Axial algebras for sporadic simple groups HS and Suz

by
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Thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

in the

School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal
October 2018

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As the candidate's supervisor I have approved this dissertation for submission.

Professor B.G. Rodrigues

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As the candidate's co-supervisor I have approved this dissertation for submission

Professor S. Shpectorov

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Dedication

To A. G. R. Stewart who first taught me algebra, and to my mother who was my first teacher.

Abstract

Motivated by the construction of the Monster sporadic simple group as a group of automorphisms of an algebra and the recent development of axial algebras as a generalization of Majorana representations, we construct axial algebras for the sporadic simple groups HS and Suz in different ways analogous to the Norton algebra construction. We study how these algebras decompose as direct sums of the adjoint action of an axis. Fusion rules, that is the rules with which eigenvectors from various eigenspaces of the adjoint action multiply, are found. This places these groups in a general framework of groups acting on algebras hence giving a common theme for their origin.

Acknowledgements

I wish to thank Professor B. G. Rodrigues and Professor S. Shpectorov without whom the present work would not have been possible. Their support and guidance has been invaluable and has extended beyond the academic to the mundane. The financial support of the NRF through grant holder bursary is gratefully acknowledged.

The support, encouragement, understanding and the sacrifice of family members is gratefully acknowledged. Without this support it would have been impossible to finish this work. To my brother and sisters I am greatly indebted for their support over the years. Sadly my brother and eldest sister did not live to see the fruition of this work. Their memories will forever be cherished.

Jane Morgan has been a pillar of support throughout my graduate studies. I am grateful for her support and kindness. I would like to extend my gratitude to Simo, Yash Madanha and Xavier who have been real brothers.

Finally, I would like to thank Ezalleth for her patience.

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List of symbols

$a \mid b, a \nmid b$	a divides b, a does not divide b
$A \setminus B$	Set difference
$A \times B$	The Cartesian product of A and B
$A \dot{\cup} B$	Disjoint union of A and B
$A \wr B$	The wreath product of A and B
$\mathrm{char}(F)$	The characteristic of F
$F^{\times}$	$F \setminus \{0_F\}$
$\mathrm{Fix}(\sigma)$	The set $\{\omega \in \Omega \vert \omega^{\sigma} = \omega\}$, of fixed points of σ
FG	Group algebra of G over F
$\gcd(a, b)$	The greatest common divisor of integers a and b
$\mathrm{Gal}(K/L)$	The Galois group of the field extension
$\vert G : H\vert$	The index of H in G
$G \times H$	The direct product of G and H
$G^{\times}$	$G \setminus \{1_G\}$
G/H	Coset space or factor group if H is normal
G_x	The stabilizer of x in G
$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$\mathrm{Hom}_R(U, V), L(U, V)$	The set of homomorphisms from U to V
L/F	Field extension L of a field F.
$\mathrm{Mat}_n(\Delta_i)$	The ring of all $n \times n$ matrices over Δ_i
$\mathrm{o}(g)$	The order of an element g of a group
$\mathrm{Orb}_G(x)$	The orbit of x under G
${}_R R$	Left regular module
S_n	The symmetric group on n symbols
$\mathrm{tr}(A)$	Trace of A
$\mathrm{tr}(x, M)$	The trace of the action of x on a module M
V^*	The dual module of V
$V \oplus W$	The direct sum of V and W
$V \otimes_R W$	The tensor product of R-modules V and W
$V\vert_{FH}$	FG-module V restricted to $H \leq G$
δ_{ij}	The Kronecker symbol

Δ_i	Division ring
χ_ρ	Character afforded by ρ
χ_V	Character afforded by a representation on a vector space V
$\chi\|_H$	The restriction of the character χ of G to a subgroup H.
$\chi_S, \mathrm{Sym}^2\chi$	The symmetric square of the character χ
χ_A	The antisymmetric square of the character χ
ω^σ	The image $\sigma(\omega)$ of ω under σ
ρ	Representation of G
$[\rho]$	Matrix representation of ρ w.r.t some basis
ζ	Primitive root of unity
Ω	A set
$\|\Omega\|$	The cardinality of Ω
$\mathfrak{p}$	An ideal in R
$\mathcal{O}_K$	The field of fractions of K
2^Ω	The power set of Ω
$\mathbb{F}_p$	The finite residue field $\mathbb{Z}/p\mathbb{Z}$
$\mathscr{B}_\mu(a)$	A basis for the μ-eigenspace of ad_a

Declaration

This dissertation, in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used in the text, proper reference has been made.

Tendai Makope Mudziiri Shumba

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Introduction

The sporadic simple groups have been constructed via ad hoc methods. A unified theory explaining their existence has been sought and none has been forthcoming to date. Recently, a new class of algebras resembling the Conway-Griess algebra for the Monster sporadic simple group have been introduced which could provide such an explanation. Physicists have long studied vertex operator algebras (VOAs), which Bocherds used to settle the Moonshine conjecture via the construction of the Moonshine module $V^\sharp$ for the Monster group.

VOAs, useful as they are, are however bulky and cumbersome to work with. Miyamoto [Miy96a] noticed that VOAs admit automorphisms canonically associated with idempotents of the associated Griess algebra. He noticed that there was a certain class of vectors in the weight 2 part of the Moonshine module which correspond to certain transpositions of the Monster. Such vectors are called $2A$-axes.

Norton [Con85] studied the subalgebras of the Griess algebra generated by 2 $2A$-axes, called *dihedral algebras*. In 2007 Sakuma [Sak07] proved a result that classified all the algebras that are generated by pairs of such vectors.

In a quest to abstract from the VOAs, Ivanov [Iva09] axiomatised some properties the class of algebras classified by Sakuma and called them Majorana algebras. Subsequently, small examples have been constructed. In [IPSS10] such algebras for the symmetric group of degree $4, S_4$, were constructed, and in [IS12] for $L_3(2)$. Seress [Ser12] developed an algorithm for constructing Majorana algebras from various groups and many of the previously known examples as well as new ones, were obtained from his GAP implementation of this algorithm. Unfortunately, after his death, this GAP implementation was lost. Subsequently, Pfeiffer and Whybrow have worked on implementing and improving his algorithm [PW18]. Many workers in the field have continued in this particular direction. See for example

[CR13, Why18, FIM16a, Iva11, FIM16b].

Shpectorov noted that his implementation of an algorithm to construct Majorana algebras did not use many of the axioms for this class of algebras and together with Hall and Rehren [HRS15a], introduced a more general class of algebras called axial algebras. Many results in this new direction have been obtained, for example [HRS15b], [Reh16, Reh17], [HSS18].

Recently, McInroy and Shpectorov [MS18] have described a new expansion algorithm for the construction of axial algebras which only uses fusion rules. Astonishingly, they have found out that all the algebras they have constructed admit Frobenius forms and many of the Majorana algebra examples that are found in the literature have been constructed this way by a MAGMA implementation of this algorithm. De Medts and Van Couwenberghe [DV17] have even investigated modules over axial algebras while De Medts and Rehren have studied the connection between Jordan algebras and 3-transposition groups [DMR17].

In the late 1970s, Norton, in his PhD thesis, a part of which has subsequently been published as [Nor88], constructed algebras admitting 3-transposition groups. In a project in which they sought to explain some phenomenon relating to Smith's work [Smi75] as well as Scott's work on products of characters [Sco77], Cameron, Goethals and Seidel [CGS78] showed that primitive rank 3 groups of even order admit a commutative Norton algebra. Other workers in the field have since constructed such algebras for various groups: Frohardt [Fro85] provided an outline of the construction of an algebra admitting O'Nan's group, Ryba [Ryb07, Ryb96] constructed algebras for the Baby Monster as well as the Harada-Norton group, while Kitazume [Kit87] constructed algebras for some groups with triple covers. In [Har85], Harada studied certain commutative algebras associated with permutation groups.

In this thesis we construct various Norton algebras which admit the almost simple and simple groups HS:2, Suz:2, HS and Suz. We show that these algebras are axial and we study the fusion rules that arise. We use the unital extensions of these algebras to try and investigate if we can find "useful" fusion rules.

Our starting point is representation-theoretic. For small dimensional irreducible modules V for the groups HS and Suz (respectively HS:2 and Suz:2), we check if these modules support a commutative algebra structure, by checking if V is a constituent of its own symmetric square $\mathrm{Sym}^2 V$. Furthermore, we require our algebras to support non-degenerate symmetric bilinear forms so we require that V is self-dual as a module, i.e., $V^* = V$. Finally, it is desirable that for some conjugacy class of involutions, the centralisers have non-trivial fixed subspaces in V of dimension 1 ideally, or 2. That is, for some class $2X$ of involutions, we desire that $\dim(V \cap \mathrm{inv}_K(W)) \in \{1, 2\}, p \in 2X$, where W is the permutation module of G and $K = C_G(p)$, the centralizer of p in G.

All these checks can best be carried out by passing to characters. The first check is equivalent to checking that if the module V affords an irreducible character χ then the inner product $(\mathrm{Sym}^2\chi, \chi)$, of the symmetric square of χ and χ is nonzero. To check the existence of a symmetric bilinear form, we need to check if the Frobenius-Schur indicator is 1. Finally to check the dimension of the space in V fixed by the centraliser K, of an involution, we compute the inner product of the restriction of χ to K and the principal character 1_K, that is, $(\chi|_K, 1_K)$.

We note that all these checks can be done by hand, from the Atlas [CCN+85], or by using a computational algebra package like MAGMA [BC94]. We summarise these data in the Table 2 below. In the table we record the dimensions of the irreducible modules, their sources, whether they arise from a permutation or an induced representation. If an irreducible module comes from a permutation representation , we put a check mark under "perm" in the source column, and if the module arises as an induced representation we put a check mark under "ind" in the source column.

For the group HS:2, we see that it is possible to impose commutative algebra structure on irreducible modules of dimensions $77, 154, 175, 693, 770, 825$ and 1056 while we can potentially do the same on irreducible modules of dimensions $143, 364, 780$ and 1001 for Suz:2. However, using the techniques described in Chapter 2, we can only use a construction of Norton algebras on the modules of dimension 780 and 1001. Due to the scale of computations involved, no attempt has been made to construct an algebra on the 1001-dimensional module of Suz:2 at this stage.

0.1 Basics and useful definitions

We will fix here some common notation which we will need throughout the thesis. We assume that groups are finite, that all algebras are commutative and that the reader is familiar with notions of linear algebra, groups, their actions and basic representation theory. We will recall a few definitions from the theory of graphs.

Group	dim(V)	Source Perm	Source Ind	V a constituent of $\mathrm{Sym}^2(V)$?	$V^* = V$?	$\dim(V \cap \mathrm{inv}_K(W))$ $K = C_G(p), p \in 2X$ involutions
	22		✓	×		
	22	✓		×		
	77		✓		✓	
	77	✓		✓	✓	$1, 2$
	154		✓	×		
	154		✓	✓	✓	1
	175		✓	✓	✓	$0, 1$
	175		✓	✓	✓	$1, 2$
	231		✓	✓	✓	1
HS:2	231		✓	×		
	308		✓	×		
	693		✓	×		
	693		✓	✓ $(8 - \mathrm{dim})$	✓	0
	770		✓	✓ $(5 - \mathrm{dim})$	✓	0
	770		✓	✓ $(2 - \mathrm{dim})$	✓	0
	825		✓	✓ $(8 - \mathrm{dim})$	✓	0
	825		✓	✓ $(2 - \mathrm{dim})$	✓	$0, 1, 2$
	1056		✓	✓ $(5 - \mathrm{dim})$	✓	0
	1056		✓	✓ $(13 - \mathrm{dim})$	✓	$0, 1, 2$
	143		✓	✓	✓	$0, 1$
	143		✓	×		
	364		✓	✓	✓	$0, 1$
Suz:2	364		✓	×		
	780	✓		✓(2-dim)	✓	$0, 1, 2$
	780		✓	×		
	1001		✓	×		
	1001	✓		✓	✓	$0, 1$

Table 2: Prelimininary check to see if algebra structure can be imposed on an irreducible module of the groups HS:2 and Suz:2 with small dimensions.

Let X be a finite set. Then an **incidence relation** E on X is a subset of the Cartesian product $X \times X$. A **graph** Γ is a pair (X, E) where X, E are finite sets of points, called **vertices**, and E is an incidence relation on X, respectively. The ordered pairs in the incidence relation are called **edges**. If $u, v \in X$ are joined by an edge $e \in E$, we write $e = uv$ instead of $e = \{u, v\}$, and say that u and v are adjacent and that e is incident with u and v. We also say that u and v are neighbours. The number of neighbours a vertex has is called its **valency**, or **degree**. Throughout we will only consider simple graphs, that is, we will only concern ourselves with graphs which do not have loops, in other words, for all $v \in X, vv \notin E$, and in which no multiple edges are allowed, that is, for any pair u, v of vertices, there is at most one edge e with $e = uv$. Thus, we may consider a graph as a set X admitting an irreflexive, symmetric binary relation $\sim$, where $u \sim v$ if and only if $uv \in E$.

Certain classes of graphs will be of use to us so we will say more. We say that a graph $\Gamma = (X, E)$ is k-**regular**, $k \in \mathbb{N}$, if every vertex $v \in X$ has exactly k neighbours. The class of strongly regular graphs and distance regular graphs will be used in what follows. We say that Γ is **strongly regular** with parameters (ν, k, λ, μ) if X has size ν, the graph is k-regular, any pair u, v of adjacent vertices have λ common neighbours, while any pair of non-adjacent vertices has exactly μ common neighbours. A u-v path is a sequence $(u = v_0, v_1, \ldots, v = v_d)$ of vertices such that $v_{i-1} \sim v_i$ for $1 \leq i \leq d$, and the vertices are distinct, except possibly for the first and last. We say that a graph is connected if for any pair u, v of vertices there exists a u-v path. In a connected graph we introduce a distance function $d : X \times X \to \mathbb{N}$ where $d(x, y)$ is the length of the shortest x-y path. The **diameter** of a graph is the maximum distance between two vertices, that is, $\text{diam}(\Gamma) = \max_{u,v \in X}\{d(u, v)\}$. A **complete** graph is the graph with a given set as vertex set in which all pairs of distinct vertices are adjacent. Opposite to this is the null graph in which no vertices are joined, i.e., a graph with no edges. Given a graph $\Gamma = (X, E)$, one can obtain many new graphs from it but we will concern ourselves with forming the complement of Γ, denoted $\bar{\Gamma}$, which is a graph on the same vertex set X as Γ but $u \sim_{\bar{\Gamma}} v$ if and only if $u \not\sim_{\Gamma} v$.

Given a graph $\Gamma = (X, E)$, we consider the symmetric $(0, 1)$ square matrix A formed by indexing the rows and the columns of the matrix by X, such that the (u, v)-th entry of A, A_{uv}, is given by

$$A_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

We can think of A as a function $X \times X \to \mathbb{F}$, where $\mathbb{F}$ is the field we use. It is well known [BCN89] that the complement of a strongly regular graph

(ν, k, λ, μ) is again strongly regular, with parameters $(\nu, \nu - k - 1, \nu - 2k + \mu - 2, \nu - 2k + \lambda)$. Finally, a connected graph Γ is **distance-regular** if there are constants $c_i, a_i, b_i, 0 \leq i \leq d$ such that if u and v are at a distance i then the number of vertices w such that $v \sim w$ and w is at distance $i-1, i, i+1$ from u is c_i, a_i, b_i respectively, where $c_0 = b_d = 0$, and d is the diameter of the graph. We note that the distance regular graphs of diameter 2 are precisely the strongly regular graphs. We will use these classes of graphs in subsequent chapters to construct association schemes and Norton algebras. Because we will be interested in group actions on graphs from now on, we finish off this section by introducing the notion of distance transitive graphs. By an **automorphism** σ of a graph $\Gamma = (X, E)$, we mean a bijection $\sigma : X \to X$ that is adjacency preserving, that is,

$$\sigma(x)\sigma(y) \in E \text{ if and only if } xy \in E, \text{ for all } x, y \in X.$$

The set of all automorphisms of Γ forms a group denoted $\mathrm{Aut}(\Gamma)$, under composition. A graph $\Gamma = (X, E)$ is **distance-transitive** if $\mathrm{Aut}(\Gamma)$ acts transitively on sets of pairs of vertices at distance i, for $i = 0, 1, \ldots, d := \mathrm{diam}(\Gamma)$. In other words, for each $0 \leq i \leq d$, for each $(x_1, y_1), (x_2, y_2)$ such that $d(x_1, y_1) = i = d(x_2, y_2)$, there exists an automorphism $\sigma \in \mathrm{Aut}(\Gamma)$ such that $\sigma(x_1) = x_2$ and $\sigma(y_1) = y_2$. This property can easily be seen to imply distance regularity; however, we note that the converse is not always true.

The structure of the thesis is as follows: in Chapter 1 we discuss the background from axial algebras, in Chapter 2 we discuss general algebras, Norton algebras, as well as their generalizations. In Chapter 3, we discuss the link between Norton algebras and algebraic graph theory. We set up the stage for Chapters 4 and 5 where we discuss the axial algebras for HS, HS:2, as well as for Suz and Suz:2, respectively.

All the MAGMA code used throughout the thesis can be found at [MS19].

Chapter 1

Background from axial algebras

In this chapter we discuss the background material from axial algebras that we will need in the remainder of the thesis. We start by discussing fusion rules, axes and axial algebras. We close the chapter with a discussion of Miyamoto involutions and Frobenius forms. Throughout this chapter, R is a commutative unital ring. We make our discussion as general as possible though in the chapters following this one we will work mostly over fields of characteristic 0, in particular $\mathbb{Q}$ and $\mathbb{C}$.

1.1 Basic definitions

We start by defining algebras. Since we are assuming that our algebras are commutative, it does not matter whether we multiply on the left or the right. For simplicity we restrict our discussion to algebras over fields.

Definition 1.1.1. An **algebra** is a module A over a field $\mathbb{F}$ with a distributive $\mathbb{F}$-bilinear multiplication, that is, for all $a, b, c \in A$ and $\lambda \in \mathbb{F}$,

$$\lambda(ab) = (\lambda a)b = a(\lambda b),$$

$$(a + b)c = ac + bc \text{ and } a(b + c) = ab + ac.$$

The focus of our study in this thesis are commutative nonassociative algebras so we will assume throughout that the algebras under study are commutative and we will not require associativity. Every nonzero element of an algebra induces an endomorphism by its adjoint action, that is, multiplication. Thus, for $0 \neq x \in A$, there is an endomorphism $\mathrm{ad}_x : A \to A$,

$$u \mapsto xu,$$

for all $u \in A$. We will denote the μ-eigenspace of ad_x by

$$A_\mu(x) := \{u \in A | xu = \mu u\}.$$

If $\Lambda \subset \mathbb{F}$, then

$$A_\Lambda(x) = \bigoplus_{\mu \in \Lambda} A_\mu(x)$$

with the convention that $A_\emptyset(x) = 0$.

Certain distinguished elements in an algebra will play an important role in our study of axial algebras, namely idempotents. We will give a definition here. We will also define the closely related concept of nilpotent elements though we do not require them.

Definition 1.1.2. Let A be an algebra. An element $a \in A$ is said to be an **idempotent** if $a^2 = a$. The additive identity 0 is called the **trivial idempotent** and if A is unital, $1 \in A$ is also an idempotent. We say a is **nilpotent** if $a^k = 0$ for some positive integer k.

1.2 Fusion laws

A common theme in the study of nonassociative algebras is the placing of emphasis on some relations to specify the multiplication of elements. In the case of Lie algebras, the device is the Jacobi identity, for Jordan algebras it is the Jordan identity and in the case of axial algebras, we use fusion laws.

Definition 1.2.1. Let $X \subset \mathbb{F}$ be a set. The pair $\mathcal{F} = (X, \star)$, where $\star : X \times X \to 2^X$ is a symmetric binary operation, is called a **fusion law** and a single instance $\lambda \star \mu$ is called a **fusion rule**.

Sometimes a fusion law is called a fusion table since the values of $\star$ can be arranged in a symmetric square table. Often we abuse notation and simply write $\mathcal{F}$ for the fusion law $(\mathcal{F}, \star)$. We will discuss how this is related to algebras in the next paragraph.

Definition 1.2.2. Let A be an algebra and $\mathcal{F}$ a fusion law of Definition 1.2.1 above. Then an $\mathcal{F}$**-decomposition** of A is a decomposition

$$A = \bigoplus_{x \in X} A_x$$

such that

$$A_x A_y \subseteq A_{x \star y} = \bigoplus_{z \in x \star y} A_z.$$

Definition 1.2.3. An idempotent $a \in A$ is said to be $\mathcal{F}$**-diagonalisable** for the fusion law $\mathcal{F} = (X, \star)$ if

$$A = \bigoplus_{\mu \in X} A_\mu(a)$$

and $A_\phi(a) A_\mu(a) \subseteq A_{\phi \star \mu}(a)$ for all $\phi, \mu \in X$. In other words, the product of a ϕ-eigenvector and a μ-eigenvector is a sum of κ-eigenvectors, κ running over $\phi \star \mu$. In this definition and the previous one, we say that a and A satisfy the fusion law $\mathcal{F}$.

The following example illustrates the concept of fusion laws.

Example 1.2.4. A Jordan algebra $\mathfrak{J}$ over a field $\mathbb{F}$ is a commutative nonassociative algebra such that elements satisfy the Jordan identity

$$(xy)x^2 = x(yx^2).$$

If $a \in \mathfrak{J}$ is an idempotent, then ad_a has eigenvalues $\subseteq \{1, 0, \frac{1}{2}\}$. Furthermore, setting

$$\mathfrak{J}_\mu(a) = \{u \in \mathfrak{J} | \mathrm{ad}_a u = \mu u\},$$

then $\mathfrak{J}$ has a decomposition

$$\mathfrak{J} = \mathfrak{J}_1(a) \oplus \mathfrak{J}_0(a) \oplus \mathfrak{J}_{\frac{1}{2}}(a),$$

called the Peirce decomposition. The Peirce decomposition satisfies the fusion law with

$$\star : \{1, 0, \frac{1}{2}\}^2 \to 2^{\{1,0,\frac{1}{2}\}}$$

given by the table below. See [Jac68, p.119] for more details.

$\star$	1	0	$\frac{1}{2}$
1	1	$\emptyset$	$\frac{1}{2}$
0	$\emptyset$	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1, 0$

Table 1.1: Fusion rules for Jordan algebras.

The fusion law in Table 1.1 has been generalised in [HRS15b] to $\mathcal{F} = (\{1, 0, \alpha\}, \star)$ where a general class of primitive axial algebras of Jordan type are studied. For this generalisation, in the fusion table $\frac{1}{2}$ is replaced by α.

1.2.1 Fusion laws with distinguished unit

In this subsection we consider fusion laws $\mathcal{F} = (X, *)$ with a distinguished unit $1 \in X$. Suppose that $1 \in X$ is a unit, i.e.,

$$1 \star x \subseteq \{x\}$$

for all $x \in X$.

Definition 1.2.5. Let A be an algebra and $\mathcal{F} = (X, \star)$ a fusion law with distinguished unit 1. We say that an $\mathcal{F}$-decomposition $A = \oplus_x A_x$ is **$\mathcal{F}$-axial** if there exists $0 \neq v \in A_1$ such that v acts as a scalar $\lambda(x)$ on each $A_x, x \in X$ and $\lambda(1) \neq 0$. That is, if $w \in A_x$, then $vw = \lambda(x)w$.

From this definition, it is clear that $vA_x \subseteq A_x$, for if $w = vw' \in vA_x, w' \in A_x$ then $w = vw' = \lambda(x)w' \in A_x$. Since $v \in A_1$, we have $vv = \lambda(1)v$. Consequently, we can scale v to be an idempotent, i.e., $\lambda(1) = 1$. The following result about the uniqueness of such a v is immediate.

Lemma 1.2.6. *An element v satisfying the conditions of Definition 1.2.5 is unique.*

Proof. Suppose that we have an axial decomposition $A = \oplus_x A_x$ with distinguished unit 1, i.e., there exists $v \in A_1$ such that v acts as a scalar $\lambda(x)$ on each $A_x, x \in X$. Suppose that $v' \in A_1$ also satisfies the same conditions. In other words, for all $x \in X$, for all $w \in A_x, v'w = \lambda(x)w$. Then, in particular,

$vv' = \lambda(1)v'$ and reversing the roles of v, v' gives $v'v = \lambda(1)v$. Commutativity of the algebra then gives $\lambda(1)v' = vv' = v'v = \lambda(1)v$. Since $\lambda(1) \neq 0$, we get $v = v'$ as claimed. □

Remark 1.2.7. The motivation of this definition is that, unlike the one found in the literature, we allow distinct parts A_x of an axial decomposition to have the same $\lambda(x)$ value. We will see examples of this in Chapter 4.

Definition 1.2.8. The vector v of Definition 1.2.5 and Lemma 1.2.6 is called an **axis**. An axis v is called **primitive** if v spans A_1.

Since we can scale v so that it is an idempotent, or equivalently, $\lambda(1) = 1$, this is equivalent to stating that v spans its own 1-eigenspace. We now define an axial algebra.

Definition 1.2.9. Let A be an algebra, $\mathcal{A} \subset A$ be a set of axes. The pair $(A, \mathcal{A})$ is called an axial algebra if A is generated as an algebra by axes $a \in \mathcal{A}$.

We note that in the definition of an axial algebra above, we often abuse notation and call A an axial algebra if the context is clear.

1.3 Gradings

In this section we will discuss gradings which are important in the study of axial algebras. Whenever the fusion rules of an axial algebra are graded, corresponding to each axis there is an automorphism of the algebra. Let A be an algebra over a field $\mathbb{F}, \mathcal{F} = (X, \star)$ a fusion law and T an abelian group. We will define the notion of grading of the fusion law.

Definition 1.3.1. The fusion law $\mathcal{F} = (X, \star)$ is **T- graded** if X has a partition $X = \dot\cup_{t \in T} X_t$ such that

$$X_t \star X_s \subseteq X_{st}$$

for all $s, t \in T$.

Let A be an algebra and $a \in A$ an $\mathcal{F}$-axis. If $\mathcal{F}$ is T-graded, then there is an induced grading on A with respect to the axis a. The weight t subspace A_t of A is defined as

$$A_t = A_{X_t} = \bigoplus_{\lambda \in X_t} A_\lambda$$

This gives rise to automorphisms of A as follows. Let T^* be the group of all linear characters of T over $\mathbb{F}$, i.e., $T^* = \mathrm{Hom}(T, \mathbb{F}^\times)$, the set of homomorphisms from T to the multiplicative group $\mathbb{F}^\times$ of $\mathbb{F}$. For an axis a and $\chi \in T^\star$, consider the map $\tau_a(\chi) : A \to A$ defined by

$$u \mapsto \chi(t)u \text{ for } u \in A_t(a)$$

extended linearly to A. Because A is T-graded, the map $\tau_a(\chi)$ is an automorphism of A.

We are interested in T-graded fusion laws where $T = \mathbb{Z}/2\mathbb{Z}$. In this case we require that $\mathrm{char}(\mathbb{F}) \neq 2$ since $T^* = 1$ otherwise. Furthermore, we have $T^* = \{\chi_1, \chi_2\}$, where χ_1 is the trivial character. The automorphism $\tau_a(\chi_2)$ is then denoted by τ_a since it is usually non-trivial.

Definition 1.3.2. A fusion law $\mathcal{F} = (X, \star)$ is $\mathbb{Z}/2\mathbb{Z}$-graded if X can be written as $X_+ \cup X_-$, necessarily disjoint, such that

$$x \star y \subseteq X_+ \text{ whenever } x, y \in X_+,$$

$$x \star y \subseteq X_+ \text{ whenever } x, y \in X_-,$$

$$x \star y \subseteq X_- \text{ if either } x \in X_+ \text{ and } y \in X_- \text{ or vice versa.}$$

Example 1.3.3. (i) The fusion law for Jordan algebras is $\mathbb{Z}/2\mathbb{Z}$-graded. Set $X = \{1, 0, \frac{1}{2}\}, X_+ = \{1, 0\}$ and $X_- = \{\frac{1}{2}\}$. In general Jordan fusion rules are $\mathbb{Z}/2\mathbb{Z}$-graded with α replacing $\frac{1}{2}$.

(ii) The 196884-dimensional Conway-Griess real algebra has fusion law $\mathcal{F} = (X, \star), X = \{1, 0, \frac{1}{4}, \frac{1}{32}\}, \star : X^2 \to 2^X$ shown in the Table 1.2.

$\star$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	$\emptyset$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\emptyset$	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

Table 1.2: Fusion rules for the Conway-Griess Monster algebra.

Table 1.2 has been drawn to be suggestive, set $X_+ = \{1, 0, \frac{1}{4}\}, X_- = \{\frac{1}{32}\}$. Then we see clearly that the fusion rules $\mathcal{F}$ are $\mathbb{Z}/2\mathbb{Z}$-graded. It turns out that the fusion rules obtained by omitting $\frac{1}{32}$ from X are $\mathbb{Z}/2\mathbb{Z}$-graded. Set $X = \{1, 0, \frac{1}{4}\}$. We obtain fusion rules of Jordan-type, with $\alpha = \frac{1}{4}$ and the grading follows.

1.3.1 Miyamoto involutions

Let $\mathcal{F} = (X, \star)$ be $\mathbb{Z}/2\mathbb{Z}$-graded fusion law and $a \in A$ an axis, A an axial algebra. We associate an involutory map τ_a to a, called a Miyamoto involution defined as

$$u^{\tau_a} = \begin{cases} u & \text{if } \ u \in A_{X_+} \\ -u & \text{if } u \in A_{X_-}. \end{cases}$$

Because of the grading, this defines an automorphism of A. We note that it is allowed to have A_{X_-} trivial, in which case τ_a is trivial. If, however, A_{X_-} is non-trivial, then τ_a is a non-trivial automorphism of A.

Remark 1.3.4. The Miyamoto involutions corresponding to axes in the Conway-Griess algebra correspond to $2A$ involutions of the Monster, which generate the whole group.

Definition 1.3.5. A fusion law $\mathcal{F} = (X, \star)$ is said to be **Seress** if $0 \in X$ and for all $\mu \in X, 0 \star \mu \subseteq \{\mu\}$ if $\mu \neq 1$ and $0 \star 1 = \emptyset$.

Remark 1.3.6. Suppose that A is a primitive axial algebra satisfying the fusion law $\mathcal{F} = (X, \star)$ and $0 \in X$. Then for a primitive axis a and any $\mu \in X$,

$$A_\mu(a) \cdot A_1(a) = \begin{cases} A_\mu(a) & \lambda(\mu) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Of interest is the existence of a certain invariant bilinear form on an axial algebra. This has been taken as an axiom in the theory of Majorana algebras though we do not assume its existence in axial algebras.

Definition 1.3.7. Let A be an axial algebra. A nonzero bilinear form $(\cdot, \cdot) : A \times A \to R$ is said to be a Frobenius form if

(1) The form is symmetric, that is $(u, v) = (v, u)$ for all $u, v \in A$,

(2) The form associates with the algebra product in the sense

$$(uv, w) = (u, vw) \text{ for all } u, v, w \in A.$$

It has been shown in [MS18], [KMS18, Lemma 4.17], that if such a form exists, then it is unique, up to a scalar. We give a result which illustrates the usefulness of Frobenius forms.

Lemma 1.3.8. *Let A be an n-dimensional real axial algebra admitting a Frobenius form $(\cdot, \cdot)$. Then for an axis a, the matrix representing ad_a relative to some orthonormal basis $\mathscr{B} = \{v_i\}_{i=1}^n$, $n = \dim A$, is symmetric.*

Proof. Define m_{ij} via

$$\mathrm{ad}_a(v_i) = \sum_j m_{ij} v_j.$$

Then we have

$$\begin{aligned} m_{ij} &= (v_j, \sum_k m_{ik} v_k) = (v_j, \mathrm{ad}_a(v_i)) \\ &= (v_j, av_i) = (av_j, v_i) \text{ by associativity of } (\cdot,\cdot) \text{ and commutatitivity ,} \\ &= (v_i, av_j) = (v_i, \mathrm{ad}_a(v_j)), \text{ using symmetry of } (\cdot,\cdot) \text{ and the definition} \\ &= (v_i, \sum_k m_{jk} v_k) = m_{ji}, \text{ by orthonormality of } (\cdot,\cdot). \end{aligned}$$

□

Chapter 2

Norton Algebras and generalisations

In this chapter we discuss Norton algebras and their generalizations. This chapter will form a foundation for Chapters 4 and 5. Recall from the introduction that Norton algebras were introduced by Norton and the term was coined by J.H Conway. In his paper [Nor88] Norton notes that there is no precise definition of Norton algebras. Indeed, Hall [Hal18] notes that this is the case even to date. We follow the definition given by Cameron, Goethals and Seidel in [CGS78] and give the theoretical formulation of this class of algebras in the context of axial algebras.

We begin the chapter by recalling some representation- (and character-) theoretic facts we need for our discussion. We do not claim completeness here and many results will be stated without proof. Throughout by group we mean a finite group and all modules and vector spaces are finite-dimensional. Unless explicitly stated otherwise, fields are of characteristic zero.

2.1 Representation-theoretic background

In this section we will gather results about bilinear forms, tensor products and representations and characters of groups that are fundamental to the study in the subsequent chapters. We will also discuss Norton algebras and generalizations. Recall from the introduction that Norton [Nor88] constructed an algebra for the group Fi_{24} which admits a certain invariant triple form. In that paper, he showed that similar algebra structures could be constructed on modules admitting other groups with triple covers as automorphisms. Again all these suggested constructions rested on the existence of a G-invariant trilinear form.

2.1.1 Bilinear forms and tensor products

We discuss bilinear forms and their links to tensor products in this subsection. We assume that the reader is familiar with notions of modules and G-modules and refer to [Lan02a] for further details. We begin by defining various terms related to bilinear forms.

Definition 2.1.1. Let V be a module over a commutative ring R which has an automorphism of order 2, written as $a \mapsto \bar{a}$. By a bilinear form we mean a map $f : V \times V \to R$ which is linear in each argument, that is, satisfies the following conditions:

(i) $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$ for all $x_1, x_2, y \in V$,

(ii) $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$ for all $x, y_1, y_2 \in V$,

(iii) $f(ax, y) = f(x, ay) = af(x, y)$, for all $x, y \in V, a \in R$.

Furthermore,

(a) We say that f is **symmetric** if $f(x, y) = f(y, x)$ for all $x, y \in V$.

(b) The bilinear form f is said to be **skew-symmetric** if

$$f(y, x) = -f(x, y) \text{ for all } x, y \in V.$$

(c) If R is the field of complex numbers, then the form is **Hermitian** if $f(y, x) = \overline{f(x, y)}$ for all $x, y \in V$, where $\overline{z}$ denotes the complex conjugate of z.

A bilinear form is **nondegenerate** if $f(x, y) = 0$ for all $y \in V$ implies that $x = 0$, that is, the left annihilator of V with respect to the form is trivial. If a form is real-valued, then we say that f is positive semidefinite if $f(x, x) \geq 0$ for all $x \neq 0$. If the inequality is strict, we say that f is **positive definite**.

Let $E_1, E_2, \ldots, E_n$ be modules over a ring R, and suppose $f : E_1 \times E_2 \times \cdots \times E_n \to R$ is a map. We extend the notion of bilinearity to n-linearity by saying f is n-linear if it is linear in each of the n arguments.

Definition 2.1.2. Let $M_1, \ldots, M_n, N$ be R modules for a ring R. Denote by $L^n(M_1, \ldots, M_n; N)$ the module over R of all n-linear maps. If $n = 1$, we will drop the use of the semicolon so that $L(M, N) := \{f : M \to N | f \text{ is linear}\}$. If $n = 2$ and $M = N$, we define $B(M)$ the module of all bilinear forms on M by $B(M) := L^2(M, M; R)$.

We have the following standard elementary result which will be useful in what follows.

Lemma 2.1.3. *Let V be a module over $R \ni \frac{1}{2}$ and denote by $B(V)$ the module $L^2(V, V; R)$ of all bilinear forms. Define $S(V) = \{f \in B(V) | f \text{ is symmetric}\}$, and $A(V) := \{f \in B(V) | f \text{ is skew-symmetric}\}$. Then $B(V) = S(V) \oplus A(V)$.*

2.1.2 Tensor products

We discuss ideas about tensor products and their links to bilinear forms in this section. Throughout R will be a commutative unital ring. Let E, F be R-modules and consider R-bilinear mappings f of $E \times F$ into an R-module G, that is, mappings such that for all $x \in E, y \in F$ and $\lambda \in R$,

$$f(x, \lambda y) = f(\lambda x, y) = \lambda f(x, y) \tag{2.1}$$

linear in each argument. Consider the free R-module $C = R^{(E \times F)}$ of formal linear combinations of elements of $E \times F$ with coefficients, most of which are zero, in R. This has a basis made up of pairs $(x, y), x \in E, y \in F$. Let D be a submodule of C generated by elements of the type

$$\begin{cases} (x_1 + x_2, y) - (x_1, y) - (x_2, y) \\ (x, y_1 + y_2) - (x, y_1) - (x, y_2) \\ (x\lambda, y) = (x, \lambda y) \end{cases} \tag{2.2}$$

where $x, x_1, x_2 \in E,\ y, y_1, y_2 \in F$ and $\lambda \in R$.

Definition 2.1.4. The tensor product of the right R-module E and the left R-module F, denoted by $E \otimes_R F$ is the quotient R-module C/D. For $x \in E$ and $y \in F$, the element of $E \otimes F$ which is the canonical image of the element (x, y) of $C = R^{(E \times F)}$ is denoted by $x \otimes y$ and is called the tensor product of x and y.

Remark 2.1.5. Since we are going to apply tensor products of modules over fields which are commutative rings, every module is a left and right module or a *bimodule*.

We will state without proof, some facts about tensor products which we will need for the rest of the chapter. We refer the reader to a standard algebra text such as [Lan02b] for details. The following result is called the universal property of tensor products and is sometimes used as a definition of tensor products by some authors.

Proposition 2.1.6. *(i) Let g be an R-linear mapping of $E \otimes_R F$ into an R-module G. The mapping $(x, y) \mapsto f(x, y) := g(x \otimes y)$ of $E \times F$ into G is R-bilinear.*

(ii) Conversely, let f be an R-bilinear mapping of $E \times F$ into G. Then there exists one and only one R-linear mapping g of $E \otimes F$ into G such that $f(x, y) = g(x \otimes y)$ for all $x \in E, y \in F$.

Remark 2.1.7. We note here that if E and F are vector spaces with bases $\{e_i | 1 \leq i \leq m\}, \{f_j | 1 \leq j \leq n\}$ respectively, then

$$\{e_i \otimes f_j | i \in I, j \in J, I = \{1, \dots, m\}, J = \{1, \dots, n\}\}$$

is a basis for $E \otimes F$. This is sometimes used as a definition of tensor product.

The following theorem gives some properties that are immediate from the definition and the universal property of tensor products above.

Theorem 2.1.8. *Let E, F, G be modules over a commutative ring R. We have*

$$L(E, L(F, G)) \cong L^2(E, F; G) \cong L(E \otimes F, G).$$

We give some definitions of special tensors here.

Definition 2.1.9. (i) Tensors of the form $v \otimes w + w \otimes v$ which are images of the symmetric functions in Theorem 2.1.8 generate the R-module denoted $E \vee E$. The submodule $E \vee E$ is called the **symmetric tensor square** and the vectors $v \otimes w + w \otimes v$ are called **symmetric tensors**. Sometimes the notation $\mathrm{Sym}^2(E)$ is used for $E \vee E$.

(ii) The vectors of the form $v \otimes w - w \otimes v$, are called **antisymmetric tensors**. The submodule denoted $E \wedge E$, called the **antisymmetric tensor square** is the submodule generated by antisymmetric tensors.

From now on, we construct algebras over fields and thus we are interested in the case where $E = F, G = R$ where R is a field. We will discus how tensor products are linked to algebras but we will first decompose tensor products.

Corollary 2.1.10. *(i) In the hypothesis of Theorem 2.1.8, let $E = F$ and $G = R$, the ring over which the module E is defined. Assume further, that 2 is a unit in R, i.e* $\text{char} R \neq 2$. *Then*

$$L^2(E, F; G) = L^2(E, E; R) = B(E).$$

(ii) $L^2(E, E; E) \cong L(E \otimes E, E)$ and hence the bilinear maps of $E \times E$ to E can be identified with linear maps of the tensor product $E \otimes E$ to E.

An immediate consequence of the identification of part (ii) of Corollary 2.1.10 is that if E is a module, the existence of an algebra structure on E is equivalent to the existence of a linear map $E \otimes E \to E$. This is sometimes used as a third definition of an algebra.

Remark 2.1.11. (i) A linear map $\text{Sym}^2(E) \to E$ corresponds necessarily to a commutative algebra product while a linear map $E \wedge E \to E$ corresponds necessarily to an anti commutative algebra product.

(ii) It follows that $E \otimes E = \text{Sym}^2(E) \oplus (E \wedge E)$, where $\text{Sym}^2(E)$ consists of symmetric tensors and $E \wedge E$ is made up of antisymmetric tensors.

2.1.3 Change of scalars

Many times in algebra it is desirable when a module over a ring is replaced by a related module over another ring. For instance, in linear algebra, it is sometimes useful to enlarge $\mathbb{R}$-vector spaces to $\mathbb{C}$-vector spaces. In general, it is often nice to work with algebraically closed fields. Let R, S be commutative, unital rings and $f : R \to S$ be a ring homomorphism. Consider an S-module N as an R-module by the action

$$rn := f(r)n,$$

for all $r \in R$ and $n \in N$. In particular, S is an R module via $rs = f(r)s$. Passing from N as an S-module to N as an R-module is called **restriction of scalars**. The process of extension of scalars reverses the process described above.

Suppose that $R \subset S$ and M is an R-module. We wish to form an S-module related to M. We call the module $S \otimes_R M$ defined via $s(s' \otimes m) = (ss') \otimes m$, for all $s \in S$, **a change of rings**. This is useful when it is desirable to multiply elements of M by scalars from a bigger ring. When we discuss rationality questions of group representations and characters, we will need the following case. Set $R = \mathbb{R}$ and $S = \mathbb{C}$. If V is a vector space over $\mathbb{R}$, then $\mathbb{C} \otimes_{\mathbb{R}} V$ is a vector space over the complex numbers. In particular, if $\dim_{\mathbb{R}} V = n$, then $\mathbb{C} \otimes_{\mathbb{R}} V$ is a $2n$-dimensional space over $\mathbb{R}$ as we can regard $\mathbb{C}$ as a 2-dimensional vector space over $\mathbb{R}$ with basis $\{1, i\}$.

2.2 Tensors and powers of characters

In this section we will discuss the link between tensors and powers of characters of groups. Let V be a G-module for a finite group G. Suppose that the module V affords a character χ. Basic results in representation theory show that $V \otimes V$ affords the character χ^2 while the symmetric part $\mathrm{Sym}^2(V)$ of $V \otimes V$ affords the character χ_S, also denoted $\mathrm{Sym}^2(\chi)$, and the antisymmetric part $V \wedge V$ affords the character χ_A such that $\chi^2 = \chi_S + \chi_A$.

We begin by giving basic definitions of the relevant concepts and state without proof some results which will be used in what follows. Throughout this section G is a finite group and every vector space is assumed to be finite-dimensional.

Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and let $GL(V)$ be the group of isomorphisms of V onto itself. By a **representation** of G we mean a homomorphism $\rho : G \to GL(V)$. The **degree** of the representation is the dimension of V as a vector space, in particular, if the degree is 1, the representation is said to be **linear**. The space V is said to be a representation space for G. If U is a subspace of V such that U is invariant under $\rho(g)$ for all g, then U is said to **admit** G or to be **G-invariant**. If $V \neq 0$ and the only G-invariant subspaces of V are 0 and V itself, then we say that ρ is **irreducible**, otherwise it is **reducible**. If $H \leq G$ is a subgroup of G, the restriction of ρ to H is a representation of H. Let V be a one-dimensional vector space. The representation ρ such that $\rho(g) = \iota$ for all $g \in G$, where ι is the identity of $GL(V)$, is called the **trivial** representation.

As an example, let Ω be a finite G-set and R a commutative ring (we do not lose much if we consider rings instead of fields). Let M be a free R-module with basis $\{m_\omega | \omega \in \Omega\}$, where the basis elements are symbols corresponding bijectively to the elements of Ω. Then, each $x \in G$ defines an R-linear map $T(x) : M \to M$, where $T(x)m_\omega = m_{x\omega}$ for each $\omega \in \Omega$. It is clear that $T(1) = 1_M, T(xy) = T(x)T(y)$, for all $x, y \in G$. Thus, there is a homomorphism from G to the group of R-automorphisms of M. We call the map $x \to T(x)$ associated with a fixed G-set Ω a **permutation representation** defined by Ω.

If $(e_i) = \{e_1, \ldots, e_n\}$ is a basis for the representation space V, then each $\tau \in GL(V)$ can be represented by a matrix as follows. Write $\tau(e_i) = \sum_j a_{ij} e_i$, for each e_i. Then the matrix representing τ, denoted by $\tau_{(e_i)}$, is given by

$$\tau_{(e_i)} = [a_{ij}].$$

Each of these matrices is invertible, so there is an isomorphism $GL(V) \cong GL(n, \mathbb{F})$ between the group of all invertible linear transformations and the group of all invertible $n \times n$ matrices over $\mathbb{F}$. Thus, equivalently, a representation of a group is a homomorphism $\rho : G \to GL(n, \mathbb{F})$. Such a representation is called a **matrix representation**.

Given a representation $\rho : G \to GL(V), V$ can be endowed with a module structure via

$$g \cdot m = \rho(g)m.$$

By a G-**module** we mean a module M such that $gm \in M$ for all $g \in G, m \in M$. Given a G-module M, we may obtain a representation $\sigma : G \to GL(M)$ by defining $\sigma(g)m = g \cdot m$. Thus representations and G-modules are equivalent and we will freely switch from one to the other.

We have a more general setting of representations by considering algebras over fields. Let A be an algebra over a field $\mathbb{F}$, and let M be an A-module. Then each $x \in A$ defines an $\mathbb{F}$-linear map $x_l : M \to M$ where $x_l(v) = xv, v \in M$. In general, we have the following definition of a representation of an algebra.

Definition 2.2.1. Let A be a finite-dimensional algebra over a field $\mathbb{F}$ and M a finite-dimensional vector space over $\mathbb{F}$. A **representation** of A with representation space M is an algebra homomorphism

$$T : A \to \mathrm{Hom}_{\mathbb{F}}(M, M),$$

that is, a mapping T such that

$$T(a + b) = T(a) + T(b),\ T(ab) = T(a)T(b),$$

$$T(\alpha a) = \alpha T(a), T(e) = 1_{End(M)}, a, b \in A, \alpha \in \mathbb{F}$$

and e is the identity of A if it exists.

In light of the last paragraph, group representations fit in the general setting as follows. Consider the set of all formal linear sums

$$\mathbb{F}G = \{\sum_{x \in G} \alpha_x x, \alpha_x \in \mathbb{F}\}.$$

This can be turned into an algebra by defining addition and multiplication as

$$\sum_{x \in G} \alpha_x x + \sum_{x \in G} \beta_x x = \sum_{x \in G} (\alpha_x + \beta_x) x,$$
$$(\sum_{x \in G} \alpha_x x)(\sum_{y \in G} \beta_y y) = \sum_{x, y \in G} \alpha_x \beta_y xy.$$

The algebra $\mathbb{F}G$ is called the **group algebra**. Elements x of G can be identified with the elements $1 \cdot x$ of $\mathbb{F}G$ so that G can be regarded as embedded in $\mathbb{F}G$. Representations of G then correspond to left $\mathbb{F}G$-modules.

Closely related to representations are characters. We start from the general setting of algebras and proceed to the case of groups. Let A be an $\mathbb{F}$-algebra and M an A-module. For every $f \in \mathrm{End}_{\mathbb{F}} M$, choosing an $\mathbb{F}$-basis for M means that f can be represented by a matrix $\mathbf{f}$ over $\mathbb{F}$. Define

$$\mathrm{Tr}(f, M) = \text{Trace of } \mathbf{f}.$$

It is clear from basic linear algebra that this is independent of the choice of basis.

Let M be a left A-module. The function $\chi : A \to \mathbb{F}$ defined by

$$\chi(a) = \mathrm{Tr}(a, M), a \in M$$

is called the **character** of A **afforded** by M. The **degree** of χ, denoted $\deg\chi$, is defined as the the dimension $\dim_{\mathbb{F}} M$, of M. Because χ is the trace map, $\deg\chi = \chi(1)$. In what follows, we define a **basic set** of simple left A-modules as a set of representatives of the isomorphism classes of simple left A-modules. In other words, $\{M_1, \ldots, M_s\}$ is a basic set of simple A-modules if every simple A-module is isomorphic to some $M_j, 1 \leq j \leq s$ and $M_i \neq M_j$ if $i \neq j$. If $\{M_1, \ldots, M_s\}$ is a basic set of simple left A-modules, and χ_i is the character afforded by M_i, the characters $\{\chi_1, \ldots, \chi_s\}$ are called a basic set of **irreducible** characters. We use the notation $\operatorname{Irr}(A) = \{\chi_1, \ldots, \chi_s\}$, following [CR62]. Sometimes the notation χ_M is used to show that χ is afforded by the module M. The **principal** character is the character whose value is 1 at each group element. This is afforded by a module of dimension one.

Let G be a group and $\mathbb{F}$ a field of characteristic 0, and let $\chi : \mathbb{F}G \to \mathbb{F}$ be the character afforded by the left $\mathbb{F}G$-module M. By definition,

$$\chi(a) = \operatorname{Tr}(a, M), a \in \mathbb{F}G.$$

Restricting χ to G, we obtain a function $\chi : G \to \mathbb{F}$ given by

$$\chi(x) = \operatorname{Tr}(x, M), x \in G.$$

It can be shown that the elements of G form a basis for $\mathbb{F}G$ so that the trace function $\mathbb{F}G \to \mathbb{F}$ can be recovered from the values $\{\chi(x) : x \in G\}$ by using the formula

$$\chi(\sum_{x \in G} \alpha_x x) = \sum_{x \in G} \alpha_x \chi(x), \alpha_x \in \mathbb{F}.$$

If $M^* := \operatorname{Hom}(M, \mathbb{F})$ is the dual of M, we denote by χ^* the character afforded by M^*.

For the remainder of this section, we will be concerned with characters of groups over the field $\mathbb{C}$, of complex numbers. A complex-valued character χ is said to be **real** if $\chi(g) \in \mathbb{R}$ for all $g \in G$. The following result will be useful in what follows. For the proof we refer the reader to [CR].

Proposition 2.2.2. *Let M and M' be $\mathbb{C}G$-modules. Then :*

(i) $\chi_{M \oplus M'} = \chi_M + \chi_{M'}$,

(ii) $\chi_{M \otimes M'} = \chi_M \chi_{M'}$,

(iii) $\chi_M^*(g) = \chi_M(g^{-1}) = \overline{\chi(g)}$,

(iv) If $\chi_M = \chi_{M'}$, *then* $M \cong M'$.

For other facts about characters, multiplicities, inner products and the arithmetic properties of characters, we will refer the reader to the standard references given below.

On occasion we will need to refer to Clifford's theorem which gives a relationship between the action of a group on an irreducible module with the restriction to a normal subgroup. We state the theorem here without proof. Again the reader is referred to [Col90],[Isa76],[CR62],[CR90],[Gor07] or [Ser77].

Theorem 2.2.3 (Clifford)**.** *Let V be an irreducible G-module over a field $\mathbb{F}$ and let N be a normal subgroup of G. Then V is a direct sum of N-invariant subspaces $V_i, 1 \leq i \leq r$, which satisfy the following conditions:*

(i) *$V_i = X_{i1} \oplus X_{i2} \oplus \cdots \oplus X_{it}$, where each X_{ij} is an irreducible N-submodule, $1 \leq i \leq r$, t is independent of i, and $X_{ij}, X_{i'j'}$ are isomorphic N-submodules if and only if $i = i'$.*

(ii) *For any N-submodule U of V, we have $U = U_1 \oplus U_2 \oplus \cdots \oplus U_r$ where $U_i = U \cap V_i, 1 \leq i \leq r$. In particular, any irreducible N-submodule of V lies in one of the V_i.*

(iii) *For $x \in G$, the mapping $\pi(x) : V_i \to xV_i, 1 \leq i \leq r$, is a permutation of the set $S = \{V_1, V_2, \ldots, V_r\}$ and π induces a transitive permutation representation of G on S. Furthermore, $NC_G(N)$ is contained in the kernel of π.*

2.2.1 Realizability questions

In this subsection we will discuss the fundamental questions of realizability of representations. Our exposition follows that of Curtis and Reiner [CR87]. Throughout G is a finite group and k is a field.

Definition 2.2.4. A kG-module M is **realizable** in a subfield F of k if there exists an FG-module M_0 such that $M = k \otimes_F M_0$. The module M_0 is called an F-form of M.

2.2.2 Rationality over $\mathbb{R}$

We are interested in the case where $k = \mathbb{C}$ and $F = \mathbb{R}$. Consider a $\mathbb{C}G$-module M. We will in general be interested in subfields K of $\mathbb{C}$ such that M is realisable in K, that is, in the spirit of Definition 2.2.4, there exists an isomorphism of $\mathbb{C}G$-modules

$$M \cong \mathbb{C} \otimes_K M_0$$

for some KG-module M_0. A K-basis for M_0 is then a $\mathbb{C}$-basis for M and relative to this basis, the matrices $\{M(g) : g \in G\}$ have entries in K.

Recall that the contragredient module M^* of a $\mathbb{C}G$-module M is the dual space $M^* = L(M, \mathbb{C})$ on which G acts according to

$$(g \cdot f)m = f(g^{-1} \cdot m), \text{ for all } f \in M^*, m \in M$$

and $g \in G$. An elementary result in the theory of characters show that if M affords a character χ, then M^* affords the character χ^* where $\chi^*(g) = \bar{\chi}(g) = \chi(g^{-1})$ for all $g \in G$. It follows that χ is real-valued if $\chi^* = \chi$. We also have that χ is real-valued if M has an $\mathbb{R}$-form. We will state some results which are relevant to the discussion in the rest of the thesis. Though we aim for self-containment, we will not prove all results. Where proofs are not provided, relevant references are given for the reader to check the proof for themselves. It turns out that the question of whether a character χ afforded by a $\mathbb{C}G$-module M is real is equivalent to the question of whether or not there exists a G-invariant bilinear form on M. We have the following theorem.

Theorem 2.2.5 (Frobenius-Schur)**.** *Let M be a $\mathbb{C}G$-module with character χ. Then*

(i) χ is real-valued if and only if M admits a nondegenerate G-invariant bilinear form.

(ii) M has an $\mathbb{R}$-form if and only if M admits a nondegenerate symmetric G-invariant bilinear form (over $\mathbb{C}$). In this case M affords a representation in which each $x \in G$ is represented by a real orthogonal matrix.

Proof. This is [CR87, Theorem 73.3]. □

Recall the notion of restriction of scalars from Section 2.1.3. We have the following.

Definition 2.2.6. Let M be a finitely generated KG-module, where G is a finite group, and K a field. For any subfield $F \subset K$ such that $\dim_F K < \infty$, the natural embedding $FG \hookrightarrow KG$ defines the structure of a finitely generated FG-module on M, called the FG-module obtained by restriction of scalars, and denoted by $M|_{FG}$ (or simply M_{FG}).

We have the following result which we will state without proof.

Lemma 2.2.7. *Let M be a KG-module affording a matrix representation $x \to M(x) = (m_{ij}(x)), x \in G$, with entries $\{m_{ij}(x)\}$ in K. Then for any subfield $F \subset K$ with $\dim_F K < \infty$, M_{FG} affords a matrix representation $x \to \rho_{K/F}(m_{ij}(x))$ where $\rho_{K/F}$ is the regular matrix representation of K over F.*

Proof. This is [CR87, Lemma 73.6, p. 722]. □

Corollary 2.2.8. *With the same notation as above, assume that K/F is a Galois extension with Galois group $\Gamma = \mathrm{Gal}(F/K)$. Then for all $x \in G$,*

$$\mathrm{Tr}(x, M_{FG}) = \sum_{\sigma \in \Gamma} \sigma \mathrm{Tr}(x, m).$$

Proof. See [CR87, Corollary 73.7, p. 723]. □

An immediate consequence of this is the following.

Corollary 2.2.9. *Let M be a finitely generated $\mathbb{C}G$-module with character χ. Then the character afforded by the $\mathbb{R}G$-module $M|_{\mathbb{R}G}$ is given by*

$$x \mapsto \mathrm{Tr}(x, M|_{\mathbb{R}G}) = \chi(x) + \overline{\chi(x)},\ x \in G,$$

where bar denotes complex conjugation.

Proof. The extension $\mathbb{C}/\mathbb{R}$ is a Galois extension with Galois group $\Gamma = \langle \sigma \rangle$, where σ is complex conjugation and so it follows that

$$\mathrm{Tr}(x, M|_{\mathbb{R}G}) = \overline{\mathrm{Tr}(x, M)} + \overline{\overline{\mathrm{Tr}(x, M)}}$$

and the result follows. □

The following theorem gives a relationship between the results just stated with some aspects of character theory.

Theorem 2.2.10. *Let M be a simple $\mathbb{C}G$-module with character χ, and let $n = \dim_{\mathbb{C}} M$. Then exactly one of the following possibilities occurs:*

(i) χ is not real-valued and the $\mathbb{R}G$-module $M|_{\mathbb{R}G}$, obtained from M by restriction of scalars is a simple $\mathbb{R}G$-module of dimension $2n$ over $\mathbb{R}$, affording the character $\chi + \bar{\chi}$. In this case the endomorphism algebra $\mathrm{End}_{\mathbb{R}G} M_{\mathbb{R}G}$ is isomorphic to $M_n(\mathbb{C})$.

(ii) χ *is real-valued, and* M *has an* $\mathbb{R}$*-form* M_0*. Then* M_0 *is absolutely simple with character* χ*. Moreover,* $\mathrm{End}_{\mathbb{R}G}M_0 \cong \mathbb{R}$ *and the Wedderburn component of* $\mathbb{R}G$ *corresponding to* M_0 *is isomorphic to* $M_n(\mathbb{R})$*. In this case,* $M_{\mathbb{R}G}$ *is a direct sum of two copies of* M_0*.*

(iii) χ *is real-valued, but* M *does not have an* $\mathbb{R}$*-form. Then the* $\mathbb{R}G$*-module* $M_{\mathbb{R}G}$ *obtained from* M *by restriction of scalars is a simple* $\mathbb{R}G$*-module of dimension* $2n$*, whose character is* 2χ*. The endomorphism algebra* $\mathrm{End}_{\mathbb{R}G}M_{\mathbb{R}G}$ *is isomorphic to* $\mathbb{H}$*, and the corresponding Wedderburn component of* $\mathbb{R}G$ *is isomorphic to* $M_n(\mathbb{H})$*, where* $\mathbb{H}$ *is the real division algebra of Hamilton's quaternions.*

Proof. This is proved in [CR87], p.723. □

Remark 2.2.11. The theorem above gives the so-called "kinds" of representations. If M affords a real-valued character and is realisable over $\mathbb{R}$, it is called of the first kind, if it affords a real-valued character but does not have an $\mathbb{R}$-form, then we say it is of the second kind and if χ is not real-valued, M is a representation of the third kind.

The next couple of results will give other ways of distinguishing between the three possibilities of the theorem above. The first is a kind of geometric way which employs the existence of G-invariant bilinear forms on M.

Theorem 2.2.12. *Let* M *be a simple* $\mathbb{C}G$*-module affording the character* χ*.*

(i) If M *does not admit a nonzero* G*-invariant bilinear form, then* χ *is not real-valued, and* M *has no* $\mathbb{R}$*-form. Call* M *unitary in this case.*

(ii) If M *admits a nonzero* G*-invariant bilinear form, then this form is unique up to scalar multiples. It is nondegenerate and is either symmetric or skew symmetric. If the form is real-valued, then* χ *is real-valued, and* M *has an* $\mathbb{R}$*-form. We call* M *orthogonal in this case (since* M *affords a real representation by orthogonal matrices). If the form is skew-symmetric, then* χ *is real-valued, but* M *has no* $\mathbb{R}$*-form. We call* M *sympletic, since* M *affords a representation by sympletic*[1] *transformations.*

Proof. See [CR87], Theorem 73.10, p.724. □

[1] A sympletic transformation of a vector space over a field is a linear map preserving a nondegenerate skew-symmetric bilinear form.

Remark 2.2.13. We note that if M is unitary, then M admits a positive definite G-invariant Hermitian form.

We close this subsection with an important test of the type of a simple $\mathbb{C}G$-module as in Theorem 2.2.12 in terms of the character table. We first set up some notation. Let M be an arbitrary finitely generated $\mathbb{C}G$-module with character χ. Starting with the isomorphism of vector spaces over $\mathbb{C}$,

$$L_{\mathbb{C}G}(M^*, M) \cong \mathrm{inv}_G(M \otimes M),$$

where $\mathrm{inv}_G(M)$ is $\{m \in M | gm = m, \ \forall g \in G\}$, we define the intertwining number:

Definition 2.2.14. Let M be a simple $\mathbb{C}G$-module and suppose that M^* is the dual of M. Then the intertwining number is defined as

$$i(M, M^*) = \mathrm{dim}_{\mathbb{C}}\mathrm{inv}_G(M \otimes M).$$

We also define the symmetric and antisymmetric intertwining numbers.

Definition 2.2.15.

$$i_s(M^*, M) = \mathrm{dim}_{\mathbb{C}}\mathrm{inv}_G(M \vee M), i_a(M^*, M) = \mathrm{dim}_{\mathbb{C}}\mathrm{inv}_G(M \wedge M),$$

respectively, where $M \vee M, M \wedge M$ are the symmetric and antisymmetric parts of $M \otimes M$.

Since $M \otimes M \cong (M \vee M) \oplus (M \wedge M)$ as $\mathbb{C}G$-modules, it follows that

$$i(M^*, M) = i_s(M^*, M) + i_a(M^*, M).$$

We define the Frobenius-Schur indicator which we will use to determine the type of a simple $\mathbb{C}G$-module M.

Definition 2.2.16. Let M be a $\mathbb{C}G$-module. Then the Frobenius-Schur indicator $c(M)$ is defined by

$$c(M) = i_s(M^*, M) - i_a(M^*, M).$$

It can easily be shown that $c(M) \in \{0, \pm 1\}$ for a simple module M. The next result gives a method of computing $c(M)$ in terms of the character afforded by M. The Frobenius-Schur indicators $\{c(Z_i)\}$ of a basic set of simple $\mathbb{C}G$-modules can be regarded as basic character table data. In fact, where the computer algebra package MAGMA [BC94] can compute the character table of a group, the Frobenius-Schur indicators of all irreducible characters come precomputed with the character table.

Theorem 2.2.17 (Frobenius-Schur). *Let M be a simple $\mathbb{C}G$-module with character χ. Then $c(M) \in \{0, \pm 1\}$, and is given by*

$$c(M) = |G|^{-1} \sum_{x \in G} \chi(x^2). \tag{2.3}$$

Moreover, M is orthogonal if $c(M) = 1$, sympletic if $c(M) = -1$ and unitary if $c(M) = 0$.

Proof. See Collins [Col90, Ch. 2, §8, p. 85–87] for details, or [CR87, Theorem 73.13, p. 726]. □

We are ready now to discuss the class of algebras dubbed Norton algebras.

2.3 Norton algebras and generalizations

In this section we discuss the class of nonassociative Norton algebras and generalisations. As discussed in the preamble to the chapter, in [Nor88], Norton constructed an algebra for the group Fi_{24} via the 783-dimensional constituent of the permutation representation of the triple cover $3 \cdot Fi_{24}$ on its 3×306936 transpositions. This algebra supported a unique nonzero, nondegenerate G-invariant bilinear form. In fact, it supported an invariant symmetric cubic form. The group $3 \cdot Fi_{24}$ is a typical example of a class of finite groups called 3-transposition groups which are generated by a class of involutions, called transpositions, such that the product of any two has order at most 3. Let G be a 3-transposition group. One obtains a graph, called a rank 3 graph, from the action of the group on its transpositions. We refer the reader to [Wil09] for further details. Take a unitary space V generated by the points of the rank three graph, which are the transpositions. Suppose $(\cdot, \cdot) : V \times V \to \mathbb{C}$ is an inner product on V. Then we can obtain a cubic form as follows. Let t_i be a transposition and let v_i be the corresponding vector. Then $(v_i, v_j, v_k) = (v_i, \overline{v}_k)(v_j, \overline{v}_k) + (v_i, \overline{v}_j^{t_k})$ is a G-invariant cubic form. See [Nor88, p.488] for more details.

From the character table, Norton noticed that there is a 783-dimensional orthogonal constituent of the permutation module which is a summand in its own symmetric square affording a character χ of the "first kind". This means that the character is real-valued and the constituent is realisable over the reals. Furthermore, according to Theorem 2.2.12, this constituent supports a unique bilinear form, up to scalar. By Remark 2.1.11 (iii), appearing in the symmetric part of the tensor square means that the constituent supports a commutative algebra structure.

In the same paper, Norton constructs a 26-dimensional algebra for ${}^3D_4(2)$ from its rank 4 graph on the action on its transpositions. This algebra also supports a unique bilinear form and Norton showed that the form can be extended to a trilinear form. This allows viewing the group as a group of linear transformations of the algebra fixing the trilinear form. Using a construction which we discuss in detail below, Norton extended the algebra to a unital 27-dimensional algebra by adjoining an algebra unit, where multiplication in this extended algebra involved both the algebra product and the bilinear form on the 26-dimensional algebra. The extended 27-dimensional algebra turns out to be the exceptional 27-dimensional Jordan algebra.

Norton also showed that it was possible to construct algebras using a similar technique for the groups $\Omega(7,3), O_7(3)$ and $\Omega^-(6,3)$ all of which have triple covers. Kitazume [Kit87] subsequently carried out these constructions.

We also note that other workers in the field have since constructed Norton algebras for other groups. In particular, D. Smith [Smi77] constructed commutative nonassociative algebras for triple covers of 3-transposition groups. Ryba [Ryb96] constructed a Norton algebra for the Harada-Norton group as well as one for the Baby Monster, $\mathbb{BM}$ [Ryb07]. Frohardt [Fro85] has also described a construction of an algebra for O'Nan's group. Cameron, Goethals and Seidel [CGS78] have shown that any group with a rank 3 action supports a Norton algebra on that rank 3 representation.

Recall that we have mentioned that Norton made an allusion to the fact that Norton algebras were loosely defined. The apparent definitions are too general to be satisfactory. Jon Hall [Hal18] mentions that indeed this is still the case. We use a different definition of Norton algebras to the one that Norton gives in his thesis. We will devote a subsection to the discussion of Norton algebras and Krein parameters.

2.3.1 Krein parameters

Krein parameters were introduced by Scott [Sco77] in his work on character products. Cameron, Goethals and Seidel [CGS78] made use of these parameters together with association schemes to define Norton algebras. Our notation here, with slight alterations, follows that paper. For a more comprehensive treatment of association schemes, we refer the reader to the inimitable

[BI84].

Definition 2.3.1. Let Ω be a finite set and consider a partition of the edge set of the complete graph on Ω into s classes. Let $A_1, A_2, \ldots, A_s$ be the adjacency matrices of each of the classes. Each of these matrices is a symmetric $(0,1)$ matrix. We say that the partition is an association scheme if the linear span of $\{A_0 = I, A_1, A_2, \ldots, A_s\}$ over $\mathbb{R}$, say, is an algebra $\mathscr{A}$. Such an algebra is called the Bose-Mesner algebra of the scheme.

If $s = 2$ for example, then we get a pair of complementary strongly-regular graphs. Conversely, if one has a strongly regular graph which is not null or complete, a theorem in the theory of strongly regular graphs shows that the complement is also strongly regular so we get an association scheme. We will show that from distance regular graphs, one obtains an association scheme which recasts the claim made above since the distance regular graphs of diameter 2 are precisely the strongly regular graphs.

The product of any two matrices A_i, A_j is again in the algebra $\mathscr{A}$ so that

$$A_i A_j = \sum_{k=0}^{s} p_{ij}^k A_k$$

for some parameters p_{ij}^k.

Definition 2.3.2. The parameters p_{ij}^k are called intersection numbers for the scheme.

The intersection numbers are non-negative integers with well known combinatorial interpretation [CGS78]. The definition above is restrictive and gives symmetric schemes only. The formulation given in the subsection below is a more general definition of an association scheme.

2.3.2 Alternative formulation of association schemes

Using our interpretation of graphs as sets supporting symmetric relations, we may define an association scheme as follows. Let Ω be a finite set of points and suppose that the ordered pairs of the points i.e., $\Omega \times \Omega$ can be partitioned into $s+1$ classes $C_0, C_1, \ldots, C_s$ such that

(i) The diagonal $\{(\omega, \omega) | \omega \in \Omega\}$ is a single class C_0,

(ii) Given $i, j, k \in \{0, \ldots, s\}$ and $(u, v) \in C_i$, the number p^i_{jk} of $w \in \Omega$ such that $(u, w) \in C_j$ and $(w, v) \in C_k$ depends on i, j, k but not on u, v.

Then $\mathfrak{X} = (\Omega, \{C_0, C_1, \ldots, C_s\})$ is an association scheme. If in addition, each class is symmetric, that is, if $(x, y) \in C_i$, then $(y, x) \in C_i$ for all i, we say that the scheme is **symmetric**. We next give a definition of the notion of group actions on association schemes.

Definition 2.3.3. A permutation group G on Ω **preserves** an association scheme $\mathfrak{X} = (\Omega, \{C_0, C_1, \ldots, C_s\})$ if all the sets C_i in the partition are fixed setwise by G. An **automorphism group** of an association scheme is the group of permutations which preserve it. The association scheme is then said to **admit** a permutation group G if G is a subgroup of its automorphism group.

Proposition 2.3.4. *A distance transitive graph (hence strongly regular graph in particular) gives rise to an association scheme.*

Proof. We will use the alternative formulation of the association schemes in this proof. Suppose that $\Gamma = (X, E)$ is distance regular of class d. Partition $X \times X$ as follows. Let $R_i \subset X \times X$ be defined by $(x, y) \in R_i$ if and only if $d(x, y) = i, 1 \leq i \leq d$, and set $R_0 := \{(x, x) | x \in X\}$. It is clear that the R_i partition $X \times X$, that R_i are symmetric since $\sim$, the adjacency relation, is, and the last requirement follows from the definition of distance regular graphs. □

Because $\mathscr{A}$ consists of pairwise commuting symmetric matrices, these matrices can be simultaneously diagonalised and there is an orthogonal decomposition

$$V = \mathbb{R}\Omega = V_0 \oplus V_1 \oplus \cdots \oplus V_s,$$

where each V_i is an eigenspace for all matrices in $\mathscr{A}$.

Since the spaces V_i have constant row sums, it can be assumed that V_0 is spanned by the all 1 vector $\mathbf{j}$. By π_i we mean the usual orthogonal projection of V onto V_i. Set E_i to be the matrix representing π_i. Then $E_0 = \frac{1}{n}\mathbf{J}$, where $\mathbf{J}$ is the all 1 matrix and $n = |\Omega|$. Furthermore, since $\pi_i^2 = \pi_i$, $\{E_i | 0 \leq i \leq s\}$ is the set of minimal idempotents of $\mathscr{A}$, i.e.,

$$E_i E_j = \delta_{ij} E_i.$$

The Bose-Mesner algebra is also closed under Hadamard multiplication, that is, point-wise multiplication where matrices of the same dimensions have a product formed by multiplying corresponding entries. This is because

$$A_i \circ A_j = \delta_{ij} A_i,$$

where $\circ$ is the Hadamard product. It follows that

$$\{A_0, A_1, \ldots, A_s\}$$

is the set of minimal idempotents for the Hadamard multiplication. Thus,

$$E_i \circ E_j = \sum_{k=0}^{s} q_{ij}^k E_k$$

for some real numbers q_{ij}^k. These real numbers q_{ij}^k are the eigenvalues of $E_i \circ E_j$. Because $E_i \circ E_j$ is a principal submatrix of the Kronecker product $E_i \otimes E_j$ of two idempotents,

$$0 \leq q_{ij}^k \leq 1$$

for all i, j, k. This is called the Krein condition [Sco77]. The parameters q_{ij}^k are called Krein parameters of the scheme. The following lemma will be useful in setting up Norton algebras.

Lemma 2.3.5 ([CGS78, Lemma 4.2])**.** *Let* $\mathbf{j}$ *be the all* 1 n*-long row vector. Then* $q_{ij}^0 = \delta_{ij}\mu_i/n, q_{ij}^k \mu_k = \mathbf{j}^T(E_i \circ E_j \circ E_k)\mathbf{j}$, *where* $\mu_i = \mathrm{dim} V_i$.

2.4 Norton algebras admitting permutation groups

In this section we introduce Norton algebras admitting group actions which we are most interested in for the rest of this thesis. We are interested in the following situation. Let G be a transitive permutation group on $\Omega, |\Omega| = n$. Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_s$ be the orbits of G on $\Omega \times \Omega$, called **orbitals**. Set $C_i = \Lambda_i$. We set C_0 to be the diagonal orbital, that is $C_0 = \{(\omega, \omega) | \omega \in \Omega\}$. It is easy to see that $\mathfrak{X} = (\Omega, \{\Lambda_i\}_{0 \leq i \leq s})$ is a possibly non-symmetric association scheme. Recall from the theory of group actions that the number of orbitals is called the rank of G in its action on Ω. Because the groups HS:2 and Suz:2 have rank 3 actions on 100 and 1782 points respectively, $s = 2$ in Chapters 4 and 5. In what follows, suppose that G is a transitive permutation group on $\Omega = \{1, 2, \ldots, n\}$, and θ is the permutation character of G. Furthermore, assume that θ is multiplicity free: $\theta = \chi_0 + \chi_1 + \cdots + \chi_s, \chi_i$ distinct irreducible characters of G. Then $\mathfrak{X}$ is a symmetric scheme. Let $\mathscr{A}$ be the adjacency (Bose-Mesner) algebra of $\mathfrak{X}, A_i, 0 \leq i \leq d$ the incidence matrices and E_i, the primitive idempotents of $\mathscr{A}$. For $\alpha \in \Omega = \{1, 2, \ldots, n\}, x_\alpha$ denotes the row vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the α^{th} position. Set $X = \{x_\alpha | \alpha \in \Omega\}$ and $V = \mathbb{C}^n$ be the space with X as orthonormal basis. Let V_i be the image of V under the projection E_i. We have an orthogonal decomposition

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_s,$$

and

$$uE_i = \begin{cases} u & \text{if } u \in V_i \\ 0 & \text{if } u \in V_j, j \neq i. \end{cases}$$

Let $\rho : g \mapsto A(g)$ be the permutation representation of $G :$

$$x_\alpha A(g) = x_\beta$$

if $\beta = \alpha^g, \alpha \in \Omega, g \in G$, i.e., $(A(g))_{\alpha\beta} = \delta_{\alpha^g\beta}$ where δ is the Kronecker symbol. It can be shown that the matrices in $\mathscr{A}$ commute with all $A(g), g \in G$. As in the previous sections of this chapter, by $L(V, V)$ we mean the set of all linear mappings from V to itself (endomorphisms of V). The operators $A(g)$ can be regarded as elements of $L(V, V)$ where we "forget" the basis X of V, and write

$$uA(g) = ug, u \in V, g \in G$$

and

$$uE_i = u\pi_i, \ \pi_i \text{ the projection of } V \text{ to } V_i.$$

The subspace V_i is invariant under G since for $u \in V_i$, we have $ug = u\pi_i g = ug\pi_i, g \in G$. This action of G on V_i induces a representation φ_i of G on V_i. It follows that $\rho = \varphi_0 + \varphi_1 + \cdots + \varphi_s$ and the φ_i are non-equivalent irreducible representations of G, since $\theta = \chi_0 + \chi_1 + \cdots + \chi_S$ with $\chi_i \neq \chi_j$ if $i \neq j$. We may, therefore, assume that φ_i affords χ_i. Thus, χ_i corresponds to E_i, and $\overline{\chi_i}$ corresponds to $\overline{E_i}$.

Let $V \otimes V$ be the tensor product of V, i.e., the n^2-dimensional vector space spanned by $u_\alpha \otimes u_\beta$. In other words,

$$(\sum_\alpha \lambda_\alpha u_\alpha) \otimes (\sum_\beta \lambda_\beta u_\beta) = \sum_{\alpha,\beta} \lambda_\alpha \lambda_\beta (u_\alpha \otimes u_\beta)$$

where $\lambda_\alpha, \lambda_\beta \in \mathbb{C}$ and $u_\alpha, u_\beta \in V$. We define a Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that $\{x \otimes y | x, y \in X\}$ is an orthonormal basis for $V \otimes V$. Then

$$\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle = \langle u_1, u_2 \rangle \langle v_1, v_2 \rangle.$$

Similarly, $V \otimes V \otimes V$ becomes a Hermitian space with

$$\langle u_1 \otimes v_1 \otimes w_1, u_2 \otimes v_2 \otimes w_2 \rangle = \langle u_1, u_2 \rangle \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle$$

for $u_i, v_i, w_i \in V, i = 1, 2$. G acts on $V \otimes V$ diagonally, i.e., for $u, v \in V, g \in G$,

$$(u \otimes v)g = (ug) \otimes (vg).$$

The tensor product $V_i \otimes V_j$ is invariant under the action of G and affords the representation $\varphi_i \otimes \varphi_j$ of G. We claim that E_i is the Gram matrix of $\{x\pi_i | x \in X\}$, i.e.,

$$(E_i)_{xy} = \langle x\pi_i, y\pi_i \rangle \text{ for } x, y \in X,$$

$(E_i)_{xy}$ being the (α, β) entry of E_i such that $x = x_\alpha, y = x_\beta, \alpha, \beta \in \Omega$.

Lemma 2.4.1. *With the notation of the preceding section, E_i is the Gram matrix of $\{x\pi_i | x \in X\}$.*

Proof. Our proof is based on the one in [BI84, Lemma 8.2]. Notice that since the vectors $x\pi_i, y\pi_i$ are the $\alpha^{\text{th}}, \beta^{\text{th}}$ rows of E_i, we have $\langle x\pi_i, y\pi_i \rangle = (E_i \overline{E_i}^T)_{xy}$. But E_i is idempotent and $\overline{E_i}^T = E_i$ so that $E_i \overline{E_i}^T = E_i E_i = E_i$ and the claim follows. □

The following results allow us to associate to each non-vanishing Krein parameter q_{ij}^k a map σ_{ij}^k from $V_i \otimes V_j$ to V_k and consequently, to define Norton algebras which are commutative algebras on V_i provided that q_{ii}^i does not vanish.

Proposition 2.4.2 ([CGS78, Proposition 5.1])**.** *Let $\sigma_{ij}^k : V_i \otimes V_j \to V_k$ be defined by*

$$(v_i \otimes v_j)\sigma_{ij}^k = \sum_{\omega \in \Omega} \langle v_i \otimes v_j, \omega\pi_i \otimes \omega\pi_j \rangle \omega\pi_k.$$

Then $\sigma_{ij}^k = 0$ if and only if $q_{ij}^k = 0$.

Proposition 2.4.3 ([CGS78, Proposition 5.2]). *The eigenspace V_i of an association scheme on Ω is a commutative algebra under the product defined by*

$$u \star v = \sum_{\omega \in \Omega} \langle u, \omega \rangle \langle v, \omega \rangle \omega \pi_i.$$

This product is zero if and only if $q_{ii}^i = 0$.

Definition 2.4.4. Let V_i be an eigenspace of an association scheme on a finite set Ω. If the Krein parameter q_{ii}^i does not vanish, then the commutative (non-associative) algebra defined in Proposition 2.4.3 is called a **Norton algebra**.

We have the following result due to Scott [Sco77]. The proof of the theorem makes very clear the ideas we use in our construction of Norton algebras and we give greater details than we would normally have.

Theorem 2.4.5 (Scott). *Let G be a transitive permutation group on $\Omega = \{1, 2, \ldots, n\}$ and θ the permutation character of G in its action on Ω. If θ is multiplicity free: $\theta = \chi_0 + \chi_1 + \cdots + \chi_s, \chi_i$ distinct irreducible characters of G, q_{ij}^k the Krein parameters of the association scheme $\mathfrak{X} = (\Omega, \{\Lambda_i\}_{0 \le i \le s})$ with the scheme set up as in the preceding discussion, then*

(i) χ_k appears in $\chi_i \chi_j$ i.e., $(\chi_k, \chi_i \chi_j) \neq 0$, whenever $q_{ij}^k \neq 0$, and

(ii) χ_k appears in $\mathrm{Sym}^2 \chi_i$, whenever $q_{ii}^k \neq 0$, where $\mathrm{Sym}^2 \chi_i$ is the symmetric part of χ_i^2.

Proof. Our proof is based on that given for Theorem 8.1 in [BI84]. Recall from Proposition 2.4.2 that the map $\sigma_{ij}^k : V_i \otimes V_j \to V_k, u \otimes v \mapsto \sum_{x \in X} \langle u \otimes v, x\pi_i \otimes x\pi_j \rangle x\pi_k$ is nonzero whenever $q_{ij}^k \neq 0$. Consider W_{ij}, the subspace of $V_i \otimes V_j$ spanned by $\{x\pi_i \otimes x\pi_j | x \in X\}$. We have, for $x \in X$ and $g \in G$, $(x\pi_i \otimes x\pi_j)g = (x\pi_i)g \otimes (x\pi_j)g = (xg\pi_i) \otimes (xg\pi_j) = y\pi_i \otimes y\pi_j$ where $x = x_\alpha, y = x_\beta$ and $\beta = \alpha^g$. We conclude that W_{ij} is G-invariant . By Proposition 2.4.2,

$$\begin{aligned}
(x\pi_i \otimes x\pi_i)\sigma_{ij}^k g &= \frac{1}{n} q_{ij}^k x\pi_k g \\
&= \frac{1}{n} q_{ij}^k xg\pi_k \\
&= ((xg)\pi_i \otimes (xg)\pi_j)\sigma_{ij}^k, \ xg \in X \text{ and by Proposition 2.4.2} \\
&= (x\pi_i g \otimes x\pi_j g)\sigma_{ij}^k \\
&= (x\pi_i \otimes x\pi_j) g \sigma_{ij}^k,
\end{aligned}$$

whence the mapping $\sigma_{ij}^k : W_{ij} \to V_k$ commutes with the action of G. Since the image of σ_{ij}^k is nonzero when $\sigma_{ij}^k \neq 0$, it is a subspace of V_k which is simple, so by Schur's lemma the irreducible representation of G afforded by V_k appears in that afforded by W_{ij}. But then $W_{ij} \subseteq V_i \otimes V_k$, so whenever $q_{ij}^k \neq 0$, χ_k appears in $\chi_i \chi_j$. If $i = j$, then $W_{ii} \subseteq V_i \vee V_i = \mathrm{Sym}^2(V_i)$, the symmetric square of V_i. Hence, when $q_{ij}^k \neq 0$, χ_k appears in $\mathrm{Sym}^2 \chi_i$, which is the character afforded by $V_i \vee V_i$. □

Remark 2.4.6. When $i = j = k$, we have a commutative, nonassociative algebra structure on V_i admitting the group G, where the product given in Definition 2.4.4 and the proof of Theorem 2.4.5 is point-wise multiplication followed by projection. The irreducible characters of HS:2 and Suz:2 afforded by the spaces over which we define algebras are real and are realisable over $\mathbb{R}$, thus we will work over this field in the next chapters. In fact, the spaces of interest have representations realisable over $\mathbb{Q}$.

We next show that the Norton algebra product associates with the usual bilinear form.

Lemma 2.4.7. *A Norton algebra product of point-wise multiplication and projection over an association scheme eigenspace associates with the usual symmetric bilinear form* $\langle \cdot, \cdot \rangle$ *over* $\mathbb{R}$.

Proof. Let $u, v, w \in V_i$. Then

$$\begin{aligned} \langle u \star v, w \rangle &= \langle \sum_{x \in X} \langle u, x\pi_i \rangle \langle v, x\pi_i \rangle x\pi_i, w \rangle \\ &= \sum_{x \in X} \langle u, x\pi_i \rangle \langle v, x\pi_i \rangle \langle x\pi_i, w \rangle \\ &= \langle u, v \star w \rangle, \end{aligned}$$

by symmetry. □

We present a different point of view proposed by J.I Hall in [Hal18].

Definition 2.4.8 (Hall)**.** A local Norton algebra $(A, \mathcal{P})$ over a field $\mathbb{F}$ is an $\mathbb{F}$-algebra A spanned by a set $\mathcal{P}$ of vectors with the following properties:

(i) for $x \in \mathcal{P}, x^2 = \epsilon x$ for a fixed $\epsilon \in \mathbb{F}$;

(ii) for each $a, b \in \mathcal{P}$ the 2-generated algebra $l_{a,b} = \langle a, b \rangle$ has isomorphism type from $\mathcal{A}$, a given set of $\mathbb{F}$-algebra isomorphism classes. We refer to A as the algebra with generating set $\mathcal{P}$.

Remark 2.4.9. Hall uses the following geometric notation:
The vectors of $\mathcal{P}$, or sometimes the 1-spaces they span, are called **points** while the 2-generated spaces of $\mathcal{L} = \{l_{a,b} | a, b \in \mathcal{P}\}$ are called **lines**.

We note that in the definition above, if $\epsilon \neq 0$, then the elements $\frac{1}{\epsilon}p$ for $p \in \mathcal{P}$ are idempotents in the algebra, or in an algebra over a quadratic extension of $\mathbb{F}$, so that we will choose $\epsilon = 1$. Since if $\epsilon = 0$, then $p \in \mathcal{P}$ is nilpotent, we will avoid this choice since we are interested in finding axes. However, we note that other researchers in the field sometimes make other choices of ϵ as appropriate in the context of their work, for example; *c.f.* [Con85, Miy96b, Mat05].

We are going to discuss the generalised Norton algebras in terms of an eigenvalue decomposition of association schemes. For the algebras we construct using this technique, we find that our algebras are local Norton algebras as defined by Hall.

2.5 Construction

In this section we give the construction of algebras which we will use in Chapters 4 and 5. We show how we can identify an idempotent in a Norton algebra constructed over an irreducible constituent of a permutation module for a permutation group acting on some association scheme. We also how we can extend such algebras to be unital and such that 0 is an eigenvalue of a specially selected idempotent. Throughout this section fields are of characteristic 0.

Let G be a transitive permutation group on a finite set $\Omega = \{1, \ldots, n\}$ which is not doubly transitive and let W be the permutation module over a field $\mathbb{F}$, in this action. Suppose that V is an irreducible constituent of W such that V is a summand in its own symmetric square $V \vee V$, and is an eigenspace of the association scheme arising from the action of G on Ω. Suppose further, that V is an orthogonal module, so that we can define a real algebra structure on V, in particular, a Norton algebra.

Let $2X$ be a conjugacy class of involutions of G, and let $z \in 2X$. Consider the centralizer, $K = C_G(z)$, of z in G. Set $U = V \cap \mathrm{inv}_K(W)$, the space of vectors fixed by K in V. Recall that $\text{inv}_K(W) = \{w \in W | k \cdot w = w \text{ for all } k \in K\}$. We choose V such that U has dimension 1, ideally, or 2. Let $* : V \times V \to V$ be the Norton product. Since the discussion in the last section shows that $*$ commutes with the group action, $u * u \in U$ for all $u \in U$.

If $\dim_{\mathbb{F}} U = 1$, then $U = \langle u_1 \rangle$, for some $u_1 \in U$. Thus, for all $u \in U, u * u = \alpha_u u_1, \alpha_u \in \mathbb{F}$. In particular, $u_1 * u_1 = \alpha_{u_1} u_1, \alpha_{u_1} \in \mathbb{F}$. It follows that $a = \frac{1}{\alpha_{u_1}} u_1$ is an idempotent in V.

On the other hand, if $\dim_{\mathbb{F}} U = 2$, we can consider U as a projective line $\mathcal{L} = \langle \alpha_1 u_1 + \alpha_2 u_2 \rangle$, where $\{u_1, u_2\}$ is a basis for U as a vector space. If $w = \alpha_1 u_1 + \alpha_2 u_2$ is an idempotent, then we have

$$\begin{aligned} \alpha_1 u_1 + \alpha_2 u_2 &= (\alpha_1 u_1 + \alpha_2 u_2) * (\alpha_1 u_1 + \alpha_2 u_2) \\ &= \alpha_1^2 u_1^2 + 2\alpha_1 \alpha_2 (u_1 * u_2) + \alpha_2 u_2^2. \end{aligned}$$

Expressing $u_1^2, u_1 * u_2$ and u_2^2 as linear combinations of u_1 and u_2 and substituting in the equation above, we obtain a pair of quadratic equations in α_1 and α_2. If the system has a non-trivial solution over $\mathbb{F}$, then we have an idempotent, otherwise we will have to work over a quadratic extension of $\mathbb{F}$ which contains a solution to the system.

In the cases which we consider in Chapter 4 and Chapter 5, the modules under consideration are realizable over $\mathbb{Q}$, so we will work over $\mathbb{Q}$ or its quadratic extensions.

Let $(\cdot, \cdot) : V \times V \to \mathbb{F}$ be a symmetric bilinear form on V, which is unique, up to scalars by Theorem 2.2.12, scaled so that $(a, a) = 1$. We can extend the algebra structure on V using both the algebra product $*$ and $(\cdot, \cdot)$. This lends a degree of freedom in the algebra and the structure of the extended algebra can be controlled in such a way that 0 is an eigenvalue of an idempotent in the extended algebra. The next subsection gives more details of this extension.

2.5.1 Algebra extension by unit

Let $(V, *)$ be a commutative algebra over a field $\mathbb{F}$ equipped with a symmetric bilinear form $(\cdot, \cdot) : V \times V \to \mathbb{F}$. Let $\tilde{V} = \mathbb{F} \oplus V$, where elements of $\tilde{V}$ are of the form $\alpha + v, \alpha \in \mathbb{F}, v \in V$. Then $\tilde{V}$ is clearly a vector space over $\mathbb{F}$ with addition defined by

$$(\alpha + v) + (\alpha' + v') = (\alpha + \alpha') + (v + v').$$

Throughout this thesis, our algebras are commutative and nonassociative. We will drop the adjectives henceforth. Scalar multiplication is done componentwise, that is $\lambda(\alpha + v) = \lambda\alpha + \lambda v$. We turn $\tilde{V}$ into an algebra via the multiplication

$$(\alpha + v) \star (\alpha' + v') = (\alpha\alpha' + \gamma(v, v')) + (\alpha v' + \alpha' v + v * v')\,, \tag{2.4}$$

where $\gamma \in \mathbb{F}$ is a structure-controlling parameter. The bilinearity follows from that of $*$. As mentioned in the preamble to this chapter, the presence of this parameter γ in the multiplication affords a degree of freedom in our algebra. The element $\mathbb{1} = 1 + 0_V$ is the algebra unit in $\tilde{V}$.

A well known result on unital rings shows that if A is a unital ring with unit $\mathbb{1}$, then whenever a is an idempotent, so is $\mathbb{1} - a$. Since idempotents are important in our study of algebras, we will discuss how we can get idempotents of $\tilde{V}$ from those of V.

2.5.2 Idempotents of the extended algebra

Let V be a commutative non-associative algebra, $a \in V$ an axis and $\tilde{V}$ be a unital extension of V as set up in §2.5.1. In this subsection we derive necessary and sufficient conditions for $w = \alpha + \beta a \in \tilde{V}, \beta \neq 0$, to be an idempotent and obtain restrictions on γ so that these hold.

Lemma 2.5.1. *Let V be a commutative non-associative algebra over a field $\mathbb{F}$ equipped with a symmetric bilinear form $(\cdot,\cdot) : V \to \mathbb{F}$ and let $a \in V$ be an axis such that $(a, a) = 1$. Suppose that $w = \alpha + \beta a \in \tilde{V}$ is an idempotent, where $\beta \neq 0$. Then:*

(i) $\beta = 1 - 2\alpha$.

(ii) $\gamma = (\alpha - \alpha^2)\beta^{-2}$.

Proof. Let $a, V, \tilde{V}$ and w be as in the hypothesis and suppose that w is an idempotent. Then

$$\begin{aligned} \alpha + \beta a &= (\alpha + \beta a) \star (\alpha + \beta a) \\ &= (\alpha^2 + \gamma\beta^2(a, a)) + (2\alpha\beta a + \beta^2 a * a) \\ &= (\alpha^2 + \gamma\beta^2) + (2\alpha\beta + \beta^2)a, \end{aligned}$$

where in the last line we have used the fact that $(a, a) = 1$ and $a^2 = a$. We have the following equations:

$$\alpha = \alpha^2 + \gamma\beta^2 \tag{2.5}$$

and

$$\beta = 2\alpha\beta + \beta^2. \tag{2.6}$$

We see that these are functions of α and β but from Equation (2.6), we see that there is a linear relationship between α and β, namely, $\beta = 1 - 2\alpha$ so that we can characterize idempotents of $\tilde{V}$ corresponding to a by the single variable α. It follows that we can write γ as $\gamma = (\alpha - \alpha^2)\beta^{-2}$, since $\beta \neq 0$. $\square$

Because β is a function of α, in order to describe an algebra product in $\tilde{V}$ so that $w = \alpha + \beta a$ is an idempotent, we need only specify the parameter α to describe this multiplication. The eigenvalues of $w = \alpha + \beta a$ are obtained from a linear transformation which we will derive shortly. Let M be the ad_a matrix and let $\mathrm{Spec}(M)$ be its spectrum. Then a well-known theorem from elementary linear algebra tells us that if $\mu \in \mathrm{Spec}(M)\backslash 1$, then $(V_\mu(a), a) = 0$, i.e., all μ-eigenvectors are orthogonal to a. Thus, let $v_\mu \in V_\mu$. Then we have

$$\begin{aligned}
w \star \tilde{v}_\mu &= (\alpha + \beta a) \star (0 + v_\mu) \\
&= (\alpha \cdot 0 + \gamma(v_\mu, a)) + (\alpha \cdot v_\mu + 0 \cdot \beta a + (\beta a) * v_\mu) \\
&= (\alpha v_\mu + \beta(a * v_\mu)), \text{ since } V \text{ is an algebra and } (a, v_\mu) = 0, \\
&= (\alpha v_\mu + \beta(\mu v_\mu)), \text{ since } v_\mu \in V_\mu(a), \\
&= (\alpha + \beta\mu)v_\mu = (\alpha + \beta\mu)\tilde{v}_\mu.
\end{aligned}$$

We conclude that if $w = \alpha + \beta a$ is an idempotent in $\tilde{V}$ corresponding to an idempotent a, and if $v_\mu \in V_\mu(a)$, then $\tilde{v}_\mu$ is an ad_w-eigenvector corresponding to the eigenvalue $\mu' = (\alpha + \beta\mu)$. Because we will use this result in the rest of thesis, we record it as a lemma.

Lemma 2.5.2. *Let a be an axis of the commutative non-associative algebra V and $v_\mu \in V_\mu(a)$, with $\mu \in \mathrm{Spec}(a) \setminus 1$. Then the map $\mu \mapsto \alpha + \beta\mu$ maps $\mathrm{Spec}(a)\backslash 1$ to $\mathrm{Spec}(\alpha+\beta a)$, where $\alpha+\beta a$ is an idempotent of $\tilde{V}$ corresponding to a.*

We have mentioned that the extension of the algebra by adjoining a unit allows us a degree of freedom. We use this degree of freedom to ensure that 0 is an eigenvalue of the idempotent $w = \alpha + \beta a$. We have the following result which we will use subsequently in many cases.

Lemma 2.5.3. *Let V be a commutative non-associative algebra over a field $\mathbb{F}$ and let a be an axis of V. Further, let $\tilde{V}$ be the unital extension of V and $w = \alpha + \beta a, \beta \neq 0$ be an idempotent of $\tilde{V}$ corresponding to a. If λ is an eigenvalue of a which transforms to the eigenvalue 0 of w, then $\lambda \neq 1/2$ and*

$$\alpha = \frac{-\lambda}{1-2\lambda}, \ \beta = \frac{1}{1-2\lambda}.$$

Proof. Recall that $\beta = 1 - 2\alpha$, by Lemma 2.5.1, so that the map of Lemma 2.5.2 transforms the eigenvalue λ of a to the eigenvalue $\lambda' = \alpha + \beta\lambda = \alpha + (1-2\alpha)\lambda$ of $\alpha + \beta a$. Since we require that $\lambda' = 0$, we have

$$0 = \lambda' = \alpha + (1-2\alpha)\lambda = (1-2\lambda)\alpha + \lambda$$

whence

$$\alpha = \frac{-\lambda}{1-2\lambda}. \tag{2.7}$$

The expression for β follows easily. □

Consequently, if the eigenvalues of an idempotent a in V are known, the structure constant γ for the extended algebra $\tilde{V}$ can be expressed in terms of these eigenvalues. The following proposition gives sufficient conditions for vectors of the form $\kappa + b \in \tilde{V}$ to be eigenvectors for $w = \alpha + \beta a$ corresponding to eigenvalues $\rho = \alpha + \beta\mu$. The algebra V satisfies the conditions set out in the previous results of this section.

Proposition 2.5.4. *Let $\tilde{V}$ be the unital extension of V making the original eigenvalue λ for an axis a into the eigenvalue 0 for the new idempotent $w = \alpha + \beta a$. Then:*

(i) $\gamma = \lambda(\lambda - 1)$ *using the notation of this section.*

(ii) *Let $u = \kappa + b \in \tilde{V}$ be an eigenvector for w with eigenvalue $\rho = \alpha + \beta\mu$. Then $\kappa = (\mu - 1) \cdot (a, b)$ and $b \in \{v | a * v = \mu v - \kappa a\}$.*
If $(a, b) = 0$, then also $\kappa = 0$ and $a * b = \mu b$, so $\mu \in \operatorname{Spec}(a)$ and b is an eigenvector of a in the original algebra V.
If $(a, b) \neq 0$, then $\gamma = \mu(\mu - 1)$, hence $\mu = \lambda$ or $\mu = 1 - \lambda$.

Proof. (ithe 0) By Lemma 2.5.1 (2), $\gamma = (\alpha - \alpha^2)\beta^{-2}$. Thus

$$\begin{aligned}
\gamma &= \left(\frac{-\lambda}{1-2\lambda} - \left(\frac{-\lambda}{1-2\lambda}\right)^2\right)\left(\frac{1}{1-2\lambda}\right)^{-2} \\
&= \left(\frac{-\lambda}{1-2\lambda} - \frac{\lambda^2}{(1-2\lambda)^2}\right)(1-2\lambda)^2 \\
&= \left(\frac{-\lambda(1-2\lambda)-\lambda^2}{(1-2\lambda)^2}\right)(1-2\lambda)^2 \\
&= -\lambda + 2\lambda^2 - \lambda^2 = -\lambda + \lambda^2 \\
&= \lambda(\lambda - 1)
\end{aligned}$$

as required.

(iithe 0) Set $\rho = \alpha + \beta\mu$ and suppose that $u = \kappa + b \in \tilde{V}_\rho(w)$. Then $\rho u = u \star w$ so that

$$\begin{aligned}
(\alpha + \beta\mu)(\kappa + b) &= (\kappa + b) \star (\alpha + \beta a) \\
&= (\kappa\alpha + \gamma(b, \beta a)) + (\alpha b + \kappa(\beta a) + b * (\beta a)) \\
&= (\kappa\alpha + \gamma\beta(a, b)) + (\alpha b + \kappa\beta a + \beta(a * b)).
\end{aligned}$$

It follows that

$$(\alpha + \beta\mu)\kappa + (\alpha + \beta\mu)b = (\kappa\alpha + \gamma\beta(a, b)) + (\alpha b + \kappa\beta a + \beta(a * b))$$

which occurs if and only if

$$(\alpha + \beta\mu)\kappa = \kappa\alpha + \gamma\beta(a, b)$$

and

$$(\alpha + \beta\mu)b = \alpha b + \kappa\beta a + \beta(a * b).$$

The first equation gives $\beta\mu\kappa = \gamma\beta(a, b)$ hence $\mu\kappa = \gamma(a, b)$. From the second equation, we have $\beta\mu b = \kappa\beta a + \beta(a * b)$ from which it follows that $\mu b = \kappa a + a * b$, i.e., $a * b = \mu b - \kappa a$. Thus, $b \in \{v | a * v = \mu v - \kappa a\}$. The equation $\mu\kappa = \gamma(a, b)$ shows that if $(a, b) = 0$, then so is κ. We conclude that κ has the form $\kappa = \theta \cdot (a, b)$, where θ is a function of μ. Therefore, $\mu \cdot (a, b)\theta = \gamma(a, b) = \lambda(\lambda - 1)(a, b)$ and $\mu\theta = \lambda(\lambda - 1)$, if $(a, b) \neq 0$. This suggests that $\theta = \mu - 1$, so $\kappa = (\mu - 1) \cdot (a, b)$ and $\mu(\mu - 1) = \lambda(\lambda - 1)$. If $\mu = \lambda$, then clearly equality holds. Furthermore, $\lambda(\lambda-1) = -\lambda(1-\lambda)$ so that we can take $\mu = 1-\lambda$ and $\mu(\mu-1) = \lambda(\lambda-1)$ as required. It follows that if $(a,b) \neq 0$, then $\gamma = \mu(\mu-1)$ and hence $\mu = \lambda$ or $\mu = 1-\lambda$. In the case $(a,b) = 0$, $\kappa = 0$ so that $a * b = \mu b - \kappa a = \mu b$, so $\mu \in \mathrm{Spec}(a)$ and b is an eigenvector of a in the original algebra V. □

The following remark will be useful in the next chapters.

Remark 2.5.5. (i) If μ is not equal to λ or $1-\lambda$, then all eigenvectors of w are of the form $0+b \in \tilde{V}$, b an eigenvector of a in V. This follows from Proposition 2.5.4 (2).

(ii) If $b = a$, then $\kappa = \mu - 1$.

The following result is used to choose a basis for an extended unital algebra $\tilde{V}$. From now on, we fix the following notation. For a vector $v \in V$ and $\mu \in \mathrm{ad}_v$, by $\mathscr{B}_\mu(v)$ we mean a basis for μ-eigenspace $V_\mu(v) = \{u \in V | u * v = \mu u\}$. If $v \in V$, then $\tilde{v} = 0 + v \in \tilde{V}$ is the image of v under the embedding $V \hookrightarrow \tilde{V}, v \mapsto 0 + v$. The following result can be used to pick a basis for the extended unital algebras $\tilde{V}$.

Theorem 2.5.6. *Let V be an algebra over a field $\mathbb{F}$ equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{F}, a \in V$ an axis such that $(a,a) = 1$ and $\tilde{V}$ the unital extension of V sending the eigenvalue λ of a to the eigenvalue 0 of the idempotent $w = \alpha + \beta a$, in the notation of §2.5. Then*

1. *If a is primitive, then*

   (a) *If $\lambda = 1$ and $0 \in \mathrm{Spec}(a)$, then a basis for the ad_w 1-eigenspace is $\mathscr{B}_1(w) = \{-1+a\} \cup \{\tilde{v} | v \in \mathscr{B}_0(a)\}$, the ad_w 0-eigenspace has basis $\mathscr{B}_0(w) = \{\tilde{a}\}$ and the rest of the ad_w eigenvalues are of the form $\rho = 1-\mu, \mu \in \mathrm{Spec}(a) \setminus \{1,0\}$ with corresponding eigenvectors $\tilde{v}, v$ a μ-eigenvector. It follows that*

$$\mathscr{B}_1(w) \cup \mathscr{B}_0(w) \bigcup_{\mu \neq 1,0} \{\tilde{v} | v \in \mathscr{B}_\mu(a)\}$$

   is a basis for $\tilde{V}$ made up of eigenvectors.

(b) *If* $\lambda = 1$ *and* $0 \notin \mathrm{Spec}(a)$*, then the* ad_w 1*-eigenspace has basis* $\mathscr{B}_1(w) = \{-1+a\}$*, the* 0*-eigenspace* $\tilde{V}_0(w)$ *has a basis* $\mathscr{B}_0(w) = \{\tilde{a}\}$ *and the rest of the* ad_w *eigenvalues are of the form* $\rho = 1-\mu, \mu \neq 1$ *with corresponding bases* $\mathscr{B}_{1-\mu}(w) = \{\tilde{v}|v \in \mathscr{B}_\mu(a)\}$*. Consequently,*

$$\{-1+a\} \cup \{\tilde{a}\} \bigcup_{\mu \in \mathrm{Spec}(a)\setminus\{1\}} \{\tilde{v}|v \in \mathscr{B}_\mu(a)\}$$

is a basis for $\tilde{V}$ *made up of eigenvectors.*

(c) *If* $\lambda \neq 1$ *and* $1-\lambda \in \mathrm{Spec}(a)$*, then a basis for the* ad_w 1*-eigenspace is* $\mathscr{B}_1(w) = \{-\lambda + a\} \cup \{\tilde{v}|v \in \mathscr{B}_{1-\lambda}(a)\}$*, the* ad_w 0*-eigenspace has a basis* $\mathscr{B}_0(w) = \{(\lambda-1)+a\} \cup \{\tilde{v}|v \in \mathscr{B}_\lambda(a)\}$ *and all other eigenvalues of* ad_w *are of the form* $\rho = \alpha + \beta\mu, \mu \notin \{1, \lambda, 1-\lambda\}$*, with corresponding eigenvectors* $\tilde{v}, v$ *being* ad_a μ*-eigenvectors. In particular,*

$$\mathscr{B}_1(w) \cup \mathscr{B}_0(w) \bigcup_{\substack{\mu \in \mathrm{Spec}(a) \\ \mu \neq \lambda, 1-\lambda, 1}} \{\tilde{v}|v \in \mathscr{B}_\mu(a)\}$$

is a basis for $\tilde{V}$ *consisting of eigenvectors.*

(d) *If* $\lambda \neq 1$ *and* $1-\lambda \notin \mathrm{Spec}(a)$*, then a basis for* $\tilde{V}_1(w)$ *is* $\mathscr{B}_1(w) = \{-\lambda+a\}, \tilde{V}_0(w)$ *has basis* $\mathscr{B}_0(w) = \{(\lambda-1)+a\} \cup \{\tilde{v}|v \in \mathscr{B}_\lambda(a)\}$*. All other eigenvalues of* w *have the form* $\rho = \alpha + \beta\mu$*, where* μ *is an eigenvalue of* a *which is not equal to* 1*, and the corresponding eigenvalues are of the form* $\tilde{v}, v$ μ*-eigenvalues of* a*. Thus,*

$$\mathscr{B}_1(w) \cup \mathscr{B}_0(w) \bigcup_{\substack{\mu \in \mathrm{Spec}(a) \\ \mu \neq 1}} \{\tilde{v}|v \in \mathscr{B}_\mu\}$$

is a basis for $\tilde{V}$ *made up of eigenvectors.*

2. *If* a *is imprimitive, then let* X *be the basis for* $a^\perp \cap V_1(a)$*, where* $a^\perp := \{v \in V|(a,v) = 0\}$*. We have the following:*

(a) *If* $\lambda = 1$ *and* $0 \in \mathrm{Spec}(a)$*, then* $\mathscr{B}_1(w) = \{-\lambda+a\} \cup \{\tilde{v}|v \in \mathscr{B}_0(a)\}$*,* $\mathscr{B}_0(w) = \{\tilde{a}\} \cup \{\tilde{v}|v \in X\}$*. The rest of eigenvalues of* w *are of the form* $\rho = 1-\mu, \mu \neq 1, 0$ *and a basis for* $\tilde{V}$ *consisting of eigenvectors is*

$$\mathscr{B}_1(w) \cup \mathscr{B}_0(w) \bigcup_{\substack{\mu \in \mathrm{Spec}(a) \\ \mu \neq 1, 0}} \{\tilde{v}|v \in \mathscr{B}_\mu(a)\}.$$

(b) *If* $\lambda = 1$ *and* $0 \notin \mathrm{Spec}(a)$, *then* $\mathscr{B}_1(w) = \{-\lambda + a\}, \mathscr{B}_0(w) = \{\tilde{a}\} \cup \{\tilde{v} | v \in X\}$. *For* $\rho \in \mathrm{Spec}(w) \setminus \{1\}, \rho = 1 - \mu$ *with* μ *an eigenvalue of* a *different from* 1. *Further, for every such* ρ, *the corresponding eigenvectors are of the form* $\tilde{v}, v \in V_\mu(a)$. *The set*

$$\mathscr{B}_1(w) \cup \mathscr{B}_0(w) \bigcup_{\substack{\mu \in \mathrm{Spec}(a) \\ \mu \neq 1}} \{\tilde{v} | v \in \mathscr{B}_\mu\}$$

is a basis for $\tilde{V}$ *consisting of eigenvectors.*

(c) *If* $\lambda \neq 1$ *and* $1 - \lambda \in \mathrm{Spec}(a)$, *then* $\mathscr{B}_1(w) = \{-\lambda + a\} \cup \{\tilde{v} | v \in \mathscr{B}_{1-\lambda}(a)\}, \mathscr{B}_0(w) = \{(\lambda - 1) + a\} \cup \{\tilde{v} | v \in \mathscr{B}_\lambda\}, \mathscr{B}_{\alpha+\beta}(w) = \{\tilde{v} | v \in X\}$. *All other eigenvalues of* w *are of the form* $\rho = \alpha + \beta\mu, \mu \neq 1, \lambda, 1 - \lambda$ *and have eigenvectors* $\tilde{v}$ *such that* v *is a* μ*-eigenvector of* a. *Thus,*

$$\mathscr{B}_1(w) \cup \mathscr{B}_0(w) \cup \mathscr{B}_{\alpha+\beta}(w) \bigcup_{\substack{\mu \in \mathrm{Spec}(a) \\ \mu \neq 1, \lambda, 1-\lambda}} \{\tilde{v} | v \in \mathscr{B}_\mu(a)\}$$

is a basis for $\tilde{V}$ *made up of eigenvectors.*

(d) *If* $\lambda \neq 1$ *and* $1 - \lambda \notin \mathrm{Spec}(a)$, *then* $\mathscr{B}_1(w) = \{-\lambda + a\}, \mathscr{B}_0(w) = \{(\lambda - 1) + a\}, \mathscr{B}_{\alpha+\beta}(w) = \{\tilde{v} | v \in X\}$ *and all other eigenvalues of* w *take the form* $\rho = \alpha + \beta\mu$ *with* $\mu \notin \{1, \lambda\}$ *and have corresponding eigenvectors of the form* $\tilde{v}$, *where* $v \in \mathscr{B}_\mu(a)$. *It follows that*

$$\mathscr{B}_1(w) \cup \mathscr{B}_0(w) \mathscr{B}_{\alpha+\beta}(w) \bigcup_{\substack{\mu \in \mathrm{Spec}(a) \\ \mu \neq 1, \lambda}} \{\tilde{v} | v \in \mathscr{B}_\mu\}$$

is a basis for $\tilde{V}$ *consisting eigenvectors.*

Before proving the theorem, we make a few observations.

(i) The vector $0 + a$ is never an eigenvector of $w = \alpha + \beta a$ unless $\lambda = 1$ or $\lambda = 0$ is the eigenvalue to be made into the 0 eigenvalue for w. Furthermore, if the idempotent a is imprimitive, vectors from $V_1(a) \setminus a$ become $\alpha + \beta$-eigenvectors. We have

$$\begin{aligned}(0+a)\star(\alpha+\beta a) &= (\gamma(a,\beta a))+(\alpha\cdot a+a*(\beta a))\\ &= \gamma\beta(a,a)+\left(\alpha a+\beta a^2\right)\\ &= \gamma\beta+(\alpha+\beta)a\\ &= \lambda(\lambda-1)\cdot\beta+(\alpha+\beta)a, \text{by Proposition 2.5.4.}\end{aligned}$$

Thus $\tilde{a}=0+a$ is an eigenvector if and only if $\lambda=1$. The case $\lambda=0$ yields the same multiplication in $\tilde{V}$ as in V.

(ii) Since $w=\alpha+\beta a$ is an ad_w 1-eigenvector, so is $\alpha/\beta+a$. But $\alpha=\frac{-\lambda}{1-2\lambda}$ and $\beta=\frac{1}{1-2\lambda}$ whence $(\alpha/\beta)=-\lambda$.

(iii) By construction, if v is a λ-eigenvector of a, then $\tilde{v}\star w=0_{\tilde{V}}$.

(iv) The vector $(\lambda-1)+a$ is a zero eigenvector for w. We have

$$\begin{aligned}((\lambda-1)+a)\star(\alpha+\beta a) &= ((\lambda-1)\alpha+\gamma(a,\beta a))+\left((\lambda-1)\beta a+\alpha a+\beta a^2\right)\\ &= ((\lambda-1)\alpha+\gamma\beta)+((\lambda-1)\beta+\alpha+\beta)\,a\\ &= \left((\lambda-1)\cdot\left(\frac{-\lambda}{1-2\lambda}\right)+\lambda(\lambda-1)\cdot\frac{1}{1-2\lambda}\right)\\ &\quad+\left(\frac{(\lambda-1)}{1-2\lambda}+\frac{-\lambda}{1-2\lambda}+\frac{1}{1-2\lambda}\right)a\\ &= 0+0\cdot a.\end{aligned}$$

(v) If $1-\lambda$ is an ad_a eigenvector, then for every $(1-\lambda)$-eigenvector $v, \tilde{v}$ is an ad_w 1-eigenvector. A direct computation shows this. We have

$$\begin{aligned}(0+v)\star(\alpha+\beta a) &= (\gamma\beta(a,v))+(\alpha v+v*(\beta a))\\ &= \lambda(\lambda-1)\cdot(a,v)+(\alpha v+\beta(v*a))\\ &= \lambda(\lambda-1)\cdot(a,v)+(\alpha v+\beta(1-\lambda)v)\\ &= \lambda(\lambda-1)\cdot(a,v)+\left(-\frac{\lambda}{1-2\lambda}+\frac{1}{1-2\lambda}\cdot(1-\lambda)\right)v\\ &= \lambda(\lambda-1)\cdot(a,v)+v\\ &= \begin{cases}1(1-1)\cdot(a,v)+v, & \text{if } \lambda=1\\ \lambda(\lambda-1)\cdot 0+v, & \text{otherwise}\end{cases}\\ &= 0+v\end{aligned}$$

in either case, where in the second case we used the fact that a is orthogonal to all eigenspaces $V_\mu(a), \mu \neq 1$.

Proof of Theorem 2.5.6. 1. All the cases (a)-(d) follow from Proposition 2.5.4 and the observations above. Dimensions add and so the various sets listed are bases for $\tilde{V}$.

2. The cardinality of X is $\dim(V_1(a))-1$. We can apply the Gram-Schmidt orthogonalisation process to get a basis $\mathscr{B}$, for $V_1(a)$ extending a where every pair of vectors is mutually orthogonal. In particular, for each $v \in \mathscr{B} \setminus a, (a, v) = 0$. The number of such v is $\dim(V_1(a)) - 1$. Set $X = \mathscr{B} \setminus a$. This together with an application of Proposition 2.5.4 and the observations above yield the result.

□

For each $\lambda \in \mathrm{Spec}(a) \setminus 1$, we investigate the structure of the extended algebra $\tilde{V}$ in turn and determine the fusion rules for such algebras. The full details of these investigations will be given in Chapters 4 and 5 for the sporadic almost simple groups HS:2 and Suz:2.

Chapter 3

Links with Algebraic Graph Theory

In this chapter we discuss some links between algebraic graph theory and the construction of axial algebras. Even though some of the results given in this chapter can be stated in the more general setting of distance regular graphs, we restrict our attention to strongly regular graphs, in particular, those strongly regular graphs which admit rank three permutation groups.

We adopt the following notation throughout. By Γ we refer to a strongly regular graph with parameters $(\nu, k, \lambda, \mu), \Omega = \{1, .., \nu\} = E(\Gamma)$ and $G = \mathrm{Aut}(\Gamma)$ acting transitively and rank three on Ω. We show that if we identify the vertices of a graph Γ with standard vectors, we can impose an algebra structure on some orthogonal projection space for the adjacency matrix A of Γ, subject to conditions mentioned in §2.4.

3.1 Basic facts about strongly regular graphs

In this section we recall some basic facts about strongly regular graphs which we will use for the rest of the chapter. We begin with introducing distance distribution diagrams, where vertices are grouped together in partitions of Ω; a line is drawn from one partition to the other if there is an edge connecting a vertex from the first to the second. A number is labeled on such a line coming out of a partition indicating the number of neighours a typical vertex in the partition has in an adjacent partition. If a number is written above or below a partition, then it indicates the number of neighbours a vertex in the partition has in the same partition. A number inside a partition gives the size of such a partition.

It can easily be seen that the the vertices in a strongly regular graph can be represented by the following distribution diagram:

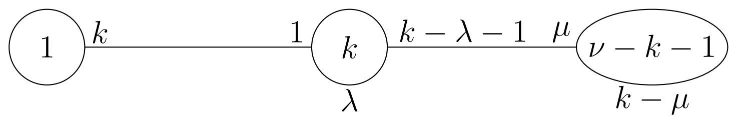

Figure 3.1: Distribution diagram for a strongly regular graph

Let $H < G$. Then the action of H on Ω partitions Ω. A diagram for the distribution of vertices in such a case is then said to be an **orbit diagram**.

Because the graphs under discussion are regular, the adjacency matrix A of such a graph has constant row and column sum. From this, it is clear that the all-one row (or column) vector $\mathbf{j}$ is an eigenvector with corresponding eigenvalue k. We call k the **principal eigenvalue**. Sometimes k is called the trivial eigenvalue. Strongly regular graphs have three distinct eigenvalues. The other eigenvalues different from k are called **non-principal eigenvalues**.

The parameters of strongly regular graphs satisfy several restrictions. We will describe the ones relevant to our discussions here. The first result, which we state without proof, gives the multiplicity of non-principal eigenvalues.

Theorem 3.1.1. *Suppose that* Γ *is a strongly regular graph with parameters* (ν, k, λ, μ)*. Then the numbers*

$$F, F' = \frac{1}{2}\left(\nu - 1 \pm \frac{(\nu-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right)$$

are non-negative integers.

We remark that strongly regular graphs are classified into two types according to :

(a) *Type I* (Called conference graphs): These have $(\nu-1)(\mu-\lambda) - 2k = 0$.

(b) *Type II*: For these graphs, $\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}$ is a perfect square, d^2, say, with d dividing $(\nu-1)(\mu-\lambda) - 2k$.

The non-principal eigenvalues r and s of a strongly regular graph can be shown to be integers with opposite signs. The parameters can be expressed in terms of the eigenvalues as:

$$\lambda = k + r + s + rs, \quad \mu = k + rs.$$

We now discuss the geometry of the eigenvalues.

3.2 Geometry of eigenvalues

We recall that the adjacency matrix A of Γ has three distinct eigenvalues k, r, s, with multiplicities $1, F, F'$ respectively. We thus can write $A = kE_0 + rE_1 + sE_2$, where E_0, E_1, E_2 are the orthogonal projections of $W = \mathbb{Q}^\nu$ onto the three eigenspaces W_0, W_1, W_2 of A.

In the ensuing discussion, we fix our focus on one of the non-trivial eigenspaces, say $V = W_1$. Suppose that x_ω is the vector which we identify with $\omega \in \Omega$, and consider its projection $v_\omega = x_\omega E_1$. We will identify E_1 with the projection $\pi : W \to V$. We first want to describe the coefficients of the vectors v_ω. By ω_i^G we denote an orbit or partition containing ω_i.

Fix a vertex ω. We use the following notation for the neighbours of ω. Define

$$N_\Gamma(\omega) = \Delta(\omega) = \{\alpha \in \Omega | \omega \sim \alpha\}$$

to be the set of neighbours of ω. Further, set

$$N'_\Gamma(\omega) := \Omega \setminus (N_\Gamma(\omega) \cup \{\omega\}).$$

Then $\{\omega\}, N_\Gamma(\omega), N'_\Gamma(\omega)$ partition Ω.

The adjacency of vertices from each of these sets is summarised by the diagram in Figure 3.1. We seek to find the projections of the standard vectors x_ω of W to V. Consider the three-dimensional subspace U of W spanned by the partition sums, i.e.,

$$U = \langle w_1, w_2, w_3 \rangle,$$

where

$$w_1 = x_\omega, w_2 = \sum_{\alpha \in N_\Gamma(\omega)} x_\alpha$$

and

$$w_3 = \sum_{\alpha \in N'_\Gamma(\omega)} x_\alpha.$$

We find the action of A on the subspace U as follows. The jth entry of w_iA, denoted $(w_iA)_j$, is the sum

$$(w_iA)_j = \sum_{l=1}^{\nu} w_{il}A_{lj},$$

where both w_{il} and A_{lj} belong to $\{0,1\}$. The summands are non-zero when $l \in \omega_i^G$ and $l \sim j$. Now, this means that the value of $(w_iA)_j$ is dependent on whether $j \in \{\omega\}, N_\Gamma(\omega)$ or ω_3^G. We have

$$(w_iA)_j = \sum_{l=1}^{\nu} w_{il}A_{lj} = |\{l \in \omega_i^G | l \sim j\}|.$$

That is, the sum is equal to the number of those k, for which both w_{il} and A_{lj} are equal to 1. We consider the action of A on each w_i each case separately.

(i) Set $i = 1$. If $j \in \{\omega\}$, then we are looking for the number of vertices in $\{\omega\}$ adjacent to the vertices in $\{\omega\}$. Thus, in this case, this is zero since our graph is simple. It follows that w_1A is zero on $\{\omega\}$. If on the other hand $j \in N_\Gamma(\omega)$, then we need the number of vertices of $\{\omega\}$ adjacent with a vertex in $N_\Gamma(\omega)$. The orbit diagram gives us this number as one. It follows that w_1A is one on $N_\Gamma(\omega)$. Finally, if $j \in N'_\Gamma(\omega)$, then the sum we seek is the number of vertices in $\{\omega\}$ adjacent with a vertex in $N'_\Gamma(\omega)$. This number is zero, from the orbit diagram. We conclude that $w_1A = w_2$.

(ii) Set $i = 2$. Then we have that the coefficient of w_2A on $\{\omega\}$ is

$$|\{l \in N_\Gamma(\omega) | l \sim j, j \in \{\omega\}\}| = k.$$

The coefficient of w_2A on $N_\Gamma(\omega)$ is

$$|\{l \in N_\Gamma(\omega) | l \sim j, j \in N_\Gamma(\omega)\}| = \lambda,$$

while the coefficient of w_2A on $N'_\Gamma(\omega)$ is

$$|\{l \in N_\Gamma(\omega) | l \sim j, j \in N'_\Gamma(\omega)\}| = \mu.$$

Thus, $w_2A = kw_1 + \lambda w_2 + \mu w_3$.

(iii) Similar arguments show that $w_3A = (k-1-\lambda)w_2 + (k-\mu)w_4$.

Thus, the action of A on the subspace U has matrix

$$C = \begin{bmatrix} 0 & 1 & 0 \\ k & \lambda & \mu \\ 0 & (k-1-\lambda) & (k-\mu) \end{bmatrix}.$$

Since the matrix C has constant column sum, it is trivial to check that the row vector $(1\ 1\ 1)$ is an eigenvector, with corresponding eigenvalue k. The characteristic polynomial of this matrix is

$$\begin{aligned} p(x) &= -x^3 + (\lambda + k - \mu)x^2 + (-\lambda k + \mu k - \mu + k)x - k(k-\mu) \\ &= (x-k)\left(-x^2 + (\lambda - \mu)x + (k-\mu)\right). \end{aligned}$$

It follows that the other eigenvalues of C are the roots of the quadratic equation $-x^2 + (\lambda-\mu)x + (k-\mu)$ which non-principal eigenvalues of A have to satisfy. Thus, using the fact that k is one eigenvalue and that the sum of of the eigenvalues, including multiplicities, equals the trace, we have that the other eigenvalues of C are

$$\frac{(\lambda-\mu) \pm \sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}{2},$$

each with multiplicity one. Consequently, if we so wished, we could use this information to find the multiplicities F and G of the eigenvalues in the original adjacency matrix A. Without loss of generality, we set $s = \frac{(\lambda-\mu)+\sqrt{(\mu-\lambda)^2+4(k-\mu)}}{2}$ and its multiplicity as an eigenvalue of A to be m. Let $v = (1\ \xi_1\ \xi_2)$ be an eigenvector of C corresponding to s. The coefficients of the projection of x_ω to V are constant on $\{\omega\}, N_\Gamma(\omega), N'_\Gamma(\omega)$. The length of the projection onto V is $\frac{m}{\nu}$, thus the projection of x_ω to V becomes $v_\omega = \frac{m}{\nu}x_\omega + \frac{m}{\nu}\cdot\xi_1 w_2 + \frac{m}{\nu}\cdot\xi_2 w_3$. We have shown:

Proposition 3.2.1. *The projection v_ω of x_ω to V has coefficients $c_0 = \frac{m}{\nu}$ on ω, $c_1 = \frac{m}{\nu}\xi_1$ on $N_\Gamma(\omega)$ and $c_2 = \frac{m}{\nu}\xi_2$ on $N'_\Gamma(\omega)$, where m is expressible in terms of graph parameters.*

We will also state the following result which generalises our analysis above of the action of the adjacency matrix on subspaces of W spanned by sums of sets which partition Ω.

Proposition 3.2.2. *Let G be a rank three permutation group on a finite set Ω and Γ be a graph on which G acts transitively. Let $H < G$ be a subgroup of G, and let $O_1, O_2, \dots, O_k$ be the orbits of H on Ω. Identify points ω in Ω with standard vectors x_ω and let*

$$w_i = \sum_{\alpha \in O_i} x_\alpha.$$

Suppose that A is the adjacency matrix of Γ. Then the action of A on $U := \langle w_i \rangle_{i=1}^k$ has matrix $B = [b_{ij}]$ where b_{ij} is the number of neighbours in O_i a fixed vertex in O_j has. That is, the matrix B has entries precisely the numbers in an orbit diagram.

In subsequent chapters, we use the fact that the algebras we construct support G-invariant scalar products. We conclude this section by proving a result on the inner products of the projections of the vertices.

Lemma 3.2.3. *Let Γ be a simple connected strongly regular graph with vertex set Ω and parameters (ν, k, λ, μ); $\omega \in \Omega$ and v_ω be the projection of x_ω to V with the same notation as before. Then the following holds.*

(i) The inner product (v_ω, v_ω) of v_ω with itself is given by

$$(v_\omega, v_\omega) = c_0^2 + kc_1^2 + (\nu - k - 1)c_2^2.$$

(ii) If $\omega \sim \alpha$, then

$$(v_\omega, v_\alpha) = 2c_0c_1 + \lambda c_1^2 + 2(k - \lambda - 1)c_1c_2 + (\nu - 2k + \lambda)c_2^2.$$

(iii) If $\omega \not\sim \alpha$, then

$$(v_\omega, v_\alpha) = 2c_0c_2 + \mu c_1^2 + 2(k - \mu) + (\nu - 2k - 2 + \mu)c_2^2.$$

Proof. We note that for the standard vectors $(x_\omega, x_\alpha) = \delta_{\omega\alpha}$, the Kronecker delta.

(i) Write

$$v_\omega = c_0 x_\omega + c_1 \sum_{\alpha \sim \omega} x_\omega + c_2 \sum_{\eta \not\sim \omega} x_\eta.$$

Note that we use symmetry and linearity of the inner product in the calculations that follow. We have

$$\begin{aligned}
(v_\omega, v_\omega) &= (c_0 x_\omega + c_1 \sum_{\alpha\sim\omega} x_\omega + c_2 \sum_{\eta\not\sim\omega} x_\eta, c_0 x_\omega + c_1 \sum_{\alpha\sim\omega} x_\omega + c_2 \sum_{\eta\not\sim\omega} x_\eta) \\
&= c_0^2(x_\omega, x_\omega) + c_0 c_1 \sum_{\alpha\sim\omega}(x_\omega, x_\alpha) + c_0 c_2 \sum_{\eta\not\sim\omega}(x_\omega, x_\eta) \\
&\quad + c_1 c_0 \sum_{\alpha\sim\omega}(x_\alpha, x_\omega) + c_1^2 \sum_{\alpha\sim\omega}\sum_{\alpha'\sim\omega}(x_\alpha, x_{\alpha'}) + c_1 c_2 \sum_{\alpha\sim\omega}\sum_{\eta\not\sim\omega}(x_\alpha, x_\eta) \\
&\quad + c_2 c_0 \sum_{\eta\sim\omega}(x_\eta, x_\omega) + c_2 c_1 \sum_{\eta\not\sim\omega}\sum_{\alpha\sim\omega}(x_\eta, x_\alpha) + c_2^2 \sum_{\eta\not\sim\omega}\sum_{\eta'\not\sim\omega}(x_\eta, x_{\eta'}) \\
&= c_0^2 + c_1^2 \sum_{\alpha\sim\omega}\sum_{\alpha'\sim\omega} \delta_{\alpha\alpha'} + c_2^2 \sum_{\eta\not\sim\omega}\sum_{\eta'\not\sim\omega} \delta_{\eta\eta'} \\
&= c_0^2 + c_1^2 \sum_{\alpha\sim\omega} 1 + c_2^2 \sum_{\eta\not\sim\omega} 1 \\
&= c_0^2 + c_1^2 |N_\Gamma(\omega)| + c_2^2 |N'_\Gamma(\omega)| = c_0^2 + k c_1^2 + (\nu - 1 - k) c_2^2.
\end{aligned}$$

(ii) Suppose that $\alpha \sim \omega$. We write

$$v_\omega = c_0 x_\omega + c_1 x_\alpha + c_1 \left(\sum_{\theta\sim\omega, \theta\neq\alpha} x_\theta \right) + c_2 \sum_{\eta\not\sim\omega} x_\eta,$$

$$v_\alpha = c_0 x_\alpha + c_1 x_\omega + c_1 \left(\sum_{\substack{\kappa\sim\alpha \\ \kappa\neq\omega}} x_\kappa \right) + c_2 \left(\sum_{\gamma\not\sim\alpha} x_\gamma \right).$$

The inner product (v_ω, v_α) becomes:

$$
\begin{aligned}
(v_\omega, v_\alpha) &= c_0^2(x_\omega, x_\alpha) + c_0c_1(x_\omega, x_\omega) + c_0c_1\left(\sum_{\substack{\kappa\sim\alpha\\ \kappa\neq\omega}}(x_\omega, x_\kappa)\right)\\
&\quad +c_0c_2\left(\sum_{\gamma\not\sim\alpha}(x_\omega, x_\gamma)\right) + c_1c_0(x_\alpha, x_\alpha) + c_1^2(x_\alpha, x_\omega)\\
&\quad +c_1^2\left(\sum_{\substack{\kappa\sim\alpha\\ \kappa\neq\omega}}(x_\alpha, x_\kappa)\right) + c_1c_2\left(\sum_{\gamma\not\sim\alpha}(x_\alpha, x_\gamma)\right)\\
&\quad +c_1c_0\left(\sum_{\theta\sim\omega,\theta\neq\alpha}(x_\theta, x_\alpha)\right) + c_1^2\left(\sum_{\theta\sim\omega,\theta\neq\alpha}(x_\theta, x_\omega)\right)\\
&\quad +c_1^2\left(\sum_{\theta\sim\omega,\theta\neq\alpha}\sum_{\substack{\kappa\sim\alpha\\ \kappa\neq\omega}}(x_\theta, x_\kappa)\right) + c_1c_2\left(\sum_{\substack{\theta\sim\omega\\ \theta\neq\alpha}}\sum_{\gamma\not\sim\alpha}(x_\theta, x_\gamma)\right)\\
&\quad +c_2c_0\sum_{\eta\not\sim\omega}(x_\eta, x_\alpha) + c_2c_1\sum_{\eta\not\sim\omega}(x_\eta, x_\omega) + c_2c_1\sum_{\eta\not\sim\omega}\sum_{\substack{\kappa\sim\alpha\\ \kappa\neq\omega}}(x_\eta, x_\kappa)\\
&\quad +c_2^2\sum_{\eta\not\sim\omega}\sum_{\gamma\not\sim\alpha}(x_\eta, x_\gamma)\\
&= 2c_0c_1 + |N_\Gamma(\omega)\cap N_\Gamma(\alpha)|c_1^2 + |N_\Gamma(\omega)\cap N'_\Gamma(\alpha)|c_1c_2\\
&\quad +|N'_\Gamma(\omega)\cap N_\Gamma(\alpha)|c_2c_1 + |N'_\Gamma(\omega)\cap N'_\Gamma(\alpha)|c_2^2\\
&= 2c_0c_1 + \lambda c_1^2 + (k-1-\lambda)c_1c_2 + (k-1-\lambda)c_2c_1\\
&\quad + (\nu - (k-\lambda-1) - (k-\lambda-1) - \lambda - 2)\, c_2^2\\
&= 2c_0c_1 + \lambda c_1^2 + 2(k-1-\lambda)c_1c_2 + (\nu - 2k + \lambda)c_2^2,
\end{aligned}
$$

since (x_θ, x_κ) in the sum

$$\left(\sum_{\theta\sim\omega,\theta\neq\alpha}\sum_{\substack{\kappa\sim\alpha\\ \kappa\neq\omega}}(x_\theta, x_\kappa)\right)$$

is nonzero when $x_\theta = x_\kappa$ meaning that θ is a common neighbour of ω

and α; (x_θ, x_γ) in the sum

$$\left(\sum_{\substack{\theta\sim\omega\\ \theta\neq\alpha}}\sum_{\gamma\not\sim\alpha}(x_\theta, x_\gamma)\right)$$

is one when $\theta=\gamma$ and θ is adjacent with ω while not adjacent with α; (x_η, x_κ) in the sum

$$\sum_{\eta\not\sim\omega}\sum_{\substack{\kappa\sim\alpha\\ \kappa\neq\omega}}(x_\eta, x_\kappa)$$

is one when $\eta=\kappa$, η is simultaneously a non-neighbour of ω and a neighbour of α; and (x_η, x_γ) in the sum

$$\sum_{\eta\not\sim\omega}\sum_{\gamma\not\sim\alpha}(x_\eta, x_\gamma)$$

is one when $\eta=\gamma$ and η is a non-neighbour of both ω and α.

(iii) Suppose that $\omega\not\sim\alpha$. We write

$$v_\omega = c_0x_\omega + c_1\sum_{\alpha'\sim\omega}x_{\alpha'} + c_2x_\alpha + c_2\sum_{\substack{\eta\not\sim\omega\\ \eta\neq\alpha}}x_\eta,$$

$$v_\alpha = c_2x_\omega + c_1\sum_{\sigma\sim\alpha}x_\sigma + c_0x_\alpha + c_2\sum_{\substack{\xi\not\sim\alpha\\ \xi\neq\omega}}x_\xi.$$

Expanding, the inner product (v_ω, v_α) reduces to :

$$\begin{aligned}
(v_\omega, v_\alpha) &= c_0c_2(x_\omega, x_\omega) + c_1^2\sum_{\alpha'\sim\omega}\sum_{\sigma\sim\alpha}(x_\omega, x_\alpha) + c_1c_2\sum_{\alpha'\sim\omega}\sum_{\substack{\xi\not\sim\alpha\\ \xi\neq\omega}}(x_{\alpha'}, x_\xi)\\
&\quad + c_2c_0(x_\alpha, x_\alpha) + c_2c_1\sum_{\substack{\eta\not\sim\omega\\ \eta\neq\alpha}}\sum_{\sigma\sim\alpha} + c_2^2\sum_{\substack{\eta\not\sim\omega\\ \eta\neq\alpha}}\sum_{\substack{\xi\not\sim\alpha\\ \xi\neq\omega}}(x_\eta, x_\xi)\\
&= 2c_0c_2 + c_1^2|N_\Gamma(\omega)\cap N_\Gamma(\alpha)| + c_1c_2|N_\Gamma(\omega)\cap N'_\Gamma(\alpha)|\\
&\quad + c_1c_2|N'_\Gamma(\omega)\cap N_\Gamma(\alpha)| + c_2^2|N'_\Gamma(\omega)\cap N'_\Gamma(\alpha)|\\
&= 2c_0c_2 + \mu c_1^2 + ((k-\mu)+(k-\mu))\,c_1c_2\\
&\quad + (\nu-\mu-(k-\mu)-(k-\mu)-2)\,c_2^2\\
&= 2c_0c_2 + \mu c_1^2 + 2(k-\mu)c_1c_2 + (\nu-2k-2+\mu)c_2^2
\end{aligned}$$

as required.

□

3.3 Norton algebra products on certain eigenspaces

In this section we describe the Norton algebra products on some eigenspaces of strongly regular graphs. Recall the notation of the previous section. A strongly regular regular graph Γ with parameters (ν, k, λ, μ) has three eigenvalues: k and two non-principal eigenvalues r and s, which for our purposes, are distinct. We have that the space $W = \mathbb{Q}^\nu$ has an orthogonal decomposition

$$W = W_0 \oplus W_1 \oplus W_2.$$

For the remainder of the section, we fix our attention to a particular non-trivial eigenspace V. Without any loss of generality, say $V = W_1$. We set v_w,to be the projection of x_ω, for the vertex ω. By Proposition 3.2.1, we can write

$$v_\omega = c_0 + c_1 \sum_{\alpha \sim \omega} x_\alpha + c_2 \sum_{\eta \not\sim \omega} x_\eta.$$

The following result is useful in proving that the Norton product is not identically zero on some subspace V of W. We will make use of it in Chapters 4 and 5. We have:

Lemma 3.3.1. *The square v_ω^2 of the projection v_ω to V of the standard vector x_ω has coefficients*

$$c_0^3 + kc_1^3 + (\nu - k - 1)c_2^3 \quad \text{on } \omega;$$

$$c_0^2 c_1 + c_0 c_1^2 + (k - \lambda - 1)(c_1 + c_2)c_1 c_2 + (\nu - 2k + \lambda)c_2^3 \quad \text{on } N_\Gamma(\omega);$$

and

$$c_0^2 c_2 + \mu c_1^3 + (k - \mu)(c_1 + c_2)c_1 c_2 + c_0 c_2^2 + (\nu - 2k + \mu - 2)c_2^3 \quad \text{on } N'_\Gamma(\omega).$$

Proof. The square of v_ω under the point-wise product is

$$(v_\omega^2)^* = c_0^2 x_\omega + c_1^2 \sum_{\alpha \sim \omega} x_\alpha + c_2^2 \sum_{\eta \not\sim \omega} x_\eta.$$

We use the diagram in Figure 3.1 to find the projections of $\sum x_\alpha$ and $\sum x_\eta$ to V. We use π to denote the projection map $W \to V$. For the coefficient of the projection of $\sum x_\alpha$ on $N_\Gamma(\omega)$, we consider a single vertex $\alpha \sim \omega$. We have that α and ω have λ common neighbours, leaving the rest $(k-\lambda-1)$ neighbours of ω as non-neighbours of α. Thus, $\pi(\sum x_\alpha)$ has coefficient kc_1 on ω and $c_0+\lambda c_1+(k-\lambda-1)c_2$ on $N_\Gamma(\omega)$. A typical vertex $\eta \not\sim \omega$ has μ neighbours in $N_\Gamma(\omega)$. Therefore, the remaining $(k-\mu)$ neighbours of ω are non-neighbours of η. We conclude that $\pi(\sum x_\alpha)$ has coefficient $\mu c_1+(k-\mu)c_2$ on $N'_\Gamma(\omega)$. Consequently,

$$\pi(\sum_{\alpha\sim\omega} x_\alpha) = kc_1x_\omega+(c_0+(k-\lambda-1)c_2+\lambda c_1)\sum_{\alpha\sim\omega}x_\alpha+(\mu c_1+(k-\mu)c_2)\sum_{\eta\not\sim\omega}x_\eta.$$

The projection of a typical vertex $\eta \not\sim \omega$ has coefficient c_2 on ω and since there are $\nu-k-1$ such, the coefficient of $\pi(\sum x_\eta)$ on ω is $(\nu-k-1)c_2$. A vertex $\alpha \in N_\Gamma(\omega)$ has $k-\lambda-1$ neighbours in $N'_\Gamma(\omega)$. It follows that $\pi(\sum x_\eta)$ is

$$(k-\lambda-1)c_1+(\nu-k-1-(k-\lambda-1))c_2 = (k-\lambda-1)c_1+(\nu-2k+\lambda)c_2$$

on $N_\Gamma(\omega)$.

A typical vertex $\eta \in N'_\Gamma(\omega)$ has μ common neighbours with ω, so has $k-\mu$ neighbours in N'_Γ and $\nu-k-1-(k-\mu)-1 = \nu-2k+\mu-2$ non-neighbours in the same set. The coefficient of $\pi(\sum x_\eta)$ is therefore

$$c_0+(k-\mu)c_1+(\nu-2k+\mu-2)c_2$$

on $N'_\Gamma(\omega)$. Thus,

$$\begin{aligned}\pi(\sum_{\eta\not\sim\omega} x_\eta) &= (\nu-k-1)c_2x_\omega+((k-\lambda-1)c_1+(\nu-2k+\lambda)c_2)\sum_{\alpha\sim\omega}x_\alpha \\ &+ (c_0+(k-\mu)c_1+(\nu-2k+\mu-2)c_2)\sum_{\eta\not\sim\omega}x_\eta.\end{aligned}$$

Thus, the square of v_ω is $\pi((v_\omega^2)^*)$. That is,

$$\begin{aligned}
v_\omega^2 &= c_0\pi(x_\omega) + c_1^2\pi\left(\sum_{\alpha\sim\omega} x_\alpha\right) + c_2^2\pi\sum\left(\sum_{\eta\not\sim\omega} x_\eta\right) \\
&= c_0^2\left(c_0x_\omega + c_1\sum_{\alpha\sim\omega} x_\alpha + c_2\sum_{\eta\not\sim\omega} x_\eta\right) \\
&\quad +c_1^2\left(kc_1x_\omega + (c_0+\lambda c_1+(k-\lambda-1)c_2)\sum_{\alpha\sim\omega} x_\alpha + (\mu c_1+(k-\mu)c_2)\sum_{\eta\not\sim} x_\eta\right) \\
&\quad +c_2^2\left((\nu-k-1)c_2x_\omega + ((k-\lambda-1)c_1+(\nu-2k+\lambda)c_2)\sum_{\alpha\sim\omega} x_\alpha\right. \\
&\quad \left. + (c_0+(k-\mu)c_1+(\nu-2k+\mu-2)c_2)\sum_{\eta\not\sim\omega} x_\eta\right).
\end{aligned}$$

On simplifying, the coefficients are $c_0^3 + kc_1^3 + (\nu-k-1)c_2^3$ on ω;

$$\begin{aligned}
& c_0^2c_1 + c_1^2\left(c_0+\lambda c_1+(k-\lambda-1)c_2\right) + c_2^2\left((k-\lambda-1)c_1+(\nu-2k+\lambda)c_2\right) \\
=\ & c_0^2c_1 + c_0c_1^2 + (k-\lambda-1)c_1^2c_2 + \lambda c_1^3 + (k-\lambda-1)c_1c_2^2 + (\nu-2k+\lambda)c_2^3 \\
=\ & c_0^2c_1 + c_0c_1^2 + (k-\lambda-1)(c_1+c_2)c_1c_2 + (\nu-2k+\lambda)c_2^3
\end{aligned}$$

on $N_\Gamma(\omega)$; and

$$\begin{aligned}
& c_0^2c_2 + c_1^2(\mu c_1+(k-\mu)c_2) + c_2^2(c_0+(k-\mu)c_1+(\nu-2k+\mu-2)c_2) \\
=\ & c_0^2c_2 + \mu c_1^3 + (k-\mu)(c_1+c_2)c_1c_2 + (\nu-2k+\lambda-2)c_2^3
\end{aligned}$$

on $N'_\Gamma(\omega)$. □

Chapter 4

Algebras from HS

In this chapter we discuss the details of axial algebras for the Higman-Sims group HS and its automorphism group HS:2. The Higman-Sims sporadic simple group was constructed by Higman and Sims in [HS68] as an index 2 subgroup of the full automorphism group of a strongly regular graph with parameters $(100, 22, 0, 6)$, which they constructed as an extension of Witt's triple Steiner system $S(3, 6, 22)$, though this graph had been constructed as early as 1956 in Dale Mesner's PhD thesis using different techniques. The group was constructed as a primitive permutation group on 100 points in which the stabilizer of a point has three orbits of length $1, 22$ and 77 respectively. The permutation character of the group in this action decomposes as $1 + 22 + 77$, where each number represents the degree of the corresponding irreducible constituent. This shows that this group and its extension are amenable to the Norton algebra treatment given in Chapter 2.

4.1 Character-theoretic checks

From the character table, either from the ATLAS or MAGMA, there are two 77-dimensional irreducible modules. Both characters afforded by these modules have intertwining number one and so the representations affording these characters are real orthogonal. Furthermore, for one of the characters, $(\mathrm{Sym}^2\chi, \chi) = 1$ whence the space of commutative products $V \otimes V \to V$ is one-dimensional and the particular module supports a commutative nonassociative algebra structure. In fact, all the character values for the 77-dimensional character under consideration are rational. Thus, we will construct our algebras over the rationals. Let $K = C_G(z)$ be the centraliser of an involution. Then $\dim_{\mathbb{C}} \mathrm{inv}_K(V) = (\chi|_K, 1_K) \in \{1, 2\}$. It follows that the centraliser of an involution fixes a one- or two-dimensional space in V. All character theoretic invariants can easily be computed by MAGMA [BC94]. Because we will use this argument often, we will record it in the following result.

Theorem 4.1.1. *Let G be a finite permutation group, V an irreducible module of G over a field $\mathbb{F}$ endowed with a commutative algebra product $* : V \otimes V \to V$ which commutes with the group action. Equivalently, suppose that $(\mathrm{Sym}^2\chi_V, \chi_V) \neq 0$. Suppose that for an involution $z \in G, K = C_G(z)$ is the centralizer of z in G. Suppose further, that W is the permutation module for G. Set $U = V \cap \mathrm{inv_K(W)}$, the space of fixed points of K in V. If $\mathrm{dim_{\mathbb{F}}U} = 1$ or 2, or equivalently, $(\chi_V|_K, 1_K) \in \{1, 2\}$, then the following holds.*

(i) If $\mathrm{dim_{\mathbb{F}}U} = 1$, then K fixes an idempotent in V.

(ii) If $\mathrm{dim_{\mathbb{F}}U} = 2$, then K fixes an idempotent in the algebra $Q \otimes_{\mathbb{F}} V$, where Q is a quadratic extension of $\mathbb{F}$.

Proof. (i) Suppose that $* : V \otimes V \to V$ is a G-invariant commutative algebra product with respect to the diagonal action of G, that is

$$g \cdot (u \otimes v) = (g \cdot v) \otimes (g \cdot w), \text{ for all } g \in G.$$

We will denote by ab the image $*(a \otimes b)$ of $a \otimes b$. Because U is invariant under K,

$$k \cdot (uv) = *((k \cdot u) \otimes (k \cdot v)) = *(u \otimes v) = uv, \text{for all } u, v \in U.$$

It follows that U is closed under the algebra product and is a one-dimensional subalgebra fixed by K. Suppose that $U = \mathrm{span(u_1)}$ as a vector space. It follows that $uv = \kappa u_1$ for all $u, v \in V$. In particular, $u_1^2 = \alpha(u_1)u_1, \alpha(u_1) \in \mathbb{F}$ and $a = \frac{1}{\alpha(u_1)}u_1$ is an idempotent in V fixed by K.

(ii) If $\dim_{\mathbb{F}} = 2$, then, choosing a basis $\mathscr{B} = \{u_1, u_2\}$ for U, we have for some scalars $\alpha_1, \alpha_2 \in \mathbb{F}$, if $w = \alpha_1 u_1 + \alpha_2 u_2$, then for all $k \in K, k \cdot w = w$. In particular,

$$\begin{aligned} k \cdot w^2 &= k \cdot (\alpha_1^2 u_1^2 + (2\alpha_1\alpha_2)u_1u_2 + \alpha_2^2 u_2^2) \\ &= (\alpha_1^2 u_1^2 + (2\alpha_1\alpha_2)u_1u_2 + \alpha_2^2 u_2^2) \\ &= w^2 \end{aligned}$$

Let $U_1 = \langle w \rangle$. Then U_1 is a one-dimensional subalgebra of V fixed by K. Suppose that $w^2 = w$. Then we have

$$\alpha_1 u_1 + \alpha_2 u_2 = \alpha_1^2 u_1^2 + (2\alpha_1\alpha_2)u_1u_2 + \alpha_2^2.$$

Let

$$u_1^2 = \mu_1 u_1 + \mu_2 u_2, u_1u_2 = \mu_1' u_1 + \mu_2' u_2, u_2^2 = \mu_1'' u_1 + \mu_2'' u_2$$

be expressions for u_1^2, u_1u_2 and u_2^2 as linear combinations of elements of $\mathscr{B}$, respectively. Then,

$$\begin{aligned} \alpha_1 u_1 + \alpha_2 u_2 &= \alpha_1^2(\mu_1 u_1 + \mu_2 u_2) + 2\alpha_1\alpha_2(\mu_1' u_1 + \mu_2' u_2) + \alpha_2^2(\mu_1'' u_1 + \mu_2'' u_2) \\ &= \left(\mu_1\alpha_1^2 + 2\mu_1'(\alpha_1\alpha_2) + \alpha_1''\alpha_2^2\right) u_1 + \left(\mu_2\alpha_1^2 + 2\mu_2' + \mu_2''\alpha_2^2\right) u_2. \end{aligned}$$

Linear independence of u_1, u_2 implies that

$$\alpha_1 = \mu_1\alpha_1^2 + 2\mu_1'(\alpha_1\alpha_2) + \mu_1''\alpha_2^2 \tag{4.1}$$

and

$$\alpha_2 = \mu_2\alpha_1^2 + 2\mu_2'(\alpha_1\alpha_2) + \mu_2''\alpha_2^2. \tag{4.2}$$

Thus we have a system of quadratic equations in α_1 and α_2 which has algebraic roots. In particular, $(0, 0)$ is a solution and in a suitable extension of $\mathbb{F}$, a non-trivial solution can be found.

□

The group HS has two classes of involutions, labelled, $2A$ and $2B$ while HS:2 has four classes $2A, 2B, 2C$ and $2D$. Because the classes of HS can be recovered from those of HS:2, we will work with those of the latter. In the following sections we will present the results of our construction of Norton algebras via eigenspace decomposition of the association scheme that arises from the strongly regular graph srg$(100, 22, 0, 6)$. There are three classes in this association scheme and these have sizes $1, 22$ and 77. We will construct 77-dimensional algebras on the 77-dimensional eigenspace using the Norton product, i.e., point-wise multiplication followed by projection into the space. It can easily be show that the adjacency matrix of the Higman-Sims graph Γ has three eigenvalues, namely, $22, -8$ and 2, with multiplicities $1, 22$ and 77.

In the fashion of §2.4, set $\Omega = \{1, 2, \ldots, 100\}$, $X = \{x_\alpha | \alpha \in \Omega\}$, where x_α is a 100-long vector with zeros everywhere except the α^{th} position, which has a "1". Set $W = \mathbb{Q}X = \langle X \rangle$, the rational vector space with X an orthonormal basis. Note that W corresponds to the permutation module. We use the orthogonal decomposition

$$W = W_0 \oplus W_1 \oplus W_2,$$

where W_2 is the 77-dimensional eigenspace. We shall set $V = W_2$.

We first show that the Norton product is not identically zero on our space of interest. We begin by recalling some facts about the Higman-Sims graph Γ. The stabiliser of a vertex in this graph is the double cover of the Mathieu group M_{22}, that is, M_{22}:2, and the stabiliser of an edge is $L_3(4)$:2^2. We show the distribution diagram for Γ in Figure 4.1.

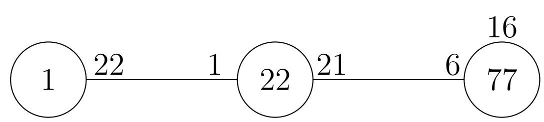

Figure 4.1: Distribution diagram for the Higman-Sims graph

In what follows, we fix the following notation. Fix $\omega \in \Omega := E(\Gamma)$ and let $H = G_\omega$. Then the orbits of H in its action on Ω are $\{\omega\}, N_\Gamma(\omega)$ and $N'_\Gamma(\omega)$, of lengths $1, 22$ and 77 respectively. Let $w_1 = x_\omega$,

$$w_2 = \sum_{\alpha \in N_\Gamma(\omega)} x_\alpha$$

and

$$w_3 = \sum_{\gamma \in N'_\Gamma(\omega)} x_\gamma.$$

Moreover, let $U = \langle w_1, w_2, w_3 \rangle$ and A be the adjacency matrix of Γ. Then the action of A on U has matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 22 & 0 & 6 \\ 0 & 21 & 16 \end{bmatrix}.$$

The matrix B has eigenvalues $22, -8$ and 2, which are the eigenvalues of A. The 2-eigenspace is the 77-dimensional space V of interest to us in this chapter. It can easily be checked that the vector $(1\ \frac{1}{11}\ -\frac{3}{77})$ is an eigenvector corresponding to the eigenvalue 2. The multiplicity of 2 as an eigenvalue of A is 77 so that the length of a projection to the 77-dimensional space V is $\frac{77}{100}$. Thus, the projection v_ω of x_ω to V is :

$$\begin{aligned} v_\omega &= \frac{77}{100}x_\omega + \frac{1}{11} \cdot \left(\frac{77}{100}\right) w_2 + \frac{77}{100} \cdot \left(-\frac{3}{77}\right) \\ &= \frac{77}{100}w_1 + \frac{7}{100}w_2 - \frac{3}{100}w_3. \end{aligned}$$

Thus, we have shown:

Lemma 4.1.2. *Let V be the 77-dimensional eigenspace of the adjacency matrix of the Higman-Sims graph Γ. Then the projection v_α of a vertex α has coefficient $\frac{77}{100}$ on α, $\frac{7}{100}$ on $N_\Gamma(\alpha)$ and $-\frac{3}{100}$ on $N'_\Gamma(\alpha)$.*

We now show that the Norton product is nonzero on V.

Lemma 4.1.3. *Let $\pi : W \to V$ be the projection of W to the subspace V. The Norton product $V \times V \to V$ given by*

$$(u_1\ u_2 \ldots\ u_{100}) \cdot (v_1\ v_2 \ldots v_{100}) = \pi(u_1v_1\ u_2v_2 \ldots\ u_{100}v_{100})$$

is not identically zero.

Throughout this chapter $c_0 = \frac{77}{100}, c_1 = \frac{7}{100}$ and $c_2 = -\frac{3}{100}$.

Proof. By Lemma 3.3.1 with c_0, c_1 and c_2 given above, $\nu = 100, k = 22, \lambda = 0$ and $\mu = 6$, we have

$$v_\omega^2 = \frac{231}{500}w_1 + \frac{21}{500}w_2 - \frac{9}{500}w_3 \neq 0_V.$$

□

We also give details of how adjacent and non-adjacent vertices multiply under the product. We first give the product of the projections of two vertices ω and α which are adjacent. We write

$$v_\omega = c_0 x_\omega + c_1 x_\alpha + c_1 \sum_\xi x_\xi + c_2 \sum_\eta x_\eta + c_2 \sum_\gamma x_\gamma,$$

$$v_\omega = c_1 x_\omega + c_0 x_\alpha + c_2 \sum_\xi x_\xi + c_1 \sum_\eta x_\eta + c_2 \sum_\gamma x_\gamma,$$

where ξ runs over the neighbours $N_\Gamma(\omega)$ of ω distinct from α; $\eta \in N_\Gamma(\alpha)\backslash\{\omega\}$, the set of neighbours of α distinct from ω and γ runs over the mutual non-neighbours $N'_\Gamma(\omega) \cap N'_\Gamma(\alpha)$ of ω and α. We use the following diagram to determine the projections of $\sum x_\xi, \sum x_\eta$ and $\sum x_\gamma$ to V.

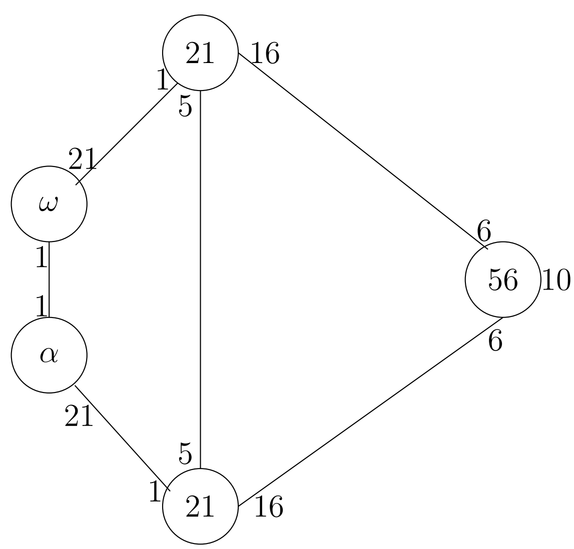

Figure 4.2: Diagram for a pair of adjacent vertices for $HS{:}2$

From the diagram, we deduce that

$$\pi\left(\sum_{\xi\sim\omega} x_\xi\right) = 21c_1 x_\omega + 21c_2 x_\alpha + (c_0 + 20c_2)\sum_\xi x_\xi + (6c_1 + 15c_2)\sum_\eta x_\eta + (6c_1 + 15c_2)\sum_\gamma x_\gamma,$$

$$\pi\left(\sum_{\eta} x_\eta\right) = 21c_2x_\omega + 21c_1x_\alpha + (5c_1+16c_2)\sum_{\xi} x_\xi + (c_0+20c_2)\sum_{\eta} x_\eta + (6c_1+15c_2)\sum_{\gamma} x_\gamma,$$

$$\pi\left(\sum_{\gamma} x_\gamma\right) = 56c_2x_\omega + 56c_2x_\alpha + (16c_1+40c_2)\sum_{\xi} x_\xi + (16c_1+40c_2)\sum_{\eta} x_\eta + (c_0+10c_1+45c_2)\sum_{\gamma} x_\gamma.$$

We denote the pointwise multiplication by $*$ and the Norton product of v_ω and v_α by juxtaposition $v_\omega v_\alpha$. The pointwise product is

$$v_\omega * v_\alpha = c_0c_1x_\omega + c_0c_1x_\alpha + c_1c_2\sum x_\xi + c_1c_2\sum x_\eta + c_2^2\sum x_\gamma.$$

Projecting, we obtain the Norton product as

$$\begin{aligned}
v_\omega v_\alpha &= c_0c_1v_\omega + c_0c_1v_\alpha + c_1c_2\pi\left(\sum_\xi x_\xi + \sum_\eta x_\eta\right) + c_2^2\pi\left(\sum_\gamma x_\gamma\right)\\
&= c_0c_1\left(c_0x_\omega + c_1x_\alpha + c_1\sum_\xi x_\xi + c_2\sum_\eta x_\eta + c_2\sum_\gamma x_\gamma\right)\\
&\quad +c_0c_1\left(c_1x_\omega + c_0x_\alpha + c_2\sum_\xi x_\xi + c_1\sum_\eta x_\eta + c_2\sum_\gamma x_\gamma\right)\\
&\quad +c_1c_2\left(21c_1x_\omega + 21c_2x_\alpha + (c_0+20c_2)\sum_\xi x_\xi + (5c_1+16c_2)\sum_\eta x_\eta\right.\\
&\quad \left.+(6c_1+15c_2)\sum_\gamma x_\gamma\right) + c_1c_2\left(21c_2x_\omega + 21c_1x_\alpha\right.\\
&\quad \left.+(5c_1+16c_2)\sum_\xi x_\xi + (c_0+20c_2)\sum_\eta x_\eta + (6c_1+15c_2)\sum_\gamma x_\gamma\right)\\
&= (c_0^2c_1 + c_0c_1^2 + 21c_1c_2(c_1+c_2) + 56c_2^3)x_\omega +\\
&\quad (c_0c_1^2 + c_0^2c_1 + 21c_1c_2(c_1+c_2) + 56c_2^3)x_\alpha\\
&\quad +(c_0c_1c_2 + c_0c_1^2 + (5c_1+16c_2)c_1c_2 + (c_0+20c_2)c_1c_2 + (16c_1+40c_2)c_2^2)\sum_\xi x_\xi\\
&\quad +(c_0c_1c_2 + c_0c_1^2 + (5c_1+16c_2)c_1c_2 + (c_0+20c_2)c_1c_2 + (16c_1+40c_2)c_2^2)\sum_\eta x_\eta\\
&\quad +(2c_0c_1c_2 + (12c_1+30c_2)c_1c_2 + (c_0+10c_1+45c_2)c_2^2)\sum_\gamma x_\gamma\\
&= \big(c_0c_1(c_0+c_1) + 21c_1c_2(c_1+c_2) + 56c_2^3)\big)\,x_\omega\\
&\quad + \big(c_0c_1(c_0+c_1) + 21c_1c_2(c_1+c_2) + 56c_2^3)\big)\,x_\alpha\\
&\quad + \big(c_0c_1(c_1+c_2) + (5c_1+16c_2)c_1c_2 + (c_0+20c_2)c_1c_2 + (16c_1+40c_2)c_2^2)\big)\sum_\xi x_\xi\\
&\quad + \big(c_0c_1(c_1+c_2) + (5c_1+16c_2)c_1c_2 + (c_0+20c_2)c_1c_2 + (16c_1+40c_2)c_2^2)\big)\sum_\eta x_\eta\\
&\quad + \big(2c_0c_1c_2 + (12c_1+30c_2)c_1c_2 + (c_0+10c_1+45c_2)c_2^2\big)\sum_\gamma x_\gamma\\
&= \frac{21}{500}x_\omega + \frac{21}{500}x_\alpha + \frac{1}{500}\sum_\xi x_\xi + \frac{1}{500}\sum_\eta x_\eta - \frac{3}{100}\sum_\gamma x_\gamma.
\end{aligned}$$

We conclude this section by giving the product of the projections of non-adjacent vertices. Let α be a vertex which is not adjacent with ω. We write

$$v_\omega = \pi(x_\omega) = c_0 x_\omega + c_2 x_\alpha + c_1 \sum_\theta x_\theta + c_1 \sum_\xi x_\xi + c_2 \sum_\eta x_\eta + c_2 \sum_\gamma x_\gamma,$$

$$v_\alpha = \pi(x_\alpha) = c_2 x_\omega + c_0 x_\alpha + c_1 \sum_\theta x_\theta + c_2 \sum_\xi x_\xi + c_1 \sum_\eta x_\eta + c_2 \sum_\gamma x_\gamma,$$

where θ ranges over the common neighbours of ω and α, ξ ranges over neighbours of ω which are not neighbours of α, η ranges over neighbours of α which are not neighbours of ω and γ ranges over mutual non-neighbours of ω and α. From the diagram below, we have that

$$\begin{aligned}
\pi\left(\sum x_\theta\right) &= 6c_1 x_\omega + 6c_1 x_\alpha + (c_0 + 5c_2) \sum_\theta x_\theta + 6c_2 \sum_\xi x_\xi + 6c_2 \sum_\eta x_\eta \\
&\quad + (2c_1 + 4c_2) \sum_\gamma x_\gamma \\
\pi\left(\sum_\xi x_\xi\right) &= 16c_1 x_\omega + 16c_2 x_\alpha + 16c_2 \sum_\theta x_\theta + (c_0 + 15c_2) \sum_\xi x_\xi \\
&\quad + (6c_1 + 10c_2) \sum_\eta x_\eta + (4c_1 + 12c_2) \sum_\gamma x_\gamma; \\
\pi\left(\sum_\eta x_\eta\right) &= 16c_2 x_\omega + 16c_1 x_\alpha + 16c_2 \sum_\theta x_\theta + (6c_1 + 10c_2) \sum_\xi x_\xi \\
&\quad + (c_0 + 15c_2) \sum_\eta x_\eta + (4c_1 + 12c_2) \sum_\gamma x_\gamma \\
\pi\left(\sum_\gamma x_\gamma\right) &= 60c_2 x_\omega + 60c_2 x_\alpha + (20c_1 + 40c_2) \sum_\theta x_\theta \\
&\quad + (15c_1 + 45c_2) \left(\sum_\xi x_\xi + \sum_\eta x_\eta\right) + (c_0 + 12c_1 + 47c_2 0 \sum_\gamma x_\gamma.
\end{aligned}$$

The pointwise product is

$$v_\omega * v_\alpha = c_0 c_2 x_\omega + c_0 c_2 x_\alpha + c_1^2 \sum_\theta x_\theta + c_1 c_2 \sum_\xi x_\xi + c_1 c_2 \sum_\eta x_\eta + c_2^2 \sum_\gamma x_\gamma.$$

Projecting this product to V, we have that

$$v_\omega v_\alpha = c_0c_2v_\omega + c_0c_2v_\alpha + c_1^2\pi\left(\sum_\theta x_\theta\right) + c_1c_2\pi\left(\sum_\xi x_\xi + \sum_\eta x_\eta\right) + c_2^2\pi\left(\sum_\gamma x_\gamma\right).$$

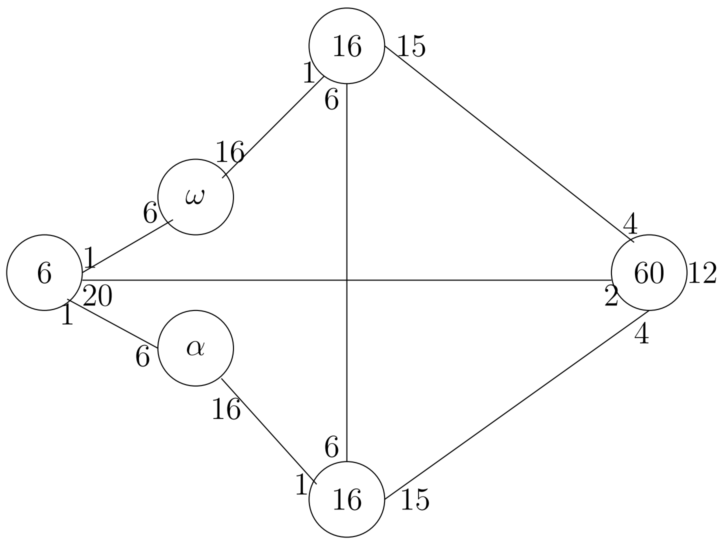

Figure 4.3: Diagram for a pair of non-adjacent vertices for HS:2

That is,

$$\begin{aligned}
v_\omega v_\alpha &= \left(c_0^2c_2 + c_0c_2^2 + 6c_1^3 + 16(c_1c_2(c_1+c_2)) + 60c_2^3\right) x_\omega \\
&\quad + \left(c_0^2c_2 + c_0c_2^2 + 6c_1^3 + 16(c_1c_2(c_1+c_2)) + 60c_2^3\right) x_\alpha \\
&\quad + \left(c_0c_2(2c_1) + c_1^2(c_0+5c_2) + 32c_1c_2^2 + (20c_1+40c_2)c_2^2\right) \sum_\theta x_\theta \\
&\quad + \left(c_0c_1c_2 + c_0c_2^2 + 6c_1^2c_2 + (c_0+15c_2)c_1c_2 + (6c_1+10c_2)c_1c_2\right. \\
&\quad \left.+(15c_1+45c_2)c_2^2\right) \sum_\xi x_\xi + \left(c_0c_2^2 + (6c_1+10c_2)c_1c_2 + c_0c_1c_2\right. \\
&\quad \left.+6c_1^2c_2 + (c_0+15c_2)c_1c_2 + (15c_1+45c_2)c_2^2\right) \sum_\eta x_\eta \\
&\quad + \left(2c_0c_2^2 + (2c_1+4c_2)c_1^2 + 2(4c_1+12c_2)c_1c_2 + (c_0+12c_1\right. \\
&\quad \left.+47c_2)c_2^2\right) \sum_\gamma x_\gamma \\
&= -\frac{9}{500}(x_\omega + x_\alpha) + \frac{1}{500}\sum_\theta x_\theta - \frac{3}{1000}\left(\sum_\xi x_\xi + \sum_\eta x_\eta\right) + \frac{1}{500}\sum_\gamma x_\gamma.
\end{aligned}$$

The following result will be useful in our discussion of algebras for the groups HS:2 and HS.

Proposition 4.1.4. *Let* $G = \mathrm{HS{:}2}$, *the automorphism of the sporadic simple group* HS *of Higman and Sims, and* W *be its permutation module in its natural action on* $\Omega = \{1, \ldots, 100\}$. *Then* W *has a* 77*-dimensional constituent* V *of* W *such that* $(\mathrm{Sym}^2\chi_{\mathrm{V}}, \chi_{\mathrm{V}}) = 1$. *Furthermore,* V_{HS}, *the restriction of* V *to* HS, *is an irreducible* HS*-module which supports a commutative rational algebra structure. The following assertions hold.*

- *(i) For the conjugacy classes* $2A, 2B$ *and* $2C$ *of involutions of* G, *let* $K = C_G(z)$ *be the centraliser of an involution* z *in each class in turn. Then,* $(\chi_V|_K, 1_K) = 1$ *and so* K *fixes a* 1*-dimensional subalgebra of* V.
- *(ii) For the class* $2D$, *let* z *be an involution in the class and* K *be its centraliser in* G. *Then* $(\chi_V|_K, 1_K) = 2$ *and* K *fixes a* 2*-dimensional subalgebra of* V.

Proof. A direct verification with MAGMA [BC94], shows that there is a 77-dimensional character χ such that $(\mathrm{Sym}^2\chi, \chi) = 1 = (\mathrm{Sym}^2\chi|_{\mathrm{HS}}, \chi|_{\mathrm{HS}})$, and both χ and $\chi|_{\mathrm{HS}}$, the restriction of χ to HS, have Frobenius-Schur indicator 1. By Theorem 4.1.1, V, the module which affords χ, supports a commutative algebra structure and the centralizer K, of an involution fixes a 1- or 2-dimensional subalgebra in V. This is also true for $V|_{\mathrm{HS}}$. In fact, by Clifford's Theorem 2.2.3 (ii), $V|_{\mathrm{HS}}$ lies in V, and since $\dim V|_{\mathrm{HS}} = \dim V, V|_{\mathrm{HS}} = V$. By Theorem 2.2.17, V is an orthogonal module, in fact, V has an irreducible $\mathbb{Q}$-form so must be realizable over $\mathbb{Q}$. From the ATLAS, [CCN+85], G has 39 irreducible characters and of these, 38 are integer-valued, the only exception being the character of degree 2050. A direct computation with MAGMA shows that the number of irreducible $\mathbb{Q}$-representations is 38, the number of conjugacy classes of cyclic subgroups of G, by [Ser77, Corollary 1, p.103]. Thus all the 38 integer-valued irreducible characters of G have $\mathbb{Q}$-forms. It follows that V is a rational algebra.

(i) Direct verification with MAGMA establishes this part.

(ii) This part also follows by a direct computation using MAGMA. □

4.2 Algebras from the class $2A$

In this section we present fusion laws for algebras for HS:2 where K is the centraliser of an involution in the conjugacy class $2A$. Let $z \in 2A$ be an involution and $K = C_G(z)$ be its centralizer in G. The centraliser has order 15360, is maximal in G and has shape $2^{1+6}.S_5$. K fixes a one-dimensional subspace in V and in particular, an idempotent a. This group has two orbits in its action on Ω, of lengths 20 and 80. The orbit diagram is depicted in Figure 4.4. Throughout this section, u_1 and u_2 are the orbit sums of these

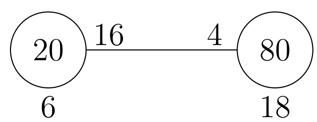

Figure 4.4: The orbit diagram relative to the centraliser of a $2A$ involution

orbits respectively. Let $U = \langle u_1, u_2 \rangle$ as a linear subspace of W. We will show that $U \cap V$ is a one-dimensional subspace of V, and consequently a one-dimensional subalgebra of V thereby identifying axes fixed by K. By

Proposition 3.2.2, we have the adjacency matrix A of Γ has the following matrix in its action on U:

$$B = \begin{bmatrix} 6 & 4 \\ 16 & 18 \end{bmatrix}.$$

The matrix B has eigenvalues 2 and 22. Since the space V of interest in this chapter corresponds to the 2-eigenspace of the adjacency matrix A, we conclude that $U \cap V$ is 1-dimensional. We fix a basis for this space here. The vector $(1 \quad -\frac{1}{4})$ is a 2-eigenvector of B, and it is not difficult to check that $a = u_1 - \frac{1}{4}u_2$ is a 2-eigenvector of A. Thus, a is a basis for the 1-dimensional fixed subspace.

From the diagram, we have the projections $\pi(u_i)$ of $u_i, i = 1, 2$ are:

$$\begin{aligned} \pi(u_1) &= (c_0 + 6c_1 + (20 - 1 - 6)c_2)u_1 + (4c_1 + (80 - 4)c_2)u_2 \\ &= -\frac{4}{5}u_1 + \frac{1}{5}u_2; \end{aligned}$$

$$\begin{aligned} \pi(u_2) &= [16c_1 + (80 - 16)c_2]u_1 + [c_0 + 18c_1 + (80 - 1 - 18)c_2]u_2 \\ &= -\frac{4}{5}u_1 + \frac{1}{5}u_2 = -\pi(u_1). \end{aligned}$$

The square of a under the pointwise product is $(a^*)^2 = u_1 + \frac{1}{16}u_2$. Projecting, the square under the Norton product becomes

$$\begin{aligned} a^2 &= a^{*2} \\ &= \pi(u_1) + \frac{1}{16}\pi(u_2) = \pi(u_1) + \frac{1}{16}\left(-\pi(u_1)\right) = \frac{15}{16}\pi(u_1) \\ &= \frac{3}{4}u_1 - \frac{3}{16}u_2 = \frac{3}{4}a. \end{aligned}$$

Consequently, for the remainder of this section, we scale the Norton product by $\frac{4}{3}$ in order to make a an idempotent. For the purposes of use in the extended algebra, it is convenient to scale the invariant inner product in order to have $(a, a) = 1$. Recall that

$$\begin{aligned} (a, a) &= (u_1, u_1) - \frac{1}{2}(u_1, u_2) + \frac{1}{16}(u_2, u_2) \\ &= 20 - \frac{1}{2} \cdot 0 + \frac{1}{16} \cdot 80 = 25. \end{aligned}$$

Thus, we scale the invariant form by $\frac{1}{25}$.

4.2.1 The fusion rules for the 77-dimensional algebra for class 2A

Picking a basis for V, we find using MAGMA, that the ad_a matrix has spectrum $\Lambda = \{1^{11}, -\frac{1}{3}^{57}, \frac{1}{3}^{4}, \frac{4}{3}^{5}\}$ which clearly shows that a is not primitive. Setting up a basis for V made up of eigenvectors, the $-\frac{1}{3}$-eigenspace splits into two orbits under the action of the Miyamoto involution τ_a corresponding to a, one even and one odd. We note here that τ_a is chosen so as to coincide with the involution z. We use MAGMA to investigate how the eigenspaces multiply and we get the following fusion law.

$\star$	1	$-\frac{1}{3}^E$	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{1}{3}^O$
1	$1, -\frac{1}{3}^E, \frac{1}{3}, \frac{4}{3}$	$1, -\frac{1}{3}^E, \frac{1}{3}$	$1, -\frac{1}{3}^E, \frac{1}{3}$	$1, \frac{4}{3}$	$-\frac{1}{3}^O$
$-\frac{1}{3}^E$		$1, -\frac{1}{3}^E, \frac{1}{3}$	$1, -\frac{1}{3}^E$	$\emptyset$	$-\frac{1}{3}^O$
$\frac{1}{3}$			$1, \frac{1}{3}, \frac{4}{3}$	$1, \frac{4}{3}$	$-\frac{1}{3}^O$
$\frac{4}{3}$				$1, \frac{1}{3}, \frac{4}{3}$	$\emptyset$
$-\frac{1}{3}^O$					$-\frac{1}{3}^E, 1, \frac{1}{3}$

Table 4.1: Fusion law for the 77-dimensional algebra for class 2A

The even part $V^E_{-\frac{1}{3}}(a)$, denoted $-\frac{1}{3}^E$ in Table 4.1, is spanned by all basis vectors of the $-\frac{1}{3}$-eigenspace of the form $\frac{1}{2}(v + v^{\tau_a})$ while the odd part, $-\frac{1}{3}^O = V^O_{-\frac{1}{3}}(a)$ is spanned by basis vectors of the form $\frac{1}{2}(v - v^{\tau_a})$.

4.2.2 The extended algebras from the class 2A

Let $\lambda \in \mathrm{Spec}(\mathrm{ad}_a) \setminus 1$ be the eigenvalue of a sent to the eigenvalue 0 of $w = \alpha + \beta a$. The eigenvalues ρ of ad_w corresponding to the eigenvalues μ of ad_a are obtained from the map $\mu \mapsto \alpha + \beta\mu = \rho$, as shown in Lemma 2.5.2. For different values of λ, we show the corresponding values of α, β and for various eigenvalues μ of ad_a we list the corresponding eigenvalue $\rho = \alpha + \beta\mu$ of w. We present these in Table 4.2. For each of the eigenvalues λ displayed in Table 4.2, we form the extended algebra $\tilde{V}$ and find a fusion law satisfied

			μ			
			$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	1
λ	$\alpha = -\frac{\lambda}{1-2\lambda}$	$\beta = 1-2\alpha$	$\rho = \alpha + \beta\mu$			
$-\frac{1}{3}$	$\frac{1}{5}$	$\frac{3}{5}$	0	$\frac{2}{5}$	1	$\frac{4}{5}$
$\frac{1}{3}$	-1	3	-2	0	3	2
$\frac{4}{3}$	$\frac{4}{5}$	$-\frac{3}{5}$	1	$\frac{3}{5}$	0	$\frac{1}{5}$
1	1	-1	$\frac{4}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	0

Table 4.2: The eigenvalues of ad_w corresponding to those of an axis a fixed by the centralizer of a $2A$ involution.

by $\tilde{V}$.

4.2.3 The extended algebra from class $2A$ with $\lambda = -\frac{1}{3}$

From line 1 of Table 4.2, the eigenvalues ρ for ad_w corresponding to the ad_a-eigenvalues are $0, 1, \frac{2}{5}$ and $\frac{4}{5}$ when $\lambda = -\frac{1}{3}$. In this case, from the same line of the table referred to above, we have $\alpha = \frac{1}{5}, \beta = \frac{3}{5}$ and hence $\gamma = \frac{4}{9}$. Thus we have $w = \frac{1}{5} + \frac{3}{5}a$. Applying Proposition 2.5.4 part (2), we find out that the ad_w-eigenvalues are $0^{58}, 1^6, \frac{4}{5}^{10}, \frac{2}{5}^4$. Theorem 2.5.6 (2) again is used to set up a basis for $\tilde{V}$ comprising eigenvectors. We use this decomposition to find the fusion rules and we obtain the following table.

$\star$	1	0^E	$\frac{2}{5}$	$\frac{4}{5}$	0^O
1	$1, \frac{2}{5}$	$\emptyset$	$1, \frac{2}{5}$	$\frac{4}{5}$	$\emptyset$
0^E		$0, \frac{2}{5}, \frac{4}{5}$	$0, \frac{2}{5}, \frac{4}{5}$	$0, \frac{2}{5}, \frac{4}{5}$	0^O
$\frac{2}{5}$			$1, 0^E, \frac{2}{5}$	$0, \frac{4}{5}$	0^O
$\frac{4}{5}$				$1, 0^E, \frac{2}{5}$	0^O
0^O					$0^E, \frac{2}{5}, \frac{4}{5}$

Table 4.3: A fusion law for the extended algebra from the class 2A with $\lambda = -\frac{1}{3}$.

For the rest of the section, we use Theorem 2.5.6 part (2) to pick a basis for each extended algebra discussed in the following subsections. We present fusion laws for each case.

4.2.4 The extended algebra from class $2A$ with $\lambda = \frac{1}{3}$

For this case, $\alpha = -1, \beta = 3$ so $w = -1+3a$ is an idempotent obtained from a. The constant γ which controls the algebra structure has value $\gamma = \lambda(\lambda-1)-\frac{2}{9}$. Using Proposition 2.5.4, we find that ad_w has spectrum

$$\Lambda = \{1^1, -2^{57}, 3^5, 0^5, 2^{10}\}$$

so that the algebra $\tilde{V}$ is primitive.
Picking a basis for $\tilde{V}$ consisting of eigenvectors using Theorem 2.5.6 (2), we obtain the following fusion law.

$\star$	1	0	-2^E	2	3	-2^O
1	1	$\emptyset$	-2^E	2	3	-2^O
0		$0, 3$	$-2^E, 2$	$-2^E, 2$	$0, 3$	-2^O
-2^E			$1, 0, -2^E, 2$	$0, -2^E, 2$	$\emptyset$	-2^O
2				$1, 0, -2^E, 3$	2	-2^O
3					$1, 0, 3$	-2^O
-2^O						$1, 0, -2^E, 2$

Table 4.4: A fusion law for the extended algebra with $\lambda = \frac{1}{3}$.

In this case, we set $\tilde{V}_+ = \tilde{V}_1(w) \oplus \tilde{V}_0 \oplus \tilde{V}^E_{-2}(w) \oplus \tilde{V}_2(w) \oplus \tilde{V}_3(w), \tilde{V}_- = \tilde{V}^O_{-2}$, to obtain a $\mathbb{Z}/2\mathbb{Z}$-grading. We conclude this subsection by giving the extended algebra obtained from setting $\lambda = \frac{4}{3}$.

4.2.5 The extended algebra from class $2A$ with $\lambda = \frac{4}{3}$

In this case $\alpha = \frac{4}{5}, \beta = -\frac{3}{5}$ and hence the algebra structure controlling parameter γ has value $\frac{4}{9}$. It follows that $w = \frac{4}{5} - \frac{3}{5}a$ is an idempotent in

$\tilde{V}$. By applying Proposition 2.5.4, the matrix of the adjoint action of w has spectrum

$$\Lambda = \left\{ \frac{3}{5}^4, \frac{1}{5}^{10}, 0^6, 1^{58} \right\}.$$

We choose a basis for $\tilde{V}$ made up of eigenvectors using Theorem 2.5.6 from which we obtain the following fusion law using MAGMA.
From the table, we see that $\tilde{V}_+ = \tilde{V}_1^E(w) \oplus \tilde{V}_0(w) \oplus \tilde{V}_{\frac{1}{5}}(w) \oplus \tilde{V}_{\frac{3}{5}}(w)$, $\tilde{V}_- = \tilde{V}_1^O(w)$ gives a $\mathbb{Z}/2\mathbb{Z}$-grading.

$\star$	1^E	0	$\frac{1}{5}$	$\frac{3}{5}$	1^O
1^E	$1^E, \frac{1}{5}, \frac{3}{5}$	$\emptyset$	$1^E, \frac{1}{5}, \frac{3}{5}$	$1^E, \frac{1}{5}, \frac{3}{5}$	1^O
0		$0, \frac{3}{5}$	$\frac{1}{5}$	$0, \frac{3}{5}$	$\emptyset$
$\frac{1}{5}$			$1, 0, \frac{3}{5}$	$1^E, \frac{3}{5}$	1^O
$\frac{3}{5}$				$1^E, 0, \frac{3}{5}$	1^O
1^O					$1^E, \frac{1}{5}, \frac{3}{5}$

Table 4.5: Fusion rules for the extended algebra with $\lambda = \frac{4}{3}$.

4.3 Algebras from the class $2B$

In this section we discuss axial decompositions of V using an idempotent fixed by the centralizer of an involution, where the involution is from the class $2B$ of HS:2. Let z be such an involution. Then the centralizer K, of z in G has order 5760, is maximal in G and has isomorphism type $(2 \times A_6.2^2).2$. The orbit diagram relative to K is shown in Figure 4.5 below.

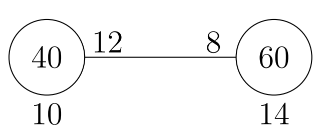

Figure 4.5: Orbit diagram with respect to the centraliser of a $2B$ involution of HS:2

Let u_1 and u_2 be the orbit sums of the orbits of K, respectively. By Proposition 3.2.2, the action of the matrix A on the space$U$ spanned by u_1 and u_2 is

$$B = \begin{bmatrix} 10 & 8 \\ 12 & 14 \end{bmatrix}$$

which has eigenvalues 2 and 22. Thus, $U \cap V$ is a 1-dimensional subspace. We have that $(1 \ \ -\frac{2}{3})$ is a 2-eigenvector of B. We will set $a = u_1 - \frac{2}{3}u_2$ for the rest of the section.

The pointwise product of a with itself is $a^{*2} = u_1 + \frac{4}{9}u_2$. From the diagram shown in Figure 4.5, we have the projections $\pi(u_i), i = 1, 2$ are:

$$\begin{aligned} \pi(u_1) &= (c_0 + 10c_1 + (40 - 1 - 10)c_2)\, u_1 + (8c_1 + (40 - 8)c_2)\, u_2 \\ &= \frac{3}{5}u_1 - \frac{2}{5}u_2; \end{aligned}$$

$$\begin{aligned} \pi(u_2) &= (12c_1 + (60 - 12)c_2)\, u_1 + (c_0 + 14c_1 + (60 - 1 - 14)c_2)\, u_2 \\ &= -\frac{3}{5}u_1 + \frac{2}{5}u_2. \end{aligned}$$

We conclude that

$$\begin{aligned} a^2 &= \pi(a^{*2}) = \pi(u_1) + \frac{4}{9}\pi(u_2) \\ &= \pi(u_1) + \frac{4}{9}\left(-\pi(u_1)\right) = \frac{5}{9}\pi(u_1) \\ &= \frac{1}{3}u_1 - \frac{2}{9}u_2 = \frac{1}{3}a. \end{aligned}$$

For the remainder of the section, we scale the Norton algebra product by three in order to make a idempotent.

The squared norm of a is

$$\begin{aligned} (a, a) &= (u_1, u_1) - \frac{4}{3}(u_1, u_2) + \frac{4}{9}(u_2, u_2) \\ &= 40 - \frac{4}{3} \cdot 0 + \frac{4}{9} \cdot 60 = \frac{200}{3}. \end{aligned}$$

We will scale the invariant inner product by $\frac{3}{100}$ in order to make a unit.

4.3.1 The 77-dimensional algebra from the class $2B$

A direct computation with MAGMA shows that the adjoint action of the idempotent a has eigenvalues $1^1, 2^9, -2^{38}, 0^{10}, 3^{19}$, that the -2- and 3-eigenspaces split into even and odd parts under the action of the Miyamoto involution τ_a corresponding to a. The 1- and 2-eigenspaces are even while the 0-eigenvector is odd. Thus V can be written as

$$V = V_1(a) \oplus V_2(a) \oplus V_{-2}(a)^E \oplus V_3^E(a) \oplus V_0(a) \oplus V_{-2}^O \oplus V_3(a).$$

Setting $V_+ = V_1(a) \oplus V_2(a) \oplus V_{-2}^E(a) \oplus V_3^E(a)$ and $V_- = V_0(a) \oplus V_{-2}^O(a) \oplus V_3^O(a)$, we have a $\mathbb{Z}/2\mathbb{Z}$-grading on V. The fusion rules are shown in Table 4.6 below.

$\star$	1	2	-2^E	3^E	0	-2^O	3^O
1	1	2	-2^E	3^E	$\emptyset$	-2^O	3^O
2		$1, 2, -2^E$	$2, -2^E$	3^E	$-2^O, 0, 3^O$	$-2^O, 0$	$0, 3^O$
-2^E			$1, 2, -2^E$	$\emptyset$	$-2^O, 0$	$-2^O, 0$	$\emptyset$
3^E				$1, 2$	$0, 3^O$	$\emptyset$	$0, 3^O$
0					$2, -2^E, 3^E$	$2, -2^E$	$2, 3^E$
-2^O						$1, 2, -2^E$	$\emptyset$
3^O							$1, 2, 3^E$

Table 4.6: Fusion rules for the 77-dimensional algebra from class 2B

The eigenvalues μ of a transform to eigenvalues $\rho = \alpha + \beta\mu$ of w as shown in Table 4.7

			μ			
			-2	0	2	3
λ	$\alpha = -\frac{\lambda}{1-2\lambda}$	$\beta = 1 - 2\alpha$	$\rho = \alpha + \beta\mu$			
-2	$\frac{2}{5}$	$\frac{1}{5}$	0	$\frac{2}{5}$	$\frac{4}{5}$	1
0	0	1	-2	0	2	3
2	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{2}{3}$	0	$-\frac{1}{3}$
3	$\frac{3}{5}$	$-\frac{1}{5}$	1	$\frac{3}{5}$	$\frac{1}{5}$	0

Table 4.7: ad_w eigenvalues corresponding to non unit eigenvalues of ad_a with $z \in 2B$

4.3.2 Extended algebra with $\lambda = 2$

As shown in Table 4.7, in this case $\alpha = \frac{2}{3}, \beta = -\frac{1}{3}$ and hence $\gamma = 2$. We get an idempotent $w = \frac{2}{3} - \frac{1}{3}a$ and by Theorem 2.5.6, the spectrum for ad_w is $\Lambda' = \{1^1, \frac{2}{3}^{10}, -\frac{1}{3}^{19}, 0^{10}, \frac{4}{3}^{38}\}$. This shows that w is a primitive idempotent. A basis picked using Theorem 2.5.6 is used to express $\tilde{V}$ as a direct sum of eigenspaces and the fusion law shown in the table below gives how the various eigenspaces multiply.

$\star$	1	0	$-\frac{1}{3}^E$	$\frac{4}{3}^E$	$\frac{2}{3}$	$-\frac{1}{3}^O$	$\frac{4}{3}^O$
1	1	$\emptyset$	$-\frac{1}{3}^E$	$\frac{4}{3}^E$	$\frac{2}{3}$	$-\frac{1}{3}^O$	$\frac{4}{3}^O$
0		$0, \frac{4}{3}^E$	$-\frac{1}{3}^E$	$0, \frac{4}{3}^E$	$\frac{2}{3}, -\frac{1}{3}^O, \frac{4}{3}^O$	$\frac{2}{3}, -\frac{1}{3}^O$	$\frac{2}{3}, \frac{4}{3}^O$
$-\frac{1}{3}^E$			$1, 0$	$\emptyset$	$\frac{2}{3}, -\frac{1}{3}^O$	$\frac{2}{3}, -\frac{1}{3}^O$	$\frac{4}{3}^O$
$\frac{4}{3}^E$				$1, 0, \frac{4}{3}^E$	$\frac{2}{3}, \frac{4}{3}^O$	$\emptyset$	$\frac{2}{3}, \frac{4}{3}^O$
$\frac{2}{3}$					$1, 0, -\frac{1}{3}^E, \frac{4}{3}^E$	$0, -\frac{1}{3}^E$	$0, \frac{4}{3}^E$
$-\frac{1}{3}^O$						$1, 0, -\frac{1}{3}^E$	$\emptyset$
$\frac{4}{3}^O$							$1, 0, \frac{4}{3}$

Table 4.8: Fusion rules for extended algebra with $\lambda = 2$.

Here a $\mathbb{Z}/2\mathbb{Z}$-grading is obtained by setting $\tilde{V}_+ = \tilde{V}_1(w) \oplus \tilde{V}_0(w) \oplus \tilde{V}^E_{-\frac{1}{3}}(w) \oplus \tilde{V}^E_{\frac{4}{3}}(w)$ and $\tilde{V}_- = \tilde{V}_{\frac{2}{3}}(w) \oplus \tilde{V}^O_{-\frac{1}{3}}(w) \oplus \tilde{V}^O_{\frac{4}{3}}(w)$.

4.3.3 Extended algebra with $\lambda = -2$

If $\lambda = -2$, then $\alpha = \frac{2}{5}, \beta = \frac{1}{5}$ and $\gamma = 6$. The idempotent obtained from a is $w = \frac{2}{5} + \frac{1}{5}a$. Applying Theorem 2.5.6 to see how the eigenvaluess of ad_a

transform, we have that the spectrum of ad_w is $\Lambda = \{0^{39}, 1^{20}, \frac{2}{5}^{10}, \frac{4}{5}^{9}\}$. We use the same result to set up a basis for $\tilde{V}$ comprising eigenvectors. The fusion law shown in the table below showcases how the eiegnspaces multiply. We have the fusion rules shown in Table 4.9.

$\star$	1^E	0^E	$\frac{4}{5}$	1^O	0^O	$\frac{2}{5}$
1^E	$1^E, \frac{4}{5}$	$\emptyset$	$1^E, \frac{4}{5}$	$1^O, \frac{2}{5}$	$\emptyset$	$1^O, \frac{2}{5}$
0^E		$0^E, \frac{4}{5}$	$0^E, \frac{4}{5}$	$\emptyset$	$0^O, \frac{2}{5}$	$0^O, \frac{2}{5}$
$\frac{4}{5}$			$1^E, 0^E, \frac{4}{5}$	$1^O, \frac{2}{5}$	$0^O, \frac{2}{5}$	$1^O, 0^O, \frac{2}{5}$
1^O				$1^E, \frac{4}{5}$	$\emptyset$	$1^E, \frac{4}{5}$
0^O					$0^E, \frac{4}{5}$	$0^E, \frac{4}{5}$
$\frac{2}{5}$						$1^E, 0^E, \frac{4}{5}$

Table 4.9: The fusion rules for extended algebra with $\lambda = -2$.

A $\mathbb{Z}/2\mathbb{Z}$-grading is obtained by setting $\tilde{V}_+ = \tilde{V}_1^E(w) \oplus \tilde{V}_0^E(w) \oplus \tilde{V}_{\frac{4}{5}}(w)$ and $\tilde{V}_- = \tilde{V}_1^O(w) \oplus \tilde{V}_0^E(w) \oplus \tilde{V}_{\frac{2}{5}}(w)$.

4.3.4 Extended algebras obtained by setting $\lambda = 3$

In this case Table 4.7 shows that $\alpha = \frac{3}{5}, \beta = -\frac{1}{5}$ and $\gamma = 6$. The spectrum of ad_w is $\Lambda' = \{0^{20}, \frac{1}{5}^{9}, \frac{3}{5}^{10}, 1^{39}\}$, applying Theorem 2.5.6. We use the same theorem to find a basis for $\tilde{V}$ comprising eigenvectors. A MAGMA computation with this basis yields the following fusion law shown in Table 4.10.

$\star$	1^E	0^E	$\frac{1}{5}$	1^O	0^O	$\frac{3}{5}$
1^E	$1^E, \frac{1}{5}$	$\emptyset$	$1^E, \frac{1}{5}$	$1^O, \frac{3}{5}$	$\emptyset$	$1^O, \frac{3}{5}$
0^E		$0^E, \frac{1}{5}$	$0^E, \frac{1}{5}$	$\emptyset$	$0^O, \frac{3}{5}$	$0^O, \frac{3}{5}$
$\frac{1}{5}$			$1^E, 0^E, \frac{1}{5}$	$1^O, \frac{3}{5}$	$0^O, \frac{3}{5}$	$1^O, 0^O, \frac{3}{5}$
1^O				$1^E, \frac{1}{5}$	$\emptyset$	$1^E, \frac{1}{5}$
0^O					$0^E, \frac{1}{5}$	$0^E, \frac{1}{5}$
$\frac{3}{5}$						$1^E, 0^E, \frac{1}{5}$

Table 4.10: The fusion law for extended algebra with $\lambda = 3$.

4.3.5 Extended algebra from class $2B$ with $\lambda = 0$

This case is quite interesting in that we have exactly the same multiplication in $\tilde{V}$ as in V. We have $\alpha = \gamma = 0$ and $\beta = 1$, from Table 4.7. It follows that $w = \tilde{a}$ is the idempotent corresponding to a. The spectrum for ad_w is

$$\Lambda' = \{1^1, 2^9, 0^{11}, -2^{38}, 3^{19}\},$$

applying Theorem 2.5.6, which is the same as the one for ad_a the only exception being that the 0-eigenspace has one additional dimension. The fusion rules for this case are shown in Table 4.11. We use Theorem 2.5.6 to set up a basis made up of eigenvectors and implement the multiplication of the eigenspaces in MAGMA. We have the following fusion table.

$\star$	1	0^E	2	-2^E	3^E	0^O	-2^O	3^O
1	1	$\emptyset$	2	-2^E	3^E	$\emptyset$	-2^O	3^O
0^E		0^E	2	-2^E	3^E	0^O	-2^O	3^O
2			$1, 2, -2^E$	$2, -2^E$	3^E	$-2^O, 0^O$	$-2^O, 0^O, 3^O$	$0^O, 3^O$
9								
-2^E				$1, 2, -2^E$	$\emptyset$	$-2^O, 0^O$	$-2^O, 0^O$	$\emptyset$
3^E					$1, 2$	$-2^O, 0^O$	$\emptyset$	$0^O, 3^O$
0^O						$2, -2^E, 3^E$	$2, -2^E$	$2, 3^E$
-2^O							$1, 2, -2^E$	$\emptyset$
3^O								$1, 2, 3^E$

Table 4.11: Fusion rules for $\lambda = 0$.

4.4 Algebras from the class $2C$

In this section we discuss algebras obtained by extending the 77-dimensional algebra V by using eigenvalues of an idempotent fixed by the centralizer of an involution in class $2C$ of the group HS:2. Let $z \in$ 2C be an involution. Then the centralizer $K = C_G(z)$ has order 80640, is maximal in G and has isomorphism type $S_8 \times 2$. The space fixed by the centralizer of an involution has dimension one, by Proposition 4.1.4, and an idempotent a from this space can be found.

Central to our discussion for the rest of this section will be the orbit diagram relative to K. We discuss the details of this diagram here. The group K has two orbits in its action on Ω, of lengths 30 and 70 respectively. A vertex in the orbit of length 20 has eight neighbours in the same orbit and 14 in the other orbit; while a vertex in the orbit of length 70 has six neighbours in the first and 16 in the orbit of length 70. The diagram is shown in Figure 4.6 below.

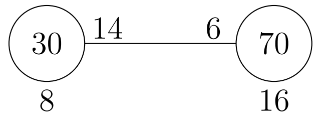

Figure 4.6: Orbit diagram with respect to the centraliser of a $2C$ involution of $HS{:}2$

For the rest of the section, we set u_1, u_2 to be the orbit sums, i.e., u_1, u_2 are $0, 1$ vectors which are nonzero precisely at coordinates lying in the respective orbits. Thus, the action of the adjacency matrix A, of Γ on the space $U = \langle u_i \rangle_{\{1,2\}}$ has matrix

$$B = \begin{bmatrix} 8 & 6 \\ 14 & 16 \end{bmatrix}.$$

The matrix B has eigenvalues 2 and 22. We fix a basis for the 2-eigenspace. It can easily be checked that $(1 \quad -\frac{3}{7})$ is a 2-eigenvector. Since our space of interest corresponds to the eigenvalue 2 of A, we conclude that U and V meet in a 1-dimensional subspace. From the above, we can easily show that $a = u_1 - \frac{3}{7}u_2$ is a basis of this subspace. We will show that this subspace is a subalgebra, and hence that we have axes fixed by K.
As usual we will denote the projection of W to V by π. We give the projections $\pi(u_i), i = 1, 2$ of u_1 and u_2 here. From the diagram, we have

$$\begin{aligned} \pi(u_1) &= \left[\frac{77}{100} + 8\left(\frac{7}{100}\right) + (30-1-8)\left(-\frac{3}{100}\right)\right] u_1 \\ &\quad + \left[6\left(\frac{7}{100}\right) + (30-6)\left(-\frac{3}{100}\right)\right] u_2 \\ &= \frac{7}{10}u_1 - \frac{3}{10}u_2; \end{aligned}$$

$$\begin{aligned} \pi(u_2) &= \left[14c_1 + (70-14)c_2\right] u_1 + \left[c_0 + 16c_1 + (70-1-16)c_2\right] u_2 \\ &= -\frac{7}{10}u_1 + \frac{3}{10}u_2 = -\pi(u_1). \end{aligned}$$

The square a^{*2} of the element a under point-wise multiplication is:

$$a^{*2} = u_1 + \frac{9}{49}u_2.$$

Projecting, we have

$$\begin{aligned}
a^2 &= = \pi(a^{*2}) \\
&= \pi(u_1) + \frac{9}{49}\pi(u_2) = \pi(u_1) + \frac{9}{49}\left(-\pi(u_1)\right) = \frac{40}{49}\pi(u_1) \\
&= \frac{4}{7}u_1 - \frac{12}{49}u_2 = \frac{4}{7}a.
\end{aligned}$$

For the remainder of this section, we scale the Norton product by $\frac{7}{4}$ in order to make a an idempotent. Since u_1, u_2 are orbit sums, $\{u_1, u_2\}$ is an orthogonal linearly independent set. Thus, the squared norm of a is

$$\begin{aligned}
(a,a) &= (u_1 - \frac{3}{7}u_2, (u_1 - \frac{3}{7}u_2) \\
&= (u_1,u_1) - \frac{6}{7}(u_1,u_2) + \frac{9}{49}(u_2,u_2) \\
&= 30 - \frac{6}{7}\cdot 0 + \frac{9}{49}\cdot 70 = \frac{300}{7}.
\end{aligned}$$

Thus, we scale the invariant form by $\frac{7}{300}$ to make a unit.
A MAGMA computation shows that the spectrum for ad_a is $\Lambda = \{1^1, -\frac{3}{4}^{14}, \frac{3}{4}^{48}, \frac{7}{4}^{14}\}$. The $-\frac{3}{4}$-eigenspace splits into two parts, one odd and one even, under the action of the corresponding Miyamoto involution $p := \tau_a$. The even part,

$$V^E_{-\frac{3}{4}} = \mathrm{Span}\left(\{\frac{1}{2}(v + v^p) | v \in \mathscr{B}_{-\frac{3}{4}}(a)\}\right)$$

is 20-dimensional while the odd part $V^O_{-\frac{3}{4}} = \mathrm{Span}\left(\{\frac{1}{2}(v - v^p) | v \in \mathscr{B}_{-\frac{3}{4}}(a)\}\right)$ is 28-dimensional. In the fusion table below, the even part of the $-\frac{3}{4}$-eigenspace is denoted $-\frac{3}{4}^E$ while the odd part is denoted $-\frac{3}{4}^O$.

$*$	1	$-\frac{3}{4}^E$	$\frac{3}{4}$	$\frac{7}{4}$	$-\frac{3}{4}^O$
1	1	$-\frac{3}{4}^E$	$\frac{3}{4}$	$\frac{7}{4}$	$-\frac{3}{4}^O$
$-\frac{3}{4}^E$		$1, -\frac{3}{4}^E, \frac{3}{4}$	$-\frac{3}{4}^E, \frac{3}{4}$	$\emptyset$	$-\frac{3}{4}^O$
$\frac{3}{4}$			$1, -\frac{3}{4}^E, \frac{3}{4}, \frac{7}{4}$	$\frac{3}{4}, \frac{7}{4}$	$-\frac{3}{4}^O$
$\frac{7}{4}$				$1, \frac{3}{4}, \frac{7}{4}$	$\emptyset$
$-\frac{3}{4}^O$					$1, -\frac{3}{4}^E, \frac{3}{4}$

Table 4.12: Fusion rules for the 77-dimensional algebra from class 2C

In the next subsections we will study unital extensions of the algebra V by sending in turn, eigenvalues λ, of ad_a to the 0-eigenvalue of $w = \alpha + \beta a$. The resulting eigenvalues ρ of idempotents $w = \alpha + \beta a \in \tilde{V}$ are shown in Table 4.13.

			μ		
			$\frac{3}{4}$	$-\frac{3}{4}$	$\frac{7}{4}$
λ	$\alpha = -\frac{\lambda}{1-2\lambda}$	$\beta = 1 - 2\alpha$	$\rho = \alpha + \beta\mu$		
$\frac{3}{4}$	$\frac{3}{2}$	-2	0	3	-2
$-\frac{3}{4}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{3}{5}$	0	1
$\frac{7}{4}$	$\frac{7}{10}$	$-\frac{2}{5}$	$\frac{2}{5}$	1	0

Table 4.13: The ad_w-eigenvalues corresponding to a fixed by $z \in 2C$.

4.4.1 Fusion rules for extended algebra with $\lambda = -\frac{3}{4}$

For this case $\alpha = \frac{3}{10}, \beta = \frac{2}{5}$ and $\gamma = \frac{21}{16}$. Applying Theorem 2.5.6, the idempotent $w = \alpha + \beta a$ corresponding to a has spectrum $1^{15}, 0^{49}, \frac{3}{5}^{14}$ so is

not primitive. For $\lambda = -\frac{3}{4}$, the parameter γ equals $\frac{21}{16}$ and $w = \frac{3}{10} + \frac{2}{5}a$, as shown in Table 4.13. We fix a basis for $\tilde{V}$ by applying Theorem 2.5.6. Thus, we have the decomposition $\tilde{V} = \tilde{V}_1(w) \oplus \tilde{V}_0^E(w) \oplus \tilde{V}_{\frac{3}{5}}(w) \oplus \tilde{V}_0^O(w)$ for $\tilde{V}$.
We investigate how the various eigenspaces multiply amongst themselves and we obtain the following fusion rules.

$\star$	1	0^E	$\frac{3}{5}$	0^O
1	$1, \frac{3}{5}$	$\emptyset$	$1, \frac{3}{5}$	$\emptyset$
0^E		$0, \frac{3}{5}$	$0, \frac{3}{5}$	0^O
$\frac{3}{5}$			$1, 0, \frac{3}{5}$	0^O
0^O				$0, \frac{3}{5}$

Table 4.14: Fusion rules for extended algebra for the class 2C with $\lambda = -\frac{3}{4}$

A $\mathbb{Z}/2\mathbb{Z}$-grading is obtained by setting $\tilde{V}_+ = \tilde{V}_1(w) \oplus \tilde{V}_0^E(w) \oplus \tilde{V}_{\frac{3}{5}}^E(w)$ and $\tilde{V}_- = \tilde{V}_0^O(w)$.

4.4.2 Fusion rules for the extended algebra with $\lambda = \frac{3}{4}$

As can be seen from the eigenvalue transformation table, Table 4.13 $\alpha = \frac{3}{2}$ and $\beta = -2$. It follows that $\gamma = -\frac{3}{16}$. The idempotent a is transformed to $w = \frac{3}{2} - 2a$. By applying Theorem 2.5.6, we find that the eigenvalues of ad_w are $0, 1, 3$ and -2 with multiplicities $15, 1, 48$ and 14 respectively. Since w is an idempotent, we have $w \in \tilde{V}_1(w)$ and so w spans its own 1-eigenspace so is a primitive idempotent. We set up a basis for $\tilde{V}$ consisting of 0-, 1-, 3- and -2-eigenvectors by applying Theorem 2.5.6.
Having chosen this basis for $\tilde{V}$, we investigate how the various eigenvectors multiply among themselves and we get the following fusion table.

$\star$	1	0	3^E	-2	3^O
1	1	$\emptyset$	3^E	-2	3^O
0		$0, 3^E, -2$	$0, 3^E$	$0, -2$	3^O
3^E			$1, 0, 3^E$	$\emptyset$	3^O
-2				$1, 0, -2$	$\emptyset$
3^O					$1, 0, 3^E$

Table 4.15: Fusion rules for extended algebra for the class 2C with $\lambda = \frac{3}{4}$

4.4.3 Fusion rules for extended algebra with $\lambda = \frac{7}{4}$

Table 4.13 shows that $\alpha = \frac{7}{10}, \beta = -\frac{2}{5}$ and so $\gamma = \frac{21}{16}$. Therefore, the idempotent $w = \frac{7}{10} - \frac{2}{5}a$. An application of Theorem 2.5.6 shows that the 1-eigenspace in this extended algebra is 49-dimensional so the idempotent w is imprimitive. Fixing a basis for $\tilde{V}$ using the same theorem, ad_w has eigenvalues $0, \frac{2}{5}$ and 1 with multiplicities $15, 14$ and 49 respectively.

Using this basis, verification by MAGMA shows that the eigenspaces multiply according to the fusion rules shown in the table below.

$\star$	1^E	0	$\frac{2}{5}$	1^O
1^E	1^E	$\emptyset$	$1^E, \frac{2}{5}$	1^O
0		$0, \frac{2}{5}$	$0, \frac{2}{5}$	$\emptyset$
$\frac{2}{5}$			$1^E, 0, \frac{2}{5}$	1^O
1^O				$1, \frac{2}{5}$

Table 4.16: Fusion rules for extended algebra with $\lambda = \frac{7}{4}$.

We set $\tilde{V}_+ = \tilde{V}_1^E(w) \oplus \tilde{V}_0(w) \oplus \tilde{V}_{\frac{2}{5}}(w)$ and $\tilde{V}_- = \tilde{V}_1^O(w)$ to obtain a $\mathbb{Z}/2\mathbb{Z}$ grading.

4.4.4 The case $\lambda = 1$

The case $\lambda = 1$ is not usually interesting because whenever A is a unital algebra with unit $\mathbb{1}$, then provided that $a \in A$ is idempotent, so is $\mathbb{1} - a$. The fusion rule in this case mirrors that in the 77-dimensional algebra. However, for this case, we obtain fusion rules which are Seress. Substituting $\lambda = 1$ in Equations (2.7) and (2.5), we have $\alpha = \frac{\lambda}{2\lambda - 1} = 1, \beta = 1 - 2\alpha = -1$ and whence $\gamma = \frac{\alpha - \alpha^2}{\beta^2} = 0$. Multiplication in the extended algebra becomes

$$(\alpha + \beta v) \star (\alpha' + \beta' v') = \alpha\alpha' + \alpha\beta' v' + \alpha'\beta v + \beta\beta' v * v'.$$

Applying Theorem 2.5.6, the eigenvalues are $1^1, \frac{7}{4}^{48}, -\frac{3}{4}^{14}, 0^1, \frac{1}{4}^{14}$. We use MAGMA to investigate how the eigenspaces multiply and we have the fusion law shown in Table 4.17.

$\star$	1	0	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{7}{4}^E$	$\frac{7}{4}^O$
1	1	$\emptyset$	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{7}{4}^E$	$\frac{7}{4}^O$
0		0	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{7}{4}^E$	$\frac{7}{4}^O$
$-\frac{3}{4}$			$0, -\frac{3}{4}, \frac{1}{4}$	$-\frac{3}{4}, \frac{1}{4}$	$\emptyset$	$\emptyset$
$\frac{1}{4}$				$0, -\frac{3}{4}, \frac{1}{4}, \frac{7}{4}^E$	$\frac{1}{4}, \frac{7}{4}$	$\frac{7}{4}^O$
$\frac{7}{4}^E$					$0, \frac{1}{4}, \frac{7}{4}^E$	$\frac{7}{4}^O$
$\frac{7}{4}^O$						$0, \frac{1}{4}, \frac{7}{4}^E$

Table 4.17: Fusion rules for 2C extended algebra with $\lambda = 1$.

We note here that as mentioned above, the fusion rules are Seress.

4.5 Algebras from the class 2D

In this section we discuss the axial algebras from the class $2D$ of involutions for the double cover of the Higman-Sims group. The stabiliser K of an involution in this class has order 3840 and is isormorphic to $2^{1+4} : S_5$. This group has four orbits in its action on Ω, of lengths two, six, 32 and 60. The diagram relative to K is shown in Figure 4.7. Let $u_i, i = 1, 2, 3, 4$ be the

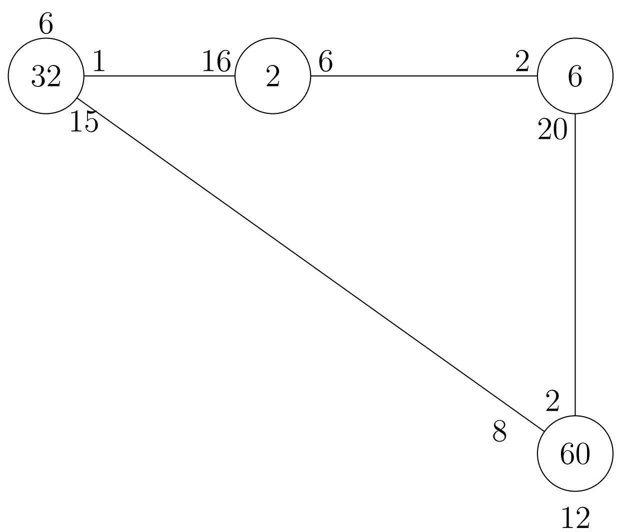

Figure 4.7: Orbit diagram for the centraliser of a $2D$ involution

orbit sums of the orbits of K in the order given above. Then the action of the adjacency matrix A of Γ on the space U spanned by the u_i has matrix:

$$B = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 6 & 0 & 0 & 2 \\ 16 & 0 & 6 & 8 \\ 0 & 20 & 15 & 12 \end{bmatrix}.$$

The eigenvalues of B are 2, -8 and 22, with multiplicities $2, 1$ and 1, respectively. It follows that the subspace $U \cap V$ of V fixed by the centraliser of a $2D$ involution is 2-dimensional. It can easily be checked that $\{(1\ 0\ \frac{1}{8}\ -\frac{1}{10}), (0\ 1\ -\frac{3}{8}\ \frac{1}{10})\}$ forms a basis for the two-eigenspace. Then $a_1 = u_1 + \frac{1}{8}u_3 - \frac{1}{10}u_4, a_2 = u_2 - \frac{3}{8}u_3 + \frac{1}{10}u_4$ forms a basis for the linear subspace $U \cap V$.

From the diagram shown in Figure 4.7, we have the projections $\pi(v_i)$ of the vectors v_i are:

$$\begin{aligned}\pi(u_1) &= (c_0+c_2)\,u_1 + 2c_1u_2 + (c_1+c_2)\,u_3 + 2c_2u_4 \\ &= \frac{37}{50}u_1 + \frac{7}{50}u_2 + \frac{1}{25}u_3 - \frac{3}{50}u_4;\end{aligned}$$

$$\begin{aligned}\pi(u_2) &= (6c_1)u_1 + (c_0+5c_2)\,u_2 + (6c_2)u_3 + (2c_1+4c_2)\,u_4 \\ &= \frac{21}{50}u_1 + \frac{31}{50}u_2 - \frac{9}{50}u_3 + \frac{1}{50}u_4;\end{aligned}$$

$$\begin{aligned}\pi(u_3) &= (16c_1+16c_2)\,u_1 + (32c_2)\,u_2 + (c_0+6c_1+25c_2)\,u_3 + (8c_1+24c_2)\,u_4 \\ &= \frac{16}{25}u_1 - \frac{24}{25}u_2 + \frac{11}{25}u_3 - \frac{4}{25}u_4;\end{aligned}$$

and

$$\begin{aligned}\pi(u_4) &= (60c_2)\,u_1 + (20c_1+40c_2)\,u_2 + (15c_1+45c_2)\,u_3 + (c_0+12c_1+47c_2)\,u_4 \\ &= -\frac{9}{5}u_1 + \frac{1}{5}u_2 - \frac{3}{10}u_3 + \frac{1}{5}u_4.\end{aligned}$$

We find the ad_{a_1} and ad_{a_2} matrices on $U \cap V = \langle a_1, a_2\rangle$. The square of a_1 under the point-wise multiplication is $a_1^{*2} = u_1 + \frac{1}{64}u_3 + \frac{1}{100}u_4$. Projecting, we have the square under the Norton product as :

$$\begin{aligned}a_1^2 = \pi(a_1^{*2}) &= \pi(u_1) + \frac{1}{64}\pi(u_3) + \frac{1}{100}\pi(u_4) \\ &= \frac{37}{50}u_1 + \frac{7}{50}u_2 + \frac{1}{25}u_3 - \frac{3}{50}u_4 \\ &\quad + \frac{1}{64}\left[\frac{16}{25}u_1 - \frac{24}{25}u_2 + \frac{11}{25}u_3 - \frac{4}{25}u_4\right] \\ &\quad + \frac{1}{100}\left[-\frac{9}{5}u_1 + \frac{1}{5}u_2 - \frac{3}{10}u_3 + \frac{1}{5}u_4\right] \\ &= \frac{183}{250}u_1 + \frac{127}{1000} + \frac{351}{8000}u_3 - \frac{121}{2000}u_4.\end{aligned}$$

The pointwise product of a_1 and a_2 is:

$$\begin{aligned}a_1 * a_2 &= \left(\frac{1}{8}\right)\left(-\frac{3}{8}\right)u_3 + \left(-\frac{1}{10}\right)\left(\frac{1}{10}\right)u_4 \\ &= -\frac{3}{64}u_3 - \frac{1}{100}u_4.\end{aligned}$$

Thus, the Norton product is

$$\begin{aligned}
a_1a_2 &= \pi\left(a_1 * a_2)\right) \\
&= -\frac{3}{64}\pi(u_3) - \frac{1}{100}\pi(u_4) \\
&= -\frac{3}{64}\left[\frac{16}{25}u_1 - \frac{24}{25}u_2 + \frac{11}{25}u_3 - \frac{4}{25}u_4\right] \\
&\quad -\frac{1}{100}\left[-\frac{9}{5}u_1 + \frac{1}{5}u_2 - \frac{3}{10}u_3 + \frac{1}{5}u_4\right] \\
&= -\frac{3}{250}u_1 + \frac{43}{1000}u_2 - \frac{141}{8000}u_3 + \frac{11}{2000}u_4.
\end{aligned}$$

Finally, we compute the square of a_2. The pointwise product is

$$a_2^{*2} = u_2 + \frac{9}{64}u_3 + \frac{1}{100}u_4.$$

Upon projecting to V, we have

$$\begin{aligned}
a_2^2 &= \pi(a_2^{*2}) \\
&= \pi(u_2) + \frac{9}{64}\pi(u_3) + \frac{1}{100}\pi(u_4) \\
&= \left[\frac{21}{50}u_1 + \frac{31}{50}u_2 - \frac{9}{50}u_3 + \frac{1}{50}u_4\right] \\
&\quad +\frac{9}{64}\left[\frac{16}{25}u_1 - \frac{24}{25}u_2 + \frac{11}{25}u_3 - \frac{4}{25}u_4\right] \\
&\quad +\frac{1}{100}\left[-\frac{9}{5}u_1 + \frac{1}{5}u_2 - \frac{3}{10}u_3 + \frac{1}{5}u_4\right] \\
&= \frac{123}{250}u_1 + \frac{487}{1000}u_2 - \frac{969}{8000}u_3 - \frac{1}{2000}u_4.
\end{aligned}$$

In order for us to get $[\mathrm{ad}_{a_1}]$ and $[\mathrm{ad}_{a_2}]$, we need to express $a_ia_j, i, = 1, 2$ in terms of a_1 and a_2. Suppose that $\sum_{i=1}^4 s_iu_i$ lies in the 2-space $\langle\langle a_1, a_2\rangle\rangle$. Then

$$\begin{aligned}
\sum_{i=1}^4 s_iu_i &= \alpha_1a_1 + \alpha_2a_2 \\
&= \alpha_1\left(u_1 + \frac{1}{8}u_3 - \frac{1}{10}u_4\right) + \alpha_2\left(u_2 - \frac{3}{8}u_3 + \frac{1}{10}u_4\right) \\
&= \alpha_1u_1 + \alpha_2u_2 + \left(\frac{\alpha_1}{8} - \frac{3}{8}\alpha_2\right)u_3 + \left(-\frac{1}{10}\alpha_1 + \frac{1}{10}\alpha_2\right).
\end{aligned}$$

Thus, $\alpha_1 = s_1$ and $\alpha_2 = s_2$. It follows that

$$\begin{aligned} a_1^2 &= \frac{183}{250}a_1 + \frac{127}{1000}a_2, \\ a_1a_2 &= -\frac{3}{250}a_1 + \frac{43}{1000}a_2, \text{ and} \\ a_2^2 &= \frac{123}{250}a_1 + \frac{487}{1000}a_2. \end{aligned}$$

Therefore,

$$[\mathrm{ad}_{a_1}] = \begin{bmatrix} \frac{183}{250} & \frac{127}{1000} \\ -\frac{3}{250} & \frac{43}{1000} \end{bmatrix}$$

$$[\mathrm{ad}_{a_2}] = \begin{bmatrix} -\frac{3}{250} & \frac{43}{1000} \\ \frac{123}{250} & \frac{487}{1000} \end{bmatrix}.$$

We wish to discuss how to get possible axes. Suppose that $a = \beta_1 a_1 + \beta_2 a_2$ is an idempotent. Then we have

$$\begin{aligned} \beta_1 a_1 + \beta_2 a_2 &= \beta_1^2 a_1^2 + 2\beta_1\beta_2 a_1 a_2 + \beta_2^2 a_2^2 \\ &= \beta_1^2 \left[\frac{183}{250}a_1 + \frac{127}{1000}a_2\right] \\ &\quad +2\beta_1\beta_2 \left[-\frac{3}{250}a_1 + \frac{43}{1000}a_2\right] \\ &\quad +\beta_2^2 \left[\frac{123}{250}a_1 + \frac{487}{1000}a_2\right] \\ &= \left[\frac{183}{250}\beta_1^2 - \frac{3}{125}\beta_1\beta_2 + \frac{123}{250}\beta_2^2\right] a_1 \\ &\quad + \left[\frac{127}{1000}\beta_1^2 + \frac{43}{500}\beta_1\beta_2 + \frac{487}{1000}\beta_2^2\right] a_2. \end{aligned}$$

That is,

$$\beta_1 = \frac{183}{250}\beta_1^2 - \frac{3}{25}\beta_1\beta_2 + \frac{123}{250}\beta_2^2 \tag{4.3}$$

and

$$\beta_2 = \frac{127}{1000}\beta_1^2 + \frac{43}{500}\beta_1\beta_2 + \frac{487}{1000}\beta_2^2. \tag{4.4}$$

Since neither a_1 nor a_2 is idempotent, any non-trivial solution of the systems of equations must have both β_1 and β_2 nonzero. Under this assumption, we can set β_1 or β_2 to one and it is not difficult to see that in either case, one of the two resulting quadratic equations has a negative discriminant. It follows that both equations are irreducible over $\mathbb{Q}$.
The fact that one of the equations has negative discriminant in either case necessitates working over some complex extension of $\mathbb{Q}$ but we do not pursue this line of action here.

Chapter 5

Algebras for the Suzuki sporadic simple group

The sporadic simple Suzuki group Suz was constructed in [Suz69] as an index two subgroup in the full automorphism group of a rank three graph Γ on 1782 points in which the stabiliser of a point is the group $G_2(4)$. The graph Γ is distance regular, of diameter two, so in particular, it is strongly regular. Its parameters are $(1782, 416, 100, 96)$. The spectrum of the graph can be shown to be $416^1, 20^{780}, (-16)^{1001}$, using the results of Chapter 3. This information can also be obtained from the literature, for example [BHKN09]. In this chapter we discuss the axial algebras for the groups Suz and $Suz{:}2$.

5.1 Preliminaries

Throughout this chapter we fix the following notation. Let
$\Omega = \{1, 2, \ldots, 1782\}$ and $G = Suz{:}2$. By Γ we refer to the strongly regular graph whose vertex set is Ω and has parameters $(1782, 416, 100, 96)$. From the information summarised in Table 2 of the introduction, an irreducible character $\chi = \chi_8$ of degree 780 appears with multiplicity two in its symmetric square, that is, $(\mathrm{Sym}^2\chi, \chi) = 2 = \dim(L(V \vee V, V)$; where the ordering of characters follows that of the GAP Character Table Library [Bre12]. The same character is labeled χ_7 in MAGMA [BC94]. We will set V to be the 780-dimensional module affording χ. Similar arguments as those used to show that the 77-dimensional module for the group $HS{:}2$ affords a rational representation also show that the 780-dimension module V under discussion

affords a rational representation.
The fact that $(\mathrm{Sym}^2\chi, \chi)$ equals two implies that the space of commutative algebra products on the 780-dimensional module is 2-dimensional. Furthermore, the Frobenius-Schur indicator of this irreducible character is one, or equivalently, the intertwining number $i(V, V) = 1$; so that up to scalar factors, there is a unique non-degenerate bilinear form supported by V which is invariant under the action of G. The group G acts transitively on Ω. The stabiliser G_ω of a point $\omega \in \Omega$ is $G_2(4)$. The stabiliser has three orbits in its action on Ω, of lengths one, 416 and 1365. It is not difficult to see that the orbit of length 416 consists of the neighbours of ω, and the orbit of length 1365 comprises non-neighbours of ω.
Consider the action of G on $\Omega \times \Omega$. There is a one-to-one correspondence between the orbitals of G and and the orbits of the stabiliser G_ω, of the point ω, given by $\Delta \leftrightarrow \Delta(\omega) := \{\alpha \in \Omega | (\omega, \alpha) \in \Delta\}$. This action partitions $\Omega \times \Omega$ into three orbitals; the diagonal orbital $\mathcal{O}_1$, of size 1782, and two other orbitals, $\mathcal{O}_2, \mathcal{O}_3$ of sizes 1782×416 and 1782×1365, respectively.
A vertex in the orbit $\{\omega\}$ is adjacent to 416 vertices in the orbit of length 416, a vertex in the orbit of length 416 is adjacent to ω, 100 vertices in the same orbit and 315 in the orbit of length 1365. A vertex in the orbit of length 1365 is adjacent to 320 vertices in the same orbit and 96 vertices in the orbit of length 416. We summarise this information in the orbit diagram shown in Figure 5.1.

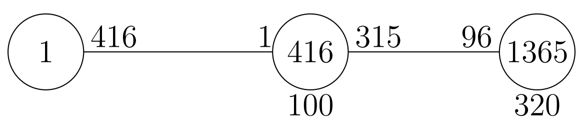

Figure 5.1: The orbit diagram for a point stabiliser

For each $\alpha \in \Omega$ define $x_\alpha \in \mathbb{Q}^{1782}$ as the row vector with one in the α^{th} position and zeros elsewhere. Let X be the totality of all the x_α's, i.e.,

$$X = \{x_\alpha | \alpha \in \Omega\}.$$

We define $W = \mathbb{Q}X \cong \mathbb{Q}^{1782}$. By $\mathbf{j}$ we mean the all one row vector of length 1782. Let A be the adjacency matrix of Γ. Then A has three eigenvalues $k = 416, 20, -16$ of multiplicities $1, 780, 1001$. Because the Suzuki graph Γ is strongly regular, we have an orthogonal decomposition

$$W = W_0 \oplus W_1 \oplus W_2,$$

where $W_0 = \langle \mathbf{j} \rangle, \dim(W_1) = 780$ and $\dim(W_2) = 1001$, i.e., W_0 is the eigenspace of A corresponding to the eigenvalue k, W_1 is the eigenspace of A corresponding to the eigenvalue 20 and W_2 is the eigenspace of A corresponding to the eigenvalue -16. Henceforth, we set $V = W_1$.

We seek to find the projections of the standard vectors x_ω of W to V. Consider the three-dimensional subspace U of W spanned by the orbit sums, i.e.,

$$U = \langle w_1, w_2, w_3 \rangle,$$

where

$$w_1 = x_\omega, w_2 = \sum_{\alpha \in N_\Gamma(\omega)} x_\alpha$$

and

$$w_3 = \sum_{\alpha \in N'_\Gamma(\omega)} x_\alpha.$$

Let A be the adjacency matrix of Γ. From § 3.2, the action of A on the subspace U has matrix

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 416 & 100 & 96 \\ 0 & 315 & 320 \end{bmatrix}.$$

Since the matrix C has constant column sum, it is trivial to check that the row vector (1 1 1) is an eigenvector, with corresponding eigenvalue 416. The other eigenvalues of this matrix are -16 and 20, all with multiplicity one. Thus, the subspace U intersects V in a one-dimensional space. We find an eigenvector corresponding to the 20-eigenvalue. Such an eigenvector is $(1\ \frac{5}{104}\ \ -\frac{1}{65})$, *i.e.*, the vector $(1, \xi_1, \xi_2)$ of § 3.1 is $(1\ \frac{5}{104}\ \ -\frac{1}{65})$. Following the notation of Chapter 3, we write v_ω for the projection of x_ω to the 780-space V. We get the following result about the projections of standard vectors in W to V.

Proposition 5.1.1. *The projection v_ω of x_ω to V has coefficients $c_0 = \frac{130}{297}$ on ω, $c_1 = \frac{25}{1188}$ on neighbours $N_\Gamma(\omega)$ of ω and $c_2 = -\frac{2}{297}$ on non-neighbours $N'_\Gamma(\omega)$ of ω.*

Proof. Apply Proposition 3.2.1 with $m = 780, \nu = 1782, \xi_1 = \frac{5}{104}$ and $\xi_2 = -\frac{1}{65}$. □

5.2 A basis of algebra products

In this section we choose a basis for the space of algebra products on the 780-dimensional space V for the Suzuki group. The following is the main result of this section.

Theorem 5.2.1. *Let V be the 780-dimensional subspace V of W and $f : V \vee V \to V$ be the Norton product. Then f is nonzero on V. Furthermore, if $\pi : W \to V$ is the projection of W onto V, then the map $g : V \vee V \to V$ defined on $X = \{x_\alpha\}$ by*

$$g(x_\omega, x_\alpha) = \begin{cases} \pi\left(\sum_{\theta \in N_\Gamma(\omega) \cap N_\Gamma(\alpha)} x_\theta\right) & \text{if } \omega \sim \alpha, \\ 0_V & \text{otherwise} \end{cases}$$

and extended linearly, is nonzero. Additionally, $\{f, g\}$ forms a basis for the space $L(V \vee V, V)$ of commutative algebra products on V.

We prove the theorem in a series of lemmas. First we show that the Norton algebra product f is not identically zero on V.

Remark 5.2.2. Since the number of common neighbours λ a pair of adjacent vertices has is nonzero for the Suzuki graph Γ, it is clear that g is not identically zero.

Lemma 5.2.3. *The Norton product f is nonzero on V.*

Proof. Fix $\omega \in \Omega$, and let v_ω be the projection of x_ω to V. Set $\nu = 1782, k = 416, \lambda = 100, \mu = 96, c_0 = \frac{130}{297}, c_1 = \frac{25}{1188}$ and $c_2 = -\frac{2}{297}$. Then an application of Lemma 3.3.1 yields

$$\begin{aligned} f(v_\omega, v_\omega) = v_\omega^2 &= \frac{5135}{58806} x_\omega + \frac{1975}{470448} \sum_{\alpha \sim \omega} x_\alpha - \frac{79}{58806} \sum_{\eta \not\sim \omega} x_\eta \\ &\neq 0_V. \end{aligned}$$

□

We also discuss how the projections of a pair of adjacent vertices and a pair of non-adjacent vertices multiply. We start with a pair of adjacent vertices. Let $\alpha \in \Omega$ be adjacent with ω. We write

$$v_\omega = c_0 x_\omega + c_1 x_\alpha + c_1 \sum_{\substack{\kappa \sim \omega \\ \kappa \neq \alpha}} x_\kappa + c_2 \sum_{\eta \not\sim \omega} x_\eta,$$

$$v_\alpha = c_1 x_\omega + c_0 x_\alpha + c_1 \sum_{\substack{\alpha' \sim \alpha \\ \alpha' \neq \omega}} x_{\alpha'} + c_2 \sum_{\eta' \not\sim \omega} x'_\eta.$$

The pointwise product is

$$v_\omega * v_\alpha = c_0 c_1 x_\omega + c_1^2 \sum_\theta x_\theta + c_0 c_1 x_\alpha + c_1 c_2 \sum_\xi x_\xi + c_1 c_2 \sum_\kappa x_\kappa + c_2^2 \sum_\gamma x_\gamma,$$

where $\theta \in N_\Gamma(\omega) \cap N_\Gamma(\alpha)$; $\xi \in N_\Gamma(\omega) \cap N'_\Gamma(\alpha)$; κ ranges over $N'_\Gamma(\omega) \cap N_\Gamma(\alpha)$, and γ ranges over $N'_\Gamma(\omega) \cap N'_\Gamma(\alpha)$, the set of mutual non-neighbours of ω and α. We use the diagram below to express the projections of $\omega, \alpha, \sum_\theta x_\theta, \sum_\xi x_\xi$, $\sum_\kappa x_\kappa$ and $\sum_\gamma x_\gamma$ in terms of each of these vectors.

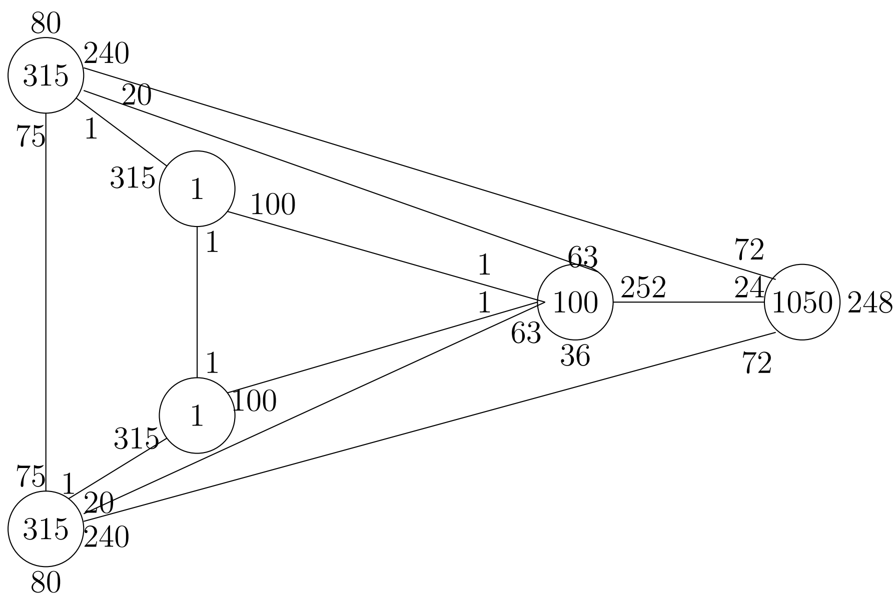

Figure 5.2: The diagram for the Suzuki graph relative to an adjacent pair

We have

$$\begin{aligned} v_\omega = \pi(x_\omega) &= c_0 x_w + c_1 x_\alpha + c_1 \sum_\theta x_\theta + c_1 \sum_\xi x_\xi + c_2 \sum_\kappa x_\kappa + c_2 \sum_\gamma x_\gamma \\ &= \frac{130}{297} x_w + \frac{25}{1188} x_\alpha + \frac{25}{1188} \sum_\theta x_\theta + \frac{25}{1188} \sum_\xi x_\xi - \frac{2}{297} \sum_\kappa x_\kappa \\ &\quad - \frac{2}{297} \sum_\gamma x_\gamma; \end{aligned} \tag{5.1}$$

$$
\begin{aligned}
v_\alpha = \pi(x_\alpha) &= c_1 x_\omega + c_0 x_\alpha + c_1 \sum_\theta x_\theta + c_2 \sum_\xi x_\xi + c_1 \sum_\kappa x_\kappa + c_2 \sum_\gamma x_\gamma \\
&= \frac{25}{1188} x_\omega + \frac{130}{297} x_\alpha + \frac{25}{1188} \sum_\theta x_\theta - \frac{2}{297} \sum_\xi x_\xi + \frac{25}{1188} \sum_\kappa x_\kappa \\
&\quad - \frac{2}{297} \sum_\gamma x_\gamma; \qquad (5.2)
\end{aligned}
$$

$$
\begin{aligned}
\pi\left(\sum_\theta x_\theta\right) &= 100c_1 x_\omega + 100 c_1 x_\alpha + (c_0 + 36c_1 + (100 - 1 - 36)c_2) \sum_\theta x_\theta \\
&\quad + (20c_1 + 80c_2) \sum_\xi x_\xi + (20c_1 + 80c_2) \sum_\kappa x_\kappa \\
&\quad + (24c_1 + (100 - 24)c_2) \sum_\gamma x_\gamma \\
&= \frac{625}{297} x_\omega + \frac{625}{297} x_\alpha + \frac{229}{297} \sum_\theta x_\theta - \frac{35}{297} \sum_\xi x_\xi - \frac{35}{297} \sum_\kappa x_\kappa \\
&\quad - \frac{2}{297} \sum_\gamma x_\gamma; \qquad (5.3)
\end{aligned}
$$

$$
\begin{aligned}
\pi\left(\sum_\xi x_\xi\right) &= 315c_1 x_\omega + 315 c_2 x_\alpha + (63c_1 + (315 - 63)c_2) \sum_\theta x_\theta \\
&\quad + (c_0 + 80_c 1 + (315 - 1 - 80)c_2) \sum_\xi x_\xi + (75c_1 + (315 - 75)c_2) \sum_\kappa x_\kappa \\
&\quad + (72c_1 + (315 - 72)c_2) \sum_\gamma x_\gamma \\
&= \frac{875}{132} x_\omega - \frac{70}{33} x_\alpha - \frac{49}{132} \sum_\theta x_\theta + \frac{6}{11} \sum_\xi x_\xi - \frac{5}{132} \sum_\kappa x_\kappa - \frac{4}{33} \sum_\gamma x_\gamma \qquad (5.4)
\end{aligned}
$$

$$
\begin{aligned}
\pi\left(\sum_\kappa x_\kappa\right) &= 315c_2x_\omega + 315c_1x_\alpha + (63c_1 + (315-63)c_2)\sum_\theta x_\theta \\
&\quad +(75c_1 + (315-75)c_2)\sum_\xi x_\xi \\
&\quad +(c_0 + 80c_1 + (315-1-80)c_2)\sum_\kappa x_\kappa \\
&\quad +(72c_1 + (315-72)c_2)\sum_\gamma x_\gamma \\
&= -\frac{70}{33}x_\omega + \frac{875}{132}x_\alpha - \frac{49}{132}\sum_\theta x_\theta - \frac{5}{132}\sum_\xi x_\xi + \frac{6}{11}\sum_\kappa x_\kappa \\
&\quad -\frac{4}{33}\sum_\gamma x_\gamma; \qquad (5.5)
\end{aligned}
$$

$$
\begin{aligned}
\pi\left(\sum_\gamma x_\gamma\right) &= 1050c_2x_\omega + 1050c_2x_\alpha + (252c_1 + (1050-252)c_2)\sum_\theta x_\theta \\
&\quad +(240c_1 + (1050-240)c_2)\sum_\xi x_\xi \\
&\quad +(240 + (1050-240)c_2)\sum_\kappa x_\kappa \\
&\quad +(c_0 + 248c_1 + (1050-1-248)c_2)\sum_\gamma x_\gamma \\
&= -\frac{700}{99}x_\omega - \frac{700}{99}x_\alpha - \frac{7}{99}\sum_\theta x_\theta - \frac{40}{99}\sum_\xi x_\xi - \frac{40}{99}\sum_\kappa x_\kappa \\
&\quad +\frac{26}{99}\sum_\gamma x_\gamma. \qquad (5.6)
\end{aligned}
$$

Projecting, the Norton product becomes

$$
\begin{aligned}
v_\omega v_\alpha &= c_0c_1v_\omega + c_0c_1v_\alpha + c_1^2\pi\left(\sum_\theta x_\theta\right) + c_1c_2\pi\left(\sum_\xi x_\xi\right) + c_1c_2\pi\left(\sum_\kappa x_\kappa\right) \\
&\quad +c_2^2\pi\left(\sum_\gamma x_\gamma\right).
\end{aligned}
$$

Substituting the expressions for the various projections given in Equations (5.1) through (5.6) in the Norton product, we have

$$\begin{aligned}
v_\omega v_\alpha &= \left(c_0^2c_1 + c_0c_1^2 + \frac{625}{297}c_1^2 + 315c_1^2c_2 - \frac{70}{33}c_1c_2 - \frac{700}{99}c_2^2\right)x_\omega \\
&+ \left(c_0c_1^2 + c_0^2c_1 + \frac{625}{297}c_1^2 - \frac{70}{33}c_1c_2 + \frac{875}{132}c_1c_2 - \frac{700}{99}c_2^2x_\alpha\right)x_\alpha \\
&+ \left(c_0^2c_1 + c_0c_1^2 + \frac{229}{297}c_1^2 - 2\left(\frac{49}{132}\right)c_1c_2 - \frac{7}{99}c_2^2\right)\sum_\theta x_\theta \\
&+ \left(c_0c_1^2 + c_0c_1c_2 - \frac{35}{297}c_1^2 + \frac{6}{11}c_1c_2 - \frac{5}{132}c_1c_2 - \frac{40}{99}c_2^2\right)\sum_\xi x_\xi \\
&+ \left(c_0c_1c_2 + c_0c_1^2 - \frac{35}{297}c_1^2 - \frac{5}{132}c_1c_2 + \frac{6}{11}c_1c_2 - \frac{40}{99}c_2^2\right)\sum_\kappa x_\kappa \\
&+ \left(2c_0c_1c_2 - \frac{2}{297}c_1^2 - 2\left(\frac{4}{33}\right)c_1c_2 + \frac{26}{99}c_2^2\right)\sum_\gamma x_\gamma \\
&= \frac{1975}{470448}x_\omega + \frac{1975}{470448}x_\alpha + \frac{391}{470448}\sum_\theta x_\theta - \frac{5}{470448}\sum_\xi x_\xi \\
&\quad -\frac{5}{470448}\sum_\kappa x_\kappa - \frac{19}{235224}\sum_\gamma x_\gamma.
\end{aligned}$$

We also discuss how a pair of non-adjacent vertices multiplies. In the ensuing discussion, we fix ω and consider α which is non-adjacent with ω. We have the diagram shown in Figure 5.3. We note that the group G does not act transitively on the set of edges $\epsilon\delta$ with $\epsilon \in N'_\Gamma(\omega) \cap N'_\Gamma(\alpha)$, the set of mutual non-neighbours of ω and α. We split the set of mutual non-neighbours into two parts according to the orbits of edges; with one part having cardinality 20 and the other having cardinality 1024.
We will write

$$v_\omega = c_0x_\omega + c_2x_\alpha + c_1\sum_\theta x_\theta + c_1\sum_\xi x_\xi + c_2\sum_\kappa x_\kappa + c_2\sum_\gamma x_\gamma + c_2\sum_{\gamma'} x_{\gamma'},$$

$$v_\alpha = c_2x_\omega + c_1x_\alpha + c_1\sum_\theta x_\theta + c_2\sum_\xi x_\xi + c_1\sum_\kappa x_\kappa + c_2\sum_\gamma x_\gamma + c_2\sum_{\gamma'} x_{\gamma'},$$

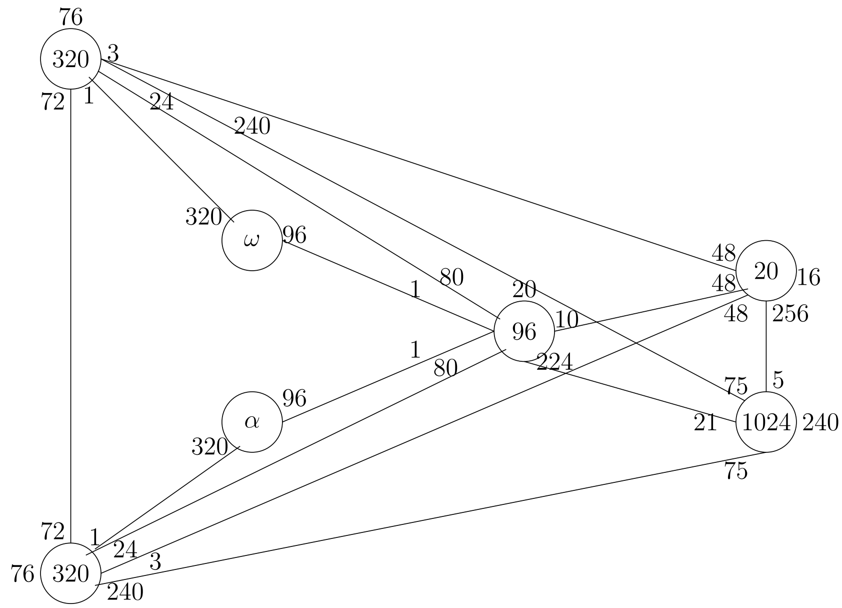

Figure 5.3: Diagram for a pair of non-adjacent vertices

where θ ranges over the set $N_\Gamma(\omega) \cap N_\Gamma(\alpha)$ of common neighbours of ω and $\alpha; \xi$ ranges over the neighbours of ω which are not neighbours of α; $\kappa \in N'_\Gamma(\omega) \cap N_\Gamma(\alpha)$, the neighbours of ω which are non-adjacent with ω, and γ, γ' range over the subsets of $N'_\Gamma(\omega) \cap N'_\Gamma(\alpha)$ of sizes 1024 and 20, respectively. The projections of the vectors $\sum x_\theta, \sum x_\xi, \sum x_\kappa, \sum x_\gamma$ and $\sum x_{\gamma'}$ are given below.

We have:

$$
\begin{aligned}
\pi\left(\sum_\theta x_\theta\right) &= 96c_1x_\omega + 96c_1x_\alpha + (c_0 + 20c_1 + (96-1-20)c_2)\sum_\theta x_\theta \\
&\quad +(24c_1 + (96-24)c_2)\sum_\xi x_\xi + (24c_1 + (96-24)c_2)\sum_\kappa x_\kappa \\
&\quad +(21c_1 + (96-21)c_2)\sum_\gamma x_\gamma + (48c_1 + (96-48)c_2)\sum_\gamma x_{\gamma'} \\
&= \frac{200}{99}x_\omega + \frac{200}{99}x_\alpha + \frac{35}{99}\sum_\theta x_\theta + \frac{2}{99}\sum_\xi x_\xi + \frac{2}{99}\sum_\kappa x_\kappa \\
&\quad -\frac{25}{396}\sum_\gamma x_\gamma + \frac{68}{99}\sum_{\gamma'} x_{\gamma'}, \qquad (5.7)
\end{aligned}
$$

$$
\begin{aligned}
\pi\left(\sum_\xi x_\xi\right) &= 320c_1x_\omega + 320c_2x_\alpha + (80c_1 + (320-80)c_2)\sum_\theta x_\theta \\
&\quad +(c_0 + 76c_1 + (320-1-76)c_2)\sum_\xi x_\xi \\
&\quad +(72c_1 + (320-72)c_2)\sum_\kappa x_\kappa + (75c_1 + (320-75)c_+2)\sum_\gamma x_\gamma \\
&\quad +(48c_1 + (320-48)c_2)\sum_{\gamma'} x_{\gamma'} \\
&= \frac{200}{297}x_\omega - \frac{640}{297}x_\alpha + \frac{20}{297}\sum_\theta x_\theta + \frac{119}{297}\sum_\xi x_\xi - \frac{46}{297}\sum_\kappa x_\kappa \\
&\quad -\frac{85}{1188}\sum_\gamma x_\gamma - \frac{244}{297}\sum_{\gamma'} x_\gamma, \qquad (5.8)
\end{aligned}
$$

$$
\begin{aligned}
\pi\left(\sum_{\kappa} x_{\kappa}\right) &= 320c_2x_\omega + 320c_1x_\alpha + (80c_1 + (320-80)c_2)\sum_{\theta} x_\theta \\
&+ (72c_1 + (320-72)c_2)\sum_{\xi} x_\xi \\
&+ (c_0 + 76c_1 + (320-76-1)c_2)\sum_{\kappa} x_\kappa + (75c_1 + (350-75)c_2)\sum_{\gamma} x_\gamma \\
&+ (48c_1 + (320-48)c_2)\sum_{\gamma'} x_{\gamma'} \\
&= -\frac{640}{297}x_\omega + \frac{2000}{297}x_\alpha + \frac{20}{297}\sum_{\theta} x_\theta - \frac{46}{297}\sum_{\xi} x_\xi + \frac{119}{297}\sum_{\kappa} x_\kappa \\
&\quad -\frac{85}{1188}\sum_{\gamma} x_\gamma - \frac{244}{297}\sum_{\gamma'} x_{\gamma'}; \qquad (5.9)
\end{aligned}
$$

$$
\begin{aligned}
\pi\left(\sum_{\gamma} x_{\gamma}\right) &= 1024c_2x_\omega + 1024c_2x_\alpha + (224c_1 + (1024-224)c_2)\sum_{\theta} x_\theta \\
&\quad +(240c_1 + (1024-240)c_2)\sum_{\xi} x_\xi + (240c_1 + (1024-240)c_2)\sum_{\kappa} x_\kappa \\
&\quad +(c_0 + 240c_1 + (1024-1-240)c_2)\sum_{\gamma} x_\gamma \\
&\quad +(256c_1 + (1024-256)c_2)\sum_{\gamma'} x_{\gamma'} \\
&= -\frac{2048}{297}(x_\omega + x_\alpha) - \frac{200}{297}\sum_{\theta} x_\theta - \frac{68}{297}\left(\sum_{\xi} x_\xi + \sum_{\kappa} x_\kappa\right) \\
&\quad +\frac{64}{297}\left(\sum_{\gamma} x_\gamma + \sum_{\gamma'} x_{\gamma'}\right); \qquad (5.10)
\end{aligned}
$$

$$\begin{aligned}
\pi\left(\sum_{\gamma'} x_{\gamma'}\right) &= 20c_2(x_\omega + x_\alpha) + (10c_1 + (20-10)c_2)\sum_{\theta} x_\theta \\
&\quad +(3c_1 + (20-3)c_2)\sum_{\xi} x_\xi \\
&\quad +(3c_1 + (20-3)c_2)\sum_{\kappa} x_\kappa + (5c_1 + (20-5)c_2)\sum_{\gamma} x_\gamma \\
&\quad +(c_0 + 16c_1 + (20-1-16)c_2)\sum_{\gamma'} x_{\gamma'} \\
&= -\frac{40}{297}(x_\omega + x_\alpha) + \frac{85}{594}\sum_{\theta} x_\theta - \frac{61}{1188}\left(\sum_{\xi} x_\xi + \sum_{\kappa} x_\kappa\right) \\
&\quad +\frac{5}{1188}\sum_{\gamma} x_\gamma + \frac{224}{297}\sum_{\gamma'} x_{\gamma'}. \qquad (5.11)
\end{aligned}$$

The pointwise product is

$$\begin{aligned}
v_\omega * v_\alpha &= c_0c_2x_\omega + c_0c_2x_\alpha + c_1^2\sum_{\theta} x_\theta + c_1c_2\sum_{\xi} x_\xi + c_1c_2\sum_{\kappa} x_\kappa \\
&\quad +c_2^2\left(\sum_{\gamma} x_\gamma + \sum_{\gamma'} x_{\gamma'}\right). \qquad (5.12)
\end{aligned}$$

Projecting the point-wise product to V, we have the Norton product is

$$\begin{aligned}
v_\omega v_\alpha &= c_0c_2v_w + c_0c_2v_\alpha + c_1^2\pi\left(\sum_{\theta} x_\theta\right) + c_1c_2\pi\left(\sum_{\xi} x_\xi\right) + c_1c_2\pi\left(\sum_{\kappa} x_\kappa\right) \\
&\quad +c_2^2\pi\left(\sum_{\gamma} x_\gamma + \sum_{\gamma'} x_{\gamma'}\right). \qquad (5.13)
\end{aligned}$$

Substituting the expressions for v_ω, v_α as well as the other projections given in Equations (5.7) through (5.11) into Equation (5.13), we have

$$\begin{aligned}
v_\omega v_\alpha &= -\frac{79}{58806}(x_\omega + x_\alpha) - \frac{5}{470448}\sum_{\theta} x_\theta - \frac{19}{235224}\left(\sum_{\xi} x_\xi + \sum_{\kappa} x_\kappa\right) \\
&\quad +\frac{79}{1881792}\sum_{\gamma} x_\gamma + \frac{73}{117612}\sum_{\gamma'} x_{\gamma'}.
\end{aligned}$$

We next show that the product g is independent with the Norton product f. Before we discuss this result, we will recast the product in a group theoretic setting. Let ω, α be adjacent vertices. Set $H = G_{\{\omega,\alpha\}}$, the set-wise stabiliser of the set in G. Then H has four orbits in its action on Ω, namely, $\{\omega, \alpha\}, N_\Gamma(\omega) \cap N_\Gamma(\alpha), (N_\Gamma(\omega) \cap N'_\Gamma(\alpha)) \cup (N'_\Gamma(\omega) \cap N_\Gamma(\alpha))$ and $N'_\Gamma(\omega) \cap N'_\Gamma(\alpha)$ of lengths $2, \lambda, 2(k - \lambda - 1)$ and $(n - 2k + \lambda)$, respectively. Here we assume that $\lambda \neq 0$, otherwise the product g is identically zero, as is the case when $G = HS{:}2$. For our case here $\lambda = 100$ so that our product is nonzero and the orbit lengths are $2, 100, 630$ and 1050. We will label the orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ and $\mathcal{O}_4$, respectively. Then we can rewrite the product g as

$$g(x_\alpha, x_\beta) = \begin{cases} \pi\left(\sum_{\theta \in \Omega_2} x_\theta\right) & \text{if } (\alpha, \beta) \in \mathcal{O}_2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.14}$$

It can easily be proven, from the definition that $G_{\{\alpha,\beta\}^h} = (G_{\{\alpha,\beta\}})^h, h \in G$. This fact, together with the fact that the orbits of $G_{\{\alpha,\beta\}^h}$ are permutations of the orbits of $G_{\{\alpha,\beta\}}$ by h, shows that g commutes with the group action. We will compute the square of the projection v_ω of a vertex ω under the second product. For convenience, we write

$$v_\omega = c_0 x_\omega + c_1 \sum_{\alpha \sim \omega} x_\alpha + c_2 \sum_{\gamma \nsim \omega} x_\gamma.$$

Furthermore, we set

$$w_1 = x_\omega, w_2 = \sum_{\alpha \sim \omega} x_\alpha, w_3 = \sum_{\gamma \nsim \omega} x_\gamma.$$

we have $v_\omega^2 = c_0^2 w_1^2 + 2c_0c_1w_1w_2 + 2c_0c_2w_1w_3 + c_1^2 w_2 + 2c_1c_2w_2w_3 + c_2^2 w_3^2$. Since the graph is simple, $w_1^2 = 0_W$. If we pick a neighbour α of ω, all the common neighbours of ω and α lie in the orbit of length 416 consisting of the neighbours of ω, so that $w_1 w_2 = 416\left(\frac{100}{416}\right) = 100 w_2$. A vertex $\gamma \in N'_\Gamma$ is non-adjacent with ω so that $w_1 w_3 = 0_W$. There are $416 \times 100 = 41600$ ordered pairs $\alpha\alpha'$ with $\alpha, \alpha' \in N_\Gamma(\omega)$. The end points α, α' of such an ordered pair are adjacent with ω, have 36 common neighbours in $N_\Gamma(\omega)$ and 63 in $N'_\Gamma(\omega)$. Thus,

$$\begin{aligned} w_2^2 &= 41600\left[1 \cdot w_1 + \frac{36}{41} w_2 + \frac{63}{1365} w_3\right] \\ &= 41600 w_1 + 3600 w_2 + 1920 w_3. \end{aligned}$$

From the diagram given in Figure 5.1, a vertex in $N_\gamma(\omega)$ has 315 neighbours in $N'_\Gamma(\omega)$, so there are $416 \times 315 = 131040$ ordered pairs $\alpha\gamma$ with $\alpha \in N_\Gamma(\omega)$ and $\gamma \in N'_\Gamma(\omega)$. Of the common neighbours of the endpoints of a typical adjacent pair $\alpha\gamma, 20$ lie in $N_\Gamma(\omega)$ and the rest lie in N'_Γ. Thus,

$$\begin{aligned} w_2 w_3 &= 131040 \left[\frac{20}{416} w_2 + \frac{80}{1365} w_3 \right] \\ &= 6300 w_2 + 7680 w_3. \end{aligned}$$

From the same diagram, a vertex $\gamma \in N'_\Gamma(\omega)$ has 320 neighbours in $N'_\Gamma(\omega)$, therefore there are $1365 \times 320 = 436800$ ordered pairs both of whose end points lie in $N'_\Gamma(\omega)$. If $\gamma\gamma'$ is such a pair, then γ and γ' have 24 common neighbours in $N_\Gamma(\omega)$ and 76 common neighbours in $N'_\Gamma(\omega)$. We conclude that

$$\begin{aligned} w_3^2 &= 436800 \left[\frac{24}{416} w_2 + \frac{76}{1365} w_3 \right] \\ &= 25200 w_2 + 24320 w_3. \end{aligned}$$

Thus, before projecting,

$$\begin{aligned} v_\omega^{*2} &= c_0^2 0_W + 2c_0c_1(100w_2 + 0_W) + 2c_1c_2(6300w_2 + 7680w_3) \\ &\quad + c_1^2(41600w_1 + 3600w_2 + 1920w_3) + c_2^2(25200w_2 + 24320w_3) \\ &= 41600c_1^2 w_1 + (200c_0c_1 + 12600c_1c_2 + 3600c_1^2 + 25200c_2^2)w_2 \\ &\quad (15360c_1c_2 + 1920c_1^2 + 24320c_2^2)w_3 \\ &= \frac{1625000}{88209} w_1 + \frac{246425}{88209} w_2 - \frac{19720}{88209} w_3. \end{aligned}$$

Recall that

$$\begin{aligned} \pi(w_2) &= 416c_1w_1 + (c_0 + 100c_1 + (416 - 1 - 100)c_2)w_2 \\ &\quad + (96c_1 + (416 - 96c_2))w_3 \\ &= \frac{2600}{297} w_1 + \frac{125}{297} w_2 - \frac{40}{297} w_3, \end{aligned}$$

and

$$\begin{aligned} \pi(w_3) &= (\nu - k - 1)c_2 x_\omega + ((k - \lambda - 1)c_1 + (\nu - 2k + \lambda)c_2)w_2 \\ &+ (c_0 + (k - \mu)c_1 + (\nu - 2k + \mu - 2)c_2)w_3 \\ &= -\frac{910}{99} w_1 - \frac{175}{396} w_2 + \frac{14}{99} w_3, \end{aligned}$$

from the proof of Lemma 3.3.1. Projecting, we have,

$$\begin{aligned}
v_\omega^2 &= \frac{1625000}{88209}\pi(w_1) + \frac{246425}{88209}\pi(w_2) - \frac{19720}{88209}\pi(w_3) \\
&= \left(\frac{1625000}{88209}\left(\frac{130}{297}\right) + \left(\frac{246425}{88209}\right)\left(\frac{2600}{297}\right) - \left(\frac{19720}{88209}\right)\left(-\frac{910}{99}\right)\right) w_1 \\
&\quad + \left(\frac{1625000}{88209}\left(\frac{25}{1188}\right) + \left(\frac{246425}{88209}\right)\left(\frac{125}{297}\right) - \left(\frac{19720}{88209}\right)\left(-\frac{175}{396}\right)\right) w_2 \\
&\quad + \left(\frac{1625000}{88209}\left(-\frac{2}{297}\right) + \left(\frac{246425}{88209}\right)\left(-\frac{40}{297}\right) - \left(\frac{19720}{88209}\right)\left(\frac{14}{99}\right)\right) w_3 \\
&= \frac{1016600}{29403}w_1 + \frac{48875}{29403}w_2 - \frac{15640}{29403}w_3.
\end{aligned}$$

This product, is however, a multiple of the corresponding product under the Norton product. From the proof of Lemma 5.2.3, the square of v_ω under the Norton product is $\frac{5135}{58806}w_1 + \frac{1975}{470448}w_2 - \frac{79}{58806}w_3$. The following calculation establishes the linear independence of the products f and g.

Recall the diagram for an adjacent pair given in Figure 5.2. For convenience, we use the following notation for the various vectors $x_\omega, x_\alpha, \sum_\kappa x_\kappa, \sum_\gamma x_\gamma, \sum_\xi x_\xi$ and $\sum_\theta x_\theta$. Each of these vectors is denoted by $w_i, i = 1, 2, \ldots, 6$, in order. Then we have

$$v_\omega = c_0w_1 + c_1w_2 + c_2w_3 + c_2w_4 + c_1w_5 + c_1w_6;$$

$$v_\alpha = c_1w_1 + c_0w_2 + c_1w_3 + c_2w_4 + c_2w_5 + c_1w_6$$

and consequently,

$$\begin{aligned}
v_\omega v_\alpha &= c_0c_1w_1^2 + (c_0^2 + c_1^2)w_1w_2 + (c_0c_1 + c_1c_2)w_1w_3 + (c_0c_2 + c_1c_2)w_1w_4 \\
&+ (c_0c_2 + c_1^2)w_1w_5 + (c_0c_1 + c_1^2)w_1w_6 + c_0c_1w_2^2 + (c_0c_2 + c_1)^2 w_2w_3 \\
&+ (c_0c_2 + c_1c_2)w_2w_4 + (c_0c_1 + c_1c_2)w_2w_5 + (c_0c_1 + c_1^2)w_2w_6 + c_1c_2w_3^2 \\
&+ (c_1c_2 + c_2^2)w_3w_4 + (c_1^2 + c_2^2)w_3w_5 + (c_1^2 + c_1c_2)w_3w_6 + c_2^2w_4^2 \\
&+ (c_1c_2 + c_2^2)w_4w_5 + 2c_1c_2w_4w_6 + c_1c_2w_5^2 + (c_1^2 + c_1c_2)w_5w \\
&\quad + c_1^2w_6^2. \qquad (5.15)
\end{aligned}$$

For the remainder of the section, we adopt the following notation. Let $O_1 = \{\omega\}, O_2 = \{\alpha\}, O_3 = N'_\Gamma(\omega) \cap N_\gamma(\alpha), O_4 = N'_\Gamma(\omega) \cap N'_\Gamma(\alpha)$,

$O_5 = N_\Gamma(\omega) \cap N'_\Gamma(\alpha)$ and $O_6 = N_\Gamma(\omega) \cap N_\Gamma(\alpha)$. Furthermore, we denote the setwise stabiliser of the set O_3 by H. We will give details of how H acts on the sets of edges of type O_iO_j for all i, j in the table below in order to be able to calculate the products of w_i, w_j. From the table, we have that $w_1^2 = w_1w_3 = w_1w_4 = w_2^2 = w_2w_4 = w_2w_5 = 0_W$,

$$w_1w_2 = \frac{1}{100}(100w_6) = w_6;$$

$$\begin{aligned} w_1w_5 &= 315\left(80 \cdot \frac{1}{315}w_5 + 20 \cdot \frac{1}{100}w_6\right) \\ &= 80w_5 + 63w_6; \end{aligned}$$

$$\begin{aligned} w_1w_6 &= 100\left(w_2 + 63 \cdot \frac{1}{315}w_5 + 36 \cdot \frac{1}{100}w_6\right) \\ &= 100w_2 + 20w_5 + 35w_6; \end{aligned}$$

$$\begin{aligned} w_2w_3 &= 315\left(80 \cdot \frac{1}{315}w_3 + 20 \cdot \frac{1}{100}w_6\right) \\ &= 80w_3 + 63w_6; \end{aligned}$$

$$\begin{aligned} w_2w_6 &= 100\left(w_1 + 63 \cdot \frac{1}{315}w_3 + 36 \cdot \frac{1}{100}w_6\right) \\ &= 100w_1 + 20w_3 + 36w_6; \end{aligned}$$

$$\begin{aligned} w_3^2 &= 25200\left(w_2 + 30 \cdot \frac{1}{315}w_3 + 46 \cdot \frac{1}{1050}w_4 + 17 \cdot \frac{1}{315}w_5 + 6 \cdot \frac{1}{100}w_6\right) \\ &= 25200w_2 + 2400w_3 + 1104w_4 + 1360w_5 + 1512w_6; \end{aligned}$$

$$\begin{aligned} w_3w_4 &= 25200\left(14 \cdot \frac{1}{315}w_3 + 62 \cdot \frac{1}{1050}w_4 + 18 \cdot \frac{1}{315}w_5 + 6 \cdot \frac{1}{100}\right) \\ &\quad +50400\left(16 \cdot \frac{1}{315}w_3 + 60 \cdot \frac{1}{1050}w_4 + 20 \cdot \frac{1}{315}w_5 + 4 \cdot \frac{1}{100}w_6\right) \\ &= 3680w_3 + 4368w_4 + 4640w_5 + 3528w_6; \end{aligned}$$

Type	Orbit length	Distribution (common neighbours)					
		O_1	O_2	O_3	O_4	O_5	O_6
O_1O_1	0						
O_1O_2	1						100
O_1O_3	0						
O_1O_4	0						
O_1O_5	315					80	20
O_1O_6	100		1			63	36
O_2O_2	0						
O_2O_3	315			80			20
O_2O_4	0						
O_2O_5	0						
O_2O_6	100	1		63			36
O_3O_3	25200		1	30	46	17	6
O_3O_4	25200			14	62	18	6
	50400			16	60	20	4
O_3O_5	315				80		20
	3150			8	72	8	12
	20160			20	60	20	
O_3O_6	6300		1	24	56	7	12
O_4O_4	16800			12	64	12	12
	16800			12	64	12	12
	25200			22	54	22	2
	201600			18	58	18	6
O_4O_5	25200			18	62	14	6
	50400			20	60	16	4
O_4O_6	25200			14	66	14	6
O_5O_5	25200	1		17	46	30	6
O_5O_6	6300	1		7	56	24	12
O_6O_6	3600	1	1	21	42	21	14

Table 5.2: Distribution of common neighbours of end points of edges

$$\begin{aligned} w_3w_5 &= \frac{1}{315}(3150 \cdot 8 + 20160 \cdot 20)w_3 + \frac{1}{1050}(315 \cdot 80 + 3150 \cdot 72 + 20160 \cdot 60)w_4 \\ &\quad + \frac{1}{315}\left(3150 \cdot 8 + 20160 \cdot 20\right) w_5 + \frac{1}{100}(315 \cdot 20 + 3150 \cdot 12)w_6 \\ &= 1360w_3 + 1392w_4 + 1360w_5 + 441w_6; \end{aligned}$$

$$\begin{aligned} w_3 w_6 &= 6300\left(w_2 + 24\cdot\frac{1}{315}w_3 + 56\cdot\frac{1}{1050} + 7\cdot\frac{1}{315}w_5 + 12\cdot\frac{1}{100}\right) \\ &= 6300w_2 + 480w_3 + 336w_4 + 140w_5 + 756w_6; \end{aligned}$$

$$\begin{aligned} w_4^2 &= \frac{1}{315}\left(33600\cdot 12 + 25200\cdot 22 + 201600\cdot 18\right)w_3 \\ &\quad + \frac{1}{1050}\left(33600\cdot 64 + 25200\cdot 54 + 201600\cdot 58\right)w_4 \\ &\quad + \frac{1}{315}\left(33600\cdot 12 + 25200\cdot 22 + 201600\cdot 18\right)w_5 \\ &\quad + \frac{1}{100}\left(33600\cdot 12 + 25200\cdot 2 + 201600\cdot 6\right)w_6 \\ &= 14560w_3 + 14480w_4 + 14560w_5 + 16632w_6; \end{aligned}$$

$$\begin{aligned} w_4 w_5 &= \frac{1}{315}\left(25200\cdot 18 + 50400\cdot 20\right)w_3 + \frac{1}{1050}\left(25200\cdot 62 + 50400\cdot 60\right)w_4 \\ &\quad \frac{1}{315}\left(25200\cdot 14 + 50400\cdot 16\right)w_5 + \frac{1}{100}\left(25200\cdot 6 + 50400\cdot 4\right)w_6 \\ &= 4640w_3 + 4368w_4 + 3680w_5 + 3528w_6; \end{aligned}$$

$$\begin{aligned} w_4 w_6 &= 25200\left(\frac{14}{315}w_3 + \frac{66}{1050}w_4 + \frac{14}{315}w_5 + \frac{6}{100}w_6\right) \\ &= 1120w_3 + 1584w_4 + 1120w_5 + 1512w_6; \end{aligned}$$

$$\begin{aligned} w_5^2 &= 25200\left(w_1 + \frac{17}{315}w_3 + \frac{46}{1050}w_4 + \frac{30}{315}w_5 + \frac{6}{100}w_6\right) \\ &= 25200w_1 + 1360w_3 + 1104w_4 + 2400w_5 + 1512w_6; \end{aligned}$$

$$\begin{aligned} w_5 w_6 &= 6300\left(w_1 + \frac{7}{315}w_3 + \frac{56}{1050}w_4 + \frac{24}{315}w_5 + \frac{12}{100}w_6\right) \\ &= 6300w_1 + 140w_3 + 336w_4 + 480w_5 + 756w_6 \end{aligned}$$

and

$$\begin{aligned} w_6^2 &= 3600\left(w_1 + w_2 + \frac{21}{315}w_3 + \frac{42}{1050}w_4 + \frac{21}{315}w_5 + \frac{14}{100}w_6\right) \\ &= 3600(w_1 + w_2) + 240w_3 + 144w_4 + 240w_5 + 504w_6. \end{aligned}$$

Substituting the various products $w_i w_j$ in Equation (5.15), this simplifies to

$$\begin{aligned} v_\omega v_\alpha &= (100c_0c_1 + 10000c_1^2 + 31500c_1c_2)w_1 + (100c_0c_1 + 10000c_1^2 + 31500c_1c_2)w_2 \\ &\quad +(20c_0c_1 + 80c_0c_2 + 2320c_1^2 + 14940c_1c_2 + 24240c_2^2)w_3 \\ &\quad +(2208c_1^2 + 14784c_1c_2 + 24608c_2^2)w_4 \\ &\quad +(20c_0c_1 + 80c_0c_2 + 2320c_1^2 + 14940c_1c_2 + 24240c_2^2)w_5 \\ &\quad +(c_0^2 + 72c_0c_1 + 126c_0c_2 + 2656c_1^2 + 14616c_1c_2 + 24129c_2^2)w_6 \\ &= \frac{78125}{88209}(w_1 + w_2) - \frac{3715}{88209}w_3 - \frac{118}{88209}w_4 - \frac{3715}{88209}w_5 + \frac{60206}{88209}w_6. \end{aligned}$$

Recall that the projection $\pi(w_i)$ are given in Equations (5.1) through (5.6), with the necessary adjustments. We will recall the projections here for the ease of access to the reader. We have

$$\pi(w_1) = c_0w_1 + c_1w_2 + c_2w_3 + c_2w_4 + c_1w_5 + c_1w_6;$$

$$\pi(w_2) = c_1w_1 + c_0w_2 + c_1w_3 + c_2w_4 + c_2w_5 + c_1w_6;$$

$$\pi(w_3) = -\frac{70}{33}w_1 + \frac{875}{132}w_2 + \frac{6}{11}w_3 - \frac{4}{33}w_4 - \frac{5}{132}w_5 - \frac{49}{132}w_6;$$

$$\pi(w_4) = -\frac{700}{99}(w_1 + w_2) - \frac{40}{99}w_3 + \frac{26}{99}w_4 - \frac{40}{99}w_5 - \frac{7}{99}w_6;$$

$$\pi(w_5) = \frac{875}{132}w_1 - \frac{70}{33}w_2 - \frac{5}{132}w_3 - \frac{4}{33}w_4 + \frac{6}{11}w_5 - \frac{49}{132}w_6$$

and

$$\pi(w_6) = \frac{625}{297}(w_1 + w_2) - \frac{35}{297}w_3 - \frac{2}{297}w_4 - \frac{35}{297}w_5 + \frac{229}{297}w_6.$$

After projecting, we have

$$\begin{aligned}
v_\omega v_\alpha &= \left(\left(\frac{78125}{88209}\right)(c_0+c_1) - \left(\frac{3715}{88209}\right)\left(-\frac{70}{33}\right) - \left(\frac{118}{88209}\right)\left(-\frac{700}{99}\right)\right. \\
&\quad \left. - \left(\frac{3715}{88209}\right)\left(\frac{875}{132}\right) + \left(\frac{60206}{88209}\right)\left(\frac{625}{297}\right)\right) w_1 \\
&\quad + \left(\left(\frac{78125}{88209}\right)(c_0+c_1) - \left(\frac{3715}{88209}\right)\left(\frac{875}{132}\right) - \left(\frac{118}{88209}\right)\left(-\frac{700}{99}\right)\right. \\
&\quad \left. - \left(\frac{3715}{88209}\right)\left(-\frac{70}{33}\right) + \left(\frac{60206}{88209}\right)\left(\frac{625}{297}\right)\right) w_2 \\
&\quad + \left(\left(\frac{78125}{88209}\right)(c_1+c_2) - \left(\frac{3715}{88209}\right)\left(\frac{6}{11}\right) - \left(\frac{118}{88209}\right)\left(-\frac{40}{99}\right)\right. \\
&\quad \left. - \left(\frac{3715}{88209}\right)\left(\frac{-5}{132}\right) + \left(\frac{60206}{88209}\right)\left(-\frac{35}{297}\right)\right) w_3 \\
&\quad + \left(\left(\frac{78125}{8820}\right)(2c_2) - \left(\frac{3715}{8820}\right)\left(-\frac{4}{33}\right) - \left(\frac{118}{8820}\right)\left(\frac{26}{9}\right)\right. \\
&\quad \left. - \left(\frac{3715}{8820}\right)\left(-\frac{4}{33}\right) + \left(\frac{60206}{8820}\right)\left(-\frac{2}{297}\right)\right) w_4 \\
&\quad + \left(\left(\frac{78125}{88209}\right)(c_1+c_2) - \left(\frac{3715}{88209}\right)\left(-\frac{5}{132}\right) - \left(\frac{118}{88209}\right)\left(-\frac{40}{99}\right)\right. \\
&\quad \left. - \left(\frac{3715}{88209}\right)\left(\frac{6}{11}\right) + \left(\frac{60206}{88209}\right)\left(-\frac{35}{297}\right)\right) w_5 \\
&\quad + \left(\left(\frac{78125}{88209}\right)(2c_1) - \left(\frac{3715}{88209}\right)\left(-\frac{49}{132}\right) - \left(\frac{118}{88209}\right)\left(-\frac{7}{99}\right)\right. \\
&\quad \left. - \left(\frac{3715}{88209}\right)\left(-\frac{49}{132}\right) + \left(\frac{60206}{88209}\right)\left(\frac{229}{297}\right)\right) w_6 \\
&= \frac{48875}{29403}(w_1+w_2) - \frac{2605}{29403}w_3 - \frac{196}{29403}w_4 - \frac{2605}{29403}w_5 + \frac{17492}{29403}w_6.
\end{aligned}$$

Lemma 5.2.4. *Let f, g be products defined on the space V of this chapter. Then f and g are linearly independent.*

Proof. It is easy to see that the products

$$-\frac{79}{58806}(x_\omega + x_\alpha) - \frac{5}{470448}\sum_\theta x_\theta - \frac{19}{235224}\left(\sum_\xi x_\xi + \sum_\kappa x_\kappa\right)$$
$$+\frac{79}{1881792}\sum_\gamma x_\gamma + \frac{73}{117612}\sum_{\gamma'} x_{\gamma'} = -\frac{79}{58806}(w_1 + w_2) - \frac{19}{235224}w_3$$
$$+\frac{79}{1881792}w_4 - \frac{19}{235224}w_5 - \frac{5}{470448}w_6$$

and

$$\frac{48875}{29403}(w_1 + w_2) - \frac{2605}{29403}w_3 - \frac{196}{29403}w_4 - \frac{2605}{29403}w_5 + \frac{17492}{29403}w_6$$

of v_ω, v_α where ω, α are adjacent, under the products f and g respectively, are linearly independent. We conclude that the products f and g are linearly independent. □

Proof of Theorem 5.2.1. By Lemma 5.2.4, $\{f, g\}$ is a linearly independent set, and since $\dim(L(V \vee V, V)) = 2$, it is a basis for $L(V \vee V, V)$. □

We note that general algebra products $\phi = \alpha_1 f + \alpha_2 g$ where not all of α_1 and α_2 are zero can be parametrised using the points of a projective line. If $\alpha_2 \neq 0$, identify $\alpha_1 f + \alpha_2 g$ with $\alpha_1\alpha_2^{-1} \in \mathbb{Q}$, and set $\infty := f$.

5.3 Algebras from the class $2A$

In this section we discuss the algebras which have axes fixed by the centraliser of an involution from the class $2A$. Let $z \in 2A$ and set $K := C_G(z)$. Then K is a maximal subgroup of G of order 6635520 and shape $2^{1+6}.U_4(2).2$. The dimension of the space fixed by the centralizer of an involution is one since $1 = (\chi|_K, 1_K)$. The group K has two orbits in its action on Ω, of lengths 54 and 1728 respectively. For ease of reference, we will denote the orbit of length 1728 by O_1 and the orbit of length 54 by O_2. We use the following diagram (Fig 5.4) in what follows.
We let

$$u_1 = \sum_{\omega \in O_1} x_\omega, u_2 = \sum_{\alpha \in O_1} x_\alpha.$$

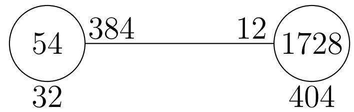

Figure 5.4: The diagram for $Suz{:}2$ relative to $2^{1+6}.U_4(2).2$

Then the action of the adjacency matrix A of the Suzuki graph Γ on the subspace $U = \langle u_1, u_2 \rangle$ has matrix

$$C = \begin{bmatrix} 404 & 384 \\ 12 & 32 \end{bmatrix}.$$

The matrix C has eigenvalues 416 and 20. Therefore, the space U fixed by the group K meets V in a one-dimensional space. We will fix a basis for this space in the next paragraph.

We find an eigenvector corresponding to the 20-eigenvalue. It can easily be seen that $(1\ \ -32)$ is a 20-eigenvector of C. Thus, we will let $a = u_1 - 32u_2$. We find the projections of u_1 and u_2 to V. From the diagram, we have

$$\begin{aligned} \pi(u_1) &= (c_0 + 404c_1 + (1728 - 1 - 404)c_2)\, u_1 + (384c_1 + (1728 - 348)c_2) \\ &= \frac{1}{33}u_1 - \frac{32}{33}u_2, \end{aligned}$$

$$\begin{aligned} \pi(u_2) &= (12c_1 + (54 - 12)c_2)\, u_1 + (c_0 + 32c_1 + (54 - 1 - 32)c_2)\, u_2 \\ &= -\frac{1}{33}u_1 + \frac{32}{33}u_2 = -\pi(u_1). \end{aligned}$$

We find the square of a under the Norton product. Before projecting, the pointwise product is

$$a^{*2} = u_1 + (-32)^2 u_2 = u_1 + 1024u_2.$$

Thus, under the Norton product,

$$\begin{aligned} a^2 &= \pi(u_1) + 1024\pi(u_2) = \pi(u_1) + 1024(-\pi(u_1)) \\ &= -1023\pi(u_1) \\ &= -1023\left(\frac{1}{33}u_1 - \frac{32}{33}u_2\right) \\ &= -31u_1 + 992u_2 = -31(u_1 - 32u_u) = -31a. \end{aligned}$$

Consequently, we scale the Norton product f by $-\frac{1}{31}$ to make the vector a an idempotent. In what follows, we use MAGMA to compute ad_a under the Norton algebra product. The ad_a-eigenvalues are $1, \frac{41}{93}, \frac{10}{31}, \frac{21}{31}, -\frac{1}{31}$, where all the eigenspaces save the $-\frac{1}{31}$-space are even, while the latter splits into an even and odd part. We use MAGMA to investigate how the eigenspaces multiply and find a fusion law. The 780-dimensional algebra is axial, obeying the fusion law shown in Table 5.3.

	1	$\frac{41}{93}$	$\frac{10}{31}$	$\frac{21}{31}$	$-\frac{1}{31}^E$	$-\frac{1}{31}^O$
1	1	$\frac{41}{93}$	$\frac{10}{31}$	$\frac{21}{31}$	$-\frac{1}{31}^E$	$-\frac{1}{31}^O$
$\frac{41}{93}$		$1, \frac{10}{31}, \frac{21}{31}, -\frac{1}{31}^E$	$\frac{41}{93}, -\frac{1}{31}^E$	$\frac{41}{93}, -\frac{1}{31}^E$	$\frac{41}{93}, \frac{10}{31}, \frac{21}{31} - \frac{1}{31}^E$	$-\frac{1}{31}^O$
$\frac{10}{31}$			$1, \frac{10}{31}, \frac{21}{31}$	$\frac{10}{31}, \frac{21}{31}$	$\frac{41}{93}, -\frac{1}{31}^E$	$-\frac{1}{31}^O$
$\frac{21}{31}$				$1, \frac{10}{31}$	$\frac{41}{93}, -\frac{1}{31}^E$	$-\frac{1}{31}^O$
$-\frac{1}{31}^E$					$1, \frac{41}{93}, \frac{10}{31}, \frac{21}{31}, -\frac{1}{31}^E$	$-\frac{1}{31}^O$
$-\frac{1}{31}^O$						$1, \frac{41}{93}, \frac{10}{31}, \frac{21}{31}, -\frac{1}{31}^E$

Table 5.3: The fusion law for the 780-dimensional algebra from the class $2A$ for Suz:2

5.3.1 Fusion laws for the extended algebras

In this section we give fusion laws for extended algebras for different values of λ. We will not give a lot of details as most of this follows closely the work presented for HS. Recall from Chapter 2, §2.5.1 that given an algebra V over a field $\mathbb{F}$ endowed with an inner product $(\cdot,\cdot) : V \times V \to \mathbb{F}$, we can form a unital extension $\tilde{V} := \mathbb{F} \oplus V$ of V by combining the algebra multiplication with the inner product as follows:

$$(\eta + v) \star (\eta' + v') = (\eta\eta' + \gamma(v, v')) + (\eta v' + \eta' v + v * v)\,,$$

where $\eta + v, \eta' + v' \in \tilde{V}, \gamma$ is a structure-controlling parameter in $\tilde{V}$ and $*$ denotes multiplication in V. Furthermore, γ is chosen so that $w = \alpha' + \beta' a$ is an idempotent where $\alpha' = \frac{-\lambda}{1-2\lambda}, \beta' = \frac{1}{1-2\lambda}, \gamma = \lambda(\lambda - 1)$ and λ is the ad_a-eigenvalue chosen to be mapped to the 0-eigenvalue of $\tilde{V}$. Recall also that the inner product $(\cdot, \cdot)$ is scaled so that $(a, a) = 1$.

We begin with the case $\lambda = 1$ and note that the resulting fusion law is Seress. From Theorem 2.5.6 1(b), the eigenvalues of ad_w, where $w = 1 - a$ are as follows: $1, 0, \frac{52}{93}, \frac{21}{31}, \frac{10}{31}, \frac{32}{31}$. The 1-, 0-, $\frac{52}{93}$-, $\frac{21}{31}$- and $\frac{10}{31}$-eigenspaces are entirely even, while the $\frac{32}{31}$-eigenspace splits into an even part $\frac{32}{31}^E$, and an odd part $\frac{32}{31}^O$, of dimensions 342 and 384 respectively. We can use the fusion law shown in Table 5.3 to obtain a fusion law for this extended algebra, or alternatively, a direct computation with MAGMA. We obtain the fusion law illustrated in Table 5.4.

	1	0	$\frac{52}{93}$	$\frac{21}{31}$	$\frac{10}{31}$	$\frac{32}{31}^E$	$\frac{32}{31}^O$
1	1	$\emptyset$	$\frac{52}{93}$	$\frac{21}{31}$	$\frac{10}{31}$	$\frac{32}{31}^E$	$\frac{32}{31}^O$
0		0	$\frac{52}{93}$	$\frac{21}{31}$	$\frac{10}{31}$	$\frac{32}{31}^E$	$\frac{32}{31}^O$
$\frac{52}{93}$			$0, \frac{21}{31}, \frac{10}{31}, \frac{32}{31}^E$	$\frac{52}{93}, \frac{32}{31}^E$	$\frac{52}{93}, \frac{32}{31}^E$	$\frac{52}{93}, \frac{21}{31}, \frac{10}{31}, \frac{32}{31}^E$	$\frac{32}{31}^O$
$\frac{21}{31}$				$0, \frac{21}{31}, \frac{10}{31}$	$\frac{21}{31}, \frac{10}{31}$	$\frac{52}{93}, \frac{32}{31}^E$	$\frac{32}{31}^O$
$\frac{10}{31}$					$0, \frac{21}{31}$	$\frac{52}{93}, \frac{32}{31}^E$	$\frac{32}{31}^O$
$\frac{32}{31}^E$						$0, \frac{52}{93}, \frac{21}{31}, \frac{10}{31}, \frac{32}{31}^E$	$\frac{32}{31}^O$
$\frac{32}{31}^O$							$0, \frac{52}{93}, \frac{21}{31}, \frac{10}{31}, \frac{32}{31}^E$

Table 5.4: Fusion law for the extended algebra with $\lambda = 1$

Next we consider the case $\lambda = \frac{41}{93}$. The eigenvalues for ad_w are $1, 0, -1, 2, -4$, by Theorem 2.5.6 1(d). A MAGMA computation shows that all the eigenspaces save the -4-eigenspace are even. The -4-eigenspace splits into an even part and an odd part of dimensions 342 and 384 respectively. The fusion law is shown it Table 5.5 below.

	1	0	-1	2	-4^E	-4^O
1	$1,0$	$1,0$	-1	2	-4^E	-4^O
0		$0,1,-1,2,-4^E$	$0,-4^E,-1$	$2,0,-4^E$	$0,-1,2,-4^E$	-4^O
-1			$1,0,-1,2$	$-1,2$	$0,-4^E$	-4^O
2				$1,0,-1$	$0,-4^E$	-4^O
-4^E					$1,0,-1,2,-4^E$	-4^O
-4^O						$1,0,-1,2,-4$

Table 5.5: Fusion law for the extended algebra with $\lambda = \frac{41}{93}$

For the case $\lambda = \frac{10}{31}$, the eigenvalues for ad_w are $1, 0, 1/3, -1$, by Theorem 2.5.6 1(c). The 1-eigenspace, the 0-eigenspace and the 1/3-eigenspace are entirely even, while the -1-eigenspace splits into an even part and an odd part of dimension 342 and 384 respectively. The fusion law satisfied by $\tilde{V}$ is shown in Table 5.6

The next extended algebra arises from the extension by unit where the eigenvalue $\lambda = \frac{21}{31}$ is mapped to the 0-eigenvalue of the extended algebra. Theorem 2.5.6 1(c). gives the eigenvalues for the idempotent w as $1, 0, 2/3$ and 2. All the eigenspaces are completely even save for the 2-eigenspace which splits into an even part and an odd part of dimension 342 and 384 respectively. A fusion law satisfied by $\tilde{V}$ is shown in Table 5.7.

For the algebra obtained by setting $\lambda = -\frac{1}{31}$, an application of Theorem 2.5.6 1(d) shows that the eigenvalues for ad_w are $1, 0, 4/9, 1/3, 2/3$. The 0-eigenspace splits into an even part and an odd part of dimension 343 and 384 respectively. Table 5.8 illustrates a fusion law that the multiplication in this extended algebra obeys.

	1	0	1/3	-1^E	-1^O
1	1, 0	1, 0	$1/3, -1^E$	$1/3, -1^E$	-1^O
0		1, 0	$1/3, -1^E$	$1/3, -1^E$	-1^O
1/3			$1, 0, -1^E$	$1, 0, 1/3, -1^E$	-1^O
-1^E				$1, 0, 1/3, -1^E$	-1^O
-1^O					$1, 0, 1/3, -1^E$

Table 5.6: Fusion law for the extended algebra with $\lambda = \frac{10}{31}$

5.3.2 Algebras from $2A$ under the product g

We conclude this section by discussing the algebras obtained from the class $2A$ under the product g. We call an edge $\omega\alpha$ of Γ to be of type O_iO_j if $\omega \in O_i$ and $\alpha \in O_j$.

The group K acts on the set of ordered pairs whose end points both lie in O_1 with four orbits, of lengths $34560, 110592, 276480$, and 276480, respectively. Let $\omega\alpha$ be an ordered pair from the orbit of length 34560. Then ω and α have eight common neighbours in O_2 and 92 in O_1. An ordered pair in the orbit of length 110592 has endpoints that have all the 100 common neighbours lying in O_1.

For an ordered pair from the first orbit of length 276480, the end points have 98 common neighbours in O_1. A typical ordered pair $\omega\alpha$ in the second orbit of length 276480 is such that ω and α have four common neighbours in the orbit O_2, and 96 common neighbours in O_1. Thus, using the fact that coefficients are uniformly distributed over O_1 and O_2, the square of u_1 (we will abuse notation and denote this by u_1^2) before projecting is :

$$\begin{aligned} u_1^2 &= \frac{1}{1728}\left(34560 \cdot 92 + 110592 \cdot 100 + 276480(98+96)\right) u_1 \\ &\quad + \frac{1}{54}\left(34560 \cdot 8 + 276480(2+4)\right) u_2 \\ &= 39280 u_1 + 35840 u_2. \end{aligned}$$

The group K has a transitive action on the set of edges of type O_2O_2 and has

	1	0	2/3	2^E	2^O
1	$1, 0$	$1, 0$	$2/3, 2^E$	$2/3, 2^E$	2^O
0		$1, 0$	$2/3, 2^E$	$2/3, 2^E$	2^O
2/3			$1, 0, 2^E$	$1, 0, 2/3, 2^E$	2^O
2^E				$1, 0, 2/3, 2^E$	2^O
2^O					$1, 0, 2/3, 2^E$

Table 5.7: Fusion law for the extended algebra with $\lambda = \frac{21}{31}$

	1	0^E	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{3}$	0^O
1	$1, 0^E$	$1, 0^E$	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{3}$	$\emptyset$
0^E		$1, 0^E, \frac{4}{9}, \frac{1}{3}, \frac{2}{3}$	$\frac{4}{9}, 0^E, \frac{1}{3}, \frac{2}{3}$	$0^E, \frac{4}{9}, \frac{1}{3}$	$0^E, \frac{4}{9}, \frac{2}{3}$	0^O
$\frac{4}{9}$			$1, 0^E, \frac{1}{3}, \frac{2}{3}$	$0^E, \frac{4}{9}$	$0^E, \frac{4}{9}$	0^O
$\frac{1}{3}$				$1, 0^E, \frac{1}{3}, \frac{2}{3}$	$\frac{1}{3}, \frac{2}{3}$	0^O
$\frac{2}{3}$					$1, 0^E, \frac{1}{3}$	0^O
0^O						$1, 0^E, \frac{4}{9}, \frac{1}{3}, \frac{2}{3}$

Table 5.8: Fusion law for the extended algebra with $\lambda = -\frac{1}{31}$

.

two orbits of lengths 3456 and 17280 in its action on the set of edges of type O_1O_2. An adjacent pair $\omega\alpha$ is such that ω and α have 80 common neighbours in O_1 and 20 in O_2. Of those ordered pairs of type O_1O_2 lying in the orbit of length 3456, the end points have all their common neighbours in O_1; while the end points of those edges of type O_1O_2 lying in the orbit of length 17280 have 92 common neighbours in O_1 and eight in O_2. Thus, before projecting,

$$\begin{aligned} u_2^2 &= \frac{80 \cdot 32 \cdot 54}{1728} u_1 + \frac{20 \cdot 32 \cdot 54}{54} u_2 \\ &= 80u_1 + 640u_2; \end{aligned}$$

$$\begin{aligned} u_1 u_2 &= \frac{1}{1728} (3456 \cdot 100 + 17280 \cdot 92)\, u_1 + \left(\frac{17280 \cdot 8}{54} \right) u_2 \\ &= 1120u_1 + 2560u_2. \end{aligned}$$

Now,

$$\begin{aligned} a^2 &= (u_1 - 32u_2)^2 \\ &= u_1^2 - 64u_1u_2 + 1024u_2^2 \\ &= 39280u_1 + 35840u_2 - 64\,(1120u_1 + 2560u_2) + 1024\,(80u_1 + 640u_2) \\ &= 49520u_1 + 527360u_2. \end{aligned}$$

Projecting, we have

$$\begin{aligned} a^2 &= 49520\pi(u_1) + 527360\pi(u_2) \\ &= 49520\pi(u_1) - 527360\pi(u_1) \\ &= -477840\pi(u_1) \\ &= -477840 \left(\frac{1}{33} u_1 - \frac{32}{33} \right) \\ &= -14480\,(u_1 - 32u_2) = -14480a. \end{aligned}$$

Using MAGMA, we get the fusion law in Table 5.9, where the eigenvalues $\frac{7}{905}, -\frac{25}{181}, \frac{129}{181}, \frac{62}{905}, \frac{41}{181}, \frac{49}{362}, -\frac{26}{905}$ are assigned the symbols a-h respectively.

5.4 Algebras from the class $2B$

In this section we present fusion laws for algebras obtained from the class $2B$ of involutions in Suz:2. Let z be an involution in this class. Then the centralizer $K = C_G(z) \cong 2^2.L(3,4).2^2$ and has order 322 560. The restriction of χ to K contains the principal character and so the dimension of the intersection of the space fixed by the centralizer of an involution with V is $1 = (\chi|_K, 1_K)$. The group K has three orbits in its action on Ω, of lengths

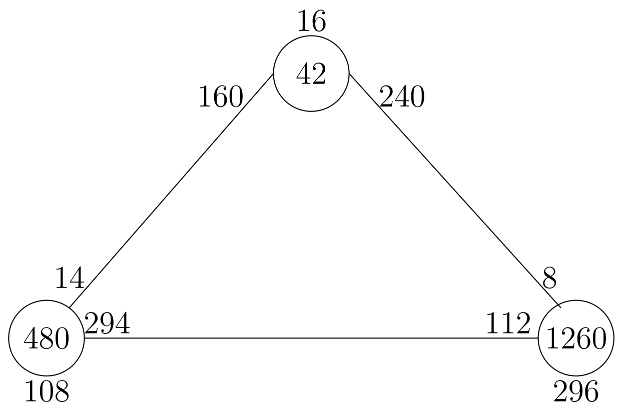

Figure 5.5: Orbit diagram: vertices

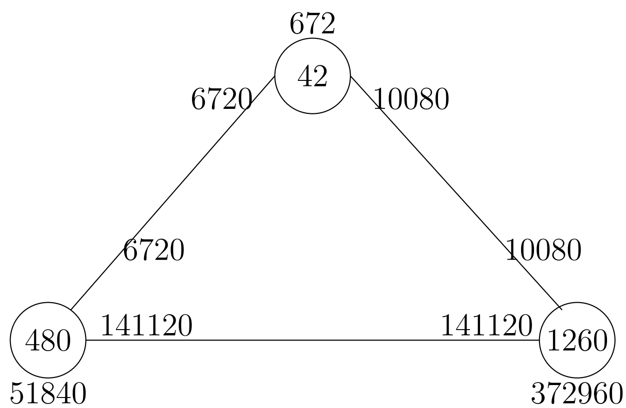

Figure 5.6: Orbit diagram: edges

$42, 480$ and 1260 respectively. We present diagrams (Figure 5.5 and Figure 5.6) showing how the various orbits connect, as well as showing how many edges of each type there are.

For the rest of the section, O_1 denotes the orbit of length 42, O_2 is the orbit of length 480 and O_3 is the orbit of length 1260. In the following table, we summarise information about how K acts on the sets of edges of types $O_iO_j, i, j \in \{1, 2, 3\}$.

In the next table, we summarise the distribution of common neighbours of a typical edge in each orbit over the orbits O_1, O_2, O_3 of K in its action on Ω.

From Figure 5.5, we have by Proposition 3.2.2, the action of the adjacency matrix A of the graph Γ on the fixed space of K has matrix :

$$B = \begin{bmatrix} 16 & 14 & 8 \\ 160 & 108 & 112 \\ 240 & 294 & 296 \end{bmatrix}.$$

The matrix B has eigenvalues $-16, 416$ and 20, so that the fixed space meets the algebra V in a one-dimensional subspace. It is easy to verify that $(1 \ \frac{7}{40} \ -\frac{1}{10})$ is a 20-eigenvector of B. It follows that $a = w_1 + \frac{7}{40}w_2 - \frac{1}{10}w_3$, where

$$w_i = \sum_{\alpha \in O_i} x_\alpha, i \in \{1, 2, 3\}.$$

We consider the projections of the vectors w_i to V. We make use of the diagram in Figure 5.5. We have:

$$\begin{aligned} \pi(w_1) &= (c_0 + 16c_1 + (42 - 1 - 16)c_2)w_1 + (14c_1 + (42 - 14)c_2)w_2 \\ &\quad +(8c_1 + (42 - 8)c_2)w_3 \\ &= \frac{20}{33}w_1 + \frac{7}{66}w_2 - \frac{2}{33}w_3; \end{aligned}$$

$$\begin{aligned} \pi(w_2) &= (160c_1 + (480 - 160)c_2)w_1 + (c_0 + 108c_1 + (480 - 1 - 108)c_2)w_2 \\ &\quad +(112c_1 + (480 - 112)c_2)w_3 \\ &= \frac{40}{33}w_1 + \frac{7}{33}w_2 - \frac{4}{33}w_3 = 2\pi(w_1); \end{aligned}$$

$$\begin{aligned} \pi(w_3) &= (240c_1 + (1260 - 240)c_2)w_1 + (294c_1 + (1260 - 294)c_2)w_2 \\ &\quad +(c_0 + 296c_1 + (1260 - 1 - 296)c_2)w_3 \\ &= -\frac{20}{11}w_1 - \frac{7}{22}w_2 + \frac{2}{11}w_3 = -3\pi(w_1). \end{aligned}$$

We find the square of a under the Norton product. The pointwise product is :

$$\begin{aligned} a^{*2} &= w_1 + \left(\frac{7}{40}\right)^2 w_2 + \left(-\frac{1}{10}\right)^2 w_3 \\ &= w_1 + \frac{49}{1600}w_2 + \frac{1}{100}w_3. \end{aligned}$$

Projecting, we have that

$$\begin{aligned}
a^2 &= \pi(w_1) + \frac{49}{1600}\pi(w_2) + \frac{1}{100}\pi(w_3) \\
&= \pi(w_1) + \frac{49}{1600}(2\pi(w_1) + \frac{1}{100}(-3\pi(w_1) \\
&= \left(1 + \frac{49}{100} \cdot 2 - \frac{3}{100}\right)\pi(w_1) \\
&= \frac{33}{32}\left(\frac{20}{33}w_1 + \frac{7}{66}w_2 - \frac{2}{33}w_3\right) \\
&= \frac{5}{8}w_1 + \frac{7}{64}w_2 - \frac{1}{16}w_3 \\
&= \frac{5}{8}(w_1 + \frac{7}{64}w_2 - \frac{1}{10}w_3) = \frac{5}{8}a.
\end{aligned}$$

Thus, in order to have a to be an idempotent, we scale the Norton product f by $\frac{8}{5}$. Using MAGMA, we find the eigenvalues of an idempotent a from the V-subspace fixed by the centralizer of an involution. We find that a is primitive and has eigenvalues $1, \frac{18}{25}, -\frac{1}{75}, -\frac{1}{20}, -\frac{4}{25}, \frac{29}{300}$ and $\frac{7}{25}$. The $-\frac{1}{75}$-, $-\frac{4}{25}$-, $\frac{29}{300}$- and $\frac{7}{25}$-eigenspaces split into even and odd parts, while the rest of the eigenspaces are entirely even. As usual we we follow the convention of writing μ^E for the even part of the μ-eigenspace and μ^O for the odd part. We have the fusion law shown in Table 5.12 for the algebra in which the eigenvalues $\frac{18}{25}, -\frac{1}{75}, -\frac{1}{20}, -\frac{4}{25}, \frac{29}{300}, \frac{7}{25}$ are assigned the symbols a, b, c, d, e and f, respectively.
In Table 5.14, the eigenvalues $5/3, 7/4, 17/12$ are assigned the values a, b and c, respectively.
The eigenvalues $17/22, -3/22, -2/11, -7/22, 5/22$ are assigned the symbols a, b, c, d and e in Table 5.18.

5.4.1 The extended algebras for the class $2B$

In this subsection we present the fusion laws for the extended algebras. Since a is primitive, the eigenvalues of the extended algebras can be found using Theorem 2.5.6 (1). We use MAGMA routines to obtain the fusion laws for each unital extension. We will present the fusion tables without comment from now on. The eigenspaces which split can be inferred from the tables, and all details follow from Chapter 4 and the discussion for the $2A$ case. The fusion laws are given in Tables 5.13-5.18.

In Table 5.13, the eigenvalues $\frac{1}{25}, \frac{76}{75}, \frac{21}{20}, \frac{29}{25}, \frac{271}{300}, \frac{18}{25}$ are each assigned the symbols a, b, c, d, e and f.
The eigenvalues $5/7, -1/28, -\frac{1}{7}, \frac{3}{28}, \frac{2}{7}$ are assigned the symbols a, b, c, d and e, respectively, in Table 5.15.
In Table 5.16, the eigenvalues $7/10, 1/30, -1/10, 2/15, 3/10$ are assigned the symbols a through e respectively.
The eigenvalues $2/3, 1/9, 1/12, 7/36, 1/3$ are assigned the symbols a, b, c, d and e in Table 5.17.

5.4.2 Algebras under the product g

In this subsection we discuss the algebras obtained by considering the algebra product g. We use the diagram in Figure 5.6 and Table 5.11 to find the products w_1^2, w_1w_2, w_2w_3 and w_3^2. We have:

$$w_1^2 = 672\left(\frac{40}{480}\right)w_2 + 60\left(\frac{672}{1260}\right)w_3 = 56w_2 + 32w_3;$$

$$\begin{aligned} w_1w_2 &= 6720\left(\frac{4}{42}\right)w_1 + 6720\left(\frac{36}{480}\right)w_2 + 6720\left(\frac{60}{1260}\right)w_3 \\ &= 640w_1 + 504w_2 + 320w_3; \end{aligned}$$

$$\begin{aligned} w_1w_3 &= 10080\left(\frac{4}{42}\right)w_1 + 10080\left(\frac{40}{480}\right)w_2 + 10080\left(\frac{56}{1260}\right)w_3 \\ &= 960w_1 + 840w_2 + 448w_3; \end{aligned}$$

$$\begin{aligned} w_2^2 &= \frac{1}{42}\left(14 \cdot 480 + 14 \cdot 960 + 6 \cdot 10080 + 20 \cdot 160 \cdot 2 + 6 \cdot 20160\right)w_1 \\ &\quad + \frac{1}{480}\left(44 \cdot 480 + 44 \cdot 960 + 28 \cdot 10080 + 20 \cdot 20160 + 28 \cdot 20160\right)w_2 \\ &\quad + \frac{1}{1260}\left(42 \cdot 480 + 42 \cdot 960 + 66 \cdot 10080 + 66 \cdot 20160 + 78 \cdot 20160\right)w_3 \\ &= 5760w_1 + 2736w_2 + 2880w_3; \end{aligned}$$

$$
\begin{aligned}
w_2 w_3 &= \frac{1}{42}\left(4\cdot 20160+4\cdot 40320+2\cdot 80640\right)w_1\\
&\quad +\frac{1}{480}\left(28\cdot 20160+28\cdot 40320+24\cdot 80640\right)w_2\\
&\quad +\frac{1}{1260}\left(68\cdot 20160+68\cdot 40320+74\cdot 80640\right)w_3\\
&= 9600w_1+7560w_2+8000w_3;
\end{aligned}
$$

and

$$
\begin{aligned}
w_3^2 &= \frac{1}{42}\left(8\cdot 10080+2\cdot 80640+2\cdot 161280\right)w_1\\
&+ \frac{1}{480}\left(40\cdot 10080+24\cdot 40320+28\cdot 80640+24\cdot 80640=28\cdot 161280\right)w_2\\
&+ \frac{1}{1260}\left(52\cdot 10080+76\cdot 40320+(70+76)\cdot 80640+70\cdot 161280\right)w_3\\
&= 13440w_1+21000w_2+21152w_3.
\end{aligned}
$$

We now compute a^2 under the product g. Before projecting, we have

$$
\begin{aligned}
a^2 &= \left(w_1+\frac{7}{40}w_2-\frac{1}{10}w_3\right)^2\\
&= w_1^2+2\left(\frac{7}{40}\right)w_1w_2-2\left(\frac{1}{10}\right)w_1w_3+\frac{49}{1600}w_2^2\\
&\quad -2\left(\frac{7}{400}\right)w_2w_3+\frac{1}{100}w_3^2\\
&= (56w_2+32w_3)+\frac{7}{20}\left(640w_1+504w_2+320w_3\right)\\
&\quad -\frac{1}{5}\left(960w_1+840w_2+448w_3\right)+\frac{49}{1600}\left(5760w_1 2737w_2+2880w_3\right)\\
&\quad -\frac{7}{200}\left(9600w_1+7560w_2+8000w_3\right)\\
&\quad +\frac{1}{100}\left(13440w_1+21000w_2+21152w_2\right)\\
&= \frac{34}{5}w_1+\frac{9359}{100}w_2+\frac{1853}{25}w_3.
\end{aligned}
$$

After projecting, we have

$$\begin{aligned} a^2 &= \frac{34}{5}\pi(w_1) + \frac{9359}{100}\pi(w_2) + \frac{1853}{25}\pi(w_3) \\ &= \frac{34}{5}\pi(w_1) + \frac{9359}{100}\left(2\pi(w_1)\right) + \frac{1853}{25}\left(-3\pi(w_1)\right) \\ &= \left(\frac{34}{5} + \frac{9359}{50} - \frac{3(1853)}{25}\right)\pi(w_1) \\ &= -\frac{1419}{50}\pi(w_1) = \frac{1853}{25}\left(\frac{20}{33}w_1 + \frac{7}{66}w_2 - \frac{2}{33}w_3\right) \\ &= -\frac{86}{5}w_1 - \frac{301}{100}w_2 + \frac{43}{25} = -\frac{86}{5}a. \end{aligned}$$

Thus, we scale the product g by $-\frac{5}{86}$ to make a an idempotent. Using MAGMA, we obtain the fusion law in Table 5.19 for the algebra where axes are fixed by the centraliser of a $2B$ involution. For reasons of space, we drop the superscript E for the even part of a space which splits. The eigenvalues $\frac{302}{43}, \frac{521}{129}, \frac{511}{43}, -\frac{468}{43}, -\frac{563}{172}, \frac{38}{43}, -\frac{853}{516}, -\frac{281}{43}, \frac{554}{43}, \frac{81}{43}, -\frac{853}{516}$ are assigned symbols a-k in order.

5.5 Algebras from the class $2C$

In this section we discuss the subalgebra of dimension two fixed by the centraliser of an involution in the class $2C$ of involutions. Let $z \in 2C$ be an involution. Then the centraliser $K := C_G(z)$of z in G has order 2 419 200 and is isomorphic to $J_2 : 2 \times 2$, which is the stabiliser of an ordered pair in the Suzuki graph Γ.

The group K has four orbits on Ω of lengths $2, 100, 630$ and 1050. We fix the following notation. By O_1, O_2, O_3 and O_4 we refer to the orbits of lengths two, 100, 630 and 1050 respectively. As before we identify $\omega \in \Omega$ with the standard vector x_ω. We want to study the subalgebra U that is fixed by the subgroup K. This has the orbit sums as a basis in W. We will use the following for the orbit sums:

$$w_i = \sum_{\omega \in O_i} x_\omega.$$

We first give information about how K acts on the graph. The two vertices in O_1 are adjacent, a vertex in O_1 has 100 neighbours in O_2 and 315 in O_3. A vertex in O_2 has two neighbours in $O_1, 36$ in $O_2, 126$ in O_3 and 252 neighbours in O_4. A vertex in O_3 has exactly one neighbour in $O_1, 20$ in $O_2, 155$ in O_3 and 240 neighbours in O_4; while a vertex in O_4 has 24 neighbours in $O_2, 144$ in O_3 and 248 neighbours within O_4. We summarise this information in the orbit diagram depicted in Figure 5.7. It follows by Proposition 3.2.2 that the

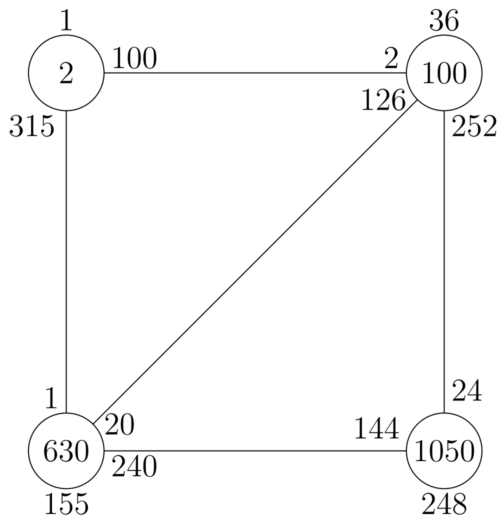

Figure 5.7: Orbit diagram for the centraliser of a $2C$ involution

action of A on U has matrix:

$$B = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 100 & 36 & 20 & 24 \\ 315 & 126 & 155 & 144 \\ 0 & 252 & 240 & 248 \end{bmatrix}.$$

The matrix B has constant column sum so that $(1\ 1\ 1\ 1)$ is an eigenvector with corresponding eigenvalue 416. The other eigenvalues are -16 and 20, with multiplicities 1 and 2 respectively. It follows that the subspace U intersects V in a two-dimensional subspace. We will fix a basis for this subspace in the ensuing discussion.

It can be easily checked that the vectors $(1\ \frac{2}{5}\ -\frac{1}{15}\ 0), (1\ \frac{19}{100}\ 0\ -\frac{1}{50})$ form a basis for the 20-eigenspace of B. We let

$$a_1 = w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3,\ a_2 = w_1 + \frac{19}{100}w_2 - \frac{1}{50}w_4.$$

We can easily check that $a_1, a_2 \in V$. We perform the calculation for a_1. We have :

$$\begin{aligned}
a_1 A &= \left(w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3\right) A \\
&= w_1 A + \frac{2}{5}w_2 A - \frac{1}{15}w_3 A \\
&= (w_1 + 2w_2 + w_3) + \frac{2}{5}(100w_1 + 36w_2 + 20w_3 + 24w_4) \\
&\quad -\frac{1}{15}(315w_1 + 126w_2 + 155w_3 + 144w_4) \\
&= 20w_1 + 8w_2 - \frac{4}{3}w_3 = 20\left(w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3\right) = 20a_1.
\end{aligned}$$

We conclude that $a_1 \in V$. Similarly, $a_2 \in V$. Thus, $\{a_1, a_2\}$ is a basis for $U = \mathrm{inv}_K(W) \cap V$.

We discuss the projections of each of the vectors $w_i, i \in \{1, \ldots, 4\}$ to V. The projection of w_1 to O_1 has coefficient $\frac{130}{297} + \frac{25}{1188} = \frac{545}{1188}$, since a vertex in O_1 has exactly one neighbour in O_1. Since a vertex in O_2 has two neighbours in O_1, w_1 has coefficient $2\left(\frac{25}{1188}\right) = \frac{25}{294}$. The projection of w_1 to O_3 has coefficient $\frac{25}{1188} + \left(-\frac{2}{297}\right)$, since a vertex in O_3 has one neighbour in O_1. Because both vertices in O_1 are not adjacent to any vertex in O_4, the coefficient of the projection of w_1 to w_4 is $2\left(-\frac{2}{297}\right) = -\frac{4}{297}$. Let $\pi : W \to V$ be the projection of W to V. Then we have shown that

$$\pi(w_1) = \frac{545}{1188}w_1 + \frac{25}{594}w_2 + \frac{17}{1188}w_3 - \frac{4}{297}w_4.$$

A vertex in O_1 is adjacent with 100 vertices in O_2 so that the projection of w_2 to O_1 has coefficient $100\left(\frac{25}{1188}\right) = \frac{2500}{1188}$. Because a vertex in O_2 has 36 neighbours in O_2 and hence 63 non-neighbours in the same orbit, the coefficient of w_2 on O_2 is $\frac{130}{297} + 36\left(\frac{25}{1188}\right) + 63\left(-\frac{2}{297}\right) = \frac{229}{297}$. A vertex in O_3 has 20 neighbours in O_2, leaving behind 80 non-neighbours in O_2. Thus, the coefficient of w_2 on O_3 is

$$20\left(\frac{25}{1188}\right) + 80\left(-\frac{2}{297}\right) = -\frac{35}{297}.$$

The projection of w_2 has coefficient $24\left(\frac{25}{1188}\right) + 76\left(-\frac{2}{297}\right) = -\frac{2}{297}$ on O_4. We conclude that

$$\pi(w_2) = \frac{2500}{1188}w_1 + \frac{229}{297}w_2 - \frac{35}{297}w_3 - \frac{2}{297}w_4.$$

The coefficient of the projection of w_3 to O_1 is $315\left(\frac{25}{1188}-\frac{2}{297}\right)=\frac{595}{132}$, since a vertex in O_1 has 315 neighbours and 315 non-neighbours in O_3. The coefficient on O_2 is

$$126\left(\frac{25}{1188}\right)+(630-126)\left(-\frac{2}{297}\right)=-\frac{49}{66}.$$

A vertex in O_3 has 155 neighbours in O_3 and $630-1-155=474$ non-neighbours in the same orbit. The coefficient of the projection of w_3 on O_3 is

$$\frac{130}{297}+155\left(\frac{25}{1188}\right)+474\left(-\frac{2}{297}\right)=\frac{67}{132}.$$

The projection of w_3 to O_4 has coefficient

$$144\left(\frac{25}{1188}\right)+(630-144)\left(-\frac{2}{297}\right)=-\frac{8}{33}.$$

Consequently,

$$\pi(w_3)=\frac{595}{132}w_1-\frac{49}{66}w_2+\frac{67}{132}w_3-\frac{8}{33}w_4.$$

Because O_1 and O_4 are non-adjacent, the projection of w_4 to O_1 has coefficient $1050\left(-\frac{2}{297}\right)=-\frac{700}{99}$. The coefficient of the projection of w_4 on O_2 is

$$252\left(\frac{25}{1188}\right)+(1050-252)\left(-\frac{2}{297}\right)=-\frac{7}{99}.$$

On O_3, the projection of w_4 has coefficient $240\left(\frac{25}{1188}\right)+(1050-240)\left(-\frac{2}{297}\right)=-\frac{40}{99}$. Finally, the projection of w_4 to O_4 has coefficient

$$\frac{130}{297}+248\left(\frac{25}{1188}\right)+(1050-1-248)\left(-\frac{2}{297}\right)=\frac{26}{99}.$$

Therefore, the projection of w_4 to V is:

$$\pi(w_4)=-\frac{700}{99}w_1-\frac{7}{99}w_2-\frac{40}{99}w_3+\frac{26}{99}w_4.$$

We next find the ad_{a_i}-matrices, $i=1,2$ on U with respect to the two algebra products. We start with the Norton product. The square of a_1 before projecting, is

$$a_1^{*2}=w_1+\frac{4}{25}w_2+\frac{1}{225}w_3.$$

Projecting to V, we have :

$$\begin{aligned}
\pi(a_1^{*2}) &= \pi(w_1) + \frac{4}{25}\pi(w_2) + \frac{1}{225}\pi(w_3) \\
&= \left(\frac{545}{1188}w_1 + \frac{25}{594}w_2 + \frac{17}{1188}w_3 - \frac{4}{297}w_4\right) \\
&\quad + \frac{4}{25}\left[\frac{2500}{1188}w_1 + \frac{229}{297}w_2 - \frac{35}{297}w_3 - \frac{2}{297}w_4\right] \\
&\quad + \frac{1}{225}\left[\frac{595}{132}w_1 - \frac{49}{66}w_2 + \frac{67}{132}w_3 - \frac{8}{33}w_4\right] \\
&= \frac{1211}{1485}w_1 + \frac{1204}{7425}w_2 - \frac{17}{7425}w_3 - \frac{116}{7425}w_4.
\end{aligned}$$

To express this product as a linear combination of a_1 and a_2, note that for some scalars $\alpha_1, \alpha_2 \in \mathbb{Q}$,

$$\begin{aligned}
\alpha_1 a_1 + \alpha_2 a_2 &= \alpha_1\left(w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3\right) + \alpha_2\left(w_1 + \frac{19}{100}w_2 - \frac{1}{50}w_4\right) \\
&= (\alpha_1 + \alpha_2)w_1 + \left(\frac{2}{5}\alpha_1 + \frac{19}{100}\alpha_2\right)w_2 - \frac{\alpha_1}{15}w_3 - \frac{\alpha_2}{50}w_4.
\end{aligned}$$

We conclude that the coefficients α_1 and α_2 for a_1^2 are

$$\alpha_1 = (-15)\cdot\left(-\frac{17}{7425}\right) = \frac{17}{495}, \alpha_2 = (-50)\cdot\left(-\frac{116}{7425}\right) = \frac{232}{297}.$$

Thus, $a_1^2 = \frac{17}{495}a_1 + \frac{232}{297}a_2$. Before projecting, the pointwise product of a_1 and a_2 is:

$$\begin{aligned}
a_1 * a_2 &= \left(w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3\right)\left(w_1 + \frac{19}{100}w_2 - \frac{1}{50}w_4\right) \\
&= w_1 + \frac{19}{250}w_2.
\end{aligned}$$

Projecting to V, we obtain

$$\begin{aligned}
\pi(a_1 * a_2) &= \pi(w_1) + \frac{19}{250}w_2 \\
&= \left(\frac{545}{1188}w_1 + \frac{25}{594}w_2 + \frac{17}{1188}w_3 - \frac{4}{297}w_4\right) \\
&\quad + \frac{19}{250}\left(\frac{2500}{1188}w_1 + \frac{229}{297}w_2 - \frac{35}{297}w_3 - \frac{2}{297}w_4\right) \\
&= \frac{245}{396}w_1 + \frac{1246}{12\ 375}w_2 + \frac{53}{9\ 300}w_3 - \frac{173}{12\ 375}w_4 \\
&= -\frac{53}{660}a_1 + \frac{346}{495}a_2,
\end{aligned}$$

where $-\frac{53}{660} = (-15) \cdot \left(\frac{53}{9\ 300}\right)$ and $\frac{346}{495} = (-50) \cdot \left(-\frac{173}{12\ 375}\right)$.
Finally, under the Norton product, the pointwise product a_2^{*2} before projecting is

$$a_2^{*2} = w_1 + \frac{361}{10\ 000}w_2 + \frac{1}{2500}w_4.$$

Projecting, we have that:

$$\begin{aligned}
\pi(a_2^{*2}) &= \pi(w_1) + \frac{361}{10\ 000}\pi(w_2) + \frac{1}{2\ 500}\pi(w_4) \\
&= \left(\frac{545}{1188}w_1 + \frac{25}{594}w_2 + \frac{17}{1188}w_3 - \frac{4}{297}w_4\right) \\
&\quad + \frac{361}{10\ 000}\left(\frac{2500}{1188}w_1 + \frac{229}{297}w_2 - \frac{35}{297}w_3 - \frac{2}{297}w_4\right) \\
&\quad + \frac{1}{2\ 500}\left(-\frac{700}{99}w_1 - \frac{7}{99}w_2 - \frac{40}{99}w_3 + \frac{26}{99}w_4\right) \\
&= \frac{7\ 021}{13\ 200}w_1 + \frac{4\ 613}{66\ 000}w_2 + \frac{653}{66\ 000}w_3 - \frac{449}{33\ 000}w_4 \\
&= -\frac{653}{4\ 400}a_1 + \frac{449}{660}a_2.
\end{aligned}$$

Therefore, the adjoint action of a_1 and a_2 relative to the Norton product f have matrices:

$$[\mathrm{ad}_{a_1}]^f = \begin{bmatrix} \frac{17}{495} & \frac{232}{297} \\ -\frac{53}{660} & \frac{346}{495} \end{bmatrix}, \ [\mathrm{ad}_{a_2}]^f = \begin{bmatrix} -\frac{53}{660} & \frac{346}{495} \\ -\frac{653}{4\ 400} & \frac{449}{660} \end{bmatrix}.$$

We find the ad_{a_i}-matrices , $i = 1, 2$, with respect to the second algebra product g. We first give more information about the Suzuki graph Γ. As mentioned at the beginning of section, the stabiliser of an ordered pair $\omega\alpha \in E(\Gamma)$ is the group $K = J_2{:}2 \times 2$, and this group acts on Ω with four orbits. The lengths of the orbits are $2, 100, 630$ and 1050 respectively. We discuss the way the vertices from these orbits connect with vertices from the same orbit and from different orbits.
There are two ordered pairs with end points in the orbit O_1. There are 200 ordered pairs $\omega\alpha$ with $\omega \in O_1$ and $\alpha \in O_2$. We remark that if $i \neq j$, then the number of ordered pairs $\omega\alpha$ with $\omega \in O_i$ and $\alpha \in O_j$ is equal to the number of ordered pairs $\alpha\omega$ with $\alpha \in O_j$ and $\omega \in O_i$, since the graph Γ is undirected. The number of ordered pairs $\omega\alpha$ with $\omega \in O_1$ and $\alpha \in O_2$ is 630.
There are 3600 ordered pairs both of whose end points lie in O_2. Ordered pairs $\omega\alpha$ with $\omega \in O_2$ and $\alpha \in O_3$ number 12600. There are 25200 ordered pairs $\omega\alpha$ with $\omega \in O_2$ and $\alpha \in O_4$.
The number of ordered pairs $\omega\alpha$ with both end points in O_3 is 97650, while the number of ordered pairs $\omega\alpha$ with $\omega \in O_3$ and $\alpha \in O_4$ is 151200.
Finally, there are 260400 ordered pairs with both end point lying in the orbit O_4. We summarise this information in the orbit diagram shown in Figure 5.8, where we collect the edges according to the orbits they link; and the number of edges of each type emanating from a typical point of each orbit is displayed.

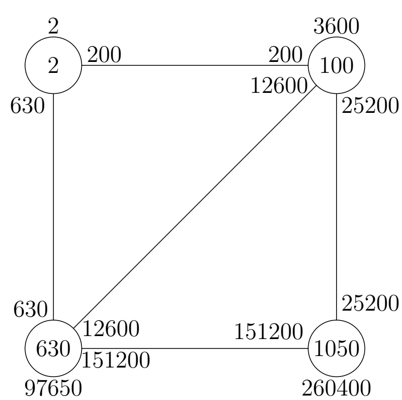

Figure 5.8: Orbit diagram for the centraliser of a $2C$ involution: edges

To compute the products of the vectors w_i, it is necessary for us to know the distribution of the common neighbours of adjacent vertices of each type.

The ordered pairs in O_1 have all the common neighbours of the end points lie in O_2. An adjacent pair $\omega\alpha, \omega \in O_1, \alpha \in O_2$ is such that the pair has one common neighbour in O_1, 36 in O_2 and 63 in O_3. An edge $\omega\alpha$ with $\omega \in O_1$ and $\alpha \in O_3$ is such that ω and α have 20 common neighbours in O_2, and 80 in O_3.
An adjacent pair both of whose end points lie in O_2 is such that the end points have two common neighbouirs in O_1, 14 in O_2, and 42 each in O_3 and O_4. If $\omega \in O_2$ and $\alpha \in O_3$ are adjacent, then they have one common neighbour in O_1, 12 in $O_2, 31$ in O_3 and 56 in O_4. For $\omega\alpha \in E(\Gamma)$, with $\omega \in O_2$ and $\alpha \in O_4$, the vertices ω, α have six common neighbours in $O_2, 28$ in O_3 and 66 in O_4.
The end points $\omega, \alpha \in O_3$ of an edge $\omega\alpha$ have one common neighbour in O_1, six in $O_2, 47$ in O_3 and 46 in O_4. For $\omega\alpha \in E(\Gamma), \omega \in O_3$ and $\alpha \in O_4$, the vertices ω, α have four common neighbours in $O_2, 36$ in O_3 and 60 in O_4.
Finally, two adjacent vertices from O_4 have twelve common neighbours in $O_2, 24$ in O_3 and 64 in O_4. Table 5.20 is a summary of this information. The edges are catergorised under types $O_iO_j, 1 \leq i \leq j \leq 4$, where this notation denotes that the first endpoint ω of an ordered pair $\omega\alpha$ lies in the orbit O_i, while the second point α lies in O_j. We note that for $i \neq j$, there are edges of the form O_jO_i which we do not display as they are captured in O_iO_j. We are now ready to give the products of the vectors w_i amongst themselves under the second product g.
From the preceding discussion, and the fact that coefficients of products are equally distributed over orbits, we have the following products:

$$w_1^2 = (2 \times 100)/100 w_2 = 2w_2, \tag{5.16}$$

$$\begin{aligned} w_1 w_2 &= \frac{(200 \times 1)}{2} w_1 + \frac{(36 \times 200)}{100} w_2 + \frac{(63 \times 200)}{630} w_3 \\ &= 100w_1 + 72w_2 + 20w_3, \end{aligned} \tag{5.17}$$

$$\begin{aligned} w_1 w_3 &= \frac{630 \times 20}{100} w_2 + \frac{630 \times 80}{630} w_3 \\ &= 126w_2 + 80w_3, \end{aligned} \tag{5.18}$$

$$w_1 w_4 = 0_W, \tag{5.19}$$

$$\begin{aligned} w_2^2 &= \frac{3600 \times 2}{2} w_1 + \frac{3600 \times 14}{100} w_2 + \frac{3600 \times 42}{630} w_3 + \frac{3600 \times 42}{1050} w_4 \\ &= 3600w_1 + 504w_2 + 240w_3 + 144w_4, \end{aligned} \tag{5.20}$$

$$
\begin{aligned}
w_2w_3 &= \frac{1\cdot 12600}{2}w_1 + \frac{12\cdot 12600}{100}w_2 + \frac{31\cdot 12600}{630}w_3 + \frac{56\cdot 12600}{1050}w_4 \\
&= 6300w_1 + 1512w_2 + 620w_3 + 672w_4. \qquad (5.21)
\end{aligned}
$$

The rest of the products, namely w_3^2, w_3w_4 and w_4^2 are not straight forward to obtain, as the group K does not act transitively on the sets of edges of the form O_3O_3, O_3O_4 and O_4O_4. We will look at these separately.
The group K has four orbits in its action on the set $\{\omega\alpha|\omega,\alpha \in O_3\}$ of edges both of whose end point lie in O_3. The lengths of the orbits are $630, 6300, 40320$ and 50400, respectively. A typical edge $\omega\alpha$ with $\omega, \alpha \in O_3$ lying in the orbit of length 630 is such that ω and α have 20 common neighbours in O_2 and 80 common neighbours in O_4. An ordered pair in the orbit of length 6300 has end points which have twelve common neighbours in O_2, 16 in O_3 and 72 common neighbours in O_4. An edge in the orbit of length 40320 has endpoints which have 40 common neighbours in O_3 and 60 common neighbours in O_4. Finally, adjacent vertices ω, α with $\omega\alpha$ lying in the orbit of length 50400 have one common neighbour in O_1, six in O_2; 47 and 46 common neighbours in the orbits O_3 and O_4, respectively. Thus the product w_3^2 is:

$$
\begin{aligned}
w_3^2 &= \frac{1\cdot 50400}{2}w_1 + \left(\frac{20\cdot 630 + 12\cdot 6300 + 6\cdot 50400}{100}\right)w_2 \\
&\quad + \left(\frac{16\cdot 6300 + 40\cdot 40320 + 47\cdot 50400}{630}\right)w_3 \\
&\quad + \left(\frac{80\cdot 630 + 72\cdot 6300 + 60\cdot 40320 + 46\cdot 50400}{1050}\right)w_4 \\
&= 25200w_1 + 3906w_2 + 6480w_3 + 4992w_4. \qquad (5.22)
\end{aligned}
$$

The action of K splits the set of ordered pairs $\omega\alpha, \omega \in O_3$ and $\alpha \in O_4$ into two orbits of lengths 50400 and 100800, respectively. The end points ω, α of an edge $\omega\alpha$ lying in the orbit of length 50400 have six common neighbours in O_2, 32 in O_3 and 62 common neighbours in O_4. The end points of an edge in the orbit of length 100800 have four common neighbours in O_2, 36 in O_3

and 60 common neighbours in O_4. Consequently,

$$\begin{aligned} w_3w_4 &= \left(\frac{6\cdot 50400+4\cdot 100800}{100}\right)w_2+\left(\frac{32\cdot 50400+36\cdot 100800}{630}\right)w_3 \\ &\quad +\left(\frac{62\cdot 50+60\cdot 100800}{1050}w_4\right) \\ &= 7056w_2+8320w_3+8736w_4. \end{aligned} \tag{5.23}$$

To get the product w_4^2, we consider the action of K on the set of the 260400 edges both of whose end points lie in O_4. The group has three orbits in this action. The lengths of the orbits are $25200, 33600$ and 201600. A typical edge $\omega\alpha$ in the orbit of length 25200 has end points ω and α sharing two common neighbours in O_2, 44 in O_3 and 64 in O_4.
The end points ω, α of an edge in the orbit of length 33600 have twelve common neighbours in $O_2, 24$ in O_3 and 64 in O_4.
An edge in the orbit of length 201600 is adjacent with vertices ω, α which share six common neighbours in $O_2, 36$ in O_3 and 58 in O_4. Thus, we have,

$$\begin{aligned} w_4^2 &= \left(\frac{2\cdot 25200+12\cdot 33600+6\cdot 201600}{100}\right)w_2 \\ &\quad +\left(\frac{44\cdot 25200+24\cdot 33600+36\cdot 201600}{630}\right)w_3 \\ &\quad +\left(\frac{54\cdot 25200+64\cdot 33600+58\cdot 201600}{1050}\right)w_4 \\ &= 16632w_2+14560w_3+14480w_4. \end{aligned} \tag{5.24}$$

We find the the products a_1^2, a_1a_2, a_2^2 under the second product. Before projecting to V, we have

$$\begin{aligned}
a_1^2 &= w_1(w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3) + \frac{2}{5}w_2(w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3) \\
&\quad -\frac{1}{15}(w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3) \\
&= w_1^2 + \frac{2}{5}w_1w_2 - \frac{1}{15}w_1w_3 + \frac{2}{5}w_2w_1 + \frac{4}{25}w_2^2 - \frac{2}{25}w_2w_3 - \frac{1}{15}w_3w_1 \\
&\quad -\frac{2}{75}w_3w_2 + \frac{1}{225}w_3^2 \\
&= w_1^2 + \frac{4}{5}w_1w_2 - \frac{2}{15}w_1w_3 + \frac{4}{25}w_2^2 - \frac{4}{75}w_2w_3 + \frac{1}{225}w_3^2, \text{by commutativity,} \\
&= 2w_2 + \frac{4}{5}(100w_1 + 72w_2 + 20w_3) - \frac{2}{15}(126w_2 + 80w_3) \\
&\quad +\frac{4}{25}(3600w_1 + 504w_2 + 240w_3 + 144w_4) \\
&\quad -\frac{4}{75}(6300w_1 + 1512w_2 + 620w_3 + 672w_4) \\
&\quad \frac{1}{125}(25200w_1 + 3906w_2 + 6480w_3 + 4992w_4) \\
&= 32w_1 + \frac{1504}{25}w_2 + \frac{592}{15}w_3 + \frac{704}{75}w_4.
\end{aligned}$$

Projecting to V, we have

$$\begin{aligned}
\pi(a_1^2) &= 432\pi(w_1) + \frac{1504}{25}\pi(w_2) + \frac{592}{15}\pi(w_3) + \frac{704}{75}\pi(w_4) \\
&= 432\left(\frac{545}{1188}w_1 + \frac{25}{594}w_2 + \frac{17}{1188}w_3 - \frac{4}{297}w_4\right) \\
&\quad +\frac{1504}{25}\left(\frac{2500}{1188}w_1 + \frac{229}{297}w_2 - \frac{35}{297}w_3 - \frac{2}{297}w_4\right) \\
&\quad +\frac{592}{15}\left(\frac{595}{132}w_1 - \frac{49}{66}w_2 + \frac{67}{132}w_3 - \frac{8}{33}w_4\right) \\
&\quad +\frac{704}{75}\left(-\frac{700}{99}w_1 - \frac{7}{99}w_2 - \frac{40}{99}w_3 + \frac{26}{99}w_4\right) \\
&= \frac{129584}{297}w_1 + \frac{256928}{7425}w_2 + \frac{22768}{1485}w_3 - \frac{98944}{7425}w_4 \\
&= -\frac{22768}{99}a_1 + \frac{197888}{297}a_2,
\end{aligned}$$

where $-\frac{22768}{99} = \frac{22768}{1485}(-15)$ and $\frac{197888}{297} = -\frac{98944}{7425}(-50)$.
Likewise, the square of a_2 is

$$\begin{aligned}
a_2^2 &= w_1(w_1 + \frac{19}{100}w_2 = \frac{1}{50}w_4) + \frac{19}{100}w_2(w_1 + \frac{19}{100}w_2 - \frac{1}{50}w_4) \\
&\quad -\frac{1}{50}w_4(w_1 + \frac{19}{100} - \frac{1}{50}w_4) \\
&= w_1^2 + \frac{19}{100}w_1w_2 - \frac{1}{50}w_1w_4 + \frac{19}{100}w_2w_1 + \left(\frac{19}{100}\right)^2 w_2^2 - \frac{19}{100}\cdot\frac{1}{50}w_2w_4 \\
&\quad -\frac{1}{50}w_4w_1 - \frac{1}{50}\cdot\frac{19}{100}w_4w_2 + \left(-\frac{1}{50}\right)^2 w_4^2 \\
&= \frac{4199}{25}w_1 + \frac{10679}{250}w_2 + \frac{1697}{125}w_3 - \frac{131}{125}w_4.
\end{aligned}$$

Projecting, we have

$$\begin{aligned}
\pi(a_2^2) &= \frac{4199}{25}\pi(w_1) + \frac{10679}{250}\pi(w_2) + \frac{1697}{125}\pi(w_3) - \frac{131}{125}\pi(w_4) \\
&= \frac{4199}{25}\left(\frac{545}{1188}w_1 + \frac{25}{594}w_2 + \frac{17}{1188}w_3 - \frac{4}{297}w_4\right) \\
&\quad +\frac{10679}{250}\left(\frac{2500}{1188}w_1 + \frac{229}{297}w_2 - \frac{35}{297}w_3 - \frac{2}{297}w_4\right) \\
&\quad +\frac{1697}{125}\left(\frac{595}{132}w_1 - \frac{49}{66}w_2 + \frac{67}{132}w_3 - \frac{8}{33}w_4\right) \\
&\quad -\frac{131}{125}\left(-\frac{700}{99}w_1 - \frac{7}{99}w_2 - \frac{40}{99}w_3 + \frac{26}{99}w_4\right) \\
&= \frac{194327}{825}w_1 + \frac{247499}{8250}w_2 + \frac{19321}{4125}w_3 - \frac{25229}{4125}w_4 \\
&= -\frac{19321}{275}a_1 + \frac{50458}{165}a_2.
\end{aligned}$$

The final piece to get the required matrices is to determine the product of

a_1 and a_2. Thus,

$$\begin{aligned}
a_1a_2 &= \left(w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3\right) \cdot \left(w_1 + \frac{19}{100}w_2 - \frac{1}{50}w_4\right) \\
&= w_1^2 + \frac{19}{100}w_1w_2 - \frac{1}{50}w_1w_4 + \frac{2}{5}w_2w_1 + \frac{2}{5} \cdot \frac{19}{100}w_2^2 - \frac{2}{5} \cdot \frac{1}{50}w_2w_4 \\
&\quad - \frac{1}{15}w_3w_1 - \frac{1}{15} \cdot \frac{19}{100}w_3w_2 + \frac{1}{15} \cdot \frac{1}{50}w_3w_4 \\
&= w_1^2 + \frac{59}{100}w_1w_2 - \frac{1}{50}w_1w_4 + \frac{19}{250}w_2^2 - \frac{1}{125}w_2w_4 - \frac{1}{15}w_1w_3 \\
&\quad - \frac{19}{1500}w_2w_3 + \frac{1}{750}w - 3w_4 \\
&= 2w_2 + \frac{59}{100}(100w_1 + 72w_2 + 20w_3) - \frac{1}{50}0_W \\
&\quad + \frac{19}{250}\left(3600w_1 + 504w_2 + 240w_3 + 144w_4\right) \\
&\quad - \frac{1}{125}\left(1512w_2 + 1120w_3 + 1584w_4\right) - \frac{1}{15}\left(126w_2 + 80w_3\right) \\
&\quad - \frac{19}{1500}\left(6300w_1 + 1512w_2 + 620w_3 + 672w_4\right) \\
&\quad + \frac{1}{750}\left(7056w_2 + 8320w_3 + 8736w_4\right) \\
&= \frac{1264}{5}w_1 + \frac{6568}{125}w_2 + \frac{1424}{75}w_3 + \frac{176}{125}w_4.
\end{aligned}$$

Finally, projecting to V, we have

$$\begin{aligned}
\pi(a_1a_2) &= \frac{1264}{5}\left(\frac{545}{1188}w_1 + \frac{25}{594}w_2 + \frac{17}{1188}w_3 - \frac{4}{297}w_4\right) \\
&\quad + \frac{6568}{125}\left(\frac{2500}{1188}w_1 + \frac{229}{297}w_2 - \frac{35}{297}w_3 - \frac{2}{297}w_4\right) \\
&\quad + \frac{1424}{75}\left(\frac{595}{132}w_1 - \frac{49}{66}w_2 + \frac{67}{132}w_3 - \frac{8}{33}w_4\right) \\
&\quad + \frac{176}{125}\left(-\frac{700}{99}w_1 - \frac{7}{99}w_2 - \frac{40}{99}w_3 + \frac{26}{99}w_4\right) \\
&= \frac{149576}{495}w_1 + \frac{457352}{12375}w_2 + \frac{16072}{2475}w_3 - \frac{98896}{12375}w_4 \\
&= -\frac{16072}{165}a_1 + \frac{197792}{495}a_2.
\end{aligned}$$

We conclude that the matrices of $\mathrm{ad}_{a_i}, i = 1, 2$ relative to the second product are:

$$[\mathrm{ad}_{a_1}]^g = \begin{bmatrix} -\frac{22768}{99} & \frac{197888}{297} \\ -\frac{16072}{165} & \frac{197792}{495} \end{bmatrix}, [\mathrm{ad}_{a_2}]^g = \begin{bmatrix} -\frac{16072}{165} & \frac{197792}{495} \\ -\frac{19321}{275} & \frac{50458}{165} \end{bmatrix}.$$

Recall that a general algebra product $\phi = \alpha_1 f + \alpha_2 g$, for some constants α_1 and α_2. We wish to identify possible axes in the subalgebra U. We set $u = \beta_1 a_1 + \beta_2 a_2, \beta_1, \beta_2 \in \mathbb{Q}$. For u to be an idempotent,

$$\begin{aligned} \beta_1 a_1 + \beta_2 a_2 = \mathrm{ad}_u(u) &= u \times u \\ &= (\beta_1 a_1 + \beta_2 a_2)(\beta_1 a_1 + \beta_2 a_2) \\ &= \beta_1^2 a_1^2 + 2\beta_1\beta_2 a_1 a_2 + \beta_2^2 a_2^2 \\ &= s a_1 + t a_2, \text{ say.} \end{aligned}$$

Thus, $s - \beta_1 = 0$ and $t - \beta_2 = 0$. But we have

$$\begin{aligned} \mathrm{ad}_u(u) &= \beta_1^2 \phi(a_1, a_1) + 2\beta_1\beta_2\phi(a_1, a_2) + \beta_2^2\phi(a_1, a_2) \\ &= \beta_1^2 [\alpha_1 f(a_1, a_1) + \alpha_2 g(a_1, a_1)] + 2\beta_1\beta_2 [\alpha_1 f(a_1, a_2) + \alpha_2 g(a_1, a_2)] \\ &\quad + \beta_2^2 [\alpha_1 f(a_2, a_2) + \alpha_2 g(a_2, a_2)] \\ &= \beta_1^2 \left[\alpha_1 \left(\frac{17}{495} a_1 + \frac{232}{297} a_2\right) + \alpha_2 \left(-\frac{22768}{99} a_1 + \frac{197888}{297} a_2\right)\right] \\ &\quad + 2\beta_1\beta_2 \left[\alpha_1 \left(= \frac{53}{660} a_1 + \frac{346}{495} a_2\right) + \alpha_2 \left(-\frac{16072}{165} a_1 + \frac{197792}{492} a_2\right)\right] \\ &\quad + \beta_2^2 \left[\alpha_1 \left(-\frac{653}{4400} a_1 + \frac{449}{660} a_2\right) + \alpha_2 \left(-\frac{19321}{275} a_1 + \frac{50458}{165} a_2\right)\right] \\ &= \left(\beta_1^2 \left[\frac{17}{495}\alpha_1 - \frac{22768}{99}\alpha_2\right] + 2\beta_1\beta_2 \left[-\frac{53}{660}\alpha_1 - \frac{16072}{165}\alpha_2\right] + \right. \\ &\quad \left. \beta_2^2 \left[-\frac{653}{4400}\alpha_1 - \frac{19321}{275}\alpha_2\right]\right) a_1 \\ &\quad + \left(\left[\frac{232}{297}\alpha_1 + \frac{197888}{297}\alpha_2\right] \beta_1^2 + 2 \left[\frac{346}{495}\alpha_1 + \frac{197792}{495}\alpha_2\right] \beta_1\beta_2 \right. \\ &\quad \left. + \left[\frac{449}{660}\alpha_1 + \frac{50458}{165}\alpha_2\right] \beta_2^2\right) a_2. \end{aligned}$$

The conditions for u to be an idempotent become :

$$\begin{aligned} 0 &= \left(\frac{17}{495}\alpha_1 - \frac{22768}{99}\alpha_2\right)\beta_1^2 + 2\left(-\frac{53}{660}\alpha_1 - \frac{16072}{165}\alpha_2\right)\beta_1\beta_2 \\ &\quad + \left(-\frac{653}{4400}\alpha_1 - \frac{19321}{275}\alpha_2\right)\beta_2^2 - \beta_1, \end{aligned} \tag{5.25}$$

$$\begin{aligned} 0 &= \left(\frac{232}{297}\alpha_1 + \frac{197888}{297}\alpha_2\right)\beta_1^2 + 2\left(\frac{346}{495}\alpha_1 + \frac{197792}{495}\alpha_2\right)\beta_1\beta_2 \\ &\quad + \left(\frac{449}{660}\alpha_1 + \frac{50458}{165}\alpha_2\right)\beta_2^2 - \beta_2. \end{aligned} \tag{5.26}$$

Since Equations (5.25) and (5.26) are linear in α_1 and α_2, we have the matrix equation:

$$\begin{bmatrix} \frac{17}{495}\beta_1^2 - \frac{53}{330}\beta_1\beta_2 - \frac{653}{4400}\beta_2^2 & -\left(\frac{22768}{99}\beta_1^2 + \frac{32144}{165}\beta_1\beta_2 + \frac{19321}{275}\beta_2\right) \\ \frac{232}{297}\beta_1^2 + \frac{692}{495}\beta_1\beta_2 + \frac{449}{660}\beta_2^2 & \frac{197888}{297}\beta_1^2 + \frac{395584}{495}\beta_1\beta_2 + \frac{50458}{165}\beta_2^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}. \tag{5.27}$$

We discuss the cases where $u = \beta_1 a_1$ or $u = \beta_2 a_2$, with β_1, β_2 nonzero in the respective cases. In the first case, $\beta_2 = 0$. The matrix equation becomes :

$$\begin{bmatrix} \frac{17}{495}\beta_2^2 & -\frac{22768}{99}\beta_1^2 \\ \frac{232}{297}\beta_1^2 & \frac{197888}{297}\beta_1^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ 0 \end{bmatrix}. \tag{5.28}$$

Thus, we have $\alpha_1 = \frac{23190}{7049\beta_1}, \alpha_2 = -\frac{435}{112784\beta_1}$.

If on the other hand, $u = \beta_2 a_2, \beta_2 \neq 0$, then $\beta_1 = 0$ and Equation (5.27) becomes:

$$\begin{bmatrix} -\frac{653}{4400}\beta_2^2 & -\frac{19321}{275}\beta_2^2 \\ \frac{449}{660}\beta_2^2 & \frac{50458}{165}\beta_2^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} \tag{5.29}$$

which has solution $\alpha_1 = \frac{772840}{26537\beta_2}, \alpha_2 = -\frac{3265}{53074\beta_2}$.

We consider the case $a = \beta_1 a_1 + \beta_2 a_2$ with $\beta_1 \neq 0, \beta_2 \neq 0$. In this case, we want the algebra product $\phi = \alpha_1 f + \alpha_2 g$ to associate with the unique bilinear form. In particular, we want to have $(a_1 a_2, a) = (a_1, a_2 a)$. But, we have

$$\begin{aligned} a_2 a &= \phi(a_2, a) \\ &= \beta_1\phi(a_2, a_1) + \beta_2\phi(a_2, a_2) \\ &= \beta_1\left[\alpha_1 f(a_2, a_1) + \alpha_2 g(a_2, a_1)\right] + \beta_2\left[\alpha_1 f(a_2, a_2) + \alpha_2 g(a_2, a_2)\right]. \end{aligned}$$

Expanding, we have

$$\begin{aligned}
a_2 a &= \beta_1 \left[\alpha_1 \left(-\frac{53}{660}a_1 + \frac{346}{495}a_2\right) + \alpha_2 \left(-\frac{16072}{165}a_1 + \frac{197792}{495}a_2\right)\right] \\
&\quad +\beta_2 \left[\alpha_1 \left(-\frac{653}{4400}a_1 + \frac{449}{660}a_2\right) + \alpha_2 \left(-\frac{19321}{275}a_1 + \frac{50458}{165}a_2\right)\right] \\
&= \left[-\frac{53}{660}\alpha_1\beta_1 - \frac{16072}{165}\alpha_2\beta_1 - \frac{653}{4400}\alpha_1\beta_2 - \frac{19321}{275}\alpha_2\beta_2\right] a_1 \\
&+ \left[\frac{346}{495}\alpha_1\beta_1 + \frac{197792}{495}\alpha_2\beta_1 + \frac{449}{660}\alpha_1\beta_2 + \frac{50458}{165}\alpha_2\beta_2\right] a_2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
a_1 a_2 &= \phi(a_1, a_2) \\
&= \alpha_1 f(a_1, a_2) + \alpha_2 g(a_1, a_2) \\
&= \alpha_1 \left(-\frac{53}{660}a_1 + \frac{346}{495}a_2\right) + \alpha_2 \left(-\frac{16072}{165}a_1 + \frac{197792}{495}a_2\right) \\
&= \left(-\frac{53}{660}\alpha_1 - \frac{16072}{165}\alpha_2\right) a_1 + \left(\frac{346}{495}\alpha_1 + \frac{197792}{495}\alpha_2\right) a_2.
\end{aligned}$$

The inner products become:

$$\begin{aligned}
(a_1 a_2, a) &= \left(\left(-\frac{53}{660}\alpha_1 - \frac{16072}{165}\alpha_2\right) a_1 + \left(\frac{346}{495}\alpha_1 + \frac{197792}{495}\alpha_2\right) a_2, \beta_1 a_1 + \beta_2 a_2\right) \\
&= \left(\left(-\frac{53}{660}\alpha_1 - \frac{16072}{165}\alpha_2\right) a_1, \beta_1 a_1\right) \\
&\quad + \left(\left(\frac{346}{495}\alpha_1 + \frac{197792}{495}\alpha_2\right) a_2, \beta_1 a_1\right) \\
&\quad + \left(\left(-\frac{53}{660}\alpha_1 - \frac{16072}{165}\alpha_2\right) a_1, \beta_2 a_2\right) \\
&\quad + \left(\left(\frac{346}{495}\alpha_1 + \frac{197792}{495}\alpha_2\right) a_2, \beta_2 a_2\right) \\
&= -\left(\frac{53}{660}\alpha_1 + \frac{16072}{165}\alpha_2\right) \beta_1 (a_1, a_1) + \left(\frac{346}{495}\alpha_1 + \frac{197792}{495}\alpha_2\right) \beta_2 (a_2, a_2) \\
&\quad + \left(\left(\frac{346}{495}\alpha_1 + \frac{197792}{495}\alpha_2\right) \beta_1 - \left(\frac{53}{660}\alpha_1 + \frac{16072}{165}\alpha_2\right) \beta_2\right) (a_1, a_2);
\end{aligned}$$

$$
\begin{aligned}
(a_1, a_2 a) &= \Bigg(a_1, \left[-\frac{53}{660}\alpha_1\beta_1 - \frac{16072}{165}\alpha_2\beta_1 - \frac{653}{4400}\alpha_1\beta_2 - \frac{19321}{275}\alpha_2\beta_2\right] a_1 \\
&\quad \left[\frac{346}{495}\alpha_1\beta_1 + \frac{197792}{495}\alpha_2\beta_1 + \frac{449}{660}\alpha_1\beta_2 + \frac{50458}{165}\alpha_2\beta_2\right] a_2\Bigg) \\
&= \left[-\frac{53}{660}\alpha_1\beta_1 - \frac{16072}{165}\alpha_2\beta_1 - \frac{653}{4400}\alpha_1\beta_2 - \frac{19321}{275}\alpha_2\beta_2\right](a_1, a_1) \\
&+ \left[\frac{346}{495}\alpha_1\beta_1 + \frac{197792}{495}\alpha_2\beta_1 + \frac{449}{660}\alpha_1\beta_2 + \frac{50458}{165}\alpha_2\beta_2\right](a_1, a_2).
\end{aligned}
$$

Now,

$$
\begin{aligned}
(a_1, a_1) &= \left(w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3, w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3\right) \\
&= 1 + \frac{4}{5} + \frac{1}{225} = \frac{406}{225};
\end{aligned}
$$

$$
\begin{aligned}
(a_1, a_2) &= \left(w_1 + \frac{2}{5}w_2 - \frac{1}{15}w_3, w_1 + \frac{19}{100}w_2 - \frac{1}{50}w_4\right) \\
&= 1 + \frac{2}{5}\cdot\frac{19}{100} = \frac{269}{250};
\end{aligned}
$$

and

$$
\begin{aligned}
(a_2, a_2) &= \left(w_1 + \frac{19}{100}w_2 - \frac{1}{50}w_4, w_1 + \frac{19}{100}w_2 - \frac{1}{50}w_4\right) \\
&= 1 + \left(\frac{19}{100}\right)^2 + \left(-\frac{1}{50}\right)^2 = \frac{2073}{2000},
\end{aligned}
$$

using the fact that $(w_i, w_j) = \delta_{ij}$, since the entries of $w_i, i = 1, 2, 3$ are $0, 1$. We conclude that

$$
(a_1 a_2, a) = \frac{225427}{371250}\alpha_1\beta_1 + \frac{47182912}{185625}\alpha_2\beta_1 + \frac{52643}{82500}\alpha_1\beta_2 + \frac{6380458}{20625}\alpha_2\beta_2
$$

and

$$
(a_1, a_2 a) = \frac{225427}{371250}\alpha_1\beta_1 + \frac{47182912}{185625}\alpha_2\beta_1 + \frac{28723}{61875}\alpha_1\beta_2 + \frac{12515477}{61875}\alpha_2\beta_2.
$$

Equality of (a_1a_2, a) and (a_1, a_2a) gives the equation:

$$\frac{52643}{82500}\alpha_1\beta_2 + \frac{6380458}{20625}\alpha_2\beta_2 = \frac{28723}{61875}\alpha_1\beta_2 + \frac{12515477}{61875}\alpha_2\beta_2. \tag{5.30}$$

That is,

$$\begin{aligned} 0 &= \frac{43037}{247500}\alpha_1\beta_2 + \frac{6625897}{61875}\alpha_2\beta_2 \\ &= \left(\frac{43037}{247500}\alpha_1 + \frac{6625897}{61875}\alpha_2\right)\beta_2. \end{aligned}$$

It follows that $\alpha_1 = \frac{26503588}{43037}\alpha_2$.
For the case $\beta_2 = 0$, we set $\beta_1 = 1$ and hence $\alpha_1 = \frac{23190}{7049}$ and $\alpha_2 - \frac{435}{112784}$. Computation with MAGMA gives the eigenvalues of the corresponding idempotent as : $\frac{2106}{7049}, 1, \frac{681}{28196}, \frac{5346}{7049}, -\frac{1111}{7049}, -\frac{702}{7049}, -\frac{1633}{28196}, -\frac{696}{7049}, \frac{533}{14098}, -\frac{2271}{7049}, \frac{5619}{7049}$ with multiplicities $63, 1, 160, 1, 28, 90, 160, 140, 36, 28, 1$, respectively. Thus, the orthogonal decomposition arising from the adjoint action of the idempotent does not span the whole space. A similar result is obtained for the case where $\beta_1 = 0$ and $\beta_2 \neq 0$. Since in the last case discussed β_1 plays no role, we conclude that the orthogonal decompositions of spaces arising from idempotents arising for the class $2D$ do not span our algebra V. Thus, we need to pass to an extension field to get all the eigenvalues.

5.6 Algebras from the class $2D$

In this section we show that the fixed space of the centraliser of a $2D$ involution intersects V in the trivial subspace. Let K be the centraliser of an involution in the class $2D$. Then K has order 380160 and has isomorphism type $M_{12}{:}2 \times 2$. The group K has two orbits in its action on Ω, of lengths 792 and 990, respectively. The distribution diagram is shown in Figure 5.21. By Proposition 3.2.2, the matrix of the action of A on the subspace spanned

Table 5.21: Distribution diagram relative to a $2D$ centraliser

by the orbit sums is :

$$C = \begin{bmatrix} 176 & 192 \\ 240 & 224 \end{bmatrix}.$$

The matrix C has eigenvalues 416 and -16. We conclude that the intersection $\mathrm{inv}_K(W) \cap V = \{0_W\}$. Thus, we cannot identify axes from this class of involutions.

	1	a	b	c	d	e	f	g	h
1	1	a	b	c	d	e	f	g	h
a		$1,b,c,d,e$	a,b,f	a,f	a,f	a,f	b,c,d,e	g,h	g,h
b			$1,a,b,c,d,e$	b	b	b,e	a,f	g,h	g,h
c				$1,d$	c,d	e	a,f	g,h	g,h
d					$1,c,d$	e	a,f	h	g,h
e						b,e	a,f	g,h	g,h
f							$1,b,c,d,e$	g,h	g,h
g								$1,a,b,c,e,f$	a,b,c,d,e,f
h									$1,a,b,c,d,e,f$

Table 5.9: Fusion law for axes fixed by $2A$ involutions under the product g

Type	Size	Action transitive or not
O_1O_1	672	Yes
O_1O_2	6720	Yes
O_1O_3	10080	Yes
O_2O_2	51840	No, five orbits
O_2O_3	141120	No, three orbits
O_3O_3	372960	No, five orbits

Table 5.10: Summary of action of the centraliser of a $2B$ involution on edges

Type	Orbit length	Distribution (common neighbours)		
		O_1	O_2	O_3
O_1O_1	672	0	40	60
O_1O_2	6720	4	36	60
O_1O_3	10080	4	40	56
O_2O_2	480	14	44	42
	960	14	44	42
	10080	6	28	66
	20160	2	20	78
	20160	6	28	66
O_2O_3	20160	4	28	68
	40320	4	28	68
	2	80640	24	74
O_3O_3	10080	8	40	52
	40320	0	24	76
	80640	2	28	70
	80640	0	24	76
	161280	2	28	70

Table 5.11: Distribution of common neighbours for end points of edges

	1	a	b^E	c^E	d^E	e^E	f^E	b^O	d^O	e^O	f^O
1	1	a	b^E	c^E	d^E	e^E	f^E	b^O	d^O	e^O	f^O
a		$1, a, c^E, f^E$	b^E, d^E, e^E	a, c^E, f^E	b^E, d^E, e^E	b^E, d^E, e^E	a, c^E, f^E	b^O, d^O, e^O	b^O, d^O, e^O	b^O, d^O, e^O	$\emptyset$
b^E			$1, a, c^E, d^E, f^E$	b^E, d^E, e^E	a, b^E, c^E, d^E, e^E	a, c^E, d^E, f^E	b^E, e^E, f^E	b^O, d^O, e^O, f^O	b^O, d^O, e^O	b^O, d^O, e^O	b^O
c^E				$1, a, c^E, f^E$	b^E, d^E, e^E	b^E, d^E, e^E, f^E	a, c^E, f^E	b^O, d^O, e^O	b^O, d^O, e^O	b^O, d^O, e^O, f^O	e^O
d^E					$1, a, b^E, c^E, d^E, e^E$	a, b^E, c^E, d^E, e^E	$\emptyset$	b^O, d^O, e^O	b^O, d^O, e^O	b^O, d^O, e^O	$\emptyset$
e^E						$1, a, c^E, d^E, f^E$	b^E, e^E, c^E	b^O, d^O, e^O	b^O, d^O, e^O	b^O, d^O, e^O, f^O	e^O
f^E							$1, a, c^E, f^E, b^E$	b^O, e^O, f^O	$\emptyset$	b^O, e^O	b^O, f^O
b^O								$1, a, b^E, c^E, d^E, e^E, f^E$	a, b^E, c^E, d^E, e^E	$a, b^E, c^E, d^E, e^E, f^E$	b^E, f^E
d^O									$1, a, b^E, c^E, d^E, e^E$	a, b^E, c^E, d^E, e^E	$\emptyset$
e^O										$1, a, b^E, c^E, d^E, e^E, f^E$	c^E, e^E
f^O											$1, f^E$

Table 5.12: The fusion law for the 780 dimensional algebra with axis fixed by the centralizer of an involution in class $2B$

	1	0	a	b^E	c	d^E	e^E	f^E	b^O	d^O	e^O
1	1	$\emptyset$	a	b^E	c	d^E	e^E	f^E	b^O	d^O	e^O
0	0	a	b^E	c	d^E	e^E	f^E	b^O	d^O	e^O	f^O
a			$0, a, c, f^E$	b^E, d^E, e^E	a, c, f^E	b^E, d^E, e^E	b^E, d^E, e^E	a, c, f^E	b^O, d^O, e^O	b^O, d^O, e^O	b^O, d^O, e^O
b^E				$0, a, c, d^E, f^E$	b^E, d^E, e^E	a, b^E, c, d^E, e^E	a, c, d^E, f^E	b^E, e^E, f^E	b^O, d^O, e^O, f^O	b^O, d^O, e^O	b^O, d^O, e^O
c					$0, a, c, f^E$	b^E, d^E, e^E	b^E, d^E, e^E, f^E	a, c, f^E, e^E	b^O, d^O, e^O	b^O, d^O, e^O	b^O, d^O, e^O, f^O
d^E						$0, a, b^E, c, d^E, e^E$	a, b^E, c, d^E, e^E	$\emptyset$	b^O, d^O, e^O	b^O, d^O, e^O	b^O, d^O, e^O
e^E							$0, a, c, d^E, f^E$	b^E, c, e^E	b^O, d^O, e^O	b^O, d^O, e^O	b^O, d^O, e^O, f^O
f^E								$0, a, b^E, c, f^E$	b^O, e^O, f^O	$\emptyset$	b^O, e^O
b^O									$0, a, b^E, c, d^E, e^E, f^E$	a, b^E, c, d^E, e^E	a, b^E, c, d^E, e^E, f^E
d^O										$0, a, b^E, c, d^E, e^E$	a, b^E, c, d^E, e^E
e^O											$0, a, b^E, c, d^E, e^E, f^E$
f^O											

Table 5.13: Fusion law for the extended algebra obtained from $2B$ with $\lambda = 1$

	1^E	0	a^E	b	2^E	c^E	1^O	a^O	2^O	c^O
1^E	$1^E, 0, b$	$1^E, 0, b$	$a^E, 1^E, c^E$	$1^E, 0, b$	2^E	$c^E, a^E,$	$1^O, a^O$	$1^O, a^O, c^O$	2^O	c^O, a^O
0		$1^E, 0, b$	$a^E, 2^E, c^E$	$1^E, 0, b$	$a^E, 2^E, c^E$	$a^E, 2^E, c^E$	$\emptyset$	$a^O, 2^O, c^O$	$a^O, 2^O, c^O$	$c^O, a^O, 2^O$
$a^E,$			$1^E, 0, b, 2$	$a^E, 2^E, c^E$	$0, a^E, b, 2^E, c^E$	$1^E, 0, b, 2^E$	a^O	$1^O, a^O, 2^O, c^O$	$a^O, 2^O, c^O$	$a^O, 2^O, c^O$
b				$1^E, 0, b$	$a^E, 2^E, c^E$	$1^E, a^E, 2^E, c^E$	c^O	$a^O, 2^O, c^O$	$a^O, 2^O, c^O$	$1^O, a^O, 2^O, c^O$
2^E					$1^E, 0, a^E, b, 2^E, c^E$	$0, a^E, b, 2^E, c^E$	$\emptyset$	$a^O, 2^O, c^O$	$a^O, 2^O, c^O$	$a^O, 2^O, c^O$
c^E						$1^E, 0, b, 2^E$	c^O	$a^O, 2^O, c^O$	$a^O, 2^O, c^O$	$1^O, a^O, 2^O, c^O$
1^O							$1^E, 0$	$1^E, a^E,$	$\emptyset$	b, c^E
a^O								$1^E, 0, a^E, b, 2^E, c^E$	$0, a^E, b, 2^E, c^E$	$1^E, 0, a^E, b, 2^E, c^E$
2^O									$1^E, 0, a^E, b, 2^E, c^E$	$0, a^E, b, 2^E, c^E$
c^O										$1^E, 0, a^E, b, 2^E, c^E$

Table 5.14: Fusion law for the extended algebra obtained from $2B$ with $\lambda = 18/25$

	1	0^E	a	b	c^E	d^E	e^E	0^O	c^O	d^O	e^O
1	$1, 0^E$	$1, 0^E$	a	b	c^E	d^E	e^E	$\emptyset$	c^O	d^O	e^O
0^E		$1, 0^E, a, b, c^E, e^E$	$a, 0^E, c^E, d^E$	$0^E, c^E, d^E, b$	$c^E, 0^E, a, b, d^E$	d^E, a, b, c^E, e^E	$0^E, d^E, e^E$	$0^O, c^O, d^O, e^O$	$0^O, c^O, d^O$	$0^O, c^O, d^O$	$e^O, 2$
a			$1, 0^E, a, b, e^E$	a, b, e^E	$0^E, c^E, d^E$	$0^E, c^E, d^E$	a, b, e^E	$0^O, c^O, d^O$	$0^O, c^O, d^O$	$0^O, c^O, d^O$	$\emptyset$
b				$1, 0^E, a, b, e^E$	$0^E, c^E, d^E$	$0^E, c^E, d^E, e^E$	a, b, d^E, e^E	$0^O, c^O, d^O$	$0^O, c^O, d^O$	$0^O, c^O, d^O, e^O$	d^O
c^E					$1, 0^E, a, b, c^E, d^E$	$0^E, a, b, c^E, d^E$	$\emptyset$	$0^O, c^O, d^O$	$0^O, c^O, d^O$	$0^O, c^O, d^O$	$\emptyset$
d^E						$1, 0^E, a, b, c^E, e^E$	$0^E, d^E, b$	$0^O, c^O, d^O$	$0^O, c^O, d^O$	$0^O, c^O, d^O, e^O$	d^O
e^E							$1, 0^E, a, b, e^E$	$0^O, d^O, e^O$	$\emptyset$	$0^O, d^O$	$0^O, e^O$
0^O								$1, 0^E, a, b, c^E, d^E, e^E$	$0^E, a, b, c^E, d^E$	$0^E, a, b, c^E, d^E, e^E$	$0^E, e^E$
c^O									$1, 0^E, a, b, c^E, d^E$	$0^E, a, b, c^E, d^E$	$\emptyset$
d^O										$1, 0^E, a, b, c^E, d^E, e^E$	b, d^E
e^O											$1, 0^E, e^E$

Table 5.15: Fusion law for the extended algebra obtained from $2B$ with $-\frac{1}{75}$

	1	0	a	b^E	c^E	d^E	e^E	b^O	c^O	d^O	e^O
1	$1, 0$	$1, 0$	a	b^E	c^E	d^E	e^E	b^O	c^O	d^O	e^O
0		$1, 0, a, e^E$	$0, a, e^E$	b^E, c^E, d^E	c^E, b^E, d^E	d^E, b^E, c^E, e^E	$0, a, e^E, d^E$	b^O, c^O, d^O	b^O, c^O, d^O	d^O, b^O, c^O, e^O	e^O, d^O
a			$1, 0, a, e^E$	b^E, c^E, d^E	b^E, c^E, d^E	b^E, c^E, d^E	$0, a, e^E$	b^O, c^O, d^O	b^O, c^O, d^O	b^O, c^O, d^O	$\emptyset$
b^E				$1, 0, a, c^E, e^E$	$0, a, b^E, c^E, d^E$	$0, a, c^E, e^E$	b^E, d^E, e^E	b^O, c^O, d^O, e^O	b^O, c^O, d^O	b^O, c^O, d^O	b^O
c^E					$1, 0, a, b^E, c^E, d^E$	$0, a, b^E, c^E, d^E$	$\emptyset$	b^O, c^O, d^O	b^O, c^O, d^O	b^O, c^O, d^O	$\emptyset$
d^E						$1, 0, a, c^E, e^E$	$0, b^E, d^E$	b^O, c^O, d^O	b^O, c^O, d^O	b^O, c^O, d^O, e^O	d^O
e^E							$1, 0, a, e^E, b^E$	b^O, d^O, e^O	$\emptyset$	b^O, d^O	b^O, e^O
b^O								$1, 0, a, b^E, c^E, d^E, e^E$	$0, a, b^E, c^E, d^E$	$0, a, b^E, c^E, d^E, e^E$	b^E, e^E
c^O									$1, 0, a, b^E, c^E, d^E$	$0, a, b^E, c^E, d^E$	$\emptyset$
d^O										$1, 0, a, b^E, c^E, d^E, e^E$	$0, d^E$
e^O											$1, 0, e^E$

Table 5.16: Fusion law for the extended algebra obtained from $2B$ with $-\frac{1}{20}$

	1	0^E	a	b^E	c	d^E	e^E	0^O	b^O	d^O	e^O
1	$1, 0^E$	$1, 0^E$	a	b^E	c	d^E	e^E	$\emptyset$	b^O	d^O	e^O
0^E		$0^E, 1, a, b^E, c, d^E$	$0^E, b^E, d^E, a$	$0^E, a, b^E, c, d^E$	$0^E, b^E, d^E, c$	$0^E, a, b^E, c, d^E$	e^E	$0^O, b^O, d^O$	$b^O, 0^O, d^O$	$0^O, b^O, d^O$	e^O
a			$1, 0^E, a, c, e^E$	$0^E, b^E, d^E$	a, c, e^E	$0^E, b^E, d^E$	a, c, e^E	$0^O, b^O, d^O$	$0^O, b^O, d^O$	$0^O, b^O, d^O$	$\emptyset$
b^E				$1, 0^E, a, c, e^E$	$0^E, b^E, d^E$	$0^E, a, c, e^E$	b^E, d^E, e^E	$0^O, b^O, d^O$	$0^O, b^O, d^O, e^O$	$0^O, b^O, d^O$	b^O
c					$1, 0^E, a, c, e^E$	$0^E, b^E, d^E, e^E$	a, c, e^E, d^E	$0^O, b^O, d^O$	$0^O, b^O, d^O$	$0^O, b^O, d^O, e^O$	d^O
d^E						$1, 0^E, a, c, e^E$	b^E, c, d^E	$0^O, b^O, d^O$	$0^O, b^O, d^O$	$0^O, b^O, d^O, e^O$	d^O
e^E							$1, 0^E, a, b^E, c, e^E$	$\emptyset$	b^O, d^O, e^O	b^O, d^O	b^O, e^O
0^O								$1, 0^E, a, b^E, c, d^E$	$0^E, a, b^E, c, d^E$	$0^E, a, b^E, c, d^E$	$\emptyset$
b^O									$1, 0^E, a, b^E, c, d^E, e^E$	$0^E, a, b^E, c, d^E, e^E$	b^E, e^E
d^O										$1, 0^E, a, b^E, c, d^E, e^E$	c, d^E
e^O											$1, 0^E, e^E$

Table 5.17: Fusion law for the extended algebra obtained from $2B$ with $-\frac{4}{25}$

	1	0^E	a	b^E	c	d^E	e^E	0^O	b^O	d^O	e^O
1	$1, 0^E$	$1, 0^E$	a	b^E	c	d^E	e^E	$\emptyset$	b^O	d^O	e^O
0^E		$1, 0^E, a, c, d^E, e^E$	$a, 0^E, b^E, d^E$	a, c, d^E, e^E, b^E	$0^E, b^E, d^E, e^E, c$	$0^E, a, b^E, c, d^E$	$0^E, b^E, e^E, c$	$0^O, b^O, d^O, e^O$	$b^O, 0^O, d^O$	$0^O, b^O, d^O$	$e^O, 0^O$
a			$1, 0^E, a, c, e^E$	$0^E, b^E, d^E$	a, c, e^E	$0^E, b^E, d^E$	a, c, e^E	$0^O, b^O, d^O$	$0^O, b^O, d^O$	$0^O, b^O, d^O$	$\emptyset$
b^E				$1, 0^E, a, c, d^E, e^E$	$0^E, b^E, d^E$	$0^E, a, b^E, c, d^E$	$0^E, b^E, e^E$	$0^O, b^O, d^O$	$0^O, b^O, d^O, e^O$	$0^O, b^O, d^O$	b^O
c					$1, 0^E, a, c, e^E$	$0^E, b^E, d^E$	$a, c, e^E, 0^E$	$0^O, b^O, d^O, e^O$	$0^O, b^O, d^O$	$0^O, b^O, d^O$	0^O
d^E						$1, 0^E, a, b^E, c, d^E$	$\emptyset$	$0^O, b^O, d^O$	$0^O, b^O, d^O$	$0^O, b^O, d^O$	$\emptyset$
e^E							$1, 0^E, a, c, e^E, b^E$	$0^O, b^O$	$0^O, b^O, e^O$	$\emptyset$	b^O, e^O
0^O								$1, 0^E, a, b^E, c, d^E, e^E$	$0^E, a, b^E, c, d^E, e^E$	$0^E, a, b^E, c, d^E$	$0^E, c$
b^O									$1, 0^E, a, b^E, c, d^E, e^E$	$0^E, a, b^E, c, d^E$	b^E, e^E
d^O										$1, 0^E, a, b^E, c, d^E$	$\emptyset$
e^O											$1, 0^E, e^E$

Table 5.18: Fusion law for the extended algebra obtained from $2B$ with $\lambda = \frac{29}{300}$

	1	a	b	c	d	e	f	g	h	b^O	i	j	k
1	1	a	b	c	d	e	f	g	h	b^O	i	j	k
a		$1,a,d,e,h$	b,c,f,g	b	a	a	b,f,g	b,f,g	a	b^O,c^O,f^O,k	b^O	b^O,f^O,k	b^O,f^O,k
b			$1,a,d,e,h$	a,h	b,f,g	b,f,g	a,d,e,h	a,d,e,h	b,c,f,g	b^O,c^O,f^O,g^O	b^O	b^O,f^O,k	b^O,f^O,k
c				1	$\emptyset$	g	$\emptyset$	e	b	b^O	i	j	k
d					$1,d,e,h$	d,e,h	b,f,g	b,f,g	d,e,h	b^O,f^O,k	$\emptyset$	b^O,f^O,g^O	b^O,f^O,k
e						$1,d,e,h$	b,f,g	b,c,f,g	d,e,h	b^O,f^O,k	k	b^O,f^O,k	b^O,c^O,f^O,g^O
f							$1,a,d,e,h$	a,d,e,h	b,f,g	b^O,f^O,k	j	b^O,c^O,f^O,k	b^O,f^O,k
g								$1,a,d,e,h$	b,f,g	b^O,f^O,k	g^O	b^O,f^O,g^O	b^O,c^O,f^O,k
h									$1,d,e,h$	b^O,c^O,f^O,g^O	b^O	b^O,f^O,k	b^O,f^O,k
b^O										$1,a,b,c,d,e,f,g,h$	a,b,h	a,b,d,e,f,g,h	a,b,d,e,f,g,h
i											$1,c$	f	e,g
j												$1,a,b,c,d,e,f,g,h$	a,b,d,e,f,g,h
k													$1,a,b,c,d,e,f,g,h$

Table 5.19: Fusion law for axes fixed by $2B$ involutions under the product g

Type	O_1	O_2	O_3	O_4
O_1O_1	0	100	0	0
O_1O_2	1	36	63	0
O_1O_3	0	20	80	0
O_2O_2	2	14	42	42
O_2O_3	1	12	31	56
O_2O_4	0	6	28	66
O_3O_3	1	6	47	46
O_3O_4	0	4	36	60
O_4O_4	0	12	24	64

Table 5.20: The distribution of common neighbours of edges of various types

Chapter 6

Conclusion

In this chapter we summarize the whole thesis and discuss some possibilities for future work as well as some limitations faced in the work.

6.1 Overview

In this thesis we have described ways of constructing axial algebras admitting the sporadic simple groups HS and Suz, together with their automorphism groups. The aim of the constructions was to provide a common source of these sporadic simple groups. G. Michler [Mic06, Mic10] has attempted such a scheme by way of an algorithm which he used to construct many of the sporadic simple groups. We have mentioned in the introduction that the sporadic simple groups have been obtained by ad hoc methods under different circumstances and that an open problem post classification, is to find a unifying theory explaining their existence.
The theory of axial algebras is another possible attempt for such a unifying theory. The present work fits into this framework, and as a first attempt, tries to construct axial algebras for the sporadic groups. In particular, it will be very desirable to construct axial algebras for the pariahs. In work not presented in this thesis, we have shown, using methods described in the thesis, that axial algebras for Janko's group J_1 exist.
From the rank 3 graph of Livingstone, backed with character theory, we see that a 56-dimensional module of J_1 supports a commutative algebra structure and that such an algebra admits a unique, up to scalar, symmetric

J_1-invariant bilinear form which associates with the algebra product.
An algebra invariant under J_1 was constructed over the quadratic field $\mathbb{Q}(\sqrt{5})$ by the eigenvector decomposition of the permutation module of the rank 3 permutation representation of J_1 via the association scheme arising from the rank 3 graph. However, in identifying axes, we find that the eigenvalues of idempotents are irrational and do not have nice representations over $\mathbb{Q}(\sqrt{5})$. This creates a computational problem since the computations are messy and consume a lot of memory.

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