, Blunk, Charalombous, Lewis, Phelps & Sleep (2008), and employing different methods of generating data, the study sought to investigate how teachers use content knowledge of functions (SMK) to teach this topic, taking into account the classroom context (PCK) and the effect that this knowledge has on the quality of instruction. SMK and PCK are combined by Ball et al., (2008) to form Mathematical Knowledge for Teaching (MKT). The study further employed, as supplementary conceptual framework, the Variation Theory by Marton, Runesson & Tsui (2004).

1.4 The organisation of the thesis

The rest of the chapters including the first chapter are organized as follows:

CHAPTER 1:

This chapter introduces the thesis and provides the background and the aim of the study. The chapter also highlights the objectives, the critical questions and the statement of the problem.

CHAPTER 2:

This chapter gives an account of international and South African literature reviewed to support the study.

CHAPTER 3:

Chapter three begins with a discussion of variation theory and the suitability of this theory as a framework underpinning this study. The MKT model of Ball et al. (2008) is also introduced as a conceptual framework for the thesis and a further discussion of the knowledge strands forming this conceptual model is provided.

CHAPTER 4:

The methodology chapter presents the research design, methodology and procedures undertaken to conduct the study. The chapter also highlights the ethical considerations and steps followed to ensure compliance.

CHAPTER 5:

The fifth chapter provides the first phase of analysis. This involves the analysis of data obtained from the written MKT (pen-and-paper) items and of data generated from the coding of video recordings from lesson observations.

CHAPTER 6:

Chapter six is the continuation of the analysis process. In this chapter, further analysis of coded video recordings is done as well as the analysis of curriculum documents.

CHAPTER 7:

The aim of this chapter is to present results of data analysis and to introduce the findings.

CHAPTER 8:

This chapter provides a discussion of the findings and continues to propose a model that can be used to facilitate reflective practice in teaching. The chapter ends with a conclusion and recommendations.

CHAPTER 2 : Literature Review

2.1 Introduction

This chapter provides an account of literature reviewed in the area of teacher knowledge in mathematics. The chapter begins with an outline of international comparative mathematics assessment studies and what this research reveals about the possible causes of the differences in learner attainment. The study continues to evaluate literature on the topic of teacher knowledge, taking into account constructs closely linked to the concept of mathematical knowledge for teaching which include attitudes and beliefs, pre-service teacher training and reflective practice. Finally, I conclude by offering a discussion of functions as a topic in the GET syllabus and literature reviewed in this area.

2.2 Factors influencing learner attainment

The teaching of school mathematics has remained a topic of great disputation over many decades across the globe and to this day there is no consensus on what constitutes the best teaching approach (Pournara, Hodgen, Adler & Pillay, 2015). Mathematics is considered to be one of the most important yet severely difficult subjects taught to school children (Dundar, Guvendir, Kocabiyik & Papatga, 2014).This is evidenced by the low levels of attainment by learners in classrooms around the world. With the exception of some Asian countries among which are China, Japan, Taiwan and Singapore, it is emerging that even the most technologically advanced and well developed countries like the United States (US) are concerned about learner attainment in mathematics (Conference Board of the Mathematical Sciences, 2012; Grady, Watkins & Montalvo, 2012). International comparative mathematics assessments reveal that of the 39 countries which participated in the TIMMS, South Africa was placed number 38, one position above the lowest performing Saudi Arabia (Spaull, 2015).

Two decades ago Ma (1999) pointed out that there was enough evidence from previous research to suggest the existence of a knowledge gap between Chinese and US learners. International studies of mathematics achievements were showing that learners from Asian countries such as China and Japan were consistently outperforming their counterparts in the US. This led to Ma (1999) conducting a study in order to document the differences between Chinese and the US teachers' knowledge of mathematics for teaching and to suggest the reason why Chinese students' success in mathematics could be attributed to the teachers' understanding of the subject. The study also documented the factors that contribute to the Chinese teachers' growth of mathematical knowledge. The findings revealed that even though

American teachers had been exposed to more advanced mathematics during high school or college education, Chinese teachers displayed a more comprehensive knowledge of the mathematics taught in primary school, outperforming these US teachers. More recent comparative studies confirm that Chinese mathematics teachers continue to show superior content and pedagogical knowledge compared to their counterparts in the US (Cai & Wang, 2010; Huang et al., 2016) and Europe (Yang etal., 2019).

The continued success of Chinese teachers compared to their counterparts in the rest of the world is attributed to the teaching methods which focus on conceptual understanding (Lai, 2012). Procedural variation prevalent in Chinese classrooms has been misunderstood and misinterpreted by Western scholars as rote learning, when in fact it is the reason why Chinese learners perform better in international comparative assessments (Marton, Runesson & Tsui 2004; Lai, 2012; Huang et al, 2016). In another comparative study, Laschke (2013) concluded that cultural differences between the West and the East or 'individualism' and 'collectivism' account for differences in mathematical achievements of teachers and learners in Germany and Taiwan. Similarly, Cai & Wang (2010) showed that teachers from two different countries held different beliefs on what constitutes effective mathematics teaching. Opportunities to learn mathematics and mathematics pedagogy were also found to have an influence on the differences in mathematical achievements of teachers (Laschke, 2013), and this confirmed what Ma (1999) had already noted. Mellor, Clark & Essien (2018) found that learners in Germany were afforded opportunities to learn functions at a deeper level by using a textbook with a higher percentage of content that promoted the development of conceptual knowledge compared to a textbook used to teach learners in South Africa.

The same trend reported in China and the US has been reported in South Africa, with the SACMEQ data indicating that compared to countries like Kenya and Tanzania, South Africa has the highest proportion of teachers with degrees and the second highest (below Sychelles) average of teacher training (Macro Indicator Report, 2013). This high level of training, however, does not translate into better content knowledge for South African teachers who are reported to have the worse content knowledge compared to many African countries which include Kenya, Zimbabwe, Uganda and Tanzania amongst others (Spaul, 2013). Perhaps the answer lies in what Kwenda (2014) identifies as lack of rigour in pre-service teacher training programmes. He argues that compared to what happens in Zimbabwe, South African prospective teachers are not given enough time to practice what they learn in theory and "the

Bachelor of Education (Bed) programme appears to be complicated by an over-indulgence in policy frameworks, curricular jargon and terminology" (p.232).

Other comparative studies have been conducted comparing South African learners to their counterparts in the neighbouring African countries (Spaull, 2013). One such study was conducted by Carnoy (2012) with the aim of investigating the reasons for the differences in learner performance in mathematics between Botswana and South Africa. The study found that even though the different historical backgrounds of the two countries may have been a contributing factor, it was apparent that Grade 6 learners in Botswana had higher mathematics achievement gains than a very similar set of Grade 6 learners in South Africa. Furthermore it was concluded that learners in both countries were learning mathematics at relatively low levels and making relatively small gains during the Grade 6 year .This finding was consistent with other research which revealed that South African learners achieve below average in international benchmark assessments (Spaull, 2013; McCarthy & Oliphant, 2013). The reason for higher achievement gains amongst the learners in Botswana was attributed to the ability of the country to provide better resourced teachers in terms of knowledge and skills. Compared to South African teachers, Tswana teachers delivered education more effectively by teaching more lessons within a year and closely following the curriculum (Carnoy, 2012)

Similarly, the 2011 Trends in International Mathematics and Science Study (TIMSS), conducted by the International Association for the Evaluation of Educational Achievement (IEA) aimed to determine the resources factors that influence South African learners' performance in mathematics (Visser, Juan & Feza, 2015). Data were collected from a stratified random sample of 298 schools and a total of 11969 Grade 9 learners participated. The sample was stratified by province, language of instruction (Afrikaans, English, or dual medium – both Afrikaans and English) and type of school (independent, public-and-not-Dinaledi, public-and-Dinaledi). Using data obtained from the TIMMS (2011), Visser et al (2015), found that both the school and the home environment play significant roles in learners' mathematics performance. These findings suggested that it is not only the socio-economic factors of schools that impact learners' mathematics performance, but other factors like the levels of parental education and the language of instruction have a significant influence (TIMMS, 2015). Adler & Venkat (2014) included teachers' Mathematical Discourse in Instruction (MDI) amongst factors which contribute to learners' poor performance.

The literature reviewed in the above discussion has provided a picture that learner attainment in mathematics is influenced by school- and home-related factors. The findings of this literature also reveal that the teaching of mathematics is at the core of how learners perform in international comparative assessments. There is therefore a need for research which focuses on the teaching of school mathematics including the mathematical knowledge that teachers bring into the classroom. The next section reviews literature on teacher knowledge.

2.3 Mathematical knowledge for teaching

Research for more than two decades has been shifting from the focus on learner attainment towards educational quality (Visser et al., 2015; Charalombus, 2015; Maoto et al., 2018; Vagi, Pivovarova & Barnard, 2019). Teacher knowledge is at the heart of educational quality. Research contributions on teacher quality have taken the form of various approaches, with most studies either focusing on teacher knowledge or teacher beliefs (Charalombus, 2015). Interventions aimed at increasing mathematics performance have shifted towards investigations of the kind of knowledge needed by teachers of mathematics (Gardee, 2015; Luneta, 2015; Myers & Rivero, 2019). There has been a growing body of literature on teacher professional noticing (Barnhart & van Es, 2015; Ding & Dominguez, 2016; Yang et al 2019; Hoynes, Klemp & Nilssen, 2019). This literature is based on the premise that teacher belief and knowledge which includes both general pedagogy and mathematical content knowledge influence the way in which teachers notice learners' responses during instruction (Yang et al., 2019). Teacher noticing of learners' mathematical understandings is described as on the moment expertise of decision making that is both complex and challenging (Hoynes et al., 2019). The three aspects of teacher noticing include attending to learners' strategies, interpreting learners' understandings and deciding how to respond based on learners' understandings (Ding & Dominguez, 2016; Yang et al., 2019). This is what Ball et al. (2008) termed, knowledge of content and students (KCS).

Studies on teacher beliefs and attitudes in mathematics suggest that teacher training programmes need to incorporate teachers' views and beliefs about how mathematics should be taught at schools (Busi & Jacobbe, 2018). Teacher beliefs go hand in hand with the concept of teacher knowledge and it is almost impossible to discuss one concept without the other. Lannin et al., (2013) state that the challenge for many researchers is to delineate the relationship between knowledge and beliefs. There seems to be an agreement amongst researchers that what teachers believe about the teaching and learning of mathematics influences the way they teach (Cai & Wang, 2010; Cao, Postareff, Lindblom-Ylanne & Too, 2019; Rodriguez-Izquierdo, Falcon & Permisan, 2019). Researchers have approached the area of beliefs from various angles, for instance, Cai & Wang, (2010) compared cultural beliefs of Chinese and US teachers with respect to effective mathematics teaching. Similarly, Cao et al., (2019) investigated perceptions of teacher educators' approaches to teaching and research. Ham & Dekker (2019) focused on the role played by teachers' beliefs in the implementation of educational reform while others examined the connection between teacher beliefs and self-efficacy (Choi, Lee & Kim, 2019; Civitillo, Juang, Badra & Schachner, 2019). It is clear from the literature reviewed that teacher beliefs do play a significant role in the understanding of teacher knowledge for teaching.

I continue the discussion on teacher knowledge by reviewing literature on pre-service teachers' beliefs about the teaching and learning of mathematics.

2.4 Teacher beliefs and attitudes

Table 2-1 is a summary of findings about what practicing and pre-service teachers believe about the teaching and learning of mathematics.

Table 2- 1: Beliefs about the teaching and learning of mathematics (Busi & Jacobbe, 2018, p.4)

Category	Belief(s)
Beliefs about mathematics:	(1) Mathematics is a web of interrelated concepts and procedures (and school mathematics should be too).
Beliefs about learning or knowing mathematics:	(2) One's knowledge of how to apply mathematical procedures does not necessarily go with understanding the underlying concepts. (3) Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures. (4) If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn the procedures first, they are less likely ever to learn the concepts.
Beliefs About Children's Learning and Doing Mathematics	(5) Children can solve problems in novel ways before being taught how to solve such problems. Children in primary grades generally understand more mathematics and have more flexible solution strategies than adults expect. (6) The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. For example, real-world contexts support children's initial thinking whereas symbols do not. (7) During interactions related to the learning of mathematics, the teacher should allow the children to do as much of the thinking as possible.

There are those who believe that teachers' beliefs about the best ways to teach or learn mathematics is linked to the methods teachers were exposed to as learners of mathematics at school (Ham & Dekker, 2019; Feldman, 2020) while other evidence seem to suggest that personal factors outside of education could also significantly support beliefs (Sawyer, 2018). Busi & Jacobbe (2018) offer a summary of what pre-service teachers believe about the teaching and learning of mathematics (Table 2-1). The table shows that views about the best ways of learning mathematics differ and that these views pertain to whether procedural or conceptual knowledge is more important. The table shows that there is a consensus regarding the importance of the role of teaching and exposing children to problem solving skills. Researchers like Charaloumbus (2015) have sought to shed light on the link between teacher beliefs and teacher knowledge. The focus of this thesis is on teacher knowledge, however, as mentioned before, it is difficult to discuss teacher knowledge without linking it to teacher beliefs about what constitutes the best teaching approaches.

There is an understanding that knowledge is a social construct, while beliefs are viewed as individual constructs, in reality, however, people are guided by their own beliefs about what they consider as truth (Liljedahl, 2008). In the same way, pre-service and practicing or qualified mathematics teachers are guided by what they believe to be true about the teaching and learning of mathematics. Cai & Wang (2010, p.266) argue that " a teacher's conception of the nature of mathematics can be viewed as the teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" . Beliefs are both abstract and complex constructs, often shaped by personal and cultural experiences and practices (Ham & Dekker, 2019). The influence of teacher beliefs on the classroom practice cannot be underestimated. A duality exists as a result of a constant tension between literature on knowledge and beliefs (Liljedahl, 2008) about what teachers have (knowledge) and what teachers should have (beliefs). Aljaberi & Gheith (2018) argue that teacher beliefs and perceptions are divided according to the two main teaching-learning approaches, namely, traditional and constructivist.

The next few paragraphs provide a discussion of these two teaching approaches.

2.5 Traditional vs Constructivist approach

The traditional, teacher-centred approach has been the target of many critics who have described it as a discourse which provides learners with fewer opportunities to participate in critical thinking and leads them to reproduce rather than create knowledge (Kiramba & Smith

(2019), and a method which puts more emphasis on the content than on the process of learning (Aalto, Tarnanen & Heikkinen, 2019). The constructivist approach or inquiry based learning (IBL) is the ideal approach and the focus of many educational theorists (Inoue, Asada, Maeda & Nakamura, 2019; Ham & Dekker, 2019). It is seen as the approach necessary in order to develop in learners crucial skills needed for participation in the 21st century (Choi et al., 2019). These skills include cooperation, independent thinking, negotiation, collaboration and communication.

Mathematical pedagogy was suggested by Ball (1988) as a theoretical framework conceptualized around teacher knowledge, beliefs, and dispositions in the domains of subject matter, teaching and learning, students, and context. This theory was developed as a vision for what true mathematical teaching and learning should look like in response to the traditional methods of teaching which have often been described as lacking of meaning and understanding (Kiramba & Smith, 2019). The approach is said to be embedded in the very core of disciplinary mathematics, because it seeks to afford learners the opportunity to develop both understanding and power in mathematics while emphasizing sense-making rather than rote learning (Awan, 2013). I will use this theory to briefly highlight the ideals of the constructivists' approach in the teaching of mathematics.

The ideals of mathematical pedagogy are two-fold. On the one hand, learners are seen as having the power to make sense of the body of knowledge that exists which consists of concepts and procedures that have been invented by others. On the other hand learners must also have the experience of doing mathematics by developing mathematical hunches themselves, inventing ideas and learning acceptable methods of justifying their arguments (Maass, Swan & Aldorf, 2017). This view of doing mathematics falls under the constructivist paradigm.

Constructivist epistemology takes a view that learners should be "actively involved in constructing their own understandings, in discovering and inventing mathematics" (Ball, 1988, p.9). Learner-centred pedagogical frameworks are characterised as flexible and responsive to learner needs (Ham & Dekker, 2019), where learners "construct their knowledge as they engage in whole class discussions of diverse ideas with their teachers and peers" (Inoue et al., 2019, p.366). In the classroom, teachers must be aware that a learner is not a *tabula rasa*[2], but that they also come into learning with prior knowledge on which to build for the new learning experiences to be effective. Not only should teachers assess the existence of this prior

[2] Tabula rasa is a Latin word for clean slate

knowledge but they should also be aware of errors and misconceptions that are regarded by learners as prior knowledge (Moru, Qhobela, Wetsi & Nchejane, 2014; Gardee, 2015; Luneta, 2015). The constructivist view of learning requires "experiences to challenge students' current conceptions (i.e., misconceptions) and ample time and activities that facilitate the reconstruction of their ideas and abilities" (Bybee, 2014, p.1).

Progressive approaches are not always easy to implement and there are a number of potential hindrances to mathematical pedagogy. In their summary of literature on teacher beliefs about reform failure, Ham & Dekker (2019), found that school structure which includes assessment demands, lack of teacher training, curriculum design, access to resources, lack of administration support, lack of monitoring and increased work load without proper compensation were the major hindrances to the implementation of learner-centred practices. I will discuss two of these hindrances which pertain to teacher and learner knowledge in the context of South Africa.

Various assumptions and misconceptions bring up questions about the compatibility or applicability of progressive approaches to the South African context. The first assumption I address pertains to teacher knowledge. There appears to be a notion that all or most teachers possess the necessary knowledge needed in the construction of learning. This assumption is a myth because research shows that most pre-service teacher training institutions are falling short when it comes to producing teachers who are competent to handle classroom demands especially with regard to subject matter knowledge, and even more so in subjects like mathematics (Mudaly & Moore-Russo, 2011; Adler, 2017). It is clear according to du Plessis (2020) that pre-service teachers need to be trained to use learner centred methods in their classrooms in order to provide learners with a better learning experience. Umugiraneza et al. (2018, p.1) argue that "learning in the 21st century requires the collaboration of well-trained teachers, working in well-equipped classrooms and using technology innovatively to support a constructive learning atmosphere".

Most teachers who teach in underprivileged communities were either educated under the apartheid regime or have inherited the legacy of poor education left by this system which sought to promote racial divisions. These teachers find themselves lagging behind their counterparts who teach in the more affluent, privileged schools (van der Berg & Gustafsson, 2017; Reddy, 2018). Unequal opportunities to learn mathematics for teaching promote imbalances in the quality of teaching and learning amongst these varying schooling contexts

(Spaull, 2013; Bohlmann, Prince & Deacon, 2017). This happens because the availability of financial and other material resources enables the more privileged schools to offer quality professional development geared towards improving teachers' content knowledge. Previous research has also found evidence of unequal opportunities to learn in South African universities in the training of mathematics teachers (Parker & Adler, 2012).

The second misconception is that learners possess the necessary knowledge and skills to construct their own knowledge. Often there is disparity between the knowledge that learners bring into the classroom and the knowledge required in order to participate in the mathematical discourse. In this regard, research (Gardee, 2015; Luneta, 2015) shows that many learners come with misconceptions as prior knowledge which undermines the acquisition of new knowledge. In inquiry-based learning, most learners are not experts in distinguishing which information should be kept for later use and which information should be ignored. As a result "much erroneous information will be acquired and germane information may be lost" (Vogel-Walcutt, Gebrim, Bowers, Carper & Nicholson, 2010, p.136). Furthermore, some learners come from illiterate home backgrounds and may be disadvantaged by lack of exposure to reading materials and other resources which might provide mental stimulation especially in early years of development (TIMMS, 2015). Language and cultural backgrounds are part of the contextual barriers encountered in ordinary pedagogy (Christansen & Aungamuthu, 2012; Naidoo & Govender, 2014; Prince & Frith, 2017; Ledibane, Kaiser & van der Walt, 2018) and so it is to be expected that these barriers will manifest even more in learner-centred classrooms.

I end this discussion by showing that while literature and school policies across the globe clearly, advocate for a learner-centred approach to teaching-learning and many criticize the traditional, teacher-centred approach which is viewed as void of meaningful learning, there are some who see value in both approaches. For instance, Cao, et al (2019) posit that transmission elements of the traditional approach can be incorporated in the learner-focused approach. Similarly, Cai & Wang (2010) and Huang et al. (2016) found that Chinese teachers display a teacher-centred approach primarily even though some elements of the progressive approach is evident in their teaching. Large classrooms pose a challenge for these teachers to shift their teaching towards a more learner-centred approach (Huang et al., 2016).

Huang et al., (2016) found that through the use of variation in the teaching of patterns, the Chinese lesson was more cognitively demanding than the US lesson, however, fewer opportunities were afforded to learners to work together and share their thoughts. The US lesson which also employed the theory of variation demonstrated the elements of progressive approach which included the establishment of mathematics goals, to focus learning as well as the implementation of meaningful collaborative tasks that promoted reasoning and problem solving. The authors argue that these two lessons defied "the culturally binding belief regarding effective teaching" (p.155). Mhlolo (2013) also found that the use of variation in the teaching of functions was effective in a classroom which employed a traditional style of teaching. Lai (2012) addresses the issue observed by some writers as a contradiction in Chinese teaching which has been observed to be teacher-centred but still produces a high percentage of learners achieving above international standards in difficult subjects like mathematics. This contradiction has been referred to as the 'paradox of the Chinese learner'. He argues that the answer to the paradox of the Chinese learner lies in the misconception by Western scholars regarding the teaching methods used by Chinese teachers. According to Lai (2012), procedural variation prevalent in Chinese classrooms (Marton et al., 2004) has been misconstrued and misinterpreted by Western scholars as rote learning when in fact the success of this approach should be investigated.

I conclude this discussion with a quote from Sfard (2000), who reminds us that the learning of mathematics must be the main focus in the classroom, and indeed attention should be paid to the various approaches at our disposal to use as means to achieving this pivotal goal.

> The student who arrives in a mathematics classroom is supposed to learn participation in a discourse that, so far, was inaccessible to him or her, and this means, among others, getting used to acting according to a new set of meta-discursive rules. The new discursive behaviors of the learner develop gradually as a result of classroom interactions. The way this happens deserves attention (Sfard, 2000, p.171)

2.6 Perspectives on teaching and learning

Psychological research identifies two distinct views that influence teaching and learning, namely, the fixed mind-set or entity view of intelligence and the growth mind-set or incremental view (Conference Board of the Mathematical Sciences, 2012). The former sees cognitive abilities as unchanging or fixed from birth while the latter considers cognitive abilities as expandable. These views or beliefs have an effect on how learners perform in mathematics and it is believed that instructional practices can influence or change how learners perceive themselves. In a study which explored learners' perceptions on mathematics performance in Polokwane, Mutodi & Ngirande (2014) found a strong positive correlation between performance and perception constructs like self-confidence, interests in mathematics, teacher and learning materials. It was found that lack of self-confidence may lead to different perceptions about mathematics and attitudes towards the learning of mathematics and this in turn affects performance in the subject. To develop mathematical knowledge for teaching, it is imperative that teacher beliefs and attitudes about the teaching and learning of mathematics be considered (Cai & Wang, 2010). This should be done from two perspectives, firstly, teacher beliefs about how the teaching of specific content should be explored and secondly, consideration should be given to what teachers believe about the learning of mathematics and learner capability. Teachers who believe that mathematics is a set of rules and procedures to be memorised through rote learning will teach mathematics the same way (Huang et al, 2016). Similarly, teachers who have a view of cognitive abilities as unchanging or fixed from birth will not expect certain learners to achieve beyond a certain level in mathematics. In contrast, teachers who believe that learners learn best through self-discovery and participatory methods will employ instructional strategies which encourage inquiry-based learning (Inoue et al, 2019). The remainder of this chapter is a presentation of literature reviewed on teacher knowledge and the teaching of functions.

2.7 Teacher knowledge of mathematics

My discussion begins with some crucial points raised by Adler (2017) regarding the place of a mathematics teacher amongst possible careers associated with studying mathematical sciences at university. Adler (2017) asked two questions, firstly, where in a discussion of the future of the mathematical sciences in a rapidly changing, challenging and exciting world do we locate the career of a future school mathematics teacher? Secondly, what does this location mean for mathematical sciences curricula or education at university? Similarly, the Conference Board of Mathematics and Science (CBMS, 2012) seems to point towards the need to define clearly the meaning of a properly qualified mathematics teacher. This report expresses concerns about the failure of the mathematics courses offered at universities in the US to meet knowledge needs of high school teachers (CBMS, 2012). Furthermore, a recommendation is made that Grades 5 – 8 learners must be taught by teachers who have specialized in the teaching of mathematics, as few undergraduates get a chance to develop mathematical knowledge as mathematicians or teachers. The report seems to suggest that prospective teachers should be exposed to the same mathematical rigour as that acquired by mathematicians or at least be exposed to a certain level of research in mathematics so as to acquire mathematical competency even prior to becoming mathematics teachers.

Similarly, Mhlolo, Schäfer & Venkat (2012) state that South Africa like other developing and developed countries, has revised its policies in recent years in order to accommodate the knowledge and skills needed for learners to participate in the global twenty-first century economy. Spaull (2015) calls for a review of the curriculum advisors' qualifications in South Africa and recommends that all curriculum advisors must be required to write a subject specific test as some of them lack the necessary subject knowledge. Goal 16 of the national development plan 2030 states that the Department of Basic Education aims "to improve the professionalism, teaching skills, subject knowledge and computer literacy of teachers throughout their entire careers" (Action Plan 2019, 2015).

A lot of research has been dedicated towards researching mathematical knowledge of pre-service teachers in training. I present this research in the next discussion.

2.8 Pre-service teacher knowledge of mathematics for teaching

In South Africa there are two routes into teaching mathematics at a secondary school level. In the first route pre-service teachers obtain a bachelor's degree either in mathematics or at least with some mathematics taught by a Mathematics Departments at a university, followed by a Postgraduate Certificate in Education (PGCE) taught by an Education Department (National Policy Framework for Teacher Education and Development in South Africa, 2006). The primarily aim of this qualification is to develop teaching skills or pedagogy and not knowledge of the subject/s (SMK). Students are therefore expected to have the necessary subject disciplinary knowledge in their chosen subjects for teaching in a school before embarking on a PGCE. The second route is a Bachelor of Education degree, with Mathematics and Education courses taught predominantly in Education Departments.

Comparing the two routes offered by South African teacher training institutions, Adler (2017) states that while on the one hand, obtaining a bachelor's degree, followed by a PGCE may provide sufficient knowledge for teachers to teach calculus and algebra, on the other hand, the training which is condensed into a one year programme may not be enough to support the needs of the school curriculum. She continues to argue that research already shows that the calculus taught at university does not support quality teaching of geometry and other topics within a school curriculum. Smilarly, the four year Bed programme which is rich in pedagogical knowledge has its own limitations considering that students entering the programme come from various educational backgrounds (Bohlmann, et al., 2017). While many of these students have ritualised knowledge of mathematics which is rich in procedure and skills, a lot of them may not be able to grasp the mathematical principles underlying the procedures involved (Adler, 2017).

The research on mathematical knowledge for teaching in South Africa has been shifting more and more towards finding ways of ensuring that the right kind of knowledge is passed on to teachers in their initial training period (Tavil, 2014; Modipabe & Kibirige, 2015; Spangenberg, 2017; Oswald, 2019). According to (Nel & Luneta, 2017) if the quality of teacher education is to improve, there needs to be an emphasis on subject matter knowledge linked closely to pedagogical knowledge in the design of the teacher training programmes. Like Adler (2017), who raises concerns about the effectiveness of the PGCE programme to prepare teachers to meet the needs of the school mathematics curriculum, (Kwenda, 2014) raises the issue of time required to train teachers in a PGCE programme, stating that the one year training is skewed towards pedagogical knowledge to the neglect of subject matter knowledge. Combining subject matter and pedagogical knowledge is not a new concept. This is what Shulman (1986) referred to as Pedagogical Content Knowledge (PCK). Hashweh (2013, p.120-121) defines PCK as "the set or repertoire of private and personal content-specific general event-based as well as story-based pedagogical constructions that the experienced teacher has developed as a result of repeated planning, teaching, and reflection on the teaching of the most regularly taught topics"

2.9 Studies involving PCK

Research on the topic of pre-service teacher knowledge of mathematics has shifted to focus heavily on student teachers' PCK for teaching (Essien, 2010; Lannin, 2013; Adler & Venkat, 2014). This emphasis on PCK results from the need to move towards placing content knowledge (CK) and pedagogical content knowledge (PCK) within a network of relations between teachers and other teachers and between teachers and learners (Askew, 2014). This enables teachers to study their own knowledge of mathematics education in order to gain better understanding of mathematics teaching (Askew, 2014). The level of knowledge attained depends on the length, intensity, and quality of the teacher-training programme attended (Baumert, Kunter, Blum, Brunner, Voss, Jordan, Klusmann, Krauss, Neubrand & Tsai, 2010).

Empirical evidence of PCK in teachers is not an easy construct to research and there have been concerns about the viability of PCK as a practical construct for examining teacher knowledge (Lannin et al., 2013, p.5). However, it is clear from previous research that this knowledge domain does exist and that it can be studied empirically (Ball et al., 2008). Studies involving PCK vary and depend on researchers' choice of constructs to input as units of analysis which will enable them to describe the teachers' PCK and hence illuminate the mathematics made available to learn. Adler & Venkat (2014) for instance, used examples and accompanying explanations that teachers offer in the classroom to study a teacher's mathematical discourse in instruction (MDI). The findings contradicted previous research which had been regarded as the norm in many South Africa's disadvantaged schools concerning the slow pacing of lessons and limited examples. Instead the study observed that the teacher offered a variety of examples displaying successive variation and a sequential progression. There is a growing body of literature on teacher noticing and Ding & Dominguez (2016) found that a strong PCK can enhance prospective teachers' opportunities to notice learners' mathematical ideas. Similarly, Llinares (2020) highlights the importance of connecting specific mathematical knowledge with the teaching practices and Myers & Rivero (2019) argue that learning experiences in teacher education should include rich conceptual knowledge. According to Zhu, Yu & Cai (2018) teachers' knowledge of learner thinking significantly impacts on how teachers teach and how learners learn. Depaepe, Verschaffel & Kelchtermans (2013) stated that it was not surprising that most studies on PCK involved fractions at the primary school level or algebra and functions at the secondary school level since these were considered to be amongst the most difficult topics in school mathematics.

This current study sought to investigate how teacher knowledge influences the quality of instruction in the teaching of functions. The more recent research on teacher knowledge in South Africa focuses on error analysis which forms part of the knowledge of content and students (KCS), an aspect of pedagogical content knowledge (Moru et al., 2014; Sorto, Sapire & Shalem, 2014; Luneta, 2015; Gardee & Brodie, 2015). Sorto et al (2014) argue that teachers' participation in error analysis is an integral aspect of teacher knowledge, similarly, Moru et al, (2014) states that when teachers manage to identify student errors, they possess a component of SMK known as common content knowledge (CCK). Sorto et al (2014) found that teachers seem to draw on different kinds of knowledge when dealing with student errors. The study concluded that procedural and conceptual explanations require teachers' knowledge of mathematics while diagnosing errors depends on teachers' familiarity with learner thinking and reasoning. Error analysis plays an important role in the teaching of mathematics, and how errors are handled will either support or deprive learners access to germane mathematical knowledge (Gardee, 2015).

Other studies have looked at how teachers handle processes central to the teaching of mathematics which include the use of authentic representation and making mathematical connections (Mhlolo et al., 2012; Gierdien, 2012; Maoto et al., 2018). The findings from this South African research reveal that learners lack conceptual knowledge and often fail to interpret questions (Luneta, 2015), make a wide variety of errors in basic algebra (Pournara, Hodgen, Sanders and Adler, 2016) and show poor quantitative competencies (Nel, 2020). Sorto et al (2014) reported that teachers experienced difficulty with responding to errors in context which could be attributed to mathematical knowledge gap, linguistic ability or lack of experience with responding to learners' utterances. However, Moru et al (2014) found that while some teachers have familiarised themselves with some of the common errors made by learners (SMK), a lot of work is still needed in the area of knowledge of content and teaching (PCK). Similarly, Maoto et al (2018) reported that teachers rushed through representations, focusing more on derivation of symbolic representations, while Maoto, Masha & Maphutha, (2016) also confirmed the tendency by teachers to view mathematics as a collection of rules to be memorised. Other studies reveal the increasing need to use technology in the teaching of mathematics (Els & Ellis, 2013; Naidoo & Govender, 2014; Leendertz, Blignaut, Nieuwoudt, Rosenberg & Koehler, 2017; Umugiraneza et al., 2018). Mishra & Koehler (2006) developed Technological, Pedagogical, and Content Knowledge (TPACK) as a framework for using technology in teaching. (Tondeur, Scherer, Siddiq & Baran, 2017, p.4) state that "teachers' Technological, Pedagogical, and Content Knowledge (TPACK) facilitates the meaningful use of technology for educational purposes"

Lannin et al. (2013) provide insight into the development of PCK predicted by Shulman (1986) as knowledge that grows in the minds of teachers, as well as a platform for other researchers to study how PCK for teaching mathematics develops for individual teachers. The study investigated PCK growth of two beginning mathematics teachers over a two year period and sought to answer the question: what PCK develops over 2 years for beginning mathematics teachers? The researchers developed a PCK model for teaching mathematics to use as a framework for generating and analysing data. Data were generated through interviews and lesson observations and the study observed that the four components of PCK for teaching developed differently in the two participants. The four components are represented in Figure 2-1. The study also observed that an increase in the knowledge of assessment led to an increase in knowledge of instructional strategies, knowledge of student understanding and curriculum knowledge. A close relationship was observed between the teacher's knowledge of assessment strategies and knowledge of student understanding. It was concluded that by improving his knowledge of assessment strategies this participant was able to gain better knowledge of how the learners understood the mathematics they were learning and this increase in the knowledge of student understanding had a positive impact on the other components of PCK.

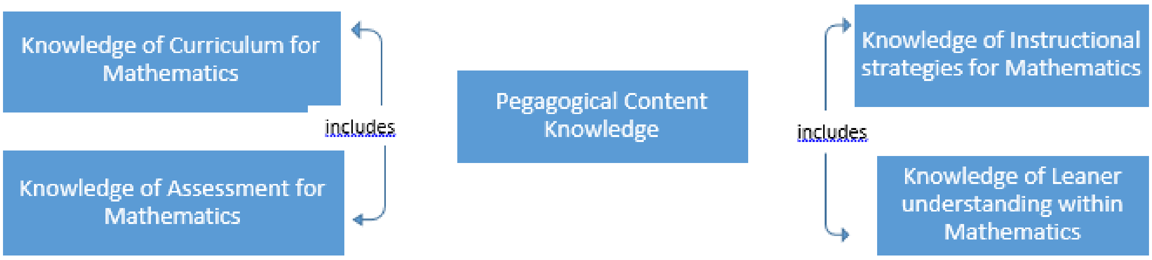


Figure 2-1: A model of PCK for teaching mathematics (Lannin et al.,2013, p.4)

The study led to the conclusion that there is a need to individualise the professional development of teachers during the induction years.

2.10 'Professionalization' of mathematical knowledge for teaching

Almost a decade ago Wilson, Rozelle & Mikeska (2011) made a compelling argument for the establishment of organized systems of opportunity to learn for teachers in their analysis of the opportunities for professional development offered to teachers in America. This evaluation painted a bleak picture of what was referred to as a '(non)-system' of professional learning opportunities. It described an incoherence that existed between teacher preparation, induction and professional development programmes which lacked a continuum and which followed an apparent 'cacophony of pathways'. These American authors called for a system of teacher development that is relevant, flexible and effective for both teachers and learning. They also highlighted the need for a more coherent system that would be "explicit about the theories of teacher learning that drive decision making about the design of substantial learning opportunities for teachers" (Wilson et al., 2011, p.10).

Not much appears to have changed since Wilson et al. (2011) wrote the report about their analysis of the opportunities for professional development offered to teachers in America. Tooley & Connally (2016) in their review of the policy for professional teacher development argue that pre-service training lacks theory and a shared vision and criticise the 'egg-crate' culture in which teachers work in isolation tucked away in their own classroom. The authors advocate for Professional Learning Communities (PLCs) that are properly implemented in order to provide effective feedback. Similarly, (Darling-Hammond, Hyler & Gardner, 2017, p.10) noted that "when professional development utilizes effective collaborative structures for teachers to problem-solve and learn together, it can positively contribute to student achievement".

Internationally and locally, the Higher Education sector is paying more attention to the quality of teaching (Brijllal & Isaac, 2011; Mobarak, 2019; Ellis & Childs, 2019). This calls for teacher education that is adapted to teachers' changing roles (Oswald, 2019). It also calls for the reconsideration of the interconnections between pre-service teacher education, induction into the workplace, and the continued professional development of teachers with a focus on improving pedagogy and content knowledge (Spangenberg, 2017).

In keeping with international standards, South Africa introduced a policy document on the development of Professional Learning Communities (PLCs). This policy was introduced due to the need to organize professional development around networks of teachers in their communities. It was said at the time "much professional development is still organized as isolated and one-time trainings, lacking a coherent strategy, monitoring and follow-up… which often fail to have durable effects on teaching and learning" (DBE, Republic of South Africa, 2015, p.4). Nel & Luneta (2017) also raise a concern that most professional development initiatives do not focus on mathematical content and instructional challenges faced by individual teachers in their classrooms and argue that this shortfall might account for South African learners' poor performance in international assessments. A lot of research has been conducted in the area of professional learning communities in South Africa (Nel & Luneta, 2017; Chauraya & Brodie, 2018; Umugiraneza, Sarah Bansilal & North, 2018; Ngcoza & Southwood, 2019; Okeke & Westhuizen, 2020; Feldman, 2020).

Professional Learning Communities are about professionals working in community for the purpose of improving learning. Other researchers have referred to professional networks, or networked learning communities as "webs of interaction, appropriate environments for enabling and enacting processes of collaborative professional development" (Ngcoza & Southwood, 2019, p.3). Feldman (2020) notes that teacher learning and development in PLCs focuses on collective reflective inquiry into becoming betters teachers and practitioners. Many have pointed out the need to move teaching and learning towards 21st century pedagogy which includes the use of technology. It has been argued that most teacher education programmes across the world still do not provide preservice teachers with the knowledge and real-world skills for teaching a global community (Myers & Rivero, 2019) and continuous professional development is necessary if teachers are to integrate the newly acquired technological knowledge into their pedagogical knowledge (Umugiraneza, Sarah Bansilal & North, 2018; Choi et al, 2019). Furthermore, PLCs need to take into account the social setting in which professional development takes place (Feldman, 2020) and some have pointed out the challenge that exists when it comes to aligning teacher education programmes to the realities of the actual classroom (Frick, Carl & Beets, 2010; Mobarak, 2019). To this regard, the need to consider context and multilingualism (Essien, 2010; Chirinda & Barmby, 2017) cannot be overlooked when PLCs are formed. The final point I want to make is that participating in professional learning networks allows teachers to gain new shared meanings of concepts, thus deepening their subject content knowledge as Chauraya & Brodie (2018) discovered. It is therefore important to involve research experts on teacher learning in the formulation of PLCs in order to enhance teacher learning in specific areas (Okeke & Westhuizen, 2020)

2.10.1 Continued Professional Teacher development in South Africa (CPTD)

It can be argued that the same disjuncture observed within the American system of teacher professional development exists in South Africa. The National Policy Framework for Teacher Education and Development (NPFTED, 2007) states that South African Council for Educators (SACE), as a statutory body for professional educators will have overall responsibility for the implementation, management and quality assurance of the Continued Professional Teacher Development (CPTD) system. It is further stated that SACE will be provided with the necessary resources and support to undertake this role. It is, however, the responsibility of individual teachers to identify their own areas of need and to approach the Department of Education for assistance to access training opportunities. On completion of such training, teachers should then earn points towards fulfilling the CPTD requirements of SACE. Since SACE cannot provide any form of professional development, the policy document stipulates that "there is a need to allow a wide range of training providers to offer professional development courses for teachers, subject to approval by SACE" (NPFTED, 2007, pp.307-308). The policy also recognises the role of Unions in providing professional development for teachers and a commitment is made to assist Unions in developing capacity to implement CPTD strategies for their members.

Working in conjunction with the National Policy Framework for Teacher Education and Development, the Integrated Strategic Planning Framework for Teacher Education and Development in South Africa (ISPFTEDSA, 2011-2015) is a 15 year plan which seeks to "improve the quality of the Teacher Education and Development (TED) system in order to improve the quality of teachers and teaching" (p.1). The policy document states that teachers will be assisted to identify professional development needs through analysis of learner assessment results and by taking friendly diagnostic tests based on the theoretical and practical framework of the content within the school curriculum. Teachers will be helped to identify and address their own professional development needs by: interpreting their own learners' performance in national (and other) assessments; assessing themselves by taking user-friendly diagnostic tests based on the content (theory and practice) frameworks of the school curriculum; and by using the results from the diagnostic tests to identify appropriate ways to address their individual needs. Furthermore, teachers will be encouraged to join Professional Learning Communities (PLCs) to facilitate the identification of learning needs.

The National Education Evaluation & Development Unit (NEEDU, 2018) came up with a paper on Effective school-based professional development for teachers in which the aims of the National Policy Framework for Teacher Education and Development (NPFTED) and the Integrated Strategic Planning Framework for Teacher Education and Development in South Africa (ISPFTEDSA) are implemented in "schools that work". Three key areas of staff development for the purpose of identifying areas of need found to be prevalent in these schools were: the establishment of the internal quality management system (IQMS), informal class visits by the school management team (SMT) and the analysis of learner assessment results. It is not clear however, where the research was done considering South Africa's complex schooling context.

While it is clear that steps are being taken to facilitate the implementation of the mandated continued professional development system for teachers as shown by the paper produced by NEEDU (2018), many questions still arise concerning South Africa's teacher professional development programme. The first question pertains to capability on the part of the teacher on the one hand and capacity on the part of the Department of Education on the other hand. It is not clear whether teachers approach the government on their own upon identification of a need or whether this is done through the school or via a union especially as there is mention of funding that can be accessed (ISPFTEDSA, 2011-2015). Given the number of teachers employed by the government it is not clear how every individual teacher needs can be met and if this were possible, what time constraints exist in terms of meeting that need. The plan indicates that teachers will be encouraged to join PLCs in order to identify their individual developmental needs, however, nothing is mentioned regarding mandatory training of new and inexperienced teachers. Other questions that may arise concern the actual earning of CTPD points. Firstly, accessing computers and the internet may prove a challenge for some teachers and secondly, how will SACE ensure validity of claims that teachers have indeed embarked on these professional development trainings?

As has been pointed out by many writers in the literature reviewed in this study, teacher preparation programmes should be closely linked to induction and continued professional development programmes. This will achieve the objective of preparing teachers to teach while ensuring that these teachers are supported in their development of the necessary knowledge and skills needed in the teaching profession. Reflecting on practice is one of the most crucial skills needed in the development of daily activities of professionals.

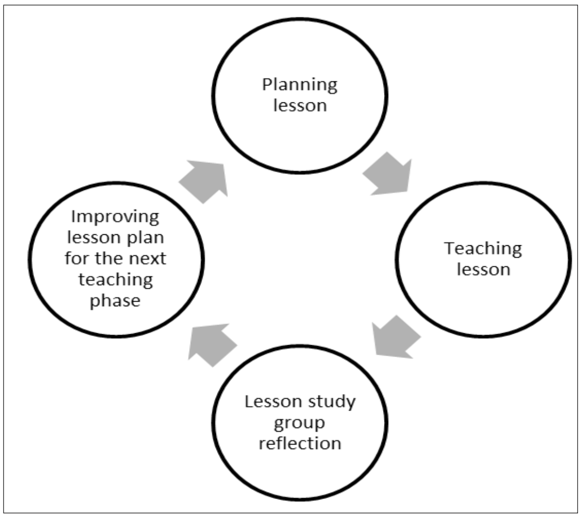
2.11 Reflexive practice in teacher education

According to Schon (1991, p.69) "the dilemma of rigor and relevance may be dissolved if we can develop an epistemology of practice which places technical problem solving within a broader context of reflective inquiry, shows how reflection in action may be rigorous in its own right and links the art of practice in uncertainty and uniqueness to the scientists art of research" Similarly, Stingu (2012) states that developing teachers' professional identity is about the deconstruction, construction and reconstruction of assumptions about the profession through everyday interactions. Furthermore, Stingu (2012) states that "reflexive practice is being used in initial and in-service teacher education to enhance teachers' capacity of self-observation, self-analysis and self-evaluation".

Fook (2015) makes a distinction between 'reflective practice' and 'critical reflection' even though these terms are often used interchangeably while Finlay & Gough (2008) describes 'reflective practice' as the amalgam of self-awareness, reflection and critical thinking. Referring to the initial conceptualisation of reflective practice by Schon (1983), Fook (2015) states that the concept of 'reflective practice' emerged as a result of a gap that was evident between theory and practice in professional practice. Reflective practice became a way of promoting professional practice by providing theory to support the work that professionals do versus what they say they do. Through reflective practice, teachers become aware of theories or assumptions involved in professional practice on the one hand and on the other hand, it allows teachers to develop theory by researching what happens in practice. Critical reflection like reflective practice is concerned with improving professional practice. It is the subset of reflective practice concerned with the understanding of power dimensions in order to create social change (Fook, 2015). Anderson (2019) states that despite extensive discussion of Schön's (1983) reflection-in-action concept in teacher education literature, very few studies have attempted to document it during interactive teaching. There is therefore a need to look closer into this conceptualisation if the teaching profession considers seriously the question of professionalism. Critical practice also involves personal research of one's practice for the purpose of understanding of how one develops to become a knower or creator of knowledge (Barnhart & van Es, 2015; Hoynes et al., 2019).

In order to facilitate the acquisition of skills which enhance the ability to reflect on practice, most pre-service teacher training institutions incorporate micro teaching in their programmes (Kwenda, 2014; Modipabe & Kibirige 2015). Micro teaching allows transference of skills and knowledge as well as offers opportunities for pre-serving teachers to reflect on their practice (Jong, Meirink & Admiraal, 2019). Micro teaching often takes a form of a lesson study, where pre-service teachers plan and reflect on their teaching by studying videos of lessons taught in collaboration with peers under the supervision of experts like university lectures or mentor teachers (Hoynes, Klemp and Nilssen, 2019). The lesson study method is an established Japanese system for teacher development (Rasmussen, 2016; Akerson, Pongsanon, Park Rogers, Carter & Galindo, 2017) which has been used in various contexts such as South Africa (Posthuma, 2012), Denmark (Rasmussen, 2016), China and the U.S. (Huang, Barlow and Prince, 2016) for school improvement initiatives and is associated with a positive effect on student outcomes (Hadfield & Jopling, 2016). According to (Mhakure, 2019, p.2), a lesson study is a "collaborative teacher-inquiry CPD with specific emphasis on reflection on practice and learners' cognition, leading to the development of a teacher's expertise and learning within the context of their work environment" and Zhou et al. (2017, p.88), define it as "a professional development process that engages teachers in collaboratively examining their practice with a goal of becoming more effective". Other collaborative methods including reflective journals and developmental portfolios are also increasingly being used globally to engage pre-service and in-service teachers in reflective practice (Pournara, 2013; Hoynes, Klemp and Nilssen ,2019; Civitillo, Juang Badra and Schachner, ; Chye et al., 2019).

In South Africa as well, a number of research studies have been conducted in the area of pre-service teacher training and in-service professional development (Ndlovu, 2014; Nel & Luneta, 2017; Ngcoza & Southwood, 2019; Moloi, Kanjee & Roberts, 2019; Feldman, 2020; Okeke & Westhuizen, 2020). Some of this research employed a lesson study as a tool for micro teaching (Posthuma, 2012; Mhakure, 2019), while others focused on other methods of developing reflective practice such as the use of reflective writings (Costandius & Botes, 2018), keeping of reflective journals including electronic journals (Tavil, 2014) and the use of reflective interviews (Chirinda and Barmby, 2017; Nel & Luneta ,2017). The findings from this research vary, for instance, Mobarak (2019) found that graduates were generally not ready for workplace while Modipabe & Kibirige (2015) concluded that universities need to establish in-service workshops to assist mentor teachers in dealing with the needs of pre-service teachers. Posthuma (2012) conducted a study which involved five mathematics teachers from one rural school using a lesson study method to generate data. The lesson study involved a three phase cycle consisting of planning, teaching and evaluation. The participants reported that through reflecting on their own practice by studying videos of themselves and their colleagues teaching, they were able to improve their practice. Areas of improvement were reported to include the ability to self-research, improved lesson planning, teaching with confidence, a deeper awareness of learners' needs and learning from colleagues. Figure 2-2 shows a lesson study cycle.



The cycle involves planning a lessons together, one teacher teaching a lesson, the group reflecting on the lesson taught and the group revising and improving on the lesson plan which would be re-taught when the cycle begins again. The rest of the South African literature reflective practice was discussed in the first chapter.

Barnhart & van Es (2015) suggest that effective reflective practice is determined by how pre-service programmes frame problems of practice as this influences how they identify noteworthy information to be analysed for the formulation of testable theories about practice. Moreover "pre-service teachers must be provided with tools and frameworks to help guide what they attend to in teaching" (p. 85). Conceptual frameworks for professional development similar to the one developed by Chong & Cheah (2009) can provide a useful knowledge structure for teachers in training and for novice teachers. A Values, Skills and Knowledge (VSK) framework for initial teacher preparation programmes is shown in Figure 2-3.

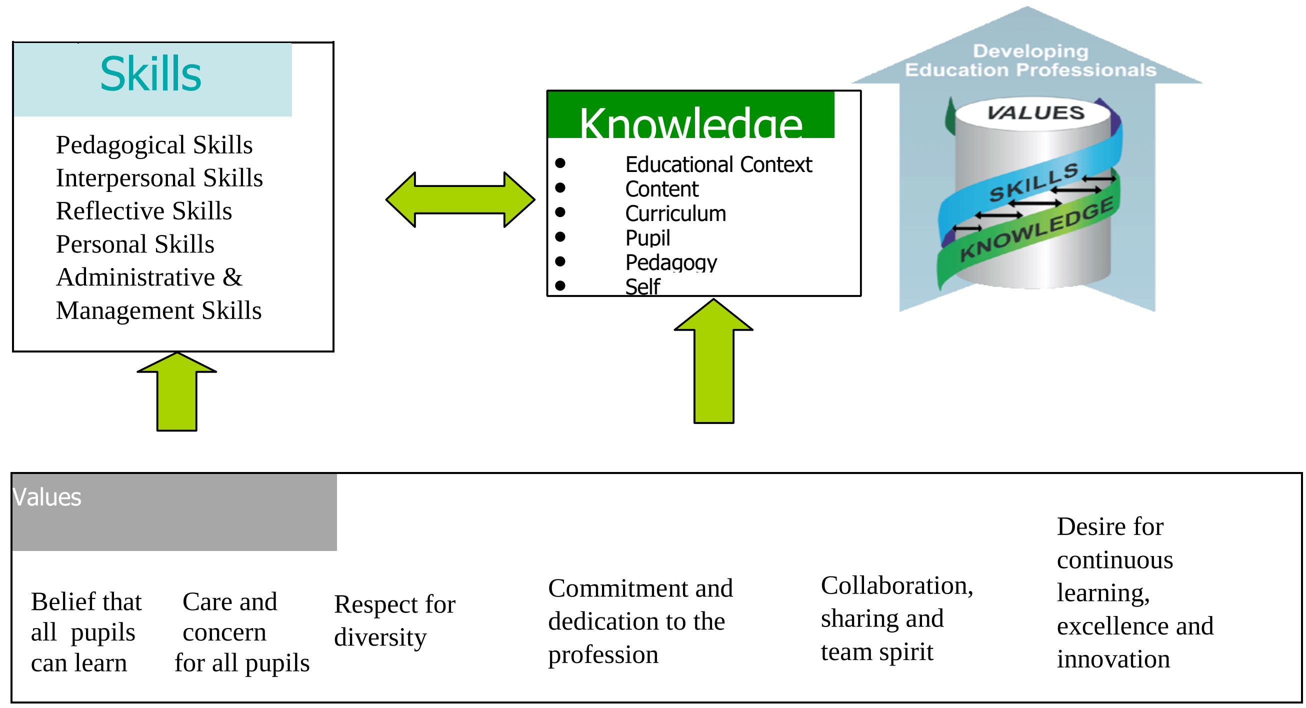


Figure 2-3: NIE's VSK Framework (Chong & Cheah, 2009, p.4)

The model focuses on the development of teachers who have the necessary values, skills and knowledge to participate authentically in the profession of education. The knowledge domains are similar to the ones introduced by Shulman (1986) as knowledge necessary for the profession of teaching except for the last one (knowledge of self). The model includes knowledge of self as knowledge that enables teachers to examine critically their beliefs and values about teaching in order to form visions about conceptions of teaching so as to identify areas of development. Using this conceptual framework would enable teachers to understand their own strengths and weaknesses for the goal of meeting learners' diverse needs (Chong & Cheah, 2009). Reflection in practice has also been associated with self-efficacy or the confidence in one's ability to perform a task effectively. Tavil (2014) found that writing e-journals during the practicum period increased the self-efficacy levels of pre-service teachers and Frick, Carl and Beets (2010) agree that reflection is a process in which students learn about self in context. Close to the concept of self-efficacy is the notion of expertise. An expert teacher would be expected to reflect differently to a novice teacher due to the differences in the levels of teaching experience and possibly content knowledge possessed by the two teachers.

2.12 Becoming an expert teacher through reflective practice

The focus of initial teacher preparation should be to lay a foundation for reflective practice upon which teachers will build their knowledge with the aim of becoming experts and specialists in the subjects they teach. Some writers have contributed towards what they consider to be characteristics of an expert teacher. For instance, Findell (2009) characterises an expert teacher as one who identifies, presents and displays the ability to connect key ideas, listens and assists learners to think for themselves by providing low threshold, high ceiling problems, remains flexible and never stops learning. Similarly, Barnhart & van Es (2015) state that when faced with a complex situation, expert teachers are able to distinguish between important and unimportant information, can reason about what they observe and can analyse these observations to make more informed decisions about teaching. Furthermore, Tsui (2003) asserts that while novices plan their lessons following rules and models, expert teachers exercise more autonomy, spending less time in planning and yet their planning is often much more effective. Other research shows that expert teachers have the ability to reflect with ease (Beswick, Fraser & Crowley, 2016) and rely on their teaching experience to understand and handle learners' thinking (Zhu et al., 2018).

It should then be the aim and focus of every initial teacher-training programme to develop teachers who are reflective practitioners and who understand and share the core values and goals of educational ends; teachers who will grow to become experts in their teaching subjects. Expert teachers show personal understanding by integrating their knowledge of the curriculum, learners, teaching methods and strategies as well as contextual factors in their planning (Tsui, 2003). Personal understanding of a concept determines the level at which a teacher can make mathematical connections (Bansilal, 2014).

2.13 The need for contextualised pre-service training programmes

Supporting knowledge acquired during the initial teacher training period within a conceptual framework is a plausible notion, however, the challenge would be to identify the model that works for individual institutions. Some writers have mentioned the need for individualisation of knowledge taught to pre-service teachers and for professional development that is tailored to individual teacher needs (Lannin et al., 2013).The need to educate teachers on issues like proper use of textbooks and other curriculum materials has been highlighted by some researchers. In a comparative study of Germany and South Africa, Mellor, et al., (2018) concluded that the German textbook included more content that promoted the development of conceptual knowledge. The study also found that, while the South African textbook presented a broader range of opportunities to interact with the different representations of functions, the German textbook, included more links to the real world. This study speaks to the importance of careful consideration when choosing textbooks in mathematics. Mwadzaangati (2019) concluded that while teachers are able to design high cognitive tasks with the availability of good textbooks, it is more important, however, that teachers develop conceptual ability to make effective use of these textbooks. Similarly, Christiansen (2012) points out that teachers' content and pedagogical knowledge has less impact when the school is more disadvantaged and failure by teachers to turn resources into educational advantage which, happens in some poorer schools, undermines the government's initiatives for school improvement.

There has been other research which has highlighted the need to introduce conceptual frameworks geared towards the teaching of specific content. For instance, Marton et al. (2004) in their discussion of variation theory as a framework for teaching mathematics argue that there is no legitimacy to the notion that knowledge of progressive approaches allows teachers to teach all content using any method. They argue that specific conditions are necessary for learning specific content, and that these conditions differ from one object of learning to another. Mhlolo (2013) also agrees that variation theory allows teachers who use traditional approaches to the teaching of mathematics to teach conceptually and to make connections procedurally. This current study also observed the use of variation by some participants as the results will show.

Having started this discussion by reviewing literature on mathematical knowledge for teaching focusing on pre-service teacher training and how this education can be used to improve the level of mathematics taught at schools, I now move on to present the review of literature on how research on teacher knowledge has evolved over time.

2.14 Research on teacher knowledge over time

2.14.1 Teacher knowledge in the medieval era and the 19th century

Making reference to Ong's (1958) chapter 'The Pedagogical Juggernaut' in Ramus, Method and Decay, Shulman (1986) gives an account of how teacher knowledge was defined by medieval universities. This account portrays universities as schools that did not separate content from pedagogy. What is known and how to teach that knowledge were treated as part of one distinguishable body of understanding. To drive this point home Shulman (1986) emphasizes that just as graduation is the beginning of a teaching career, so were the medieval universities seen as normal schools. These schools were medieval societies composed of various teachers' faculties. The bachelor degree was admission to the body of apprentice teachers, while the master or doctorate degree was the formal admission into the guild or society. The highest degree obtained therefore allowed the candidate to teach and the purpose of the examination was so the candidate could demonstrate that they possessed the highest levels of subject matter competence in the domain of that degree. The ability to teach the subject was the only way to demonstrate understanding of the subject matter. There was therefore no separation of content from pedagogy. Kilpatric (1992, p.4), however, states that "although through the 19th century, universities graduated teachers of mathematics for secondary school, instruction in the teaching of mathematics was at best a separate and minor part of the teacher's preparation". According to Kilpatric (1992, p.4), only at the end of the century were the attempts made to "establish didactics as a discipline dealing with school knowledge as against a more general pedagogy" and only then did university students in Germany begin to receive practical training in mathematics teaching.

Shulman (1986) highlights clearly that the medieval universities were not a pedagogical utopia. Universities were pedantic on how the curriculum was handled by those entrusted with the responsibility to teach. Punctuality was enforced to the extent that lecturers were fined for going over time or of not adhering to the end of the lesson bell. Lecturers were also not permitted to skip a chapter in a book or to postpone a difficult section to the end of the lecture. Teacher knowledge in the medieval era required a stringent demonstration of subject matter knowledge through the art of teaching that subject. There was no separation between the known and the knower.

The medieval era appears to have been the perfect age for teacher knowledge. By the nineteenth century a demarcation between content and pedagogy started to appear. Analysing the 1875 California Teachers Examination items, Shulman (1986) makes an argument that teacher knowledge in the 19th century was characterized by the emphasis on subject matter to be taught or knowledge base assumed to be needed by teachers to the neglect of pedagogy. Although attempts were made to cover pedagogy, Shulman (1986) shows that out of a total of 1000 possible points to be scored on the examination paper, only 50 points were given over to theory and practice of teaching. According to Shulman (2004, p.191), "ninety to ninety-five percent of the test is on the content, the subject matter to be taught, or at least on the knowledge base assumed to be needed by teachers, whether or not is taught directly". It is not clear what brought about this distinction between the subject matter and the teaching thereof.

2.14.2 The 1950s to the 2000s

In the 1960s most research associated with learner gains in mathematics was designed as comparative studies on instructional programmes. In his review of this research, Romberg (1992) states that such studies attempted to control factors like teachers and other procedures which relate to the instructional situation. One of the biggest comparative cross-cultural, research studies conducted in 1964 was The International Study of Achievements in Mathematics. This involved school children in Australia, England, Belgium, France, Finland, Japan, Israel, Sweden, Scotland, The Netherlands, United States and West Germany. In total 130 000 learners, 13 000 teachers and 5000 schools from 12 countries participated in the study (Robitaille & Travers, 1992). The main objective of the study was to identify social and educational practices which influence learner achievement in mathematics. The results revealed considerable differences among learners in different countries, however, suggestions were made that if the differences were due to opportunities to learn, then these differences were a function of the differences in the curriculum rather than social or educational practices (Robitaille & Travers, 1992). This study claimed to be correlational rather than causal in nature, however, Romberg (1992) argues that the discussions and conclusions were leaning towards an uncritical acceptance of certain causal interpretations

The trend in the research in mathematics education in the 1960s further showed that Doctoral theses published on this topic were on the rise. Kilpatric (1992) offers insight into this growth in academic publications in what he refers to as the 'golden age'. According to Kilpatric (1992), in the early to mid-1950s there were proposals from many sides to reform the school mathematics curriculum. American schools were receiving pressure from the business sector, the military, colleges and the public, who were accusing the education system of watering down the curriculum in response to progressivism and life-adjustment education. This watering down of the curriculum was said to have resulted in schools graduating young adults who lacked basic computational skills and who were ill-prepared for college mathematics (Robitaille & Travers, 1992). New doctoral programmes in mathematics education were established in response to this pressure, which saw a rise in dissertations (Kilpatric, 1992).

The 1960s and 1970s, saw much growth in mathematics education research studies and this increase was in terms of numbers and scope as researchers moved across cultural boundaries and countries. Romberg (1969) predicted that the increase in mathematics education research would result in better and basic research dealing with basic problems about human acquisition of concepts and skills and in the development of better tests for use in maths education. Since then researchers have been striving to develop better tests and to enhance the quality of research in mathematics education by considering variables previously controlled (Ball et al., 2008).

By 1970 studies were conducted to ascertain the relationship between teacher knowledge and student learning in mathematics and conclusions were drawn that no direct relationship existed. According to Fennema & Franke (1992), teacher knowledge was defined in these studies as the number of university level courses completed successfully. The use of these university courses as a proxy measure for teacher knowledge resulted in very little evidence presented regarding the integration of teacher knowledge and little was known about the relationship between the courses taken at university and classroom teaching.

From the 1980s a shift was observed in the approach of research to the relationship between teacher knowledge and student learning (Hashweh, 1986; Fenemma and Franke, 1992). The major shift was in the methodology used by different researchers. The focus of research became the teaching itself and in particular, what teachers do in the classroom. This paradigm shift placed research on teacher knowledge in the interpretive paradigm for the most part and away from the correlational techniques previously used to measure the relationship between components of teacher knowledge and student learning. Rich descriptions of what teachers do in the classroom were now viewed as the mediator between teachers' content knowledge and student learning (Fennema & Franke, 1992). This research has taken the form of case studies where individual or small number of teachers are observed and inferences are drawn between teachers' subject-matter knowledge and various aspects of the classroom. One study cited by Fennema & Franke (1992) was the observation of Ms Jackson, an expert primary school teacher, who was observed over a two year period in the areas in which her content knowledge differed from the norm. The study concluded that in the area in which the teacher was more knowledgeable, classroom instruction and subsequently learning was richer than in the area where Ms Jackson was lacking in knowledge. Unlike the research conducted in the previous decade, these results revealed that a relationship did exist between a teacher's knowledge of the content and student learning as observed through the classroom interactions and learner assessment of the content taught.

Teachers' subject matter knowledge continues to be the focus of research in mathematics education and the direction of future research points towards professionalism in mathematics teaching as already discussed in this chapter. The question of the kind of knowledge required for teaching others was Shulman's (1986) focus of discussion. I continue to trace the evolution of teacher knowledge as a subject of research by summarising Shulman's (1986) discussion and his introduction of pedagogical content knowledge as a knowledge domain.

Shulman (1986) points out that efforts to simplify the complexities of classroom teaching and in order to narrow the scope of research focus, teacher effectiveness studies neglected one central aspect of classroom life: the subject matter. Shulman referred to the absence of focus on subject matter as the 'missing paradigm' problem. The problem arose as research was no longer focused on how subject matter was transformed from teacher knowledge to the content of instruction. Questions about how particular formulations of that content affected how learners received it as knowledge were also neglected by researchers. This missing paradigm problem had a spill over into the designs and structures of teaching programmes, which began to treat teaching more or less generically. This was a radical departure from research of the day, which focused almost exclusively on general aspects of teaching.

2.14.3 Teacher knowledge in the twenty-first century

Shulman correctly predicted that the direction of research post-1986 would steer towards subject matter knowledge. The introduction of PCK as a knowledge domain produced widespread research from 1990 onwards (this literature was reviewed in section 2.9 of this chapter). A body of knowledge exists from a plethora of research on the knowledge needed to teach mathematics or mathematical knowledge for teaching (Hill, Ball & Schilling, 2008; Ball & Schilling, 2008; Christiansen, 2012; Pournara, 2014; Mudaly, 2015). Many more researchers are responsible for the existing knowledge on the topic of mathematical knowledge for teaching (MKT) which also happens to be the focus of this thesis. As has been mentioned before, Ball and colleagues have been amongst the most prominent writers in this area.

2.15 Teacher knowledge and the quality of instruction

There is enough evidence from literature on educational production function which focuses on the effect that learners, teachers and school resources have on learner achievement as well as literature on teacher knowledge, to suggest that strong teacher knowledge has a positive effect on classroom instruction (Kiramba & Smith, 2019; Inoue et al., 2019; du Plessis, 2020).

This claim was illuminated in the 2008 exploratory study by Ball and colleagues. The study employed qualitative and quantitative methods with the aim of investigating the association between the Mathematical Knowledge for Teaching (MKT) and the Mathematical Quality of Instruction (MQI). The authors state that MKT includes both the mathematical knowledge that is common and used by individuals in diverse professions as well as the specialised knowledge of mathematics used for the purpose of teaching. For this paper, the mathematical knowledge for teaching, refers to the knowledge domains under Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) as formulated in the conceptual framework by Ball et al. (2008). Five cases were selected to detail the association between MKT and MQI in order to answer five questions. The first question the study sought to answer was, what is the overall strength of the relationship between teachers' mathematical knowledge for teaching and the mathematical quality of their instruction? Secondly, how does MKT affect instruction? Thirdly, how does a lack of MKT constrain instruction? Fourthly, what factors mediate the expression of MKT in instruction? Finally, which teaching tasks are most affected by MKT, whether strongly or weakly? (Ball et al., 2008)

The study's findings revealed a strong and positive association between teacher knowledge and quality of instruction, mathematical errors in particular were found to be strongly related to teacher knowledge. Other factors contributing to either the support or to the hindrance of the teachers' use of knowledge in practice included the richness of mathematics taught, connection of classroom activities to mathematics and responding to learner contributions. Avoidance of errors and more rigorous mathematics in instruction were found to mediate the influence of MKT on instruction. The study also found that lack of mathematical knowledge for teaching resulted in instruction that was lacking in explanations, justifications and alternate representations. Other factors that mediated teacher knowledge in teaching were the curriculum materials used, beliefs about the teaching of mathematics and the effect of teacher professional development.

Cases were chosen to examine teachers whose pencil-and-paper responses (MKT) and video rubric scores (MQI) converged and those that showed divergence. It was hypothesised that teachers whose scores diverged would be the most likely to have the quality of their mathematical instruction thwarted by other factors.

2.16 Relevance of Ball et al. (2008)'s research to the current study

The study conducted by Ball et al. (2008) has a huge bearing on the current study which also sought to investigate the effect of teachers' mathematical knowledge in the teaching of functions. The five cases in particular provided the ground for comparison in the analysis of video recordings and in the presentation of the findings. The MKT model by Ball et al., (2008) also provide a theoretical framework which underpins this study. I now present the five cases analysed according to the model advocated by Ball et al. (2008).

Lauren

Convergent cases were those in which a teacher either scored high on both the pencil-and-paper items and video rubric scores or maintained a low score on both accounts. Two such cases are discussed in the article by Ball et al. (2008). Lauren's is a case of a teacher whose written MKT and observed lesson scores were high. Her lessons were characterised by few mathematical errors, and she spent more time engaging learners in worthwhile mathematical activities, never misinterpreted a learner's idea or failed to respond productively to a learner's mistake, provided insight into additional classroom tasks in which mathematical knowledge was apparent and necessary, chose and sequenced mathematical tasks building ideas in a logical way from a variety of curriculum materials. Lauren's lessons also revealed equitable practices and contextualisation of classroom mathematics into real-life problems. Lauren's overall instruction was described as "mathematically rich, linguistically clean, and responsive to students' ideas and misunderstandings" (Ball et al., 2008, p.446). Lauren's case lead to the conclusion that a teacher with a high level of MKT offers a better than average quality of instruction to the learners being taught in the classroom.

Zoe

Contrary to Lauren's instruction, Zoe's teaching was full of technical and general language errors. Her explanations and recording of the mathematics of the lesson on the board also contained errors, she failed to use multiple models to demonstrate mathematical ideas, neither Zoe nor her learners correctly explained a mathematical idea or procedure, there was no mathematical justification or proof in her teaching and there were missed opportunities to use learner errors and misconceptions in explanations.

Noelle

Noelle is a case of both convergence and divergence. Like Lauren, Noelle's pen-and-paper MKT score was high, however, video rubric scores revealed that she did better in some areas but scored very low in other aspects of her teaching. Closer analysis of the results showed that although Noelle selected rich mathematical tasks for her students, her explorations lacked telos or closure. Her lessons lacked direction and she did not make attempts to summarize the mathematical purpose or goal of the explorations. Noelle's learners were not assisted to arrive at some closure after engaging in a mathematical activity and no mathematical connections were made across lessons or to connect various activities. This lack of a telos in Noelle's teaching lead the authors to believe that when planning lessons, Noelle's focus was more on the nature of the activities she selected than on the content goals that these activities served. In the interviews, Noelle had been very critical of the mandated curriculum and preferred to consult a variety of materials as sources for her lesson preparation. Lauren had also worked across materials, however, in her case this had resulted in high quality teaching. Noelle's case led to a conclusion that "even strong MKT might not be sufficient to support teachers as curriculum developers" (Ball et al, 2008, p.51).

Anna

Anna like Noelle is both a divergent and a convergent case. Anna's MKT score was moderate and her mathematical quality of instruction (MQI) scores were also average in part. Anna scored the lowest in connecting classroom work to mathematical ideas and procedures, and richness of the mathematics. Anna's CCK faired moderately and this was reflected in her teaching which in some instances contained mathematical errors, errors in language, and she was weak in responding to students. Her poor performance in the SCK items was mirrored by her very low score for the richness of the mathematics knowledge in the lessons. The results revealed that when Anna followed the prescribed textbook, the quality of her instruction improved compared to when she used her own activities collected from professional development experience.

Anna preferred to use these activities over a textbook due to a belief that mathematics has to be made fun for learners. The authors argue that this supplemental curriculum material lacked mathematical content and degraded Anna's mathematical quality of instruction. In most of the observed lessons, learners in Anna's class spent more than half of the lesson time doing non-mathematics related work like cutting, pasting and colouring pictures. Anna's case led to a conclusion that the relationship between MKT and MQI is affected by teachers' beliefs about mathematical teaching and learning and the context within which lessons are constructed. Like Noelle's case, Anna's case also raises questions about the teachers' use of curriculum material.

Rebecca

The last case is a divergent Rebecca who was the lowest scoring teacher in the sample. At the time of data collection, Rebecca did not have a teaching certificate. Rebecca's teaching focused on the mechanics of the procedures with no corresponding explanations and had some errors. Her teaching was described as mathematically thin with poorly designed tasks, not responding to learner productions and errors and a lack of technical language. Rebecca's idea of assisting struggling learners was to repeat the procedure until a learner mastered it. Instead of mathematical explanations, Rebecca would resort to the use of mnemonics or metaphors. Unlike, Zoe, however, Rebecca is a divergent case because her lack of attempt to teach conceptually resulted in purely procedural mathematics with very few explanations and hence fewer errors. The authors argue that Rebecca's mistakes were not as damaging to the mathematical content compared to Zoe who offered incorrect explanations. Unlike Anna, Rebecca's learners were always on task, doing real mathematics which would have resulted in her class gaining proficiency in specific mathematical procedures.

Rebecca's reliance on the textbook and following the sequencing laid out in the book allowed her to make germane connections between lessons. In the interview, Rebecca revealed that she embraced "drill and skills" or the memorisation method in her teaching which she believed is what her learners needed in order to be successful in mathematics. Her focus on procedural knowledge, however, caused her to miss out on opportunities afforded by the textbook to expose learners to conceptually rich instruction. Rebecca's class had a visible structure and learners had an understanding of the teacher's expectation to focus on doing mathematics from the start to the end of the lesson. The conclusion drawn from Rebecca's case is that additional factors such as teacher beliefs about instruction; equity; the use of pedagogical routines and textbooks, shape the relationship between MKT and MQI. Ball et al. (2008) maintain, however, that although additional factors mitigate the influence of MKT on MQI, these in totality play a minor role compared to the effect of MKT on its own. Ma (1999), was also of the opinion that teacher knowledge unlike other external factors has a direct effect on teaching and learning and might be much easier to change or adjust compared to these other factors which have an effect on learner attainment.

2.17 The effect of MKT on the mathematical quality of instruction

The cases summarised in this study highlight the importance of MKT on the teaching of mathematics. The study shows that without mathematical knowledge, teachers cannot offer mathematically rich instruction by providing explanations, justifications and making use of multiple representations. Teachers with poor MKT also lack the ability to respond correctly to learner questions, to interpret learner ideas correctly, or to address errors and misconceptions as well as the ability to use these to enhance the quality of the mathematical instruction. A simple task like choosing an appropriate example becomes a challenge for a teacher with a low MKT. The study also found that linking the mathematics and making connections between lessons and within concepts proved to be a challenge among teachers with lower knowledge. A conclusion was drawn that when teachers with poor knowledge follow the prescribed curriculum material like in Rebecca's case, learners are exposed to real mathematics, time is used effectively and learners acquire skills which help them to be proficient in specific procedures.

In contrast, the study shows that teachers like Lauren who portray sound knowledge of the subject matter are able to offer high quality mathematical instruction. The results documented in this study offer some insight into how teacher knowledge influences classroom interaction which indirectly leads to learner attainment in mathematics. Lauren's case provides a window into what teacher knowledge affords instruction.

In an attempt to explain how teachers' subject matter knowledge develops with reference to why Chinese teachers portrayed superior knowledge of mathematics compared to their counterparts in the West, Ma (1999) showed that there are three periods of such growth as depicted in a cyclical process diagram in Figure 2-4:

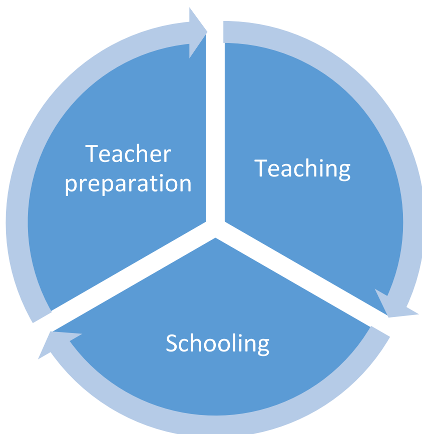


Figure 2-4: Three periods during which teachers' subject matter knowledge develops (Ma, 1999, p.145)

Ma (1999) claimed that the way Chinese children learn mathematics is the first period of their preparation as teachers of mathematics. Ma was in fact proposing that Chinese children are taught mathematics at the level which prepares them to teach others, thus justifying her contention that Chinese teachers spend less time in training compared to teachers in the U.S. and yet these teachers performed much better than their counter-parts. The schooling section therefore represents the subject matter knowledge acquired during a teacher's period as a learner. The second phase of development happens during teacher training. Since then many studies have been attributed to the knowledge gained during pre-service teacher training (French, 2003; Niess, 2011; 2007; Kwong, Joseph, Eric, Khoh, Gek & Eng, 2007; Kilic, 2011; Hobden & Mitchell, 2011; Lannin et al., 2013; Aalto, Tarnanen & Heikkinen, 2019) and conclusions drawn that knowledge acquired during this period is not enough to produce teachers ready to teach mathematics in South African classrooms (Mudaly, 2015; Kwenda, 2014). The last development stage takes place during classroom interaction as teachers empower learners to become mathematically competent individuals. Ma (1999) further asserts that the development of subject matter knowledge in this way results in Profound Understanding of Fundamental Mathematics (PUFM) where teachers value connectedness, multiple perspectives, basic ideas and longitudinal coherence.

2.18 Enactment of teacher knowledge in the classroom

Teachers with sound knowledge of the content should be able to support the soundness of this knowledge. Anthony & Walshaw (2009) developed a set of principles based on research findings from the West regarding the characteristics of effective (quality) pedagogy. These principles embedded in knowledge, skills and values, claim to incorporate classroom essential components of practice which include classroom environment, the kind of tasks that enhance learner involvement and the role of teacher knowledge. Figure 2-5 has been adapted to meet the needs of the current study:

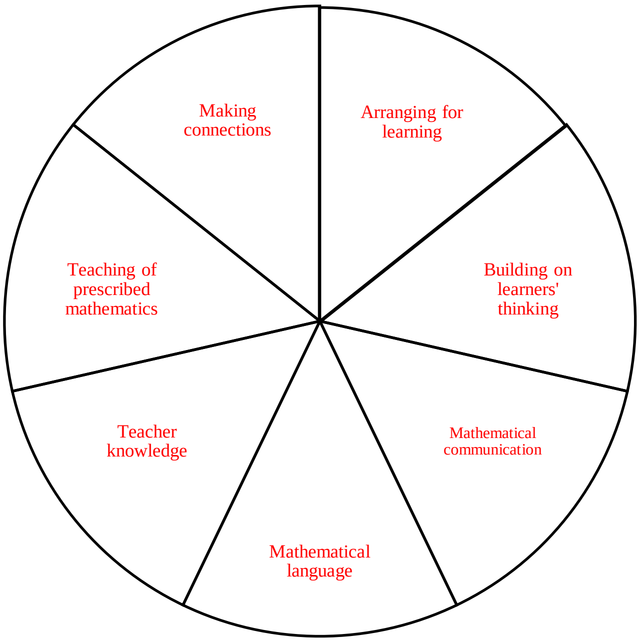


Figure 2-5: Principles of quality pedagogy: Adapted from Anthony & Walshaw (2009, p.148)

These principles have been used in this study to develop codes for analysing the mathematical knowledge for teaching items and data analysis rubrics for analysing video recordings. Most of these have been grouped together to form subject matter knowledge and pedagogical content knowledge strands in the conceptual framework by Ball et al. (2008). Combined, these principles talk to the ability of a teacher to use their knowledge of content and teaching (KCS), knowledge of content and students (KCS) and knowledge of the curriculum to design instructional strategies which are inclusive and that cater for diverse learner needs. There is an overlap in the way in which these principles operate in practice, therefore they cannot be viewed in isolation from all of the others.

Knowledge of various tools connected to different content and the use of multiple representations in teaching does not only cater for diverse learner needs but ensures that content is communicated in the manner that enhances learning. Arranging for learning is about providing opportunities for independent thinking as well as creating an environment in which learners can work as a community. In the same way, making connections between content and linking concepts to content also builds on student thinking and assessment for learning which includes assessing learners' prior knowledge. The use of technical language in the teaching of mathematics will be discussed extensively in the next section. Learners should be encouraged to communicate their ideas clearly using different modes which include oral and written communication. Teachers have a task of listening carefully to what learners communicate, decipher misconceptions, encourage the use of mathematical language and foster the development of mathematical discourse through the evaluation of mathematical phenomena using sound mathematical principles to support their arguments (Ball et al., 2008). Teachers are also responsible for responding appropriately to learner contributions and for using these to enhance the learning experience.

Designing tasks and activities to enact the mathematics prescribed in the curriculum is the ultimate goal of teaching any content in the classroom. Worthwhile tasks do not only focus on procedural and routine knowledge of algorithms. They are designed to enhance knowledge of the concepts and to enable learners to do original thinking about concepts, understand relationships, see connections, apply existing knowledge to solve unfamiliar problems and to allow for mastery of skills (Anthony & Walshaw, 2009).

2.19 Teacher knowledge and learner achievement in mathematics

It has been well documented that learners in South Africa perform poorly in mathematics assessments (Taylor, 2011; van der Berg, Taylor, Gustafsson, Spaull, & Armstrong, 2011; Maher, 2015). Various factors have been cited by different researchers as causes of this poor performance. These factors include, amongst others, socio-economic background of the learner, home influence, language of instruction, availability of resources and teachers' knowledge of the subject. Pournara et al. (2015) make a claim that although teacher knowledge has been included amongst factors which contribute towards learner attainment, the evidence for this claim is quite weak. Similar claims are made in a report by the Organisation for Economic Co-operation and Development (OECD, 2017) which states that despite long discussions and debates surrounding the connection between teacher knowledge and quality instruction, there is still not enough empirical research to test this hypothesis or to even connect teacher knowledge to student learning.

What is clear, however, from research (Ball et al., 2008; Inoue et al., 2019) is that there is knowledge that exists for teaching mathematics and that this knowledge can be developed in teachers through professional development programmes which include initial teacher training, induction and continued professional development courses (Myers & Rivero, 2019; Wolf & Peele, 2019). While it is apparent that more research needs to be conducted to support claims of a link between teacher knowledge and learner gains, there are studies that have shown that this link does exist and these studies are discussed in the next section.

2.20 Factors influencing learner gains in mathematics

2.20.1 Teacher knowledge

Pournara et al. (2015) conducted a study and posed the following question: can improving teachers' knowledge of mathematics lead to gains in learners' attainment in Mathematics? This quasi-experimental study involved 803 Grade 10 learners and 21 teachers. The findings showed that learners who were taught by teachers who had attended a year-long professional development course outperformed their peers taught by other teachers in the same school. The same authors claim that although these results reflected small learner gains, they were statistically significant. In Germany Baumert et al., (2010) also found that there was a substantially positive effect of pedagogical content knowledge on learning gains in a one year longitudinal study which involved a total of 181 mathematics teachers from 194 classrooms and 4,353 learners from Grade 10.

The results of a study by Maher (2015) shed some light on some of the most effective ways in which learner gains in mathematics can be maximised. This study reported some learners improving their mathematics results from 30 to 80 per cent. Maher (2015) did an analysis of the results from a Khanyisa Project which is the initiative of various stakeholders. This partnership includes a Project coordinator, 20 teachers from 20 schools from previously Black only and historically disadvantaged schools in KwaZulu-Natal, Grade 12 learners from these schools, and funders (Maher, 2015). The study documents how by empowering both learners and teachers through the development of their mathematical knowledge, the schools recorded an unprecedented turn-around in their mathematical performances which raised both teacher and learner motivation to excel. Although this finding raises hopes that with concerted effort, the strongholds left by the past regime can be redressed, it is clear that without continued funding this project alone will not survive. Maher (2015) claims that the findings highlight the importance of pre-service and ongoing professional development for novice teachers.

2.20.2 Socio-economic status (SES)

There is an indication from research which analyses the effect of SES on learner achievement that an achievement gap exists between learners who come from privileged homes and those from disadvantaged backgrounds and that this gap widens with each year of schooling (Taylor, 2011; Venkat & Spaull, 2015). In South Africa, it has been estimated that the achievement gap is equivalent to over two years of learning and the majority of learners in the system fall within the disadvantaged sector due to the policies of the previous apartheid regime (Taylor, 2011).

There have been suggestions that this achievement gap results from the differences in the calibre of teachers who teach in privileged schools compared to the calibre of those who teach the majority of learners from low income backgrounds (Ball, Hill & Bass, 2005; Baumert et al., 2010; Venkat & Spaull, 2015). From anecdotal experience, the ability to retain highly qualified mathematics teachers and to continuously provide quality training for these teachers is linked to a school's socio economic status (SES). Financially capable schools spend more money on teaching and learning resources and towards in-service training for teachers. As a result of this financial commitment, parents who are fee payers demand value for their investments, which, in turn, puts pressure on teachers to work harder than is ordinarily the case in schools from low socio-economic contexts.

Other factors linked to socio economic status include the types of conversations which occur in homes and in social circles. The SES also affects access to books and computers, opportunities to travel and to be exposed to other sources of information. According to Taylor & Vinjevold (1999, p.119), these factors enable learners from middle-income homes to "have ready entry into the principles which underlie school knowledge". However, learners from working-class homes have poor exposure to the same elaborated language codes and to the structures and principles of formal schooling as their peers from privileged homes. This differential access to formal knowledge is the biggest obstacle to equity, even more damaging than the unequal distribution of physical resources and of quality teachers.

The number of books available at home is also associated with socio-economic status. Recent studies show that educational levels of parents in South Africa have improved substantially between 2003 and 2015 (TIMMS, 2015). Learner performance is positively associated with higher educational level of the household. TIMMS (2015) reported that the differences in learner performance between those who come from a post-matric household and those without matric was "43 points for mathematics and 55 points for science" (TIMMS, 2015, p.11). Baumert et al. (2010) found that socio-political factors create unequal opportunities for teachers to learn which were caused mainly by interactions between institutional structures. This resulted in weaker learners from lower SES and immigrant families being taught by teachers who were less competent in knowledge.

2.20.3 Language of instruction

Christiansen & Aungamuthu (2012) conducted a study which included a sample of 30 Public primary schools in the Umgungundlovu Education district in KwaZulu-Natal, chosen amongst the poorest 76 per cent of the schools, and 10 schools chosen amongst the more affluent schools. The aim of this research was to analyse responses of Grade 6 learners to 40 questions across the mathematics curriculum collected in 2009. A quantitative analysis of the results revealed English home language learners had an advantage in the learning of mathematics over non-English home language learners. These learners were expected to score 14.5 percentage points higher on the learner maths test than their counterparts after socio-economic status had been accounted for. Discussing these findings, Christiansen & Aungamuthu (2012), identified three factors contributing to this disparity in learner gains as a result of the language of instruction. Firstly, English home language learners are likely to be placed in historically advantaged schools. Secondly, learners are faced with a difficult transition from being taught in their mother tongue in Grades 1-3 to being taught in English or Afrikaans. Thirdly, accessing media or reading material written in English proves to be a challenge in low income households where the parents may also be illiterate. TIMMS (2015) also reported that learners performed better when the language of learning and teaching corresponded to the learner's frequently spoken language, especially when it came to language intensive subjects.

Similarly, in the United States it was shown that learners from a Spanish background were disadvantaged in the 2003 Massachusetts Comprehensive Assessment System (MCAS) math test. Through the use of think-aloud protocols Martiniello (2008) conducted a textual analysis of the learners' responses to illustrate some of the linguistic characteristics of math word problems in the test. She concluded that the learners' limited vocabulary, background knowledge, and unfamiliarity with the usage of the English language hindered their comprehension of the problems posed in the test. In her discussion she illustrates how the majority of the learners interpreted the sentence: exactly 3/4 of the marbles in the bag are blue, to mean: 3/4 of the marbles in the blue bag. The study found that learners battled mainly with word problems due to their unfamiliarity with the context behind the problems especially with regard to the use of vocabulary. This finding is also seen in (Prince & Frith, 2017) regarding South African school leavers preparing to enter high education.

Rodriguez-Izquierdo et al. (2019) argue that teachers need to adopt a multi-lingual approach to teaching. In a study that sought to investigate teacher beliefs and approaches to linguistic diversity, Rodriguez-Izquierdo et al. (2019), found that teachers viewed the mother tongue as an obstacle to learning Spanish and that none of the immigrant learners were referred to as bilingual, instead the study observed the tendency to ignore learners' mother tongue. Chirinda & Barmby (2017) found that in cases where learners did not understand the language of instruction, teachers code-switched. The study further recommends that the professional development programmes must include a segment on code-switching in order for teachers to know how to support learners with language.

Ball (2011) highlights three reasons in support of the claims that language is important in the teaching of mathematics. Firstly, she states that mathematical language supports mathematical knowledge and reasoning, secondly, mathematical language supports the teaching and learning of mathematics and thirdly, mathematical language by nature translates to mathematical content to be learned and medium for learning that content. Ball (2011) further illustrates that there are additional dimensions for learners who find themselves navigating between home, school and mathematical languages as shown in the diagrams in Figure 2-6:

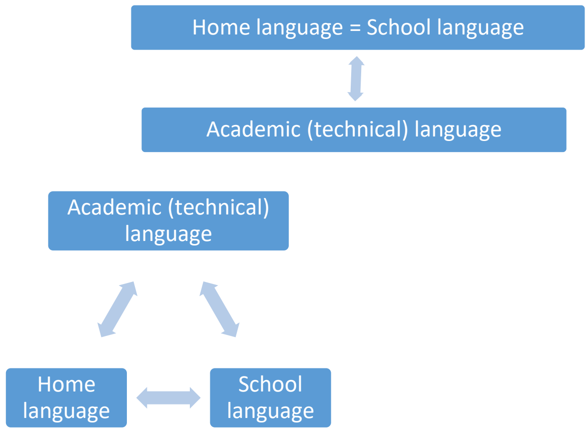


Figure 2-6: The effect of language on learning (Ball, 2011, p.52)

Studies conducted in South Africa show that school leavers who are preparing to enter Higher Education struggle with quantitative language, interpreting data displayed in table form and with tasks associated with unfamiliar context (Frith & Lloyd, 2016; Prince & Frith, 2017). The academic language involved in mathematics has been referred to as a third language by some. According to Ledibane et al (2018), research shows that native English-speaking learners learning academic language face similar challenges to learners learning English Second Language and, as a result, they should be paired during group work activities. Considering that most of the text books are written in English, this statement is quite a revelation as one thinks about the plight of English Second Language learners (ESL) in mathematics classrooms.

Researchers like Macdonald & Burroughs (1991) have asserted that success in a second language seems to depend on the learner's first language. Table 2-2 illustrates the interaction between the required level of thinking, the thinking skills and learners' behaviour required in the teaching of science. These skills are language intensive and progress from lower order to higher order thinking and reasoning in accordance with the Blooms Taxonomy levels (Forehand, 2010). These skills are explained briefly in the next paragraph.

The first level on top is about recollection of knowledge like stating that the equation defining a linear function is $y = mx + c$ and in mathematics. The second level is also lower order and involves explaining ideas or concepts. This can be in the form of classifying shapes in geometry. The last lower order level involves applying concepts and skills which may include drawing and interpreting graphs and tables. The first of the higher order thinking skills is about analysing information and drawing conclusions amongst a set of ideas. Topics like data handling and probability are rich with content which provides learners with opportunities to acquire these skills. Synthesizing skills allow learners to combine or fuse together concepts learned in order to create or build a new concept. In mathematics this could involve a complex procedure or a problem solving activity like modelling a real life situation which involves identifying and choosing an appropriate representation. The last level is about verifying or making scientific judgements which can involve making conjectures or discovering and proving new theories. In mathematics, the phrasing of instructions is important and the teachers need to have this knowledge in order to teach equitably.

Table 2- 2: A model of thinking skills for Science teaching (Macdonald & Burroughs, 1991, p.39)

A model of thinking skills for Science teaching

Level of thinking	**Thinking skills named**	**Pupils' behaviour**
Knowledge Retention of concepts	Recalling	Observe, match, repeat, memorize, label, name, recall, cluster, recount, sort, record, define
Comprehension Understanding concepts	Explaining, comparing	Recognize, report, find, express, identify, explain, restate, review, paraphrase, tell, describe, summarize
Application Using concepts	Applying concepts and skills	Select, show, demonstrate, use, apply, sequence, dramatize, organize, illustrate, test out, solve, imagine
Analysis Breaking down science concepts into smaller parts of information	Analysing	Examine, classify, compare, contrast, outline, interpret, debate, defend, question, draw conclusions, research, analyse
Synthesis	Synthesizing	Propose, construct, plan, emulate,

Putting information together to form a new concept		compose, speculate, create, design, invent
Evaluation Judging the value of a science concept	Verifying	Judge, decide, rank, persuade, evaluate, assess, criticize, value, predict, justify, verify, convince

2.20.4 Mathematical language

Language in the mathematics classroom is linked largely to vocabulary and it is not possible to discuss one without addressing the other. Mathematic language is rich with symbols and vocabulary which seems formal but different to everyday language. Teachers try their utmost best to make this language accessible to learners. Some have even defined mathematics itself as a language because "like other languages, it has its own grammar, syntax, vocabulary, word order, synonyms, conventions, idioms, abbreviations and sentence and paragraph structures" (Ledibane et al., 2018, p.1). Mathematical literacy can be described as the means by which learners' access mathematics through language, and it is important that natural everyday language is used to teach formal mathematical language (Meiers & Trevitt, 2010). The argument lies at the endeavour to keep the balance between the use of informal everyday language and mathematical language in the classroom. Sfard (2000) argues that the difficulty is often with ineffective communication rather than with an attempt to communicate. She further asserts that language is about participating in a discourse with meta-discursive rules and agreed upon meaning which allows people to participate meaningfully in the discourse and that there is knowledge that learners cannot assimilate with prior or everyday discourse because it is knowledge distinct to mathematics as a discourse. Berger (2013) states that participants in a mathematical discourse use visual mediators such as symbols, graphs and diagrams to identify the objects of their thinking or communication and to bring these objects into focus. Teachers have to work at finding ways of representing this knowledge in a manner comprehensible to learners without giving up the mathematical rigor.

Some of the hindrances to the use of mathematical discourse in teaching may stem from previous encounters with mathematics at high school level. Bohlmann et al.(2017) state that at university level, students' prior domain knowledge and previous learning experiences that they bring to their studies are acknowledged as important in their success as students of mathematics. Another limitation is teachers' beliefs that learners are not able to comprehend mathematical language either due to language barriers or due to age. Wanjiru (2015) found that teachers in Kenya believed that learners would not be able to understand mathematical vocabulary, verbally express mathematical concepts or participate fully in the classroom discourse and achieve mathematics proficiency. In a study that involved 216 grade 9 learners and 6 mathematics teachers Wanjiru (2015) found a positive association between mathematical vocabulary instruction and learners' performance in mathematics.

To conclude this discussion on factors that influence learner gains in mathematics, I fully support the ideas shared by many writers regarding the need to promote social justice by narrowing the learner achievement gap through ensuring that every learner has a teacher who is equipped with necessary knowledge for teaching. The argument returns to the need for teacher development programmes that focus on increasing quality of teaching by empowering teachers who teach in disadvantaged schools with content knowledge for teaching mathematics. There is a need for courses which target specific areas like code switching and teaching with equity. Other areas include the empowerment of teachers to acquire skills and knowledge to teach conceptually by focusing on mathematical vocabulary and definitions rather than on memorising procedures.

I conclude this chapter by discussing function as a topic in the CAPS document and the teaching of this topic.

2.21 Function as a topic in the mathematics syllabus

The three core ideas that define a function are the input-output relationship, the rule or pattern that connects this relationship and the idea that each input is only assigned to one output (Swarthout, Jones, Klespis & Cory, 2009). Kjeldsen & Lutzen (2015) trace the development of a function concept back to Dirichlet-type functions at least 4,000 years ago. According to these writers, the first introduction of the word function appeared in the geometric paradigm of curves. The word was used by Leibniz in 1673 "to denote a quantity that varies from point to point on a curve, such as the tangent, the normal, or the ordinate" (Kjeldsen & Lutzen, 2015, p.4). In their investigation of the main stages of the development of a function concept, Kjeldsen & Lutzen (2015) discovered four historical changes. The first development began with Euler's analytical expression [a formula denoted $f(x)$] expressing y in terms of x, second was the Dirichlet concept: y is a function of x if to each x there is associated one value of y [called $f(x)$]. Thirdly, the Bourbaki concept: a function from a set A to a set B is a subset C of the Cartesian product A $\times$ B with the property that for each x in A there is exactly one y in B such that $(x;y)$ is in C and finally, the distributions or generalized functions.

Engelke, Oehrtman & Carlson (2005) discuss four levels of viewing functions, namely, pre-action, action, process and object levels. Each level is said to represent the extent of learner understanding of the concept. Pre-action refers to a level of very little understanding of the concept, action is the level restricted to actual physical or mental operations on specific numerical values while process level involves a view of the entire transformation of quantities independent of any procedure. The object is at a level whereby a function is seen as a concrete entity on which other operations may be performed. A quantitative study which involved large numbers of college students after the completion of a pre-calculus course revealed that many students operate on the action level where algorithms are performed with very little conceptual understanding (Engelke et al., 2005). Only when students have a process view of a function, do they reflect a deeper understanding of the concept. Font, Bolite & Acevedo (2010) state that metaphorical expressions of the object metaphor occur when mathematics teachers refer to the graph of a function as an object with physical properties. The use of verbal expressions and gestures suggests the possibility of manipulating mathematical objects as if they were objects with a physical existence.

Bohlmann et al. (2017) are of the opinion that the definition of a function, different representations and the terminology associated with functions need to be revised. They argue that "ideally, generic graphs should be used to clarify function terminology such as domain, range, function graph (where the graph lies above, on or below the x-axis), function value (which is then positive, zero or negative), turning points, asymptotes, intercepts (of a graph) and roots (when the equation representing the graph is equal to zero), period and amplitude" (p.7). White & Mitchelmore (1993) assert that if a general concept of a variable is to be developed, there must be a shift in focus towards what the letters mean rather than on how they can be manipulated. They also argued that in order to build the knowledge and skills of defining variables in a mathematical context including real life problems, a functional approach to teaching needs to be adopted. Adopting this approach does not only provide opportunities to teach learners how to define variables properly, but it also provides the much needed reinforcement knowledge of a function as a concept. Areas in the mathematics syllabus in which a functional approach can be applied include number patterns, financial mathematics, ratios & proportions and measurements.

2.22 Policy on the teaching of patterns and functions in the GET phase in South Africa.

Kabael & TanisIi (2010) talk about a 'functional relationship' when describing a relationship that exists between patterns and functions. According to these authors, a functional relationship is introduced early in the form of patterns and gradually develops during the algebraic process in contexts like word problems and eventually becomes an abstract concept in the form of a 'function'. According to CAPS (2011), in Grades 7, 8 and 9, South African learners are expected to acquire four basic skills from the learning of patterns and functions. The first skill is that of investigating and extending numeric and geometric patterns looking for relationships between numbers including patterns. In Grade 7, these patterns must firstly, be represented in physical or diagramatic form (in tables), not limited to sequences involving a constant difference or ratio and must be of learners own creation and represented in tables. Secondly, learners are expected to be able to describe and justify the general rules for observed relationships between numbers in their own words. Thirdly, learners must be able to determine input values, output values or rules for patterns and relationships using: flow diagrams, tables and formulae. Fourthly, learners should be able to determine, interpret and justify equivalence of different descriptions of the same relationship or rule: presented verbally, in flow diagrams, in tables, by formulae and by number sentences (CAPS, 2011).

In Grade 8, the skill is extended to algebraic language and algebraic representations of patterns and to the use of equations in functions and relationships. Progression of the concept of a function is done by including graphical representations on a Cartesian plane in Grade 9. The total number of hours allocated to the teaching of patterns and functions is 9 hours in Grade 7, 7,5 hours in Grade 8 and 8,5 hours in Grade 9. The total of 25 hours is therefore required for leaners to be taught the concepts of patterns and functions in the GET phase.

From this policy statement it is clear that the teaching of patterns and functions requires vast knowledge of multiple representations. When one considers the time allocation for mastery of the four basic skills within the phase and between the Grades, one wonders if the objectives of the policy are not too ambitious. If the same learners are to progress to understanding calculus and inverse functions in Grade 12, surely mastery of these basic concepts and skills is not optional. A lot has been said about how South African learners lag behind their peers in other countries including neighbouring countries in basic mathematics skills (Spaull, 2013). On the one hand there is a justifiable need to ensure that South African learners perform competently to meet global standards. On the other hand there seems to be a disjuncture between the ideals of policy and the actual implementation of these policies. Teachers often find themselves facing a dilemma between teaching concepts in depth and ensuring that curriculum coverage is satisfied (Shulman, 2004). This creates a dilemma because there is often not enough time to do both.

The question of curriculum coverage is addressed in detail by Shulman (2004). By introducing the principle of coverage, Shulman seeks to address the question: how do we create a curriculum that is intellectually honest i.e. one that covers all topics adequately. Firstly, Shulman asserts that teachers will do all they can to cover the curriculum as an act of avoiding feelings of guilt. If teachers know that they have covered everything, the onus is now on the learner to learn everything, and if learners fail the subject, the fault lies with them and not with the teacher. Secondly, political pressure comes from various stakeholders who value different kinds of knowledge and in an attempt to satisfy everyone, the system surrenders to the principle of coverage. Commenting on the issue of curriculum coverage, Pritchett & Beatty (2012, p.10) state that "the usual question is 'why are students so far behind the curriculum?', but the more telling question is 'why is the curriculum so far ahead of the students'?".

It is clear, however, that the principle of coverage sacrifices depth for width. Sacrificing of depth for width in a way ensures that more learners participate in the process of learning because to go deeper would mean teaching less and less is more difficult to learn (Shulman, 2004). Having said all that, I do believe that when teachers teach for conceptual understanding by focusing on developing in learners the ability to think for themselves and skills to interrogate content independently, the pace of learning is accelerated naturally without compromising on quality. This will only happen in classrooms where teachers possess the necessary knowledge and skills to teach this way.

2.23 The teaching of functions

The concept of a function is fundamental to many first-year mathematics courses at university across disciplines (Bohlmann et al., 2017). There is therefore, a perceived need for all mathematics teachers to have a good understanding of the concept of a function (Nyikahadzoyi, 2013). As with any other mathematical topic, working with functions requires teachers to make use of multiple representations in order to reach learners at various cognitive levels. Depending on the context and need, mathematics teachers should be able to choose appropriate representations and demonstrate an understanding of the inherent weaknesses and strengths of the various representational forms (Ball et al., 2008). The various types of representations include verbal, set diagrams, function boxes, a table of values and graphs (CAPS, 2011). Teachers need to be familiar with how functions are presented and interpreted for instance Nyikahadzoyi (2013), highlights that in set diagrams functions are presented and interpreted as prototypes to represent general ideas while in graphs and formulae, functions are represented as clusters (linear, quadratic, exponential etc.). It is also important for teachers to emphasize the connections among the various types of representations in order to create a deeper picture of a function as a concept.

In the teaching of functions teachers need to ensure that each learner has a well-developed understanding of this topic, and that they are able to use different representations to investigate, describe, and communicate a pattern or connections recognised between two sets (Swarthout et al. 2009). The only way teachers would know if learners have acquired conceptual understanding of functions is to create for learners, opportunities to make their own representations. Assessment for learning as opposed to assessment of learning requires that teachers design assessment programmes that provide opportunities for learners to acquire deeper conceptual understanding rather than a mere demonstration that concepts taught in class have been understood. When the principle of multiple representations is applied consistently, deeper and complete learning of concepts is emphasized. To do this, teachers have to possess a special kind of knowledge.

Hill & Ball (2004) state that teachers need to make a convincing claim for the existence of professional knowledge needed for quality instruction. Teaching with skill is about demonstrating convincingly that educators possess knowledge which others do not have and that this knowledge matters for learning (Ball et al., 2004). Subject matter knowledge (SMK) and pedagogical content knowledge (PCK) are combined to form Mathematical Knowledge for Teaching (MKT). In the teaching of functions, knowledge of content and students (KCS) and knowledge of content and teaching (KCT) require teachers to take what is common knowledge about functions and decompress it into knowledge that will be comprehensible to learners at their cognitive level. To do this, teachers will need to know what makes the learning of functions easy or difficult for learners. This knowledge will in turn inform the instructional design in terms of the choice of teaching methods, type of representations and assessment strategies employed. The fusion of teacher knowledge, belief and experience is what determines the meaningfulness of the learning experience created for learners. Teachers who know that learners bring with them misconceptions will design their instruction in ways that cater for these misconceptions to be addressed. Teachers who understand what makes the teaching of particular content easy or difficult will also design their instruction with this knowledge in mind e.g. item 8 of data generation was designed to test this kind of knowledge. A Teacher in the process of planning to introduce simultaneous equations would have to take into account what makes learning of this content easy or difficult for learners when planning to introduce this topic for the first time.

Knowledge of the curriculum enables teachers to make connections and link related topics in the syllabus while planning carefully the sequencing of these topics. The choice of textbooks and other teaching and learning materials is of utmost importance. The skill of drawing a graph requires learners to be hands on with the work, hence it is important that learner books are chosen carefully. Furthermore teachers will use horizon knowledge to make horizontal and vertical connections when teaching skills and concepts not easy to grasp like defining a gradient, drawing graphs, plotting points on a Cartesian plane etc. Successful teaching of functions will require more than mere knowledge of mathematics, it calls for teachers within a grade, across grades and across subjects to work in collaboration.

2.24 Chapter Summary

I end this discussion by restating the purpose of this chapter. The chapter began with an account of literature reviewed in the area of teacher knowledge in mathematics by outlining the international comparative mathematics assessment studies to reveal the possible causes of differences in learner attainment. The study continued to evaluate literature on the topic of teacher knowledge, taking into account constructs closely linked to the concept of mathematical knowledge for teaching which include attitudes and beliefs, pre-service teacher training and reflective practice. Finally, I concluded by offering a discussion of functions as a topic in the GET syllabus and literature reviewed in this area.

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CHAPTER 3: Theoretical Framework

3.1 Introduction

This chapter provides the theoretical framework supporting the study. The chapter begins by presenting and offering a lengthy critical discussion of the variation theory and offers justification for using this theory in the study. The model of Mathematical Knowledge for Teaching (MKT) proposed by Ball et al. (2008), is then introduced as a supporting conceptual framework.

3.2 Introduction of variation theory

Knowledge is increasingly becoming an important commodity which needs to be developed to suit the needs of the communities who use it (Marton et al., 2004). Addressing knowledge growth can be approached from various perspectives which include the political, economic, social and pedagogical approach. The theory of variation offers a pedagogical view which is more concerned with teaching and learning. The use of variation in learning and awareness was initially proposed by Marton & Booth (1997) and subsequently developed by Marton et al. (2004) as a generic learning theory (Mhlolo, 2013).

The starting point of the variation theory is the importance of addressing the question of what it is that should be learned in each case. Thereafter it is important to find different conditions that would be conducive to different kinds of learning. The theory claims to provide an understanding of what learners are expected to learn in specific situations, what they actually learn under those circumstances and why they learn something in a particular situation but not in another. Variation as a theory also emphasizes the importance of the content in teaching. It is argued that for learning to take place, three elements must be present. These elements are: the teacher, the content and the learner. The learner is the indirect object while content is the direct object of learning. The teacher creates the space of learning which promotes learner capabilities to discern the intended object of learning. The object of learning as a concept will be outlined as this chapter progresses.

The main approaches to learning either envisage a traditional or a progressive view of education. The traditionalist approach as discussed earlier puts the teacher and the mastery of the content at the core of teaching and learning. This approach, it is argued, emphasizes the importance of what is covered in teaching. In contrast, the progressive view, is a learner-focused orientation which advocates for teaching methods which are adjusted to suit the needs of the learner. Marton et al. (2004) argue that these two orientations have been used to replace each other pedagogically in a recurring cycle to fulfil political agendas. Variation theory claims to have as its point of departure the belief that pedagogical practices will improve only when both the learner and the content are given equal consideration (Marton et al, 2004).

3.3 The object of learning

The authors define the object of learning as the acquired knowledge of something. Lo (2012, p.43) defines object of learning as that which "the students need to learn to achieve the desired learning objectives". The object of learning is not the same as the learning objective. Lo (2012) states that while the learning objectives refer to the kinds of behavioural changes expected of learners as a result of learning activities, the object of learning is defined in terms of capabilities or what learners are expected to become capable of doing. The function of school is viewed as that of developing capabilities in learners. Marton et al. (2004) refer to these capabilities as the object of learning.

The object of learning represents the content that brings substance to the act of learning. In order to learn, there must be something to be learned. Lo (2012) states that, in order to talk about learning, we must clarify what we are learning. This 'what' is the 'object of learning' The programme of learning is seen as more than mere categories of content in terms of different parts of various learning areas or school subjects. Lo (2012) argues that when the focus is on the learning objective, there is an expectation of a desired learning outcome which is predetermined but only specifies the end result and does not give direction on how to get there. This according to Ling, creates two problems. Firstly, when teachers know what the learning outcomes are they may feel under pressure to gear their teaching towards assessment needs and thus undermining the true purpose of education. Secondly, by specifying the end result, teachers undermine the dynamic nature of the object of learning, which according to the variation theory should be negotiated with the learners. This has the effect of limiting the learners' learning outcomes. The object of learning is therefore seen as pointing to the starting point of the learning journey rather than to the end of the learning process.

The object of learning is also defined by the critical aspects which must be discerned in order to have learning take place (Marton et al., 2004). The intended object of learning is the starting point as seen from the teacher's perspective of learning. This object of learning is then realised in the classroom in terms of the 'space of learning' or classroom interactions. The space of learning is what constitutes the enacted object of learning or what is possible to learn. Through the enacted object of learning, the curriculum, teacher's intentions and other factors are mediated or made concrete. What the learners remain with beyond the lesson is the lived object of learning.

The object of learning is differentiated according to acts and what is acted upon. These are referred to as the general and specific aspects of learning. The general aspect describes the nature of the capability, the acts of learning carried out in activities like interpreting graphs or remembering a formula while the specific aspect has to do with the subject on which these acts are carried out like formulae and the various topics in the curriculum. The specific aspect can also be viewed as the direct object of learning, and this is what learners are normally focused on. This can be viewed as the way in which learners handle or interpret the content. The teacher, however, focuses on the general aspect or indirect object of learning as well as the specific aspect. Teachers strive for the intended object of learning which is evidenced by what they do and say during the course of learning.

The concept of the object of learning puts equal emphasis on the learner and the content rather than on the importance of the teacher's subject knowledge alone. How the teacher structures the conditions of learning should allow the learners to be aware of or discern the intended object of learning. What the learners encounter is the enacted object of learning, which then defines what can be learned in the actual setting, from the point of view of the specific object of learning. What learners discern and focus on may not necessarily be the critical aspect of the object as intended by the teacher but another aspect. What learners actually learn is the lived object of learning. The outcome or result of learning is the object of learning as seen from the learner's point of view. Figure 3-1 depicts Lo's (2012) interpretation of the relationship between the object of learning and its attributes.

The object of learning refers to what learners need to learn to achieve the desired learning objectives. This refers to meeting the long term goal associated with developing a learner as a social and global citizen. This is knowledge that should guide teachers and Shulman (1986) referred to it as knowledge of educational ends. The object of learning is divided into the general and specific aspects. The general aspects refer to the capabilities that learners acquire through the specific aspect. For instance through linear graphs, learners are able to draw other types of graphs in order to achieve the object of learning which is to be able to present and organise data. The structure represents the topic within which the specific object is classified. Linear graphs belong to functions, a topic in the curriculum which deals with relations where there is only one output for every input value. The structure therefore represents the whole. The other types of graphs are on the external horizon in relation to the linear graph (the specific aspect) i.e. they are different according to the shape and the general formula which defines each type. The characteristics or features of a straight line graph which include the gradient and the y-intercept which gives the graph its shape are on the internal horizon and provide critical features that can be studied about a linear function. The meaning aspect refers to how the specific object is viewed in relation to the whole. Linear graphs are of the form $y = mx^1 + c$ as opposed to quadratic graphs which are represented by $y = ax^2 + bx + c$. The meaning of this general formula is that a straight line graph will be produced by plotting on a Cartesian Plane, the input, output coordinates obtained from the equation. The object is understood according to its meaning.

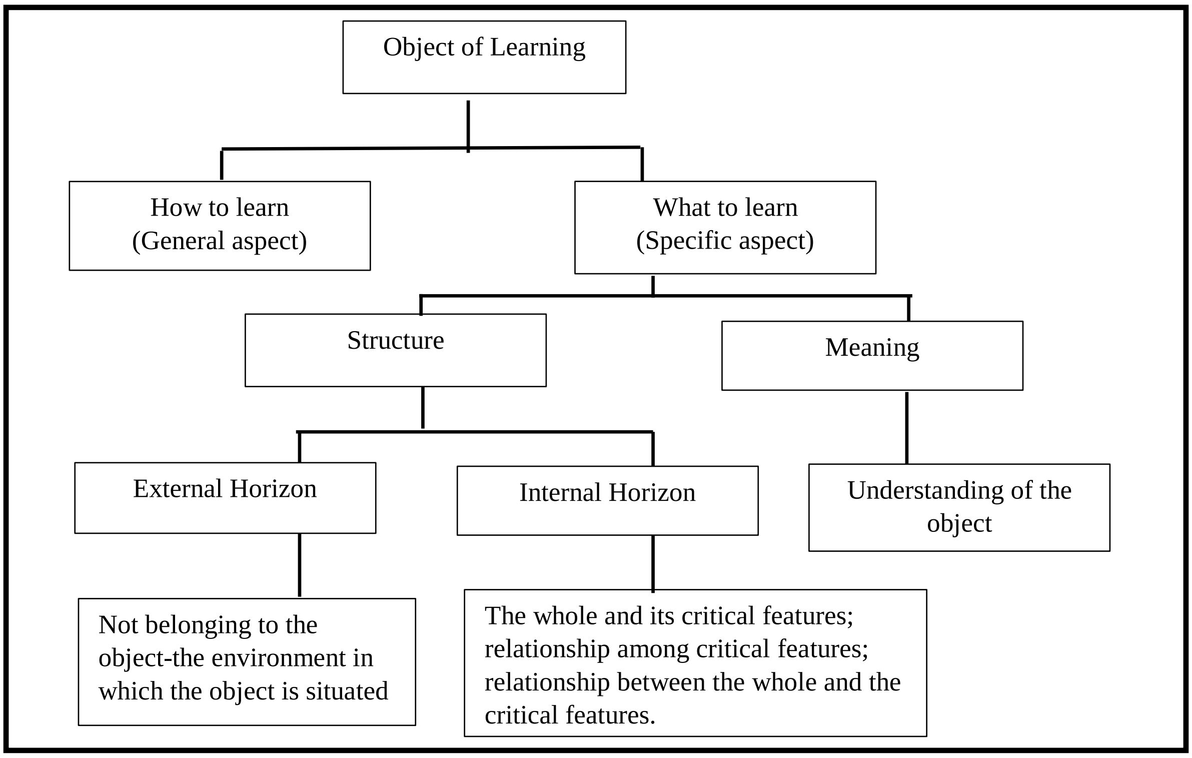

Figure 3-1: The relationship between the object of learning and its attributes (Lo, 2012, p.63)

3.4 The illusions of the art of teaching all things to all men and general capabilities

Quoting the phrase from *The Great Didactics* published in 1657, Marton et al. (2004) refute as a fallacy the attempts to teach anything to anyone as long as we can find a suitable method. They criticise the ideals of progressive thinkers who advocate for cooperative learning, IT-supported learning, project work and problem-based learning as long as these offer an ‘illusion’ of general approaches that provide the specific conditions necessary for the learning of specific objects of learning. Their contention is that there are specific conditions necessary for learning specific objects of learning, and these differ from one object of learning to another. Similarly, Lo (2012, p.107) states that “no single teaching approach or strategy will be effective for teaching all objects of learning and their critical features”. It is therefore necessary in each case to investigate the conditions conducive to learning specific content.

Furthermore, Marton et al. (2004) condemn and label as an illusion the idea of emphasising generic capabilities. This view according to Marton et al. (2004) emanates from arguments that the schooling system is faced with an exponential growth of knowledge and in order to assist learners to cope with the evolving world, there is a need to equip learners with capabilities for obtaining knowledge instead of equipping them with knowledge. General capabilities should be viewed as the ways of dealing with different topics, knowledge and content. They are domain-specific and refer to the ways in which people act rather than to what people have or what they are. A person’s capability in one domain, for example language, does not imply a capability in dealing with numbers. Knowledge should rather be viewed as fundamentally ways of seeing the world, therefore generic capabilities cannot be seen as developing independently of knowledge and content.

The argument offered here is that learning should be viewed in terms of the learner and the content or object of learning as designed and communicated by the teacher. There are specific conditions for learning one thing or another. It is important to justify why certain conditions are chosen for learning specific content because there is no such a thing as ‘a one size fit all’ when it comes to learning.

3.5 The view of learning and the role of the teacher

The role of the teacher is seen as that of promoting effective learning. A certain pattern of variation is necessary to ensure that certain learning takes place (Marton & Pang, 2006). The role of the teacher is therefore, firstly, to be clear about the capabilities to be developed in learners. Secondly, the teacher must identify what is critical for learning that particular content. Lastly, the teacher has to make it possible for learners to discern those features that are critical for that particular learning. The theorists believe that there is no causality between teaching and learning. What is taught by the teacher is not always what is learned by the learners. The intended object of learning is not necessarily the same as the enacted object of learning which may also differ from the lived object of learning (Marton et al., 2004). Teachers can therefore only create the possibility of learning something in a certain way. One cannot assess conclusively what learners learn by simply observing how the teacher has structured the lesson. However, it is possible to say what learners are afforded to learn in that lesson (Lo, 2012).

3.6 Powerful ways of acting

Marton et al. (2004) further introduce the phrase "powerful ways of acting". The notion of this phrase is that, people are acting in powerful ways when they engage in acts instrumental to efficiently achieving their goals. According to Marton and Pang (2006), "one of the main functions of schooling today is to enable students to handle novel situations in powerful ways". The variation theory proposes that when learners view an arithmetic problem in terms of part-whole relations they have a more powerful view of the problem than those who see it in terms of an arithmetic operation. An example to illustrate: Sam arrives at school with very little pocket money and his friend Lihle gives him R5. Sam is able to buy a cool drink for R9. How much money did he have when he arrived at school? The authors discuss a similar problem posed to seven year old children and argue that those children who did not find the problem difficult made part-whole connections. Those children had as their starting point the R9 and understood that only R5 came from Lihle. They then took away the R4 using their fingers. These children, it is further argued, did not see the problem as an addition/subtraction problem but rather as a part-whole problem. It is argued that teachers can act in ways that are powerful when they predict what learners' responses will be, based on knowledge of learners and content.

This concept of knowledge of content and learners is not new, Shulman (1986) introduced domains of knowledge necessary in teaching. One of these domains is Knowledge of Content and Students (KCS). This and other knowledge domains are discussed further on in this chapter.

3.7 Ways of seeing

A way of seeing has to do with the fact that people will discern certain aspects of something in different ways, that is, two people will view the same thing in different ways. Marton et al., (2004) define the way of seeing in terms of the aspects that are discerned at a specific point in time or in terms of the critical features of what is seen. An aspect of a thing is the way in which that thing is perceived to differ from or experienced as similar to other things. In the previous example, different learners would have perceived the same problem differently. For some, the problem is viewed in terms of the sum of 9, while for others, R4 is one of the parts. Other children, however, would have understood the problem in terms of the arithmetic operation involving the addition or subtraction of two numbers. Attending to different aspects of the problem is the result of comparing the new with what is already known from earlier experience. The aim of variation theory is to highlight the need to empower learners to develop capabilities that enable them to deal with situations in powerful ways.

3.8 Discernment

Marton et al. (2004, p.10) state that "a way of seeing can be characterised in terms of the aspects discerned that are attended to simultaneously". Discerning something is not the same as being told about that thing. Discernment is a way of seeing that has to do with understanding certain features of the thing being discerned. A feature refers to an attribute or value such as colour or height etc. In order to discern features of something, a person must have an experience of those features, and this experience is only gained by varying the features so as to compare. Experiencing variation allows people to discern certain aspects of their environment and become sensitised to those aspects. This sensitisation means that future events are likely to be seen in terms of these aspects. While it is important to discern features which are critical in a general sense, it is also important to discern features which are critical in a specific sense. Discernment of new features is as important as discerning of features that have been encountered in past experience.

3.9 Discerning in context

An important part of discerning is to discern the relation of parts within wholes and discerning the whole from the context and the way the wholes relate to context. The way the parts relate to the whole is shaped by the way the whole relates to the context. In teaching, it is important that both the learner and the teacher have the same contextual understanding of the object of learning. In the teaching of linear graphs, the object of learning could be that parallel lines have the same gradient. The discernment of the critical component of parallel lines would be brought about by discerning the context of parallel lines as belonging to linear functions. It is the discernment of how the whole (parallel lines) relates to the context (linear functions of the form $y = mx + c$) that enabled the discernment of the parts ($m_1 = m_2$, where m_1 and m_2 are gradients of parallel line 1 and parallel line 2).

3.10 Variation

Variation theory suggests that relationships between objects should be perceived as instantiations of general properties which can apply in many different situations. The critical features are detected as a juxtaposition of variation in close proximity of time or place, in a range that is comprehensible with few dimensions (Mason, Stephen & Watson, 2009). Varying four or more different aspects at the same time, and using elements which are unfamiliar, is unlikely to promote awareness of possible variation.

The theory makes the hypothesis that the pattern of variation inherent in the learning situation allows the experiencing of features that are critical for a particular learning as well as the development of certain capabilities. These critical features must be experienced as the dimensions of variation. Critical features refer to ways in which an object can be made known in order to be recognised and distinguished from other objects (Marton et al., 2004). To know a triangle, it is imperative that critical features of a triangle are discerned. These features include, the shape, number of sides and the relationship between angles. However, merely, pointing out these features is not enough for learning to take place. Some kind of variation has to be made if learners are to discern the critical features of a triangle or any geometric figure for that matter effectively.

Variation refers to the act of varying certain aspects while keeping certain aspects invariant in order to discern the critical features of the object of learning. Variation in terms of shapes and sizes of triangles will allow for contrast to take place in the discernment of the critical features of a triangle. Similarly, the formula $y = mx + c$ contains critical features of a linear function which vary from other functions like a parabola or a hyperbola. These features must be discerned simultaneously as a pattern of dimensions of variation. The capability to draw a linear graph is improved by varying the features e.g. the gradient or the y-intercept. Varying the gradient shows clearly the different shapes of a linear graph. In this case, the gradient is viewed as the critical feature because according to the theory, that which varies is likely to be discerned.

3.11 Patterns of Variation

In a learning situation it is necessary to consider closely what varies and what is invariant. The following are considered patterns of variation which are necessary conditions for perceptual learning (Marton & Pang, 2006):

3.11.1 Contrast

In order to experience what something is, one needs to have an experience of what it is not. Contrast is about comparing something with other things that are different to it. Instead of offering examples of things that are similar to the object of focus, contrast can be used by giving examples of things that differ from the object of learning.

3.11.2 Generalisation

In order to understand fully what something is, we need to experience varying appearances of that thing. This variation in appearance is meant to separate its critical features from other irrelevant features. To understand what a linear graph is, it is important to experience various forms of linear graphs in order to distinguish it from other types of graphs. By varying the formulae or functions that produce different graphs, the critical or invariant features can be identified.

3.11.3 Separation

In separation, differences between different aspects of the same object can be discerned. In order to experience certain aspect of something and be able to separate this aspect from other aspects, the aspect being studied must vary while other aspects remain invariant. We can study the effects of the variables m and c on the formula $y = mx + c$. In other words, the various aspects of a linear function are discerned while the general formula remains invariant.

3.11.4 Fusion

Fusion is based on the conjecture that a more effective basis for a powerful action is viewing a certain class of phenomena in terms of a set of aspects that are analytically separated but simultaneously experienced than if a global, undifferentiated way of seeing the same class of phenomena is used. "If there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously." (Marton et al., 2004, p.16).

3.12 Simultaneity

It has been argued so far that in order to discern a feature, variation must be experienced in that feature. Diachronic simultaneity "is the simultaneous experience of different instances at the same time, which is necessary for experiencing variation in a certain dimension and for discerning the aspect of an instant corresponding to the dimension".(Marton et al., 2004, p.17). Diachronic simultaneity is more about discerning aspects of the same thing separately by experiencing variation in the dimensions of variation matched to each aspect, while synchronic simultaneity is about the simultaneous discernment of various critical features of an instance defined as a way of seeing something. It is about experiencing "different co-existing aspects of the same thing at the same time". (Marton et al., 2004, p.18). Both diachronic and synchronic simultaneity are a function of discernment because they must be discerned and a person must have a focal awareness of them. Therefore, diachronic simultaneity is necessary for the experience of synchronic simultaneity.

Synchronic simultaneity can also be viewed as the simultaneous experience of the whole and its parts. In the teaching of functions, the whole can be 'functions' as a topic. Parts would refer to the different functions that are covered in the High School syllabus which may include, linear, quadratic, exponential and cubic functions. Simultaneity can be seen as the fusion of previous experiences of variation which allows the learner to diachronically or synchronically focus on critical features simultaneously for achieving a certain aim.

Simultaneity empowers learners to act in powerful ways. Diachronic simultaneity is more about discerning aspects of the same thing separately by experiencing variation in the dimensions of variation matched to each aspect. In a pedagogical situation, these four types of variation interaction act together in a concerted way to bring about discernment. A practical demonstration of how variation theory can be applied in the teaching of functions is offered in Figure 3-2. The flow diagrams show a visual, intuitive classification of plane figures and a pedagogical time sequence. The circular arrows and the dotted rectangle indicate that a mutually enhancing interaction between contrast and generalisation is at work to bring about awareness of dimensions of variation and/or critical features.

Figure 3-2: A pedagogical time sequence on the understanding of plane figures (Leung, 2012, p.438)

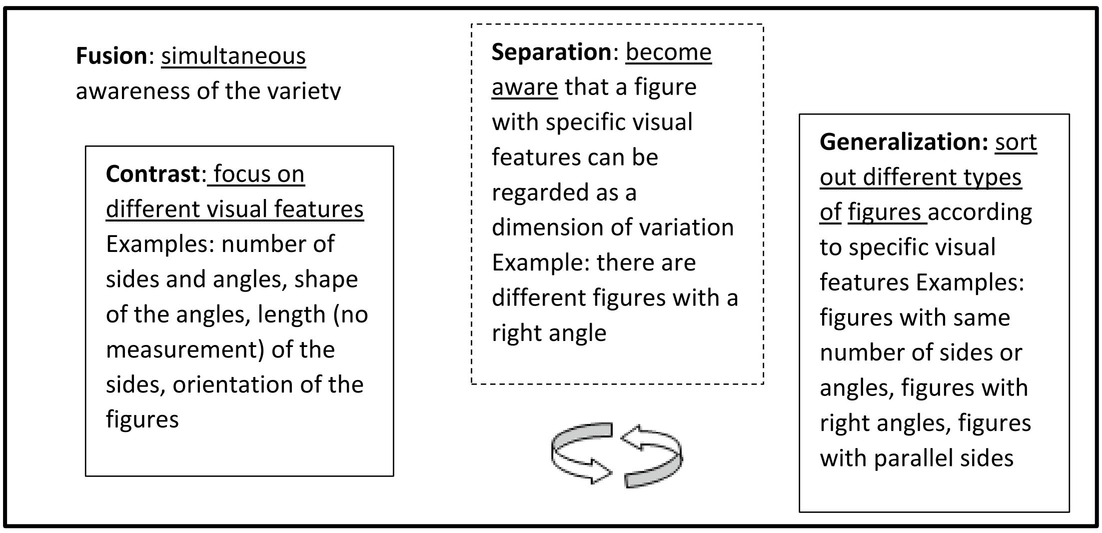

Figure 3-3: A visual intuitive classification of plane figures utilizing the four types of variation interactions (Leung, 2012, p.437)

The following is an example of discernment where contrast is achieved by representing the mathematical concept in multiple ways.

Discernment unit 1: Classification of linear functions by the shape of the graph

Type of variation: Contrast

Classification of straight line graphs by shape i.e. increasing or decreasing. The focus is on visual features.

Learners are given six straight line graphs with the equations written in standard form. The values representing the gradient and the y-intercept are varied. By drawing the graphs, learners discover that all graphs are straight lines of different slopes and shapes (increasing and decreasing). Contrast is used here by representing straight line graphs in multiple ways.

Type of variation: Separation

The focus is on the awareness that graphs can be increasing and either go through the origin or cut on the y-axis above or below the x-axis. The goal of variation is to show for instance that if the gradient is kept positive the resultant graph can be increasing and go through the origin or increasing and cut on the y-axis. In this case c is varied and the sign of the co-efficient of x is the invariant part for increasing and decreasing graphs. There is a separation between increasing and decreasing graphs and the focus is on the critical features of variation.

Type of variation: Generalisation

The focus is to identify the invariant part by decontextualizing the pattern observed. Under contrast and separation, it is clear that while other parts can be varied, the highest power or degree of x , the input variable is consistently 'one'. Therefore it can be generalized that all straight line graphs whether increasing or decreasing are of the form $y = mx^1 + c$, where m and c are elements of real numbers.

Type of variation: Fusion

Fusion is the perception that a graph of a linear function is a representation of a relationship between variable x and variable y in the general form $y = mx + c$. Different functions are represented by unique graphs depending on what features have been varied.

Discernment unit 2: Classification of relations by type

Type of variation: Contrast

Counter examples are used to discern the critical features of a linear function. The following three relations are representations of a parabolic function, a hyperbolic function and a circle centred at the origin.

(a) $y = x^2$ (b) $xy = 12$ (c) $x^2 + y^2 = 25$

The three relations (a), (b) and (c) differ from the linear function in their presentations of general formulae. Contrast is used to teach what a linear function is not versus teaching what a linear function is.

3.13 The two types of variation

3.13.1 Conceptual Variation

Conceptual variation comes from the idea that concepts can be presented in multiple ways in order for optimum learning to take place. This allows connections to take place and creates a space to move between concrete and abstract experiences. In conceptual variation, concepts are varied in standard and non-standard ways. The standard variation refers to connections between concrete and abstract experiences and how teachers vary the ways in which tasks are presented in order to obtain the best possible learning experience. Non-standard variation, however, is about presenting a concept using non-standard examples like contrasting a triangle with a square in order to best explain what a square is not. Both of these variations are positive because they are representations of a concept. Negative or non-concept represents the way in which learning is presented making use of non-concepts such as counter examples. Charles (1980) stated that the ideal examples to use are those that are 'just examples' and ideal non-examples are those that are very nearly examples like a penguin and a bat as examples of a bird and a 'non-bird' than a robin and an elephant. This choice of examples, it is, argued, helps in the elimination of irrelevant features. Furthermore, Charles (1980, p.19), conjectured that "non-examples are more instructive for learning difficult concepts, whereas examples are more instructive for learning easy concepts".

3.13.2 Procedural Variation

Procedural variation is the idea that learning or mastery of concepts takes place through the process of unfolding mathematical activities step-by-step from multiple approaches. The multiple approaches involve the unfolding of concepts using one key point with different numbers, scaffolding using one key point with different applications and using one key point, one problem but different solutions. Figure 3-4 demonstrates procedural variation.

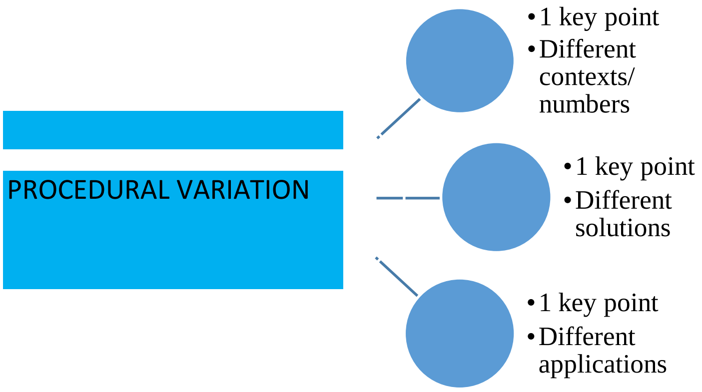

Figure 3-4: Procedural Variation (Source:Adpted from https://view.vzaar.com/13945697/video)

It has been argued that the use of procedural variation accounts for the reason behind Chinese learners' superior achievements over their Western counterparts in many international comparative studies (Lai, 2012). The argument is based on the notion that when non- arbitrary and substantive connections between various items of teaching content are well structured, learners can be assisted to understand the essential features of different mathematical concepts, and Chinese teachers understand this. Unlike rote learning which employs the same strategy in similar problems, procedural variation advocates for the organisation of interrelated tasks to enable the learner to make connections between interrelated concepts (Lai, 2012).

3.14 The cognitive construction process

The study will make use of variation theory, as a tool to analyse the participants' understanding of functions and how this knowledge is used in instruction. Understanding is a cognitive construction process at four levels (Dionne, 1993) as demonstrated in Table 3-1:

Table 3-1: The cognitive construction process (Dionne (1993, p.4)

Intuitive Understanding	Procedural Understanding	Abstraction	Formalisation

3.14.1 Intuitive Understanding

Intuitive understanding is characterised by informal knowledge and preconception and "a type of thinking based on visual perception and primitive unquantified actions limited to rough approximations" (Dionne, 1993, p.5). This type of understanding allows one to make estimations based on visual perception. Intuitive understanding has limitations and gives rise to errors due to its dependence on visual perception, however, it is useful as an initial level of understanding as it provides meaning and relevance to the acquisition of new knowledge.

3.14.2 Procedural Understanding

This is the second level of understanding which is foundational to much more sophisticated mathematical procedures. It is the assimilation of knowledge and procedures resulting from pedagogical interventions. This assimilation may involve basic knowledge such as reciting numbers to simple procedures such as comparing two quantities.

3.14.3 Abstraction

Dionne (1993, p.6) states that “abstraction is initially characterized by the separation of the concept from the procedure and then by its generalization, or by its conservation which reflects the invariance of the mathematical object”. Abstraction is seen as the construction of mathematical activity at a more advanced phase.

3.14.4 Formalisation

“Formalization is characterized either by the use of symbolism, or, as often interpreted in mathematics, by the logical justification of operations or the discovery of axioms” (Dionne, 1993, p.7). For formalisation to take place, some level of abstraction must have happened.

3.15 Justification of variation as the theoretical framework for the current study

The complex nature of learning in the classroom makes it impossible to find a single suitable method of generating reliable data to explain the relationship between the knowledge of the teacher and the effect that this knowledge can have on the quality of learning. How does one for instance tell with certainty what learners have attained during a particular lesson? How does one judge for quality where one lesson is presented using a traditional, teacher-centred approach versus one where a progressive, learner-centred approach is used? The focus of variation theory as has been argued is not on any single approach to teaching or learning. Variation is about creating opportunities for learners to learn specific content. It is also about an understanding that there are no generic approaches to teaching as the goal of teaching is to create opportunities for learners to develop certain capabilities. The question for this study is, how is the space of learning used to enhance the capabilities of discerning the critical features of the object of learning?

3.16 Limitations of variation theory

Theories are not ‘truths’, they all have limitations, and “no single theory can be used to explain all kinds of learning” (Lo, 2012, p.1). Different learning theories have their own special features and purposes. The complex nature of learning in classrooms, makes it impossible to find one theory that suits all purposes.

While variation theory has a place for the current study, like all other theories, it cannot on its own do justice to every aspect of the inquiry. It does, however, support this study as a theoretical lens. Some of the reasons for this claim are that, the theory is vague in its covering of the nature of knowledge needed for teaching of specific content. The premise of the variation theory is that teachers present the intended object of learning while learners discern the specific object of learning. What is learned is therefore not necessarily what is taught. The aim of any lesson is to communicate the intended object of learning and to create opportunities for learning to take place such that the learners discern the same critical features of the object as intended by the teacher. The theory fails, however, to offer guidance on how teachers can enhance the space of learning such that learners move closer to discerning the intended object of learning.

Research into the area of teacher knowledge reveals that while it is vital to know the content or subject matter to be taught, it is equally important to have knowledge of how to teach that content. Shulman (1986) introduced Pedagogical Content Knowledge (PCK) which he defined as the amalgam of subject matter (content) knowledge and how to teach that subject matter (pedagogy). Variation does allude to the use of learners' prior learning as a way of enhancing the quality of discernment. I use the word quality here because in as much as the theory states that discernment or learning always takes place, the difference is that what learners discern might be different to what the teacher had in mind. In the classroom, however, the teacher will assess the intended object of learning. Teachers must therefore be in possession of certain kinds of knowledge about learners and about the teaching of that particular content if they are to empower learners to develop capabilities to act in powerful ways.

It is not to say that the theory does not make attempts to offer explanations about quality teaching. The authors do refer to the patterns of variation which include separation, contrast, generalisation and fusion as ways in which teachers can ensure that learners use prior knowledge and make connections in their discernment of the critical features of the object of learning. However, one cannot argue with the fact that teachers still need to know how to use these patterns of variation in their teaching. In choosing examples for the purpose of separation or contrast, teachers need to use knowledge and insight. Charles (1980) argued that the choice of examples helps in the elimination of the irrelevant features (See example of an ideal non-example mentioned earlier on page 69).

The role of the teacher is to investigate and discover the necessary conditions needed for the learning of specific content in specific cases. The theorists state that student learning should not be accidental but that teachers need to take into account characteristics like the age and general capability of the learner, class size and the equipment available. What teachers do with the curriculum matters and this determines how the object of learning is dealt with in the classroom. Teachers are actively involved in the process of discernment by asking questions to determine what it is that learners are noticing. Taking create the space of learning taking into account learners' prior learning and misconceptions about specific content.

The theory's special focus on learners and the content to the neglect of the teachers' subject matter knowledge does create limitations in terms of supporting the current study theoretically. This study's focus is investigating the kind of relationship that exists between teacher knowledge and the quality of classroom instruction. For this reason, the study adopted as a supplementary conceptual framework, Mathematical Knowledge for Teaching (MKT) model offered by Ball et al. (2008).

3.17 The kind of knowledge needed in teaching

Teacher knowledge is a widely researched phenomenon (Hill & Ball, 2004; Hill et al., 2008; Niess, 2011; Mudaly & Moore-Russo, 2011; Christiansen, 2012). Shulman (1986) presented seven categories of teacher knowledge and these include: content knowledge; general pedagogical knowledge; curriculum knowledge; pedagogical content knowledge; knowledge of learners and their characteristics; knowledge of educational contexts and knowledge of educational ends, purposes and values.

A myriad of studies have been conducted since Shulman's first presentation of teacher knowledge. A number of models have also emerged employing Shulman's model as their foundation and adding constructs like teacher beliefs and educational contexts (Frith & Lloyd, 2016; Prince & Frith, 2017; Aljaberi & Gheith, 2018;). Building on these models, Cogill (2008) developed a three dimensional model to illustrate that there is a constant two-way flow between base elements consisting of various knowledge forms and the vertex element depicting pedagogical change. It can be argued that the most influential contributions in the subject of teacher knowledge has been the works of Ball and colleagues (Ball, 1988; Hill & Ball & Rowan, 2004; Hill, Ball & Schilling; Ball & Rowan, 2004; Thames, 2006; Ball, Thames & Phelps, 2008; Hill, 2008; Ball, Hill, Blunk, Charalombous, Lewis, Phelps & Sleep, 2008; Ball, 2011). Ball et al. (2008) developed an oval shaped model separating knowledge domains into two main categories, namely, Subject Matter Knowledge (SMK) and pedagogical Content Knowledge (PCK). The six knowledge strands are discussed in Figure 3-5.

Mathematical Knowledge for Teaching (MKT)

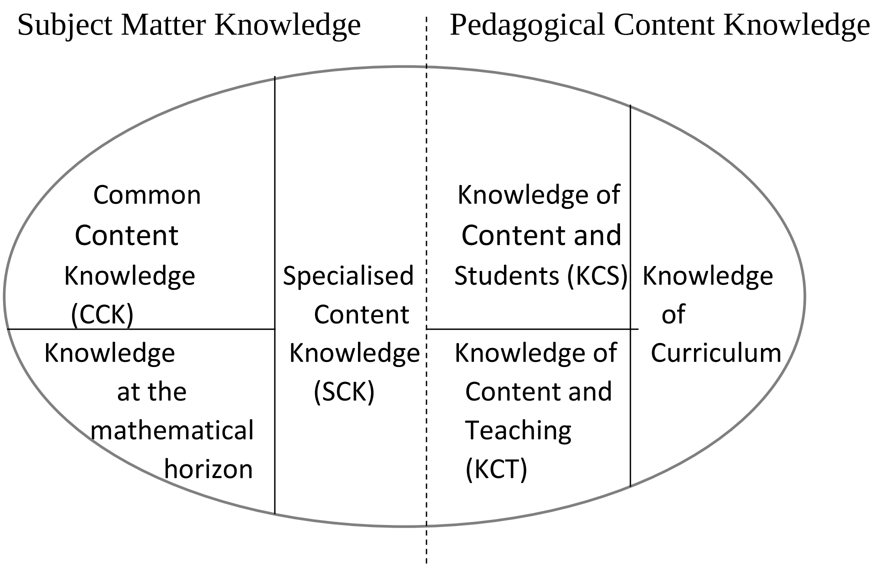

Figure 3-5: Domain map for mathematical knowledge for teaching (Ball et al, 2008, p.377)

3.18 Discussion of the Mathematical Knowledge for Teaching (MKT) model

Based on the analysis of the mathematical demands of teaching, the researchers hypothesised that Shulman's content knowledge could be subdivided into common content knowledge (CCK) and specialised content knowledge (SCK) and that his pedagogical content knowledge could be divided into knowledge of content and students (KCS) and knowledge of content and teaching (KCT). Knowledge at the mathematical horizon (Horizon knowledge) was included as a subset of subject matter knowledge (SMK) while knowledge of curriculum was placed as a strand on the right hand side under pedagogical content knowledge (PCK). The two main knowledge domains and their sub-divisions fall under the mathematical knowledge for teaching (MKT) framework. This model fits in with the purpose of this thesis which sought to investigate how teachers use their knowledge of the content or subject matter in the teaching of functions.

3.19 Strands under SMK

3.19.1 Common Content Knowledge (CCK)

The first knowledge domain CCK, is defined as the mathematical knowledge used in other settings other than teaching. Ball et al., (2008) further assert in their definition that this is knowledge that any reasonably educated person using mathematics is expected to possess as a skill. In teaching, this is the knowledge that allows teachers to see whether a solution to a mathematical problem is correct or not. Teachers however, unlike other people using mathematics in their vocations, have to know more than just whether the solution to a problem is correct or not. CCK is therefore not enough if it only allows teachers merely to assess the correctness of a mathematical production.

3.19.2 Specialised Content Knowledge (SCK)

SCK was introduced as a new knowledge domain to cater for the knowledge and skills unique to teaching. Shulman had commented that it is not enough for a teacher to know that the solution is incorrect but teachers should be able to know also why that it so, because there should not be a separation between the knower and the known. Teachers use this knowledge to assess routine tasks that learners embark on daily which demand unique mathematical understanding and reasoning. This knowledge goes beyond what teachers teach to learners. It requires understanding different interpretations of the operations in ways that learners need not explain or distinguish explicitly. Some have, however, questioned the existence of SCK as knowledge domain exclusive to SMK. I discuss this criticism in the next section:

3.19.3 Criticism of SCK as knowledge exclusive to SMK

Specialised content knowledge is defined by Ball, Thames & Phelps (2008) as knowledge that is purely mathematical and specific to the profession of teaching. It has been referred to by some writers as the knowledge that all teachers should have (Mudaly, 2015). Medrano, Escudero & Yanez (2013), however, question the exclusivity of specialised content knowledge to the mathematical domain. The argument given is that, the advocates of the definition of SCK as that knowledge unique to the ambit of education, fail to explain how they know that this knowledge is knowledge not used in other professions. They posit that the definition of SCK by certain writers makes it clear that the knowledge required is demonstrated as skills in terms of what a person in possession of that knowledge is able to do. The argument continues that there is no clear demonstration that the knowledge that enables teachers to perform certain tasks is not shared by other professions.

The writers further question the definition of SCK that compares this knowledge domain to CCK. They especially interrogate the definition offered by Markworth, Goodwin & Glisson, 2009, p. 69**)** which refers to SCK as content knowledge needed for the teaching of mathematics, beyond the common content knowledge needed by others, of particular concern is the intended meaning of the word ‘beyond’. Does beyond mean a deeper or amplified kind of CCK or is some kind of intention required for knowledge to be categorised as SCK? Bearing in mind that SCK is defined as knowledge not taught to learners, what if the educational intention was to extend knowledge of a topic, would this not form part of CCK that is taught to learners? How is this knowledge separate to knowledge used by mathematicians? What value is there in separating CCK from SCK?

The authors did an in-depth analysis of the two most commonly used representative examples of SCK reported as specific educational tasks in the literature. One of the specific tasks required from a mathematics teacher is to detect errors made by learners. SCK is identified as that knowledge needed to analyse procedures leading to the detection of errors. Ball et al. (2008) illustrate that teachers often encounter unconventional methods used by learners which result in correct answers. The generalizability or mathematical validity of these methods are not immediately clear but to be effective, teachers are expected to be able to engage in inner mathematical dialogue in order to evaluate and size up these mathematical productions fluently and quickly.

The following example of CCK is provided by Ball, Thames & Pelphs (2008):

	Yes	No
a) 0 is an even number.	1	2
b) 0 is not really a number. It is a placeholder in writing big numbers.	1	2
c) The number 8 can be written as 008.	1	2

This is an example of knowledge that any adult using mathematics should have, including teachers. In the case of student errors, recognising a wrong answer is common content knowledge (CCK). Any person who knows mathematics can recognise a wrong answer but they do not have to know the source of error. Teachers, however, have to size up the nature of an error, even more so if the error is not common and this "typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge" (Medrano et al., 2013, p.3058). Common errors and deciding which errors learners are most likely to make before teaching a content, fall under knowledge of content and students (KCS).

An analysis of a subtraction problem by looking at the typical algorithm for solving it together with two common errors that teachers may encounter in their teaching is used to make a case for indistinct lines between CCK and SCK.

(a) 307	(b) 307	(c) 307
-168	- 168	-168
139	261	169

According to Ball et al., (2008) any adult familiar with mathematics including mathematicians will recognise that incorrect algorithms were used to solve the problem in (b) and (c). However, teachers would have to go a step further and realise that in (b) the learner has failed to grasp the importance of the relationship between the top and the bottom rows which is brought about by lack of understanding of subtraction as the distance between two numbers. By analysing the error the teacher should come to the conclusion that the source of error was caused by subtracting the smaller digits from the bigger ones in each column. In (c) the learner fails to understand the position of zero during regrouping. The teacher who analyses this error would know that the problem represents: 300 + 0 + 7 minus 100 + 60 + 8. 100 is borrowed from 300 to yield 10 tens (100) and a 10 is borrowed from the 10 tens to lend to 7. In the end the problem is regrouped to be: 200 + 90 + 17 minus 100 + 60 + 8. The learner however, fails to understand that zero represents the absence of tens and simply brings down the six. The question to ask is, would any other person who understands this method other than a teacher have recognised this error?

Medrano et al. (2013) argue that the case for SCK discussed in the two examples does not provide sufficient evidence to guarantee that the knowledge used is exclusive to mathematics teachers. They posit that all knowledge involved in these examples can be categorised as CCK depending on the researcher’s belief in what learners should know about the content. This position seems to suggest that what is regarded as common or specialised is subjective. It is argued that, the existence of the need to analyse the error does not give justification that the knowledge is special to teachers, it simply means that others do not need to analyse the error or to know why something is incorrect.

One of the skills representative of SCK is interpreting mathematical productions whether by learners, other teachers or from curriculum materials. The case for lack of distinction between CCK and SCK is extended to examples that include the correct answer but incorrect procedure to produce the answer. Using the example found in Suzuka, Sleep, Ball, Bass, Lewis & Thames, (2009) in which a teacher has to analyse a story which apparently contains errors in the way it is set up but has the correct answer, the writers claim that their case is further strengthened as there is lack of evidence of purely mathematical knowledge exclusive to mathematics teachers in this example. The problem involves the division of two fractions $1\frac{3}{4} \div \frac{1}{2}$

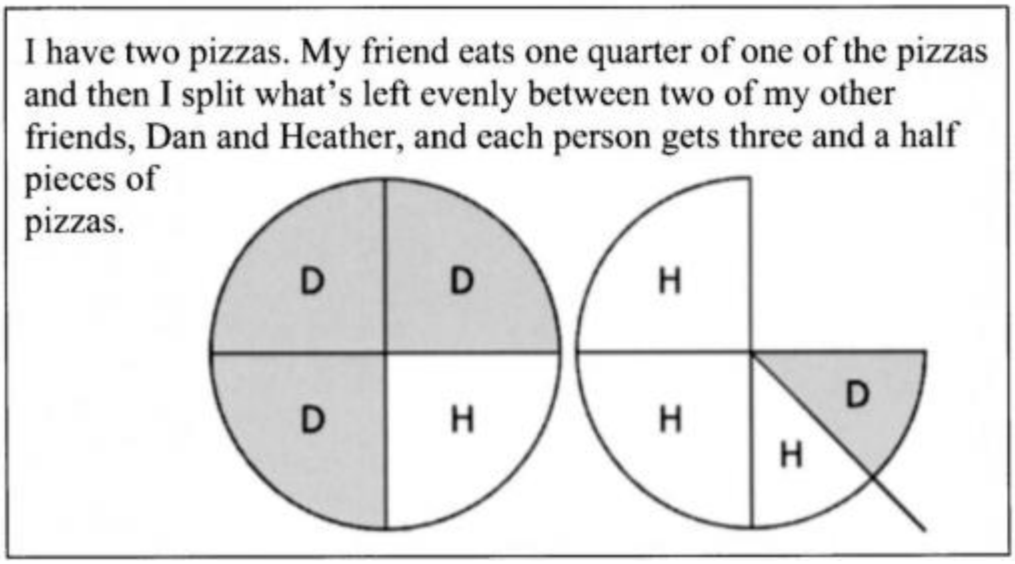

Figure 3-6: Incorrect story problem to represent $1\frac{3}{4} \div \frac{1}{2}$ (Medrano, Escudero & Yanez, 2013, p. 3060)

Using the commonly accepted distinction between CCK and SCK most writers would say the problem represents two types of mathematical knowledge. The first is knowing that the production is correct (CCK) and the second is understanding why the story representation is incorrect (SCK). The argument made is that anyone with knowledge of equivalent fractions (including learners) would know that the answer is correct by simply computing the operation: $1\frac{3}{4} \div \frac{1}{2} = \frac{7}{4} \div \frac{2}{4} = \frac{7\times4}{2\times4} = \frac{7}{2}$, therefore this is common content knowledge. The second part involves SCK and seeks to address why the setup of the story is incorrect. This requires knowledge of the use of divisions as quantifier and as sharing out. The story answers the question, how many times does 2 go into $1\frac{3}{4}$? (quantifier) instead of how many times does $\frac{1}{2}$ go into $1\frac{3}{4}$. It is also argued that anyone with knowledge of mathematics should realise that the question story set up represents the question: how many 2 wholes does one get out of 1 whole and $\frac{3}{4}$? (sharing out) instead of how many halves does one get out of 1 whole and $\frac{3}{4}$? It is apparent from analysing the picture representation that it is the 7 slices of pizza (the size of each being $\frac{1}{4}$) that are divided by 2. From this argument, Medrano et al. (2013) refute SCK as knowledge that is unique to the work of teaching.

The claim put forward by the advocates of SCK puts the teacher in a superior position in terms of content knowledge taught to learners. The teacher seemingly knows more about the question, the solution, possible ways of getting to the solution and also possesses the unique skill or insight to analyse barriers to getting the correct solution. It is assumed that other people who use mathematics for other purposes other than teaching do not need to know anything else other than that the solution is correct or incorrect. It is argued that this kind of knowledge is specialised to the work of teaching. The critics of this notion claim that the teacher is not the superior being in possession of this 'special' content knowledge, because other people who know mathematics including learners can also possess the skill to analyse and interpret the question in order to arrive at the correct solution. It is further argued that the skill to achieve this end can be taught to children.

The critics state clearly that they are not arguing against the existence of mathematics teachers' specialised knowledge, however, they question the claims that this knowledge is exclusive to the mathematical domain (Medrano et al., 2013). They are comfortable with the view of SCK as knowledge about mathematics teaching, such as ways of constructing the subject, the development of complexity within topics and features involved in the learning of the content. It would therefore seem that the critics are proposing that all knowledge required in teaching is specialised content knowledge.

This study will adopt the view of SCK as knowledge about mathematics teaching which includes ways in which teachers analyse and interpret learners' mathematical productions (content) as well as ways in which these interpretations are communicated to the learners (pedagogy).

3.19.2.2 Other studies alluding to the existence of SCK

Ball et al. (2008) were not the first researchers to mention the existence of SCK and CCK in the teaching of mathematics. Dionne (1993, p.8), designed a questionnaire reflecting four types of problems with four possible solutions by learners:

Type 1: a right answer stemming from a correct reasoning;

Type 2: a right answer by itself, without any trace of reasoning;

Type 3: a right answer but originating from a faulty reasoning; and

Type 4: a wrong answer in spite of a good reasoning.

The study hypothesised that a teacher who placed a higher value on learners' reasoning would assign low marks for type 2 and type 3 solutions and a high mark for a type 4 solution. The skill involved in this analysis of learner productions is the same as that described by Ball et al., (2008) as SCK. Teachers who only focus on type 1 solutions and assign a high mark for type 3 solutions reflect someone using CCK to evaluate learner productions.

Ma (1999) in her design of the pen-and-paper items included scenarios in which teachers had to demonstrate their knowledge of responding to a unique or unconventional method or idea raised by a learner. The skill to analyse learner productions in which unconventional methods have been applied, is not common to everyone who uses mathematics for any reason other than teaching others. As Ball, Thames & Pelphs (2008) pointed out, this knowledge is unique for the specialised work of teaching.

3.19.4 Knowledge in Mathematical Horizon or Horizon Content Knowledge (HCK)

The third knowledge domain categorised is horizon content knowledge or knowledge at mathematical horizon (HCK). This knowledge domain has to do with teachers' awareness of how mathematical topics are related over the span of mathematics included in the curriculum. Ball & Bass (2009, p.5) refer to this domain as a 'peripheral vision', or a view of the larger mathematical landscape, that teaching requires. It allows instruction to incorporate mathematical foresight while maintaining mathematical integrity. Horizon knowledge is the idea that any mathematical concept is related to larger mathematical ideas, structures and principles. This includes vertical knowledge useful in seeing connections between mathematical ideas taught now and those that are taught much later as well as lateral or horizontal knowledge which links mathematical content taught across various learning areas or subjects within a grade. Having this sort of knowledge can help teachers anticipate or decipher learners' thinking patterns. Ball et al., (2008) admitted that Horizon knowledge still needed more exploration in order to be certain how exactly to categorise it, and Gencturk (2012) is of the opinion that the place of this category is not yet fixed.

3.20 Strands under PCK

PCK was an unfamiliar concept prior to its introduction by Shulman. He defined it as "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own form of professional understanding" Shulman (2004, p.227). PCK goes beyond mere subject matter knowledge and incorporates knowledge of the most useful ways of representing ideas and concepts in teaching, the knowledge of what makes topics easy or difficult as well as strategies most likely to be fruitful in addressing misconceptions. Shulman further argued that there should not be a distinction between pedagogy and content, or a separation of what is known from how to teach it. Therefore a teacher's ability to transform knowledge into teachable form is directly linked to how much of that knowledge was comprehended.

3.20.1 Knowledge of Content and Students (KCS)

KCS as a subset of PCK is knowledge that "combines knowing about students and knowing about mathematics" (Ball et al, 2008, p.401). It requires that teachers have an understanding of the interaction between specific mathematical understanding and learners and their mathematical thinking. Teachers also need to predict what learners will find interesting, motivating, confusing, easy or challenging. Knowledge of common errors, conceptions and misconceptions and deciding before assigning a task which errors learners are mostly likely to make all form part of KCS. Furthermore, teachers must possess the ability to hear and interpret learners' emerging and incomplete thinking expressed in language form. KCS is therefore more concerned with how learners learn particular content.

3.20.2 Knowledge of Content and Teaching (KCT)

KCT is knowledge that "combines knowing about teaching and knowing about mathematics" (Ball et al, 2008, p.401). It deals with the interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning. When teachers design instructional tasks they consider how sequencing of particular content will be done, which examples will be appropriate, which representations will be suitable and how these tasks will be evaluated. Without KCT, teachers would not be able to cope with these instructional demands.

3.20.3 Curriculum knowledge (CK)

While KCS and KCT focus on the content, students and pedagogy, knowledge of the curriculum deals with a teacher's grasp of materials and programmes needed in teaching. As professionals, teachers are expected to know how to choose good text books, software for effective teaching and how to use available technology to enhance teaching and learning (de Freitas & Spangenberg, 2019). There is also an expectation that mature teachers possess understanding about the curricular alternatives available for instruction. Such knowledge according to Shulman should be taught in pre-service teacher training programmes. Furthermore, Shulman (2004) states that teachers need to possess lateral and vertical knowledge of the curriculum. Lateral knowledge deals with the integration of knowledge across subjects within a grade. An example would be teachers of mathematics and natural science in grade 9 collaborating to discuss how the content related to graphs overlaps across the two subjects. Vertical knowledge refers to how knowledge progresses across the grades. Teachers ought to be familiar with how the teaching of content relating to various topics progresses from previous grades to the grade ahead.

3.21 Linking teacher knowledge to classroom instruction

Since this study is about linking teacher knowledge to classroom instruction, it is important that I provide a framework within which this relationship operates. Pedagogical reasoning is the process by which teachers use instructional processes to transform their knowledge of the content into a form comprehensible to a learner. I will present Shulman's (2004) model of pedagogical reasoning which is a cycle involving comprehension, transformation, instruction, evaluation and reflection.

3.21.1 Comprehension

This activity involves a critical understand of a set of ideas to be taught. Teachers need to comprehend the knowledge they teach in different ways. They should be able to observe how a given idea relates to other ideas within the same subject as well as understand how knowledge is integrated horizontally across subjects within a grade and how vertical progression of ideas takes place across the grades. The teacher as a primary source of knowledge is expected to understand the structures of subject matter, the principles of conceptual organisation and of inquiry. The comprehension of this knowledge helps to answer questions relating to the important ideas and skills within the domain as well as questions about rules and procedures of good scholarship and inquiry. How a teacher communicates knowledge about the subject to the learners gives an indication of the value and the extent to which that knowledge is understood. The diversity of the classroom environment places the responsibility on teachers to have a flexible and multifaceted comprehension of a knowledge base which enables them to use a repertoire of representations to communicate the same principles or ideas in order to cater for this learner diversity.

Comprehension also involves understanding of the goals and purposes of education. Teaching of concepts and ideas within subjects are often a means to achieving educational purposes (the end) which transcend comprehension of those ideas. It is important to understand that these educational purposes will be impossible to achieve without comprehension of both content and purpose. Teachers possess a special kind of knowledge that their peers who have majored in the same subject do not. This special knowledge is what Shulman (1986) referred to as pedagogical content knowledge. It is according to Shulman, knowledge that grows in the minds of teachers. PCK is teachers' SMK adapted to the general characteristics of the learners to be taught. The process of this adaptation is known as transformation.

3.21.2 Transformation

Transformation is the process whereby teachers prepare, represent, select and adapt knowledge for teaching. Teachers have a duty to prepare and to interpret personally comprehended ideas critically in such a manner that they can be taught to others. Transformation also involves the use of multiple representations to clarify the content, selecting instructional teaching strategies from among a repertoire of strategies and adapting these strategies to the characteristics of the learners in the classroom. Shulman asserts that PCK also includes knowledge of what makes that subject easy or difficult to learners. Teachers therefore also have to consider conceptions, preconceptions and misconceptions in their preparation of teaching the content. Transformation can be viewed as the process of preparing for the act of teaching which is referred to as instruction.

3.21.3 Instruction

This is the active management of the transformation process, through various forms of interactions with learners, using instructional methods like discovery or inquiry and other observable forms of classroom teaching. The transformation process is acted out in instruction and it continues during this process. The style of teaching employed is closely linked to teacher comprehension and transformation of understanding. Teachers with sound comprehension of the content will use various and appropriate methods of passing on knowledge to the learners they teach. Teacher knowledge also plays a major role in responding to learner questions and in making judgements about mathematical arguments and evaluating claims. Classrooms become more interactive environments of learning when there is greater teacher comprehension of the content.

3.21.4 Evaluation and reflection

Retrospectively, teachers evaluate learner understanding during interactive teaching and reflect on their own classroom performance after the teaching process has taken place. Checking for learner understanding of the concept takes place during and after the instructional process. Teachers need to have deep comprehension of the subject matter and be able to transform that understanding for teaching in order to evaluate effectively for learner understanding. For

Shulman this extends to knowledge about what makes the subject or content easy or difficult for learners to grasp as well as what misconceptions and misunderstanding render it difficult for learner comprehension. Teachers as professionals are capable of reflecting on their teaching in ways that lead to a better understanding of themselves and of the reasons for their actions. Reflection entails explaining to others various choices made and justifying why other alternatives were left out in favour of those particular choices. This kind of reflection leads to new comprehension.

3.21.5 New comprehension

Although new comprehension may happen after evaluation and reflection it should be understood that this process represents new knowledge gained as a result of teaching that content. This could be new knowledge pertaining to learners, the subject matter, the purposes of the subject to be taught, or new comprehensions about teachers themselves. 'Aha' moments represent new comprehensions and may occur at any stage during the process of pedagogical reasoning.

3.22 Chapter Summary

This third chapter has provided a discussion of Marton et al.'s (2004) theory of variation and Ball et al., (2008)'s MKT conceptual framework as well as literature reviewed to substantiate the use of these two theories and their suitability for this study. The chapter ended with a discussion of Shulman's (2004) pedagogical reasoning process which links the theoretical framework of teacher knowledge to classroom instruction. The next chapter presents and discusses the study methodology.

CHAPTER 4: Methodology

4.1 Introduction and Overview

I would like to begin this chapter by restating the purpose of the research and the critical questions posed at the beginning of the study. The study sought to explore how GET teachers use their mathematical knowledge for teaching (MKT) to teach functions. MKT refers to subject matter (SMT) and pedagogical content knowledge (PCK). It was my hope that this exploration will shed light on how a mathematics teacher's knowledge of the content influences their instructional choices and activities in the classroom.

The following critical questions were posed:

1) How do teachers perceive their mathematical knowledge for teaching functions?
2) How does a teacher's content knowledge of functions influence the way they teach?
3) What other factors influence the quality of instruction?
4) Why does content knowledge influence instruction in the way it does?

The aim of this methodology chapter is to document the rationale behind the research approach, describe the setting, context and sample as well as present data generation and analysis methods. The chapter will also provide an overview of the participants and give a detailed account of how and why the chosen methodology was suitable for the study. To conclude the chapter, a comprehensive summary of the topics covered in the methodology chapter will be presented.

4.2 The Research paradigm

The study was located within the interpretive paradigm. Neuman (2011) explains that this paradigm seeks to explain why people act the way they do and how they interact with each other. He defines it as "the systematic analysis of socially meaningful actions through the direct detailed observation of people in natural settings in order to arrive at understandings and interpretations of how people create and maintain their social worlds". The interpretive approach seeks to answer questions such as why does the phenomenon come about and how does it unfold over time (Romberg, 1992, p.54).

Working within Neuman's (2011) definition it was felt that the interpretive paradigm was suitable for this study because it allowed for the incorporation of approaches and methods that enabled the observation of teachers in their natural settings i.e. classrooms. In seeking to understand how teachers apply their content knowledge in practice to teach a specific topic, it was necessary to include verbal data which provided insight into these teachers' perceptions of their mathematical knowledge for teaching. Studying and analysing the interview transcripts allowed the researcher to gain a deeper understanding of entrenched and implied meaning attached to various attributes of teacher knowledge. Using direct observations puts a researcher in a position to observe some relevant social or environmental conditions and this serves as yet another source of evidence for the study (United States Agency for International Development USAID, 1996). Indeed other social and environmental factors were observed and documented in a field journal during school visits and lesson observations. Observation allows for a validity check on whether or not people do what they claim to do (Guthrie, 2010).

4.3 The research design and approach

As mentioned before, the study was located within an interpretive paradigm. A qualitative design and case study approach were used in line with the interpretive paradigm. Qualitative data is information that is presented using words instead of numbers. The distinction between qualitative and quantitative data is that the former involves subjective perceptions which are difficult to analyse and justify scientifically whereas the latter is objective and involves numbers that can be analysed using scientific methods. Qualitative data is just as demanding as its counterpart in research, and careful measurement is a necessary part of this research. In qualitative research "the rules of the game are not as transparent as they are in quantitative research", however, "in researching people's subjective perceptions, we build up scientific knowledge about their personal knowledge by objectifying their perceptions systematically" (Guthrie, 2010, p.157).

Qualitative designs are suitable for researchers who are more concerned about the process than the product. For this reason, the questions posed by this study would best have been answered by qualitative methods. In choosing the approach and data generation methods, the researcher took cognisance of methods that would minimise problems of reliability. For that reason, a case study approach was utilised.

Yin (1994, p.13) defines a case study as "an empirical inquiry that investigates a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident". Case studies allow researchers to study complex social phenomena and to "retain the holistic and meaningful characteristics of real-life events" (Yin, 1994, p.3). The teachers represented the cases with their knowledge of mathematics as units of analysis. As explained before, the aim of the study was to gain an understanding of how teacher knowledge of the subject matter was used in the teaching of functions. The case study approach allowed the researcher to answer the ‘how’ and ‘why’ questions by employing multiple sources to generate and analyse data. As a research strategy, the case study relies on multiple sources of evidence to allow data convergence and triangulation. Methods employed to generate data are discussed in the section following sampling.

4.4 Sampling

Teachers under the GET band were targeted. The reason for targeting these teachers is that the mathematics they teach is foundational to the concepts that will later be covered in the Further Education and Training (FET) band for those learners who continue with core mathematics up to matric. I also chose to investigate up to Grade 9 classrooms because learners have not yet made subject choices and this will give a clearer picture of the type of learner that most teachers deal with in South African classrooms as well as how teachers handle learners with diverse abilities in mathematics.

The choice of the location of the four schools was influenced by the location of the researcher’s own place of work. As a teacher in Pietermaritzburg it would have been difficult to travel to schools outside of the area as most of the data were collected during school hours. However, in choosing the schools, care was taken to select schools from various contextual backgrounds in order to generate richer data due to the belief that the context of a school influences the work of teaching (Christiaansen & Aungamuthu, 2012). The four contextual backgrounds were rural, township, former Model C and an independent school.

Purposive sampling was therefore used in choosing teachers and the schools. In choosing the participants, the researcher targeted teachers with diverse levels of qualification or education in mathematics or mathematics education. Ball et al., (2008, p.431) stated that “large-scale educational production function studies never peer inside classrooms to compare the practice of higher knowledge and lower-knowledge teachers”. The researcher was also aware that experience does not always lead to improved learner achievement (Zhang, 2008) since all the participants were teachers with less than five years of experience in the classroom, teaching mathematics. One of the participants had been a qualified mathematics teacher for over 15 years. However, she had only worked in the classroom (with more than one learner) for less than five years and it was felt that including her would deepen the data generated as literature shows that a relationship exists between teacher knowledge and the classroom dynamics (Ball et al, 2008). This participant possessed a Bachelor of Science degree as well a Post Graduate

Certificate and had only worked as a tutor for individual learners. In the South African context, I believe it is important to look not only at what experienced teachers or knowledgeable teachers bring into the classroom but that teachers on the other end of the spectrum should also be considered. It is a fact that teachers with very little or no mathematical training teach in classrooms across the country due to a shortage of qualified mathematics teachers. How does this lack of knowledge or lack of training affect the teaching of mathematics? I identified this lack of knowledge comparison as a gap in literature especially in the South African context. The study targeted four teachers three of which had less than five years of experience. Two of these teachers had a high level of training or qualifications in mathematics and two of them had very little training in the teaching of mathematics (out of field teachers). All participants were referred to the researcher by fellow educators who work in these schools.

4.4.1 Profiling the participants

Table 4-1 shows the biographical information of the four participants.

Table 4-1 Biographical information of participants

CATEGORY	**Terry**	**Amanda**	**Lily**	**Brian**
Highest qualification	Bed. Hons (Maths & Science	BSc. (Genetics & Microbiology)	BSc. (Chemistry)	Social Science
Highest level of mathematics at University	Mathematics III	Mathematics I	Mathematics III	Mathematics I
Institution of Higher learning	Same as two other participants	Same as two other participants	Different to the other 3 participants	Same as two other participants
Teaching grades	8 - 10	8 - 10	8 - 10	6 - 7
Language of instruction	English	English	English	English
Home language	English	IsiZulu	English	IsiZulu

Type of school	[3]Former Model C	[4]Township	[5]Independent	[6]Rural
Experience of teaching maths	3 – 5 years	3 – 5 years	15 – 20 years	3 – 5 years

4.5 The scope of the study

When researching knowledge that teachers bring into the classroom it is important to elaborate on the extent to which the study will link teacher knowledge to other closely linked constructs. Learner achievement and teachers' perceptions or views regarding their own knowledge or the teaching of the subject matter are closely related to teacher knowledge. The focus of this study was teacher knowledge, and teachers themselves were the cases and their knowledge units of analysis not learners. However, the study collected data on learners' interactions with the teacher and learning materials were analysed together with the curriculum materials used by the teacher. Teachers' views or perceptions and the school context were predicted to be amongst the factors influencing teacher knowledge. The study focused on the teaching of functions, however, during coding other minor contents linked to the teaching of functions were coded.

The study sought to investigate MKT of GET mathematics teachers. It is important to mention that the participants did not necessarily only teach GET mathematics, however, only GET mathematics was observed.

4.6 Data generation methods and instruments

As mentioned before, the paradigm chosen influences the approaches used and the choice of data collection methods. Case studies are best designed using multiple instruments and for this study no single method would have done justice to the research question, hence a variety of methods were employed to generate data and these are discussed next:

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[3] Former Model C schools are previously Whites only schools. Most of these schools are still well resourced and accommodate learners from various cultural and racial backgrounds.

[4] A Township school is ordinarily located in a previously Blacks only residential zone where low to middle income families live. The majority of the schools are under-resourced and accommodate mainly learners from low-income families.

[5] Independent or Private schools generally accommodate learners from middle- to high-income families. Teachers are paid by the Board of Governors using the money obtained from parents in the form of school fees. This particular school followed the Independent Examination Board (IEB) curriculum.

[6] Rural schools generally house learners from amongst the poorest communities in the country and they are known as non-fee paying schools because they receive full support from the government.

4.6.1 Interviews

In-depth semi-structured interviews were used. A series of interviews took place at various stages of the research process. The first interview was conducted with each one of the four participants at the beginning of the study in order to gather information about the individual's education history, work experience, reasons for studying teaching, perceptions about teaching and learning of mathematics, perceptions about and thoughts regarding their own knowledge of mathematics and their ability to teach it. During these preliminary interviews, the researcher took time to establish trust with each participant. Of particular interest was the participants' attitude towards the researcher whom they regarded as possessing superior knowledge of teaching mathematics. This was apparent from their keenness to receive feedback from lesson observations and this eagerness was expressed explicitly by all participants during the preliminary interview and later during post-lesson observation meetings. Post-observation debriefings were done either immediately after lesson observation or at a later stage with the aim of gaining insight into aspects of the observed lessons. These were also used as opportunities by the participants and the researcher to receive or give feedback on the lesson observed. The researcher kept hand written notes of these debriefings in the field-research journal. All interviews were also audio-recorded and later transcribed. Silverman (2004) mentions that some scholars have argued that it is important that the researcher be a member of the same group they study if they are to have the subjective knowledge needed to truly understand the life experiences of their interviewees. I concur with this view and as a mathematics teacher myself I found it easy to relate to the experiences of the participants in the study. I aimed to build trust by encouraging the participants to share openly their experiences without fear of being judged. There was mutual understanding or 'a maths teachers' code' based on the consideration that all of us are faced with similar challenges of teaching a very challenging school subject (Dundar et al., 2014) and we often feel ill-equipped to teach it.

4.6.2 Pen-and-paper items

The use of pen-and-paper or written items has been documented in other studies (Ma, 1999 & Ball et al., 2008). These items were designed to represent classroom scenarios and to probe teachers about knowledge of mathematics in the context of teaching. These were also designed for the purpose of coding teachers' mathematics knowledge for teaching (MKT) and to corroborate their views regarding their level of subject matter knowledge (SMK). The items were based on the content related to functions as prescribed in the Content and Assessment Policy Statement (CAPS, 2011). These were taken or answered by the participants in the presence of the researcher and the participants (those who consented) were video recorded during this process. Later on the researcher was able to study the individual participant's reaction to each question. Since teachers are expected to answer on the spot questions posed by learners, this tool and method were used to study how confident a teacher was in their approach to a question related to content taught in class as a measure of their content knowledge. I will later discuss how these were analysed.

4.6.3. Lesson Observations

Levin & Wadmany (2006) suggested that studies on teacher belief should involve classroom observation because a teacher's actual classroom practice cannot be predicted from what they expressed earlier regarding teaching and learning. Based on this caution and on other similar research, the study included classroom observation as a method of generating data. Lesson observations took place in the classrooms across the four chosen schools. The researcher made arrangements with each participant to observe lessons on functions. Where clashes were noted, participants were able to accommodate the researcher by adjusting their planning so that no two participants would be teaching functions at the same time. A total of 28 lessons were observed between May and September 2017. The lessons lasted between 35 minutes to 1 hour in length and the researcher made attempts to be present from the start to the end of each lesson observed. On occasions where the researcher had to leave before the end of the lesson, care was taken to ensure that she was present at the start of the lesson and that at least 80 per cent of the lesson was observed. In these instances, a follow up interview would be arranged in combination with the teacher's lesson plan, the researcher would be able to get a picture of how the lesson progressed to the end. These lessons were all video recorded and a journal was also used to jot down the researcher's thoughts at the time.

During lesson observation the researcher was able to capture moment by moment interaction between teacher and learner. As teacher knowledge was the unit of analysis, issues of classroom management were not coded even though these issues would have influenced the lesson flow. However, these were considered as other factors influencing classroom instruction. The use of mathematical language, representations, responding to learner questions, quality of explanations, use of appropriate definitions, type of analogies, class organisation as well as the use of time in the lesson were the main areas of focus.

4.6.4 Document analysis

Document analysis of the curriculum materials used by the teacher and the learners was done so that the researcher was familiar with how the concepts to be taught were organized and the extent to which these materials were helpful to teachers. Curriculum knowledge is one of the knowledge domains envisaged by the theoretical framework of this study, hence it was important to do an analysis of these materials and of how the participants used them in the classroom. Analysis of formative assessment material like homework and class exercises was also done in order to study the progression of knowledge and concepts taught in class.

4.6.5 Field note journal

This journal was kept at all times during field visits. It was used to document every minute detail that captured the researcher's attention which would be useful during the data analysis process. The journal was also used to complement the video and audio recordings in interviews and during lesson observations.

4.7 Limitations of data collection methods

Some participants were observed more times than others due to the travel distance and factors beyond the researcher's control. However, the researcher believes that the lessons observed from each participant were adequate for data analysis and even if more lessons had been observed this would not have significantly altered the data that was analysed or the subsequent findings and conclusions arrived at. The researcher was also not able to stay for the duration of the lesson at times due to travelling distance and other dynamics related to her own work demands as a full time teacher. It is however, felt that this limitation too was countered by the fact that a number of lessons were observed with individual participants and that follow-up interviews were conducted.

The use of one field researcher meant that there was only one video camera used for lesson observations. This rendered it difficult to capture fully classroom interactions as the camera was focused on the teacher the majority of the time. The researcher did keep a journal in which she attempted to record what she felt was important to remember about the lesson which the camera would have missed. The audio recordings from the video camera also compensated for the missing visuals.

A journal was kept for the purposes of making notes of the researcher's thoughts and observations during field visits and interviews. This journal proved to be very useful during data analysis as a way of complimenting other methods of data generation.

4.8 Data analysis and Synthesis

Computer-aided analysis was used to organize data obtained from interviews as well as linking these to the readings from the journal articles. The process began with the importing of the interview transcripts into the Nvivo computer software program. The material was then organized into nodes which represented themes, concepts, ideas, opinions and experiences as they emerged from the interview transcripts. Sub-folders were in turn created to organize the nodes. Memos and annotations were created to capture observations, insights and interpretations linking sources and nodes. Content analysis was further used to create nodes to represent knowledge domains as themes based on the theoretical framework.

The advantage of using Nvivo for data analysis was that it allowed data to be organized into manageable chunks. Without the use of this software it would have taken longer to code the large volumes of data obtained from interviews. Furthermore the program allowed for the importation and coding of other sources like journal articles from literature review materials. The program was therefore used more for organising data into themes and for preliminary coding. At the end of the day, however, the software cannot replace the human intuition in identifying and analysing the coded themes. It was therefore the researcher's job to interpret and make sense of the coded data. This was done by identifying links and patterns in the data and by using these to create themes for further analysis.

To analyse data obtained from classroom observation videos, a coding instrument was developed informed by previous research (Ball et al., 2008; Anthony & Walshaw, 2009; Maniraho, 2017). The codes were classified into 5 main sections or categories so as to record and analyse the content taught, the curriculum materials used, how teacher knowledge manifested in instructional activities and how learners were included in the mathematics taught.

The tool was designed to capture the knowledge domains which form the theoretical framework of mathematical knowledge for teaching. These knowledge domains have already been introduced in the theoretical framework chapter and they consist of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) as the main domains. The strands under SMK are common content knowledge, specialised content knowledge and horizon content knowledge. The sub-sets of PCK are knowledge of content teaching, knowledge of content and students and curriculum knowledge. Combined these knowledge domains capture content knowledge, curriculum knowledge, student knowledge, as well as how content knowledge is used in actual teaching. While the pen-and-paper items were designed to record teachers' SMK and PCK, the observation coding tool was designed to capture mainly PCK. Codes were therefore developed to capture amongst others: use of mathematical language, representations, responses to learner questions, quality of explanations, use of appropriate definitions, type of analogies and examples used, learner inclusion, responding to learner comments and questions as well as assessing how the curriculum material is used to assist the teacher.

To analyse the video recordings, each lesson was divided into 5 minutes time slot to allow for more effective coding.

4.8.1 Knowledge categories

To analyse data generated from the study, Ball et al.'s (2008) mathematical knowledge for teaching (MKT) theory was used to categorise the different knowledge strands. The interview questions, the pen-and-paper items and the observation schedule were all designed around these knowledge domains. The study also utilised variation theory as a tool to analyse the curriculum materials used as well as how these were used to enhance the quality of mathematics taught in the classroom. Variation theory was further used to analyse how content connection was made in the teaching of functions. The instrument presented in Table 4-2 was used to categorise teacher knowledge and knowledge domains.

Table 4-2: The instrument categorising teacher knowledge and knowledge domains

TEACHER KNOWLEDGE TO BE OBSERVED	CCK	SCK	HCK	KCS	KCT	CK
Content connections	x		x		x	x
Content construction through variation or other approaches		x		x	x	x
Knowledge transformation				x	x	x
Engaging learners' (systematic) errors		x		x	x	
Nature of feedback					x	
Focus of feedback		x		x	x	
Unpacking of content	x	x	x	x	x	x
Problem-solving engagement activities	x	x	x	x	x	
Engagement of learners' prior knowledge				x	x	
Engagement of learners' preferred methods				x	x	
Correct use of mathematical arguments		x	x		x	
Application / use of definitions	x	x			x	

Correct application of rules, procedures and calculation methods	x				x	
Understanding of what makes learning easy or difficult		x		x	x	x

4.8.2 Coding of video recordings

The following knowledge was observed when coding the video recordings:

A. RICHNESS OF MATHEMATICS

This was about how teachers connected the classroom practice to mathematics which included:

1) Links

The focus was on how a teacher made links among symbols and representations including justification about why these links were made and why the representations were the best ones chosen to solve that particular problem.

2) Explanations

The quality of explanations which did not include errors. These included explicit talk about meaning and use of mathematical language, ways of reasoning and mathematical practices.

3) Justifications

Justification about mathematical reasoning including why proof is valid or a mathematical statement is true.

4) Multiple representations

The use of multiple models including graphs, equations, tables and pictures for the sake of explaining content or to clarify a concept.

5) Responding to learners

This included the way in which the teacher responded to learners' questions, interpreted learners' mathematical productions and the way learner errors were used in teaching.

B. LANGUAGE USE

This code included both technical and general or everyday language use.

1) General language

This is the use of language for expressing mathematical ideas including the use of analogies, metaphors, stories etc.

2) Technical language

This is the language used to explain mathematical terms and concepts. It includes the use of spelling but not pronunciation or incorrect use of grammar. Other studies have included pronunciation and grammar, however, this was not the focus of coding in this study due to the understanding of the South African education context and the current debates about language use in schools. Research also shows that it is still not clear how language should be considered in such a study.

C. ERRORS

These included computational errors, oversight, inappropriate use of conventional notation etc.

D. OPPORTUNITIES TO LEARN / INCLUSIVITY

This was about the teacher's attempt to include all learners by considering various cognitive abilities, learning styles and contextual factors.

E. ORGANISATION OF THE CLASSROOM

This was the observation of the seating arrangement, the topic covered, teaching and learning resources as well as lesson progression from introduction to conclusion.

4.8.3 The coding rubric

Coding of video recordings was done by dividing the teaching episodes into 5 minutes slots using the coding table as illustrated in Table 4-3. Evident (E) indicates an observation that was made during that 5 minute time interval. The nature of the observation is explained using comments on the right hand side. If more than one observation was made the nature of each is commented on but only one indication is made using (x). Not evident (N) is only used when opportunities are missed, however, when no observations were made for that criterion due to the nature of lesson progression, blank spaces are left. This coding and analysis tool was useful because it allowed for qualitative analysis to be done rather than ticking of boxes. Later on this information was analysed and further summaries made in order to extract themes that emerged

Table 4-3: The video coding and analysis instrument

A. Use of maths with learners: Evident (E) Not evident (N)

Teacher: Amanda																
	5min		5min		5min		5min		5min		5min		5min		5min	
	E	N	E	N	E	N	E	N	E	N	E	N	E	N	E	N
Learners' explanations encouraged																
Learner contributions solicited	x		x		x		x		x		x		x		x	
Scaffolding of learner explanations/efforts							x									
Recognition of errors made by learners							x									
Addressing of errors							x									
Addressing of misconceptions																
Correct interpretation of learner thinking																
Worthwhile mathematics	x		x		x		x		x		x		x		x	
Correct tasks chosen	x						x		x				x			
Responds to learner questions													x			

4.9 Access and consent

The first step in gaining access was to contact each participant upon referral. The researcher introduced herself to each participant and explained the purpose of the study to them. The second step was to obtain consent from the prospective participants' school principals. Upon permission being granted to use the school as a site for conducting research, a letter was then delivered to each participant detailing the purpose of the study, the right to withdraw from the study at any stage during the process and written consent was obtained from each one. The final stage was to seek permission from the Department of Education to conduct research in the government schools and this permission was also granted. Upon receiving ethical clearance from the University, the researcher started making arrangements with the participants to start collecting data.

Pseudonyms were used for participants and school names in order to ensure confidentiality.

4.10 Challenges encountered

As a member of the School Management Team, the researcher is on a reduced teaching load and the school agreed to release her to travel to the research sites when she did not have lessons to teach. One of the schools was 30 km away and on some days the researcher had to rush back so as to be on time for her own lessons. One of the schools is on a ten day cycle and if a lesson is interrupted, that lesson would have to be recovered or revived the following day. It became difficult at times to know when the next lesson would be observed as the participants themselves would sometimes not have that knowledge.

4.11 Validity, Reliability and Rigour

Morse, Barrett, Mayan, Olson & Spiers (2002, p.2) state that "without rigor, research is worthless, becomes fiction, and loses its utility". By nature case studies allow for richer data to be collected. This study made use of a variety of data generation instruments which included, lesson observation, semi-structured interviews, pen-and-paper items, field notes and document analysis. Triangulating data from multiple sources was used to corroborate evidence and prevented the potential problems of construct validity while allowing for the provision of multiple measures of the same phenomenon,

Working with transcribed data from the interviews ensured that data was verbatim thus preventing significant extent of loss in meaning. Although some responses were in the participant's mother tongue, there was no loss of meaning during translation because the researcher is both fluent and proficient in the language used by the participants. Taking videos of the observed lessons and audio recordings of interviews ensured that the researcher was not only relying on her memory but had stored data for later retrieval at any stage during data analysis.

4.12 Chapter Summary

In conclusion, this chapter was aimed at documenting the rationale behind the research approach, describing the setting, defining the context and the sample as well as at presenting data generation and analysis methods. The chapter also provided an overview of the participants and provided a detailed account of how and why the chosen methodology was suitable for the study. The chapter began by reintroducing the critical questions and proceeded to cover a variety of topics which support research methodology. These topics included research paradigm, design and approach, sampling methods, issues of consent, validity, limitations and data analysis. The next chapter is the presentation and analysis of results.

CHAPTER 5: Presentation and analysis of results

5.1. Introduction

As part of the process of data generation about teachers' knowledge of the subject matter and how this knowledge is taught in the classroom, 10 questions were designed and given to the participants to answer on paper. I shall refer to these questions as pen-and-paper or written items. As these were numbered from one to ten, the first question shall be referred to as item 1, the second question, item 2 etc. The written items were designed to encourage the respondents to explain mathematical concepts and to think about the teaching of certain concepts relating to functions. The plan had been to video record the respondents as they answered the questions. The reason for recording this process was to try and capture the actual communication that took place as the teachers tried to make sense of the content in the question. It was also hoped that utterances and gestures similar to what happens in the classroom would be observed on video and analysed in greater detail later. In describing the interaction between thinking and communication, Sfard (2009) introduced a new term which she coined: 'commognition'. This term combines the words 'communication' and 'cognition' and describes the process of communicating in thinking.

Sfard (2009, p.193) puts forward the following conjecture:

> "The property of self-reference (or recursivity), unique to human languages, plays the decisive role in making us able to transcend the concrete and to proceed to ever more advanced levels of mathematical abstractions. In this process, gestures and other visual mediators constitute the material of which the abstractions (e.g., mathematical objects) are produced, one layer after another"

Two views of thinking and communication are highlighted by Sfard (2009). The first view is dualistic and sees thinking and communication as two separate processes coming from different sources and running in parallel. The second perspective views thinking and communication as two manifestations of the same phenomenon. According to the dualistic view, thinking is an internal process while communication is interpersonal, done for the purpose of conveying thoughts to interlocutors. In contrast, the non-dualists have a view of thinking as developing gradually through interpersonal communication resulting in a person learning to communicate with self. Under this perspective, talking and gestures are no longer mere expressions of thinking but are rather viewed as the actual process of thinking.

Sfard (2009) uses the term commognition to bring together interpersonal communication and thinking. Both utterances and gestures are important for commognition. Gestures are used to realise (make real) words while words can be used as indices for gestures. A combination of gestures with a verbal description of such gestures is more effective than gestures or words on their own. When it comes to mathematical discourse, 'recursion', which refers to thinking about thinking or communicating about communication is an essential tool. In analysing the responses from the pen-and-paper items, the aim was not to merely analyse the correctness of the mathematics or comment on the teachers' knowledge but the interest was also on analysing commognition. The focus of the study is on the effect of mathematical knowledge for teaching on the classroom instruction. Specialised Content Knowledge (SCK), which is a subset of SMK, can be demonstrated as a skill used for analysing student's or learner's thinking processes during the production of a mathematical presentation resulting in an erroneous answer. Some type of commognition is taking place during this analysis process as the teacher thinks or focuses on the learner's thinking which resulted in them producing an incorrect answer or making an error.

The items were also designed with the aim of tracking the relationship between the teachers' SMK of functions and the way in which they view the teaching of the concepts relating to this topic (Pedagogical Content Knowledge or PCK). Teachers' responses to the items would also give an indication of whether or not they understand the topic and the related concepts. The results obtained from this data could then be triangulated with data obtained from classroom observations and the interviews. Table 5-1 displays the summary of knowledge strands represented in the MKT items on pen and paper items which the respondents answered. The items represented on the right column were designed to accommodate the SCK knowledge strands on the left column. The numbers on the right column correspond to the item number.

Table 5-1: Knowledge strands represented in the MKT items

Criteria related to SCK	**Pen-and-Paper item**
Correct application of definitions	5, 6,2,11
Valid mathematical arguments and reasoning	8
Correct application of rules, procedures and calculation methods	8,2
Progression and linkage to other content	7, 1,9,6,10
Knowledge of what makes learning easy or difficult	3,4
Engagement of learners' misconceptions & errors	3
Use of correct notation	7

5.2 Teachers' responses to the pen-and-paper items

Brian, after studying the items for a considerable amount of time, made a decision not to participate in the completing of the written items. He declared that he was not comfortable with answering any of the questions because his knowledge of mathematics referred to in the items was not that good. He, however, was happy to do the interviews and to be observed in the classroom. Amanda was also not comfortable with being recorded but did not have a problem with completing the written items. Terry and Lily were the two participants who did not object to being video recorded during the completion of the pen-and-paper items. The next section is a discussion and analysis of how the three participants responded to the pen-and-paper items.

5.2.1 Presentation and analysis of the three teachers' responses to the written items

In this section I will present and discuss how each participant responded to each one of the ten written items. This item-by-item discussion will be followed by an analysis of each participant's response linking this data to classroom observations where necessary.

Item 1

A question was posed in which teachers were to help a learner who was struggling to differentiate between a decreasing and an increasing function from a given parabolic function:

Joshua, a boy in your class asks you to explain to him the difference between a decreasing and a negative function. How would you use the graph below to explain to Joshua between which values the function is decreasing? Also explain to him the interval where the function is negative.

In designing this item it was taken into account that learners often struggle to tell the difference between a decreasing/increasing function and a negative/positive function. Teachers had to demonstrate the ability to explain to learners the idea that functions increase from left to right i.e. as x values increase, y values also increase. A function is positive above the x-axis and negative below the x-axis. Teachers were also expected to indicate an understanding of the turning point and the x-intercepts as the critical points where graphs change from positive to negative or from increasing to decreasing.

In the GET band or senior phase, quadratic functions are not explicitly taught, however, the concept of increasing and decreasing functions is taught. This item was designed anecdotally from the researcher's experience of how learners generally struggle with differentiating between negative/positive and decreasing/increasing functions. How teachers respond to this item would be a demonstration of their SCK and other knowledge domains.

5.2.2 Discussion of how Amanda, Lily and Terry responded to item 1

Joshua, a boy in your class asks you to explain to him the difference between a decreasing and a negative function. How would you use the graph below to explain to Joshua between which values the function is decreasing. Also explain to him the interval where the function is negative.

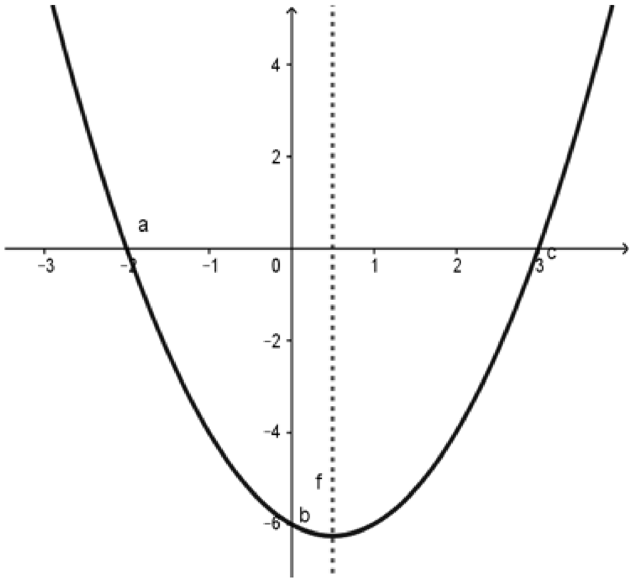


Figure 5-1: Item 1

5.2.2.1 Analysis of Amanda's response to item 1

Between a and b, the values are decreasing, and between b and c are increasing.

a, b: $\frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-6-0}{0--2}$

$= \frac{-6}{2}$

$= -3$

The gradient is negative, therefore decreasing

b, c : $M = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{0-(-6)}{3-0}$

$= \frac{6}{3}$

$= 2$

The gradient is positive, therefore increasing

The parabolic graph contained the coordinates of the x and y intercepts and the equation of the axis of symmetry. Amanda refers to the coordinates of the x-intercepts as a and c and the y -intercept as *b*. Amanda then states that the function is decreasing between the first intercept

a and the y -intercept b and increasing between the y -intercept b and the second x -intercept c.

To answer the second part of the question about showing the learner the interval at which the graph is negative, Amanda calculates the average gradient of the graph between a and b and between b and c. Her response is based on the concept of the gradient of a straight line graph which is the concept she is familiar with from teaching Grade 9 mathematics. She is transferring her knowledge of linear functions into quadratic functions by picking the points given as these allow her to calculate the gradient of the graph at the intervals between the x and the y-axes.

Amanda associates the word 'negative' with the concept of a negative gradient. She finally arrives at the conclusion that when graphs have negative gradients it indicates that they are decreasing and positive gradients are associated with increasing functions. In the lessons observed, there was no evidence that Amanda had knowledge that in linear functions, the sign of the gradient is indicative of a function's shape. It can only be assumed that this was her attempt to link the concept of a decreasing function to the interval where the graph is negative. In her mind, the negative part of the graph refers to the negative gradient rather than the portion of the graph below the x-axis. An episode from one of the observed lessons further strengthens the case for this assumption:

In this particular lesson, Amanda had drawn two straight lines on the board with the aim of explaining to the learners the difference between an increasing and a decreasing function. She had copied the two graphs from the learners' workbook which had labelled the vertical axis as the -x axis and the y-axis as the horizontal axis. The workbook did not give answers to the exercise, hence Amanda had to rely on her own knowledge to explain the difference between the two graphs by pointing out which graph was decreasing and which one was increasing.

> Amanda: This graph (pointing to the graph drawn below), is it decreasing or increasing?

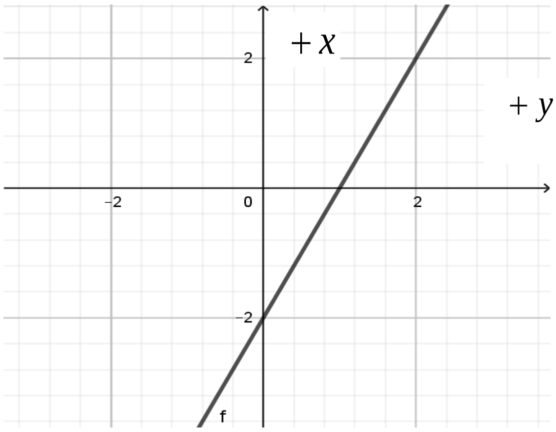


Figure 5-2: A Graph from a teaching episode

Chorus: decreasing!

Amanda: yes decreasing. Why?

Learners remain silent and wait

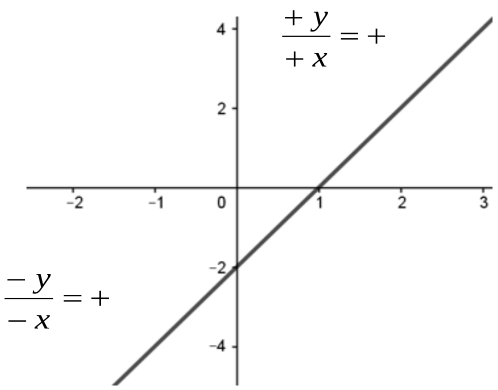
Amanda: It means it started from the positive numbers where x was positive 2 and ended up where x is negative 2, therefore it means the graph is getting smaller. Angithi? (Isn't it)?

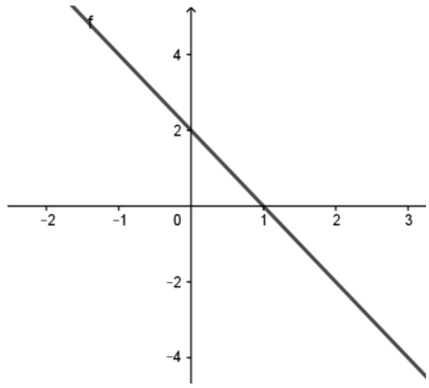
Learners: Yes!

Later on in the same lesson Amanda asked the class about the same graph and the chorus answered 'increasing!' This time she approached the graph from -2 and explained that the values were increasing therefore the graph was increasing. None of the learners picked up or commented on this contradiction, they simply went along with their teacher's explanation and when asked if they understood, in chorus they shouted out a unified "Yes!"

The purpose of the exercise was to represent direct and inverse proportions graphically. A straight line by convention increases from left to right on the x axis and upwards on the y-axis. This is a direct proportion as both variables are increasing. Consequently, a function conventionally decreases left to right on the x-axis and downwards on the y-axis. This is an inverse proportion because x increases as y decreases. Amanda's explanations due to a lack of this crucial knowledge was likely to create confusion when learners in her class re-visited the concepts of increasing and decreasing graphs. It is worth noting that the incorrect labelling of the axes in the curriculum material, in my opinion may have been an error, which partly added to the poor quality of the lesson.

The quickest way of checking whether a linear graph is increasing or decreasing in this confusing case due to incorrect labelling, would have been to find the ratio of $\frac{y}{x}$ in the first and third quadrants and alert learners to the sign of this ratio. A positive sign would mean that the line is increasing and a negative sign in the second and fourth quadrants as shown in the two graphs would indicate a decreasing line. This is consistent with what learners would have observed had the axes been labelled correctly as demonstrated in the following two graphs:





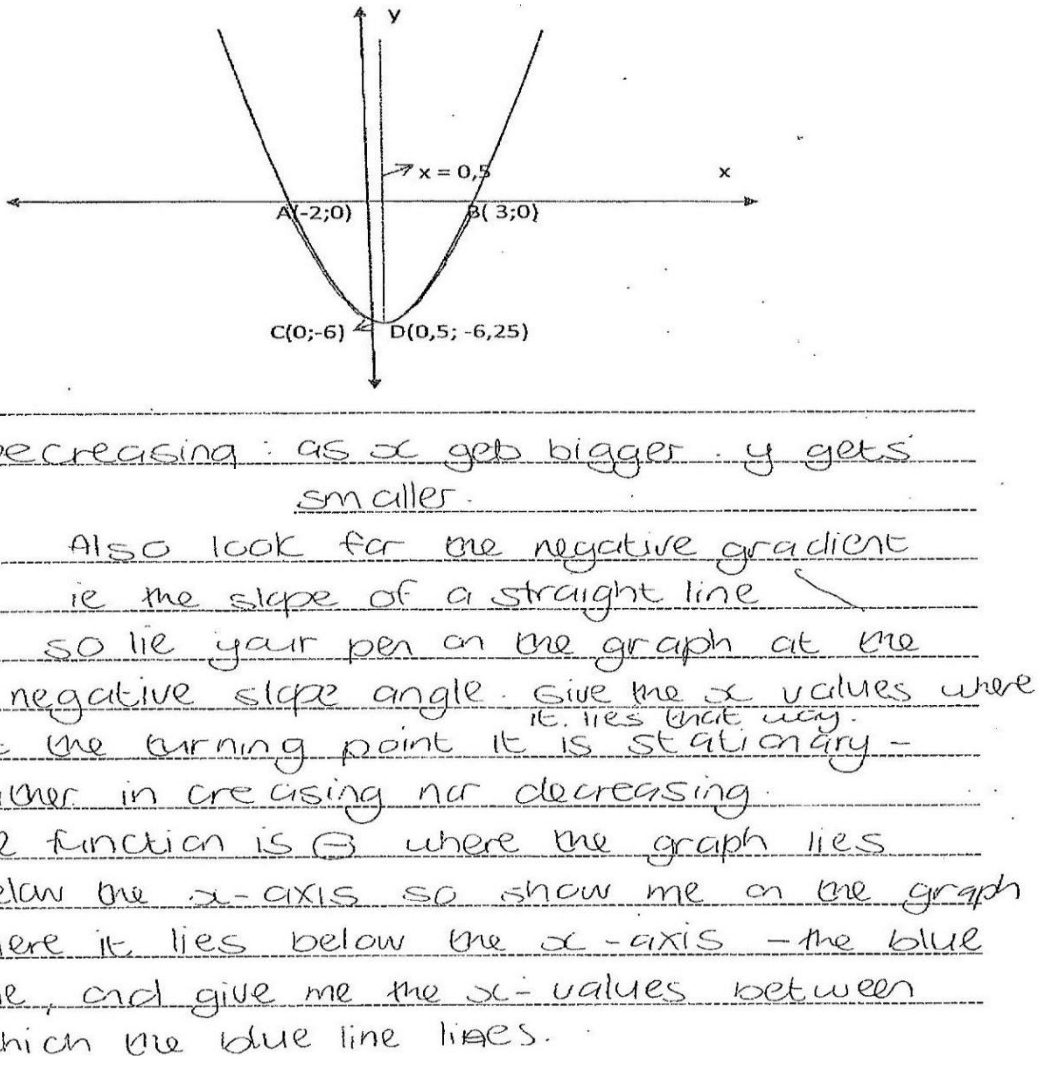
This knowledge of multiple representations is crucial in mathematics and can be used by teachers to bring clarity in situations like the one observed in Amanda's classroom. This class episode does therefore strengthen the case that Amanda's calculation of the gradient in her response was linked to the graph being positive/negative rather than increasing/decreasing.

Amanda's response to item 1 also reveals her limited knowledge of the x-intercept as the critical point at which the graph changes from negative to positive and *vice-versa* and of the turning point as the point at which the graph changes its direction (from decreasing to increasing).

Amanda showed partial understanding of the concepts relating to positive/ negative functions and increasing / decreasing functions, her explanation contained errors and lacked the type of analysis needed when teachers apply their SCK.

5.2.2.2 Analysis of Lily's response to Item 1

Lily was video recorded while completing the questions as mentioned previously. The analysis of the video recordings shows Lily engaging silently with the items. She neither used gestures nor any utterances except in the last question about composite functions in which she muttered and repeated the question to herself a couple of times before making a remark that it was not an easy question. Occasionally she would make a comment like "my kids also struggle with this" or would jokingly ask a question like "how bright is Kayla?" At times she would look dissatisfied with her response and make a comment along those lines "…I don't think I have answered this adequately. There is so much more I could say, but unless a child asks me another question, I don't think I can go further". This gave an indication that in her quietness, Lily was communicating with herself and thinking about what approach she would use if the child in question was weak or strong in mathematics. This will become evident as this analysis chapter progresses. Lily did not write lengthy explanations in her responses but rather stuck to the facts. When probed about this, she explained that all that needed to be said had been said and she did not see the reason for writing lengthy explanations. Her comment that unless the child asks her a follow up question revealed that Lily would rather have the learner drive the direction of the lesson by asking questions rather than assume that the learner did not understand her initial explanation.



Decreasing: as x gets bigger y gets smaller.
Also look for the negative gradient ie the slope of a straight line
so lie your pen on the graph at the negative slope angle. Give the x values where it lies that way.
At the turning point it is stationary - neither increasing nor decreasing.
The function is ⊖ where the graph lies below the x-axis so show me on the graph where it lies below the x-axis - the blue line, and give me the x-values between which the blue line lies.

In terms of teacher knowledge, Lily's response shows more conceptual understanding of increasing/decreasing and negative/positive intervals in a graph than Amanda. She starts by defining a decreasing function as that in which x gets bigger while y gets smaller thus defining an inverse proportion. She continues to draw a negative straight line and states that the gradient of this line is negative thus indicating that the function is decreasing.

In her lessons Lily also demonstrated her ability to use this knowledge to teach this concept:

> Lily: Sihle, would you expect a negative or positive gradient?
>
> Sihle: Negative… Positive!
>
> Lily: Why?
>
> Sihle: using hand gestures to indicate an increasing function.
>
> Lily: Also as x increases, y also increases that's why you know you have a positive gradient.

r

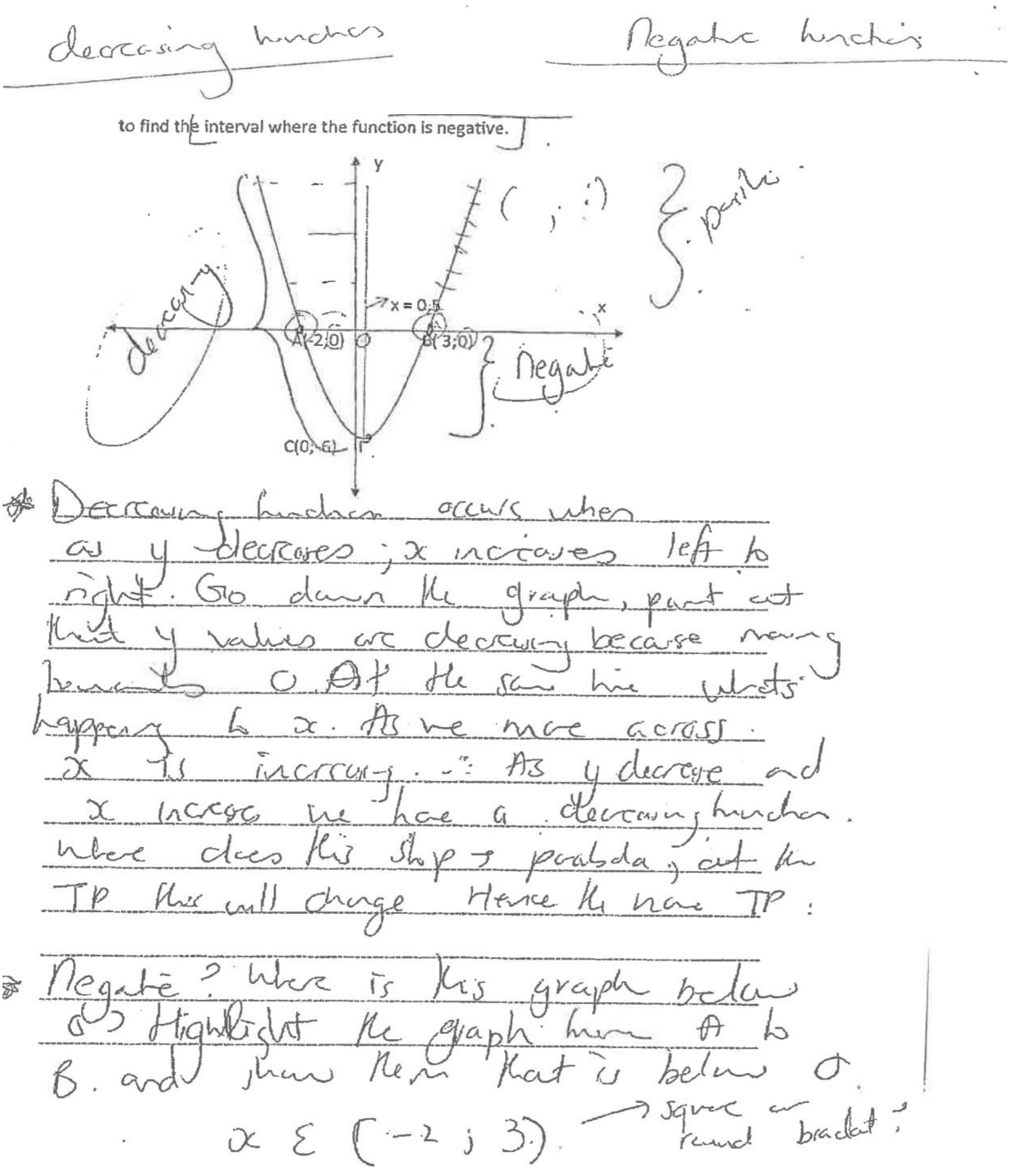
which the slope is negative. She then mentions that the graph is stationery (neither increasing nor decreasing) at the turning point.

Lily also shows understanding of the x-intercepts as the critical values where the graph changes from being negative to positive. She draws a blue line below the x-axis and asks the learner to show her where the graph lies below the x-axis by giving the x-values between which the blue line lies.

The analysis of Lily's response reveals the use of multiple representations in the form of (1) a working definition of a concept which allows the learner to draw or see a decreasing function (2) The drawing or a sketch of a decreasing function (3) Demonstration, by asking the learner to lie his pen at the negative slope angle and using a blue pen to show the values of x for which the graph is negative. This demonstration enabled Lily to explain in depth where the function is decreasing or negative.

to realise the signifier slope, Lily draws a negative line and instructs the learner to lie his pen at a negative gradient in order to show a decreasing line. Also it can be concluded that by highlighting with a blue pen she was attempting to ensure that Joshua realised the concept "negative interval" in the same way as she intended it to be realised, hence ensuring that the intended object of learning was the same as the lived object of learning.

5.2.2.3 Analysis of Terry's response to questions 1



Terry begins by writing *decreasing function* and *negative function* on top of the page. She continues to read and underlines "also explain to him" and puts square brackets around the phrase [interval where the function is negative]. She starts talking to herself and says

> "now we know a decreasing function occurs when as y decreases, x-increases. We do this from left to right. So we start from the far left of the Cartesian plane and we work our way right and we look at the values. What is the value of this graph? Is the graph

> increasing? Is it decreasing? And how we determine this is we start at one point and we seek where the y values are decreasing and how do we know this? We can look at the y-values of the graph and we can see that the y-values are getting smaller and smaller as the graph approaches zero". As she speaks she draws horizontal lines between the graph and the left side of y axis. "As you go down the graph, point out that y values are decreasing because they are moving towards zero. At the same time, what's happening to x ? As we move across x is increasing, therefore as y decreases x increases we have a decreasing function. Where does this stop? Because it is a parabola, at the turning point this will change, hence the name turning point. Where is the graph negative? You look at where the graph is below zero?"

She puts her hand on the x-axis and indicates that the values below her hand are negative values. She then puts two dots on the x-axis and writes "highlight the graph from A to B and show them that this is below 0" pointing at the values below the x-intercepts. She then goes back to the graph and circles the two dots she had drawn on the x-intercepts and continues to write: $x \in (-2; 3)$ *square or round bracket?* At this stage she starts talking to herself and says

> "I'm deciding whether to say included or not included. What do we know about A and about B? They are zero. Anything above this point is positive. At those points what do we know? Well they are x-intercepts and our y-value is zero. Can you see? We look at the coordinates of all the points on the graph here [putting small lines on the portion of the graph above the point (3; 0)]. All the y-values are positive, hence this part here is positive. What do we know about the y-values below the x-axis here? They are all negative. Now would I include A and B? No! Why wouldn't I include A and B? Because at that point it's zero. Everything above there, is positive, at that point it is zero and below it, everything is negative. So this is what a negative function is or where the negative part of my function is.
>
> Decreasing and negative. No they are not the same thing. Decreasing means…now I have to make sure I get the words right. Negative means below zero. Decreasing means it is just…, it is going down. Decreasing and negative does not mean the same thing. Decreasing means it could be negative at some point but it could also be positive at some point. We're going down, decreasing the values. Negative on the other hand implies that it is below zero. And where is this graph below zero? It is between A and B (pointing to the region below the x-axis), but I can't include A and B because at that point it's zero".

Terry displays a good grasp of the concepts relating to decreasing and negatives functions and is also clear about the difference between the two concepts. The analysis reveals the use of multiple representations or approaches in her explanation. (1) Like Lily, she starts by giving a working definition of a decreasing function which helps Joshua understand or even recognise when a function is decreasing. (2) Terry also makes use of gestures in her explanation with the use of her hands and demonstrating by making marks with a pen. (3) Question and answer method is used to show links between the graph and the concepts being taught. (4) Finally, Terry makes use of interval notation to highlight the interval between which the graph is negative. She is connecting content from functions with content from the number system. This concept is used to express the domain and range of a graph and Terry probably knows that this knowledge is crucial.

Like Lily, Terry shows a good understanding of the critical points (the x-intercepts and the turning point). Not only does she circle the x-intercepts on the actual graph but she continuously repeats that at these two points the graph has a value of zero and that the graph is neither positive nor negative at the roots.

Commognition is quite evident in the way Terry presents her explanation. She starts off by writing down what is required of her in the question and underlines the words *decreasing function* and *negative function*. Her entire explanation flows from concepts around decreasing and negative functions. In the end she makes a conclusion which offers an explanation about the difference between decreasing and negative functions. It can be concluded that Terry did not take her eye off the ball in her explanation and writing down the two key concepts boldly on the top of the page may have assisted her to stay focused on task. Terry is also clarifying concepts for herself to ensure that she is clear about the concepts she is explaining for the sake of the learner. As a teacher she is not content with giving vague explanations but wants to ensure that there are no errors or misconceptions created by her teaching.

Terry's thinking process shows that she is visualising the actual lesson and pre-empting what a learner would be finding difficult to understand. This commognition was also observed in Lily's response. Both Terry and Lily express their SCK through this use of commognition. For a non-teacher it would suffice to simply define the terms or concepts and possibly explain the difference between the two thus demonstrating Common Content Knowledge (CCK). However, a commognitive analysis is encountered when one analyses Lily and Terry's responses. This virtual analysis has its focus on the learner's thinking. It is the pre-emption of the errors that a learner is likely to make which would have given rise to the design of the question at hand.

It could further be argued that Terry's PCK grows during the process of commognition. Shulman (1986) defines PCK as knowledge that grows in the minds of teachers. This growth is observed in the utterances that Terry makes to herself about the difference between the use of the terms decreasing and negative functions. She starts off not too clear about her own understanding. "Decreasing means…now I have to make sure I get the words right" As this internal dialogue expressed in words continues, Terry is thinking deeper about the difference between the two concepts. This indicates that making this distinction is probably not something she has put much thought into before but this does not stop her from using her existing knowledge to give clarity to the learner about how these two concepts differ. Terry's actions confirm Ma's (1999) theory that teacher's SMK grows during teaching as they explain concepts to learners. Teachers encounter questions from learners all the time and this requires that a teacher goes beyond a lesson plan in their thinking but also that they have to be very clear about the concepts they teach in order to answer these questions adequately. Although like Terry, Lily portrays a good SMK, Terry's explanation also portrays both knowledge of content and students (KCS) and knowledge of content and teaching (KCT).

5.3 Teachers' responses to item 2

This item consisted of a positive linear graph drawn on a Cartesian plane. The respondents were asked to explain if it was possible for this graph to have a gradient of 2.

Is it possible for this graph to have a gradient of 2? Explain

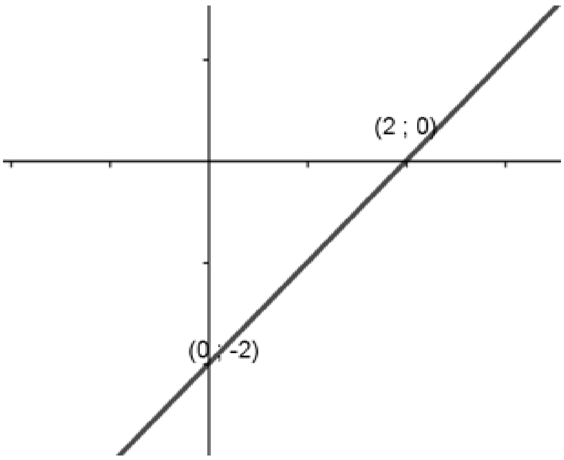
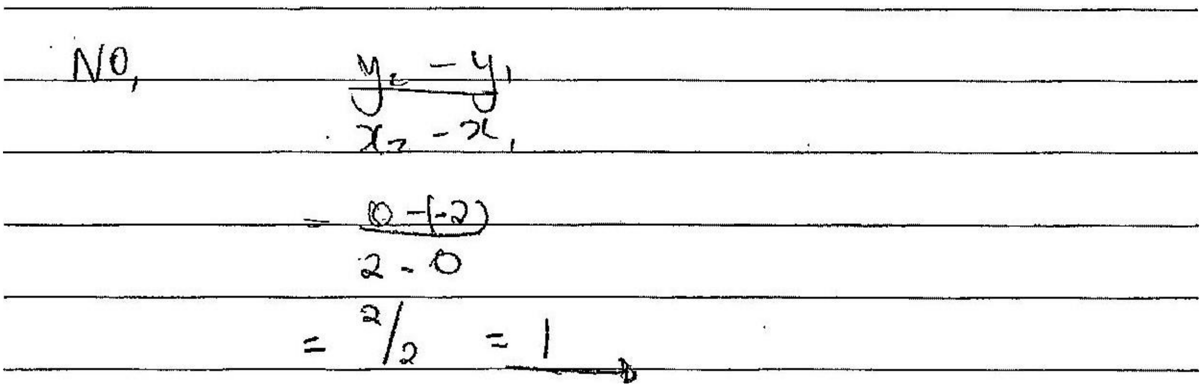


Figure 5-3: Item 2

5.3.1 Analysis of Amanda's response to item 2


No,

$$\frac{y_2 - y_1}{x_2 - x_1}$$

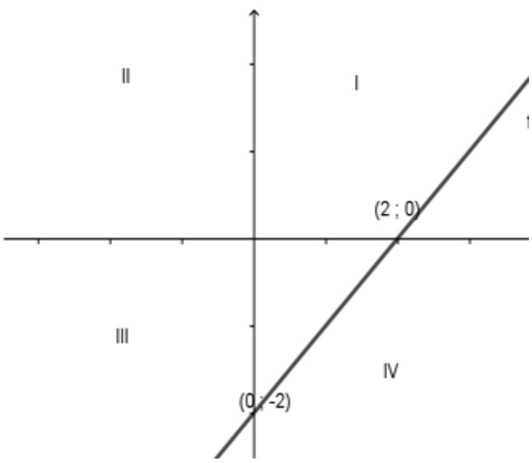
$$= \frac{0-(-2)}{2-0}$$

$$= 2/2 = 1$$

because x & y have the same value just different signs

Amanda immediately puts values for the x and y intercepts. She uses these coordinates to calculate the gradient of the line as shown in her response. Her conclusion is that the line cannot have a gradient of 2. The reason for this conclusion is based on the values she has chosen for the two intercepts. She states that because x and y have the same value i.e 2, but different signs i.e $x = 2$ and $y = -2$, it is not possible for the gradient to be a value of 2.

Amanda's response lacks critical thinking required in mathematics. Firstly, she neglects the fact that diagrams are often not drawn to scale in mathematical problems. She uses inspection to speculate the values of the x and y intercepts. She then formulates her argument based on an assumption rather than factual knowledge. Secondly, the statement that x and y have the same value just different signs does indicate that she is probably thinking about the ratio of the rise over run as calculated from her chosen coordinates of the intercepts. However, this statement does show Amanda's limited knowledge of x and y as input and output values anywhere on the line. Her explanation reflects intuitive understanding. The following diagram is an illustration of the three quadrants that the graph occupies.



In the first and third quadrants the x and y values have the same signs i.e. both positive in the first quadrant and both negative in the third quadrant. The line is continuous, and regardless of the signs of the x and y coordinates, the gradient will remain positive. Amanda's own answer confirmed this. Her response therefore reveals that she has misconceptions about calculations of gradient. Amanda's misconception about the output y and the y intercept c in the equation: $y = mx + c$ was observed in the classroom as illustrated:

> Amanda: Yes, we need to find c. What do you remember about the c? It is equal to?
>
> Chorus: y!
>
> Amanda: $c = y$, they are equal (this misconception was observed in both Terry and Lily's lessons but these two teachers kept emphasising the difference between the output value y and the y -intercept represented by c)
>
> Amanda: we find y from c and c from y, they are always equal.

Comment:

Amanda had explained to the learners in one of the earlier lessons that to find c, x must be made zero and demonstrated using values that $y = m(0) + c$. The result of this equation is $y = c$. Her limited knowledge that x represents changing input values caused her to conclude that y is always equal to c. Her learners seemed to be content with this conclusion and in subsequent lessons they began to state in their own responses that $y = c$ as can be seen in the extract. Both this classroom episode and Amanda's response on paper show a lack of conceptual knowledge of graphs and a deficiency in applying analytical skill for critical thinking.

5.3.2 Analysis of Lily's response to item 2

All I know is that the gradient is
positive because it is an increasing
function.
So it is possible to be 2.

Lily's response shows a good understanding of linear graphs and that she has applied critical thinking in her short explanation. In most of her responses, Lily tends to write only what she deems necessary, however, in the interviews and during classroom observation, it was clear that Lily has a good SMK of functions. Her response is consistent with the definition she had offered in her response to Item 1. This shows that her knowledge is solid and her explanations are consistent. In practice, however, it was shown that although Lily has sound knowledge of the concepts she teaches, she is not always able to give clear explanations to her learners:

Lily: As we can see, in the straight line equations, there is a plain x but if you look at these two, the x is not plain. There is a fancier X here (pointing at the two equations).

Siya: Mam what do you mean by fancier x?

Lily: It is not squared like these two (pointing at a quadratic and a circle equation). It is just plain and multiplied to something, writing down $y = 2x + 1$

Tayla: But Mam, that x is also multiplied by something (pointing at the equation of the hyperbola, $xy = 12$).

Lily: But it has a y, it is not plain.

Plain x or plain old x was the term used by Lily and understood by the class in most of the observed lessons. This next exchange took place a few lessons later:

Lily: Jason, are you happy that this is a linear function? (pointing at an equation on the board).

Jason: Yes

Lily: Why?

Jason: Because the x is not squared

Lily: Yes it is just plain.

Comment:

Lily casually accepted Jason's answer that the equation represented a linear function because x is not squared, but this explanation is mathematically incorrect. Although the workbook was full of definitions and technical terms, Lily did not always emphasize the need to use this technical language in her teaching. The use of the terms input and output for instance was not part of the everyday teaching of functions in the classroom even though these terms are part of the definition of a function. During the observation of lessons it was clear that learners could do procedures when it came to solving problems relating to linear functions but there was no evidence that there was clear understanding about the relationship between the input, the output and the rule or that learners even understood the meaning of the word function. This is not to say that this word was not defined at some stage during the teaching of the topic.

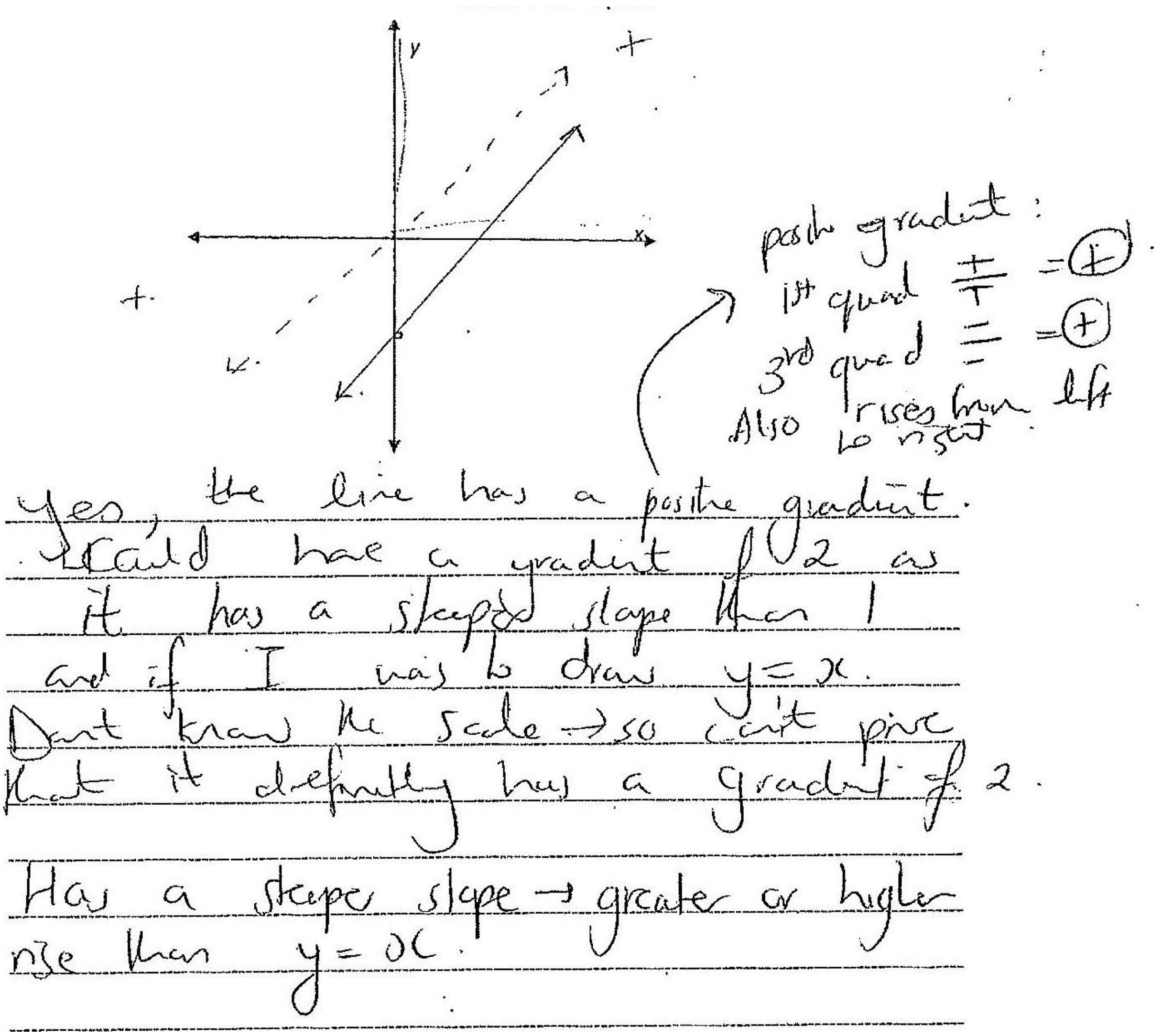
In the interview, Lily had mentioned that learners fear mathematical language and one of the ways she tried to ease that fear was to avoid using technical language:

> "I've noticed that no one likes it if you say, show me where $f(x) > g(x)$, and that freaks them out, they don't know what they are saying. So I will explain to them in English, not in maths language. Like in English, what is this saying to you, come show me, because the maths language is scary, but all they're asking you is where is the one graph above the other graph, that's all they're asking. So I do try to take the fear away and help them to understand what the maths language is saying".

The use of informal language, unfortunately, although it may be understood by both teacher and learners, does have a tendency to undermine the quality of teaching and learning.

I will proceed to present Terry's response to Item 2 and thereafter offer an analysis and comparison of all three responses.

5.3.3 Analysis of Terry's response to item 2



Terry after looking at the problem for a few minutes immediately starts writing: yes, the line has a positive gradient. She then draws a dotted line passing through the origin and continues to write: It could have a gradient of 2 as it has a steeper slope than 1 and if I were to draw $y = x$. After this sentence she stops and starts talking to herself "if we had to move this line up by c units, it definitely does have a steeper gradient. Yes, it is really steep but whether it is 2, we can't determine 'cause we don't have a scale" She continues to talk and write at the same time "So we can't determine definitely whether it is 2 but 2 is positive, so I can say to you that it definitely has a positive gradient, because I can see the line and where it is going. How we teach is, this has a positive gradient because it seats in these quadrants" She places her writing pen in the first and third quadrants. She then traces an L shape in the first and third quadrants and explains using the concept of rise over run that the graph is positive because it occupies the first quadrant (positive y over positive x) and the third quadrant (negative y over negative x). She continues to emphasize that the line has a positive gradient because it is increasing from left to right and can definitely have a gradient of 2 because it is steeper than the line $y = x$.

Terry's knowledge of this concept was observed in one of her lessons as she addressed a learner's misconception about the shape of a straight line and the quadrants:

Reece: Mam I notice that when the gradient is positive the line lies in the first, second and third quadrants.

Terry: Let's see, (she immediately starts moving the m slider and the graph remains in the first, second and third quadrants. She then carries on to move the c slider to the right and the graph still occupies the first, second and third quadrants).

Reece: You see

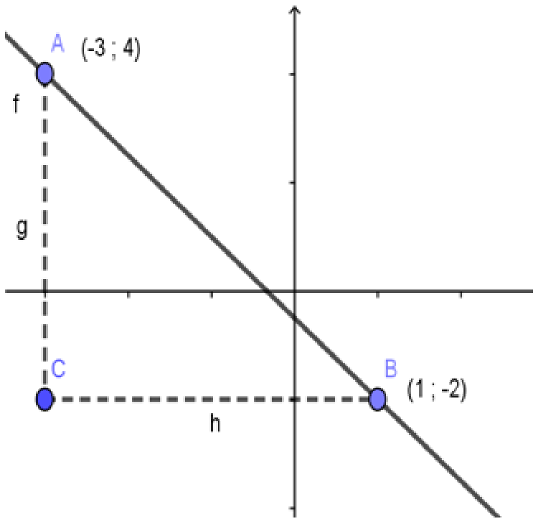
Terry: Let's see what happens when we move the c slider this way (moving the slider to values below zero). You see the quadrants change as the y-intercept goes below zero. So I don't want you to look at it in terms of the quadrants. Look at it in terms of the shape.

Comment:

The use of technology created an environment conducive to various investigations and Terry had told her learners to go ahead and make their own discoveries with the *Geogebra App*. Terry's sound MKT was used to guide these self-discoveries so that learners like Reece did not arrive at incorrect conclusions.

Terry's response to item 2 contains three various representations. Firstly, she draws a dotted line to compare and contrast the steepness of the two lines in order to support her argument that the line can have a possible gradient of 2. Terry also uses the concept of rise over run and shows that the line has a domain $x \in (-\infty; \infty)$ in the third and first quadrants respectively, hence it will always be positive. Lastly, Terry uses the definition of an increasing graph as the line that rises from left to right to emphasize that the gradient of the line is positive. The use of multiple representations was also observed in Terry's teaching:

Learners are working on a worksheet in which they are expected to find gradients of given linear graphs as shown in the diagram:



Terry explains the answer using $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. Some learners are still struggling with the concept of grouping the coordinates to substitute into the equation even though she has explained the process a couple of times.

Terry: Another way of looking at it would be to say, my y value here is 4 and my y value here is negative 2. What is the distance between 4 and negative 2. The distance between 0 and 4 is 4 units plus another 2 units, therefore the vertical distance is 6. Again you look at the run and say, what is the distance between 1 and negative 3? Well, the distance -3 to 0 is 3 units and then another 1 unit which is 4. Therefore my rise over run is 6 over 4. With this method you still have to look at the shape of the graph and decide whether the gradient is negative or positive and a lot of people don't like this method because of that.

Comment:

Terry had switched to this method in order to better clarify the concept she was teaching. She later taught this method explicitly and gave her leaners opportunities to practice it. This ability to spontaneously switch from one representation to another was common practice in Terry's lessons. This skill does not necessarily grow with experience only but also with confidence because the teacher knows the concepts she is teaching that well. Terry's knowledge gave her the confidence to venture out into multiple representations. From what was observed the other teachers would wait to bring in a new method until it was time to teach it under its own heading. Terry was also the only teacher who used real life representation in her teaching. To illustrate the meaning of the word gradient or slope, Terry projected two pictures, one of a steep mountain and another of a flat hill.

Comparing Lily's and Terry's responses to item 2

The approach used by each of these teachers is very similar to the approaches used to respond to the first item. Lily does not write or say too much but does offer a direct response which shows a sound understanding of the concept under discussion. Her response is in a way similar to Terry's response because they both state that the line has a positive gradient and that it is not easy to tell from the given information whether it has a gradient of 2, however, it is possible that the line could have a gradient of 2.

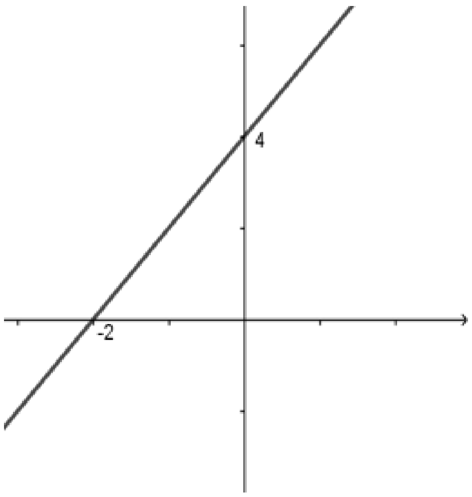
The difference between Lily's and Terry's responses is that Terry goes into an in-depth explanation to substantiate her claim that the line is positive and that it can have a gradient of 2. Both Lily and Terry had enquired if they were expected to talk out loud during the taking of these items and they had been informed that they should be free to do what they felt comfortable with. Both had started off reading and writing quietly, however, Terry began to talk aloud and to use gestures as if explaining the concept both to herself and to someone she could see. Explaining the concept in different ways was Terry's way of ensuring that her explanation was consistent and mathematically sound.

It was clear that both Terry and Lily had good SMK of the concept they were explaining, however, their individual approaches differed. Terry was thinking more like a teacher in her response and it can be further argued that she was using SCK to analyse the question and pre-empting areas of confusion and possible learner misconceptions. It can be argued however, that Lily approached the question using CCK which did not require her to engage much with the question as did Terry.

Neither Terry nor Lily mentioned the angle of inclination in their responses to item 2, however, Terry did give her learners a worksheet which contained lines at 45 and 135 degrees. She went on to inform her learners that 45 degrees indicated a gradient of 1 while 135 degrees signified a gradient of -1. Terry later gave her learners an assignment to do a concept map and some of her learners indicated this concept in their summary. Terry did allude to this representation in her explanation using the quadrants. A positive line (line with positive gradient) creates an acute angle with the x-axis in an anti-clockwise direction. This knowledge is taught in analytical geometry in the upper grades (FET), however, there is nothing stopping teachers in the GET phase from using it to show learners the difference between a positive and a negative linear graph. This is what Shulman (1986) was referring to when he spoke about teachers needing to have vertical knowledge of the subject matter. Ball et al., (2008) refer to this as horizon content knowledge (HCK).

5.4 Teachers' responses to item 3

This item presents a problem in which a learner gives an incorrect equation to a linear graph and the teacher is expected to identify the source of error. Teachers are expected to possess a different kind of knowledge (SCK) which is embedded in skills to enable them to analyse learners' mathematical productions for the sources of errors and misconceptions.



Cebo proudly shows Mr Liao his solution: $y = -2x + 4$. Being unsure of Cebo's source of error Mr Liao draws another graph and asks Cebo to find the equation:

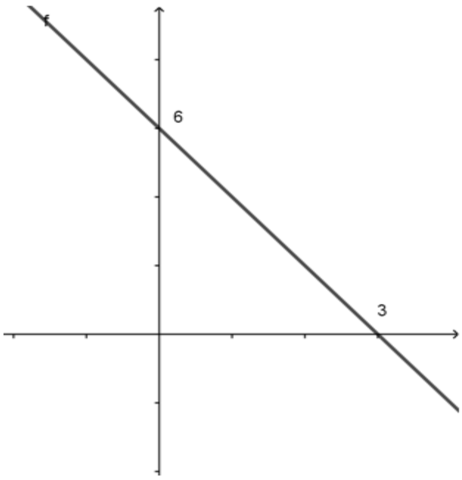


Figure 5-4: Item 3

Cebo's solution: $y = 3x + 6$. If you were Mr Liao, how would you explain to Cebo what he is doing wrong with the hope that he does not make the same error in future?

5.4.1 Analysis of Amanda's response to item 3

I would introduce the method of first explaining what the equation means $y = mx + c$. *Then tell him to first find the gradient* $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-0}{0-3} = -2$

Then tell him in order to find y *let* $x = 0$, *to find* x *let* $y = 0$

In the observed lessons Amanda spent a lot of time teaching about the equation of a linear function. She emphasized the equation of the gradient $m = \frac{y_2 - y_2}{x_2 - x_1}$ and on making x zero in order to find the y-intercept. In her response to this item, Amanda focused on the second part of the problem and on the procedure for calculating the equation of a linear graph. She did not make any attempt to analyse the learner's initial solution as per instruction. In her teaching Amanda emphasized mastery of procedure to find the equation of a linear graph. She made attempts to make her teaching accessible to her learners by using an analogy of teams. The idea of teams is based on sports, like soccer and netball, which all the learners would have been familiar with. The excerpt from her teaching demonstrates this:

> Amanda: To work out the equation of the graph, we start by choosing teams.
>
> Amanda writes the two coordinates with the help of the learners, always shouting in chorus or finishing off the teacher's sentences.
>
> $T_1(-6;-13)$
>
> $T_2(6;11)$
>
> Amanda: What do we use these teams for? Can you remember what I said we need them for?
>
> Chorus: To find the gradient!
>
> Amanda: Yes to find the gradient
>
> She continued to work out the gradient on the board
>
> Amanda: So now we have our equation as $y = 2x + c$
>
> Amanda: So now what do we need to do now?
>
> Random learner shouting: We need to find c

As is evident from this teacher-learner exchange, not much learner individual thinking was encouraged in the lessons which were teacher driven. Procedural knowledge was promoted over conceptual understanding, however, learners were given opportunities to write on the chalkboard and to do individual work in the workbook.

5.4.2 Analysis of Lily’s response to item 3

I would ask Cebo what the m-value in the equation y=mx+c stands for.
If he says gradient I will ask him how he found the m value and also how he got a positive value for an increasing and a decreasing value.

Lily responded by stating that she would ask the learner to explain what the value of *m* in the equation stands for and continue to ask the learner to show how they calculated a positive *m* value. As with Amanda, Lily’s response reflects her teaching style in the classroom. Lily’s lessons were characterised by interactions between the teacher and the learner and between learners themselves. These were in the form of questions and answers as well as comments. Learners engaged in a high level of mathematics from the workbook guided by the teacher. Although Lily also tended to focus on procedural knowledge in her teaching, her learners were exposed to a variety of problem-solving procedures and explanations provided by the teacher which were based on sound knowledge of the concepts taught. Learners themselves were expected to learn and demonstrate knowledge of these procedures through homework and class exercises. The following extract has been taken from the transcripts of video recordings to illustrate a typical lesson in Lily’s class and her strategy of handling errors. In this episode, this learner was giving his answers to a class exercise by writing on the board:

Andy : $3y = -6 + 9x$

$y = -2 + 9x$

Lily: can you write it in the form $y = mx + c$?

Andy (writes $m =$ and hesitates)

Another learner: Just write it as $y =$

Andy: oh!

$y = 9x - 2$

Lily has seen the error in this equation but ignores it as she normally does, hoping that one of the other seated learners will spot it.

Lily: so they say right at the beginning, group the equations of the graphs that are parallel

Lily: Andy, why do you think I asked you to write it in the form of $y = mx + c$

Andy looks unsure.

Lily: Tyla? (his hand is up)

Tyla: Mam because you taught us that in standard form we can read off the gradient and the y-intercept.

Lily: (Turning to Andy) in this equation m =

Andy: the gradient

Lily: and c?

Andy: the y-intercept

Lily: In this equation pointing at $y = -2 + 9x$, what is your gradient?

Andy: oh I made it -2

Lily: you didn't make it -2 because the gradient is the coefficient of x

Andy: I mean the position.

Lily: You could have still written it like this, you wouldn't be incorrect but I prefer it if you write it in standard form $y = mx + c$.

A hand goes up from one of the seated learners.

Lily: Why is Courtney's hand in the sky? What do you think she is going to query?

Lily: Lulu (she has raised her hand) why is her hand up?

Lulu: Mam because 9 divided by 3 is not 9.

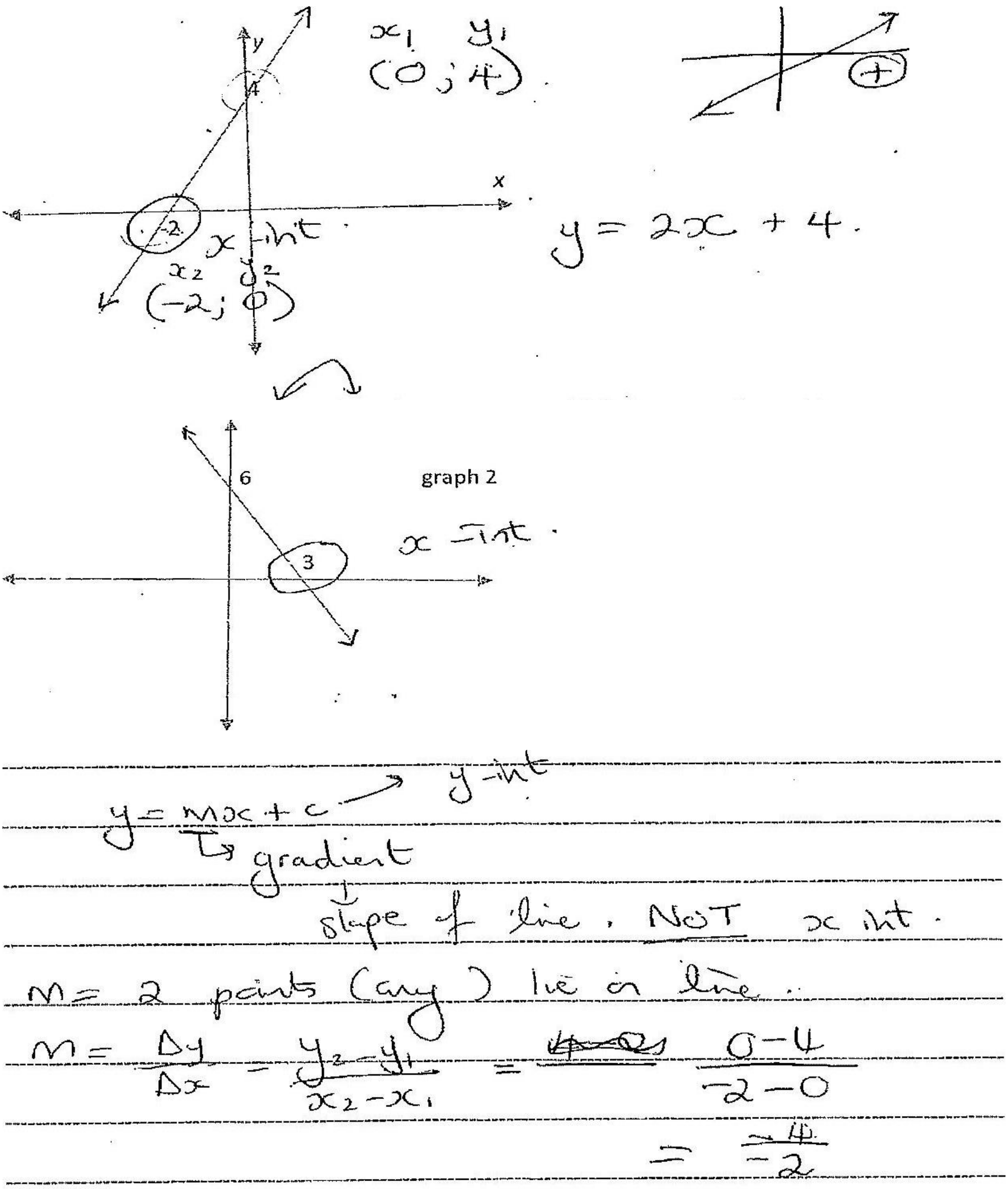
Lily: Correct, if you divided by 3, every term must be divided by 3.

Comment:

Lily does not immediately point out to the error made by Andy, but waits for one of the other learners to see the mistake before addressing it. In all the lessons observed, Lily used this strategy to address errors. In cases where no one spotted the error she would ask the class to study the production on the board and evaluate it, thus bringing their attention to a possible error. If the error was still not spotted, she would highlight it and would correct it together with the class. Of the four teachers observed, Lily's lessons had the highest level of learner participation.

A closer look at Lily's response to item 3 reveals that she can identify that the learner does not have a clear understanding of the gradient (she was using CCK). That being said, however, Lily's explanation does not indicate that she was able to analyse the source of error correctly. The learner's mistake is consistent on both productions. It shows that his thinking is that m is replaced with the x-intercept. Lily does not comment on this error.

5.4.3 Analysis of Terry's response to item 3



After looking at the problem briefly, Terry exclaims that this is a common error and continues to say, "most of them assume that the *x*-intercept is the gradient, that's what they assume, but you've got to start by writing down that $y = mx + c$, where m and c stand for something. Most importantly, you've got to make them understand that m stands for gradient and what the gradient means. The gradient is the slope of the line and how I explain slope is, you're climbing a hill or a mountain, and c is the y-intercept. Now what I can see here is that he understands the concept of y-intercept because he has put in the 4 and he has put in the 6 but the problem [is that] he is assuming that the gradient of the line is the *x*-value, the *x*-intercept. He doesn't understand that the *x*-intercept is just a coordinate of the line that we can use to find the gradient of a line. What we have to do is to break down and explain that m is the gradient, the slope of the line and not the *x*-intercept".

Comment

At this point Terry is explaining what error the learner has made. She has correctly analysed the mathematical production and, not only does she point out that the learner's production is incorrect (CCK), but proceeds to identify the source of the error (SCK). Terry also makes a statement that this is a common error (KCS). The next extract is Terry's explanation aimed at helping the learner understand what he did wrong, what he should have done instead, and how to avoid making the same mistake in future (KCT). Terry proves to be the only teacher who correctly interprets the instruction on the item. Her explanation resembles her classroom teaching which is characterised by constant questions and answers, emphatic explanations, demonstrations and tips on how to check for mistakes in order to ensure that the answer is correct.

Terry's explanation continues:

> "What is an *x*-intercept? It is where the graph cuts the x-axis. So we can say yes this is the *x*-intercept" Terry circles the -2 and the 3 while she speaks. "But to find the gradient, (depending on what level the student is at,) to find the gradient we will have to find the slope. <u>Can you see the difference in the slope?</u> This one is going upwards from left to right while the second one is going down (using gestures). So surely there must be a difference in the slopes, the one is higher the one is lower and we can deal with the differences in the slope, but most important thing, (depending on the student) we've got to teach that gradient is rise over run and some of the students can automatically see it and they use rise over run but what I prefer is to say that to find the gradient we use two points. Any two points that lie on the line. I believe in using the formula because the formula is something you can always go to. So we have to use the formula and show that *m* stands for gradient. Most people think that *m* stands for something else, they

automatically assume that m stands for something like midpoint but we say that ‘no’ the small *m* stands for gradient.

The gradient of a line is the change in y over the change in *x* which is where we get the rise over run, what we say it as $y_2 - y_1$ over $x_2 - x_1$. If a student can automatically read off the difference of let’s say this is 4 over 2 (tracing an L shape from the y axis to the *x*-axis in the first diagram), and see that the gradient is negative, that is fine. But if they can’t, we always say, stick to what you know. Now I always say to find the gradient you need any two points on the line, makes no difference what those points are. Now because this is an *x*-intercept, we write the coordinates as (-2; 0), the y-value if you look along this line (tracing on the *x*-axis) will always be zero. Same thing applies for the y-intercept, if you go along this line, the x-values will always be zero. No matter what you decide, it will depend on the person. If you make that y_1 that will have to be x_1. This is important, you will never have $x_1 y_2$ or $x_2 y_1$ it has to be the same number so this is $x_1 y_1$; $x_2 y_2$. Now we’re taking that and putting it into our formula.”

She then writes this on the paper. “If you look at this graph on top you can see that it has a positive gradient, this one must have a negative gradient. If the gradient is not negative, then check again you must have made a mistake.

The formula works but it has to be systematic and I always say to them do you see that the slope is positive 2? Let’s look at the line, do you see that it has a positive slope? Therefore I must have a positive gradient (she draws a Cartesian plane with a positive line and writes a plus sign). How do I know? I then go back to the graph and see that I have a positive slope.

This graph is $y = 2x + 4$. A lot of the time they see this *x* as the *x* value of the *x* intercept but it is not! The *x* is standing there, it is a function which mean it is y with regard to *x* and *x* with regard to y. We will never get rid of those, we always have to have an *x* and y. The *m* stands for gradient and not the *x* point given on the graph”.

Comment:

Once more Terry demonstrates a very good understanding of a linear function. As in her previous explanations, Terry is not content with giving a vague or superficial answer, but goes deeper as if talking to one of her own learners in the classroom. She once again pre-empts what the learner is likely to do wrong, like pairing the x and the y coordinates incorrectly. In class during one of her observed lessons, she spent a considerable amount of time teaching one of her learners how to form a pair of coordinates using $x_1 y_1$ and $x_2 y_2$. This lesson took place much later in the year after she had written the item above. This is an indication that at the time of the writing of this item, Terry was aware of this common error and was anticipating that any learner would most likely make this error. Amanda’s analogy of teams seemed to have helped her learners when it came to choosing coordinates. When these learners were given individual work to complete in class, they did not seem to struggle with choosing and arranging coordinates to calculate the gradient.

The next discussion pertains to how the three participants responded to Item 4. Ball et al. (2008) state that as part of Knowledge of Content and Students (KCS), teachers need to know what makes learning of certain topics easy or difficult. In designing this item, it was borne in mind that each teacher would interpret the question based on their own teaching experience.

5.5 Teachers' responses to item 4

Mrs Sishi wants to introduce simultaneous equations to her Grade 8 class. Which one of the following set of graphs would you recommend she uses for her introduction? Explain

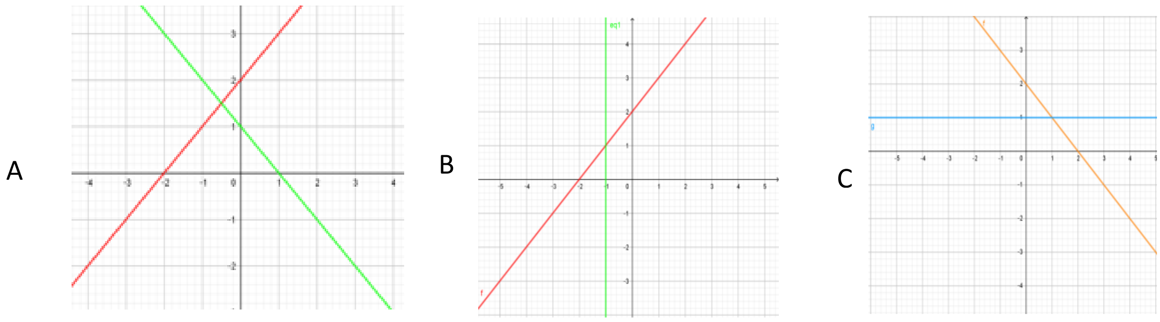


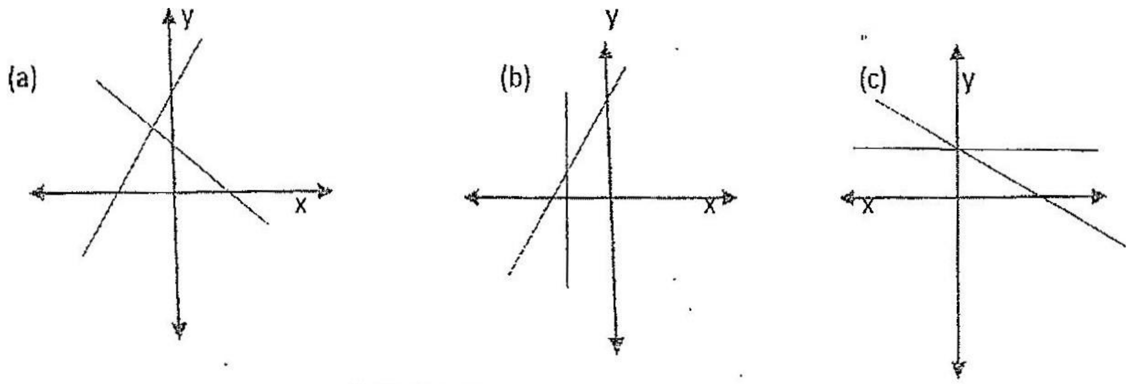
Figure 5-5: Item 4

5.5.1 Analysis of Amanda's response to item 4

B & C because both of these graphs each has a constant axis therefore it is easy to compare them and solve them simultaneously.

Amanda's response is that graphs b and c would be more effective to use for introduction of simultaneous equations because these contain a vertical and a horizontal line. She refers to these graphs as 'constant axes'. In her teaching, Amanda did touch on these constant graphs as they appeared in the learners' workbooks, however, she did not elaborate much on their equations or on how these equations relate to the general formula for linear graphs, $y = mx + c$ These graphs were introduced as specific functions, $y = 1$ and $x = 1$ rather than in their general form $y = c$ or $x = a$. Amanda's conclusion that these graphs are easy to compare with graphs of the form $y = mx + c$ where neither m nor c is zero is based on her own encounter with these graphs which appeared to be easy during teaching. On the contrary both Lily and Terry seemed to believe that learners struggle to grasp the concepts of vertical and horizontal lines and Terry referred to these as 'special lines'.

5.5.2 Analysis of Lily’s response to item 4

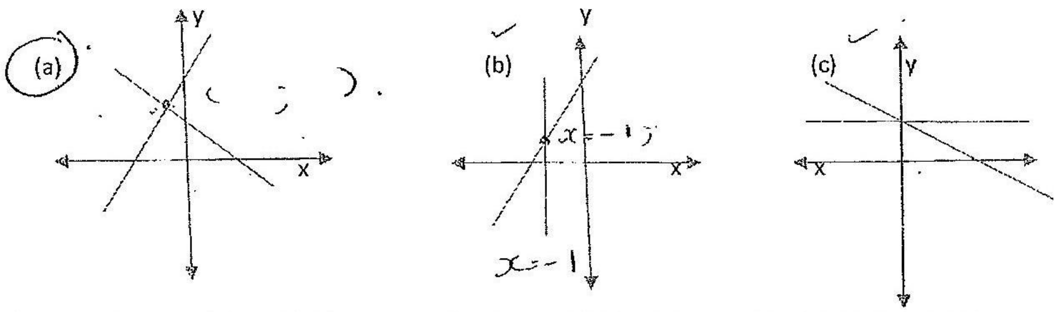


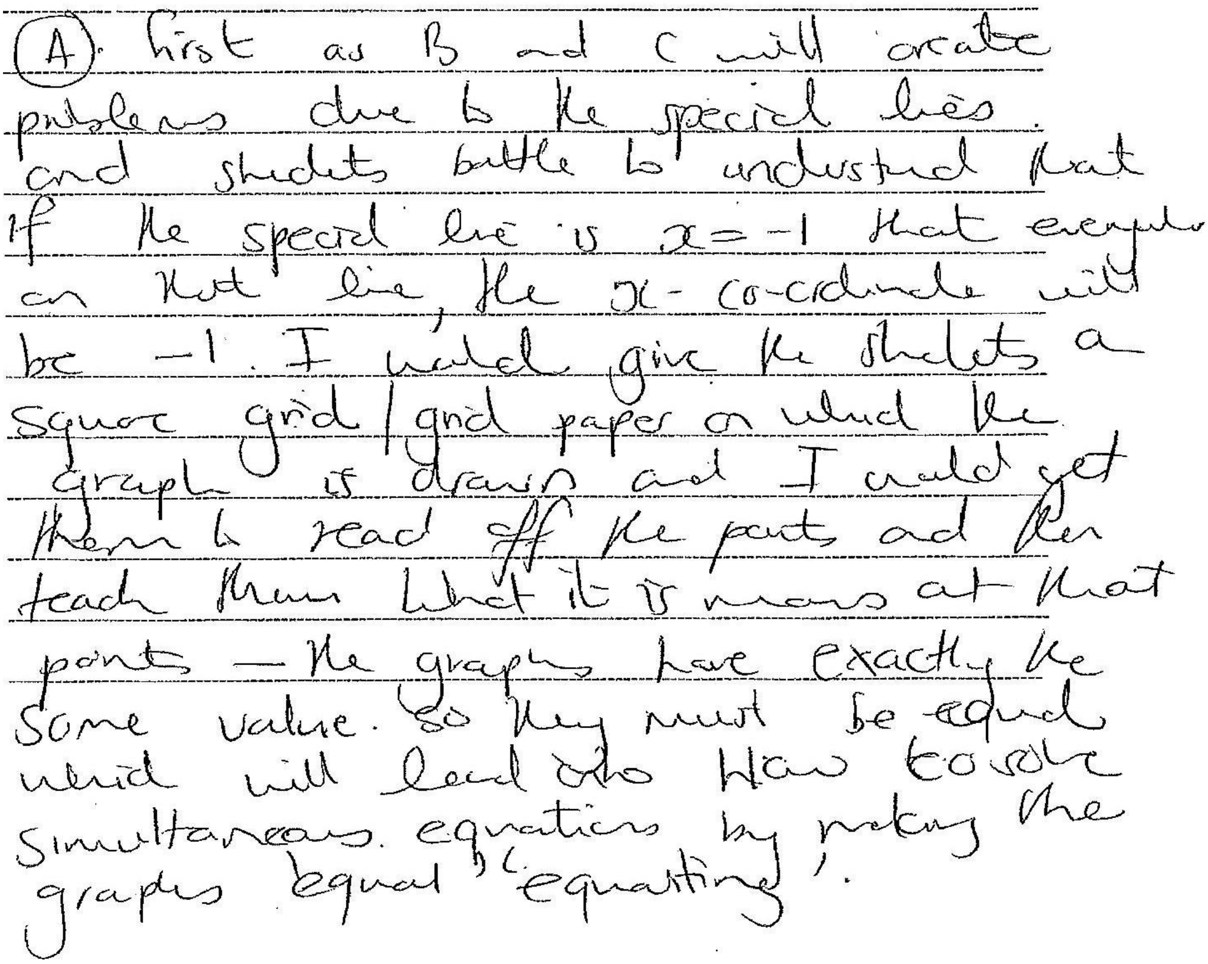
a) because it looks the more difficult one to me. But the weaker kids struggle with vertical and horizontal lines.

For the brighter kids I would start with c as the y-intercept is an easy concept and it is simple to make the connection.

Lily’s response is a bit difficult to follow because she starts off by stating that she would advise Mrs Sishi to use the graphs in (a) for the introduction of simultaneous equations because to her this looks more difficult. One would hope that teachers would pick easy examples to use in their introduction of new concepts or topics. Lily continues to state that the weaker kids struggle with vertical and horizontal lines, a sentiment shared by Terry. She concludes by writing that for the brighter kids she would start with c as the y-intercept is an easy concept and it is simple to make the connection. Based on this response, it is clear that Lily does have an idea that some learners find the concept of vertical and horizontal lines difficult to understand, however, she has not put much thought into the sequence of teaching these concepts. This also came up in our post-lesson discussion after her introduction of linear graphs. I questioned Lily why she had chosen to teach the drawing of oblique lines before vertical or horizontal lines and in her response she had indicated that she had not put much thought into the sequence of teaching these lines.

5.5.3 Analysis of Terry's response to item 4




(A). First as B and C will create problems due to the special lines and students battle to understand that if the special line is $x = -1$ that everywhere on that line, the x-co-ordinate will be -1. I would give the students a square grid / grid paper on which the graph is drawn and I would get them to read off the points and then teach them what it is means at that points — the graphs have exactly the same value. So they must be equal which will lead into How to solve simultaneous equations by making the graphs equal 'equating'.

Terry's response is that graphs in (a) should be used to introduce simultaneous equations as (b) and (c) would create problems due to special lines and learners battle to understand that if the special line is $x = -1$ that everywhere on that line the x-coordinate will be -1. She continues to explain that by giving learners grid paper to draw graphs she would get them to find the parts where the two graphs meet and then continue to teach them that those are points where the graphs have exactly the same value, so they must be equal. That will lead to the teaching of simultaneous equations by making the graphs equal 'equating'.

The problem often arises when teachers have to explain horizontal and vertical lines because unlike $y = mx$ which still obeys $y = mx + c$ because it can be written as $y = 2x + 0$, so it is clear that the input x is multiplied by the gradient 2 and zero is added to the product to obtain the output value. It is different when $y = c$ is given. The reason for this difficulty is that it is not clear for many learners what the input is since there is no x variable. Terry's explanation in one of her lessons was that the equation obeys the rule $y = mx + c$ because m is zero which creates $y = 0x + 4$ and therefore $y = 4$.

In her introduction to linear functions while using the *Geogebra App*, Terry was aware that learners generally battle with horizontal and vertical lines. In her design of the investigation in each case the variable under investigation was varied while the other variable was kept at zero, for example when investigating the effect of m in $y = mx + c$, m was varied while c was kept constant at zero. In order to investigate how horizontal lines are created, Terry cleverly varies c while keeping m constant at zero as demonstrated in this next investigation:

Instructions

a) Move the m slider (red one) to 0. Draw a rough sketch of the graph on Diagram

b) Keep the m slider at 0 and now change the c slider (blue one) to 1 and draw a rough sketch of the graph on Diagram C.

c) Again, keep the m slider at 0 and now change the c slider to -l and draw a rough sketch of the graph on Diagram C.

Diagram C

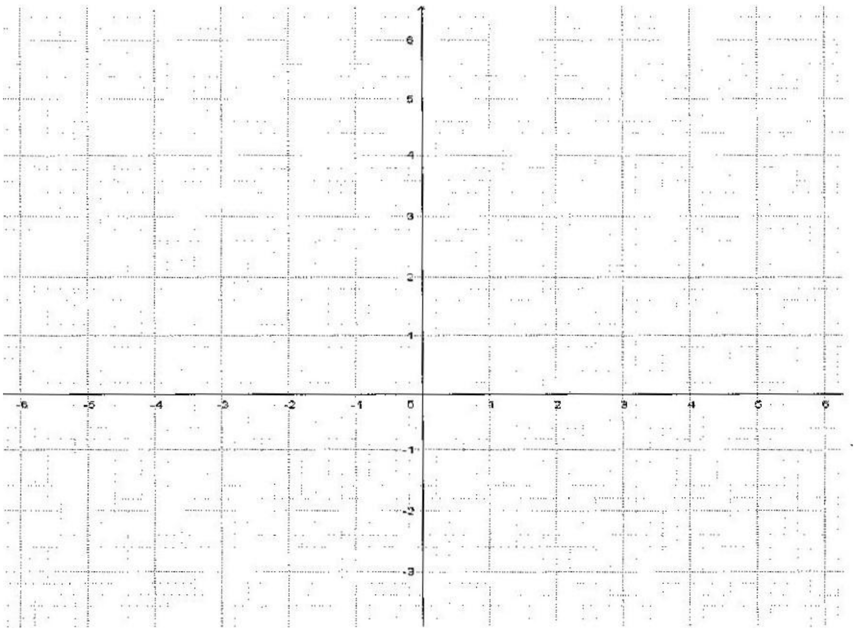


Figure 5-6: Geogebra investigation 1

During this investigation learners were able to draw horizontal lines and to associate them with $y = 0x + c$. Terry later taught this concept using change in y over change in x thus showing that the change in y is zero hence the gradient of zero.

5.6 Teachers' responses to item 5

Item 5 presented two parallel lines and the participants were expected to demonstrate their knowledge of the concepts taught in grade 9 relating to gradients of parallel lines. Both Lily and Terry showed competence in their SMK and in their teaching of parallel lines. Amanda did not respond to this and latter items.

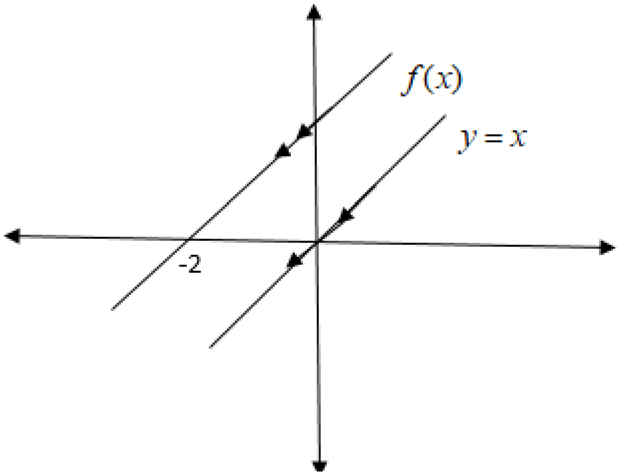


Figure 5-7: Item 5

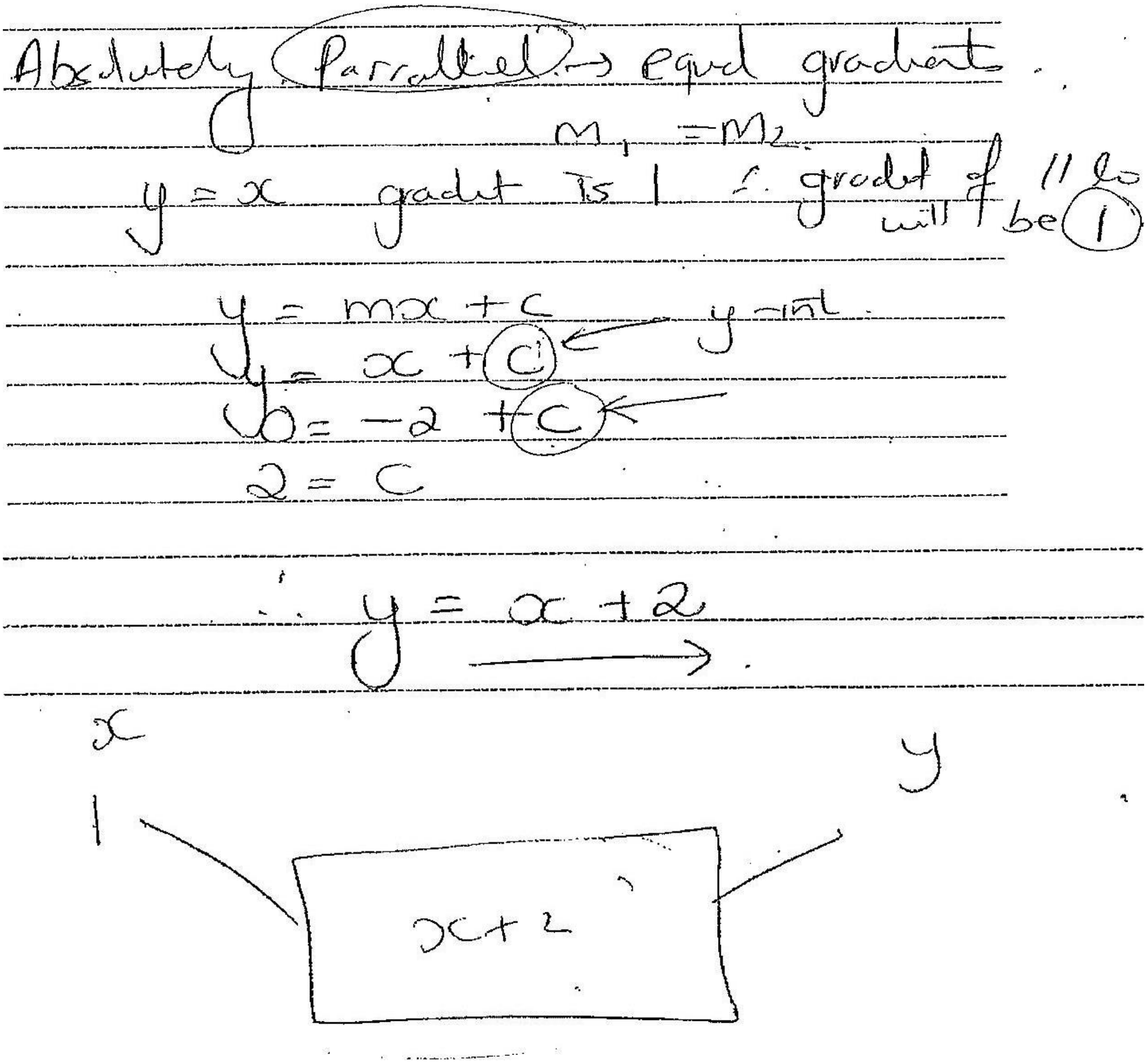
A straight line defined by the equation $y = x$ and another straight line $f(x)$ are drawn on the same set of axes parallel to each other, $f(x)$ intercepts the x - axis at -2. Is there enough information provided to find the equation of $f(x)$? Explain.

5.6.1Analysis of Lily's response to item 5

yes. The graphs have the same gradient of 1.
Once you have fand the gradient, substitute
the point (-2;0) into my favourite equation
y=mx+c ☺ and find c.

As she had done in her approach to previous items, Lily writes very little and goes straight to the point. Her response is accurate, however, Lily makes no effort to explain why both graphs have the gradient of 1. Later on in this analysis chapter, it will be shown how Lily approached the teaching of parallel lines in her lessons.

5.6.2 Analysis of Terry's response to item 5



Terry makes an attempt to give more detail than Lily, and tries to link her response coherently to the information provided in the question. Once again, Terry thinks like a teacher, imagining that she was explaining to a learner and hence the need to break things down and to make clear links. She makes a point of showing that $m_1 = m_2$ for parallel lines before proceeding to show that because $y = x$ for one graph, the gradient of both graphs must therefore be equal to 1. She then continues to find the equation of $f(x)$ by substituting a point. Terry's response is consistent with what was said in earlier discussions that she communicates with the question at a level of SCK. Her thinking approach is systematic, as if pre-empting what a learner would be thinking and addressing possible questions or concerns in advance. Terry's approach to all items is consistent and indicates that she is thinking like a mathematics teacher. Even though she is not actually teaching a lesson, her approach to the items reveals her passion for teaching and her desire to explain mathematical concepts in a manner that is clear and leaves no room for ambiguity. Lily's responses are short and precise and are consistent with her idea of doing mathematics for fun or therapy rather than linking it to real life. This excerpt is taken from the interview transcript:

> H: What is your idea of what a function is?
>
> Lily: It's a graph and you draw a graph and you answer all the little questions about that graph.
>
> H: If you want to apply it to real life?
>
> Lily: I hate that maths, I hate maths that's applied to real life. I love maths that's never practical, that's why I can't stand the questions that are related to a picture (real life context) and I ignore the picture, I just want to do the maths.
>
> H: What is your strategy in teaching then, when you teach functions or any other content, what is your strategy when preparing to teach a concept?
>
> Lily: To show them that when you have this equation, you can make it into a picture (graph) and you can move the picture around, up and down, left and right, and how that will affect the equation, and so we are just playing around with the maths, as soon as we take maths out of the classroom and put it into real life, I've lost interest because one of the reasons I love maths is because maths obeys all the rules and there's no problem, that's why it's like a therapy to me. So I do maths for the therapy side.

5.7 Teachers' responses to item 6

Kayla, a girl in your grade 8 class brings you a parabolic graph of the type drawn below and wants to know how to find its gradient. What explanation would you give Kayla.

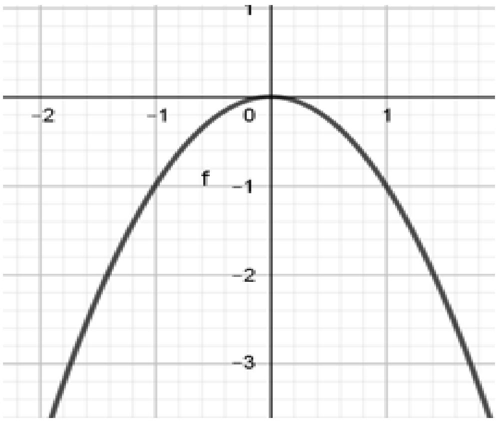


Figure 5-8: Item 6

Like item 3, this item was also designed with learners' misconceptions in mind. As was shown in Amanda's case, if the concept of gradient is not properly explained, learners develop misconceptions which result in errors in their mathematical productions. This excerpt shows how these misconceptions can emanate from teaching:

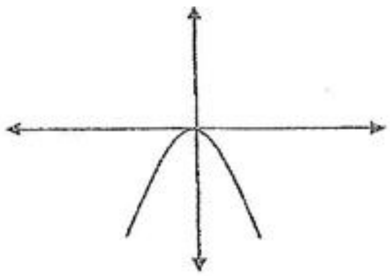
> Amanda: You will come across situations where you calculate your gradient and you find that it starts off as zero then one then minus one. If that happens don't worry, you are not doing something wrong, it means your graph is not linear but another type (pointing at a parabola on the board).
>
> Amanda: do you understand?
>
> Chorus: Yes!

This statement was made during a lesson in which contrast was used to explain the difference between a linear function and a parabolic or non-linear function. This statement is misleading because it gives the impression that all functions have gradients, and the process of drawing any graph begins with finding the gradient. If the gradient is constant, the graph is of a linear function and if the gradient is not constant then the graph is of another type other than a linear function.

Only Lily and Terry responded to item 6 and their responses are presented and discussed in the next section. Both these teachers use their vertical knowledge of what is taught in latter grades to respond to this item. The item presented a case in which a grade 8 learner needed assistance with finding 'the gradient' of a parabola. It should be borne in mind that equations of parabolas are taught in grade 10 and therefore only a teacher with knowledge of parabolas would be able to answer this question adequately. This item was designed with learners' misconceptions in mind. What was observed in Amanda's lesson as demonstrated in the previous extract supports the thinking behind the design of this item.

5.7.1 Analysis of Lily's response to item 6

Lily begins by asking questions which aim to point out that the process of calculating the gradient of a linear function is not the same in parabolic functions. While Lily's explanation is accurate, she does not consider why this grade 8 learner is asking this question. Since this topic falls beyond the scope of grade 8 syllabus, it would have been more appropriate to first ascertain what the learner's thought process was especially because it has been shown that misconceptions do arise in this area. It is also not clear if this learner knows what a turning point is. Lily's explanation is fit for a learner in grade 10 who has been introduced to concepts surrounding a quadratic function. During the writing of this item Lily made a comment that one of the grade 10 learners had asked if in the equation $y = ax^2 + x$, a stands for the gradient. This further strengthens the case that learners do not always understand that a gradient is a property of linear functions, average gradient can be calculated for parabolas in grades prior to grade 12. In grade 12 learners are introduced to the concept of a derivative.



How do you work out the gradient of a straight line?
Can we apply that equation to this graph?
Do you think the gradient remains constant?

I think we would reach the conclusion that you can work out the gradient over small intervals to be most accurate. And that the gradient is constantly changing.
At the turning point it changes from a ⊕ to a ⊖ gradient.

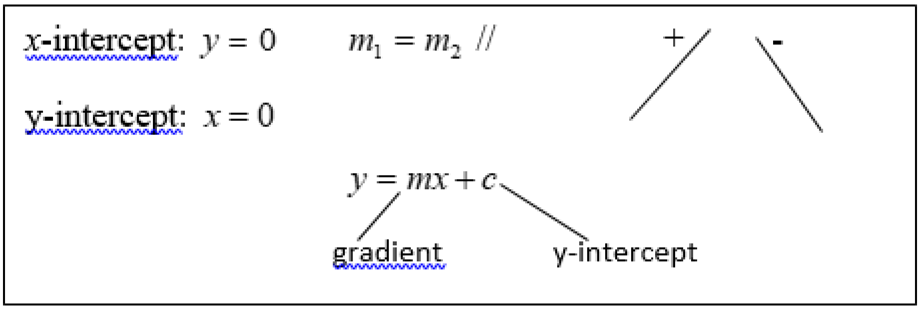
Terry’s approach is very similar to Lily’s. Both these teachers miss out on the opportunity to find out the basis of this question by a grade 8 learner. Base line assessment is foundational to any teaching activity and is linked to KCS and KCT. During lesson observations however, both teachers would begin their lessons by writing a summary of key concepts on the white board and would often end their lessons in a similar fashion. This is shown in the extract that follows: Learners had been calculating the x-and y-intercepts and drawing graphs and at the start of each lesson, Lily would begin with a summary of these important concepts:

Lily: Tell me everything you know about linear functions

L 1: On the x-axis, y equals zero and on the y-axis, x-equals zero

L2: the point where x and y meet is called the origin

Lily writes on the board:



5.7.2 Analysis of Terry's response to item 6

Terry also begins by stating that a parabola is a dynamic graph. She continues to explain that the learner will have to draw a tangent to the graph to find the gradient or find average gradient. This graph will not have a set gradient as it changes. First it is an increasing function and then changes to a decreasing function. Where does it change? At the turning point. Finding the average gradient is a much simpler explanation to a grade 8 learner than drawing a tangent. To draw a tangent and finding the gradient of that tangent linking it to the gradient of a parabola requires knowledge of the derivative taught in calculus. Since the question is about finding the gradient of the parabola, one would have to find the function for the derivative at any point not just the equation of one tangent. Like Lily, Terry does not consider the age of the learner asking the question. Both these teachers however, use their vertical knowledge in their explanations. SCK should also be about understanding the level at which the learner operates and offering explanations appropriate to their cognitive level. In the lessons observed, neither Terry, nor Lily, taught concepts beyond grade 9 level. However, Terry was mindful of how certain content was taught in the latter grades and emphasized the need for learners to stick to methods they would encounter in grade 10. This sentiment she also shared in the interview:

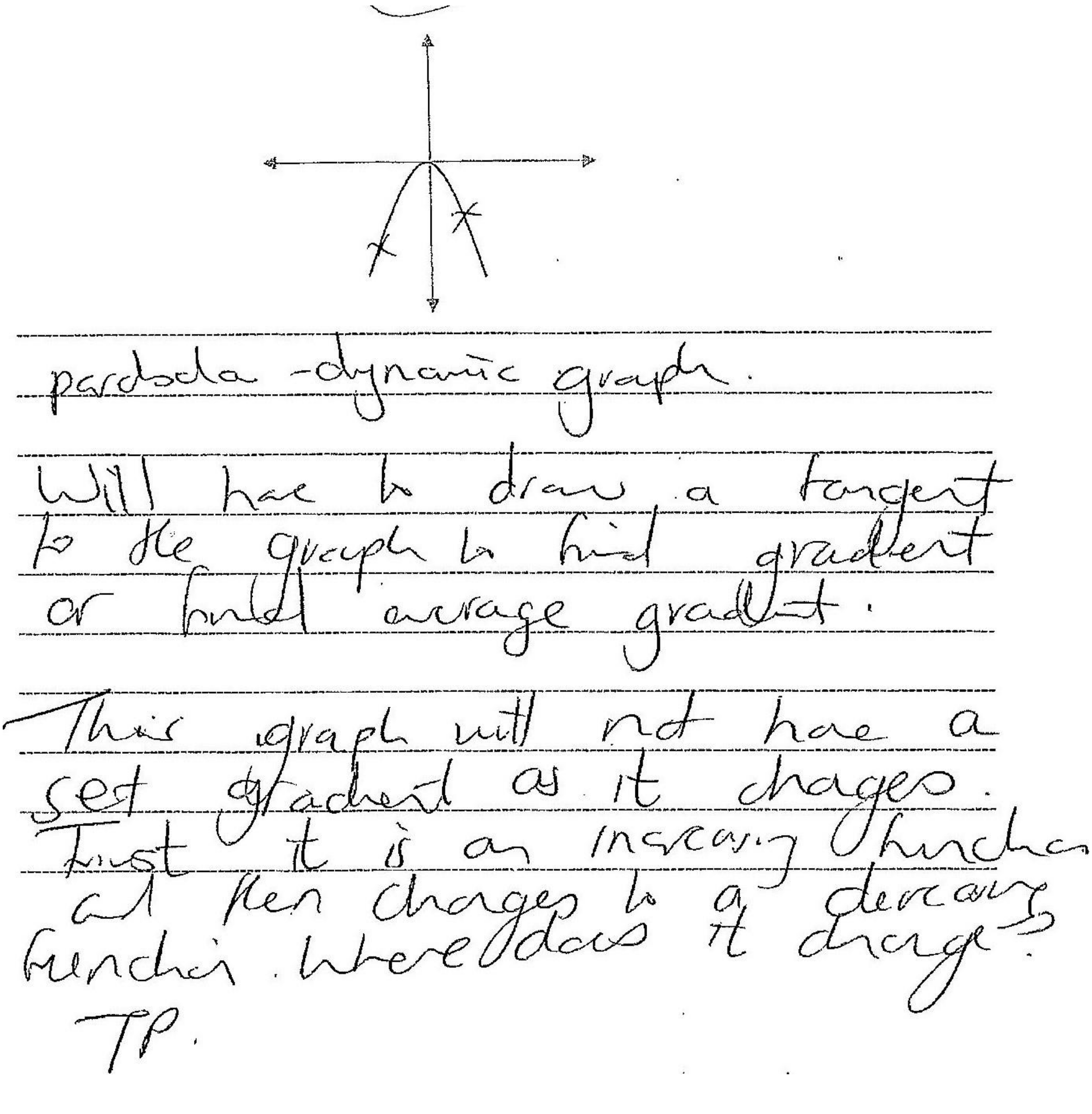
H: In your own opinion, how much knowledge must a teacher have?

Terry: I think it's really important that teachers know the subject matter in the previous and the next grades. Especially for grade 9 when we teach straight lines, I understand that the kids should be taught the gradient intercept method but the kids will never use it again so when we get to grade 10, 11 and matric, the kids only use the dual intercept method.

> So you've got to know where you're going forward to and we often have a teacher come to us and say oh I want to do it like that and we say but we never do it like that when we get to grade 10, 11 and 12 and ultimately we want to be teaching kids so that they can go on to do Maths Core. So the teachers need to know where the subject is headed. What helps is having someone higher up in the department saying "we are going to teach it like this because this is how we go forward".

This way of thinking was also observed in the classroom:

Using the white board Terry writes down $y = mx + c$ and draws an arrow from m and labels gradient and draws another arrow from c and labels it y-intercept. She then draws an increasing line on a Cartesian plane and next to it writes: $m > 0$ and draws a decreasing line on another



set of axes and next to it writes: $m < 0$.

> Terry: What does greater than zero mean?
>
> Chorus: Positive
>
> Terry: Yes, I use inequalities because in form four (grade 10), that's how we teach.
>
> Terry: Why did I not write greater or equal to zero (writes $\geq$) ? Reece?
>
> Reece: Because zero is a special case
>
> Terry: Correct, $m = 0$ is a special line, it is a horizontal line.

5.8 Teachers' responses to item 7

This item required teachers to show their ability to link functions to other topics taught in the GET phase. There is a need to re-enforce the teaching of functions by linking it to other topics which may include measurements, finance, the number system, ratios and proportions etc.

7. Sam was practicing Mathletics at home in his computer and wants to know why his answer was incorrect to one of the questions. The question required him to identify the graph of $y = 2x$, where x is an integer. Two of the 4 options are shown below, one of them is correct. How would you explain to Sam what the difference is between the two graphs, and why his choice is incorrect?

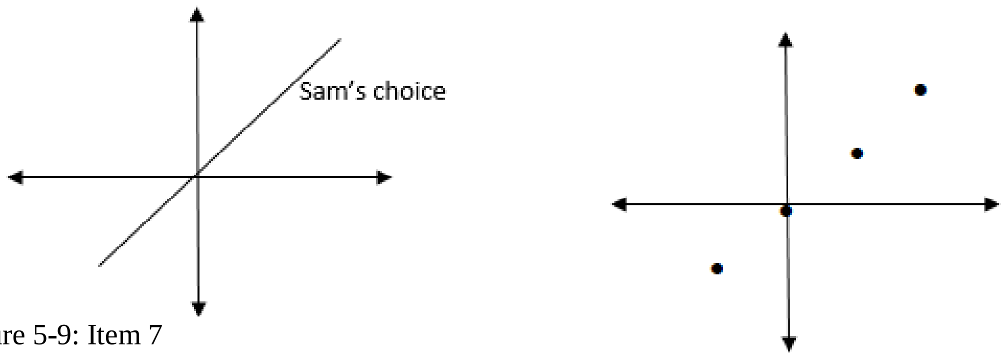


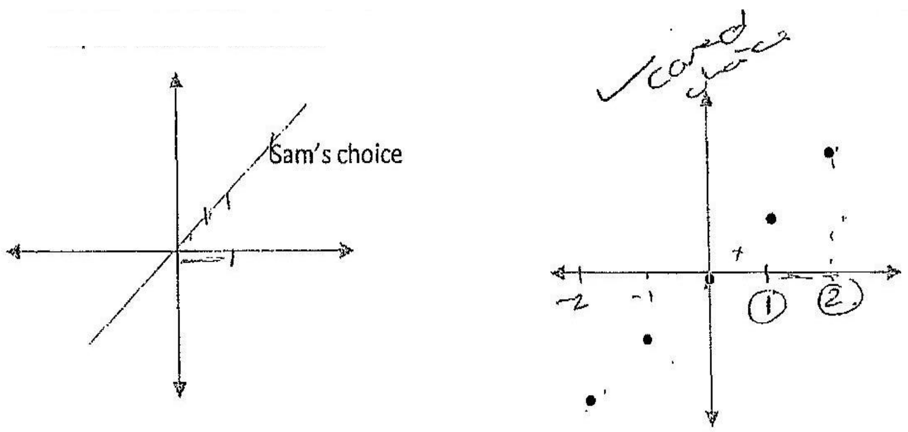
Figure 5-9: Item 7

5.8.1 Analysis of Lily's response to item 7

I like Sam's choice, but x is an integer so does not include fractions or decimals but has to be ⊕ and ⊖ whole numbers.

Lily's response again shows that she is well acquainted with the topics and concepts taught in the GET phase. Although her answer is short, it is also precise and includes a definition of integers. Lily, however, does not answer the question which required her to explain why a learner's choice was incorrect and to point out the differences between the two graphs. Once again Lily thinks at a level of any person who knows or uses mathematics (CCK) because her answer could have been given by anybody who is not a teacher but understands mathematics. In the classroom this learner would not have understood why his answer was incorrect based on Lily's explanation. Lily could have pointed out the difference between a solid line, which shows continuous data versus points which indicate discrete data.

5.8.2 Analysis of Terry's response to item 7



$y = 2x$; x is an integer.

Sams choice $y = 2x$ BuT . x is real number/ Rational number.

Terry like Lily demonstrates an understanding of the difference between the two graphs. She shows that Sam's choice cannot be correct because his graph represents the domain of real numbers which also includes rational numbers which are not integers. Unlike Lily who does not show any evidence of having given much thought to the question, Terry engages with the question and talks to herself in the process. Terry like Lily does not explain the difference between the two graphs even though both teachers show understanding of the concepts posed in the scenario.

5.9. Teachers' responses to item 8

Like item 4, this item also dealt with vertical and horizontal lines. Teachers were asked to state how they would explain to a class the difference between a line with a zero gradient and one for which the gradient is undefined. Both Lily and Terry responded to this item.

8. How would you explain to a learner why GRAPH A has a gradient of zero while GRAPH B's gradient is undefined?

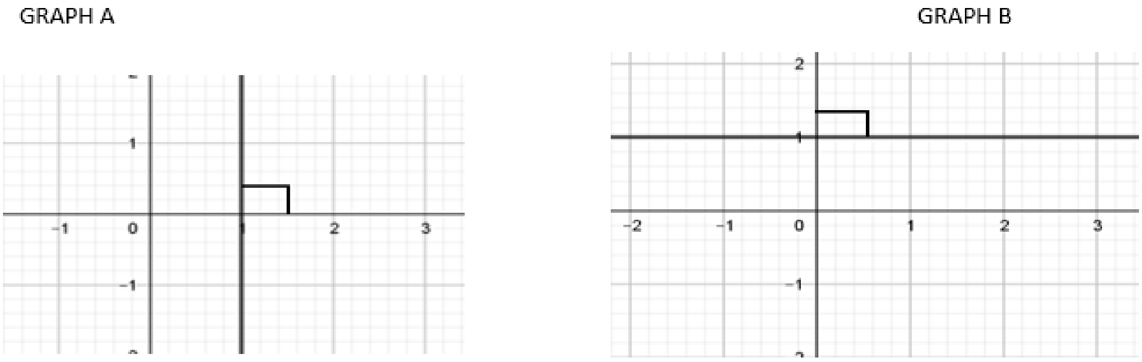
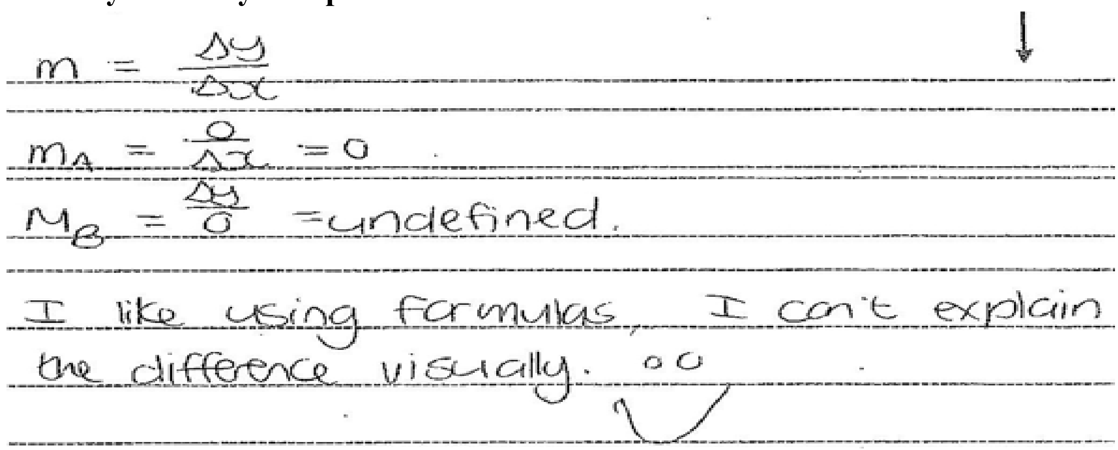


Figure 5-10: Item 8

5.9.1 Analysis of Lily's response to item 8



$m = \frac{\Delta y}{\Delta x}$

$m_A = \frac{0}{\Delta x} = 0$

$m_B = \frac{\Delta y}{0}$ = undefined.

I like using formulas, I can't explain the difference visually.

Although Lily can demonstrate why a horizontal line has a zero gradient using a formula and that a vertical line is undefined because of a division by zero, she clearly states that she cannot explain the difference visually. At the completing of this item, after noticing her hesitation and the time it took Lily to respond compared to the previous items, I decided to interact with her to find out more about what she was thinking at the time:

H: Are these the kind of questions you think about when you plan your lessons?
Lily: …this does not interest me, I don't care what the gradient is
H: What do you mean you do not care what the gradient is?
Lily: I don't care that the gradient is zero, I always go for the easiest thing like y equals or x equals. So what the gradient is does not make a difference because you don't need it.
H: If you explain to a learner the concept of gradient, do you ever use vertical or horizontal lines as your starting point?
Lily: I think I never would
H: Why not?
Lily: Because to me they don't have a gradient. Which is probably not a correct thing to say. I always think of gradient as rise over rise, the change in y over change in x. So because these don't have that, I don't think of them, I would rather do a slope. Because a slope is more visual
H: How would you relate the steepness or shallowness of a slope to a vertical or horizontal line?
Lily: I would await that question if it comes.

Consequently, a question about the slope of a vertical line did come in one of her observed lessons. Lily had randomly chosen an example and asked learners to give her two points that might lie on the line. With the help of the class, she started calculating the gradient, using the two points: The two points were (-11; 1) and (-11; -5).

$$m = \frac{1-(-5)}{-11-(-11)}$$

Lily: David what is one minus negative five?

David: positive six

Lily: and negative eleven minus negative eleven?

Another learner: zero!

Lily: for a vertical gradient they actually say the gradient is undefined

Andy: Why mam?

Lily: because you are getting a zero at the bottom

After this Lily immediately instructs the class to move on to another exercise.

Lily's attitude and lack of knowledge surrounding the teaching of slopes of vertical and horizontal lines resulted in a poor explanation to a learner's question.

The use of 'they' when referring to the content in the text books was quite common in the lessons observed from all four participants. It was as if the teachers were distancing themselves from ownership of the content presented. Lily does this as well when she refers to the gradient of vertical lines in the previous extract. Mudaly (2015) found that pre-service teachers' poor understanding of the concept of gradient was linked to their inability to form links between the gradient, shape of graph, ratio of the changes in x and y values, angle the line makes with the x-axis and the sign of the gradient value.

Unlike Lily, Terry did take time to explain about gradients of special lines. She showed the class that although the equation of a horizontal line was $y = 4$, the equation still obeyed the rule $y = mx + c$ because m is zero which creates $y = 0x + 4$ which resulted in $y = 4$. Terry also made use of analogies in her explanation:

> Terry: A horizontal line has zero gradient (moving her hand from left to right, indicating a flat surface). I think of it like walking on a flat surface (walking in straight line). There is zero effort needed for me to do this whereas climbing Kilimanjaro, because it has a steep slope (showing steep slope with her hand) now that would require a lot of energy.

The analogy used by Terry had been mentioned in the interview prior to classroom observations:

> H: Does it surprise you that most learners struggle with functions?
>
> Terry: Kids can't see functions, I explained it using a factory and they finally understood it. They can't see the relationship between the graph and the function. I often tell them to use the table. They get the idea and then they forget it a week later. I make them write notes. One of the boys came up with something more interesting, he said it's like climbing a wall, it's impossible to climb a vertical wall but when you walk in straight line, it takes no effort.

Comment:

The overall observation of how teachers teach or understand vertical and horizontal graphs was that there is a lack of application knowledge or knowledge of how these lines relate to real life contexts. This may be the reason why learners find these concepts difficult to grasp. The analogies used by Terry of walking in a straight line to illustrate a line with zero gradient and of climbing a vertical wall to illustrate an undefined gradient seem plausible at face value and one can understand how a learner can think this way. However, a closer analysis of these analogies reveal a certain level of inconsistency with the definition of a gradient. These inconsistencies are discussed below:

Discussion of Analogy 1

The very act of walking implies that there is change in distance covered over time, therefore the gradient cannot be zero since the gradient here is the ratio of the change in the distance covered over time. The drawing of the horizontal line on a Cartesian plane contradicts the analogy given by the teacher in the explanation. The line shows that the object is stationery because the vertical axis which displays the distance covered shows that there is no change in the distance, therefore the rise is zero. In terms of the distance/time relationship, a horizontal line shows a stationery object.

Discussion of Analogy 2

The idea of climbing a wall reflects a change in position from point A to point B on the wall. This change takes place over time, therefore there is rise over run present which means the gradient cannot be undefined. Secondly, the idea that climbing a steep wall is associated with something difficult or an impossible motion is misleading. A very steep slope shows a motion that happens rapidly. More distance is covered in a very short space of time, hence the steepness of the slope and the shape of the graph. This is in contradiction with Terry's analogy which implied that steep means difficult, in fact in kinematics, a steep slope indicates that the object is moving faster and therefore this motion cannot be difficult or impossible. The vertical line can be described in terms of what happens to the object at an instant or at that particular point when $x = a$.

The two analogies at face value do give learners the ability to differentiate between the gradient of horizontal line and that of a vertical line and may even achieve the intended result of helping them remember how to draw these graphs. This teaching however lacks the depth needed for conceptual understanding and is likely to create gaps between concepts taught in mathematics and those taught in other subjects. The Physical Science syllabus covers extensively graphs of motion as one of the topics. Mathematics teachers in collaboration with their peers who teach Physical Science can come up with a repertoire of ideas to teach the concept of gradient in linear functions effectively. Instantaneous rate of change, which is an example of a vertical line with undefined gradient, is a concept not only covered in Physical Science but also in calculus in the latter grades, hence the need for teachers to be able to link horizontal and vertical knowledge in their teaching (HCK).

The study did not observe the teaching of the concept of inclination of a line as this is taught in grade 11 in analytical geometry. Trigonometry is only introduced in grade 10 and the angle of inclination is linked to a tangent function. This vertical knowledge would empower teachers to offer richer explanations in their teaching of grade 9 linear functions. Terry did introduce her learners to the basic angles of inclination 45º and 135º and gave her learners simple exercises to practice this knowledge. When a line is drawn at an angle of 45º, all points that lie on the line will have x and y coordinates that are equal, therefore rise over run will always be equal to 1. The line will have an equation $y = x$ and the same can be shown of a line drawn at an angle of 135º which has a gradient of -1. Using this knowledge as a starting point, learners could also be shown that 90º represents a vertical line where the run is zero, hence the calculations of gradient will reveal a division by zero, hence an undefined gradient.

Another evidence of an undefined gradient is when one approaches a vertical line from a point of a moving object. Since the horizontal axis represents time, when one draws a vertical line, there is an indication that there is a change in distance without a corresponding change in time, in other words, time is standing still. Teachers could explain to learners that this represents an impossible scenario, hence the gradient of a vertical line is undefined. Using the same scenario it can be shown that the gradient of a horizontal line is zero because the line represents a stationery object but not an impossible scenario.

5.9.2 Analysis of Terry's response to item 8

Using change in y over change in x, Terry demonstrates that while the run changes by 3 units, the rise remains constant at 2, hence the gradient of zero. On the vertical line she also demonstrates that the run stays constant at 2 while the rise changes by 6 units. Since the ratio requires that the run be the denominator and the change in the denominator is zero, the result is an undefined gradient. Like Lily, Terry uses the formula in her explanation. Terry's explanation is more concrete and practical as she uses real values to calculate the rise over run, while Lily's explanation is more theoretical. Both teachers show a good understanding of how this formula works.

5.10 Teachers' responses to items 9 and 10

The last two items were related to composite functions, a concept not taught in the GET, however, teachers were expected to demonstrate their level of SMT in the way they responded to these items. Both Lily and Terry said that although item 9 was familiar to them, they had not come across a graphical representation of composite functions similar to item 10 before. By using their knowledge of functions, both teachers were able to work out the answer to a question they had not encountered prior. Teachers encounter this experience on a daily basis in their teaching as they are expected to think on the spot and make sense of learners' questions and opinions.

9. Two functions $f(x) = x^2$ and $g(x) = 2x - 1$ are given. How would you work out $f(g(2))$?

----

10. Two functions f and g are given graphically below. How would you work out f(g(2))?

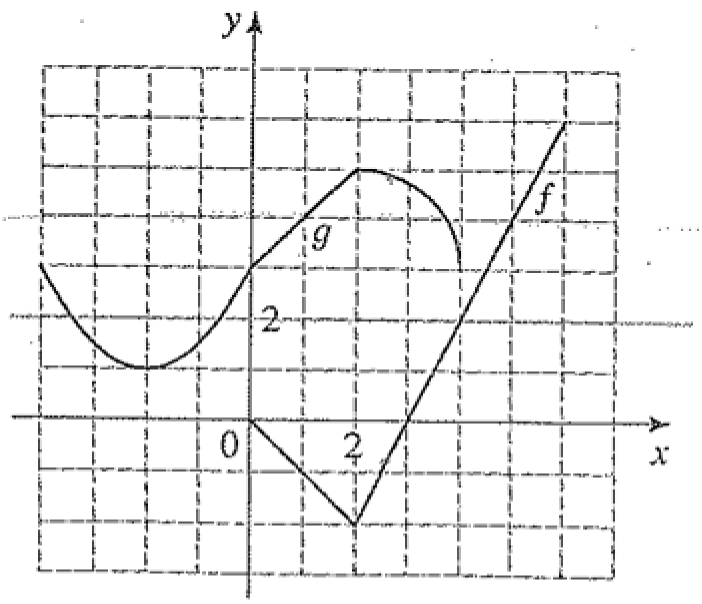
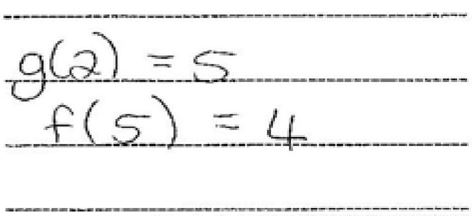
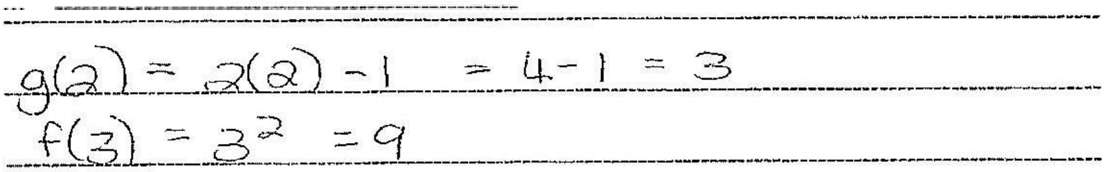


Figure 5-11: Items 9 and 10, Engelke, Oehrtman & Carlson (2005, p.4)

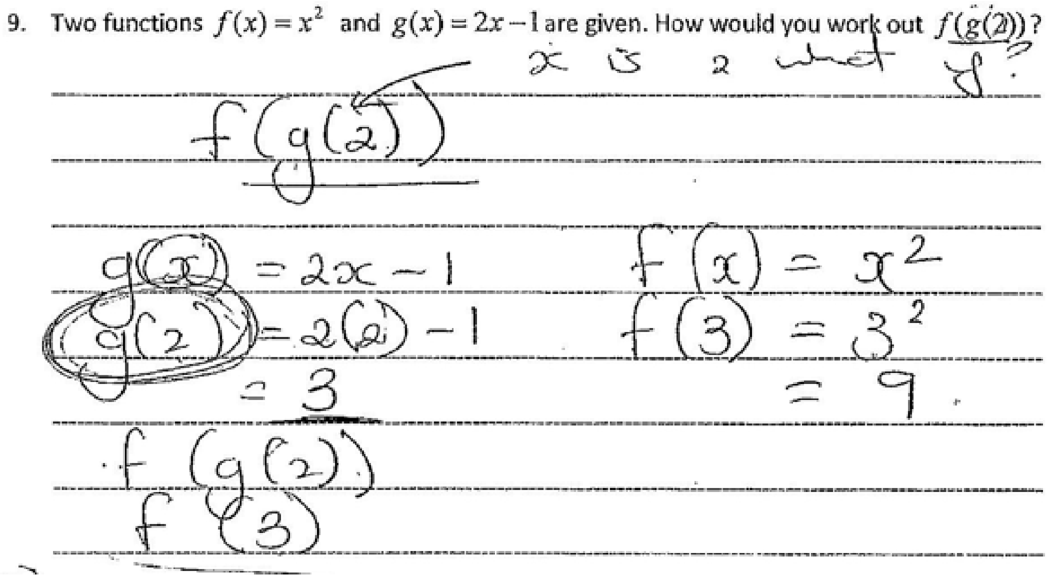
5.10.1 Analysis of Lily's response to items 9 and 10





Lily gives her answer in two lines which indicates that the thought process was internal. She had expressed that she does not see the need for lengthy responses when one can get straight to the answer in a few steps. To her this is what maths is all about: solving a problem without having to relate it to a real life context or give lengthy explanations. The items did not require the participants to offer any explanations, therefore there is nothing wrong with Lily's approach to these items.

5.10.2 Analysis of Terry's response to items 9 and 10



Terry arrives at the same answers as Lily, but shows a step-by-step solution to the problem. She starts by asking a question: x is to what y? She answers item 9 similar to how she would teach a learner. She shows that there is a link between $g(x) = 2x - 1$ and $g(2) = 2(2) - 1$. She then demonstrates that the input of $f(x)$ is the answer to $g(2)$. She uses the same reasoning in her approach to item 10.

Chapter summary

This chapter has been a presentation and analysis of data from the three participants' MKT obtained from pen-and-paper items and transcribed video recordings of classroom observations. The analysis also included data obtained from interview transcripts. The analysis has revealed that the participants' knowledge observed from the analysis of written items is consistent with knowledge observed in teaching. Data also shows many similarities between what the participants claim or believe about the teaching of mathematics and what was actually observed in the lessons. Further summary of these results will be presented in chapter seven.

In the next chapter, I continue to present and analyse data which include data obtained from the fourth participant. This data will provide results from classroom observations, interview transcripts and document analysis.

CHAPTER 6: Further analysis of data

6.1 Introduction

This chapter presents an analysis of video recordings and document analysis. This includes the analysis of the fourth participant's teaching. Most of the analysis on teachers' MKT and how this was translated into classroom teaching was covered in the previous chapter. The purpose of this chapter is to offer a holistic picture of the contexts and classroom dynamics. The chapter ends with an analysis of documents used by the participants including learner textbooks and workbooks. The aim of this document analysis was to investigate the effect that the use of these documents might have had on the quality of instruction. The use of these documents was analysed making use of variation theory.

6.1.1 Brian

Brian taught at a rural school and his class consisted of 28 grade 7 learners from an IsiZulu language background. Most of the teaching was done on the chalkboard using different coloured chalks. In his teaching Brian used both English and IsiZulu to explain concepts. Learners were familiar with this bilingual teaching and they also switched between IsiZulu and English when they answered questions. The teacher did most of the talking and would often stop to ask if learners understood before carrying on talking. Learners mainly answered in chorus unless the teacher insisted on hands being raised. Brian's learners sat in pairs facing forward with the desks arranged in rows. This seating arrangement was observed in all four classrooms. The lessons began with the teacher writing notes or homework on the chalkboard which would take between 20 and 30 minutes. These lessons were an hour long. The stationery that was visible on top of each learner's desk consisted of a textbook, a note-taking book, a calculator and a writing pen. Like Amanda and Lily, Brian allowed learners to write answers on the board. Calculators were also frequently used to do calculations.

Brian had opted not to do the pen-and-paper items as he did not feel confident about answering the questions posed. Brian had indicated in the interview that he had good knowledge of functions and that his previous learners had enjoyed learning this topic. The following interview extract shows that Brian was familiar with some of the concepts covered in the items, however, he still did not feel confident enough to attempt any of the questions:

> H: And did you use any sort of graphs during the teaching of functions?
>
> Brian: Oh yes we did and the graphs were another thing that they enjoyed, so we did do graphs and the increasing, decreasing, the linear, the nonlinear, we did do that and they really enjoyed it.

In the lessons observed, Brian's class was well-managed, with learners always keen to offer answers to questions posed by the teacher. Learners were not observed to ask any questions in these lessons. Brian taught formulae and procedures to solving problems and learners used their calculators to do even the most basic calculations. I shall give an illustration of a typical exchange that was observed in most of Brian's lessons:

The following table has been drawn on the board:

Position	1	2	3	4	5	?	10
Term	12	22	32	42	?	82	?

Brian: Remember when we spoke about the constant difference? What did we say it is?

Learner 1: Ayishintshi (it does not change).

Learner 2: It is the number that does not change, it stays the same throughout the sequence (speaking in IsiZulu).

Brian: Yes, the constant difference stays the same, in other words ayishintshi! Injani?

Chorus: Ayishintshi!

This definition of a constant difference is not conceptually accurate, however, both the teacher and learners had an understanding of what the constant difference does to a sequence and they could use this concept to generate linear sequences and to answer routine questions.

Brian's class was quite familiar with the inverse operations required to solve problems relating to functions. However, there is evidence to suggest that most of these procedures were memorised and did not involve much conceptual understanding.

Brian: What is the constant difference in this sequence? (Referring to the table drawn on the board).

Learner 3: It is 10.

Brian: What would be the next term be?

Learner 1: 52.

Brian: For term 82, what would be the position?

Learner 4: 8.

Many hands go up

Learner 4: 92.

Learner 5: 102.

The debate starts in the class, some learners saying the answer is 92 and others saying 102. The teacher allows this debate to continue for a considerable amount of time and it is not clear from his behaviour whether he agrees or disagrees with any of the answers shouted out by learners who are frantically punching numbers on their calculators and shouting out answers.

> Brian: How will we find the position given the term? What is the rule?
>
> *It is not clear whether the learners understood the question or not.*
>
> Brian: How did they find the position given the term? (keeps repeating this question a number of times while moving around the classroom).
>
> Brian: What did they multiply or add?

The class had no problems with filling in the table up to the 5^{th} term. From then on the confusion begins and problem-solving ability is required, since some numbers in the sequence have been skipped. The first learner who had given 8 as the answer was correct and showed the ability to think logically as this answer is not obvious from the table. It is not clear whether Brian knew that 8 was, in fact, correct. From a non-response from the teacher, this learner started to doubt herself and changed the answer to 92. By engaging this learner, Brian would have enriched the learning experience of the other learners in class. This leaner's explanation of how she had arrived at the answer could have changed the course of this lesson, instead, too much time was wasted in this exercise with the lesson not going in any direction.

The teacher experienced challenges with pacing the lesson and too much time was spent waiting for learners to respond. The issue of 'wait' time has been addressed by researchers in the past (Shulman, 2004; Dillon, 1985). It has been shown that increased 'wait' time can either negatively or positively influence learner engagement. When used in higher level questioning, increasing 'wait' time is beneficial. However, if too much 'wait' time is allowed, it can also yield adverse results as the lesson loses its momentum. The type of questioning methods used also have an effect on learner achievement in mathematics. Asking more demanding or higher level questions has a positive outcome on learning and learner achievement in any area of learning, but even more so in mathematics.

Learners in Brian's class were enthusiastic and used their calculators with confidence, however, a closer inspection into the actual activities done in class revealed that not much individual and critical thinking was taking place. This does not mean that learners were not learning any valuable skills as a class. The question is that of learning with understanding the concepts being taught and being able to apply this knowledge in other settings without the help of a teacher. It is not clear how much, if any, conceptual knowledge taught in class would allow these grade 7 learners to participate individually in mathematical discourse. The effect of Brian's knowledge deficiency is demonstrated in the following extract from a lesson following the one illustrated in the previous discussion.

In this lesson, learners were given an activity after much time had been spent by the teacher, writing down notes on the board. The lesson progressed as follows:

> Brian: (reading from the board). The price of a certain number of DVDs = number of DVDs times price of one DVD, the price of one DVD is R65. Rewrite the formula and complete the diagram.
>
> Brian: What is the formula? Price of a certain number of DVDs = number of DVDs times price of one DVD.

The teacher keeps repeating the same question for quite some time while moving around the classroom, expecting learners to answer. Eventually he writes down $y = R65x$. Immediately learners start using their calculators to complete the flow diagram.

The teacher continues to give them another table with only input values filled in and learners are instructed to work out the output values and, using their calculators, most of them cope well with this easy activity. Brian reads out the last question of the activity:

> Brian: You pay a total of R4 615. How many DVDs do you buy? How will we work it out?
>
> Brian: Lindiwe! (This is the only time that a learner is addressed by name)
>
> Lindiwe: We will say 4615 divided by 65 equals 71 DVDs.
>
> Brian: How many DVDs?
>
> Chorus: 71 DVDs!

Brian misses out on the opportunity to follow up on Lindiwe's answer and to use this as a learning opportunity for the class. Most of the activities done in class or for homework lacked depth, and were routine and similar in nature.

The one activity that was clearly challenging was given to the class with no discernment from the teacher about the cognitive level of the task. Learners had found this task too challenging to do at home for homework. The following day, Brian started the lesson by going through the homework which none of the learners had managed to do. The task read as follows:

The formula to convert from inches (imperial system) to centimetres (metric system) is: Number of centimetres = number of inches $\times$ 2,54

Learners were instructed to draw a square with each side 2 inches long. On top of the square, learners were to draw a triangle with base 3 inches long and a perpendicular height of 1 inch long.

With Brian's help, learners calculated the area of the square in inches. The next question stated: In the imperial system the area will be 4 $inches^2$, write down a formula that converts $inches^2$ to cm^2 .

More than 20 minutes of the lesson was spent with different learners coming up to the board to try and solve the problem. Eventually, at the end of the lesson, the teacher tells the learners that the answer is:

Number of inches = Number of centimetres divided by 2,54.

During the briefing session I discussed this answer with Brian and together we relooked at the question. He expressed his fear that his lack of knowledge might be having a negative effect on the learners' achievement. The absence of a teacher's guide put Brian in a predicament where he had to use his own knowledge, or lack thereof, to provide the answer to this task. By spending time studying the worked-out solution in the guide, Brian would have realised that this was a higher order question and would have not spent so much time expecting learners to answer it. Brian had memorised the procedure based on the previous class exercises done with the class. The quality of Brian's lessons was compromised because of his failure to use a teacher's guide during lesson preparation.

6.1.2 Amanda

Amanda taught in a township school and her Grade 9 class consisted of 28 learners from the surrounding neighbourhood. The seating arrangement was similar to that in Lily, Terry and Brian's classrooms. Like Brian, Amanda tended to switch between IsiZulu and English in her teaching. She mostly wrote on the chalkboard while learners used workbooks. Learners were given opportunities to write their solutions on the board with the help of the teacher and the rest of the class.

Amanda had given herself a 6 out of 10 in the interview, explaining that this score was due to her having only done one year of mathematics at university and having attended only Anglican workshops as form of professional development. For the pen-and-paper items, Amanda had either left some questions unanswered or displayed a limited understanding of functions.

Observation of her lessons showed that Amanda had good procedural knowledge of solving routine problems and her class was kept busy most of the lesson time. Time was wasted mostly when she had to write work and draw graphs on the chalk board. Amanda had good classroom management skills and learners responded well to her instructions. Learners had the freedom to collaborate with classmates close by. During my visits there would sometimes be noise coming from the next door classroom because of the practice for a cultural festival. Amanda and her class did not appear phased by this noise, as teaching and learning continued as usual.

Most of the lessons would begin with the marking of homework using a similar strategy to that observed in Lily's class where learners took turns to write homework on the board. Drawing graphs on the chalkboard proved to be a challenge and, in most cases, the line would be more of a curve than straight, but everyone understood that it was meant to be straight and sometimes Amanda would point out to the class that the lines were meant to be straight.

Although Amanda's explanations contained errors, at times, as illustrated in the previous chapter, her lessons were, however, well-planned and she made attempts to structure and pace her teaching. Most of the lessons began with the marking of homework and this often lasted for up to 20 minutes. The next phase would be to introduce a new concept followed by opportunities to practice the new content from the workbooks. Amanda would walk along the rows assisting learners as they completed the class exercises. Lastly, new homework would be given or learners would be instructed to finish off a class exercise for homework.

Amanda was also committed to improving her mathematics knowledge. In the interview she revealed that her qualification was in teaching Life Science, but she had found herself teaching Grade 9 mathematics because she had taken a mathematics module during her first year of study at university. Her love for teaching mathematics had grown and she had taken the initiative to improve her understanding and teaching of the subject through the Anglican Maths Teachers' Initiative workshops. These were designed to focus on sharpening teachers' knowledge related to various topics covered in the GET phase, as well as empowering teachers with knowledge of technology and problem-solving skills in mathematics.

Like Lily and Terry, Amanda also took time to consolidate the concepts taught so that learners would see the fusion of concepts and their interrelatedness as demonstrated here:

> Amanda: I think everyone was able to find the x and the y intercepts. The equation which we were given for the graph was $y = 3x - 1$. What does 3 represent? It is our m which is

our gradient. Do we remember that? In the equation $y = mx + c$, m represents the gradient and x is still x and c is the y intercept.

Learners keep saying, "Yes!"

Amanda: So now if we are asked for the gradient, there are two ways of calculating the gradient. Sometimes they can ask you to calculate the gradient in order to see that you can calculate the gradient from a given equation. Someone else might use the equation which shows the change in y over change in *x*.

In this equation $y = 2x + 3$, what is our gradient?

Chorus: 2!

Amanda: So you don't even need to calculate the gradient, remember that our equation is $y = mx + c$

The overall observation shows that required level of mathematics took place in Amanda's lessons despite her knowledge deficiency when it came to conceptual understanding of functions. The use of workbooks by learners, together with the teacher's commitment to ensuring that time was used optimally during the lesson, gave learners opportunities to engage in mathematics at a level required by the CAPS.

6.1.3 Lily

Lily taught in an independent suburban school and her class consisted of 22 grade 9 learners from around the city. Learners came from a diverse racial and cultural background with English being the only language spoken in the classroom. Lily's class has two whiteboards, whiteboard markers of different colours, an air-conditioner, an overhead projector, a teacher's table and learners' desks. Most of the work is done on the whiteboard with learners having opportunities to do exercises in their workbooks individually or in pairs. Occasionally learners would get up from their seats and walk to other classmates to discuss the work, or to check if they were on the right track. Learners also put up their hands to ask for the teacher's help. There is a lot of freedom in these lessons and most learners display confidence in answering questions and are not afraid to interrupt the lesson flow to ask questions or point out an error. It is also common practice in this classroom for learners to go up to the whiteboard and write their mathematical productions which could either be answers to homework task or class exercises.

Lily's competency in procedural knowledge was evident in all the lessons observed. Her learners were encouraged to come up to the white board and solve problems. She also monitored her learners' progress to see that they were competent in solving problems to completion. This was observed, for instance, when learners were given a linear equation and asked to draw a graph of that equation. Learners would have to use a suitable method of choice to generate coordinates and draw graphs with little help from the teacher as illustrated in these solutions:

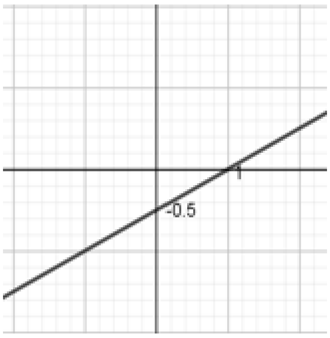
One of the boys in class works out the *x* and y intercepts as shown:

$$x - 2y - 1 = 0$$
$$0 - 2y - 1 = 0$$
$$0 - 0 - 1 = 2y$$
$$-\frac{1}{2} = \frac{2}{2}y$$
$$-0{,}5 = y$$

$$x - 2y - 1 = 0$$
$$x - 2(0) - 1 = 0$$
$$x - 0 - 1 = 0$$
$$x = 0 + 0 + 1$$
$$x = 1$$

Lily: Can you draw it on the graph?

Learner: Yes



Learner draws a sketch

Lily: Is everyone happy with what Tyla did? It's called suiting yourself because it is a rough sketch.

Unlike Terry, who always insisted that equations be written in standard form before attempting any calculations, Lily did not penalise her learners if they did not follow this procedure.

No use of any visual representations or manipulatives[7] was observed in Lily's lessons. Only one narrative was used in one lesson out of all observed lessons. This lack of use of alternative teaching approaches or representations may have been due to Lily's belief that mathematics is all about rules and following procedure, rather than simulation of real life.

Lily and Terry were the only teachers who made attempts to ensure equity by considering various abilities in their teaching. Lily's strategy for ensuring that all learners were catered for in her class was to allow those learners who did not struggle with the concepts taught to carry on with more challenging tasks from the workbook, while she kept her focus on the rest of the class. From the observed lessons, these learners who sat quietly at the back did not engage much in class discussions, however, they remained accessible to their classmates who would occasionally find their way to the back of the class to ask for help. Undoubtedly, the class could have benefitted more if at times Lily had asked these learners to explain to their peers their approach to solving problems. This would have brought about a different dynamic into the classroom. In the interview, Lily had explained her approach in an inclusive classroom:

> So with my Grade 9 class, with the kids that have got it, I will give them the memo, they're allowed to go on and do a ton of worked examples and carry on, and if they ever need my help then they can come and ask me. So in Grade 9 I think I tend to leave the clever kids a bit because there are way weaker kids, but I'm hoping I'm stimulating the clever kids, they have the memo to refer to and so they can actually teach themselves, good or bad?

6.1.4 Terry

Terry taught in a former Model C suburban school. Her class consisted of twenty-seven boys from different racial groups with a class average of between 85 and 91% each term and learners' marks in the class ranged from 75 per cent to 99 per cent on average. According to Terry, learners in this class are normally taught using a traditional style of teaching where the teacher explains the content to them for half to three quarters of the lesson and the learner's role is to passively take in information. My observation of lessons in this class did reveal a teacher-centred approach, however, learners were engaging with the content, with the teacher and with each other. Terry did most of the talking and spent a lot of the lesson time writing and explaining concepts on the whiteboard. Learners were given opportunities to practice the concepts being taught and, during this time, Terry would walk around asking and answering questions and marking learners' work. Terry's classroom set-up was quite similar to Lily's class, with visible furniture being the whiteboard, teacher's and learners' desks and an overhead

[7] Manipulatives in this paper refers to the use of concrete or physical objects to teach or explain a concept.

projector. All the boys had an IPAD which contained an electronic textbook and some learners also had a hard copy of the book.

Terry did a lot of work in the background. Her learners were given worksheets which they would complete at home. These contained challenging exercises, some of which included the work done in grade 10. She then provided solutions to the homework tasks and each one marked his own work. Terry also stuck the answers with step-by-step solutions on the back wall in the classroom. Learners were given permission to take these down and check them against their own work. If they still had problems with the worksheet, then it was up to them to approach her and ask for help. This strategy was almost similar to the one applied by Lily towards more capable learners.

Terry was very big on learners taking responsibility for their own work. Her learners wrote tests prepared by other Grade 9 maths teachers and they found these far too easy according to Terry who was thinking of setting her own tests. Terry's learners scored full marks in grade tests and her class average was normally above 90%. Terry explained that there were some learners in her class who did not easily grasp many concepts in mathematics, however, Terry was dedicated to seeing them reach a certain level of success in mathematics and these learners pushed themselves to perform at a high level. Terry had also informed me that some learners who had been recruited to join her class had refused to do so on the basis of too much pressure and high expectation to keep performing at a high academic level.

Terry did not experience any classroom management problems as her learners were kept busy the entire lesson time. As in Lily's class, these learners had the autonomy to move around and consult with friends, but unlike Lily, whose approach was always soft, Terry's approach was that of a strict teacher. She would often raise her voice when she was unhappy and all her boys were addressed by their surnames. The class was on task at all times and like Amanda, who carried on teaching through the noise coming from the next door classrooms, Terry's class also appeared not to be perturbed by the intercom making announcements from time to time.

Like Lily, Amanda and Brian, Terry also emphasized the mastering of procedures. She emphasized the need to always write the linear equation in standard form and provided the class with many opportunities to help them practice this skill. I present six equations to demonstrate this. Learners were instructed to rewrite the equations in standard form and give the gradient and y intercept in each case.

Equation 1: $y = 2x + 4$

Equation 2: $y = -x + 8$

Equation 3: $2y = x + 4$

Equation 4: $-y = x + 2$

Equation 5: $4y + x = 6$

Equation 6: $y - 2x + 8 = 0$

Like Lily, Terry portrayed a good understanding of the subject matter. Her teaching was systematic and well-planned. Part of her teaching strategy was to spontaneously come up with problems for learners to do in class as a way of driving a point home or ensuring that learners had a proper understanding of a particular concept or mastery of a procedure. These equations were written on the spot. Terry had come up with these equations within a few seconds, after realising that some of the learners were still struggling to write equations in standard form. One can see how cleverly Terry tries to achieve her teaching goal with these equations.

An overall analysis of Terry's lessons showed her to be a competent teacher who had the ability to transfer her SMK into PCK. Terry also made use of analogies, counter examples and also used other creative ways to help her learners to achieve mastery of procedure. One of the methods she used to help learners remember to simplify correctly, was to use the analogy of fairness. In rewriting $2y = 4x - 8$ for instance, she would emphasize that the 2 must be divided fairly to all the terms. Terry was also very strict in the way she expected the boys in her class to present their solutions. She explained to her class that $y = 4 - 2x$ is not in standard form and she would not be accepting it as the answer. Likewise, Lily preferred her learners to present their solutions in standard form, however she used a softer approach than Terry, as demonstrated in this quote: "You could still have written it like this, you wouldn't be incorrect, but I prefer it if you write it in standard form $y = mx + c$". Terry's strict approach was consistent with what she had said in the interview about her teaching style:

> Terry: A large part of why I'm a maths teachers is because of my high school teacher, she was amazing, absolutely amazing. She never raised her voice, she was calm and always collected. You did it a certain way but [if] it wasn't done that way, that was it, don't expect to get a mark. So I kind of have that approach. If you get it done and it's not how I would like it done, I tear it up or put a red line through it and say, do it again!

> Because maths, I believe, is systematic, it's logical, so if you're going to be writing here and everywhere, nothing is going to be systematic, so it's sink or swim.

To introduce linear graphs, Terry used technology in the form of a dynamic and interactive GeoGebra App. The App allowed her to create sliders to change values of *m* and *c* which allowed learners to make observations and conclusions. At the start of the lesson she explained to the class why they had to do the work themselves.

> Terry: Educational theory states that if a kid does the work themselves, they learn better than if a teacher has to do it for them.

This method was new to these learners and this is the reason Terry started by explaining to them the benefits of learning through self-discovery. Learners were seated in groups of four facing each other with their IPads. Worksheets were handed out and the procedure for the investigation explained. Figure 6-1 is an investigation worksheet given to learners at the start of the lesson to explain the purpose of the investigation and to introduce important terms and concepts. Figure 6-2 is one of the worksheets used by the learners to investigate the effect of *m*, the gradient of a line.

<u>Objectives of the Investigation and Task:</u>

1. Student is able to describe the general equation of a straight line graph.
2. Student is able to read off the values of m and c when given the equation of a straight line graph.
3. Student is able to describe what the property m stands for and its function in a straight line graph.
4. Student is able to describe what the property c stands for and its function in a straight line graph.
5. Student is able to apply the properties of m and c in answering questions about straight line graphs.

<u>The Straight line Graph</u>

The straight line graph is a linear graph which is formed by joining, with a ruler, two or more points together on the Cartesian plane.

Example of a straight line graph: $y = 2x + 2$ is drawn below.

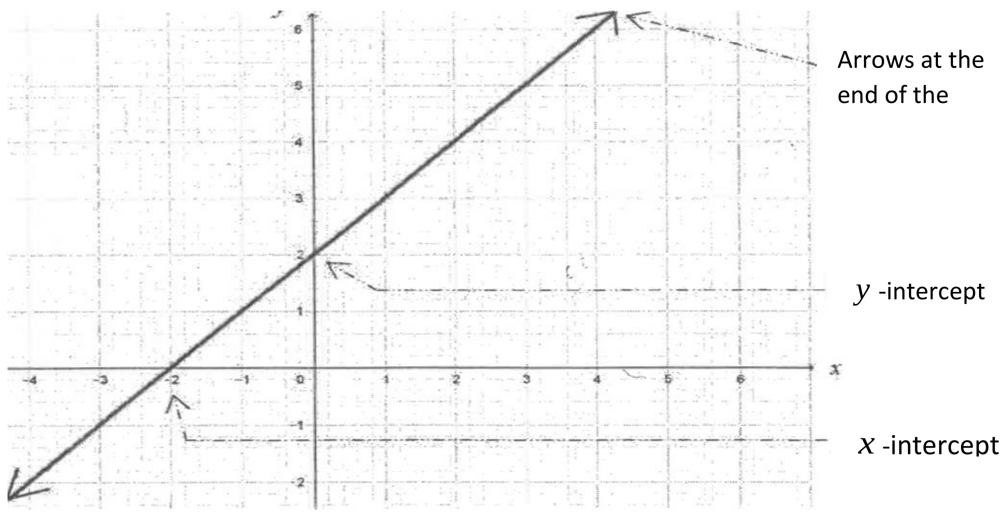


Figure 6-1: Geogebra investigation 2

It is important to note that m and c each stand for a specific property within the equation and each have a specific function. You will now investigate each property independently the use of the Geogebra App.

Open up the file, found on Moodle, called (Form 3 Investigation- properties of straight line graphs) on your Geogebra App. You are to follow the instructions (below) together with working on the App to answer the following questions. Write your answers in the space provided.

Investigating m

a) Move the m slider (red one) to 1 and draw a rough sketch of the graph on the Cartesian plane provided below labelled Diagram A.
b) Move the m slider to 2 and draw a rough sketch of the graph on Diagram A.
c) Move the m slider to 3 and draw a rough sketch of the graph on Diagram A.
d) Move the m slider to 4 and draw a rough sketch of the graph on Diagram A.

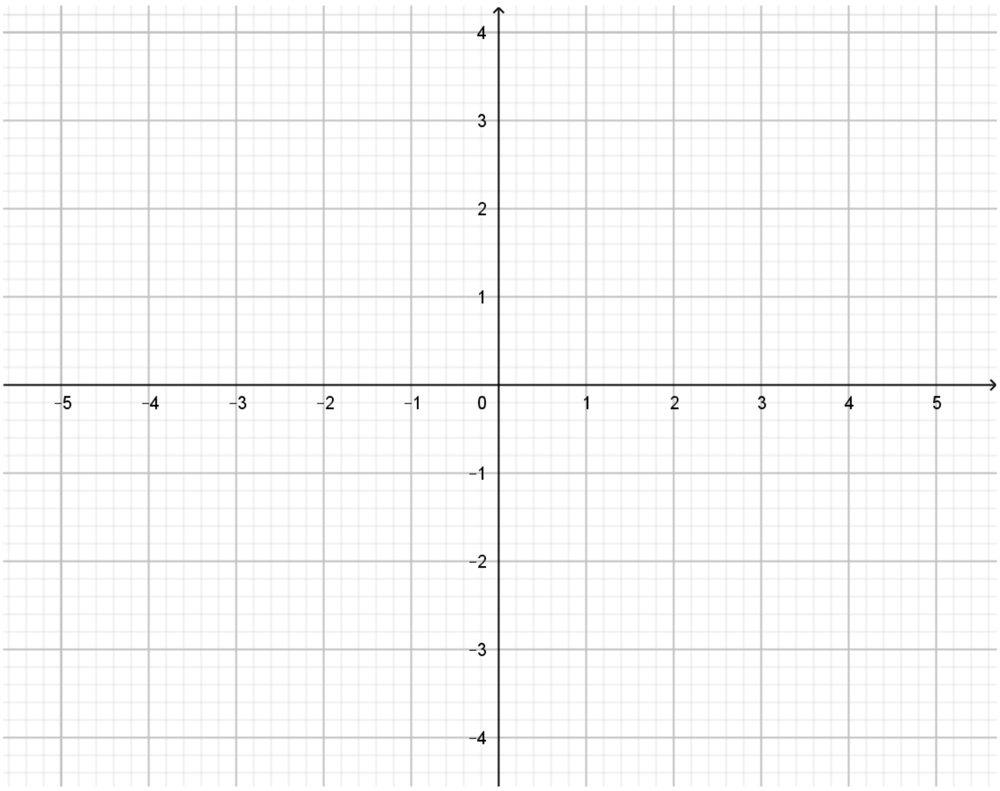


Figure 6-2: Geogebra investigation 3

During this period, learners were actively engaged and helping each other with the Geogebra App. Terry had indicated in the interview that most of her teaching adopts lecture style, but ideally, she would prefer to teach learners in groups:

> H: What is your ideal classroom?
>
> Terry: The ideal situation would be grouped tables with, I wouldn't say complete silence, but when I'm teaching, they are asking questions to fully understand the concept and when they do the activity to work together, so I can walk around helping those who need to be helped.

In this lesson when learners were using technology, Terry was teaching in her ideal classroom. Learners were discussing the concepts with each other and Terry was moving around, listening to their discussions and asking questions to help steer the discussions in the right direction. She had not set out to teach an ideal lesson, but the use of technology made this possible for her.

The advantage of using technology was that Terry and her class were able to cover more in one lesson than the learners in the other two observed classrooms where technology was not used. Moving the *m* slider from left to right, learners were able to make 4 important observations. Firstly, they were able to see that when *m* is negative the line decreases from left to right. Secondly, at $m = 0$, learners were able to observe that the line was horizontal therefore parallel to the x-axis. Thirdly, it was observed that positive values of *m* resulted in a graph that increases from left to right. Lastly, the slider allowed learners to see that the bigger the value of m (disregarding the negative sign) , the steeper the line and vice versa.

In the lessons that followed, Terry used a projector and a laptop to demonstrate the use of the *App* on the whiteboard. Starting with $y = x$, she varied *m* while fixing *c* at zero. Learners were able to see the effect of *m*, the gradient, on the graph. She continued to ask questions as she varied the values of *m* from small to big positive values. Learners could see that the slope was getting steeper with increasing values of *m*. She explained that it is like having a little hill which starts off with a gradient of one, but as it got steeper, the gradient increased to ten. In each case the variable under investigation is varied, while the other variable is kept at zero. Using the *App*, the class was able to investigate the effects of m and c on the function $y = mx + c$.

6.2 Document analysis

Curriculum knowledge is one of the knowledge domains under PCK. Shulman (1986) states that teachers need to have knowledge of how the sequencing of topics is done, as well as how topics are linked within the curriculum. The choice of curriculum material is important and should be done with careful consideration. Areas of focus in the analysis of documents were: the language used, sequencing of topics, type of tasks and activities, degree to which guidance is given to the teacher regarding the purpose of the lesson, linking of related topics, type of examples used, linking of tasks to real life context, guidance offered to check for common errors and misconceptions and types of representations used. The main focus of this analysis was the topic on functions. The theory of variation was further applied as the analysis tool on how the curriculum material was used for content construction in the lessons.

6.2.1 Lily

The workbook used by Lily was both a textbook and a workbook. The teacher used this book to plan and carry out teaching. The book is designed and authored by fellow teachers from a neighbouring independent school. The concepts are covered according to the prescriptions of the CAPS and there are some similarities in the presentation of the content in this workbook and the content found in the workbook supplied by the Department of Education (DOE) to the public schools and used by Amanda. The main difference is that this workbook does not give many step-by-step examples. The book covers a number of topics presented in different headings which highlight and introduce methods and procedures, definitions, examples, problems and solutions. Each of these mini topics is followed by tasks covering concepts at various levels of difficulty.

New knowledge is investigated and discovered through the process of variation. No scaffolding is done in the workbook and teachers who use the book are expected to possess the necessary knowledge and skills to scaffold the content and to assist learners in working through the tasks. At the end of every section, there is a revision exercise which assesses knowledge of the concepts covered. Answers to the revision exercises are given without workings. This revision exercise offers opportunities for extension for those learners who need this cognitive stimulation. Teachers need to have sound knowledge of the concepts presented in the book in order to assist learners who use this workbook.

Each topic ends with a 60 minute test followed by a detailed model answer. Learners are exposed to formative and summative assessments of the concepts covered within a topic. The test also covers various cognitive levels. Past exam papers are also included at the end of the workbook.

The observation of lessons showed that Lily relied solely on the workbook for her teaching. The only time she deviated from the use of the workbook was when she introduced functions as a topic. She linked the new topic to linear number patterns before turning to the workbook. Lily would often be spontaneous and encourage learners to come up with their own examples.

The workbook made use of variation to show connections and to offer opportunities for learners to do investigations that led towards generalisation. This use of variation is discussed in detail in the next section.

6.2.2 Amanda

Amanda made use of a government supplied learner workbook, a government prescribed text book and a planning document provided by the government. The planning document stipulated that on top of the books listed, teachers also needed to have the Sasol Inzalo workbook which Amanda did not have. This workbook could however be downloaded online in electronic form. The planning document is a very detailed book showing daily lesson plans per topic. The document also shows links to prior knowledge and other topics. For some topics, common errors and misconceptions are highlighted and teachers are given recommendations on how to teach these topics taking into account these errors and misconceptions.

The planning document also gives exercises to do at each stage of the lesson plan including investigations and homework. Some of these exercises are directly linked to the three prescribed books i.e. the learner workbook, teacher reference book which contains a teacher's guide and the Sasol Inzalo workbook. The combination of this teaching and learning curriculum material can be highly effective if used appropriately. The learner workbook gives definition of terms and step-by-step solution to examples given. Learners are then given similar problems to solve and by following the step-by-step solutions to examples they should be able to solve these problems on their own. Learners can solve these problems by rote learning without much conceptual understanding. However, the use of the other complimentary teaching and learning material caters for the problem of rote learning. The exercises in the Sasol Inzalo workbook, specifically, do require that learners apply mathematical thinking and reasoning and vary in levels of difficulty. This workbook gives exercises in which learners have to link flow diagrams, tables and graphs as representations of functions (this workbook was not used at all by Amanda or her learners, I downloaded the copy on line).

The planning document states explicitly that teachers need to emphasize that when plotting a graph, the input values are represented by the x-axis and output values by the y-axis. On analysing the learner workbook, it was discovered that these books contain an error in the labelling of the axes on the Cartesian plane. The x-axis is represented on the vertical axes and the y-axis on the horizontal axes. This error had serious consequences as observed in one of the lessons discussed in the previous chapter.

6.2.3 Terry

During the interview with Terry, it was revealed that she and her colleagues designed their own textbooks.

> Terry: We've got a text book that is designed by the maths teachers. We hated the Grade 8 book we had used before. Because it was not set by us we found that the levels changed so drastically, that you had to go from very simple to very hard all in the space of a question. And we found the boys battled with this and we found that because the year wasn't perfect, that year we had a very bad year (learner attainment). So now we design our own text book. We've looked at how other text books are designed and what we do is we would design a chapter from scratch. How we would set it is how we would teach it and then everyone gets the same text book but all six forms have their own text books set by the school.

In the lessons observed, Terry would give learners work from the worksheets she had created herself and would occasionally refer them to the textbook which many of them had stored on their devices in electronic form. The analysis of the textbook designed by Terry and her colleagues shows that the book is designed with learners in mind. Each chapter shows a progression of concepts. The levels move gradually from easy to more challenging. The book uses lots of demonstrations and explanations which follow definitions of terms. It also uses technical language and arrows are used to show links between concepts. Step-by-step solutions to examples are given followed by exercises for learners to do the work on their own. At the end of each chapter, solutions to these practice exercises are given. The chapter on linear functions was being revised and the book pages had some underlining and circling where changes needed to be made.

6.2.4 Brian

When I first interviewed Brian it was revealed that he made use of a teacher's guide and a policy document in his teaching. The learners were supplied with the textbooks and workbooks similar to the ones used by Amanda.

> Brian: There's a workbook and there is a textbook, and the government did say that there should be a rule that one learner, one textbook, so each and every learner must have a textbook so that it would be easier to do homework. But to be honest with you, starting from January up until June, the learners of this school were not given these textbooks, the only thing we did was, if someone needed it then they would use it at school. The reason being that we did not have textbooks that are equal to their number.

During the observation of lessons, I noticed that each learner in class had a textbook. Brian explained to me that they had received money to buy more books after June as the number of learners had increased. Even though Brian had mentioned workbooks, learners did not make use of these during the lessons observed. During a visit to the school the following year, I noticed that the workbooks were stored in the staffroom and Brian explained to me that the school had made a decision not to hand any of these out to the learners because there were not enough books for each learner to receive a copy.

The analysis of the textbook and the workbook revealed that without the workbook, learners would not find it easy to do work on their own without the help of a teacher. Understanding that these Grade 7 learners came from a non-English speaking background, the textbook lacked scaffolding in terms of the technical language used for definitions and the context of exercises given for practice. There was also no coherence in the layout of the tasks given, for instance, in one exercise the task moved from finding a rule of a simple function to converting from inches^2 (imperial system) to centimetres^2 (metric system) in order to find the area of a compound figure. This problem was also cited by Terry as one of the reasons her school had opted to design their own curriculum material. In the lessons observed, there was no evidence of the use of a teacher's guide in conjunction with this textbook and Brian indicated in a later discussion that he had misplaced it.

In the absence of a teacher's guide, the textbook and the policy document were the only books available for Brian to use in his teaching. The policy document taken from the CAPS includes a planning document designed in a similar way to the one Amanda was using. There was no evidence that Brian had used this document to plan for any of the lessons observed. The document has detailed lesson plans, examples of exercises to use in class, indication of how to give homework, multiple representation of content, linking of content to other topics, investigations etc.

Some of the tasks from the textbook were too complex even for the teacher to understand as was evident from one of the lessons observed. The workbook which was not used is colourful and written in simple language, contains step-by-step solutions to problems and offers multiple representations to the concepts relating to functions as well as opportunities for learners to practice skills taught by doing individual work.

6.3 Variation theory as the analysis tool

The theory of variation is discussed in greater length in the theoretical framework section. This theory was used as a tool to analyse how variation was used in the teaching of linear functions. The analysis focused on how separation, contrast, generalisation and fusion were used in the curriculum materials and interpreted by teachers in the classroom.

Figure 6-3 illustrates a pedagogical process driven by the four types of variation interaction namely, contrast, separation, generalisation and fusion.

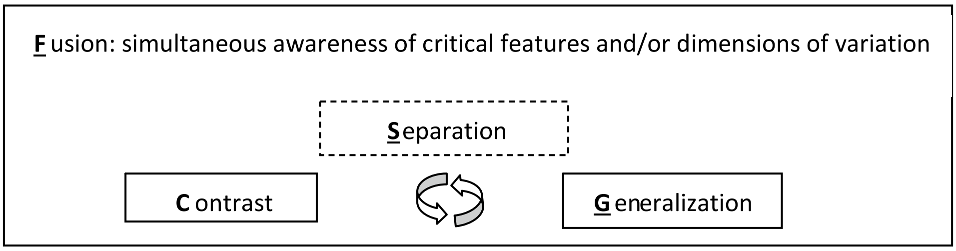


Figure 6-3: A discernment unit driven by types of variation interaction (Leung, 2012, p.437)

6.3.1 The use of variation in Lily's teaching

To illustrate how variation theory was used to analyse the use of curriculum material to construct content and make connections in the teaching of functions, I will make use of discernments.

The following six functions were given and learners were instructed to draw and compare graphs represented by the equations.

$y = 2x$	$y = 2x - 2$	$y = \frac{1}{2}x$
$y = -2x + 1$	$y = x - 1$	$y = -x + 3$

Discernment unit 1: Classification of linear functions by the shape

Type of variation: Contrast

Classification of straight line graphs by shape, i.e. increasing or decreasing. The focus is on visual features.

Learners are given six straight line graphs with the equations written in standard form as shown. The values representing the gradient and the y-intercept are varied. By drawing the graphs, learners discover that all graphs are straight lines of different slopes (increasing or decreasing). Contrast is used here by representing straight line graphs in multiple ways.

It is the teacher's duty to make clear the object of learning. In this case although both the gradient m and the y-intercept c are varied, the teacher needs to emphasize to the learners that the focus is on the shape of the line.

Type of variation: Separation

Increasing: $y = 2x$; $y = 2x - 2$; $y = \frac{1}{2}x$; $y = x - 1$

Decreasing: $y = -2x + 1$; $y = -x + 3$

The focus is on the awareness that increasing or decreasing graphs can vary in steepness and can either go through the origin or cut on the y-axis. For increasing graphs 'm' and c are varied while the positive sign of the co-efficient of x is the invariant part. There is a separation between increasing and decreasing graphs and the focus is on the critical features of variation.

Type of variation: Generalisation

The focus is to identify the invariant part by decontextualising the pattern observed. Under contrast and separation, it is clear that while other parts can be varied, the highest power or degree of x, the input variable remains the same. Therefore it can be generalized that all straight line graphs whether increasing or decreasing are of the form $y = mx^1 + c$, where m and c are variables representing elements of real numbers.

Type of variation: Fusion

Fusion is the perception that a graph of a linear function is a representation of a relationship between variable x and variable y in the general form $y = mx + c$. Different functions are representations of unique graphs having both invariant and varied features. For a linear function, x^1 is the invariant feature while m and c can be varied to produce different types of linear graphs.

In the second part of the investigation learners are instructed to draw with the help of the teacher, three more graphs and to compare their equations to the six straight line graphs referred to earlier. The three graphs represent equations of a parabola of the type $y = ax^2$, a hyperbola with general equation $xy = k$ and a circle centred at the origin with the equation $x^2 + y^2 = r^2$.

I shall continue to demonstrate how variation applies to this investigation task. Because this is a continuation of the variation process described in the preceding illustration, I will refer to this process as Discernment unit 2.

Discernment unit 2: Classification of linear functions by type

Type of variation: Contrast

Counter examples are used to discern the critical features of a linear function. The three relations are represented by a parabolic function, a hyperbolic function and a circle which passes through the origin. The aim is to show the critical features of a linear graph by revealing what a linear graph is not. This is evident, first of all from the shapes of the other graphs and secondly, from the defining equations of these other graphs i.e. not $y = mx + c$, which is the defining equation for a linear function. When the six graphs defined by $y = mx + c$ were drawn, they all produced straight line graphs, the invariant was $y = mx + c$ however, when the equations were varied, different types of graphs were produced.

Lo (2012) argues that the process of variation can either lead to a separation or generalisation depending on the focus of learning. In Discernment unit 1, separation was done with the aim of revealing that straight line graphs themselves differ in steepness, height on the Cartesian plane and shape, i.e. direct or indirect relationship between x and y. It can therefore be argued that the second part of the investigation was to generalize the invariant part of $y = mx + c$. The dialogue between the teacher and her learners below clearly supports this argument.

Type of variation: Generalisation

> Lily: As we can see, in the straight line equations, there is a plain x but if you look at these two, the x is not plain. There is a fancier x here (pointing at the two equations).
>
> Learner: Mam what do you mean by fancier x?
>
> Lily: It is not squared like these two (pointing and the quadratic and a circle equations). It is just plain old x and multiplied to something, writing down $y = 2x + 1$

From this exchange we can clearly see the teacher's attempt to generalize x^1 (plain x) as the invariant part of all linear functions. The question we ask is whether the focus is on the superordinate or on the subordinate. In this case it is clear that the focus is on the superordinate, therefore we can conclude that the variation pattern leads to generalisation because separation had already been done. The two investigations are therefore part of the same variation process.

Type of variation: Fusion

In her introduction of linear graphs, Lily had done a link between a linear number pattern of the form $T_n = a + bn$ and $y = mx + c$, the equation of a straight line graph. She had gone to great lengths to explain to her learners that the common difference d and the gradient m represent the same thing. Diachronic simultaneity is seen when the variation experiences gained during the introductory stages of learning about linear functions are connected to the final generalisation about x^1 as the invariant part of all linear functions. Similarly, synchronic simultaneity is seen in the pattern of variation observed in the two discernments as learners focused at different aspects of the linear function at the same time.

It is worth mentioning that, although learners embarked on investigations from the workbook, Lily did not always succeed in making explicit the object of learning. The absence of a teacher's guide caused Lily to approach the workbook from her own interpretation of the tasks and investigations. Lily's approach was to give her learners the investigations to do as class exercises or homework. The following example shows how the objective of an investigation was misinterpreted by the teacher and miscommunicated to the class:

The investigation required learners to draw on the same set of axes graphs of:

$y = 2x + 2$ $y = 2x - 2$ $y = 2x$

The next instruction was for learners to identify first the value in $y = mx + c$ that determines the slope of the graph and then identify the variable that affects the y-intercept.

Since the slope is the same, it is easy to see that the coefficient of x determines the slope as this is the only value that remains the same. Lily used this investigation to conclude that the lines were parallel, which is true. This observation or conclusion was, however, not the intended goal of the investigation.

Overall, Lily's class benefitted immensely from the use of variation in the workbook as these learners were hands on with drawing graphs, doing investigations and making connections which resulted in them embarking on mathematics according to predetermined outcomes. The absence of a teacher's guide resulted in misinterpreting of the purpose of some investigations, however, it is debatable if this misinterpretation would have affected the quality of the lessons. Lily's good knowledge of functions resulted in learners not being deprived of good learning and the choice of curriculum material enhanced the quality of her lessons.

6.3.2 The use of variation in Amanda's teaching

The workbook used by learners in Amanda's class made use of variation to explain and generalize about the effect of variables in the general formula $y = mx + c$ and to show that parallel lines have equal gradients. Learners worked through the investigations guided by the teacher. The design of the workbook was such that, detailed examples showing step-by-step solutions were given, followed by exercises for learners to do in a similar fashion. Learners in Amanda's class were hands-on with drawing graphs and doing comparisons to draw conclusions. In one of the investigations, learners were given an example with three graphs drawn on the same set of axes on the Cartesian plane, using the table method.

Learners were then given four more functions to draw following the procedure used in the example. In all four functions, the gradient was the invariant while the y-intercept was varied. Learners, using the table method drew the graphs and it was generalized as in Lily's lesson that the lines were parallel because of equal gradients. Using variation improved the quality of the lessons in this class as learners were actively involved with the required level of mathematics at various levels. Not only were these learners acquiring the skills to draw graphs using the table method, they were also investigating important concepts about straight line graphs.

One of the shortcomings of the workbook used by the class was that the object of learning was not clearly stated in any of the investigations. This problem was also encountered in the analysis of Lily's lessons. The teacher and learners had to conclude at the end of the investigation what the investigation was about. The variation theory discussed in this paper states that the object of variation should be clearly stated prior to embarking on the process of variation. The object of variation in the previous investigation could have been (1) to generalize about the coefficient of *x* (2) to generalize about equal gradients (3) to generalize about the y-intercept.

The first generalisation would have been that the coefficient of *x* represents the slope of the graph since the y-intercept is varied. In the second case a conclusion would have been drawn that equal gradients produce parallel lines. Lastly, varying the y-intercept could also have led to the generalisation that c in the equation $y = mx + c$ represents the y-intercept of the graph since the slope was kept constant at 2. Amanda and her class chose to go with the second generalisation and other possibilities were not considered. The next two examples further demonstrate how the object of learning was not clearly stated.

Example 1

Draw and compare graphs of: A. $y = 3x$ B. $y = -2x$ C. $y = -x$

In functions, A, B and C, *m* is varied but c is kept constant at zero. The use of positive and negative signs seems to suggest that the object of variation was separation in order to generalize that positive gradients produce increasing functions while negative gradients produce decreasing functions.

Example 2

Draw and compare graphs of: $y = 3x$; $y = 4x$ and $y = 5x$

Two possibilities that can be explored in this investigation are: (1) the object of variation is the use of separation to generalize that the coefficient of *x* represents the slope since the y-intercept is kept constant at zero (2) separation is used to generalize that the greater the value of the gradient, the steeper the graph and vice versa. Smaller values including fractions between 0 and 1 could have been used in the second case to produce clearer results that the smaller the gradient the flatter the graph.

Apart from the error in the labelling of the axes, the workbook proved to be a good source from which learners learned and engaged in required level of mathematics. Amanda also made fewer errors when she followed the workbook closely. The problem arose when she had to explain concepts from her own understanding.

6.3.3 The use of variation in Terry's teaching

In Terry's lessons, variation was done through the use of technology. Figure 6-4 is a screenshot of the *App* used in investigations. Sliders were created using $y = mx + c$.

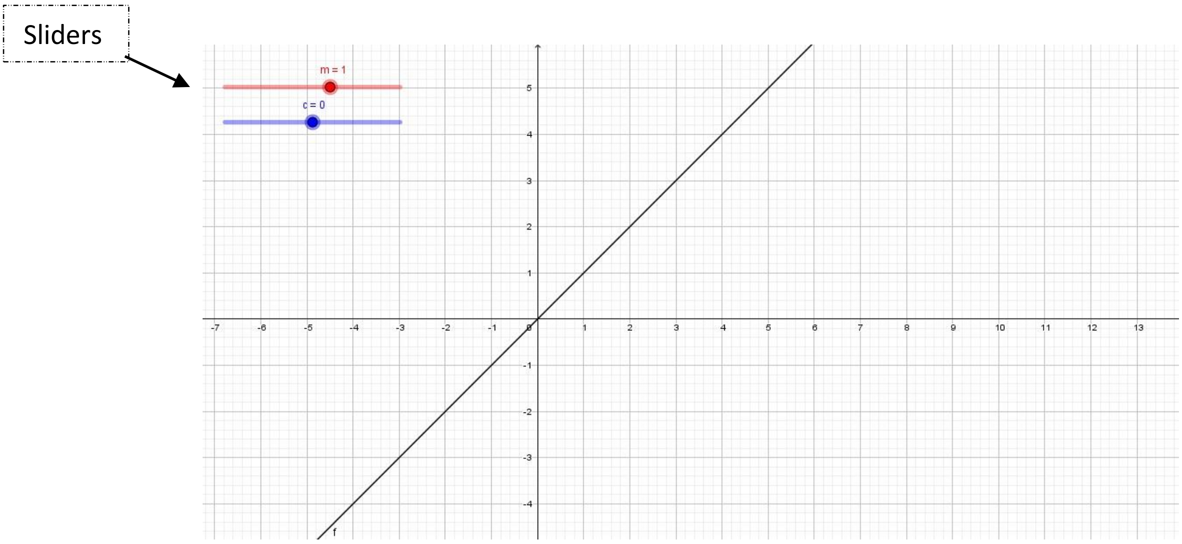


Figure 6-4: A screenshot of Geogebra App used to discover properties of straight lines

The sliders were created in different colours and learners used their IPads to investigate the effects of *m* and *c* on the graphs of linear functions. Figure 6-5 shows how *c* was investigated:

<u>Investigating c</u>

<u>Instructions</u>

(a) Move the m slider (red one) to I and change the c slider (blue one) to 0 and draw a rough sketch of the graph on the Cartesian plane provided below labelled Diagram D
(b) Move the c slider to 1 and draw a rough sketch of the graph on Diagram D.
(c) Move the c slider to 2 and draw a rough sketch of the graph on Diagram D.
(d) Move the c slider to 3 and draw a rough sketch of the graph on Diagram D.

Diagram D

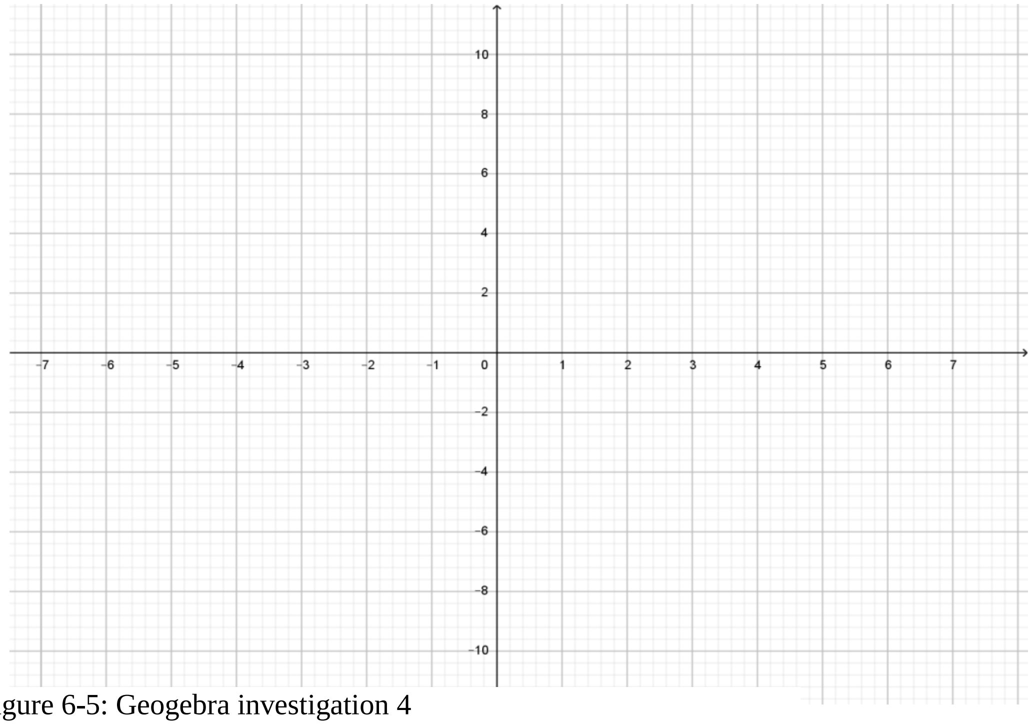


Figure 6-5: Geogebra investigation 4

Questions: 1.1. What happens to the graph as the value of c increases?

The object of variation is made explicit, learners know that they are investigating *c* in the equation $y = mx + c$. In the investigation, *m* is the invariant part, being kept constant at 1. As the learners move their fingers to increase the value of *c* on the slider, they can see that the point at which the graph intercepts with the y-axis increases by the same value each time. The effect of *m* was investigated in a similar way by varying m and keeping c constant.

The use of technology made variation far more effective than in the classroom, where variation was done without technology. Learners were hands-on and were able to do more self-discoveries using the *App*:

> Reece: Mam I notice that when the gradient is positive the line lies in the first, second and third quadrants.

Furthermore, the time it took Terry to cover most of the concepts related to linear functions was less than the time it would have taken in the absence of the *App*. Learners appeared to have more fun during these investigations, probably because using technology kept them actively involved and they were also given opportunities to discuss their discoveries with each other. In the latter lessons, Terry explicitly taught them how to draw graphs using the table and dual intercept methods, however, the fundamental concepts relating to the gradient and the y-intercepts had been covered in the investigations through the use of the *App*.

The use of the *App* produced unexpected results which surprised Terry. In a post-lesson interview she shared that the lesson had taken a completely different turn to what she had planned. Learners were the ones driving the lesson, asking questions about concepts she had planned to teach at a later stage and seeming to grasp otherwise challenging concepts. Terry's discovery shows that the use of technology is one of the necessary knowledge domains needed in the teaching of mathematics.

Towards the end of the topic on functions, Terry gave her learners the following exercise:

(a) (3; -2) lies on the graph of $y = 4x + k$, find k

(b) (5; *a*) lies on the graph of $y = \frac{1}{x} - 1$, find a

(c) Determine whether (-1; 7) lies on the graph of $y + 2x = 5$

It was amazing to see the number of learners who shouted, 'I've got it!' to the first problem within the first minute. To be sure, Terry asks one of them to shout out the answer and he says 'minus fourteen'! When she asked if the other boys had arrived at the same answer, they all affirmed this answer. Learners continued to do the second problem with no difficulty. Some of them used inspection and did not show any workings. The class got stuck on the third problem and Terry had to demonstrate on the board how to solve this type of problem.

This exercise required demonstration of conceptual understanding especially because Terry had not done similar problems prior to giving the class this exercise. More worksheets were distributed and, in one of the problems, learners needed to find the equation of lines given a line at an angle of 45º and another at an angle of 135º. When learners were unable to use the given information to find the equations, Terry simply told them that angle of 45º meant the gradient was 1 and -1 at 135º and did not give any reasons with that information. Learners did not question their teacher but continued with the worksheet. The angle of inclination of a line is taught in Grade 11 after learners have encountered trigonometric ratios in Grade 10, but Terry introduced her Grade 9 learners to this knowledge as a way of exposing them to a variety of problems at various levels of difficulty.

Learners in Terry's class were able to handle challenging work because of the way in which foundation had been laid on the rudiments of linear functions.

6.3.2.1 Conceptual variation through learner productions in Terry's class

Terry was always worried about whether her learners would be able to comprehend the 'puzzle' that was being built in the teaching of linear functions. She brought the teaching of the entire topic into a conclusion by giving her learners an assignment involving a cognitive map of the concepts covered in the topic on an A3 poster. Terry is the only teacher who made an attempt to ensure that there was fusion of the concepts which had been introduced and taught in various headings. Figure 6-6 is a display of one of the learners' work..

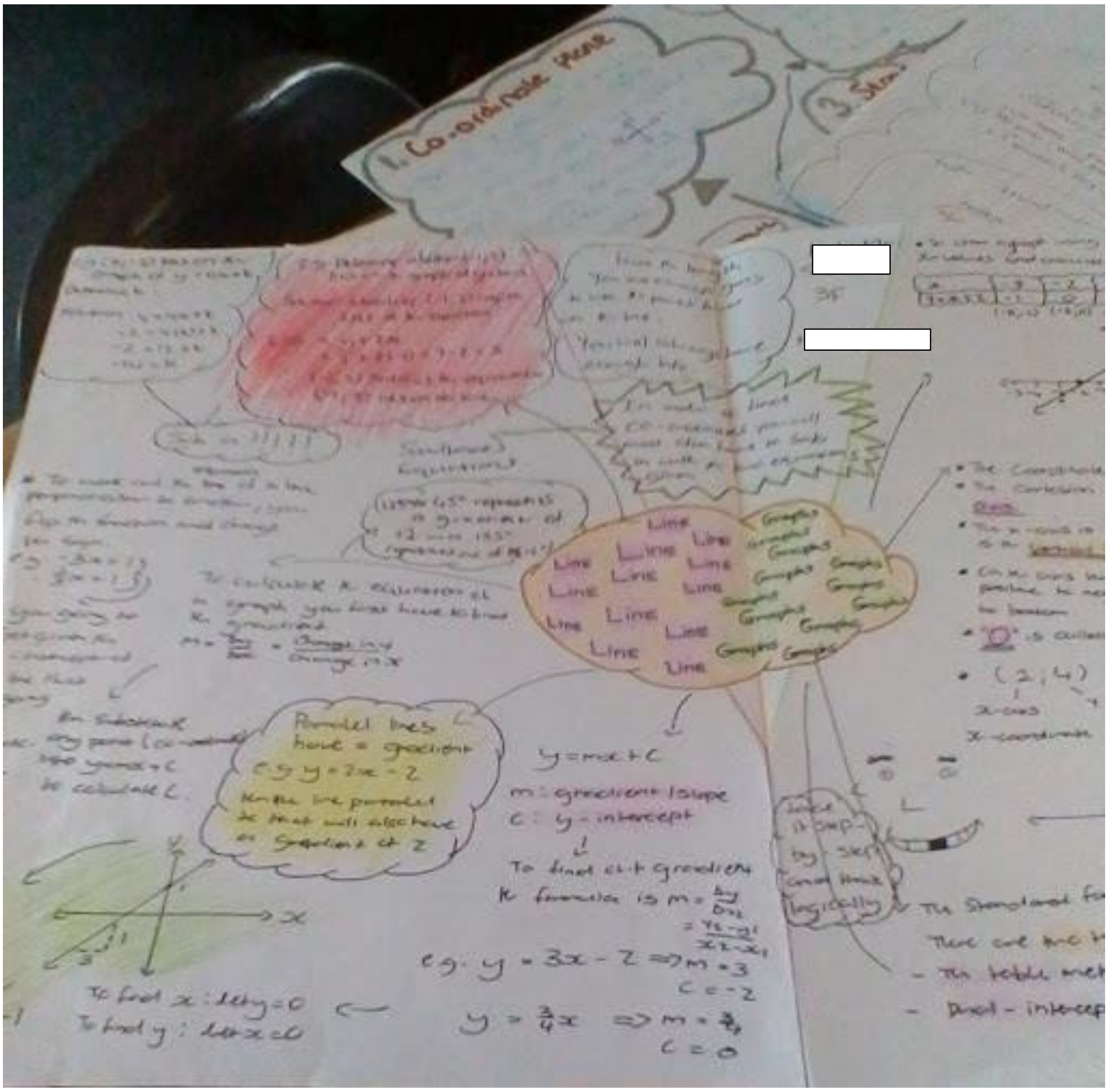


Figure 6-6: Conceptual variation through learner productions (Part 1)

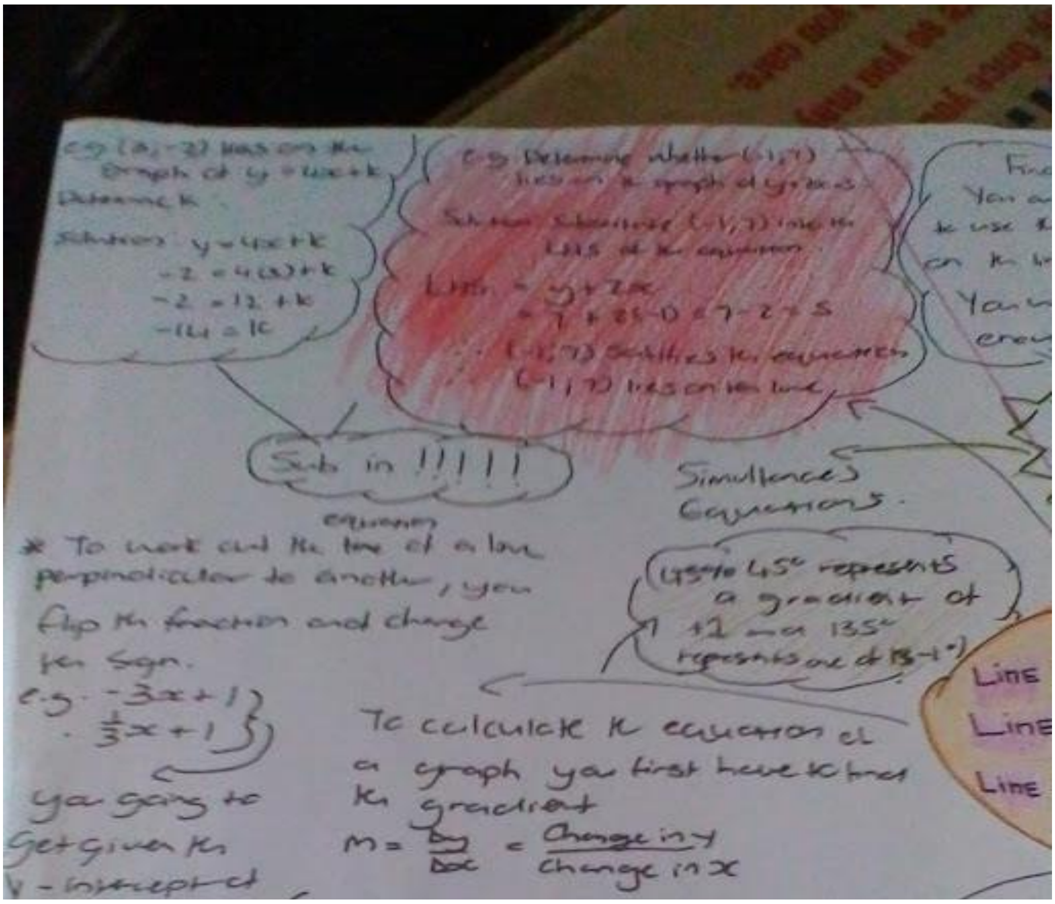


Figure 6-7: Conceptual variation through learner productions (Part 2)

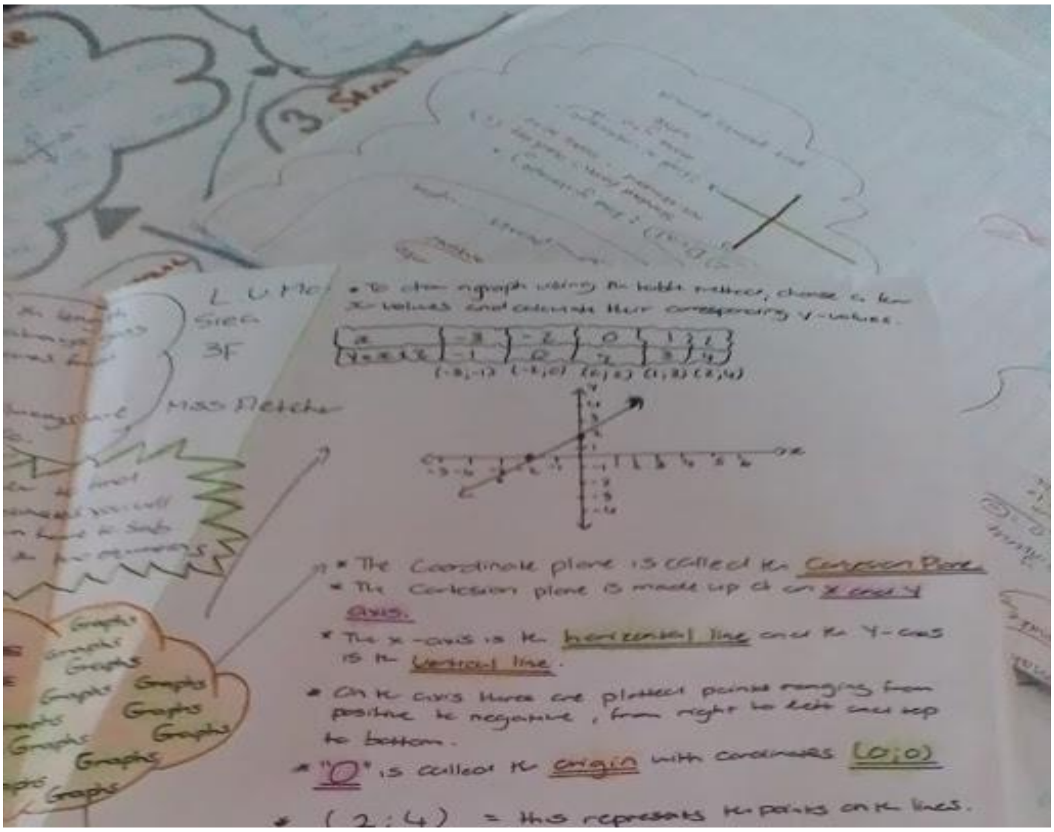


Figure 6-8: Conceptual variation through learner productions (Part 3)

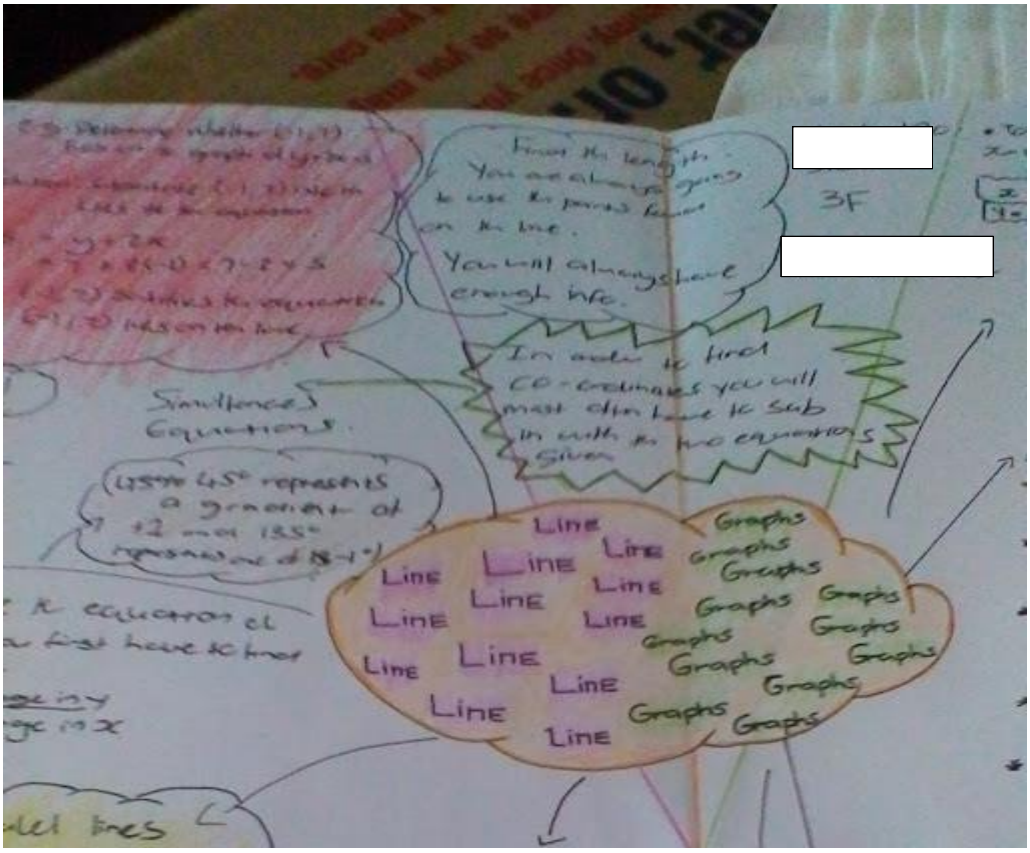


Figure 6-9: Conceptual variation through learner productions (Part 4)

These pictures represent four parts of an A3 poster and are a depiction of the content taught in Terry's class. Conceptual variation is about different representations of the same concept in order to strengthen the understanding of that concept. Careful study of this learner's production reveals that the leaner has offered various representations of a linear function which include an equation, a graph and a table of values. Similarly, the concept of a gradient is represented as the rise over run, the change in y over the change in *x* and the use of the angle of inclination.

6.4 Chapter summary

This chapter provided an analysis of data obtained from interview transcripts, video recordings, documents and field journals. The use of variation theory as an analysis tool was also demonstrated in the analysis of curriculum documents and of lessons observed. In the next chapter I present findings from these results.

CHAPTER 7: Presentation and discussion of findings

7.1 Introduction

In this chapter I present and discuss the study's findings as emerged from the results of the data presented in the fifth and sixth chapters. The chapter concludes with a comparison of the four participants with the cases from Ball et al. (2008) discussed in chapter two.

7.2 Teacher knowledge from rubric coding tables

Tables 7-1 to 7- 6 provide a summary of findings from rubric coding tables. Teacher knowledge is represented on the left column and teachers: Lily (L), Terry (T), Amanda (A) and Brian (B) are on the right hand column.

Table 7-1: Teacher knowledge as evident from observed lessons

	Evident in one or more lessons				Not evident			
	L	T	A	B	L	T	A	B
Use of manipulatives[i]					✓	✓	✓	✓
Use of narratives					✓	✓	✓	✓
Use of analogies		✓	✓		✓			✓
Part whole connections	✓	✓	✓	✓				
Pre-requisite connections	✓	✓	✓					✓
Progression and linkage to other content	✓	✓	✓	✓				
Presence of errors			✓	✓	✓	✓		
Pictorial representations		✓			✓		✓	✓
Use of drawings other than graphs					✓	✓	✓	✓
Use of Illustrations including graphs	✓	✓	✓					✓
Correct use of procedure for various algorithms	✓	✓	✓	✓				
Engages different methods	✓	✓	✓	✓				
Engages different methods and compares them		✓			✓		✓	✓

Table 7-1 offers a summary of teacher knowledge observed in the lessons. It is evident from the table that there was very little use of practical teaching aids to make new knowledge relatable to learners in the observed lessons.

Table 7-2: Use of mathematics with learners

	Evident in one or more lessons				Not evident			
	L	T	A	B	L	T	A	B
Learner explanations encouraged	✓	✓					✓	✓
Scaffolding of learner explanations	✓	✓	✓					✓
Recognition of errors made by learners	✓	✓	✓	✓				
Visible strategy for addressing of errors	✓					✓	✓	✓

Addressing of misconceptions	✓	✓					✓	✓
Correct interpretation of learner efforts /thinking	✓	✓					✓	✓
Use of learner errors to enhance teaching	✓	✓					✓	✓
Learner participation encouraged	✓	✓	✓	✓				
Learner efforts recognised and affirmed/praised	✓	✓	✓					✓

Table 7-2 shows how mathematics was used with learners. The table focuses on how learner efforts were recognised and addressed by the teachers. Recognition of errors and using these to enhance instruction was one of the focus of study and the results are tabulated here.

Table 7-3: Teaching with equity

	Evident in one or more lessons				Not evident			
	L	T	A	B	L	T	A	B
Use of real life examples in explanation		✓	✓		✓			✓
Pacing of the lesson supports diversity	✓		✓	✓		✓		
Learners work autonomously	✓	✓	✓					✓
Multiple contributions are supported	✓	✓	✓	✓				
Real life examples are appropriate		✓	✓		✓			✓
The level of class tasks accommodate diverse abilities.	✓	✓					✓	✓
Teaching accommodates learners with language barrier	✓		✓	✓		✓		

Teaching with equity concerns acts of inclusivity in the classroom to ensure that all learners participate equally or have the same access to learning activities. Table 7-3 displays results of how teachers attempted to cater for diverse learner needs in the observed lessons.

Table 7-4: Curriculum materials used

	Evident in one or more lessons				Not evident			
	L	T	A	B	L	T	A	B
Offers guidance on how to check for learner understanding					✓	✓	✓	✓
Highlights common misconceptions			✓		✓	✓		✓
Offers opportunities for learners to discover content on their own	✓	✓	✓					✓
Exposes teachers to the use of variation in the teaching of functions	✓	✓	✓					✓
Exposes learners to the use of variation in the learning of functions	✓	✓	✓					✓
Explains purpose of variation		✓			✓		✓	✓
Offers guidance on choice of models and representations			✓		✓	✓		✓
Offers definitions and use of technical language	✓	✓	✓	✓				

Table 7-4 presents results of research on how effective the curriculum materials used were in guiding teachers to enhance the quality of their instruction. The table also analyses if these materials offered opportunities for the use of variation in the teaching of functions.

Table 7-5 Class configuration

	Evident in one or more lessons				Not evident			
	L	T	A	B	L	T	A	B
Learners work in groups		✓			✓		✓	✓
Seating arrangement fosters learner collaboration		✓			✓		✓	✓
Lesson progression is systematic	✓	✓	✓					✓
Lesson has telos[8]	✓	✓	✓					
Class routines are observable	✓	✓	✓	✓				
Content covered is obvious to all	✓	✓	✓	✓				
The class is well managed	✓	✓	✓	✓				

Class configuration presented in Table 7-5 is a presentation of classroom management and structure in the observed lessons. The table does reveal a pattern of established class routines and good management of learning overall.

Table 7-6 Teacher knowledge of mathematics as evident from MKT items

	Evident				Not evident			
	L	T	A	B	L	T	A	B
Correct application of definitions	✓	✓					✓	
Valid mathematical arguments and reasoning	✓	✓					✓	
Correct application of rules, procedures and calculation methods	✓	✓	✓					
Knowledge progression	✓	✓					✓	
Awareness of what makes learning easy or difficult	✓	✓	✓					
Linkage to other content	✓	✓					✓	
Use of correct notation	✓	✓					✓	
Analysis of errors and misconceptions		✓			✓			
Teacher makes no errors	✓	✓					✓	

The last table represents data from coding of teachers' mathematical knowledge for teaching (MKT) of functions obtained from pen-and-paper items. Brian's column is left blank because he did not participate in these written items. This table reveals how the three participants demonstrated their knowledge of content in answering the pen and paper items.

[8] A telos is closure or finality with regard to the progression of the lesson from introduction to conclusion.

7.3 Emerging themes

Combining data from the initial coding table with that obtained from the comparative tables paints a more complete picture of each participant's knowledge as well as offers an indication of how each participant's knowledge compares to other participants. From this data it was possible to extract themes. Table 7-1 for instance reveals that participants in the study did not make use of visuals or manipulatives to make concrete abstract concepts in their teaching (**Emerging theme 1**). Taken together tables 7-1 and 7- 6 it was revealed that the participants with strongest knowledge of mathematics for teaching functions are Terry and Lily. However data shows from table 7- 1 that although Amanda's knowledge was deficient, she still managed to prepare and teach lessons which were mathematically rich in knowledge and skills. Table 7-4 offers insight into Amanda's practice. This data shows that the curriculum materials Amanda used assisted her to expose her learners to a teaching approach which allowed self-discovery of knowledge and put an emphasis on mastery of procedures (**Emerging theme 2**). These resources also gave Amanda exposure to a variety of models and representations while offering guidance on which ones were suitable for particular content. Even though Amanda's explanations contained errors at times, her teaching was still richer because of the use of technical language including definition of terms afforded by the curriculum materials (**Emerging theme 3**).

This data also shows that Brian who did not use the prescribed curriculum materials taught lessons which were lacking in mathematical richness in terms of knowledge and skills. With regard to classroom management, data shows that all four participants had established routines in their classrooms. However, this data also reveals that participants in the study did not create learning environments conducive to learner collaboration and this observation coincides with the teaching approaches observed which were mainly teacher-centred. Terry and Lily reported that they took into consideration learners' diverse needs in their teaching. This endeavour to teach with equity was observed in both these classrooms. Both participants offered opportunities for the more mathematically capable learners to be enriched by exposing them to more challenging tasks. These high achievers were also given the autonomy to work on their own, while the teacher spent more time assisting those who struggled to grasp concepts.

7.4 More Emerging Themes

7.4.1 Emerging theme 4: Compatibility of perceived and observed knowledge according to the participants' level of knowledge of mathematics for teaching.

Compatibility of perceived and observed knowledge varied according to the participants' level of knowledge of mathematics for teaching. Comparison of data obtained from semi-structured interviews and that from video recordings revealed that the two participants who had studied a higher level of mathematics at university were able to convey their knowledge of a function as well as their approach to the teaching this content. The study, however, failed to observe the same from participants who had received minimal exposure to university mathematics. The four tables present a summary of participants' perception of their knowledge, views about learners and mathematics, actual knowledge observed and the ability to reflect on MKT and teaching.

Table 7-7 Terry's perceived and observed knowledge

Teacher Knowledge of functions for teaching	**Terry**	**Notes**
perception of knowledge	…linear graphs, my content knowledge is amazing at that, I will be like a 9…	Terry reported that she was confident about her overall knowledge of the subject and her ability to teach functions.
Knowledge observed	Knowledge observed in Terry's teaching and the analysis of MKT items revealed that she possessed sound knowledge of functions taught in GET as well as content knowledge of functions taught in the FET.	Not only was Terry's knowledge of functions good, but she was also able to use this knowledge to enhance the quality of her instruction.

Ability to self-reflect	…I would say a 3, I'm still learning. I pick up new things every day, when you're so mathematical in your thinking you never think a child would think that and you think 'how is that possible?'. I think I should look up more misconceptions but time isn't there. From the questions they ask, I pick up misconceptions and I say oh that's why you're doing it. I prefer the transmission method because it is the easiest where you stand in front and you provide a lecture… using social constructivism is a much better method but at the end of the day it's not always feasible because we have time frame by when to finish.	Reflecting on her knowledge of learner misconceptions, Terry scored herself a 3. She also portrayed the ability to reflect on her teaching style which she described as transmission approach. Observation of her lessons revealed that although her teaching style was for the most part teacher- centred, her use of IPads resulted in self-discovery learning.
Perception of learners and mathematics	Maths is not easy to teach because it takes work and patience. I always say 'when in doubt cross out'; they don't understand that because they want to get it right the first time. Half the problem is that they don't want to try, they just want to rush it and get on with life, it's like there are more important things to do.	Terry taught the top set in the grade which consisted of learners with exceptional mathematical abilities. Her perception of these learners was that they lacked the patience to go through their own mathematical productions and evaluate these for errors because they wanted to get the correct answer on their first attempt.

Table 7-8 Lily's perceived and observed knowledge

Teacher Knowledge of functions for teaching	**Lily**	**Notes**
Perception of knowledge	I like functions in the higher grades, from grade 10 to grade 12, things are quite nice because in grade 10 you've introduced them already and then in grade 11 and 12, you're just expanding on it, the questions are always the same. So I like them because it's very comfortable, it's like an old friend and you're just growing the friendship.	Lily reported that her knowledge of functions was good and she did not experience any challenges with teaching this topic.
Knowledge observed	In Lily's lessons, learners were actively involved and Lily had a strategy of using learner mistakes to enhance teaching.	Observation of Lily's lessons and analysis of pen-and-paper items, revealed that her knowledge of functions is sound. Her teaching was enhanced by her vertical knowledge.
Ability to self-reflect	…at the end of the day you would have that day where you feel like 'gosh that lesson was a failure' and then you'll remember one kid that got something out of the lesson and then it'll be alright. My teaching style is probably more teacher centred. I'm a tutor, so I haven't been exposed to letting people discover.	Lily portrayed the ability to reflect at her teaching and like Terry, Lily was able to diagnosis her teaching style as being teacher-centred. Lessons observed did confirm that Lily used a traditional style in her teaching even though learners had freedom to engage in the lessons.
Perception of learners and mathematics	Probably every kid could do maths but as soon as you decided you can't or someone has told you, you can't, there's no ways you can do it. So maths is a lot about your belief in yourself. So I think that when you take the fear away and it's not for marks and it's not something they have to learn and it's kind of just for fun a bit, then that's the best way. Maths language is scary, but all they're asking you is where is the one graph above the other graph, that's all they're asking. So I do try to take the fear away and help them to understand what the maths language is saying.	Lily's perception is that learners fear mathematics. Her strategy is to give learners work to do but to take away fear, she would let them know that the work would not be for marks. Lily also teaches in a way that addresses the fear of maths language.

Table 7-9 Brian's perceived and observed knowledge

Teacher Knowledge of functions for teaching	Brian	Notes
Perception of knowledge	Relations is a very interesting topic, it was good and even myself when I was preparing for lessons every time. It is just one of those topics that made me love teaching because it gave me no problems … and even with my learners in class with relations would do much better than any other topic. We have used flow diagrams a lot as well as tables…graphs was another topic that they enjoyed, increasing, decreasing, the linear, the nonlinear, the learners really enjoyed them.	When Brian was given pen-and-paper SMK and PCK items to answer, he spent a considerable amount of time studying the items. Eventually he made a decision not to participate in the answering of these items. He stated that he did not have confidence in his knowledge to even attempt answering the questions. Some of these questions were on linear, increasing and decreasing functions. This did not corroborate his earlier claims that he and his previous learners had enjoyed this topic.
Knowledge observed	Question: In the imperial system the area will be 4 inches^2 , write down a formula that converts inches^2 to cm^2 . Brian's answer: Number of inches = Number of centimetres divided by 2,54.	This and other observed teaching episodes in Brian's lessons revealed that he had memorised the procedures involved in generating tables from flow diagrams. Observation of lessons showed that learners had also memorised these procedures. As a teacher, Brian lacked the ability to discern the cognitive level of this question which he treated as a routine rather than a complex problem.
Ability to self-reflect	None was observed	
Perception of learners and mathematics	some of them when they hear the word "Mathematics" they think it's the monster that they don't know how to deal with so I think that the teacher's role, number one, is to prepare their minds so that by the time, you start to learn maths they are all ready, because some of them, before they even go to a mathematics class, before they even start practising, they have failed mathematics in their minds.	Brian's view is that the way mathematics is taught determines the learners' attitudes towards the subject. Like Lily, Brian also believes that learners' fear of mathematics affects their ability to understand this subject.

Table 7-10 Amanda's perceived and observed knowledge

Teacher Knowledge of functions for teaching	**Amanda**	**Notes**
Perception of knowledge	I would rate my knowledge at 6 because besides the workshops, not much has been done by the Department. I try to develop myself further by using different books and attending Anglican workshops.	Unlike Terry and Lily who rated themselves according to various content, Amanda rated her knowledge of teaching mathematics at 60%. It was not easy to quantify this knowledge as Amanda herself was not able to specify how she had arrived at this rating.
Knowledge observed	Knowledge observed through analysis of pen-and-paper items including video recordings revealed that Amanda had challenges in her understanding of concepts related to the teaching of functions. At times her explanations contained errors, however, she was competent in solving routine procedural problems involving simple algorithms.	Analysis of pen-and-paper items and of video recordings revealed that Amanda's SMK and PCK of functions taught in Grade 9 was deficient. Amanda's teaching showed gaps in conceptual understanding which resulted in erroneous explanations at times. Although it is clear that with time and experience, Amanda's knowledge will grow, at the time of this study her knowledge was at an average level.
Ability to self-reflect	Amanda: Remember yesterday I said that to find the gradient, you have to use the *x* and y intercepts? Chorus: Yes! Amanda: you can actually use any two points not just the *x* and y intercept.	Amanda was able to come back and correct her own explanation which was incomplete. This reflection took place after she had taught a lesson on finding the gradient of a line. She probably made this discovery as she was preparing to teach the next lesson.
Perception of learners and mathematics	We are dealing with kids that do not care about maths and the Department is not helping with its low standards of promoting learners. You find kids who do not care about working hard because either way they will pass.	Amanda's perception of her learners was that they did not work hard enough to understand the subject. This however, was partially evident in the lessons observed. Observation of her lessons showed that there were about four learners who appeared not interested in the lessons and had not brought their workbooks to class. The majority of learners were engaged the entire time and these learners also produced completed homework.

7.4.2 **Emerging theme 5:** Opportunities to learn mathematics influenced the level of teachers' mathematical knowledge for teaching and the quality of instruction.

Table 7-11 provides a summary of each participant's opportunities to learn mathematics and the effect of their acquired knowledge on the quality of instruction. Ma (1999) concluded that a teacher's knowledge grows over a three period cycle which involves schooling, teacher preparation and teaching. This summary indicates that opportunities to learn had an effect on the participants' MKT.

Table 7-11 Linking teacher knowledge to MQI

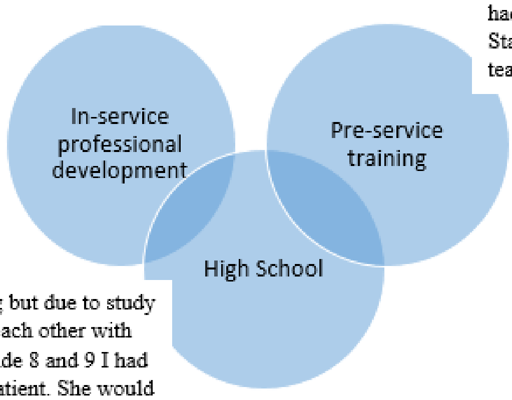
Linking teacher knowledge to MQI	**Terry**	**Amanda**	**Lily**	**Brian**
High school	Enjoyed maths, was a competent learner, had an excellent teacher	Loved maths, found it challenging, feared and did not understand the teacher	Enjoyed maths, was a competent learner, had an excellent teacher	Loved maths, found it challenging, did not have a competent teacher
Prior education including Pre-service training	Prepared her enough to teach conceptually using multiple approaches.	Did not prepare her to teach GET mathematics.	Enabled her to develop SMT but not PCK	Did not prepare him to teach GET mathematics
In-service professional development	Collaborated with colleagues through a school initiated programme.	Pursued her own professional development outside of the school.	Was involved in a mentorship relationship with a colleague.	Was not part of a continued professional development programme.
Performance in the KMT items	Displayed sound knowledge of functions.	Knowledge displayed revealed gaps in understanding.	Displayed sound knowledge of functions	Opted not to participate after studying the items.
MQI observed	Used technology to teach with variation, learners engaged in knowledge discovery, displayed good conceptual and procedural knowledge.	Used curriculum material to teach with variation, learners engaged in knowledge discovery, displayed good procedural skills. Explanations contained errors at times revealing lack of conceptual understanding.	Used curriculum material to teach with variation, learners engaged in knowledge discovery, displayed good conceptual and procedural knowledge.	Used a textbook to teach, learners not engaged in knowledge discovery, displayed good procedural skills, lacked ability to differentiate between lower and higher order knowledge content. Explanations contained errors at times.

7.4.3 Emerging theme 6: The participants' knowledge growth was linked to their opportunities to learn mathematics.

Figure 7-1 to 7-4 are a representation of each participant's knowledge growth.

I try to develop myself further by using different text books and attending Anglican workshops but besides that there is nothing else that has been done to help. I find the workshops useful but sometimes they clash with my personal plans and I am not able to attend.

I ended up doing a PGCE due to the lack of job opportunities. As I had a Science degree and had done Stats in my first year I was able to teach maths in GET phase.

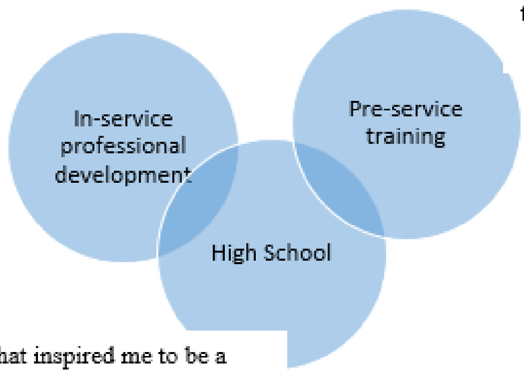


Maths in high school was challenging but due to study groups we had, we managed to help each other with concepts we didn't understand. In grade 8 and 9 I had a great teacher. She was gentle and patient. She would explain the concept in different ways we can understand and motivated us in liking maths and practicing it more often. But in FET we were more afraid of our teacher than understand him.

Figure 7-1: Amanda's knowledge growth cycle and opportunities to learn

I went to a workshop, I think it wasn't as productive as it should have been. I've never had help before, yes I did get documents and hand-outs and then I had to do exactly what they instructed me to do but I did not benefit from that.

…at varsity I did just an introduction to mathematics in my first year but then it just ended there.

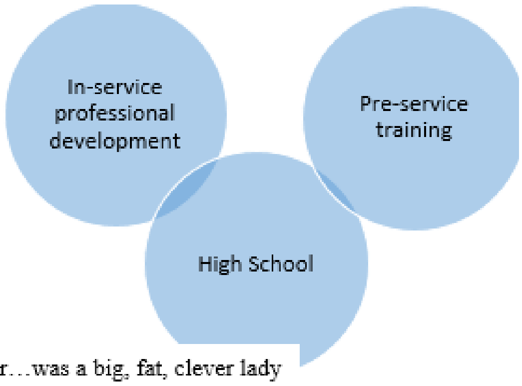


The high school I learnt at that inspired me to be a teacher, was very disadvantaged. We had to improve ourselves as much as we could such as attending programs... and this program helped kids with subjects especially in Maths, English, Physics and other subjects like Biology

Figure 7-2: Brian's knowledge growth cycle and opportunities to learn

…one of the reasons I love maths is because maths obeys all the rules and there's no problem… but my mentor is slowly raking me out of that but it's deeply entrenched in me.

When I did my PGCE through … and because of where I did it in rural areas, I never had an opportunity to study other methods of teaching there. I was looking at first year teachers, I haven't been exposed to other people's methods because there wasn't really anyone else so then I would fall back to my teacher, who I thoroughly enjoyed.

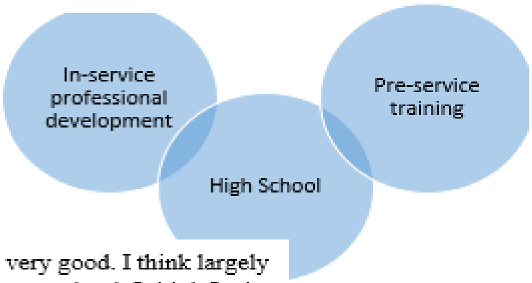


My high school teacher…was a big, fat, clever lady who just made you love maths. I remember like we were, our classes were streamed so we were split up into the best class and my friends were always jealous that I had that teacher and everyone wanted her.

Figure 7-3: Lily's knowledge growth and opportunities to learn

…for maths we have meetings every two weeks and we do professional development for instance we have done marking. One of our teachers is a senior marker so he brought past exam papers. We have done probability using logic and doing Honours in Maths and Science has been quite helpful.

So I think varsity is good, I think the maths modules are good because we learned different circle geometry theorems and things like that which we hadn't done at school but other modules were just rather idealistic, inapplicable. Where varsity did help was with using different programs so we learned a lot from Prof …, he taught us how to use sketch pad and things like that.



I enjoyed maths, maths was very good. I think largely it was my own maths teacher at school. I think I take a lot of her principles and I've applied it in my own classroom so it's not necessarily varsity, it's taking the staff, how I was taught is how I apply because for me it really, really worked. I understood what was going on, we did examples… So a large part of why I'm a maths teacher is because of my high school teacher, she was amazing, absolutely amazing. She never raised her voice, she was calm and always collected.

Figure 7-4: Terry's knowledge growth and opportunities to learn

7.4.4 **Emerging theme 7:** The use of variation enhanced the mediation of teacher knowledge

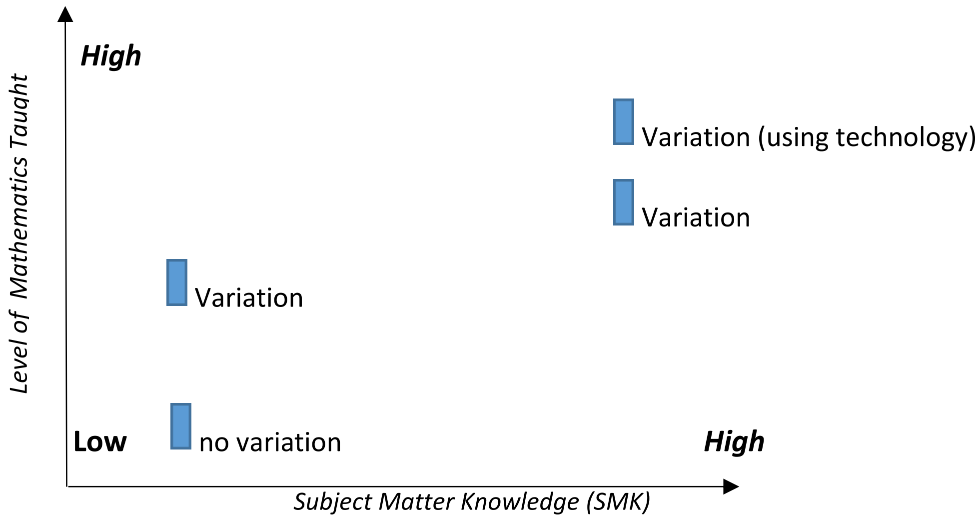


Figure 7-5: A summary of the relationship between teacher knowledge, variation and quality of mathematics observed

The quadrant is a representation of the relationship between a teacher's subject matter knowledge of functions and the level of mathematics taught in comparison with the expected level of mathematics. The expected level of mathematics refers to the mathematics that is prescribed in the CAPS. High level would indicate that the mathematics taught and learned in the classroom matched the required level and vice versa. Using the theory of variation allowed the researcher to check for quality in terms of the purpose of the lesson, how this was communicated to the class, learners' engagement with mathematics, appropriateness of explanations, generalisation and fusion of the concepts learned so as to bring a telos to the lesson. The diagram shows the findings of the study.

7.4.5 **Emerging theme 8**: The participants' approach to the pen-and-paper items was compatible with the teaching approach in the observed lessons.

Table 7-12 shows a comparison of the participants' approach to the written items and their teaching approach from classroom observations:

Table 7-12 A comparison of the participants' approach to MKT items and observed teaching approach

Knowledge vs Teaching	Terry	Lily	Amanda	Brian
Approach to the MKT items	Used lengthy explanations, gestures, technical language and multiple representations.	Used few words, but these went to the heart of the matter. Preferred the use of formulae.	Answered some questions intuitively, explanations contained errors.	
Approach to teaching	Used lengthy explanations, gestures, technical language and multiple multiple representations.	Engaged learners more, made use of everyday language and emphasized mastery of procedure.	Used intuitive understanding to explain some concepts, explanations contained errors.	

7.5 Discussion of research question findings

In this section, I present the research question findings. This discussion appears here because chapter 8 and chapter 9 have been combined into a single chapter.

7.5.1 How do teachers perceive their mathematical knowledge for teaching of functions?

Table 7-7 to Table 7-10 present findings on the participants' perceptions of their mathematical knowledge for teaching.

7.5.1.1 A belief that mathematical knowledge for the teaching of functions was above average (acceptable level)

What emanates from this presentation and the discussion that follows is that the participants believed that they possessed adequate or fairly sound mathematical knowledge for the teaching of functions as illustrated in the excerpts:

Amanda: I would rate my knowledge at 6 because besides the workshops, not much has been done by the Department

Terry: …linear graphs, my content knowledge is amazing at that, I will be like a 9…

Lily: … I like them (functions) because it's very comfortable, it's like an old friend and you're just growing the friendship.

Brian: *Relations is a very interesting topic, it was good and even myself when I was preparing for lessons every time. It is just one of those topics that made me love teaching because it gave me no problems*

The study found that only one of the four participants was able to reflect on her knowledge of content and students (KCS). This perception was also observed in her teaching: This participant rated herself low and stated that she felt she still needed to grow in this knowledge:

Terry: I would be a 3, I am not really familiar with misconceptions. I pick up new things everyday, when you're so mathematical in your thinking you never think a child would think that and you think how is that possible? I think I should look up more misconceptions but time isn't there. From the questions they ask, I pick up misconceptions and I say oh that's why you're doing it.

This statement was corroborated in Terry's teaching and it was therefore concluded that she was able to realistically reflect on her knowledge of content and students. Terry was also clear about the teaching approach she was using in the classroom (KCT).

Terry: I prefer the transmission method because it is the easiest where you stand in front and you provide a lecture but where I bring in an interactive role is, I will go through one with them and explain it and then give them another one and say you do it yourself or I would alter the question and say you do it.

Again this knowledge transmission method was observed in Terry's teaching.

None of this ability to reflect on knowledge was observed in any of the other participants. This is demonstrated in this next excerpt. Lily, the other participant with good knowledge of the subject matter attempted to explain why she had given herself a high rating for knowledge of learner misconceptions:

Lily: I think of it like in Functions, the one area I've noticed that no one likes is if you say, show me where "f (x) > g (x)", and that freaks them out, they don't know what they are saying, so I will explain to them in English, not in maths language, like in English, what is this saying to you, come show me, because the maths language is scary, but all they're asking you is where is the one graph above the other graph, that's all they're asking. So I do try to take the fear away and help them to understand what the maths language is saying.

This explanation proves that Lily despite having rated herself high on the knowledge of learner misconceptions, was not able to explain what learner misconceptions she was referring to. Instead this explanation reveals that she is answering the question about what makes the learning of functions difficult. In this case she is showing that the technical language used in teaching makes the learning of this topic difficult.

Lily was therefore not able to reflect on her knowledge of learner misconceptions. However, her perception about language and student learning was enacted in practice.

The objective of this question was to document the participants' ability to reflect on their mathematical knowledge for teaching. The study sought to compare the participants' declared knowledge with the knowledge observed in the lessons. In this regard the study found that in most cases, teachers' declared knowledge was not the same as knowledge observed in the lessons and that only one participant was able to clearly justify the ratings (perceptions) on their mathematical knowledge for teaching.

7.5.1.2 A belief in the importance of improving mathematical knowledge for teaching

The study found that there was a belief amongst the participants that their mathematical knowledge for teaching needed to grow. This conclusion is based on the participants' views about the need to embark on continued professional development activities and on the reports of involvement in pedagogical professional development or the pursuit of higher academic degrees in the teaching of mathematics. Figure 7-1 to Figure 7-4 illustrate this belief and finding.

7.5.2 How does a teacher's content knowledge of functions influence their teaching?

Tables 7-1 to 7-6 and Tables 7-11 to 7-12 offer insight into how the study answered this question. Two out of the four participants were highly qualified to teach GET (grades 7-9) mathematics while the other two had merely taken one year of mathematics at university. Although the study saw very little use of practical teaching aids in all the 28 lessons observed, the participants still managed to use simple procedures and basic algorithms, making content connections with the help of curriculum materials in some cases.

7.5.2.1 Teaching with depth and avoidance of errors

In the case of the two qualified teachers the study concluded that their content knowledge of functions enabled them to teach with depth while avoiding errors. The content knowledge allowed these teachers to prepare and teach lessons which were procedurally and conceptually sound and to engage learners in worthwhile rather than superficial learning of mathematics.

7.5.2.2 Exposing learners to inquiry based learning

In the case of Terry who used technology, the study found that the lesson design enabled learners to embark on investigations which resulted in self-discovery of new knowledge. This Geogebra App allowed learners to focus on critical ideas and to obtain immediate feedback. Teacher knowledge of functions played a vital role in guiding this learner-centred approach to linear functions. More content was also covered in this classroom compared to the other observed lessons.

7.5.2.3 Using learner errors to enhance teaching and learning

In all the lessons observed, teachers were able to recognise errors made by learners. The difference however, was in the manner in which these were used in teaching. The study found that the two qualified mathematics teachers were able to design lessons around these learner errors and misconceptions as well as devise strategies to use them to engage other learners by offering for the class to spot the errors. In this case the knowledge of the content was used not only to avoid errors but to use these errors to enhance instruction.

7.5.2.4 Correct interpretation of learner thinking

A good content knowledge of functions also enabled teachers to correctly interpret learner thinking while encouraging learners to explain their answers for the benefit of the class. Learners were given the freedom to express themselves even if they were unsure of the correctness of their thinking process.

7.5.2.5 Flexibility of teaching methods

The study also observed flexibility of teaching approaches in the classrooms where teachers were in possession of sound content knowledge.

7.5.2.6 Goal orientated teaching

Both Terry and Lily showed clarity of focus in their planning which was goal orientated and the ability to teach with the end in mind.

7.5.2.7 A lack of clarity about the content on the teachers' part deprived learners of meaningful learning of mathematics. In these classrooms, the bulk of the lesson time was spent marking homework and teachers kept to strict classroom routines and inflexible teaching methods. The lessons observed were procedurally sound, however, teacher explanations were marked with various conceptual errors. Most of the errors observed were an indication of the teachers' lack of understanding or grasp of the content being taught. The study concluded that scarcity of content knowledge of functions deprived learners in these classrooms of dynamically rich mathematical experiences, more so in the class where the teacher's knowledge or lack thereof was not supplemented by the curriculum materials used.

7.5.3 What other factors influence the quality of instruction?

The study found that there were other factors which influenced how teachers enacted their pedagogical content knowledge of functions in practice. These are the factors which influenced the quality of instruction other than teacher knowledge discussed in 8.3.

7.5.3.1 Opportunities to learn influenced teachers' PCK

Figure 7-1 to 7-4 summarises teachers' opportunities to learn as transcribed from the interviews. Opportunities to learn included the type of mathematics learned in high school, mathematics learned in pre-service training and continued professional development or in-service training that teachers were embarked in with the aim of enhancing their knowledge of teaching mathematics.

7.5.3.2 High school learning of mathematics

The study found that mathematics taught in high school was influential to the way participants taught and understood the content. The participants reported having being influenced by their high school mathematics teachers which were regarded as role models. These participants used their former teachers as points of reference for good teaching. The study also found that the two participants who went on to qualify as mathematics teachers came from privileged high school backgrounds compared to the other two participants who had only done one year of mathematics at university.

7.5.3.3 Pre-service training experience

The level of pre-service training received was also found to have a bearing on the participants' ability to teach functions. Terry who was found to be the most competent of the four participants had completed a four year Bed programme and was trained in the area of high school mathematics. She also reported having to draw on her university professors' methods in her teaching, including the use of technology. The other three participants had done a year post graduate certificate in teaching and reported having had less than exciting pre-service training experience.

7.5.3.4 Involvement in continued professional development

The study found that the participants were engaged in various forms of in-service training which included collaborating with colleagues in pedagogical discussions, attending content specific workshops and pursuing a post graduate degree. Participants reported that pursuing continued professional development enhanced their content knowledge and gave them ideas for teaching various content.

7.5.3.5 Choice of curriculum materials enhanced teachers' PCK

Choosing and using curriculum materials which offered opportunities for learners to embark in worthwhile tasks was enabled teachers to make connections and teach with variation. This was the case even in the classroom where the teacher lacked the necessary knowledge and skills to teach rich mathematics, avoiding errors. Some textbooks highlighted common errors and misconceptions, thus assisting teachers to cater for these in their planning and teaching.

7.5.3.6 Teaching functions with variation enhanced the process of learning

Using variation in the teaching of functions whether through learner workbooks or making use of technology, was found to be more effective in promoting learner engagement in the investigation of knowledge. Variation as a teaching and learning approach enabled learners to contrast, separate and arrive at generalisation of concepts as well as make connections across different representations of the same concept.

7.6 Chapter Summary

A summary of the findings linking teacher knowledge to classroom observations: a comparison of each participant's knowledge with cases from Ball et al. (2008) discussed in chapter two.

TERRY

Terry can be compared to the convergent case of Lauren from Ball et al (2008), discussed in chapter two. Terry's performance in the pen-and-paper items revealed a good mathematical knowledge for teaching. The analysis of Terry's response to various questions showed that her approach and interaction with the questions was at a level of a teacher who was using SCK. The analysis further showed that using 'commognition', Terry was analysing learner responses using her knowledge of content and students (KCS) and that her SMK and PCK were growing during this process in the same way as it would have happened in the classroom as she interacted with learners (Ma, 1999; Shulman, 1986).

Terry's MKT enabled her to prepare and teach mathematically rich lessons. Her explanations were clear and precise and did not contain any errors and her choice of examples was mathematically intelligent. In the pen-and-paper items, Terry showed the ability to link content of various topics across the curriculum. Like Lily, Terry's knowledge of functions was not restricted to functions taught in grade 9. Her HCK allowed her to teach intellectually rich lessons, taking into account learner thinking. This knowledge was observed first in how Terry approached the MKT items. Her response to these items showed that she was anticipating what the learners were thinking or the mistakes they would make and was trying to cater for these in her explanations. This was more than mere KCS, but knowledge of broader concepts surrounding the teaching of functions, including content taught in grades 10 -12. Observation of Terry's lessons showed that she was competent to handle learner questions and to address their errors and misconceptions.

Terry's classroom was the only one in which technology was used. The use of technology created an environment in which learners were authentically engaged in mathematical investigations and discovering knowledge for themselves. Terry was able to use her knowledge to design and guide these classroom investigations which resulted in rich mathematical experiences for her learners. The use of technology also enabled the class to cover more content in a short space of time compared to the other three classrooms. The study observed that using technology to teach with variation enhanced the level of investigations and the learning of functions. Unlike in Lily's and Amanda's teaching, the object of learning was made explicitly clear in the design of Terry's investigations.

Terry like Lily, used a whiteboard in her teaching. The use of a whiteboard was compatible with her style of teaching as she spontaneously came up with examples and drew graphs to explain and clarify concepts. Unlike the chalkboard used in Brian's and Amanda's classrooms, the whiteboard was easy to erase and enabled the teachers to write equations and draw graphs with ease. Terry was also the only teacher who strategically gave her learners an opportunity to fuse all the concepts (conceptual variation) related to the learning of functions thus bringing the whole experience of teaching and learning functions into a telos. She did this by giving learners an assignment to do a metacognitive or concept map to summarise the content covered and other concepts linked to functions.

The study found Terry to be highly motivated and dedicated to her teaching. She did a lot of preparation in the background to ensure that her learners were kept busy at all times and that they were exposed to high quality mathematical tasks. Like Amanda, Terry was pursuing further studies and met regularly with colleagues for professional development. This collaboration involved discussions and debates about the teaching of certain content as well as the reviewing of the curriculum materials. Although the school had published a mathematics textbook to be used by learners, Terry still used a variety of curriculum material and like Lauren in Ball et al. (2008), her use of various textbooks enhanced the quality of her instruction. Terry also used multiple representations in her teaching and was the only teacher who was observed using a picture to explain a concept.

The conclusion of this study is that Terry's mathematical knowledge for teaching (MKT) resulted in classroom instruction of high quality. Amongst factors that influenced Terry's mediation of her knowledge to enhance the quality of instruction were: her dedication to seeing every learner in her class succeed, her commitment to continuous professional development, the availability of teaching and learning resources, collaboration with colleagues and the use of technology in teaching. The fact that Terry taught the top set in the grade (high achieving learners) might have enhanced her high quality teaching.

LILY

Lily's case was both convergent and divergent. She can be compared to a divergent Anna. Lily's approach and response to the pen-and paper items revealed that her subject matter knowledge was good. This knowledge was seen in her teaching in the way she explained concepts and procedures to her learners. Lily's teaching and response to learners showed that she was able to use her knowledge in the classroom. The majority of the lessons observed were of good quality and her overall teaching was marked with no conceptual errors.

Lily's belief that mathematics should not be linked to real life context may have affected the quality of her lessons. Some might argue that there is no evidence that Lily's failure to relate mathematical concepts to real life context lowered the quality of her instruction. As illustrated in the analysis chapter, Lily's approach to mathematical procedure was on the one hand more theoretical and abstract which might have added quality to the mathematical rigour of the lesson. On the other hand, Lily's lack of use of manipulatives and other visual materials might have deprived her learners of a rich mathematical learning experience, especially those learners whose learning styles are tactile and visual.

Lily avoided using technical language in her teaching as a result of her endeavour to simplify the mathematical language to a form comprehensible to learners. This poses a contradiction to Lily's statement or belief that mathematics should not be related to real life context. Lily had stated in the interview that she had not attended any professional development programmes or courses while working as a tutor. This lack of professional development would have meant that Lily had not been exposed to any training for good teaching of mathematics since her initial pre-service training. Lily might not have had any understanding of the effect of technical language use or (non-use) in the teaching of mathematics. Had she used mathematical terms in her explanations, her quality of instruction would have improved greatly. Her failure to use mathematical terms resulted in vague and incomplete explanations. This had a ripple effect in that learners themselves were not able to articulate their thinking process and ended up providing illogical answers, lacking in rational reasoning.

As a qualified teacher with 17 years experience in the teaching of mathematics, Lily was found to be a teacher with a sound knowledge of functions (SMK). However, compared to Terry who only had 5 years of teaching experience, the study found that Lily's ability to transform her knowledge for instructional purposes was not as good due to the lack of technical language use in her teaching. It can be argued that the difference between Terry and Lily was that Terry's PCK had been growing during the five years of teaching in the classroom. Shulman (1986) defined PCK as knowledge that grows in the minds of teachers while Ma (1999) theorised that a teacher's subject matter knowledge grows during teaching as teachers interact with learners.

Lily had spent 15 years working as a tutor with individual learners and only 2 years in the classroom. Lily's approach in the classroom was that of an approachable teacher with an interest in individual learner progress and a belief that all learners can master mathematical procedures. This transference of mathematical skills improved Lily's quality of instruction.

Lily's use of variation to construct knowledge for teaching functions resulted in learners being hands-on with investigations. The study observed the use of contrast, separation, generalisation and fusion of concepts due to the use of variation. The use of variation was affected by the teacher's focus on or interpretation of the object of learning. The quality of teaching would have been enhanced had the curriculum material made explicit the object of learning. It was nonetheless concluded that the use of variation enhanced the quality of instruction.

It is therefore this study's conclusion that exposure to or familiarity with the classroom dynamics has an effect on how teachers enact their mathematical knowledge for teaching. Mathematical knowledge for teaching grows more in the classroom environment than in other settings like tutoring individual learners, and this growth has a positive effect on the quality of instruction. Another conclusion drawn from Lily's case is that professional development or involvement in further studies has an effect on a teacher's knowledge for teaching and its relationship to the quality of instruction. The study also confirms Ball et al's (2008) finding that teacher beliefs have an effect on how teacher knowledge influences the quality of instruction.

AMANDA

Amanda is both a convergent and a divergent case. Amanda's response to the pen-and-paper items revealed that her knowledge of functions was very limited. Amanda was also deficient in the knowledge of concepts taught in high school beyond grade 9. This lack of knowledge progression affected the quality of the explanations offered to learners and resulted in serious conceptual errors and a re-enforcement of misconceptions. Some of these errors were serious and damaging to learners' opportunities to acquire new knowledge. Amanda is a convergent case in that her poor SMK was also observed in the explanations and teaching which lacked depth and mathematical rigour as well as in her inability to engage learners in mathematical arguments and discussions. Amanda's procedural knowledge was, however, good and her learners benefitted from her knowledge of various algorithms and skills to solve routine mathematical problems as she emphasized mastery of procedures.

Amanda is also considered a divergent case like Rebecca because although her subject matter knowledge was weak and her explanations contained errors, her learners were engaged in the mathematics prescribed in the curriculum all the time. The learner workbooks enabled learners to be exposed to investigations making use of contrast, separation, generalisation and fusion. The use of variation as a tool to construct the content in the teaching of functions compensated for Amanda's deficient content knowledge. As in Lily's case, the study found that the curriculum material did not specify the object of learning in variation and this lack of clarity resulted in conclusions that were based on the teacher's interpretation of the lesson. Since the object of learning is defined by the critical aspects which must be discerned in order to have learning take place, the teacher must specify beforehand the intended object of learning. When a teacher is unable to do this, learners also fail to attend to the critical aspects of the linear function that need to be discerned as was observed in Amanda's and Lily's teaching. It can be argued that the curriculum designers have their focus on the general aspects or the indirect object of learning, however, they fail to communicate this to the teachers. Through the space of learning, the teacher and learners focus on the specific or direct aspects based on their interpretation of the content and this is their lived object of learning or outcome of the lesson.

Data shows that by sticking to the use of prescribed curriculum material, like Rebecca, Amanda was able to establish classroom routines which created a learning space conducive to engaging in real or worthwhile mathematics. The quality of Amanda's instruction was enhanced by the use of this curriculum material. Like Terry, Amanda was also pursuing professional development in order to improve her knowledge of the content and her ability to teach mathematics. Unlike Lily, Amanda made attempts to relate classroom mathematics to real life contexts. Her use of the analogy of teams to assist learners in choosing correct coordinates proved to be effective as in all the lessons observed, learners were consistently accurate in their application of this knowledge. Even strong learners in Terry's class were battling to pair up coordinates, however, this difficulty was not observed in Amanda's class.

The use of the chalkboard in Amanda's teaching, resulted in linear graphs that were skewed. To rectify these graphs would have been time consuming. The writing of notes either from the homework activities or for new learning proved to be time-consuming. This was also observed in Brian's classroom. However, unlike Brian whose learners were expected to take down all the notes from the board, Amanda would give her learners some work to do from the text book while she attended to writing notes on the chalkboard. These notes contained work already covered either in the textbooks or workbooks and were written for the purpose of introducing new concepts or marking homework as already mentioned. Amanda and Brian used different coloured chalks in their teaching and these often produced writing that was difficult to read or erase. Learners in particular found it difficult to write on the board as part of the classroom interactions. This was not observed in Terry's and Lily's classroom where white boards and overhead projectors were used.

From the analysis of Amanda's case, the study concludes that a teacher with poor or limited SMK will not be able to offer mathematically rich explanations and engage learners in rigorous mathematical discourse. The study also found that the use of prescribed curriculum material enhances the relationship between a teacher's MKT and the quality of instruction. The use of variation to construct knowledge in the teaching of functions led to meaningful and expected level of mathematics which would not have been possible given the teacher's limited knowledge of the subject. The effect of this variation was, however, thwarted by the curriculum designers' lack of guidance to the teacher on how to use variation in individual activities. Variation is not a one size fits all approach, the object of learning has to be specified and communicated in every discernment unit. The study identified other factors which mediate the relationship between MKT and the quality of instruction and these included a teacher's commitment to continuous professional development, the establishment of classroom routines and the availability of teaching and learning resources.

With regard to professional development, it was found that Amanda's commitment to personal knowledge growth in understanding and teaching of mathematics improved the relationship between her MKT and the quality of her instruction. Amanda herself had stated in the interview that she had found the Anglican courses to be highly beneficial to her knowledge growth. The colleague who had referred her for the study had also described her as a hardworking and dedicated teacher with a lot of potential. The observation of Amanda's teaching also supported this fellow teacher's view. Like Rebecca, Amanda had established classroom routines which necessitated the efficient use of lesson time. The study found that the use of a chalkboard hindered greatly the flow of Amanda's lessons as was the observation in Brian's teaching. The flow of the lesson was enhanced by the use of the white board and the availability of the overhead projector in Lily's and Terry's classrooms. On the availability of resources, the study also found that supply of learner workbooks and textbooks for every learner by the DOE, enhanced the quality of Amanda's instruction. The opposite was observed in Brian's class where learner workbooks were not given to learners on the premise that these were not enough to be used by every learner in class.

Amanda's response to the written (MKT) items was consisted with the knowledge observed in class. When Amanda was not clear about a concept, the study observed that she relied on intuitive understanding. This was demonstrated in the analysis chapter.

BRIAN

Brian can be compared to the convergent case of Zoe. Brian had not participated in the writing of pen-and-paper items because he was not confident about answering the questions posed in various scenarios. Although Brian had stated in the interview that his knowledge of functions was good, the observation of his lessons showed otherwise. Brian's explanations and definitions contained errors. He also had difficulty with interpreting classroom activities which resulted in erroneous solutions resulting from memorised procedures. The observation of Brian's teaching revealed that learners had memorised mathematical operations and their inverses. When filling in tables, for instance, learners had memorised that multiplication was used to find the output values and to find the input, division was used. Brian was also unable to discern the cognitive level of tasks embarked on in class and tended to treat as routine complex and problem-solving tasks.

As in Amanda's class, learners in Brian's class did not ask any questions during the observed lessons. Brian used a single approach in his teaching and his way of clarifying concepts was to repeat the same explanation, using a different tone or to write the same explanation on the board. Brian also found it difficult to rephrase questions, instead he would offer learners clues on how to answer the question. Brian's overall teaching was consistent with that of a teacher with very poor conceptual understanding.

No use of variation was observed in Brian's teaching and he did not make any attempts to use contrast, separation, generalisation or fusion of concepts in his content construction. In the majority of the lessons observed, learners were using routine procedural knowledge to find input / output values, given a function. Brian's assessment of prior knowledge involved asking learners about definitions covered in previous lessons and the learners were also reminded occasionally that a function consists of the input and output variables. This knowledge was, however, not re-enforced during the activities, hence, it was not clear whether learners understood the difference between input and output values. It was also not clear whether learners could identify these given an equation or a table of values generated from a function.

The review of curriculum material revealed that had Brian used the learner workbooks supplied by the government, he would have been able to teach with variation which would have resulted in better quality instruction. These workbooks were stored in the staffroom and it was clear from inspection that the stock supply of workbooks would have benefitted the majority of the learners in the grade 7 class. Instead none of the learners received this learning material based on the policy which stated that there must be a book for every learner. Since these workbooks were not enough to supply all learners, none were handed out, even though only a few were missing.

Brian used a chalkboard in his teaching and spent the majority of the lesson time writing notes for learners to copy from the board. Learners would copy the notes into their writing books while the teacher continued to write on the board. The teacher would then read the notes and emphasize the important or underlined concepts. These notes contained activities for learners to do in class or to take home for homework. Brian's school did own a brand new projector of a very high quality brand. None of the teachers, however, used this projector in their teaching, in fact some of the teachers were not even aware of its existence as it was kept in the principal's office. Lily was another teacher whose class was furnished with a projector which she did not make use of in teaching.

Brian, like Lily was not pursuing any further studies or involved in any form of professional development. He had stated in the interview that he would like to take up mathematical modules at university, not for degree purposes but simply to improve his mathematical knowledge for teaching. Brian had also mentioned that he had attended a professional development programme organized by the government which in his opinion, was a waste of time.

The study draws some conclusions based on Brian's case. The first conclusion is that Brian's poor knowledge of the subject matter led to low quality classroom instruction. The second conclusion is that Brian was exposed to teaching and learning resources which would have enhanced the quality of his instruction had he made use of these resources in his teaching. Thirdly, lack of an appropriate approach like variation or any other form of discovery in the teaching of functions, deprived learners of a rich mathematical learning experience. The fourth conclusion is that Brian's lack of involvement in professional development was a factor which influenced how his knowledge was mediated in the teaching of functions. This in turn affected the quality of his instruction. The study also observed that Brian was not able to discern cognitive levels of the tasks embarked on in class.

CHAPTER 8: Discussion and Conclusion

8.1 Introduction

This chapter begins by offering a discussion of findings followed by a proposed model of reflective practice which adds new knowledge to the study. The study then offers a conclusion and ends with recommendations.

8.2 Discussion of findings

Using qualitative methods enabled the study to meet the previously formulated objectives which were:

- To investigate GET teachers' ability to reflect on their mathematical knowledge for teaching specific content
- To explore how teachers' knowledge of specific content (functions) influences mathematical quality of their instruction;
- To explore other factors that may have an influence on the quality of instruction
- To make recommendations as to the measures that can be put in place in order to enhance the quality of instruction in mathematics

Working within the framework of mathematical knowledge for teaching (MKT) enabled the study to formulate principles to guide the process of data generation. These principles which included amongst others: making connections, arranging knowledge for teaching, building on learners' thinking, mathematical language, teaching of prescribed mathematics, mathematical communication, tools & representations and assessment for learning. This theoretical framework allowed the study to analyse data with regard to teachers' perceptions about their knowledge for teaching, observe how this knowledge is enacted in practice, compare the observed knowledge with that declared by the participants and arrive at the conclusion that teachers' declared knowledge was not the same as knowledge observed in the lessons, particularly in teachers with no prior training to teach mathematics (the out of field teachers). Using the principles formulated based on the domain knowledge by Ball et al (2008), the study confirmed the hypothesis made that a teacher with a sound knowledge of the subject matter is likely to use this knowledge to design pedagogically sound instruction. With regard to the question of quality of the study found that a high level of MKT which includes both subject matter and pedagogical content knowledge improves the overall mathematical quality of instruction (MQI).

The use of Variation Theory provided a lens through which the study observed the actual teaching of functions in the classroom. The theory has as its premise, the focus on specific content, the space of learning provided by the teacher and instruction which is guided by the act of fostering and nurturing in learners, the capability to discern or notice that which is invariable (object of learning). This theory was effectively applied during lesson observation to gather data about how teachers created opportunities for learners to investigate the concept of a function using contrast and separation in order to arrive at generalization about linear graphs. Conceptual and procedural variation was also observed which allowed the fusion of concepts leading to content connection was also analysed through this theory. The two theoretical frameworks worked in tandem to provide a fuller picture of the knowledge possessed by each participant and the factors promoting or hindering quality in instruction as mediated by this knowledge in practice. The study concluded that when teachers use conceptual and procedural variation in the teaching of functions, learners are provided with opportunities to investigate knowledge hence improving their generic capabilities to discern the critical features of the object of learning in the learning of specific content. This finding again speaks to the quality of instruction observed. A divergent case was discovered in which an out of field teacher produced a prescribed level of mathematics through the use of variation in teaching.

Some of the findings were also supported by previous research. For instance, Mudaly (2015) had found that pre-service teachers' poor understanding of the concept of gradient was linked to the inability to form links between a gradient, the angle of inclination, the shape of the graph and the sign of the gradient value. This study also found that the concept of the angle of inclination was not taught explicitly in relation to the shape of a linear graph and that most explanation errors were observed as a result of a lack of clarity concerning the concept of a gradient. Similarly, the study found that the use of "Angithi?" or "Injani?" in chorus were used in the study as generalized invariant tags which limited learner participation thus creating fewer opportunities for the study to observe how the teachers dealt with learner questions. This was observed more in the lessons where the teachers came from previously disadvantaged schooling backgrounds. Literature does show that the use of the invariant tag question *isn't it?* is common in African English and undermines quality pedagogy (Kiramba & Smith, 2019).

The study concluded that teacher knowledge of the subject matter is important in the teaching of mathematics and that the type of resources available does influence the quality of instruction. In this regard, the divergent case was observed due to the availability of textbooks and learner workbooks which were used to supplement teacher knowledge or lack thereof. A growing body of literature points to the need to train teachers on how to choose textbooks, however, the study found that teachers also need training in our how to apply the variation theory used in the design of activities in learner workbooks. This will result in lessons which put the focus on the specific content and enhance in learners the ability to notice important knowledge. When the teacher is no longer the focus of the lesson, fewer errors based on incorrect explanations will be observed in teaching.

The study arose as a result of a gap in literature which focus on how specific content is taught in the classrooms in South Africa taking into account multiple schooling contexts. The study found that the teachers teaching in previously advantaged backgrounds were trained in the mathematics they taught and portrayed a sound knowledge of the topic observed. However, in the rural and township schools, the study found out of field teachers teaching GET mathematics. As a result the study observed fewer opportunities for learners to participate in worthwhile mathematics, however, as mentioned already, the lessons were enhanced by the use of variation in the design of activities from learner workbooks. The implication of this finding is that while the use of out of field teachers to teach mathematics is not a uniquely South African phenomenon, there is hope that this does not have to result in learners learning poor quality mathematics. Careful consideration is needed in the choice and review of learner workbooks to ensure that these are of good quality with absence of errors. The study found that errors contained in a workbook used in a Grade 9 classroom hindered effective teaching.

Other factors which were found to thwart the quality of instruction were failure to use prescribed learning materials, absence of manipulatives and concrete tools including analogies and examples to promote learning, lack of technical language in explanations, lack of the use a teacher's guide and the use of chalkboards. Factors which were found to have a positive effect on instruction included the use of technology and teachers' involvement in continued professional development and pursuit of higher academic degrees in mathematics teaching and the use of whiteboards.

Further implications arising from the findings are for pre-service educators and school management teams. The findings indicate a need to introduce pre-service teachers to knowledge domains reflecting the type of knowledge needed in the work of teaching. This knowledge will empower these prospective teachers with skills to state with clarity the meaning of their declared knowledge for teaching specific content. This skill is closely linked to the ability to reflect on one's capabilities to perform or teachers' self-efficacies. In schools this skill can be developed to meet the needs of out of field and qualified novice teachers. This points towards the need to promote professional learning communities (PLCs) as envisaged by the Department of Basic Education's policy on PLCs. Through these professional learning networks teachers can be taught how to identify knowledge gaps and other areas of need for continued professional development in line with the South African Council of Educators' (SACE) needs. Carefully formulated structures should involve teachers of the same subject as well as teachers of other subjects within the grade to cater for both horizontal and vertical knowledge in lesson planning. It is therefore concluded that the incompatibility of teachers' declared and observed knowledge found in the study can be reduced when pre-service and in-service teachers work with more knowledgeable mentors to develop skills to identify and reflect on their own mathematical knowledge for teaching.

The research findings address the study's aims and objectives by providing insight into what happens in the classrooms of knowledgeable and less knowledgeable teachers. The findings also provide a platform to make conclusions and recommendations about how resources can be used in the classrooms effectively. The use of a variety of methods to generate data allowed the study to focus on the objectives and hence arrive at findings which directly answer the study's research questions and hence address the aims and objectives formulated at the beginning.

I end this discussion with a summary of the four participants' knowledge and the effect this knowledge would have had on the quality of instruction by using the analogy of a soccer game.

Table 8-1 Summary of teacher knowledge observed from the four participants

Hear and forget (listen to a soccer game announced on the radio).	***See and remember (watch soccer players show off their skills on television).***
• Teacher is the dominant figure • Superficial learner engagement • Explanations may contain errors • The mathematics is routine • Procedures are memorised • Learners use calculators for simple procedures • Ineffective use of lesson time • Learners speak in chorus • Utilises a single teaching approach • Lessons lack direction and structure • Very little if any worthwhile mathematics is done. • Makes no attempt to cater for diverse learning abilities • Lacks ability to check for learner understanding • Seating arrangement does not foster group work • Learners have some freedom to consult with peers	• Teacher is the dominant figure • Mathematics is routine • Teacher follows a ready designed structured curriculum • Learners answer in chorus • Lacks ability to check for learner understanding • Explanations may contain errors • Builds on prior knowledge • Ineffective use of lesson time • Turns to logic to compensate for lack of conceptual understanding • Utilises a single teaching approach • Lacks ability to interpret curriculum material • Learners engage in the expected level of mathematics • Displays a good knowledge of procedures to routine problems • Seating arrangement does not foster group work • Learners have freedom to consult with peers • Makes no attempt to cater for diverse learning abilities

<table>
<tr><th>Do <u>and</u> understand (be one of the soccer players on the pitch).</th><th>Be present <u>and</u> semi-understand (watch a live soccer game from the stadium).</th></tr>
<tr><td>

Teacher is the dominant figure
Learners engage in guided discovery learning
Teacher uses technology
Learners use technology
The object of learning is made clear
Teacher designs own curriculum in collaboration with colleagues
Learners are meaningfully engaged
Learners engage freely in class discussions
Builds on prior knowledge
Learners engage in high level of mathematics
Displays good conceptual knowledge
Is spontaneous and can link topics with ease
Learners’ questions show depth of understanding
Teacher is adept at applying a variety of teaching approaches
Uses definitions to teach concepts
Teacher is enrolled for further study in mathematics education.
Teacher approaches the teaching content with confidence
Provides detailed summary of important concepts at every lesson
Teacher’s communication mostly includes the use of gestures
Learners are encouraged to use gestures to communicate abstract concepts.
Seating arrangement does not foster group work
Learners have freedom to consult with peers
Teacher’s focus is specialised content knowledge

</td><td>

Teacher is the dominant figure
Learners engage freely in class discussions
Good conceptual understanding displayed
Follows ready designed structured curriculum
Builds on prior knowledge
Is spontaneous and can use knowledge with ease
Comes up with own examples
Ignores the importance of definitions
Individual learners are recognised
Displays good procedural knowledge
Does not cater for diversity (equity)
Does not utilise multiple teaching approaches
Approaches the teaching content with confidence
Provides detailed summary of important concepts at every lesson
Teacher’s communication sometimes includes the use of gestures
Seating arrangement does not foster group work
Learners have freedom to consult with peers
Teacher’s focus is common content knowledge

</td></tr>
</table>

8.3 Towards understanding individual teacher knowledge (a new model)

It is a conclusion of this study based on the finding which shows a discrepancy between teachers' perceived and actual knowledge, that teachers are not able to identify exactly what they know or do not know about their own ability to teach specific content. I have also proposed that this knowledge is necessary and should be taught specifically to teachers. The variation theory alludes to a need to understand what specific approach can be used to teach specific content. The study noted a growing body of literature which reveals that reflective practice is being recognised by various teacher training institutions across the globe as a necessary vehicle to develop in prospective teachers the ability to ask relevant questions pertaining to their own knowledge of teaching and self-efficacy. Microteaching which includes lesson studies and other methods practiced within professional learning communities is used to model perceptions of what good teaching looks like to those who are entering the profession. Other researchers have also suggested that pre-service training programmes need to be tailored for individual needs of each student teacher.

It is difficult to talk about professionalism in mathematics teaching without considering how teachers rate their own knowledge as either good or bad. How do they know which areas they need to focus on for development as per SACE requirements? I suggest that a segment of teacher development training be dedicated to training teachers to develop the ability to diagnose their own mathematical knowledge for teaching. I propose a model within which this knowledge can be framed. Teachers working within professional learning communities in various schools can continue to use this model to identify areas of continued professional development. Anderson (2019) notes that not many studies in teacher education literature have attempted to document the conceptualisation of reflection-in-action during interactive teaching despite widespread discussions of this process. Moreover, according to Hartley, (2010, p.39) "knowing what questions to ask and when to ask them are difficult skills to develop and should be incorporated into teacher preparation programs so that teachers have the opportunity to hone their reflection skills with the support and guidance of mentors, teacher educators, and peers before being in a situation where they are required to use them on their own". This study recognised this gap in literature and suggests a model of reflective practice discussed in 8.4 – 8.5 of this chapter.

8.4 A new model of reflective practice

I propose a model of self-assessment knowledge for teaching. The model illustrates how teachers can reflect on their own knowledge of mathematics for teaching in order to improve their quality of instruction. This model is presented in figure 8-1.

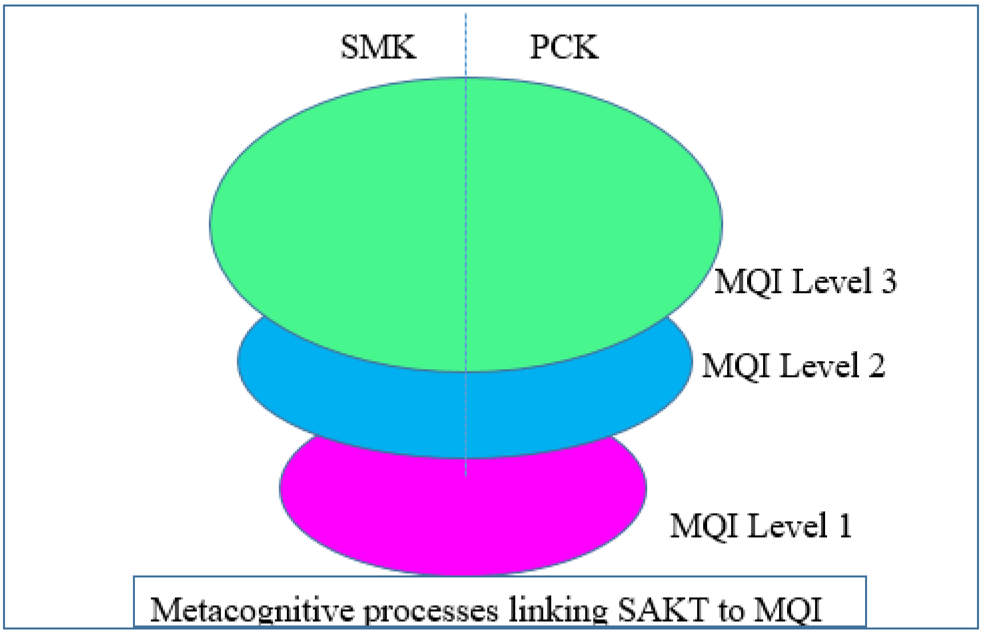


Figure 8-1: An illustration of the relationship between reflective practice and the quality of instruction

The model assumes that teachers begin their knowledge journey during pre-service training and that they continue to grow in their knowledge of various aspects of mathematics within the school curriculum through professional development programmes which include induction, mentorship, post-graduate degree studies and other subject-specific courses. The mathematical quality of instruction (MQI) levels are associated with the level of reflection required. Reflective questions are developed at each level of MQI starting with basic personal reflective questions at MQI level 1. It is proposed that MQI level 1 provides exit level questions which should be developed at pre-service teacher training. It should be the goal of pre-service teacher education programmes to develop in student teachers the ability to reflect on their current knowledge of the subject focusing on specific content. Marton et al (2004) suggested that when the space of learning is created during the use of variation, teachers continually raise the following questions (Marton,et al, 2004, p.179):

What is actually learned, when learning about fractions?

Why is learning fractions difficult?

What is it that makes learning fractions difficult?

How can we conceptualize these difficulties?

The study proposes the use of exit level reflective questions similar to the ones developed in tables 8-2 to 8-4:

8.5 The critical reflective process

Level 1 reflection

Level 1 reflection is the type of reflection that all teachers should be able to do because it seeks to answer basic and foundational questions about the teacher's MKT prior to teaching a lesson. This stage can be enhanced when teachers work collaboratively because this will enable them to come up with a range of questions specific to the content they are preparing to teach. However, both novice and experienced teachers should be able to embark on this reflection as this knowledge would have been encountered at pre-service training.

Table 8-2: Exit level critical reflection linked to MQI Level 1

Type of knowledge	Critical reflective questions
Self-assessment knowledge (SAKT) with regard to specific content	What is the specific outcome of this content?
	How does it link to previous topics?
	What prior knowledge do I need?
	What is my strength with regard to prior knowledge?
	What is my weakness with regard to prior knowledge?
	What are my beliefs about how learners learn this content?
	What do I find easy about this content?
	What do I find difficult about this content?
	What are the important concepts linked to this content?
	How do I enhance my knowledge and understanding of this content?
	What would hinder my ability to teach this topic?

Knowledge of content and teaching (KCT)	What teaching approach is best suited for this content?
	How will I connect this topic with daily life?
	How can I apply multiple representations in my teaching of this content?
	How will I use differentiation in my teaching?
	What other instructional strategies will I need to consider?
Knowledge of content and students (KCS)	How will I check for learner understanding?
	How will I bridge any gaps that emerge in learner understanding?
	What misconceptions exist in learner understanding of this topic?
	How will I use learner misconceptions to enhance teaching?
	How will I make content relevant to learners?
Curriculum knowledge (CK)	What curriculum material will I need?
	How will I connect this topic with other content within the curriculum?
	Which resource would I start with?
Horizon knowledge	How does this content link to content taught in other subjects within the same grade?
	Which colleagues will I need to collaborate with?
	How does this content progress in the grades ahead?

Level 2 reflection

Critical reflective questions associated with level 2 enable teachers to reflect on the actual instruction during teaching. While level 1 reflection takes place during the transformation process, level 2 reflection happens during the actual instruction process. MQI level 2 is reached when teachers engage in both level 1 and level 2 critical reflection. When teachers learn to reflect on their practice while they are engaged in the teaching process, their quality of instruction is enhanced as this reflection enables them to gather enough information to gauge the effectiveness of their teaching and whether they are achieving the intended outcome. This results in teachers determining the closeness of the intended object of learning as enacted in the space of learning, to the lived object of learning. Are the learners discerning what they need to discern? This reflection allows for flexibility to adjust instructional strategies to allow for maximum learning to take place, hence improving the quality of instruction.

Table 8-3 Level 2 critical reflective questions

Type of knowledge	Critical reflective questions
Self-reflection during teaching	Are the learners engaged?
	What misconceptions are arising?
	Which curriculum resource should I consult?
	How do I check for understanding?
	How am I doing with time?
	How can I use their prior knowledge?
	How do I ensure maximum participation?
	Why am I using this approach?
	Am I achieving the goal of this teaching?
	Is my teaching differentiating in terms of learner ability?
	Are they noticing what they should notice?

Level 3 reflection

Level 3 reflection happens when teachers do post-lesson evaluation of their teaching after the instructional process. This reflection allows teachers to look back at their teaching resulting in what Shulman (2004) referred to as new comprehension, for the purpose of refining future lesson plans to improve teaching. This new comprehension pertains to teachers themselves, learners and the subject matter. MQI level 3 is reached when teachers use lesson plans to reflect on their knowledge of the subject, prior to teaching particular content, during instruction and after the teaching of specific content. MQI level 3 therefore involves all 3 levels of reflection. Expert teachers are able to do this reflection on practice with ease (Beswick, Fraser & Crowley, 2016)

Table 8-4 Level 3 critical reflective questions

Type of knowledge	**Critical reflective questions**
Post-lesson reflection	What worked well?
	What went wrong?
	What needs to stay the same?
	What needs to change?
	Why did this strategy not work?
	Were the learners engaged?
	How close was I to achieving the curriculum goals?
	How can I be sure that learners acquired the necessary capabilities?
	What would I do differently in future?
	How well did I use the time?

8.6 Opportunities for mathematical knowledge for teaching to grow

I suggest that teachers continuously assess their understanding of content during lesson preparation and that this knowledge leads to growth in MKT during the transformation of content for instructional purposes. The analysis of Terry's MKT items in Chapter five revealed that she was reflecting on her own knowledge or understanding of the content by asking questions and clarifying concepts to herself to ensure that she was clear about what she was explaining or about to explain to the learner. This provides opportunities for knowledge growth as teachers consult with colleagues and research the content prior to teaching it, the study did find that the participants collaborated with teachers who taught in the FET. Opportunities for MKT growth also takes place during pre-service training and this was confirmed by Terry and Lily who had done work experience by observing high school teachers or mentor teachers teach mathematics. Lesson studies and other forms of micro teaching implemented at the pre-service training programmes all provide opportunities for growth in MKT. During the process of teaching, teachers grow in their knowledge of content and students (KCS) as they interact with learners and do assessment for learning. Teachers also grow in their knowledge of content and teaching (KCT) as they apply various approaches and continuously assess the effectiveness of representations in each lesson. Post teaching reflection happens when teachers individually or in collaboration with colleagues review the lesson for purposes of identifying areas of future development. Figure 8-2 illustrates the four stages of growth in MKT through reflective practice.

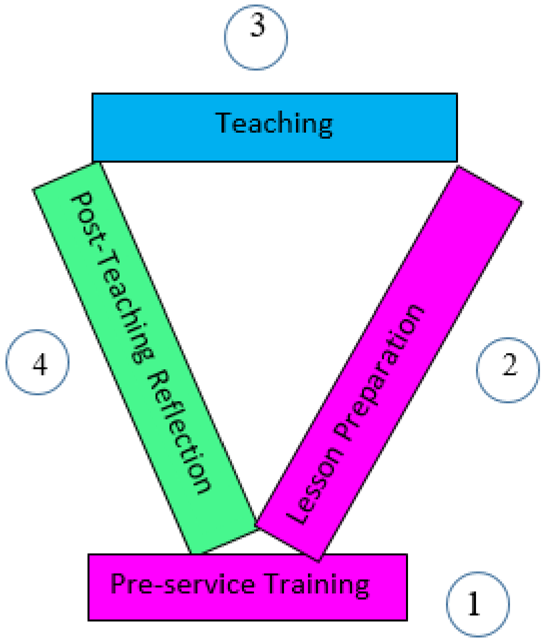


Figure 8-2: Opportunities for growth in MKT through reflective practice

The study proposes that pre-service training and lesson preparation are associated with level 1 MQI and level 1 reflection because pre-service and in-service teachers encounter knowledge that is foundational to teaching. Both these stages are viewed as rehearsal phases to classroom teaching which takes into account all pedagogical elements of teaching. During teaching the MKT growth results in better instruction. As teachers re-adjust their teaching accordingly during new comprehensions, the mathematical quality of instruction grows. This was observed in Terry's class when a learner pointed out that the graphs occupied certain quadrants, and Terry had not anticipated this misconception. Working with the learner she investigated the validity of the learner's claim and in so doing discovered a misconception which she later addressed. It can therefore be argued that Terry's MKT grew during this teaching episode and that her MQI was improved when she addressed the learner's misconception to the benefit of the whole class. Data generated from the interviews corroborate what was observed in the classroom.

H: On a scale of 1 to 10, how well would you say you are familiar with the errors and misconceptions that learners make in maths?

Terry: I think I should look up more misconceptions but time isn't there. From the questions they ask, I pick up misconceptions and I say oh that's why you're doing it.

Teachers use the new knowledge gained from post-teaching reflection to design improved lesson plans. The implementation of these plans results in improved teaching thus enhancing their MQI. The study observed that the participants were keen to receive post teaching feedback and that this feedback was used to design future lessons. Another evidence of this growth in MKT was observed in Amanda's teaching. In the lesson that was observed, Amanda corrected her own previous explanation as illustrated in the following excerpt:

> Amanda: Remember yesterday I said that to find the gradient, you have to use the x and y intercepts?
> Chorus: Yes!
> Amanda: You can actually use any two points not just the x and y intercept.

Amanda came across this knowledge while preparing to teach her next lesson. She then realised that while her previous explanation had been correct, it was, however, limited and not consistent with the work which learners would encounter in the following and subsequent lessons. A number of studies reviewed on reflective practice indicate that lesson studies are the best way of assessing and improving the knowledge of teaching specific content.

8.7 Conclusion

The study posed four questions in order to generate and analyze data from four participants. The aim of the research was to investigate how GET teachers use their mathematical knowledge for teaching to influence the quality of instruction in the teaching of functions. The study asked: 1) How do teachers perceive their mathematical knowledge for teaching functions? 2) How does a teacher's content knowledge of functions influence their teaching? 3) What other factors influence the quality of instruction? 4) Why does teacher knowledge influence instruction in the way it does?

The first chapter introduced the research questions and highlighted the background and the purpose of the study. Chapter two offered a review of the literature consulted to place the study within a broader context of previous research conducted in the area of teacher knowledge. In the third chapter I presented the theoretical framework underpinning this current research. The fourth chapter highlighted the methodology used to generate data in order to answer the research questions. This chapter also covered issues of consent and validity. Chapters five and six presented and analyzed data generated from interview transcripts, written items, video recordings, document analysis and the field journal. Chapter seven presented the results and findings. This chapter also showed how the study answered the three research questions. I began Chapter 8 with the discussion of findings, offered a new model of reflective practice developed from literature reviewed and from the findings of the study, and the chapter ends with the conclusion and recommendations.

Data generated from pen and paper items corroborated the results of the interviews and the data generated from the classroom observations. This suggests that teacher knowledge does influence the quality of classroom instruction. The study had hypothesised that teachers with a sound knowledge of the subject matter as demonstrated in the pen and paper items will also demonstrate this knowledge by designing and exposing learners to high quality instructional activities.

The findings support the literature which shows that teachers' subject matter knowledge hugely impacts on the quality of instruction. The study, however, concluded that a lack of subject matter knowledge does not stop teachers from delivering lessons of acceptable level as required by the CAPS if they follow readily designed lesson plans and make use of prescribed curriculum materials including learner workbooks. It was concluded that when teachers do this, they are able to involve learners in worthwhile learning of mathematics similar to that made available to learners in classrooms where the teacher has a sound knowledge of the subject matter.

With regard to the gap in the literature identified earlier, the study documented that studying teacher knowledge taking into account the context of the school, provides useful insight into how the curricular materials supplied by the government are used in various schools. The study also shed some light regarding the knowledge of out of field teachers (with no prior training in mathematics teaching). A useful finding for policy makers is that government supplied teaching and learning resources can be used to supplement the mathematical knowledge of out of field teachers, resulting in better quality instruction in schools from previously disadvantaged background.

The traditional teacher-centred method was found to be the more dominant approach in the majority of the lessons observed. The theory of variation is critical of the view of education as being either progressive, because it puts the learner at the centre or traditional, because it regards the content as more important. The theory also opposes the notion that a particular orientation can be used to teach all content in all classrooms. A teacher in the study who had declared that she was using traditional transmission style in her teaching was surprised by the direction taken by a lesson designed using variation theory. Through the space of learning created by the teacher, the lived object of learning far exceeded the intended object of learning because of technology. The study also observed how an out of field teacher was able to create opportunities for learners to do worthwhile mathematics through the use of workbooks provided by the government. This led to the conclusions that when the focus is on creating in learners capabilities to discern or notice in the learning of specific content, the focus is no longer about whether the teaching approach is progressive or traditional. The focus is shifted towards empowering learners with skills to identify knowledge that is germane.

It is therefore the conclusion of this study that pedagogical content knowledge is necessary for quality instruction but quality instruction does not always equal learner-centred pedagogy as defined by many writers. Learner capabilities are enhanced when learners work on their own, following carefully designed activities which include investigations where variation of procedure and concepts leads to generalisation and learning of key concepts. This learning experience does not necessarily rely on the teacher's knowledge or familiarity with progressive methods in order to make connections, but it relies on the space of learning created. The implication of this finding is that professional development programmes need to focus more on developing in teachers the ability to choose carefully what textbooks and other teaching and learning materials to use. Teachers should also be trained on how to use these resources effectively in the classroom. The focus should also be on investigating and discovering ways to maximise the learning of specific content regardless of whether the learner is present in the classroom or learning remotely at home. Teachers should be taught how to design or choose activities which promote in learners the ability to discern or notice that which is invariable.

In relation to the study's objectives and research questions, the findings indicate that teachers are finding it difficult to specify exactly what it is that they know or do not know about their knowledge of teaching functions and other content. The skill and the ability to identify this knowledge needs to be taught and developed. The findings further indicate that subject matter knowledge is necessary in order to avoid errors in instruction which might hinder further understanding of key concepts. However, other factors like the use of technical language, technology and the choice of instructional materials also have an effect on the quality of instruction delivered in the classroom.

The study proposes a new knowledge domain based on the model of reflective practice which aims to assist teachers with identifying individual knowledge areas of need for continued professional development. This is the study's contribution to teacher knowledge. This knowledge domain which is framed within the mathematical knowledge for teaching and quality of instruction model was presented and discussed in 8.3 – 8.6. In this model I have proposed that content specific reflective practice skills can be taught at pre-service teacher training and to out of filed and qualified novice teachers. Through this model, these prospective and in-service teachers can be empowered with skills to ask the right kind of questions pertaining to their own knowledge of specific content. I have argued that through a framework of reflective practice within professional learning communities, teachers can design lesson studies informed by the reflective questions. Teachers can also identify areas of need for continued professional development as required by the SACE.

The study has limitations in that data was generated from four participants teaching in different schools. It is therefore difficult to generalize on the findings due to the size of the sample and also because the participants came from four various schooling contexts.

8.8 Recommendations

I would like to make the following recommendations based on the findings of the study:

A follow up research project should be undertaken concerning how the government supplied teaching and learning resources are being utilised by the schools and individual teachers. The study found a wealth of electronic resources which none of the teachers in the study mentioned or used in their teaching.

I further recommend that an awareness campaign be instituted to alert teachers to the availability of electronic resources and that workshops should be conducted in order to equip and educate teachers on how to access these online materials. Teachers also need to be educated on how to use text books to enhance the use of variation in teaching which should enhance the procedural and conceptual knowledge and skills of both learners and teachers. It is also important that textbooks and learner workbooks are checked thoroughly for errors before being printed out and distributed to schools as this can have an adverse effect on learning especially in subjects like mathematics.

Studies should also be conducted which investigate the effect of teacher knowledge on learner achievement with a focus on comparing knowledgeable and less knowledgeable teachers.

References

Abdulhamid, L., & Venkat, H. (2014). Research-led development of primary school teachers' mathematical knowledge for teaching: A case study. *Education as Change, 18,* S137–S150. https://doi-org.ukzn.idm.oclc.org/10.1080/16823206.2013.877355

Aalto, E., Tarnanen, M., Heikkinen, H.L.T. (2019). Constructing a pedagogical practice across disciplines in pre-service teacher education. Teaching and Teacher Education, 85, 69 – 80.

Adler J. (2017). Mathematics in mathematics education. S Afr J Sci, 113(3/4), Art. #a0201, 3 pages. http://dx.doi. org/10.17159/sajs.2017/a0201

Adler, J., & Venkat, H. (2014). Teachers' mathematical discourse in instruction: Focus on examples and explanations. In Venkat, H., Rollnick, M., Loughran, J., & Askew, M. (eds). (pp 132-146). London: Routledge.

Akerson, V. L., Pongsanon, K., Rogers, M. A. P., Carter, I., & Galindo, E. (2017). Exploring the use of lesson study to develop elementary preservice teachers' pedagogical content knowledge for teaching nature of science. *International Journal of Science and Mathematics Education*, *15*(2), 293-312.

Akew, M. (2014). Mathematics teachers' content knowledge. In Venkat, H., Rollnick, M., Loughran, J., & Askew, M. (eds). (pp 3-14). London: Routledge.

Aljaberi, N. M., & Gheith, E. (2018). In-Service Mathematics Teachers' Beliefs about Teaching, Learning and Nature of Mathematics and Their Mathematics Teaching Practices. *Journal of Education and Learning*, *7*(5), 156–173. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ1185907&site=ehost-live

Anderson, J. (2019). In search of reflection-in-action: An exploratory study of the interactive reflection of four experienced teachers. *Teaching and Teacher Education*, *86*, 102879.

Anthony, G., & Walshaw, M. (2009). Characteristics of effective teaching of mathematics: A view from the West. *Journal of Mathematics Education*, *2*(2), 147-164.

Anthony, M., & Walshaw, G. (2009). Characteristics of Effective Teaching of Mathematics: A View from the West *Journal of mathematics education*, *2*(2), 147-164.

Awan, S.A. (2013). Comparison between Traditional Text-book Method and Constructivist Approach in Teaching the Concept "Solution." *Journal of Research & Reflections in Education (JRRE)*, *7*(1), 41–51. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eue&AN=91269290&site=ehost-live

Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education* (Doctoral dissertation, Michigan State University).

Ball, D. L., & Rowan, B. (2004). Introduction: measuring instruction. *The Elementary School Journal*, *105*(1), 3-10.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide?.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide?.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special? *Journal of teacher education*, *59*(5), 389-407.

Ball, D.L, & Bass, H. (2009). With an Eye on the Mathematical Horizon. *Beiträge zum Mathematikunterricht 2009*.

Ball, D.L. (2011). *Mathematics Teaching and Learning to Teach*. School of Education: University of Michigan

Ball, D.L., Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., & Sleep, L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and instruction*, *26*(4), 430-511.

Bansilal, S. (2014). Recognising and illuminating connections in proportional relationships. In Venkat, H., Rollnick, M., Loughran, J., & Askew, M. (eds). (pp 49-64). London: Routledge.

Barnhart, T., & van Es, E. (2015). Studying teacher noticing: Examining the relationship among pre-service science teachers' ability to attend, analyze and respond to student thinking. Teaching and Teacher Education, 45, 83 – 93.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., … Tsai, Y.-M. (2010). Teachers' Mathematical Knowledge, Cognitive Activation in the Classroom, and Student Progress. *American Educational Research Journal*, *47*(1), 133–180. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ883791&site=ehost-live

Berger, M. (2013). Examining mathematical discourse to understand in-service teachers' mathematical activities. *pythagoras*, *34*(1), 1-10.

Beswick, K., Fraser, S., & Crowley, S. (2016). " No Wonder Out-of-Field Teachers Struggle!": Unpacking the Thinking of Expert Teachers. *Australian Mathematics Teacher*, *72*(4), 16-20.

Bohlmann, C. A., Prince, R. N., & Deacon, A. (2017). Mathematical errors made by high performing candidates writing the National Benchmark Tests. *Pythagoras*, *38*(1), 1-10.

Brijlall, D., & Isaac, V. (2011). Links between content knowledge and practice in a Mathematics Teacher Education course: A case study. *South African Journal of Higher Education*, *25*(4), 660–679. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eue&AN=82584913&site=ehost-live

Busi, R., & Jacobbe, T. (2018). The Impact of Analyzing Student Work on Preservice Teachers' Content Knowledge and Beliefs about Effective Mathematics Teaching. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, *1*. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ1189602&site=ehost-live

Busi, R., & Jacobbe, T. (2018). The Impact of Analyzing Student Work on Preservice Teachers' Content Knowledge and Beliefs about Effective Mathematics Teaching. *Issues in the Undergraduate Mathematics Preparation of School Teachers, 1*.

Bybee, R. W.(2014) .The BSCS 5E Instructional Model: Personal Reflections and Contemporary Implications. *Science & Children*, *51*(8), 10–13. https://doi-org.ukzn.idm.oclc.org/10.2505/4/sc14_051_08_10

Cai, J., & Wang, T. (2010). Conceptions of effective mathematics teaching within a cultural context: Perspectives of teachers from China and the United States. *Journal of Mathematics Teacher Education*, *13*(3), 265-287.

Cao, Y., Postareff, L., Lindblom-Ylänne, S., & Toom, A. (2019). Teacher educators' approaches to teaching and connections with their perceptions of the closeness of their research and teaching. *Teaching and Teacher Education*, *85*, 125-136.

Carnoy, M. (2012). Comparing learner performance in southern Africa: A natural experiment. In. doi:DOI 10.1007/s11125-012-9248-4

Charalambous, C. Y. (2015). Working at the Intersection of Teacher Knowledge, Teacher Beliefs, and Teaching Practice: A Multiple-Case Study. *Journal of Mathematics Teacher Education*, *18*(5), 427–445. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ1075081&site=ehost-live.

Charles, R. I. (1980). Exemplification and Characterization Moves in the Classroom Teaching of Geometry Concepts *Journal for Research in Mathematics Education*, *11*(1), 10-21.

Chauraya, M., & Brodie, K. (2018). Conversations in a professional learning community: An analysis of teacher learning opportunities in mathematics. *pythagoras*, *39*(1), 1-9.

Chirinda, B., & Barmby, P. (2017). The development of a professional development intervention for mathematical problem-solving pedagogy in a localised context. *Pythagoras*, *38*(1), 1-11.

Choi, J., Lee, J. H., & Kim, B. (2019). How does learner-centered education affect teacher self-efficacy? The case of project-based learning in Korea. *Teaching and Teacher Education*, *85*, 45-57.

Chong, S. C.; Cheah H.M. (2009). A Values, Skills and Knowledge Framework for Initial Teacher Preparation Programmes *Australian Journal of Teacher Education*, *34*(3), 1-17

Christiansen, I. M. (2012). Ready to teach? Reflections on a South African mathematics teacher education programme. *Journal of Education*(56), 164-195.

Christiansen, I., & Aungamuthu, Y. A. (2012). Language issues, '"misconceptions"' and confusion: A qualitative analysis of KZN grade 6 learners' responses on a mathematics test. *Education as Change*, *16*(1), 51–67. https://doi-org.ukzn.idm.oclc.org/10.1080/16823206.2012.691713

Chye, S., Zhou, M., Koh, C., & Liu, W. C. (2019). Using e-portfolios to facilitate reflection: Insights from an activity theoretical analysis. *Teaching and Teacher Education*, *85*, 24-35.

Civitillo, S., Juang, L.P., Badra, M., & Schachner, M, J. (2019). The interplay between culturally responsive teaching, cultural diversity beliefs, and self-reflection: A multiple case study. Teaching and Teacher Education, 77, 341 – 351.

Cogill, J. A. (2008). *Primary teachers' interactive whiteboard practice across one year: changes in pedagogy and influencing factors* (Doctoral dissertation, University of London).

Conference Board of the Mathematical Sciences (2012). The Mathematical Education of Teachers II. Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.

Costandius, E., & Botes, H. (Eds.). (2018). *Educating citizen designers in South Africa*. African Sun Media.

Courtey-Clarke, M., & Wessels, H. (2014). Number sense of final year pre-service primary school teachers. *pythagoras*, *35*(1), 1-9.

Darling-Hammond, L., Hyler, M. E., & Gardner, M. (2017). Effective teacher professional development.

Das, G. C. (2015). Pedagogical Knowledge in Mathematics: A Challenge of Mathematics Teachers in Secondary Schools. *International Journal of Information and Education Technology*, *5*(10), 789.

de Freitas, G., & Spangenberg, E. D. (2019). Mathematics teachers' levels of technological pedagogical content knowledge and information and communication technology integration barriers. *Pythagoras*, *40*(1), 13.

de Jong, L., Meirink, J., & Admiraal, W. (2019). School-based teacher collaboration: Different learning opportunities across various contexts. *Teaching and Teacher Education*, *86*, 102925.

Depaepe, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concept has pervaded mathematics educational research. *Teaching and teacher education*, *34*, 12-25.

Department of Basic Education. (2011). Curriculum and Assessment Policy Statement (CAPS). *Grades 7–9. Natural Sciences*.

Department of Basic Education. (2015). Action Plan to 2019: Towards the realisation of schooling 2030. Taking forward South Africa's National Development Plan 2030.

Department of Education. (2006). The national policy framework for teacher education and development in South Africa.

Ding, L., & Domínguez, H. (2016). Opportunities to notice: Chinese prospective teachers noticing students' ideas in a distance formula lesson. *Journal of Mathematics Teacher Education*, *19*(4), 325-347.

Dionne, J. J. (1993). Third Misconceptions Seminar Proceedings.

Dolittle, P.E. (1997). Vygotsky's zone of proximal development as a theoretical foundation for cooperative learning. Journal on Excellence in College Teaching. 8(1),83-103.

du Plessis, E. (2020). Student teachers' perceptions, experiences, and challenges regarding learner-centred teaching. *South African Journal of Education*, *40*(1).

Dündar, S., Güvendir, M. A., Kocabiyik, O. O., & Papatga, E. (2014). Which Elementary School Subjects Are the Most Likeable, Most Important, and the Easiest? Why?: A Study of Science and Technology, Mathematics, Social Studies, and Turkish. *Educational Research and Reviews*, *9*(13), 417–428. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ1041142&site=ehost-live

Ellis, V., & Childs, A. (2019). Innovation in teacher education: Collective creativity in the development of a teacher education internship. *Teaching and Teacher Education*, *77*, 277-286.

Engelke, N., Oehrtman, M., & Carlson, M. M. C. (2005). Composition of Functions: Precalculus Students' Understandings. *Conference Papers -- Psychology of Mathematics & Education of North America*, 1–8. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eue&AN=31572536&site=ehost-live.

Essien, A. A. (2010). Mathematics Teacher Educators' Account of Preparing Pre-service Teachers for Teaching Mathematics in Multilingual Classroom: The Case of South Africa. *International Journal of Interdisciplinary Social Sciences*, *5*(2), 33–44. https://doi-org.ukzn.idm.oclc.org/10.18848/1833-1882/CGP/v05i02/51603

Essien, A. A. (2010). What teacher educators consider as best practices in preparing pre-service teachers for teaching mathematics in multilingual classrooms. *Perspectives in Education*, *28*(4), 32-42.

Feldman, J. (2020). The role of professional learning communities to support teacher development: A social practice theory perspective. *South African Journal of Education*, *40*(1).

Fennema, E., & Franke, M.L. (1992). Teachers' knowledge and its impact. In D.A.Grouws (Ed), Handbook of research on mathematics teaching and learning (pp 147-164). Macmillan Publishing Company: New York.

Findell, C. R. (2009). What Differentiates Expert Teachers from Others? *Journal of Education*, *188*(2), 11–23. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eue&AN=44087961&site=ehost-live

Finlay, L., & Gough, B. (Eds.). (2008). *Reflexivity: A practical guide for researchers in health and social sciences*. John Wiley & Sons.

Font, V., Bolite, J., & Acevedo, J. (2010). Metaphors in mathematics classrooms: analyzing the dynamic process of teaching and learning of graph functions. *Educational Studies in Mathematics*, *75*(2), 131–152. https://doi-org.ukzn.idm.oclc.org/10.1007/s10649-010-9247-4

Fook, J. (2015) 'Reflective practice and critical reflection', in Handbook of Theory for Practice Teachers: A New Updated Edition, ed. J. Lishman, Jessica Kingsley, London

Forehand, M. (2010). Bloom's taxonomy. *Emerging perspectives on learning, teaching, and technology*, *41*(4), 47-56.

French, D. (2003). Subject knowledge and pedagogical knowledge. *Mathematics Education Review: Journal of The Association of Mathematics Education Teachers*, *16*, 3-11.

French, D. (2005). Subject knowledge and pedagogical knowledge. Mathematics Education Review, 16, 3-11.

Frick, L., Carl, A., & Beets, P. (2010). Reflection as learning about the self in context: Mentoring as catalyst for reflective development in pre-service teachers. *South African Journal of Education*, *30*(3).

Gardee, A. (2015). A teacher's engagement with learner errors in her Grade 9 mathematics classroom. *Pythagoras*, *36*(2), 1-9.

Gencturk, Y. C. (2012). *Teachers' mathematical knowledge for teaching, instructional practices, and student outcomes* (Doctor of Philosophy), Illinois

Gierdien, F. (2012). Pre-service teachers' views about their mathematics teacher education modules. *Pythagoras*, *33*(1), 1-10.

Grady, M., Watkins, S., & Montalvo, G. (2012). The Effect of Constructivist Mathematics on Achievement in Rural Schools. *Rural Educator*, *33*(3), 37–46. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ987623&site=ehost-live

Guthrie, G. (2010). Basic research methods. Sage publications: California

Hadfield, M., & Jopling, M. (2016). Problematizing lesson study and its impacts: Studying a highly contextualised approach to professional learning. *Teaching and teacher education*, *60*, 203-214.

Ham, M., & Dekkers, J. (2019). What role do teachers' beliefs play in the implementation of educational reform?: Nepali teachers' voice. *Teaching and Teacher Education*, *86*, 102917.

Hartley,A.(2010). You can't teach what you don't know: Examining and improving teacher preparation: A thesis submitted to the faculty of wesleyan university.

Hashweh, M. (1986). Towards an explanation of conceptual change. International Journal of Science Education, 8(3), 229 – 249. Retrieved from DOI: 10.1080/0140528860080301

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for research in mathematics education*, 330-351.

Hill, H.C., Ball, D.L., & Schilling, S.G. (2008). *Unpacking pedagogical content knowledge: Conceptualizing and Measuring Teachers' Topic-Specific knowledge of students*: Journal for research in Mathematics Education, 39(4) 372-400.

Hobden, S., & Mitchell, C. (2011). Maths and me: Using mathematics autobiographies to gain insight into the breakdown of mathematics learning. *Education as Change*, *15*(1), 33–46. https://doi-org.ukzn.idm.oclc.org/10.1080/16823206.2011.566572

Høynes, S. M., Klemp, T., & Nilssen, V. (2019). Mentoring prospective mathematics teachers as conductors of whole class dialogues–Using video as a tool. *Teaching and Teacher Education: An International Journal of Research and Studies*, *77*(1), 287-298.

Høynes, S. M., Klemp, T., & Nilssen, V. (2019). Mentoring prospective mathematics teachers as conductors of whole class dialogues–Using video as a tool. *Teaching and Teacher Education: An International Journal of Research and Studies*, *77*(1), 287-298.

Huang, R., Barlow, A.T., & Prince, T. (2016). The same tasks, different learning opportunities: An analysis of two exemplary lessons in China and the U.S. from a perspective of variation. The journal of Mathematics Behaviour, 41, 141 – 158.

Hurrell, D.P. (2013). *What teachers need to know to teach mathematics: An argument for a reconceptualized model:* Australian Journal of Teacher Education, 38(11). Jossey-Bass.

Inoue, N., Asada, T., Maeda, N., & Nakamura, S. (2019). Deconstructing teacher expertise for inquiry-based teaching: Looking into consensus building pedagogy in Japanese classrooms. Teaching and Teacher Education, 77, 366 – 377.

Kabael, T., & Tanışlı, D. (2010). Teaching from Patterns to Functions in Algebraic Thinking Process. Elementary Education Online, 9(1), 213-228.

Kilic, H. (2011). Preservice Secondary Mathematics Teachers' Knowledge of Students. *Online Submission*, *2*(2), 17-35.

Kılıç, H. (2011). Preservice Secondary Mathematics Teachers' Knowledge of Students. *Turkish Online Journal of Qualitative Inquiry*, *2*(2), 17–35. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eue&AN=65721681&site=ehost-live

Kilpatrick, J. (1992). A history of research in mathematics education. In D.A.Grouws (Ed), Handbook of research on mathematics teaching and learning (pp 3-38). Macmillan Publishing Company: New York.

Kiramba, L. K., & Smith, P. H. (2019). "Her sentence is correct, isn't it?": Regulative discourse in English medium classrooms. *Teaching and Teacher Education*, *85*, 105-114.

Kjeldsen, T. H., & Lützen, J. (2015). Interactions Between Mathematics and Physics: The History of the Concept of Function—Teaching with and About Nature of Mathematics. *Science & Education, 24*(5-6), 543-559. doi:10.1007/s11191-015-9746-x

Kwenda, C. (2014). In search of a model for best practice in student teaching practice: A comparative study of South Africa and Zimbabwe. *Journal of Educational Studies*, *13*(2), 216-237.

Kwong, C. W., Joseph, Y.K.K.,Eric, M.C.C., Khoh, L.S., Gek, C.K., & Eng, L.N. (2007). Development of mathematics pedagogical content knowledge in student teachers: *The Mathematics Educator*, *10*(2), 27-54.

Lai, M. Y. (2012). Teaching with Procedural Variation: A Chinese Way of Promoting Deep Understanding of Mathematics

Lannin, J.K., Webb, M., Chval, K., Arbaugh, F., Hicks, S., Taylor, C., & Brutonet, R. (2013). The development of beginning mathematics teacher pedagogical content knowledge: *Journal of Mathematics Teacher Education*, 16(6): 403-426.

Laschke, C. (2013). Effects of Future Mathematics Teachers' Affective, Cognitive and Socio-Demographic Characteristics on Their Knowledge at the End of the Teacher Education in Germany and Taiwan. *International Journal of Science and Mathematics Education*, *11*(4), 895–921. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ1037557&site=ehost-live

Ledibane, M., Kaiser, K., & van der Walt, M. (2018). Acquiring mathematics as a second language: A theoretical model to illustrate similarities in the acquisition of English as a second language and mathematics. *Pythagoras*, *39*(1), 1-12.

Leendertz, V., Blignaut, A. S., Nieuwoudt, H. D., Els, C. J., & Ellis, S. M. (2013). Technological pedagogical content knowledge in South African mathematics classrooms: A secondary analysis of SITES 2006 data. *Pythagoras*, *34*(2), 1-9.

Leendertz, V., Blignaut, A.S., Nieuwoudt, H.D., Els, C.J., & Ellis, S.M. (2013). Technological pedagogical content knowledge in South African mathematics classrooms: A secondary analysis of SITES 2006 data. Pythagoras, 34(2), Art. #232, 9 pages. http://dx.doi. org/10.4102/pythagoras. v34i2.232

Leung, A. (2012). Variation and Mathematics Pedagogy. *Mathematics Education Research Group of Australasia*.

Levin, T., & Wadmany, R. (2006). Teachers' Beliefs and Practices in Technology-Based Classrooms: A Developmental View. *Journal of Research on Technology in Education*, *39*(2), 157–181. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ768874&site=ehost-live

Liljedahl, P. (2008). Teachers‘ beliefs as teachers‘ knowledge. Paper presented at the Symposium on the Occasion of the 100th Anniversary of ICMI (Rome, 5–8 Mar 2008).

Llinares, S. (2020). Promoting explicit connections in mathematics teaching: scopes for the teachers learning in context. *Journal of Mathematics Teacher Education*, *23*(1), 1-4.

Lo, M.L. (2012). Variation theory and the improvement of teaching and learning. Gothenburg, Sweden: Acta Universitatis Gothoburgensis. Available from http://gupea.ub.gu.se/handle/2077/29645

Luneta, K. (2015). Understanding students' misconceptions: an analysis of final Grade 12 examination questions in geometry. *Pythagoras*, *36*(1), 1-11.

Luneta, K. (2015). Understanding students' misconceptions: an analysis of final Grade 12 examination questions in geometry. *Pythagoras*, *36*(1), 1-11.

Ma, L. (1999). Knowing and teaching elementary mathematics: Teachers understanding of fundamental mathematics in China and the United States. Hillsdale, NJ: Erlbaum

Maass, K., Swan, M., & Aldorf, A.-M. (2017). Mathematics Teachers' Beliefs about Inquiry-Based Learning after a Professional Development Course--An International Study. *Journal of Education and Training Studies*, *5*(9), 1–17. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ1150550&site=ehost-live

MacDonald, C., & Burroughs, E. (1991). Eager to talk and learn and think. *Cape Town: Maskew Miller Longman*.

Maher, M. (2015). Lighting up learning: mathematics becoming less of a'killer subject'in KwaZulu-Natal, South Africa. *International Journal for Mathematics Teaching and Learning*.

Maniraho, J. F. (2017). *The pedagogical content knowledge (PCK) of Rwandan grade six mathematics teachers and its relationship to student learning* (Doctor of Philosophy), University of KwaZulu-Natal, Pietermaritzburg.

Maoto, S., Masha, K., & Maphutha, K. (2016). Where is the bigger picture in the teaching and learning of mathematics?. *Pythagoras*, *37*(1), 1-8.

Maoto, S., Masha, K., & Mokwana, L. (2018). Teachers' learning and assessing of mathematical processes with emphasis on representations, reasoning and proof. *Pythagoras*, *39*(1), 1-10.

Markworth, K., Goodwin, T., & Glisson, K. (2009). The development of mathematical knowledge for teaching in the student teaching practicum. *AMTE monograph*, *6*, 67-83.

Martiniello, M. (2008). Language and the Performance of English-Language Learners in Math Word Problems. *Harvard Educational Review*, *78*(2), 333–368. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ800934&site=ehost-live

Marton, F., & Booth, S. A. (1997). *Learning and awareness*. Psychology Press.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *The Journal of the Learning sciences*, *15*(2), 193-220.

Marton, F., Tsui, A.B.M & Runesson, U. (2004). Classroom Discourse and the Space of Learning. Lawrence Erlbaum Associates, Publishers: London

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal*, *21*(2), 10-32.

McCarthy, J., Oliphant, R. (2013). *Mathematics outcomes in South African schools: What are the facts? What should be done?* Center for Development and Enterprise: Johannesburg.

Medrano, F. E., Escudero, D. I., & Yáñez,C. J. (2013). A theoretical review of specialised content knowledge.

Meiers, M. & Trevitt, J. (2010) Language in the mathematics classroom. The Digest, NSWIT, 2010 (2). Retrieved from http://www.nswteachers.nsw.edu.au

Meiers, M. (2010). Language in the mathematics classroom. *QCT Research Digest*, *7*.

Mellor, K., Clark, R., & Essien, A. A. (2018). Affordances for learning linear functions: A comparative study of two textbooks from South Africa and Germany. *Pythagoras*, *39*(1), 1-12.

Mhakure, D. (2019). School-based mathematics teacher professional learning: A theoretical position on the lesson study approach. *South African Journal of Education*, *39*(4).

Mhlolo, M. (2013). The merits of teaching mathematics with variation. *Pythagoras, 34*(2), Art. #233, 8 pages.http://dx.doi.org/10.4102/pythagoras.v34i2.233

Mhlolo, M. (2013). The merits of teaching mathematics with variation. *Pythagoras*, *34*(2), 1-8.

Mhlolo, M. K., Schafer, M., & Venkat, H. (2012). The nature and quality of the mathematical connections teachers make. *pythagoras*, *33*(1), 1-9.

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers college record*, *108*(6), 1017-1054.

Mobarak, K. (2019). Reflections of employed graduates on the suitability of their skills and knowledge for workplace-readiness. *South African Journal of Higher Education*, *33*(4), 186-202.

Modipane, M. C., & Kibirige, I. (2015). Experiences of teaching practice at the University of Limpopo: Possibilities for curriculum improvement. *south African Journal of Higher education*, *29*(2), 202-217.

Moloi, Q. M., Kanjee, A., & Roberts, N. (2019). Using standard setting to promote meaningful use of mathematics assessment data within initial teacher education programmes. *Pythagoras*, *40*(1), 14.

Morse, J. M., Barrett, M., Mayan, M., Olson, K., & Spiers, J. (2002). Verification strategies for establishing reliability and validity in qualitative research. International Journal of Qualitative Methods 1 (2), Article 2. Retrieved DATE from http://www.ualberta.ca/~ijqm/

Moru, E. K., Qhobela, M., Wetsi, P., & Nchejane, J. (2014). Teacher knowledge of error analysis in differential calculus. *pythagoras*, *35*(2), 1-10.

Mudaly, V. & Moore-Russo, D. (2011). South African teachers' conceptualisations of gradient: A study of historically disadvantaged teachers in an advanced certificate in education programme. Pythagoras, 32(1), 27–33.

Mudaly, V. (2015). Pre-service Teacher Knowledge: Thinking About Conceptual Understanding. Journal of Social Sciences, 44 (2,3), 160-168.

Mutodi, P., & Ngirande, H. (2014). Exploring mathematics anxiety: Mathematics students' experiences. *Mediterranean Journal of Social Sciences*, *5*(1), 283.

Mwadzaangati, L. (2019). Comparison of geometric proof development tasks as set up in the textbook and as implemented by teachers in the classroom. *Pythagoras*, *40*(1), 14.

Myers, J. P., & Rivero, K. (2019). Preparing globally competent preservice teachers: The development of content knowledge, disciplinary skills, and instructional design. *Teaching and Teacher Education: An International Journal of Research and Studies*, *77*(1), 214-225.

Naidoo, J., & Govender, R. (2014). Exploring the use of a dynamic online software programme in the teaching and learning of trigonometric graphs. *pythagoras*, *35*(2), 1-13.

Ndlovu, M. (2014). The effectiveness of a teacher professional learning programme: The perceptions and performance of mathematics teachers. *Pythagoras*, *35*(2), 1-10.

Nel, B. P. (2020). Implications of the quantitative literacies test results of the National Benchmark Test Project (NBTP) for teachers. *South African Journal of Education*, *40*(1).

Nel, B., & Luneta, K. (2017). Mentoring as professional development intervention for mathematics teachers: A South African perspective. *Pythagoras*, *38*(1), 1-9.

Neuman, W. L. (2011). *Social research method: Qualitative & Quantitative approaches.* (7th ed.). Boston: Allyn & Bacon.

Ngcoza, K., & Southwood, S. (2019). Webs of development: Professional networks as spaces for learning. *Pythagoras*, *40*(1), 1-7.

Niess, M. L. (2011). Investigating TPACK: Knowledge growth in teaching with technology. *Journal of educational computing research*, *44*(3), 299-317.

Nyikahadzoyi, M. (2013). Teachers' Knowledge of the Concept of a Function: A Theoretical Framework. *International Journal of Science & Mathematics Education*, *13*, 261–283. https://doi-org.ukzn.idm.oclc.org/10.1007/s10763-013-9486-9

Okeke, C. C., & van der Westhuizen, G. (2020). Learning from professional conversation: A conversation analysis study. *South African Journal of Education*, *40*(1).

ORGANIZATION FOR ECONOMIC COOPERATION AND DEVELOPMENT. (2017). *Education at a glance 2017: OECD indicators*. OECD.

Oswald, M. M. (2019). Teachers need scientific and experiential knowledge on learning support: Recommendations for a teacher educator. *South African Journal of Higher Education*, *33*(4), 237-252.

Parker, D., & Adler, J. (2012). Sociological tools in the study of knowledge and practice in mathematics teacher education. *Educational Studies in Mathematics*, *87*(2), 203–219. https://doi-org.ukzn.idm.oclc.org/10.1007/s10649-012-9421-y

Posthuma, B. (2012). Mathematics teachers' reflective practice within the context of adapted lesson study. *Pythagoras*, *33*(3), 1-9.

Posthuma, B. (2012). Mathematics teachers' reflective practice within the context of adapted lesson study. *Pythagoras*, *33*(3), 1-9.

Pournara, C. (2013). Teachers' knowledge for teaching compound interest. *Pythagoras*, *34*(2), 1-10.

Pournara, C. (2014). Mathematics-for-teaching: Insights from the case of annuities. *pythagoras*, *35*(1), 1-12.

Pournara, C., Hodgen, J., Adler, J., & Pillay, V. (2015). Can Improving Teachers' Knowledge of Mathematics Lead to Gains in Learners' Attainment in Mathematics? *South African Journal of Education*, *35*(3).

Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ1134890&site=ehost-live

Pournara, C., Sanders, Y., Adler, J., & Hodgen, J. (2016). Learners' errors in secondary algebra: insights from tracking a cohort from Grade 9 to Grade 11 on a diagnostic algebra test. *Pythagoras*, *37*(1), 1-10.

Prince, R., & Frith, V. (2017). The quantitative literacy of South African school-leavers who qualify for higher education. *Pythagoras*, *38*(1), 1-14.

Pritchett, L., & Beatty, A. (2012). The negative consequences of overambitious curricula in developing countries. *Center for Global Development Working Paper*, (293).

Rasmussen, K. (2016). Lesson study in prospective mathematics teacher education: didactic and paradidactic technology in the post-lesson reflection. *Journal of Mathematics Teacher Education*, *19*(4), 301-324.

Reddy, V. (2018). TIMSS in South Africa: making global research locally meaningful.

Robitaille, D.F., & Travers, K.J. (1992). International studies of achievement in mathematics. In D.A.Grouws (Ed), Handbook of research on mathematics teaching and learning (pp 687-709). Macmillan Publishing Company: New York.

Rodríguez-Izquierdo, R. M., Falcón, I. G., & Permisán, C. G. (2020). Teacher beliefs and approaches to linguistic diversity. Spanish as a second language in the inclusion of immigrant students. *Teaching and Teacher Education*, *90*, 103035.

Rodríguez-Izquierdo, R. M., Falcón, I. G., & Permisán, C. G. (2020). Teacher beliefs and approaches to linguistic diversity. Spanish as a second language in the inclusion of immigrant students. *Teaching and Teacher Education*, *90*, 103035.

Romberg, T. A. (1969). 7: Current Research in Mathematics Education. *Review of Educational Research*, *39*(4), 473-491.

Romberg, T.A. (1992). Perspectives on scholarship and research methods. In D.A.Grouws (Ed), Handbook of research on mathematics teaching and learning (pp 49-64). Macmillan Publishing Company: New York.

Romberg, T.A. (2003). Creating a research community in mathematics education. University of Wisconsin-Madison (Working Paper No. 2003-10).

Rosenberg, J., M. & Koehler, J,M. (2017). Context and Technological Pedagogical Content Knowledge (TPACK): A Systematic Review. Journal of Research on Technology in Education, 47(3), 24. DOI: 10.1080/15391523.2015.1052663

Sapire, I., Shalem, Y., Wilson-Thompson, B., & Paulsen, R. (2016). Engaging with learners' errors when teaching mathematics. *Pythagoras*, *37*(1), 1-11.

Sawyer, A. G. (2018). Factors Influencing Elementary Mathematics Teachers' Beliefs in Reform-Based Teaching. *The Mathematics Educator*, *26*(2), 26–53. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eric&AN=EJ1165807&site=ehost-live

Schon, A. (1983). Reflective practitioner: how professionals think in action. United States: Basic books.

Schön, D. A. (Ed.). (1991). *The reflective turn: Case studies in and on educational practice*. New York: Teachers College Press.

Sfard, A. (2000). On reform movement and the limits of mathematical discourse. *Mathematical thinking and learning*, *2*(3), 157-189.

Sfard, A. (2009). What's all the fuss about gestures? A commentary. Educational Studies in Mathematics, 70(2), 191–200. doi:10.1007/s10649-008-9161-1

Shulman, L. S., & Hutchings, P. (2004). *Teaching as community property: Essays on higher education*. Jossey-Bass.

Shulman, L.S. (1986). *Those who understand: Knowledge Growth in Teaching*: Educational Researcher, 15(2) 4-14.

Sikhwari, T. D., Ravhuhali, F., Lavhelani, N. P., & Pataka, F. H. (2019). Students' perceptions of some factors influencing academic achievement at a rural South African university. *South African Journal of Higher Education*, *33*(4), 291-306.

Silverman, D. 2004. Qualitative research. Sage publications: London

Sorto, M. A., Shalem, Y., & Sapire, I. (2014). Teachers' explanations of learners' errors in standardised mathematics assessments. *Pythagoras*, *35*(1), 1-11.

South Africa. Department of Basic Education. (2011). *Integrated Strategic Planning Framework for Teacher Education and Development in South Africa, 2011-2025: Technical Report*. Department of Basic Education.

South African Council for Educators. (2002). Handbook for the code of conduct of professional ethics.

Spangenberg, E. D. (2017). The interplay between theory and practice: mathematics pre-service teachers' learning experiences at a teaching school. *The Independent Journal of Teaching and Learning*, *12*(2), 92-112.

Spaull, N. (2013). Poverty & privilege: Primary school inequality in South Africa. *International Journal of Educational Development*, *33*(5), 436-447.

Spaull, N. (2013). *South Africa's Education Crisis: The quality of education in South Africa 1994-2011*: Johannesburg: Center for Development and Enterprise.

Spaull, N. (2015). Improving Education Quality in South Africa: Stellenbosch: National Planning Commission.Van Schaik Publishers.

Stîngu, M. M. (2012). Reflexive practice in teacher education: facts and trends. *Procedia-Social and Behavioral Sciences*, *33*, 617-621.

Stylianides, A. J., & Stylianides, G. J. (2006). Content knowledge for mathematics teaching: The case of reasoning and proving. In J. Novotna, H. Moraova, M. Kraka & N. Stehlikova (Eds.) Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education (Vol. 5, pp. 201-208). Prague: PME.

Suzuka, K., Sleep, L., Ball, D. L., Bass, H., Lewis, J., & Thames, M. (2009). Designing and using tasks to teach mathematical knowledge for teaching. *Scholarly practices and inquiry in the preparation of mathematics teachers*, 7-24.

Swarthout, M. B ., Jones, D., Klespis, M., & Cory, B. (2009). Sneaking a Peek at Students' Understanding of Functions: Why Not Concept Maps? *Ohio Journal of School Mathematics*, (60), 24–27. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eue&AN=45615568&site=ehost-live

Tavil, Z. M. (2014). The effect of self reflections through electronic journals (e-journals) on the self efficacy of pre-service teachers. *South African Journal of Education*, *34*(1), 1-20.

Taylor, N., & Vinjevold, P. (1999). *Getting Learning Right: Report to the President's Education Initiative Research Project* Johannesburg: The Joint Education Trust.

Taylor, S. (2011). Uncovering indicators of effective school management in South Africa using the National School Effectiveness Study. *Stellenbosch Economic Papers*, (10/11).

Thames, H,T. (2006). Using Maths to teach Maths. Mathematical Science Institute: Berkely, California.

Tondeur, J., Scherer, R., Siddiq, F., & Baran, E. (2017). A comprehensive investigation of TPACK within pre-service teachers' ICT profiles: Mind the gap!. *Australasian Journal of educational technology*, *33*(3).

Tooley, M., & Connally, K. (2016). No Panacea: Diagnosing What Ails Teacher Professional Development before Reaching for Remedies. *New America*.

Tsui, A. B. M. (2003). Distinctive qualities of expert teachers. *Teachers & Teaching*, *15*(4), 421–439. https://doi-org.ukzn.idm.oclc.org/10.1080/13540600903057179

Umugiraneza, O., Bansilal, S., & North, D. (2018). Exploring teachers' use of technology in teaching and learning mathematics in KwaZulu-Natal schools. *Pythagoras*, *39*(1), 1-13.

USAID. (1996). Performance Monitoring and Evaluation: USING DIRECT OBSERVATION TECHNIQUES. *USAID Center for Development Information and Evaluation, 4*.

Vagi, R., Pivovarova, M., & Barnard, W. (2019). Dynamics of preservice teacher quality. *Teaching and Teacher Education*, *85*, 13-23.

Van der Berg, S., & Gustafsson, M. (2017). Quality of basic education. A report to working group 1 of the high level panel on the assessment of key legislation. *Unpublished. Stellenbosch. August*.

Van der Berg, S., Taylor, S., Gustafsson, M., Spaull, N., & Armstrong, P. (2011). *Improving Education Quality in South Africa*: Stellenbosch: National Planning Commission.Van Schaik Publishers.

Van Staden, S., & Motsamai, P. (2017). Differences in the quality of school-based assessment: Evidence in Grade 9 mathematics achievement. *Pythagoras*, *38*(1), 1-10.

Venkat, N., Spaull, H. (2015). *What do we know about primary teachers' mathematical content knowledge in South Africa? An analysis of SACMECQ 2007*: International Journal of Educational Development, 41 121-130.

Visser, M., Juan, A., & Feza, N. (2015). Home and school resources as predictors of mathematics performance in South Africa. *South African Journal of Education*, *35*(1), 1–10. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eue&AN=101304644&site=ehost-live

Vogel-Walcutt, J. J. 1. jvogel@ist. ucf. ed., Gebrim, J. B. ., Bowers, C. ., Carper, T. M. ., & Nicholson, D. . (2011). Cognitive load theory vs. constructivist approaches: which best leads to efficient, deep learning? *Journal of Computer Assisted Learning*, *27*(2), 133–145. https://doi-org.ukzn.idm.oclc.org/10.1111/j.1365-2729.2010.00381.x

Wang, H. (2016). *From Mathematics to Philosophy (Routledge Revivals)*. Routledge.

Wanjiru, B. (2015). Effects of Mathematical Vocabulary Instruction on Students' Achievement in Mathematics in Secondary Schools of Murang'a County, Kenya. *Journal of education and Practice*, *6*(18), 201-207.

White, P., & Mitchelmore, M. (1993). Aiming for variable understanding. *Australian Mathematics Teacher*, *72*(3), 50–52. Retrieved from http://search.ebscohost.com.ukzn.idm.oclc.org/login.aspx?direct=true&db=eue&AN=118270999&site=ehost-live.

Wilson, S. M., Rozelle, J. J., & Mikeska, J. N. (2011). Cacophony or embarrassment of riches: Building a system of support for quality teaching. Journal of Teacher Education 62(4), 383–394. DOI: 10.1177/0022487111409416.

Wolf, S., & Peele, M. E. (2019). Examining sustained impacts of two teacher professional development programmes on professional well-being and classroom practices. *Teaching and Teacher Education*, *86*, 102873.

Yang, X., Kaiser, G., König, J., & Blömeke, S. (2019). Professional Noticing of Mathematics Teachers: a Comparative Study Between Germany and China. *International Journal of Science and Mathematics Education*, *17*(5), 943-963.

Yin, R.K. (1994). *Case study research* (2nd ed.). London: Sage publications.

Zhang, D. (2008). The effect of teacher education level, teaching experience and teaching behaviour on student science achievement. (Doctor of Philosophy), Utah State University, Utah, Logan.

Zhou, G., Xu, J., & Martinovic, D. (2016). Developing pre-service teachers' capacity in teaching science with technology through microteaching lesson study approach. *EURASIA Journal of Mathematics, Science and Technology Education*, *13*(1), 85-103.

Zhu, Y., Yu, W., & Cai, J. (2018). Understanding students' mathematical thinking for effective teaching: A comparison between expert and nonexpert Chinese elementary mathematics teachers. *Eurasia Journal of Mathematics, Science and Technology Education*, *14*(1), 213-224.

Zuze, L., Reddy, V., Visser, M., Winnaar, L., & Govender, A. (2017). TIMSS 2015 GRADE 9.

Appendix A

DATA GENERATION INSTRUMENTS

A. INTERVIEW SCHEDULE (*The aim of this section is to gather as much information as possible about the participant including their views about the teaching of mathematics. This information will be used during observation and will also be crucial during the analysis stage*).

1. Tell me about your background (The aim of this question is to find out more about the participant, where did they grow up, which high school they attended, what they studied, what is their teaching background).
2. Why did you choose teaching?
3. Why did you choose to teach mathematics?
4. What are your views about the teaching of mathematics in South Africa?
5. In your opinion, what makes the teaching of mathematics easy or difficult?
6. In your opinion, what makes the learning of mathematics easy or difficult?
7. Do you consider yourself a good maths teacher? Why?
8. What challenges have you encountered in your career as a maths teacher?
9. What have been the highlights of your career as a math teacher?
10. What teaching methods do you use in your classroom? Please elaborate on each.
11. On a scale of 1-10, where do you rate your knowledge of the concepts you teach?
12. On a scale of 1-10, how well do you understand the errors and misconceptions made by the learners in mathematics?
13. What do you consider when you plan your lessons?
14. What curriculum material do you use in your teaching? Why?
15. What other resources do you use in your teaching? Why?
16. What learning resources are available for your learners?
17. What do you think the role of the teacher is in the mathematics classroom?
18. What do you think the role of the learner is in the mathematics classroom?
19. What professional development have you received as a maths teacher?
20. Are there topics in the math syllabus that you are not confident about? Explain?
21. If yes, what attempts have you made to improve your understanding?
22. Do you feel that the teacher training programme prepared you well enough for the challenges you have encountered in teaching?

23. What could have been done differently to help you prepare for the teaching challenges?

B. POST LESSON OBSERVATION INTERVIEW GUIDE

1. What are your views about the topic you have just taught? Did you find it easy or difficult to teach?
2. What are your feelings about the learners having to learn this topic?
3. What in your opinion do leaners find easy about the topic?
4. What do the learners find challenging about the topic?
5. Is there a particular reason why you have chosen to teach this topic at this time?
6. What did you hope to achieve at the end of today's lesson?
7. Do you feel that you achieved your objective? Why?
8. If you had to re-teach this lesson, what would you do differently?

C. DOCUMENT ANALYSIS

Curriculum material which includes teacher's book (if different from leaners'), learners' text books and note books will be analyzed during lesson observation. The reason for including this type of analysis is that prescribed text books are designed with the curriculum objectives in mind. Most of these text books further highlight areas of misconception and common errors made by learners which teachers need to take into account during planning. How teachers use these materials will provide insight into the planning that goes on beforehand as well as indicate whether teachers take into account curriculum objectives before embarking on teaching. Some teachers prefer to use their own supplementary materials on top of the prescribed text book. How this materials are used has a bearing on the quality of instruction as Ball et al (2008) show. As mathematics is a practical subject, I believe what learners write down in their note or homework books is a crucial indication of the quality of work done in the classroom.

D. OBSERVATION SCHEDULE

- How is the class organized?
- How are the learners working? (in groups or individually).
- What is the goal of the lesson?
- What content is covered in the lesson?
- What is the role of the teacher?
- What is the role of the learner?
- How much time is spent on teaching vs classroom management?
- What kind of questions does the teacher ask?
- What kind of examples does the teacher provide?
- What is the teacher's level of mathematical explanation?
- Does the teacher have knowledge of leaners' misconceptions and errors pertaining to the topic?
- How does the teacher handle learner errors and misconceptions?
- How does the teacher respond to leaner questions, ideas, suggestions, comments and solutions?
- What is the teachers' knowledge of the content?
- How does the teacher use technical and general language?
- Is the teacher explicit in his/her explanation and questions asked?
- How does the teacher use conventional notation?
- Does the teacher justify why a particular algorithm or method works, making use of deductive reasoning or other mathematically sound justification?
- What is the level of learner involvement in the lesson?
- How does the teacher attempt to include all learners in the lesson?
- Are the learners encouraged to think for themselves?
- Are learners given freedom to express themselves, challenge each other, agree or disagree with an idea, suggestion or solution?
- What representations does the teacher make use of?
- How appropriate are the representations chosen?
- Does the lesson reach its objective i.e is there telos?

BIOGRAPHICAL INFORMATION

1. My age is :

☐ 20 – 29 ☐ 30 - 39 ☐ 40 - 49 ☐ above 50 years

2. My teaching experience is:

☐ 0 - 4 ☐ 5 – 10 ☐ 11 – 15 ☐ 20 or more years

3. My teaching grades are: ---

4. I teach

☐ Only maths ☐ Other--- -------------

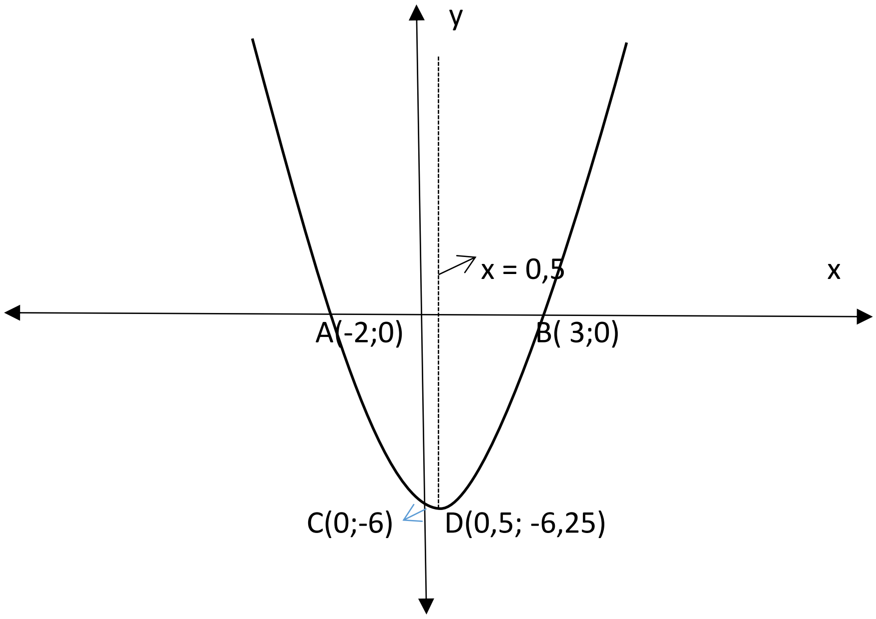
5. I am:

☐ Male ☐ Female

==

Appendix B

1. Joshua, a boy in your class asks you to explain to him the difference between a decreasing and a negative function. How would you use the graph below to explain to Joshua between which values the function is decreasing. Also explain to him how to find the interval where the function is negative.



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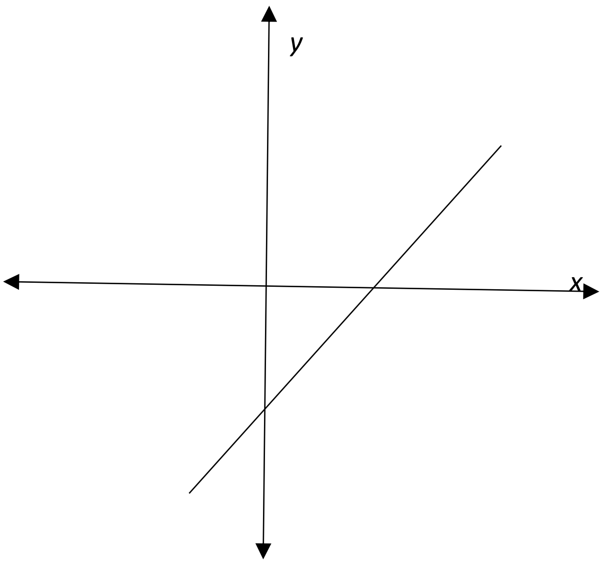
2. Is possible for this graph to have a gradient of 2? Explain



Mudaly (2015, p.164)

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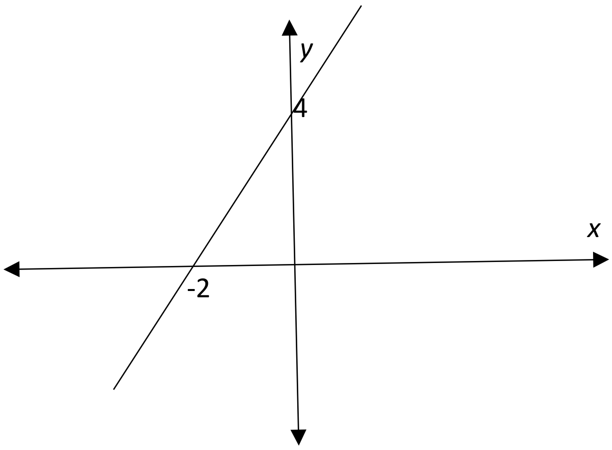
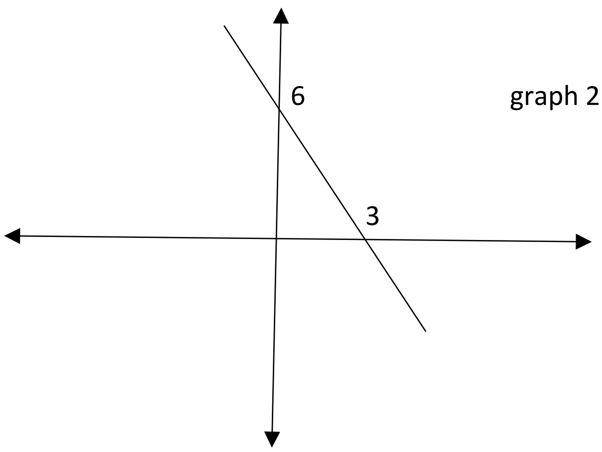
3. Mr Liao asked his grade 9 class to find the equation of the following graph:



Cebo proudly shows Mr Liao his solution: $y = -2x + 4$. Being unsure of Cebo's source of error Mr Liao draws another graph and asks Cebo to find the equation:



Cebo's solution: $y = 3x + 6$. If you were Mr Liao, how would you explain to Cebo what he is doing wrong with the hope that he does not make the same error in future?

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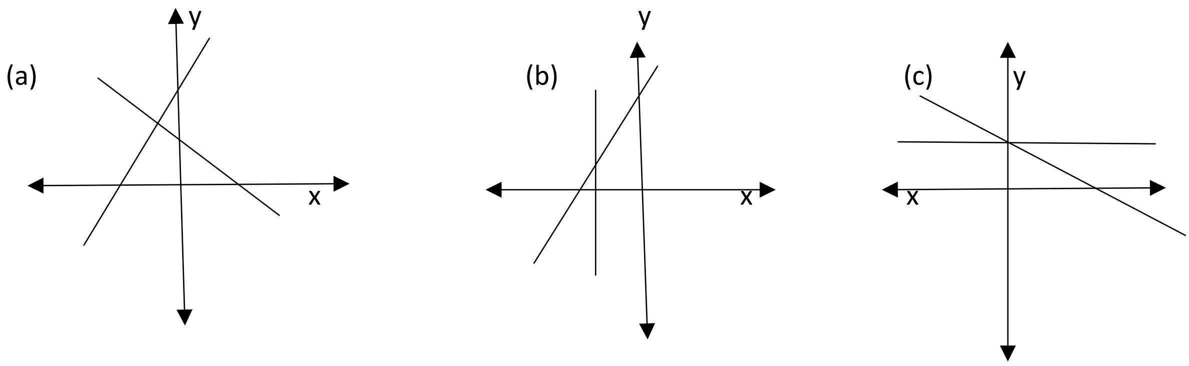
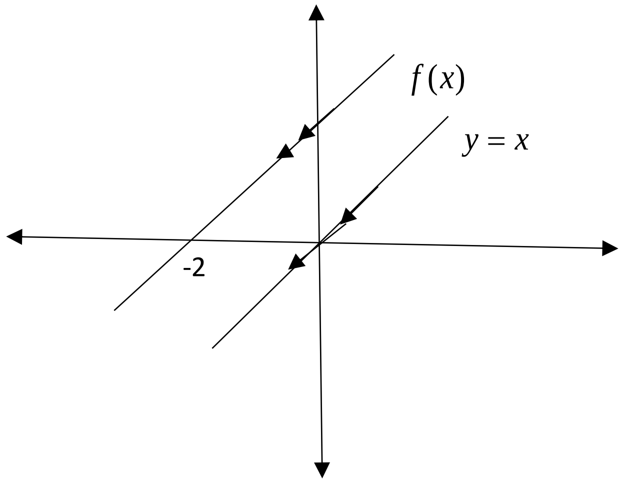
4. Mrs Sishi wants to introduce simultaneous equations to his grade 8 class. Which one of the following set of graphs would you recommend she uses for her introduction? Explain

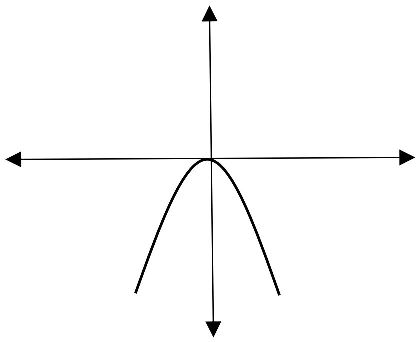


5.

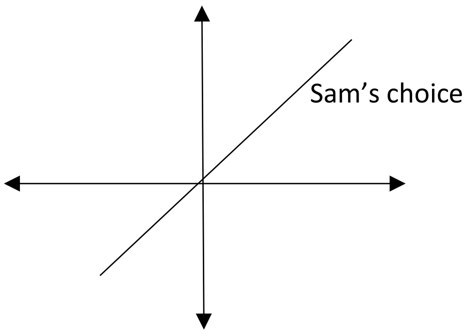


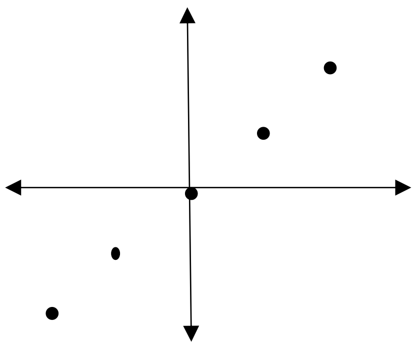
A straight line defined by the equation $y = x$ and another straight line $f(x)$ are drawn on the same set of axes parallel to each other, $f(x)$ intercepts the $x-axis$ at -2. Is there enough information provided to find the equation of $f(x)$? Explain

6. Kayla, a girl in your grade 8 class brings you a parabolic graph of the type drawn below and wants to know how to find its gradient. What explanation would you give Kayla?



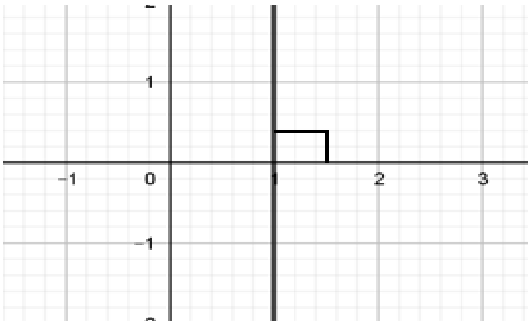
7. Sam was practicing Mathletics at home in his computer and wants to know why his answer was incorrect to one of the questions. The question required him to identify the graph of $y = 2x$, where x is an integer. Two of the 4 options are shown below, one of them is correct. How would you explain to Sam what the difference is between the two graphs, and why his choice is incorrect?



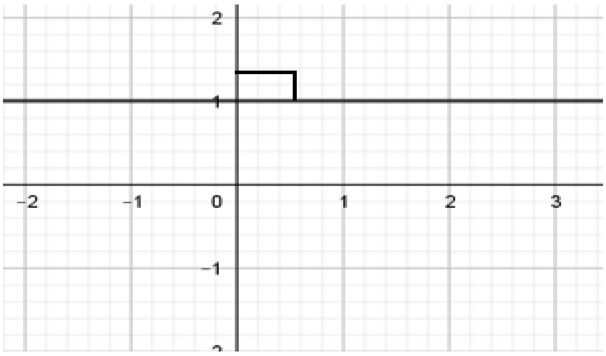


8. How would you explain to a learner why GRAPH A has a gradient of zero while GRAPH B's gradient is undefined?

GRAPH A



GRAPH B



9. Two functions $f(x) = x^2$ and $g(x) = 2x - 1$ are given. How would you work out $f(g(2))$?

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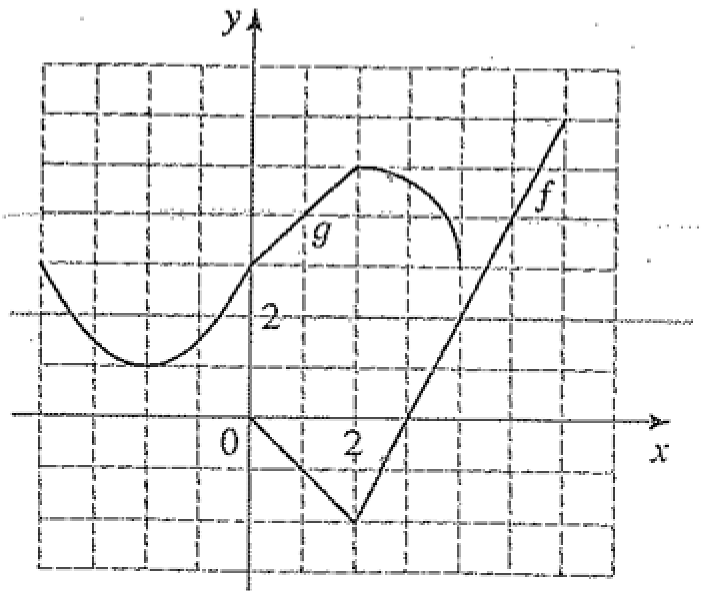
10. Two functions f and g are given graphically below. How would you work out $f(g(2))$?



Engelke, Oehrtman & Carlson (2005, p.4)

Appendix C

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

08 September 2016

Mrs Hlengiwe Abigail Nguse (952059614)
School of Education
Edgewood Campus

Dear Mrs Nguse,

Protocol reference number: HSS/0938/016D
Project title: An exploration of General Education and Training (GET) Mathematics teachers' specialised Mathematics content knowledge of functions in selected schools in Pietermaritzburg

Full Approval – Expedited Application
In response to your application received on 28 June 2016, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and the protocol have been granted **FULL APPROVAL**.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

Dr Shamila Naidoo (Deputy Chair)

/ms

Cc Supervisor: Dr V Mudaly
Cc Academic Leader Research: Dr SB Khoza
Cc School Administrator: Ms Tyzer Khumalo

Humanities & Social Sciences Research Ethics Committee
Dr Shenuka Singh (Chair)
Westville Campus, Govan Mbeki Building

Appendix D

education

Department:
Education
PROVINCE OF KWAZULU-NATAL

Enquiries: Phindile Duma Tel: 033 392 1004 Ref.:2/4/8/762

Mrs H Nguse
PO Box 100824
Scottsville
Pietermaritzburg
3209

Dear Mrs Nguse

PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: **"AN EXPLORATION OF MATHEMATICS TEACHERS' SPECIALIZED CONTENT KNOWLEDGE OF FUNCTIONS IN THE GENERAL EDUCATION AND TRAINING (GET) PHASE IN SELECTED SCHOOLS IN PIETERMARITZBURG",** in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the Intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from 13 April 2016 to 30 June 2017.
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Miss Connie Kehologile at the contact numbers below
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report / dissertation / thesis must be submitted to the research office of the Department. Please address it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to schools and institutions in KwaZulu-Natal Department of Education.

uMgungundlovu District

Nkosinathi S.P. Sishi, PhD
Head of Department: Education
Date: 13 April 2016

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beyond the call of duty