A VISUALISATION INTERVENTION IN A GRADE 11 TRIGONOMETRY CLASS

BY

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DEDICATION

This dissertation is dedicated to my late father, to whom I owe everything. He was my fountain of knowledge and guiding light.

AND

To my amazing wife who always motivated and supported me throughout this journey.

ACKNOWLEDGEMENTS

All praise and thanks to the Almighty, who guided me and enabled me to strive for excellence. I am truly indebted by his grace, for giving me the strength and fortitude to complete this work.

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DECLARATION

Submitted in fulfilment of the requirements for the degree of

Master's in Education, in the Graduate Programme in, the College of Education

University of KwaZulu-Natal, Durban, South Africa.

I, **Ashraf Khan**, declare that:

1. The research reported in this thesis, except where otherwise indicated, is my original research.

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ACRONYMS

AI	Artificial Intelligence
AR	Augmented Reality
CAPS	Curriculum and Assessment Policy Statement
DBE	Department of Basic Education
HOD	Head of Department
KZN	KwaZulu-Natal
MPI	Mathematical Processing Instrument
MV	Mathematical Visuality
NSC	National Senior Certificate
PCK	Pedagogical Content Knowledge
STEM	Science, Technology, Engineering and Mathematics
TIMMS	Trends in Mathematics and Science Study
TUI	Tangible User Interfaces
VR	Virtual Reality

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ABSTRACT

The study investigated the impact of a visualisation intervention programme in a grade 11 trigonometry classroom. The literature confirms that visual strategies bring about much-needed improvement in trigonometry at secondary schools. Working within the interpretivism research philosophy, the study involved participants from a secondary school in the Umlazi District. To determine the impact of the visualisation intervention, distributed cognition was employed as the theoretical framework to understand its effectiveness. This research used a qualitative approach to gain deep insight into how visualisation impacts the effective teaching and learning of trigonometry. Purposive sampling was used to select participants and the data was generated through research instruments that included a questionnaire, pre-and post-tasks, structured and semi-structured interviews, and observation. The data were analysed following a thematic approach and findings were established using a thematic analysis of the data. The emerging themes included the strong need for collaboration and visual dynamism with beliefs and attitudes that are centred on learner development in trigonometry. This study showed a strong positive association between the use of visual representations and trigonometry achievement. The existing corpus of literature regarding this topic is in its nascent stages hence the need for this study was critical. Based on these findings, the study recommends the need for employing visual techniques to mitigate the abstraction of concepts in a sustained manner and to create and promote a culture for learning trigonometry, in addition to prioritising professional development of educators regarding visual strategies in pedagogical instructional design. Educators must encourage the use of visual strategies, not to supplant algebraic manipulation but to supplement it. For the general trend to change in the South African educational landscape, dynamic visual strategies must be implemented to transcend the microcosm of mathematics teaching and learning.

CHAPTER 1

BACKGROUND AND OVERVIEW

In the past decade, South Africa has been noticeably declining in the quality of its National Senior Certificate (NSC) mathematics results (Motha, 2021). From 2019 to 2022, only 55% of grade 12 mathematics candidates on average obtained a pass mark of 30% or above, from which 2,7% achieved distinctions (NSC diagnostic report, 2022). However, the dismal state of mathematics education became internationally known when South Africa was rated 142 out of 148 countries in grade nine mathematics education in 2014 according to the World Economic Forum Global Information Technology Report (Erasmus, 2015). This poor performance in grade nine mathematics persisted in Trends in Mathematics and Science Study (TIMMS) 2019 where South Africa was rated 37 out of 39 countries (Reddy, 2021). The current dysfunctional state of the education system as evidenced by Robertson and Graven (2020) is invariably linked to pedagogy. The traditional method of teaching trigonometry, hinged on the chalk and talk approach, favours algebraic manipulation over visual methods and this augments the challenge of improving pedagogical practices and learner performance (Bansilal et al., 2017; Brijlall & Ndlovu, 2013; Brijlall, 2017). Adhikari and Subedi (2021) noted that learners encountered serious difficulties in conceptualising and visualising abstract concepts because of poor foundational knowledge in trigonometry. In many cases, learners could not link the definitions and written statements to diagrams, and this led to an ever-widening gap in their understanding. The current literature was found to be in its infancy stage regarding how to alleviate this learning challenge and it is to this end, that the researcher embarked on this current work.

Trigonometry involves the study of the relationship between the sides and angles of a triangle thus making it an indispensable tool in mathematical modelling. It was for this reason that an amendment in the trigonometry mark allocation was implemented which saw an increase from approximately 40 to 50 marks in the 2019 NSC mathematics paper two (CAPS abridged section 4, p.136). It was noted that the vast majority of learners performed poorly in this section and according to the NSC diagnostic report 2023 on mathematics paper 2, a plethora of weak areas was identified. The report highlighted learners' conceptual and procedural errors which invariably exposed their foundational incompetencies. According to Mudaly (2014), the meaning and construction of mathematical thought are inextricably linked to semiotic mediation which is the use of symbols that have assigned meanings to coalesce external reality to the internal mental process. Mathematical thinking is sequentially associated with internalising the information resulting in a cognitive process that makes the study of trigonometry inextricably concomitant with visualisation (Mokotjo & Mokhele, 2021). The art of forming mental imageries is regarded as a lens of the mind's eye and this requires a psychological process where learners must allow themselves to be creative in thinking about the abstract concept (Adhikari & Subedi, 2021). The learning of trigonometry can be less daunting if learners are taught to visualise more and to reason within themselves.

1.1 Purpose of the study

Dorier and Maass (2020) suggest mathematical inquiry begins with curiosity when something is unsatisfactory and inadequate. The formulation of questions becomes necessary to address uncertainties, namely the gaps in understanding, flaws in reasoning, ambiguities and unresolved problems that may surface when existing knowledge is insufficient to find solutions. To this extent, the current education system lacks strategic measures to steer the minds of learners in the mathematical direction, and this could spell disaster for a developing South Africa (Akon-Yamga, 2024). The Umlazi District and the country in general require competent engineers, statisticians, and computer scientists. Given this situation, it became vital for this study to address the impact of a visualisation intervention on effective teaching and learning of trigonometry at the grade 11 level. The egregious situation of mathematics education in the country requires bold initiatives to improve learner performance in mathematics. By emphasising mathematics education, the economic landscape will be reshaped, providing a fertile ground for much needed development and prosperity (Vasilevska, 2024).

Mudaly and Rampersad (2010) assert that the advancing of learners' conceptual understanding of mathematics is greatly influenced by visual representations be it in the form of a physical or mental process. Trigonometry being a fundamental tool in solving real-life problems requires such visual thinking. This research explores the current pedagogical trends in a grade 11 trigonometry class and the impact of a visualisation intervention in the trigonometric application. A radical shift from abstract rote learning to an engaging substantive and meaningful learning environment is a priority since the grounding of mathematical skills must be firmly established.

1.2 Location of the Study

The study was conducted at a secondary school in a Durban suburb called Chatsworth. It is affiliated with the Umlazi District in KwaZulu-Natal. The school has a learner population of approximately one thousand learners with a staff of forty-one members. The school fee is R1500, and learners come from different socioeconomic backgrounds. The research study was conducted over three weeks.

1.3 Aim of the study

The study aimed to investigate the effects of a visualisation intervention in effective teaching and learning of trigonometry in a grade 11 class.

1.4 Objectives

The objectives of this study are:

1. To understand the different strategies available from prior research for determining ways to improve the understanding of trigonometry using visual stimuli.
2. To administer an intervention programme, implementing visualisation.
3. To measure the impact of using visualisation in trigonometry.

1.5 Research questions

1. What visual strategies are available for the teaching of grade 11 trigonometry?
2. What intervention programmes can be developed in trigonometry to make the content and solutions more visual?
3. What is the impact of such an intervention programme?

1.6 Approach to Study

Research methodology is essential in formulating a path to conducting the study. The study is streamlined by appropriate procedures and techniques used in the selection, collation, and analysis process. Almazan (2021) asserts that this is critical for the evaluation of the findings as it affects trustworthiness and credibility. This study follows a rigorous methodology as outlined and explained.

1.7 Significance of the Study

The current study sought to explore the effects of a visualisation intervention in a grade 11 trigonometry class. The outcomes of this study may benefit the mathematics learners, Head of Departments (HODs), and principals, who require much-needed strategies and recommendations to improve teaching and learning in mathematics education. The enrichment of teaching and learning trigonometry and providing solutions to the problems haunting the school mathematics community was core to this study.

Furthermore, the outcomes of this study advocated a reformation in the policy and procedures for strategic planning in educational institutions, as well as increasing calls for accountability. The findings may benefit education administrators in the KZN Education Department by identifying ways to work with different school leadership, educators, learners, and other education stakeholders in improving the effective teaching and learning of mathematics at secondary schools in the Umlazi District. The study further helped educationists on ways to become more innovative and creative in developing and promoting a vibrant mathematics learning environment.

Finally, this study adds to the body of knowledge regarding the teaching and learning of trigonometry which is critical in the effective teaching and learning of mathematics at secondary schools. Such transformational leadership skills have been found to be successful in the health sector as well as in higher education institutions internationally and in the sub-Saharan African region. Thus, this study contributes to the literature on how visualisation intervention improves the effective teaching and learning of trigonometry at secondary schools and provides a framework for achieving better results.

1.8 Rationale of the study

The researcher's motivation was sparked by the perennial decline in mathematics education over the past decade. My professional work experience, spanning over fifteen years, allowed me to have a holistic understanding of the dynamics of the classroom, the conditioning of learners in the mathematics arena, the nuances of the curriculum, and the challenges of the mathematics education fraternity in general. This provided an impetus to perform a detailed analysis of the Grade 12 mathematics results in the NSC examinations from 2014 to 2023 because this exit level is heavily predicated on a firm foundation of Grade 11 skills. Fluency in rudimentary Grade 11 mathematical skills is therefore an imperative for an improvement in Grade 12 learner performance. I discovered a chronic negative trend specifically in trigonometry where the conceptual, procedural, and problem-solving abilities of learners were issues of epidemic proportion. In my opinion, a radical shift in pedagogy was necessary to bring about a diametrical change.

1.9 Limitations of the study

Siahaan (2023) defines research limitations as a systematic bias that could affect the outcomes of the study. Limitations that influenced the research include:

- Time constraints impacted the findings. The duration of interviews, assessments, and the filling out of questionnaires must be convenient to mitigate the spread of Covid-19.
- Participants lack the interest to go further in the research.
- Financial constraints. The rising cost of petrol, data, and stationery had to be factored in.
- The causal relationship between visualisation and the use of technology was challenged. Loadshedding affected the use of technology in the visualisation intervention.

1.10 Chapter outline

The structure and streamlining of the research study are discussed with an overview of the six chapters as follows:

Chapter 1

This chapter presents the background, purpose, significance, location, and aim of the study. The rationale of the study provided the motivation and impetus for conducting the research. This chapter also shows the careful alignment and consideration of the research objectives to the research questions. The limitations of the study were highlighted to show which characteristics impacted the research study.

Chapter 2

A critical review of literature deconstructed the curriculum showing the inadequacies of the current system in the South African landscape. The initial focus of the literature review was to understand the current pedagogical methods in the teaching and learning of trigonometry. The seminal work in the field of visualisation and visual thinking was then carefully examined, analysed, and evaluated with the intent of showing the relevance of this study.

Chapter 3

The philosophy that underpins the research problem is discussed. Distributed cognition being the theoretical framework used in this study, provided the foundation for guiding the processes in trigonometric understanding, with visualisation strategies being the bedrock.

Chapter 4

This chapter on research methodology delineates the processes and techniques used in the selection, collation, and analysis of the study. This all-encompassing systematic plan was a key component in providing an effective resolution to the research problem.

Chapter 5

This chapter provides a discussion of the data collected and highlights the significance of the outcomes. The analysis and interpretation of the findings were closely connected to the research objectives.

Chapter 6

The inferences and recommendations were outlined, illustrating the strong positive impact of the visualisation intervention in the grade 11 trigonometry classroom.

1.11 Synopsis

This chapter introduced the background of the research problem and outlined the purpose of this study. The research objectives were closely linked to the aim of the study. The research questions were formulated in close consideration of the aims and objectives. The dire need for this study was highlighted in the benefits and significance of this research. The rationale of the study was fuelled by my professional work experiences considering the crisis in mathematics education in South Africa. The limitations of the study accentuated the variables and constraints that affected the length and breadth of this research. The subsequent chapter will investigate the literature review, drawing upon the seminal research done in the field of visualisation in mathematics education.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In the mathematics realm, there is an extant connection between the complex and the concrete, the theory, and the visual. As learners meander through the Grade 11 trigonometry curriculum, they find themselves engaging with the intricate relationships between angles and sides of triangles, moving between the application of trigonometric concepts and computational fluency (Ngu & Phan, 2020). This critical juncture marks an innate transition in their mathematical journey, demanding a more intense engagement with concepts that transcend the conventional boundaries of the classroom (Spangenberg, 2023). This introduction seeks to illuminate the role of visualisation in the Grade 11 trigonometry class, uncovering how visualisation not only bridges the gap between abstraction and reality but also magnifies the understanding and appreciation of the complex world of trigonometric principles. However, alongside its potential lies a terrain riddled with challenges (Baidoo & Luneta, 2023). This literature review will embark on a journey to explore the undulating landscape of a visualisation intervention in a Grade 11 trigonometry class, unpacking the available visual strategies, benefits, drawbacks, and future trajectories that can radically transform the path ahead.

The contemporary corpus of literature was exhaustively explored with the intent of showing the relevance of this study concerning the teaching and learning trigonometry using visual representations. The road map used in this chapter sheds light on educationist perspectives on visualisation, current visual strategies used to enhance the trigonometric understanding that validates its impact, its challenges, and considerations. The initial focus of the literature review was to understand the objectives of the grade 11 trigonometry curriculum, then explore the dynamic interplay between visualisation and trigonometry, dissect the best pedagogical practices, and examine the inextricable challenges. The literature review concludes with strategies of transformational visualisation measures that underscore their significance in trigonometry and mathematics in general.

2.2 Grade 11 Trigonometry

In the rich tapestry of mathematics education, trigonometry emerges as a thread that weaves together the intricate realm of geometry, algebra, and modelling (Arhin & Hokor, 2021). As learners venture into Grade 11 trigonometry, their study blossoms into a transformative journey that empowers them to comprehend the fundamental relationships between angles and sides of geometric shapes. Guided by the Curriculum and Assessment Policy Statement (CAPS), Grade 11 trigonometry establishes a firm connection between theory and practical applications (Ozudogru, 2019).

2.2.1 Navigating the Trigonometry Syllabus

Grade 11 trigonometry marks a considerable advancement of the foundational concepts introduced in earlier grades. It serves as a bedrock for the more advanced mathematical concepts that await learners in their academic journey. At its core, trigonometry unlocks profound relationships between angles and the ratios of sides of triangles, thereby providing deeper geometric insight into the natural world and specific scientific disciplines.

According to Tombs (2018), the CAPS syllabus for Grade 11 trigonometry is a harmonious blend of theoretical understanding and practical application. Learners not only engage with the complex principles of trigonometric functions, identities, equations, and 2-dimensional problems but also investigate the invariable manifestations of these concepts in mathematical modelling (Noreen & Rana, 2019; Jelatu et al., 2019). According to Jahan et al. (2020), the merging of theory with practical application serves to empower learners with the discerning ability to problem solve, thereby crystallising the relevance of trigonometry in mathematics education. The grade 11 CAPS syllabus includes the study of the following topics with a concomitant timeframe:

Table 2.1

Grade 11 Mathematics Annual Teaching Plan

Topic	Curriculum Statement	Duration
Identities and Reduction formulae	✓ Theorem of Pythagoras ✓ Trig ratios (SOH-CAH-TOA) ✓ CAST diagram ✓ Reduction formulae for: ✓ $\sin(90^\circ \pm \theta)$, $\cos(90^\circ \pm \theta)$, $\sin(180^\circ \pm \theta)$, $\cos(180^\circ \pm \theta)$, $\tan(180^\circ \pm \theta)$, $\sin(360^\circ \pm \theta)$, $\cos(360^\circ \pm \theta)$, $\tan(360^\circ \pm \theta)$, $\sin(-\theta)$, $\cos(-\theta)$, $\tan(-\theta)$ ✓ identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$ ✓ Special Angles	12 days
Trig equations and general solutions	✓ General solutions and ✓ specific solutions in a given interval	4 days
Functions	✓ Sketch and interpret graphs of the form: ✓ $y = a\sin k(x + p) + q$, ✓ $y = a\cos k(x + p) + q$, ✓ $y = a\tan k(x + p) + q$ At most two parameters at a time	7 days
Solving triangles	Apply sine, area, and cosine rules in 2-dimensional problems	7 days

Note. From KwaZulu-Natal Mathematics ATP*,* department of education, 2024, p. 2.

According to the CAPS syllabus, the Grade 11 trigonometry curriculum consists of a range of goals intended to provide students with procedural and conceptual fluency.

2.2.2 Conceptual Understanding

Conceptual understanding of trigonometry involves acquiring comprehensive knowledge regarding concepts and relationships of sides and angles of triangles that underlie this branch of mathematics (Dundar, 2015). The CAPS curriculum insists on analytical skills, spatial insight, and logical reasoning which promotes an explorative approach in contrast to rote learning (Gilligan et al., 2017). Asomah (2023) asserts that conceptual fluency requires a deep understanding of the fundamentals of trigonometry to attain flexibility and ease in problem-solving. Dewi et al. (2024) insist that generative learning, which involves the integration of real-world problems into the curriculum, enhances conceptual understanding and critical thinking, with its core being the use of multiple visual representations. This is supported by Muhali et al. (2021) who further argue for a rigorous visualisation intervention to achieve a guided discovery in forming solutions.

2.2.3 Procedural Understanding

Learner competence in understanding procedures and steps is the primary goal required to solve problems. The algorithmic calculation of sides, angles, and areas of triangles requires fluency, and this can be achieved by linking algebraic, geometric, and graphical reasoning (Urrutia et al., 2019). Learners must be able to manipulate relevant formulae and graphical representations to derive solutions with ease. The associated formulae must relate to abstract concepts which in turn should be linked to visual representations (Ngu & Phan, 2020). This is supported by the National Senior Certificate (NSC) diagnostic report 2022, which states that learners predominantly grappled with basic calculations, incorrect application, and conceptual inadequacies. This problem is rooted in the inability to use visual representations to aid in preventing such errors and the invariable question becomes what does visualisation in trigonometry entail?

2.3 Visualisation

The consensus amongst educationists is that visualisation involves the process of creating mental images or physical representations to understand and solve problems (Cooper et al., 2017). Enhancing understanding using visual aids, diagrams, graphs, charts, technologically mediated tools, and semiotics lies at its core (Moleko, 2021). Since logical reasoning and deduction is innate to navigate the manifolds of trigonometry (Krutetskii, 1976), visualisation becomes essential to support conceptual clarity and memory retention in addition to promoting an exploratory approach.

In addition to augmenting trigonometric understanding, visualisation makes learning more gripping and enjoyable for learners (Mudaly, 2021). According to Mainali (2021), the use visual tools in improving learner proficiency and performance in mathematics is beyond reproach, and in recognition of its importance, empahsises the use of construction, charts, diagrams, graphical representations etcetera. He further distinguishes between the tangible (physical external visualisation) and insightful (images in the mind for internal visualisation) to effectively navigate mathematical challenges. This is consistent with Arcavi's (2003, p.26) definition that visualisation requires "reflection upon pictures, images, and diagrams in our mind, on paper…". The seminal work of Mudaly (2010) supplants Arcavi's (2003) definition by further expounding on visual literacy, placing much emphasis on the thinking and analysis that is inherent to visualisation be it physical or mental. Mudaly and Narriadoo (2023) posit that the interpretation of the object is dependent on how an object is viewed using our sense of sight. They further argue that visualisation mitigates the abstraction of mathematical concepts in a way that is simple for learners to grasp. Presmeg (1986, p.45) alludes to this notion as "concretising the referent" which she defines as the "embodiment of an abstract idea into a concrete image". Complex concepts become more comprehensible in this concretising process by leveraging visual tools to enhance understanding. This is supported by Fischbein (1987, as cited in Arcavi, 2003, p.28) who regards this concretising process as an "essential factor for creating the feeling of self-evidence and immediacy". Learners' perception of mathematical truths being self-evident and immediate, concretised by visual techniques, provides a channel of accessibility to deeper insight and understanding. Rif'at (2020) asserts that concrete visual representations precipitate relatability which supports the effective communication of complex ideas and concepts. However, Presmeg (1985) recognises that learners could be placed on a "continuum" regarding their propensity for the use of visual imagery. This is supported by Mainali (2021) who broadens the discourse of learners having diverse cognitive preferences by underscoring different levels of engagement when dealing with visual representation. This propensity invariably paved the formation of a new construct, mathematical visuality (MV).

Presmeg (1986) defined MV as "the extent a person prefers to use visual methods" when problem-solving in mathematics. Suwarsono (1982) developed the Mathematical Processing Instrument (MPI) to measure the individual's mathematical visuality, providing invaluable insight into the learner's cognitive styles and learning modalities. The benefits of MPI include increased learner motivation and engagement, improved spatial reasoning, conceptual understanding etcetera as evidenced by Mainali (2021) who further acknowledges the potential of MPI in instructional design, where the needs of the individual learner based on their strengths and weaknesses can be addressed.

2.4 Types of visual imagery used in trigonometry

Several forms of imagery are employed in trigonometry instruction to support learning and problem-solving. The many forms of visual imagery aid in the recognition of patterns, provide rise to analogical reasoning, and aid in the acquisition of abstract concepts. Here is an exploration of some of the popular visual forms used in trigonometric instruction.

2.4.1 Concrete Imagery

Concrete imagery pertains to the use of physical objects or manipulatives to represent and perceive concepts (Widada, 2016). In trigonometry, concrete imagery is a nexus between abstraction and understanding, through the use of visual and concrete representations. A practical classroom example is to manipulate the sides and angles of right-angled triangles with protractors and other measuring tools, whereby learners can explore the ratios of special angles concretely. According to Helske et al. (2021), visual representations in trigonometry play an invaluable role in highlighting its spatial nature, as such, concrete imagery, such as diagrams, manipulatives, and illustrations, supplements the comprehension of spatial relationships and transformation. Mosese and Ogbonnaya (2021) contend that the introduction of abstract trigonometric representations before concrete imagery enhances learners' understanding, underscoring its role in solidifying understanding after grasping abstract concepts. Technology-mediated tools such as the GeoGebra software, provide an interactive visualisation mechanism of various trigonometric concepts, enabling learners to deeply engage with concrete representations to enhance their understanding (Wijaya et al., 2024). Additionally, Yavuz and Yuca (2017) emphasised that a blend of abstract, visual, and concrete representations in trigonometry is required for a thorough comprehension of trigonometric concepts, suggesting that concrete imagery plays a crucial role in enhancing abstract concepts. This invariably implies that a link between abstract mathematical ideas and their real-world applications is created by tangible images. In summary, the concept of concrete imagery in trigonometry refers to the utilisation of observable and palpable depictions to augment comprehension, establish a connection between abstract and concrete ideas, and ease the visualisation of spatial correlations and transformations intrinsic to the subject.

2.4.2 *Pictorial Imagery*

Pictorial imagery is the process of explaining trigonometric principles by the use of visual aids such as charts, graphs, diagrams, or drawings (Pope, 2023). This can comprise graphs of functions, number lines for trigonometric inequalities, geometric shapes etcetera. Understanding trigonometric functions and their graphical representations is greatly aided by the use of pictorial imagery. An in-depth knowledge of trigonometry necessitates the capacity to transition between abstract concepts, and visual, and tangible representations (Yavuz & Yuca , 2017). Intending to help with understanding the nature of functions, graphical representations of trigonometric functions, for instance, offer an observable study of the distinctive characteristics of trigonometry. As a practical demonstration, educators may instruct learners to colour trigonometric functions differently to distinguish between them, highlighting key points such as turning points, points of intersection with the axes, and other graphs as well as asymptotes where applicable. Other examples may include drawing flowcharts, diagrams, and perhaps a balancing scale to convey concepts relating to identities and equations. According to Maulyda et al. (2019), the utilisation of technology tools such as GeoGebra has proven to be advantageous in augmenting learners' understanding of trigonometry by employing visual aids. Research has indicated that interactive software tools are beneficial in characterising the types of mathematical representation that appear when learners attempt to comprehend how variations in the parameter values of trigonometric functions correlate to their graphical functions. Additionally, research has revealed that using technology-mediated tools like Desmos and GeoGebra assists learners in becoming more immersed in developing their conceptual command of trigonometric concepts (Hamsah et al., 2021). The study by Hokor et al. (2022) also posits that the perceptions and pedagogical content knowledge of pre-service educators are greatly influenced by pictorial imagery since it boosts their motivation and confidence levels to effectively communicate trigonometric concepts to learners. Polat and Akgün (2020) point out that pictorial imagery can also progress atextual arguments, which are used in trigonometry. Without depending exclusively on the precision of algebraic techniques, these visual proofs are useful for communicating mathematical concepts. Smith et al. (2022) posit that pictorial representation might be interpreted as nonverbal proof, both as a product and a crucial knowledge construct. Further underscoring the vital function that visual aids serve in communicating trigonometric facts, Polat and Demircioğlu (2021) state that pictorial proofs can be applied in the field of trigonometry. Mastery of trigonometric concepts requires the ability to correctly interpret visual proofs and apply the pertinent mathematical skills.

Trigonometry hinges largely on the use of pictorial imagery, which is made attainable by interactive resources like GeoGebra and Desmos. These instruments are essential for ensuring that learners understand the distinctive traits and behaviours of functions. Trigonometry is better understood when classes are more engaging and richer thanks to pictorial representations.

2.4.3 Symbolic Imagery

Symbolic imagery is defined as the utilisation of symbols and notation to precisely communicate concepts and express relationships in a clear, concise depiction (Boaler, 2022). Considering mathematics is a means of expressing ideas and conveying thought, it is essential to comprehend and manipulate symbols, equations, and expressions (Rohid et al., 2019). In trigonometry in particular, where the study of relationships between angles and sides of triangles is represented by algebraic expressions and equations, symbolic imagery plays an integral part in assisting learners to grasp and work with complex ideas. Its significance for problem-solving and overcoming time constraints is highlighted by the comprehension and manipulation of these symbolic representations (Oyoo, 2022). According to Yavuz and Yuca (2017), a thorough mental grasp of trigonometry is demonstrated by the significance of using symbols and notation. This emphasises the magnitude of symbolic imagery in trigonometry since learners must be competent in translating between abstract ideas and symbolic algebraic expressions fluently (Bates et al., 2021). Rosjanuardi and Jupri (2020) emphasised a unit circle-based didactical design for the graph creation and analysis of trigonometric functions. Drawing from fundamental trigonometric ideas, this study highlighted the inherent connection of symbolic imagery used to sketch trigonometric functions, which concretises the relationship between symbolism and visualisation. Symbolic imagery is essential for helping learners engage with abstract mathematical ideas. To gain a thorough understanding of trigonometry, one must be able to navigate between abstract algebraic methods and visual representations. As the foundation for computations and problem-solving, trigonometric symbols are essential for expressing trigonometric functions, identities, angles, and relationships.

2.4.4 Pattern Imagery

Pattern imagery in mathematics is the mental or visual depiction of periodic arrangements, symmetries, or patterns seen in mathematical objects or systems (Towsey et al., 2018). It entails identifying and deciphering visual patterns that result from properties or relationships in mathematics. These patterns can take many different forms, including algebraic structures, sequences, and geometric shapes. In trigonometry, pattern imagery refers to the application of dynamic models, multimodal dynamic representations, and visual representations to help comprehend spatial connections, transformations, and trigonometric functions. Studies have shown that improving learners' trigonometry performance requires the use of interactive technology settings, including graphing calculators or technology-mediated tools such as Geogebra (Nordlander, 2022). Furthermore, it has been observed that using multiple dynamic representations using tactile input supports trigonometry learning, underscoring the importance of merging several sensory modalities in pattern visualisation (Cuturi et al., 2022). According to Mosese and Ogbonnaya (2021), the order in which visual aids are introduced in trigonometry is to first convey the abstract and then offer meaning to it by visual imagery. This emphasises how crucial it is to have physical, visual, and abstract representations in pattern imagery to promote a thorough comprehension of trigonometric ideas. To further emphasise the connection between various forms of imagery in trigonometry, a thorough grasp of the subject necessitates the capacity to shift between abstract, visual, and concrete representations of mathematical objects (Yavuz & Yuca, 2017). Through the integration of pattern imagery into trigonometric discourse, learners can foster a more profound comprehension of intricate ideas, expanding the reach of their trigonometric inquiry. Additionally, by promoting a sense of richness in the complex structures that underpin trigonometric theories and proofs, the aesthetic appeal of pattern imagery can enhance learners' broader appreciation and delight of the subject.

2.4.5 Spatial Imagery

The internal visualisation and manipulation of entities in space are known as spatial imagery (Hoogland et al., 2018). This helps to grasp spatial relationships and transformations and is particularly significant in the realms of 2D and 3D trigonometry. Geometrising trigonometry requires the manipulation and visualisation of spatial configurations, a task that is optimally achieved through spatial imaging techniques (Gilligan et al., 2019; Ilhan & Aslaner, 2020; Kotsopoulos et al., 2017). The research of Fang et al. (2018) acknowledges that articulating pertinent constructions requires visualising patterns and structures as well as using trigonometry concerning spatial imaging. This shows that the ability to seamlessly transition between several representations is facilitated by spatial imagery, which improves comprehension of trigonometric functions and their cyclic characteristics. The visual inquiry into trigonometric concepts, according to Rosjanuardi and Jupri (2020), further accentuated the substantial impact that spatial imagery makes on the process of learning. Additionally, the use of dynamic interactive software such as GeoGebra in teaching trigonometry is invaluable in the explorative process (Pope, 2023; Maulyda et al., 2019). Spatial imagery in mathematics involves the cognitive manipulation and representation of spatial data, playing a crucial role in mathematical development and achievement. Its profound impact on trigonometry performance is evidenced by various studies, underscoring its role in enriching educational delivery and knowledge acquisition.

2.4.6 Temporal Imagery

Understanding and visualising mathematical concepts as they vary over time is known as temporal imagery (Wang et al., 2022). In trigonometry, temporal aspects including amplitude, periodicity, and phase shifts are cognitively visualised and manipulated inside trigonometric functions and equations through the concept of temporal imaging (Fan et al., 2017). According to Daud and Sudirman (2022), music has a significant impact on a variety of mental tasks, including trigonometry activities, indicating that rhythmic auditory stimuli may have a bearing on cognitive processes related to trigonometry, which could include temporal imagery. Though there is limited research on temporal imagery concerning trigonometry, temporal imagery is useful in mathematical analysis and problem-solving (Delahunty et al., 2020).

2.4.7 Transformational Imagery

In trigonometry, the concept of transformational imagery refers to the visualisation and comprehension of spatial transformations, such as rotations, reflections, translations, and dilations, which are essential concepts in this field (Lima et al., 2019). This kind of visualisation helps learners grasp trigonometry by enabling them to manipulate and comprehend shapes and angles cognitively. For learners to understand the links between various angles and sides within a triangle and the impact of trigonometric functions on graphs, transformational imagery is required. Transformational imagery is used in trigonometry to provide dynamic and interactive visualisations as a supplement to more conventional representations.

Some applications in transformation imagery may include trigonometric functions in grade 11 as shown in Table 2.2.

Table 2.2

Parametric changes to three trigonometric functions.

Trigonometric function	Parametric change			
$f(x) = a\sin k(x+p)+q$	Amplitude	Period	Horizontal	Vertical
$f(x) = a\cos k(x+p)+q$	Amplitude	Period	Horizontal	Vertical
$f(x) = a\tan \mathrm{k}(x+p)+q$	Shape	Period	Horizontal	Vertical

Note. Adapted from KwaZulu-Natal Mathematics ATP by the department of education, 2024, p. 4.

2.4.7.1 Translation

The trigonometric functions according to the CAPS curriculum can be shifted left or right at the grade 11 level.

The parameters p and q are affected as follows:

If $p > 0$, the graph shifts left by |p| units.

If $p < 0$, the graph shifts right by |p| units.

If $q > 0$, the graph shifts up by |q| units.

If $q < 0$, the graph shifts down by |q| units.

2.4.7.2 Reflection

The trigonometric functions according to the CAPS curriculum can be reflected across the $x-axis$ or $y-axis$ at the grade 11 level.

$x-axis$ reflection, replace y with $-y$

$y-axis$ reflection, replace x with $-x$

According to the CAPS curriculum, variations can occur with a maximum of two parametric changes in a combined transformation.

2.4.7.3 Dilation

The trigonometric function according to the CAPS curriculum can be dilated horizontally and vertically at the grade 11 level.

For $y = f(mx)\ where\ 0 < m\ < 1, f(x)$ stretches horizontally by a factor of $\frac{1}{m}$.

For $y = f(mx)\ where\ m > 1, f(x)$ shrinks horizontally by a factor of $\frac{1}{m}$.

For $y = nf(x)\ where\ 0 < n\ < 1, f(x)$ shrinks vertically by a factor of n.

For $y = nf(x)\ where\ n > 1, f(x)$ stretches vertically by a factor of n.

Encouraging students to engage in transformational imagery not only enhances their spatial reasoning skills but also deepens their understanding of trigonometric concepts. This visualisation approach fosters a more intuitive grasp of how parametric variations impact the behaviour of trigonometric functions, laying a solid foundation for further mathematical exploration. According to Tella and Sulaimon (2022), learning trigonometry can be improved by origami exercises that enhance understanding of spatial transformation. The call for educators to incorporate both dynamic and visual proofs when teaching trigonometry is in line with the use of transformative imagery. Concepts related to trigonometry and spatial transformations can be better understood with the use of visual and dynamic proofs in addition to symbolic formal reasoning (Mudaly & Reddy, 2016). This method of instruction not only makes learning easier but also improves learners' capacity to manipulate and visualise trigonometric relationships mentally.

2.4.8 Dynamic Imagery

According to Atit et al. (2020), the concept of dynamic imagery in trigonometry entails the interactive manipulation of visual information to comprehend and resolve problems. Abstract conceptual learning is significantly supported by the use of dynamic, interactive software like GeoGebra (Young, 2023). By employing these resources, learners have the opportunity to engage with and enhance their understanding of complex trigonometric principles, including the intricacies and nuances of two-dimensional and three-dimensional problem-solving (Liburd & Jen, 2021). Laudano (2021) argues that learners can efficiently comprehend the spatial linkages and transformations that are inherent in trigonometry with the integration of dynamic proofs and visual manipulatives. The advantage of dynamic imagery is conspicuous in correspondence learning environments, where its ease of use and degree of interactivity aid the process of visual manipulation to enhance the learning experience. For instance, a right-angled triangle with variable angles or side lengths can undergo parametric changes in real time showing learners what changes occur by manipulating specific variables. Dynamic imagery enables learners to observe how trigonometric ratios respond to these changes. This visual exploration promotes an intuitive understanding and concretises the abstraction in trigonometric relationships. Such a learning strategy is in alignment with the developmental needs of learners regarding visuospatial reasoning.

2.4.9 Kinaesthetic Imagery

Kinaesthetic imaging is a technique for learning and remembering mathematical concepts that integrate movement, tactile, and physical experiences. Research indicates kinaesthetic imagery is positively related to verbal learning (Peters, 2019). This is supported by Ali et al. (2019) who further assert that the use of kinaesthetic experiences with visual, auditory, and tactile aids, supplements the learning of trigonometry. This type of imaging is aligned with the cognitive processes required for trigonometry learning since it is linked to the improvement of mental rotation skills, which enhances visual-spatial reasoning and problem-solving skills (Kahraman, 2023). According to Simarmata (2018), the educator as facilitator and motivator must incorporate kinaesthetic imagery into trigonometry lessons since it greatly influences learners' interest, motivation, and performance. Lauer et al. (2016) recognise that kinaesthetic imagery is a priceless tool as a method of instruction for immersive trigonometric encounters because it transcends the limits of visual perception to inspire motion as a tool. As an example, educators can simulate varying degrees of angles of elevation and depression by raising and lowering their arms. This is in line with the interdisciplinary teaching approach that emphasises the diverse uses and consequences of kinaesthetic imagery (Cece et al., 2018). In addition to strengthening comprehension through practical activities, kinaesthetic imagery in trigonometry produces memorable and captivating learning experiences.

2.4.10 Memory Imagery

According to Guarnera et al. (2019), memory imagery in the mathematics domain is the process of forming associations or mental images to help in memorisation of mathematical concepts, formulae, and procedures. Trigonometric mnemonic devices, like SOHCAHTOA, are a prime instance of memory imagery, which is critical to comprehending and remembering relationships, angles, and transformations. Holmes and Dunning (2017) claim that memory imagery, which uses a retrieval style that fosters imagery-based representations, has been specifically found to facilitate the encoding, storage, and retrieval of particular memories. Furthermore, Dijkstra et al. (2017) argue that memory imagery aids in the recall of trigonometric concepts and relationships by demonstrating how vivid visual representations as a performance predictor, rely on neuronal overlap with perception in visual areas, a critical component for mathematical retention and recall. A practical example of this is relating special angles and a unit circle to a Ferris wheel or a clock, the movement of which influences the trigonometric ratios in question. Learners can relate to these recognisable objects thereby reinforcing fundamental concepts. Visual imagery is a fundamental process for cognitive processes associated with episodic and visual working memory, both of which are critical to memory retention of trigonometric concepts (Dawes et al., 2020; Slinn et al., 2023). Effective memory imagery development depends on the control of visual imagery generation, which is based on strata stimulus-response training. A more nuanced visual picture is created by the hierarchical layers of stimuli, spanning from straightforward visual cues to more elaborate and detailed visuals which boosts memory retention (Haj-Yahya, 2023). Jacobs et al. (2022) elaborate on how learners' academic performance is negatively impacted by poor visual working memory, underscoring the functional role memory imagery has on cognition. It is through active engagement to create indelible visual associations, memory imagery surfaces as a priceless tool in the teaching repository for deepening understanding and retention of trigonometry at the grade 11 level. The types of visual imagery used in a classroom setting should be dependent on the level of abstraction of specific subtopics taught in trigonometry therefore integrating a plethora of imagery types to cater to diverse learning modalities can enhance understanding.

2.5 Current visualisation tools used to teach trigonometry

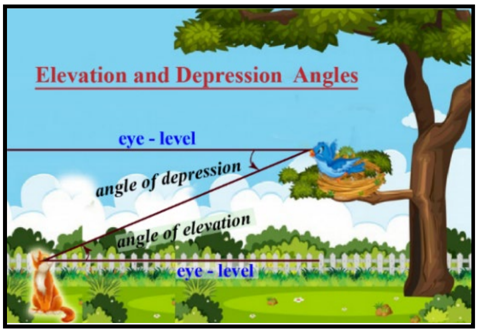
Trigonometry is a mathematical discipline that has both the elements of fascination and frustration for learners worldwide (Khoza & Biyela, 2019). The intrigue lies in its elegant ability to solve real-life problems, but owing to its abstract nature, trigonometry elicits the feeling of frustration in many (Ferrer, 2016; Maula & Nu'man, 2021). However, the light at the end of the tunnel in enhancing the comprehension of trigonometry is the integration of visual stimuli. Visualisation being core to the genetic make-up of the 5E instructional model, namely engagement, exploration, explanation, elaboration, and evaluation, predominates the best pedagogical practices in teaching and learning trigonometry (Adu & Folson, 2023). Prior research firmly establishes that immersive experiences in learning trigonometry are deeply rooted in the use of visual representations (Wijaya et al., 2020). The highly spatial nature of trigonometry is intertwined with visualisation (Gilligan et al., 2017; Ocampo et al., 2023). Yavuz and Yuca (2016) accentuate the importance of teaching abstract concepts with the use of visual representation for grasping the geometric relationships encoded in trigonometric concepts. To this end, the current use of the best visual strategies in the teaching of trigonometry is now explored.

2.5.1 Animated Tutorials and Virtual Reality

The advancement of technological achievements has dawned a new era for enhancing trigonometric understanding. Animated tutorials and virtual reality (VR) experiences immerse learners in trigonometric scenarios, enabling them to interact with the subject matter on a more intense level, producing positive learning outcomes (Rahmawati et al., 2023). Research indicates that such immersive experiences can significantly enhance comprehension by allowing learners to visualise trigonometric principles dynamically and engagingly. Martin and Martin (2015) argue that animated tutorials are highly effective in conveying trigonometric concepts because they create a multisensory dynamic environment. This is further underscored by the research done by Rissanen and Costello (2023) who showed that animated tutorials provided supplementary assistance to learners, which helped them improve their results. Figure 2.1 illustrates a problem-based animated tutorial regarding the angle of elevation and depression.

Figure 2.1

The concept of Angle of Elevation and Depression



Note. From Seifert, K. (2022, p. 46).

Animated tutorials can illustrate real-world applications of trigonometry, such as 2-D and 3-D problems, the periodic nature of trigonometric functions, trigonometric equations, and identities in a visually arresting way (Wijaya et al., 2020). By contextualising trigonometric concepts, animated tutorials make the subject matter more relevant and accessible to learners. The dynamic and visually attractive nature of animated tutorials can enhance learner engagement and motivation, making trigonometry learning more enjoyable and stimulating. Kwon and Lee (2016) highlight the efficacy of animations which allow learners to observe and understand complex trigonometric relationships in a visually intuitive manner. The potential impact of animated tutorials in facilitating the teaching and learning of trigonometry is overwhelming because of the interactive and engaging visual content (Setyowati & Fida, 2021). Vakaliuk et al. (2020) assert that the use of 3D virtual reality can effectively complement traditional teaching methods as shown in Figure 2.2.

Figure 2.2

Virtual reality introduced to a South African classroom

Note. From Solomon et al. (2018, p. 301).

In the realm of blended reality, which hybridises the physical and virtual worlds, VR has shown great promise in its contribution to synergetic learning. Learners can leverage the capabilities of virtual world technology to create high-level learning designs and engage in interactive and immersive learning experiences (Bower et al., 2020). This approach promotes active participation and collaboration among students, enhancing their understanding of trigonometry concepts. Virtual reality (VR) has gained traction in numerous domains, including mathematics education and its increasing utilisation in mathematics education has shown positive results in enhancing learning experiences and outcomes. Contemporary research indicates the pronounced effectiveness of VR-based trigonometry learning albeit with an ethnomathematics strategy (Rahmawati et al., 2022; Su et al., 2022). Lai and Cheong (2022) highlight the positive strides made by virtual reality in the educational domain, associating it with several benefits, such as enhancing learners' motivation levels, analogical-reasoning skills, and multimodal learning. Moreover, VR technologies' immersive capabilities enable users to interact with virtual spaces, meeting the requirement for immersion and collaboration, which is pertinent in mathematics education (Wu et al., 2021). VR has also been found to be beneficial for learners with learning disabilities, providing them with an alternative route for learning through virtual manipulatives (Shin et al., 2023). Virtual reality (VR) technology offers the potential to personalise the instructional design, modify interventions for students with attention deficit disorders, and provide immediate feedback in mathematics teaching (Shanley et al., 2019). Freitas et al. (2020) posit that using VR simulators to teach complex topics has been acknowledged as a viable choice since it makes abstract pedagogical content concrete, gives possibilities for completing tasks, and simulates real-world scenarios. VR has been successfully utilised in secondary school Life Science Education, revealing its flexibility to a variety of educational domains, further supporting the efficacy of VR in mathematics education (Su et al., 2022). Additionally, the use of virtual reality technology to support pre-service mathematics educators has demonstrated great promise in fostering competence and confidence (Lvov & Popova, 2019). VR technology in mathematics education personalises instruction and offers different learning paths to learners with varying degrees of academic potential. VR's advantages in teaching mathematics span across a wide array of disciplines, such as Physical Sciences and special education, demonstrating how versatile it is in an assortment of learning environments.

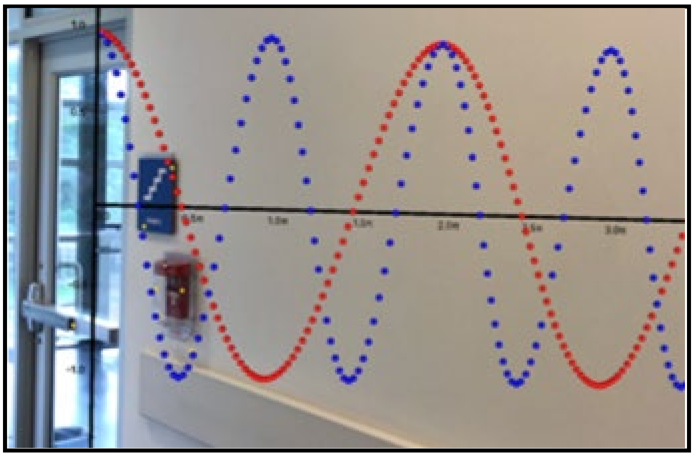
2.5.2 Augmented Reality

Augmented reality (AR) has been found to assist in visualisation in trigonometry and enhance learning experiences. AR technology allows for the coalescing of digital material with the real world, providing interactive visualisation systems that can enhance trigonometry learning (Chao & Chang, 2018). AR technology allows learners to engage with trigonometric concepts on a deeper level and in a more dynamic way, leading to improved understanding and retention of the material. AR visualisation technology allows learners to overlay virtual models to real-life applications which positively impacts learners' conceptual understanding.

AR can also be utilised in educational settings to support spatial awareness and manipulation of holographic objects (Thigpen et al., 2019). AR interfaces allow learners to interact with virtual objects in 3D, allowing them to visualise and manipulate trigonometric concepts more concretely. Digital device applications that utilise AR technology have been developed to facilitate trigonometry learning (Palanci & Turan, 2021). These applications provide interactive and engaging experiences where learners can visualise and manipulate mathematical concepts using AR technology that simulates the natural environment. AR technology greatly influences trigonometry fluency and assists in the transferability and adaptation of skills to other situations (Singh et al., 2023).

Figure 2.3

AR technology being used to teach trigonometric functions



Note. From Hidayat and Wardat (2024, p. 74).

AR has been identified as an invaluable tool for secondary school trigonometry education (Touel et al., 2020). According to Hidayat and Wardat (2024), visualisation is the cornerstone of AR technology which provides a space to be creative within the confines of trigonometric rules. Digital content interfaced with the real environment creates an opportunity for spatial adjustment, application of intuitive reasoning, and immersive interactive experiences.

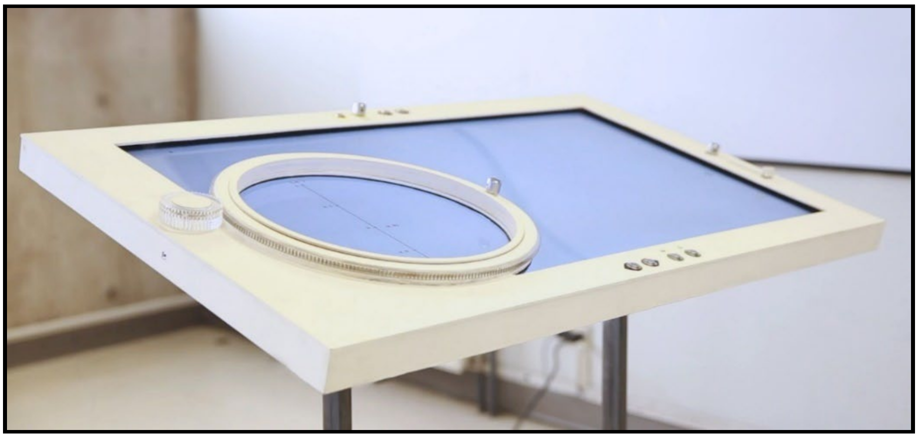
2.5.3 Interactive Software Tools

Interactive software tools may help learners harness visual imagery to connect with abstract ideas, thereby making concepts more tangible and meaningful (Moleko, 2021). Some of the common dynamic mathematics software include GeoGebra, Desmos, Khan Academy, Wolfram Alpha, Trigsted, and many others. Mosese and Ogbannaya (2021) emphasises the importance of learning tools such as GeoGebra in combining pictorial and algebraic thinking. It was found that graphical explanations using trigonometric functions, mediated by mathematics software and graphing calculators, have a significant positive impact on trigonometry learning (Reginald, 2023).

In addition to software, there are also visual tools such as tangible user interfaces (TUIs) which is a more recent approach to teaching trigonometry. TUIs enable learners to interact with digital information by manipulating physical objects tangibly and intuitively (Urrutia et al., 2019). TUIs involve the use of physical objects and manipulatives such as circular objects for exploring special angles coupled with digital information. In contrast to traditional methods which are subject to iterative procedures and cyclic observations, tangible interfaces allow for intuitive, playful, and collaborative experiences (Pires et al., 2022; Patten & Ishii, 2000). The visual stimuli, influenced by prior motor and cognitive skills foster an exploratory approach that helps learners internalise abstract concepts (Sudsanguan et al., 2021).

Figure 2.4

*Tangible mathematics interface designed to learn trigonometry***.**



Note. Sourced from *Technologies in Learning* (2019).

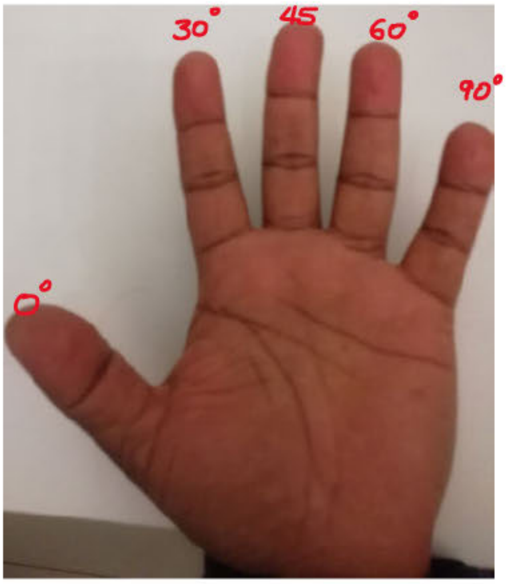
The use of software apps and visual tools may also help in providing visual proofs in trigonometry. According to Laudano (2021), the use of visual proofs instead of formal argumentation can play a significant role in building a stronger understanding of trigonometric concepts. This is supported by Mudaly (2013) who further asserts that through visual proofs, learners have a high level of conviction in the results and this hands-on experience may help to change negative perceptions of learning mathematics.

2.5.4 Visual Mnemonics and Memory Tools

Visual mnemonics, such as mnemonic diagrams and memory aids, provide students with memorable visual cues to recall trigonometric relationships and formulas (Cooper et al., 2018). These aids can simplify complex concepts and make them more meaningful and accessible. Research suggests that didactical designs that incorporate visual mnemonics such as SOH-CAH-TOA and CAST diagrams, can be particularly effective for students struggling with trigonometry, as they offer a simplified framework for understanding and retaining key information (Ureta, 2019). Educators may use a unit circle, table, or right-angled triangles to explain special angles. Hand tricks in trigonometry may also provide an exciting visual method for remembering special angles.

Figure 2.5

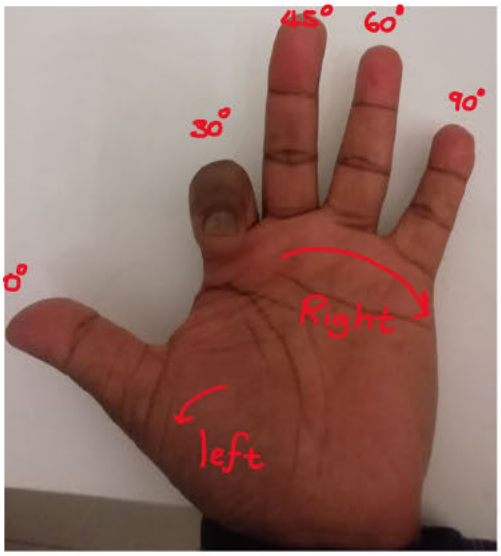
Left hand with an assigned special angle on each finger.



Note. Own construction.

Figure 2.6

Bent finger of chosen special angle.



Note. Own construction.

Special angles are calculated as follows:

$$\sin\,(special\ angle\ represented\ by\ bent\ finger) = \frac{\sqrt{number\ of\ fingers\ on\ the\ left\ of\ bent\ finger}}{2}$$

$$\cos\,(special\ angle\ represented\ by\ bent\ finger) = \frac{\sqrt{number\ of\ fingers\ on\ the\ right\ of\ bent\ finger}}{2}$$

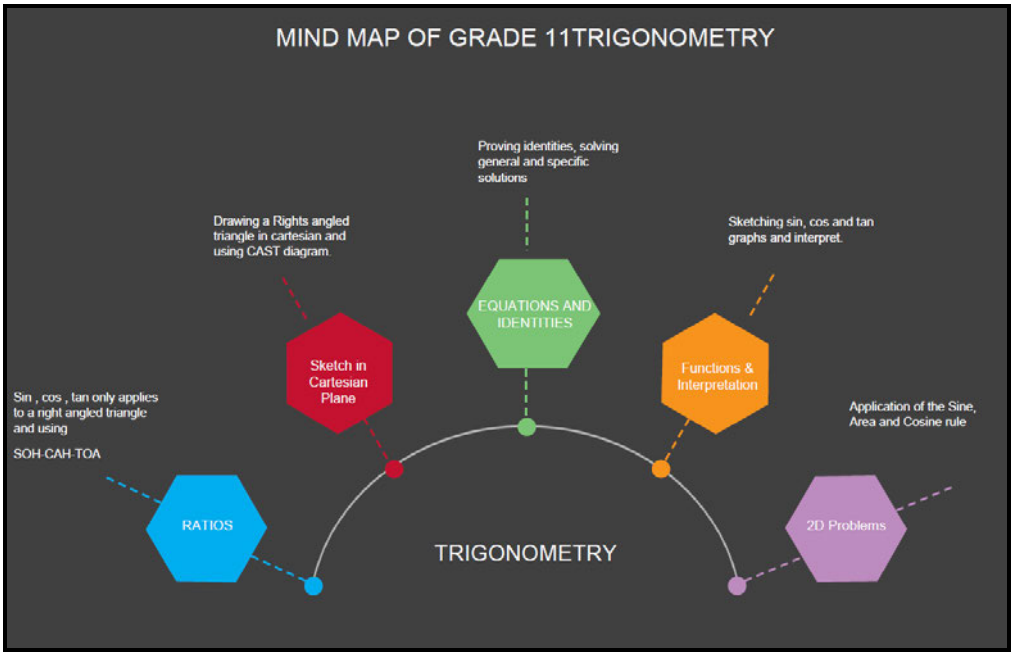
The mnemonic visualisation of the hand is a simple yet effective method of recalling the special angles (Gunadi et al., 2023). Gunadi et al. (2023) further argues that mastering trigonometric concepts requires much practice and visual mnemonics may provide a "helping hand" in this process.

2.5.5 Concept Mapping and Mind Mapping

Concept and mind mapping, be it handwritten on paper or digitally involves visually organising information and relationships between trigonometric concepts. According to Zhang et al. (2024), visual tools help students see the connection between different trigonometric ideas, provide sustainable creativity, and facilitate a more comprehensive understanding. Research has demonstrated the effectiveness of concept mapping in promoting comprehensive learning and enhancing trigonometric concept retention (Buchori & Puspitasari, 2023).

Figure 2.7

Mind Map of Trigonometry



Note. Adapted from the CAPS mathematics document (2011, p. 12).

Improving the understanding of trigonometry through visual stimuli is a promising avenue in mathematics education. Various strategies, as highlighted through prior research, offer valuable insights into how educators and learners can harness the power of visuals to enhance comprehension and information retrieval (Bandera et al., 2018).

Geometric models, animated tutorials, interactive software, real-world applications, visual mnemonics, and concept mapping all contribute to a more robust and engaging learning experience for students. As we continue to explore innovative ways to teach and learn trigonometry, these strategies will undoubtedly play a crucial role in helping students master this fascinating yet challenging topic.

2.6 Analysis of visualisation techniques

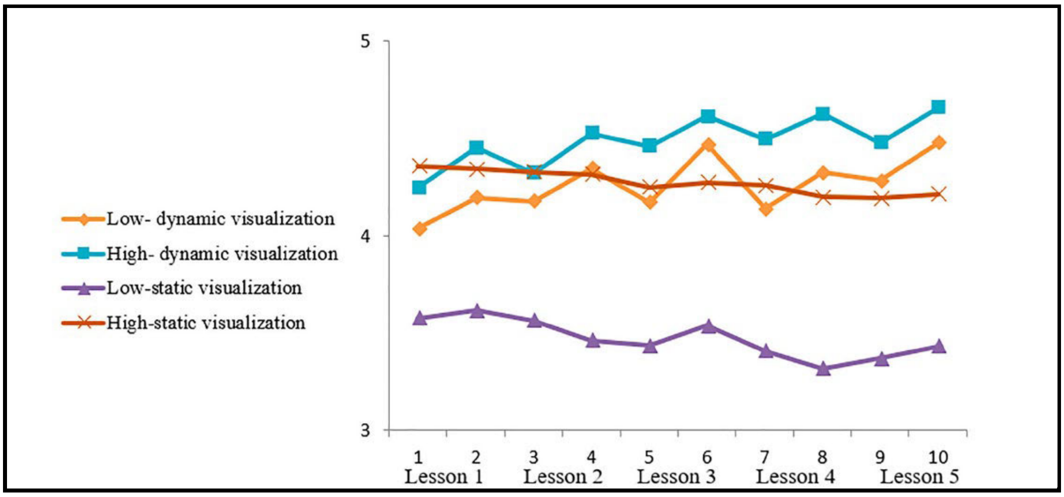
Visual information can be presented in different ways depending on factors such as educational context, cultural diversity in the classroom, varying educator and learner academic competencies, and available resources. The efficacy of teaching trigonometric concepts in a meaningful way is pivoted on the choice of visualisation techniques used by educators. This comparative analysis seeks to discuss the impact of static and dynamic visualisation techniques to enhance learner engagement and understanding.

Static visualisation in trigonometry refers to the use of visual representations of trigonometric concepts in a fixed and non-changing way (Murray, 2017). The absence of dynamic elements in static visualisation includes still images that are non-changing over time such as unit circle diagrams, trigonometric functions, and 2D and 3D models to illustrate trigonometric relationships. This snapshot approach is an indispensable tool to convey specific information only, and due to its relatively low cognitive load in relating fundamental concepts, learners may find it useful (Laudano, 2021). Static visualisation reinforces key concepts in a clear concise manner, making them appropriate for printed resources and under-resourced schools (Lowrie et al., 2021). However, due to its lack of interactivity, and over-reliance on educator explanations, learners find it less engaging. The restricted single-framed illustrations cannot capture the essence of transforming trigonometric functions which further constrains an explorative approach when variations arise. Interpreting 3D models is also confusing for learners since, static visualisation does not effectively communicate spatial relationships, leading to reduced motivation and engagement.

In stark contrast, dynamic visualisation uses interactive visuals to represent and explore trigonometric concepts. Being a powerful and indispensable didactic tool, dynamic visualisation allows learners to manipulate trigonometric parameters instantly, enabling them to observe variations in real-time, thereby concretising abstract concepts (Mahayani et al., 2023). This immersive hands-on experience leads to a positive impact on learner competence (Bornstein, 2020). An instructional design that includes dynamic visualisation was found to increase the attention span of learners, assist in creative reasoning, and accelerate cognitive processing (Granberg & Olsson, 2015; Kaplar et al., 2022). Dynamic visualisation personalises the learning process, allowing learners the freedom to make mistakes and correct themselves without embarrassment (Naidoo & Govender, 2019). Vallo (2021) argues that dynamic visualisation can be adapted to accommodate learners with varying skill levels, where low achievers can focus on basic concepts and high achievers can explore more abstract topics, making it a versatile tool in the mathematics milieu. Mudaly and Budaloo (2016) extol the use of simultaneous visual strategies, encouraging educators to explore the benefits of this strategy in improving learners' analogical reasoning skills. A study by Kohen et al. (2022) shows a significant improvement in mathematical conceptual efficacy among high and low achievers as shown in Figure 2.8 below.

Figure 2.8

The efficacy of dynamic versus static visualisation among high and low achievers.



Note. From Kohen et al. (2022, p. 766).

The high and low achievers who used dynamic visualisation showed an increasing trend in efficacy whereas the high and low achievers who used static visualisation did not show any significant trend of note. It was also observed that the gap between the high and low achievers using static visualisation grew noticeably. The gap between high and low achievers using dynamic visualisation decreased considerably. This study showed that the low achievers in the dynamic visualisation group benefited substantially. However, Palanci and Turan (2021) advocating for static visualisation, insist that the overreliance on dynamic visualisation may inhibit comprehensive understanding if educators ignore diverse learning approaches. Studies by Kohen et al. (2022) indicate that introducing static visualisation before dynamic visualisations produces outcomes that are more systematic and explorative. Dynamic visualisation overemphasises the use of technology, with a lack of technical training and support in addition to time constraints that may pose an overbearing challenge. Disparities in education may become more apparent with some learners from historically marginalised communities not having access to technology.

The educator's preference for a certain visualisation technique significantly influences learning achievements. The learning objectives should dictate the appropriate use of static or dynamic visualisation methods. Fixed representations are effective in conveying foundational concepts but may be unsuccessful for parametric changes. Dynamic visualisation introduces an interactive dimension in trigonometry which allows learners to manipulate variables and observe instantaneous mathematical phenomena. To prevent overreliance and distractions from the use of technology, educators must balance instructional design in consideration of learning objectives and outcomes. Both static and dynamic visualisation has a part to play in creating immersive and meaningful experiences.

2.7 Cultivating Mathematical Thinking Using Visualisation

Roslan and Ahmad (2017) posit that visualisation skills are imperative for successful outcomes in science, technology, engineering, and mathematics (STEM) subjects. Learners who are sturdy visualisers are better equipped to comprehend and manipulate 3D objects, understand, and apply trigonometric concepts, and solve complex problems in the real world.

Cultivating trigonometric thinking using visualisation involves fostering learners' ability to explore, analyse, and reason about trigonometric concepts visually. Visualisations provide a concrete way to connect abstract mathematical ideas with real-world representations. Educators may begin with an introduction relating to real-life examples. This will harvest curiosity about how trigonometry is applied in mathematical modelling, prompting learners to observe patterns and relationships (Lestiana et al., 2022). Educators must ignite interest by asking key questions, encouraging learners to make practical connections, and sharing multiple visual representations to accentuate the various trigonometric concepts. Presenting minor variations in visual representations will assist learners in seeing its impact on concepts. Visual metaphors and analogies will help learners connect abstract concepts to individual experiences. Collaboration is also an essential aspect of teaching and learning, allowing learners to self-regulate and share their interpretation of visualisation ideas and to present arguments for their conclusions (Sholihah & Firdaus, 2023). This is supported by Jelatu et al. (2019) who further identify technology as a supporting tool for a visualisation intervention in learning trigonometric concepts.

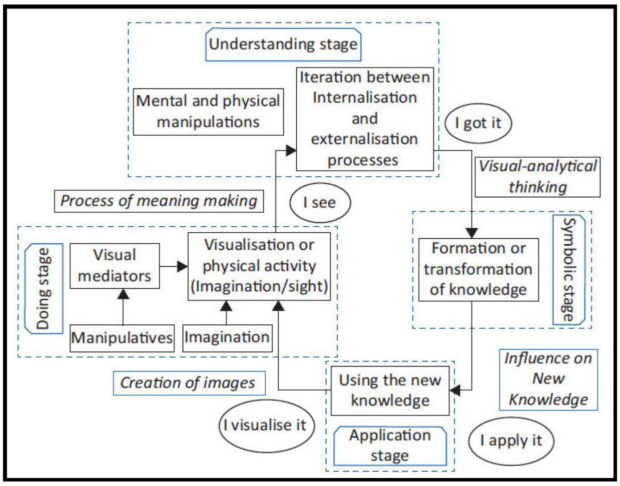
By incorporating dynamic visualisation strategies, educators can create an environment where students engage deeply with mathematical concepts through meaningful experiences. This approach encourages critical thinking, and problem-solving, and fosters a deeper understanding of abstract mathematical ideas.

2.8 Iteration in visualisation

Iteration in trigonometry involves the repetitive and adaptive process of producing, honing, and revising visual depictions to enhance understanding. The progressive attempts for visual clarity and precision seek to harmonise mathematical abstraction and problem-solving (Marita et al., 2021). The iterative process includes adjustment to visual elements until it enhances its communicative power (Walton, 2018). The refinement allows learners to change the specifications according to the different variables at play, filtering the essentials from non-essentials. Bekene et al. (2022) posit that iteration provides learners with the opportunity of identifying and correcting errors which directly affects and influences the accuracy and integrity of the visual elements. Figure 2.9 shows the thinking cycle used in the iterative process.

Figure 2.9

Thinking cycle in the iterative process



Note. From *Constructing mental diagrams during problem-solving in mathematics*, Mudaly, 2021, p. 3.

Visual imagery, be it physical or mental, stimulates the reflection process using prior knowledge, thereby influencing meaning construction (Mudaly, 2010). The new knowledge assimilates and corrects prior assumptions using visual analytical thinking which aids in the internalisation process. The cyclic iterative process, predicated on appropriate visual stimuli, plays a significant role in fostering conceptual understanding and developing problem-solving skills.

2.9 Response to the visualisation intervention

Learners with different learning modalities namely visual, auditory, and kinesthetic, respond to visualisation interventions in diverse ways. It is crucial to understand the learning preferences of individual learners and design instructional strategies accordingly since they influence learner engagement and understanding (Mostert & Roberts, 2022). According to Reid (1987), visual learners are inclined to excel in a visualisation-rich environment because they engage deeply with diagrams, charts, graphs, and other visual representations. Laudano (2021) asserts that visual representations are the primary tools used by visual learners to process information, and as such are heavily dependent on visual cues for effective memory retention. Visualisation helps them identify patterns and relationships, allowing for a more immersive experience.

Auditory learners, however, find visual approaches less natural, and primarily use verbal explanations and discussions to process information (Purbaningrum & Setyaningrum, 2023). Educators can still strive to fuse verbal explanations with visual representations to show a connection. The use of audiovisual resources mitigates such challenges by accommodating visual and auditory learners (Sell & Kaschak, 2009).

Kinesthetic learners prefer hands-on activities to learn concepts. The use of physical manipulatives, movement, and touch is the primary means of processing information (Baykan & Nacar, 2007). Educators can include kinesthetic elements in visualisation by using physical models or simulations. Learners can engage in these tactile experiences by manipulating and experimenting to enhance their understanding (Nancekivell et al., 2020). According to Hernandez et al. (2020), multimodal learners who prefer multiple modalities can benefit from educators incorporating visual representations, and verbal explanations with physical activities.

Creating an all-inclusive personalised learning environment is necessary to foster learner engagement and achieve positive outcomes in learning trigonometry.

Although learners may have their individual learning preferences, they can also benefit from exposure to different learning styles. A well-balanced approach that includes various teaching methods, tailored for individual needs, ensures that all learners can engage, learn, and succeed. It is worthwhile to note that the effectiveness of a visualisation strategy is dependent on specific features of the visualisation instructional design, the individual learning style, and *a priori* knowledge. Further research is required that focuses on the response of different types of learners to visualisation interventions in trigonometry.

2.10 Culturally relevant and inclusive visualisations

The use of culturally relevant and inclusive visualisations is crucial for ensuring that the subject matter resonates with a diverse range of learners and promotes positive learning outcomes (Spangenberg, 2023). Visualisations that include relatable cultural contexts, examples, and references are more likely to capture learners' attention and interest. When learners study content that mirrors their background and experiences, they can see the relevance and applicability, making the learning experience more meaningful (Utami & Hwang, 2022). Han (2019) argues for indigenous knowledge to be integrated into the curriculum in a more robust, appropriate, and acceptable way to make learners feel more valued and included. According to Mudaly (2018), the CAPS curriculum for mathematics does acknowledge the value of indigenous knowledge systems but fails to harvest knowledge from the national context. It therefore becomes expedient to decolonise the curriculum by introducing culturally responsive pedagogy.

Figure 2.10

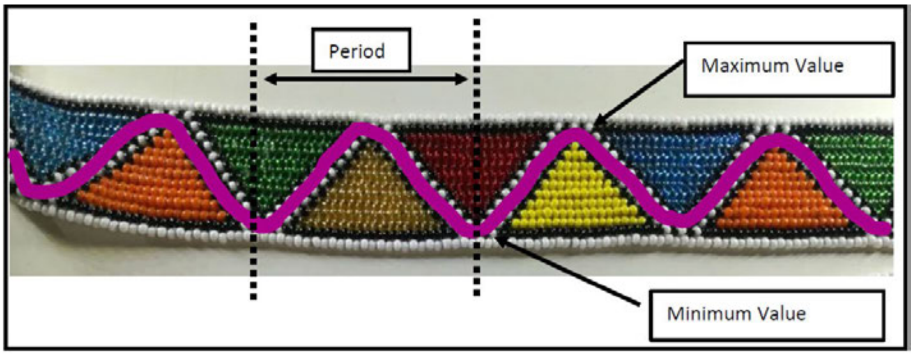
African beadwork

Note. From Kahts, Africa beadwork (2023).

The African bead work can be adapted to educate learners about sine and cosine functions having a maximum and minimum value as well as the concept of period and amplitude as illustrated in Figure 2.11.

Figure 2.11

Cosine function



Note. Adapted from *Africa beadwork* (2023).

Culturally inclusive visualisations help challenge stereotypes and biases that can perpetuate dichotomies in education. By presenting a wide range of visual representations, educators can challenge preconceived notions and advance a more accurate depiction of various cultures. Learners may find it easier to grasp abstract concepts when they are presented in familiar contexts that are compatible with their cultural backgrounds. This supportive environment mitigates trigonometric anxiety and promotes positive attitudes (Barroso et al., 2021). The learner's cultural connection to the content influences their levels of motivation and engagement (Andrews, 2020). This is supported by Mudaly (2018) who challenges the current curriculum to be recontextualised without diminishing the importance of Eurocentric mathematics.

Visualisations that are culturally inclusive, allow learners to collaborate whilst being sensitive to each other's beliefs and values, fostering greater teamwork and a broader understanding when they share their interpretation and perspectives (Prabowo et al., 2022). This helps prepare students to be informed and open-minded global citizens. Creating visualisations that highlight and celebrate diversity sends a message that all students belong and have a place in the classroom. This contributes to a more inclusive and equitable learning experience. When learners see themselves and their cultures represented in educational materials, it can positively impact their self-esteem and motivation to learn. This is especially important for students from historically marginalised backgrounds. Inclusive visualisations can encourage students to critically analyse and question their assumptions and biases. They promote discussions about cultural differences, stereotypes, and the importance of respectful interactions.

Creating culturally relevant and inclusive visualisations is a powerful way to ensure that education is accessible, engaging, and meaningful to all students. It supports diversity, equity, and inclusion goals, fostering a learning environment where every student can thrive.

2.11 Educators perspectives on the impact of incorporating visualisations

Educators' views and perspectives on the effectiveness and prudence of including visualisation strategies in their teaching can be directly linked to their age, experiences, teaching practices, available resources, and personal pedagogical ideologies (Opie, 2019). Although opinions may vary, the consensus amongst educators who embrace the constructivist approach sees visualisation as a means to concretising complex trigonometry concepts (Agormor et al., 2022).

According to Urrutia et al. (2019), learners tend to be more attentive and enthusiastic during lessons that involve interactive and visually stimulating content as observed by educators. Furthermore, the value of visualisation for their ability to show real-world applications of trigonometry cannot be over-emphasised (Laudano, 2021). Learners see visualisations as a bridge between theoretical concepts and practical situations. Masela and Subekti (2020) noted whilst educators recognise that visual learners, in particular, benefit from this approach, learners from all learning modalities appreciate the visualisation approach. Schafer (2023) argues that the role of the educator in using appropriate visual aids strongly influences the success of the learning process. Jelatu et al. (2019) insist that visualisation must be used to encourage active engagement and must be seen as a tool to prompt learners to ask questions, stir curiosity, explore, and experiment.

However, challenges such as the availability of technology and resources needed for effective visualisation strategies are highly concerning. Resource disparities, among schools and learners add to the woes (Raw & Ismail, 2021). Limited access to digital devices, internet connectivity, training, technical support, and loadshedding are major impediments to technology-induced visualisation. Furthermore, educators are overwhelmed with completing the syllabus, leading to reluctance to use technology-mediated tools for visualisation due to time constraints. Pedagogical alignment between the visualisation intervention and teaching objectives is another factor that influences the integration of visualisation approaches, if educators think that it does not contribute to positive learning outcomes (Bannerjee & Murthy, 2018). Technology may potentially be distracting and serve as a hindrance, thereby making its use in the classroom more daunting.

A large body of educators believes that visualisations are most effective when integrated strategically, complementing multiple pedagogical approaches rather than making them obsolete (Lau et al., 2022; Fauziah, 2020). Educators often emphasise the relevance of flexibility and situational adaptation, in preference of having a repository of teaching strategies, visualisation being the centrefold tool, to accommodate different topics and learner needs (Astuti et al., 2020). Fokuo et al. (2023) further argue for differentiated instruction but acknowledge educators' lack of pedagogical content knowledge (PCK) as a possible setback.

Reflective teaching, based on learner feedback, may also help gauge the impact of visualisation strategies, with positive and constructive responses from learners being a motivation (Bornstein, 2020). Educators use such responses to adjust the instructional design to improve engagement in lessons. Professional development workshops are also viewed as opportunities for guidance and support on effectively including visualisation approaches in teaching (Botha et al., 2023). Although the views on the impact and practicality of visualisation approaches are varied, the consensus amongst educators is that it provides meaningful and engaging learning experiences when tailored visual strategies contextually address the needs of individual learners. According to the study by Nabie et al. (2018), visualisation strategies amongst teacher trainees were seen positively. This study indicates, based on pre-service educator perceptions and insights, a trajectory that is geared toward the use of visual strategies in mathematics classrooms.

2.12 Disadvantages of a visualisation intervention in trigonometry

Mathematical cognitive ability is a crucial determining factor in conceptual understanding, procedural fluency, and problem-solving. One approach to cultivating deeper mathematical thought is through visual stimuli, which involves using visual imagery, be it physical or mental, to enhance understanding. In the context of trigonometry, visualisation augments the comprehension of geometric relationships and solving of trigonometry problems. While visualisation strategies can be incredibly effective for enhancing the understanding of trigonometry, there are also drawbacks and considerations, one of which is saturated visual-spatial memory.

Visual-spatial working memory is defined as the cognitive ability to encode, store, preserve, and retrieve information from memory regarding spatial relationships (McAfoose & Baune, 2009). It is particularly essential for the cognitive processes that include recognition, navigation, and mental imagery (Haji, 2019). Individuals can mentally manipulate spatial information and hold it temporarily in memory to problem-solve. In as much as visualisation has the potential to enhance mathematical understanding, it significantly depends on strong visual-spatial working memory. In contrast, learners with weak visual-spatial memory are not adept at manipulating visual representations to problem-solve (Stillman, 2017). However, research indicates that the bounded capacity of visual-spatial working memory can be a drawback in complex problem-solving situations (Weijer-Bergsma et al., 2015). This limitation can impede the efficacy of cultivating mathematical thought through the exclusive use of visualisation strategies in trigonometry which is a sense of overreliance.

An overreliance on visual representations may also limit learners' ability to develop complex mathematical thinking. Trigonometry involves working with abstract concepts such as angles, ratios, functions, identities, equations, and 2D problems and as such, showing complete dependence on visual representations may restrict learners' ability to think semiotically and manipulate abstract mathematical ideas (Alshwaikh, 2018). This overreliance on visual representations may restrict learners' ability to think critically, generalise, and transfer their understanding of trigonometry beyond a given visual context (Urrutia et al., 2019). Learners may become exceedingly dependent on visualisation and struggle to grasp important concepts without them. This problem is exacerbated when learners are exposed to trigonometry problems in text-based or non-visual forms. Visualisations may also trivialise conceptual understanding because rote learners may choose to memorise visual content, without a thorough understanding of underlying concepts. Some learners exhibit poor or incorrect use of visual representations, struggling to make connections, especially in high-order questions. Basic errors regarding interpreting and incorrect construction of visual imagery, if not vetted timeously, reinforce misconceptions. Financial constraints and poor school infrastructure are serious impediments to a technology-based visualisation intervention. Technology has become an integral part of the 21^{st} century mathematics classroom and schools that are privy to such resources have been proven to be more successful (Karim & Zoker, 2023). Research has shown that the lack of technology in historically disadvantaged schools due to budget constraints exposes educational disparities. Having limited or no access to sophisticated technology such as interactive smartboards, tablets, computers, and even internet connectivity, negates the use of advanced technology-mediated visualisation strategies thereby limiting the learning process. Many learners also have low socioeconomic status, making digital devices with appropriate software less accessible for education purposes, resulting in poor academic achievement (Vadivel et al., 2023).

Educators also require training and technical support to effectively integrate technology into their teaching (Barroso et al., 2021). However, poor infrastructure, inadequate budgets, and/or apathy towards technology, coupled with limited access to professional development, workshops, or technical support, lead to underutilisation and devaluation of technology-based visualisation strategies (Yu et al., 2022). Resistance to change by some "orthodox" educators to use technology, possibly due to low confidence or fear, is also highly concerning as it significantly reduces learning opportunities and engagement (Li & Tsai, 2022). Furthermore, some educators have an increased reliance on traditional methods, namely the chalk-and-talk approach, thereby inhibiting innovative pedagogical approaches.

Visualisation in trigonometry can sometimes be highly demanding on the cognitive loading of learners. Cognitive load in the context of visualisation is defined as the mental effort required to process, interpret, and manipulate spatial information (van Nooijen et al., 2024). While visual representations can augment understanding, complex visualisations can often overwhelm, divert, and discourage learners from salient mathematical concepts (Khalil et al., 2024). This excessive cognitive load can harm learners' ability to engage in higher-order thinking and abstract problem-solving questions. It becomes imperative to not only depend on visualisation, as it may not always optimise trigonometric thinking.

While visualisation can be invaluable in cultivating trigonometric thinking, it is important to recognise its limitations and drawbacks. Visual-spatial working memory limitations, overreliance on visual representations, lack of transferability, and excessive cognitive load are some of the challenges associated with cultivating mathematical thinking using visualisation in trigonometry. To obviate these limitations, educators should incorporate a spectrum of instructional strategies that promote both visual and analytical thinking, allowing students to develop a balanced understanding of trigonometry from all learning modalities. By integrating multiple approaches, students can cultivate mathematical thinking that extends beyond visual representations and enhances their problem-solving abilities in trigonometry and other mathematical domains.

2.13 Overcoming disadvantages of a visualisation intervention in trigonometry

Mitigating the drawbacks of visualisations in Grade 11 trigonometry is challenging. It requires a thoughtful approach that balances the benefits of visuals with pedagogical strategies to address the potential challenges. Emphasis must be placed firmly on fundamental concepts and principles of trigonometry alongside visualisations leading to engaging discussions and interactive collaboration that highlights visual patterns and relationships.

To overcome drawbacks, educators must have the "know-how" of multiple pedagogical practices which can integrated with visual strategies to problem-solve real-world manifestations (Lowrie et al., 2021). Providing such an indelible practical learning experience ultimately leads to positive learning outcomes. According to Dockendorff and Solar (2018), the collaborative approach, prompting learners to actively interact with visualisations, encourages critical thinking and deeper exploration of the concepts. Educators may feel the need to incorporate multiple representations with minor variations to explain concepts in varying contexts alongside textual explanations and hands-on activities to support different learning modalities (Shoba, 2020). Educators need to identify common misinterpretations and misconceptions of visualisations and unequivocally correct them, whilst encouraging learners to question, verify, and critically reflect on their construction and interpretation of the visuals.

Skilled educators begin with elementary visualisation strategies like using charts, easy-to-follow diagrams, and basic manipulatives and gradually introduce more complex ones to avoid overwhelming students. The instructional design needs to include scaffolding to support complex visuals, in addition to learners being constantly reminded about the limitations and assumptions of visualisations. Class discussions should be centred around the creation of accurate visual depictions to cultivate mathematical thought to concretise understanding (Sun et al., 2021). Learners must be encouraged to explore additional primary and secondary resources and interrogate different visualisation techniques.

Financial resources are scarce, therefore the involvement of local businesses, communities, and other private organisations is crucial (Samuel, 2020). Schools need to engage in aggressive fundraising campaigns to gather funds to procure and upkeep appropriate technological equipment. Grant opportunities from educational foundations must be explored and all efforts must be made to partner with tech companies and non-profit organisations for assistance. The school must harness the strengths of the local community to assist with technical support and provide security measures. Educators from academically successful schools should be invited to host professional development workshops to train educators on effectively using visualisation strategies in the classroom. By integrating these visual strategies, educators can create an exciting and engaging learning experience. The benefits of visual learning with appropriate instructional methods must be highlighted, encouraging the buy-in from all educators. Educators who are resistant to change must have a platform to raise concerns and objections (Abas & David, 2019). These concerns must be tactfully addressed to offer solutions, guidance, and support to quelle anxiety and fear about incorporating visualisations with technology-mediated tools.

2.14 Future Trajectories Regarding Visualisation

The immense advancement in technology presents an opportunity for profound reform in trigonometric visualisation. Emerging trends indicate that the future of learning will be shaped by innovative pedagogy and visualisation tools integrated with technology (Mosese & Ogbonnaya, 2021). Many affluent educational institutions have already started the transformation journey by becoming fully digital.

The future trajectories of the educational fraternity include the use of tangible user interfaces (TUIs), sophisticated mathematics software such as GeoGebra, problem-based learning approaches linked to the real world, graphical calculators for verifying functions, and appropriate didactical designs (Borba, 2021). Li et al. (2022) provides a strong case for the leveraging of visual representations with interactive and simulation tools to enhance learner engagement. The introduction of virtual reality (VR) and augmented reality (AR) is expected to revolutionise the visualisation of trigonometry by creating an immersive learning experience, allowing learners to explore trigonometric concepts in an unimaginable way (Luo et al., 2024). Singh and Riedel (2016) posit that artificial intelligence (AI) platforms will dominate the personalisation of individual learning experiences by tailoring visuals for individual learning modalities. Learners' interactions with visual materials using interactive simulations and gamification are also showing growing interest, both of which make learning more fun and enjoyable (Lee et al., 2022).

Collaborative and virtual social platforms have already materialised due to the advent of Covid 19. Future visualisation strategies will prioritise and emphasise collaborative learning, allowing learners to interact in virtual spaces. These platforms will be an impetus for dynamic discourses, fostering a community-based approach to learning trigonometry. By integrating these advancements, the future trajectory of visualisation in trigonometry is expected to drastically transform learning by making it more interactive, personalised, and applicable to real-world scenarios. These technological innovations will not only enhance understanding but also foster a deeper appreciation for trigonometry by making it more accessible and engaging for learners of all backgrounds and learning styles.

The advancement and design of trigonometric visualisations will be greatly facilitated by current developments in research on interactive technologies and mathematics education. Interdisciplinary research collaborations that encompass insights from cognitive science, psychology, computer science, and education are plausible features of future trends in trigonometry visualisation. These collaborations aim to enhance our understanding of visual learning processes and optimise visualisation techniques for mathematics education. The continuous development of more enriching, didactically sound, and multi-faceted visualisation strategies hold exciting prospects for the future. As these dynamic trends unfold, researchers, educational practitioners, and curriculum developers must collaborate in shaping the future of mathematics learning.

2.15 Synopsis

The review explored the wide corpus of literature regarding a visualisation intervention intended to invigorate and enrich the learning experience in trigonometry. Initially, the grade 11 trigonometry curriculum was analysed and assessed, paving the way for defining visualisation in the educational context and discussing current visual strategies used in the teaching and learning of trigonometry. Drawing upon contemporary research and best pedagogical practices, a range of visualisation strategies were investigated. These strategies included interactive software, augmented and virtual reality, conceptual and mind mapping, and visual mnemonics. The review went further to investigate the use of culturally relevant visual representations. This was followed by the challenges of a visualisation strategy and possible ways to overcome such disadvantages. The review culminated in the future trajectory of visualisation strategies in the trigonometry classroom.

CHAPTER 3

THEORETICAL FRAMEWORK

3.1 Introduction

The preceding chapter presented a comprehensive and critical review of the extant literature that informed this study. This chapter focuses on the theoretical framework that underpins the conceptual structure which guides the research process. Attention now converges to the mainstay of the research design, data collation, and analysis process, intending to unpack the research outcomes, namely Distributed cognition theory, but firstly I deal with the relevance of a theoretical framework.

3.2 Relevance of a Theoretical Framework

According to Kivunja (2018), a theoretical framework provides a foundation that supports the research approach, analysis, interpretations, and conclusions drawn from the study. It clearly defines salient terms and parameters for a methodical and reasoned inquiry. Since research is not marooned, a theoretical framework establishes the research in a wider context. Tafahomi (2022) asserts that the research validity and reliability are further enhanced by following a structured and coherent process. Relevant theories and studies can be manipulated from prior knowledge to gain new insight into a specific domain in addition to establishing, developing, and modifying contemporary models. This is supported by Radmehr (2023) who further asserts that researchers are then able to leverage existing theories to identify various characteristics, relationships, and patterns in data. Hence, a common theoretical foundation can foster collaboration between researchers which can invariably lead to cumulative progress in the field of study (Hendriana et al., 2022). The theoretical framework that I found most relevant to this study is distributed cognition.

3.3 Genesis of Distributed Cognition Theory

Distributed cognition is a theoretical framework in cognitive science that was pioneered by Edwin Hutchins in the 1990s. Hutchins (1991) proposed the idea that cognition is not reserved for an individual’s brain, but rather it extends to the realms of sociocultural environments, collaboration amongst individuals, and the use of tools in shaping cognitive processes and achieving specific learning outcomes. In the introduction to his seminal work “*Cognition in the Wild”*, he stated, “The emphasis on finding and describing knowledge structures that are somewhere inside the individual encourages us to overlook the fact that human cognition is always situated in a complex sociocultural world and cannot be unaffected by it’’ (Hutchins, 1991, p. 2). In contrast to orthodox neuroscience, which confines cognition to be the sole property of an individual’s mind, Hutchins argued for the interplay and integration of distributed knowledge, harnessing tools, artifacts, and social interactions in acquiring an epistemic outcome (Pinheiro et al., 2023).

3.4 Overview of Distributed Cognition in Mathematics Education

Hora (2015) highlights the interconnectedness between cognitive processes and the environment in knowledge construction. As such, distributed cognition can be integrated with the mathematics education fraternity as a strategy to optimise learning outcomes through meaningful experiences. This theoretical framework emphasises the role of collaboration and the use of technology-based tools in the cognitive processes to enhance the acquisition of mathematical skills (Aruna & Swarna, 2022).

3.5 Distributed Cognition in Visualising Grade 11 Trigonometry

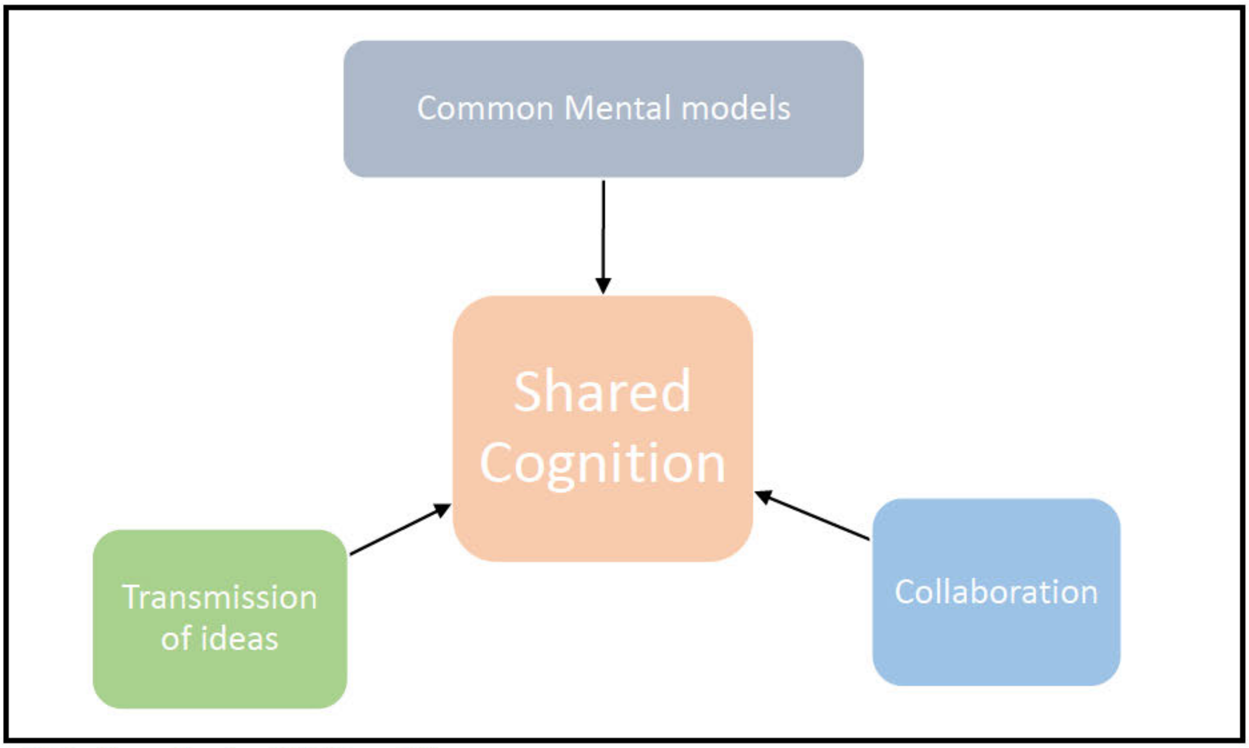
The study of trigonometry is critical in mathematics education where learners explore the relationship between sides and angles of a triangle. Mosese and Ogbonayya (2021) highlight the importance of visualising trigonometric concepts by showing the enhancement of learners' understanding and application. The theoretical foundation of distributed cognition which emphasises the significance of social interaction and technology provides an invaluable platform for enhancing visualisation in trigonometry (Hackett & Proctor, 2018). Salomon (1993) argued that distributed cognition which is deeply connected to visual representations has two aspects, namely shared cognition and cognitive offloading.

3.5.1 Shared Cognition

Shared cognitions acknowledge the coalescing of collaboration with other individuals and the use of external aids. Figure 3.1 illustrates the key features of shared cognition.

Figure 3.1

The key components of shared cognition



Note. From Levine (2018, p. 32).

3.5.2 Common Mental Models

When group discussions occur, a common understanding among members will assist in making coordinated decisions. A shared mental model will foster good teamwork which invariably improves the interactive learning process. Pea (1993) recognises that the complex landscape of human intelligence is distributed, as such, having a shared understanding in this social context will prove invaluable. McDevitt et al. (2021) agree with this concept of situated learning by emphasising the importance of contextual awareness and social cohesion.

3.5.3 Transmission of ideas through communication

Effective communication is crucial for learners to develop a deep and meaningful understanding of mathematical concepts. Qohar and Fazira (2022) explain that mathematical communication involves the exchange of ideas, thoughts, and reasoning to clarify their understanding and embark on critical thinking. Shodiquin and Waluya, (2020) assert that communicating ideas promotes metacognition because learners become more aware of their thought processes when they explain their thinking and reasoning to others. Effective communication of ideas leads to feedback and alternative perspectives from other learners. Ching et al., (2020) state that communication involving effective questioning and attentive listening plays an important role in mathematical engagement. Learners can now claim ownership of their learning and create an all-inclusive learning environment. However, educators must be mindful of challenges such as language barriers, cultural differences, and varying levels of mathematical proficiency. Planas et al. (2018) insist that educators must be proactive in creating an inclusive and supportive environment by using accessible multilingual resources to accommodate diverse language backgrounds. Culturally responsive teaching must also be considered which acknowledges and values cultural identities.

3.5.4 Collaboration

Collaboration among learners allows for constructive and productive discussions. Working together promotes active learning where insightful discussions can cultivate critical thinking, invariably leading to knowledge gaps being filled. Hwang and Ham (2021) assert that collaboration coalesces diverse perspectives which is useful in solving complex problems. This is further supported by Alibali and Knuth (2018) who recognises the development of mathematical skills because of active engagement. A collaborative learning environment provides a platform where the exchange of ideas concomitant with peer review can augment proficiency in mathematical skills.

3.6 Cognitive offloading

Cognitive offloading is a process of mitigating the effects of mental saturation by utilising external aids and resources (Morrison & Richmond, 2020). These resources and aids may include textbooks, calculators, visual aids, and educational software to reduce mental labour to focus on more complex problems. In the context of a grade 11 trigonometry class, educators can leverage cognitive offloading techniques to enhance learner experiences and performance by employing the following strategies:

3.6.1 Semiotic approach

Mudaly (2014) defines this approach as the use of signs and symbols to convey and understand mathematical concepts. Semiotics presents a framework for perceiving how signs and symbols are used in distributed cognition (Ata & Queiroz, 2023). Signs and symbols are pivotal to the representation and communication of information. Signs are manifested in various forms, including physical objects, gestures, symbols, or language, to which meaning is attached by individuals. These signs are interpreted and understood by individuals, allowing for the construction and dissemination of knowledge. This assists learners in simplifying complex ideas thereby making knowledge construction more achievable. One aspect of semiotics that is highly beneficial in cognitive offloading is the use of visual representations. The geometrisation of trigonometry with the use of graphs and diagrams, and the assistance of technology-mediated tools, may help learners grasp abstract trigonometric concepts. Distributed cognition accentuates the dynamic and interactive features of cognitive processes, highlighting the role of signs and symbols in shaping cognition.

3.6.2 Mnemonics and Acronyms

Memory aids are key tools used to remember important concepts, formulae, and relationships. In the context of visualising grade 11 trigonometry, diagrams associated with mnemonics and acronyms may include:

3.6.2.1 CAST diagram

The CAST diagram is a visual aid that helps learners identify the quadrants in which the trigonometric ratios are positive and negative.

Quadrant 1: All ratios are positive

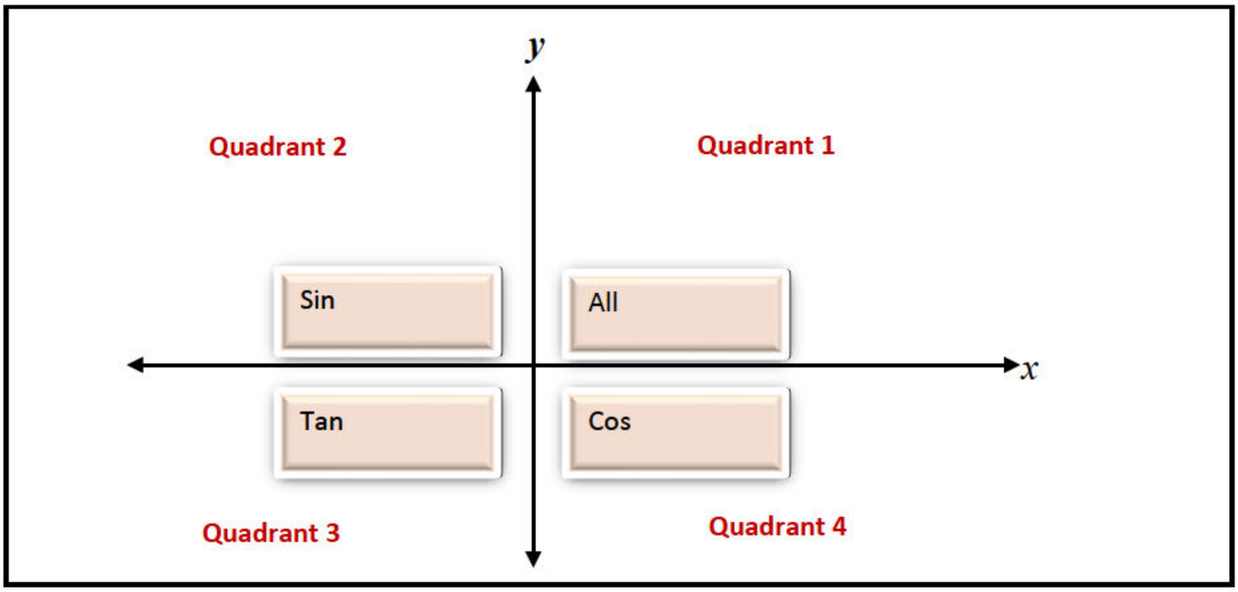
Quadrant 2: Only the sin ratio is positive whereas cos and tan are negative.

Quadrant 3: Only tan is positive whereas sin and cos are negative.

Quadrant 4: Only cos is positive whereas sin and tan are negative.

Figure 3.2

Traditional use of the CAST diagram using the chalk and talk technique.

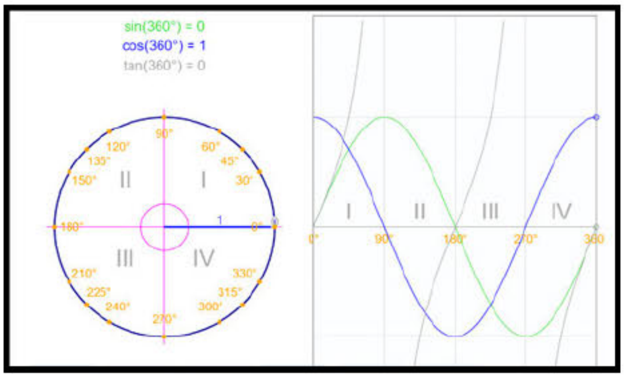


Note. Own construction.

Daher (2020) asserts that the integration of visualisation into mathematics education will enhance spatial reasoning thereby simplifying complex problems. Madaki and Zainon (2022) propose that visual software can be adapted to show the relationships and connections in trigonometry. Complex trigonometric concepts can now be visualised in engaging and dynamic ways. The use of interactive mathematics software allows learners to work at their own pace which personalises the learning process. According to Adhikari and Subedi (2021), interactive software such as Desmos, GeoGebra, Mathematica, etcetera, supports active learning in a dynamic environment which provides support for learners to manipulate variables and analyse data, to make connections and derive conclusions. However, Kaplan (2019) argues that educators need highly skilled training and support to adequately use such technology and consideration must be given to the accessibility and inclusivity of all learners. Figure 3 shows the use of an interactive unit circle to show special angles and their relationship to trigonometric functions. This further enhances the understanding of the CAST diagram because the trigonometric functions associated with the different angles show the correlation of the change of sign of the ratios.

Figure 3.3

An interactive unit circle of special angles with side-by-side trigonometric functions.



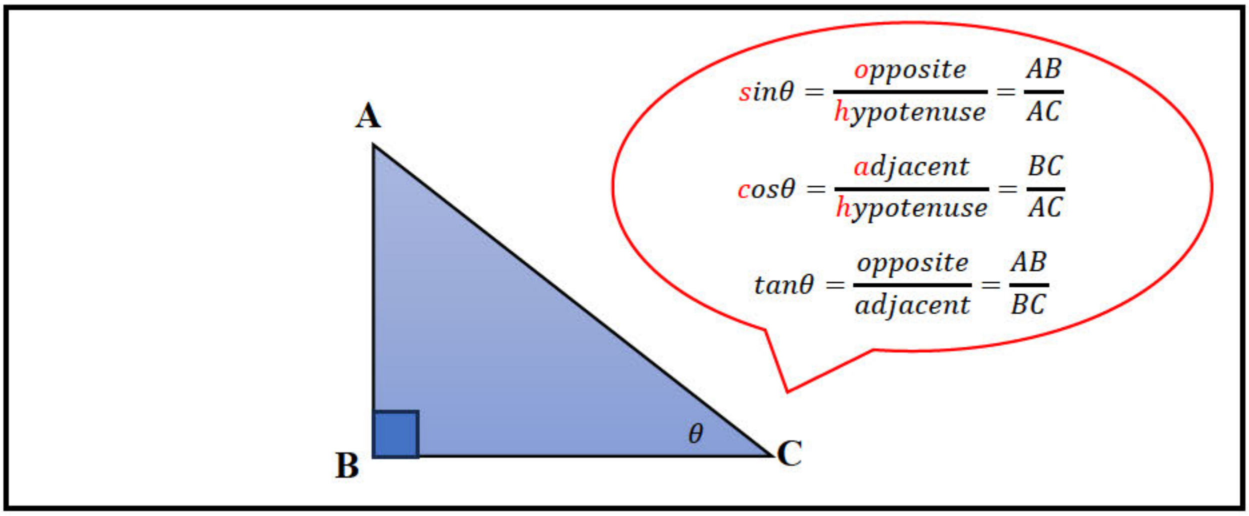
Note. From *Desmos* interactive unit circle.

3.6.2.2 SOH-CAH-TOA

This mnemonic is popular for recalling the three basic ratios namely sine, cosine, and tangent ratios. The associated right-angled triangle must be used in the context of visualisation.

Figure 3.4

Use of SOH-CAH-TOA

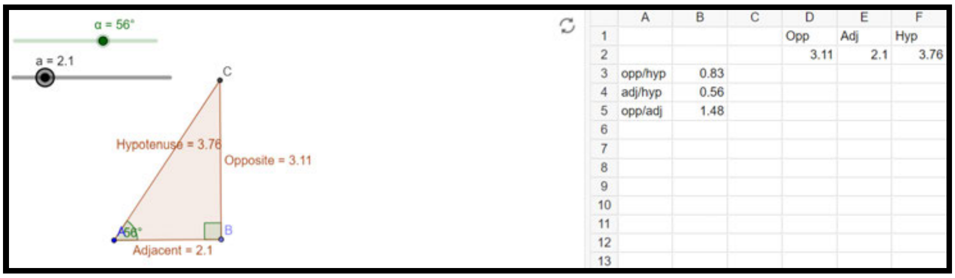


Note. Own construction.

The concrete representation of SOH-CAH-TOA bridges the gap between the theory and the real-life application. This mnemonic significantly aids in memory retention and is used as a recall tool for problem-solving. The diagrams below show the use of GeoGebra, showing that for a fixed angle and by using a slide to change the lengths, the ratios of sin, cos, and tan remain constant. This is directly related to the properties of similar triangles. This shows that the geometrisation of trigonometry can be used to show links between topics thereby enhancing the understanding of fundamental trigonometric concepts.

Figure 3.5

Slides of $\alpha = 56^0$ *and* $a = 2{,}1$



Note. From Algebron (2024, GeoGebra).

Figure 3.6

Slides of $\alpha = 56^0$ *and* $a = 2{,}7$

Note. From Algebron (2024, GeoGebra).

Fig 3.7

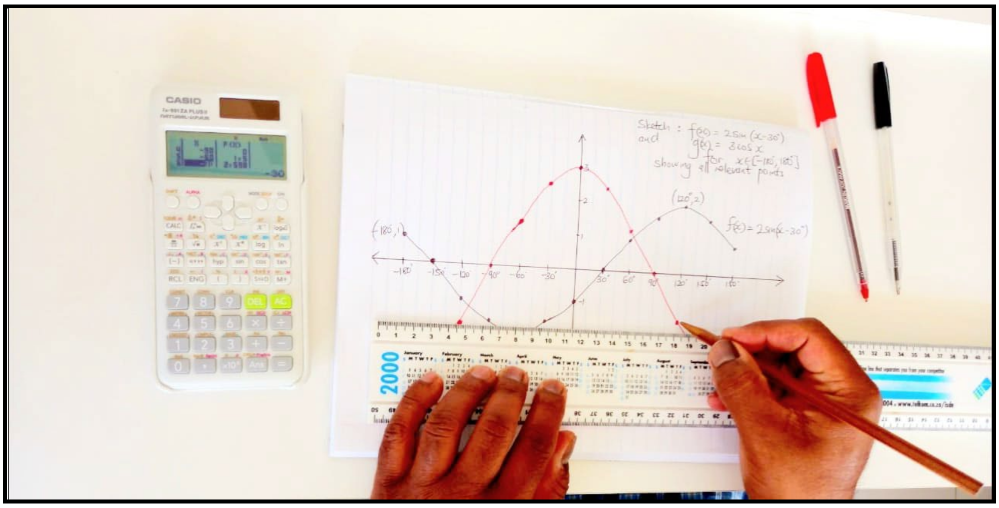
Slides of $\alpha = 56^0$ *and* $a = 3{,}3$

Note. From Algebron (2024, GeoGebra).

A simple example demonstrating the application of distributed cognition in a trigonometry class is the use of a book, pencil, ruler, and calculator to sketch a trigonometric function is shown in Figure 3.8.

Figure 3.8

The use of a book, pencil, ruler, and calculator



Note. Own construction.

The book serves as an external representation of the internal cognition that has taken place. Cognitive offloading occurs when the learner uses the book as a repository of information which can be used for information retrieval and scaffolding for new ideas (de Bruin et al., 2020). The pencil is a medium through which the cognitive process is externalised. Klein and Boscolo (2016) posit that the physical writing of information can augment the organization, assimilation, and coherence of thoughts and ideas. The ruler, being a simple tool for measurement, incorporates the concept of distributed cognition by extending cognitive processes into spatial recognition (Park & Jee, 2019). Learners can make accurate sketches which is inextricable from further interpretation such as trigonometric functions. The calculator is a computational tool that is used for quick and precise complex calculations that mitigate cognitive saturation (Browne & Garnham, 2022). Carlino (2023) further argues that cognitive skills are significantly enhanced by cognitive offloading resulting in improved critical reasoning.

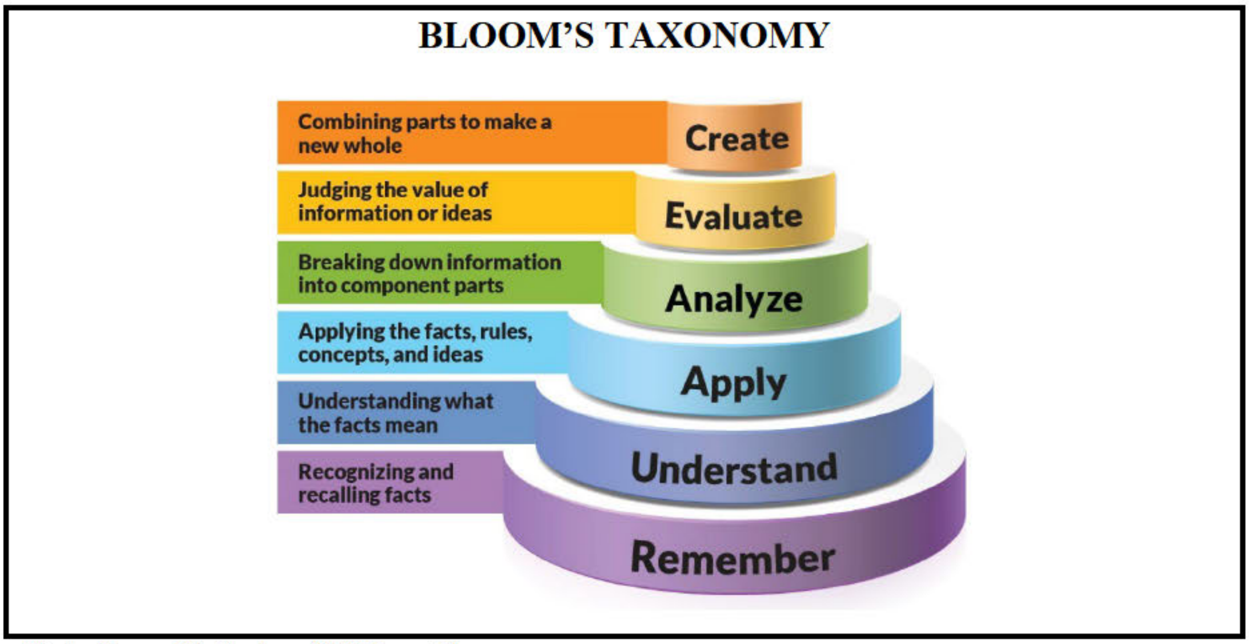
The book, pencil, ruler, and calculator scenario are a simple yet effective demonstration of distributed cognition in a classroom setting. The transformative benefits of distributed cognition cannot be overstated. This approach stimulates critical thinking and assists in the development of problem-solving skills that can be applied in different mathematical contexts (Maries & Singh, 2023).

3.7 Blooms taxonomy application in grade 11 trigonometry

Bloom's Taxonomy is a sequential and systematic hierarchical objective that is widely used in education to develop a structured parameter for meeting learner goals and achievement. This was developed by Benjamin Bloom, an educationist in 1956 and later revised in 2001 (Banda, 2023). Bloom's taxonomy and its variations is now a household tool in the National Curriculum and Assessment Policy Statement (CAPS) for mathematics curriculum design, assessment development, and instructional tasks that target various levels of cognitive complexity. The taxonomy categorises the cognitive processes into six progressive levels usually in a pyramid, with the baseline skills forming the foundation and the higher-order skills towards the apex as shown in Figure 8.

Figure 3.9

The six levels of Bloom's taxonomy in a pyramid.



Note. From Shabatura (2022, p. 2).

3.7.1 Remembering

This level involves recalling or recognising facts, information, or concepts. It is the foundation of learning and includes tasks such as memorisation, identification, and recall. In the context of trigonometry, learners have to memorise vocabulary, reduction formulae, the CAST diagram, and other salient concepts and procedures (Samo, 2017). According to Koksal et al. (2023), the sole focus on the basic retrieval of information is highly limiting to deeper understanding. This view is supported by Setyowati et al. (2018) who further argue that a short-sighted approach of this level will hinder the critical thinking required in complex problem-solving. Level 1 of Bloom's taxonomy revolves around low order thinking skills, depending on rote and passive learning. Critics argue that educators must move away from basic knowledge acquisition to integrate activities that require abstract and critical thinking skills (Rahman & Manaf, 2017).

3.7.2 Understanding

At this level, learners must demonstrate comprehension by explaining ideas or concepts in their own words. They should be able to interpret and summarise information and grasp the meaning of concepts beyond simple recall (Samo, 2017). Educators need to design instructional activities to encourage learners to explain and apply mathematical concepts to problem solve. Instructional design that is aligned with level 2 of Bloom's taxonomy should engage learners to a deeper level and promote the analysis and evaluation of various strategies for learners to gain more insight. Visual aids and manipulatives can be used to effectively communicate the concepts used in trigonometry. SivaKumar (2023) observed the vagueness of the different levels of Bloom's taxonomy regarding understanding and as such makes it challenging for educators to assess and gauge learners' levels of understanding. Another criticism is the overemphasis on lower-level cognitive skills such as remembering and understanding which impedes the process of deep mathematical reasoning (Agarwal, 2019).

3.7.3 Applying

This level involves the application of learned information in new and concrete situations. It focuses on applying knowledge, concepts, and principles to solve real-life problems. The primary purpose of this level is to narrow the divide between abstract and mathematical modelling. Banda (2023) states this stage develops higher-order thinking skills and can be used as a measure to evaluate learners' understanding and proficiency skills. This is also supported by Rahman and Manaf (2017) who assert that the process of application prompts learners to interpret, analyse, and synthesise information to formulate solutions. This is evident in solving 3D problems where learners have to distinguish between planes, solve triangles, and thereafter link solutions via the common side. This nurtures critical thinking skills, empowering learners to approach problems with greater self-confidence. Active engagement with mathematical skills has been found to significantly improve memory retention, concretising deep learning, and supporting the development of mathematical skills (Warren & Ursini, 2016). However, some educationists and academics have expressed concerns about level 3 being applied to only familiar and well-defined contexts. In this situation, learners may not be adequately prepared to approach unstructured and open-ended mathematical problems (Drakpa et al., 2021). Another matter of concern is the use of the “Apply” level to homogenise problem-solving and ignore the unique needs of diverse learners. Educators must exercise caution in not promoting a generic instructional design that fails to address the spectrum of learning abilities (Campbell et al., 2022).

3.7.4 Analysing

It is intended that learners at this level break down complex concepts into manageable pieces to understand their organisational structure. They must be able to identify patterns and relationships and make connections within the information. Level 4 seeks to transform passive learning to active engagement thereby fostering an application-oriented approach. Mualem et al. (2018) posit that this level has the potential to improve analytical skills in mathematical reasoning. The mathematical landscape demands that learners possess the discerning ability to deconstruct open-ended problems to explore potential solutions. Given the technological evolution, resources allow for improved visualisation, which learners can harness to ask Socratic questions to engage in a deeper and more meaningful analysis. Thompson and O’Loughlin (2015).

3.7.5 Evaluating

This level involves making judgments about the value of information, ideas, or solutions. Learners assess the credibility, relevance, and quality of arguments or theories. The evaluation level in mathematics requires considerable abstract thinking and problem-solving skills. The primary strength of this level is to help learners discern efficient methods and identify errors in mathematical reasoning (Al Haydary & Majeed, 2021). However, some learners may grapple with this level owing to the abstract nature of mathematical concepts, leading to the potential widening of gaps in their understanding (Samo, 2017). Many educators also experience challenges guiding learners to develop a critical approach to problem-solving and to think independently. Furthermore, assessment instruments, such as tests, assignments, and quizzes, may not fully capture the depth of thinking and understanding required at this level. Evaluating critical thinking and analytical skills requires more open-ended questions which can be time-consuming and challenging to evaluate but is necessary for learners to innovate and create. The effectiveness of problem-based learning on higher-order mathematical thinking skills has been explored, underscoring the importance of evaluating learners' mathematical skills in the context of varying pedagogical approaches (Simanjuntak et al., 2022).

3.7.6 Creating

At the pinnacle of Bloom's taxonomy, learners synthesise new ideas, products, or solutions by combining or reorganising extant elements and exhibiting originality and creativity in their mathematical thinking. Level 6 has the objective of engaging learners in higher-order thinking and problem-solving, and as such, educators need to encourage and motivate learners to be creative in their mathematical thinking. Learners must think 'outside the box', to see beyond basic procedures and approach problems in dynamic and creative ways. Creating a level 6 is centered around an application-oriented approach that enhances the relevance of mathematics, helping learners see the practical application of the curriculum. However, assessing creativity in mathematics can be highly subjective since the notion of creativity within the mathematics realm may vary from educator to educator. This subjectivity poses challenges in evaluating learners' progress accurately and consistently. Creating opportunities for students to engage in Level 6 activities can be resource-intensive, which requires educators to design complex problems, provide diverse materials, and support a range of potential solutions. This can be challenging in resource-constrained schools. Level 6 lacks instructional design clear guidelines and educators require ongoing professional development. Professional development programs should focus on fostering creativity in mathematical thinking, providing educators with the tools and methodologies to guide learners through the creation process, and developing reliable and valid assessment methods. Assessing creativity should also include diverse approaches, such as rubrics, and peer assessments. Zorluoglu and Guven (2020) highlight the usefulness of Bloom’s taxonomy in the analysis of mathematics learning outcomes and the setting of exam papers in South Africa. This is supported by Machisi (2023) who additionally impresses on the learning from shared experiences to target Bloom’s hierarchical objectives. In contrast, Long and Dunne (2014) argue that Bloom’s taxonomy is inadequate in providing reliable and meaningful feedback from what are over-generalised cognitive objectives. The contestation of cognitive levels of exam questions is further argued due to the lack of a relationship between the learner’s background experience and the problem. Berger (2018) challenges the current use of Bloom’s taxonomy, stating that learning is not hierarchical or linear but rather an integrated process. In the South African milieu, resource constraints, lack of professional development for educators, and language and cultural diversity are just a few challenges that impede the use of Bloom’s taxonomy. The problem is further compounded by the misconception amongst many educators to devalue basic skills by emphasising a certain cognitive level and ignoring the others. It is in this context that distributed cognition can play a significant role in aligning and balancing the way teaching and learning take place.

3.8 Integrating Bloom’s Taxonomy and Distributed Cognition

Integrating Bloom's Taxonomy with distributed cognition in grade 11 trigonometry involves aligning the six cognitive levels of Bloom's Taxonomy with the collaborative and external cognitive aids of distributed cognition. This integration enhances the learning experience, promotes deeper understanding, and fosters collaborative problem-solving. The integration process may use the following:

Remembering (Bloom's Taxonomy) can be aided by external tools (Distributed Cognition) to recall trigonometric ratios and definitions using mnemonic devices or interactive software. Using digital tools and resources to access and retrieve trigonometric information is essential in the 21st-century classroom. Understanding (Bloom's Taxonomy) and collaborative learning (Distributed Cognition) should be explored where relationships between angles and sides in triangles are discussed through a collaborative approach and peer teaching. Learners must engage in group activities using visual aids and interactive software to collectively interpret trigonometric ratios and unit circle concepts. Applying (Bloom's Taxonomy) and problem-solving (Distributed Cognition) to solve trigonometric equations as a group, discussing visualisation strategies, and collaborating on complex problems must simulate real-world scenarios for meaningful experiences. The analysing (Bloom's Taxonomy) and collaboration (Distributed Cognition) level must explore patterns and trends in trigonometric graphs as a collective, identifying how variations in parameters impact solutions. Learners must decompose complex trigonometric problems into smaller manageable components and collaboratively analyse their significance. Evaluating (Bloom's Taxonomy) and Peer Review (Distributed Cognition) must be used to validate trigonometric solutions through peer reviews and group discussions in consideration of different perspectives. When learners collaboratively critique the appropriateness of using specific trigonometric methods for diverse problem types, positive learning outcomes can be achieved. Creating (Bloom's taxonomy) and collaborative design (distributed cognition) must be accompanied by the use of visual aids, interactive software, or physical models to teach trigonometric concepts to peers. Learners need to engage in group discussions to devise innovative and creative ways in the applications of trigonometric concepts.

By integrating Bloom's Taxonomy and distributed cognition, grade 11 trigonometry education becomes a dynamic and interactive process. Students collaboratively explore, analyse, and create with the support of external cognitive tools such as technology, visual aids, and peer interactions. This integration enhances their ability to engage with complex trigonometric concepts, promotes collaborative problem-solving, and prepares them for practical applications of trigonometry in the real world.

3.9 The Role of Distributed Cognition in the Visualisation of Trigonometry

Distributed cognition plays a vital role in the visualisation of mathematics, specifically in the domain of trigonometry (Bower et al., 2020). Distributed cognition is predicated on the fact that cognitive processes are dispersed across external tools, social interactions, and environmental settings. According to Young (2023), this theoretical framework is pertinent to trigonometry, due to cognitive stresses of understanding convoluted relationships between sides, angles, and graphs often exceed what learners can manage without external assistance. This is supported by Nordlander (2022) who further recognises the use of visualisation to be cognitively demanding at times since it involves the manipulation of diagrams, symbols and spatial relationships. At this critical juncture, the intersection of distributed cognition and visualisation becomes more apparent in the cognitive offloading process by harnessing external resources which invariably optimises the effectiveness of visual representations. The leveraging of artefacts, objects or tools is inextricably connected to visualisation by aiding in creating charts, diagrams, and influencing the semeiotic process. This is achieved by externalising abstract concepts, making them more accessible and easier to manipulate, understand and internalise. Moreover, educators must integrate external resources into their instructional design to facilitate the distributed cognitive process. Examples may include the use of an interactive computer software that allows learners to link a unit circle to functions or the construction of diagrams to mitigate the complexity of the trigonometry problem. By encouraging discussions, prompting learners to articulate their visualisations, and providing feedback, educators can facilitate the scaffolding of cognitive work distributed across the classroom. Distributed cognition in the context of learning trigonometry, provides a powerful framework that complements the visualisation of trigonometry by enhancing and enriching the classroom space.

3.10 Synopsis

Distributed cognition is a groundbreaking theory that challenges the traditional view of cognition as confined solely to the individual mind. It explores the dynamic interaction between individuals, their surrounding environment, and external tools or artifacts in shaping cognitive processes, problem-solving, and learning. This theory recognises that cognitive activities often extend beyond the boundaries of a single mind, incorporating interactions with tools, technologies, social networks, and cultural contexts. In educational settings, distributed cognition advocates for the integration of external tools and resources to enhance learning experiences. In essence, distributed cognition having such a profound impact on the learning environment, challenges us to rethink the boundaries of the educational ecosystem.

CHAPTER 4

RESEARCH METHODOLOGY

4.1 Introduction

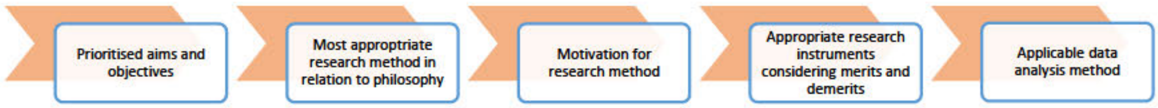
Research methodology is essential in formulating a path to conducting a research study. Almazan (2021) asserts that philosophical assumptions are critical for the evaluation and legitimacy of the findings as they affect trustworthiness and credibility. Discussion on the research approach, paradigm, and philosophy set the boundaries so that the process could be organised and manageable. Peel (2020) also asserts that research methodology is an inquiry framework that harmonises the interconnected components to reach quality inferences. This study is streamlined by appropriate procedures and techniques used in the selection, collation, and analysis process. The sampling methods and administration of the research instruments were paramount to the validity and integrity of the research. This study follows a rigorous methodology as outlined and explained.

4.2 Research Approach

Dzogovic and Bajrami (2023) define qualitative research as a means to address questions relating to experiences and emotions in the human realm. It involves the collection and collation of non-numeric data to understand the perspectives, opinions, and experiences of groups or individuals. According to Crossman (2021), qualitative research is the development of concepts that help us to understand social phenomena in natural settings rather than experimental settings, giving due emphasis to the meanings, experiences, and views of the participants. A qualitative research approach was used as a mode of inquiry in this study. The reason for this choice was that it provides an opportunity for generating an in-depth analysis of the experiences of relatively small numbers of participants and the researcher would be in a position to address the problem in a meaningful and non-reductionist way (Smith & Bell, 2011). Qualitative research proved advantageous as it provided valuable data concerning the needs, experiences, and patterns of behaviour of the participants. The study was carried out in a natural setting, which gave it tangible results since the participants were directly involved in it. The researcher observed the hierarchy and operations of the school, carrying on with their daily duties and operations in a real-world setting. The interaction between the principal, mathematics head of department, and mathematics educators indicated the type of relationship fostered. The study was conducted in its natural setting, without manipulation or interference. The participants' suggestions and contributions were highly honoured within the community of study. The rigorous steps used in this research methodology are shown in Figure 4.1.

Figure 4.1

Flow chart showing the research methodology process



Note. Own construction.

The above flow chart was used to sequentially navigate the operational process of the research. It provided a systematic and controlled approach to the spectrum of my endeavour.

4.3 Research Philosophy

The use of an interpretivist research philosophy was adopted. Interpretivism is defined as access to reality via a social construct (Ndlovu, 2021). The researcher is seen as an explorer who acknowledges the dichotomies in the participants. The use of language, perceptions, and views plays an integral part in the research. Observing the participants in their natural environment without manipulation allowed for profound insight into the complexities of the social construct (Irshaidat, 2022). Non-empirical data provided a clear understanding of human behaviour and attached meanings to it. Potrac et al. (2014) regard the epistemological underpinnings of interpretivism as subjectivist as it has to manage the emotions of participants. The advantage of this philosophy is that the data generated cannot be generalised since it is heavily impacted by personal viewpoints and values. For this reason, the data gathered was representative of the community under study to a larger extent.

4.4 Research Strategy

This research emphasises the importance of understanding individual perspectives in a natural context. As such, interpretivism offers invaluable insight into subjective learning influenced by contextual and cultural factors. According to Potrac et al. (2014), interpretive researchers assume that access to reality is only through social constructions, such as shared meanings. Meanings can be shared through interactive, cooperative, and participative methods of data gathering (Smith & Bell, 2011). Thus, the commonly used data collection methods are interviews, questionnaires, and observation (Potrac et al., 2014). In this study, data collection was based on structured and semi-structured interviews, pre- and post-tasks, in addition to a questionnaire.

4.5 Research Design

The framework of this study was informed by the positive effects of a visualisation intervention in a grade 11 trigonometry class (Smith & Bell, 2011). The research design included a descriptive. The descriptive research served to define the problem with the objective of describing it. The study was focused on describing the visualisation practices from purposive sampling. Descriptive research design was an appropriate choice because this research aimed to provide an in-depth description of visual strategies that educational leadership could utilise to improve the effective teaching and learning of grade 11 trigonometry. The research design allowed the researcher to get acquainted with the strategies to be utilised and suggested ways to improve mathematics results. The design helped the researcher to understand the best strategies learners could utilise to improve the effective learning of trigonometry using visualisation techniques at secondary schools in the Umlazi District.

4.6 Target Population and Sampling

4.6.1 Target population

This study was conducted in the Umlazi District, KwaZulu-Natal. The 21 public high schools of Chatsworth formed the sample frame of which one school was purposively selected due to the characteristics of interest which included socio-economic diversity, grade level learner performance, and accessibility since I am an educator at the school. The target population of interest was the complete group of 79 learners from the grade 11 trigonometry class of which eight participants were selected from varying tiers of academic excellence, ranging from excellent, mediocre, and unsatisfactory according to their Grade 10 mathematics results.

The school is relatively large with forty-one staff members and a learner population of approximately one thousand. It is regarded as historically disadvantaged, experiencing many challenges such as resource constraints, poor socio-economic conditions, limited access to technology, and many others. The participants spoke English and isiZulu as their main language, and the official language of instruction at the school is English. There has been historically poor mathematics performance at the school due to various factors.

4.6.2 Kinds of Sampling

The research used non-probability sampling. Non-probability sampling involved a sample selection using purposive techniques which was advantageous since judgement was used in the process (Smith & Bell, 2011). Purposively selection was relevant to this research due to cost effectiveness and accessibility to specific characteristics that aligned with the research objectives. Grade 11 core mathematics learners were a matter of interest of varying degrees of academic achievement, some of whom made use of visualisation tools and some who did not. A stratified purposive sampling technique as shown below was used.

Criteria for selection:
0 - 40% (38% of learners fit in this category),
40% - 80% (52% of learners fit in this category) and
80% - 100% (10% of learners fit in this category).

Rai and Thapa (2015, p. 2) recommend purposive sampling as it requires judgement from the researcher in the selection process, making it a "non-representative subset of some larger population". The main objective of purposive sampling was to target specific schools with specific needs.

4.6.3 Sample size

Krejcie and Morgan (1970) state that the sample size needs to be representative of the population. Braun et al. (2017) argue for eight to ten participants for qualitative research to reach data saturation. The selected participants have aided the desired characteristics of visual learning since some of them had prior visualisation experience and some did not. This was further aided by their diverse mathematical skills together with their levels of motivation and interest. This inclusion criteria assisted in examining the effects on mathematics education. This study includes eight participants based on ranked grade 10 mathematics results concerning their MV. Participants' MV was based on the frequency and success with which they used diagrams in answering their Grade 10 mathematics examination. The results were stratified into three tiers as shown in Table 4.1.

Table 4.1

Selection of participants

Mark (m)	Number of participants
$0 \leq m \leq 40$	3
$40 < m \leq 80$	3
$80 < m \leq 100$	2

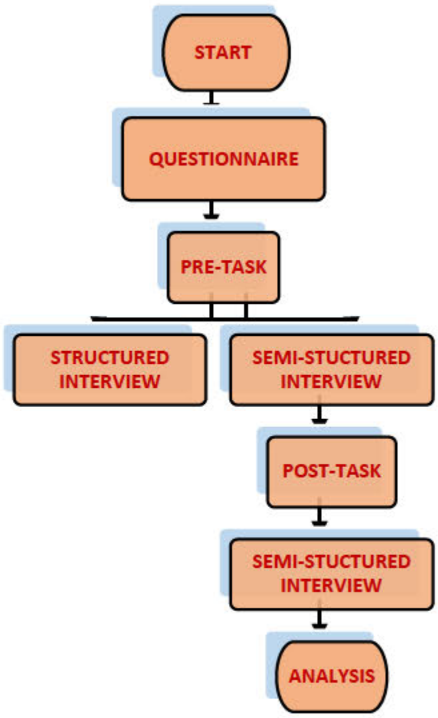
From each mark category, one learner who used diagrams with success, one learner who used diagrams without success, and/or one learner who did not use diagrams were chosen.

4.7 Research Instruments

The re-imagining of data collection, in the post-COVID-19 era, was informed by a blended approach that made use of traditional and remote means (Khirwadkar et al., 2020). Considering this emerging reality, the study was conducted using the following research instruments as shown in Figure 4.2.

Figure 4.2

Flow chart showing the research instruments



Note. Own construction.

4.7.1 Questionnaire

The questionnaire being the preliminary research instrument, primarily constituted of multiple-choice questions that were sent to participants via Google Forms, which took no longer than 20 minutes to complete. This inexpensive and efficient way of obtaining data characterised the learners' thoughts, attitudes, and experiences in the context of the current visualisation strategies implemented (Almanthari et al., 2020).

4.7.2 Pre-Task

The pre-task followed the administering of the questionnaire which established the competency of learners regarding trigonometry before the visualisation intervention. The pre-task highlighted the gaps in learner's understanding and provided a bearing for an intervention strategy. Hendriana and Fadhillah (2019) assert that the pre-task is a reference point for the growth and development of the learner. This benefited the process of developing more visual content and solutions for trigonometry.

4.7.3 Structured interview

The structured interview was 45 minutes in duration and utilised predetermined questions that were aligned with the research questions as explained by Roos (2023). The standardised ordered questions mitigated the effects of biases and allowed for direct comparison and evaluation of participants' skills and abilities. This was beneficial in mapping an appropriate visualisation strategy. It must be noted that the structured interviews alongside semi-structured interviews offered a comprehensive strategy for data collection. The structured component provided standardised and elicited responses, whilst the semi-structured component allowed for flexibility and rich in-depth insight into participants' experiences and perspectives.

4.7.4 Semi-structured interviews

The ensuing semi-structured interviews included open-ended and follow-up questions which provided a protracted understanding of the participant's thoughts, feelings, and beliefs in the school milieu (Sbaffi & Rowley, 2017). This two-way communication provided awareness of the unique needs of each participant and took no longer than 30 minutes.

4.7.5 Post-Task

Reid and Reid (2017) posit that a post-task also serves as an indispensable diagnostic tool to gauge the effect of the intervention programme. The post-task bounded by the semi-structured interviews, indicated the level of improvement in each segment of trigonometry in the context of a visualisation programme.

These instruments provided sufficiently worthy data which was used to gain significant insight into visualisation in the grade 11 trigonometry classroom from a commognitive perspective.

4.8 Trustworthiness

The trustworthiness of content data analytics uses concepts of credibility, dependability, conformability, and transferability. Stahl and King (2020) define trustworthiness as having a characteristic of vibrancy which indicates the level of confidence and trust in the research results. Since qualitative research does not utilise instruments with established metrics, it is of vital importance to firmly establish credibility, conformability, transferability, and dependability. Polit and Beck (2017) further add that confidence, interpretation, and methods used to analyse data are crucial to the quality of the study.

4.8.1 Credibility

Self and Roberts (2019) define credibility as a measure of belief and trust in the study. Rycroft-Smith and Stylianides (2022) consider credibility as the level of confidence in the truth. Credibility can be achieved by having a more robust engagement with participants, enriching observations, member checking, and reflective journaling (Sbaffi & Rowley, 2017). This study was objective in using eight participants in the data generation. The structured and semi-structured interview questions, and the closed questionnaire allowed for meaningful research, answering the objectives of the study, thereby establishing the quality of the study.

4.8.2 Transferability

Tobin and Begley (2004) define transferability as the generalisability of research. This involves the transferring of the findings of one study to another. Merriam and Grenier (2019) add that transferability refers to the process of external validity so that the findings of one study can be applied to another. Erlandson (1993) argue that a qualitative researcher's generalisability is never truly possible since each situation is contextual. Denscombe (2017) suggests a contrasting view by acknowledging that although each case is unique, it is also an example within a broader group. This study detailed the research context and outlined all assumptions. In doing so, the applicability of the findings could be transferred to other settings. Schools with similar characteristics can use the findings and recommendations of this research to benefit their institution. The congruencies of beliefs, experiences, and settings could mean that results from this study can be transferred and adapted.

4.8.3 Dependability

The purpose of qualitative research was to achieve dependability. Dependability is defined as being consistent and reliable in research findings (Sandelowski & Barroso, 2006). Gioia (2021) notes that the dynamic nature of a qualitative research design will render consistency problematic. Lincoln and Guba (1982) stress the sustainability of the findings to ensure dependability. The pilot study was used as an audit to establish the consistency of this research. The ethics applied together with a positivist approach made the results dependable. The evaluation and interpretation of the findings established the dependability of this study and provided critical recommendations that proved invaluable in this process.

4.8.4 Confirmability

Confirmability emphasises the need for research objectivity. Patton (2014) insists that to prevent bias, the researcher must ensure that he or she only utilises the participants' experiences and ideas and does not contaminate them with their own. The importance of triangulation must be emphasised to mitigate prejudice and bias. Beck (2019) recommends that the findings must not be concocted by the researcher. To achieve confirmability, researchers must embark on a reflexive approach in which a journal is kept by the researcher to reflect on the research process. This research documented all procedures adopted in the checking process and underwent rigorous analysis to identify contradictions and biases. This aided in eliminating contamination from the research.

4.9 Data Analysis

Data analysis involves collecting, organising, reviewing, and exploring the data (Evans & Field, 2020). This study involved thematic forms of data analysis. According to Kansteiger and Konig (2020), thematic analysis utilises the evaluation of patterns in the research. The specific use of words, phrases, or imagery in the content together with its frequency can be used to form meaningful interpretations. Vaismoradi and Snelgrove (2019) add that thematic analysis is repetitive and delinearised in nature. However, this can be minimised by being more flexible, allowing for greater analytic choices and interpretation and inferences as supported by the data (Coy, 2019). Additionally, thematic analysis involved analysing the data and searching for patterns. The patterns were studied for similarities and differences which were formulated into themes. Themes helped the researcher to attach meanings from which inferences were made.

Codes were also formed which led to a thematic structure. Priority was given to the accuracy and integrity of the analysis and to this end, coding was a good approach. Coding, also known as categorising data, is a very important phase in the analysis process. Mohajan and Mohajan (2022) add that coding is used to locate, reduce, manipulate, and transcribe information. This was done in an interactive exploratory manner to understand the dynamics of the content from data collected via structured and semi-structured interviews via Google Meet and a questionnaire using Google Forms.

4.10 Data Storage and Protection

Hard copies of the interview transcriptions and questionnaires are stored in a hidden safe. Computer storage of Google Meet interviews was password protected. Only the researcher has access to all data. Hard copies will be shredded, and digital storage will be deleted after 5 years.

4.11 Limitations and Delimitations of the Study

Limitations of the study were due to time constraints and lack of financial resources. Both the researcher and participants were only available for this study after work and on weekends. This was navigated by reorganising the researcher's daily schedule, prioritising the study. Load-shedding was a significant setback to a fixed interview roster as virtual Google Meet interviews could not be conducted on some days, however, the use of an inverter mitigated some delays. Exorbitant data costs as well as intermittent unstable internet connectivity were curbed on some days by working at school A which allowed the researcher to use their WI-FI hotspot.

Delimitations of this study only allowed eight participants from one school due to the time factor. However, this allowed for more substantive interviews that assisted with a greater understanding of the topic. The scope of this research was restricted to only the Umlazi District and the findings should be confined to this context. Although the research study only applied two theoretical frameworks and three research instruments due to a tight schedule, the effects were mitigated by carefully selecting appropriate instruments to obtain sufficiently worthy data. The post-COVID-19 era allowed for face-to-face interviews provided a mask was worn and a safe distance was maintained as safety was prioritised hence Google Meet interviews were also conducted. The Google Meet interviews replicated the natural setting thereby allowing the researcher to gauge the non-verbal communication.

4.12 Elimination of Bias

The research optimised objectivity by using gender-neutral words. The questionnaire prompted mostly a binary response of “Yes or No”. The researcher conducted the interviews with decorum and did not form opinions and expectations from the responses. The participants were not defined by race, religion, ethnic group, culture, or political affiliation. The language used in this research avoided assumptions and stereotyping of groups or individuals.

4.13 Research Ethics: Key Considerations

Breetzke and Hedding (2020) define research ethics as a moral code in a scientific inquiry. The rights of people and intellectual property must be protected as enshrined in the Bill of Rights of the Republic of South Africa 1996. The researcher's guide in terms of their conduct should extend to responsibilities to the participants, publications, and dissemination. The principles of good research ethics can be achieved by researchers leading by example by maintaining exemplary ethical conduct. The researcher must be proactive in ensuring that no improper actions are taken to avoid any anticipated ethical conflicts.

A gatekeeper's permission letter (see Appendix C) was sought from the school’s principal, informed consent letters from parents (see Appendix B) and participants consent forms (see Appendix A) were obtained for the research to be conducted. An ethical clearance approval was also obtained from the University of KwaZulu-Natal (see Appendix E) which ensured that the study was conducted responsibly and the protection of all participants from any harm was ensured. The informed consent of participants was done at the beginning of the study. The researcher was sensitive to the possibility that the participant may withdraw their consent at any time during the study. This was made clear to the participants and under no circumstances should they be under duress (Ramrathan & Le Grange, 2017). This study prioritised the dissemination of information to participants concerning the nature and details of the research. It was emphasised that being a participant is voluntary and they are not forced to answer any questions. It was imperative to ensure that no harm should come to the participant due to the nature of the study. Vanclay et al. (2013) state that beneficence which stresses the obligation to not harm is an overarching norm in the circle of scientific inquiry. The participant's safety and welfare were prioritised. Beneficence also included the protection of participants from exploitation. The researcher also considered the protection of the identity of the research participants, though known by the researcher. Some participants may also wish to remain anonymous, in other words, they are not known to the researcher. Confidentiality is a condition where the researcher knows the identity of the participant but protects it to ensure an honest response. The researcher used codes instead of names to label data. Breach of confidentiality was always avoided since trust and credibility were paramount (Gibson et al., 2013). In this study, stringent steps were taken to ensure confidentiality. This was achieved by ensuring that the participants understood the details of the study with emphasis that they were volunteers. No penalties were to be incurred if they at any time decided to not participate. The "no harm" policy ranked foremost on the list of priorities. All relevant information was communicated to the participants to obtain informed consent. Minors were involved in this study, so parents were consulted, and all protocols were strictly adhered to.

4.14 Synopsis

This chapter explains the research process. The issue of research philosophy was addressed, and the details of the research design were presented. The pilot study was an important part as it helped with the planning and modification of the study. The level of confidence and trustworthiness of the data and its interpretation were essential regarding the quality of the study. Data analysis involved a thematic approach which proved flexible due to the limitations and delimitations.

CHAPTER 5

DATA PRESENTATION, ANALYSIS AND FINDINGS

5.1 Introduction

The previous chapter on research methodology functioned as the study's compass and offered a clear methodical framework for examining the effects of a visualisation intervention in a grade 11 trigonometry class. This chapter being the core narrative of the research, focuses on the presentation, analysis, and findings of the collected and collated data.

In the mathematics education arena, specifically within the domain of trigonometry, educators and educational researchers constantly seek progressive and novel approaches to enhance learners' comprehension and engagement. In recognition of these challenges where learners frequently encounter difficulties with the complexities and nuances of trigonometric concepts, this research seeks to explore the metamorphic potential of a visualisation intervention in a Grade 11 trigonometry class. This chapter embarks on a comprehensive data analysis journey, delving into the outcomes of a meticulously designed visualisation intervention program implemented in a Grade 11 trigonometry classroom. Through the judicious use of a qualitative approach, this research aims to unravel the impact of these visualisations on learners' comprehension, retention, and overall engagement with trigonometric principles. This analysis involves not only the deconstruction of pre- and post-intervention tasks but also the nuanced exploration of learners' perceptions and experiences through a questionnaire, and structured and semi-structured interviews. By triangulating these disparate data instruments, this research aspires to offer a heterogeneous understanding of the efficacy of visualisation intervention from a trigonometric perspective. The collected data was organised and thematised to address the subsequent primary research questions:

1 What visual strategies are available for the teaching of grade 11 trigonometry?

2 What intervention programs can be developed in trigonometry to make the content and solutions more visual?

3 What is the impact of such an intervention program?

5.2 Creating Themes and Subthemes

The emerging themes resulting from coding chains of the qualitative data enhanced the depth and richness of the analysis. It offered a lens into participants' involvement, perceptions, and overarching experiences of the visualisation intervention. This was achieved by the detailed analysis of the questionnaire, verbatim quotations from interview transcripts, and responses to pre- and post-tasks followed by an in-depth exploration into patterns that materialised.

To preserve confidentiality and anonymity, the identity of each participant was protected by using pseudonyms as shown in Table 5.1.

Table 5.1

Pseudonyms of Participants

Participant Number	Pseudonym
1	Sanaya
2	Siya
3	Shriav
4	Nora
5	Precious
6	Meena
7	Gift
8	Sanav

The main aim of themes and subthemes in this data analysis was to provide a methodical approach to organising and interpreting qualitative data. The structured data were categorised into meaningful subsets to facilitate a clear understanding. Themes and subthemes played a vital role in supporting me as they provided several key purposes such as detecting recurrent patterns and investigating trends within the data as shown in Table 5.2. The methodical classification of data into themes and subthemes aided in its analysis and interpretation, leading to a profound and meaningful understanding of the underlying phenomena which included the efficacy of visualisation, participant's adaptability to using visual tools, and their experiences in using these tools (Bostrom, 2019). The significance of themes and subthemes extended to the condensation of voluminous amounts of information into manageable and meaningful segments. These summaries facilitated the unpacking of principle findings in addition to the development of coherent interpretations (Peck & Mummery, 2018). Themes and subthemes also contributed to the evolution of the said theoretical framework and aided in the design of novel conceptual models where the identification of recurrent themes provided new theoretical insights (Liu et al., 2024). The credibility and trustworthiness of the data analysis were further enhanced with the delineation of the primary ideas with groupings emerging from the data. The iterative process I used regarding theme development promoted an increased methodological rigour through a process of constant reflection and refinement of the data. Furthermore, the themes and subthemes provided a clear and systematic approach to data interpretation, contextualising the analysis within the boundaries of extant literature and relevant frameworks thereby ensuring firm grounding and reliability (Erdem, 2023). The process of thematising data not only helped to keep the analysis coherent and well-organised, but it also produced insightful evidence-based conclusions.

Table 5.2

Summary of the themes and subthemes of the visualisation intervention

Themes	Subthemes
Application of Visual Aids	- Graphical representations - Interactive simulations/animations (using software) - Diagrams, manipulatives, etc (Dynamic/Static)
Visualisation Efficacy	- Level of participation and attention - Comprehension and understanding - Ability to recall
Harmonising with the curriculum	- Alignment with learning objectives - Integration into lesson plans - Reinforcement of key concepts
Learner Feedback	- Experiences with visualisation - Preferences for a visualisation approach - Methods of enhancement

5.3 Analysis of Questionnaire regarding prior experiences and perceptions of trigonometry concerning visualisation

Participants rating their understanding of trigonometry

Only Nora responded with a rating of "*excellent*". She attributed her solid grounding to a lot of practice. In her response to this question she said, "*I always loved mathematics and my curiosity into wanting to know more led me on a journey which entailed studying hard".* She recalled asking her educator a lot of questions and her inquisitive mindset assisted her in enhancing her procedural fluency and conceptual understanding. Her penchant for numbers and mathematics in general, as noted in the reflective journal, motivated her to spend a lot of time with her work. In an interview Nora said:

> *My fondness for mathematics and getting to know the universe emboldens me to study trigonometry. I am a stickler for detail, accuracy and efficiency and studying trigonometry allows me to appreciate the patterns and structure of the universe itself. Trigonometry has attached substantive meaning to the environment I belong, as such my passion for trigonometry and mathematics in general grows.*

Siya was more cautious in his response of "*good*" although he performed well in his previous assessments in trigonometry. His reserved outlook alluded to his shortcomings in certain subtopics, one of which was solving 2D problems. In a discussion Siya said:

> *Trigonometry is very interesting, but I get confused with complex questions like 2D problems. I am unable to visualise the problem and link it to real-life problems.*

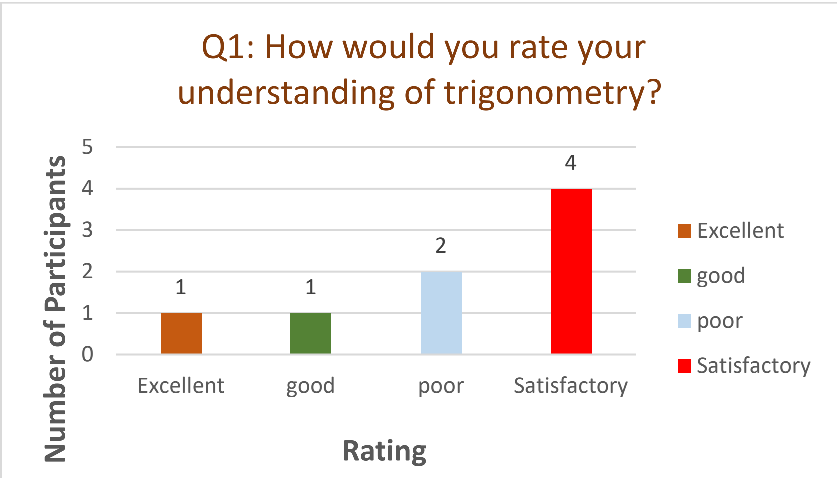
Precious, Shriav, Gift, and Sanav responded with a "*satisfactory*" because they felt unsure about a plethora of fundamental concepts. They mentioned the following in a discussion that provided a lens into their experiences with this topic.

> *Precious: I am keen on studying trigonometry but some concepts I just cannot understand.*
>
> *Shriav: I understand the basics of trigonometry, but I cannot I apply it well.*
>
> *Gift: I cannot identify and connect concepts.*
>
> *Sanav: The educator does not explain things to us with diagrams or pictures and this makes things really confusing for me.*

A common thread in the semi-structured interview was an indifference towards mathematics, some of whom cited pressure from parents to choose mathematics. Furthermore, they complained about this topic being loaded with abstract concepts which was further compounded by the educator implementing a dry chalk-and-talk approach. Meena and Sanaya being the two learners who wrestled with this topic historically, always performed poorly as attested by previous assessments given by their educator. In an interview Meena said, "*I just cannot picture these concepts in the real world"*, whilst Sanaya said, "*I am used to getting bad marks for trigonometry because I do not understand it*". A noteworthy observation was none of these learners elicited a lack of resources or other potential drawbacks as excuses.

Figure 5.1

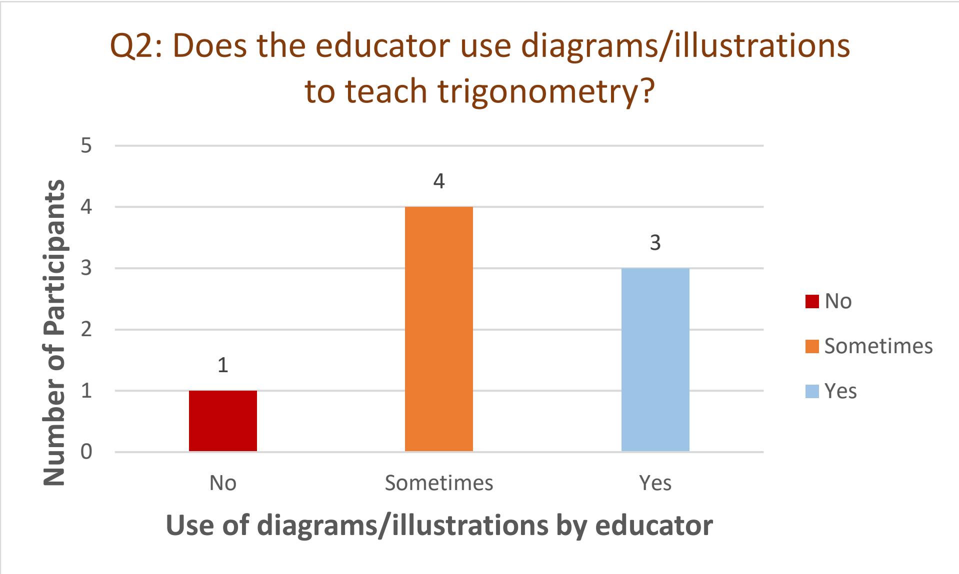
Bar graph representing the self-rating of participants regarding trigonometry understanding.



The CAPS document does not guide the pedagogical approach to delivering the curriculum (Hughes & Lewis, 2020). It however grants freedom to the educator, depending on preference and effectiveness to embark on an appropriate instructional design (Fair & Stott, 2021). Educators must draw on their experience and training to effectively deliver the curriculum in a way that considers the diverse classroom. However, Tsakeni et al. (2021) assert that this lack of guidance by the CAPS document leads to inconsistent pedagogical approaches by educators. Educators may not be equipped with adequate teaching skills to effectively translate the curriculum into an effective instructional design. It was in this light, that the responses were varied since the participants were exposed to different approaches from their respective educators. Sanav responded with a "No" although he claimed familiarity with the unit circle from mathematics tuitions he attended. Shriav, Nora, and Siya answered in the affirmative and the semi-structured interview evidenced their confidence in a visual understanding of basic concepts. All other participants were exposed to intermittent use of a visual pedagogical approach. They also spoke of YouTube links that were provided by the educator to facilitate and enhance the teaching and learning process. However, some learners who were part of the group found it time-consuming and boring to watch those videos.

Figure 5.2

Bar graph representing the participants' experience regarding the use of diagrams/illustrations in trigonometry



The efficacy of diagrams/illustrations in understanding concepts

The significance and effectiveness of diagrams, be it static or dynamic, in explaining trigonometric concepts is inextricably connected with pedagogical instructional design, the intrinsic nature of trigonometric concepts, and experiences and perceptions of learners among the various possible factors. Ngu and Phan (2020) argue that the use of diagrams may not always be an enjoyable and effective tool for learners especially if they cannot link the abstract theory to real-life application. Meena and Sanaya are the only participants who did not benefit much from the use of diagrams for reasons that included a lack of conceptual awareness, lack of clarity and connection of the diagram and question, and little or no collaboration with peers. In a discussion these two participants said:

Meena: Although visual tools made lessons exciting and enjoyable, I find it very difficult to formulate solutions on my own. I always make errors and cannot picture the given problem.

Sanaya: Trigonometry is a nightmare because my diagrams are mostly wrong. When things get complicated, I am at a loss.

Gift regarded himself as a multi-modal learner and to some extent found diagrams beneficial in as much as it was presented and explained to him. In a discussion, explaining the extent to which diagrams assisted him he said:

Diagrams help me understand the problem but in class, my educator did not use this method a lot.

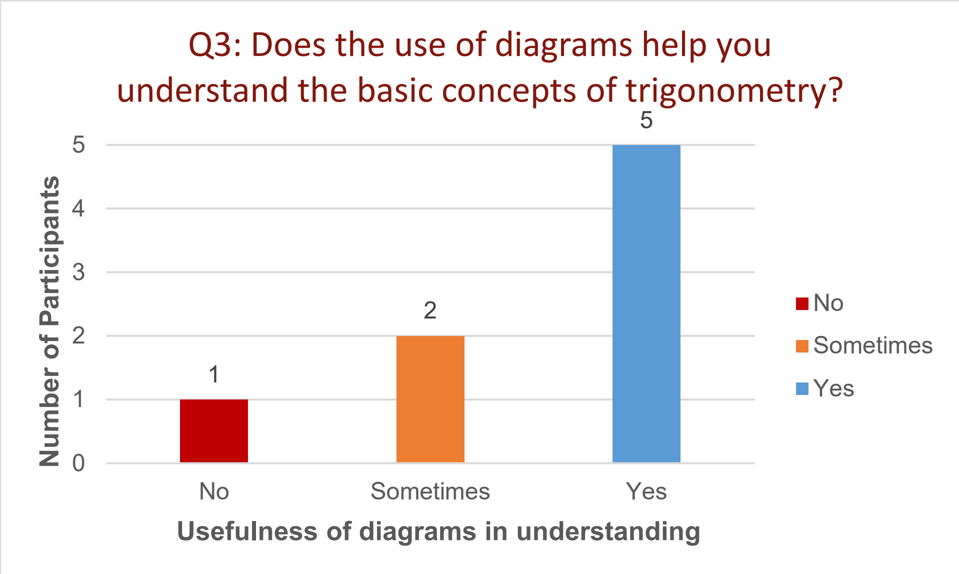
As shown in Figure 5.3, the remaining learners found the diagrams useful to supplement the theory albeit at various levels. Through my initial interactions and semi-structured interviews, it was discovered that the five learners who felt diagrams assisted them were accustomed to an activity-based approach to learning that integrated some collaboration. In an interview, Shriav regarding working with peers and using diagrams said:

When we occasionally worked in groups, we discussed different perspectives using sketches. We discovered from each other whether our diagram is correct or not and this gave us confidence in attempting the problem.

Collaboration allowed the learners to refine their use of diagrams by sharing ideas and taking corrective measures. Complex ideas were communicated through this process, enabling them to understand salient concepts. This learning approach is supported by Acharya (2023) who further assert that learners are more likely to develop robust understanding of concepts through insightful discussions and sharing of visual interpretations.

Figure 5.3

Bar graph representing the efficacy of diagrams/illustrations in trigonometry

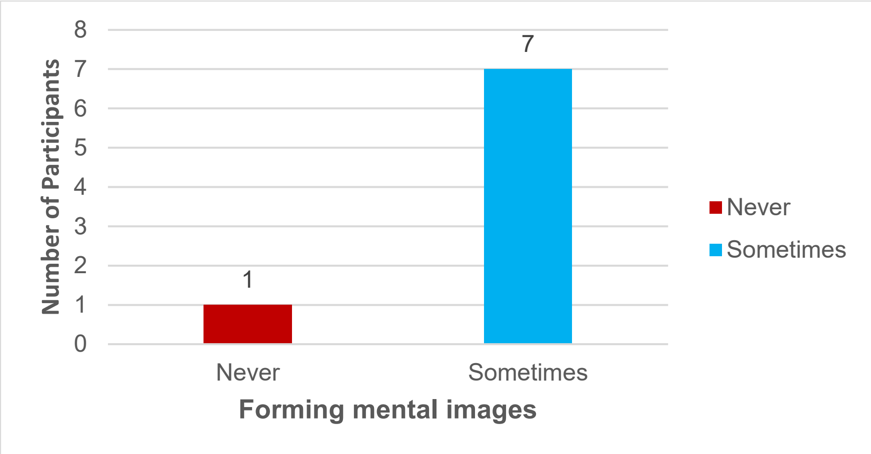


The use of mental imagery in problem-solving

None of the participants answered "always" indicating that forming mental images is not necessarily a spontaneous process in trigonometry. Palmiero and Piccardi (2020) assert that visualisation does assist in cognitive scaffolding but not all learners can be classified as visualisers. Although this is supported by Presmeg (1986), she further contends that learners of all preferences may show reluctance or altogether avoid visual representations because of discouragement by educators. An alternate explanation however lies in the fact that some learners may formulate solutions that may not suggest the use of imagery due to lack of confidence in its accuracy. It is with this in mind that the Krutetskii (1976) model distinguishes between learners that are "analytic" and "geometric" minded. As shown in Figure 5.4, seven participants said "sometimes" suggesting varied preferences and abilities in opting for a visual approach to solving problems. This is consistent with Presmeg's (1986) concept of mathematical visuality (MV) which she defines as the degree to which visual methods are used. However, the semi-structured interviews revealed that some participants ascribed their occasional use of diagrams to misconceptions about trigonometric complexity. Gift in particular stated that "*the intimidating nature of trigonometry made me feel that diagrams would possibly further complicate the problem*". This was partly due to his lack of skills and fear of being unsuccessful as with previous experiences. Nora expressed a certain level of comfort with algebraic manipulation because she felt it was efficient, less time-consuming and it fits into her learning approach. In explaining this choice of method she said, "*I prefer algebraic methods over using diagrams because it is faster, and I make less mistakes. My diagrams tend to be incorrect, and in fact my educator hardly used diagrams, that is why I do not use it*". It was noted from this discussion with her, that not all learners are inclined to using visual methods in solving problems. According to Asomah (2023), redundancy, overreliance of educators on symbols and equations, and uncertainty from learners were reasons for not using visual techniques in the teaching and learning of trigonometry.

Figure 5.4

Bar graph representing the frequency of using diagrams/illustrations in trigonometry

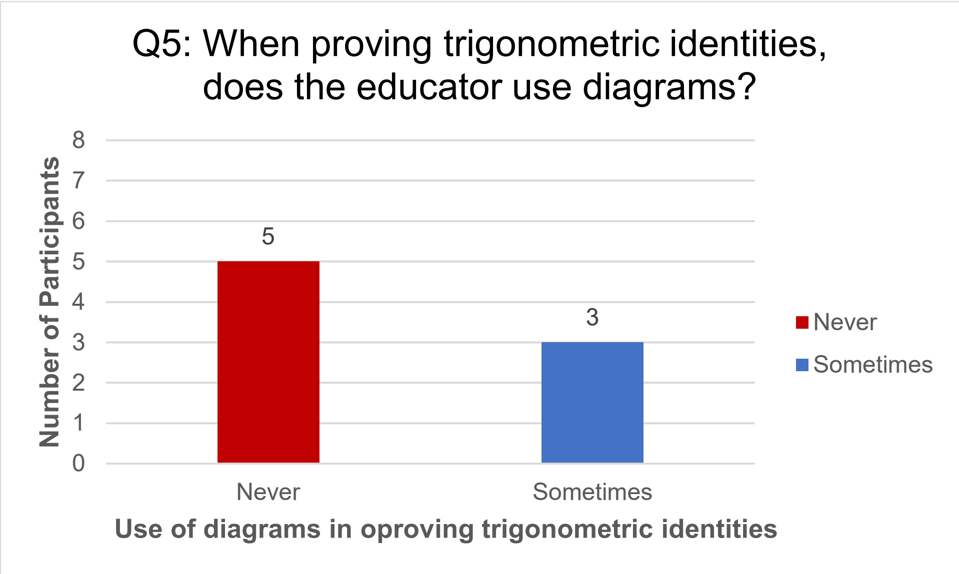


The use of diagrams in proving trigonometric identities

None of the participants replied with "always" suggesting that the educators have opted for what they perceived to be a more rigorous approach namely algebraic manipulation. Despite proving trigonometric identities are an Achilles heel for the majority of learners, the data analysis suggests that educators refrain from cultivating a visual approach although visual imagery does mitigate the complexities of trigonometric ideas (Manalao & Fukuda, 2024) as noted by five learners who replied "never". However, Uesaka et al. (2022) observed that many educators express caution in using diagrams because not all learners are adept at interpreting diagrams especially when it is loaded with abstraction, and further exacerbated in some cases where learners are taught in their second language. Another reason offered by Nabie et al. (2018) is that some educators themselves experience difficulty in cultivating visual strategies which invariably discourages them from embarking on this route. Precious was amongst the three participans who answered "sometimes". She explained in our interactions "*I am not sure about the link between diagrams and proving identities*". A prevalent connection amongst these three learners was that the educator used diagrams only in the introduction of trigonometric identities. The assessment of their workbooks indicated that higher cognitive level identities were proven using algebraic manipulation only. Figure 5.5 shows that in some cases where visual aids were used, it was done solely for an intuitive grasp and the acquisition of beginner skills.

Figure 5.5

Bar graph representing the use of diagrams/illustrations in trigonometric identities.

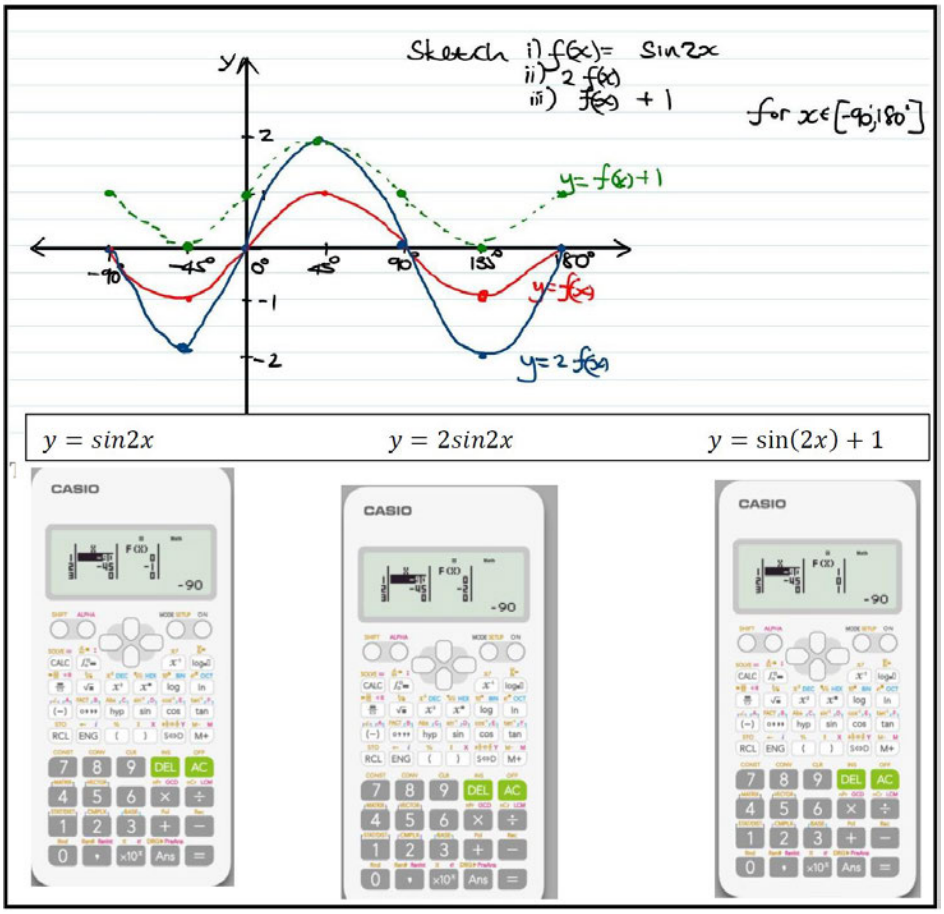


Graphical transformation of functions

Three participants answered "no" suggesting that they were not exposed to parametric changes concerning the mother graphs. Sanaya being one of them, made it a case in point to stress dependency on the calculator table mode to sketch graphs without relating it to possible variations. According to Nabie et al. (2018), educators may prefer simplicity and clarity over parametric alteration due to establishing firm grounding before delving into advanced concepts. It was noted from participants that although this school does have a smartboard and projectors, there was minimal use of software such as GeoGebra by mathematics educators who had the software installed to show them how to transform trigonometric graphs using the sliding scale. According to the participants, educators opted for a nuanced approach showing static transformation in some cases as evidenced by the remaining responses. An example that was apparent from the semi-structured interview with Nora was the calculator table mode which generated points to sketch each of the three graphs as shown in Figure 5.6.

Figure 5.6

Sketching of trigonometric functions using the calculator table mode.

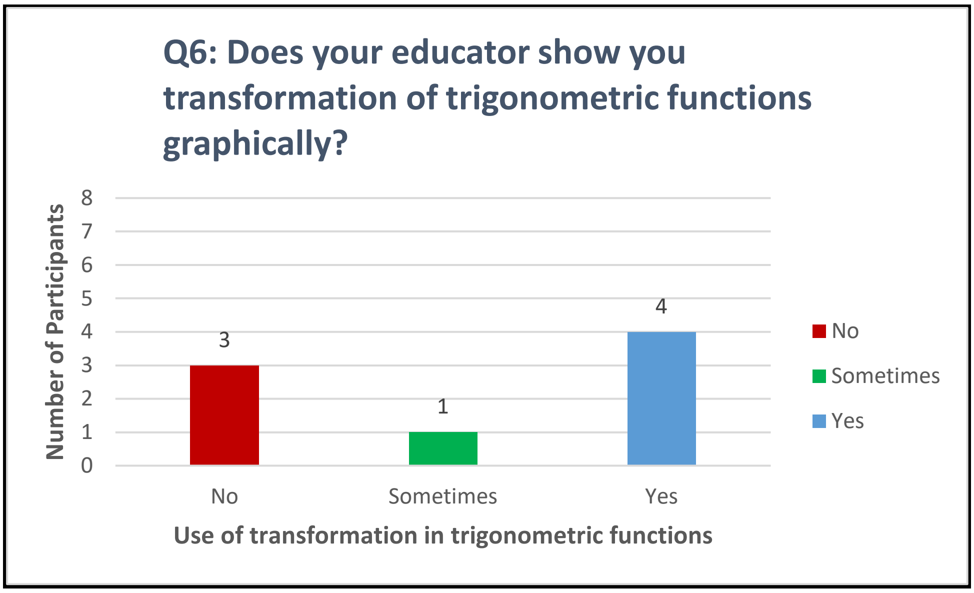


Note. Own construction.

This approach does exhibit traces of distributed cognition although it has the absence of collaboration and use of dynamic technology-mediated tools. The use of visualisation in the above example did provide a stable reference point for the four learners who answered "yes" as shown in Figure 5.7. It can be construed that visual strategies breaks down the concepts into manageable pieces, promotes clarity to a reasonable extent and obviates potential distractions from the hasty introduction of complex parametric variations (Urrutia et al., 2019). This is supported by Bal and Kapucu (2022) who further assert that the deconstruction of concepts into digestible components using visualisation methods allows learners to engage with mathematics in a more substantive way. Leung (2023) endorses the deconstruction process by using visual strategies and further argues that analysis of mathematical content in manageable pieces stimulates variational thinking thereby fostering profound conceptual understanding. Another factor that may contribute to this methodology of deconstructing problems into manageable pieces using visualisation is the time constraint, which in part is due to a lengthy syllabus barring other disruptions from the school calendar, thereby pressurising educators to do a superficial completion.

Figure 5.7

Bar graph representing the use of transformation in trigonometric functions.

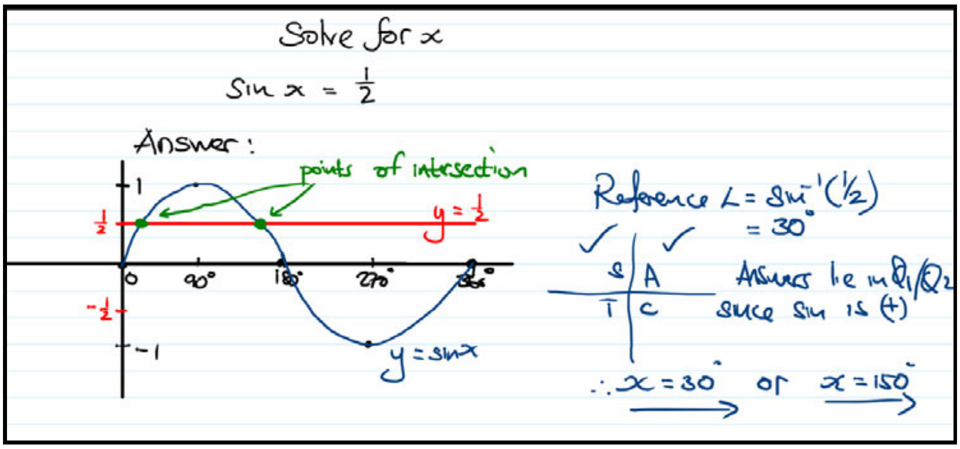


Relationship between trigonometric equations and graphs

The responses as shown in Figure 5.9 indicate that more than fifty percent of learners were exposed to a visualisation approach to solving trigonometric equations. However, it was discovered later that trigonometric graphs were confined to basic equations only. Siya explained that the educator stressed an intuitive understanding of basic concepts and shifted to analytical methods as they attempted higher cognitive-level equations. According to Maknun et al. (2020), many educators tend to favour mathematical rigour and theoretical understanding over graphical representations. In a study conducted by Manoharan and Kaur (2023), the findings showed that some educators refrained from graphical approaches due to a lack of pedagogical content knowledge (PCK) which hindered effective communication with learners. It was also noted by Spangenberg (2023) that time limitations and diverse learning modalities influenced didactical approaches. Precious pointed out, *"Graphs were initially used to show* *the points of intersection of the left-hand side function to that of the right-hand side function of an equation without specifying the periodicity and specific solutions for a given interval"*, as shown in Figure 5.8.

Figure 5.8

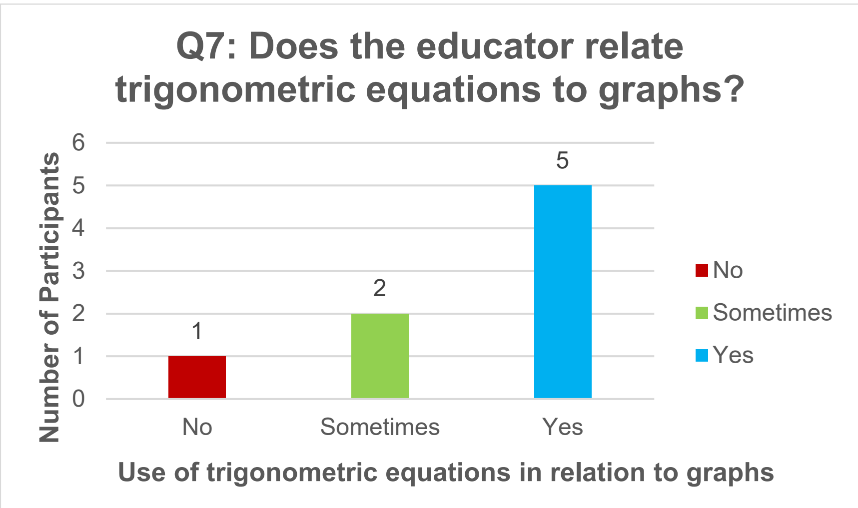
Solving a basic equation with the aid of a trigonometric graph.



Note. Own construction.

Figure 5.9

Bar graph representing the use of graphs when teaching trigonometric equations.

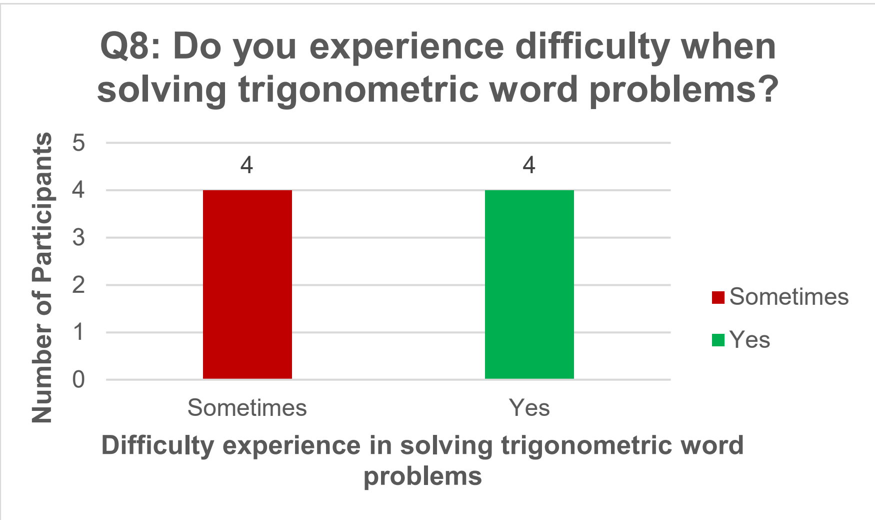


Challenges associated with solving word problems

It is evident from the data presented in Figure 5.10 that none of the respondents selected "no", with fifty percent responding "sometimes" and "always" respectively, indicating that they experience a varying extent of difficulty with trigonometric word problems. This is supported by Wulandri (2020), explaining that learners grapple with conceptual understanding and the interplay of abstract concepts within this domain, leading to inherent challenges in application. Other reasons include learners' prior experiences, negative attitudes, lack of spatial visual skills, overwhelming abstraction, exclusive dependence on algebraic manipulation of formulae and identities, poor procedural skills, inability to translate the words into appropriate equations, and lack of sufficient practice amongst others. Salam and Salim (2020) argue that heuristic learning models integrating interactive visual pedagogical methods can be the missing link that connects all concepts in providing a practical and relatable experience. Gunadi et al., (2023) assert that active participation in word problem activities links favourably with positive outcomes in the learning process. The 2022 mathematics NSC examination's DBE diagnostic analysis revealed the deteriorating state of trigonometric word problems, at a time when South Africa is desperately lacking in critical thinkers and problem solvers. This serves as a further validation of this study.

Figure 5.10

Bar graph representing the difficulty in solving word problems.

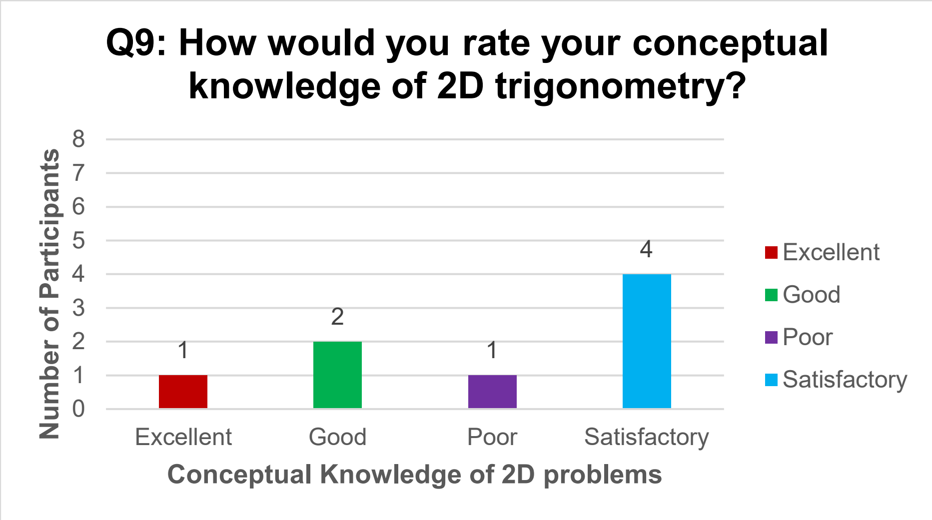


Participants rating their own understanding of 2D problems

Meena responded with "poor" citing reasons for weak foundational understanding. The explanation she gave was that her teacher rushed through the syllabus because they wrote provincial examinations and needed to finish the syllabus on time. She believed that the instructional design was not tailored according to her educational requirements, and she did not get sufficient practice time. Maphutha et al. (2023) recognise 2D trigonometry as a complex subtopic, posing significant challenges for learners. These challenges include the transition from basic to abstract questions. Learners struggle to find trigonometric relationships in various planes let alone comprehend the intricacies and nuances of trigonometric terminology. Half of the participants indicated a "satisfactory" understanding of the fundamental concepts. Shriav in particular was adamant about not having clarity about how to approach 2D problems and not having detailed feedback about his mistakes. He felt that visual aids would have been useful to link this section with practical day-to-day application. This is supported by Ocampo et al. (2023) who asserts that the integration of visual tools such as diagrams, charts and interactive software assists learners in understanding abstract ideas in trigonometry. Shriav was poignant about not achieving results that he was capable of, and this negatively impacted his motivational levels. The lack of collaboration and the use of technology for a visual approach added to the mental labour of learners. It is in this context that distributed cognition would have been most beneficial. Nora, Siya, and Precious as shown in Figure 5.11, exhibited more confidence in conceptual understanding and attributed this to working in their group where they exchanged ideas and watched YouTube videos in which educators used visual simulations and animations to explain concepts. They found that drawing diagrams, forming mental pictures, and using an iterative process helped enormously. Ngu and Phan (2020) support the use of visualisation in trigonometry and emphasised its importance in the understanding and retention of key concepts. Although Gift was more inclined to a verbal-logic approach to learning, he highlighted the value of having a dynamic visual approach to consolidate his understanding.

Figure 5.11

Bar graph representing the level of conceptual understanding of 2D problems.

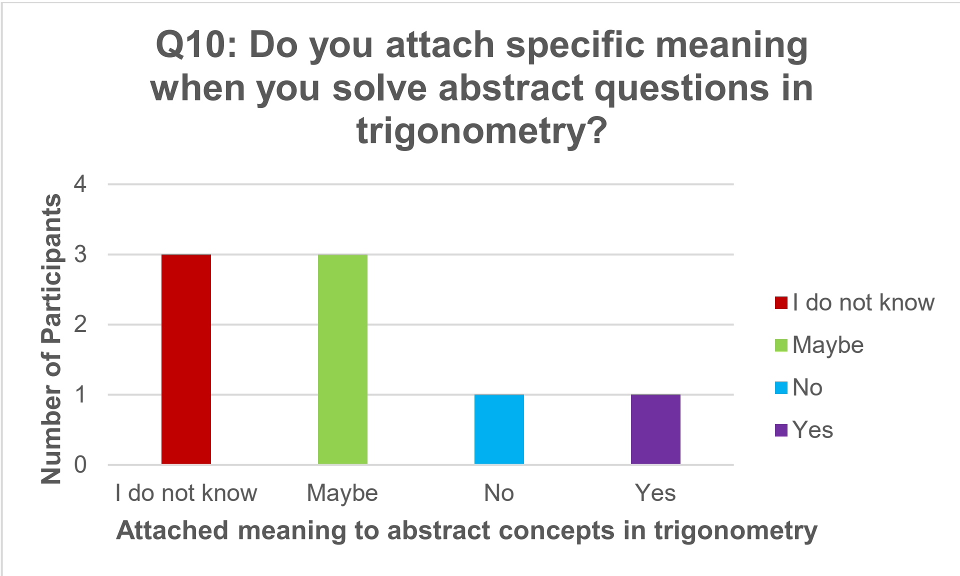


Interpretation of trigonometric concepts

Only Siya responded in the affirmative as shown in Figure 5.12, indicating that the pursuit of meaning in learning trigonometry needs to be prioritised. He alluded to the importance of attaching meanings to solving seemingly mundane equations in that it has far-reaching consequences on learners because they must be able to cultivate skills that can assist in real life. The purpose of trigonometry and mathematics, in general, is to echo the profundity of problem-solving in various disciplines ranging from physics to architecture (Ma'ruf et al., 2023). It is to this end that the essence of trigonometry lies in the synthesis of algebraic, analytical, and geometric skills to equip learners with a toolkit to be problem solvers. The remaining participants expressed uncertainty and frustration citing a disconnect of trigonometric concepts from the physical world. This can be attributed to unfamiliarity with foundational concepts, inadequate instructional design devoid of contextual relevance, and possible rote learning disregarding the internalisation of key principles (Hidayati & Prahmana, 2022). Arhin and Hokor (2021), in light of poor performance in trigonometry, argue for a robust visuospatial prioritisation in pedagogical practices to deepen comprehension and application.

Figure 5.12

Bar graph representing the ability to attach meaning to abstract questions.



5.4 Analysis of the Pre-Task

The pre-task allowed for the collection of baseline data to measure performance levels before the intervention. This aided in assessing the impact of the visualisation intervention in addition to discerning whether the impact was attributed to external factors rather than the intervention itself. Further scrutiny was possible to the detection of inherent characteristics and pre-existing conditions that influenced responses which provided a solid foundation for evaluation. The pre-task also reinforced the credibility and rigour of the study thereby making for a robust analysis of the collected data.

5.4.1 Curriculum alignment of pre-task

The CAPS document served as the bedrock to ensure that the task was consistent with the expectations and objectives of the curriculum. The taxonomy applied, rigidly adhered to the guidelines in the aforementioned document. The pre-task also considered the diversity of participants by including a range of questions that highlighted targeted skills and competencies that were relevant to the research. The alignment of the pre-task to the curriculum allowed for the comparison of learners' performance with established standards which additionally allowed for the results to be extrapolated to more extensive educational settings. The curriculum alignment also ensured that participants' experiences and goals were respected, consequently promoting trust between themselves and the researcher. This alignment reassured participants that every question was pertinent to their academic growth. Once the strengths and weaknesses were determined, the appropriate intervention was then distributed. The taxonomy was in accordance with the TIMSS study of 1999 as stipulated in the CAPS document as shown in Table 5.3.

Table 5.3

Taxonomy of assessments

Cognitive levels	Description of skills to be demonstrated
Knowledge **20%**	• Straight recall • Identification of correct formula on the information sheet (no changing of the subject) Use of mathematical facts • Appropriate use of mathematical vocabulary
Routine Procedures **35%**	• Estimation and appropriate rounding of numbers • Proofs of prescribed theorems and derivation of formulae • Identification and direct use of correct formula on the information sheet (no changing of the subject) • Perform well known procedures • Simple applications and calculations which might involve few steps • Derivation from given information may be involved • Identification and use (after changing the subject) of correct formula • Generally similar to those encountered in class
Complex Procedures **30%**	• Problems involve complex calculations and/or higher order reasoning • There is often not an obvious route to the solution • Problems need not be based on a real world context • Could involve making significant connections between different representations • Require conceptual understanding
Problem Solving **15%**	• Non-routine problems (which are not necessarily difficult) • Higher order reasoning and processes are involved • Might require the ability to break the problem down into its constituent parts

Note. Adapted from the CAPS document (2011, p. 53).

5.4.2 Findings of pre-task

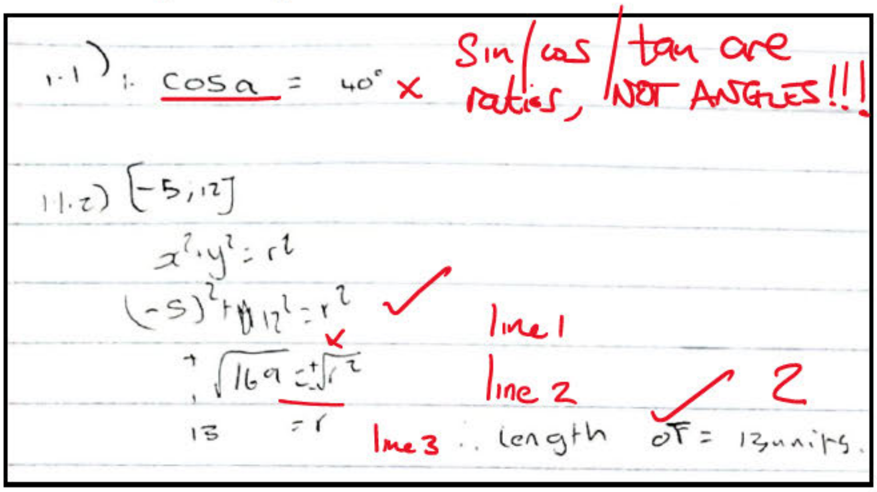
Each question integrates the six major subtopics in trigonometry as stipulated in the CAPS document. The findings are discussed with regard to each of those subtopics.

5.4.2.1 Sketch in Cartesian Plane

Question 1.1 was aligned with the trigonometry curricula as defined by the CAPS document regarding conceptual understanding of trigonometric ratios, reduction formulae, knowledge of the Cartesian Plane, and use of the Theorem of Pythagoras. This question is considered to be a combination of level one standard (Basic knowledge) and level two standard (routine procedure).

Figure 5.13

Meena's response to question 1



It was noted that Meena as shown in figure 5.13, did not draw any diagram either on the task sheet or question paper. In addition to being regarded as a non-visualiser, she started with a conceptual error of ratios. Sin, cos, and tan are ratios and not angles, yet her answer was in degrees. Q1.1.2 was correctly answered showing her strength in algebraic manipulation although she made a minor error in solving the quadratic equation in line 2 of her solution. She correctly identified that lengths (r) are non-negative quantities which establishes a level of mental reasoning related to a real-life scenario.

Researcher: I noticed you haven't used a diagram in this question. Any particular reason for that.

Meena: Umh…I struggle with drawing diagrams and I'm not sure of myself at times.

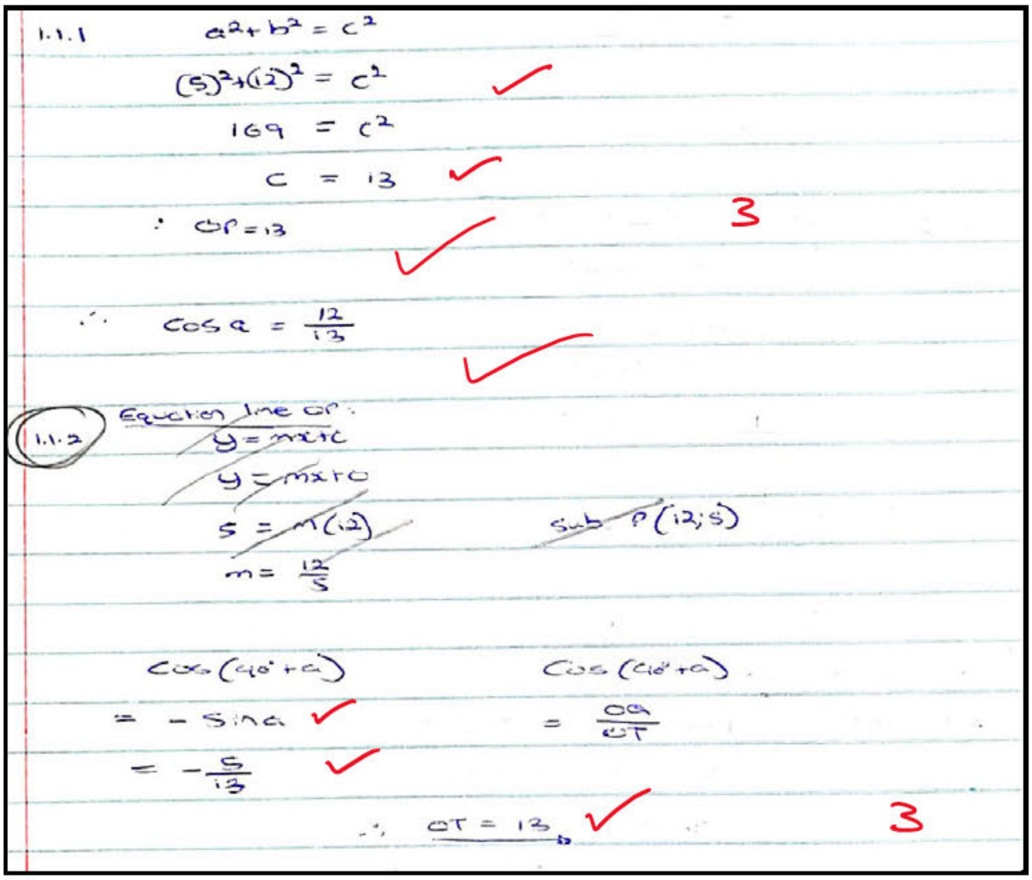
Researcher: Have you been taught trig ratios with a diagram?

Meena: Not really.

Meena's response indicated hesitancy in using visual representations in her answer because she was afraid of getting it wrong. Her apathy was understandable considering her experience with using pictures was all but non-existent.

Figure 5.14

Precious's response to question 1



It was particularly interesting to note that Precious used the sequential order of the letters of the English alphabet to remember the Pythagorean statement as evidenced in figure 5.14. She constructed right-angled triangles on her question paper as a ploy to save time. She accurately performed co-ratio reduction and formed a link to the initial triangle configured to angle α. Her previous experience with visualisation formed the bedrock for her solutions as evidenced in the structure of her answer. Ngu and Phan (2020) state that incorporating visual elements in the teaching and learning of trigonometry enhances the comprehension of abstract concepts. Michalik et al. (2023) further adds that visual representations in trigonometry creates a dynamic learning environment and facilitates learning across diverse educational levels.

Researcher: It was really smart of you to use the question to complete the triangles.

Precious: Eish…I remember my teacher telling us to always save time, so I used the question paper to my advantage. In my mind's eye, I could see the triangle relative to the *quadrants and it made things easy.*

Researcher: Have you always taken a liking to visual methods?

Precious: At my previous school, the teacher used the smartboard, and that inspired me to be more visual in my approach. I developed the attitude that 'visual memories are the best memories".

Researcher: Your co-ratio reduction was accurate. Did you memorise it or did you use visual techniques?

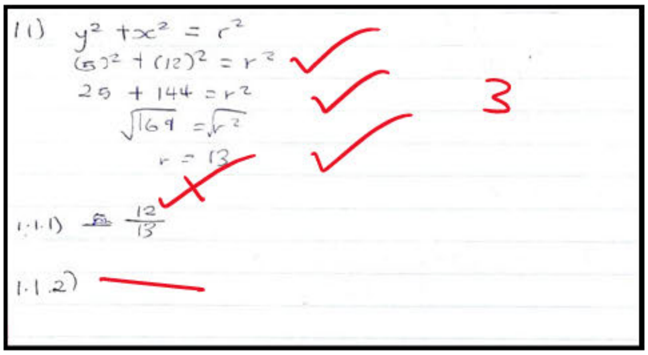
Precious: I sketched a composite CAST diagram with all my reduction formulae. I was then able to precisely work it out.

Her answer to Question 1.1.2 began with her finding the gradient of the terminal arm which indicates the tangent ratio. However, she realised her mistake and used the angle in the standard position which showed that her conceptual understanding of the Cartesian Plane is exceptional. She used the CAST diagram on which she wrote her complementary reduction formulae which aided her in cognitive offloading.

Sanaya's approach as shown in Figure 5.15 was diametrically opposite to Precious's. There was no indication of her using any visual constructions, and her question paper also was devoid of filtering key information. A noteworthy observation though, was the correct use of the Theorem of Pythagoras and trigonometric ratios. Question 1.1.2 was left blank since she could not geometrise similar triangles in this context. She could not contextualise the problem in the standard position which led to a co-ratio reduction.

Figure 5.15

Sanaya's response to question 1



Researcher: I noticed you have good algebraic skills.

Sanaya: Thank you, sir. I am more comfortable with algebra than other methods.

Researcher: Is there any particular reason for this?

Sanaya: Umh…I was always taught that way. Although my teacher occasionally used a sketch, it was never reinforced as a significant tool in answering questions.

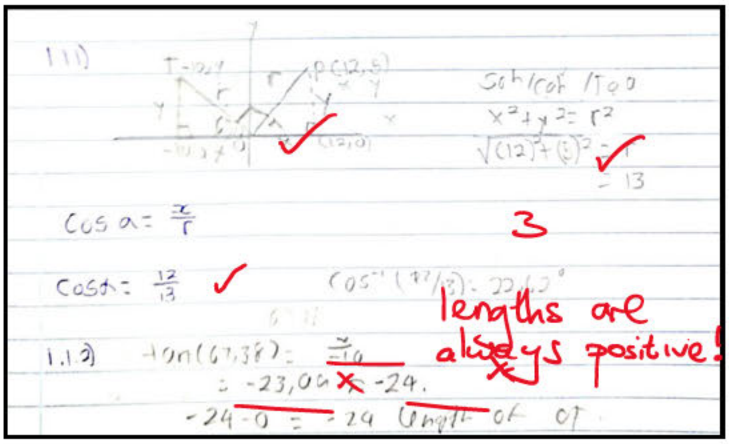
Researcher: Why did you leave Question 1.1.2 blank?

Sanaya: I could not understand what was being asked. My algebra was unable to help me here.

Algebraic skills seem to be Sanaya's armour which shows that such a skill by itself is not sufficient. Her lack of conceptual understanding and procedural fluency concerning reduction formulae were exposed. It must be emphasised that visual representations are a complement to algebraic skills, but unfortunately, algebraic prowess has become the predictor of mathematics achievement (Kennedy & Ebuwa, 2022). Shriav on the other hand showed a good visualisation attempt in Question 1.1.1 by constructing two right-angled triangles, both in standard position. This helped him work out the cos ratio accurately but unfortunately, he could not relate the two triangles via co-ratios although it was indicated on his diagram. His analysis of the diagram as shown in Figure 5.16, was devoid of pattern recognition and angle relationships, indicating a lack of spatial reasoning.

Figure 5.16

Shriav's response to question 1.1



Researcher: Your diagram is excellent. You applied the theorem of Pythagoras correctly and applied your ratios correctly. Have you always used sketches?

Shriav: Actually, this method is what helps me understand the complexities of trigonometry.

Researcher: Once you sketched your diagram, how do you go about using it?

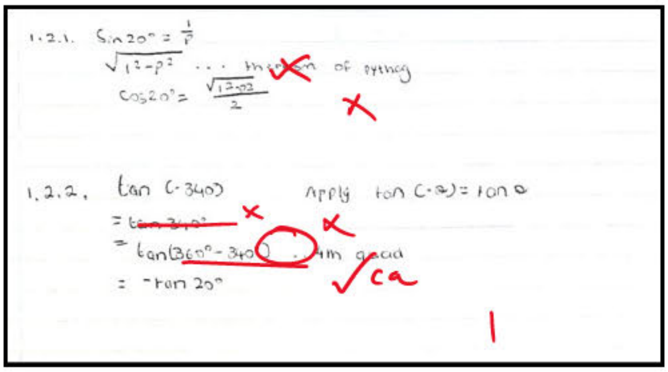
Shriav: I try to filter out what I need and then apply the basics.

Shriav's answers indicated a disconnect between abstract trigonometric concepts and real-life applications. According to Maphuta et al., (2022), learners need to understand the interconnectedness of algebra, geometry, and trigonometry so that it helps them to effectively solve problems. The overall analysis of this question strongly suggests that visualisation is not being used effectively due to educators showing an overreliance on algebra. Learners are not adequately prepared to use visual tools to navigate the intricacies of trigonometry (Ismail et al., 2022). This is supported by Fendrik and Elvina (2018) who further argue that visual thinking approaches notably enhance critical thinking and problem-solving abilities. The challenges experienced by learners in utilising visual techniques require a holistic approach that integrates technology-mediated tools, manipulatives, semiotics, problem-based instructional design, and algebraic comprehension amongst other things to better equip learners (Alqahtani et al., 2022).

Question 1.2 provided a similar lens into the participants' mathematical thinking. Spatial visualisation skills were not optimally used when presented and, in most cases, not used at all as shown in Figure 5.17.

Figure 5.17

Sanav's response to question 1.2



Researcher: Have you tried drawing diagrams to assist you in unpacking the question?

Sanav: I don't because I am not used to it.

It was evident that Sanav did not utilise a CAST diagram for reduction purposes, or any diagrams related to the trigonometric ratio. Regarding this question, four of the eight learners did not attempt to use a visual approach, all of whom scored less than thirty percent. The results of the semi-structured interview showed that the teachers were not well-prepared with visual techniques because they did not include visual elements in their pedagogy. The main reason why learners found it difficult to understand the significance of visual aids was the false belief that trigonometry had no practical applications. This impeded the participants' conceptual understanding resulting in poor performance (Ghannam & Ansari, 2020; Siregar, 2023). Research by Talli et al. (2022) further revealed that learners look toward their teacher as their primary source of information and as such try to emulate what was shown to them. Consequently, educators who neglect visuospatial reasoning in their instructional design hinder learners' cognitive development (Judd & Klingberg, 2021; Ericson & Klingberg, 2023).

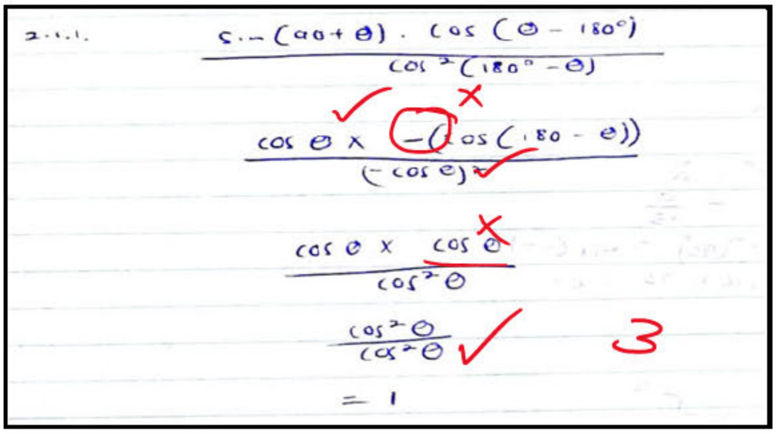
5.4.2.2 Application of reduction formulae

Question 2 required learners to reduce ratios to positive acute angles in addition to knowing special angles, without using a calculator. Understanding and recalling important trigonometric concepts, being the centerpiece of this question, was in alignment with the CAPS document (2011), where learners had to draw on their knowledge of the CAST diagram in relation to applying reduction formulae. The cognitive levels varied from level 1 (knowledge) and level 2 (routine procedure) according to TIMSS (1999) based on the four cognitive dimensions (Sriyanti & Pusphita, 2022).

Gift's attempt to question 2.1.1 showed errors in the reduction application, all of which were done without using a CAST diagram. There was no evidence of any physical visual techniques as evidenced by the absence of imagery.

Figure 5.18

Gift's response to question 2.1.1



Researcher: I noticed that you did not draw a CAST diagram to reduce ratios. Do you picturise it mentally or did you just memorise reduction?

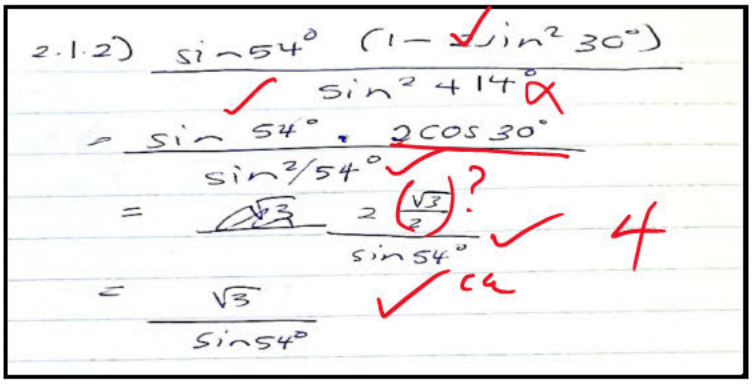
Gift: My teacher told us to learn it by heart.

The response above unmistakably demonstrates rote learning without understanding as attested by Gift's solution as shown in Figure 5.18. While the basic reduction was accurate, the advanced reduction of $\cos(\theta - 180^0)$ was incorrect, indicating that Gift did not fully comprehend the derivation, which was connected to a thorough comprehension of the CAST diagram. Research by Subedi (2023) supports that insufficient focus on conceptual learning shows that trigonometry was not taught in a meaningful context. Ma'rufi et al. (2020), assert that in some instances the educator's interpretation of the curriculum may not prioritise the connection between visual imagery and trigonometric concepts. This is supported by Abduh and Taqwa (2022) who favour inquiry-based approaches with the use of dynamic mathematics software such as GeoGebra. It was also alarming to note that six out of the eight participants made similar errors in the advanced reduction, most of whom cited limited exposure to visual learning as a major drawback.

The performance in question 2.1.2 was poor, exposing multiple flaws in reducing numerical angles. It stemmed from the conceptual inaccuracies, lack of practice, and cognitive challenges. This is supported by Jones and Tarr (2017), who emphasised the challenges learners have with visualising numerical angles in connection with the unit circle regarding special angles which induced deficiencies in reduction.

Figure 5.19

Siya's response to question 2.1.2



Siya's solution as shown in Figure 5.19, indicated a minor mistake in his knowledge of co-ratios. He did not make use of a CAST diagram or a unit circle. It was evident that rote learning again manifested misapplications. However, Sanaya's response as shown in Figure 5.20, experienced difficulties in applying fundamental concepts, and her inability to manipulate multiple trigonometric ratios further compounded problems.

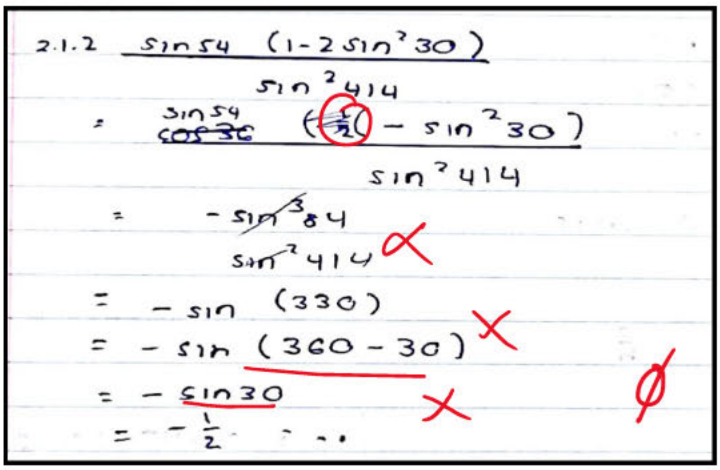
Researcher: What method do you use for reduction?

Sanaya: I memorised the notes my teacher gave us.

The above response revealed that learning without understanding took place. It was noted that no attempts were made to use visual methods to mitigate the complexities of basic concepts.

Figure 5.20

Sanaya's response to question 2.1.2



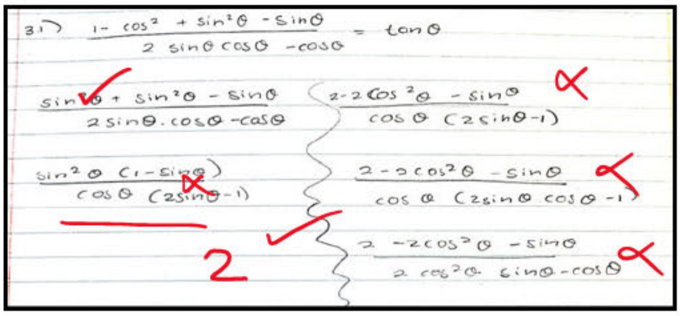
The results for this question showed that three out of eight learners failed to score a single mark. Only one learner, Precious scored full marks, whilst the others scored between thirty and seventy percent. A recurring pattern of non-visual techniques emerged amongst all learners that had below sixty percent. It was concerning to note that the above question, regarded as a level two in cognitive standards, produced such a poor performance.

5.4.2.3 Proving Identities

Question 3.1 dealt with proving identities which relied on learners' application of the quotient and square identity (CAPS, p.33) together with algebraic manipulation. This topic in trigonometry generally poses great difficulty for learners (DBE NSC diagnostics report, 2022).

Figure 5.21

Meena's response to question 3.1



Meena managed to correctly substitute for the square identity, as shown in Figure 5.21, but was unsuccessful in proving the identity, primarily because she failed to see the trigonometric relationships. Her solution did not make use of any diagrams that could have possibly assisted her.

Researcher: What method do you use when proving identities?

Meena: I split the LHS from RHS and then try to prove they are the same.

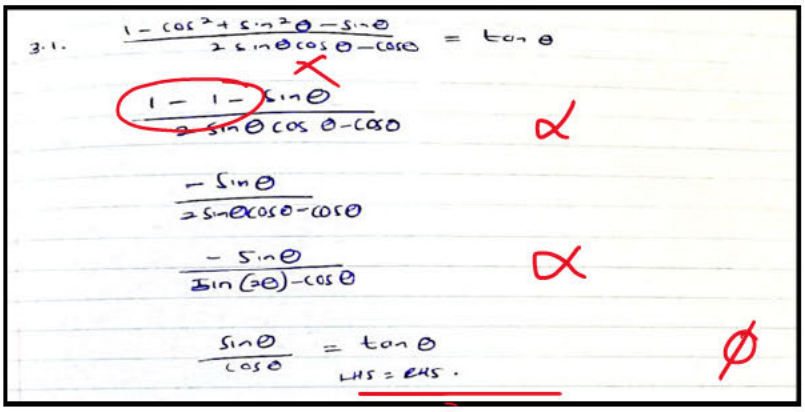
Researcher: Excellent. I noticed you did not use any diagrams to help you answer.

Meena: Yes, I use my symbolic and algebraic knowledge coz I am not used to anything else.

Meena's response clearly showed a complete dependence on algebra and memorisation, resulting in inaccurate solutions. Nurmeidina et al. (2021) primarily attribute learners' difficulty in proving identities to misunderstandings and uncertainty in utilising algebraic and trigonometric properties. Sari and Nurfauziah (2019) argue that poor basic trigonometry skills constrain learners' ability to prove simple identities, leading to their incapacitation in proving more complex identities. Shriav's attempt as shown in Figure 5.22 reveals a lack in conceptual understanding of trigonometric properties and mathematical reasoning. There is a complete lack of visual representation, and similar to Meena's solution, this approach was also unrewarding. Only two participants obtained a perfect score, while the rest scored below seventy percent. A common thread among the mistakes made by the participants suggested a lack of familiarity with visual methods and a misunderstanding regarding the superiority of algorithms.

Figure 5.22

Shriav's response to question 3.1



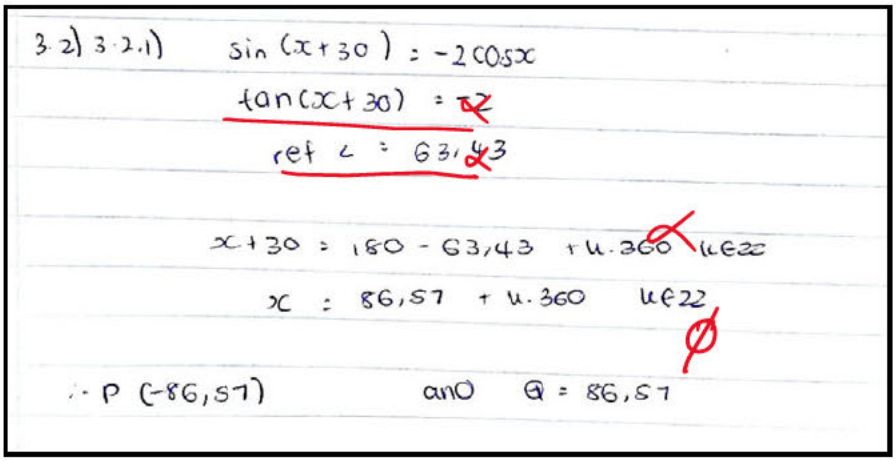
Schafer (2023) argue that visual aids can enhance the mathematical approach to proving identities by significantly improving comprehension. In contrast, Polat and Akgun (2023) intimates that the impact of a visual strategy will fluctuate according to different learning modalities but does acknowledge its benefit to some degree.

5.4.2.4 Trigonometric Functions and equations

Question 3.2 dealt with trigonometric functions with sub-question one being a trig equation in this context. The $sin\ (A + B)$ expansion was given to participants in class sessions with the primary objective being pattern recognition as a prelude to grade 12 trigonometry. Participants had to realise that points of intersection meant that they had to solve equations simultaneously which meant that algebraic procedures together with graphical analysis were crucial.

Figure 5.23

Sanav's response to question 3.2.1



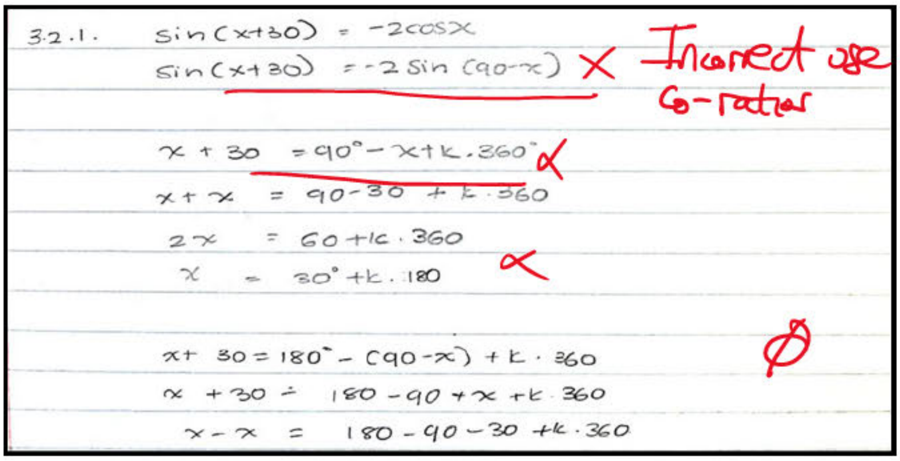
Sanav's attempt shown in Figure 5.23, did not utilise the given expansion thereby making a fundamental mistake with the quotient identity application. He made a conceptual error by dividing through by the cos ratio not realising the ratio of angles were different. This inevitably led to the general and specific solutions being incorrect. The graphical representations were not used to give him an approximate interval that contained the solution, indicating a visuo-spatial deficiency in his analysis of the question. An alarming statistic was that six out of eight participants failed to score any marks. Dahal et al. (2022) underscores the significance of visualising trigonometric concepts by creating interactive learning experiences. A study by Sofianingsih and Ahmadi (2023) corroborates the transformative nature of using visual elements in the teaching and learning of trigonometry and firmly establishes that good visuospatial skills develops positive learning attitudes and assists in overcoming learning challenges.

Researcher: Did you analyse the graphs before solving this problem?

Sanav: Not really. I realised it was simultaneous equations.

Figure 5.24

Nora's response to question 3.2.1



Nora's answer was also marred with conceptual errors as shown in Figure 5.24, with her using co-ratios but ignoring the coefficient of -2 in line one. Her misconception and misapplication of complementary ratios emanated from her lack of comprehension of the unit circle and the transformation of functions.

Researcher: Why did you change the sine ratio to its co-function?

Nora: My teacher told us to make both ratios the same.

Researcher: Have you tried a visual approach to assist you in solving?

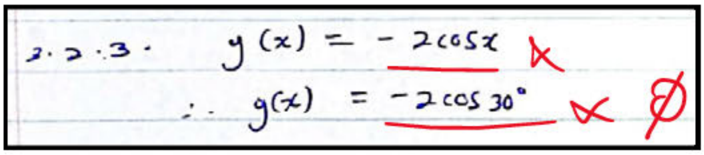
Nora: No because this is the method we are accustomed to.

Nora did not understand the parametric variations in trigonometric equations in addition to never using a graphical approach to assist her in the solving process. From her response, it was clear that she did not know that solving an equation is about determining the points of intersection of the left-hand side function in relation to the right-hand side function. According to Tanisli and Dur (2018), learners who are not adept at problem-solving lack visual and spatial skills, posing serious obstacles in mathematical reasoning. Rohimah and Prabawanto (2020) assert that the abstract nature of trigonometric equations overwhelms learners, thereby creating more confusion and anxiety. The DBE mathematics diagnostic report (2023) affirms that a significant number of learners do not know how to begin solving equations, implying a deficiency in methodology and conceptual understanding.

Question 3.2.2 was scaffolded from the previous question, accentuating the need for learners to compare graphs in a restricted domain. This question was poorly answered, Precious being the only participant to score maximum marks. Participants failed to understand that they were required to read off x values for which $f(x)$ had to be above $g(x)$, indicating a lack of semiotic fluency and graphical interpretation skills. Question 3.2.3 dealing with the transformation of functions, proved no different in terms of outcomes. Learners could not visualise the left horizontal shift and the parametric change to the functional notation as evidenced by Sanaya's response as shown in Figure 5.25.

Figure 5.25

Sanaya's response to question 3.2.3



It was noted that only three out of eight participants scored full marks. According to Eynde et al. (2019), learners often grapple with interpreting notation and symbols which leads to erroneous mathematical reasoning. Budak and Ozkan (2022) further contend that the traditional pedagogical approach of simply plotting and joining points to draw graphs may not be sufficient to impart the complexities associated with translations.

5.4.2.5 2D problems in pre-task

The primary reason for incorporating 2D problems in trigonometry was to equip learners with the tools to solve real-life problems dealing with angles, distances, and geometric shapes (Agormor et al., 2022). The CAPS document (2011, p. 37) requires that learners be able to prove and apply the sine, cosine, and area rules, thereby honing their visuospatial abilities. The integration of these concepts is meant to foster critical thinking and analytical reasoning.

It was disconcerting to note that four out of eight participants left this question either blank or partially completed. It was evident from Sanaya's and Shriav's responses in Figure 5.26 and Figure 5.27 that they struggled to visualise trigonometric concepts in the context of two-dimensional problems. Baidoo and Luneta (2023) point out that learners' lack of conceptual understanding regarding the interconnectedness between geometry and trigonometry, invariably leads to difficulties in problem-solving.

Figure 5.26

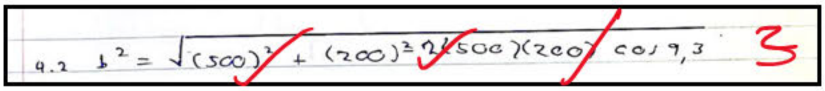
Sanaya's response to question 4

Figure 5.27

Shriav's response to question 4

Figure 5.28

Gift's response to question 4.2



Question 4.2 required a sketch, but six out of eight participants either made no attempt or partial attempt, as shown in Gift's response in Figure 5.28. This question posed great difficulty for learners due to its abstract nature, all of which could have been mitigated if a visualisation strategy had been used (Riccomin et al., 2022). Additional challenges included a lack of conceptual understanding and spatial reasoning, both of which could have contributed to the inability to translate verbal descriptions into diagrams (Powell et al., 2021).

5.5 Analysis of the interview

Do you find trigonometry challenging? If so, what are the challenges?

This question was asked to establish the background knowledge of each participant. The individual experiences, perspectives, and perceptions prompted unique responses, and this provided a lens into their mindset and attitudes. The responses also provided a contextual understanding and shed light on various influences. This assisted me in managing ethical considerations such as cultural sensitivity and the dynamics at play in homes and communities. Gift’s response:

Yes, because it is not always clear what path would lead to the solution and I find it hard to fully understand and interpret the meaning of the answer. The 2D problems are very challenging because I get confused with the terminology and this leads to misinterpretation. I also find it difficult to apply the correct formulae. I wish the educator could relate these concepts to everyday reality.

Sanaya’s response:

Yes. I have difficulty in incorporating and identifying algebraic rules in trigonometry. I often expand and reduce incorrectly and in certain situations, reduction is not required. This work is so abstract, and the challenge lies in the fact that I don’t know what the meaning of all this is. I cannot understand the relevance of trigonometry in real life, and this adds to my fear and anxiety about this topic. I am the first amongst my siblings to do pure mathematics because of my career choice but I wish sometimes that I could get some help from them but unfortunately, I can’t. My dream is to become a civil engineer and I know I have to perform well in mathematics and physics, that is why I am pushing myself constantly to overcome the challenge

Nora’s response:

I cannot say I find trigonometry challenging as a whole. Certain topics for example trigonometric functions is challenging. I cannot understand how certain variables in the function affect the graph. Another thing is that I cannot see how the subtopics in trigonometry as a whole are linked to each other and mathematics in general. Equations are also tricky, but the rest is manageable. I am now trying to watch YouTube videos to get a richer understanding.

Meena’s response:

Yes, I do. Trigonometry is a nightmare for me. I am a learner who prefers pictures to help me but unfortunately, that doesn’t happen. I am not blaming my teacher because I know he is also under pressure to finish the syllabus and he is trying his best. I am so stressed out with this topic that sometimes I want to give up. My marks were always bad for this topic and at this *stage, I am overwhelmed. My problem is that I just cannot unpack questions successfully and I cannot reason within myself because of the complexities of the problems. I confuse the concepts especially when it's a hard question and it takes me a long to process all the information. I am wondering if there is a magic wand to make things easier.*

Siya's response:

Not really. I kind of enjoy trigonometry more than some other topics in paper 2. I feel as if the more practice examples I do, the more I can understand trigonometry. I practice a lot picturising the problems, using some tricks I learned from the internet. However, truth be told I still have some issues with 2D problems once the diagram becomes complicated. I do not know where to start.

This question uncovered a plethora of challenges faced by learners. A common thread was the inability to understand the concepts and applications. Only two learners exuded confidence in their response, whilst the others had a waning level of motivation. The insight gained from the responses indicated that the pedagogical approach must be revised. Siya's response in particular alluded to imagery, mitigating the abstraction of some questions which indicated that visual aids do have an impact. However, he also mentioned that once the diagrams became more complex, they became increasingly difficult to understand. A common thread amongst all responses was that the learners felt a disconnect between trigonometry and reality. They also grapple with visualising trigonometric functions and their relationship with the unit circle and equations, thereby affecting their computational precision.

Explain how you go about formulating solutions in trigonometry.

This question was of paramount importance regarding its effect on pedagogical improvement. It was used as a springboard to identify common misconceptions, difficulties, and effective teaching and learning approaches, given the diversity of learning modalities. Deep insight was gained into the mathematical cognition of individual learners, contributing to the understanding of how learners think and reason. Upon evaluation of responses, it was determined where revision was required in teaching methodology to adequately achieve educational outcomes.

Sanav's response:

I tried to read the question several times trying to filter key information. I sometimes highlight important information on the question paper in an attempt to draw out as much as I can from the question. In some instances, I try to make sketches to help me break down the problem for better understanding. However, I am reluctant to make sketches because I fear I am putting all my eggs in one basket in the sense that if I do not make an accurate depiction of the problem then I may take the wrong route in problem-solving.

Nora's response:

I try to understand what was being asked and then I attempt to put things into context in relation to the curriculum. I make little notes alongside the question to get some direction. In most cases, I seek algebraic methods because that's what I am comfortable with. I am hesitant to draw diagrams because I am not used to it. Perhaps it is because I am not exposed to such teaching methods. I sometimes watched some trigonometry videos on YouTube and got some ideas on acronyms to remember concepts. This helps a lot in formulating solutions.

Precious response:

The nature of the question seems to steer me in a certain direction. When the question is very wordy, I kind of get lost. I try to give it my all, but I guess it's not good enough. I am not happy with my trigonometry marks and I'm wondering what I can do to improve. I feel like I am in a cloud of uncertainty regarding the way forward. My mum told me to start extra lessons with a tutor so I hope that may help me formulate more successful solutions. At the moment my solutions are kind of garbled especially for hard questions. I don't know where to start and what logic I should apply. I feel like I know the basics, but my problem-solving strategies let me down and I cannot form reasonable solutions.

Shriavs's response:

After some thought, I consider various possibilities in strategising a solution. At every step I check for appropriate rules I have to work with and if it will apply. I tend to take a long in writing solutions and more often than not that's the reason why I don't finish my paper. I also *look for clues in the question to guide me and then I try to relate it to the concept the examiner is trying to test. This is the juncture that I find most difficult because I have to "connect the dots" at this stage to find a coherent and meaningful answer. I hardly use diagrams or any visual imagery because I am not used to this method since my teacher never emphasised it.*

Siya's response:

I pretend that I am the teacher so that I can break up the problem into little pieces to explain to the class. I connect all the information presented in the question using a mind map. After sifting the relevant info, I get a holistic idea of the way ahead. I seek help from my older brother who always reminds me that the mind thinks in pictures, so little diagrams guide me regarding the understanding and plausibility of my solution.

It is clear from the responses that learners have their individual preferences on how to form solutions. Their experiences and perceptions played a vital role in their engagement with trigonometry. As evidenced by the responses, there is no clear overriding approach that indicates the influence of a central pedagogical intent. The learners do not have a uniform, coherent structure in the way they think about answering trigonometry. All the responses revealed the under-reliance of a visual approach further underscoring the importance of this intervention. Shriav categorically stated he was not accustomed to forming pictures in deriving solutions whereas Nora expressed hesitancy in using diagrams because she feared making a mistake in assimilating information. According to Setiawan and Surahmat (2021), learners produce more errors in trigonometry than in any other topic in mathematics which indicates major challenges faced by learners in processing skills. This is supported by Gradini et al. (2024) who further elaborates on the misunderstanding of learners formulating and implementing strategies in solving trigonometric problems.

How did your teacher introduce trigonometry to you?

This question was intended to investigate learners' initial experience with trigonometry. The saying "first impressions last" is so apt, especially in the teaching of mathematics. The foundational skill in any topic provides the groundwork for more advanced things to come. Learners' initial engagement with trigonometry has a long-term effect on their motivational levels, attitude, and mindset. The excitement of being taught something new, which has invaluable significance in everyday life such as trigonometry, should pique curiosity and intrigue, the resonance of which should be far-reaching (Eze, 2023). The fascination of this topic in relation to its historical and cultural framework must harness the intrinsic appeal amongst learners to solve an array of problems. It was to this end that this interview question was at the core of understanding the learner's trigonometric approach.

Nora's response:

My teacher saturated the lesson with an algebraic approach. The central focus was on formulae and their application. I felt a bit dismayed that a topic like trigonometry would have had more of a practical dimension. After watching a few YouTube videos, I realised that the world around us is laden with trigonometric applications. It is used in such diverse contexts, and this inspired me to study it more. The dynamic visuals related to trigonometry was unfortunately not the bedrock of its introduction in my class. However, I do understand that educators are also under tremendous pressure to cover syllabus so perhaps the shortcomings can be attributed to time constraints.

Siya's response:

My teacher mentioned a right-angled triangle and the theorem of Pythagoras. He then drew the triangle and started speaking about ratios. In the beginning I was a bit overwhelmed by all this new information which I could not relate to. I started to understand things more when sir explained the CAST diagram by drawing the diagram in colour. He used his finger to indicate ratios and moved it from one quadrant to another to tell us about the sign changes. I was fascinated by this, and that lesson was memorable.

Meena's response:

My teacher used SOH-CAH-TOA for us to understand and when he drew the Cartesian Plane, he used the acronym "Shine Your Racer Cos X-Rays Tan Your X-terior" to help us remember the trigonometric ratios. I remember this lesson because of the interesting ways that were used to calculate heights and angles. I wish all the lessons were like that, but it got more complex and that's were things started to go wrong.

Precious response:

I came from another school that had smartboards in all classrooms. My teacher used an animation to explain trigonometry which was about a boat and an observer standing on a cliff. It was amazing because it was so simple to understand. It was a captivating experience, and the entire class was so immersed in that lesson. Once the animation ended, the masked right-angled triangle appeared in a dynamic PowerPoint presentation where he spoke about the sides of the triangle compared to each other to form ratios. My teacher at that school was quite an expert with technology and always used it as a tool to make lessons interesting. I miss that!

Sanaya's response

*The introduction to trigonometry was very vague. The teacher drew a triangle on the chalkboard and explained sine, cosine, and the tangent ratio. He told us to use the mnemonic "**T**ommy **o**n **a** **S**hip **o**f **h**is **C**aught **a** **h**erring" to remember the three trigonometric ratios. I thought it was a useful trick for memory retention. My teacher also constantly spoke about positive and negative angles and theorem of Pythagoras. It was a lot to digest.*

The results revealed that, with the exception of Precious's class, a visualisation technique was not given prominence. For Precious, using the smartboard was a memorable experience. Her response demonstrated that she had a clear and vivid memory of the class and a strong grasp of the key ideas. In order to supplement the instruction and aid in the understanding of abstract topics, several learners took the initiative to watch YouTube videos that provided visual representations. Many learners felt intimidated by this topic from the onset and the waning confidence was inevitable. While some educators employed creative mnemonic devices to help learners remember the ratios, the interest and enthusiasm were not maintained long enough to have a meaningful impact. Twomey and Kroneisen (2021) posit that the waning confidence levels of learners studying trigonometry must be addressed by focussing on innovative ways to invigorate the learning space using visual strategies. The introduction must stimulate intellectual curiosity in addition to leveraging engaging instructional pedagogical methods. It was noted that apart from Precious, no attempt was made to communicate key concepts using interactive visual tools to cultivate a fertile ground for the more complex concepts. Unfortunately, the learning environment in most cases was not characterised by fascination, curiosity, creativity, and inquiry-based experiences.

Did your educator use visual aids to supplement your understanding of trigonometry? Explain.

This question was asked as an extension to the previous one to establish if a visualisation instructional design was given priority. In recent years, visualisation in mathematics education has gained increasing recognition in helping learners understand important concepts. However, in many instances, visualisation strategies in the mathematics classroom are relegated for reasons such as educators opting for traditional chalk-and-talk methods, time constraints, lack of educator training, access to resources and resistance to change etcetera.

Siya's response:

My teacher used the whiteboard to draw a few diagrams initially but never sustained that approach. I remember he drew the Cartesian Plane and a right-angled triangle. Most subsequent lessons were driven by algebraic methods and no emphasis was made for us to picturise the problems. The only other time was when he sketched a few trigonometric functions.

Nora's response:

I do remember once my teacher brought a 3D model of a pyramid to explain the concepts of different planes, angles of elevation, and angles of depression. It was quite enjoyable because we got an opportunity to feel and interact with the model. Other than that, my teacher occasionally sketched diagrams to explain a few concepts.

Sanaya's response:

Not really. In all honesty, I find trigonometry boring because all my teacher spoke of was ratios and angles. I could not relate any of the information to the real world. Although I understood the basics, I could not apply them correctly in different situations. Perhaps if the lessons were more in tune with reality, I could do better.

Precious's response:

The use of the smartboard at my previous school was quite amazing. I felt drawn to the technology because it is now part of our generation. My current educator is old school and more verbal than visual. He talks from siren to siren in the hope he gets through to us. Although I appreciate all that he does for us, perhaps he should allow the information to simmer with us by using group work where we could discuss concepts amongst ourselves.

Shriav's response:

We had a change of teacher in the middle of the year. The teaching styles were significantly different. My first educator depended on formulae because he said that solid knowledge of algebra was the key to success in trigonometry. My second educator was more subtle with algebra but tried to make us see what was below the surface by using visual imagery. I remember him saying that the mind is a repository of pictures. I felt that it made a world of difference because in every lesson my teacher breaks the abstract concepts into pieces trying to connect them with our everyday experiences and then emphasises algebraic techniques to further reinforce them.

The geometrisation of trigonometry enhances learners' engagement and understanding (Hackett & Proctor, 2018). Formal arguments when supplemented by dynamic visual approaches simplify learning and serve as a means of encouragement to participate in lessons (Arhin & Hokor, 2021). These myriads of benefits of a visualisation pedagogical approach are profound and far-reaching, yet it seems to be ignored or relegated to the "fringes of a footnote" in the classroom. The analysis of the responses indicates this to be true. Neglecting the use of visualisation tools or worse still the total absence of them, has created a serious impediment to the understanding and retention of conceptual knowledge. Apart from Precious who had a brief encounter with novel visual aids, it was apparent that the rest of the learners had a minimalistic experience with this powerful tool. I observed in my interaction with the participants regarding this question, that their body language indicated a high degree of indifference to the instructional approach and their tone of voice expressed much the same.

Where the abstract concepts such as $sin^2\theta + cos^2\theta = 1$ linked to visualisation techniques? Elaborate.

The proof of the above identity is dependent on learners' understanding of the Cartesian Plane in relation to right-angled triangles. The concept of the unit circle is reinforced with the Pythagorean identity thereby cultivating a fertile mind for trigonometric understanding. This basic identity is inextricably linked to visualisation and geometric constructions which contributed to the validation of this question.

Nora's response:

No. The teacher wrote the identity on the board and told us it is true for all values of angle θ. He said that we can test it by replacing θ with any value of our choice by using a calculator. I never knew that this proof required the use of a diagram.

Sanaya's response:

I can't remember because that lesson was a blur to me. However, I do recall that this identity was used to prove other identities, but the teacher said we must prove left-hand side equals the right-hand side. From my classroom experience, we were never taught to prove identities using visualisation. I wonder how that is possible.

Gift's response:

Yes, to some extent. The teacher drew a diagram and explained the identity. I think that helped me to get a clue about what was going on. I don't recall him drawing diagrams though for any other identities.

Shriav's response:

I guess to a certain level. The teacher showed us a right-angled triangle where we had to determine the ratios in terms of x, y, and r. It made me wonder at first if the answers will always be true. He then said that the letters in algebra can represent any number.

The responses revealed varying levels of visualisation usage in teaching identities. It was noted that the school had a smartboard and projectors, which was accessible to permanently employed mathematics educators, yet none used it for lessons except for PowerPoint presentations used in staff meetings. This is partly due to educational paradigms, pedagogical strategies, and lack of training or possibly what many mathematics educators describe as "going through the motion". The latter indicates that some educators deliver the curriculum as a mundane activity without searching for novel ways to impress upon the minds of their learners. An interesting phenomenon in the responses was the fact that educators never used visual aids in more complex identities since algebra was the mainstay of their approach. According to Mpofu et al. (2023) educators lack training and familiarity of integrating visual tools into their pedagogical instructional design. Algebraic approaches were favoured to the neglect of visual aids thus limiting learning opportunities to develop understanding through visualisation (Urrutia et al., 2019). This led me to conclude that their confidence and competence in employing visual methods are largely responsible for that. A pattern of routine delivery of the syllabus by educators emerged from the responses. This nonchalant attitude in discharging the curriculum as attested by the responses seemed to contribute negatively to the learning process. The responses to this question further validated the dire need for a visualisation intervention in the trigonometry class.

Were you taught the art of reasoning with yourself to link concepts and connect them with mental imagery? Explain.

The art of reasoning and connecting concepts, fundamental in mathematical inquiry, indicates a clear interplay between cognition and mental imagery. The internal dialogue where the individual explores ideas and arguments, in the realm of mental manipulation, must be a guided experience. Although this skill cannot be taught in the traditional sense, providing learners with diverse methods and perspectives can stimulate curiosity. Harnessing visual tools can enhance the learner's capacity for deep mathematical thought, thereby stirring cognitive reason with imagery be it mental or physical. Carden and Cline (2015) further argue that visualisation promotes effective communication in the problem-solving process, which further serves as a catalyst for reasoning within oneself. In the context of trigonometry education, this research question is imperative to find ways to develop methods to enhance the teaching and learning process.

Gift's response:

Not exactly yes. After struggling with trying to understand trigonometry with formulae and text as I was taught, I decided to take the initiative of linking problems visually. This process of using imagery did help me understand some work but I need support and guidance on this approach.

Sanaya's response:

No. I was just asked to memorise and apply. Now that this question was asked, I am curious to know how reasoning and forming mental images can help me in achieving better marks.

Meena's response:

I was not taught this, but I picked it up myself while trying to understand a topic. I tried to visualise the CAST diagram and relating to ratios in each quadrant. I am not good at linking concepts but when I saw the diagram in my mind's eye, I could see a connection with ratios being positive and negative in each quadrant.

Siya's response:

Unfortunately, not. I am excited to learn this new method though.

Sanav's response:

No, I find it extremely difficult to link concepts with mental images in trigonometry, more especially with trigonometric identities and simplifying. I like trigonometric functions because of the drawings. I get to see the final product and when I sketch, it becomes obvious if I make a mistake then I correct my answer.

The responses were unanimous in terms of not being taught the linking of reasoning with mental imagery. Siya's response was particularly interesting because he showed great interest in wanting to learn techniques to foster better understanding. The process of cognitive development involves much introspection and deductive and inductive reasoning. Mudaly

(2010) captures its essence by contending that a priori knowledge assists in transforming the imagery into meaningful constructs. The shunting of internalisation and externalisation proceeds inexorably when acted upon a stimulus, thereby resulting in the development of new constructs. Sanav's response also highlighted the importance of an iterative process in relation to visual aids. His mistakes became evident because of the visuals and then he corrected it. Meena's response indicated that she internalised the CAST diagram and when she had to solve problems, she tried to find a link. In contrast, Sanaya was made to rote learn and apply. It must be noted that rote learning supported by visual representations has shown promise (Popova et al., 2022).

Do you think a picture or mental image can assist in your understanding of trigonometry? Explain.

The subject of interest should not just be the discipline the educator teaches. It must include first and foremost the recipient of the knowledge, namely the learner. It therefore becomes necessary to be informed how each of them thinks. This interview question was meant to investigate learner perceptions in anticipation of a visualisation intervention.

Nora's response:

Well, I want to explore new ways of learning and only then I can decide what's best for me. I am not that inclined to use diagrams and symbols, but I know it has so much to offer. The mathematical language is undeniably stricken with signs, symbols, diagrams, and formulae. This reflects the genius of its language and I want to know and experience all its nuances and subtleties.

Precious's response:

I would regard myself as more of a visual person so I would say yes. The pictures would add a new dimension to problems in trigonometry. I suppose it will bring more meaning and it will make things a lot easier to understand.

Gifts's response:

Honestly, I am a bit sceptical because of my experience in the classroom. I could not keep up with the diagrams the teacher used. Perhaps I was confused with the explanations and the terminologies. However, I am open to new ventures.

Siya's response:

Eish, from what I know, a picture is worth a thousand words. So, I think it will benefit me because I don't have to think about things I don't have to. I can perhaps channel my energy and effort to things that require it. It can unpack the deeper meaning of the problem. You know that I am so proud to say that the richness of my culture lies in the fact that drawings and design, the use of rainbow colours and patterns play an important part. I see a lot of mathematics in my environment so that is why I am inclined to picturising.

Sanaya's response:

I am hesitant to think that pictures or forming mental imagery would help. I don't want to sound like a pessimist but what if my diagram is incorrect, it would mean that I did not interpret the question properly or I am afraid that I will try to draw pictures for everything and that could mean me not finishing the paper.

Firstly, the range of responses supports the research's use of a qualitative methodology. The responses strongly indicate a plethora of learning modalities and to this end, educators need to be flexible in their instructional design. Educators frequently impart knowledge to groups rather than to individuals. In other words, the pedagogical approach cannot be a "one size fits all". The corpus of extant literature overwhelmingly supports the use of visualisation in trigonometry in the sense that it mitigates abstraction. Nora's response alluded to this by mentioning semiotics in which signs and symbols convey profound meaning. Educators must impress on the power of semiosis since this type of visual can be used to construct and convey mathematical ideas. According to Mudaly (2014), the semiotic analysis must be emphasised in order to assist learners in studying patterns and structures in search of meaning. It was intriguing to hear Sanaya's response. After all, she expressed a strong hesitation to utilise visualisation because she was worried that her images might be inaccurate. Her argument was relevant because a wrong diagram would indicate a wrong way to interpret the question. Her response also hinted at the possibility of overreliance which can have a negative impact. Gift also alluded to the possibility that the diagram may be too complex to interpret. Their lack of familiarity with visualisation techniques is said to have contributed to their degree of pessimism and their reluctance to employ visual ways to solve problems. Siya's and Precious's responses indicate a diametrically opposite view since they are keener to learn more about visualisation. Siya's choice to learn more about using visual aids to solve trigonometry issues appears to have been influenced by both his cultural background and milieu. Precious on the other hand, has had positive influences with her engagement with visualisation from her previous school in addition to being visual naturally.

5.6 The visualisation intervention

To achieve a positive impact, it was important to design a tailored visualisation pedagogical approach that was not only exciting and engaging but would accommodate learners with varying degrees of visual literacy. In order to achieve this, I employed a range of visual aids such as the use of geometric diagrams, GeoGebra interactive software, manipulatives, and Phet simulations. Since the only smartboard at this school was accessible to only the permanently employed mathematics educators, I compensated for this setback by using the projector. The instructional design was in alignment with the CAPS curriculum and was coordinated to meet the needs of the Annual Teaching Plan (ATP). The intervention was spread out over 12 equivalent school periods on different consecutive days, each 50 minutes long and included:

- The pre-task (one period) determined participants current knowledge at the time.
- Core visualisation intervention which occurred over six periods. This focussed on all the main concepts according to the ATP, namely the unit circle with reduction formulae (two periods), trigonometric equations and identities (two periods), functions (one period) and real-world application problems (one period).
- Practice and application lessons which occurred over two periods during which the participants were required to apply concepts they learnt with a continued emphasis on visual techniques.

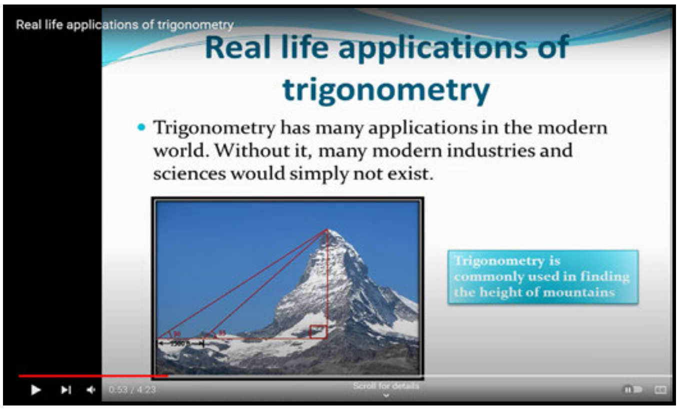
- Review and reinforcement constituted two periods during which participants consolidated the learning that took place. I addressed misconceptions and summarised possible misinterpretations using visual aids and ways to obviate them.
- The post-task (one period) which was used to measure the effectiveness of the visualisation intervention.

5.6.1 The Introductory lesson

The introduction to any topic is the key so I endeavoured to arouse interest and intrigue that was sustainable by making learners aware of trigonometry in the real world. I played a YouTube video that captured the "thunder of trigonometry" by using real-life examples to demonstrate its practical application as shown in Figure 5.29. Nilimaa (2023) asserts that the integration of real-world problems into instructional design is imperative in developing creative and analytical skills.

Figure 5.29

Real-life problems in trigonometry

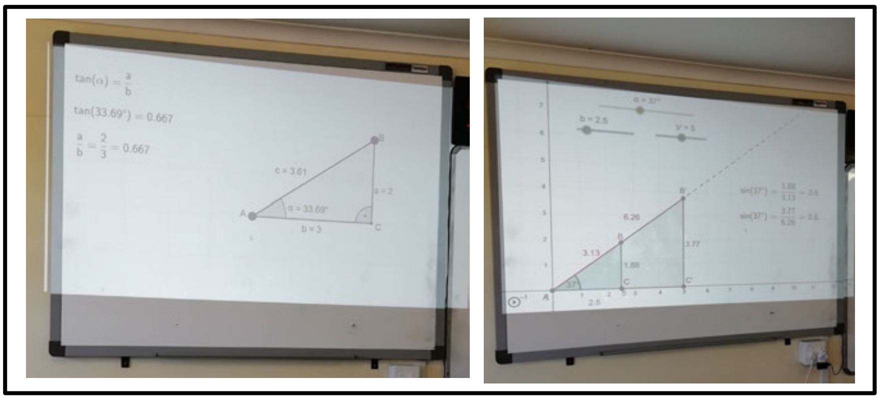


Note. Enjoy Math (2021, March 14). *Real applications of trigonometry*. https://youtu.be/bGPIcQftgYI

The intervention included the gradual conceptual development that geometrised trigonometric ratios and similar triangles using GeoGebra as shown in figure 5.30. The intent was to impart the use of right-angled triangles to determine the values of trigonometric ratios. According to Munir et al. (2022), a successful learning outcome is predicated on the mastering of foundational concepts, and it was with this in mind that great emphasis and care was dedicated to the entrenchment of fundamental trigonometric thought.

Figure 5.30

The use of GeoGebra to explain ratios



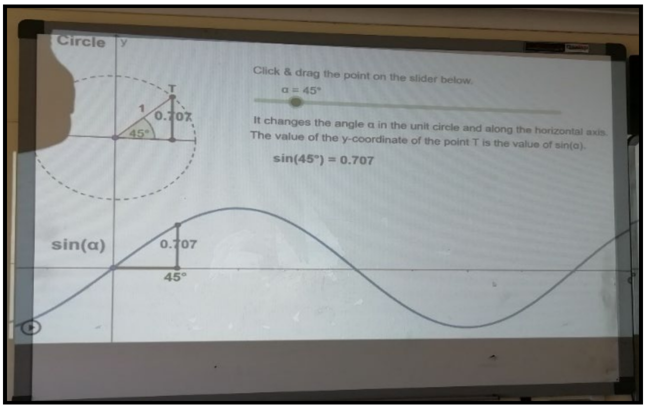
The mnemonic SOHCAHTOA was invaluable in the explanation of ratios. Learners were asked to independently work out the ratios before the solution being unmasked. It was observed that levels of engagement, motivation, and confidence were significantly improved. The visual experience proved highly engaging and immersive due to the multimodal representations which included the use of a coloured interface, symbols, numeric values, and dynamic graphics. This introduction to ratios was a prelude to the unit circle on the Cartesian Plane on which special angles were linked to trigonometric functions.

5.6.2 The unit circle

The unit circle (radius of one unit) is of cardinal importance in the geometric construct of trigonometric functions, as such, allows for the intuitive visualisation of concepts (Downing & Black, 2023). The unit circle centred at the origin of the Cartesian Plane shows a direct relationship between the sine function and the y-coordinates and the cosine function and the x-coordinates. This visual representation abates abstraction by giving greater insight into the behaviour of trigonometric principles. Special angles can be derived from the unit circle, mitigating rote learning, and easing the complexities of calculations in solving equations as shown in Figure 5.31.

Figure 5.31

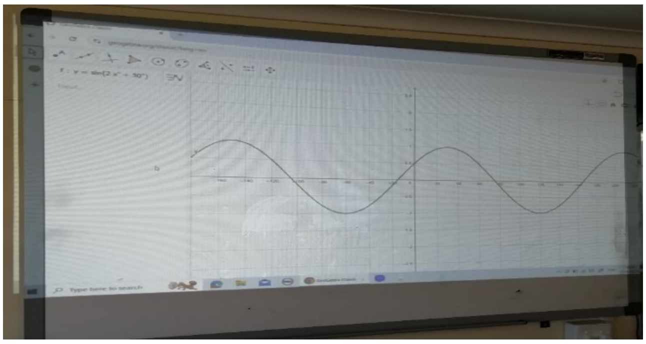
Unit circle linked to trigonometric functions



The slide allowed for angle variation and point tracing on the graph. Learners were able to observe the periodicity of trigonometric functions together with other parametric variations that affected vertical shifts, horizontal shifts, and reflections as shown in Figure 5.32.

Figure 5.32

Use of GeoGebra in demonstrating parametric variations



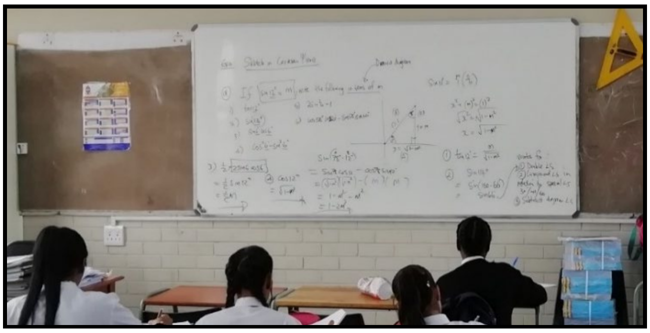
The participants were prompted to observe the signs of the ratios in the four quadrants which visually established the use of the CAST diagram and reduction formulae. This culminated in the use of diagrams which was a visual tool linking all concepts.

5.6.3 Use of Diagrams

The Cartesian Plane provides a reference framework for angles, terminal arms, and right-angled triangles. The trigonometric ratios are defined in accordance with this coordinate system, making it a significant visual aid for understanding key trigonometric concepts such as equations, functions, and geometric shapes. The CAPS curriculum (CAPS document, p. 47) emphasises its importance with exam questions centred around the use of diagrams as shown in Figure 5.33.

Figure 5.33

Use of a diagram drawn in the Cartesian Plane

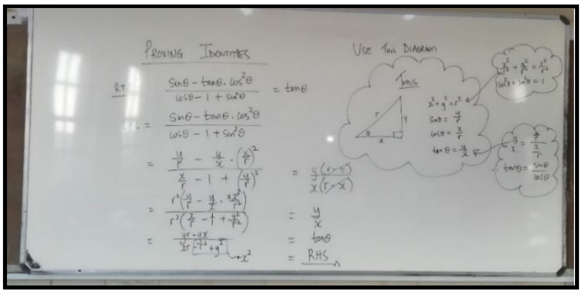


5.6.4 Proving complex identities using a diagram

The use of diagrams to prove trigonometric identities is helpful in providing profound insight and clarity in the proving process. It is a visual tool that assists in demystifying many complex identities by allowing learners to navigate through algebraic manipulations by fostering greater understanding (Lestyanto et al., 2022). Figure 5.34 demonstrates a novel method of using a diagram concomitant with algebra to prove a complex identity. Learners can use the right-angled triangle in relation to the Theorem of Pythagoras and trigonometric ratios to rewrite the identity in algebra and manipulate.

Figure 5.34

Using a diagram to prove identities

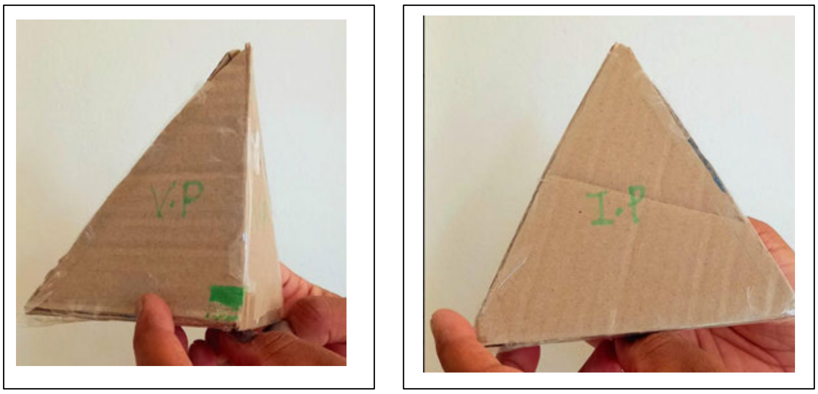


5.6.5 Use of manipulatives

The use of manipulatives is highly beneficial for fostering greater understanding by providing a hands-on experience to visualise complex concepts (Ulyani & Qohar, 2021). Kinesthetic learners, in particular, find tactile engagement useful to internalise fundamental principles. The geometric model as shown in Figure 5.35 was made from cardboard. It assisted learners in understanding the concept of planes, angles of elevation, and depression and also played a pivotal role in developing spatial reasoning. The use of manipulatives concretises the learning process by enhancing conceptual understanding and geometric intuition.

Figure 5.35

Sanav's interaction with the cardboard model



Researcher: Does the model help you understand the concept of a plane?

Sanav: Definitely sir. I can touch and feel the difference of the different flat surfaces.

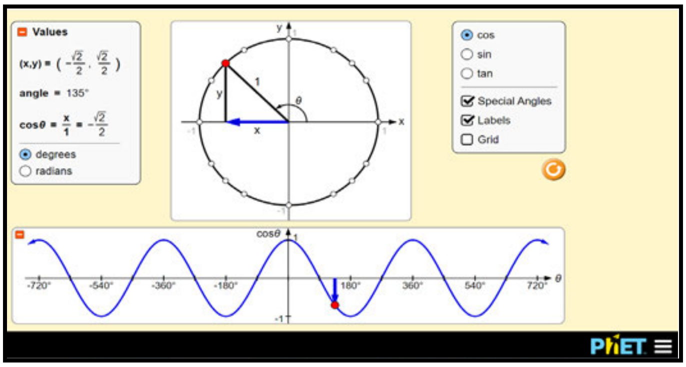
Siya: I never thought that a physical experience of just playing with this model could allow me to discover so many mathematical truths. Just amazing.

5.6.6 Learners' interaction with Phet simulations and GeoGebra

Phet (trig tour) is a free simulation where learners can investigate the unit circle in relation to trigonometric functions. In Figure 5.36, the unit circle was programmed to show the special angle values and its associated function values on the graph leading to improved understanding (Syafriyanti, 2023). Participants were given the URL so that they could explore the simulations through dynamic and interactive visualisation. The learners were allowed to work at their own pace to experiment with various trigonometric concepts. My iPad, tablet, laptop, and data resource were made available to participants who did not have access to a device. I observed great enthusiasm, increased motivation, and heightened engagement when participants were using the simulation.

Figure 5.36

Noras interaction with Phet



Note. Phet interactive simulations (2024). https://phet.colorado.edu/en/simulation/trig-tour

Researcher: Does the simulation help you understand the connection between the unit circle and trigonometric functions?

Nora: Yes sir. I can visually make the connection between special angles and the graph.

Siya: Absolutely. It allowed me to watch how the ratios changed from positive to negative in the different quadrants.

Precious: Yes. I was able to work at my own pace until I understood all the work without pressure. It was really exciting.

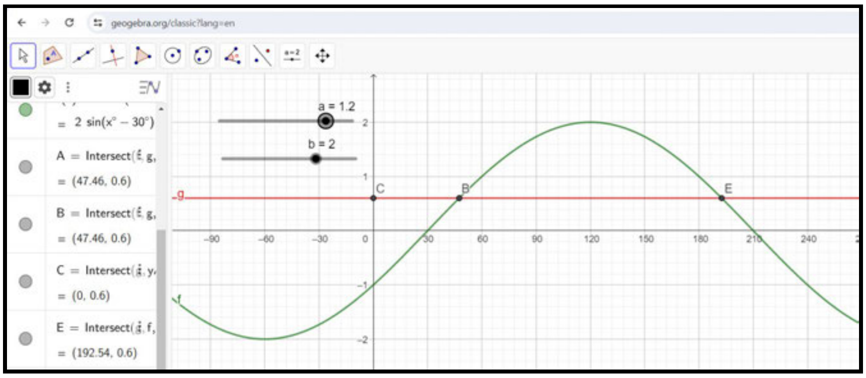
According to Mardiana and Prasetya (2022), simulations are beneficial to all learning modalities. It allows users to self-explore using dynamic and static representations at their own pace to accommodate their individual needs. This is supported by Anisa and Astriani (2022) who also contend that such simulations improve critical thinking and enhance conceptual understanding.

GeoGebra was another free interactive tool that I used extensively. Its wide variety of features and functionality served as an indispensable mechanism for a visualisation intervention.

Participants enjoyed the privilege of a dynamic interactive explorative experience with trigonometric concepts thereby enhancing understanding (Bedada & Machaba, 2022). Parametric changes to functions were observed in real-time and learners were able to establish the consequences of variations regarding amplitude, period, translations, and reflections. A notable observation was that participants were immersed in the graphical analysis and the instant feedback they received. Figure 5.37 shows Gift's interaction with GeoGebra where he is establishing a connection with trigonometric equations and equations by manipulating slides *a* and *b*.

Figure 5.37

Gifts interaction with GeoGebra



Note. GeoGebra (2024). https://www.geogebra.org/

Researcher: What is your impression of the interactive tool regarding your understanding of fundamental concepts?

Sanav: I am amazed by how easy it makes trigonometry. I can finally understand the effects of changing values on graphs and equations.

Gift: Awesome. I like the instant feedback. I was able to see that the point of intersection is the solution to the equation and also the recurring nature of the solutions refers to the periodicity.

5.7 Analysis of post-task

Similar to the pre-task, all aspects of trigonometry aligned to the CAPS document and TIMSS (1999) were given as a task to the participants to gauge the impact of the visualisation intervention in a grade 11 trigonometry class. The review and analysis process shed invaluable insight into the behaviour and experiences of the participants. According to Finn et al., (2019), post-tasks are crucial in assessing and evaluating the validity, reliability, and robustness of the collected data. As an integral part of the post-task process, participants were encouraged to reflect on possible misconceptions of concepts, inaccuracies in interpretation, and ambiguities in language instruction. The outcomes of the post-task are now examined.

5.7.1 Diagram in Cartesian Plane

Question 1 was primarily based on the use of diagrams in the Cartesian Plane where learners had to use the theorem of Pythagoras, the CAST diagram, and apply relevant reduction formulae. This question visually links fundamental trigonometric concepts.

Figure 5.38

Precious's response to question 1.1

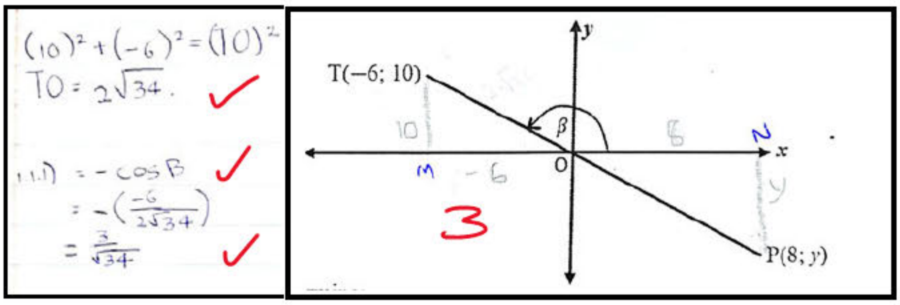
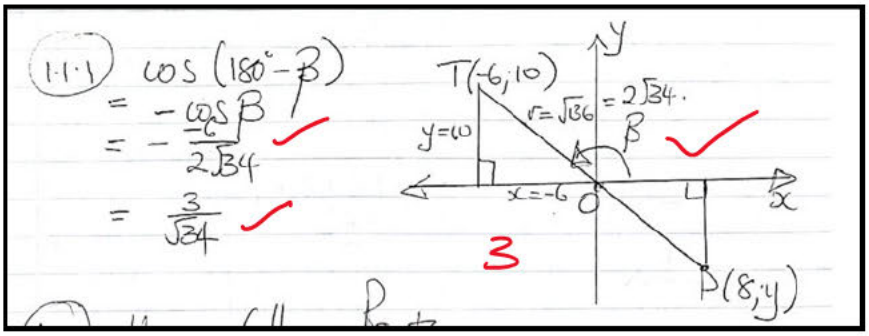


Figure 5.39

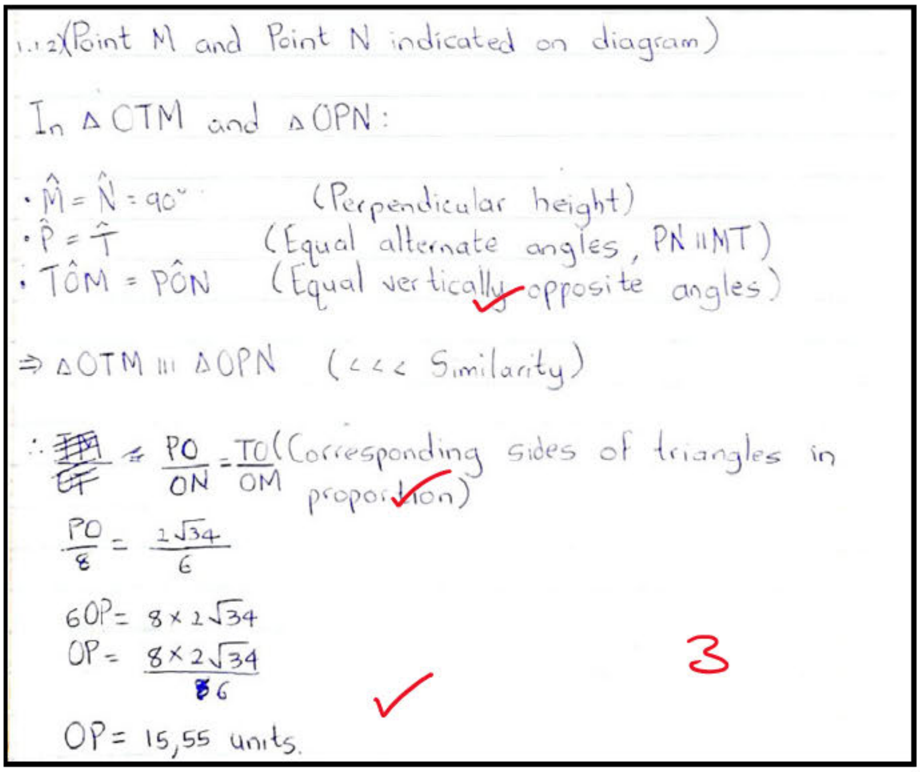
Sanav's response to question 1.1



I observed that six out of the eight learners used a sketch, having an average mark of eighty percent. The construction of the right-angled triangle was a visual tool that assisted in the solving process. Participants also used the composite reduction diagram to reduce ratios to positive acute angles. Sanav's response in particular, as shown in Figure 5.39, now employed visual techniques to cognitive offload to great success and Precious was effective in timesaving by using the diagram on the question paper as shown in Figure 5.38.

Figure 5.40

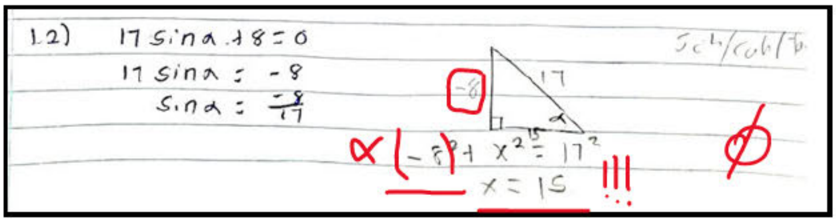
Nora's response to question 1.1.2



Nora's answer to Question 1.1.2 as shown in Figure 5.40, was creative with her visual skills because she geometrised trigonometry by introducing similar triangles. It was worthwhile to note how she connected concepts and the structure of her solution. However, some learners did not draw an accurate sketch in question 1.2 as depicted in Figure 5.41 which propagated confusion and multiple errors (Bastian et al., 2022).

Figure 5.41

Meena's response to question 1.2



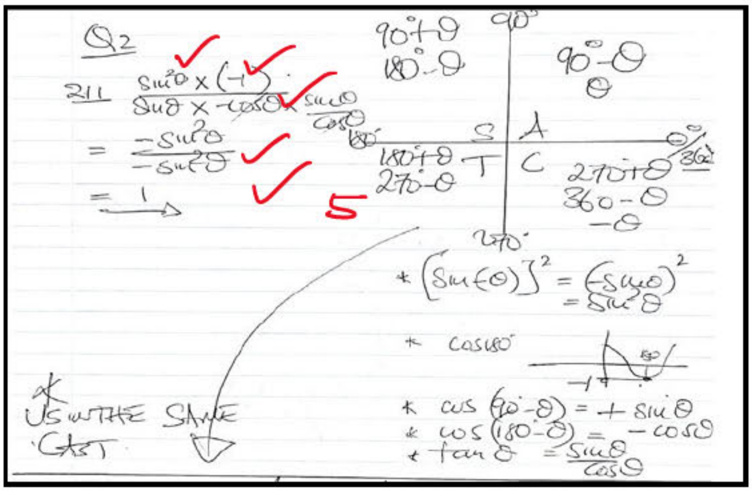
The right-angled triangle as shown in Figure 5.41 incorrectly represents the sides together with a misapplication of the theorem of Pythagoras. Such misconceptions impeded the development of the correct solution by distorting the sides and lengths, invariably resulting in an incorrect calculation. These experiences contribute negatively to learners employing visual representations in problem-solving (Mhlolo, 2022)

5.7.2 Application of reduction formulae

Question 2.1 was centred around the use of reduction formulae in terms of firstly rewriting ratios in terms of positive acute angles, in which subsequent steps made use of special angles and co-ratios to further simplify to abide by the instructions of "no calculators allowed". This instruction emphasised memory retention and the possible use of visual representations.

Figure 5.42

Siya's response to question 2.1.1



Siya used the composite CAST diagram as shown in Figure 5.42, to reduce all ratios to positive acute angles correctly. Furthermore, he reduced one ratio at a time at the side of his page so as not to clutter his work. This was a visual skill that he used so that he would not get confused by the multiple conceptual applications. He also added that he chose an arbitrary value of θ to double-check his solution, making his attempt consistent with the theoretical framework of Distributive Cognition. Siya quite cleverly also recognised time constraints and used the same composite CAST diagram for other questions that followed. It was interesting to note that 7 out of 8 participants used the CAST diagram to varying extent, all yielding positive outcomes.

5.7.3 Proving identities

Question 2.2 was based on proving identities which traditionally was challenging for many learners. Sanav's response as shown in Figure 5.43 showed a radical shift to using visual representations in relation to rudimentary algebraic skills with great success.

Figure 5.43

Sanav's response to question 2.2

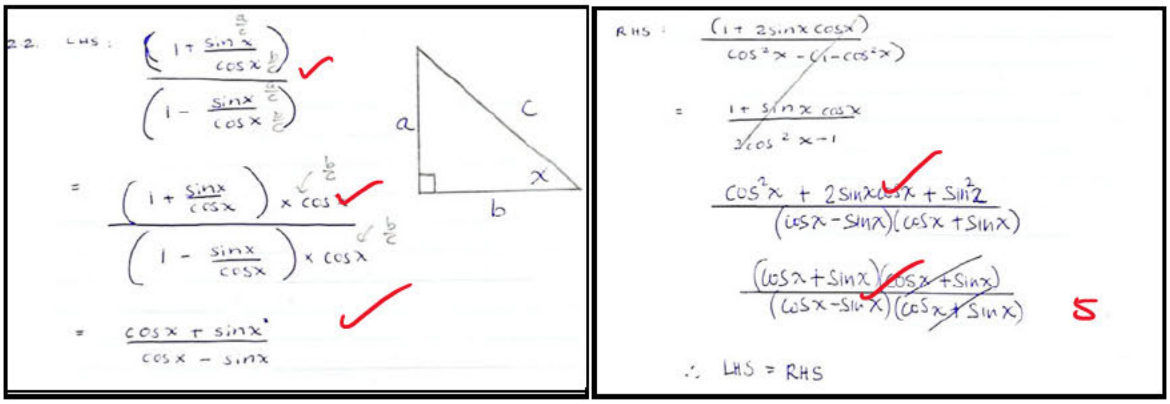
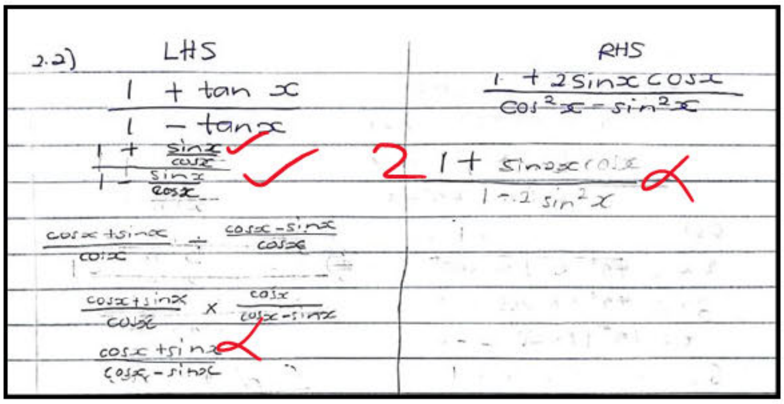


Figure 5.44

Sanaya's response to question 2.2



However, in contrast, Sanaya's response as shown in Figure 5.44, did not utilise any visual methods. She attributed her non-visual methods to her fear and anxiety for not drawing the diagrams correctly.

Researcher: What are your thoughts concerning using diagrams?

Sanaya: Honestly speaking, I struggle to make use of diagrams. Sometimes I get confused and *it makes me nervous, therefore I tend to use my teacher's method.*

Sanaya's hesitancy to adopt visual aids is the manifestation of her fear and apprehension due to her negative prior experiences, a sustained lack of exposure to effective visual strategies, and her preferred non-visual learning modality. According to Qasem (2020), learners' unwillingness to embrace a visual approach must be factored in relation to their confidence and perceived proficiency. A noteworthy statistic was that four out of eight participants gravitated towards a visual strategy in proving identities, scoring eighty percent and above which traditionally was a nemesis for many (DBE diagnostic analysis, 2023).

5.7.4 Trigonometric functions integrated with equations

Participants were asked to sketch trigonometric functions with a maximum of two parametric changes as prescribed in the CAPS document. Knowledge of transformation, variations in amplitude and period, and solving equations were imperatives. This question also integrated concepts such as inequalities and the nature of roots as an extension to assess visual skills.

Figure 5.45

Meena's response to question 3.1

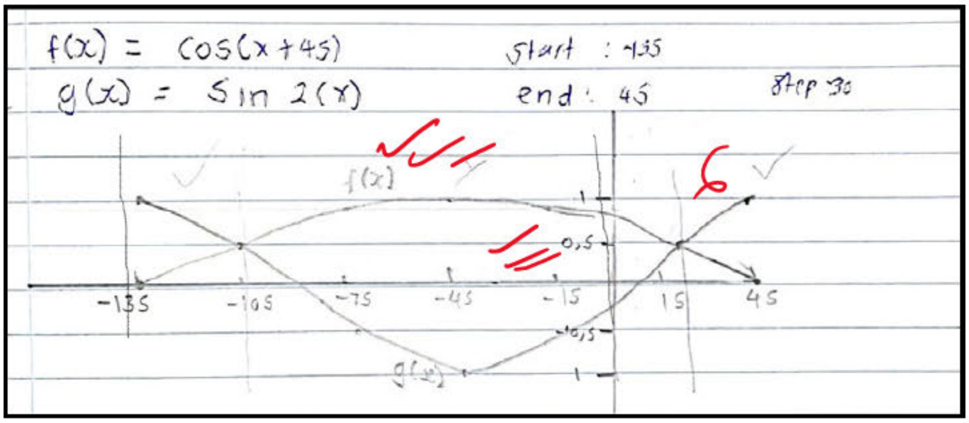
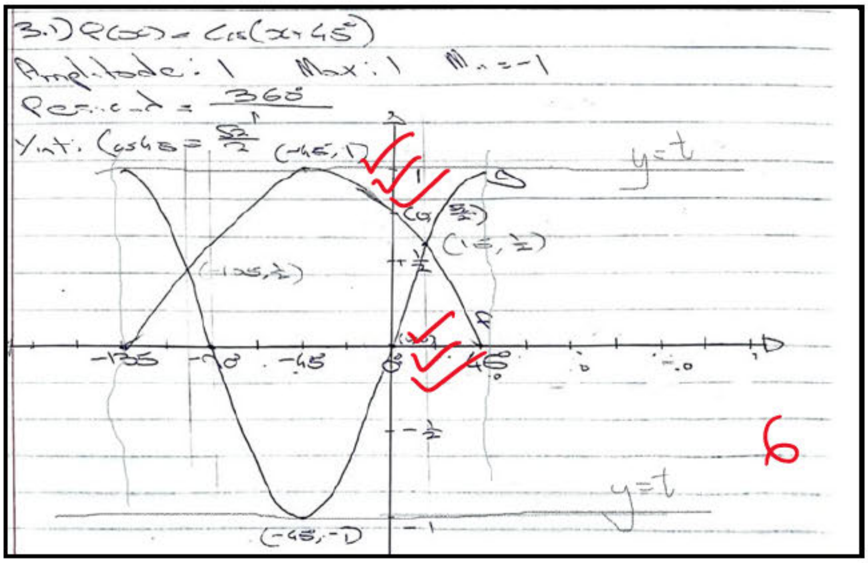


Figure 5.46

Gift's response to question 3.1



All eight participants managed to score above seventy percent on this question albeit with minor hiccups. In Figure 5.45, Meena used an incorrect scale on the x-axis which resulted in her making errors in the x-intercepts of the $f(x)$ graph and also fell short of not labelling relevant points of the functions as required by the question. I also noticed that she indicated arrows at the endpoints, not realising that these functions should only be drawn for a strictly closed domain. Gift on the other hand used a correct step in the table method of generating points which yielded precise intercepts and turning points as shown in Figure 5.46.

Researcher: What changes did you notice in these two graphs compared to the "mother graphs" of $y = sinx$ *and* $y = cosx$*?*

Gift: The graph of f was a 45^0*-phase shift to the left compared to its "mother graph". The graph of g doubled the x values, meaning that 2 basic graphs could be drawn in* 360^0 *therefore, the period is* 180^0*.*

Meena: I remembered working with the GeoGebra applet where I discovered that when adding to the angle, the graph moved left and towards the right when subtracted from the angle. I also observed from the applet that the coefficient of the angle was responsible for horizontal compression and dilation, in this case the g graph was compressed to half its normal length of 360^0*.*

The responses shed invaluable light on the effect of the visualisation intervention. Meena's response was notable in relating the positive impact GeoGebra had on her understanding of the parametric changes in functions. Gift also noticed the changes that occurred relative to the "mother graphs", establishing a positive impact of the dynamic visual representations on him. The inequality and the nature of roots question were also answered reasonably with both participants making use of their visual skills by using a partitioning method. However, only 3 of the 8 participants scored maximum marks in those areas, indicating that connections between topics must be emphasised in the pedagogical instructional design.

Question 3.2 referred to points of intersection which required solving simultaneous equations. Nora's solution was particularly interesting since she used the visual property of reflective symmetry of the cosine function in conjunction with the application of co-ratios as shown in Figure 5.47.

Figure 5.47

Nora's response to question 3.2

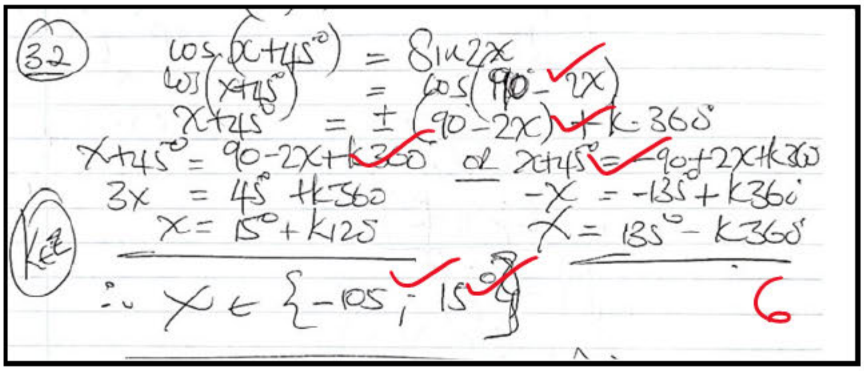
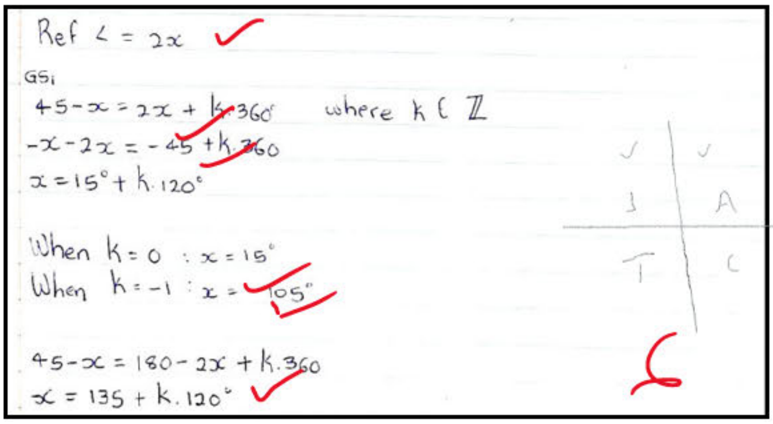


Figure 5.48

Precious's response to question 3.2



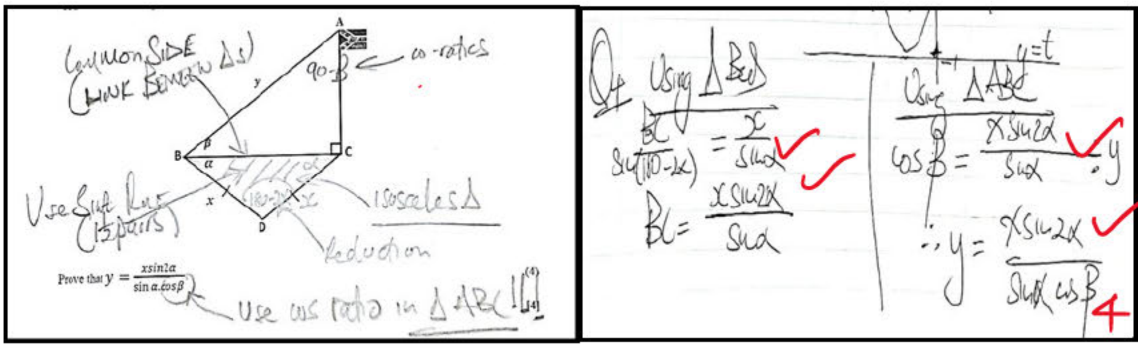
Precious opted for the traditional CAST diagram method in which she identified the quadrants that contained the solutions as shown in Figure 5.48. She also used her sketch to get expected answers, observing from her graphs that there were two points of intersection with approximate values of -100^0 and 20^0. It was quite novel for her to use an estimation technique to guide her based on her sketch which she felt confident about. Her argument was constructed on the fact that since her sketch was accurate with minimal distortion, her estimated interval would contain the actual answer. The remaining 6 participants followed the same method as that of Precious, some with conceptual flaws in the application of co-ratios and errors in algebraic manipulation. The positive taken from this question was that all learners correctly identified the quadrants which contained the solutions using the CAST diagram. This logic led me to conclude that the intervention has allowed the internalisation of many powerful concepts, making for a burgeoning mathematical discourse.

5.7.5 Solution of triangles

Question 4 was premised on solving triangles using the sine, cosine, and area rules. Participants had to also distinguish between the two planes and apply the concept of angle of elevation. Siya's response in Figure 5.49 captured the essence of the ideal approach to solving triangles. He filled in all the remaining angles in each triangle and noted the key points on the diagram which reduced mental strain. He also sought hints from the question itself by vetting salient information and then applying it with great success.

Figure 5.49

Siya's response to question 4



Gift was also creative in his strategy which included shading planes in different directions and indicating the angle of elevation with an upward arrow from the horizontal as represented in Figure 5.50.

Figure 5.50

Gift's response to question 4

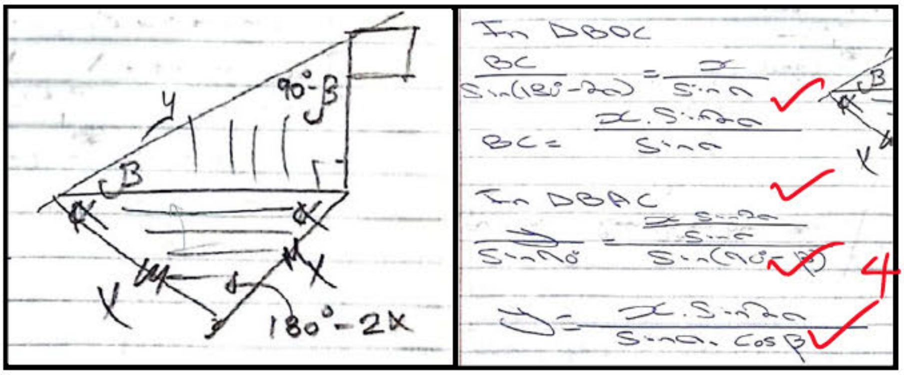
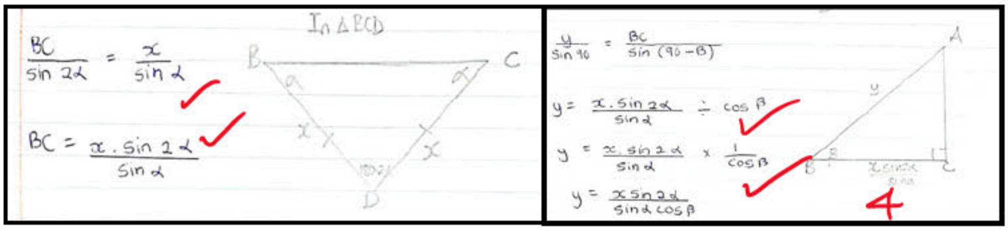


Figure 5.51

Shriav's response to question 4



Shriav drew the triangles separately in order to simplify the question as shown in Figure 5.51. He used his geometry skills to find all the angle values after which he realised which rule to apply.

Researcher: It was a good idea to draw the triangles separately.

Shriav: Yes sir, I found that it helped me visualise better. I could focus better because of the clarity of information, and this helped me by applying the correct rule.

According to Ngu and Phan (2020), learners have an enhanced understanding of complex problems when drawing the triangles separately. They are better equipped to identify the relationships at play when analysing the simpler components. It was further noted that Sanaya who preferred non-visual methods left this question blank, along with two other participants.

An improvement was noted in solutions of triangles throughout this research, indicating visual representations were beneficial.

5.8 Comparison of the Pre-Task and Post-Task

I used a five-point rating scale with a descriptor as shown in Table 5.4 to gauge the impact of the visualisation intervention programme in relation to participants' experiences, behaviours, and motivation. The participants were asked to rate the impact of this intervention irrespective of their marks in both tasks as shown in Table 5.5.

Table 5.4

Rating scale for the visualisation intervention

Rating in stars ☆	Descriptor regarding effectiveness
5	Exceptional
4	High
3	Moderate
2	Limited
1	Low

Table 5.5

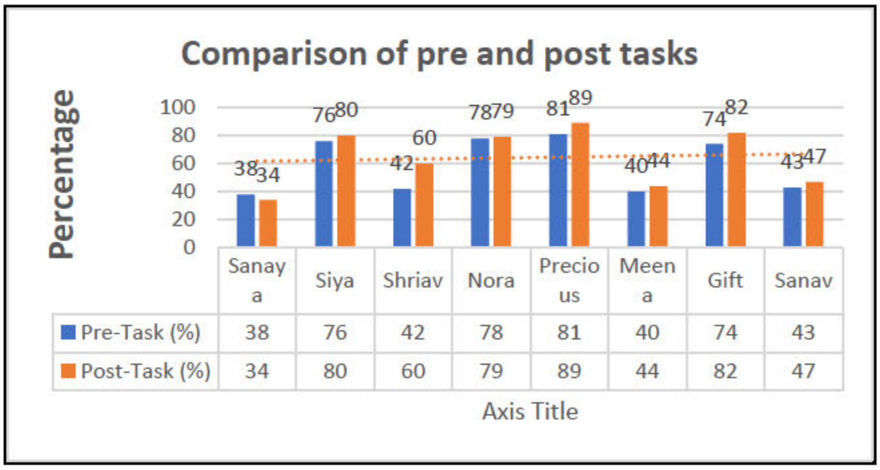
Tabulation of pre- and post-task results

Participant	Visualisation Intervention Rating
Sanaya	1
Siya	4
Shriav	5
Nora	4
Precious	5
Meena	3
Gift	5
Sanav	3

Figure 5.52 shows the comparative scores between the pre-task and post-task. It was worthwhile to note that Shriav showed an eighteen percent increase, whilst Sanaya showed a four percent decrease. All other participants showed an increase, some more than others as shown by the dotted trend line, indicating a positive correlation between the visualisation intervention and post-task. Sanaya attributes her low rating of the visualisation intervention to her being more of an auditory-verbal learner. She did not particularly enjoy her experience with GeoGebra because of her computer literacy levels since she did not own a computer. However, she did say that given more time and practice in proviso of available technology, things could change. All other participants on the other hand were complimentary of this intervention and hoped to continue on this newly routed journey.

Figure 5.52

Side-by-side bar graph comparing pre- and post-tasks



	Sanaya	Siya	Shriav	Nora	Precious	Meena	Gift	Sanav
Pre-Task (%)	38	76	42	78	81	40	74	43
Post-Task (%)	34	80	60	79	89	44	82	47

5.9 Synopsis

The analysis of the data instruments revealed an enhanced performance albeit to varying extents following the visualisation intervention program. This is supported by Gunadi et al. (2023) who assert that visual representations mitigates the abstraction of trigonometry by making concepts more tangible and comprehensible. Additionally, research by Asomah et al. (2023) confirms the findings by placing great emphasis on the exploration of trigonometric concepts using interactive visual strategies. Distributed cognition, a framework that is pivoted on external representations and visual tools in cognitive processes had a major impact on the findings. The incorporation of visual pedagogical strategies in trigonometry supported distributed cognition by extending cognitive processes beyond the individual. The feedback from participants reinforced this view by generally showing positive shifts in motivation levels, attitudes, and behaviour regarding the learning of trigonometry. Many learners expressed hope, confidence, and a sense of empowerment in the learning of trigonometry, adding that they wished to continue on this new trajectory. Furthermore, this program with the use of technology-mediated tools, highlighted the positive effect of visual representations on diverse learning modalities. The challenges and difficulties faced were grounds for tweaking the instructional design. The path going forward holds much promise and given the evolution of technology, greater benefits from a visual learning strategy can be achieved.

CHAPTER 6

RECOMMENDATIONS AND CONCLUSION

6.1 Introduction

The preceding chapter examined the efficacy of a visualisation intervention in a grade 11 trigonometry class through a rigorous analysis process of the collected data. This chapter shifts focus to the distillation of the analysis and findings, leading to wielding recommendations and implications. It further draws overarching conclusions from the analysis of the findings and discusses the limitations that governed this study. But firstly, this chapter presents an overview of the research, contextualising this study in the existing literature and putting into perspective the objectives of the research study in conjunction with the methodology and theoretical framework.

6.2 Overview of the research

Whilst trigonometry is an essential topic in the grade 11 mathematics curriculum, it presents manifold challenges to learners due to its complexity and abstraction. To address these setbacks, this research study aimed at enriching the teaching and learning of trigonometry in a grade 11 class by implementing visualisation strategies. The main objectives of this study were:

4. To understand the different strategies available from prior research for determining ways to improve the understanding of trigonometry using visual stimuli.
5. To administer an intervention programme, implementing visualisation.
6. To measure the impact of using visualisation in trigonometry.

The CAPS document, being the compass of the curriculum, emphasises the need for problem-solving skills, as such, conceptual understanding and procedural fluency are imperatives. It was to this end that the visualisation intervention played a crucial role in concretising fundamental concepts by making abstract concepts more tangible. The corpus of literature was then explored for various available visual strategies to aid in the comprehension of grade 11 trigonometry.

The various types of visual forms together with their usefulness in a diverse classroom were then investigated. Interactive software such as GeoGebra which is a free applet, was a notable visual tool that is currently spearheading the visualisation pedagogical instructional design. It incorporates static and dynamic visual representations and with its striking interface and all-encompassing functionality, appeals to different learning modalities. Amongst other useful technology-mediated tools were tangible user interfaces, animations, and augmented and virtual reality, all of which provided immersive experiences thereby enhancing the learning process. However, these technological resources are expensive and not accessible to historically disadvantaged schools. In classrooms with less technological privileges, participants mentioned the use of visual aids in their pedagogical instruction, the likes of which included visual mnemonics, diagrams, mind mapping, use of manipulatives, in addition to harnessing the power of semiotic influences. To a lesser extent, participants noted the use of static visualisation when their educators drew basic diagrams to introduce concepts and as evidenced by the wide body of literature, proved beneficial in expressing trigonometric concepts in a more meaningful way.

The research methodology made use of an interpretivist philosophy in recognition of the intricate human experiences. This choice allowed me to gain deeper insight into the learners' perspectives and inclinations so that the instructional design could be tailored to suit the needs of the individual. This learner-centred approach not only provided an augmented level of motivation and engagement for the participants but also encouraged reflexivity in which participants used iteration to refine their attempt at a solution. The interpretivism philosophy, being the natural choice for this study, was validated by the trustworthiness of the research results. This process was achieved by data triangulation which involved the use of multiple data instruments namely questionnaires, structured and semi-structured interviews, pre- and post-tasks, observations, interactions, and reflexivity, all of which demonstrated an alignment of findings. It was worthwhile noting that the roadmap of this research was chartered on the Distributed cognition framework.

Distributed cognition underscored the interconnectedness of the individual's cognition and the environment. The leveraging of shared cognition which included the use of collaboration, visual aids, and other external artifacts such as the 3D manipulative provided several compelling benefits. These benefits included immersive experiences that promoted participation and raised motivational levels. Lessons were enhanced through this scaffolded approach in which participants were given as much support as they required. The use of technology-mediated tools appealed to multiple learning modalities which assisted in the gradual transition from basic to abstract concepts in trigonometry. The visual tools mitigated cognitive offloading, redirecting the participants' thinking capacity to problem-solving rather than grappling with memory retention of rudimentary concepts.

6.3 Analysis of the Findings

The focal point of this study was a visualisation intervention in a grade 11 trigonometry classroom. By utilising research instruments which included a questionnaire, pre- and post-tasks, as well as structured and semi-structured interviews, three main research questions were answered, namely:

4 What visual strategies are available for the teaching of grade 11 trigonometry?
5 What intervention programmes can be developed in trigonometry to make the content and solutions more visual?
6 What is the impact of such an intervention programme?

6.3.1 Available visual strategies for teaching grade 11 trigonometry

It was established from the questionnaire, pre-task, structured interview, and the vast body of extant literature that the available visual strategies included technology-mediated tools, the use of diagrams and charts, and to a lesser extent, manipulatives. These visual aids are designed to enhance conceptual understanding and learner performance. However, the emerging trend from the initial phase of the study showed that learners gave preference to algebraic manipulation rather than visual representations in formulating solutions. Although some technological resources such as a smartboard and projectors were available, learners cited that there was minimal use of them in the instructional design. Only one learner who had prior experience with the smartboard from her previous school, was inspired to integrate visual means in problem-solving. Another participant who showed somewhat confidence in visual methods acknowledged cultural influences for his propensity towards picturising problems.

It was noted from the pre-task that the learner's answers were largely devoid of visual representations. The interviews revealed according to participants that diagrams were occasionally used by educators to introduce concepts and thereafter they showed complete reliance on algebraic methods. Although one learner regarded herself to be a non-visualiser, the lack of exposure to visual methods was the main contributor to visualisation strategies not being used by her. Some participants also perceived that a visual approach to solving trigonometry was more difficult. Their reluctance was predicated on the fear of not drawing diagrams correctly in addition to time constraints. Many participants were also overwhelmed by the visual techniques that were intermittently taught to them which discouraged its adoption. This intermittent use of employing visual aids in pedagogical practices was primarily founded on the idea that visualisation is contextually restricted to specific subtopics in trigonometry. In many instances, I observed that visualisation was not seen as a mitigator of abstract concepts nor was it seen as a means for explorative learning.

6.3.2 Developing visual content

In answering the second research question, I considered the available resources and the diversity of the participants regarding their level of mathematical visuality, using the responses from the questionnaire, pre-tasks, and interviews as a guide. In recognition of the different learning preferences, a visual strategy was developed that leveraged a variety of visual tools to appeal to the individuals' strengths. The GeoGebra applet and Phet, both of which are free, were extensively used to show learners that harnessing dynamic visualisation methods is a means of mitigating complex concepts. Learners observed the manipulation of trigonometric concepts with the use of these dynamic tools and in doing so, they were able to link several concepts. Furthermore, the parametric variations produced real-time feedback, promoting a greater intuitive understanding of the concepts. Learners were also allowed to experiment with these visualisation tools, enabling them to have an explorative learning experience, working at their own pace. In addition to technologically mediated tools, I used diagrams drawn on the whiteboard to introduce concepts and in problem-solving contexts. This afforded learners a deeper insight into the trigonometric complexities and allowed them to leverage these visual representations to enhance their understanding.

6.3.3 The impact of a visualisation pedagogical approach

In answering the third research question, I compared the pre- and post-task scores in addition to considering the participants' experiences. The visualisation interview rating indicated that the learners were positively influenced by the visualisation intervention so much so that lessons were more immersive and meaningful, with many of them reporting an improved sensory engagement. I also observed a heightened sense of confidence and renewed vigour among learners. Moreover, it emerged from the feedback that participants appreciated the aesthetic appeal of the technology-mediated tools which created a rich dynamic learning environment.
The comparative analysis of the pre-and post-task scores on average revealed significant progress. A cognitive shift was noted where learners adopted visualisation strategies which produced positive outcomes. The visualisation intervention supplemented the traditional teaching pedagogy thereby enhancing conceptual mastery and procedural fluency. The tangible visual representations empowered learners to navigate the nuances of trigonometry with more confidence and with less dependence on rote learning.

6.4 Recommendations

Trigonometry is a fundamental pillar in the mathematics discipline and often poses great challenges to many learners due to its abstraction. Its interdisciplinary nature lends itself to solving problems in the real world, inevitably cementing its key status in mathematics, and thus making it a crucial topic for learners to understand. It requires learners to have strong visual-spatial skills as well as proficiency in algebraic manipulation (Lau et al., 2022). It was in this light a visualisation intervention in a grade 11 trigonometry classroom became necessary due to its capacity to provide tangible representations to assist learners in comprehending essential trigonometric concepts. Drawing insight from the data analysis of this study, the following recommendations are outlined:

6.4.1 Sustained visualisation strategy

The findings of this study yielded an undeniable positive outcome for a visualisation intervention in a grade 11 trigonometry classroom. Although many learners were initially not adept or accustomed to using visual tools due to a lack of exposure and encouragement by educators, it was noted that the cogency of the visualisation intervention programme paved the way for its adoption by many. Learners from diverse learning modalities and varying levels of proficiency experienced a revolutionary change in the way they formulated problem-solving strategies and to this end a sustained implementation is required. Mathematical visuality must be prioritised in pedagogical practices to enhance conceptual clarity, pique interest, and curiosity, and facilitate a cohesive understanding of trigonometry (Schafer, 2023). A sustained implementation will deepen learners' comprehension, promote active engagement, and assist learners in migrating from rote learning to attaining a meaningful knowledge construct. The consistent leveraging of visual aids will not only concretise the geometric nature of trigonometry but enable learners to tackle real-world problems with greater confidence. It is imperative that educators consistently reinforce complex concepts, and a sustained visual strategy is paramount in achieving this.

6.4.2 Use of interactive tools

Interactive software provides a dynamic tool to assist learners in grasping abstract concepts by making them more tangible. Learners are positively influenced by real-time feedback which provides a personalised diagnostic insight into learners' misconceptions and difficulties. Educators are then able to discern nuances in learner's misconceptions and correct repeated errors, allowing them to change the learning trajectory by individualising the instructional design. Furthermore, the hands-on explorative nature of using technology-mediated tools is highly beneficial for diverse learning modalities. It transforms the educational landscape into a rich vibrant active learning environment, one in which learners can self-assess and self-regulate at their own pace.

6.4.3 Visual scaffolding

Educators must use visual tools strategically to optimise learning outcomes. The pedagogical instructional design must include sequential dissemination of information, concretising key concepts in stages by using appropriate visual elements, thereby providing a tangible foundation for future lessons. Careful planning is required for the harmonious integration of visual elements to create a coherent and fulfilling lesson. Visual resources must be used in a way that scaffolds learning, appealing to learners with varying levels of cognitive abilities. A visual strategy that includes juxtaposing important elements can draw greater attention to areas of concern. It must target specific curriculum objectives in addition to providing structured support that helps learners navigate the complexities of trigonometry. The impact of visual scaffolding can be profound because it promotes active participation by allowing a gradual transition from elementary to abstract concepts.

6.4.4 *Use of iteration*

The anxiety regarding the use of visual representations by learners is primarily due to the mistakes in their construction of diagrams. They fear that if the diagram they sketched is wrong then they will not perform well. This impedes their motivation and confidence, resulting either in the abandonment of visual strategies, minimal use of them, or the use of alternate methods such as algebraic manipulation as the bedrock for their reasoning. It is therefore imperative for educators to encourage learners to use iteration, a process that includes the constant review and refinement of visual representations through repeated correction until it is aligned with the trigonometric concepts at play. This repeated search for errors in the diagram not only allows learners to validate their representations but also improves their trigonometric acumen. This re-evaluation, refinement, and reflection process leads to more clarity and greater comprehensibility of the trigonometric concepts. It is through this revisionist attitude that iteration can play a pivotal role in boosting learner confidence and sharpening the learner's trigonometric skills.

6.4.5 Encouragement to Implement Visual Strategies

The implementation of visual strategies in trigonometric pedagogy and mathematics, in general, serves as a transformative path for enhancing the understanding of concepts. Its mitigating powers are a stimulus for achieving active participation and immersive learning. Educators need not only to integrate visual aids in their instructional design but also encourage learners to adopt them in their tasks. The chalk-and-talk approach alone no longer appeals to the modern learner, as such visualisation strategies must be utilised concurrently with traditional methods to enhance understanding. A synergetic relationship must be established between algebraic manipulation and visual strategies, the former providing a structure and framework for reasoning whilst the latter enriches the learning environment with its intuitive geometric revelations.

6.4.6 Visualisation development

Educators must be trained and encouraged to use visual strategies in their classrooms because they are viewed as the main authority and source of knowledge by their learners. They need to be upskilled to use technology-mediated tools in addition to being trained in aligning curriculum objectives with appropriate visual language. Collaboration between educators must be encouraged because it can serve as an excellent support structure, one in which ideas and best practices can be shared. There is a dire need for cultivating a culture of innovation and transformation in the mathematics classroom because the current trajectory as evidenced by the NSC mathematics results is disillusioning.

6.5 Limitations

Limitations of the study were primarily due to time constraints and lack of financial resources. Both the researcher and participants had limited access to the media room for the use of technology-mediated tools in addition to load-shedding being another constraining factor. This setback was at times obviated by swapping classes thereby preventing loss of time. The load-shedding dilemma was overcome mostly by using a portable inverter. Exorbitant data costs as well as intermittent unstable internet connectivity hampered the lessons at times, but the school allowed me to use their wi-fi.

I observed erratic absenteeism from participants, and this potentially stunted their progress. At times I noticed a disengagement from certain participants with a short attention span which prompted me to revise my strategy to include a tactile approach to improve the situation. It was further discovered that those participants who did not have access to technology due to socio-economic reasons needed more technical support when they were allowed to explore the technology. In some lessons, few participants were overwhelmed by the advanced software I used but this initial excitement was not sustained throughout the intervention programme. This prompted me to tweak my strategy by using different visual tools to keep lessons interesting.

It must be noted that the scope of this research was restricted to only one school in the Umlazi District and the findings should be confined to this context. However, the undeniable positive impact of this visualisation intervention should provide encouragement and interest for future researchers to advance the use of visualisation. The research study applied an encompassing theoretical framework with five research instruments, and the enormity of this study may at times nettled participants especially when they did not get the right answers. This however posed no prolonged challenge because they were encouraged to persevere and were provided with extra support to overcome this mathematics anxiety.

6.6 Synopsis

The visualisation intervention in a grade 11 trigonometry classroom showed a positive effect as evidenced by the findings. Participants found the lessons enriching and engaging and this had a profound effect on how they approached trigonometry. The myriad of visualisation tools made lessons more inclusive and appealed to the diverse learning modalities. The visualisation strategy proved to be a dynamic avenue that educators must leverage to achieve better understanding. The recommendations are worth exploring because visual strategies are now starting to revolutionise the mathematical landscape for the better.

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APPENDIX A

LEARNER ASSENT FORM

LEARNER ASSENT FORM TO PARTICIPATE IN A RESEARCH PROJECT

School of Education, College of Humanities, University of KwaZulu-Natal, Edgewood Campus,

Information Sheet and Consent to Participate in Research

Date: 12 December 2022

Dear Participant

My name is Ashraf Khan. I am a Master's degree (Mathematics Education) candidate studying at the University of KwaZulu-Natal, Edgewood Campus, South Africa. I am interested in exploring the impact of visualisation in a grade 11 trigonometry class.

The aim and purpose of this research are to promote effective teaching and learning of trigonometry using visualisation. The study is expected to include 8 participants in total from varying tiers of academic performance. It will involve the completion of a questionnaire, interviews, and pre and post-tasks in trigonometry. The duration of your participation, if you choose to enrol and remain in the study, will be 6 months. The study is not funded by any entity.

To gather the information, I am interested in asking you some questions.
Please note that:

- Your confidentiality is guaranteed as your inputs will not be attributed to you in person, but reported only as a population member's opinion.
- The interview may last for about 45 minutes to 1 hour.
- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- You have a choice to participate, not participate, or stop participating in the research. You will not be penalised for taking such an action.
- Your involvement is purely for academic purposes only, and there are no financial benefits involved.
- If you are willing to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

Equipment	Willing	Not willing
Audio equipment		
Photographic equipment		
Video equipment		

I can be contacted at:
Email: [redacted]
Cell: [redacted]

My supervisor is Professor Vimolan Mudaly who is located at the School of Education, Edgewood campus of the University of KwaZulu-Natal.
Contact details: email: Mudalyv@ukzn.ac.za Phone number: 0312603682

You may also contact the Research Office through:

HUMANITIES & SOCIAL SCIENCES RESEARCH ETHICS ADMINISTRATION
Research Office, Westville Campus
Govan Mbeki Building
Private Bag X 54001
Durban
4000
KwaZulu-Natal, SOUTH AFRICA
Tel: 27 31 2608350/3587/4557- Fax: 27 31 2604609
Email: HSSREC@ukzn.ac.za

DECLARATION

I…………………………………………………………………………… (full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participate in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT DATE

……………………………………………… ………………………

APPENDIX B

INFORMED CONSENT LETTER

School of Education,
College of Humanities,
University of KwaZulu-Natal,
Edgewood Campus,

Dear Parent/Guardian

INFORMED CONSENT LETTER

My name is Ashraf Khan. I am a Master's degree (Mathematics Education) candidate studying at the University of KwaZulu-Natal, Edgewood Campus, South Africa. I am interested in exploring the impact of visualisation in a grade 11 trigonometry class.

The aim and purpose of this research are to promote effective teaching and learning of trigonometry using visualisation. The study is expected to include 8 participants in total from varying tiers of academic performance. It will involve the completion of a questionnaire, interviews, and pre- and post-tasks in trigonometry. The duration of participation, if you choose your child to enrol and remain in the study, will be 6 months. The study is not funded by any entity.

To gather the information, I am interested in asking your child some questions.
Please note that:

- Your child's confidentiality is guaranteed as his/her inputs will not be attributed to him/her in person but reported only as a population member's opinion.
- The interview may last for about 45 minutes to 1 hour.
- Any information given by your child/ward cannot be used against him/her, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- Your child has a choice to participate, not participate or stop participating in the research. He/She will not be penalised for taking such an action.
- Your child's involvement is purely for academic purposes only, and there are no financial benefits involved.
- If you are allowing your child to be interviewed, please indicate (by ticking as applicable) whether or not you are willing to allow the interview to be recorded by the following equipment:

Equipment	Willing	Not willing
Audio equipment		
Photographic equipment		
Video equipment		

I can be contacted at:
Email: [redacted]
Cell: [redacted]2
My supervisor is Professor Vimolan Mudaly who is located at the School of Education, Edgewood campus of the University of KwaZulu-Natal.
Contact details: email: Mudalyv@ukzn.ac.za Phone number: 0312603682

You may also contact the Research Office through:

HUMANITIES & SOCIAL SCIENCES RESEARCH ETHICS ADMINISTRATION
Research Office, Westville Campus
Govan Mbeki Building
Private Bag X 54001
Durban
4000
KwaZulu-Natal, SOUTH AFRICA
Tel: 27 31 2608350/3587/4557- Fax: 27 31 2604609
Email: HSSREC@ukzn.ac.za

DECLARATION

I………………………………………………………………………… (full names of parent/guardian) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to my child's participation in the research project.

I understand that my child is at liberty to withdraw from the project at any time, should he/she so desire.

SIGNATURE OF PARENT/GUARDIAN DATE

……………………………………………… ………………………

APPENDIX C

GATEKEEPERS LETTER

APOLLO SECONDARY SCHOOL

Tel : 031 403 5368
: [redacted]
e-mail : *apollosecondary@telkomsa.net*

PO Box 562299
Chatsworth
4030

12 December 2022

PERMISSION TO CONDUCT RESEARCH

To whom it may concern

This letter serves to confirm that Mr. Ashraf Khan is currently completing his Masters of Education (Mathematics Education) via UKZN (student number 931331943).

I, Mr. S. Moodley, Principal at Apollo Secondary hereby grant permission to the above mentioned student to conduct research on the topic "A visualisation intervention in a grade 11 trigonometry class".

Thank you.

KZN DEPT. OF EDUCATION
APOLLO SECONDARY SCHOOL
CELL: 074 803 5368
TEL: 031 403 5368
EMAIL: apollosecondary@telkomsa.net

[redacted]

Mr S. Moodley
PRINCIPAL ([redacted]

APPENDIX D

ETHICAL CLEARANCE

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

05 May 2023

Ashraf Khan (931331943)
School Of Education
Edgewood Campus

Dear A Khan,

Protocol reference number: HSSREC/00005468/2023
Project title: A visualization intervention in a grade 11 trigonometry class
Degree: Masters

Approval Notification – Expedited Application

This letter serves to notify you that your application received on 04 April 2023 in connection with the above, was reviewed by the Humanities and Social Sciences Research Ethics Committee (HSSREC) and the protocol has been granted FULL APPROVAL.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number. PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

This approval is valid until 05 May 2024.
To ensure uninterrupted approval of this study beyond the approval expiry date, a progress report must be submitted to the Research Office on the appropriate form 2 - 3 months before the expiry date. A close-out report to be submitted when study is finished.

HSSREC is registered with the South African National Health Research Ethics Council (REC-040414-040).

Yours sincerely,

[redacted]

Professor Dipane Hlalele (Chair)

/dd

Humanities and Social Sciences Research Ethics Committee
Postal Address: Private Bag X54001, Durban, 4000, South Africa
Telephone: +27 (0)31 260 8350/4557/3587 **Email:** hssrec@ukzn.ac.za **Website:** http://research.ukzn.ac.za/Research-Ethics

APPENDIX E

INTERVIEW GUIDE

1 Do you find trigonometry challenging? If so, what are the challenges?
2 Explain how you go about formulating solutions in trigonometry?
3 How did your educator introduce trigonometry to you?
4 Did your educator use visual aids to supplement your understanding of trigonometry? Explain.
5 Were the abstract concepts such as $sin^2\theta + cos^2\theta = 1$ linked to visualisation techniques? Elaborate.
6 Were you taught the art of reasoning with yourself to link concepts and connect them with mental imagery? Explain.
7 Do you think a picture or mental image can assist in your understanding of trigonometry? Explain.
8 In your opinion, how can the learning of trigonometry improve?

APPENDIX F

QUESTIONNAIRE USING GOOGLE FORMS

Visualisation in trigonometry usi

khan.ashraf14@gmail.com (not shared) Switch account

How would you rate your understanding of trigonometry?

- ○ Excellent
- ○ Good
- ○ Satisfactory
- ○ Poor

Does the educator use diagrams/illustrations to teach trigonometry?

- ○ Yes
- ○ No
- ○ Sometimes

Does the use of diagrams help you understand the basic concepts of trigonometry?

- ○ Yes
- ○ No
- ○ Sometimes

Do you form mental images when problem solving in trigonometry?

- ○ Always
- ○ Never
- ○ Sometimes

When proving trigonometric identities, does the educator use diagrams?

- ○ Always
- ○ Never
- ○ Sometimes

Does the educator show you transformation of trigonometric functions graphically?

- ○ Yes
- ○ No
- ○ Sometimes

Does the educator relate trigonometric equations to graphs?

- ○ Yes
- ○ No
- ○ Sometimes

Do you experience difficulty when solving trigonometric word problems?

- ◯ Yes
- ◯ No
- ◯ Sometimes

How would you rate your conceptual knowledge of 2D trigonometry?

- ◯ Excellent
- ◯ Good
- ◯ Satisfactory
- ◯ Poor

Do you attach specific meaning when you solve abstract questions in trigonometry?

- ◯ Yes
- ◯ No
- ◯ Maybe
- ◯ I do not know

Submit

Clear form

APPENDIX G

TRIGONOMETRY: PRE-TASK

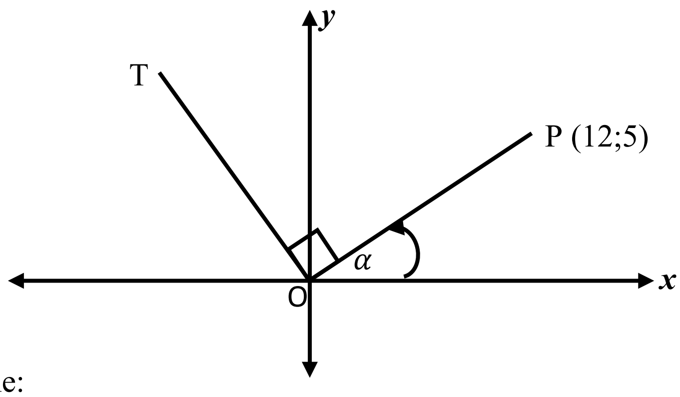
Duration: 60 minutes **Marks: 50**

Instructions

1. Clearly show **ALL** calculations, diagrams, graphs, etcetera that you have used in determining your answers.
2. The use of a non-programmable / non-graphical calculator is allowed.
3. Answers only will NOT necessarily be awarded full marks.
4. If necessary, round off answers to TWO decimal places, unless stated otherwise.
5. Number the answers correctly according to the numbering system used in this question paper. Write neatly and legibly.

Question 1

1.1 In the diagram, P is a point (12; 5) and $P\hat{O}S = \alpha$. OT $\perp$OP.



Determine:

1.1.1 $cos\alpha$ (3)

1.1.2 The length of OT (3)

1.2 If $psin20^0 = 1$, express the following in terms of p:

1.2.1 $cos20^0$ (2)

1.2.2 $\tan(-340^0)$ (3)

1.2.3 $sin110^0cos160^0$ (3)

[14]

Question 2

2.1 Simplify the following without using a calculator:

2.1.1 $\frac{sin(90^0+\theta).cos(\theta-180^0)}{cos^2(180^0-\theta)}$ (4)

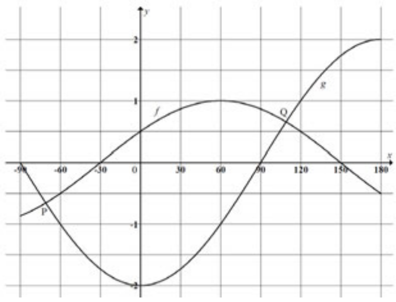
2.1.2 $\frac{sin54^0(1-2sin^2 30^0)}{sin414^0}$ (6)

[10]

Question 3

3.1 Prove the identity: $\frac{1-cos^2+sin^2\theta-sin\theta}{2sin\theta cos\theta-cos\theta}=tan\theta$ (5)

3.2 The graphs of $f(x)=\sin(x+30^0)$ and $g(x)=-2cosx$ for $x\in[-90^0;180^0]$. The graphs intersect at P and Q.



3.2.1 Calculate the *x*-coordinates of points P and Q. (5)

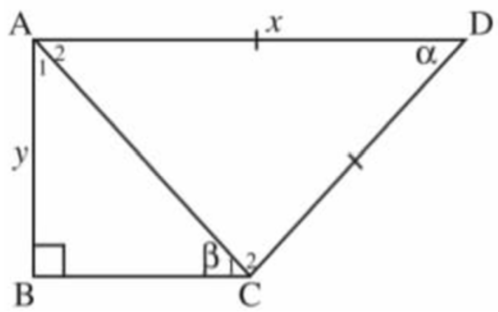
3.2.2 Determine the value(s) of *x* for which $f(x)>g(x)$ (2)

3.2.3 If the graph of *g* is shifted 30^0 to the left, determine the equation of the resulting function. (2)

[14]

Question 4

4.1 In the diagram below, D = $\alpha, \hat{C}_1=\ \beta$, AB $=y$, AD $=$ CD $=x\ and\ AB\perp BC$.



Prove that $y=xsin\beta\sqrt{2(1-cos\alpha)}$ (6)

4.2 A pilot takes off from airport A to fly to airport B, 500km away. After flying 200km away he realises he has been flying $9,3^0$ off course. At this point, how far is he from B? (6)

[12]

TOTAL MARKS: 50

APPENDIX H

TRIGONOMETRY: POST-TASK

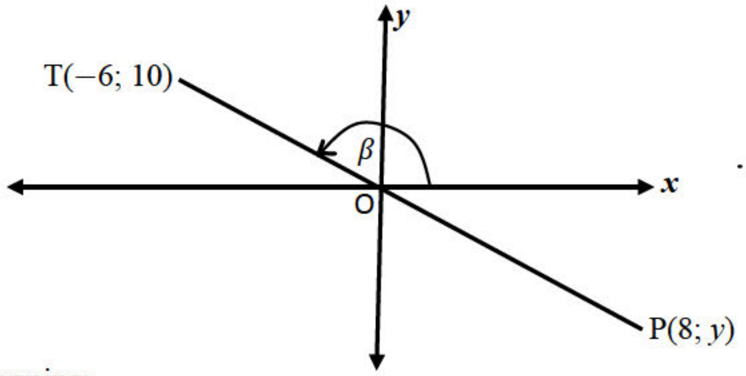
Duration: 60 minutes **Marks: 50**

Instructions

1. Clearly show **ALL** calculations, diagrams, graphs, etcetera that you have used in determining your answers.
2. The use of a non-programmable / non-graphical calculator is allowed.
3. Answers only will NOT necessarily be awarded full marks.
4. If necessary, round off answers to TWO decimal places unless stated otherwise.
5. Number the answers correctly according to the numbering system used in this question paper. Write neatly and legibly.

Question 1

1.1 In the diagram, T is a point $(-6; 10)$ and P is a point $(8; y)$
$X\hat{O}T = \beta$



Determine:

1.1.1 $\cos(180^0 - \beta)$ (3)

1.1.2 The length of OP (3)

1.1.3 The size of β to two decimal places. (3)

1.2 If $17sin\alpha + 8 = 0$, where $cos\alpha > 0$, determine without using a calculator the value of $3\cos\alpha - tan\alpha$ (6)

[15]

Question 2

2.1 Simplify the following without using a calculator:

2.1.1 $\dfrac{sin^2(-\theta).co\,s(180^0).}{cos(90^0-\theta).cos(180^0-\theta).tan\theta}$ (5)

2.1.2 $\frac{sin100^0(-sin^2 15^0 - cos^2 15^0)}{cos190^0}$ (5)

2.2 Prove the identity: $\frac{1+\tan x}{1-\tan x} = \frac{1+2\sin x cosx}{\cos^2 x - sin^2 x}$ (5)

[15]

Question 3

Given: $f(x) = \cos(x + 45^0)$ *and* $g(x) = \sin 2x$ where $x \in [-135^0; 45^0]$

3.1 Sketch the graphs of f and g on the same system of axes for the given domain showing all relevant points. (6)

3.2 Determine the points of intersection for the given domain. (6)

3.3 Determine the value(s) of x for which $f(x).g(x) > 0$ (2)

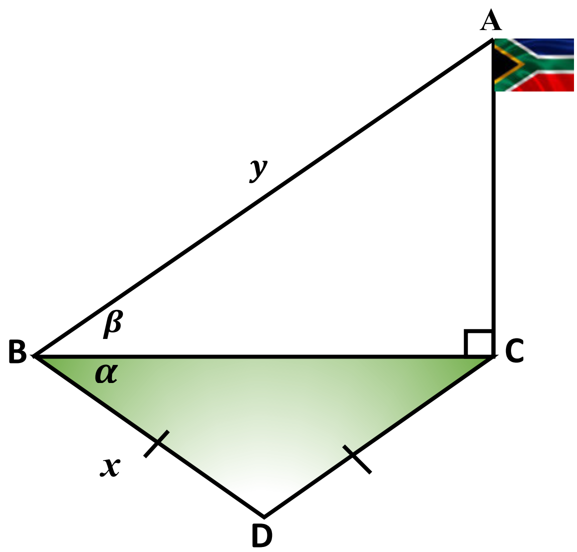
3.4 Determine the value(s) of t for which $f(x) = t$ has two negative real roots. (2)

[16]

Question 4

In the sketch below, AB represents a flagpole with height h.The angle of elevation of the top of the flagpole, A, from B is β.

$C\hat{B}D = \alpha$ and AB $= y$ and BD = CD= x.



Prove that $y = \frac{xsin2\alpha}{\sin \alpha .cos\beta}$ (4)

[4]

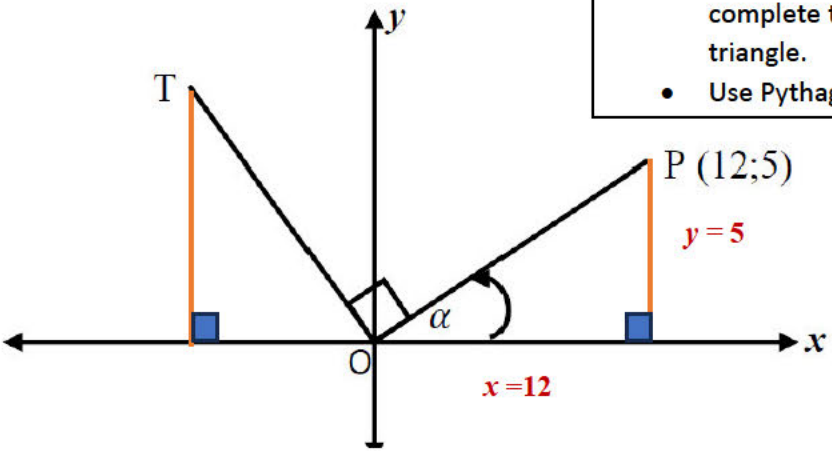
<u>TOTAL MARKS: 50</u>

APPENDIX I

MODEL SOLUTION (PRE-TASK)

Visualisation Method (using right-angled triangles)

- Construct perpendicular to x-axis from terminal points to complete the right-angled triangle.
- Use Pythagoras



$x^2 + y^2 = r^2$

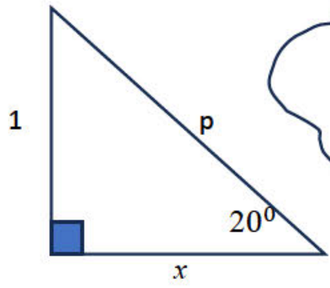
$(12)^2 + (5)^2 = r^2$

$\therefore r = 13$

1.1.1 $\quad cos\alpha = \frac{12}{13}$

1.1.2 $\quad OT = 13\ or\ any\ value\ that\ preserves\ the\ ratio$ $\sin(90^0 + \alpha) = \frac{12}{13}$ using similar Δs

1.2 $\quad sin20^0 = \frac{1}{p}$



Using Pythagoras

$x^2 + (1)^2 = p^2$

$\therefore x = \sqrt{p^2 - 1}$

1.2.1 $cos20^0 = \frac{\sqrt{p^2-1}}{p}$

1.2.2 $\tan(-340^0) = tan20^0$

$= \frac{1}{\sqrt{p^2-1}}$

1.2.3 $sin110^0 cos160^0 = sin80^0 cos20^0$

$= cos20^0 cos20^0$

$$= \left(\frac{\sqrt{p^2-1}}{p}\right)^2$$

$$= \frac{p^2-1}{p}$$

Question 2

2.1.1 $\frac{\sin(90^0+\theta).\cos(\theta-180^0)}{cos^2(180^0-\theta)}$

$$= \frac{cos\theta.-cos\theta}{cos^2\theta}$$

$$= \frac{-cos^2\theta}{cos^2\theta}$$

$$= -1$$

2.1.2 $\frac{sin54^0.(1-2sin^2 30^0)}{sin^2 414^0}$

$$= \frac{sin54^0.(1-2(\frac{1}{2})^2}{sin^2 54^0}$$

$$= \frac{1}{2sin54^0}$$

Question 3

3.1 **LHS:** $\frac{\mathbf{1-cos^2\theta+sin^2-sin\theta}}{\mathbf{2sin\theta cos\theta-cos\theta}}$

$$= \frac{sin^2\theta+sin^2\theta-sin\theta}{2sin\theta cos\theta-cos\theta}$$

$$= \frac{2sin^2\theta-sin\theta}{2sin\theta cos\theta-cos\theta}$$

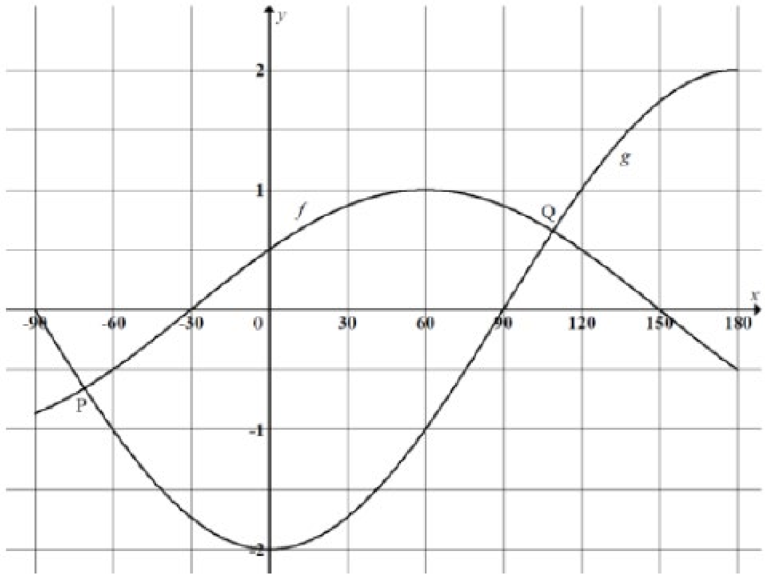
$$= \frac{sin\theta(2sin\theta-1)}{cos\theta(2sin\theta-1)}$$

$$= \frac{sin\theta}{cos\theta}$$

RHS: $tan\theta = \frac{sin\theta}{cos\theta}$

$\therefore \boldsymbol{LHS = RHS}$

3.2.1 P and Q are points of intersection. Solve equations simultaneously



$\sin(x + 30^0) = -2cosx$

$sinxcos30^0 + cosxsin30^0 = -2cosx$

$0{,}5sinx + 0{,}5cosx = -2cosx$

$0{,}5sinx = -2{,}5cosx$

$tanx = -5$

Reference $\angle = tan^{-1}(5)$

$= 78{,}69^0$

$\therefore x = 180^0 - 78{,}69^0 + k180^0$

$\therefore x = 101{,}31^0 + k180^0$

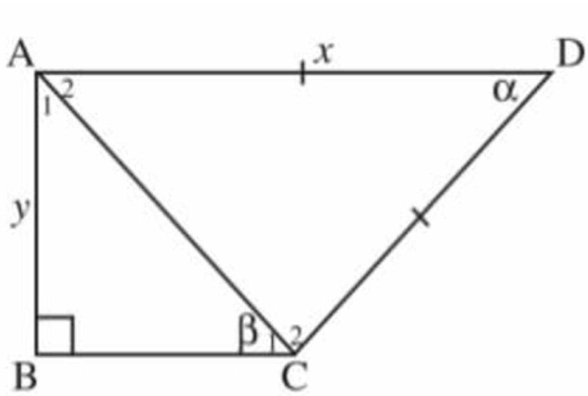
Specific solutions for the interval: $\{-78.69^0\ ;\ 101.31^0\}$

$x_p = -78.69\ ^0 \quad or \quad x_Q = 101.31^0$

3.2.2 $-78.69^0 < x < 101.31^0$

3.2.3 $y = -2\cos(x + 30^0)$

Question 4



R.T.P: $y = xsin\beta\sqrt{2(1-cos\alpha)}$

Proof: In ΔACD apply the cosine rule

$AC^2 = x^2 + x^2 - 2(x)(x)cos\theta$

$\therefore AC = 2x^2 - 2x^2cos\theta$

$AC = x\sqrt{2(1-cos\theta)}$

$In\ \Delta ABC$

$sin\beta = \dfrac{y}{AC}$

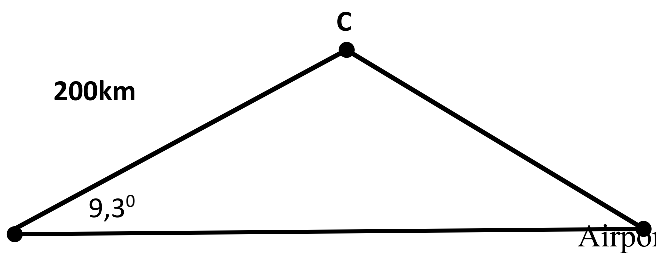
$\therefore AC = \dfrac{y}{sin\beta}$

$\dfrac{y}{sin\beta} = x\sqrt{2(1-cos\theta)}$

$\therefore y = xsin\beta\sqrt{2(1-cos\alpha)}$

4.2

Airport A



Airport B

500km

Let C represent the point 200km from airport A

$AB^2 = 500^2 + 200^2 - 2(500)(200)cos9{,}3^0$

$\sqrt{AB^2} = \sqrt{92628.86}$

$\therefore AB = 304{,}35m$

The pilot is 304,35 metres from airport B.

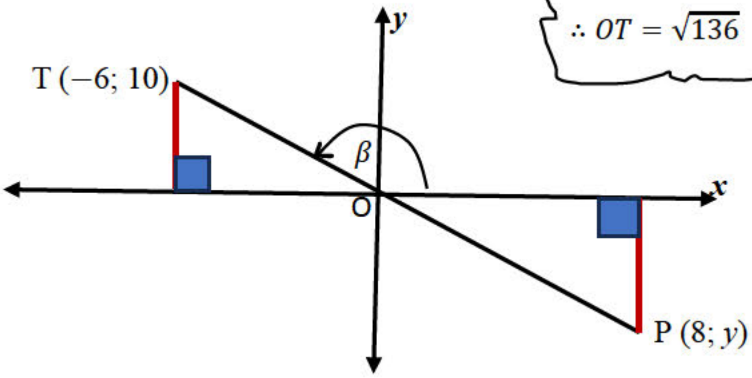
APPENDIX J

POST TASK MODEL SOLUTIONS

Question 1

$OT^2 = (-6)^2 + (10)^2$

$\therefore OT = \sqrt{136}$



1.1.1 $\cos(180^0 - \beta)$

$= -cos\beta$

$= -(\frac{-6}{\sqrt{136}})$

$= \frac{6}{\sqrt{136}}$

1.1.2 $\cos(180^0 + \beta)$

$= -cos\beta = \frac{6}{\sqrt{136}}$

$= -(\frac{-6}{\sqrt{136}})$

$= \frac{6}{\sqrt{136}}$

But $cos\beta = \frac{8}{OT}$

$\therefore \frac{8}{OT} = \frac{6}{\sqrt{136}}$

$OT = \frac{4\sqrt{136}}{3}$ units

1.1.3 $sin\beta = \frac{10}{\sqrt{136}}$

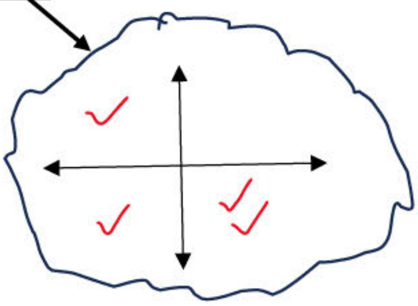
$Reference\angle = sin^{-1}(\frac{10}{\sqrt{136}})$

$\therefore Reference\angle = 59{,}04^0$

$\therefore \beta = 180^0 - 59{,}04$

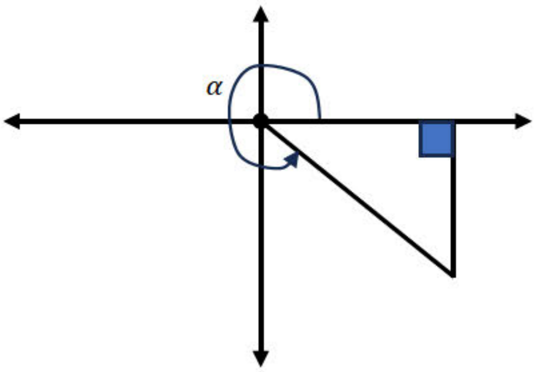
$\therefore \beta = 120{,}96^0$

CAST DIAGRAM



1.2 $17sin\alpha + 8 = 0$

$sin\alpha = \frac{-8}{17}$ $\therefore \alpha$ is possibly in quadrant 3 or 4
$since\ cos\alpha > 0$, α is possibly in quadrant 2 or 4
αin in quadrant 4



$x^2 + (-8)^2 = (17)^2$

$\therefore \sqrt{x^2} = \pm\sqrt{225}$

$\therefore x = \pm 15$ but $x > 0$

$\therefore x = 15$

$3cos\alpha + tan\alpha = 3\left(\frac{15}{17}\right) + \left(\frac{-8}{17}\right)$

$= \frac{37}{17}$

Question 2

2.1.1 $\frac{sin^2(-\theta).co\,s(180^0).}{cos(90^0-\theta).cos(180^0-\theta).tan\theta}$

$= \frac{sin^2\theta.(-1)}{sin\theta.-cos\theta.\frac{sin\theta}{cos\theta}}$

$= \frac{-sin^2\theta}{-sin^2\theta}$

$= 1$

2.1.2 $\frac{sin100^0(-sin^2 15^0 - cos^2 15^0)}{cos190^0}$

$= \frac{sin80^0.-(sin^2 15^0 + cos^2 15^0)}{-cos10^0}$

$$= \frac{cos10^0 \times -1}{-cos10^0}$$

$$= -1$$

2.2 Prove the identity: $\frac{1+\tan x}{1-\tan x} = \frac{1+2\sin x cosx}{\cos^2 x - sin^2 x}$

$$LHS: \frac{1+\frac{sinx}{cosx}}{1-\frac{sinx}{cosx}}$$

$$= \frac{cosx + sinx}{cosx - sinx}$$

$$= \frac{(cosx + sinx)(cosx + sinx)}{(cosx - sinx)(cosx + sinx)}$$

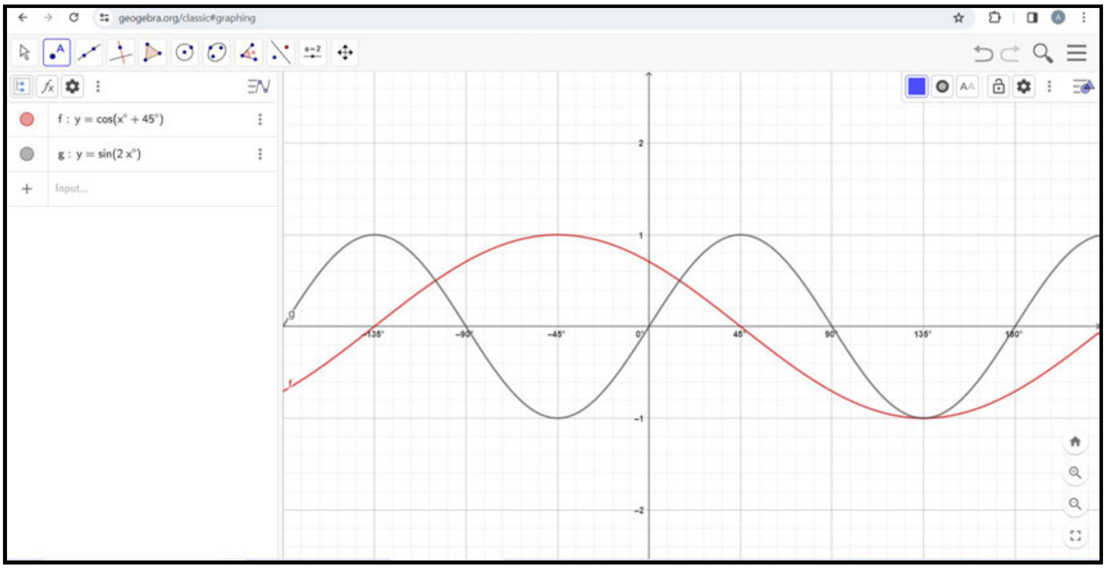
$$= \frac{cos^2x + 2sinxcosx + sin^2x}{cos^2 x - sin^2x}$$

$$= \frac{1 + 2sin\,x\,cosx}{cos^2 x - sin^2x}$$

=RHS

Question 3

3.1



https://www.geogebra.org/classic

a. $\cos(x + 45^0) = \sin 2x$

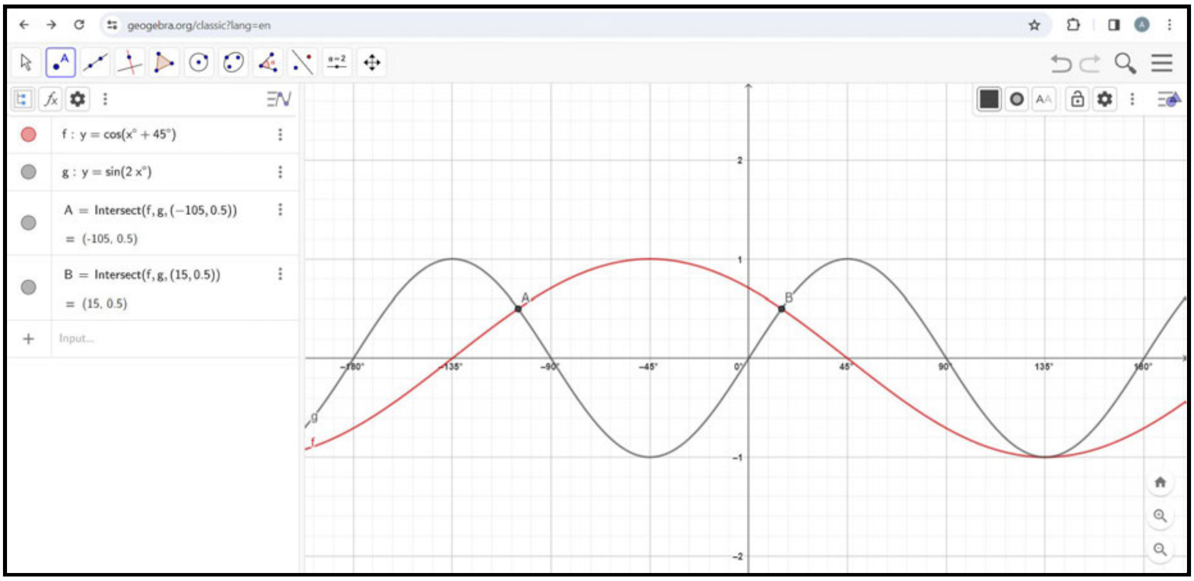
$\sin(45^0 - x) = sin2x$

$x - 45^0 = 2x + k.360\ ^0 where\ k \in Z$ or $x - 45^0 = 180^0 - 2x + k.360^0$

$-x = 45^0 + k.360^0$ $\quad x = 225^0 + k.360^0$

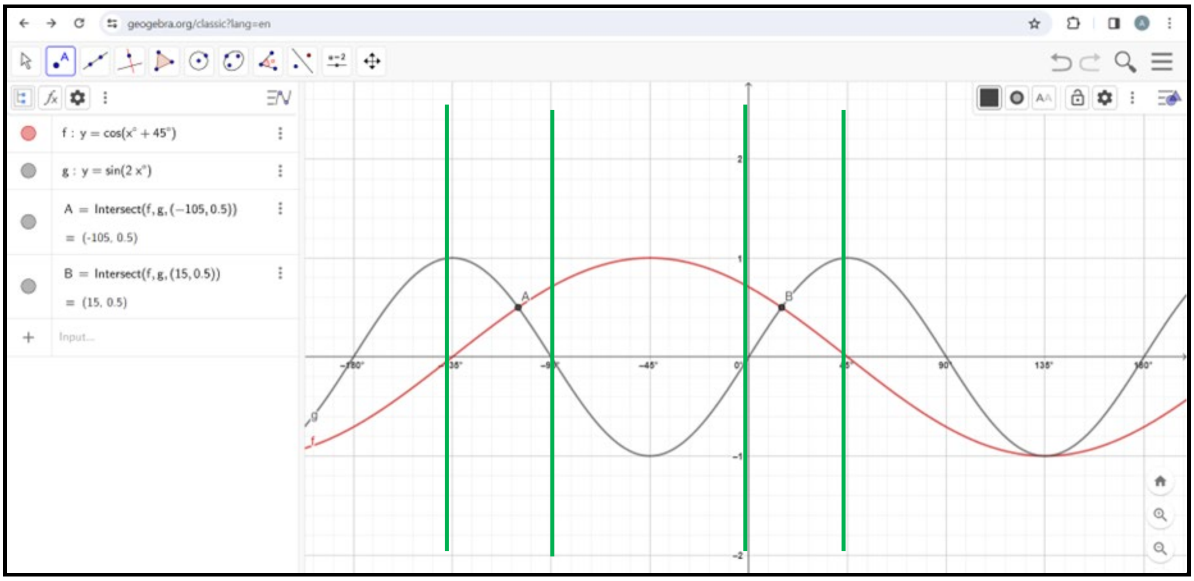
$x = 15^0 + k.120^0$

Specific Solutions: $x\epsilon\{-105^0,\ \ 15^0\}$



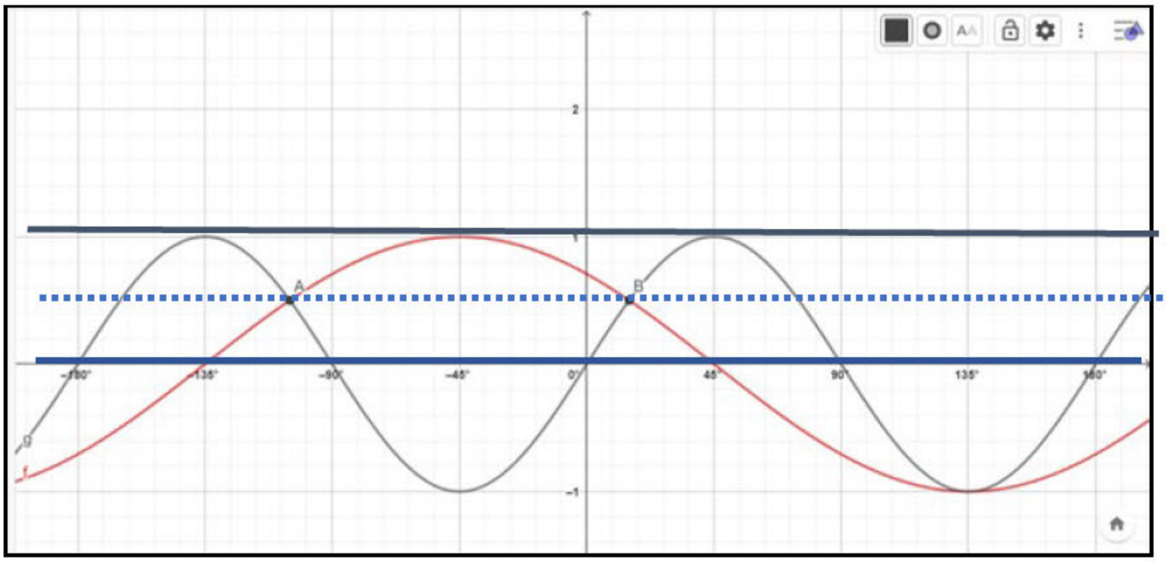
https://www.geogebra.org/classic

3.3



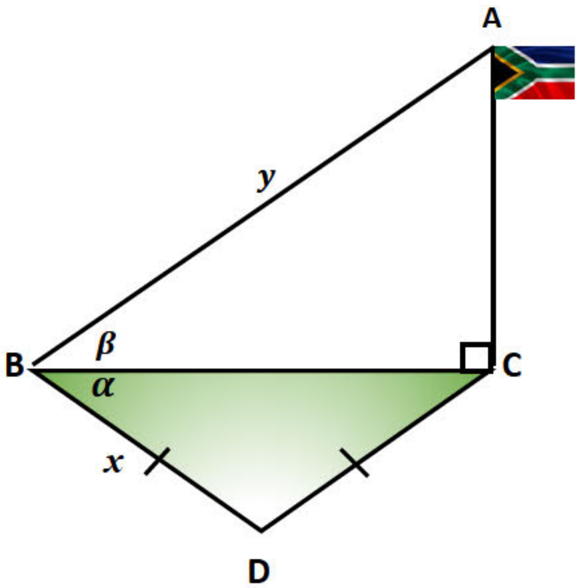
$-135^0 < x < -90^0\ or\ 0^0 < x < 45^0$

3.4



$0 < t < 1$

3.5



R.T.P: $y = \frac{x\sin 2\alpha}{\sin\alpha . \cos\beta}$

Proof: In ΔABC

$$\cos\beta = \frac{BC}{y}$$

$$\therefore BC = y\cos\beta$$

$In\ \Delta BCD$

$$\frac{BC}{\sin(180^0 - 2\alpha)} = \frac{x}{\sin\alpha}$$

$$\frac{y cos\beta}{\sin(180^0 - 2\alpha)} = \frac{x}{sin\alpha}$$

$$y = \frac{x sin2\alpha}{\sin\alpha . cos\beta}$$

APPENDIX K

CERTIFICATE OF ENGLISH EDITING

CERTIFICATE OF ENGLISH EDITING

Date: 16 April 2024

To whom it may concern

This is to certify that the Master's dissertation titled: **A visualisation intervention in a Grade 11 trigonometry class**, authored by Ashraf Khan, has been edited by myself for language and compliance.

Please contact me should you require any further information.

[redacted]

Editor: Mrs D Naidoo

B.A / HED / BEd (Hons)

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APPENDIX L

TURNITIN REPORT

A VISUALISATION INTERVENTION IN A GRADE 11 TRIGONOMETRY CLASS

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