

**An exploration of grade 11 Mathematical Literacy learner's engagement  
with start-unknown and result-unknown type problems set in a variety of  
real life contexts**

by

Martin Siphon Mbonambi

(207520775)

Supervisor

Prof. S. Bansilal

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## ABSTRACT

With the introduction in 2006 of the school subject Mathematical Literacy (ML) in the further Education and Training band, there have been expectations that such a subject might develop responsible citizens, contributing workers and self-managing people. The extent to which the subject can meet these aims is dependent on the ways in which the subject is taught and assessed, which influences the focus of ML in the classrooms. One of the differences between the respective subjects of Mathematics and Mathematical Literacy is that when it comes to the latter, there has been less emphasis on carrying out algebraic procedures, and a greater focus on working with contexts. However, algebraic skills can be advantageous even when solving problems set within contexts. One area, which surfaces the distinction between arithmetic and algebraic skills, is in the substitution and computation of a formula, as compared to the solution of equations. In this study, I focus on this distinction by examining Grade 11 ML learner skills in solving both result-unknown problems and start-unknown problems, where the former involves substituting and computing the result of a formula or equation for which the input is given. The latter involves re-arranging the equation or formula in order to solve for the input when the output is given. With this in mind, this study sets out to explore the strategies used by Grade 11 learners to solve result-unknown and start-unknown problems set in real life contexts.

This is a qualitative study, carried out with three hundred and forty Grade 11 Mathematical Literacy learners from rural and urban school in North Durban. Data was gathered from a document analysis of 340 learners' written responses to the research instrument, along with interviews with ten of these learners. There were four tasks in the research instrument, each of which had a result-unknown, a start-unknown and a reflection question. In the four tasks with the exception of Question 1.2.1 and 1.2.2 in tasks one, were set around a linear equation, while Question 1.2.1 and 1.2.2 involved a hyperbolic equation. Semi-structured interviews were conducted individually with ten learners and the audio recorded. The purpose of the interviews was to explore some of the factors that influenced their written responses. The findings revealed the solving of start-unknown questions to be a serious problem for learners. On average, the success rate at result-unknown questions was 75%, while it was 26% for start-unknown questions. For start-unknown questions based on linear equations only, the success rate was a mere 19 percent.

Some strategies used by learners in responding to start-unknown questions included number grabbing, systematic guess and test, conjoining, symbol manipulation and working backwards. On average, over the four tasks based on linear equations, only nine percent of learners successfully used strategies based on algebraic skill. Most learners who obtained correct answers in the start-unknown questions used the guess and test strategy. Strategies identified in result-unknown questions included direct arithmetic strategy.

The study recommends that for ML learners, teachers need to impress upon learners that the location of the formula in the question is not an indication that certain questions would be answered using the formula, because the formula is placed next to them. It also recommends that teachers create opportunities for learners to continue to practice the algebraic skills they learned in the GET band, particularly in the area of transforming and solving simple linear equations.

## PREFACE

The work described in this thesis was carried out in the School of Education, University of KwaZulu-Natal, from July 2011 to November 2013 under the supervision of Prof. Sarah Bansilal (Supervisor).

This study represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others, it is duly acknowledged in text.

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**Martin Sipho Mbonambi (Mr.)**

**[Candidate]**

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**Prof. S. Bansilal**

**[Supervisor]**

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## DEDICATION

To my fiancée,

Whom I deprived of quality time during the completion of this study;

To all, who never lost hope on my unavailability.

And, to my relatives, friends and colleagues and community who gave me support:

“if you take care of the small things... bigger things will automatically fall into place”

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## CHAPTER 1

### INTRODUCTION

#### 1.1 Overview

Mathematical Literacy was introduced into the South African school curriculum in 2006 at the Grade 10 level. Since 2006, learners have had the choice to study either Mathematics or Mathematical literacy between Grades 10 to 12. Learners study Mathematics from Grade 4 to Grade 9, and from Grade 10 to Grade 12 learners can choose between Mathematics and Mathematical Literacy (DoE, 2003a). Prior to this, the subject Mathematics was optional, with a choice of taking the subject either on the Higher Grade or the Standard Grade. Standard Grade Mathematics was considered easier than Higher Grade Mathematics. Furthermore, prior to 1994, when learners completed Grade 9 they could choose not to pursue any study in Mathematics. The motivation for the introduction of some form of compulsory mathematics in 2006 was partly to alleviate the high levels of innumeracy and mathematical illiteracy that existed in the country (DoE, 2003, p.9). At the end of 2008, Grade 12 learners wrote the first Department of Education (DoE) National Senior Certificate (NCS) examination in Mathematical Literacy (North, 2010). Under the new NCS system, Higher and Standard Grade Mathematics have been merged into one subject, namely Mathematics, with its alternative Mathematical Literacy. Furthermore, under the new system, it is compulsory for students to study one of the subjects between Mathematics and Mathematical Literacy.

## 1.2 Focus and Purpose of the Study

The period from 1994 to the present has seen South African Education undergoing dramatic transformation. As from 1994 till today, the South African system of education is currently going through a wave of curriculum changes and amendments. The phasing in of Outcomes-Based Education (OBE) was an attempt to make education more meaningful and relevant to society (DoE, 2006). In 1998, a new education system called Curriculum C2005 was implemented. Mathematics has received close scrutiny since 1994 and the performance of learners in Mathematics is still a problem. This, among other reasons, led to the introduction of Mathematical Literacy in 2006 in the Further Education and Training (FET) band, in an attempt to make South African citizens mathematically literate, as described below:

Being literate in Mathematics is an essential requirement for the development of the responsible citizen, the contributing worker and the self-managing person. Being mathematically literate implies an awareness of the manner in which Mathematics is used to format society and enables astuteness in the user of the products of Mathematics such as hire-purchase agreements and mathematical arguments in the media – hence the inclusion of Mathematics Literacy as a Fundamental requirement in the Further Education and Training curriculum (DoE, 2006, p.43).

The ever-changing nature of the curriculum is also likely to create further challenges to the education system, especially for teachers who struggle with mathematical content knowledge. Whilst these massive changes are on-going, one of the concerns regarding the ever-changing nature of the South African curriculum is the extent to which teachers and learners understand the demands of a curriculum, which focuses on using mathematics tools in context. Since ML is a new subject, there has been little attention paid to exactly how such a curriculum might work in developing learners' mathematical literacy skills. In fact, only a basic knowledge of mathematics is required in order to cope with the subject of ML. It is important to know how learners cope with the transition from GET Mathematics to ML.

The skill of solving linear equations, for example, is one of the first challenges encountered by learners when introduced to algebra in Grades 8 and 9, and many educators assume that learners who have completed Grade 9 are well placed to solve algebraic

equations. However, there is no research dealing directly with ML learners' skills with respect to solving equations. Hence, there is little knowledge about what learners are doing in terms of solving algebraic equations at Grade 11 level in ML. It is against this background that I decided to conduct this study, 'An exploration of Grade 11 Mathematical Literacy learner's engagement with Result-Unknown and Start-Unknown problems set in a variety of real life contexts', with the rationale of transforming our educational approaches by taking into account the potentially strong and deep-seated preconceptions and misconceptions of students (McClosely et al., 1980, p.39).

Usually in mathematics, you have an equation and you want to find a solution, in other words, you are solving a direct problem. In most cases the inverse problem evolves from direct problem. With an inverse problem, you are given a solution and you have to find the input (Groetch, 1999). Nathan and Koedinger (2000) define Result-unknown questions as that manner of question in which the unknown quantity is the results of the events or mathematical operations described in the problem. They further define the Start-unknown questions as those in which the unknown value refers to the quantity needed to specify a relationship (that is, the input value for the given result). Vergnaud's (2009, p. 87) writes that an inverse problem requires a "theorem –in –action: If  $T(I) = F$ , then  $I = T^{-1}(F)$ , where I stands for initial state, F for the final state, T for direct transformation, and  $T^{-1}$  for inverse transformation".

It is important to know how learners cope with the transition from GET Mathematics to ML. The ever-changing nature of the curriculum is also likely to bring instability to the education system, especially for teachers who begin to struggle with, among other things, mathematical content knowledge.

The aim of ML is to provide "learners with an awareness and understanding of the role that mathematics plays in the modern world... [and to enable] learners to develop the ability and confidence to think numerically and spatially in order to interpret and critically analyse everyday situations and to solve problems" (DoE, 2003, p.9). This focus on real life has led many to assume that learners do not need to know formal mathematics in ML. For example in ML, learners are generally not introduced to any more mathematics beyond what is covered at the Grade 9 level. However, dealing with real life situations with a mathematical gaze often requires the sophisticated use of basic algebraic mathematics.

A basic understanding of what an equation is would seem to be fundamental to success in any mathematically oriented endeavour (Matz, 1980).

For example, a common example, which may appear in ML textbooks and in assessments questions, may involve calculations based on cost and units of cell phone usage, premised on a formula such as:

$$\text{COST} = R1, 80 \times \text{no. of units used} + R85.$$

In an ML classroom, learners are often required to use the formula to calculate costs if a certain number of units are known, for example, to calculate the electricity or water bill based on a given consumption. Such formulae form part of mathematical literacy knowledge across the FET level. Solving these problems requires the use of Result-Unknown strategy, because the value of the variable (number of units) is given, and one is being asked to find the cost (or Result). However, a question such as ‘how many units you can get for R300?’ requires learners to calculate a number of units that correspond to a certain given cost. This question requires a renewed strategy, which can be described as a Start-Unknown Question (Nathan & Koedinger, 2000), because in this case, the result (cost is given) and the start (number of units), is required. Groetsch (1999) uses the terms ‘direct problem’ and ‘inverse problem’ to refer to result-unknown and start-unknown problems, respectively. Solving problems based on well-known everyday situations such as telephone, water or electricity bills may thus require the use of the start-unknown strategy, which is mathematically more demanding than the routine use of a result-unknown strategy. In my experience, I have noted that learners seem to struggle when faced with a start-unknown question in their examinations and tests and other assessment tasks, but that they cope well with result-unknown questions. Bansilal, Long & Debba (2013) carried out a Rasch analysis using Grade 12 ML learners’ responses to a KZN preparatory examination paper, finding that the question that had the highest difficulty level was a question which asked for the price of an item before the sales tax was added, and the final price was given. This question was a start-unknown question, which led to my interest in investigating ML learner competence in solving such problems.

I have therefore become interested in researching learners’ engagement with these types of problems in different real life settings, using formal algebraic solution strategies. Here I refer specifically to algebra instruction aimed at advancing learner conceptual knowledge and skill, by shifting attention away from focusing on symbolic manipulation and equation solving, toward analysing and interpreting solutions to problems based on generalisation formulae found in car hire such as:

$COST = R1.85 \times \text{no. of km travelled} + (\text{no. of km travelled} - 1000)$ . Accordingly, the purpose of this study is to explore Grade 11 ML learners' engagement with Result-Unknown and Start-Unknown Questions. Therefore, the key research questions of the study are as follows:

- How do Grade 11ML learners respond to the five Result-Unknown questions, and what are the strategies that they use to solve the problems?
- How do the learners respond to the five Start-Unknown questions, and what are the strategies that they use to solve the problems?
- Which of the two types of questions (Result-Unknown or Start-Unknown) do the learners experience as more difficult?
- What are learner perceptions about which questions were more challenging, and why they were experienced as more challenging?

The selection of the types of equations around which the study could be focused was guided by the ML curriculum for Grade 10 (DoBE, 2011, p.36), which specifies the study of constant (fixed), linear and inverse proportion relationships. Although my sample was taken from Grade 11 learners, the study was done in the first half of the year, so I used this Grade 10 curriculum as a guide to what they would have encountered already. In addition, these types of relationships are also studied at Grade 9 level. I chose the linear equation as my main focus, and included one task based on an inverse proportional relationship. The contexts that gave rise to the relationships were set within different contexts, similar to those used in the ML assessments.

### **1.3 Details of Study**

In Chapter 1, I describe the rationale for the introduction of ML in South Africa and the focus and purpose of this study.

In Chapter 2, I review recent studies on the teaching and learning of ML in South Africa and the role played by contexts in the teaching and learning of Mathematical Literacy. Furthermore, this chapter provides a brief overview on the thinking styles of Mathematics and ML learners, research studies on algebra ( $y = mx + c$ ) and aspects related to teaching of

algebra in schools. This chapter concludes with the theoretical framework that underpins this study.

Chapter 3 outlines the research methodology and the design of the study. It describes the contexts in which the study was undertaken and the characteristics and selection of the sample. This chapter also describes the qualitative paradigm on which this study was based, and the qualitative paradigm is strengthened by the case study approach, the design of the study corresponding to the four research questions and the data collection methods that were used. The chapter also discusses the criteria considered to ensure reliability, validity and trustworthiness of this research.

Chapter 4 provides a quantitative data analysis of the learners' responses using table and graphs generated from Excel and the SPSS. I designed research instrument personally.

Chapter 5 provides the answers to the research questions and also makes recommendation on possible future and further study.

## CHAPTER 2

### LITERATURE REVIEW

In this chapter I present results of previous related research studies, which can add insight to the findings obtained in this study. The review focuses first on the subject of Mathematical Literacy, its background and purpose. Thereafter, I look at studies that have been conducted in South Africa on understanding concepts in ML. Since this study is based on strategies used by learners in solving two different forms of equations, it is necessary to review some pertinent studies on errors or misconceptions of learners in algebra as well as strategies employed by learners in various algebraic tasks. This is then followed by studies that focus on the teaching of algebra and arithmetic. Finally, I present the conceptual framework guiding the study.

#### 2.1 Background to Mathematical literacy in South Africa

The advent of democracy in 1994 signalled a new era for South African education system and this impacted heavily on the curriculum. In less than 20 years of new dispensation since 1994, the South African education system has transformed from OBE to curriculum C2005, NCS, and RNCS. Currently, in 2014, the South African education is implementing the new revised system called CAPS for Grade 12.

After 1994, Curriculum 2005 was introduced as a curriculum that was set to improve numeracy levels in South Africa. This was because the country was emerging from a situation where poor quality education resulted in low levels of literacy and numeracy.

In the new era, education was to be oriented to the future needs of learners and students as well as society in general, and as a result, a new system called OBE was implemented to drive C2005, “to ensure that our citizens of the future are highly numerate consumers of mathematics” (DoE, 2003, p.9). However, international studies like the TIMSS showed that

South African learners perform poorly when compared to their counterparts in other countries (Howie, 2004). Therefore, the school subject ML was introduced in SA schools in 2006 from Grades 10 to 12 as an attempt to improve numeracy levels in secondary schools, and to ultimately develop informed citizens. One of the aims of ML is for learners to become participating citizens, contributing workers, and to be able to use mathematics information in everyday experiences to make informed choices. ML was specifically introduced as an intervention to improve the numeracy skills of South African citizens in response to poor performance in mathematics in the past (Bansilal, Mkhwanazi & Mahlabela, 2012). ML is necessary for those learners who do not need to pursue studies in Mathematics at FET band. ML in South African education system therefore defines the subject as:

...a subject driven by life-related applications of mathematics. It enables learners to develop the ability and confidence to think numerically and spatially in order to interpret and critically analyse everyday situations and to solve problems (DoE, 2008a, p.7).

From this quotation, it can be observed that an important element in ML is “mathematics in life related applications”, which implies the importance of contexts in ML.

According to PISA, when learners finish their compulsory mathematics learning, ML will be a subject that may help them cope with mathematical information in a reflective and insightful manner. The term ‘mathematical literacy’ therefore includes aspects of mathematical knowledge.

According to Organisation for Economic Co-operation and Development/ Programme for International Student Assessment (OECD/PISA) ML is defined as, “an individual’s capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual’s life as a constructive, concerned and reflective citizen”(Organisation for Economic Co-operation and Development/ Programme for International Student Assessment (OECD/PISA) 2003, p.24).

This view focuses on helping to improve mathematical literacy skills in order to deal with simple mathematical concepts as found in all facets of our everyday lives, such as for example municipal tariffs and cell phone contracts. ML is still a relatively new learning area in the South African education system.

ML has been a focus in other countries for a long time, however, where it has been referred to by other names such as Quantitative Literacy, Numeracy and Critical Numeracy (Stoessiger, 2002; Steen, 1999; and Madison, 2006). The USA and the UK are some of the countries that have emphasised the role of mathematical literacy skills:

In United States of America, Quantitative Literacy is defined as:

...an aggregate of skills, knowledge, beliefs, dispositions, habits of mind, communication capabilities, and problem solving skills that people need in order to engage effectively in quantitative situations arising in life and work (Madison, 2004, p. 10).

In United Kingdom, numeracy is defined as:

...the ability to process, interpret and communicate numerical, quantitative, spatial, statistical, even mathematical information, in ways that are appropriate for a variety of contexts, and that will enable a typical member of the culture or subculture to participate effectively in activities that they value (Stoessiger, 2003, p.3).

The perspective we can derive from these two quotations is that ML is a subject that entails the use of mathematical tools and resources, together with those from the contextual domain, in order to solve mathematical problems that ought to be interpreted in the context.

## **2.2 Studies Based on Understanding ML Concepts**

Mathematics and ML related to each other, but differed in terms of their nature and their aims (Spangenberg, 2012). Mathematics “enables creative and logical reasoning about problems in Mathematics itself”, which “leads to theories of abstract relations” (DoE, 2003c, p.9). In contrast to Mathematics, ML deals with ‘making sense of real-life contexts and scenarios and mathematical content should not be taught in the absence of context’ (DBE, 2011a, p. 8).

Various studies have investigated both the teaching and learning of algebra practices. These studies help to understand how teachers employ teaching practices.

Since 1994 ML has been a new subject in SA curriculum, therefore it is important to keep asking questions such as, “does student-teacher thinking styles match or mismatch matter in students’ achievement?” Zhang (as cited in Spangenberg, 2012, p. 33).

Spangenberg’s (2012) study characterises and compares the thinking styles of learners taking Mathematics, and those taking ML in Grade 10. Although the Spangenberg article reports on study that was conducted with Grade 10 learners, I am of the view that the recommendations raised with Grade 10 ML learners are also applicable to Grade 11 ML learners. The study asks which thinking styles are associated with learners taking Mathematics, and those taking ML.

Table 2.1 below shows Sternberg’s theory of mental self-management (Sternberg, 1990), which can be used to analyse thinking styles. Furthermore, research conducted by Sternberg revealed that teaching and learning could improve if teachers pay more attention to thinking styles (Spangenberg, 2012).

Category	Style	Characteristic
Functions	Legislative	Likes to create, discover, design; does things using own method; less structure.
	Executive	Likes to follow instructions; does what is requested; structure must be given.
	Judicial	Likes to criticize and evaluate people and things.
Forms	Monarchic	Likes to do one thing at a time; spends almost all the energy on it.
	Hierarchic	Likes to do many things at once; prioritizes what and when to do a thing and how much time and energy to spend on it.
	Oligarchic	Likes to do things at once, but experiences problems with prioritizing.
	Anarchic	Likes to follow an extraordinary approach to problems, hates systems, guidelines and any restrictions.
Levels	Global	Likes to work with the bigger picture, generalizations and abstract.
	Local	Likes to work with detail, specifications and concrete examples.
Scopes	Internal	Likes to work alone; focuses on the inside and is independent.
	External	Likes to work with other people; focuses on the outside and is interdependent.
Learning	Liberal	Likes to do things in a new manner and deviates from traditions.
	Conservative	Likes to do things in a proven and real manner and follows traditions.

Source: Adapted from Sternberg’s Allowing for thinking styles (1994, p. 36-40).

### **Table 2.1: Thinking style dimension.**

Teachers and learners bring individual characteristics and thinking styles to the learning environment (Zhu, 2011). Thinking style preferences lead to learning style preferences, and in turn determine learners’ dominant cognitive modes, that is, the ways in which they communicate and receive information.

According to author (Spangenberg, 2012, p.34), Grade 10 ML focuses more on contexts relevant to daily life, whereas Mathematics require learners be able to think in terms of “symbolic representation or conceptualization”.

There have been few studies that have focused on ML learners’ conceptual understanding. Venkat (2010) has written about the way in which learners’ mathematics proficiency has improved, while engaging in ML activities. In her study, she observed an ML teacher teach six lessons to a Grade 10 class based on a litter project. Venkat used Kilpatrick, Swafford and Findell’s (2001) strands of mathematical proficiency as a framework for the study. Venkat’s stated interest was to explore the extent to which Kilpatrick et al.’s five strands of mathematical proficiency could be useful for understanding the mathematical potential of a subject such as ML. She found that there was evidence in the lessons of useful mathematics that was produced in the course of the ML lessons. Furthermore, she wrote that some of strands that are usually under-represented in mainstream mathematics classrooms, such as “strategic competence, adaptive reasoning and the development of a productive disposition, feature strongly in ML lessons” (Venkat, 2010, p.66). However, Venkat did acknowledge that opportunities for mathematical abstraction and generalisation were often not taken up.

Debba (2012) carried out a study with Grade 12 Mathematical Literacy learners with the purpose of identifying possible factors that influences learners’ success in the provincial preparatory examination paper. The findings revealed that the design of the tasks were sometimes a problem. Some tasks had errors, while with other tasks, the location of the instruction appeared too far from the associated instructions, causing many learners to assume that they did not have the crucial information necessary to answer the questions. Some contexts contained complex information, which confused learners. Learner factors associated with poor performance in the examination included poor conceptual understanding, misconceptions, and language-related misinterpretation. Some of the strategies used by learners in responding to the tasks included: number grabbing; guessing without checking; scanning for crucial information; and making the task easier by making additional assumptions.

The language used to present the information in contextualised tasks has been raised in other studies in mathematics, as well as in ML. Bansilal and Wallace (2008) carried out a study with Grade 9 mathematics learners, which looked at how a classroom teacher, Vani,

mediated tasks from the national assessment tool known as the Common Tasks for Assessment (CTA). All the tasks were set in relation to the context of Robben Island. The learners' language skills were low and this affected their understanding of the contextualised information and the instructions.

Bansilal and Wallace (2008) also found that in presenting too much of unnecessary information about the context, this constituted an information overload, which distracted the learners from picking out the crucial information that they needed. Other findings showed that many learners struggled with the tasks due to a poor understanding of the mathematical concepts that were used in the CTA. The deficit in basic skills and foundational knowledge is referred to as 'gaps in knowledge' by authors. Examples of some knowledge gaps were that learners could not use the ruler accurately; learners omitted numbers that were necessary in calculations; and learners did not understand the meaning of terms such as 'mode'.

Khan's (2009) study was also carried out with Grade 9 learners. The purpose was to explore why learners performed poorly in summative assessments that were in the form of contextualised assessment tasks (CTA). The learners reported that they were frustrated with the form of the tasks, because of the amount of unnecessary information that they needed to work through. They often could not understand the instructions of the task. Learners expressed their dissatisfaction with the amount of language and literary passages that she had to wade through. Another issue that emerged was that the learners were not familiar with the specialised language, such as 'base occupancy' and 'additional person supplements', which were used in the context.

A study by Bansilal, Mkhwanazi and Mahlabela (2012) focused on 108 ML teachers and their interpretation and use of the transfer duty rule, used to calculate the duty payable when somebody buys a house. The study investigated the teachers' facility in solving problems based on using a direct version and an inverse version of the transfer duty rule. A direct problem is one that asks for an output (transfer duty payable), when given the input (cost of the house), and the process. For an inverse problem, the output is given, and the problem could ask for either the input or the process that led to the output. In this study, the inverse problem asks for the input. The findings revealed that 81% of the teachers were able to calculate the transfer duty payable when given the cost of the house, that is, they were able to solve the direct problem.

There were 12 teachers (11%), who failed to solve the direct problem because they misinterpreted the transfer duty rule. In terms of the inverse problem, the success rate dropped dramatically, with only 55% of the teachers being able to calculate the cost of the house (input) when the transfer duty (output) was given. Thirteen teachers did not recognise the mathematical demand, because they were unable to distinguish between a direct and an inverse problem; while 29 teachers recognised the inverse nature of the problem, but did not have the algebraic skill to transform the rule in order to calculate the input.

In another study carried out with ML teachers, Bansilal (2012) used a process-object framework (Sfard, 1991; Dubinsky, 1991) to understand the teachers' varying levels of engagement with the inflation rate signifier. The findings revealed that some teachers were able to move easily between using the inflation rate signifier both as a process and as an object. However, many teachers were able to access a process understanding of the inflation rate concept, but struggled to use the inflation rate signifier as an object. It was also found that some teachers held poor conceptions of the concept of percentage, and that this hindered them from attaining a dual process-object understanding of the inflation rate signifier.

I will now present a review on studies that have looked at learners' conceptions in algebra.

### **2.3 Research Studies on Algebraic Thinking and the Concept of Equation**

A key pillar of this study concerns equations, which forms the second tier of my research. According to Herscovics and Kieran (1980), an equation is an arithmetic identity with a hidden number. A box or letter of an alphabet is used to hide the number and that alphabet is usually called an unknown. Students build equations from arithmetic identities, which makes it easier for them to construct meaning for a new mathematical meaning. According to Herscovics and Kieran (1980), the construction of equations based on arithmetic identities provides students with equations that have a solution. Other equations can be invented which have no solution such as  $2x + 3 = 2x + 4$ . Herscovics and Kieran (1980, p.57) further suggest a solution strategy to solve equation, where the teacher "guid[es] working backwards" in elementary form such as, 'what you do to one side, you have to do to the other side'.

This approach of defining equations is believed to be more accessible to students as it relates closely to sequence of representations, identified by Brunner, (1963) which are as follows:

- enactive (based on concrete operations)
- iconic (involving images)
- symbolic

Nickerson (1985) describes the important role played by the concept of equation in mathematics thus:

*An area of confusion in mathematics that deserves special mention involves the understanding of simple algebraic equations. Equations, which are used to represent quantitative relationships, between two or more variables, are ubiquitous in mathematics and the quantitative sciences. A basic understanding of what an equation is would seem to be fundamental to success in any mathematically oriented endeavor [sic] (Nickerson, 1985, p.216).*

Basic understanding of equations involves understanding the transformation rules and how to apply those rules to solve the problem. These rules are dealt with in the GET Mathematics curriculum. At GET band (CAPS, Grades 7-9, p.129); under functions and relationships, the concepts and skills dealt with are as follows:

- determine input, output values or rules of patterns and relationships using: flow diagrams, tables, formulae, and equations
- determine equivalent forms, interpret and justify equivalence of different descriptions of the same relationship or rule presented: verbally, in flow diagrams, in tables, by formulae, by equations, and by graphs on a Cartesian plane
  - algebraic language, expand and simplifies algebraic expressions, and factorize algebraic expressions
  - solving equations

Learners are therefore required to continue to practice the transformation rules such asrewriting expression, collecting like terms, factoring, literal symbols, expanding, applying the same operation on both sides of an equation etc. Despite the study being focused on ML, an important tool in ML is algebraic thinking, which Ball (2003) describes thus: “algebraic thinking as a tool for representing and analysing quantitative relationships, as a technical

language which makes it possible to model situations as well as formulate and prove general statements. Furthermore, algebra provides fundamental methodology and concepts for several branches of mathematics, thus contributing profoundly to the coherence of mathematical sub-domains” (Ball, 2003, p.345).

Since this research is aimed at answering questions on solution strategies used by Mathematical literacy Learners to solve result-unknown and start-unknown questions, this quotation by Ball (2003) emphasises the importance of quantitative relationships, which serves mainly the construction of the formula in a mathematical scenario or situation. The formula in most instances remains the most important tool that can be used to solve a mathematical problem or situation.

In order to understand quantitative relationships, one needs to understand the parts of a formula including coefficient values, constant values, and the rules of manipulation of the formula. Furthermore, Ball emphasises the importance of algebraic thinking in other aspects of mathematics, and suggests that thinking algebraically leads to systematised thinking. Hewitt (1998) states that working algebraically means “awareness of awareness” (as cited in Madson, Graham, & Johnson-Wilder, 2005, p.309), which means more than simply knowing the terms of the equations to be solved. Awareness of awareness caused Fritzler and Karpinski-siebold (2012) to postulate certain components of primary age algebraic abilities, namely:

- handling operations (as objects) and their inverses (a component that deals with knowledge regarding arithmetic operations and their applications)
- establishing relationships between numbers, sets and relations (relational thinking) is a component that deals rearrangement of arithmetic expressions for an easier subsequent calculation by Warren, 2003.
- generalising (a component that deals with detaching the thought from the concrete of object is significant to algebra) by Blanton,2008
- dealing with changes (a component that deals with a variables as changeable numbers or concepts or quantities) by Zevenbergen, Dole, & Wright, 2004
- dealing with unknowns (a component that includes algebra expressions, perceptions of equal sign, and dual character of terms as processes and products) by Kieran, 2004

ML emphasises the use of contexts and requires algebraic thinking in context, such as that found or depicted in municipal tariffs and telephone contracts, as opposed to algebraic thinking based on traditional mathematics, which focuses on symbolic manipulation and equation solving.

According to Driscoll (1999), algebraic thinking is characterised as thinking about quantitative situations in ways that make relationship between variables obvious. Driscoll (1999) conceptualises algebraic thinking as including the following three habits of mind but in this study, only building rules to represent functions is discussed: building rules to represent function is a habit of mind that enables thinking processes such as:

- organising information
- predicting patterns
- chunking information
- different representation
- describing a rule
- describing change
- justifying a rule

Ideally, in this regard, algebraic experiences need to be designed to allow ML learners to see algebraic thinking as a network of knowledge and skills, rather than as the traditional algebraic thinking that focuses on isolated concepts.

NTCM declares the equal sign as an important *algebraic concept* that learners must encounter and understand in lower grades (NTCM, 2000, p.94). Kieran (1981) identified cognitive problems and stumbling blocks related to the limited understanding of the equal sign ( $=$ ) as a ‘do something’ symbol by students. Kieran states that the use of the equal sign ( $=$ ) in algebraic equations as a symbol for *equivalence* maybe concealing a fairly tenuous grasp of the underlying relationships between the equal sign and the notion of equivalence as indicated by some of the ‘shortcuts’ errors learners make when solving equations. Past studies and other investigations that have been carried out on the equal sign show that the equal sign has been misunderstood as a unidirectional symbol (Essien & Setati, 2006). When using an equation as a tool to solve a mathematical problem, understanding the equal sign as a symbol for equivalence assists in result-unknown variables or start-unknown variables. The

problem is created by the fact that some learners understand the equal sign as a sign that instructs one to do a calculation or to obtain an answer.

Past experiences and previous learning situations involving the equal sign are over-generalised, as the equal sign indicates the need to calculate the answer. Research (Ginsburg, 1977 and Behr et al., 1976) shows the equal sign (=) as a symbol for equivalence to have been less emphasised, or not taught at all. I present two examples of this.

Example one: in one of the studies by (Essien & Setati, 2006), learners were instructed to find the missing number in a box that would make the statement true  $8 + 4 = \Delta + 5$ , most learners responded by giving 12 as the missing number. This was interpreted by the researcher as indicating that learners tend to regard the equal sign (=) as a symbol that instructs a student to compute what precedes it, neglecting the other parts of the statement. Booker (1987) describes this error as a misunderstanding of the equal sign (=) as indicating *compute now*, rather than to the correct meaning of equivalence. Pirie (1998) adds to this by noting that a mathematical symbol has abstract meaning that may be considered absolute. Saenz-Luthlow and Walgamuth (1998) further warn that the acceptance of a symbol by learners is not always accompanied by the absolute meaning it harbours.

Example two, in expressions  $3x + 2$  and  $5x + 3$ , learners were instructed to determine whether the two expressions were equivalent. When students responded to the instruction, according to Liebenberg, Sasman and Olivier (1999), some students at Grade 9 level responded to this instruction by judging algebraic equivalence by first substituting with a particular numerical value in the variable, if they found a value that satisfied both expressions, they concluded the two expressions to be equivalent.

Other sources of learner error with equality, according to Essien and Setati (2006), emanated from over-generalisation of the *commutative law* by learners. For example in:  $4 + 3 = 3 + \Delta$  and  $4 + 3 = \Delta + 3$ , learners were instructed to find the missing number to make the statement true, and learners gave 4 as the missing number. Essien and Setati (2006) assert that this is over-generalised into other cases, such as  $\frac{169}{13} = 13 - \Delta$  and  $\frac{169}{13} = \Delta - 13$ , where it is not applicable, because commutative law is closed under subtraction. For this statement, many learners gave zero as the missing number. Pirie and Martin (2005) describe this error as a stumbling block, brought by introducing a negative sign in an equation. Pirie

and Martin (2005) further mention three other common stumbling blocks associated with solving linear equations with understanding:

- introduction of a multiple number of unknowns in linear equation;
- encountering the unknown on the right hand side of the equation;
- and transition to ‘letters’ for the unknown.

Vergnaud’s (1979, p.264) refers to being able to choose the right arithmetical operation “relational calculus” (“the transformation and composition of relationships given in [a] situation”) whereas carrying out the arithmetical operation is described as “numerical calculus”.

De Lima and Tall (2008) describe in their study an important aspect of symbolic development from process to object, and comment that this relationship is mirrored by a shift in focus from the steps of an action to the effect of the action. Their study (De Lima & Tall, 2008) revealed 5 areas of shift from arithmetic to algebra, as well as changes from the world of conceptual embodiment to the world of symbolism. The 5 areas of shift from arithmetic to algebra are as follows:

- notion of met-before, embodiment in arithmetic and algebra;
- evaluation algebra in equations as operations;
- more complicated equations;
- and symbolic shifting as an embodied action with a difference.

De Lima and Tall (2008) define the ‘met-before’ as a mental construct that an individual uses at a given time based on experiences they have undergone. These mental constructs are what Booth (1981 and 1984) describes as constructions of algebraic notions based on previous experience in arithmetic. Conception in mathematical situations is influenced by previous experiences encountered in arithmetic as well as underlying conceptual embodiment. As a result, a ‘met-before’ will influence the interpretation of the new knowledge in both positive and negative ways. For example,  $2 + 2 = 4$  is an important piece of knowledge, but if a generalisation is made that if you add 2 numbers you will always get a bigger number, this ‘met before’ will not hold if applied to the addition of negative numbers.

De Lima and Tall (2008) found that when students attempted to solve  $3 + 2x$ , they picked 3 and 2 and added 3 and 2 together to get 5, and placed 5 next to  $x$ . Their learners' descriptions of their actions and their strategies prompted de Lima and Tall to deduce that the learners were physically picking up and moving these symbols around, which the authors saw as a form of procedural embodiment. As students attempt to apply their previous experience in arithmetic to solve algebraic equations, Linchevski and Herscovics (1994) on the other hand noted that cognitive readiness to manipulate the numerical parts of the equation revealed misconceptions of the mathematical structure in students.

They describe these misconceptions as *detachment*. Evidence of detachment of a term from the indicated subtraction in an algebraic context is deduced in the following problem:  $4 + n - 2 + 5 = 11 + 3 + 5$ . Most learners who failed to solve this problem had simplified it to  $4 + n - 7 = 19$ . The authors thus deduced that from  $4 + n - 2 + 5 = 11 + 3 + 5$ , students detached 2 from the preceding operation, and computed with 2 it was positive from the given equation where positive.

Linchevski and Herscovics (1994) referred to this detachment as an inability to select the appropriate operations for partial sums. They further outlined the following list as obstacles identified in algebraic contexts:

- failure to perceive cancellation in an expression;
- a static view of the use of the brackets;
- lack of acceptance of the equal sign as a symbol for decomposition,
- incorrect order of operations, and jumping off with the posterior operation.

Linchevski and Livneh (1998) further stated that these misinterpretations could be rooted in arithmetical understanding, and the algebraic contexts is magnifying and revealing their existence.

Some researchers, including Matz (1980) and Lins (1990) claim that learner difficulties in algebra are in part due to their lack of understanding of various structural notions in arithmetic. Matz (1980) used common errors as a basis for building a theory of the way in which high school students solve (correctly or incorrectly) algebra problems. Matz's theory of high school algebra solving distinguishes two components:

- base rules (rule knowledge that the student has already acquired)
- extrapolation technique (methods for bridging the gap between known rules and unfamiliar problems)

There are two extrapolation techniques, namely generalisation and linear decomposition. With regards to generalisation as an extrapolation technique, student over generalise even inappropriate areas e.g.  $(x - 5)(x - 4) = 0$  is solvable because the right-hand side term is zero, but students will attempt to apply same process to  $(x - 5)(x - 7) = 3$  with the right-hand side equal to 3.

Linearity refers to decomposition of a problem into components treating each component independently. Linear decomposition works in the distributive law of multiplication over addition, as in the case  $A(B + C) = AB + AC$ . It works in  $\sqrt{A \times B} = \sqrt{A} \times \sqrt{B}$ , but it is inappropriate in the situation  $\sqrt{A + B} \neq \sqrt{A} + \sqrt{B}$ . Matz has distinguished between errors of extrapolation (planning) and errors of processing (execution). Extrapolation is considered a conceptual error, where errors are defined as being made by those who do not understand the domain. Extrapolation takes place if one attempts to inappropriately apply a technique that is appropriate only in certain limited circumstances. Processing errors are not conceptual errors, but are errors made by skilled problem solvers and novices, which are quickly detected and corrected spontaneously.

Davis (1980) raises some common arithmetic errors made by schoolchildren, and has postulated four specific 'frames' that, in combination, predict these errors.

The primary-grade undifferentiated-binary operation frame:

The operation sign is ignored and the two arguments are always added. The frame is initially acquired because, when a child first learns to add two numbers, addition is the only operation that is taught, so the addition sign is superfluous. The superfluity generalises to other signs that are introduced later.

The primary-grade addition frame:

This frame demands two digits as inputs. Consequently when it is used to do addition of multi-digit numbers trouble is encountered when two numbers do not have the same number of digits.

The primary-grade symmetrical subtraction frame:

The difference between two numbers is found by subtracting, in all cases, the smaller number from the larger number.

The label ‘frame’:

This frame, which is appropriately used to express a relationship such as 12 inches equals one foot, is erroneously used to express the relationship between number of inches in a measurement and number of feet in the same measurement, appropriately expressed as  $I = 12F$ ; but which could easily be erroneously expressed as  $12I = F$ .

Gray and Tall (1994) refer to a power in mathematical symbols to evoke either process or concept, which they coined as *procept*. According to Gray and Tall (1994), the amalgam of a process, a concept output produced by that process, and a symbol that can evoke either process or concept is called *procept*. The procept is in some manner related to the term *concept image*. Concept image consists of “all of the mental pictures and associated properties and process[es]” related to the concept in the mind of the individual (Tall and Vinner, 1981, p.152). The shift from many actions to a single thinkable concept is an act of compression that is an essential tool in making profound sense of mathematics.

The following illustration explains how *procept* is developed, where the act of counting involves pointing at successive objects and saying the number ‘three’. The word ‘three’ is not only a counting word, but also a number concept. By this, the counting process is compressed into the concept ‘three’, for example three things. If, for instance, I count, one, two and three, ‘three’ takes the form of a counting word, but ‘three’ is also a number concept. Counting in this instance refers to the ‘process’, and ‘three’ at the same times is a number and refers to the number as a concept. We conceive of numbers as mental entities that maybe operated upon. For example, with ‘ $2 + 3 = 5$ ’, the addition of two numbers begins as ‘count-all’ involving three counting stages: ‘count one set, count another set, and put them together and them all’. This action can be compressed at certain stages, for example, the first number is taken as the starting value and the second is used to count-on to obtain the result. According to Thurston (1990) compression is at the heart of human thought in general, and of mathematical thinking in particular. Learning occurs by making new connections in the brain. More powerful connections are made using words and symbols to focus on relevant aspects of experience that become ‘thinkable concepts’ used in ‘thought experiments’ to imagine new

possibilities and to grow in sophistication. This process of making links leads to a compression of knowledge from a complicated phenomenon to rich concepts.

Results may be committed to memory to give ‘known facts’, for instance  $2 + 3 = 5$ . Five becomes a ‘known fact’ and from this ‘known fact’ it can also be deduced that  $2 + 2 = 4$  is one less, namely 4.

This supposed link between algebra and arithmetic, as noted by many other authors such as Booth (1984) and Lins (1990), is also highlighted by Carracher & Schliemann (2007) when stressing the importance of continuity between arithmetic and algebra. These authors suggest that many problems that students experience with learning algebra are caused by earlier difficulties with arithmetic. Linsell (2009) has shown a strong association between equivalence and inverse operations and solving equations by formal methods.

The misconception between arithmetic and algebra is also noted by Steinberg et al. (1992), when they outline the incorrect reasons given by students when they are instructed to judge the algebraic equivalence of two equations. For example, students were given two equations  $3x = 5 + 4$  and  $3 + x = 5 + 4$  and asked to judge algebraic equivalence of the two equations. One of the reasons put forward by students for equivalence was a transformation of  $3 + x$  into  $3x$ .

## **2.4 Issues Related to the Teaching of Arithmetic Algebraic Equations**

Algebra is a generalised arithmetic of numbers and quantities, in which the concept of function assumes a major role. Algebra emerged as a generalisation of arithmetic. Arithmetic deals with operations involving particular numbers, variables and functions. Arithmetic focuses upon number facts, computational fluency and word problems involving particular values. Later, arithmetic then transits to letters used to stand for any number or sets of numbers. This sharp demarcation leads to considerable tension along the frontier between arithmetic and algebra. Many mathematics education researchers have researched this area of transition (amongst others Filloy & Rojano, 1989; Herscovics & Kieran, 1980, and Blanton, 2008). Transition between arithmetic and algebra is thought to occur during a period in which

arithmetic is ending and algebra is beginning, i.e. during a transitional or prealgebra. Pronounced difficulties in learning algebra or elementary algebra by mathematicians have been reported by a number of authors (Booth, 1984; Kuchemann, 1981; Sfard & Linchevsky, 1994; Steinberg, Sleeman, & Ktorza, 1990, and Wagner, 1981). Prealgebra is an area that has become an area of importance when it comes to helping students to bridge this transition gap, and it is the area with many implications for the teaching and learning of algebra.

Attributions of developmental constraints have been researched by many authors. Conceptual difficulties involved in learning algebra are widespread in the transition phase (Wagner, 1977). This has prompted authors such as Vergnaud (1988) to propose that to ameliorate this transition, algebra should start at elementary level in conjunction with arithmetic, so as to better prepare students to deal with epistemological issues involved in the transition from arithmetic to algebra.

According to Herscovis and Kieran (1980), the way schools introduce and teach algebra is by linking it to arithmetic e.g.  $2x - 3 = 5$ . This ( $2x - 3 = 5$ ) is verbalised as ‘twice something take away three makes five’. The image of ‘=’ is sufficient to deal with problems like  $2x - 3 = 5$  i.e. the (=) is sufficient as ‘indicating the result of an operation’. The introduction of linear equations in the form  $5 = 2x - 3$  leads to a different verbalisation of  $5 = 2x - 3$  as a meaningless arithmetic statement such as: ‘five equals two times something, take away three’. Arithmetic involves computational processes; while algebra involves potential evaluation processes, but manipulable concepts. Since algebra is introduced through linking it to arithmetic, it is unnatural that pupils seek to transfer the associated linguistic understanding. School textbooks are designed such that algebra is introduced through arithmetic. Kaput (2008) considers that traditional approaches, that treat arithmetic and algebra as separate subjects, are dysfunctional because arithmetic just focuses on computation while algebra is taught in a superficial way that leads to high failure rate. The title in Brekke’s (2001) paper, “School algebra: Primarily manipulations of empty symbols on a piece of paper?” sums up the situation for many students’ difficulties and misconceptions about algebra.

Pirie and Martin (2005) have raised three ways in which teachers introduce the solutions linear equation:

- to relate it to a real life situation –starting with a life situation and drawing the mathematics out of it

- to introduce materials to model the problem – starting with mathematics and illustrate it in some way through objects or pictures
- to teach those algorithms by which mathematically formulated problems may be solved – staying with mathematical notions as such, and finding generalisable methods of a solution

Steinberg et al. (1990) suggest that students are taught procedural steps for solving equations, but that the concepts underlying these operations and equivalent equations are not always stressed. They concede that learners master the procedural details of solving equations without understanding why they are justified. This is a procedural type of understanding in learning mathematics that Skemp (1976) has described as instrumental understanding, e.g. by memorising a procedure or formula, complete with inner ‘incantations’ that support the procedure. Steinberg et al. (1990) conducted a study that was aimed at examining algebra students’ understanding of equivalent equations, where the participant students were required to justify their answers. Many researchers have attributed many of the fundamental difficulties experienced by novice algebra students to their failure in identifying ‘all the equivalent forms of the expression’.

Learner justifications in this study likewise revealed important information about how students understand algebra, particularly the misconceptions that underpinned incorrect justifications such as:

- comparing only one side of the two equations and ignoring the other side;
- assuming that  $a + x = ax$ ;
- expecting to see the same numbers in the transformed equation as in the original equation;
- believing that subtracting a number from both sides of the equation resulted in non-equivalent equations, because the transformed equation has “less” “-4 on each side is subtracting 4 twice”;
- using surface reasons based on external syntactic clues such as “it looks longer,” or “one equation has more numbers,” or “it looks different”;

- committing an arithmetic error, usually when computing a solution for an equation.

These results are important as they indicate a pre-algebraic cognitive obstacle. The lists of incorrect justifications indicate and emphasise that students constructs their algebraic notions on the basis of their previous experience in arithmetic. Learner's algebraic system inherits structural properties associated with the number system. Booth has claimed that failure in understanding algebraic structure underpins difficulties in understanding the arithmetical structure. On the other hand, Martz and Lins have noted that those students who show difficulty learning algebra evidence a difficulty with the transition from arithmetic to algebraic formalism.

The National Council of Teachers of Mathematics (1989) has called for the “move away from a tight focus on manipulative facility to include a greater emphasis on conceptual understanding” (p.150). Nathan and Koedinger (2000) have meanwhile cited Riley, Greeno and Heller (1983) to state that algebraic problem-solving difficulty is strongly affected by the role or position of the unknown quantity within the problem statement. In an algebraic equation, there are two cases i.e. result-unknown questions and the start-unknown questions. Nathan & Koedinger (2000) provide the following example as a result-unknown question, solve for  $x$  in:  $\frac{(R81,90-R66)}{6} = x$  or, expressed verbally, “starting with R81, 90, if I subtract R66 and then divide by 6, I get a number. What is it?”

For start-unknown questions, they provide the following example: solve for  $x$  in  $x \times 6 + R66 = R81,90$ ; or expressed verbally as: “starting with some number, if I multiply it by 6 and then add R66, I get R81, 90. What did I start with?”

According to Riley et al. (1983), there is a general pattern of problem solving performance favouring result-unknown questions over start-unknown questions. Nathan & Koedinger (2000) investigated the difficulty-factor analysis (DFA) affecting students' problem solving difficulties with verbal stories with contexts, word equations with no context, and symbolic equations, along with two placements of the unknown quantity i.e. the result-unknown question or start-unknown question.

The DFA revealed that learners exhibited a lower performance on start-unknown questions than they did on result-unknown questions. The position of the unknown on the question had a significant effect on problem difficulty. Furthermore, the formats in which a

mathematical problem was presented also affected the problem difficulty. Students' success rate was then significantly higher when solving symbolic equation problems.

Symbolic equations are described as number sentences. Word-equation problems were found to be equal in difficulty. Word-equation problems verbally describe the relationship among pure quantities, with no story contexts. Furthermore, symbolic equation problems were found to be difficult to solve, either with story problems, or with word equations.

Nathan & Koedinger (2000) suggested that firstly, there could have been an inhibiting effect of the symbolic format that burdens the student's cognitive resources for arithmetic reasoning on the arithmetic equation. Contrary to word equation, they speculate that there may have been a facilitating effect of verbal product that lessened the demands of the start unknown structure found in algebra story and word-equation problems.

Their study further revealed three types of standard school taught strategy. These were arithmetic methods, algebraic methods, and guess and test strategies, respectively. In addition, for these strategies they found evidence of an informally adopted and invented strategy, which they called unwinding.

The arithmetic strategy was used overwhelmingly in solving result-unknown problems. For the word story problems, learners however preferred the use of the unwinding strategy. For word equations, students preferred the unwinding and guess and test strategy. In the study, when students respond to algebraic equations, they tended to stay within algebraic formalism and to apply symbol-manipulation methods, or to use the 'guess and test' method.

The unwinding and guess-and-test methods showed correct solutions when they were applied to start-unknown questions. Symbol manipulation was effective about 50 percent of the time.

## **2.5 Conceptual Framework**

A key pillar underpinning my study is the *inversion* principle in mathematics reasoning. After outlining this here, I present the framework devised by Nathan and Koedinger that is used to analyse the data in the study.

### 2.5.1 Inversion in mathematical reasoning

Greer (2011, p.1) has written of inversion that it

...is a fundamental relational building block both within mathematics as the study of structures and within people's physical and social experience, linked to many other key elements such as equilibrium, invariance, reversal, compensation, symmetry and balance.

Notable in this statement is the concept of *reversal*, which affects a wide range of situations modelled by arithmetic operations. Furthermore, inversion is a central structural element within mathematics. According to Greer, there are four important manifestations of inversion and they are as follows:

- The four basic operations and the growth of the number systems – i.e. disequilibrium – is a driver of cognitive growth
- Inverses in the solution of certain algebraic equations and inverse functions
- A characteristic feature of modern mathematics is the study of abstract structures, defined by certain properties, such as in addition, where it is always true that  $x + y = y + x$ , then the group is called commutative
- Inverse processes, such as the inverse to the multiplying of  $n$  factors  $x \rightarrow a_1, x \rightarrow a_2, \dots$  where  $x \rightarrow a_n$  is the factoring of a polynomial of degree  $n$ .

Out of the four important manifestations of inversion, one directly relates to this study and is used to write the results of the study:

- *inverses in the solution of certain algebraic equations and inverse functions*

If a learner reasons that  $6 + 2 = 8 \rightarrow 8 - 2 = 6$  (or  $8 - 6 = 2$ ), then that learner implicitly invokes an algebraic principle that  $a + b = c \rightarrow c - b = a$  (or  $c - a = b$ ), and similarly with multiplication and division. Equations of the form  $y = mx + c$  which form the bases of this study are related to this algebraic principle

e.g.  $6 + x = 8$  and  $x + 2 = 8$ . A characteristic feature of solving start-unknown problems is related to the understanding of the principle of inversion.

Nunes et al (2009) have stated that the teaching of arithmetic must take into consideration two different questions about the types of *cognitive* demand that is placed upon the learners when solving the kind of mathematical problems that require skill or understanding, such as inverse relations.

The two cognitive demands are: being able to carry out arithmetical calculations efficiently and accurately; and knowing the right arithmetical operations to use in order to solve a particular mathematical problem.

Vergnaud (1979) refers to these two cognitive demands as numerical calculus and relational calculus, respectively. Numerical calculus refers to operations on the numbers to find sum, product, difference and quotient. Take for example, the statement ‘I have 24 francs and I spent 6 francs. How much do I have now?’ In the context of this study, numerical calculus relates largely to solving result-unknown question.

Relational calculus means “the transformation and composition of relationships given in [a] situation” (Vergnaud, 1979, p. 264). For example, in a problem like ‘my grandmother has just given me 6 francs and I now have 24 francs. How much did I have before?’, according to Vergnaud, to solve this problem the learner must consider what transformations took place before grandmother gave 6 francs, and what transformation would bring the money back to its original state. This scenario relates to my study where I examine solving start-unknown problems. The inverse addition-subtraction relation computation prowess proves a further important key pillar in solving start-unknown questions.

Selter et al. (2011) introduced two models of subtraction that show the relevance of the inverse relation between addition and subtraction i.e. “taking away” (ta) and “determining the difference” (dd). For example,  $7 - 3 = x$  is a subtraction problem, in a take away (ta) mode, however, the same problem can be solved in another way, such as  $3 + x = 7$ ; and this becomes a subtraction problem in “determining the difference” (dd) mode.  $7 - 3 = x$  is subtraction problem in a subtraction format, while  $3 + x = 7$  is a subtraction problem in an addition format. This second mode (dd) is important and has relevance to the inverse relation between addition and subtraction.

This inverse relation between addition and subtraction is important for my study, especial when learners are required to solve start-unknown problems. Nunes et al. (2009) have stated that learners require relational calculus to solve start-unknown story problems. To solve mathematical problems sometimes requires the interpretation of mathematical operations in different ways, such as (ta) and (dd) modes.

Torbeyns et al. (2009) refer to the models of subtraction, but introduce a concept called *indirect addition*, which is a skill of using addition to solve subtractions problems. For example,  $71 - 69 = ?$  To get the answer to this problem, it is easy to count up from 69 to 71 and find the answer, which is 4.

The learner, instead of subtracting, will add from 69 upwards. This is the skill that Torbeyns et al. (2009) have called indirect addition to solve subtraction problems. Selter et al. (2011) also referred to this as application of strategy. During solving or manipulation of a linear problem, which constitute part of my study, learners deal with computation problems such as  $9 = x - 7$ . Understanding the inverse relation between addition-subtraction becomes important, especially when manipulation changes to  $9 + 7 = x - 7 + 7$  i.e. transpose and change sign. This is related to my study, because when learners transpose and change sign, they are actually attempting to solve the problem by unwinding. It is important also to note that these two abilities can be involved simultaneously in solving the same problem.

Groetch (1999) has described the result-unknown question as direct problem and the start-unknown question as an inverse problem. In multiplication, given two numbers we find their product, which is classed as a direct problem. The corresponding inverse problem is to find a pair of factors of a given number. The prevailing paradigm in direct problems may therefore be described symbolically as follows:

INPUT	→	PROCESS	→	OUTPUT
x		[ P ]		?

Inverse problems are thus immediately suggested by every direct problem. One is the causation problem, where given a model P and an effect y, the task is to find the cause of P effect. This is the inverse problem and is described symbolically as follows:

INPUT	→	PROCESS	→	OUTPUT
?		[ P ]		y

In the elementary multiplication problem, multiplication of two numbers is considered direct problem and the factorisation is then considered an inverse problem.

I now present more specific details of the conceptual framework, based on the work of Nathan and Koedinger (2000).

### **2.5.2 Strategies used to solve Result-Unknown and Start-Unknown questions (Nathan and Koedinger, 2000)**

Further refinement to conceptual framework is drawn from the study by Nathan & Koedinger (2000), which allows the researcher to analyse the written repossess of 340 learners. Researchers reviewed a study by Nathan & Koedinger (2000) in which the relative effects of certain factors on algebra and arithmetic problem-solving difficulties were considered. From their study, the problem format (story or word or symbolism) in which a mathematical problem is presented, contributes to problem difficulty (Baranges et al., 1989; Carpenter et al., 1980 and Carraher, Carraher, & Schlieman, 1987). In this study, I looked particularly at:

- Learner solution strategies of symbolic equation problems;
- And the position of the unknown quantity in a symbolic equation problem.

A symbolic equation problem is described as a number sentence, and according to Nathan and Koedinger, the positioning of the unknown quantity in a symbolic mathematical problem can be two-fold, i.e. a result-unknown symbolic problem. This is a mathematical problem in which the unknown quantity is the result of the events or mathematical operations described in the problem.

The start-unknown symbolic problem on the other hand is a mathematical problem in which the unknown value refers to the quantity needed to specify a relationship (Carpenter et al., 1994; Riley et al; 1983). Start-unknown questions subvert simple modelling and direct calculation approaches of arithmetic problems, and they often require algebraic methods or sophisticated modelling (Hall, Kibler, Wenger, & Truxaw, 1989). Furthermore, start-unknown questions are considered algebra level problems, because they can be solved through the

application of standard algebraic procedures. In my research, I am looking at strategies used by learners to solve these problems i.e. algebraic strategies, informal or formal.

With regards to learner solution strategies, Nathan & Koedinger revealed four major learners' solution strategies:

- i. guess and test method or substitution
- ii. unwinding method
- iii. arithmetic method
- iv. algebraic method

### ***Informal arithmetic and formal strategies***

Nathan & Koedinger distinguish between formal algebraic strategies and informal arithmetic strategies. Arithmetic and algebraic strategies are strategies that are taught as a matter of course in schools. Formal algebraic strategy is also termed symbol manipulation, however, it refers to the manipulation of variables in the equation, until the unknown is isolated and then calculated. The guess and test and unwinding methods are informally adopted and invented strategies.

#### *Guess & test method*

Guess & test is an informal strategy using procedure in a forward manner to solve algebra word equation iteratively once a value has been substituted for the unknown.

e.g.  $4x + 17 = 77$

test 1:  $x = 10$  does not fit,

test 2 :  $x = 20$  does not fit again

test 3 :  $x = 15$  is a solution to the problem

The formal strategy that corresponds to guess and test would be 'the solutions of  $4x + 17 = 77$  are defined as those numbers that can be inserted to make the equation true.'

*Covering-up*

Covering-up is an informal strategy. The following example illustrates the ‘covering-up’ method:

$$4x + 17 = 77$$

$$4x = 60$$

$$x = 15$$

To solve this equation, the learner starts by taking away 17 on both sides of the equation to make the equation easy, and then divides by 4 on both sides of the equation and final  $x = 15$ . The formal strategy that corresponds to ‘covering-up’ is ‘performing same operation on both sides’ as follows:

$$4x + 17 = 77$$

$$4x + 17 - 17 = 77 - 17$$

$$4x = 60$$

$$\frac{4x}{4} = \frac{60}{4}$$

$$x = 15$$

*Unwinding method*

Unwinding method untangles the quantitative relations of an algebraic problem by inverting the mathematical operations and the order of quantitative constraints (Nathan & Koedinger, 2000, p.5). The unwinding method circumvents the use of equations or symbolic placeholders for unknown quantities, but finds the learner operating directly on numbers in a computational way.

For the solver, the unwinding method maybe undertaken verbally, or through written work. When using unwinding method, the solver unwinds the mathematical relationships step-by-step; in addition to that, the solver systematically transforms a multistep problem into a smaller one step problem, unwinding a single operation at a time. Furthermore, the learner uses arithmetical skills in the unwinding procedure, using the *inverse* operations of those given in the original problem. The formal strategy that corresponds to the unwinding strategy

is ‘transposing.’ The strategy of ‘transposing’ and ‘same on both sides’ is based on the following transformation:

$$A = B \leftrightarrow A + C = B + C$$

The following example illustrates this strategy:

$$4x + 17 = 77$$

$$4x = 77 - 17$$

$$4x = 60$$

$$x = 60 / 4$$

$$x = 15$$

Nathan and Koedinger study revealed that algebra story problems elicited the unwinding strategy, whilst algebra word equations elicited either an unwinding or a guess and test method. As far as symbolic equations were concerned, 30% of learners never responded to symbolic problems and those who responded applied formal algebraic methods. The unwinding and guess and test strategy had a 70% success rate on start-unknown problems. Formal algebra methods such as symbol manipulation were effective at a 30% rate on start-unknown problems.

Formal algebraic strategy is also termed symbol manipulation, as it refers to the manipulation of variables in the equation until the unknown is isolated and then calculated.

### *Symbol Precedence View*

Nathan & Koedinger explored and investigated the relationship between teachers’ and researchers’ predictions about the development of learner algebraic reasoning. The study revealed that teachers and researchers *tend to think* that learners first develop symbolic skills in arithmetic, with instruction and a practice focused on solving result-unknown problems. Teachers and researchers tend to think that once learners acquire or develop symbolic skills in arithmetic, the next step is to learn to apply and extend these skills to other general families of problems such as start-unknown problems either verbally or symbolic. New procedures, concepts and laws will therefore be introduced to support conventional algebraic manipulation skills. Algebraic formalism is introduced after the symbolic form is developed.

Nathan & Koedinger described this trend or line of thinking in the development of learner algebraic reasoning by teachers and researchers as emanating from a *symbol precedence view*. For example, if teachers and researchers within a symbol precedence view are required to make a decision on the level of difficulty of a problem among the three formats of presentation i.e. word or story or symbol, they would tend to base their decision on how far along the developmental trajectory they are from symbolic arithmetic skills.

According to the authors, the symbol precedence view has caused certain teachers and researchers to tend to think that learners develop their algebraic reasoning by first acquiring symbolic arithmetic. Furthermore, the symbol precedence view has caused some teachers and researchers to tend towards the perception that learners are likely to find story or word problems much more difficult than symbolic problems. Textbook content planning further fosters symbol precedence view.

Textbook first presents arithmetic computations in symbolic form, and is then followed by application of these procedures to other problem formats such as verbal or word problems (Groenewald et al, 2012). According to this line of thinking, a learner first develops symbolic skills, and these skills will be extended and applied in other formats of problem presentation. Nathan & Koedinger, however, has proven the contrary, i.e. that learners find story or word problems much easier than they do symbolic problems.

The analysis of learner problem-solving process data in their study suggests an alternative trajectory for the development of learner algebraic reasoning, contrary to that suggested by other teachers and researchers. This learner trajectory shows that learners attempt to circumvent the difficulties of symbolic algebra. This alternative view is characterised by verbal competence and reliance on guess-and-test and unwinding strategies, and other informal strategies. From this trajectory, Nathan & Koedinger hypothesised that formats like word or story problems precede symbol manipulation skills for both arithmetical and algebraic problems.

As a result of these two alternative views, two competing views emerged from learners' performance data and from teachers' and researchers expectations of development of learner algebraic reasoning. These two views have given rise to two competing models.

The first model is suggested by teachers, researchers and textbook content planning arrangements, presenting that learners' symbolic reasoning skills develop first, with other formats (word or story) of problem-solving ability developing later. This hypothesis has been termed the symbolic precedence model (SPM). The second model is based on an analysis of learners' performance data, and it hypothesises that other formats of problem presentation such as word or story precede symbolic reasoning. This hypothesis is therefore called verbal precedence model (VPM). If the learner is competent in symbolic arithmetic, but not in further skills, and the learner cannot solve verbal arithmetic problems, the learner is considered to fit the SPM. On the contrary, if the learner is competent in verbal-arithmetical problem solving, but not with further skills, the learner cannot solve symbolic-arithmetic problems, and is considered to fit the VPM.

## CHAPTER 3

### METHODOLOGY

This section discusses the purpose of study and key research questions, contexts of study, research paradigm of study and research methodology. Furthermore, it describes the selection of participants, data collection procedures, data analysis procedures, issues pertaining to validity and trustworthiness, as well as ethical issues and limitations of the study.

#### 3.1 Research Purpose

This study explores Grade 11 ML learners' strategies that they use to solve result-unknown questions and start-unknown questions set in real life contexts. The underlying assumption in this study is that learners lack a careful plan for solving result-unknown questions and start-unknown questions. It is held that identifying learners' strategies will lead to a better understanding of why learners opt for certain strategies. This understanding will provide an opportunity for teachers to re-assess their teaching methods so as to meet the needs of the learners. Solving a mathematical problem forms the basic foundation of mathematics learning. Therefore, this study seeks to explore those strategies learners ordinarily use to solve mathematical problems in these selected schools.

The research question is thus formulated as below:

*How do Grade 11 ML learners respond to Result-Unknown and Start-Unknown questions set within various contexts?*

There are four sub-questions that have been drawn from the main research question and these are:

1A .How do Grade 11 ML learners respond to the five Result-Unknown Questions, and what are the strategies that they use to solve the problems?

1B. How do the learners respond to the five Start-Unknown Questions, and what are the strategies that they use to solve the problems?

1C. Which of the two types of Questions (Result-Unknown or Start-Unknown) do learners experience as more difficult?

1D. What are learners' perceptions about which questions were more challenging and why they were experienced as more challenging?

### **3.2 Research Design**

A research study is always premised on a paradigm. A paradigm describes distinct concepts and thought patterns to which the researcher may subscribe. According to (Voce, 2004), as cited by (Mahlabela, 2012. p.33:

Paradigm is a framework within which theories are built, that fundamentally influences how the researcher sees the world, determines his perspective, and shapes his understanding of how things are connected.

From the above quotation, a paradigm is a framework containing the basic assumptions, ways of thinking, and methodology that are commonly accepted by members of a scientific community. Therefore, the research paradigm provides a conceptual framework for the researcher, under which to both use and to make sense of the phenomena under exploration.

In this study of exploration of strategies used by Grade 11 ML learners to solve result-unknown questions and start-unknown questions, a constructivist paradigm has been adopted to interpret learner strategies. The goal is to understand a situation as participants construct it; what Patton (1991) describes as a constructive inquiry paradigm. A constructive perspective in education is a view that explains how knowledge is constructed in the human mind when information comes into contact with existing knowledge that had been developed from experience. The strategies adopted by the learners in trying to solve the two types of problems are based on their own knowledge constructions. I therefore conduct this study from a constructivist perspective.

Research design refers to the systematic, theoretical analysis of the methods applied to a field of study. This study utilises a qualitative research design, while also employing a case study approach, described by Denzin as “[the] study [of] things in their natural settings, attempting to make sense of, or to interpret, phenomena in terms of the meanings people bring to them”(1994).

According to Denzin (1994) a qualitative research design gathers data in the natural setting or naturalistic way. A natural setting or naturalistic way implies that the data is collected without interfering with or altering the research setting. In this instance, the learners were administered a test in their classroom, and test was suited to the subject ML which they were studying. Grade 11 ML learners are familiar with test writing, thus data collection using class tests is effective in representing their way of working.

Cresswell (2008) recommends for the use of a qualitative research design that the key research question(s) begins with the formulation of a ‘what’ or a ‘how’ question. The key research question of this study has been formulated as a ‘how’ question, and therefore, this study utilises a qualitative research design. Furthermore, this study intends to research a phenomenon in its natural settings, where Neuman (2011) recommends the use of natural settings as opposed to contrived settings, and asserts the importance of using qualitative research design, as does Cresswell (2008). Participants in this study are researched in their own natural school setting. An important aim of qualitative research is to elicit the perspective of the participant, a method that Schmid (1981) promoted, describing qualitative research as the study of the empirical world from the viewpoint of the person under the study.

In addition to qualitative research design, this study uses the case study approach, which as Yin has noted that

...the case study is an empirical inquiry that investigates a contemporary phenomenon in depth and within its real-life contexts, especially when the boundaries between phenomenon and context are not clearly evident (2009, p.18).

This above definition by Yin emphasises the importance of the use of the case study as a research method, where he can be seen to note that it involves “investigating a phenomenon within its real-life contexts”. In this study an attempt is made to conduct research by collecting data without contrivance or resorting to manipulating the participants, but where attention is paid to research the phenomenon in as naturalistic a way as possible within real-life contexts. Here the specific phenomenon under scrutiny is the strategies used to solve problems set within a variety of contexts, where ML learners are attempting to solve these ML tasks. A case study is a research study conducted in order to obtain an in-depth understanding of a particular situation. Cohen, Manion and Morrison (2007) describe a case study as a “naturalistic inquiry” that undertakes “an investigation into a specific instance or real phenomena in its real life context” (p.170). The concept of natural settings in a definition of qualitative research by (Denzin, 1994) in the first paragraph and the concept naturalistic inquiry in a definition of case study by Cohen et al. (2007) indicates the point of contact between qualitative research design and the case study approach.

This point of contact indicates that the case study strengthens the qualitative research design. According to Thomas (2011), case studies involve analyses of persons, events, decisions, periods, projects, policies, institutions, or other systems that are studied holistically by one or more methods. The subject of the enquiry is an instance of a class of phenomena that provides an analytical frame-an object-within which the study is conducted and which the case illuminates and explicates.

The research object in this study is Grade 11 ML learners, and the study seeks to explore the strategies that they use to solve result-unknown problems and start-unknown problems set in a real life context, as highlighted by researchers (Cohen et al., 2007 and Nieuwenhuis, 2007). A case study as a research strategy is an empirical inquiry that investigates a phenomenon within its real-life context. The qualitative research design aspect of the test entailed the use of the individual interviews, which took into consideration the

perceptions and experiences of the learners. Perceptions and experiences led the learner in their choice of strategies to be used.

### 3.3 Selection of the Sample

The research was conducted using seven secondary schools of diverse background. All seven schools included learners coming from disadvantaged backgrounds. The following table gives a detailed assessment of each school under ten headings. The letters (y) and (n) refer to yes and no respectively:

	No. of gr. 11	Quintile	Laboratory	Library	Computer	Medium of instruction	Fees per year	District	Location	Pipe water
School 1	45	3	Y	Y	N	English	R3000	Pinetown	rural	Y
School 2	63	2	N	Y	N	English	R1500	Pinetown	urban	Y
School 3	59	2	N	N	N	English	R200	Pinetown	rural	Y
School 4	48	4	N	N	N	English	R140	Pinetown	rural	Y
School 5	25	2	N	N	N	English	none	Pinetown	rural	Y
School 6	26	2	N	N	N	English	none	Pinetown	rural	Y
School 7	25	2	N	N	N	English	none	Ilembe	rural	Y

**Table 3.1: Contexts of the schools in which the study was undertaken.**

The summary in the table indicates the contexts of the school in which the study was carried out. As can be seen in Table 3.1, three are non-fee paying schools, only two schools

have libraries, and the only urban school is that with a science laboratory. The other six schools are rural schools.

Thus, the setting of the study may be seen to be typical of the ordinary schools in KwaZulu-Natal, which are largely rural, with many learners coming from disadvantaged backgrounds.

This study utilises qualitative research design and case study approach therefore an appropriate method for sampling of participants in this study is purposeful sampling.

Aggarwal has written that

...purposeful sampling is a form of non-probability sampling in which decisions concerning individuals to be included in the sample are taken by the researcher, based upon a wide variety of criteria that may include specialist knowledge of the research issue or capacity and willingness to participate in the research. (2013, p.27)

The above emphasises that purposeful sampling entails selecting subjects with a specific reason, or with a particular purpose in mind. According to Paton (1990), in purposive sampling subjects are selected due to particular characteristics that they might possess. The chosen participants in this study are Grade 11 ML learners. They were selected according to these criteria because the Grade 11 ML curriculum requires that they must have acquired strategies to solve result-unknown questions, and that start-unknown questions set in real life contexts in the previous grades. Similarly, schools were selected by making a mix of urban schools, semi-urban and rural schools. Table 3.1 provides a contexts of these schools.

A further reason for focusing on ML learners is the fact that Grade 11 ML learners would have last received instructions and practice on formal algebraic manipulation and equation solving skills at the Grade 9 level. Although the solution of start-unknown problems could be obtained without using algebraic techniques, learners who are able to use algebraic manipulation techniques will be able to reach the solution more easily and efficiently. It is important to note that the Grade 10 ML syllabus includes, among other strategies, formal algebraic procedure as methods or strategies to use to solve these problems. Therefore, this study seeks to explore the strategies used at Grade 11 level to solve these problems.

Furthermore, Grade 11 ML learners were chosen because I noted a disturbing trend of a high failure rate in solving start-unknown questions at the Grade 12 level. Therefore, Grade 11 was selected in order to explore what strategies they are using at grade 11, probably, they can shed light on why the performance is low at grade 12.

In total, there were 340 participants from seven schools in North Durban. The selection of participants for the interview was done after the initial analysis of the written responses to the test items. The learners were then selected for the interviews based on their solution strategies, mathematical reasoning, insight, thought processes and understanding depicted in a particular solution. During the analysis process, certain patterns of common errors or answers were identified, and learners were also selected for interview in such instances. Ten learners were interviewed.

### **3.4 Data Collection**

Since this study utilised a qualitative research design, it was important that the data provided information about the participants' realities as revealed during "social interaction and interpretation" (Neuman, 2011, p.402). In line with this, the methods that were used for data collection included document analysis and interviews. To collect data, the participants wrote an hour and a half long test under controlled conditions. The test was administered to learners during the second term of the academic calendar year. The test consisted of five tasks (or items), each of which had three questions.

Additional data was looked at through individual semi-structured interviews. The interview questions were designed to further probe the strategies that the learners used. Table 3.2 shows the key research questions of the study. Column 2 then juxtaposes the data collection instrument to the key research question in Column 1 is envisaged to answer.

How do Grade 11 ML learners respond to Result-Unknown and Start-Unknown questions set within various contexts?	
1	2
Research Question	Data Collection Method
1A. How do Grade 11 ML learners respond to the five Result-Unknown Questions, and what are the strategies that they use to solve the problems?	Analysis of the learner responses to the test items. Semi-structured interviews with ten learners to the test items.
1B. How do Grade 11 ML learners respond to the five Start-Unknown Questions, and what are the strategies that they use to solve the problems?	Analysis of the learner responses to the test items. Semi-structured interviews with ten learners to the test items.
1C. Which of the two types of questions (results-unknown or start-unknown) do the learners experience as more difficult?	1. Analysis of learner written responses to the reflection questions. 2. Semi-structured interviews with ten learners.
1D. What are learner perceptions about which questions were more challenging and why they were experienced as more challenging?	1. Analysis of learner written responses to the reflection questions. 2. Semi-structured interviews with ten learners.

**Table 3.2: Details of study**

### 3.4.1 Research instrument

The primary research instrument was a specially designed to test consisting of a series of tasks constructed and assessed by drawing upon the revised Blooms taxonomy (Anderson, 2005); which describes the general format of assessment tasks. Table 3.3 shows the structure of the assessment task:

Introductory material		
1. Written	2. Pictorial	3. Real Object
Stem		
1. Question	2. Incomplete Statement	3. Directive
Response		
1. Short Answer  Supply (fill in the blanks)  Select (multiple choice)  Matching / True-False	2. Extended Response  Written  Performance	

**Table 3.3: Structure of the assessment task (Anderson, 2005, p.108).**

According to Anderson, 2005, a task or question should consist of three elements, namely: introductory material, the stem and the response. The introduction is the information or knowledge area about the question. The stem is the actual question requiring an answer. The response is the answer provided by the learner.

In this study, one of the research instruments was a test, which contained five tasks (or items). The tasks were presented in written form and set within a real life context, which was explained in the introductory material. All four tasks with the exception of question 1.2.1 and 1.2.2, were based on an algebraic equation of the form  $y = mx + c$ . Each of the tasks posed two problems, the first of which was a result unknown problem, followed by a start-unknown problem, followed by a third reflection question. The introductory material in each case was in written form, and included a formula presented in symbolic form (Task 1); in a flow diagram (Task 2); symbolic form (Task 3); a table and equation (Task 4).

### Task One

Table 3.4 shows Task One, a *contextual task* of the form  $y = mx$ , with four problems. The first question and the third question are result-unknown questions respectively, the second and the fourth question are start-unknown questions respectively, while the third and sixth question are both reflection questions. The reflection questions required the learner to indicate with reason which of the preceding two questions (result-unknown question and the start-unknown question) was the most difficult.

1. The cost of a cell phone call from a phone on ENG pre-paid to a cell phone on ZZZ cell service provider during peak times is given by the equation:

$$C = R2,75 \times t.$$

Where  $C$  represents Cost (in rands) of a call and  $t$  represent time in (minutes) spent on call.

1.1.1 Find  $C$  if  $t = 87$  minutes. Show all working details.....(2)

1.1.2 Find  $t$  if  $C = R487.32$ . Show all working detail.....(3).

1.1.3 Between 1.1.1 and 1.1.2 above which question was harder for you to work out. Why do you say so?.....(3)

1.2 It costs Mrs Khoza R1050,63 in petrol costs to drive from Durban to Johannesburg. If she is the only person in the car, then she will have to pay the whole R1050.63, if there is another person in the car with her, then each of them will have to pay R525, 32, if there are three people in the car, then each person will have to pay R35, 21 and so on. This relationship is linked by the formula:

$$C = \frac{R1050,63}{P}$$

where  $P$  represents the number of passengers in a car.

1.2.1 Calculate how much will each person pay if there are 5 people in the car?.....(2)

1.2.2 Calculate how many people are in the car if it costs each person R150.09 to pay for petrol?.....(4)

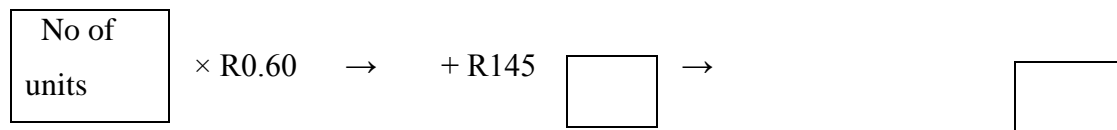
1.2.3 Between 1.2.1 and 1.2.2 above which question was harder for you to work out. Why do you say so?.....(3)

**Table 3.4: Task One.**

## Task Two

Table 3.5 shows Task Two which is a *contextual task* of the form  $y = mx + c$  presented in the form of a flow diagram. Like all four tasks, task four carried both result-unknown and start-unknown questions, along with a reflection question.

2. The city, works out the amount electricity usage due by using the formula outlined below: You take the number of units and multiply it by R0.60 and then you add R145 to get the total amount.



Work out how much is owed by Miss Khoza, if she used:

2.1 735 units .....(2)

2.2 How many units did she use, if her bill (amount due) was R415? Explain how you get your answer:.....(4)

2.3 Between 2.1 and 2.2 above which question was harder for you to work out? Why do you say so? .....(3)

**Table 3.5: Task Two.**

### Task Three

Table 3.6 shows Task Three, which is a *contextual task* of the form  $y = mx + c$  presented in the form of a formula. Like all other tasks, Task Three carried two questions i.e. result-unknown and start-unknown questions, along with a reflection question.

3. Although in South Africa we measure temperature in degrees Celsius (C), many countries measure temperature in degrees Fahrenheit (F). If you are a South African travelling overseas, it can become confusing to try to work out the temperature. Similarly, for an overseas tourist visiting South Africa it can be equally confusing.

The equation  $F = (1,8 \times C) + 32$  represent the relationship between temperature measured in degrees Fahrenheit (F) and degrees Celsius (C). Use this equation to answer the following questions.

3.1 Find F if C = 83. Show all working details. .... (2)

3.2 find C if F = 93. Show all working details. ....(3)

3.3 Between 3.1 and 3.2 which question was harder for you to work out. Why do you say so? .....(3)

**Table 3.6: Task Three.**

### Task Four

Table 3.7 shows Task Four, which is a *contextual task* of the form  $y = mx + c$  accompanied by the table of values (input and output). Like all the four tasks, tasks four carried two questions i.e. result-unknown, start-unknown and a reflection question.

4. Ms. Adams a teacher at Sunset High School in Ladysmith is organizing an educational excursion to Durban for grade 12 learners. The distance from Ladysmith to Durban is 255km. The excursion will take one day. Ms. Adams receives a quotation from a bus company (Company A). The cost is worked out by multiplying the number of kilometres travelled by R1,96 and adding a flat rental fee of R200.

The table 1 shows the cost for different distances travelled, with some details omitted.

**Table 1: Cost of hiring a bus from Company A**

<b>no. of km travelled</b>	0	63	177	366	489	592	<b>B</b>
<b>Cost(C)</b>	R200	R323.48	R546.92	R917.36	R978	<b>A</b>	R1376

4.1 Find **A** on table 1 using formula: **Cost = R1.96 × no. of km + R200**. Show all your calculations .....(3)

4.2 Find **B** on table 1 using formula: **Cost = R1.96 × no. of km + R200**. Show all your calculations .....(4)

4.3 Between 4.1 and 4.2, which question was harder for you to work out? Why do you say so? .....(3)

**Table 3.7: Task Four.**

### 3.4.2 Interviews

Interviews are a form of data collection that provide insight when using the case study or the qualitative research design. I collected data about participants using participants and direct observation and interviews. A great deal of qualitative material in a case study comes from interviews, which have the potential to reveal an additional depth to the reality of the situation. According to Cohen et al. (2000), an interview can be described as “a two person conversation initiated by the interviewer for the specific purpose of obtaining research relevant information” (p.269). Through the interview, I was able to discover subjects’ meanings and understanding. Cohen et al. (2000) state that “one advantage of interviews is that it allows for greater depth than is the case with other methods of data collection” (p.269). Through interviews, I was able to probe the participants on their written responses and participants themselves acquired the opportunity both to pose questions and to reflect.

The interview questions were semi-structured questions. Struwig and Stead (2001, p.99) state that, “by imposing predetermined questions on the participant, the interviewer does not allow the participant to express his or her opinions or views freely”. I therefore chose to use semi-structured interview questions to further analyse learner thought processes on the strategies that they had applied. Before each interview, learners were made aware that the interview would be recorded, and that each learner was also handed a copy of his or her responses to the test to refresh memory. Each interview was scheduled for 15 minutes per learner, and each learner was chosen to be interviewed for a particular strategy that they displayed; however, other interviews lasted longer than the scheduled time, because the participants were initially shy. Since they had not been in such a position before, it was not easy to solicit the desired data. I first elicited by asking them general questions about ML, and thereafter once they were comfortable, I continued to probe their responses. Semi-structured interview questions were used to probe learners on their strategies.

In order to allow learners to voice their opinions freely without fear and prejudice during the interview session, the interviews were conducted on one to one basis. The interviews were audio taped recorded on an audio data capturer, for further analysis and for future reference.

### **3.4.3 Field diary**

A further source of data was a field diary that I kept, where I made notes while watching the learner write the test, as well as during their interviews. The field notes were used to support the data obtained from the interviews and test.

## **3.5 Data Analysis**

Data analysis is a process of inspecting, cleaning, transforming and modelling data with the goal of discovering useful information. Dey (1993, p.30) describes data analysis as “a process of resolving data into its constituent components in order to reveal its characteristic elements and structure”. In a similar manner, the 340 learners’ response were broken down into

constituent parts, reflecting different strategies. This was done in order to classify and make connections across the data elements (Henning, 2004, p.128). The responses were coded, which means representing “the operations by which data are broken down, conceptualised, and put together in new ways” (Strauss & Corbin, 1998, p.120), in order to view the responses in terms of strategies as well as to identify the extent to which they were able to recognize the differences between start-unknown and result-unknown questions. Such an analysis took place on two levels, namely at the level of the actual words and symbols used by the respondents in their written responses as well as in the interview responses, and then at the level of interpretation of these words by myself as the researcher. Initially, based on the conceptual framework, I drew up descriptions of possible categories, representing the range of strategies that were expected. However, in the actual document analysis process, it was found that not all of the expected strategies were present. Furthermore, there were also some strategies that were identified that were not in the original list of categories. I was then engaged in an inductive process of moving from the written data to the interview to the initial categories, then back to the data, in order to identify and refine descriptions of further categories. After a period of time, final categories were then identified.

With respect to the result-unknown questions, the strategies that were used were:

- *Direct arithmetic*

This strategy refers to the substitution and calculation of the unknown value in the result-unknown question. When confronted with the result-unknown question, learners usually find the solution by carrying out purely arithmetical calculations using numeric operations to find the result.

With respect to the start-unknown questions, the following strategies were identified:

- *Symbol manipulation*

The symbol manipulation strategy entails working algebraically by isolating the unknown variable using inverse operations and finding the variable by purely mathematical calculation.

- *Systematic guess and test*

This is a method of finding answers to start-unknown problems by simple guessing a value for the variable in the equation and substituting that value into the given equation, and engaging in a series of guessing and testing attempts in order to determine if the selected

value makes the equation true. If the equation becomes true at a particular value, that value is therefore considered to be the solution to the problem.

- *Swopping of variables*

This is the strategy of swopping around the unknown and the given variable in a start-unknown problem, thereby (incorrectly) converting the problem into a result-unknown calculation, and then carrying out a direct arithmetic calculation, which we refer to as a swopping variables-direct calculation.

- *Number grabbing*

Schönfeld (1988) was the first to coin the phrase ‘number grabbing’. Number grabbing occurs when learners are confronted with a question they do not understand. They then resort to number grabbing to enable them to perform calculations and obtain an answer.

- *Working backwards*

In this method the learner ‘works backwards’ from the givens of the problem and ‘unwinds’ or undoes the imposed quantitative constraints in order to isolate the unknown.

- *Conjoining*

This is a tendency of learners to ‘simplify’ expressions such as ‘ $3x + 4$ ’ as equal to  $7x$ . Learners are trying to make the expression ‘ $3x + 4$ ’ as simple as possible, by disregarding the notions of like and unlike terms, i.e. learners are eager to ‘finish’ the expressions or to close the expression, and remain with one single term.

- *Working with numbers only*

Working with numbers refers to the learner work where the learner chooses to work with numbers only in the mathematical problem, and to suspend the variable for while. The variable is recalled when the answer has been found.

### 3.6 Validity and Trustworthiness

Trustworthiness is an important issue in qualitative research design and case study because subjective meanings and perception are critical issues in these two approaches. Anderson and Arsenault (1998) have written:

the internal validity of qualitative research [...] comes from keeping meticulous records of all sources of information used, using detailed transcripts, and taking field notes of all communications and reflective thinking activities during the research process (p.134)

Anderson refers to keeping meticulous records for internal validity of qualitative research. In order to comply with Anderson's recommendations, all records are readily available when required. The 340 learner scripts that were analysed to compile this research, as well as the original copies of the test interview schedule are accessible, where electronic copies of the instruments can be found in the appendices of this thesis. Audio copies as well as transcriptions of the one to one interview sessions are also available, along with other documents, such as extracts of documents from learners' responses; mark sheets with learners' marks; extracts of scanned material; verbatim quotes and other transcripts to support document analysis; as well as the spreadsheet used to during coding of data.

An important component of establishing validity in qualitative research is that of triangulation, which has been explained as follows:

To minimize the degree of specificity of certain methods to particular bodies of knowledge, a researcher can use two or more methods of data collection to test hypotheses and measure variables; this is the essence of *triangulation* (Frankfort-Nachmias and Nachmias, 1996, p. 206, cited in Reidar and Mosvold, 2005).

Triangulation refers to eliciting information by means of a number of different devices, so as to be more confident of accuracy. According to (Guba & Lincoln, 1989) triangulation refers to the use of multiple sources of data or multiple methods in order to confirm findings. Triangulation involves the use of several methods to explore an issue in order to increase the chances of both depth and accuracy. To ensure triangulation in this study, the qualitative research design strengthened a case study approach. While qualitative research design and a case study approach both have their own limitations, in order to ensure

and to guarantee triangulation, this study further uses three different methods of data collection. These are analysis of learner written responses, interviews and a field diary, respectively. In this study, triangulation was carried out by using the results of the interviews to support the results obtained from the document analysis of the written responses. Another area in which triangulation was utilised was in the responses to the third reflection question of each task. Often learners' responses here guided the research, confirming the strategy that they had adopted in solving the problem during the test.

Guba (1981) suggests a model that identified *truth value*, *applicability*, *consistency*, and *neutrality* as four criteria applicable to the assessment of qualitative research. Truth value checks whether the researcher has established confidence in the truth of the findings for the subject or informants, and contexts in which the study was undertaken. Applicability refers to the degree to which the findings can be applied to other contexts and settings. Consistency of the data refers to whether the findings would be consistent if the inquiry were replicated with the same subjects, or in similar contexts. Neutrality refers to the degree to which findings are a function solely of the informants and conditions of the research, and not of other biases, motivations and perspectives.

Guba's model further defines different strategies that are used in assessing these criteria. These strategies are credibility, transferability, dependability and confirmability. Table 3.8 below shows how this research addresses these criteria. Yin has written that "any finding or conclusion in a case study is likely to be much more convincing and accurate if it is based on several different sources of information, following a corroboratory model" (1994, p.92).

As this study uses qualitative research design and case study, especial case study it is important to use many different types of data collection in order to ensure validity and trustworthiness. In this study, three different types of data collection methods were used, namely: analysis of learner scripts, interviews and observation.

<b>Strategy</b>	<b>Criteria</b>	<b>Application</b>
Credibility	Triangulation, field journal Interviews technique, member checking, peer examination	Here I used multiple sources of data such as my field journal, interview responses and written responses analysis. Member checking and peer examination was carried out by checking with my supervisor about emerging hypotheses, and the strength of claims being made by the study.
Transferability	Dense description	Here I used verbatim quotes from interviews as well as extracts of participants' written responses. Thus the dense descriptions can allow other researchers to judge whether my methods and findings could be applicable to other situations.
Dependability	Triangulation, dense description method, stepwise replication, peer examination	Semi-structured interviews, document analysis, written responses and readily available transcripts provide a dependability audit that was used to construct a stepwise replication, which is available per peer examination.
Confirmability	Triangulation, confirmability audit, reflexivity	Semi-structured interviews, document analysis, and written responses also provide evidence of triangulation and act as a form of confirmability audit.

**Table 3.8: Criteria for trustworthiness in this study.**

### **3.7 Ethical Considerations**

Soltis has written that “researchers should observe the ‘non-negotiable’ of honesty, fairness, respect for persons, beneficence” (1989, p.129). This view guided ethical considerations, which included a letter written to the Head of the Department of Basic Education in

KwaZulu-Natal, requesting permission to conduct a study with the seven KZN schools, which was duly granted (see **Appendix C**). Letters were also written to the school principals, subject teachers, parents and learners, requesting permission to conduct the study with their learners and personnel and on their premises. Copies of these letters are readily available in appendices **A-D** of this thesis.

The letters explained clearly that participation in the study was voluntary, and that participants were at liberty to withdraw at any stage of the research if they so wished. Participants were also guaranteed that their written responses and names would be kept confidential. Anonymity was also guaranteed, by ensuring that participants that their names would never be revealed in any reports about the study. The study was conducted taking into cognisance the fact that it ought not to affect or interfere with learner academic work. The study was conducted after the principal, subject teacher, learners and parents had signed letters of consent to participate in the study; and the Faculty of Education of the University of Education of the KwaZulu-Natal issued an ethical clearance certificate, Ethical Approval Number HSS/0428/012M (see **Appendix D**).

### **3.8 Limitations of the Study**

During the interview sessions, some learners were unable to express themselves clearly in English and as a result, it was difficult for me to ascertain the reason why they chose certain strategies during the test. Furthermore, other learners when asked about certain strategies they used to solve problems, where they responded by saying that they had simply guessed. Others responded by saying that they could not remember what they did to solve a problem.

Other limitations of the study could arise due to the fact that learners knew that the test was not going to contribute to their own school assessments, which may have caused many learners not to have been motivated to do their best in providing answers to the responses. Related to this, some learners may have copied, not having taken the test conditions seriously.

## CHAPTER 4

### 4. DATA ANALYSIS

In this chapter, I present a detailed description of the strategies applied by learners to solve result-unknown questions and start-unknown questions, as revealed by data analysis. Some of the strategies applied by learners are ineffective strategies, and do not lead to a solution of the problems; nevertheless, learners continue to apply these strategies regardless. I conclude this chapter by providing a detailed analysis of learner reflections on the degree of difficulty between the result-unknown questions and the start-unknown questions.

#### 4.1 Description of Strategies Applied by Learners to Solve Result-unknown Questions

Below a list of some of the strategies I found in learner responses from the result-unknown questions, accompanied whenever possible by a sample of a learner's responses that illustrates that particular strategy.

##### 4.1.1 Direct Arithmetic Strategy

*Direct arithmetic strategy* as a strategy refers to the substitution and calculation of the unknown value in the result-unknown question. When learners are confronted with the result-unknown questions, learners usually find the solution by carrying out purely arithmetical calculations using numeric operations to find the result.

Question 4.1							
Table 1: Cost of hiring a bus from Company A							
Distance(km)	0	63	177	366	489	592	<b>B</b>
Cost(C)	R200	R323.48	R546.92	R917.36	R978	<b>A</b>	R1376

4.1 Find A on table 1 using formula:  $\text{Cost} = \text{R}1.96 \times \text{no. of km} + \text{R}200$ . Show all your calculations. (3)

$$A = 592 \text{ km} \times \text{R}1.96 + \text{R}200$$

$$= 1360$$

$$\therefore A = 1360$$

**Figure 4.1: A response illustrating the direct arithmetic strategy (321).**

As is expected when using a direct arithmetic strategy, in question 4.1, the learner correctly substituted 592 into the correct formula for the number of km and carried out purely arithmetic calculations. This method is predominant with result-unknown problems.

It is important to note that whilst learners attempted to apply a direct arithmetic strategy, they sometimes made manipulative or calculation errors. Hence, some of these errors are classified under direct arithmetic strategy, and include the following categories: incorrect substitution, incorrect formulae, and calculation errors. These are described in the paragraphs that follow.

4.1.1 A: Under *incorrect formulae*, I considered those responses of learners who have used a formula, which is evident but it may be incomplete, perhaps having the first steps only e.g. in Question 1.1.1, where Learner 013 wrote  $87 \div 2.75$ .

4.1.1 B: *Incorrect substitution* refers here to an attempt by a learner to solve a problem using direct arithmetic strategy, where the correct formula is used but the learner has substituted incorrect value.

4.1.1 C: Under the category *calculation errors*, I consider those learner responses where a learner has attempted to apply the direct arithmetic strategy, used the correct formula, and made the correct substitutions, but then made a calculation error. These

responses are classified under calculation errors e.g. in Question 1.2.1, where Learner 318 wrote  $R1050.63/5 = R21.12$ .

#### 4.1.2 Number grabbing

This strategy is evident in both result-unknown and start-unknown questions, and this is described in detail under section 4.2.6.

#### 4.1.3 Unclear answers

Learner responses were coded as unclear if the learners' intention could not be inferred from the response, or if there was not enough information in the response.

#### 4.1.4 No response

This category is used only where the learner has made no attempt at providing a solution to the problem and the answer space is left blank.

## 4.2 Description of Strategies Applied by Learners to Solve Start-unknown Questions

Below is a list of some of the strategies I found in learners' responses from start-unknown questions, and they are accompanied whenever possible by a sample of a learners' responses that illustrates that particular strategy.

### 4.2.1 Conjoining

Some learners who incorrectly solved question 3.2 exhibited an error called *conjoining*. This involves the tendency of learners to 'simplify' expressions such as ' $3x + 4$ ' as  $7x$ . In trying to make the expression such as ' $3x + 4$ ' as simple as possible, the learners disregard the notions of like and unlike terms, because they are eager to 'finish' the expressions or close the expression with a single term. Tirosh et al. (1998) have referred to this tendency as an over-arching urge to do something with ' $3x + 4$ '. According to Tirosh et al. (1998) this tendency emanates from previous experiences in arithmetic learning, or to the tension between the process and process facets of mathematical concepts. Figure 4.2 shows

an example of a learner response that falls into the conjoining category. As it can be observed from the learner response, in step 3, the learner worked with numbers only. He added 1.8 to get 33.8 and to conclude the calculation, the learner placed or attached the variable  $c$  next to 33.8 to give 33.8c.

### Question 3.2

The equation  $F = (1,8 \times C) + 32$  represent the relationship between temperature measured in degrees Fahrenheit (F) and degrees Celsius (C). Use this equation to answer the following questions. (Some details of the question are omitted).

3.2 find C if  $F = 93$ . Show all working details. (3)

$F = (1,8 \times C) + 32$

$93 = (1,8 \times C) + 32$

$93 = 1,8c + 32$  N.B, Please explain to me how you got

$93 = 33,8c$  33,8

$33,8c$  33,8

$2,75 = c$

The image shows a student's handwritten work on lined paper. The student starts with the equation  $F = (1,8 \times C) + 32$  and substitutes  $F = 93$  to get  $93 = (1,8 \times C) + 32$ . They then write  $93 = 1,8c + 32$ . A handwritten note says "N.B, Please explain to me how you got". Below this, they write  $93 = 33,8c$  and  $33,8c$  with a red circle around  $33,8c$  and the number  $33,8$  written next to it. Finally, they write  $2,75 = c$ .

**Figure 4.2: A response illustrating the ‘conjoining’ strategy (L317).**

Conjoining is an incorrect mathematical strategy as it can be seen in learner calculation above from step 3 to step 4. Learners use this strategy repeatedly when attempting to solve start-unknown questions, and in the case of this study, especially the question

$$F = (1.8 \times C) + 32.$$

#### 4.2.2 Working Backwards

Working backwards (Brown et al., 1989) as a strategy refers to the unwinding method described in the study by Nathan and Koedinger (2000), where learners systematically reversed each step or solved the equation by using inverse operations.

In this method, the learner ‘works backwards’ from the givens of the problem, and ‘unwinds’ or ‘undoes’ the imposed quantitative constraints, in order to isolate the unknown. The *unwinding method* untangles the quantitative relations of an algebra problem by inverting the mathematical operations and the order of quantities” (Nathan & Koedinger, 2000, p.5).

In this study, working backwards refers to the traditional algebraic approach such as transposing terms, while maintaining a balanced equation. Working backwards in this study is therefore regarded as formal procedure, described by Setler et al. (2011) as ‘transposing’ which, uses inverse relations of operations. Most learners who produced correct answers in this study drew upon this strategy. Figure 4.3, shows a learner response with a working backwards strategy.

Question 4.2

Table 1: Cost of hiring a bus from Company A. (Some details of the question are omitted).

Distance(km)	0	63	177	366	489	592	<b>B</b>
Cost(C)	R200	R323.48	R546.92	R917.36	R978	<b>A</b>	R1376

4.2 Find **B** on table 1 using formula:  $\text{Cost} = \text{R}1.96 \times \text{no. of km} + \text{R}200$ . Show all your calculations (4)

B.  $\text{Cost} = \text{R}1.96 \times \text{no. of km} + \text{R}200$

$1376 = (\text{R}1.96 \times \text{B}) + 200$

$1376 - 200 = \text{R}1.96 \times \text{B}$

$1176 = \text{R}1.96 \text{B}$

$\text{R}1.96 \quad \text{R}1.96$

$\text{B} = 600 \text{ km}$

**Figure 4.3: A response illustrating a ‘working backwards’ strategy (319).**

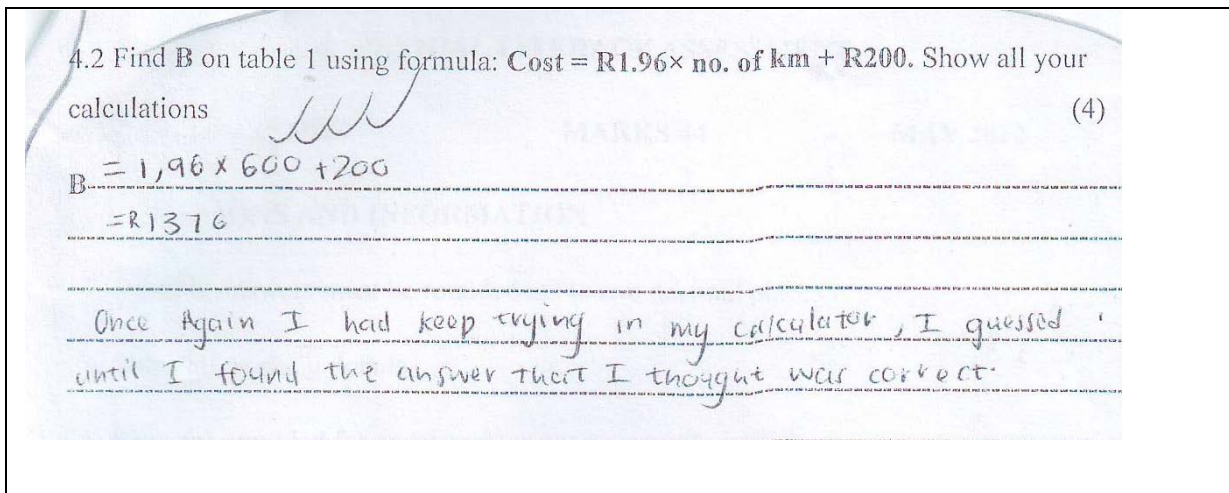
As depicted in the learner calculation in Figure 4.3, the learner made a correct substitution for the cost, which is 1376 in the second step. In the third step, the learner attempts to isolate the unknown variable by applying the inverse of the operation of addition, namely subtraction. In step 4, the learner further attempts to isolate the unknown variable by carrying out division as the inverse operation of multiplication. Finally, the unknown variable B is isolated and calculated to be 600.

### 4.2.3 Systematic Guess and Test

Nathan and Koedinger (2000) describe *guess-and-test* as using arithmetic procedures in a ‘forward’ manner to solve algebra word problems iteratively, once a value has been substituted for the unknown quantity. It is a strategy of finding answers that is dominant with start-unknown problems, by simple guessing a value for the variable in the equation, by substituting that value into the given equation, and by engaging in a series of guessing and testing attempts to determine if the selected value makes the equation true. If the equation becomes true at a particular value, that value is therefore considered to be the solution to the problem. The guess and test in this study is regarded as systematic, because it largely depends on the preceding result-unknown question.

Figure 4.4 shows a learner response that attempts to apply the systematic guess and test strategy. The learner’s response to the reflection question confirms that she has used the guess and test strategy by engaging in a series of guessing and testing moves, until she arrived at a close enough answer.

Question 4.2							
Table 1: Cost of hiring a bus from Company A. (Some details of the question are omitted).							
Distance(km)	0	63	177	366	489	592	<b>B</b>
Cost(C)	R200	R323.48	R546.92	R917.36	R978	<b>A</b>	R1376



**Figure 4.4: A response illustrating systematic guess and test (326).**

In this study, systematic guess and test is also characterised by a single correct answer that is not supported by calculations, but is found to be correct with start-unknown questions.

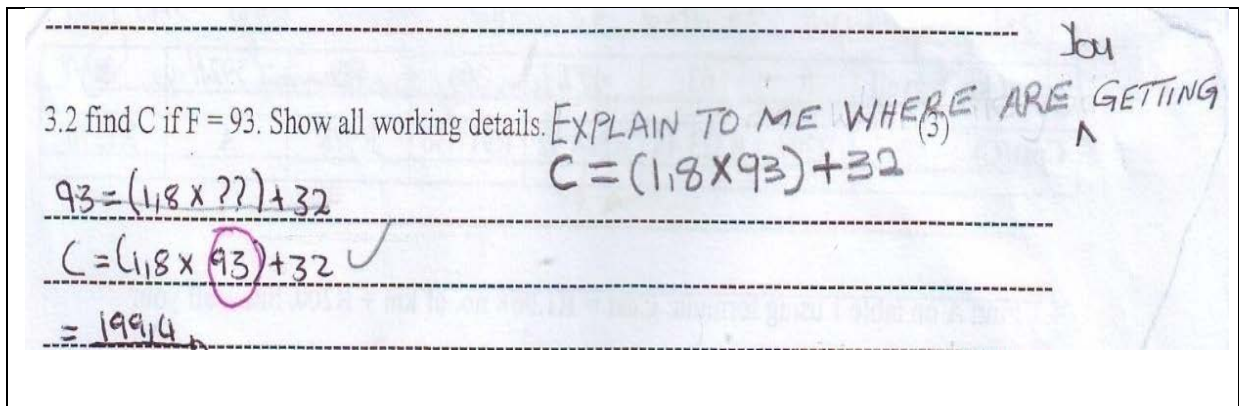
#### 4.2.4 Swopping of Variables

Swopping of variables is an incorrect strategy of swopping around the unknown variable and the given variable in a start-unknown problem, thereby (incorrectly) converting the problem into a result-unknown problem and then carrying out calculation as a direct arithmetic calculation, which is referred to in this study as *swopping variables-direct calculation*.

Figure 4.5 shows the work of a learner applying this strategy. Here we can see that the learner has changed the subject of the formula by directly swopping the value of C, which is the unknown variable, with the known variable by just placing C on the LHS of the equation. Thereafter, he has substitute the value for F, and then continued with the calculation as if it was a direct arithmetical calculation.

#### Question 3.2

The equation  $F = (1,8 \times C) + 32$  represent the relationship between temperature measured in degrees Fahrenheit (F) and degrees Celsius (C). Use this equation to answer the following questions.



**Figure 4.5: A response illustrating a ‘swopping variables-direct’ calculation (318).**

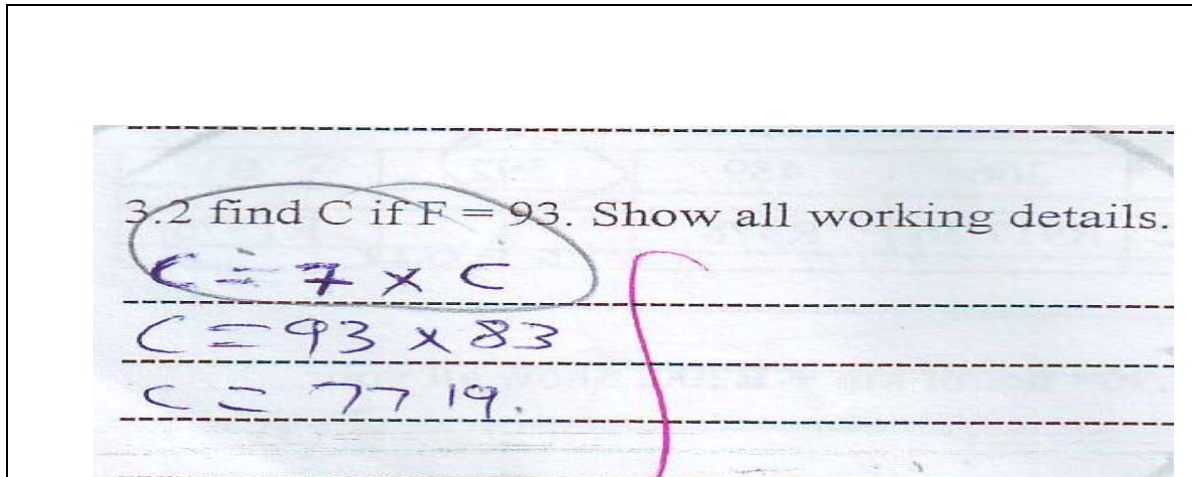
#### 4.2.5 Number Grabbing

Schönfeld (1988) was the first to coin the phrase *number grabbing*. Number grabbing occurs when learners are confronted with a question they seem not to understand. They then resort to number grabbing to enable them to perform calculations and to obtain an answer. Characteristics of number grabbing are such that the learner will grab a number appearing from the instruction or a number from the preceding instructions or answer, and will combine these numbers using various operations that seem to be arbitrary, and present that as a solution to the problem.

Number grabbing was particularly evident when learners attempted to answer the start-unknown question. Figure 4.6, shows a learner who applied the strategy of number grabbing. As it can be seen from the learner response, it appears that the learner grabbed 93 from the instruction, while 83 is the number appearing from the preceding instruction, and multiplied these numbers.

#### Question 3.2

The equation  $F = (1.8 \times C) + 32$  represent the relationship between temperature measured in degrees Fahrenheit (F) and degrees Celsius (C). Use this equation to answer the following questions. (L313) (Some details of the question are omitted).



**Figure 4.6: A response illustrating ‘number grabbing’.**

As it can be seen from the learner response in figure 4.6, it would appear that the learner grabbed 93 from the instruction, while 83 is the number from the preceding instruction, and then proceeded to multiply the two numbers.

#### 4.2.6 Working with Numbers only

According to Lincheski et al (1998), *working with numbers only* refers to the learner calculations, where the learner chose to ignore the unknown variable in the equation and to work with numbers only in the whole mathematical problem, thereby suspending the unknown variable throughout the whole of calculation. In other words, the required or the unknown variable never appears, but is recalled only as a final answer. The unknown variable is then recalled at a final stage of the calculation, when the answer has been found, and it is made equal to the answer that was found. Conjoining (strategy 4.2.1) is similar to working with numbers, except that the variable is not ignored, but is carried along in all the steps, until the final answer. Figure 4.7 shows learner calculation with the application strategy ‘working with numbers only’.

#### Question 3.2

The equation  $F = (1, 8 \times C) + 32$  represent the relationship between temperature measured in degrees Fahrenheit (F) and degrees Celsius (C). Use this equation to answer the following questions.

3.2 find C if F = 93. Show all working details. (3)

$$93 = (1.8 \times C) + 32$$

$$\frac{93}{1.8} = \frac{1.8 \times C}{1.8} + \frac{32}{1.8}$$

$$= 0.167 + 32 ?$$

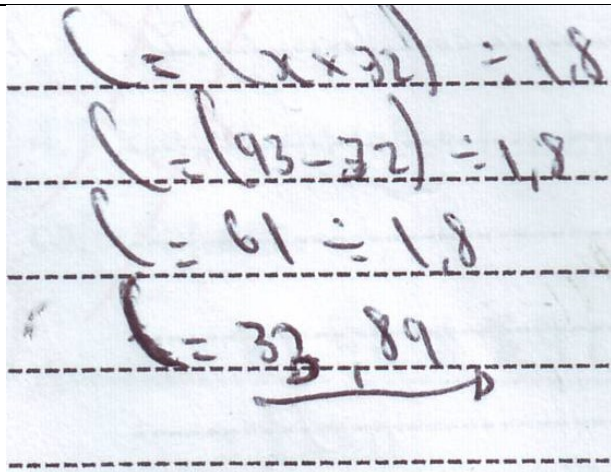
$$C = \underline{83.67}$$

**Figure 4.7: A response illustrating working with numbers only (315).**

As shown in figure 4.7, the learner manipulated the numbers and disregarded the variable as if it had never existed. The unknown variable is then recalled as the answer is found. Figure 4.7 shows the learner has completely ignored the variable until the end, when it appears as the subject of the formula.

#### 4.2.7 Symbol Manipulation

Symbol manipulation is a strategy involving the manipulation of symbols, at first in order to change the subject of the formula to the one that is required. Once that has been achieved, the learner would then substitute the given values. Although the ‘working backwards’ strategy may be seen as similar to the ‘symbol manipulation’ strategy, where the former is easier, because it involves identifying the operation and then reversing it, while all the time operating with numbers and not with variables. With the symbol manipulation strategy, a purely symbolic manipulation method would have involved working algebraically to derive a new algebraic expression. For example in Question 3.2, changing the subject of the formula from F to C would mean writing C in terms of F, that is  $C = (F-32)/1.8$ . Figure 4.8 illustrates a response of a learner who starts his solution with the new formula, and thereafter substitutes the value of F=93 to get his answer of 33.89.



Learner 156

**Figure 4.8: Symbol manipulation.**

#### 4.2.8 Incorrect formula

In the start-unknown problems, the strategy incorrect formula refers to instances where learners have changed the given formula and then made a substitution. For example in Question 3.2, a learner who wrote  $93(1.8 \times 83)$  was coded as incorrect formula.

#### 4.2.9 Incorrect substitution

Here learners have used the given formula as if this were a result-unknown question, but learners substituted the result that was given in the position of the start-unknown, then it was classified as swopping variables, which is described in 4.2.4.

#### 4.2.10 Unclear answers

In this category, the learners' intention could not be inferred from the response, or there was not enough information in the response e.g. "C cost 75 minutes", taken from Learner 005.

#### 4.2.11 No response

This category is considered only where the learner has made no attempt at providing a solution to the question; the answer space is left blank.

### 4.3 Result-unknown Questions

I now present the results of each of the result unknown questions. In each subsection, I first present the details of the task.

#### 4.3.1 Question 1.1.1

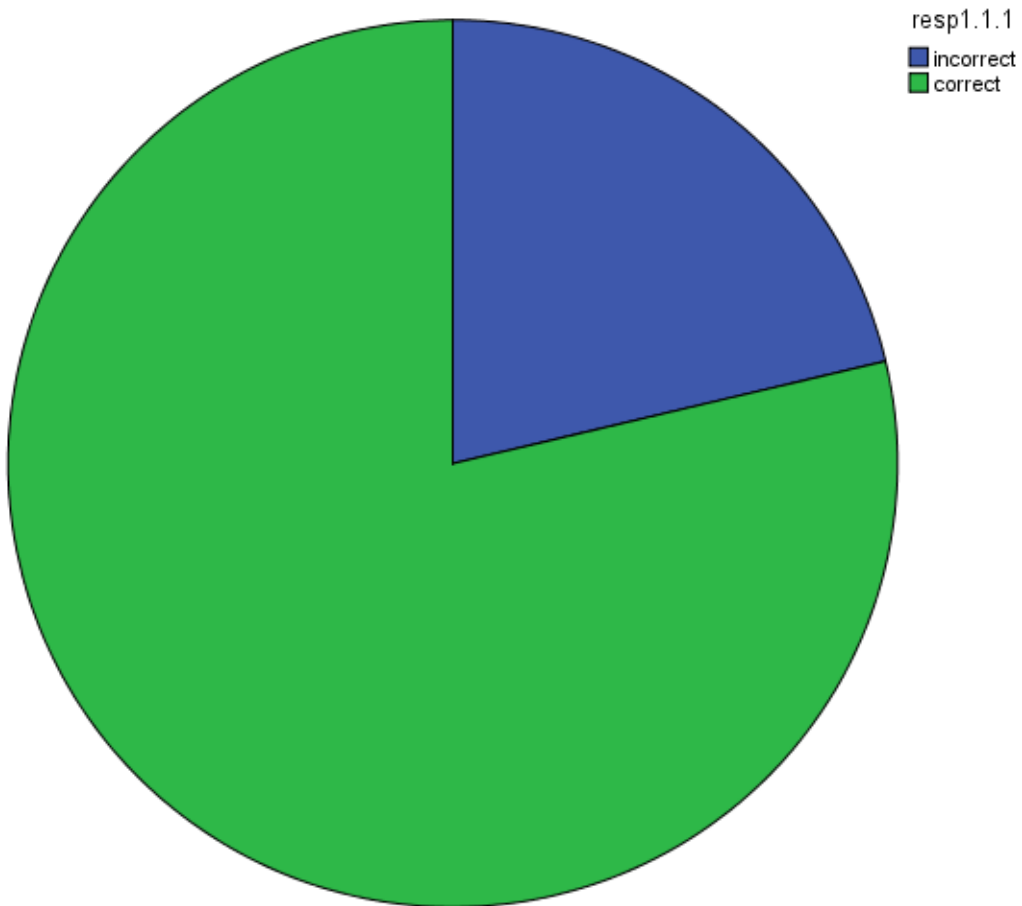
1. The cost of a cell phone call from a phone on TOK pre-paid to a cell phone on ZEE or DORY service provider during peak times is given by the equation:  $C = R2,75 \times t$ .  $C$  represents Cost (in rands) of a call and  $t$  represent time in (minutes) spent on call.

1.1.1 Use the formula:  $C = R2,75 \times t$  to find  $C$  if  $t = 87$  minutes. Show all working details.

-----(2)

#### Table 4.1a: Question 1.1.1.

Question 1.1.1 is a result-unknown question in the form of  $y = mx$ . It is set within the contexts of cell phone billing. The introductory material explains the contexts and also provides the formula in symbolic form. Figure 4.9a shows a graphical representation of learner performance in this task. The graph indicates that most learners were successful in applying the direct arithmetic strategy to solve this problem.



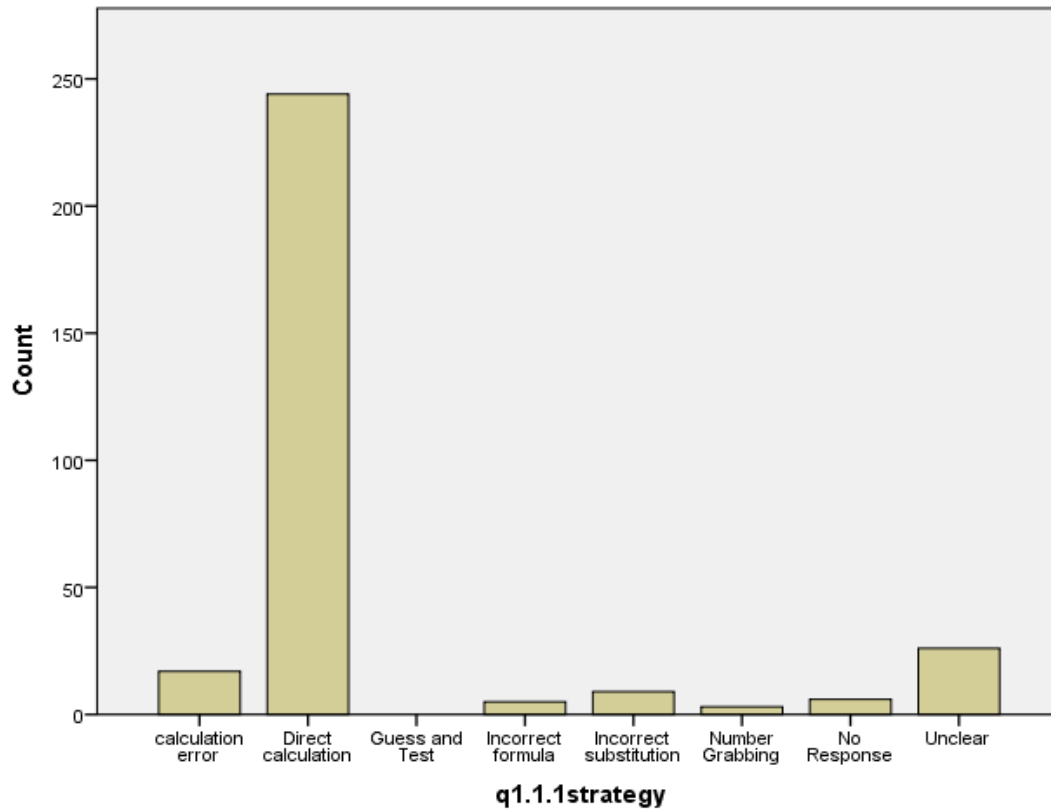
**Figure 4.9a: Graphical representation of learner performance in Question 1.1.1.**

In addition to figure 4.9a, Table 4.1b shows a detailed numerical breakdown of strategies applied by learners to solve question 1.1.1, and it further shows that learners applied the *direct arithmetic strategy* with a success rate of 80 percent.

<i>Direct arithmetic strategy</i>				<i>Unclear answers</i>	<i>No responses</i>	<i>Number grubbing</i>	<i>TOTAL</i>
<i>Correct</i>	<i>Calculation errors</i>	<i>Incorrect formula</i>	<i>Incorrect substitution</i>				
273	18	5	9	26	6	3	<b>340</b>
80%	5%	4%		8%	3%		

**Table 4.1b: learners' strategies applied to Question 1.1.1.**

With the application of direct arithmetic strategy, table 4.1b further shows that the strategy accounted for 5% of calculation errors. 4% of other errors were related to incorrect formula and incorrect substitution. Unclear answers accounted for 8% of the other responses, and number grubbing and no response were minimal in usage at 3% altogether. Figure 4.9b shows a graphical representation of strategies applied by learners to solve question 1.1.1.



**Figure 4.9b: Graphical representation of strategies applied by learners to solve Question 1.1.1.**

#### 4.3.2 Question 1.2.1

2. It costs Mrs. Khoza R1050, 63 in petrol costs to drive from Durban to Johannesburg. If she is the only person in the car, then she will have to pay the whole R1050.63, if there is another person in the car with her, then each of them will have to pay R525.32, if there are three people in the car, then each person will have to pay R35.21 and so on. This relationship is linked by the formula:

$$C = \frac{R1050.63}{p}$$

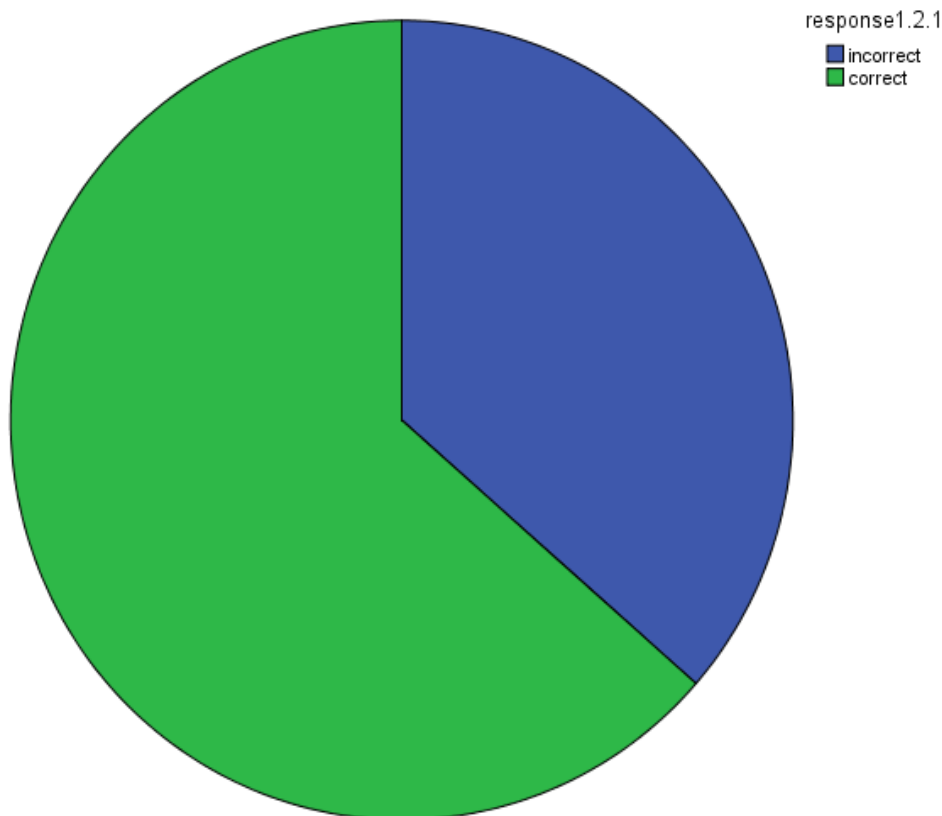
where P represents the number of passengers in a car.

1.2.1 Calculate how much will each person pay if there are 5 people in the car?.....(2)

**Table 4.2a: Question 1.2.1.**

Question 1.2.1 is designed as result-unknown question because the solution requires the learner to substitute the given value in the equation, without making any changes to the form of the equation. It is set within the context of the multiplicative relationship between cost per person ( $y$ ) and number of persons ( $x$ ) in a situation where the cost times the number of persons ( $xy$ ) is fixed. The introductory material explains the context and also provides the formula in symbolic form. There is also illustration of the relationship using various figures. The equation is of the form  $y = \frac{k}{x}$ , where  $k \in \mathbb{R}$ ,  $x \neq 0$ .

Figure 4.10a is a graphical representation of learner's performance in question 1.2.1. The graph indicates that most learners were successful in solving question 1.2.1



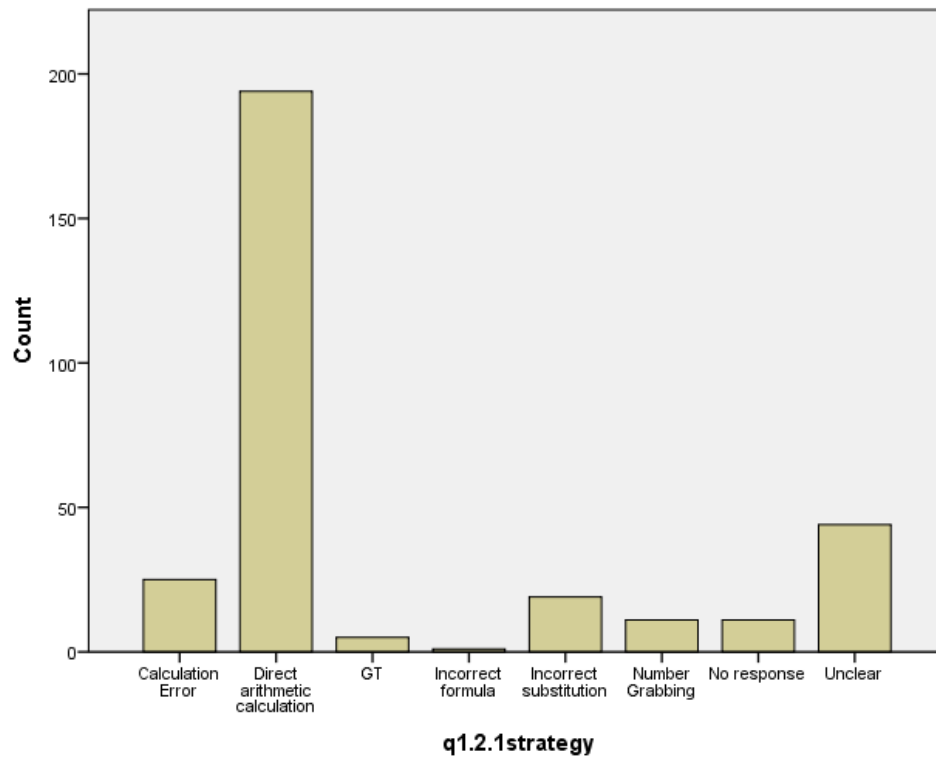
**Figure 4.10a: Graphical representation of learner performance in Question 1.2.1.**

Table 4.2b provides the detailed breakdown of different learner strategies applied in question 1.2.1, learners applied the direct arithmetic strategy with a success rate of 66%. Calculation errors in this question accounted for 8 percent. Errors related to incorrect substitution accounted for 6 percent. Other responses of 13% were classified under unclear answers, or

categorised under no response and number grabbing, at 6% altogether. Table 4.2b is followed by Figure 4.10b, which provides a graphical representation of the figures given in table 4.2b.

<i>Direct arithmetic strategy</i>				<i>Systematic guess &amp; test</i>		<i>Unclear answers</i>	<i>No response</i>	<i>Number grabbing</i>	<i>TOTAL</i>
<i>Correct</i>	<i>Calculation error</i>	<i>Incorrect formula</i>	<i>Incorrect substitution</i>	<i>Correct</i>	<i>Incorrect</i>				
222	26	1	19	3	2	45	11	11	<b>340</b>
65%	8%	0%	6%	1%	1%	13%	6%		

**Table 4.2b: Learner strategies applied in Question 1.2.1.**



**Figure 4.10b: Graphical representation of strategies applied by learners to solve Question 1.2.1.**

### 4.3.3 Question 2.1

Question 2.1

2. The city, works out the amount electricity usage due by using the formula outlined below: You take the number of units and multiply it by R0.60 and then you add R145 to get the total amount.

$$\boxed{\text{No of units}} \times \text{R0.60} \rightarrow \boxed{\phantom{000}} + \text{R145} \rightarrow \boxed{\phantom{000}}$$

2.1 Work out how much is owed by Miss Khoza, if she used  
735 units------(2)

**Table 4.3a: Question 2.1.**

Question 2.1 is a result unknown question of the form  $y = mx + c$ . It is set within the contexts of electricity billing. The introductory material explains the context and also provides the formula in verbal form, which formula is illustrated in the form of a flow diagram.

Table 4.3b, shows that learners were successful in this question, with the *direct arithmetic strategies* used by learners for question 2.1. It further shows a success rate of 86 percent.

Calculation errors occurred at a rate of 3 percent and 5% of other learner responses were classified as unclear answers, while 4% of other responses were classified under no response.

<i>Direct arithmetic strategy</i>				<i>Unclear answers</i>	<i>No responses</i>	<i>Number grubbing</i>	<i>TOTAL</i>
<i>Correct</i>	<i>Calculation error</i>	<i>Incorrect formulary</i>	<i>Incorrect substitution</i>				
295	9	3	1	17	13	2	<b>340</b>
86%	3%	1%		5%	4%	1%	

**Table 4.3b: showing learners' strategies applied to Question 2.1.**

#### 4.3.4 Question 3.1

##### Question 3.1

3. Although in South Africa we measure temperature in degrees Celsius, many countries measure temperature in degrees Fahrenheit (F). If you are a South African travelling overseas, it can become confusing to try to work out the temperature. Similarly, for an overseas tourist visiting South Africa it can be equally confusing.

The equation  $F = (1,8 \times C) + 32$  represent the relationship between temperature measured in degrees Fahrenheit (F) and degrees Celsius (C). Use this equation to answer the following questions.

Use the formula:  $F = (1,8 \times C) + 32$

3.1 Find **F** if  $C = 83$ . Show all working details. -----(2)

#### Table 4.4a: Question 3.1.

Question 3.1 (presented in Table 4.4a) is a result unknown question of the form  $y = mx + c$ . It is set within contexts of temperature conversions between degrees Centigrade (C) and degrees (F). The introductory material explains the contexts and provides the symbolic form.

Table 4.4b, shows that there was a 77% success rate using the *direct arithmetic strategy*, where 4% is classified under calculation errors, whilst 7% of responses for question 3.1 are classified under no response, and 7% were classified as unclear answers. 4% and 1% of responses are classified under incorrect formula, and incorrect substitution, respectively. The formula used in Question 3.1 is similar to that of Question 1.1.1, except that Question 3.1 includes an additional step. Question 1.1.1 is in the form  $y = mx$ , while Question 3.1 is in the form  $y = mx + c$ .

<i>Direct arithmetic strategy</i>				<i>Unclear answers</i>	<i>No responses</i>	<i>Number grubbing</i>	<i>TOTAL</i>
<i>Correct</i>	<i>Calculation error</i>	<i>Incorrect formular</i>	<i>Incorrect substitution</i>				
259	15	13	5	21	25	2	<b>340</b>
77%	4%	4%	1%	6%	7%	1%	

**Table 4.4b: showing learner strategies for Question 3.1.**

#### 4.3.5 Question 4.1

Question 4.1							
<p>4. Ms. Adams, a teacher at Sunset High School in Ladysmith, is organizing an educational excursion to Durban for Grade 12 learners. The distance from Ladysmith to Durban is 255km. The excursion will take one day. Ms. Adams receives a quotation from a bus company (Company A). The table below shows the cost for different distances, with some details omitted.</p>							
Table 1: Cost of hiring a bus from Company A							
Distance(km)	0	63	177	366	489	592	<b>B</b>
Cost(C)	R200	R323. 48	R546.9 2	R917.3 6	R978	<b>A</b>	R1376
<p>4.1 Find <b>A</b> on table 1 using formula: <b>Cost = R1.96 × no. of km + R200</b>. Show all your calculations. A------(3)</p>							

**Table 4.5a: Question 4.1.**

Question 4.1 is a result-unknown question in the form  $y = mx + c$ . It is set within the contexts of car hiring. The introductory material explains the contexts. The formula is illustrated in the form of a table as well as in symbolic form.

Table 4.5b, shows that a 65% success rate with the use of *direct arithmetic strategy* and % of calculation errors were related to the use of direct arithmetic strategy. 15% of other responses were problems related to the incorrect substitution. 11% of other responses were classified under no response and 6% were classified as unclear answers.

<i>Direct arithmetic strategy</i>				<i>Unclear answers</i>	<i>No responses</i>	<i>TOTAL</i>
<i>Correct</i>	<i>Calculation error</i>	<i>Incorrect formulary</i>	<i>Incorrect substitution</i>			
223	6	7	44	21	39	<b>340</b>
66%	1%	2%	13%	6%	11%	

**Table 4.5b: showing learners' strategies applied to Question 4.1.**

#### 4.3.6 Summary of Result-unknown Tasks

The tasks were set in real-life contexts, and presented in various formats as shown below:

- *Task 1.1.1: of the form  $y = mx$ ,*
- *Task 1.2.1 :  $y = \frac{k}{p}$ ,  $k \in R$  and  $p \neq 0$*
- *Task 2.1 : of the form  $y = mx + c$ , and in the form of a flow diagram*
- *Task 3.1 : of the form  $y = mx + c$ ,*
- *Task 4.1: of the form  $y = mx + c$ , accompanied by a table of values*

The data reveals that most learners use the direct arithmetic strategy to solve result-unknown questions. Table 4.6 shows correct responses against incorrect responses in each of the five result-unknown question formats.

Type of question	Correct	Incorrect	Total	Success rate
Question 1.1.1: $y = mx$ , formula provided in symbolic form	273	67	<b>340</b>	80%
Question 1.2.1: $y = \frac{k}{p}$ , where $k \in R$ and $p \neq 0$ formula given symbolically and verbally	225	115	<b>340</b>	66%
Question 2.1: of the form $y = mx + c$ , formula provided verbally, and in a flow diagram	295	45	<b>340</b>	87%
Question 3.1: of the form $y = mx + c$ , formula given in symbolic form	259	81	<b>340</b>	76%
Question 4.1: of the form $y = mx + c$ , formula provided symbolically and in table	223	117	<b>340</b>	66%
Average success rate with <i>direct arithmetic strategy</i>				75%

**Table 4.6: Correct responses compared to incorrect response for a result-unknown questions.**

Learners have recorded great success with direct arithmetic strategy for result-unknown questions. The task that they performed best in was Question 2.1 (87% success rate) and the Questions which they performed most poorly were Questions 1.2.1 and 4.1 with the success rate of (63%) in both questions.

#### 4.4 Start-unknown Questions

This section of the study reports on the strategies applied by learners to solve the five start-unknown questions in the four tasks.

##### 4.4.1 Question 1.1.2

1. The cost of a cell phone call from a phone on TOK pre-paid to a cell phone on ZEE or DORY service provider during peak times is given by the equation :  $C = R2,75 \times t$ .  $C$  represents Cost (in rands) of a call and  $t$  represent time in (minutes) spent on call.

Use the formula:  $C = R2,75 \times t$  to

1.1.2 Find  $t$  if  $C = R487.32$ . Show all working details. :.....(2)

**Table 4.7a: Question 1.1.2.**

With Question 1.1.2, data revealed that learners were most unsuccessful with this question because Figure 4.5 shows that the main strategies are number grabbing, unclear answers, no response and errors related to incorrect substitution.

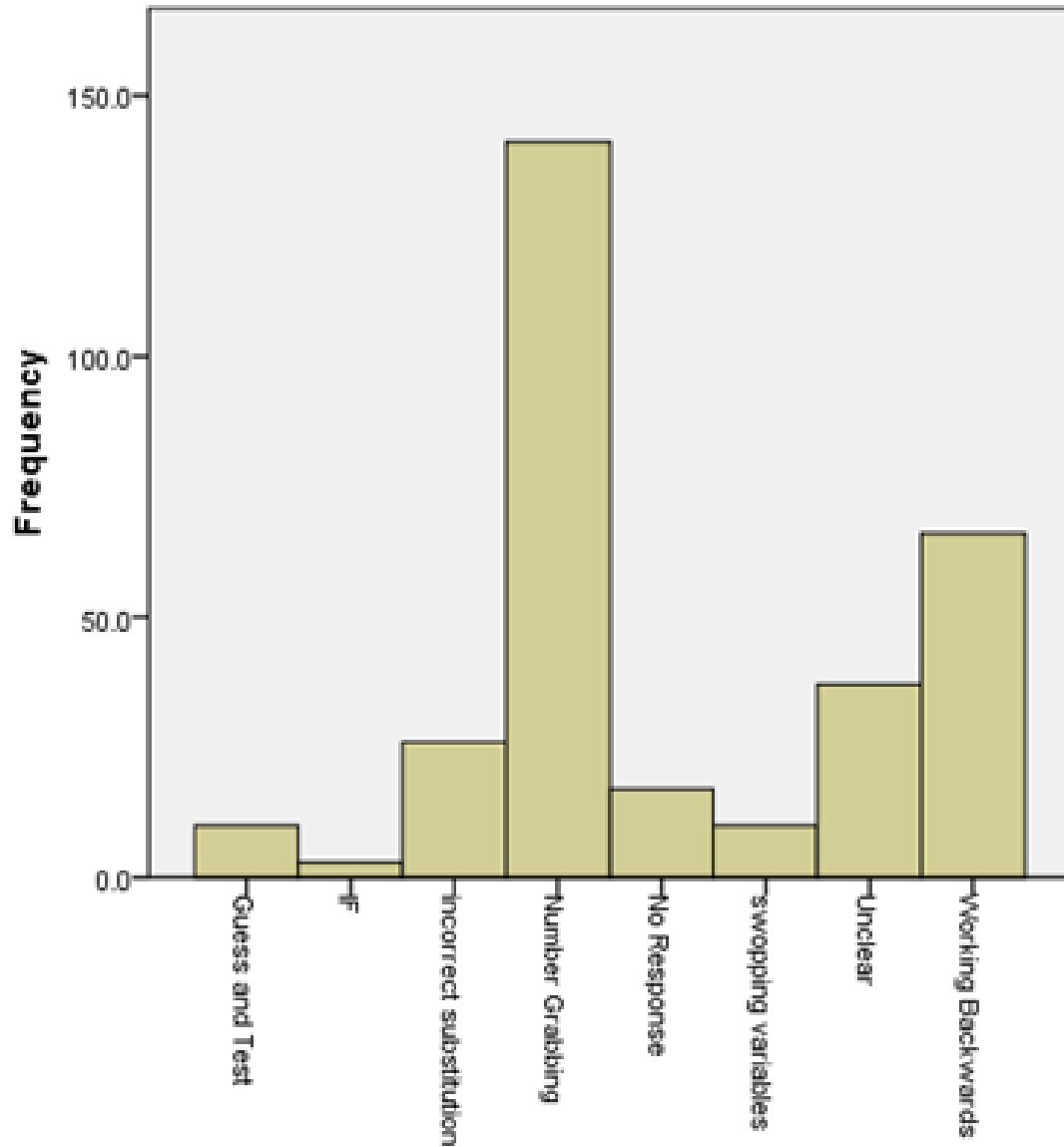
<i>Systematic guess &amp; test</i>		<i>Working backwards</i>		<i>Incorrect formula</i>	<i>incorrect substitution</i>	<i>Swooping of variables</i>	<i>Unclear answer</i>	<i>No response</i>	<i>Number grabbing</i>	<i>TOTAL</i>
Correct	Incorrect	Correct	Incorrect							
11	0	83	3	3	28	11	38	17	146	<b>340</b>
3%	0%	24%	1%	1%	8%	3%	11%	5%	43%	

**Table 4.7b: Learner strategies applied to Question 1.1.2.**

The success rate with Question 1.1.2 is 24% with ‘working backwards’, and 3% with ‘systematic guess and test’. 43% of most responses were classified as number grabbing, while 5% of other strategies were classified under no response. 11% of other learner responses are categorised as unclear answers.

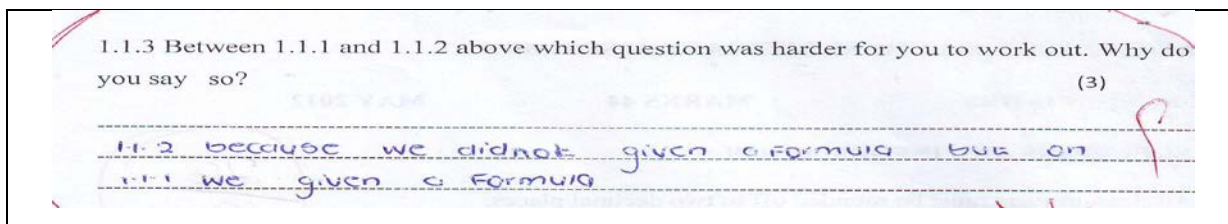
With 24% of learners, *working backwards* was employed as a strategy with successful results. Data shows the most popular correct strategy in Question 1.1.2 to have been working backwards. Incorrect substitution as a strategy in the start-unknown questions refers to a substitution into the given formula. Most of the errors occurred where learners attempted to convert minutes into seconds or hours. This is most likely due to the fact that most of the time problems that involve calculation with time involve the need to change the units first. This was an attempt by learners to try and manipulate 87 minutes by first multiplying it by 100, or by another number.

Table 4.7b, shows a more detailed analysis of all the strategies used by learners to solve Question 1.1.2 and figure 4.11a shows a graphical representation of strategies applied by learners to solve Question 1.1.2.

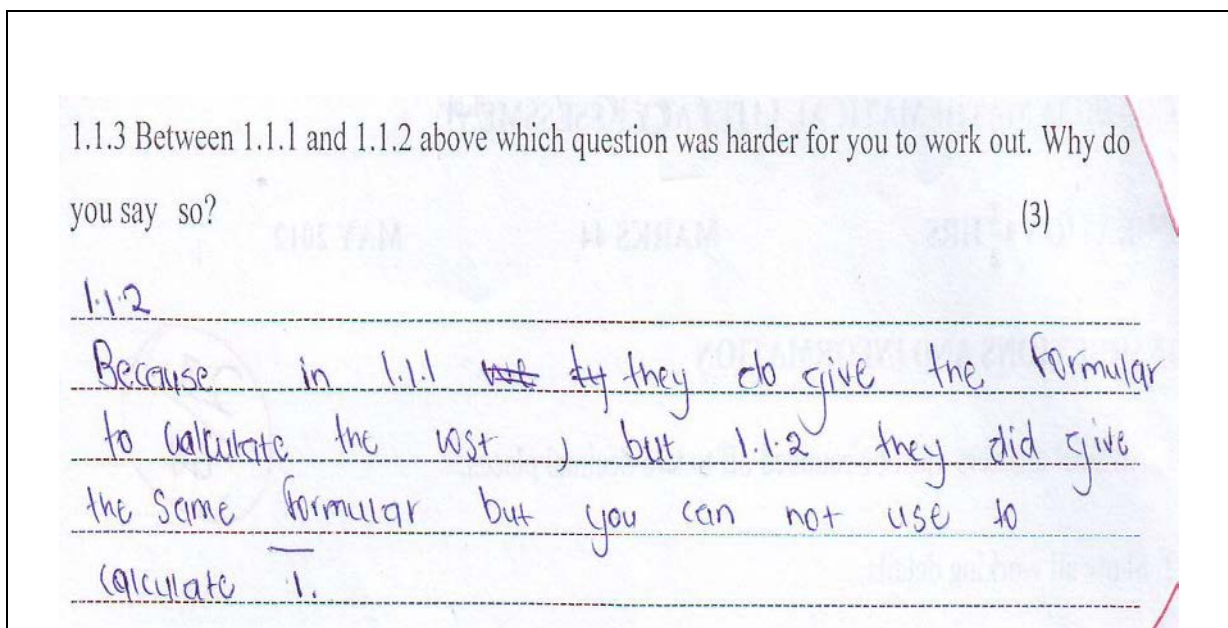


**Figure 4.11a: Graphical representation of strategies applied by learners to Question 1.1.2.**

In order to understand why there was high number of responses that are classified under number grabbing or unclear answers, and why there were so many responses in the number grabbing category, I scrutinised the learner responses to Question 1.1.3, which asked learners for their reflections as to which question they found more difficulty and why. A common response is illustrated in Figure 4.11b and 4.11c.



**Figure 4.11b: First learner response to Question 1.1.3 (333).**



**Figure 4.11c: Second learner response to Question 1.1.3 (337).**

Figure 4.11b and 4.11c similarly illustrate that a learner did not recognise that the same formula could be used to calculate the cost as well as the time.

Interviews with learners confirmed this, along with other misconceptions or assumptions relating to Question 1.1.2 that might explain why some of the learners resorted to a strategy of number grabbing. In one of the interviews, I asked L312: ‘Why do you say there is no formula for Question 1.1.2?’ This was in response to the learner script, where the learner had responded by saying that Question 1.1.2 was more difficult than Question 1.1.1, because there was no formula for Question 1.1.2. Question 1.1.1, provided the formula  $C = R2,75 \times t$  and  $C = R487,32$ , where the question required the learner to solve for  $t$ . I interviewed Learner 312 in an attempt to find out more about his written reflection given in Figure 4.11b. In response to my question to L312, namely: ‘Why do you say there is no formula for

question 1.1.2?', the learner's response is shown below as written in the transcript. According to L312, if Question 1.1.2 requires solving for  $t$  using this formula ( $C = R2, 75 \times t$ ), the subject of the formula must be  $t$ , otherwise the formula cannot be applied. The following extract from the interview with L312 reveals this misconception or assumption, where R refers to the researcher and L refers to the learner.

- R: Why are you saying there is no formula for Question 1.1.2?
- L312: There is no formula for 1.1.2.
- R: What is the question for 1.1.2?
- L312: Find  $t$  if  $C = R487, 32$  [sic].
- R: Why are you saying  $C = R2, 75 \times t$  is only for Question 1.1.1, and why not for Question 1.1.2 as well?
- L312: Because the question says find  $t$ , I cannot use the formula.
- R: Why?
- L312: [Because] its start with C.

According to L312, at the interview, the formula  $C = R2, 75 \times t$  is invalid for Question 1.1.2 and it cannot be used to find  $t$  because the subject of the formula is C, and this formula cannot be used to find  $t$ . Furthermore, an interview with L313 revealed a second misconception or assumption that could have also led to the use of incorrect strategies. According to learner L313, if the formula belongs to a particular question, that formula will have to be placed next to that particular question. For example,  $C = R2, 75 \times t$  is a formula to be used in question 1.1.1 because it is placed next to Question 1.1.1. Therefore, according to L313, the formula can only be used for Question 1.1.1 and not for Question 1.1.2. 3 other learners, who were part of the interview as well, shared this assumption.

#### 4.4.2 Question 1.2.2

##### Question 1.2.2

2. It costs Mrs. Khoza R1050, 63 in petrol costs to drive from Durban to Johannesburg. If she is the only person in the car, then she will have to pay the whole R1050.63, if there is another person in the car with her, then each of them will have to pay R525, 32, if there are three people in the car, then each person will have to pay R35, 21 and so on. This relationship is

linked by the formula:

$$C = \frac{R1050.63}{p}$$

where P represents the number of passengers in a car.

1.2.2 Calculate how many people are in the car if it costs each person R150.09 to pay for petrol? .....(4)

#### Table 4.8a: Question 1.2.2

Question 1.2.2 was taken as a start-unknown question, since C was the subject of the formula, where the value of C was given and learners were asked to solve for P. This question presents an interesting situation, because in this scenario, the swooping variable strategy – which is an incorrect strategy in all the other start-unknown questions – is actually correct here. Because of the multiplicative relationship,  $p \times C = 1050.63$ , where the variables C and p are swapped, the equation remains correct. Hence, in this question, it was not possible to identify the swooping variable as a strategy. Instead, those responses where the variables P and C were swapped for Question 1.2.2, were considered to be a direct arithmetic strategy, because once the equation was re-written as  $P = \frac{R1050.63}{c}$ , the problem was transformed into a result-unknown question, which could be solved using the direct arithmetic strategy.

Table 4.8b further shows strategies applied to solve Question 1.2.2, revealing how learners achieved a 32% success rate using *direct arithmetic strategy* in this start-unknown question, followed by a 25% success rate using the *systematic guess and test* strategy. 22% of other learner responses were classified as unclear answers. A further 12% are classified under no response, and 7% classified under number grabbing.

<i>Systematic guess &amp; test</i>		<i>Direct arithmetic strategy</i>		<i>Incorrect formula</i>	<i>incorrect substitution</i>	<i>Unclear answer</i>	<i>No response</i>	<i>Number grubbing</i>	<i>TOTAL</i>
<i>Correct</i>	<i>Incorrect</i>	<i>Correct</i>	<i>Incorrect</i>						
86	0	108	4	2	3	75	40	22	<b>340</b>
25%	0%	32%	1%	1%	1%	22%	12%	6%	

**Table 4.8b: Learner strategies applied to Question 1.2.2.**

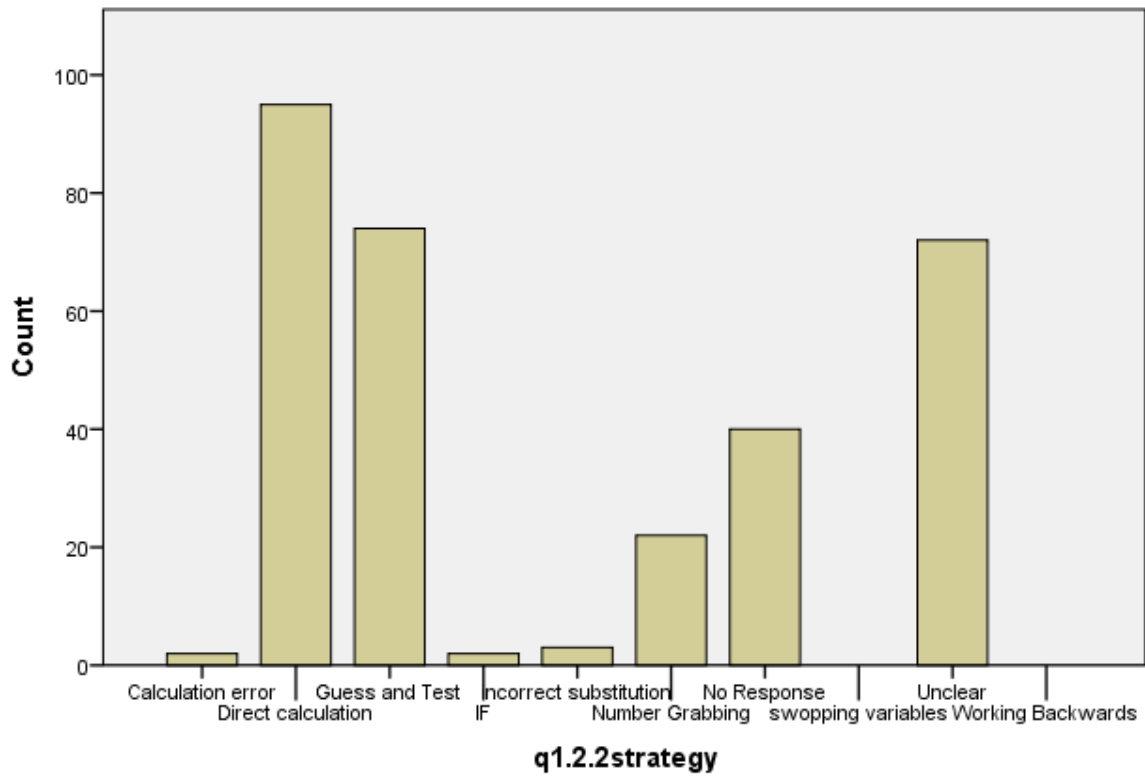


Figure 4.12: Graphical representation of strategies applied to Question 1.2.2.

4.4.3 Question 2.2

Question 2.2

2. The city works out the amount of electricity usage due by using the formula outlined below: take the number of units and multiply it by R0.60 and then add R145 to get the total amount.

No of units    ×    R0.60    +        R145    →   

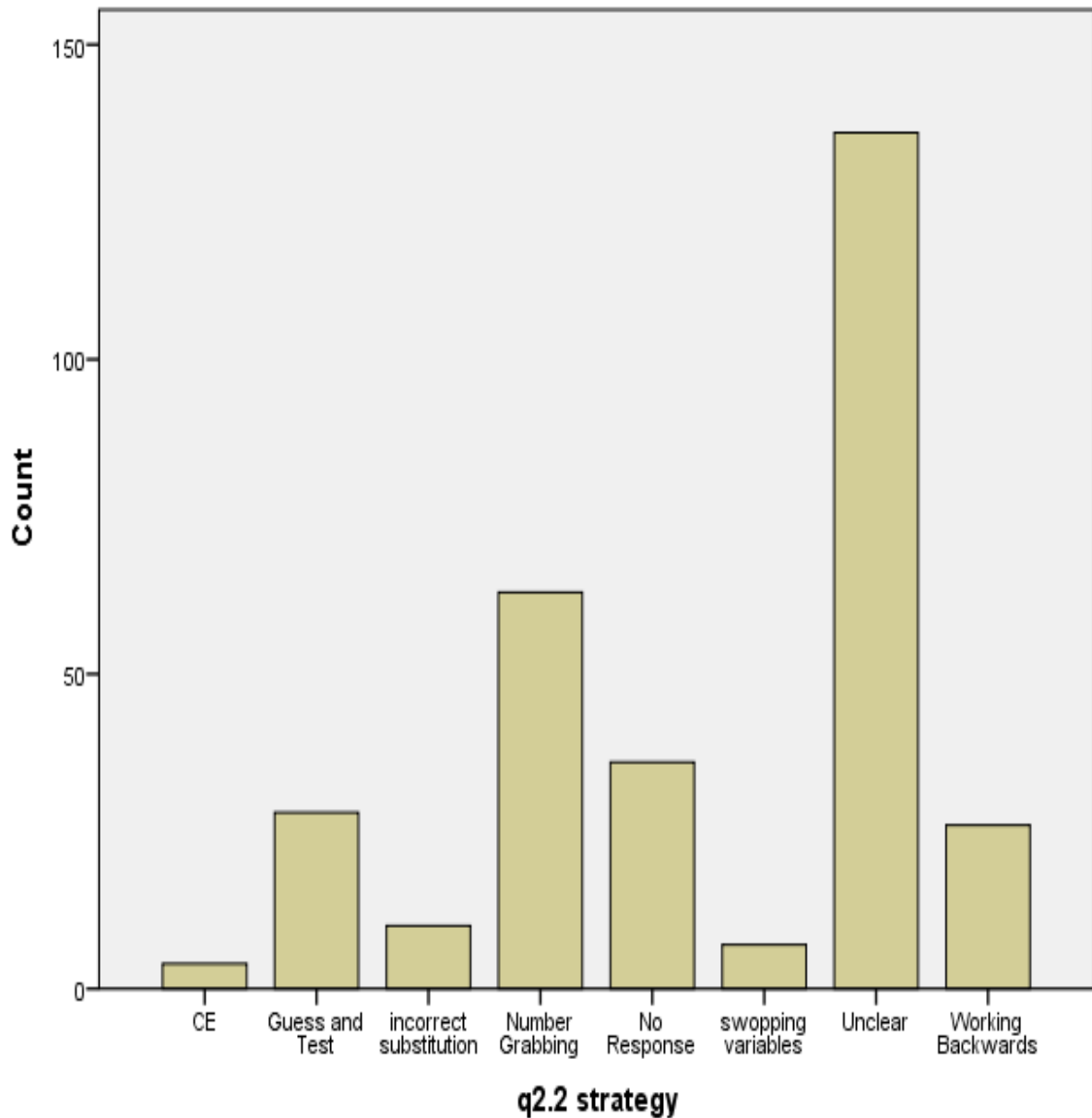
2.2 How many units did she use, if her bill (amount due) was R415? Explain how you get your answer. :-.....(4)

Table 4.9a: Question 2.2.

Question 2.2 is of the form of  $y = mx + c$  presented in the form of a flow *diagram*. The success rate in Question 2.2 with *systematic guess and test* is 10%, and with *working backwards* was 5 percent. Furthermore, 43% of learner responses were classified as unclear answers. 19% of other learner responses were classified under number grabbing, and 11% were classified under no response. See Table 4.9b.

<i>Guess &amp; test</i>		<i>Working backwards</i>		<i>incorrect substitution</i>	<i>Swopping of variables</i>	<i>Unclear answer</i>	<i>No response</i>	<i>Number grabbing</i>	<i>TOTAL</i>
<i>Correct</i>	<i>Incorrect</i>	<i>Correct</i>	<i>Incorrect</i>						
35	0	17	19	12	8	146	40	63	<b>340</b>
10%	%	5%	6%	4%	2%	43%	11.0%	19%	

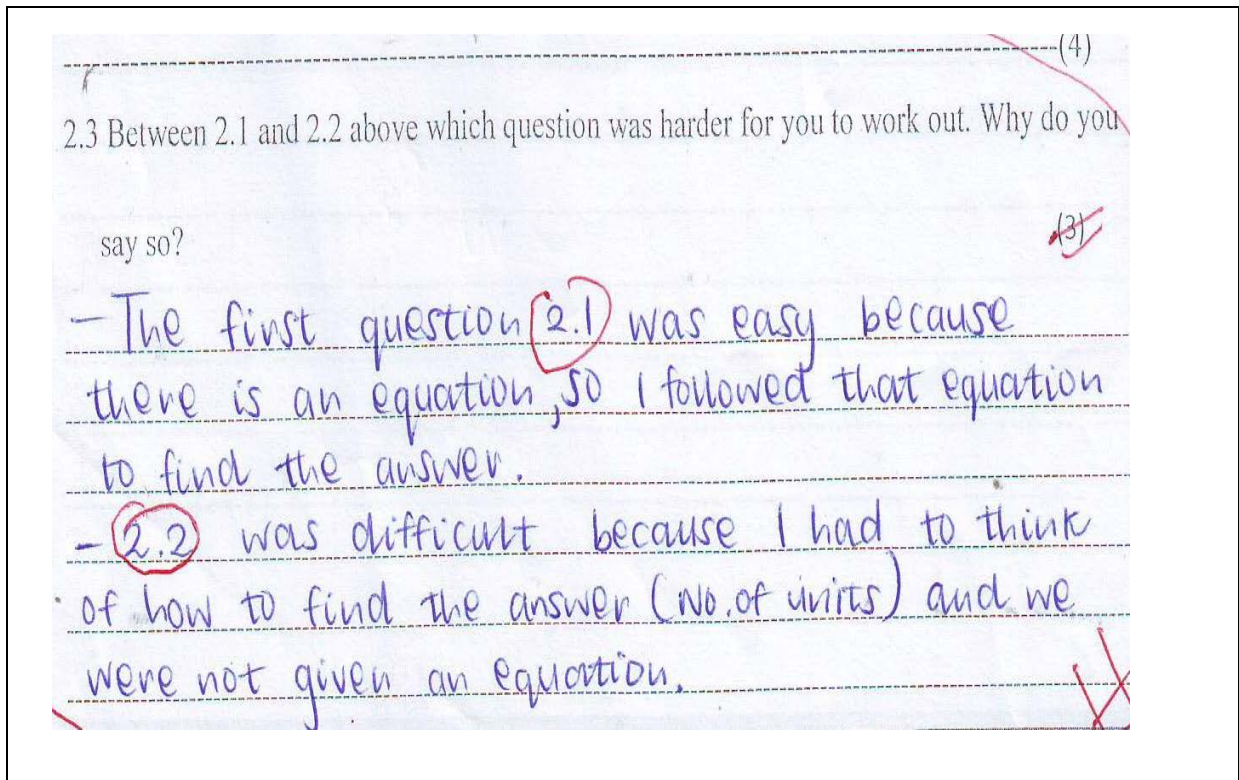
**Table 4.9b: Learner strategies applied to Question 2.2.**



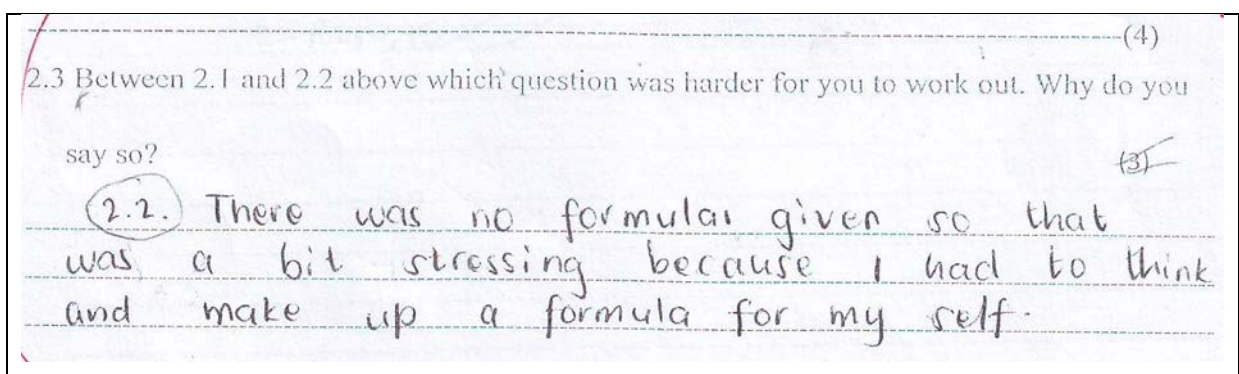
**Figure 4.13a: Graphical representation of strategies applied to Question 2.2.**

Figure 4.13a shows the strategies of working backwards and systematic guess and test to have been successful for Question 2.2. Compared to question 1.1.2, where the favoured strategy for Question 1.1.2 is working backwards and for Question 2.2, the systematic guess and test was the most commonly used. It is possible to speculate that for Question 1.1.2, there was only one operation that needed to be ‘undone’, making it easier to apply the working backwards strategy, while for Question 2.2, there were two operations that needed to be ‘undone’. A success rate of 15% indicates that learners struggled with Question 2.2.

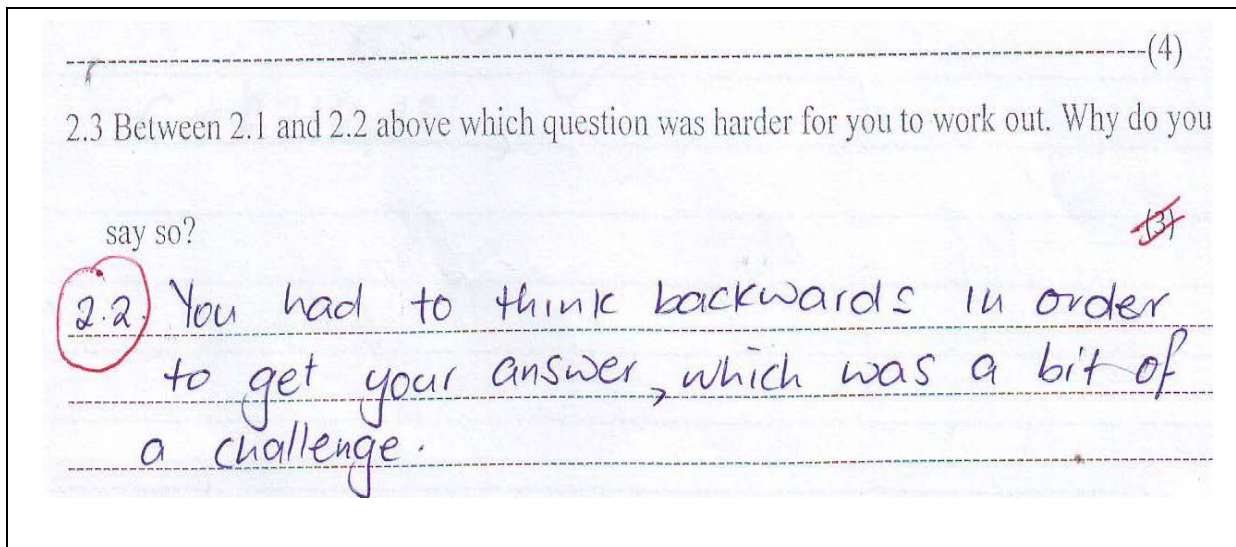
Learner reflections in the learner scripts depicted below, indicate that learners found Question 2.2 difficult. Figures 4.13b to 4.13e show learner reflections on Question 2.2. Here is a list of four learner responses that indicate that learners found Question 2.2 difficult. Figure 4.13.b to 4.13e show learner responses to the reflection Question 2.3. I interpreted these reflections as indications that learners found Question 2.2 difficult.



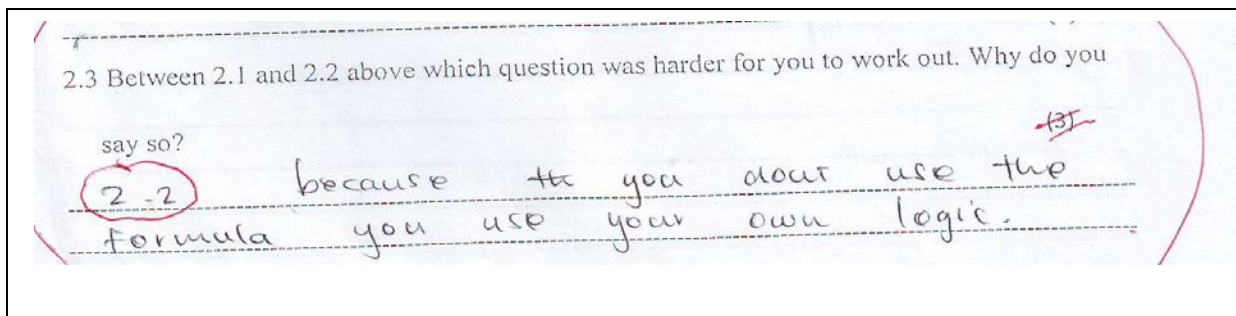
**Figure 4.13b: First learner reflection on Question 2.3: (L323).**



**Figure 4.13c: Second learner reflection on Question 2.3**



**Figure 4.13d: Third learner (L329) reflection on Question 2.3 (335)**



**Figure 4.13e: Fourth learner reflection on Question 2.3 (L328).**

The four reflections from learners that appear in the preceding figures indicate that learners generally found Question 2.2 more difficult than Question 2.1. The reflections indicate that the learners were troubled because they could find no equation or formula that was relevant for the task, and they were required to try to think of another way to get to the answers they needed in order to change the strategy for solving Question 2.2. One learner described his method of working backwards as an attempt to find a way around the problem. Another learner described her use of the guess and test method as “using [one’s] own logic”.

#### 4.4.4 Question 3.2

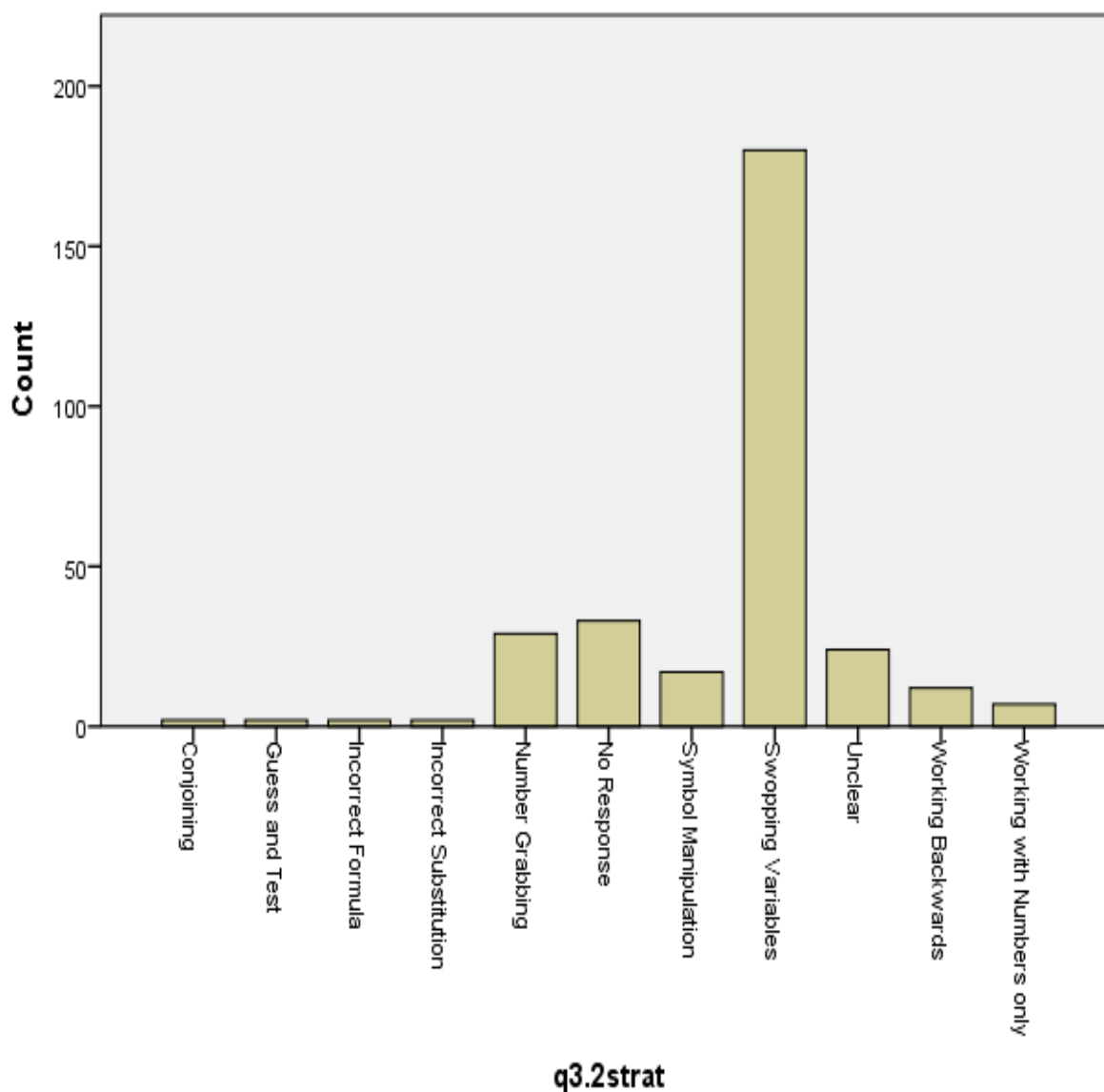
##### Question 3.2

3. The equation  $F = (1, 8 \times C) + 32$  represent the relationship between temperature measured in degrees Fahrenheit (F) and degrees Celsius (C). Use this equation to answer the following questions. Use the formula:  **$F = (1, 8 \times C) + 32$** .

3.2 Find **C** if **F** = 93. Show all necessary working details. :-------(3)

#### Table 4.10a: Question 3.2.

Question 3.2 is of the form  $y = mx + c$ . Data reveals the most popular incorrect strategy with Question 3.2 to have been *swopping variables*. Figure 4.14a shows the swooping of variables to have been applied extensively in Question 3.2.



**Figure 4.14a: Graphical representation of strategies applied by learners to solve Question 3.2.**

The results show that 56 percent of learners used swopping variables to try to solve this problem. Some learners who used the swopping variable strategy also applied a technique of swopping variables with the ‘magic rules’ of change of sign.

The phrase ‘magic rules’ is taken from De Lima and Tall (2008), who wrote about learners working with linear equations. They found that learners build their own ways of working, based on the embodied actions they seem to perform on the symbols, almost as if they were mentally ‘picking them up’ and then ‘moving them around’. In addition, they utilised the added ‘magic of rules’ such as ‘change sides, change signs’ when performing

these actions on the symbols. Figure 4.14b presents learners response with an application of this strategy.

2.2 find C if F = 93. Show all working details.

$$F = (1,8 \times C) + 32$$

$$93 = (1,8 \times C) + 32$$

$$C = [(1,8 \times (-93))] + 32$$

$$C = -135,40$$

**Figure 4.14b: Swopping variables with a ‘magic rule’ change of sign (L324)**

In the learner response in Figure 4.14b, it can be seen in the second step that the learner substituted 93 for F. Then in the third step, the learner swopped the variables F and C. While swopping the variables, he further changed the sign of 93 in response to the fact that he was transposing the 93, ‘across an equal sign’ and seemingly following the rule to ‘change sides [and] change signs’.

The correct successful strategies in this task but with minimal success were *symbol manipulation*, *working backwards* and *systematic guess and test* with %, 3% and 1%, respectively. Further to this, 56% of learner’ responses were classified under swopping of variables, as mentioned previously. Other strategies that received more usage were number grabbing, unclear answers and no response. Again, table 4.10b shows a detailed numerical breakdown of the number of learners applying each strategy.

<i>Guess &amp; test</i>		<i>Working backwards</i>		<i>Symbol manipulation</i>		<i>Conjoining</i>	<i>Working with</i>	<i>Swopping of</i>	<i>Unclear answer</i>	<i>No response</i>	<i>Number grubbing</i>	<i>TOTAL</i>
<i>Correct</i>	<i>Incorrect</i>	<i>Correct</i>	<i>Incorrect</i>	<i>Correct</i>	<i>Incorrect</i>							
3	0	9	7	14	2	6	13	18 9	26	35	32	<b>340</b>
1%	0 %	3%	2%	4 %	1%	2%	4%	56 %	8%	10 %	9%	

**Table 4.10b: Learner strategies applied to Question 3.2.**

The most popular incorrect strategy noted in Question 3.2 was swopping variables. Learner work shown on Figure 4.14c shows evidence of calculation using swopping variables calculations.

3.2 find C if F = 93. Show all working details. (3)

$$C = (1.8 \times 93) + 32$$

$$= 167.4 + 32$$

$$= 199.4$$

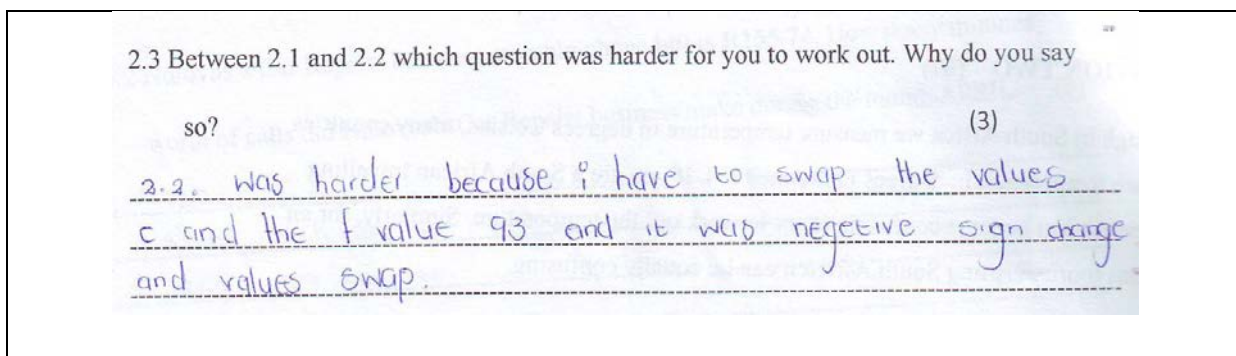
Please explain 2 me how u came up with step I of 3.2

**Figure 4.14c: Swopping variables-direct calculation strategy (L318).**

Figure 4.14c shows that the learner swapped the positions of F and C in the formula, where he or she then substituted  $F = 93$ . This step has allowed the learner to convert the start-unknown question into a result-unknown question, after which the learner could easily apply the direct arithmetic strategy.

This act of symbol swapping seems to be a physical ‘picking up’ and ‘moving around’ of symbols in an algebraic equation. This action is in the same detailed by De Lima and Tall (2008), who described it as a form of procedural embodiment.

One learner described her symbol swapping attempt in the reflection elicited following Question 3.2. Figure 4.14d proves that the learner applied the swapping of variables with a ‘magic rule’ of changing the sign even though the learner seemed to the numbering of questions.



**Figure 4.14d: Learner reflections on swapping of variables (L309).**

To probe the origin and the extent of use of this strategy, and to determine whether this is an oversight or serious underlying misconception, I selected learner 313 for an interview. The main interview question for L313 was, ‘please explain how you arrived at Step1:

$$C = (1, 8 \times 93) + 32, \text{ from } F = (1, 8 \times C) + 32'.$$

Please note that **R** refers to the researcher and **L** refers to the learner, below:

R: Okay, please read the formula for me at the top. I just want to be sure we are on the same line [sic].

L313: The equation:  $C = (1, 8 \times 93) + 32$ .

R: That was given in the formula, right... In the formula C is on your right hand side and then F is on your left hand side of the equation [sic]. Look at task 3.2, right, in your first step, your first line of work, I see that C is on the right now, but where is C in the given formula?

L313: Left hand side [sic].

R: In the given formula, C is on the left, right? But when you start working, your C has moved from the right to the left hand side. Do you see that? How did you get C to go [sic] to the left hand side?

L313: I probably made a mistake somewhere.

(note that the learner had already received marked script so she realised that his answer was incorrect).

R: What makes you think it was wrong?

L313: I didn't do it according to the formula. I changed everything.

(At this point, it seems as if the learner has now recognised that she has changed the formula by swopping the letters around).

R: You changed everything.

L313: I started with the given formula, C was on this side and then I changed everything to the left side [sic].

R: Yes.

L313: And then, I continued writing from the left to the right, and everything went wrong [sic].

The purpose of the interview was to confirm that the swopping strategy was used, and to also understand why the learner was applying this strategy. The learner responded by saying that she was required by the task to find C, but the subject of the formula to be used was F. Therefore, to be able to calculate the value of C using the formula, the learner moved C over to the left hand side of the equal sign; where F was then transposed to the right hand

side of the equal sign. F is placed at the position that was initially occupied by C in the formula. Once the physical moving of numbers or variables was completed, the learner then carried out the computations on the numbers as a result-unknown question, using direct arithmetical strategy.

The physical picking up and moving of the numbers or variables was carried out without adhering to formal mathematics rules. From the interview, L313 revealed that she did not know how to solve the problem. Swopping a variables-direct calculation seemed a plausible method for the learner. Once the learner noted the problem to have been looking for C where C is not the subject of the formula, the learner reached an impasse, and started applying the strategy of a swopping variables-direct calculation. The swopping of variables is manipulated without due regard or adherence to formal mathematical rules of manipulating values in an algebraic equation.

A further incorrect strategy with Question 3.2 is what is termed conjoining. Figure 4.8e shows the learner response to question 3.2, showing evidence of conjoining. I selected learner 317 to participate in the interview in order to determine the extent of conjoining.

Q2 find C if  $F = 93$ . Show all working details. (3)

$F = (1,5 \times C) + 32$

$93 = (1,5 \times C) + 32$

$93 = 1,5C + 32$  N.B. Please explain to me how you got

$93 = 33,8C$

$33,8C$

$2,75 = C$

**Figure 4.14e: Evidence of conjoining (L-317).**

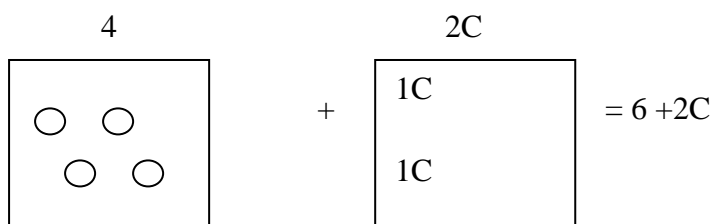
From the learner response above, particularly from step 3 to step 4, the learner added the unlike  $1.8C$  and  $32$  to arrive at  $33.8C$ . During the interview, I used a short series of examples similar to those of step 3 and 4 in order to probe the learner for logic.

An excerpt from the interview with L317 follows below, where R again refers to the researcher and L refers to Learner 317.

- R : What does  $1x + 3$  equal?
- L317 :  $4x$ .
- R : What would you say  $5x - 2$  would be equal to?
- L317 :  $3x$ .
- R :  $1,8C + 32$ .
- L317 :  $33,8C$ .

From the above, it seems that the learner was unable to identify her error, but has repeatedly made the error referred to as conjoining (as described in strategy 4.2.1); where the learner simply operated on the numbers alone, and thereafter attached the variable.

I then further probed the learner using a diagrammatic illustration of  $4 + 2C$  in an attempt to lead the learner to become aware that 4 and  $2C$  are unlike terms, or two different entities that cannot be added together.



The researcher then probed the learner regarding the way in which  $2C$  was obtained, expecting the learner to presume that  $2C$  is a result of  $1C$  plus  $1C$ . However, the learner still used the conjoining strategy to explain the way in which the result  $2C$  was obtained in this situation. In trying to add  $1C + 1C$ , the learner interpreted the sum in the erroneous formulation  $1 + 1 + C$ , by saying that this is the same as  $2C$  i.e.  $1 + 1 = 2$ , where putting  $C$  next to 2 results in  $2C$ .

#### 4.4.5 Question 4.2

##### Question 4.2

4. Ms. Adams a teacher at Sunset High School in Ladysmith is organizing an educational excursion to Durban for grade 12 learners. The distance from Ladysmith to Durban is 255km. The excursion will take one day. Ms Adams receives a quotation from a bus company (Company A). The table below shows the cost for different distances, with some details omitted.

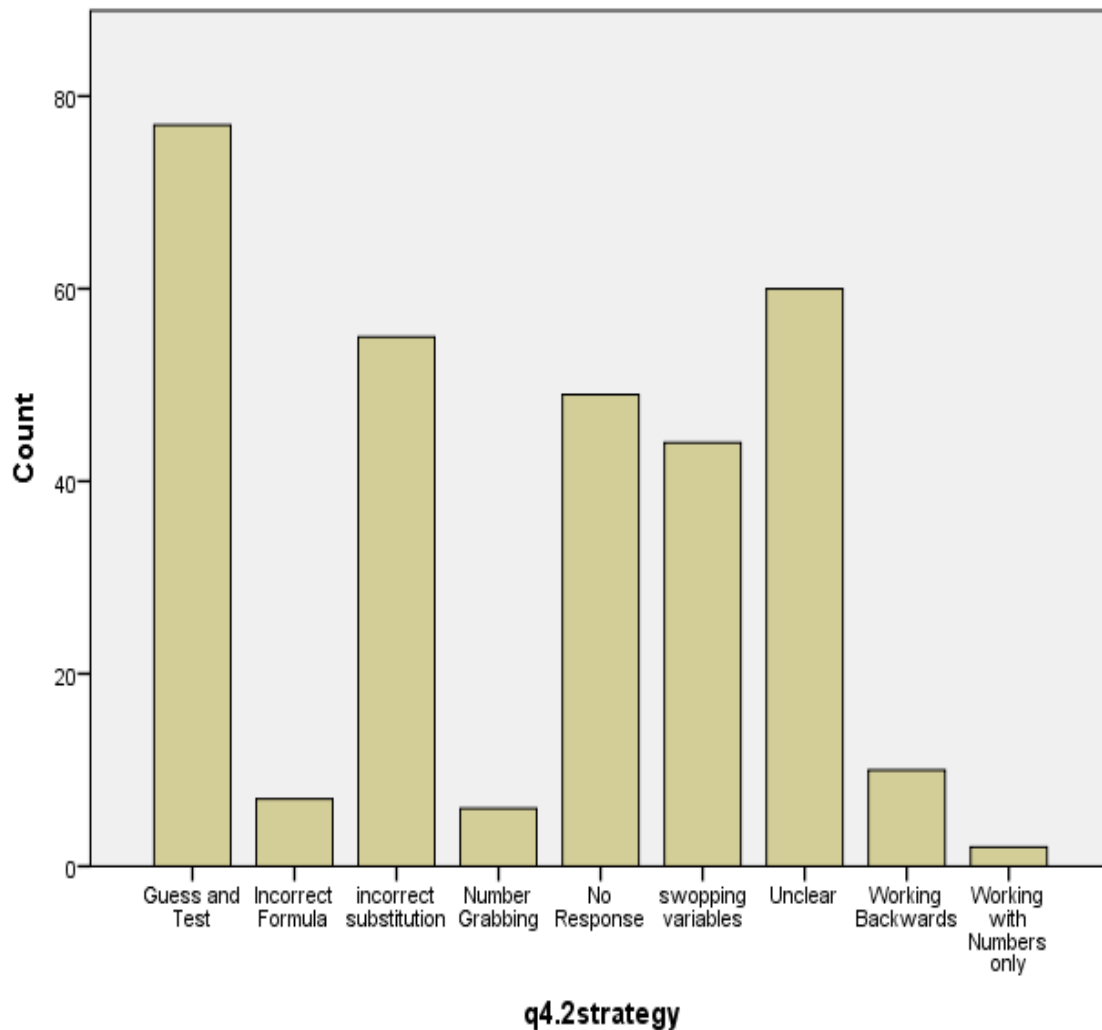
Distance(km)	0	63	177	366	489	592	<b>B</b>
Cost(C)	R200	R323. 48	R546.9 2	R917.3 6	R978	<b>A</b>	R1376

Table 1: Cost of hiring a bus from Company A

4.2 Find B on table 1 using formula:  $\text{Cost} = R1.96 \times \text{no. of km} + R200$ . Show all your calculations:.....(4)

#### Table 4.11a: Question 4.2.

In Question 4.2, 19% of learners obtained correct answers using systematic guess and test, and 3% obtained correct answers using working backwards as a strategies. Unclear answers accounted for 19% of responses, and 16% did not respond, while the smallest percentage was that of swopping of variables, which accounted for 13 percent.



**Figure 4.15a: Graphical representation of learner strategies applied to Question 4.2.**

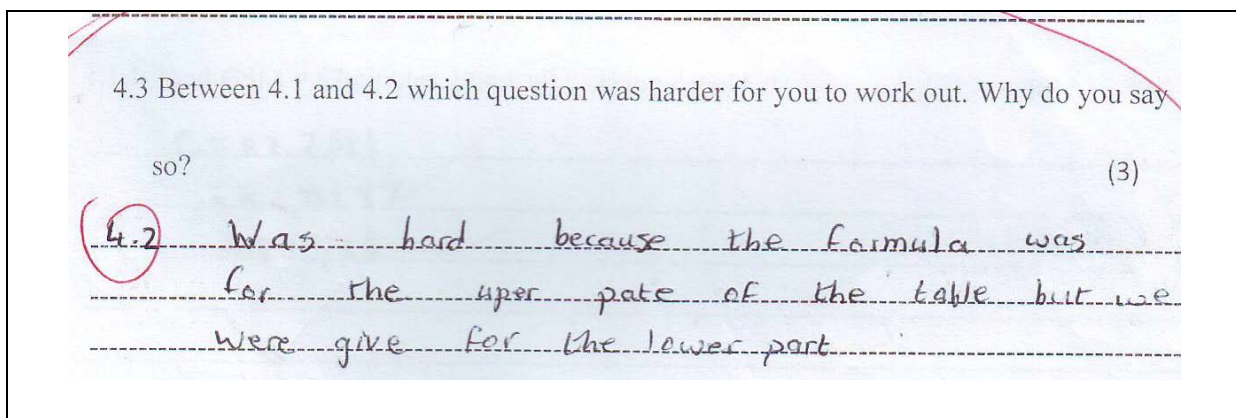
Figure 4.15a shows that the most popular correct strategy for this task to have been systematic guess and test, and it led to correct answers. Most of the incorrect answers were classified as unclear answers, or included in the ‘no response’ column. Incorrect substitution was also a big problem, especially the incorrect substitution of 255km, which came with the instruction. Many learners were distracted by the 255 km given in the instruction, and they disregarded the values provided in the table. The swopping of variables with Question 4.2 was also popular, and it received about 13% of learner responses.

<i>Systematic guess &amp; test</i>		<i>Working backwards</i>		<i>Incorrect formula</i>	<i>Incorrect substitution</i>	<i>Working with numbers only</i>	<i>Swooping of variables</i>	<i>Unclear answer</i>	<i>No response</i>	<i>Number grubbing</i>	<i>TOTAL</i>
<i>Correct</i>	<i>Incorrect</i>	<i>Correct</i>	<i>Incorrect</i>								
66	15	9	15	7	57	4	45	63	53	6	<b>340</b>
19%	4%	3%	4%	2%	17%	1%	13%	19%	16%	2%	

**Table 4.11b: Learner strategies applied to Question 4.2.**

With question 4.2, some learners did not seem to recognise the relationship presented in the form of the table i.e. that cost is dependent on the number of kilometres travelled. For instance, the value of B is linked to R1376, but at B, some learners used 255km as the number of kilometres travelled, which is not linked to B, but was given in the instruction.

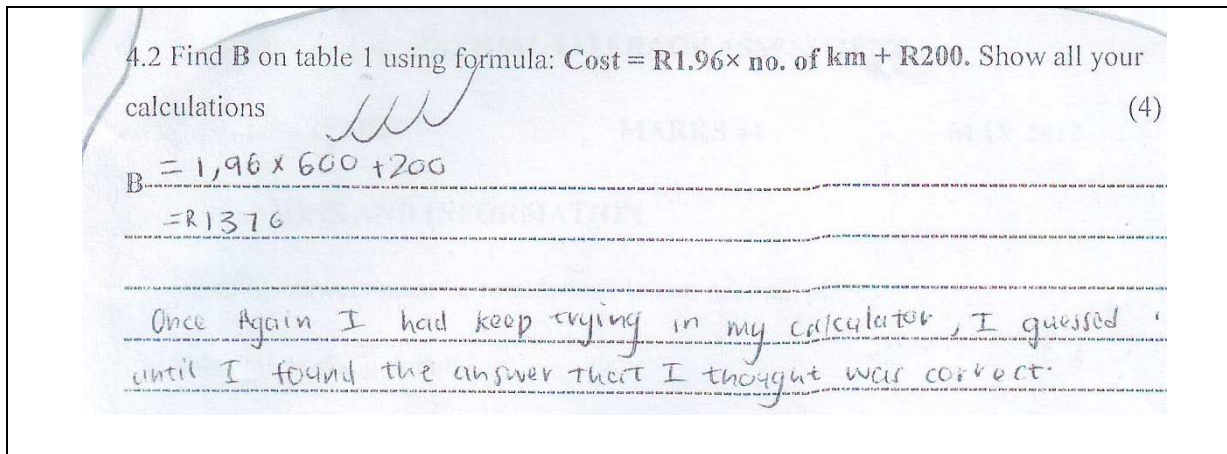
The following reflection by some of the learners shows that they found question 4.2 to be difficult as well. This is reflected in Figure4.9b and figure4.9c, respectively.



**Figure 4.15b: First learner reflection on Question 4.3 (L321).**

From the reason advanced above in figure 4.15b by the learner, the learner is making an error by thinking that the formula that is given can only be used to answer question 4.1, which is related to the first part of the table that is result-unknown question. It would appear that the learner was unable to change the subject of the formula in order to find the value of B. This line of thinking is similar to the other reasons noted earlier, where learners stated that there was no formula for Question 1.1.2 with the formula  $C = R 2.75 \times t$  despite being asked to find the value of  $t$ . Some other learners just stated this formula to have been invalid for finding  $t$  unless the formula were to start with  $t$ .

L326 applied a systematic guess and test strategy for question 4.2. The learner had a way of answering start-unknown questions without using a formal mathematical approach. Systematic guess and test is systematic in this instance because the answers that the learner obtains for the result-unknown question are further used to form a baseline for the next number that they attempt to fit into the formula. The reasons advanced by the learners for Question 4.2 indicate systematic guess and test strategies.



**Figure 4.15c: Second learner reflection on Question 4.3.**

**4.4.6 Summary on Results of Start-unknown Questions**

The data revealed that learners applied the following strategies when solving the five start-unknown questions: direct swopping of variables in the formula; swopping of variables

in the formula with magic rules; conjoining; working with numbers only; number grabbing; systematic guess and test; and working backwards. Other learner responses did not fit in with the strategies outlined above; therefore those responses were classified under unclear answers, or no response, respectively. Across all start-unknown questions, data indicates that the use of working backwards have been marginal for this study. Figure 4.16 shows evidence of a learner who applied working backwards to solve question 1.1.2, furthermore, the strategy depicted on table 4.11 shows content knowledge and skills to have been developed in order to apply working backwards or symbol manipulation. At FET ML, learners are expected work specifically with equations of relationships to solve (that is, to determine the value of the independent variable for given value(s) of the dependent variable), using simple algebraic manipulation. Figure 4.16 shows a calculation of L316, where the learner applied the strategy, working backwards.

1.1.2 Find  $t$  if  $C = R487.32$ . Show all working details. (3)

$$C = R2,75 \times t$$

$$R487,32 = 2,75 \times t$$

$$\frac{2,75}{2,75} \quad \frac{2,75}{2,75}$$

$$\therefore t = \frac{177,2042427}{2,75}$$

$$= 177,21 \text{ minutes}$$

or

$$177 \text{ minutes}$$

**Figure 4.16: showing Question 1.1.2 solved by working backwards (L316).**

Step 2 of the calculations in Figure 4.16, shows the application of the strategy of working backwards; where to find the value of  $t$ , the learner is required to use the formula and reverse the calculation that multiplies  $t$  in the formula with 2.75 by using the multiplicative inverse of multiplication, which is division. If knowledge of *inverse relations* is applied correctly, it helps to isolate the variable  $t$  and all the required calculations are transposed to the other side of the equals sign. Step 2 of the learner calculation in figure 4.16

clearly shows that the strategy demands correct application of knowledge of an inverse relation.

Table 4.12 shows the correct learner responses compared with incorrect learner responses for the five start-unknown questions, with the success rate noted for each question; further showing an average success rate of 24%, which indicates that learners performed poorly in the start-unknown questions.

Type of question	Correct responses	Incorrect responses	Total	Success rate
Question 1.1.2, in the form $y = mx$ , <i>formula provided in symbolic form</i>	95	245	<b>340</b>	28%
Question 1.2.2: in the form $y = \frac{k}{s}$ , where $k \in R$ and $x \neq 0$ and in context	194	146	<b>340</b>	57%
Question 2.2: of the form $y = mx + c$ , formula provided verbally, in and in flow diagram	52	288	<b>340</b>	15%
Question 3.2: of the form $y = mx + c$ , formula given in symbolic form	26	314	<b>340</b>	8%
Question 4.2: of the form $y = mx + c$ , provided symbolically and as table	79	261	<b>340</b>	23%
Average success rate for start-unknown questions				26%
Average success rate excluding Question 1.2.2				19%

**Table 4.12: Correct responses compared to incorrect responses for start-unknown questions.**

#### 4.5 Learners Reflections on Result-Unknown and Start-Unknown Questions

In the research instrument, there were five tasks where learners were required to reflect with reasons on the aspect of each task they found most difficult, where all five tasks were arranged such with a result-unknown question followed by a start-unknown question. The last question in the tasks was a reflection question where learners were required to reflect with reasons on the degree of difficulty between result-unknown and start-unknown questions. Table 4.13 shows a numerical breakdown of interpretations of the reflections inferred from the reflection questions in all tasks. The responses were classified into three categories, coded as follows.

A coding of zero was allocated if the learner stated that the result-unknown was more difficult, but that a coding of one was allocated if the learner stated that the start-unknown was more difficult. If the statement is unclear, or there is no response, a code of N was allocated. Data indicate that learners perceive start-unknown questions as being more difficult than result-unknown ones. Transcripts formulated from learner reflection below, further indicate that learners found start-unknown questions to be most difficult.

	Task 1a			Task 1b			Task 2			Task 3			Task 4			TOTAL
	Q 1.1.3			Q 1.2.3			Q 2.3			Q 3.3			Q 4.2			
	0	1	N	0	1	N	0	1	N	0	1	N	0	1	N	
SCORES	44	214	82	54	175	111	24	230	86	25	171	144	22	185	133	340

**Table 4.13: Coded learner reflections.**

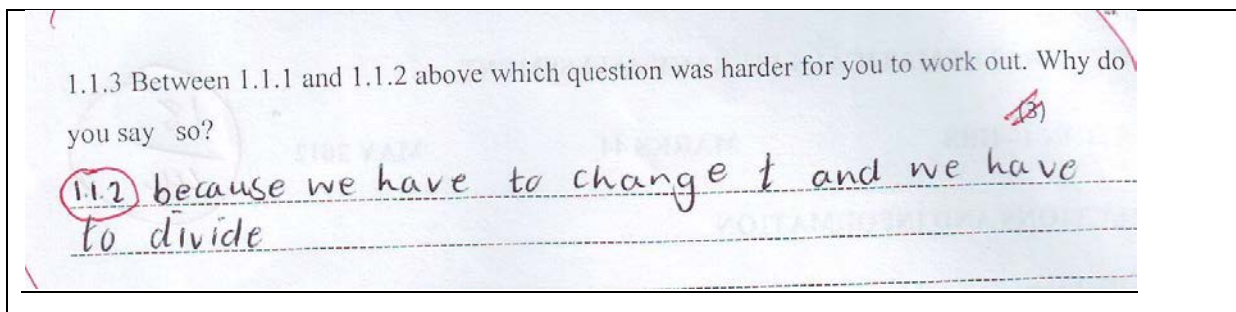
Learners' reflections about which problem is difficult between result-unknown and start-unknown questions, indicate that they found start-unknown questions to have been more difficult than result-unknown questions. For all four tasks, learners noted that the start-unknown questions were the most difficult.

The following extracts from learners' reflection responses reveal that the learners have distinguished between the two types of questions. The responses make reference to the position of the unknown, and they acknowledge that the change in position has made the

problem difficult, or has made the problem an entirely new problem, that requires a new strategy in order to reach a solution. The five responses are labelled as figures 4.11a to 4.11e and I discussed each learner response in terms of its reference to the problem format.

#### 4.5.1 Reflection on question 1.1.1 and 1.1.2

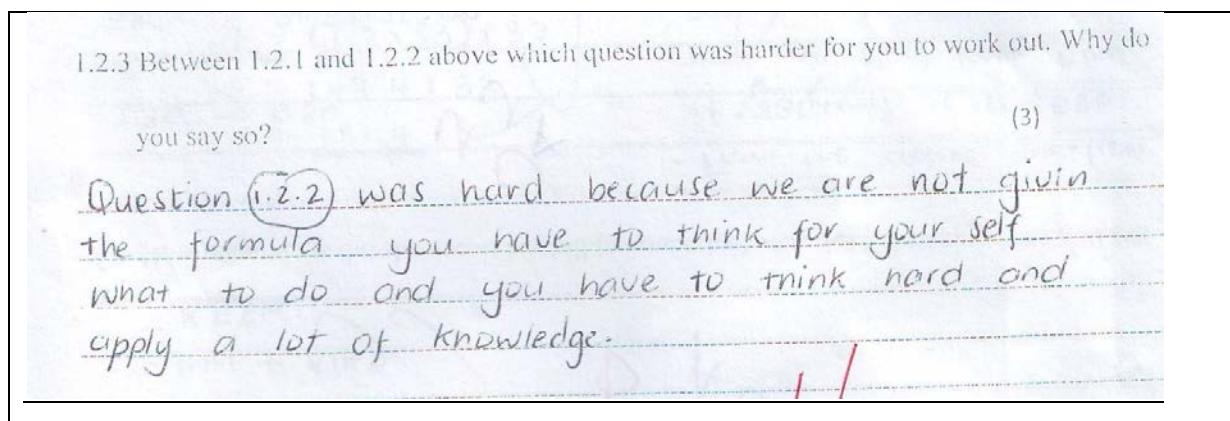
The response from the learner indicates that the learner realised that question has changed and new strategy has to be formulated in order to solve Question 1.1.2. This is evident from this response when the learner states the question was difficult because he or she needed to invert operations i.e. multiplication to division. The learner realises the distinction in the two questions.



**Figure 4.17a: First learner reflection on Question 1.1.3 (L322).**

#### 4.5.2 Reflection on Questions 1.2.1 and 1.2.2.

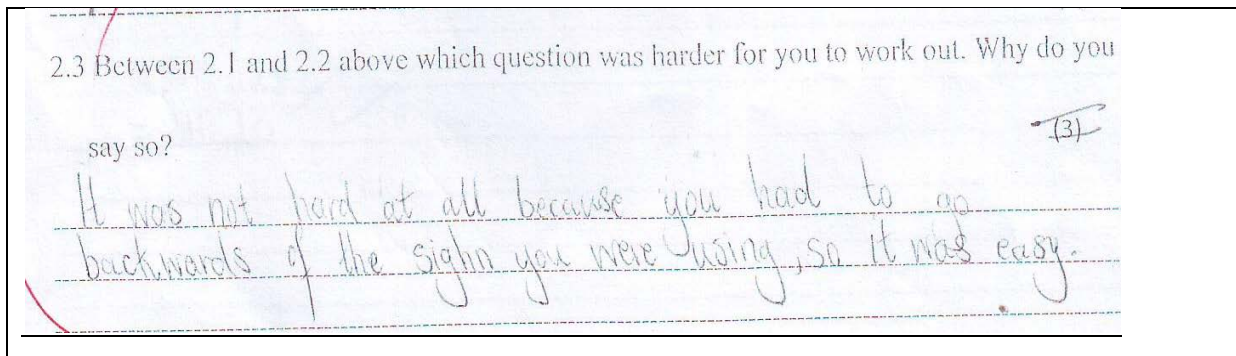
From this extract, the learner states that the question was hard because there was no formula for the question. From this response I assume two things, namely the question was hard as stated and the learner realised also that question has changed because he or she cannot use the formula in the same way as it was used to answer the preceding question.



**Figure 4.17b: Second learner reflection on Question 1.2.3.**

#### 4.5.3 Reflection on Question 2.1 and 2.2.

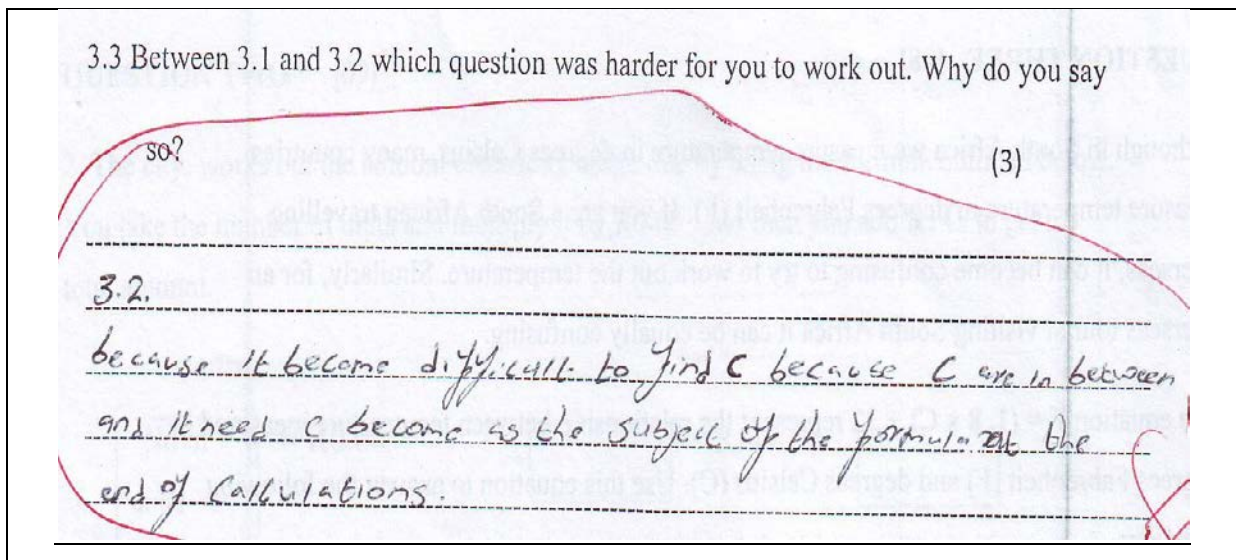
Figure 4.1c shows again the learner's reasoning with regards to 2.1 and 2.2. The learner writes: 'it was not hard at all because you had to go backwards of the sign you were using, so it was easy' [sic]. From this it is clear that the learner holds a sense of discernment with regards to the positioning of the variable. From this response, I infer that the learner implies questions 4.1 to have been easy, because it was a forward calculation. The learner writes about going backwards in the other problem (the other problem refers to question 4.2, by reversing the operation signs, which imply a forward movement in the other problem (the other problem refers to question 4.1). I have interpreted this reasoning as an indication that learner is conscious of the changes in the placement of the unknown in the problems 2.1 and 2.2. The distinction in the positioning of the unknown is evident in the learner response, and the learner is even clearer about how to solve the two problems, i.e. to use a strategy that will either reverse or undo mathematical operations.



**Figure 4.17c: Third learner reflection on Question 2.3**

#### 4.5.4 Reflection on Question 3.1 and 3.2

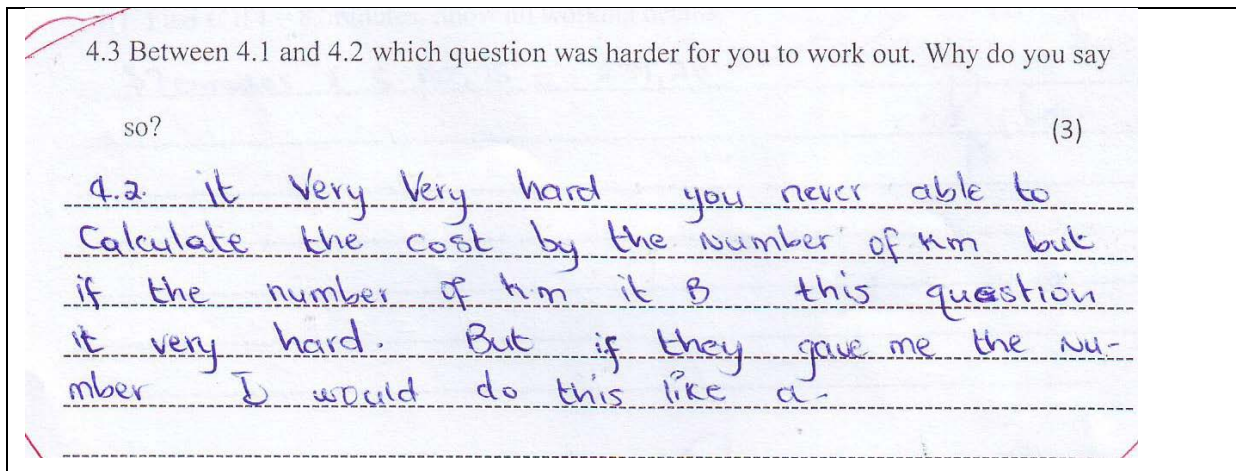
Learner reflects in this response the questions has changed, the position of the unknown, namely C, has changed places and this has made question difficult



**Figure 4.17d: Fourth learner reflection on Question 3.3.**

#### 4.5.5 Reflection on Question 4.1 and 4.2

From this response the learner clearly states that 4.2 was difficult and the learner further indicates having noticed the distinction in the type of question when she states



**Figure 4.17e: Fifth learner reflection on Question 4.3 (L332).**

Figure 4.11e reveals the ‘distinction’ in this reasoning, where the L332 states that you will never be able to calculate cost if the variable is in the number of kilometres (start-unknown) position, but it would have been easier for him if the variable was positioned as in problem 4.1 (result-unknown). Figure 4.11e clearly indicates that the learner is conscious of the variable and that the variable has changed places in the two questions. The learner further implies that the new position in which the unknown is placed has made the problem more difficult. Secondly, in this extract the learner admits that with problem 4.2 (start-unknown), the solution will never be found. It is clear from this comment that this learner does not know how to solve this problem, and that according to the learner, there is no strategy to solve Question 4.2. In this case, the learner comments clearly indicate that the learner is conscious of the fact that the position of the unknown in the formula brings some impact. The impact will either make the problem easy to solve or difficult to solve.

## CHAPTER 5

### 5. FINDINGS, DISCUSSION AND CONCLUSION

#### 5.1 Introduction

In this chapter, I provide answers to the research questions by analysing and interpreting data and linking the findings to the literature review and theoretical framework highlighted in Chapter 2.

The data analysis outlined in Chapter 4 was categorized into three sections according to the questions in the research instrument, namely result-unknown questions, start-unknown questions and reflection questions. In this chapter, I examine how the data can be used to provide answers to the research questions.

The purpose of this study was to explore the strategies used by ML learners in responding to result-unknown and start-unknown questions. The main research question related to this is: *How do Grade 11 ML learners respond to result-unknown and start-unknown questions set within various contexts?*

There are four sub-questions that have been drawn from the main research question and these are:

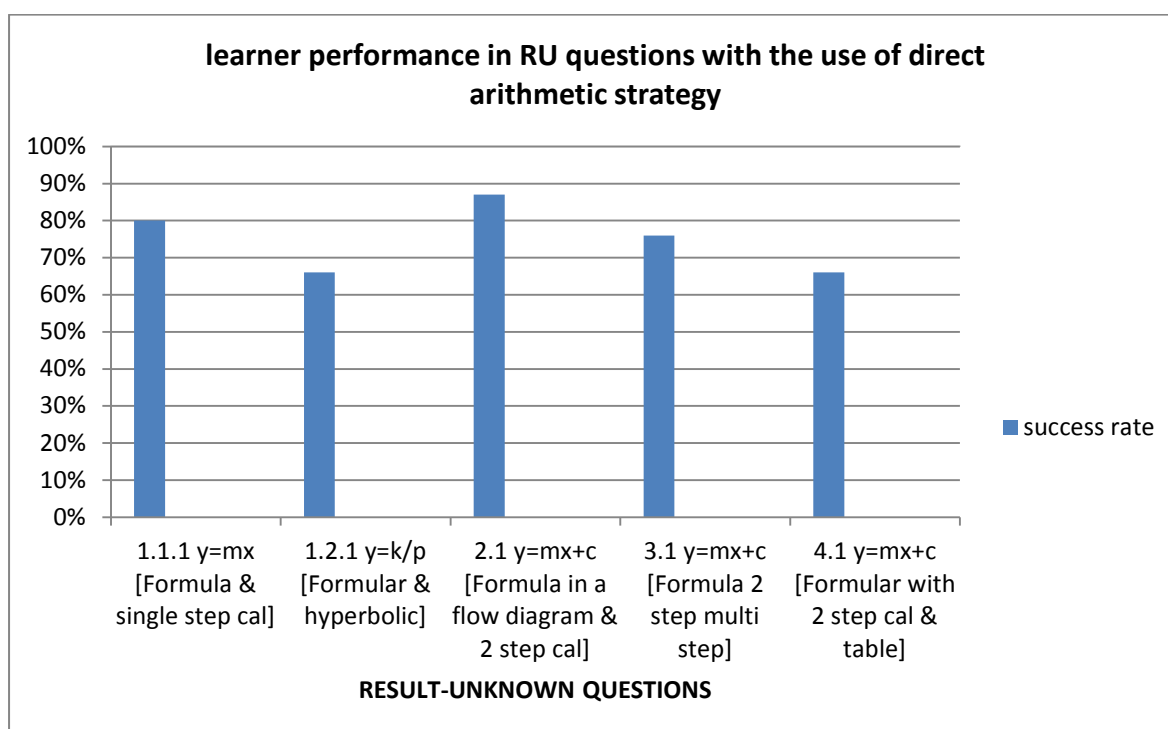
- 1A. How do Grade 11ML learners respond to the five-result-unknown questions, and what are the strategies that they use to solve the problems?
- 1B. How do the learners respond to the five start-unknown questions, and what are the strategies that they use to solve the problems?
- 1C. Which of the two types of tasks (result-unknown or start-unknown) do the learners experience as more difficult?

1D. What are learners' perceptions about which questions were more challenging, and why they were experienced as more challenging?

## 5.2 Answers to Research Question 1A

How do Grade 11 ML learners respond to the five result-unknown questions, and what are the strategies that they use to solve these problems?

Figure 5.1 below shows learner performance in all five result-unknown questions with the use of the direct arithmetic strategy.



**Figure 5.1: Learner performance in RU questions with the use of direct arithmetic strategy.**

Three out of the five result-unknown questions was based on a relationship of the form  $y = mx + c$ , one was in the form  $y = mx$  and one question was in the form  $y = \frac{k}{x}$ ,  $x \neq 0$ . All five questions included some context, and the relationship was sometimes given in symbolic form, as a flow diagram, or in a table format.

Across all five result-unknown questions in the four tasks, data reveals that the preferred strategy by learners was the *direct arithmetic strategy*. The average success rate with the application of direct arithmetic strategy is 75% across all questions. Furthermore, the direct arithmetic strategy has been used with an average rate of 4% in calculation errors. This percentage of calculation errors indicates that there were a few learners who made calculation errors, and that this indicated *strategy efficiency* by learners. There were other learner responses that I categorised under other incorrect strategies, such as number grabbing, unclear answers, no response and incorrect substitution; however, such responses were minimal. This indicated that most of the learners were able to solve these problems, which required arithmetical reasoning.

The best performance in the result-unknown questions was observed in Question 2.1, where the formula was explained in words as well as in the form of a flow diagram. It is important to note that learners performed better in Question 2.1 than in Question 1.1.1, which modelled a simpler relationship of the form  $y = mx$ . The approach of defining equations using flow diagrams makes the equation more accessible to the students, because of a closely-related sequence of representations (Brunner, 1963). Flow diagrams reflect a sequence of representations, which are based on concrete operations, and which serve as a guide to students. It is possible that the flow diagram made it easier for learners to carry out the required calculations.

This result, where the learners were revealed to be most successful with the question that was described by a flow diagram, may be linked to De Lima and Tall's (2008) explanation of a conceptual embodiment. De Lima and Tall (2008) proposed that Piaget's notions of empirical and pseudo-empirical abstraction lead to two different ways of operating on the world; on the one hand, focusing on objects and their properties; and on the other, focusing on the actions carried out on the objects of the world. The first is what they refer to as a conceptual embodied world and the second is a proceptual symbolic world (as in the process-object encapsulation development described by Dubinsky, 1991). In this situation, learners were more successful at using the flow diagram (Question 2.1) to carry out a calculation based on the cognitively more demanding equation ( $y = mx + c$ ) than the calculation based on the simpler equation ( $y = mx$ ), which did not have a flow diagram. Furthermore, the success rate of Question 2.1 was much higher than that of Question 3.1 and 4.1, whose underlying equations were all of the same form, but which were presented as a symbolic equation only, and in a table form and equation form, respectively.

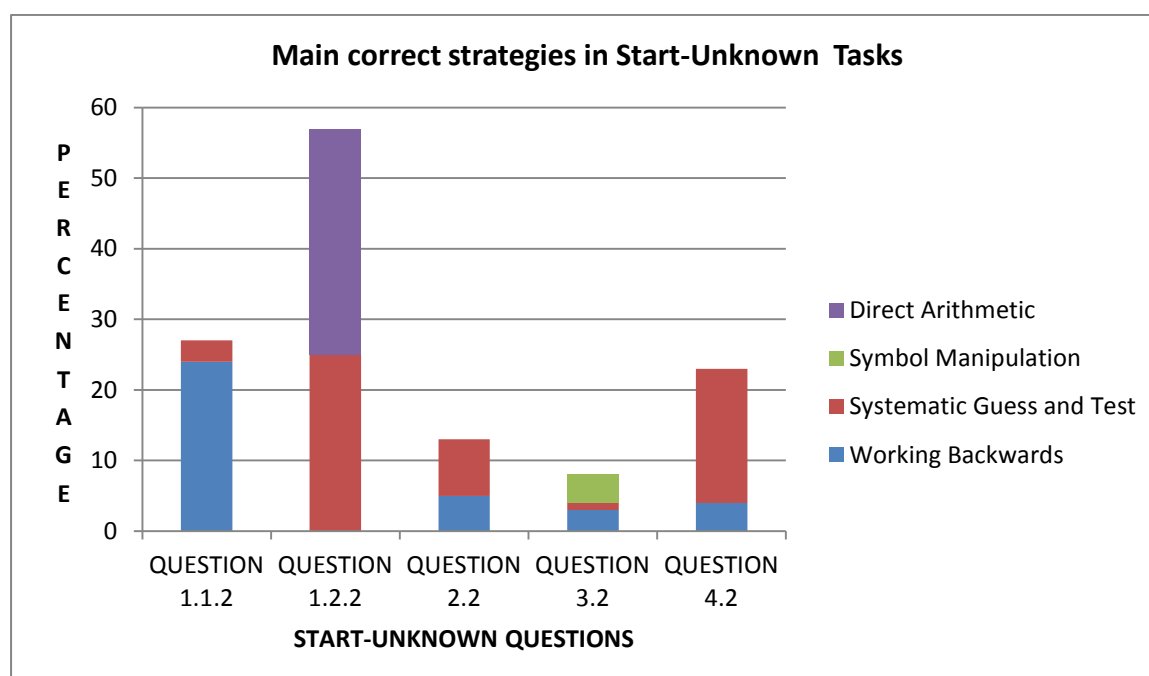
It is plausible that the flow diagram acted as a concrete embodiment, helping learners identify what operation they were supposed to carry out. The flow diagram had an empty box, which signalled that the input value should be placed there, and the operations could then be carried out as if it was a physical object that was being manipulated. However, the flow diagram can be seen as a conceptual embodiment. De Lima and Tall (2008) distinguished between conceptual embodiment and procedural embodiment, which is exhibited for example when symbols are moved without meaning. In their study, De Lima & Tall (2008) suggest that a robust, concrete embodiment may help them move to a conceptual embodiment, which might then facilitate further movement in understanding. However, many learners develop procedural embodiments, which may work for one situation, but when trying to extend it to other situations, they develop misconceptions. However in this situation, the flow diagram seems to work as a conceptual embodiment, because it provides a different but more accessible representation of the formula.

In an examination of first-grade students solving simple one-step problems, results indicated that students correctly solved 100% of result-unknown questions, and 33% of start-unknown Questions (Riley et al., 1983). Even though these students were not using algebraic methods to solve these problems, it explains the nontrivial performance on start-unknown questions. This seems to be a general pattern or trend even for college level students when they solve multi-step problems with rational numbers. The findings of their examination concluded that there is a general pattern of problem solving performance, favouring result-unknown questions over start-unknown questions. In this study, data seems to show a similar pattern to that observed in the study by Riley et al. (1983). Learners are likely to be more successful in result-unknown questions than in start-unknown questions. Furthermore, a study conducted by Nathan et al. (2000) investigated the *difficulty factor analysis* (DFA) with regards to the impact brought about by the placement of the unknown quantity in the mathematical equation. The study revealed that the position of the unknown on the question had a significant effect on the question difficulty. The difficulty factor analysis revealed learners to have exhibited lower performance on start-unknown questions than on result-unknown questions.

### 5.3 Answers to Research Question 1B

How do the learners respond to the five start-unknown questions, and what are the strategies that they use to solve the problems?

Figure 5.2 shows a graphical representation of the success rate in each of the start-unknown questions in the four different tasks. The graph indicates the success rates achieved with strategies that led to the correct answers. The next graph (figure 5.3) will show strategies that were used by learners, but which are incorrect strategies.

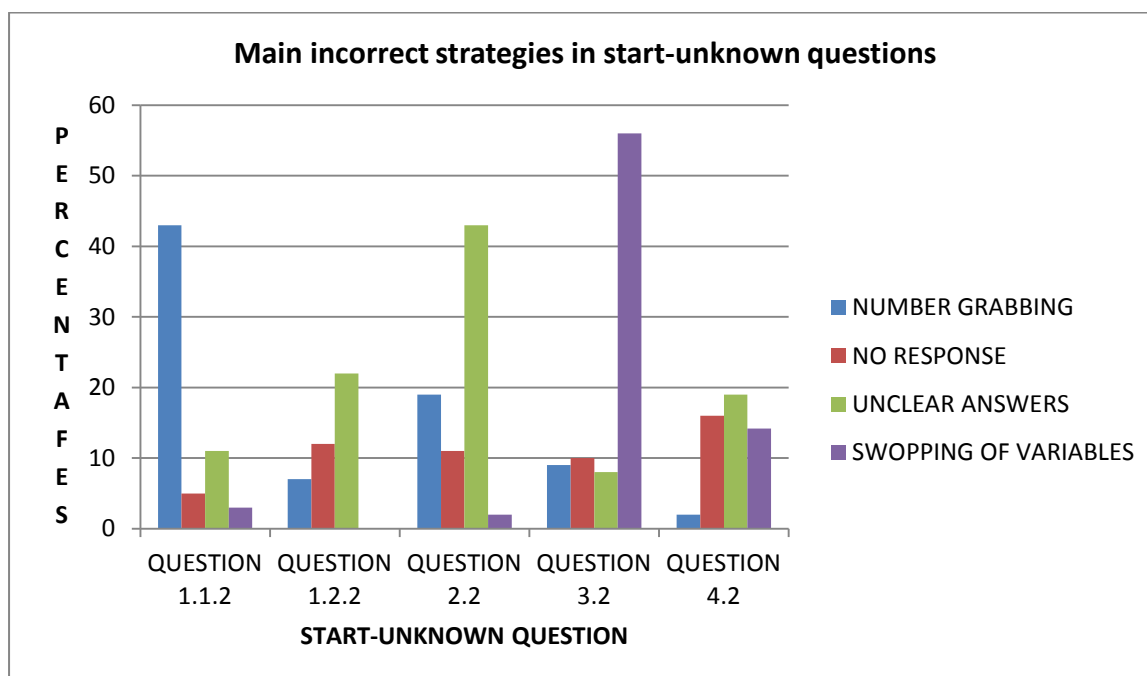


**Figure 5.2: Graphical representation of learner performance with the use of correct strategies.**

Across all five start-unknown questions, the successful strategies are working backwards, symbol manipulation and systematic guess and test, even though data indicates that these strategies show minimal or negligible success.

With question 1.1.2, data reveals that the preferred successful strategies (with minimal success) are working backwards and systematic guess and test, with the success rate of 24% and 3%, respectively. Question 1.2.2 is of the form,  $y = \frac{k}{x}$ , where  $x \neq 0$ , in context. For question 1.2.2, the preferred successful strategy (with minimal success) with a success rate of 25% is systematic guess and test, while the other preferred successful strategy for question 1.2.2 is direct arithmetic strategy with a success rate of 32 percent. With question 2.2 ( $y = mx + c$ , in contexts and in a form of a flow diagram), data shows that the preferred successful strategies to have been systematic guess and test and working backwards, with a minimal success rate of 10% and 5%, respectively. The successful strategies for question 3.2 ( $y = mx + c$ , in context) are symbol manipulation, working backwards and systematic guess and test, with a minimal success rate again of 4%, 3% and 1%, respectively. Finally, question 4.2 ( $y = mx + c$ , in context and with a table of values), the preferred successful strategies were systematic guess and test, and working backwards, with a success rate of 19% and 4%, respectively.

Figure 5.3 shows a graphical representation of incorrect strategies that were revealed across all five start-unknown questions. The main incorrect strategies are number grabbing, no response, unclear answers and swopping of variables.



**Figure 5.3: Graphical representation of learner performance with the use of incorrect strategies.**

With regards to start-unknown questions, unlike in the case of result-unknown questions, there is a wide list of solution strategies preferred by learners, including incorrect strategies, therefore, each problem will now be discussed in turn.

### 5.3.1 Question 1.1.2

Question 1.1.2 is the second question in which learners performed better with the application of formal procedure. The successful formal procedure is working backwards at a rate of 24%, which represent a higher rate of success when compared to the use of the working backwards strategy in the three start-unknown questions of the form  $y = mx + c$  (2.2; 3.2 and 4.2). The success rate of these three start-unknown questions: 2.2; 3.2 and 4.2; are 5%, 3% and 4%, respectively.

One reason for the greater success rate with the working backwards strategy in this task is the fact that there was only one operation or one step, compared to the other linear problem situations. The other three start-unknown questions carried an extra additional step i.e.  $y = mx + c$ . For question 1.1.2, data shows that the preferred successful solution strategy is working backwards, but for the other three questions, data reveals that working backwards is no longer found to be the preferred successful strategy, but instead learners opted for an informal strategy of systematic guess and test.

The additional cognitive demand required in the strategy of working backwards may also explain the change in the choice of strategy from formal to informal in the other three start-unknown questions (2.2; 3.2 and 4.2). I had expected that due to the heavier cognitive demand brought about by an extra step in the other three questions; learners would continue to attempt to apply working backwards, but with a large number of responses classified under incorrect working backwards. But data indicates the contrary, namely that there is a shift from using formal procedures to the use of informal procedures. In question 1.1.2 there were 83 learners that solved the problem using working backwards. With problems 2.2, 3.2 and 4.2, the number of learners who attempted to use working backwards were 17; 9 and 9, respectively. On the other side, the number of incorrect attempts with working backwards did not increase either.

It is also important to note that despite an attempt by some learners to apply the formal working backwards procedure with this question, most learners did not know what to do. This is evident in the fact that about 43% of the learner responses categorised as 'other'

for question 1.1.2, are classified under incorrect strategy number grabbing, where they ‘grabbed numbers and carried out arbitrary operations on these numbers. Furthermore, incorrect substitution in question 1.1.2 accounted for 8 percent of learner strategy.

As far as systematic guess and test is concerned, with question 1.1.2, data shows a low success rate at 3 percent. This figure is in contrast to the use of systematic guess and test right across all start-unknown questions of the form  $y = mx + c$ , where the average rate of success was 10 percent. The strategy was also used successfully in question 4.2, where there was a 19% success rate.

### 5.3.2 Question 1.2.2

It is in this start-unknown question that learners performed best, with a 57% success rate with combined strategies, direct arithmetic and systematic guess and test. The direct arithmetic strategy accounted for 32% of correct answers. Question 1.2.2 represented the unique situation where the swapping variable strategy was correct, and which allowed the learners to convert to a result-unknown question, upon which the direct arithmetic strategy could be applied. The relationships between the two variables can be seen as a hyperbolic problem:  $xy = k$ .

In Question 1.2.2 the formula that was given was  $C = \frac{R1050.63}{p}$ , where  $C$  represented the cost per passenger and  $p$  represented the number of passengers or  $y = \frac{k}{x}$ . In this question, learners were required to find the number of passengers in the car if each passenger paid R150.09 for petrol. Question 1.2.2 is a start-unknown question and to solve this question most learners swapped positions of the variables  $C$  and  $p$ . The variable  $p$  then became the subject of the formula, once positions of  $C$  is swapped in for  $p$ , the entire question changes from being a start-unknown question to a result-unknown question. The swapping of variables is an incorrect strategy, however, due to the multiplicative relationship between the variables, where  $C$  was swapped with  $p$  the result is still true (i.e.  $C = \frac{R1050.63}{p}$  becomes  $p = \frac{R1050.63}{C}$ , which holds true. As a result, learners were able to perform far better in this start-unknown question when compared to the other four start-unknown questions, by using a combination of strategies including the swapping of variables, direct arithmetic strategy, and then systematic guess and test. In figure 5.2 showing successful strategies, I did not include

swopping of variables, and instead reported on the second step of the learner calculation, which involved calculation by direct arithmetic strategy.

With regards to systematic guess and test, learners applied this strategy with success in question 1.2.2 at a success rate of 25 percent. In analysing the data for all systematic guesses, including learner reflections, in question 1.2.3, I concluded that the success rate with systematic guess and test was boosted by the fact that the numbers in question 1.2.2 were ‘manageable numbers’ (easy to count by hand or on a calculator). For instance, it is not difficult to guess which number to multiply with R150.09 to reach R1050.63, i.e. a few guesses requiring simple calculations leads quickly to  $7 \times R150.09 = R1050.63$ . This is most evident in learner calculations, where they answer problem 1.2.2, they simply show  $7 \times R150.09 = R1050.63$ . The mere emergence or appearance of 7 was interpreted as a systematic guess and test. Most of learner solutions show 7 as indicating the number of people in the car, without showing calculations; or learners would alternatively show that  $7 \times R150.09 = R1050.63$ .

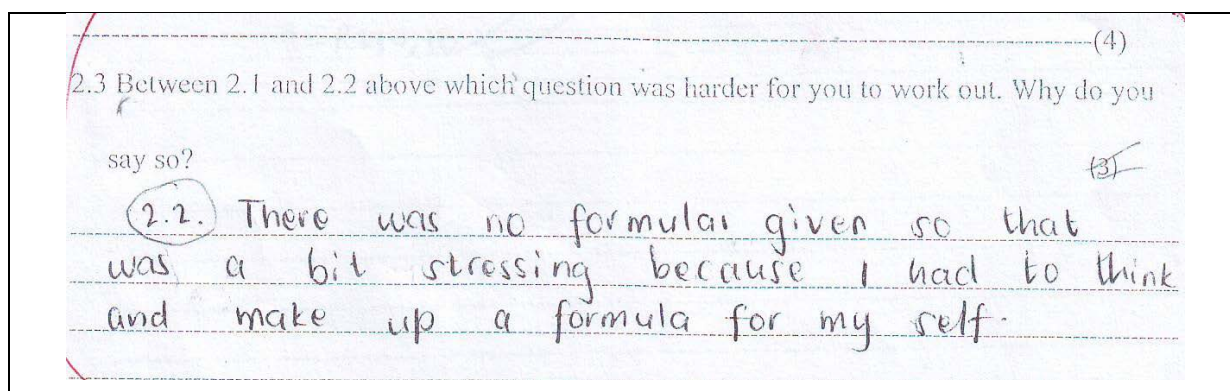
With regards to systematic guess and test in problem 3.2, the success rate of the systematic guess and test as a preferred solution strategy was just 1% (compared to 1.2.2, where it was 25 percent). This difference in success rate in using the guess and test strategy may be explained by the fact that it was much more difficult to guess and then test in Question 3.2, where the answer was 33.9, as compared to question 1.2.2, where the answer was seven. Furthermore with Question 1.2.2 there were two ways to ‘test’, where one could either work out  $\frac{R1050.63}{7}$ , or  $7 \times R150.09$ . This may illuminate what caused a low success rate with the use of systematic guess and test in question 3.2 when compared to question 1.2.2.

Continuing with the systematic guess and test strategy, learners performed better on question 4.2 than they did on 3.2 with this strategy, although both involved working with ‘messy’ numbers and both using  $y = mx + c$ . With question 4.2 however, there was a table provided, where as a result, there were a few worked examples. Perhaps it was easier to guess and test because the last value of the independent variable appearing in the table before the unknown was 592 (the correct answer was 600), which may have helped learners to narrow down the possible numbers that they could substitute. I therefore concluded that the table made it easier to use systematic guess and test for question 4.2.

### 5.3.3 Question 2.2

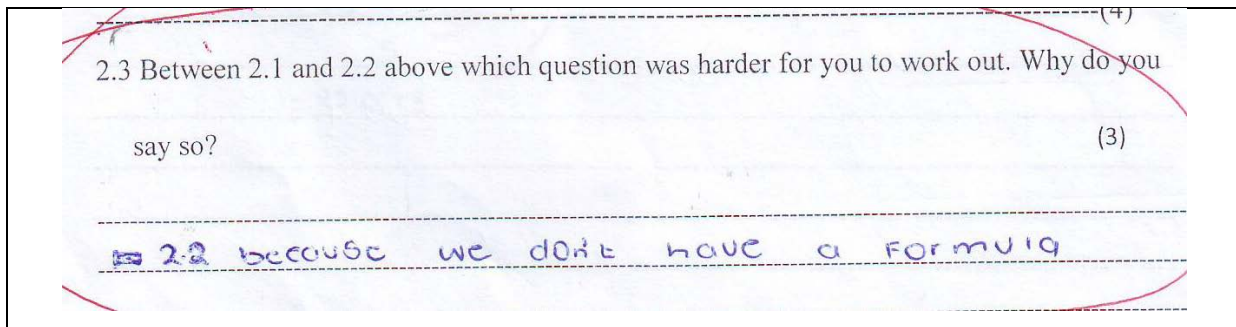
Question 2.2 is a start-unknown question, which did not carry an algebraic formula, but the formula was explained in words, and also described in a flow diagram. The rationale for the use of a flow diagram in 2.2 was that the flow diagram was going to make this question easier. The flow diagram was specifically going to serve as the ‘visual mediator’. Visual mediators are visible objects, which participants in the mathematical discourse can use as objects with which to think (Sfard, 2008). However, data seems to indicate that learners continue to struggle, despite the visual mediators.

For question 2.2 in particular, many learners did not know what to do, because most learner responses are classified under unclear answers, with no response and number grabbing at a rate of 73 percent. By studying the learner reflections regarding the difficulty of the problem, the confusion about how to go about securing a solution was apparent. The following extract (fig.5.4) from L321 indicates that the learner tried to make up her own formula in order to solve the problem. The learner statement could be interpreted as indicating there was no formula for question 2.2, or it could mean that the learner thought she ought to express the formula from the flow diagram in symbolic form, before she could solve for the unknown.



**Figure 5.4: First learner reasoning relating to Questions 2.1 and 2.2.**

Figure 5.4 shows the reasoning from the second learner about why question 2.2 was found to be difficult. L333 states clearly that in question 2.2, there was no formula to use. This is what caused the question 2.2 to be difficult. This point has been made repeatedly in Chapter 4.

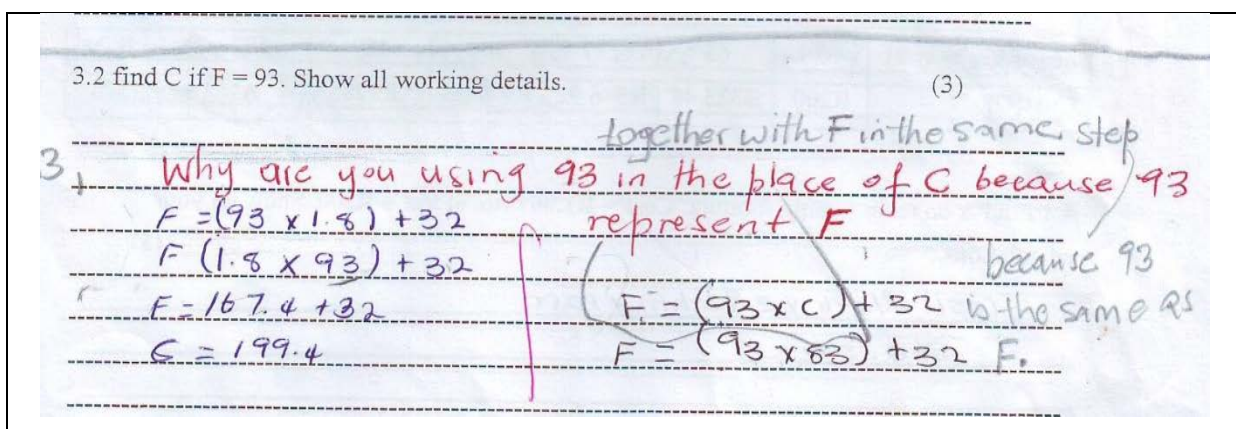


**Figure 5.5: Second learner reasoning relating to Questions 2.1 and 2.2.**

### 5.3.4 Question 3.2

Question 3.2 is taken from the famous temperature conversion formula,  $F = (1.8 \times C) + 32$ , and learners were informed that if  $F = 93$ , the formula can be used to calculate the value of  $C$ . Learners solved this problem by applying a symbol manipulation strategy at a rate of 4 percent. The second strategy that learners used to solve this problem, to a limited extent, was working backwards with a success rate of 3 percent. Only 1% of learners solved this problem by applying systematic guess and test successfully.

Firstly, I have alluded to the research that has long recognised that learners misunderstood the meaning of the equal sign as an operator, where it is instead taken as a symbol inviting them to ‘do something’ or to ‘find something’, rather than a relational symbol, signifying equivalence or quantitative sameness (Essien, 2009). This is shown in learner calculation in figure 5.6 below.



**Figure 5.6.: Misconception of the equal sign.**

Interview with L312 revealed a misconception about the equal sign in question 3.2. In question 3.2, the learner is required to find the value of C if  $F=93$  and use the formula,  $F = (1.8 \times C) + 32$ . A learner revealed in the interview that if  $F = 93$ , this meant to her that the equation ought to be restructured as follows:  $F= 93 (1.8 \times C) +32$ . According to the learner, the given part  $F = 93$  cannot be separated, which means you cannot replace F with 93, or vice versa. The equal sign is understood as a unidirectional symbol.

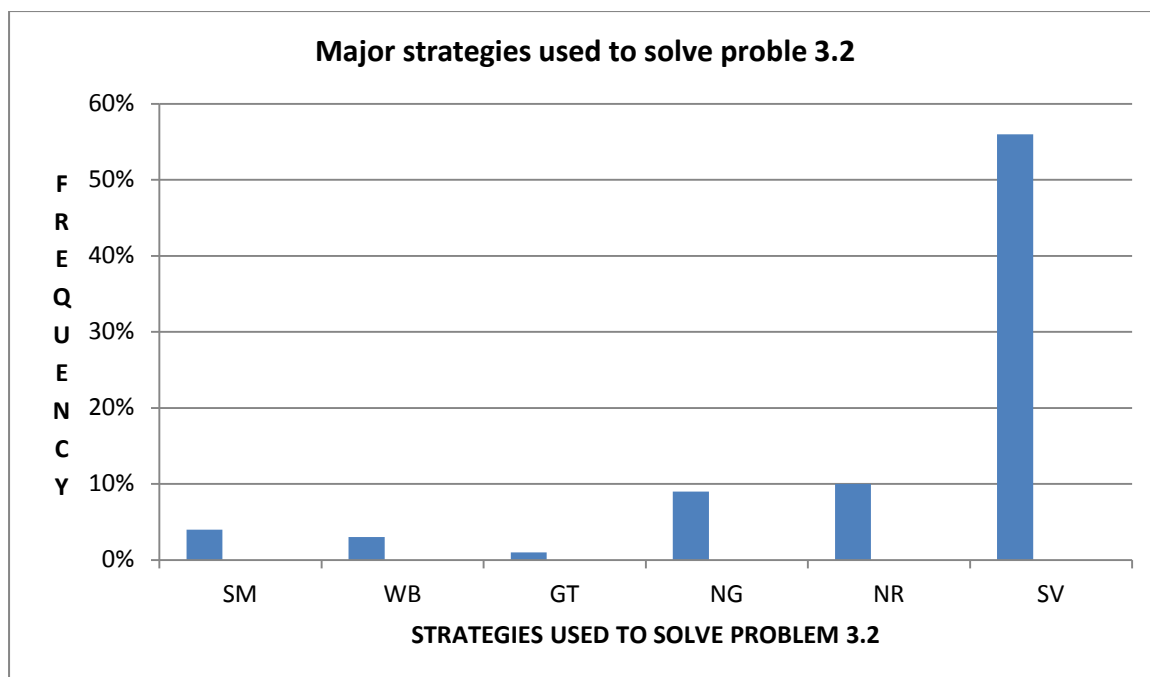
Although the study has looked at strategies employed by learners in solving particular types of questions, their struggles may also be explained by using APOS<sup>1</sup> theory, applied to the notion of a linear equation as well as a hyperbolic relationship  $y = \frac{k}{x}$ . For the purpose of the discussion, let us consider just the linear equation  $y = mx + c$ , which was the basis of the questions in Questions 1.1.1; 1.1.2; 2.1; 2.2; 3.1; 3.2; 4.1 and 4.2.

For example, when considering Question 3.1 and 3.2, learners who are able to work out the value of F, when given the value of C ( $F = (1.8 \times C) + 32$ ), may be seen as working on an action level of understanding the linear equation. They can work out, step by step, the various calculations, without interiorising them. The C-value acts as an external prompt that allows a learner to then carry out the other steps. Hence, when they are asked to find the value of C given F, due to the fact that the equation has not been interiorised into a process, they cannot access the reversal of that process, and cannot see past the given direct question; and they then try to operate on the formula to convert it into a question that can be solved in a direct calculation. Their struggles are linked to not being able to transform their understanding of the linear equation from an action to a process, or even an object. The reason for that may be because with the subject ML, the focus is not on manipulating and transforming mathematical objects. The ML curriculum document makes it clear that skill in such algebraic manipulation is not a focus of ML, but that “as a rule of thumb, if the required calculations cannot be performed using a basic four-function calculator, then the calculation is in all likelihood not appropriate for Mathematical Literacy” (DoBE, 2011, p.8).

In fact, the swapping of variables was a serious and incorrect strategy that learners used consistently to solve this problem. Figure 5.7 below provides a good view of learner performance for question 3.2. It shows the incorrect popular strategy of swapping variables.

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<sup>1</sup> The possible application of APOS theory in terms of a breakdown of action, process and object level understandings of a linear equation appears in Appendix B



**Figure 5.7: Graphical representation of major strategies used to solve Question 3.2.**

Data further reveals that with question 3.2, the incorrect popular strategy is swapping of variables at a rate of 58.9 percent. The swapping of variable is born out of a struggle to apply the necessary algorithm to convert the formula, where learners lack the necessary algebraic skills to manipulate the formula. Again, this trend is interpreted as an indication that learners lacked formal algebraic skills to solve this problem.

### 5.3.5 Question 4.2

Question 4.2 is a start-unknown question of the form  $y = mx + c$ . The formula,

$\text{Cost} = R1.96 \times \text{no. of km} + R200$  was provided, as well as table of values. Question 4.2 required learners to acquire the corresponding number of kilometres, if the cost of hiring a bus is R1375.

Figure 5.8 shows learners to have achieved minimal successes in solving this problem by applying systematic guess and test, and working backwards at a rate of 19% and 4%, respectively. With regards to errors, systematic guess and test and working backwards accounted for 4% and 4%, respectively. Figure 5.8 further shows that this question occupies the third spot in the order of success in terms of learner performance across all start-unknown

questions. Responses accounting for 17% are classified for this problem as incorrect substitution.

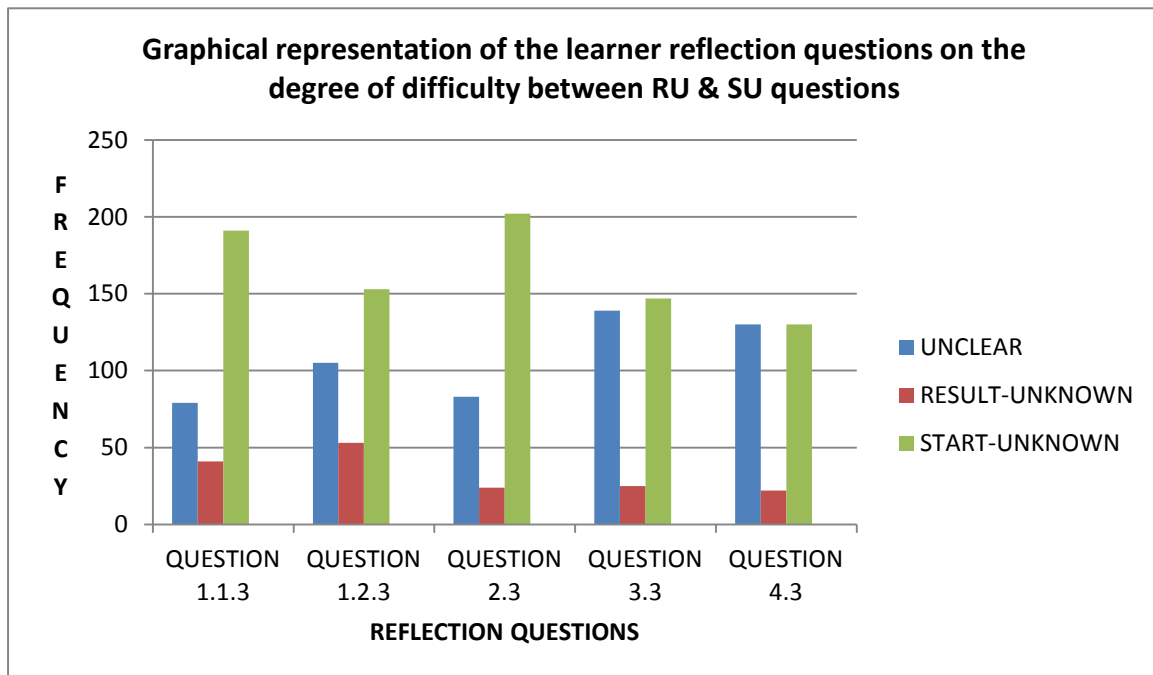
This is due to the fact that learners substituted other values, different from 1376 in the formula for cost. Those responses involving a substitution of 1376, were classified as the swopping of variables, which was done by 13% of the learners. Unclear answers and no response accounted for 19% and 16% of answers, respectively. The high number of responses in these incorrect categories of unclear answers, no response and swopping of variables, also indicates that learners seemed not to know what they were supposed to do, or that they were unable to apply the formal algebraic strategies.

#### **5.4 Answers to Research Question 1C and 1D**

1C and 1D ask: 'which of the two types of questions (result-unknown or start-unknown) do the learners experience as more difficult?' and 'what are learners' perceptions about which questions were more challenging and why?', respectively.

Figure 5.8 shows a graphical representation of the learners' reflections in the five reflection questions. In each reflection question, learners were asked to state with reason which question between the result-unknown and the start-unknown they consider difficult. Learner reflections in each question have been classified into three labels, namely '0', '1' and 'N'. If a learner regards result-unknown question as difficult, a label '0' is allocated, similarly, a learner who regards the start-unknown as difficult, a label '1' is allocated. Finally, if the

learner's response is unclear, a label 'N' is allocated.

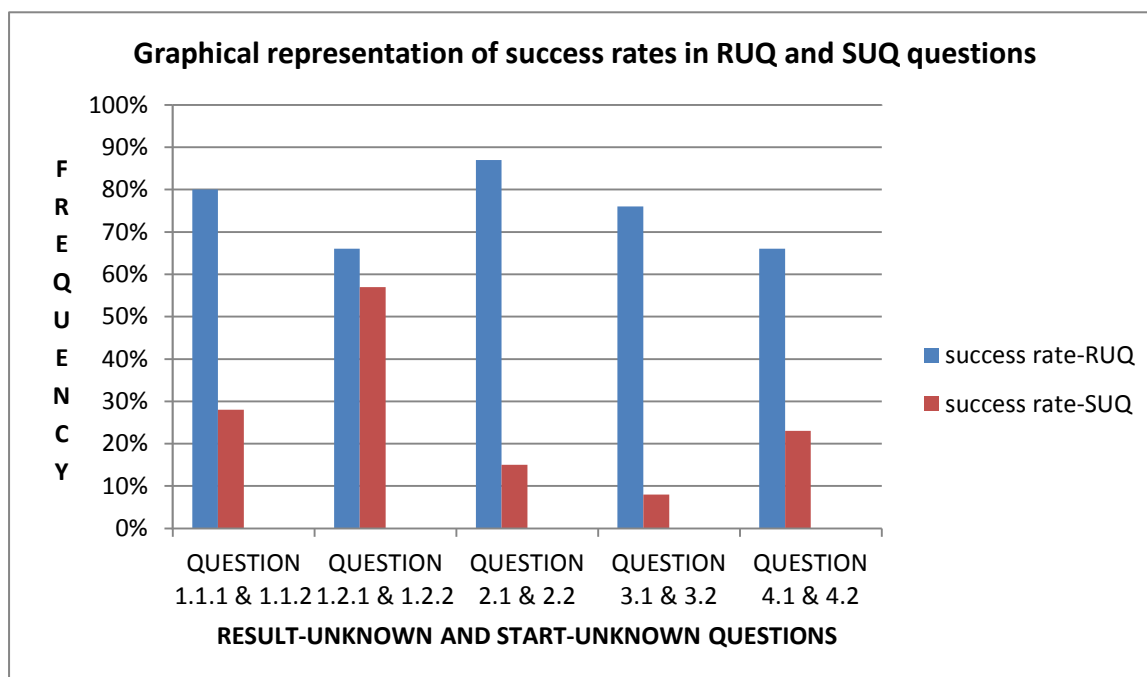


**Figure 5.8: Graphical representation of the learner reflection questions on the degree of difficulty between RU and SU questions.**

Data from this quantitative count indicates that learners feel that the start-unknown (SU) questions are more difficult than the result-unknown (RU) questions. Furthermore by comparing the success rates of each of the tasks, it can be seen that in each case the start-unknown version was always more difficult than the result-unknown. Evidence is shown by the success rate; where data reveals there was an average of 74% success rate in result-unknown questions when compared to an average success rate of 25% in start-unknown questions.

Figure 5.9 shows a graphical representation of learner performance in result-unknown questions and start-unknown questions. Data shows that learners performed well in result-

unknown questions when compared to start-unknown questions.



**Figure 5.9: Graphical representation of success rates in RUQ and SUQ problems.**

As can be seen for each question context, the learners did much more poorly at the start-unknown version than at the result-unknown version. Possible reasons for this have been put forward in section 5.3. Furthermore, in studies by Spangenberg (2012) and Spangenberg and Grigorenko (1993), the authors found that certain thinking styles correlated to a learner's success in academic tasks, whereas other thinking styles tended to correlate negatively to success in the same tasks. With regards to thinking styles, Mathematics deals with concepts as ideas or abstractions, which learners have to bring to solve a mathematical problem to enable 'to understand the world' (DBE, 2011b, p.8). Data shows a shift in or lack of application of 'creative and logical reasoning' in solving these tasks, which can be attributed to a difference in thinking style.

The data from reflection questions reported in Chapter 4 have also indicated that learners find start-unknown questions to be more difficult than result-unknown questions.

A study conducted with algebra teachers by Nathan et al. (2000), reported that teachers rank start-unknown questions as more difficult than the result-unknown ones. Furthermore, the study revealed that school textbooks arrange content material in a manner that indicates that start-unknown are more difficult than result unknown questions.

Both Mathematics teacher rankings and textbook content arrangement rankings agree that the start-unknown questions are more difficult for learners when compared to the result-unknown. The study was applicable to learners at Level 8. In this study, the learners are at Grade 11, and have completed Mathematics up to Grade 9, where they were taught how to solve linear equations, which were much more complicated than the ones which appear in this study.

According to Groetsch (1999), the positioning of the unknown in a mathematical problem makes a distinction as to whether it is a direct problem or an inverse problem. Groetsch (1999) describes result-unknown questions as direct problems and start-unknown questions as inverse problems, because of this positioning of the unknown. It was of interest to find out whether learners could discern this distinction, highlighted by Groetsch (1999), and whether Grade 11 ML learners are *conscious* of the impact that is brought by two forms of placement of the variable in the mathematical equation, which leads to result-unknown questions or start-unknown questions. The analysis of the learner reflections indicate that most learners acknowledged the differences in the type of the two problems, as well as the reflections; and this is also seen to corroborate the finding that showed the start-unknown to have been experienced as more difficult. Learner reflection responses reveal that the learners have distinguished between the two types of questions. The responses make reference to the position of the unknown, and they acknowledge that the change in position has made the question difficult, or has made the question an entirely new question, that requires a new strategy to solve. In the following extract, I quote a learner response to the reflection question 2.3. A learner writes: 'it was not hard at all because you had to go backwards of the sign you were using, so it was easy' [sic]. The learner clearly reveals here a discernment of the positioning of the variable. From this response, I infer that the learner implies question 4.1 to have been experienced as easy, because it was a forward calculation. The learner writes about going backwards in the other problem by reversing the operation signs, which imply a forward movement in the other problem. I have interpreted this reasoning as an indication that the learner is conscious of the changes in the placement of the unknown in the questions 2.1 and 2.2. The distinction in the positioning of the unknown is evident in the learner response, and the learner is even clearer about how to solve the two problems.

Learners revealed this 'distinction' in this reasoning, when a learner states that it is impossible to calculate cost if the variable is in the number of kilometres (start-unknown) position, but that it would have been easier for him if the variable was positioned as in

question 4.1 (result-unknown). The reflection shows clearly that the learner is conscious of variable, and that the variable has changed places in the two questions. The learner further implies that the new position in which the unknown is placed has made the problem more difficult. Secondly, from this extract, the learner admits that with question 4.2 (start-unknown), the solution will never be found. It is clear from this comment that this learner does not know how to solve this problem i.e. there is no solution strategy. In this case the learner's comments clearly indicates that the learner is conscious of the impact brought about by the position in which we find the unknown.

## **5.5 FURTHER ISSUES ARISING FROM THE STUDY**

The data also provided evidence of some deep-seated misconceptions, such as evidenced in the conjoining and swopping of variables. Conjoining was more evident in question 3.2 and the second misconception was most evident in question 3.2 and question 4.2. I have discussed the swopping of variables in detail in question 3.2; however, I would like to add to what I mentioned in Chapter 2 with regards to algebraic errors. Errors are assumed, in many cases, to be the result of reasonable attempts to apply previously acquired knowledge to new knowledge (Matz, 1980). According to Matz (1980), the swopping of variables might be interpreted as an attempt by a learner to bridge the gap between known rules and unfamiliar problems. This is what Matz refers to as an extrapolation technique. The errors that students make on procedural (mathematical) tasks are the consequence of rule-determine behaviour. This means that mistakes result from the application of an incorrectly coded procedure i.e. a procedure with a 'flaw' in it. This is observed in the two misconceived strategies of the conjoining and swopping of variables. Learners are applying perfectly good procedures in an inappropriate context.

Another misconception came to the fore in this study, related to the numbering of problems in the research instrument and the given formula or the positioning of the formula in relation to the question to be answered. This is related by learner 313 below. The learner was responding to reflection question 3.3, namely, between question 3.1 and 3.2 which question do you consider as difficult and why.

L-313 wrote in response to the question: ‘which question is more difficult between 3.1 and 3.2 and why you think so’

[3.2 because They didn’t provide me with a formula with a formula it was difficult to find the answer. I 3.1 They did provide me a formula it was easy to get an answer.]

The learner makes the assumption that the formula  $F = (1.8 \times C) + 32$  is to be used to solve question 3.1 since question 3.1 is placed next to the given formula. However, I interpreted this statement as the learner seem to be unaware that 3.2 is also a sub-section of Task 3 and the same given formula can be used to solve question 3.2 by means of symbol manipulation, or by means of working backwards or systematically with guess and test. This was a general misconception across all the start-unknown questions in the research instrument.

Another possible interpretation for this statement is related to what I raised in Chapter 2 about the equation. Once a problem has been expressed as an algebraic equation, the solutions can be obtained by applying a succession of transformation rules, such as: rewriting an expression; collecting like terms; factoring; expanding; applying the same operation on both sides of an equation, etc. In this instance, it seems like the learner was unable to apply a succession of transformation rules to the algebraic equation. This was further evident in all the reflection responses; where learners seem to raise the same issue about no formula for start-unknown questions.

With all the start-unknown questions of the form  $y = mx + c$  in this study, data shows that learners performed poorly with the application of formal procedures, such as working backwards, and symbol manipulation. In fact, the average success rate with using the two strategies in these problems was 12 percent. It seems as if learners attempted to circumvent the solution of start-unknown questions by using informal procedures, whilst opting for strategies such as systematic guess and test. However, it was shown that the systematic guess and test was sometimes inadequate in dealing with certain questions (such as 1.1.2, 2.2 and 3.2), because the success rate in these questions with the strategy was minimal. I have alluded to the fact that systematic guess and test sometimes relies on whether or not the given numbers are ‘manageable numbers’.

Linsell et al. (2009) describes what Herscovics and Linchevski (1994) refer to as a transition between arithmetic and algebra, which is called a ‘cognitive gap’. According to Herscovics et al. (1994, p.75), a characteristic of cognitive gap is, “the student’s inability to operate with the unknown. The cognitive gap is a void that is caused by a lack of arithmetical structural experiences. Data reveals that the Grade 11 ML learners in this study have been largely unable to operate on the unknown i.e. learners cannot manipulate the algebraic formula. The study by Spangenberg added to the research on the transition of Mathematics learners in South Africa in the General Education and Training band to Mathematics and Mathematical Literacy in the Further Education and Training Band.

Setler et al. (2011) suggest that this gap between arithmetic and algebra can hinder success when applying formal procedures to solve start-unknown questions. In this study it was shown that most learners were successful at the arithmetical level of result-unknown questions. It seems however that these learners have not yet made a transition from arithmetic to algebra.

This lack of skills is also evident in the fact that most learner responses across start-unknown questions were largely classified under number grabbing, unclear answers, swopping of variables and no response. Learners often resort, for instance, to number grabbing, if they are uncertain over what they ought to do (Khan, 2009). Data shows that with each successive question, i.e. from question 1.1.2 to question 4.2 (excluding question 2.2), the number of learners who applied working backwards, dropped from 83 learners to 9 learners, with each successive question. This drop corresponds to the move from a one-step  $y = mx$  question to the two- step  $y = mx + c$  question. The learners’ poor skills in algebra is a concern, especially in the questions of the form  $y = mx + c$ . Solving linear equations is a skill taught from Grade 7 in an informal manner, and more formally in Grades 8 and 9, yet on average, only 6% could solve the start-unknown questions, using working backwards strategies. Many learners who solved the problem used the guess and test strategy, which is cumbersome and inefficient.

Linsell et al. (2009) refer to the systematic guess and test strategy as a relatively unsophisticated strategy, and according to Nathan et al. (2000) these informal arithmetical strategies show limitations when problem complexities increases. Thus it was noted in the data that there may have been a trend where working backwards dwindled as learners progressed from question 1.1.2 to question 4.2.

The difference between question 1.1.2 and the other three start-unknown questions is that the other three problems included an additional step, and I assumed the change of strategy, from formal to informal, could have been caused by the fact that learners struggled to apply formal procedures to the other three linear question which involved an additional step. I assume that the informal strategies were inadequate to solve other three start-unknown problems and the learners at the same time were not competent enough in formal procedures.

## **5.6 Limitations of the Study**

For all participants in the interview, English was a second language. As a result, interviews took a significant amount of time, and learners were unable to articulate their ideas clearly and easily. A further limitation was that learners took part voluntarily, and the results were not used for assessment purposes. Hence, the learners may not have been motivated to apply their mind fully in trying to solve these problems, because they knew it would not affect their results in class.

## **5.7 Implications of the Study**

In an article (Pythagoras, 2012), Spangenberg suggests that one of the difficulties that new Mathematics and Mathematical Literacy curricular faces is a lack of consensus and clarity about what each of these are in terms of HG and SG mathematics curriculum. This debate still continues despite the fact that Mathematics HG and SG curricular merged in 2006. This sustained debate is caused by continuous poor performance of SA learners in Mathematics. Studies seem to indicate that the situation of Mathematics performance is not improving. This situation continues to put ML in the spotlight, since a substantial number of learners have shifted from Mathematics to ML. I will add on these by saying that, further studies need to be considered, where the issues between FET Mathematics and FET Mathematical Literacy can be more closely examined. Issues such as the supposed 'knowledge gap' between the two subjects and the 'impact of transition' from GET Mathematics to FET Mathematical Literacy ought to receive more attention. This is because,

as suggested by the author, ML continues to appear to be a practical or functional lower order Mathematics, while despite this, its learner intake is continuing to grow.

This study has provided evidence of the low levels of algebraic skills held by the 340 ML Grade 11 learners from 6 schools. Although the sample may not be fully representative of all schools in South Africa, it provides an indication that ML learners are more comfortable with working with numbers than they are working with basic algebra. The study was based on items that appear commonly in ML assessments, so they do form part of what ML learners are expected to be able to do. In those questions which could not be solved by arithmetical means, most learners opted for the guess and test strategy, which is cumbersome and inefficient, especially in situations with large numbers. In fact, across the questions based on linear equations, only 9% of the learners were able to use the working backwards or the symbolic manipulation strategies to solve the problems correctly. This means that only 9% of the learners were able to solve linear equations, a concept that is taught at Grade 8 and 9 levels, and is considered as a basic algebraic skill. Perhaps ML curriculum developers ought to consider including working with equations as a basic skills topic in the ML curriculum.

## **5.8 Conclusion**

This chapter addressed the four key research questions posed at the end of Chapter 3 as well as at the beginning of Chapter 5, by relating the findings in this study to similar findings in the literature review in Chapter 2. With regards to the first research question, it was found that learners were successful in solving result-unknown questions by using the direct arithmetic strategy. As far as the second research question is concerned, it was revealed that learners performed poorly, and achieved minimal success in solving start-unknown questions using formal strategies such as symbol manipulation and working backwards. The best performance was in result-unknowns and the worst performance was in start-unknown questions, which were set in real life contexts. Incorrect strategies such as number grabbing were found to be dominant in start-unknown questions. The study also found that learners perceived and experienced start-unknown problems as more difficult than result-unknown questions, and this finding is related to the research questions three and four. Factors that affected learner success in start-unknown questions is related to poor conceptual knowledge,

as well as errors and misconceptions such as swopping of variables, and conjoining. I concluded the final chapter by outlining limitations of the study and also recommended possible areas for further study.

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**APPENDIX A: Research Instrument**

**GRADE 11 MATHEMATICAL LITERACY ASSESSMENT**

**DURATION  $1\frac{1}{2}$  HRS**

**MARKS 44**

**MAY 2012**

**INSTRUCTIONS AND INFORMATION**

1. All final answers must be rounded off to two decimal places.
2. Show all working details.
3. Space(s) provided for answers does not necessarily indicate amount of answer required

**QUESTION ONE [17]**

1.1 The cost of a cell phone call from a phone on ENG pre-paid to a cell phone on ZZZ cell service provider during peak times is given by the equation:

$$C = R2,75 \times t.$$

Where **C** represents Cost (in rands) of a call and **t** represent time in (minutes) spent on call.

1.1.1 Find **C** if **t** = 87minutes. Show all working details. (2)

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1.1.2 Find **t** if **C** = R487.32. Show all working details. (3)

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1.1.3 Between 1.1.1 and 1.1.2 above which question was harder for you to work out. Why do you say so? (3)

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1.2 It costs Mrs. Khoza R1050, 63 in petrol costs to drive from Durban to Johannesburg. If she is the only person in the car, then she will have to pay the whole R1050.63, if there is

another person in the car with her, then each of them will have to pay R525, 32, if there are three people in the car, then each person will have to pay R35, 21 and so on. This relationship is linked by the formula:

$$C = \frac{R1050,63}{P}$$

where P represents the number of passengers in a car.

1.2.1 Calculate how much will each person pay if there are 5 people in the car? (2)

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1.2.2 Calculate how many people are in the car if it costs each person R150.09 to pay for petrol? (4)

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1.2.3 Between 1.2.1 and 1.2.2 above which question was harder for you to work out. Why do you say so? (3)

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## QUESTION TWO [09]

2. The city, works out the amount electricity usage due by using the formula outlined below: You take the number of units and multiply it by R0.60 and then you add R145 to get the total amount.

$$\boxed{\begin{array}{l} \text{No of} \\ \text{units} \end{array}} \times R0.60 \rightarrow \boxed{\phantom{0000}} + R145 \rightarrow \boxed{\phantom{0000}}$$

Work out how much is owed by Miss Khoza, if she used

2.1 735 units(2)

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2.2 How many units did she use, if her bill (amount due) was R415? Explain how you get your answer. (4)

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2.3 Between 2.1 and 2.2 above which question was harder for you to work out. Why do you say so? (3)

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### QUESTION THREE [08]

Although in South Africa we measure temperature in degrees Celsius, many countries measure temperature in degrees Fahrenheit (F). If you are a South African travelling overseas, it can become confusing to try to work out the temperature. Similarly, for an overseas tourist visiting South Africa it can be equally confusing.

The equation  $F = (1,8 \times C) + 32$  represent the relationship between temperature measured in degrees Fahrenheit (F) and degrees Celsius (C). Use this equation to answer the following questions.

3.1 Find F if C = 83. Show all working details. (2)

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3.2 find C if F = 93. Show all working details. (3)

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3.3 Between 3.1 and 3.2 which question was harder for you to work out. Why do you say so? (3)

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**QUESTION FOUR [10]**

Ms Adams a teacher at Sunset High School in Ladysmith is organizing an educational excursion to Durban for grade 12 learners. The distance from Ladysmith to Durban is 255km. The excursion will take one day. Ms Adams receives a quotation from a bus company(Company A ). The cost is worked out by multiplying the number of kilometres travelled by R1, 96 and adding a flat rental fee of R200

The table 1 shows the cost for different distances travelled, with some details omitted.

**Table 1: Cost of hiring a bus from Company A**

<b>no. of km travel</b>	0	63	177	366	489	592	<b>B</b>
<b>Cost(C)</b>	R200	R323.48	R546.92	R917.36	R978	<b>A</b>	R1376

4.1 Find **A** on table 1 using formula: **Cost = R1.96 × no. of km + R200**. Show all your calculations. (3)

**A**-----

4.2 Find **B** on table 1 using formula: **Cost = R1.96 × no. of km + R200**. Show all your calculations (4)

**B**-----

4.3 Between 4.1 and 4.2 which question was harder for you to work out. Why do you say so? (3)

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## APPENDIX B: APOS Theory

In another study carried out with ML teachers, Bansilal (2011) used a process-object framework (Sfard, 1991; Dubinsky, 1991) to understand the teachers' varying levels of engagement with the inflation rate signifier. The findings revealed that some teachers were able to move easily between using the inflation rate signifier both as a process and as an object. However many teachers were able to access a process understanding of the inflation rate concept but struggled to use the inflation rate signifier as an object. It was also found that some teachers had a poor conceptions of the concept of percentage and this hindered them from attaining a dual process-object understanding of the inflation rate signifier.

Dubinsky's Action process, object, schema (APOS) theory has been widely used in many areas of mathematics to explore students' understanding of various mathematics concepts. Although APOS is primarily a learning theory used to describe how learners 'learn' a concept, in this study the theory can also help me understand the learners' understanding of algebraic equations.

The APOS theory is a theory of learning that attempts to explain how a learner deals with a mathematical situation, Dubinsky et al (2005, p338). The description of the APOS theory detailed herein is taken from Dubinsky et al (2005) and Dubinsky and McDonald (2005). According to the APOS theory, the learner deals with a mathematical situation by using certain **mental mechanisms** to build **cognitive structures** that are applied to the mathematical situation. Piaget (1973) refers to cognitive structures as mental structures and emphasises that there is a close relationship between the nature of a mathematical concept and the concepts' development in the mind of an individual.

According to Dubinsky et al (2005), the mental structures are **action, process, objects and schema (APOS)**. There are two main mechanisms for building these cognitive mental structures. The two main mechanisms are called **interiorisation and encapsulation**. The APOS theory suggests that a mathematical concept begins to form as an individual applies a transformation on an object to obtain other objects. These concepts will be illustrated by the use of a linear equation given by  $0.79x + 200$

**Action** is regarded as the first stage of the APOS theory. Dubinsky and McDonald describes action as, "a transformation of objects perceived by the individual as essentially externally and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation." APOS theory states that a transformation is first conceived as an action. The action structure or stage is characterised by involvement of senses such as listening to instruction, observing and participating or mimicking explained procedures in building the mental structure or stage. At an action structure or stage, the transformation must be explicit and hands-on. For example, the learner, functioning at an action structure or stage in a topic of solving linear equation problems could be characterised by being able to substitute values in variables and perform computations successfully. The learner follows routine procedures as explained and demonstrated to him or her by the instructor. I will illustrate action structure or stage by using the formula:  $C(x) = 0.79x + 200$ . If a learner can substitute values for (x) in the formula and calculate the correct answers, in a step by step manner the learner is considered to have acquired the **action level understanding**.

The learner repeats and reflects on the procedures at an action level until these become automatic and familiar. Through repetition and reflection on procedures at an action level which leads to familiarity of procedures, interiorisation into a mental **process** may take place. Mental activities at this stage can be viewed from a perspective of interiorisation described by Sfard (1991), “at the stage of interiorisation a learner gets acquainted with the processes which will eventually give rise to a new concept. These processes are operations performed on lower-level mathematical objects e.g. counting that leads to natural numbers, subtraction leads to negative numbers, algebraic manipulation which turn into functions etc.”,(Sfard ,1991,p.18).

Dubinsky et al, (2005), describes process as, “a mental structure or stage that performs the same operations as the action being interiorized, but wholly in the mind of the individual, thus enabling the individual to imagine performing the transformation without having to execute each step explicitly.” At a process level, the action level is relegated to a subconscious mind. With regards to linear equations problems, the process level is characterised by mental activities in which a formula is considered to be an input and output machine i.e. the input is always linked to an output and vice versa. At this stage the notion of a variable is considered to be clear to the learner and the learner must be able to find values of the dependent variable when instructed to do so. Furthermore, the learner may be able to think about reversing operations or composing operations with other operations. At this level the learner is considered to have acquired a **process construction**.

The third stage of the APOS theory is called **object**. When the learner has reached a stage of awareness of a mental process as a totality and can actual construct such transformation at an action level and at a process level, then the individual is considered to have encapsulated the process into a **cognitive object**. At an object level and in a formula like  $C(x) = 0.79x + 200$ , the learner views  $0.79x + 200$  as an object. It is an on object that can be transformed and an action structure and process can be applied on  $0, 79x + 200$  which happens to be an object.

The fourth stage of the APOS theory is called the **schema**. Dubinsky and McDonald (2005) describes schema as, “a certain mathematical concept is an individual’s collection of actions, processes, objects and other schemas which are linked by some general principles to form a framework in the individual’s mind that may be brought to bear upon a problem situation involving that concept. This framework must be coherent in the sense that it gives, explicit or implicitly, means of determining which phenomena are in the scope of the schema and which are not.”

These mental structures i.e. action, process, object and schema describe how a learner may construct a single transformation but a mathematical situation or problem often involves many actions, processes and objects that need to be organised and linked into a coherent framework. This coherent framework which links and decides which mental structures to be used or combined or eliminated is called the **schema**.

The main mental mechanisms (interiorisation and encapsulation) and mental structures (action, process, object and schema) describe the mental activities of learners who are engaged in a mathematical situation, however, it is important to note that an individual may possess these mental structures but it does not mean that the individual will automatically use these mental structures in a mathematical situation. The use or non-usage of the available mental structures is further influenced by factors such as managerial strategies, prompts, emotional states etc.

## APPENDIX C: Permission to conduct Study from KZN DoE



**kzn education**

Department:  
Education  
KWAZULU-NATAL

Enquires: Sibusiso Alwar

Tel: 033 341 8610

Ref.:2/48/234

Mr. Martin Siphiso Mbonambi  
P.O. Box 842  
Verulam  
4340

Dear Mr. Mbonambi

### PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: **An Investigation of the Strategies Used by Learners to Solve Start – Unknown and Result –Unknown Problems in Mathematics**, in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the intended research and interviews are to be conducted.
6. The Period of investigation is limited to the period from 01 July 2012 to 31 December 2013
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Mr. Alwar at the contact numbers below.
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report / dissertation / thesis must be submitted to the research office of the Department. Please address it to The Director-Resources Planning, Private Bag X9137, Pietermaritzburg, 3200
10. Please note that your research and interviews will be limited to the following Schools and Institutions:

- 10.1 Lockhart High School
- 10.2 Ukuzamakwehu Secondary School
- 10.3 Suloman Patel Secondary School
- 10.4 Sacred Heart Secondary School

  
**Nkosinathi S.P. Sishi, PhD**  
Head of Department: Education

2012/07/10  
Date

...dedicated to service and performance  
beyond the call of duty.

KWAZULU-NATAL DEPARTMENT OF EDUCATION

POSTAL: Private Bag X9137, Pietermaritzburg, 3200, KwaZulu-Natal, Republic of South Africa

PHYSICAL: Office C 25, 188 Pietermaritzburg Street, Metropolitan Building, Pietermaritzburg 3201

TEL: Tel: +27 33 341 8610 | Fax: +27 33 3341 8612 | E-mail: sibusiso.alwar@kzndoe.gov.za |  
Web: [www.kzndoe.gov.za](http://www.kzndoe.gov.za)

## APPENDIX D: Ethical Clearance



24 June 2012

Mr Martin S Mbonambi (207520775)  
School of Education

Dear Mr Mbonambi

PROTOCOL REFERENCE NUMBER: HSS/0428/012M

PROJECT TITLE: An investigation of the strategies used by learners to solve start-unknown and result-unknown problems in mathematics

#### PROVISIONAL APPROVAL

This letter serves to notify you that your application in connection with the above has been approved, subject to necessary gatekeeper permissions being obtained from the Department of Education.

This approval is granted provisionally and the final approval for this project will be given once the above condition has been met. In case you have further queries/correspondence, please quote the above reference number.

Kindly submit your response to the Chair, Prof. S Collings Research Office as soon as possible

Yours faithfully

Professor Steven Collings (Chair)  
/s/

cc: Supervisor: Dr S Bansilal  
cc Academic Leader: Dr MN Davids  
cc School Admin: Mr N Memela / Mrs S Nalcker

Professor S Collings (Chair)  
Humanities & Social SC Research Ethics Committee  
Westville Campus, Govan Mbeki Building

Postal Address: Private Bag 354111, Durban, 4001, South Africa

Telephone: +27 (0)31 260 3687/8300 Facsimile: +27 (0)31 260 4409 Email: simba@ukzn.ac.za / s.collings@ukzn.ac.za

Founding Constituent: ■ Edgewood ■ Howard College ■ Medical School ■ Pietermaritzburg ■ Westville

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