This is to confirm that the student has carefully and thoroughly gone through the Examiners comments and reports and has carried out the corrections and comments by the examiners to my satisfaction as his supervision. I therefore approve the submission of the corrected and final version of his thesis.
Thanks

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25-01-2023

ITERATIVE APPROXIMATIONS OF CERTAIN NONLINEAR OPTIMIZATION, GENERALIZED EQUILIBRIUM AND FIXED POINT PROBLEMS IN HILBERT AND BANACH SPACES

by

Emeka Chigaemezu Godwin

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy
(Ph.D.)

in the

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Iterative Approximations of Certain Nonlinear Optimization, Generalized Equilibrium and Fixed Point Problems in Hilbert and Banach spaces

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B.Sc. (Hons.)(IMSU), M.Sc. (FUTO)

As the candidate's supervisor, I have approved this thesis for submission.

Prof. O. T. Mewomo

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Dedication

To my teachers Prof. A.M. Ette, Prof. E.N. Erumaka, Prof. S.C. Inyama, Prof. M.O. Osilike, Prof. J.N. Nnadi, Prof. C.A. Nse, Prof. B.G. Akuchu, Prof. M.C. Obi and Prof. O.T. Mewomo. Also in loving memory of my beloved mum Mrs Lucy Ndidiamaka Mbachu.

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Abstract

In this thesis, in the framework of Banach spaces, we study several iterative methods for finding the solutions of many important problems in fixed point theory and optimization. Some of these include the equilibrium problem, monotone variational inclusion problem, variational inequality problem, split common fixed point problem and split minimization problem. In addition, we study fixed point problem for some important and interesting classes of mappings such as nonexpansive mappings, pseudocontractive mappings, asymptotically demicontractive mappings, quasi-pseudocontractive mappings, demimetric mappings and multivalued demicontractive mappings in real Hilbert spaces. Furthermore, we study some other classes of mappings which include the class of Bregman quasi-nonexpansive mappings and Bregman relatively nonexpansive mappings in real p-uniformly convex Banach spaces which are also uniformly smooth. Another important problem considered is the split equality problem. The split equality problem has gained attention from authors because of its vast applications to real life problems. This problem is known to contain several other optimization problems as special cases. Based on its numerous applications, we study a multiple set split equality equilibrium problem consisting of pseudomonotone bifunctions together with fixed point problem of certain nonlinear mappings in p-uniformly convex and uniformly smooth Banach spaces. In each case, we propose and study iterative algorithms for approximating the solutions of these problems and prove strong convergence theorems under suitable conditions on the control parameters. In most cases, we incorporate the inertial term which is known to speed up the convergence rate of iterative schemes. In addition, we employ several efficient iterative techniques which include the projection and contraction method, alternative regularization method, modified Halpern's method, inertial Tseng's extragradient method and viscosity approximation method. In all the cases, we design our algorithms in such a way that the step size does not depend on the knowledge of the Lipschitz constants of the cost operator or the norm of the bounded linear operators. We present some applications of our results to solve convex minimization problems, multiple set split variational inequality problem, image restoration problem, oligopolistic market equilibrium problem, among others. Also, we present several numerical experiments to demonstrate the efficiency, applicability and usefulness of our iterative schemes in comparison with several of the existing methods in the literature. The results obtained in this thesis extend and improve many existing results in the literature in a unified way.

Contents

Declaration

I declare that this thesis in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used, proper reference has been made.

Emeka Chigaemezu Godwin

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Contributed papers from the thesis

Papers published/accepted from the thesis.

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Chapter 1

General Introduction

1.1 Background of study

Let X and Y be two linear spaces and $T : X \to Y$ be a bounded linear operator. The *Split Inverse Problem* (see [54]) (shortly, SIP) is defined as:

$$\text{Find } \; x^* \in X \; \text{ that solves } IP_1 \; \text{ such that } y^* = Tx^* \in Y \text{ solves } IP_2, \tag{1.1}$$

where IP_1 and IP_2 denotes the inverse problems in X and Y, respectively. The first known SIP (1.1) is the *Split Feasibility Problem* (shortly, SFP) introduced and studied by Censor and Elfving [51], where X, Y are Euclidean spaces and IP_1, IP_2 are convex feasibility problems. The SFP has the following definition:

$$\text{Find } x^* \in C \; \text{ such that } \; Tx^* \in Q, \tag{1.2}$$

where C and Q are nonempty closed and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively and T is an $m \times n$ real matrix. The SFP (1.2) has been deployed to model inverse problems which arise from phase retrievals, signal processing, image restorations, radiation therapy treatment planning, among others (see for example [34, 35] and the references therein). Several optimization problems such as Split Variational Inequality Problem (SVIP), Split Common Null Point Problem (SCNPP), Split Monotone Variational Inclusion Problem (SMVIP), Split Common Fixed Point Problem (SCFPP), Split Equilibrium Problem (SEP), Split Minimization Problem (SMP), among others have been defined in terms of SFP (1.2), (see for example [76, 168] and the references therein).

In the field of optimization and modern analysis, fixed point theory is one of the powerful and efficient tool used for approximating the solutions of certain nonlinear mappings [18]. Let X be any given set and $S : X \to X$ be a self-mapping. A point $x \in X$ is called a fixed point of S if $x = Sx$. On the other hand, for a set-valued mapping $S : X \to 2^X$, a point $x \in X$ is called a fixed point of S if $x \in Sx$. We denote the set of fixed points of S by $F(S)$. Fixed point theorems are theorems concerning the existence and properties of fixed points as well as conditions under which a mapping has one or more fixed points.

The study of fixed point theory commenced in the nineteenth century and has wide range of applications in several fields of science and engineering technology including computational electromagnetism [108], Fredholm and Volterra integral equations, ordinary differential equations [32], chaos theory and dynamical systems, [145, 188]. The concept of fixed point theory was initially introduced by Poincaré [190]. In 1912, Brouwer [80] proved a famous fixed point theorem "a continuous mapping on a closed unit ball for a finite dimensional space $\mathbb{R}^n$ has a fixed point". Moreover, in 1922, Banach [19] proved a fixed point theorem for a complete metric space, which guarantees the existence of a unique fixed point. Banach's fixed point theorem is well known as the *Contraction Mapping Theorem* (shortly, CMT). Thereafter, in 1930, Schauder [203] studied the compactness requirement on the operator or on the feasible set, which was an extension of Brouwer's [80] fixed point theorem to an infinite dimensional spaces. Since then, there has been several modifications and improvement in fixed point theory for both single-valued and set-valued mappings in the literature, (see [1, 99, 2, 122, 123, 205]). Afterward, in 1969, Nadler [172] studied the CMT using the concept of Hausdorff metric and extended the CMT from single-valued mappings to set-valued mappings. Since then, several authors have extended the CMT in several ways, (see [20, 38, 40, 43, 79, 103] and the references therein).

It is well known that iterative schemes play a crucial role in finding solutions of many fixed point and optimization problems. Picard's iteration is one of the earliest and most famous fixed point algorithm; that is, given $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ determined by the successive iteration method is defined by

$$x_n = S(x_{n-1}) = S^n(x_0), \quad n = 1, 2, ..., \tag{1.3}$$

where $S : X \to X$ is a self-map consisting of at least one fixed point $x^* \in F(S)$. (1.3) is referred to as the Picard's iteration. Some examples have shown that for nonexpansive mapping S, the Picard iteration may not converge to a fixed point of S, even when such fixed points exist (see for example [61]). Many fixed point iteration methods have been proposed to conquer this limitation. Take for example, the Krasnoselskii [143] iterative scheme for nonexpansive mappings finds its application in several areas of mathematics and it is known to converge weakly in infinite dimensional space. Let X be a linear space and $S : X \to X$ be a self-map. For $x_0 \in X$ and $\gamma \in [0, 1]$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1-\gamma)x_n + \gamma S x_n, \quad n = 0, 1, 2, ... \tag{1.4}$$

is called the Krasnoselskii iteration [143]. The Krasnoselskii iteration $\{x_n\}_{n=0}^{\infty}$ (1.4) is exactly the Picard iteration corresponding to the averaged operator $S_\gamma = (1-\gamma)I + \gamma S$ where I is the identity operator and $\gamma \in [0, 1]$. Moreover, we have $F(S) = F(S_\gamma)$ for all $\gamma \in (0, 1]$.

Also, the Mann [165] iterative process is defined as follows: For $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty} \in X$ generated by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S x_n, \quad n = 0, 1, 2, ..., \tag{1.5}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$ meets certain appropriate conditions.

Considering $S_n = (1-\alpha_n)I + \alpha_n S$, then we have that $F(S) = F(S_n)$, for all $\alpha_n \in (0, 1]$. If the sequence $\alpha_n = \gamma$ (constant), then the Mann iterative algorithm obviously reduces to the Krasnoselskii iteration.

The Ishikawa iterative algorithm is another fixed point algorithm that is mostly used together with the Mann iterative algorithm [116]. The Ishikawa iteration was initially used to approximate the solution of fixed point problem for a Lipschitzian and pseudocontractive self-map of a convex and compact subset of a real Hilbert space. It is defined as follows: Find $x_0 \in X$ such that

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S[(1-\beta_n)x_n + \beta_n S x_n], \quad n = 0, 1, 2, ..., \tag{1.6}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ satisfy certain appropriate conditions. The iterative scheme given by (1.6) can be written in the form

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n S x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n, \quad n = 0, 1, 2, ... \end{cases} \tag{1.7}$$

The Ishikawa iteration may be viewed as a two-step Mann iteration with two distinct parameter sequences. If $\beta_n = 0$, the Ishikawa iteration reduces to Mann iteration.

Note that the iterative algorithms given by (1.4), (1.5), (1.6) and (1.7) all give weak convergence. However, in the setting of infinite dimensional spaces, strong convergence are often more desirable than weak convergence, (see for example [21]). For this purpose, in 1967, Halpern [95] initiated and studied the following iterative algorithm which converges strongly to a fixed point of a nonexpansive mapping in the setting of real Hilbert space H :

$$\begin{cases} u, x_1 \in H, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) S x_n, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. A vital generalization of the Halpern iteration algorithm (1.8) is the viscosity iteration algorithm introduced by Moudafi [168] in real Hilbert space, as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n) S x_n, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and g is a strict contraction mapping on H. One important advantage of Algorithm (1.9) over the Halpern iteration scheme (1.8) is that it also converges strongly to a unique solution of some variational inequality problems associated with the contraction mapping g.

Several other modifications of the above algorithms and many others have been employed to solve optimization and inverse problems (see [36, 100, 101, 102, 139, 175, 185, 221, 234]).

The beauty of iteration algorithms is made manifest not only in their convergence analysis but also in the rate at which they converge. One of the best means to speed up the convergence rate of iterative schemes is to combine the iterative scheme with the inertial technique. The inertial term denoted by $\theta_n(x_n - x_{n-1})$, is a remarkable tool for significantly improving the performance of algorithms and it is known to have some nice convergence features.

In 1964, Polyak [191] introduced a two-step iterative method known as the heavy-ball method involving minimizing a smooth convex function h given by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = y_n - r\Delta h(x_n), \ n \geq 1, \end{cases} \tag{1.10}$$

where Δh is the gradient of h and $\theta_n \in [0, 1)$ is an extrapolation factor with step size r that has to be chosen sufficiently small.

Also, in 2001, Alvarez and Attouch [12] introduced an inertial forward-backward splitting method which is the modification of (1.10), and is given by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + rM)^{-1} y_n, \ n \geq 1, \end{cases} \tag{1.11}$$

where M is a multi-valued operator. They proved a convergence result for approximating solution of monotone inclusion problem under the condition $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}||^2 < +\infty$ with $\{\theta_n\} \subset [0, 1)$ in the setting of Hilbert space. Since then, there has been a growing interest by authors working in this direction (see [76, 78, 83, 123, 157, 178]).

In this thesis, we devote time to study the equilibrium problem, variational inequalities, monotone variational inclusion problem, minimization problem, fixed point problem and other optimization problems in both Hilbert and Banach spaces. In addition, we present several iterative schemes for finding the solution of these problems and we prove strong convergence theorems for the sequences generated by these algorithms in all cases. We carry out some numerical experiments to show the efficiency and applicability of our algorithms and present some theoretical applications of our results.

1.2 Research problems and motivation

In this section, we discuss the research problems and motivation of our study.

1.2.1 Research problems

We study multiple set split equality equilibrium and fixed point problems (MSSEEFPP) consisting of pseudomonotone bifunctions and the set of fixed points for two finite families of Bregman quasi-nonexpansive mappings in the framework of p-uniformly convex Banach spaces which are also uniformly smooth. Also, we consider the multiple set split equality convex minimization problems and multiple set split equality variational inequality problems. Also, we look into split equality equilibrium, monotone variational inclusion problem and fixed point problem for Bregman relatively nonexpansive mappings in p-uniformly convex and uniformly smooth Banach spaces.

The inertial extrapolation technique has been widely used by many authors in order to accelerate the rate of convergence of iterative algorithms. This method is characterized by a discrete analogue of a second order dissipative dynamical system in time (see [185]).

In this thesis, we propose several inertial-type iterative schemes in both real Hilbert and Banach spaces. Moreover, we propose and analyze an Halpern-type subgradient extragradient algorithm for approximating a common element of the set of solution of pseudomonotone equilibrium and common fixed point problem for a finite family of Bregman quasi-nonexpansive mappings in uniformly convex and uniformly smooth Banach space which is more general than related works done in real Hilbert spaces. In addition, in the framework of real Hilbert spaces, we study and analyze split generalized equilibrium problem with multiple output sets and common fixed point problem for an infinite family of multivalued demicontractive type mappings.

Furthermore, we propose a modified relaxed inertial-type method for solving monotone variational inclusion problem and common fixed point problem for nonexpansive mappings applied to image restorations. In addition, we propose a relaxed double inertial Tseng's extragradient method for solving non-Lipschitz split monotone variational inclusion problem with fixed points constraints.

Moreover, we investigate the approximation of solutions of split minimization problem with multiple output sets and common fixed point problem for a finite family of Bregman relatively nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces. The new method employs the inertial Halpern approximation technique. Also, in the framework of real Hilbert spaces, we present a new algorithm for solving split common fixed point problem for a class of asymptotically demicontractive mapping. We give an example to illustrate that the class of asymptotically demicontractive mappings and the class of demicontractive mappings are independent.

In addition, we consider the problem of finding the common element of the solution set of variational inequality and fixed point problems involving quasi-pseudocontractions. We propose a new Tseng's extragradient method, which combines the relaxation technique and inertial method with self-adaptive step size for solving the problem. Furthermore, we propose an inertial extrapolation method for solving generalized split feasibility problem in real Hilbert spaces. Also, in the framework of real Hilbert space, we study an inertial scheme for solving two-level variational inequality and fixed point problem involving pseudomonotone and ϱ-demimetric mappings.

1.2.2 Motivation

We discuss the motivation of our study as follows:

1. **Optimization Problems:** In mathematical sciences and engineering technology, optimization problem is the problem of finding the best possible solution from all feasible solutions. In convex analysis and variational analysis, optimization and fixed point theory have evolved to become a twin concept. This is known to include the equilibrium problems, variational inequality problems, minimization problems and the inclusion problems, to mention a few. These problems appear frequently in many practical problems arising, for instance, in physics, engineering, game theory, transportation, economics and network [84] and have become an attractive field for many researchers in both theory and applications (see [15, 164, 192]). All the afore-

mentioned optimization problems can be formulated in terms of the Split Feasibility Problem (SFP).

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ be a bounded linear operator. The SFP is defined as follows:

$$\text{Find } \; x^* \in C \; \text{ such that } \; Ax^* \in Q. \tag{1.12}$$

The SFP which has attracted the attention of many authors due to its applicability in various disciplines, such as image restoration, computer tomography and radiation therapy treatment planning (see [52, 53, 55]) is known to be a generalization of the convex feasibilty problem introduced by Censor and Elfving [51] in 1994. For finding a solution of the SFP, Censor and Elfving [51] investigated the use of different kinds of generalized projections in a single iterative process. Their proposed iterative algorithm involves the computation of inverse of a matrix, which is known to be a difficult task. To overcome this difficulty, Bryne [35] proposed the CQ algorithm, which generates a sequence by a recursive procedure with suitable step-size and only involves the computations of projections onto the sets C and Q, respectively. However, it is known that the computation of projection onto an arbitrary closed and convex set is also a difficult task. Yao et al. [247] introduced a self-adaptive method which permits step-sizes being selected self-adaptively to solve SFP. We observe that the normalized duality mappings in Banach spaces are nonlinear while the normalized duality mapping in Hilbert spaces is an identity mapping, and the adjoint operator A^* of the bounded linear operator A from Banach space E_1 to Banach space E_2 is a single-valued mapping from the dual space ${E_2}^*$ of E_2 to the dual space ${E_1}^*$ of E_1, but the adjoint operator A^* of the bounded linear operator A from Hilbert space H_1 to Hilbert space H_2 is a mapping from H_2 to H_1. In addition, the projection operators in Hilbert spaces are firmly nonexpansive, while the various projections in the Banach spaces are no longer nonexpansive. Therefore, it is more difficult to solve SFPs in Banach spaces than in Hilbert spaces. In view of this, we study the SFP in Banach spaces and introduce new and efficient iterative algorithms to approximate solutions of the SFPs.

Also, since the inception of the SFPs, many other optimization problems such as the split minimization problem, split variational inequality problem, split monotone variational inclusion problem and the split equality problem, (see [8, 9, 58, 71, 74, 92, 103, 124, 192, 205, 212, 218, 237]) to mention a few have been introduced. We observe that most of these split-type problems have only been studied in the framework of Hilbert spaces. Motivated by this, we extend some of the existing results in these areas to the framework of Banach spaces. However, we note that some of the complexities encountered in trying to extend results in Hilbert spaces to Banach spaces in some cases is the issue of convexity property which is easy to handle in Hilbert spaces than Banach spaces which is due to the geometry of a Banach space. In addition, it is known that the Bregman function fails to satisfy the triangular inequality property in uniformly convex and uniformly smooth Banach spaces, as well as in reflexive Banach spaces. This goes to explain why viscosity iteration method cannot be applied in Banach spaces.

2. **Spaces of interest:** The Hilbert space H is known to have the most simplest geometric structure among all Banach spaces. Some of the geometric properties that characterize Hilbert spaces which make it easier to compute in this space compared to general Banach spaces include the availability of the inner product, and the non-expansivity property of the nearest point map defined on a real Hilbert space onto a closed convex subset C of H. For instance, the well-known inequality in Hilbert spaces defined thus: $\forall\ x, y \in H$,

$$||x+y||^2 = ||x||^2 + ||y||^2 + 2\langle x, y\rangle. \tag{1.13}$$

does not hold in general Banach spaces due to the lack of inner product. This makes computing in general Banach spaces more difficult than in Hilbert spaces. However, most real life problems do not exist in Hilbert spaces. Hence, to overcome these difficulties, researchers introduced the concept of the duality mappings which can be seen as a suitable analogue of the inner product in Hilbert space. Also, the distance function (also known as the Bregman distance) has been used to make computations less difficult to handle in Banach spaces. In view of this, we considered extending some recent results from Hilbert spaces to Banach spaces as most real life problems find applications in this space.

1.3 Objectives

The following are the objectives of this work:

(i) propose a new and efficient iterative method for approximating the solution of pseudomonotone Equilibrium Problem (EP) and common fixed point problem for some nonlinear operators in Banach spaces,

(ii) introduce and study the notion of Split Generalized Equilibrium Problem (SGEP) with multiple output set. Propose a new iterative scheme for approximating the common solution of this problem and fixed point problem of certain nonlinear mappings in real Hilbert space,

(iii) extend the notion of split equality equilibrium, Monotone Variational Inclusion Problem (MVIP) and fixed point problem from real Hilbert spaces to uniformly convex and uniformly smooth Banach spaces,

(iv) to propose a Halpern type algorithm for solving Multiple Set Split Equality Equilibrium and Fixed Point Problem (MSSEEFPP) for finite families of quasi-nonexpansive mappings in Banach spaces,

(v) introduce iterative algorithms for approximating split-type problems and common fixed point problems of asymptotically demicontractive mapping in Hilbert spaces,

(vi) introduce and study the concept of split minimization problem with multiple output sets. Propose a new iterative algorithm for approximating the solution of this problem in Banach spaces,

(vii) introduce new algorithm which employs relaxation and inertial techniques for approximating the solution of variational inequalities and fixed point problem of quasi-pseudo-contractions,

(viii) propose a new relaxed double inertial Tseng's extragradient algorithm for solving non-Lipschitz split monotone variational inclusion problem with fixed points constraints,

(ix) present applications of our results to study real world problems which includes image restoration, oligopolistic market equilibrium models, signal processing, electricity distribution models among others,

(x) provide some numerical examples to illustrate the performance and behavior of our results and the comparative advantage of our algorithms with other existing algorithms in the literature.

1.4 Main Results

The following are the main results in this thesis:

(i) Throughout this research work, we proved a strong convergent result in all our convergence analysis. We remark that strong convergence of iterative schemes are often more desirable that weak convergence result.

(ii) The strong convergence of our results in most cases does not rely on the usual two cases approach often used in the literature to establish strong convergence.

(iii) In all cases, we carefully introduced and applied a self-adaptive stepsize technique which does not depend on the prior knowledge of the Lipschitz constant or the norm of the bounded linear operator.

(iv) Our results in most cases employs the inertial and relaxation terms. We know that in the field of optimization, the inertial term is known to speed up the convergence rate of iterative algorithms. Therefore, the combination of inertial and relaxation terms, will guarantee more faster convergence of our algorithms.

(v) We extended some well known results in the literature from Hilbert space settings to Banach spaces.

(vi) Our numerical analysis shows the comparative algorithm of our research with some existing works. This shows the applicability, efficiency of our proposed algorithms.

1.5 Organization of the thesis

We organize the thesis into eight chapters as follows:

Chapter 1 (General Introduction): In this chapter, we present a brief historical background of our study. We also discuss the research problems and the motivation for our study. Also, we highlight the objectives of the study and discuss in details the organization of the thesis.

Chapter 2 (Preliminaries and Literature): In this chapter, we recall some basic definitions, vital concepts, geometric properties and already established results which we shall use to obtain our main results. We also introduce some of the optimization and fixed point problems relevant in this work.

Chapter 3 (Split Generalized Equilibrium Problems and Fixed Point Problems in Banach Spaces): The main results of this thesis begin with this chapter. We consider iterative approximation of common solutions of split generalized equilibrium problem and other optimization problem. Firstly, we introduce a strong convergent inertial Halpern subgradient extragradient method for solving pseudomonotone equilibrium and common fixed point problems in Banach spaces. Our proposed algorithm is designed in such a way that it does not rely on the prior estimates of the Lipschitz constants of the pseudomonotone bifunction. Also, we introduce and study the notion of split generalized equilibrium problem with multiple output sets. We propose a new iterative method which employs viscosity approximation technique for approximating the common solution of the split generalized equilibrium problem with multiple output sets and common fixed point problem for an infinite family of multivalued demicontractive mappings in real Hilbert spaces. Under mild conditions, we prove a strong convergence theorem for the proposed method. Our method uses self-adaptive step size which does not require prior knowledge of the operator norm. Numerical examples are also presented to show the applicability of the algorithm.

Chapter 4 (Split Monotone Variational Inclusion Problems and Fixed Point Problems): This chapter comprises of two sections organized as follows. In Section 4.1, we propose a new relaxed double inertial Tseng's extragradient method with self-adaptive step sizes for solving split monotone variational inclusion problem (SMVIP) involving non-Lipschitz operators and fixed point problem of strict pseudocontractive mappings. Under more relaxed assumptions, we prove that our proposed scheme converges strongly to a minimum-norm solution of the aforementioned problem in real Hilbert spaces. We point out that while the operators are non-Lipschitz, our method does not involve linesearch procedure which is known to be time-consuming, but we employ a more efficient self-adaptive step size technique that generates non-monotonic sequence of step sizes at each iteration. Results of the numerical experiments demonstrate the comparative advantage of our method over existing methods in the literature.

In Section 4.2, we propose a new self-adaptive method and prove that it converges strongly to a minimum-norm solution of a generalized split feasibility problem in real Hilbert spaces. The proposed method originates from an explicit discretization of a dynamical system in time, which combines both the relaxation and inertial techniques for the purpose of increasing the convergence rate of the scheme. The method requires that the underlying single-valued operator A is monotone and Lipschitz continuous, and it uses some simple self-adaptive step sizes that are generated at each iteration by some simple computations. As a by-product, we obtain methods for solving other classes of generalized split feasibility problems in real Hilbert spaces. The two major merits of the proposed algorithm in solving image restoration problems over related algorithms are the higher signal-to-noise ratio value and lower CPU time for generating recovered images. Finally, we compare our methods with other related strong convergent methods in the literature.

Chapter 5 (Inertial Type Algorithms for Solving Variational Inequality Problems and Fixed Point Problems): In this chapter, we consider a general system of VIP and other optimization problems. This chapter comprises of three sections. In Section 5.1, we propose a new relaxed inertial Tseng's extragradient method with self-adaptive step size for approximating common solutions of monotone variational inequality and fixed point problems of quasi-pseudo-contraction mappings in real Hilbert spaces. We prove strong convergence result for the proposed algorithm without the knowledge of the Lipschitz constant of the cost operator. Additionally, we apply our results to approximate solution of convex minimization problem and we present some numerical experiments to demonstrate the efficiency and applicability of our method in comparison with some existing methods in the literature. Our proposed method is easy to implement as it requires only one projection onto a constructible half-space.

In Section 5.2, we introduce an iterative algorithm which approximates the solution of two-level variational inequality and fixed point problem in a real Hilbert space where the underlying operators are pseudo-monotone and ϱ-demimetric. Our proposed algorithm is shown to converge strongly to the solution set of two-level variational inequality and fixed point problem. Four numerical examples are presented to further demonstrate the usefulness and applicability of our method. The result obtained extends, generalizes and compliments several existing results in this direction of research.

In Section 5.3, we introduce a new inertial extrapolation algorithm for solving a certain class of generalized split feasibility problems in two real Hilbert spaces. We prove that the proposed method converges strongly to a minimum norm solution of the problem when the underlying operator is pseudomonotone and uniformly continuous which are much more weaker assumptions than the inverse strongly monotonicity assumption used in the literature. In addition, our method uses self-adaptive step sizes that are generated at each iteration by some simple computations, which allows it to be easily implemented without the prior knowledge of the operator norm. Furthermore, some examples and numerical experiments to show the efficiency and applicability of our method are also discussed in the framework of infinite dimensional Hilbert spaces.

Chapter 6 (Split-Type Problems Minimization and Hierarchical Fixed Point Problem): This chapter comprises of two sections. In Section 6.1, we propose a new algorithm for solving split common fixed point problem for asymptotically demicontractive mappings in two real Hilbert spaces and prove that our scheme converges strongly to a solution of the problem. We present examples to illustrate that the class of asymptotically demicontractive mappings and the class of demicontractive mappings are independent. Moreover, our numerical experiments show the efficiency and applicability of our method in comparison with a related method in the literature. The results obtained unify, improve and extend so many related results in the literature in this direction.

In Section 6.2, we propose an accelerated iterative algorithm for approximating the solution of split minimization problem with multiple output sets. We propose a new iterative method, which employs an inertial Halpern approximation technique for approximating the common solution of split minimization problem with multiple output sets and fixed point problem for an infinite family of Bregman relatively nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. Our iterative method uses step sizes which do not require prior knowledge of the operators norm and we prove strong convergence result. Moreover, we present some applications of our result and further demonstrate the efficiency and applicability of our algorithm with some numerical examples.

Chapter 7 (Split Equality Equilibrium Fixed Point and Monotone Variational Inclusion Problem): This section comprises of two sections. In section 7.1, we propose and study a new and efficient algorithm for finding a common element of the set of solutions of split equality pseudomonotone equilibrium, split equality monotone variational inclusion and fixed point problems for Bregman relatively nonexpansive mappings in p-uniformly convex and uniformly smooth Banach spaces. Our iterative approach makes use of self-adaptive step sizes which does not require prior knowledge of the operator norm. Furthermore, we apply our result to solve split equality variational inequality and split equality convex minimization problems. The result presented in this section unifies and extends several existing results in the literature.

In section 7.2, using an Halpern extragradient method, we study a new iterative scheme for finding a common element of the set of solutions of multiple set split equality equilibrium problems consisting of pseudomonotone bifunctions and the set of fixed points for two finite families of Bregman quasi-nonexpansive mappings in the framework of p-uniformly convex Banach spaces which are also uniformly smooth. For this purpose, we design an algorithm so that it does not depend on prior estimates of the Lipschitz-type constants for the pseudomonotone bifunctions. Furthermore, we present an application of our study to investigate a common element of the set of solutions of multiple set split equality variational inequality problems and fixed points set for two finite families of Bregman quasi-nonexpansive mappings. Finally, we conclude with two numerical experiments to support our proposed algorithm..

Chapter 8 (Conclusion, Contributions to Knowledge and Future Research): In this chapter, we give the conclusion of our study and highlight the contributions of our study to existing knowledge. We also identify and discuss possible areas of future research.

Chapter 2

Preliminaries and Literature Review

In this chapter, we provide some definitions and concepts that will be useful throughout our study. More so, we discuss some past and recent results on optimization problems and give detailed literature review of some geometry of Banach spaces. Lastly, we recall important results that are required in the proofs of the main results of this thesis.

2.1 Preliminaries

In this section, we state some useful definitions and important results on nonlinear mappings that are essential to this study. In addition, we present some geometric properties of the spaces considered in this thesis. For more detailed information on certain concepts and terminologies used in this section, we refer the reader to the following wonderful materials [27], [31], [45] and [61].

2.2 Some operators in Hilbert spaces.

Let H be a real Hilbert space equipped with an inner product $\langle .,. \rangle$ and an induced norm $\|.\|$. Let C be a nonempty, closed and convex subset of H and $I : H \to H$ be the identity mapping on H. Let $T : H \to H$ be a nonlinear mapping. A point $x \in H$ is called a fixed point of T if $x = Tx$. In what follows, we shall denote the fixed point set of T by $F(T)$. In addition, we denote by $'x_n \rightharpoonup x'$ and $x_n \to x'$, the weak and strong convergence of the sequence $\{x_n\}$ to a point x, respectively and $w_\omega(x_n)$ denotes the set of weak limits of $\{x_n\}$, that is; $w_\omega(x_n) = \{x \in H : x_{n_k} \rightharpoonup x \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\}$.

Definition 2.2.1. *A mapping $T : H \to H$ is called*

1. *monotone if*

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall\ x, y \in H;$$

2. *pseudo-monotone, if*

$$\langle Tx, y - x\rangle \geq 0 \Longrightarrow \langle Ty, y - x\rangle \leq 0 \ \forall \ x, y \in H;$$

3. *β-strongly monotone, if there exists a constant $\beta > 0$ such that*

$$\langle Tx - Ty, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall \ x, y \in H;$$

4. *β-inverse strongly monotone (ism), if there exists a constant $\beta > 0$ such that*

$$\langle Tx - Ty, x - y\rangle \geq \beta\|Tx - Ty\|^2, \quad \forall x, y \in H;$$

5. *β strongly positive linear bounded operator, if there exists a constant $\beta > 0$ such that*

$$\langle Tx, x\rangle \geq \beta\|x\|^2, \quad \forall x \in H;$$

6. *k-Lipschitz continuous, if there exists a constant $k > 0$ such that*

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall \ x, y \in H.$$

   If $k \in [0, 1)$, then we say that T is a contraction.

7. *nonexpansive, if*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall \ x, y \in H;$$

8. *firmly nonexpansive, if*

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall \ x, y \in H;$$

   or equivalently $\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2$.

9. *averaged, if*

$$T = (1 - \alpha)I + \alpha S, \quad \alpha \in (0, 1),$$

   where $S : H \to H$ is nonexpansive and I is an identity mapping;

10. *quasi-nonexpansive, if $F(T) \neq \emptyset$ and for any $x^* \in F(T)$, we have*

$$\|Tx - Tx^*\| \leq \|x - x^*\|, \quad \forall \ x \in H;$$

11. *k-strictly pseudo-contractive mapping in the sense of Browder and Petryshyn [33], if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall \ x, y \in H;$$

   where $k \in [0, 1)$. If $k = 1$ in the last inequality, we say that T is pseudo-contractive and if $k = 0$, then T is simply nonexpansive.

12. *k-demicontractive, if $F(T) \neq \emptyset$ and there exists a constant $k \in [0, 1)$ such that for any $x^* \in F(T)$, we have*

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2, \quad \forall\ x\ \in H;$$

13. *k-demimetric, if there exist $k \in (-\infty, 1)$, $\forall\ x \in H$ and $x^* \in F(T)$ such that*

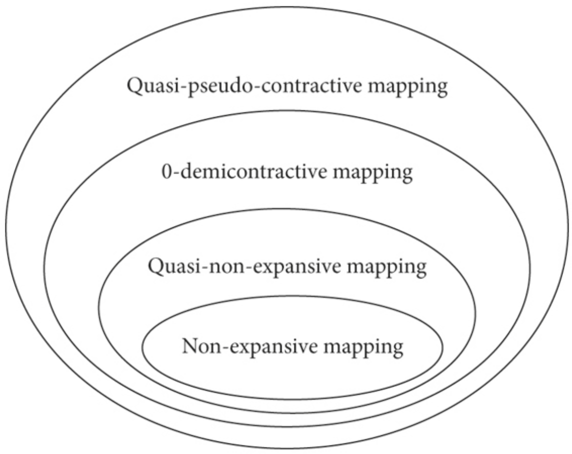
$$\langle x - x^*, x - Tx\rangle \geq \frac{1-k}{2}\|x - Tx\|^2.$$

Equivalently, T is k-demimetric, if there exists $k \in (-\infty, 1)$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2,\ \forall\ x \in H\ \text{and}\ x^* \in F(T).$$

14. *quasi-pseudocontractive (Hemicontractive), if $F(T) \neq \emptyset$ and*

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|x - Tx\|^2, \quad \forall\ x\ \in H\ \text{and}\ x^* \in F(T);$$



Remark 2.2.2. *The following can be deduced from the above definition:*

1. *Every firmly nonexpansive mapping is nonexpansive.*

2. *If T is β-inverse strongly monotone, then T is $\frac{1}{\beta}$-Lipschitz continuous.*

3. *If T is nonexpansive, then $I - T$ is monotone.*

4. *T is nonexpansive if and only if $I - T$ is $\frac{1}{2}$-inverse strongly monotone.*

5. *Every inverse strongly monotone operator is monotone and continuous.*

6. *Every firmly nonexpansive mapping is* 1*-inverse strongly monotone.*

7. *Every nonexpansive mapping with the nonempty fixed point set is quasi-nonexpansive.*

8. *Every quasi-nonexpansive mapping is k-demicontractive.*

9. *It is obvious that the class of demicontractive mapping is a subclass of the class of quasi-pseudocontractive (Hemicontractive) mappings as it generalizes the class of demicontractive mappings and quasi-nonexpansive mappings. Moreover, the class of demicontractive mappings includes the class of firmly nonexpansive and quasi-nonexpansive mappings.*

10. *Every monotone mapping is pseudomonotone but the converse is not always true. (See [129]).*

We present the following examples:

Example 2.2.3. *[167] Let $\mathbb{R}$ denote the set of real number with the usual norm and $T : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$Tx = \begin{cases} x, & \text{if} \quad (-\infty, 0); \\ -3x, & \text{if} \ [0, \infty). \end{cases}$$

Observe that $F(T) = (-\infty, 0]$. Then T is $\frac{1}{2}$-demicontractive but not quasi-nonexpansive.

Example 2.2.4. *[173] Let H be the closed interval $[0, 1]$ with the absolute value as norm. Define $T : H \to H$ by*

$$Tx = \begin{cases} \dfrac{1}{3}, & x \in [0, \frac{1}{3}], \\ \\ 0, & x \in (\frac{1}{3}, 1]. \end{cases}$$

Obviously, $F(T) = \{\frac{1}{3}\}$. Furthermore, for $x \in [0, \frac{1}{3}]$,

$$|Tx - \frac{1}{3}|^2 = 0 \leq |x - \frac{1}{3}|^2 + |x - \frac{1}{3}|^2$$

and for $x \in (\frac{1}{3}, 1]$,

$$|Tx - \frac{1}{3}|^2 = |\frac{1}{3}|^2 = \frac{1}{9} < |x - \frac{1}{3}|^2 + |x|^2.$$

Hence, for $x \in [0, 1]$,

$$|Tx - T(\frac{1}{3})|^2 \leq |x - \frac{1}{3}|^2 + |x - Tx|^2.$$

Next, we show that there is no $\alpha \in [0, 1)$ such that

$$|Tx - T(\frac{1}{3})|^2 \leq |x - \frac{1}{3}|^2 + \alpha|x - Tx|^2 \ \forall \ x \in [0, 1].$$

Let us assume on the contrary that there exists such α, then $\frac{1}{3} < \frac{1}{\alpha+1} < 1$. For such α, select x such that $\frac{1}{3} < x < \frac{1}{\alpha+1}$. It follows that $\alpha < \frac{1-x}{x}$ and so

$$|x-\frac{1}{3}|^2+\alpha|x-Tx|^2 < |x-\frac{1}{3}|^2+\frac{1-x}{x}|x-Tx|^2$$
$$=\frac{1}{9}=|Tx-T(\frac{1}{3})|^2,$$

which is a contradiction. Therefore, T is quasi-pseudocontractive but not demicontractive.

The next example shows a countable family of quasi-pseudocontractions which are not demicontractive.

Example 2.2.5. *[222] Let $H=(\mathbb{R}^4,\|.\|_2)$. For $\bar{s}=(s_1,s_2,s_3,s_4)\in\mathbb{R}^4$ and $i\in\mathbb{N}$ define*

$$T_i\bar{s}=\begin{cases}(\frac{-1-i}{4}s_1,s_2,s_3,\frac{1}{2}), & s_4\in[0,\frac{1}{2}],\\ (\frac{-1-i}{4}s_1,s_2,s_3,0), & otherwise.\end{cases}$$

Then, $F(T_i)=\{(0,0,0,\frac{1}{2})\}$. Let $x^*(0,0,0,\frac{1}{2})$. Now, for $\bar{s}=(s_1,s_2,s_3,s_4)\in\mathbb{R}^4$ such that $s_4\in[0,\frac{1}{2}]$, we obtain

$$\|T_i\bar{s}-x^*\|^2=\left(\frac{1+i}{4}\right)^2s_1^2+s_2^2+s_3^2,\quad \|\bar{s}-x^*\|^2=s_1^2+s_2^2+s_3^2+\left(s_4-\frac{1}{2}\right)^2$$

and

$$\|\bar{s}-T_i\bar{s}\|^2=\left(\frac{5+i}{4}\right)^2s_1^2+\left(s_4-\frac{1}{2}\right)^2.$$

It follows that

$$\|T_i\bar{s}-x^*\|^2\le\|\bar{s}-x^*\|^2+\|\bar{s}-T_i\bar{s}\|^2.$$

In the other case, let $\bar{s}=(s_1,s_2,s_3,s_4)\in\mathbb{R}^4$ such that $s_4\notin[0,\frac{1}{2}]$, we have

$$\|T_i\bar{s}-x^*\|^2=\left(\frac{1+i}{4}\right)^2s_1^2+s_2^2+s_3^2+\frac{1}{4},\quad \|\bar{s}-x^*\|^2=s_1^2+s_2^2+s_3^2+\left(s_4-\frac{1}{2}\right)^2$$

and

$$\|\bar{s}-T_i\bar{s}\|^2=\left(\frac{5+1}{4}\right)^2s_1^2+s_4^2.$$

We also see in this case that

$$\|T_i\bar{s}-x^*\|^2<\|\bar{s}-x^*\|^2+\|\bar{s}-T_i\bar{s}\|^2.$$

Therefore, $\forall\ \bar{s}\in H$,

$$\|T_i\bar{s}-x^*\|^2\le\|\bar{s}-x^*\|^2+\|\bar{s}-T_i\bar{s}\|^2,\quad i\in\mathbb{N}.$$

Next, we show that T_i is not demicontractive, that is, there is no $\alpha \in [0,1)$ such that $\|T_i\bar{s} - x^*\|^2 \leq \|\bar{s} - x^*\|^2 + \alpha\|\bar{s} - T_i\bar{s}\|^2, \quad \forall\ \bar{s} \in H, \quad i \in \mathbb{N}$. Now,suppose contrariwise that there exists such α, then by implication, $\frac{1}{2} < \frac{1}{\alpha+1} \leq 1$. For $x^* = \left(0,0,0,\frac{1}{2}\right)$ and such α, select $\bar{s} = (s_1, s_2, s_3, s_4)$ such that $\frac{1}{2} < s_4 < \frac{1}{\alpha+1}$. This implies that $\alpha < \frac{1-s_4}{s_4}$ and so

$$\begin{aligned}\|\bar{s} - x^*\|^2 + \alpha\|\bar{s} - T_i\bar{s}\|^2 &< s_1^2 + s_2^2 + s_3^2 + \left(s_4 - \frac{1}{2}\right)^2 + \frac{1-s_4}{s_4}\left[\left(\frac{5+i}{4}s_1^2 + s_4^2\right)\right]\\ &= s_1^2 + s_2^2 + s_3^2 + \frac{1}{4} + \frac{1-s_4}{s_4}\left(\frac{5+i}{4}\right)s_1^2.\end{aligned}$$

In particular, consider $\bar{s} = (0, s_2, s_3, s_4) \in \mathbb{R}^4$, then the last inequality gives

$$\|\bar{s} - x^*\|^2 + \alpha\|\bar{s} - T_i\bar{s}\|^2 < s_2^2 + s_3^2 + \frac{1}{4} = \|T_i\bar{s} - x^*\|^2,$$

which is a contradiction. Hence $\{T_i\}$ is a countable family of quasi-pseudocontractive mappings which are not demicontractive.

In addition, we consider an example of a monotone operator in quantum mechanics.

Example 2.2.6. *[207] Let the operator*

$$Au := -b^2\Delta u + (f(x) + c)u(x) + u(x)\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}dy,$$

where $\Delta := \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^3$, b and c are constants, $f(x) = f_0(x) + f_1(x)$, where $f_0(x) \in L^\infty(\mathbb{R}^3)$ and $f_1(x) \in L^2(\mathbb{R}^3)$. Let $A := L + B$, where the operator L which is the schrödinger operator is the linear part of A and B defined by the last term. It is known that B is a monotone operator on $L^2(\mathbb{R}^3)$, (see page 23 of [11]) which also implies that $A : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is also a monotone operator.

Definition 2.2.7. *A bifunction $f : C \times C \to \mathbb{R}$ is called*

(a) strongly monotone on C, if there exists a constant $\beta > 0$ such that

$$f(x,y) + f(y,x) \leq -\beta\|x-y\|^2, \quad \forall\ x, y \in C;$$

(b) monotone on C, if

$$f(x,y) + f(y,x) \leq 0, \quad \forall\ x, y \in C;$$

(c) strongly pseudomonotone on C if there exists a constant $\beta > 0$ such that

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \quad \forall\ x, y \in C;$$

(d) pseudomonotone on C, if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \quad x, y \in C;$$

(e) f is said to satisfy the Lipschitz-type condition, if there exists constants $c_1, c_2 > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \qquad x,y,z \in C.$$

Clearly, (a) $\Rightarrow$ (b) $\Rightarrow$ (d) and (a) $\Rightarrow$ (c) $\Rightarrow$ (d). However, the converse is not necessarily true is general.

A subset D of H is called *proximal* if for each $x \in H$, there exists $y \in D$ such that

$$||x-y|| = \text{dist}(x, D),$$

where $\text{dist}(x, D) = \inf\{\|x-y\| : y \in D\}$ is the distance from a point x to D.

Let H be a real Hilbert space. We denote by $CB(H), CC(H)$ and $P(H)$ the collections of all nonempty closed bounded subsets of H, nonempty closed convex subset of H and nonempty proximal bounded subsets of H, respectively. The Pompeiu-Hausdorff metric [28] $\mathcal{H}$ on $CB(H)$ is defined by

$$\mathcal{H}(D_1, D_2) := \max\left\{\sup_{x\in D_1} \text{dist}(x, D_2), \sup_{y\in D_2} \text{dist}(y, D_1)\right\}, \ \forall D_1, D_2 \in CB(H).$$

Let $S : H \to 2^H$ be a multivalued mapping. An element $x \in H$ is called a fixed point of S if $x \in Sx$. The set of all fixed point of S is denoted by $F(S)$. That is, $F(S) = \{x \in H : x \in Sx\}$. We say that S satisfies the *endpoint condition* if $Sp = \{p\}$ for all $p \in F(S)$. For multivalued mappings $S_i : H \to 2^H (i \in \mathbb{N})$ with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, we say S_i satisfies the *common endpoint condition* if $S_i(p) = \{p\}$ for all $i \in \mathbb{N}, p \in \cap_{i=1}^{\infty} F(S_i)$.

Recall that a multivalued (set-valued) mapping $S : H \to 2^H$ is called

(i) L-Lipschitzian if there exists $L > 0$ such that

$$\mathcal{H}(Sx, Sy) \leq L\|x-y\|, \ \forall x, y \in H. \tag{2.1}$$

In (2.55), if $L \in (0,1)$, then S is called a contraction while S is called nonexpansive if $L = 1$.

(ii) k-nonspreading if for all $x, y \in H$

$$\mathcal{H}^2(Sx, Sy) \leq k\left(\text{dist}^2(Sx, y) + \text{dist}^2(x, Sy)\right); \tag{2.2}$$

(iii) quasi-nonexpansive if $F(S) \neq \emptyset$ and

$$\mathcal{H}(Sx, Sp) \leq \|x-p\|, \ \forall x \in H, p \in F(S),$$

(iv) k-strictly pseudocontractive, if there exists a constant $k \in (0,1)$ such that for all $x, y \in H$

$$\mathcal{H}^2(Sx, Sy) \leq \|x-y\|^2 + k\|x-y-(a-b)\|^2 \ \forall \ a \in Sx, \ b \in Sy; \tag{2.3}$$

(v) generalized k-strictly pseudocontractive [60], if there exists $k \in (0,1)$ such that $x, y \in H$ and

$$\mathcal{H}^2(Sx, Sy) \leq \|x - y\|^2 + k\mathcal{H}^2(Bx, By),$$

where $B := I - S$, and I is the identity operator on H

(vi) demicontractive if $\mathrm{F}(S) \neq \emptyset$ and

$$\mathcal{H}^2(Sx, Sp) \leq \|x - p\|^2 + k\mathrm{dist}(x, Sx)^2, \ \forall x \in H, p \in F(S) \text{ and } k \in [0,1), \tag{2.4}$$

where dist is the distance between x and Sx.

Remark 2.2.8. *If $k = 1$ in (2.2), (2.3) and (2.4), we have a new set of mappings called nonspreading, pseudocontractive and hemicontractive respectively.*

Remark 2.2.9. *Clearly, every multivalued quasi-nonexpansive mapping is a multivalued demicontractive mapping. However, the following counter example demonstrates that the converse is not always true.*

Example 2.2.10. *[119] Let $H = \mathbb{R}$ (endowed with the usual metric) and $T : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by*

$$Tx = \begin{cases} [-(\alpha+1)x, \ \ -\frac{2\alpha+1}{2}x], & x \in [0, \infty) \\ \\ [-\frac{2\alpha+1}{2}x, \ \ -(\alpha+1)x], & x \in (-\infty, 0), \ \forall \alpha > 0. \end{cases}$$

Then, T is a demicontractive mapping but not quasi-nonexpansive.

The fixed point theory for multivalued mappings can be utilized in various areas such as game theory, control theory, mathematical economics, market economy, differential inclusions and constrained optimization. They are also useful in generating critical points in optimal control problems, energy management problems, signal processing and image reconstruction, to mention a few. Therefore, the existence and approximations of fixed points for multivalued mappings have been developed and investigated in the literature (see for example [172]) and the references therein.

Definition 2.2.11. *[184] Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear operator. Then, T is called*

(i) asymptotically nonexpansive if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\|, \ \forall \ \ n \in \mathbb{N}, \ \ x, y \ \ \in C;$$

(ii) asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n x - T^n y, x - y\rangle \leq k_n\|x - y\|^2, \ \ \forall n \in \mathbb{N}, \ \ x, y \in C, \tag{2.5}$$

it is easy to see that (2.5) is equivalent to

$$\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|x - y - (T^n x - T^n y)\|^2, \ \ \forall \ n \in \mathbb{N}, \ \ x, y \in C;$$

(iii) *k-strictly asymptotically pseudocontractive if there exists a sequence $\{k_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$\|T^n x - T^n y\|^2 \leq k_n\|x-y\|^2 + k\|(1-T^n)x - (1-T^n)y\|^2, \quad \forall\, n \in \mathbb{N}, \quad x, y \in C; \tag{2.6}$$

(iv) *asymptotically demicontractive if there exists a sequence $\{k_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} k_n = 1$, and for $0 \leq k < 1$,*

$$\|T^n x - x^*\|^2 \leq k_n\|x - x^*\|^2 + k\|x - T^n x\|^2, \quad \forall\, n \in \mathbb{N}, \quad x \in C, \quad x^* \in F(T);$$

or equivalently,

$$\langle x - T^n x, x - x^* \rangle \geq \frac{1}{2}(1-k)\|x - T^n x\|^2 - \frac{1}{2}(k_n^2 - 1)\|x - x^*\|^2; \tag{2.7}$$

Remark 2.2.12. *If $k_n = 1$, and $T^n = T$ in (2.6), then we obtain the class of strict pseudocontractive mappings. The class of asymptotically demicontractive mappings and the class of demicontractive mappings are independent as shown in the examples below.*

Example 2.2.13. *Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx = -3x$. Then the mapping T is demicontractive but not asymptotically demicontractive.*
It is clear that $Fix(T) = \{0\}$. For $\beta \in [\frac{1}{2}, 1)$ and $x \in \mathbb{R}$, we get

$$\begin{aligned} |Tx - 0|^2 &= 9|x|^2 \\ &\leq (1 + 16\beta)|x|^2 \\ &= |x - 0|^2 + \beta|Tx - x|^2. \end{aligned}$$

Which shows that T is β-demicontractive. Suppose that T is k-asymptotically demicontractive with sequence $\{k_n\}$. Then there exists some $N_0 \in \mathbb{N}$ such that $k_n < 2$ for $n \geq N_0$. For such n which is even,

$$|T^n x - 0|^2 = 3^{2n}|x|^2.$$

$$\begin{aligned} k_n^2|x - 0|^2 + k|T^n x - x|^2 &= k_n^2|x|^2 + k(3^n - 1)^2|x|^2 \\ &< (5 + 3^{2n} - 2(3^n))|x|^2 \\ &< 3^{2n}|x|^2 \\ &= |T^n x - 0|^2. \end{aligned}$$

Hence T is not a k-asymptotically demicontractive mapping.

Example 2.2.14. *Let $E = \ell_2 := \{x = \{x_i\}_{i=1}^{\infty} : x_i \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $B = \{x \in \ell_2 : \|x\| \leq 1\}$. Define $T : B \to \ell_2$ is defined by*

$$Tx = (0, x_1^2, a_2x_2, a_3x_3, \dots),$$

where $\{a_j\}_{j=1}^{\infty}$ is a real sequence satisfying: $a_2 > 0, 0 < a_j < 1, j \neq 2$, and $\prod_{j=2}^{\infty} a_j = \frac{1}{2}$. It is known that T is k-strictly asymptotically pseudocontractive but not k-strictly pseudocontractive (see [184]). Since $Fix(T) = \{(0, 0, 0, \dots)\} \neq \emptyset$, it then follows that T is k-asymptotically demicontractive but not k-demicontractive.

Remark 2.2.15. *Although the classes of demicontractive and asymptotically demicontractive mappings are independent, it is noteworthy that the metric projection P_C is both 0-demicontractive and 0-asymptotically demicontractive.*

2.2.1 Metric projection

In this subsection, we define the metric projection in a real Hilbert space H and present some of its basic properties with examples.

Definition 2.2.16. *Let C be a nonempty, closed and convex subset of H. Recall that for all $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that*

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

The mapping P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and satisfies

$$\langle x - z, x - P_C z\rangle \geq \|x - P_C z\|^2 \ \ \forall x \in C, \ z \in H. \tag{2.8}$$

The following are examples of metric projections of H onto different set C, some of which are used in this thesis (see for example [156]).
The following are equivalent (see [96, 182] for details);

$$\begin{cases} (i.) & P_C : H \to C \text{ is a projection of } H \text{ onto } C, \\ (ii.) & \forall\ x \in H, \quad \langle x - P_C x, z - P_C x\rangle \leq 0, \ \forall\ z \in C, \\ (iii.) & \forall\ x \in H, \ \|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2, \ \forall\ z \in C. \end{cases} \tag{2.9}$$

Example 2.2.17. *Let $C = [a, b]$ be a closed rectangle in $\mathbb{R}^n$, where $a = (a_1, a_2, ..., a_n)^T$ and $b = (b_1, b_2, ..., b_n)^T$, then for $1 \leq i \leq n$, we have*

$$(P_C x)i = \begin{cases} a_i, & x_i < a_i, \\ x_i, & x_i \in [a_i, b_i], \\ b_i, & x_i > b_i, \end{cases}$$

is the metric projection with the i^{th} coordinate.

Example 2.2.18. *Let C be the range of an $m \times n$ matrix with full column rank and A^* be the adjoint of A, then*

$$P_C x = A(A^* A)^{-1} A^* x$$

is the metric projection P_C onto C.

Example 2.2.19. *If $C = \{y \in H : \langle a, x\rangle = \alpha\}$ is a hyperplane with $a \neq 0$ and $\alpha \in \mathbb{R}$, then*

$$P_C(x) = x - \frac{\langle a, x\rangle - \alpha}{||a||^2} a.$$

Example 2.2.20. *Let $C = \{x \in H : ||x - u|| \leq r\}$ is a closed ball centered at $a \in H$ with radius $r > 0$, then*

$$P_C x = \begin{cases} a + r\frac{(x-a)}{||x-a||}, & x \notin C, \\ x, & x \in C. \end{cases}$$

The following are important inequalities that characterize the metric projection.

Proposition 2.2.21. *[220] Let C be a nonempty closed and convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. We have that $z = P_C x$ if and only if $\langle x - z, \quad z - y \rangle \geq 0, \quad \forall y \in C$.*

Proposition 2.2.22. *Let C be a nonempty closed and convex subset of a real Hilbert space H and $x \in H$. Then,*

(a) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \ \forall \ y \in C.$

(b) $\|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2, \ \forall \ y \in C.$

The interested reader should also check Section 3 of [94] for more properties of metric projection.

Let us recall the indicator function of C denoted by i_C and defined as

$$i_C := \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Definition 2.2.23. *A mapping $S : H \to H$ is said to be $I - S$ demiclosed at 0, if for any sequence $\{x_n\}$ that converges weakly to $x \in H$ and $\|x_n - Sx_n\| \to 0$, then $Sx_n = x$.*

2.3 Some geometric properties of Banach spaces.

In this section, we discuss briefly some important definitions, examples and fundamental theorems of some geometric properties of Banach spaces. Most of the results discussed in this section can be found in [31, 39, 40, 41, 61, 216].

2.3.1 Reflexive Banach spaces.

Let E be a Banach space with its dual E^*, which is the space of all continuous linear functionals on E. We define the norm on the dual space E^* of E denoted with $||.||_{E^*}$ by

$$\|f\|_{E^*} = \sup\{|\langle f, x \rangle| : ||x|| \leq 1, x \in E\},$$

where the pairing $\langle f, x \rangle$ is denotes the action of $f \in E^*$ on $x \in E$, that is $\langle f, x \rangle := f(x)$. The bidual E^{**}, which is the dual of E^* with $||.||_{E^*}$ can also be defined as

$$||\xi||_{E^{**}} = \sup\{|\langle \xi, f \rangle| : f \in E^*, \|f\| \leq 1\}, \text{ where } \xi \in E^{**}.$$

There is a canonical injection $J : E \to E^{**}$ defined as follows: given $x \in E$, the map $f \longmapsto \langle f, x \rangle$ is a continuous linear functional on E^*, thus it is an element of E^{**} which is denoted by Jx. Hence, we have that

$$\langle Jx, f \rangle_{E^{**}, E^*} = \langle f, x \rangle_{E^*, E}, \forall \ x \in E, \text{ and } f \in E^*.$$

It is obvious that J is linear and isometry, that is $\|Jx\|_{E^{**}} = \|x\|_E$. Indeed, we have that

$$\|Jx\|_{E^{**}} = \sup\{|\langle Jx, f \rangle| : f \in E^*, \|f\| \leq 1\} = \sup\{|\langle f, x \rangle| : f \in E^*, \|f\| \leq 1\} = \|x\|.$$

Definition 2.3.1. *Let E be a Banach space and $J : E \to E^{**}$ be the canonical injection from E into E^{**}. The space E is said to be reflexive if J is surjective, that is $J(E) = E^{**}$.*

Corollary 2.3.2. *A Banach space E is reflexive if and only if its dual space E^* is reflexive.*

Examples of reflexive Banach spaces are the infinite dimensional spaces, L_p and l_p spaces for $1 < p < \infty$, Hilbert and Sobolev spaces.
Below is an important result on the convergence of sequence in reflexive Banach space.

Theorem 2.3.3. *[241] Assume that E is a reflexive Banach space and $\{x_n\}$ is a bounded sequence in E. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges in the weak topology $\sigma(E, E^*)$.*

2.3.2 Uniformly convex spaces

Definition 2.3.4. *A Banach space E is said to be uniformly convex if $\forall\ \varepsilon > 0$ there exists $\delta > 0$ such that*

$$[x, y \in E, ||x|| \leq 1, ||y|| \leq 1 \text{ and } ||x - y|| > \varepsilon] \Longrightarrow \left[\left|\left|\frac{x+y}{2}\right|\right| < 1 - \delta\right].$$

The uniform convexity is a geometric property of the unit ball if we slide a rule of length $\varepsilon > 0$ in the unit ball, then its midpoint must stay within a ball of radius $(1 - \delta)$ for some $\delta > 0$.

Example 2.3.5. *The space L^p are uniformly convex $1 < p < \infty$ and the Hilbert spaces are also uniformly convex.*

Below is a very useful convergence result of uniformly convex spaces.

Proposition 2.3.6. *Assume that E is a uniformly convex Banach space. Let $\{x_n\}$ be a sequence in E such that $x_n \rightharpoonup x$ weakly $\sigma(E, E^*)$ and*

$$\limsup \|x_n\| \leq \|x\|.$$

Then $x_n \to x$.

Definition 2.3.7. *A Banach space E is said to be strictly convex if for all $x, y \in E,\ x \neq y,\ \|x\| = \|y\| = 1$, we have*

$$\|\lambda x + (1 - \lambda)y\| < 1,\ \forall\ \lambda \in (0, 1).$$

Example 2.3.8. *[91] Let $\sigma > 0$ and $k_0 = k_0(\mathbb{N})$ with norm $\|.\|_\sigma$ defined for $x = \{x_n\} \in k_0$ by*

$$\|x\|_\sigma := \|x\|_{k_0} + \sigma\Big(\sum_{i=1}^{\infty}(\frac{x_i}{i})^2)^{\frac{1}{2}},$$

where $\|.\|_{k_0}$ is the usual l_∞ norm. Then the space $(k_0, \|.\|_\sigma)$ for $\sigma > 0$ are strictly convex but not uniformly convex, while k_0 with its usual norm is strictly convex.

Definition 2.3.9. *Let E be a Banach space with (dimension) dim$(E) \geq 2$. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by*

$$\delta_E(\varepsilon) := \inf \left\{ 1 - \left|\left| \frac{x+y}{2} \right|\right| : ||x|| = ||y|| = 1,\ \varepsilon = ||x-y|| \right\}.$$

In particular, for a real Hilbert space H, we have

$$\delta_H(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

Theorem 2.3.10. *The modulus of convexity of a Banach space E given by δ_E is a convex and continuous function.*

Theorem 2.3.11. *A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.*

Theorem 2.3.12. *If E is a uniformly convex Banach space, then E is reflexive.*

Definition 2.3.13. *Let $p > 1$ be a real number, then a Banach space E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) \geq c\varepsilon^p$.*

Example 2.3.14. *If $E = L_p$ (or l_p), $1 < p < \infty$, then*

1. $\delta_E(\varepsilon) \geq \frac{1}{2^{p+1}}\varepsilon^2$, , *if* $1 < p < 2$;
2. $\delta_E(\varepsilon) \geq \varepsilon$, *if* $2 \leq p < \infty$.

2.3.3 Uniformly smooth spaces

Definition 2.3.15. *A Banach space E is said to be smooth if for every $x \in E$, $\|x\| = 1$, there exists a unique $x^* \in E^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.*

Definition 2.3.16. *A Banach space E is said to be uniformly smooth whenever given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = 1$ and $\|y\| \leq \delta$, then*

$$\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|.$$

Definition 2.3.17. *Let E be a Banach space with dim $E \geq 2$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by*

$$\begin{aligned} \rho_E(\tau) : &= \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1 : \|y\| = \tau \right\} \\ &= \sup \left\{ \frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : \|x\| = 1 = \|y\| \right\}. \end{aligned}$$

Theorem 2.3.18. *A Banach space E is uniformly smooth if and only if*

$$\lim_{t \to 0^+} \frac{\rho_E(t)}{t} = 0.$$

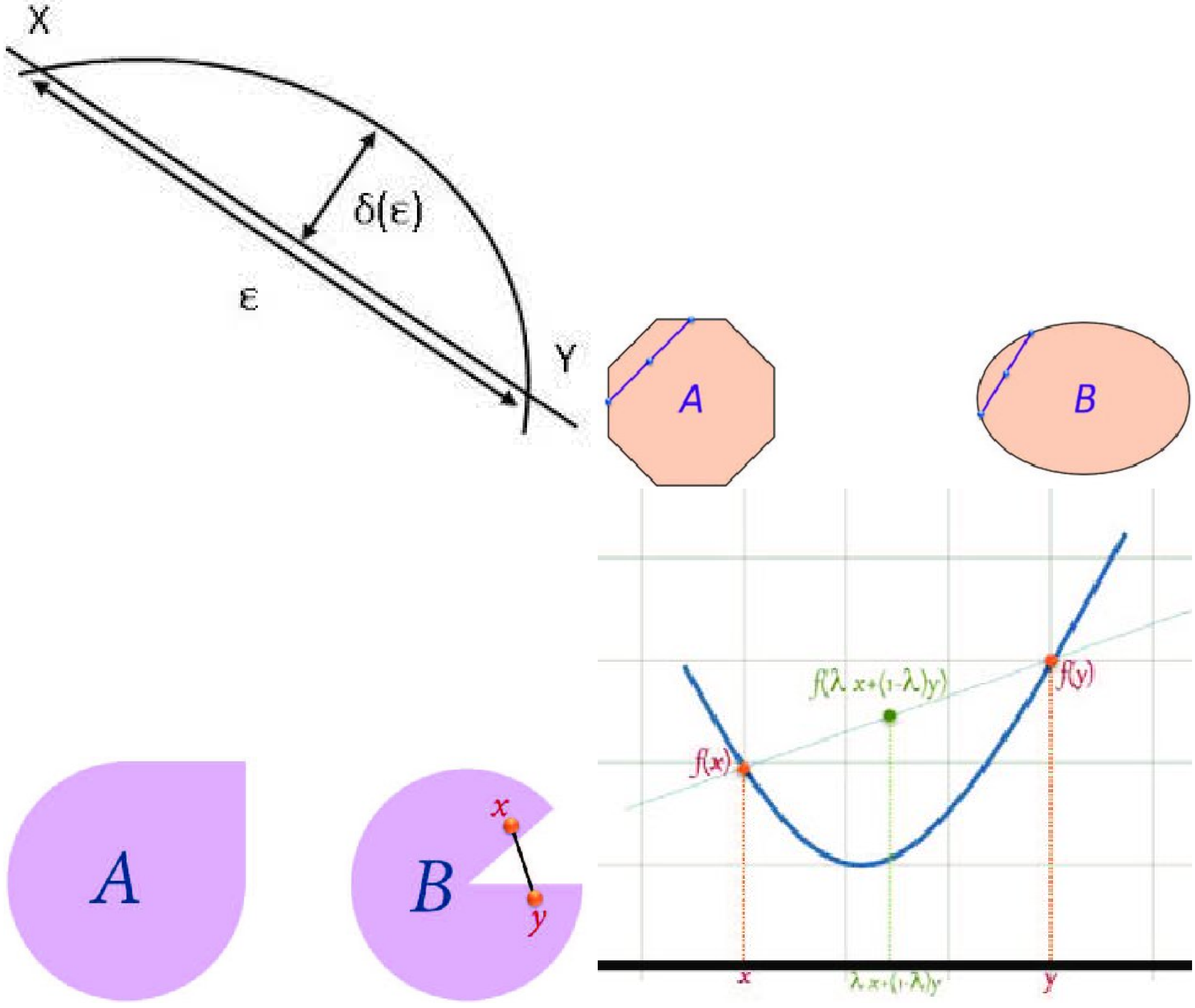


Figure 2.2. The geometric representation of convexity functions: Top Left: Uniformly convex space; Top Right: uniformly convex set; Bottom Left: Convex and Non convex set; Bottom Right: Convex function

Theorem 2.3.19. *Every uniformly smooth Banach space E is smooth.*

Theorem 2.3.20. *Let E be a Banach space with dual space E^*, then we have that*

1. *E is uniformly smooth if and only if E^* is uniformly convex.*
2. *E is uniformly convex if and only if E^* is uniformly smooth.*

Corollary 2.3.21. *Every uniformly smooth Banach space is reflexive.*

Definition 2.3.22. *For $q > 1$, a Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q,\ t > 0$.*

Proposition 2.3.23. *Let E be a real Banach space, then*

1. *E is p-uniformly convex if and only if E^* is q-uniformly smooth.*
2. *E is q-uniformly smooth if and only if E^* is p-uniformly convex, and $\frac{1}{p} + \frac{1}{q} = 1$.*

2.3.4 Duality mappings

Definition 2.3.24. *Let E be a Banach space with its dual E^*. Given a gauge function ϕ, the mapping $J_\phi : E \to 2^{E^*}$ defined by*

$$J_\phi x = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|,\ \|x^*\| = \phi(\|x\|), x \in E\}$$

is called the duality map with gauge function ϕ. In particular, if $\phi(t) = t$, the duality map $J = J_\phi$ is called the normalized duality map.

Proposition 2.3.25. *In a real Hilbert space H, the normalized duality map is the identity map.*

The normalized duality map J possess the following properties:

1. If E is smooth, then J is single-valued;
2. If E is strictly convex, then J is one-to -one;
3. If E is reflexive, then J is surjective;
4. If E is uniformly smooth, then J is uniformly norm to norm continuous on each bounded subsets of E;
5. If E^* is uniformly convex, then J is uniformly continuous on each bounded subsets of E and J is single-valued and also one to one.

Definition 2.3.26. *Let E be a real Banach space with the dual E^*, for $p > 1$, let p be a given real number. The generalized duality mapping J_p^E from E to 2^{E^*} is defined as*

$$J_p^E(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p,\ \|x^*\| = \|x\|^{p-1}, x \in E\}.$$

In a q-uniformly smooth Banach space E, the generalized duality mapping is injective and satisfies $J_p^E = (J_q^{E^*})^{-1}$ where $J_q^{E^*}$ is the generalized duality mapping of E^*, (see [10, 66]). The generalized duality mapping J_p^E is said to be weak to weak continuous if $x_n \rightharpoonup x \Longrightarrow \langle J_p^E x_n, y \rangle \to \langle J_p^E x, y \rangle$ holds for any $y \in E$.

2.3.5 Basic notions of convex analysis

We give the following important definitions which are used in subsequent chapters.
Let E be a real Banach space and $f : E \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. We denote the domain of f by $Domf = \{x \in E : f(x) < +\infty\}$.

Definition 2.3.27. *Let D be a convex subset of a vector space X and $f : D \to \mathbb{R} \cup \{+\infty\}$ be a map. Then,*
(i) f is convex if for each $\lambda \in [0, 1]$ and $x, y \in D$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y);$$

(ii) f is called proper if there exists at least one $x \in D$ such that

$$f(x) \neq +\infty;$$

(iii) $f : Dom(f) \to (-\infty, \infty]$ is lower semi-continuous at a point $x \in Dom(f)$ if

$$f(x) \leq \liminf_{n\to\infty} f(x_n), \tag{2.10}$$

for each sequence $\{x_n\}$ in $Dom(f)$ such that $\lim\limits_{n\to\infty} x_n = x$. An example of a convex and lower semicontinuous function is the indicator function $\delta_C : H \to \mathbb{R}$ of a nonempty closed and convex subset C of H defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise}; \end{cases}$$

(iv) f is upper semi-continuous at $x_0 \in D$ if

$$f(x_0) \geq \limsup_{x\to x_0} f(x).$$

Definition 2.3.28. *Let H be a real Hilbert space and $A : H \to H$ be a bounded linear map. We define a map $A^* : H \to H$ by the relation*

$$\langle Ax, y\rangle = \langle x, A^*y\rangle,$$

for all $x, y \in H$. The map A^* is called the adjoint/dual of A.

Let $x \in int\ domf$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^*\rangle \leq f(y),\ \forall\ x \in E\}.$$

The Fénchel conjugate of f is the convex function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

It is known that f satisfies the Young-Fénchel inequality

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*)\ x \in E,\ x^* \in E^*.$$

Moreover, the inequality holds if $x^* \in \partial f(x)$.
Given $x \in int\ dom f$ and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by

$$f^0(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.11}$$

The function f is said to be Gâteaux differentiable at x if (2.11) exists for any y. In this case, the gradient of f at x is the linear function $\bigtriangledown f(x)$ defined by $\langle y, \bigtriangledown f(x)\rangle := f^0(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in int\ dom f$. When the limit as $t \to 0$ in (2.11) is attained uniformly for any $y \in E$ with $||y|| = 1$, we say that f is uniformly Fréchet differentiable at x. Also, if f is Fréchet differentiable, the $\bigtriangledown f$ is norm to norm continuous (see [141]).

Definition 2.3.29. *The function $f : E \to (-\infty, +\infty]$ is called Legendre if it satisfies the following conditions:*

1. *f is Gâteaux differentiable, $int\ dom f \neq \emptyset$ and $dom \bigtriangledown f = int\ dom f$,*

2. *f^* is Gâteaux differentiable, $int\ dom f^* \neq \emptyset$ and $dom \bigtriangledown f^* = int\ dom f^*$.*

One important and interesting Legendre function is $\frac{1}{p}||.||^p$ $(1 < p < \infty)$, where the Banach space E is smooth and strictly convex, and in particular the Hilbert space. For more examples of Legendre function, (see [22, 24]).

Remark 2.3.30. *If E is a real reflexive Banach space and f is a Legendre function, then we have*

1. *f is a Legendre function if and only if f^* is a Legendre function,*

2. *$(\partial f)^{-1} = \partial f^*$,*

3. *$\bigtriangledown f = (\bigtriangledown f^*)^{-1}$, $ran \bigtriangledown f = dom \bigtriangledown f^* = int(dom f^*)$, $ran \bigtriangledown f^* = dom \bigtriangledown f = int(dom f)$;*

4. *f and f^* are strictly convex on the interior of their respective domains.*

Remark 2.3.31. *If $f : E \to \mathbb{R}$ is Gâteaux differentiable and convex, then*

$$\begin{aligned}
\langle y, \bigtriangledown f(x)\rangle = f^0(x, y) &= \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} \\
&= \lim_{t \to 0} \frac{f((1 - t)x + t(x + y)) - f(x)}{t} \\
&\leq \lim_{t \to 0} \frac{(1 - t)f(x) + tf(x + y) - f(x)}{t} \\
&= f(x + y) - f(x).
\end{aligned}$$

Definition 2.3.32. *[30] Let $f : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function, the Bregman distance with respect to f is the bifunction $D_f : domf \times int\ domf \to [0, \infty)$ defined by*

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \bigtriangledown f(y)\rangle. \tag{2.12}$$

It is worth mentioning that D_f is not a distance in the usual sense but enjoys the following properties:

1. $D_f(x, x) = 0$, but $D_f(x, y) = 0$ may not simply imply $x = y$,

2. D_f is not symmetric and does not satisfy the triangle inequality,

3. for $x \in domf$ and $y, z \in domf$, we have
$$D_f(x, y) + D_f(y, z) - D_f(x, z) \leq \langle \bigtriangledown f(z) - \bigtriangledown f(y), x - y\rangle, \tag{2.13}$$

4. for each $z \in E$, we have $D_f(z, \bigtriangledown f^*(\sum_{i=1}^{N} t_i \bigtriangledown f(x_i))) \leq \sum_{i=1}^{N} t_i D_f(z, x_i)$, where $\{x_i\}_{i=1}^{N} \subseteq E$ and $\{t_i\}_{i=1}^{N} \subseteq (0, 1)$ satisfies $\sum_{i=1}^{N} t_i = 1$.

More so, it is well known that the duality mapping J_p^E is the sub-differential of the functional $f_p(.) = \frac{1}{p}||.||^p$ for $p > 1$, (see [61]). Then, the Bregman distance D_p is defined with respect to f_p as follows:

$$\begin{aligned} D_p(x, y) &= \frac{1}{p}||y||^p - \frac{1}{p}||x||^p - \langle J_P^E x, y - x\rangle \\ &= \frac{1}{q}||x||^p - \langle J_p^E x, y\rangle + \frac{1}{p}||y||^p \\ &= \frac{1}{q}||x||^p - \frac{1}{q}||y||^p - \langle J_p^E x - J_p^E y, y\rangle. \end{aligned} \tag{2.14}$$

Bregman distance has been studied by many researchers because of its nice and effective characteristics in analyzing optimization and feasibility algorithms, (see [24, 38, 39, 40] and the references contained in). The function $V_f : E \times E^* \to [0, +\infty]$ associated with f, which is defined by [41]

$$V_f(x, x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \quad \forall\ x \in E. \tag{2.15}$$

Then $V_f(x, x^*) = D_f(x, \bigtriangledown f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover, by subdifferential inequality, we have

$$V_f(x, x^*) + \langle \bigtriangledown f^*(x^*) - x, y^*\rangle \leq V_f(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.
The modulus of total convexity at x is the bifunction $\upsilon_f : int\ domf \times [0, +\infty) \to [0, +\infty)$ defined by

$$\upsilon_f(x, t) := int\{D_f(y, x) : y \in domf, ||y - x|| = t\}.$$

The function f is said to be totally convex at $x \in$ *int domf* if $\upsilon_f(x,t)$ is positive for any $t > 0$. Let C be a nonempty subset of E, the modulus of total convexity of f on C is the bifunction $\upsilon_f : int\ domf \times [0,+\infty) \to [0,+\infty)$ defined by

$$\upsilon_f(C,t) := \{\upsilon_f(x,t) : x \in C \cap int\ domf\}.$$

The function f is called totally convex on bounded subsets if $\upsilon_f(C,t)$ is positive for any nonempty and bounded subset C and any $t > 0$.

Proposition 2.3.33. *[197] If $x \in int\ domf$, then the following statements are equivalent:*

1. *the function f is totally convex,*

2. *for any sequence $\{y_n\} \subset domf$,*

$$\lim_{n\to\infty} D_f(y_n, x) = 0 \Longrightarrow \lim_{n\to\infty} \|y_n - x\| = 0.$$

Definition 2.3.34. *[39] A function f is called sequentially consistent if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,*

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Longrightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Proposition 2.3.35. *[41] If $domf$ contains at least two points, then the function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Proposition 2.3.36. *[194] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

The Bregman projection [30] $(Proj_C^f(x))$ with respect to f at $x \in int(domf)$ onto a nonempty, closed and convex set $C \subset int(domf)$ is defined by

$$D_f(Proj_C^f(x), x) = \inf_{y\in C} D_f(y, x). \tag{2.16}$$

It is well-known that (see [39]) $z = Proj_C^f x$ if and only if $\langle \triangledown f(x) - \triangledown f(z), y - z\rangle \leq 0$ for all $y \in C$. We also have

$$D_f(y, Proj_C^f(x)) + D_f(Proj_C^f(x), x) \leq D_f(y, x),\ \forall\ x \in E, y \in C.$$

Similar to the metric projection in Hilbert spaces, the Bregman projection with respect to totally and Gâteaux differentiable functions has a variational inequality characterization, (see [39]). Using (2.15), the function $V_p : E \times E^* \to [0, +\infty)$ for $2 \leq p < \infty$ is defined by

$$V_p(x, y) = \frac{1}{q}||x||^p - \langle x, y\rangle + \frac{1}{p}||y||^p,\ \forall\ x^* \in E^*, x \in E.$$

V_p is nonnegative and $V_p(x^*, x) = D_p(J_q^{E^*}(x^*), x)$ for all $x \in E^*$ and $y \in E^*$. Moreover, by the subdifferential inequality

$$\langle \triangledown f(x), y - x\rangle \leq f(y) - f(x),$$

with $f(x) = \frac{1}{q}\|x\|^q,\ x \in E^*$, then $\bigtriangledown f(x) = J_q^{E^*}(x)$. So, we have

$$\langle J_q^{E^*}(x), y\rangle \leq \frac{1}{q}\|x+y\|^q - \frac{1}{q}\|x\|^q. \tag{2.17}$$

From (2.17), we get

$$\begin{aligned} V_p(x^*+y^*, x) &= \frac{1}{q}\|x^*+y^*\| - \langle x^*+y^*, x\rangle + \frac{1}{q}\|x\|^p \\ &\geq \frac{1}{q}||x^*||^q + \langle y^*, J_q^{E^*}(x^*)\rangle - \langle x^*+y^*, x\rangle + \frac{1}{q}||x||^p \\ &= \frac{1}{q}||x^*||^q - \langle x^*, x\rangle + \frac{1}{p}||x||^p + \langle y^*, J_q^{E^*}(x^*)\rangle - \langle y^*, x\rangle \\ &= \frac{1}{q}||x^*||^q - \langle x^*, x\rangle + \frac{1}{p}||x||^p + \langle y^*, J_q^{E^*}(x^*) - x\rangle \\ &= V_p(x^*, x) + \langle y^*, J_q^{E^*}(x^*) - x\rangle, \end{aligned}$$

for all $x \in E$ and $x^*, y^* \in E^*$.
For p-uniformly convex space, The Bregman distance also possess the following important properties

$$D_p(x,y) = D_p(x,z) + D_p(z,y) + \langle z-y, J_p^E x - J_p^E y\rangle,\ \forall\ x,y,z \in E.$$

$$D_p(x,y) + D_p(y,x) = \langle x-y, J_p^E x - J_p^E y\rangle,\ \forall\ x,y \in E.$$

It is also known that the norm metric and the Bregman distance has the following relation, (see [204]).

$$\tau||x-y||^p \leq D_p(x,y) \leq \langle x-y, J_p^E x - J_p^E y\rangle, \tag{2.18}$$

where $\tau > 0$ is some fixed number. Let C be a nonempty, closed and convex subset of E, the metric projection defined as

$$P_C x := argmin_{y\in C}||x-y||,\ x \in E$$

is the unique minimizer of the norm distance, which can be characterized by a variational inequality:

$$\langle J_p^E(x - P_C x), z - P_C x\rangle \leq 0,\ \forall\ z \in C.$$

Similar to the metric projection is the Bregman projection (the minimizer of the Bregman distance) which is defined as

$$\Pi_C x := argmin_{y\in C} D_p(x,y),\ x \in E.$$

The Bregman projection can also be characterized by a variational inequality:

$$\langle J_p^E(x) - J_p^E(\Pi_C x), z - \Pi_C x\rangle \leq 0, \forall\ z \in C, \tag{2.19}$$

from which one get

$$D_p(\Pi_C x, z) \leq D_p(x,z) - D_p(x, \Pi_C x),\ \forall\ z \in C.$$

Definition 2.3.37. *A function f is said to be*

1. *strongly coercive, if*

$$\lim_{||x_n||\to\infty} \frac{f(x_n)}{||x_n||} = \infty.$$

2. *super coercive, if*

$$\lim_{x\to\infty} \frac{f(x)}{||x||} = +\infty.$$

Definition 2.3.38. *[196] A point $u \in C$ is said to be an asymptotic fixed point of $T : C \to C$ if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup u$ and $||x_n - Tx_n|| \to 0$. We denote the asymptotic fixed point set of T by $\hat{F}(T)$.*

Definition 2.3.39. *Let C be a nonempty subset of int dom$f$. An operator $T : C \to$ int dom$f$ is said to be*

1. *Bregman firmly nonexpansive (BFNE), if*

$$\langle Tx - Ty, \bigtriangledown f(Tx) - \bigtriangledown f(Ty)\rangle \leq \langle Tx - Ty, \bigtriangledown f(x) - \bigtriangledown f(y)\rangle,$$

for any $x, y \in C$, or equivalently

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

2. *Bregman quasi firmly nonexpansive (BQFNE), if $F(T) \neq \emptyset$ and*

$$\langle Tx - p, \bigtriangledown f(x) - \bigtriangledown f(Tx)\rangle \geq 0, \ \forall\ x \in C,\ p \in F(T),$$

or equivalently,

$$D_f(p, Tx) + D_f(Tx, x) \leq D_f(p, x).$$

3. *Bregman quasi-nonexpansive (BQNE), if $F(T) \neq \emptyset$ and*

$$D_f(p, Tx) \leq D_f(p, x),\ x \in C,\ p \in F(T).$$

4. *Bregman relatively nonexpansive, if $F(T) \neq \emptyset$, and*

$$D_f(p, Tx) \leq D_f(p, x),\ \forall\ p \in F(T),\ x \in C \text{ and } \hat{F}(T) = F(T).$$

5. *Bregman Strongly Nonexpansive (BSNE) with $\hat{F}(T) \neq \emptyset$, if*

$$D_f(p, Tx) \leq D_f(p, x)\ \forall\ x \in C,\ p \in \hat{F}(T)$$

and for any bounded sequence $\{x_n\}_{n\geq 1} \subset C$,

$$\lim_{n\to\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0$$

implies that $\lim_{n\to\infty} D_f(Tx_n, x_n) = 0$.

2.4 Some important optimization problems

In this section, we briefly introduce and review some optimization problems that are relevant in this study. Throughout this section, we denote by H and E a real Hilbert and a Banach space, respectively.

2.4.1 Minimization problems

One of the most important problems in optimization theory and non-linear analysis is the problem of approximating solutions of Minimization Problem (MP) which is defined as follows: Find $x \in H$ such that

$$f(x) = \min_{y \in H} f(y), \tag{2.20}$$

where $f : H \to (-\infty, \infty]$ is a proper, convex and lower semicontinuous function. For any $\lambda > 0$, the resolvent (or Moreau-Yosida approximation) of f in H is defined as (see [199])

$$J_\lambda^f(x) = Prox_\lambda f(x) = \arg\min_{y \in H} \left[f(y) + \frac{1}{2\lambda} ||y - x||^2 \right],$$

for all $x \in H$. It is generally known that J_λ^f is well-defined and firmly nonexpansive for all $\lambda > 0$. Hence, J_λ^f is nonexpansive for all $\lambda > 0$. For simplicity, we shall write J_λ for the resolvent of a proper, convex and lower semi-continuous mapping f. Furthermore, we denote the solution set of problem (2.20) by $\underset{y \in H}{\operatorname{argmin}} f(y)$. It is also known that $F(J_\lambda)$ coincides with $\underset{y \in H}{\operatorname{argmin}} f(y)$.

In 1970, Martinet [163] introduced the well known Proximal Point Algorithm (PPA) which is a powerful and one of the most popular tools for approximating solutions of MP (1.4). Rockafellar [200] further studied the PPA in Hilbert spaces for approximating solutions of unconstrained convex minimization problem in Hilbert spaces. Let f be a proper, convex and lower semi-continuous function on H, the PPA is defined for $x_1 \in H$ by

$$x_{n+1} = argmin_{u \in H} \big[f(y) + \frac{1}{2\lambda_n} ||u - x_n||^2 \big], \forall\ n \ \geq \ 1,$$

where $\lambda_n > 0$ for all $n \geq 1$. It was shown that if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ generated by the PPA converges weakly to a minimizer of f. Recently, Moudafi and Thakur [170] considered the following MP,

$$\min\{g(x) + f_\lambda(Ax) : x \in H_1\}; \tag{2.21}$$

where $g : H_1 \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semi-continuous function, and $f_\lambda(y) := \min_{u \in H_2}\{f(u) + \frac{1}{2\lambda}||u - y||^2\}$ is the Moreau-Yosida approximate of the function f of parameter λ also called the proximal operator of f of order λ and $A : H_1 \to H_2$ is a bounded linear operator. Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, $g : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $f : H_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper and lower semi-continuous convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator, then the Split Minimization Problem (SMP) is defined as follows: Find

$$x^* \in C \text{ such that } x^* = argmin_{x \in C} g(x), \tag{2.22}$$

and such that

$$\text{the point } y^* = Ax^* \in Q \text{ solves } y^* = argmin_{y \in Q} f(y). \tag{2.23}$$

We also consider the finite families of SMP, which is defined as follows:

$$x^* \in C \text{ such that } x^* = \cap_{i=1}^{N} argmin_{x \in C} g_i(x), \tag{2.24}$$

and such that

$$\text{the point } y^* = Ax^* \in Q \text{ solves } y^* = \cap_{j=1}^{m} argmin_{y \in Q} f_j(y). \tag{2.25}$$

For $\lambda > 0$, $x \in H_1$, we define

$$h(x) := \frac{1}{2}||(I - Prox_{\lambda f})Ax||^2; \tag{2.26}$$

$$l(x) := \frac{1}{2}||(I - Prox_{\lambda g})x||^2; \tag{2.27}$$

$$\theta(x) := \sqrt{||\nabla h(x)||^2 + ||\nabla l(x)||^2}; \tag{2.28}$$

and

$$\gamma_n := \rho_n \frac{h(x_n) + l(x_n)}{\theta^2(x_n)}, \ n \geq 1, \tag{2.29}$$

where $0 < \rho_n < 4$. Then the gradient ∇h and ∇l of h and l, respectively are

$$\nabla h(x) := A^*(I - Prox_{\lambda f})Ax; \tag{2.30}$$

and

$$\nabla l(x) := (I - Prox_{\lambda g})x. \tag{2.31}$$

Using (2.26)-(2.29), Moudafi and Thakur [170] studied the following PPA and proved a weak convergence theorem for the sequence generated by their algorithm to a solution of SMP (2.22)-(2.23):
Given an initial point $x_1 \in H_1$, assume that $\{x_n\}$ has been constructed and $\theta(x_n) \neq 0$, then compute x_{n+1} as follows:

$$x_{n+1} = Prox_{\lambda} g(x_n - \mu_n A^*(I - Prox_{\lambda f})Ax_n), \ \forall \ n \geq 1. \tag{2.32}$$

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of MP (2.21) and the iterative process stops, otherwise, we set n:= n+1 and go to (2.32).

2.4.2 Split convex feasibility problem

The Split Feasibility Problem (SCFP) was first initiated by Censor and Elfving (see [51]). Simply put, the SCFP involves finding an element in a nonempty, closed and convex subset in one space (say, X) such that its image under a certain operator is in another nonempty, closed and convex subset in the image space (say, Y). Mathematically, the SCFP can be stated as follows: Let C and Q be two nonempty, closed and convex subsets of n and m dimensional Euclidean spaces. The SCFP is defined as

$$\text{Find} \quad x^* \in C \text{ such that } Ax^* \in Q, \tag{2.33}$$

where A is a given $n \times m$ real matrix. The SCFP models many problems in the real world such as signal processing, image processing and medical care (see [35, 52, 53, 54, 55, 56] for more details). For instance, the image processing problems can be modeled as a split convex feasibility problem. The vector x represents a vectorized image where the entries of x, represent the intensity levels at each voxel or pixel. The set C can be selected to incorporate properties like positivity of the entries x while the matrix A can describe a linear functional or projection measurements we have made as well as other linear combinations of entries of x on which we wish to impose restrictions. The set Q on the other hand can be the product of the vector of measured data with other convex sets, such as nonnegative cones that is used to describe the restrictions to be imposed [34]. For more on this interesting description, see the works of ([34, 50]).

To solve the SCFP, Byrne [34] suggested the following CQ algorithm whose sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \quad n \geq 1, \tag{2.34}$$

where the initial point $x_1 \in C$, and $\gamma \in (0, 2/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. He proved a weak convergence of this method to a solution of the SCFP with the assumption that the solution set is nonempty. Recently, the SCFP where C and or Q are the solution sets of some optimization problems have been considered in different settings.

2.4.3 Monotone inclusion problems

One of the most important problems in monotone operator theory is the Monotone Inclusion Problem (MIP), which is also known as the problem of finding zeros of monotone (accretive) operators. The monotone inclusion problem is to find an element $x \in H$ such that $0 \in B(x)$, where $B : E \to 2^E$ is a multi-valued operator. This problem is very important in many areas such as convex optimization and monotone variational inequalities. It is worth mentioning that every monotone operator on Hilbert spaces can be regularized into single-valued, nonexpansive, Lipschitz continuous monotone operator by means of Yosida approximation notion. For each $\mu > 0$, a nonexpansive single-valued mapping $J_\mu^B : H \to H$ defined by $J_\mu^B = (I + \mu B)^{-1}$ is called the resolvent of B. It is well-known that $B^{-1}(0) = F(J_\mu^B)$ for all $\mu > 0$ and J_μ^B is firmly nonexpansive. Thus, for any $u \in F(J_\mu^B)$, we have that

$$||u - J_\mu^B x||^2 + ||J_\mu^B x - x||^2 \leq ||u - x||^2. \tag{2.35}$$

The inclusion problem can also be defined in terms of sum of two monotone operators M and B, where one of the operators is single-valued and the other is a multi-valued operator. Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be α-inverse strongly monotone operator, the MIP is to find $x \in E$ such that

$$0 \in (A + B)x. \tag{2.36}$$

We denote by $(A + B)^{-1}(0)$ the solution set of (2.36).
Based on a series of studies in the past years, the splitting method has been known to be a popular method for solving (2.36). The splitting methods for linear equations was introduced by Peaceman and Rachford [187]. Extensions to nonlinear equations in Hilbert spaces were carried out by Lions and Mercier [151]. Since then, many authors have considered approximating solutions of variational inclusion (2.36) using this method, (see [98, 99, 76, 181, 207] and the references contained in).

We present the following definitions that are associated with monotone operators.

Definition 2.4.1. *A multi-valued (set-valued) operator $B : E \to 2^{E^*}$ with domain $Dom(B)$ and the range $R(B) = \{Bx : x \in D(B)\}$ is said to be monotone if for $x, y \in Dom(B), a, b \in R(T)$, the following inequality holds:*

$$\langle x - y, a - b \rangle \geq 0.$$

A monotone operator B is said to be maximal if its graph $Gra(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone operator.
If E is a strictly convex, reflexive and smooth Banach space and $B : E \to 2^{E^*}$ is a maximal monotone operator. Then, for any positive real number λ, we can define a nonexpansive single-valued operator $J_\lambda^B : E \to E$ by

$$J_\lambda^B(x) := (J + \lambda B)^{-1} J(x), \;\; x \in E.$$

This operator is called the resolvent of B for $\lambda > 0$. It is well known that $B^{-1}(0) = F(J_\lambda^B)$ for all $\lambda > 0$ and $B^{-1}(0)$ is a closed and convex subset of E.
Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function and B be a maximal monotone mapping from E to E^*. For any $\lambda > 0$, the mapping $Res_{\lambda B}^f : E \to domB$ defined by

$$Res_{\lambda B}^f = (\triangledown f + \lambda B)^{-1} \circ \triangledown f, \tag{2.37}$$

is called the f-resolvent of B. It is well known that $B^{-1}(0) = F(Res_{\lambda B}^f)$ for each $\lambda > 0$.
Let C be a nonempty closed and convex subset of a reflexive Banach space E, the mapping $A : E \to 2^{E^*}$ is called Bregman Inverse Strongly Monotone (BISM) on the set C if

$$C \cap (dom f) \cap (int\ dom f) \neq \emptyset,$$

and for any $x, y \in C \cap (int\ dom f)$, $\xi \in Ax$ and $\zeta \in Ay$, we have that

$$\langle \xi - \zeta, \triangledown f^*(\triangledown f(x) - \xi) - \triangledown f^*(\triangledown f(y) - \zeta) \rangle \geq 0.$$

2.4.4 Split monotone variational inclusion problem

Let H_1 and H_2 be two real Hilbert spaces. The Split Monotone Variational Inclusion Problem (SMVIP) in the sense of Moudafi [171] consists of approximating a point

$$\text{Locate } \bar{x} \in H_1 \text{ such that } 0 \in (A + B)(\bar{x}) \tag{2.38}$$

and

$$\bar{y} = T\bar{x} \in H_2 \text{ such that } 0 \in (D + G)(\bar{y}), \tag{2.39}$$

where $A : H_1 \to H_1$ and $D : H_2 \to H_2$ are single-valued mappings, $B : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ are multi-valued mappings and $T : H_1 \to H_2$ is a bounded linear operator.

When $A = D = \mathbf{0}$, then Problem (2.38) and (2.39) reduces to the split variational inclusion problem (SVIP). Problem (2.38) and (2.39) has continued to draw the attention of many researchers due to its far reaching applications in many mathematical problems as it includes naturally split variational inequalities, split feasibility problems, split minimization problems, split equilibrium problems and split hierarchical fixed point problems. In addition, several real life problems such as signal processing, image restoration problem, sensor networks, computer tomography, data compression, linear inverse problem and machine learning can all be mathematically modeled as SMVIP (2.38) and (2.39) (see for instance [13, 35, 52, 54]). We denote the solution set of SMVIP (2.38) and (2.39) by

$$\Omega = \{\hat{x} \in (A + B)^{-1}(0) : T\hat{x} \in (D + G)^{-1}(0)\}.$$

For approximating the solution of SMVIP (2.38) and (2.39) in Hilbert spaces, Moudafi [171] put forward the following iterative scheme:

$$x_{n+1} = J_\gamma^B(I^{H_1} - \gamma A)\left(x_n + \mu T^*\left(J_\gamma^G(I^{H_2} - \gamma D) - I^{H_2}\right)Tx_n\right), \quad n \geq 1; \tag{2.40}$$

where $\mu \in \left(0, \frac{1}{L}\right)$, L is the spectral radius of T^*T, I^{H_1} and I^{H_2} are the identity operators on H_1 and H_2, respectively, J_γ^B and J_γ^G are the resolvents of B and G respectively; B and G are maximal monotone, A, D are α_1, α_2-inverse strongly monotone and $\gamma \in (0, 2\psi)$, where $\psi := \min\{\alpha_1, \alpha_2\}$. Moudafi [171] proved that the sequence $\{x_n\}$ generated by (2.40), converges weakly to a solution of SMVIP (2.38) and (2.39).

2.4.5 Split hierachical monotone variational inclusion problem.

Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued mappings with nonempty values, and $f_1 : H_1 \to H_1$, $g : H_2 \to 2^{H_2}$ be mappings. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be mappings such that $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. Let $U := J_\lambda^{B_1}(I - \lambda f)$ and $V := J_\lambda^{B_2}(I - \lambda g)$. The Split Hierachical Monotone Variational Inclusion Problem (SHMVIP) is defined as follows:

$$\text{Find } x^* \in F(T) \text{ such that } x^* \in F(J_\lambda^{B_1}(I - \lambda f)), \tag{2.41}$$

and such that

$$y^* = Ax^* \in F(S) \text{ solves } y^* \in F(J_\lambda^{B_2}(I - \lambda g)). \tag{2.42}$$

We denote by Θ, the solution set of (2.41) and (2.42). Using the following iteration process, Ansari and Rehan [13] proved the following weak convergence result.
Let $\lambda > 0$ and take arbitrary $x_0 \in H_1$. For a given current $x_n \in H_1$, compute

$$x_{n+1} = TU(x_n + \gamma A^*(SV - I)Ax_n), \tag{2.43}$$

where $\gamma \in (0, \frac{1}{||A||^2})$.

2.4.6 Variational inequality problems

Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ be a mapping, the Variational Inequality Problem (VIP) which was introduced by Lions and Stampacchia [150] is defined as follows: Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \tag{2.44}$$

for all $y \in C$. We denote by $VI(C, A)$ the solution set of (2.44). The VIP is well known to include several branches of mathematical and engineering sciences with a wide range of applications in industry, finance, optimization and mechanics. Many important problems such as signal recovery, power control, bandwith allocation, optimal control and beam forming are special cases of (2.44), (see [69, 114, 115, 212]). The gradient projection method is the simplest and oldest projection method which is formulated as:

$$x_{n+1} = P_C(x_n - \lambda A(x_n)), \quad \forall n \geq 0, \tag{2.45}$$

where $\lambda \in \left(0, \frac{2}{L}^2\right)$, L is the Lispchitz constant of A and P_C is a projection of a real Hilbert space H onto a nonempty, closed and convex subset C of H is a positive real number. It is well-known that the convergence of the gradient projection method requires that the cost operator be strongly monotone (or inverse strongly monotone).

In order to weaken this assumption, Korpelevich [142] proposed the following extragradient method

$$\begin{aligned} y_n &= P_C(x_n - \lambda A(x_n)) \\ x_{n+1} &= P_C(y_n - \lambda A(y_n)) \ \ \forall n \geq 0, \end{aligned} \tag{2.46}$$

where $\lambda \in (0, \frac{1}{L})$ and L is the Lipschitz constant of the cost operator A. The extragradient method requires the computation of two projections onto the feasible set C and two evaluations of the cost operator A at each iteration. This is known to affect the effectiveness and applicability of the method especially when the cost operator A and the feasible set C have complex structures. It is worthy to mention that several authors have improved this method (see for example, [46, 48, 85, 105, 117]).

Remark 2.4.2. *If we set $B = N_C$ in (2.36), then we obtain from the definition of normal cone (see (2.5.44)) that $J_\lambda^{N_C}(I - \lambda A) = P_C(I - \lambda A)$. In this case, we know that $(A + N_C)^{-1}(0) = VIP(C, A)$. That is the resolvent $J_\lambda^B = P_C$.*

2.4.7 Equilibrium and split variational inequality problems

Another optimization problem which can be applied to solve solutions of VIP is the Equilibrium Problem (EP). This was first introduced and studied by Blum and Oettli [29]. Many problems in physics, optimization and economics can be reduced to finding the solution of EP. The EP is defined as follows: Find $x \in C$ such that

$$F(x, y) \geq 0, \ \forall \ y \in C, \tag{2.47}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction. We denote by $EP(F, C)$ (2.47), the solution set of (2.47).
Let $F : C \times C \to \mathbb{R}$ be a bifunction and $f : H \to H$ be a mapping . The Generalized Equilibrium Problem (GEP) is defined as: Find $x \in C$ such that

$$F(x, y) + \langle f(x), y - x\rangle \geq 0, \tag{2.48}$$

for all $y \in C$. We denote by $GEP(F, f)$ the solution set of (2.48).

Remark 2.4.3. *If $F \equiv 0$, the GEP (2.48) reduces to VIP (2.44) and when $f \equiv 0$, the GEP (2.48) reduces to EP (2.47).*

In 2013, Kazmi and Rizvi [134] introduced and studied the Split Generalized Equilibrium Problem (SGEP) in real Hilbert spaces, which is formulated as finding an element $x^* \in C$ such that

$$F_1(x^*, x) + \phi_1(x^*, x) \geq 0, \quad \forall x \in C \tag{2.49}$$

and

$$y^* = Ax^* \in Q \ \text{ solves } \ F_2(y^*, y) + \phi_2(y^*, y) \geq 0, \quad \forall y \in Q, \tag{2.50}$$

where $F_1, \phi_1 : C \times C \to \mathbb{R}$ and $F_2, \phi_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions and $A : H_1 \to H_2$ is a bounded linear operator. We denote the set of solutions of the SGEP (2.49)-(2.50) by

$$SGEP(F_1, \phi_1, F_2, \phi_2) := \{x^* \in C : x^* \in GEP(F_1, \phi_1) \ \text{ and } \ Ax^* \in GEP(F_2, \phi_2)\},$$

If $F_2 = 0$ and $\phi_2 = 0$, the SGEP reduces to the generalized equilibrium problem studied by Cianciruso et al. [59] which is defined as finding an element $x^* \in C$ such that

$$F(x^*, x) + \phi(x^*, x) \geq 0, \quad \forall x \in C, \tag{2.51}$$

where $F : C \times C \to \mathbb{R}$ and $\phi : Q \times Q \to \mathbb{R}$ are two bifunctions. We denote by $GEP(F, \phi)$ the solution set of the GEP (2.51). The GEP is general in the sense that it includes as particular cases, minimization problems, fixed point problems, Nash equilibrium problems in noncooperative games, mixed equilibrium problems, variational inequality problems to mention but few, see for example [47, 89, 123, 135, 166, 178]. When $\phi \equiv 0$ in (2.51), the GEP reduces to the classical Equilibrium Problem (EP) in the sense of Blum and Oettli [29].

Observe also from (2.49) and (2.50) that when $\phi_1, \phi_2 = 0$, the SGEP reduces to Split Equilibrium Problem (SEP) in the sense of [133], which is defined as finding an element $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall\, x \in C$$

such that

$$y^* = Ax^* \text{ solves } F_2(y^*, y) \geq 0, \quad \forall\, y \in Q.$$

We denote by $SEP(F_1, F_2)$, the solution set of the SEP.

The Split Variational Inequality Problem (SVIP) was introduced and studied by Censor et. al. [54] and they defined the problem as follows: Find $x^* \in C$ such that

$$\langle f_1(x^*), x - x^* \rangle \geq 0, \; \forall\, x \in C, \tag{2.52}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } \langle f_2(y^*), y - y^* \rangle \geq 0, \; \forall\, y \in Q, \tag{2.53}$$

where $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings. The SVIP have already been used in practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning and the modelling of many inverse problems arising for phase retrieval and other real-world problems such as data compression, sensor networks in computerized tomography, for example, (see [67]).

For solving EP, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following assumptions:
(A1) $F(x, x) = 0$, for all $x \in C$;
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0, \; \forall\, x, y \in C$;
(A3) for each $x, y, z \in C, \limsup_{t \to 0} F(tz + (1-t)x, y) \leq F(x, y)$;
(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semi-continuous.
Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \; \forall\, y \in C.$$

2.5 Some important lemmas

In this section, we recall some important definitions and lemmas which will be needed in the proof of our main results.

Definition 2.5.1. *Let H be a real Hilbert space and C be a nonempty closed and convex of H. A mapping $T : C \to C$ is said to be demiclosed at 0, if for any bounded sequence $\{x_n\} \subset C$ such that $\{x_n\}$ converges weakly to x and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, then $Tx = x$.*

Definition 2.5.2. *A single-valued mapping $A : H \to H$ is said to be hemi-continuous, if for any $x, y, z \in H$, the function $t \mapsto \langle A(x+ty), z\rangle$ is continuous at* 0.

Lemma 2.5.3. *[57] Let H be a real Hilbert space and $T : H \to H$ be L-Lipschitzian mapping with $L \geq 1$. Denote $K := (1-\xi)I + \xi T\big((1-\eta)I + \eta T\big)$ if $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}$, then the following conclusions holds.*
(1) $F(T) = F(T((1-\eta)I + \eta T)) = F(K)$;
(2) If T is demiclosed at 0, then K is also demiclosed at 0;
(3) In addition, if $T : H \to H$ is quasi-pseudocontractive, then the mapping K is quasi-nonexpansive, that is,

$$\|Kx - u^*\| \leq \|x - u^*\| \;\forall\; x \in H \, and \; u^* \in F(T) = F(K).$$

Definition 2.5.4. *[23] A function $c : H \to \mathbb{R}$ is said to be Gâteaux differentiable at $x \in H$, if there exists an element denoted by $c'(x) \in H$ such that*

$$\lim_{t\to 0} \frac{c(x+hv) - c(x)}{h} = \langle v, c'(x)\rangle, \quad \forall \quad v \in H, \quad h \in [0,1],$$

where $c'(x)$ is called the Gâteaux differential of c at x. Recall that if for each $x \in H$, c is Gâteaux differentiable at x, then c is Gâteaux differentiable on H.

Definition 2.5.5. *[23] A convex set $c : H \to \mathbb{R}$ is said to be subdifferentiable at a point $x \in H$ if the set*

$$\partial c(x) = \{\bar{x} \in H \mid c(y) \geq c(x) + \langle \bar{x}, y - x\rangle \quad \forall y \in H\} \tag{2.54}$$

is nonempty. Each element in $\partial c(x)$ is called a subgradient of c at x. We note that if c is subdifferentiable at each $x \in H$, then c is subdifferentiable on H. It is also known that if c is Gâteaux differentiable at x, then c is subdifferentiable at x and $\partial c(x) = \{c'(x)\}$.

Definition 2.5.6. *Let H be a real Hilbert space. A function $c : H \to \mathbb{R} \cup \{+\infty\}$ is said to be weakly lower semicontinuous (w-lsc) at $x \in H$, if*

$$c(x) \leq \liminf_{n\to+\infty} c(x_n)$$

holds for an arbitrary sequence $\{x_n\}_{n=0}^{+\infty}$ in H satisfying $x_n \rightharpoonup x$.

Definition 2.5.7. *A bounded linear operator $D : C \to H$ is called strongly positive if there exists a constant $\bar{\gamma} > 0$ such that*

$$\langle Dx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \text{for all} \;\; x \in C.$$

Lemma 2.5.8. *[106] Let C be a nonempty closed and convex subset of H, $y := P_C(x)$ and $p \in C$. Then*

$$\|y - p\|^2 \leq \|x - p\|^2 - \|x - y\|^2.$$

Lemma 2.5.9. *[77] Let C be a nonempty closed and convex subset of H. Let $A : C \to H$ be a continuous, monotone mapping and $z \in C$, then*

$$z \in VI(C, A) \iff \langle A(x), x - z\rangle \geq 0 \;\forall\; x \in C.$$

Lemma 2.5.10. *[70] Let H be a real Hilbert space and $A : C \to H$ be a continuous pseudomonotone mapping. Then, $x^* \in VI(C, A)$ if and only if*

$$\langle Ax, x - x^* \rangle \geq 0, \ \forall \ x \in C.$$

Lemma 2.5.11. *[105] Assume that the solution set VI(C, A) of the VIP (2.44) is nonempty, and C is defined as $C := \{x \in H \mid c(x) \leq 0\}$, where $c : H \to \mathbb{R}$ is a continuously differentiable convex function. Let $\bar{x} \in C$. Then, $\bar{x} \in$ VI(C, A) if and only if either*

1. *$A(\bar{x}) = 0$, or*

2. *$\bar{x} \in \partial C$ and there exists $\epsilon > 0$ such that $A(\bar{x}) = -\epsilon c'(\bar{x})$, where ∂C denotes the boundary of C.*

Lemma 2.5.12. *[62] Let H be a real Hilbert space and $\{x_i\}_{i \in \mathbb{N}}$ be a bounded sequence in H. For $\delta_i \in (0, 1)$ such that $\sum_{i=1}^{\infty} \delta_i = 1$, the following identity holds:*

$$||\sum_{i=1}^{\infty} \delta_i x_i||^2 = \sum_{i=1}^{\infty} \delta_i ||x_i||^2 - \sum_{1 \leq i < j < \infty} \delta_i \delta_j ||x_i - x_j||^2.$$

Let C be a nonempty convex subset of a real Hilbert space H. It is a known fact that the mapping $J : C \to H$ is uniformly continuous if and only if for every $\epsilon > 0$, there exists a constant $K < +\infty$ such that

$$\|Jx - Jy\| \leq K\|x - y\| + \epsilon, \quad \forall \ x, y, \in C. \tag{2.55}$$

Lemma 2.5.13. *[251] Let H be a real Hilbert space and $S : H \to H$ be k-strictly pseudocontractive mapping with $k \in [0, 1)$. Let $S_\alpha := \alpha I + (1 - \alpha)S$, where $\alpha \in [\beta, 1)$. Then,*
(i) $F(S) = F(S_\alpha)$,
(ii) S_α is a nonexpansive mapping.

Lemma 2.5.14. *[251] Let S be a k-strict pseudocontractive mapping on a closed and convex subset C of a real Hilbert space H, then $I - S$ is demiclosed at any point $y \in H$.*

Lemma 2.5.15. *[148] Suppose H be a real Hilbert space. In addition, let $A : H \to H$ be a hemicontinuous, monotone and bounded operator, and $B : H \to 2^H$ be a maximal monotone operator. Then, the operator $(A + B) : H \to 2^H$ is maximal monotone.*

Lemma 2.5.16. *[243] Let H be a real Hilbert space and $T : H \to H$ be a nonlinear mapping, then the following results hold:*

(i) f is nonexpansive if and only if the complement $I - f$ is $\frac{1}{2}$-ism,

(ii) if f is v-ism and $r > 0$, then rf is $\frac{v}{r}$-ism,

(iii) f is averaged if and only if the complement $I - f$ is v-ism for some $v > \frac{1}{2}$. Indeed, for some $\beta \in (0, 1)$, f is β-averaged if and only if $I - f$ is $\frac{1}{2\beta}$-ism,

(iv) if f_1, is β_1-averaged and f_2 is β_2-averaged, where $\beta_1,\ \beta_2 \in (0,1)$, then the composite $f_1 f_2$ is β-averaged, where $\beta = \beta_1 + \beta_2 - \beta_1\beta_2$.

Lemma 2.5.17. *[219] Let H be a real Hilbert space and let η be a real number with $\eta \in (-\infty, 1)$. Let $T : H \to H$ be an η-demimetric mapping. Then $F(T)$ is closed and convex.*

Lemma 2.5.18. *[61] Let H be a real Hilbert space, then $\forall\ x, y \in H$ and $\alpha \in (0,1)$, we have*

(i) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2$,

(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$,

(iii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$.

Lemma 2.5.19. *[56] Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \le r\}$. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$, in (0,1) with $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuously strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R} \text{ with } \ g(0) = 0,$$

such that for any positive integers i, j with $i < j$, the following inequality holds

$$\|\sum_{i=1}^{\infty} \lambda_i x_i\|^2 = \sum_{i=1}^{\infty} \lambda_i \|x\|^2 - \sum_{i,j=1}^{\infty} \lambda_i\lambda_j g(\|x_i - x_j\|).$$

Lemma 2.5.20. *[241] Let E be a 2-uniformly smooth Banach space with the best smoothness constant $k > 0$. Then, the following inequality holds:*

$$||x+y||^2 \le ||x||^2 + 2\langle y, Jx\rangle + 2||ky||^2, \ \forall\ x, y \in E.$$

Lemma 2.5.21. *[241] Given a number $r > 0$, a real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|);$$

for all $x, y \in E$ with $\|x\| \le r$ and $\|y\| \le r$ and $\lambda \in [0,1]$.

Lemma 2.5.22. *[127] Let E be a uniformly convex and smooth Banach space and let $\{x_n\}, \{y_n\}$ be two sequences in E. If $\phi(x_n, y_n) \to 0$ and either of $\{x_n\}$ or $\{y_n\}$ is bounded. Then, $||x_n - y_n|| \to 0$.*

Lemma 2.5.23. *[227] Let $\{\tau_n\}$ and $\{\rho_n\}$ be two nonnegative real sequences such that*

$$\tau_{n+1} \le \tau_n + \rho_n, \ \forall\ n \ge 1.$$

If $\sum_{n=1}^{\infty} \rho_n < \infty$, then $\lim\limits_{n\to\infty} \tau_n$ exists.

Lemma 2.5.24. *[159] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n + \gamma_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a real sequence. Assume that $\sum \gamma_n < \infty$ and $\sigma_n \leq \alpha_n M$ for some $M > 0$, then $\{a_n\}$ is a bounded sequence.

Lemma 2.5.25. *[220] Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with $T \neq 0$, and $S : H_2 \to H_2$ be a nonexpansive mapping. Then $T^*(I-S)T$ is $\frac{1}{2\|T\|^2}$-ism.*

Lemma 2.5.26. *[238] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation;*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, \quad n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ are sequences satisfying the following conditions: $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup\limits_{n\to\infty} \delta_n \leq 0$, then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.5.27. *[174] Let E be a Banach space, $r > 0$ be a constant, ρ_r be the gauge of uniform convexity of f and $f : E \to \mathbb{R}$ be a continuous uniformly convex function on bounded subset of E. Then, for any $x, y \in B_r$, we have*

$$f\left(\sum_{k=0}^{\infty} \alpha_k x_k\right) \leq \sum_{k=0}^{\infty} \alpha_k f(x_k) - \alpha_i\alpha_j\rho_r(||x_i - x_j||)$$

for all $i, j \in \mathbb{N} \cup \{0\}$, $x_k \in B_r, \alpha_k \in (0,1)$ and $k \in \mathbb{N} \cup \{0\}$ with $\sum\limits_{k=0}^{\infty} \alpha_k = 1$. Here, $B_r := \{z \in E : ||z|| \leq r\}$.

Lemma 2.5.28. *[39] Let E be a reflexive Banach space, $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and V be a function defined by*

$$V(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \ x \in E, \ x^* \in E^*.$$

The following assertions also hold:

$$D_f(x, \bigtriangledown f^*(x^*)) = V(x, x^*), \ \textit{for all } x \in E \ \ \textit{and } x^* \in E^*.$$

$$V(x, x^*) + \langle \bigtriangledown f^*(x^*) - x, y^* \rangle \leq V(x, x^* + y^*) \ \ \textit{for all} \ \ x \in E \textit{and} \ \ x^*, y^* \in E^*.$$

Lemma 2.5.29. *[214] Let E be a real Banach space with Fréchet differentiable norm and $\beta^*(t)$ be defined by*

$$\beta^*(t) = \sup\left\{\left|\frac{||x+ty||^2 - ||x||^2}{t} - 2\langle y, j(x)\rangle\right| : ||y|| = 1\right\}, \ \forall x \in E \ \textit{and} \ 0 < t < \infty. \tag{2.56}$$

Then, $\lim_{t\to 0^+} \beta^*(t) = 0$, and for all $h \in E$ such that $h \neq 0$, we have

$$||x+h||^2 \leq ||x||^2 + 2\langle h, j(x)\rangle + ||h||\beta^*(||h||). \tag{2.57}$$

Remark 2.5.30. *In Lemma 2.5.29, if $\beta^*(t) \leq ct$, for $t > 0$ and for some $c > 1$, then we obtain from (2.57) that*

$$2\langle h, j(x)\rangle \leq ||x||^2 + c||h||^2 - ||x - h||^2. \tag{2.58}$$

Lemma 2.5.31. *[39, 65] Let E be a Banach space and $f : E \to \mathbb{R}$ a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then,*

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Lemma 2.5.32. *[248] Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ a convex function which is bounded on bounded subsets of E. Then, the following assertions are equivalent:*
(i) f is strongly coercive and uniformly convex on bounded subsets of E;
(ii) dom $f^* = E^*, f^*$ is bounded on bounded subsets and uniformly smooth on bounded subsets of E^*;
(iii) dom $f^* = E^*, f^*$ is Fréchet differentiable and $\bigtriangledown f^*$ is uniformly norm-to-norm continuous on bounded subset of E^*.

Lemma 2.5.33. *[39] If dom$f$ contains at least two points, then the function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 2.5.34. *[194] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.5.35. *[248] Let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then, the following assertions are equivalent:*

(i) f is bounded on bounded subsets and uniformly smooth on bounded subsets of E,

(ii) f is Fréchet differentiable and $\bigtriangledown f^*$ is uniformly norm-to-norm continuous on bounded subset of E^*,

(iii) dom$f^* = E^*$, f^* is strongly coercive and uniformly convex on bounded subsets of E^*.

Lemma 2.5.36. *[248] Let $f : E \to \mathbb{R}$ be a continuous convex function which is bounded on bounded subsets of E. Then, the following are equivalent:*
(i) f is super coercive and uniformly convex on bounded subset of E,
(ii) dom$f^* = E^*, f^*$ is bounded and uniformly smooth on bounded subsets of E^*,
(iii) dom$f^* = E^*, f^*$ is Fréchet differentiable and $\bigtriangledown f^*$ is uniformly norm-to-norm continuous on bounded subset of E^*.

Lemma 2.5.37. *[195] Let C be a nonempty closed and convex subset of a reflexive Banach space E and $x \in E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Then,*
(i) $z = P_C^f(x)$ if and only if $\langle \bigtriangledown f(x) - \bigtriangledown f(z), y - z\rangle \leq 0,\ \forall\ y \in C$.
(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x)\ \forall\ y \in C$.

Lemma 2.5.38. *[39] Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex function whose domain contains at least two points. Then, the following statement hold:*
(i) f is sequentially consistent if and only if it is totally convex on bounded subsets.
(ii) If f is lower semicontinuous, then f is sequentially consistent if and only if it is uniformly convex on bounded subsets.
(iii) If f is uniformly strictly convex on bounded subsets, then it is sequentially consistent, and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain and the Fréchet differentiable $\bigtriangledown f$ is uniformly continuous on bounded subsets.

Lemma 2.5.39. *[160] Let $\{a_n\}$ be a sequence of non-negative number such that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n r_n$, where $\{r_n\}$ is a sequence of real numbers bounded from above and $\{\alpha_n\} \subset [0,1]$ satisfies $\sum \alpha_n = \infty$. Then $\limsup\limits_{n\to\infty} a_n \leq \limsup\limits_{n\to\infty} r_n$.*

Lemma 2.5.40. *[231] Let E be a Banach space, $s > 0$ a constant, ρ_s the gauge of uniform convexity of g and $g : E \to \mathbb{R}$ a convex function which is uniformly convex on bounded subset of E. Then*
(i) for any $x, y \in B_s$ and $\alpha \in (0,1)$, we have

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_s(||x-y||),$$

(ii) for any $x, y \in B_s$,

$$\rho_s(||x-y||) \leq D_g(x,y).$$

Here, $B_s := \{z \in E : ||z|| \leq s\}$.

Lemma 2.5.41. *[158] Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that $a_{n_i} < a_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_{k+1}} \text{ and } a_k \leq a_{k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.5.42. *[66] If f and g are two convex functions on E such that there is a point $x_0 \in dom f \cap dom g$ where f is continuous, then*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x), \ \forall \ x \in E. \tag{2.59}$$

Lemma 2.5.43. *[174] Let E be a real p-uniformly convex and uniformly smooth Banach space, $r > 0$ be a constant. Let $z, x_k \in E$ $(k = 1, 2, ..., N)$ and $\alpha_k \in (0,1)$ with $\sum_{k=1}^N \alpha_k = 1$. Then, we have*

$$\Delta_p(J_q^{E^*}(\sum_{k=1}^N \alpha_k J_p^E(x_k)), z) \leq \sum_{k=1}^N \alpha_k \Delta_p(x_k, z) - \alpha_i\alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|),$$

for all $i, j \in \{1, 2, ..., N\}$ and $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ being a strictly increasing function such that $g_r^*(0) = 0$.

For any $x^* \in C$, the subgradient of the convex function $g(x^*,.)$ at x^* is represented by $\partial_2 g(x^*,x^*)$, that is,

$$\begin{aligned}\partial_2 g(x^*,x^*) &= \{\phi \in E^* : g(x^*,v) \geq g(x^*,x^*) + \langle \phi, v - x^*\rangle, \quad \forall\, v \in C\}\\ &= \{\phi \in E^* : g(x^*,v) \geq \langle \phi, v - x^*\rangle, \quad \forall\, v \in C\}.\end{aligned}$$

Lemma 2.5.44. *[232] Let K be a nonempty convex subset of a Banach space E. Let $g : K \to \mathbb{R}$ be a convex, subdifferentiable function on K. Then g attains its minimum at $x \in K$ if and only if $0 \in \partial g(x) + N_K(x)$, where $N_K(x)$ is the normal cone of K at x, that is*

$$N_K(x) := \{\hat{x} \in E^* : \langle x - \varphi, \hat{x}\rangle \geq 0, \quad \forall\, \varphi \in K\}.$$

Lemma 2.5.45. *[208] Let E be a real p-uniformly convex and uniformly smooth Banach space, $g : E \to \mathbb{R}$ be a strong coercive Bregman function and $V_p : E^* \times E \to [0,+\infty)$ be defined by*

$$V_p(x,x^*) = \frac{1}{p}\|x\|^p - \langle x, x^*\rangle + \frac{1}{q}\|x^*\|^q, \ \forall\, x \in E, x^* \in E^*.$$

Then, the following assertions hold:
(i) V_p is nonnegative and convex in the first variable.
(ii) $\Delta_p(x, J_q^{E^*}(x^*)) = V_p(x,x^*), \ \forall\, x \in E, \ x^* \in E^*$.
(iii) $V_p(x,x^*) + \langle J_q^{E^*}(x^*) - x, y^*\rangle \leq V_p(x, x^* + y^*), \forall\, x \in E, \ x^*, y^* \in E^*$.

Lemma 2.5.46. *[245] Let $q \geq 1$ and $r > 0$ be two fixed real numbers. Then, a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^*$, $g(0) = 0$ such that for all $x, y \in B_r$ and $0 \leq \alpha < 1$,*

$$\|\alpha x + (1-\alpha)y\|^q \leq \alpha\|x\|^q + (1-\alpha)\|y\|^q - W_q(\alpha)g(\|x - y\|),$$

where $W_q(\alpha) := \alpha^q(1-\alpha) + \alpha(1-\alpha)^q$ and $B_r := \{x \in E : \|x\| \leq r\}$.

To solve pseudomonotone EP, the following assumptions will be needed:
Assumption A:
(A1) f is pseudomonotone, i.e., for all $x, y \in C$, $f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0$ and $f(x,x) = 0$, for all $x \in C$.
(A2) f satisfies the Bregman-Lipschitz type condition on C, that is, there exists two positive constants c_1 and c_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\Delta_g(y,x) - c_2\Delta_g(y,z), \quad \forall \ \ x, y, z \in C,$$

where $g : E \to (-\infty, +\infty]$ is a Legendre function. The constants c_1 and c_2 are called Bregman-Lipschitz coefficients with respect to g.
(A3) $f(x,.)$ is convex, lower semicontinuous and subdifferentiable on C for all $x \in C$.
(A4) f is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ converges weakly to x and y respectively, then $f(x_n, y_n) \to f(x,y)$ as $n \to \infty$.

Lemma 2.5.47. *[82] Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions $A_1 - A_4$ of* **Assumption A**. *For the arbitrary sequences $\{u_n\} \subset C$ and $\{\lambda_n\} \subset (0, +\infty)$, let $\{z_n\}$ and $\{t_n\}$ be sequences generated by*

$$\begin{cases} z_n = argmin\{\lambda_n f(u_n, w) + \Delta_p(w, u_n) : \quad w \in C\}, \\ t_n = argmin\{\lambda_n f(z_n, w) + \Delta_p(w, u_n) : \quad w \in C\}. \end{cases}$$

Then, for all $x^* \in EP(f)$,

1. $\langle J_p^{E_1} u_n - J_p^{E_1} z_n, w - z_n \rangle \leq \lambda_n [f(u_n, w) - f(u_n, z_n)]$

2. $\Delta_p(x^*, t_n) \leq \Delta_p(x^*, u_n) - (1 - \lambda_n c_1)\Delta_p(z_n, u_n) - (1 - \lambda_n c_2)\Delta_p(t_n, z_n)$.

Lemma 2.5.48. *[220] Let C be a nonempty closed and convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. We have that $z = P_C x$ if and only if $\langle x - z, \quad z - y \rangle \geq 0, \quad \forall y \in C$.*

Lemma 2.5.49. *[216] Let C be a closed and convex subset of H, let h be a real-valued function on H and define $K := \{x \in C : h(x) \leq 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then*

$$dist(x, k) \geq \theta^{-1} \max\{h(x), 0\}, \forall x \in C,$$

where $dist(x, K)$ denotes the distance function from x to K.

Lemma 2.5.50. *[117, 118] Let H_1 and H_2 be two Hilbert spaces. Suppose $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then $A(M)$ is bounded.*

Lemma 2.5.51. *[240] Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1 - \psi_n)a_n + \psi_n b_n + \theta_n, n \geq 0,$$

where $\{\psi_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{\theta_n\}_{n=1}^{\infty}$ satisfy the conditions:

(i) $\{\psi_n\}_{n=1}^{\infty} \subset [0, 1]$, $\sum_{n=1}^{\infty} \psi_n = \infty$ *or* $\prod_{n=1}^{\infty}(1 - \psi_n) = 0$;

(ii) $\limsup_{n\to\infty} b_n \leq 0$;

(iii) $\theta_n \geq 0$, $(n \geq 1)$, $\sum_{n=1}^{\infty} \theta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5.52. *[158, 223] Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ such that*

$$\Gamma_{n_j} < \Gamma_{n_j+1} \quad \textit{for all} \quad j \geq 0.$$

Also, consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by

$$\tau(n) := \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau_n\}_{n\geq n_0}$ is a non-decreasing sequence satisfying $\lim_{n\to\infty} \tau_n = \infty$, and, for all $n \geq n_0$, the following two estimates hold:

$$\begin{aligned} \Gamma_{\tau(n)} &\leq \Gamma_{\tau(n)+1}, \\ \Gamma_n &\leq \Gamma_{\tau(n)+1}. \end{aligned}$$

Lemma 2.5.53. *[179] Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a BISM mapping such that $(A+B)^{-1}(0^*) \neq \emptyset$. Let $g : E \to \mathbb{R}$ be a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of E. Then,*

$$\Delta_g(u, Res^g_{\sigma B} \circ A^g_\sigma(x)) + \Delta_g(Res^g_{\sigma B} \circ A^g_\sigma(x), x) \leq \Delta_g(u, x),$$
$$\textit{for any} \quad u \in (A+B)^{-1}(0^*), \quad x \in E \quad \textit{and} \quad \sigma > 0.$$

Lemma 2.5.54. *[179] Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a BISM mapping such that $(A+B)^{-1}(0^*) \neq \emptyset$. Let $g : E \to \mathbb{R}$ be a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subset of E. Then,*

1. $(A+B)^{-1}(0^*) = Fix(Res^g_{\sigma B} \circ A^g_\sigma)$;
2. $(Res^g_{\sigma B} \circ A^g_\sigma)$ *is a BSNE operator with* $Fix(Res^g_{\sigma B} \circ A^g_\sigma) = \hat{Fix}(Res^g_{\sigma B} \circ A^g_\sigma)$.

Lemma 2.5.55. *[202] Let $\{z_n\} \subset \mathbb{R}_+$, $\{\alpha_n\} \subset (0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\{d_n\} \subset \mathbb{R}$. Assuming that*

$$z_{n+1} \leq (1-\alpha_n)z_n + \alpha_n d_n, \quad \forall n \geq 0.$$

If $\limsup_{j\to\infty} d_{n_j} \leq 0$ for every subsequence $\{z_{n_j}\}$ of $\{z_n\}$ which satisfies the relation

$$\liminf_{j\to\infty}(z_{n_j+1} - z_{n_j}) \geq 0,$$

then $\lim_{n\to\infty} z_n = 0$.

Chapter 3

Split Generalized Equilibrium Problems and Fixed Point Problems in Banach Spaces.

3.1 Introduction

In this chapter, we present our results on psuedomonotone equilibrium problem and common fixed point problem of Bregman quasi-nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. Furthermore, we present our results on split generalized equilibrium problem with multiple output sets and common fixed point problem for an infinite family of multivalued demicontractive mappings in real Hilbert spaces. A review of some important works that motivate our study in this chapter was done in Section 2.4.7.

3.2 Pseudomonotone equilibrium and common fixed point problems

In this section, we study and analyze a Halpern-type subgradient extragradient algorithm for approximating a common solution of pseudo-monotone equilibrium problem and common fixed point problem for a finite family of Bregman quasi-nonexpansive mappings in the setting of p-uniformly convex Banach space which is more general than related works done in real Hilbert spaces. Moreover, we prove a strong convergent theorem for our algorithm. We apply our result to solve the classical variational inequality problem.

3.2.1 Main result

We present our algorithm and show its convergence analysis. First we begin with the following assumptions.

Assumption 3.2.1.

1. *E is a p-uniformly convex real Banach space, which is also uniformly smooth.*
2. *C is a nonempty, closed and convex subset of E.*
3. *$f : C \times C \to \mathbb{R}$ is a bifunction satisfying $A_1 - A_4$ of* **Assumption A**.
4. *$D_j : E \to E, \quad j = 1, 2, \dots, m$ is a finite family of Bregman quasi-nonexpansive mappings such that $I - D_j$ is demiclosed at zero for each $j = 1, 2, \dots, m$.*
5. *The solution set $\Omega := EP(f, C) \cap \bigcap_{j=1}^{m} Fix(D_j) \neq \emptyset$.*

In addition, let $\{\gamma_n\}, \quad \{\epsilon_n\}, \quad \{\eta_{n,j}\}_{j=0}^{m}$ be positive sequences satisfying the following conditions:

(C1) $\{\gamma_n\} \in (0, 1), \quad \lim_{n\to\infty} \gamma_n = 0, \quad \sum_{n=0}^{\infty} \gamma_n = \infty.$

(C2) $\{\epsilon_n\} \in (0, 1)$ *such that* $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\gamma_n} = 0.$

(C3) $\{\eta_{n,j}\} \in (0, 1), \quad \sum_{j=0}^{m} \eta_{n,j} = 1$ *and* $\liminf_{n\to\infty} \eta_{n,0}\eta_{n,j} > 0.$

We now present the main algorithm. The sequence $\{x_n\}$ is generated as follows:

Algorithm 3.2.2. *Calculation of the sequence $\{x_n\}$.*

Initialization: *Choose $x_0 \in C, \quad u \in E, \quad \rho_0 > 0, \quad \mu \in (0, 1), \quad \theta \geq 3$ and set $n = 1$.*

Step 1. *Given the iterates x_{n-1} and x_n $(n \geq 1)$, we choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n := \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \tag{3.1}$$

Set

$$w_n = J^{-1}\Bigg(J(x_n) + \theta_n(J(x_{n-1}) - J(x_n))\Bigg). \tag{3.2}$$

Step 2: *Compute*

$$u_n = \arg\min_{z\in C}\{\rho_n f(w_n, z) + \Delta_p(z, w_n)\}.$$

If $w_n = u_n$, set $u_n = v_n$ and go to **Step 4**. *Else, do* **Step 3**.
Step 3: *Compute*

$$v_n = \arg\min_{z\in \zeta_n}\{\rho_n f(u_n, z) + \Delta_p(z, w_n)\},$$

where $\zeta_n = \{z \in E : \langle J(w_n) - \rho_n\varphi_n - J(u_n), z - u_n\rangle \leq 0\}$, *and* $\varphi_n \in \partial_2 f(w_n, u_n)$ *satisfying* $J(w_n) - \rho_n\varphi_n - J(u_n) \in N_C(u_n)$.

Step 4: *Compute*

$$z_n = J^{-1}\left(\eta_{n,0}J(v_n) + \sum_{j=1}^{m} \eta_{n,j}J(D_j v_n)\right).$$

Step 5: *Calculate* x_{n+1} *and* ρ_{n+1} *as follows:*

$$x_{n+1} = J^{-1}\left(\gamma_n J(u) + (1-\gamma_n)J(z_n)\right),$$

and

$$\rho_{n+1} = \begin{cases} \min\{\rho_n, \frac{\mu(\Delta_p(u_n,w_n)+\Delta_p(v_n,u_n))}{f(w_n,v_n)-f(w_n,u_n)-f(u_n,v_n)}\} \\ \text{if } \; f(w_n,v_n) - f(w_n,u_n) - f(u_n,v_n) > 0 \\ \rho_n, \quad \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n = n+1$ *and revert to* **Step 1**.

Remark 3.2.3. *Indeed, from* (3.1), *it follows that*

$$\lim_{n\to\infty} \frac{\theta_n}{\gamma_n}\|x_{n-1} - x_n\| = 0. \tag{3.4}$$

In fact, we obtain the following:

$$\theta_n\|x_{n-1} - x_n\| \leq \epsilon_n,$$

which follows from the above inequality and our hypothesis (C2) that

$$\lim_{n\to\infty} \frac{\theta_n}{\gamma_n}\|x_{n-1} - x_n\| \leq \frac{\epsilon_n}{\gamma_n} = 0.$$

Consequently, by the norm-to-norm continuity of J, *it follows that*

$$\lim_{n\to\infty} \frac{\theta_n}{\gamma_n}\|Jx_{n-1} - Jx_n\| = 0. \tag{3.5}$$

Lemma 3.2.4. *The sequence* $\{\rho_n\}$ *generated by Algorithm 3.2.2 is well-defined and bounded.*

Proof. Evidently, from (3.3), it is clear that that $\rho_{n+1} \leq \rho_n \;\; \forall\, n \in \mathbb{N}$. This means that ρ_{n+1} is monotonically decreasing. Furthermore, we obtain from **Assumption A** that

$$c_1\Delta_g(u_n, w_n) + c_2\Delta_g(v_n, u_n) \geq f(w_n, v_n) - f(w_n, u_n) - f(u_n, v_n).$$

Hence

$$\begin{aligned}\frac{\mu\left(\Delta_g(u_n,w_n)+\Delta_g(v_n,u_n)\right)}{f(w_n,v_n)-f(w_n,u_n)-f(u_n,v_n)} &\geq \frac{\mu\left(\Delta_g(u_n,w_n)+\Delta_g(v_n,u_n)\right)}{c_1\Delta_g(u_n,w_n)+c_2\Delta_g(v_n,u_n)}\\ &\geq \frac{\mu\left(\Delta_g(u_n,w_n)+\Delta_g(v_n,u_n)\right)}{\max\{c_1,c_2\}(\Delta_g(u_n,w_n)+\Delta_g(v_n,u_n))}\\ &= \frac{\mu}{\max\{c_1,c_2\}}.\end{aligned}$$

Therefore, we conclude that the sequence $\{\rho_n\}$ is bounded by $\min\{\rho_0, \frac{\mu}{\max\{c_1,c_2\}}\} > 0$, which implies that $\lim_{n\to\infty} \rho_n = \rho > 0$. □

Lemma 3.2.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2.2 such that $\{x_n\}$ is bounded. Suppose conditions (C1) and (C2) hold, then for all $x^* \in \Omega$ we have:*

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\Big(\triangle_p(x^*,x_{n-1}) - \triangle_p(x^*,x_n)\Big) = 0.$$

Proof. Let $x^* \in \Omega$. Observe that

$$\begin{aligned}\Delta_p(x^*,x_{n-1}) - \Delta_p(x^*,x_n) &= \frac{1}{q}\|x_{n-1}\|^p - \langle x^*, J(x_{n-1})\rangle + \frac{1}{p}\|x^*\|^p\\ &\quad - \Big(\frac{1}{q}\|x_n\|^p - \langle x^*, J(x_n)\rangle + \frac{1}{p}\|x^*\|^p\Big)\\ &= \frac{1}{q}\Big(\|x_{n-1}\|^p - \|x_n\|^p\Big) + \langle x^*, J(x_n) - J(x_{n-1})\rangle\\ &\leq \frac{1}{q}M\|x_{n-1}-x_n\| + \|J(x_n) - J(x_{n-1})\|\|x^*\|, \end{aligned} \tag{3.6}$$

for some constant $M > 0$.

Applying (3.4) and (3.5), we obtain from (3.6) that

$$\begin{aligned}\lim_{n\to\infty}\frac{\theta_n}{\gamma_n}\Big(\Delta_p(x^*,x_{n-1}) - \Delta_p(x^*,x_n)\Big) &\leq \lim_{n\to\infty}\Big(\frac{M}{q}\cdot\frac{\theta_n}{\gamma_n}\|x_{n-1}-x_n\|\\ &\quad + \|x^*\|\frac{\theta_n}{\gamma_n}\|J(x_n) - J(x_{n-1})\|\Big) = 0,\end{aligned}$$

which is the required result. □

Lemma 3.2.6. *Suppose Assumptions 3.2.1 hold. The sequence $\{x_n\}$ generated by Algorithm 3.2.2 is bounded.*

Proof. Let $x^* \in \Omega$. We obtain from (3.2) that

$$\begin{aligned}\Delta_p(x^*,w_n) &= \Delta_p\left[x^*, J^{-1}\Big((1-\theta_n)J(x_n) + \theta_n J(x_{n-1})\Big)\right]\\ &\leq (1-\theta_n)\Delta_p(x^*,x_n) + \theta_n(x^*,x_{n-1}). \end{aligned} \tag{3.7}$$

From Algorithm 3.2.2, since $v_n \in \zeta_n$, we obtain that

$$\langle J(w_n) - \rho_n\varphi_n - J(u_n), v_n - u_n\rangle \leq 0,$$

which implies that

$$\langle J(w_n) - J(u_n), v_n - u_n\rangle \leq \rho_n\langle\varphi_n, v_n - u_n\rangle. \tag{3.8}$$

Also, since $\varphi_n \in \partial_2 f(w_n, u_n)$, we have that

$$f(w_n, z) - f(w_n, u_n) \geq \langle\varphi_n, z - u_n\rangle \quad \forall\ z \in E.$$

In particular, we have

$$f(w_n, v_n) - f(w_n, u_n) \geq \langle\varphi_n, v_n - u_n\rangle. \tag{3.9}$$

Using (3.8) and (3.9), we obtain

$$\langle J(w_n) - J(u_n), v_n - u_n\rangle \leq \rho_n\{f(w_n, v_n) - f(w_n, u_n)\}. \tag{3.10}$$

Furthermore, since $v_n = \arg\min_{z\in\zeta_n}\{\rho_n f(u_n, z) + \Delta_g(z, w_n)\}$, we obtain from Lemma 2.5.44 that

$$0 \in \partial\left(\rho_n f(u_n, z) + \Delta_g(z, w_n)\right)(v_n) + N_{\zeta_n}(v_n).$$

The above expression implies that there exists $\bar{\varphi}_n \in \partial_2 f(u_n, v_n)$ and $\phi \in N_{\zeta_n}(v_n)$, such that

$$\rho_n\bar{\varphi}_n + J(v_n) - J(w_n) + \phi = 0.$$

Furthermore, since $\phi \in N_{\zeta_n}(v_n)$, for all $z \in \zeta_n$, we have $\langle\phi, z - v_n\rangle \leq 0$. Therefore,

$$\rho_n\langle\bar{\varphi}_n, z - v_n\rangle \geq \langle J(w_n) - J(v_n), z - v_n\rangle, \quad \forall\ z \in \zeta_n. \tag{3.11}$$

Moreover, $\bar{\varphi}_n \in \partial_2 f(u_n, v_n)$ implies that

$$f(u_n, z) - f(u_n, v_n) \geq \langle\bar{\varphi}_n, z - v_n\rangle \quad \forall z \in \zeta_n.$$

Thus,

$$\rho_n\left(f(u_n, z) - f(u_n, v_n)\right) \geq \rho_n\langle\bar{\varphi}_n, z - v_n\rangle \quad \forall\ z \in \zeta_n. \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\langle J(w_n) - J(v_n), z - v_n\rangle \leq \rho_n\left(f(u_n, z) - f(u_n, v_n)\right) \quad \forall\ z \in \zeta_n. \tag{3.13}$$

Since $x^* \in \Omega$, then $x^* \in EP(f, C)$. Observe that $EP(f, C) \subset C \subset \zeta_n$. Hence, by letting $z = x^* \in \mathrm{EP}(f, C)$ in (3.13), we get

$$\langle J(w_n) - J(v_n), x^* - v_n\rangle \leq \rho_n\left(f(u_n, x^*) - f(u_n, v_n)\right).$$

Since $f(x^*, u_n) \geq 0$, then by the pseudo-monotonicity of the bifunction f, we have that $f(u_n, x^*) \leq 0$. Consequently, we have

$$\langle J(w_n) - J(v_n), x^* - v_n \rangle \leq -\rho_n f(u_n, v_n). \tag{3.14}$$

Adding (3.10) and (3.14) together gives

$$\begin{aligned} &\langle J(w_n) - J(v_n), x^* - v_n \rangle + \langle J(w_n) - J(u_n), v_n - u_n \rangle \\ &\qquad \leq \rho_n \{ f(w_n, v_n) - f(w_n, u_n) - f(u_n, v_n) \}. \end{aligned}$$

Applying Bregman three point identity, (2.13) we have

$$\begin{aligned} \Delta_p(x^*, v_n) \leq{}& \Delta_p(x^*, w_n) - \Delta_p(u_n, w_n) - \Delta_p(v_n, u_n) \\ &+ \rho_n \{ f(w_n, v_n) - f(w_n, u_n) - f(u_n, v_n) \}. \end{aligned}$$

Besides, we obtain from the definition of ρ_n that

$$\begin{aligned} \Delta_p(x^*, v_n) &\leq \Delta_p(x^*, w_n) - \Delta_p(u_n, w_n) - \Delta_p(v_n, u_n) \qquad (3.15) \\ &\quad + \frac{\rho_n}{\rho_{n+1}} \rho_{n+1} \{ f(w_n, v_n) - f(w_n, u_n) - f(u_n, v_n) \} \\ &\leq \Delta_p(x^*, w_n) - \Delta_p(u_n, w_n) - \Delta_p(v_n, u_n) + \frac{\rho_n}{\rho_{n+1}} \mu \left(\Delta_p(u_n, w_n) + \Delta_p(v_n, u_n) \right) \\ &= \Delta_p(x^*, w_n) - \left(1 - \frac{\rho_n}{\rho_{n+1}} \mu \right) \left(\Delta_p(u_n, w_n) + \Delta_p(v_n, u_n) \right). \qquad (3.16) \end{aligned}$$

Since $\lim_{n\to\infty} \left(1 - \frac{\rho_n}{\rho_{n+1}} \mu \right) = 1 - \mu > 0$, then there exists $N \in \mathbb{N}$ such that

$$\left(1 - \frac{\rho_n}{\rho_{n+1}} \mu \right) > 0, \quad \forall n \geq N.$$

Hence, we have

$$\Delta_p(x^*, v_n) \leq \Delta_p(x^*, w_n), \quad \forall n \geq N. \tag{3.17}$$

Also, from Algorithm 3.2.2 and the fact that D_j is quasi-nonexpansive for each $j \in \mathbb{N}$, we have

$$\begin{aligned} \Delta_p(x^*, z_n) &= \Delta_p \left(x^*, J^{-1} \left(\eta_{n,0} J(v_n) + \sum_{j=1}^{m} \eta_{n,j} J(D_j v_n) \right) \right) \\ &\leq \eta_{n,0} \Delta_p(x^*, v_n) + \sum_{j=1}^{m} \eta_{n,j} \Delta_p(x^*, D_j v_n) \\ &\leq \eta_{n,0} \Delta_p(x^*, v_n) + \sum_{j=1}^{m} \eta_{n,j} \Delta_p(x^*, v_n) \\ &= \Delta_p(x^*, v_n). \qquad (3.18) \end{aligned}$$

Therefore, we get

$$
\begin{aligned}
\Delta_p(x^*, x_{n+1}) &= \Delta_p\left(x^*, J^{-1}\Big(\gamma_n J(u) + (1-\gamma_n)J(z_n)\Big)\right)\\
&\leq \gamma_n \Delta_p(x^*, u) + (1-\gamma_n)\Delta_p(x^*, z_n) \qquad (3.19)\\
&\leq \gamma_n \Delta_p(x^*, u) + (1-\gamma_n)\Delta_p(x^*, v_n)\\
&\leq \gamma_n \Delta_p(x^*, u) + (1-\gamma_n)\Delta_p(x^*, w_n)\\
&\leq \gamma_n \Delta_p(x^*, u) + (1-\gamma_n)[(1-\theta_n)\Delta_p(x^*, x_n) + \theta_n(x^*, x_{n-1})]\\
&\leq \max\{\Delta_p(x^*, u)\}, \max\{\Delta_p(x^*, x_n), \Delta_p(x^*, x_{n-1})\} \qquad (3.20)\\
&\vdots\\
&\leq \max\{\Delta_p(x^*, u)\}, \max\{\Delta_p(x^*, x_N), \Delta_p(x^*, x_{N-1})\}.
\end{aligned}
$$

Therefore, $\{\Delta_p(x^*, x_n)\}$ is bounded and by Lemma 2.5.34, the sequence $\{x_n\}$ is also bounded. Consequently, $\{z_n\}$, $\{v_n\}$ $\{u_n\}$ and $\{w_n\}$ are all bounded. □

Lemma 3.2.7. *Suppose we let $r = \sup_{n\in\mathbb{N}}\{\|J(v_n)\|,\ \|J(D_j v_n)\|\}$ and let $W_q : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. It follows that*

$$
\Delta_p(x^*, z_n) \leq \Delta_p(x^*, v_n) - \frac{W_q(\eta_{n,j})}{q} g\big(\|J(v_n) - J(D_j v_n)\|\big), \qquad (3.21)
$$

where $W_q(\eta_{n,j}) = (\eta_{n,0})^q \sum_{j=1}^m \eta_{n,j} + \eta_{n,0}(\sum_{j=1}^m \eta_{n,j})^q$.

Proof. Indeed, we obtain from Algorithm 3.2.2 that:

$$
\begin{aligned}
\Delta_p(x^*, z_n) &= \Delta_p\left(x^*, J^{-1}\Big(\eta_{n,0}J(v_n) + \sum_{j=1}^m \eta_{n,j}J(D_j v_n)\Big)\right)\\
&= V_p\left(x^*, \eta_{n,0}J(v_n) + \sum_{j=1}^m \eta_{n,j}J(D_j v_n)\right).
\end{aligned}
$$

Furthermore, we get from Lemma 2.5.45 and Lemma 2.5.46 that

$$\begin{aligned}
\Delta_p(x^*, z_n) &= \frac{1}{p}\|x^*\|^p - \eta_{n,0}\langle x^*, J(v_n)\rangle - \sum_{j=1}^{m} \eta_{n,j}\langle x^*, J(D_j v_n)\rangle \\
&+ \frac{1}{q}\|\eta_{n,0}J(v_n) + \sum_{j=1}^{m} \eta_{n,j} J(D_j v_n)\|^q \\
&\leq \frac{1}{p}\|x^*\|^p - \eta_{n,0}\langle x^*, J(v_n)\rangle - \sum_{j=1}^{m} \eta_{n,j}\langle x^*, J(D_j v_n)\rangle \\
&+ \frac{1}{q}\eta_{n,0}\|J(v_n)\|^q + \frac{1}{q}\sum_{j=1}^{m} \eta_{n,j}\|J(D_j v_n)\|^q - \frac{W_q(\eta_{n,j})}{q} g\big(\|J(v_n) - J(D_j v_n)\|\big) \\
&= \frac{1}{p}\eta_{n,0}\|x^*\|^p + \frac{1}{p}\sum_{j=1}^{m} \eta_{n,j}\|x^*\|^p - \eta_{n,0}\langle x^*, J(v_n)\rangle - \sum_{j=1}^{m} \eta_{n,j}\langle x^*, J(D_j v_n)\rangle \\
&+ \frac{1}{q}\eta_{n,0}\|J(v_n)\|^q + \frac{1}{q}\sum_{j=1}^{m} \eta_{n,j}\|J(D_j v_n)\|^q - \frac{W_q(\eta_{n,j})}{q} g\big(\|J(v_n) - J(D_j v_n)\|\big) \\
&= \eta_{n,0}\left\{\frac{1}{p}\|x^*\|^p - \langle x^*, J(v_n)\rangle + \frac{1}{q}\|J(v_n)\|^q\right\} \\
&+ \sum_{j=1}^{m} \eta_{n,j}\left\{\frac{1}{p}\|x^*\|^p - \langle x^*, J(D_j v_n)\rangle + \frac{1}{q}\|J(D_j v_n)\|^q\right\} \\
&- \frac{W_q(\eta_{n,j})}{q} g\big(\|J(v_n) - J(D_j v_n)\|\big) \\
&= \eta_{n,0}\Delta_p(x^*, v_n) + \sum_{j=1}^{m} \eta_{n,j}\Delta_p(x^*, D_j v_n) - \frac{W_q(\eta_{n,j})}{q} g\big(\|J(v_n) - J(D_j v_n)\|\big).
\end{aligned}$$

Since D_j is Bregman quasi-nonexpansive for each $j \in \mathbb{N}$, we obtain

$$\begin{aligned}
\Delta_p(x^*, z_n) &\leq \eta_{n,0}\Delta_p(x^*, v_n) + \sum_{j=1}^{m} \eta_{n,j}\Delta_p(x^*, v_n) - \frac{W_q(\eta_{n,j})}{q} g\big(\|J(v_n) - J(D_j v_n)\|\big) \\
&= \Delta_p(x^*, v_n) - \frac{W_q(\eta_{n,j})}{q} g\big(\|J(v_n) - J(D_j v_n)\|\big).
\end{aligned}$$

□

Next, we prove the strong convergence theorem for the sequence $\{x_n\}$ generated by our proposed algorithm.

Theorem 3.2.8. *The iterative sequence $\{x_n\}$ generated by Algorithm 3.2.2 under Assumption 3.2.1 strongly converges to $x^* \in \Omega$, where $x^* = Proj_{\Omega}^{f}(u)$.*

Proof. Suppose $x^* = \mathrm{Proj}_{\Omega}^{f}(u)$. Then from Algorithm 3.2.2 and by applying Lemma 2.5.45

(iii), we have

$$
\begin{aligned}
\Delta_p(x^*, x_{n+1}) &= \Delta_p\left(x^*, J^{-1}\big(\gamma_n J(u) + (1-\gamma_n)J(z_n)\big)\right)\\
&= V_p\left(x^*, \gamma_n J(u) + (1-\gamma_n)J(z_n)\right)\\
&\leq V_p\bigg(x^*, \gamma_n J(u) + (1-\gamma_n)J(z_n) - \gamma_n\big(J(u) - J(x^*)\big)\bigg) \qquad (3.22)\\
&+ \gamma_n\langle J(u) - J(x^*), x_{n+1} - x^*\rangle\\
&= V_p\bigg(x^*, \gamma_n J(x^*) + (1-\gamma_n)J(z_n)\bigg) + \gamma_n\langle J(u) - J(x^*), x_{n+1} - x^*\rangle\\
&\leq \gamma_n V_p(x^*, J(x^*)) + (1-\gamma_n)V_p(x^*, J(z_n)) + \gamma_n\langle J(u) - J(x^*), x_{n+1} - x^*\rangle\\
&= (1-\gamma_n)\Delta_p(x^*, z_n) + \gamma_n\langle J(u) - J(x^*), x_{n+1} - x^*\rangle. \qquad (3.23)
\end{aligned}
$$

Applying (3.21) in (3.22), we obtain

$$
\begin{aligned}
\Delta_p(x^*, x_{n+1}) \leq (1-\gamma_n)&\left[\Delta_p(x^*, v_n) - \frac{W_q(\eta_{n,j})}{q}g\big(\|J(v_n) - J(D_j v_n)\|\big)\right]\\
&+ \gamma_n\langle J(u) - J(x^*), x_{n+1} - x^*\rangle. \qquad (3.24)
\end{aligned}
$$

Furthermore, by applying (3.7) and (3.15) in (3.24) we have

$$
\begin{aligned}
\Delta_p(x^*,x_{n+1}) &\leq (1-\gamma_n)\left[\Delta_p(x^*, w_n) - \Big(1 - \frac{\rho_n}{\rho_{n+1}}\mu\Big)\Big(\Delta_p(u_n, w_n) + \Delta_p(v_n, u_n)\Big)\right]\\
&- (1-\gamma_n)\left[\frac{W_q(\eta_{n,j})}{q}g(\|J(v_n) - J(D_j v_n)\|)\right] + \gamma_n\langle J(u) - J(x^*), x_{n+1} - x^*\rangle\\
&\leq (1-\gamma_n)\Big[(1-\theta_n)\Delta_p(x^*, x_n) + \theta_n\Delta_p(x^*, x_{n-1})\Big]\\
&- (1-\gamma_n)\Big(1 - \frac{\rho_n}{\rho_{n+1}}\mu\Big)\Big(\Delta_p(u_n, w_n) + \Delta_p(v_n, u_n)\Big)\\
&- (1-\gamma_n)\frac{W_q(\eta_{n,j})}{q}g(\|J(v_n) - J(D_j v_n)\|) + \gamma_n\langle J(u) - J(x^*), x_{n+1} - x^*\rangle\\
&= (1-\gamma_n)\Delta_p(x^*, x_n) - (1-\gamma_n)\Big(1 - \frac{\rho_n}{\rho_{n+1}}\mu\Big)\Big(\Delta_p(u_n, w_n)\Delta_p(v_n, u_n)\Big)\\
&- (1-\gamma_n)\frac{W_q(\eta_{n,j})}{q}g(\|J(v_n) - J(D_j v_n)\|) + \gamma_n\Psi_n, \qquad (3.25)
\end{aligned}
$$

where $\Psi_n := \bigg(\frac{\theta_n}{\gamma_n}\Big(\Delta_p(x^*, x_{n-1}) - \Delta_p(x^*, x_n)\Big) + \langle J(u) - J(x^*), x_{n+1} - x^*\rangle\bigg)$.

Thus, from (3.25), we have

$$
(1-\gamma_n)\left(1 - \frac{\rho_n}{\rho_{n+1}}\mu\right)\left(\Delta_p(u_n, w_n) + \Delta_p(v_n, u_n)\right) \leq \Delta_p(x^*, x_n) - \Delta_p(x^*, x_{n+1}) + \gamma_n\Psi_n, \qquad (3.26)
$$

Also, from (3.25), we obtain

$$
(1-\gamma_n)\frac{W_q(\eta_{n,j})}{q}g(\|J(v_n) - J(D_j v_n)\|) \leq \Delta_p(x^*, x_n) - \Delta_p(x^*, x_{n+1}) + \gamma_n\Psi_n. \qquad (3.27)
$$

Moreover, it follows from (3.25) that

$$\Delta_p(x^*, x_{n+1}) \leq (1-\gamma_n)\Delta_p(x^*, x_n) + \gamma_n \Psi_n. \tag{3.28}$$

We now show that the sequence $\{x_n\}$ converges to x^*. Let $a_n := \Delta_p(x_n, x^*)$, it is easy to see that (3.28) satisfies the inequality

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n \Psi_n.$$

To apply Lemma 2.5.55, it is enough to show that $\limsup\limits_{k\to\infty} \Psi_{n_k} \leq 0$ for every subsequence $\{\Delta_p(x^*, x_{n_k})\}$ of $\{\Delta_p(x^*, x_n)\}$ satisfying the relation

$$\liminf_{k\to\infty} \left(\Delta_p(x^*, x_{n_k+1}) - \Delta_p(x^*, x_{n_k})\right) \geq 0. \tag{3.29}$$

Now, we assume that $\{\Delta_p(x^*, x_{n_k})\}$ is a subsequence of $\{\Delta_p(x^*, x_n)\}$ such that (3.29) holds. Then, from (3.26) and by applying (3.29) and condition (C1), we have that

$$\begin{aligned}
\limsup_{k\to\infty}(1-\gamma_{n_k})\left(1 - \frac{\rho_{n_k}}{\rho_{n_k+1}}\mu\right)&(\Delta_p(u_{n_k}, w_{n_k}) + \Delta_p(v_{n_k}, u_{n_k})) \\
&\leq \limsup_{k\to\infty}\left(\Delta_p(x^*, x_{n_k}) - \Delta_p(x^*, x_{n_k+1}) + \gamma_{n_k}\Psi_{n_k}\right) \\
&= \limsup_{k\to\infty}\left(\Delta_p(x^*, x_{n_k}) - \Delta_p(x^*, x_{n_k+1})\right) \\
&\leq -\liminf_{k\to\infty}\left(\Delta_p(x^*, x_{n_k+1}) - \Delta_p(x^*, x_{n_k})\right) \\
&\leq 0.
\end{aligned} \tag{3.30}$$

It follows that

$$\limsup_{k\to\infty}\left(1 - \frac{\rho_{n_k}}{\rho_{n_k+1}}\mu\right)(\Delta_p(u_{n_k}, w_{n_k}) + \Delta_p(v_{n_k}, u_{n_k})) = 0. \tag{3.31}$$

Since $\lim\limits_{k\to\infty} \rho_{n_k}$ exists, then $\lim\limits_{n\to\infty} \frac{\rho_{n_k}}{\rho_{n_k+1}} = 1$. Using the fact that $\mu \in (0,1)$, it follows from (3.31) that

$$\lim_{k\to\infty} \Delta_p(u_{n_k}, w_{n_k}) = 0; \quad \lim_{k\to\infty} \Delta_p(v_{n_k}, u_{n_k}) = 0.$$

By Lemma 2.5.31, we obtain

$$\lim_{k\to\infty} \|u_{n_k} - w_{n_k}\| = 0 = \lim_{k\to\infty} \|v_{n_k} - u_{n_k}\|. \tag{3.32}$$

Consequently, we have

$$\lim_{k\to\infty} \|v_{n_k} - w_{n_k}\| \leq \lim_{k\to\infty} \|v_{n_k} - u_{n_k}\| + \lim_{k\to\infty} \|u_{n_k} - w_{n_k}\| = 0. \tag{3.33}$$

Similarly, from (3.27) and by applying (3.29), we have

$$\begin{aligned}\limsup_{k\to\infty}(1-\gamma_{n_k})\frac{W_q(\eta_{n_k,j})}{q}g(\|J(v_{n_k})-J(D_jv_{n_k})\|)\\ &\le \limsup_{k\to\infty}\Big(\Delta_p(x^*,x_{n_k})-\Delta_p(x^*,x_{n_k+1})+\gamma_{n_k}\Psi_{n_k}\Big)\\ &= \limsup_{k\to\infty}\Big(\Delta_p(x^*,x_{n_k})-\Delta_p(x^*,x_{n_k+1})\Big)\\ &\le -\liminf_{k\to\infty}\Big(\Delta_p(x^*,x_{n_k+1})-\Delta_p(x^*,x_{n_k})\Big)\\ &\le 0.\end{aligned}$$

Therefore,

$$\limsup_{k\to\infty}(1-\gamma_{n_k})\frac{W_q(\eta_{n_k,j})}{q}g(\|J(v_{n_k})-J(D_jv_{n_k})\|)=0.$$

By condition (C3), we have

$$\lim_{k\to\infty} g\|J(v_{n_k})-J(D_jv_{n_k})\|=0,\quad j=1,2,\dots,m.$$

By the property of g and the fact that J^{-1} is norm-to-norm continuous on bounded subsets of E^*, we have

$$\lim_{k\to\infty}\|v_{n_k}-D_jv_{n_k}\|=0,\quad j=1,2,\dots,m. \tag{3.34}$$

Moreover, by applying condition (C3) and (3.34) we obtain

$$\begin{aligned}\lim_{k\to\infty}\|Jz_{n_k}-Jv_{n_k}\| &= \lim_{k\to\infty}\|\eta_{n_k,0}J(v_{n_k})+\sum_{j=1}^m\eta_{n_k,j}J(D_jv_{n_k})-J(v_{n_k})\|\\ &\le \lim_{k\to\infty}\Big(\eta_{n_k,0}\|J(v_{n_k})-J(v_{n_k})\|+\sum_{j=1}^m\eta_{n_k,j}\|J(D_jv_{n_k})-J(v_{n_k})\|\Big)=0.\end{aligned}$$

By the norm-to-norm continuity of J^{-1} on bounded subsets of E^*, we get

$$\lim_{k\to\infty}\|z_{n_k}-v_{n_k}\|=0. \tag{3.35}$$

From (3.33) and (3.35), we have

$$\lim_{k\to\infty}\|z_{n_k}-w_{n_k}\|=0. \tag{3.36}$$

Recall that $w_{n_k} = J^{-1}\Big[J(x_{n_k}) + \theta_{n_k}\big(J(x_{n_k-1}) - J(x_{n_k})\big)\Big]$. Then, by using (3.5) we obtain

$$\lim_{k\to\infty} \|J(w_{n_k}) - J(x_{n_k})\| = \lim_{k\to\infty} \gamma_{n_k} \cdot \frac{\theta_{n_k}}{\gamma_{n_k}} \|J(x_{n_k-1}) - J(x_{n_k})\| = 0. \tag{3.37}$$

By the norm-to-norm continuity of J^{-1} on bounded subsets of E^*, we get

$$\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0. \tag{3.38}$$

Then, from (3.32), (3.35), (3.36) and (3.38) we obtain

$$\lim_{k\to\infty} \|z_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|v_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0. \tag{3.39}$$

Furthermore, from Algorithm 3.2.2, we have

$$\begin{aligned}\lim_{k\to\infty} \|J(x_{n_k+1}) - J(z_{n_k})\| &= \lim_{k\to\infty} \|\gamma_{n_k} J(u) + (1-\gamma_{n_k})J(z_{n_k}) - J(z_{n_k})\| \\ &= \lim_{k\to\infty} \big(\gamma_{n_k}\|J(u) - J(z_{n_k})\| + (1-\gamma_{n_k})\|J(z_{n_k}) - J(z_{n_k})\|\big) = 0.\end{aligned}$$

By the norm-to-norm continuity of J^{-1} on bounded subsets of E^*, we have

$$\lim_{k\to\infty} \|x_{n_k+1} - z_{n_k}\| = 0. \tag{3.40}$$

Therefore, from (3.39) and (3.40) we obtain

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{3.41}$$

Next, since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $\hat{x} \in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $\hat{x} \in E$. Then, by (3.39) it follows that $\{u_{n_k}\}$ converges weakly to $\hat{x} \in E$. Also, since

$$u_{n_k} = \arg\min_{z\in C}\{\rho_{n_k} f(w_{n_k}, z) + \Delta_p(z, w_{n_k})\},$$

it follows from Lemma 2.5.44 that

$$0 \in \partial\Big(\rho_{n_k} f(w_{n_k}, z) + \Delta_p(z, w_{n_k})\Big)(u_{n_k}) + N_C(u_{n_k}), \quad z \in C.$$

This means that

$$\rho_{n_k}\varphi_{n_k} + J(u_{n_k}) - J(w_{n_k}) + \phi = 0, \quad \text{where} \quad \phi \in N_C(u_{n_k}). \tag{3.42}$$

Observe that

$$\langle \phi, z - u_{n_k}\rangle \le 0, \quad \forall\, z \in C.$$

Therefore, it follows from (3.42) that

$$\langle \rho_{n_k}\varphi_{n_k}, z - u_{n_k}\rangle + \langle \phi, z - u_{n_k}\rangle = \langle J(w_{n_k}) - J(u_{n_k}), z - u_{n_k}\rangle,$$

Consequently, we have

$$\rho_{n_k}\langle \varphi_{n_k}, z - u_{n_k}\rangle \geq \langle J(w_{n_k}) - J(u_{n_k}), z - u_{n_k}\rangle. \tag{3.43}$$

Furthermore, since $\varphi_n \in \partial_2 f(w_{n_k}, u_{n_k})$, we have

$$f(w_{n_k}, z) - f(w_{n_k}, u_{n_k}) \geq \langle \varphi_{n_k}, z - u_{n_k}\rangle, \quad \forall\ z \in C. \tag{3.44}$$

Combining (3.43) and (3.44), we have

$$\rho_{n_k}\Big(f(w_{n_k}, z) - f(w_{n_k}, u_{n_k})\Big) \geq \langle J(w_{n_k}) - J(u_{n_k}), z - u_{n_k}\rangle, \quad \forall\ z \in C. \tag{3.45}$$

Since $\lim\limits_{k\to\infty} \|w_{n_k} - u_{n_k}\| = 0$ and J is uniformly continuous on bounded subsets of E, we have

$$\lim_{k\to\infty} \|J(w_{n_k}) - J(u_{n_k})\| = 0. \tag{3.46}$$

Consequently, passing limit $k \to \infty$ in (3.45) and by applying condition (A4), (3.38) and (3.46) we obtain

$$f(\hat{x}, z) \geq 0, \quad \forall\ z \in C,$$

which implies that,

$$\hat{x} \in \mathrm{EP}(f, C). \tag{3.47}$$

Also, since $\lim\limits_{k\to\infty} \|v_{n_k} - D_j v_{n_k}\| = 0, \quad j = 1, 2, \ldots, m$ and $v_{n_k} \rightharpoonup \hat{x}$ by (3.39), then by the demiclosedness of $I - D_j$, we obtain $\hat{x} \in F(D_j), \quad \forall\ j = 1, 2, \ldots, m$. Therefore, it follows that

$$\hat{x} \in \bigcap_{j=1}^{m} F(D_j). \tag{3.48}$$

Since $\hat{x} \in w_\omega(x_n)$ is arbitrary, then by combining (3.47) and (3.48), we have

$$w_\omega(x_n) \subset \mathrm{EP}(f, C) \cap \bigcap_{j=1}^{m} F(D_j). \tag{3.49}$$

To complete the proof, we need to show that $\lim\limits_{k\to\infty} \Psi_{n_k} \leq 0$. Firstly, we show that

$$\limsup_{k\to\infty} \langle J(u) - J(x^*), x_{n_{k+1}} - x^*\rangle \leq 0.$$

By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup q$ and

$$\lim_{j\to\infty}\langle J(u) - J(x^*), x_{n_{k_j}} - x^*\rangle = \limsup_{k\to\infty}\langle J(u) - J(x^*), x_{n_k} - x^*\rangle.$$

Since $x^* = \mathrm{Proj}_{\Omega}^{f}(u)$, then by (2.19) and (3.41) we get

$$\begin{aligned}
&\limsup_{k\to\infty}\langle J(u) - J(x^*), x_{n_{k+1}} - x^*\rangle \\
&\qquad = \limsup_{k\to\infty}\langle J(u) - J(x^*), x_{n_k} - x^*\rangle + \limsup_{k\to\infty}\langle J(u) - J(x^*), x_{n_{k+1}-x_{n_k}}\rangle \\
&\qquad = \lim_{j\to\infty}\langle J(u) - J(x^*), x_{n_{k_j}} - x^*\rangle \\
&\qquad = \langle J(u) - J(x^*), q - x^*\rangle \\
&\qquad \leq 0.
\end{aligned} \tag{3.50}$$

Recall that $\Psi_n = \Big(\frac{\theta_n}{\gamma_n}(1-\gamma_n)(\Delta_p(x^*, x_{n-1}) - \Delta_p(x^*, x_n)) + \langle J(u) - J(x^*), x_{n+1} - x^*\rangle\Big)$.

Now, by Lemma 6.3.6 and (3.50) we have that $\limsup_{k\to\infty}\Psi_{n_k} \leq 0$. Next, by applying Lemma 2.5.55 to (3.28), we conclude that $\Delta_p(x^*, x_n) \to 0$ as $n \to \infty$. By Lemma 2.5.31, it follows that $\lim \|x_n - x^*\| = 0$. Hence, $\{x_n\}$ converges strongly to $\hat{x} \in \Omega$ as required. □

3.2.2 Application

In this subsection, we apply our result to study the classical variational inequality problem (VIP). Given an operator $A : C \to E^*$. The VIP is formulated as finding a point

$$q^* \in C \ \text{ such that } \ \langle \bar{x} - q^*, A(q^*)\rangle \geq 0, \quad \forall\ \bar{x} \in C. \tag{3.51}$$

The solution set of VIP (3.51) is denoted by $\mathrm{VIP}(C, A)$.

Variational inequalities have been found very useful in various real-world problems such as optimization problems, minimax theorems, differential equations and in certain applications to economic theory and mechanics. For more details on variational inequalities (see, [4, 96, 97, 180] and the references contained therein).

The mapping $A : C \to E^*$ satisfies the following conditions:

(D1) A is pseudo-monotone; that is for any $x, \ \ y \in C$, we have

$$\langle Ax, y - x\rangle \geq 0 \Longrightarrow \langle Ay, y - x\rangle \geq 0,$$

(D2) A is K-Lipschitz continuous with respect to Δ_p, that is, there exists a constant $K > 0$ such that

$$\Delta_p(Ax, Ay) \leq K\Delta_p(x, y), \quad \forall x, y \in C,$$

(D3) A is sequentially weakly continuous, that is, for any sequence $\{x_n\} \subset C$, we have $x_n \rightharpoonup x \in C$ implies that $Ax_n \rightharpoonup Ax \in E^*$.

Lemma 3.2.9. *[82] Let C be a nonempty, closed and convex subset of a reflexive Banach space E, $A : C \to E^*$ be a nonlinear mapping. Then,*

$$\Pi_C\Big(J_q^{E^*}[J_p^E(x) - \theta A(y)]\Big) = \arg\min_{z\in C}\{\theta\langle z - y, A(y)\rangle + \Delta_p(z,x)\}, \tag{3.52}$$

for all $x \in E$, $y \in C$ and $\theta \in (0,+\infty)$.

If we take $f(x,y) = \langle A(x), y - x\rangle, \ \ \forall\ x,y \in C$, then since $\varphi_n \in \partial_2 f(w_n,u_n)$, we have that

$$f(w_n,z) - f(w_n,u_n) \geq \langle \varphi_n, z - u_n\rangle, \ \ \forall\ z \in E,$$

which implies that

$$\langle A(w_n), z - w_n\rangle - \langle A(w_n), u_n - w_n\rangle \geq \langle \varphi_n, z - u_n\rangle.$$

Simplifying the last inequality, we obtain

$$\langle A(w_n) - \varphi_n, z - u_n\rangle \geq 0, \ \ \forall\ z \in E.$$

From this, it follows that

$$\langle J(w_n) - \rho_n A(w_n) - J(u_n), z - u_n\rangle \leq \langle J(w_n) - \rho_n\varphi_n - J(u_n), z - u_n\rangle \leq 0, \ \ \forall\ z \in \zeta_n \subset E. \tag{3.53}$$

Now, taking $f(x,y) = \langle A(x), y-x\rangle, \ \ \forall\ x,y \in C$, then the bifunction f satisfies conditions (A1)-(A4) (see [82]). Hence, by applying Theorem 3.2.8 and Lemma 3.2.9, we obtain the following result for approximating a common solution of pseudo-monotone variational inequality problem and common fixed point problem for a finite family of Bregman quasi-nonexpansive mappings in p-uniformly convex real Banach space, which is also uniformly smooth.

Theorem 3.2.10. *Let E be a p-uniformly convex real Banach space, which is also uniformly smooth and let C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be a mapping satisfying conditions (D1)-(D3) and let $D_j : E \to E, \ \ j = 1,2,\ldots,m$ be a finite family of Bregman quasi-nonexpansive mappings such that $I - D_j$ is demi-closed at zero for each j. Suppose that conditions (C1)-(C3) hold and the solution set $\Gamma := VI(C,A) \cap \bigcap_{j=1}^m F(D_j) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm (3.2.11) below converges strongly to $x^* \in \Gamma$, where $x^* = Proj_\Gamma^f(u)$.*

Algorithm 3.2.11.

Choose $x_0 \in C, \ \ u \in E, \ \ \rho_0 > 0, \ \ \mu \in (0,1)$ and set $n = 1$. Given the iterates x_{n-1} and x_n, we have

$$
\begin{cases}
w_n = J^{-1}\Big(J(x_n) + \theta_n(J(x_{n-1}) - J(x_n))\Big) \\
u_n = \Pi_C[J^{-1}(J(w_n) - \rho_n A(w_n))] \\
\zeta_n = \{z \in E : \langle J(w_n) - \rho_n A(w_n) - J(u_n), z - u_n\rangle \le 0\} \\
v_n = \Pi_{\zeta_n}[J^{-1}(J(w_n) - \rho_n A(u_n))] \\
z_n = J^{-1}\left(\eta_{n,0}J(v_n) + \sum_{j=1}^{m} \eta_{n,j}J(D_j v_n)\right) \\
x_{n+1} = J^{-1}\Big(\gamma_n J(u) + (1-\gamma_n)J(z_n)\Big),
\end{cases} \tag{3.54}
$$

where the sequence of step sizes $\{\rho_n\}$ *is defined by* $\rho_{n+1} = \begin{cases} \min\{\rho_n, \frac{\mu(\Delta_p(u_n,w_n)+\Delta_p(v_n,u_n))}{\langle Aw_n - Au_n, v_n - u_n\rangle}\} \\ \textit{if } \ \langle Aw_n - Au_n, v_n - u_n\rangle > 0 \\ \rho_n, \quad \textit{otherwise,} \end{cases}$

and θ_n *is defined by* (3.1).

3.2.3 Numerical experiments

In this section, we provide some numerical experiments to show the convergence rate/performance of our proposed iterative method, Algorithm 3.2.2 and compare it with some related methods in the literature.

In the numerical experiments, for our proposed Algorithm 3.2.2, we consider the case for which $m = 5$ and choose $\mu = 0.9$, $\gamma_n = \frac{1}{(2n+1)}$, $\eta_{n,0} = \frac{n}{2n+1}, \eta_{n,j} = \frac{n+1}{5(2n+1)}$, $j = 1, 2, \ldots, 5$. In Appendix 9.1.1, we choose $\lambda_n = 0.3$, $\beta_n = \frac{n}{2n+1}$. In Appendix 9.1.2, we choose $\xi = 0.1$, $v = 0.4$ and in Appendix 9.1.3, we take $N = 1$. In Tables 3.1.17, 3.1.18 and 3.1.19, Iter. means the number of iterations while CPU means the CPU time in seconds.

Firstly, we consider the following two examples in finite dimensional spaces.

Example 3.2.12. *Let the bifunction* f *be defined by*

$$f(x, y) = [F(x)]^T (y - x),$$

where T *is the transpose of* $F(x)$ *and* $F(x) = Mx + P(x)$ *with* M *an* $p \times p$ *symmetric semidefinite matrix and* P *is defined by:*

$$P(x) = \arg\min\left[\frac{\|y\|^4}{4} + \frac{1}{2}\|y - x\|^2 : y \in \mathbb{R}^p\right].$$

We let $C = \{x \in \mathbb{R}^p : Ax \le b\}$, *where* $A \in \mathbb{R}^{q\times p}$ *and* $b \in \mathbb{R}^q$ *with* $q = 10$. *The bifunction* f *is pseudo-monotone and satisfies conditions (A1)-(A4) of Assumption A. In addition, for* $j = 1, 2, ...m$, *let* $D_j : \mathbb{R}^p \to \mathbb{R}^p$ *be defined by* $D_j(x) = \frac{x}{3j} = T_j(x)$. *It is easy to verify that* D_j *is Bregman quasi-nonexpansive for each* $j = 1, 2, ..., m$. *We choose* $u = x_1 = (1, 1, ..., 1)$, $x_0 = 2u \in C$ *and different cases of* p.

Example 3.2.13. *Consider the Nash-Cournot oligopolistic equilibrium model [186]. Let the bifunction $f \in \mathbb{R}^m$ be defined as*

$$f(x,y) = \langle Px + Qy + r, y - x\rangle,$$

where r is a vector in $\mathbb{R}^p$ and P, Q are two matrices of order p such that Q is symmetric positive semidefinite and $Q - P$ is symmetric negative semidefinite. We define the set C such that $C = \{x \in \mathbb{R}^p : Ax \leq b\}$, where $A \in \mathbb{R}^{q\times p}$ is a random matrix and $b \in \mathbb{R}^q$ with $q = 10$. It is easy to see that f is pseudo-monotone and satisfies conditions (A1)-(A4) of **Assumption 3.2.1** *with the Lipschitz constants $c_1 = c_2 = \frac{\|P-Q\|}{2}$.*
For the optimization program in Algorithm 3.2.2, we have the following steps:

$$u_n = \arg\min_{z\in C}\{\frac{1}{2}z^T H_n z + b_n^T z\},$$

where $H_n = 2\rho_n Q + I$ and $b_n = \rho_n\left[(P-Q)w_n + r\right] - w_n$;
and

$$v_n = \arg\min_{z\in\zeta_n}\left[\frac{1}{2}z^T \bar{H}_n z + \bar{b}_n^T z\right], \tag{3.55}$$

where $\bar{H}_n = 2\rho_n Q + I$ and $\bar{b}_n = \rho_n\left[(P-Q)w_n + r\right] - w_n$. Also, since $\varphi_n \in \partial_2 f(w_n, u_n)$, we see that $\varphi_n = 2Qu_n + (P-Q)w_n + r$. Furthermore, letting $a_n = (I - \rho_n(P-Q))w_n - (2\rho_n Q + I)u_n - \rho_n r$, we get that $\zeta_n = \{x \in \mathbb{R}^p : \langle a_n, x\rangle \leq \langle a_n, u_n\rangle\}$.
Equation (3.55) is a quadratic convex program which can be solved efficiently using Matlab Optimization Toolbox. In addition, for $j = 1, 2, ...m$, let $D_j : \mathbb{R}^p \to \mathbb{R}^p$ by $D_j(x) = \frac{x}{5j} = T_j$. It is easy to verify that D_j is Bregman quasi-nonexpansive for each $j = 1, 2, ..., m$. Also, r, P and Q are selected randomly and $u = x_0 = x_1 = (1, 1, ..., 1) \in C$ with different cases of p.

We next consider the following example in infinite dimensional space.

Example 3.2.14. *Let $E = \ell_2(\mathbb{R})$ be the linear spaces whose elements are all 2-summable sequences $\{x_i\}_{i=1}^\infty$*

$$\ell_2 = \{x : x = (x_1, x_2, ..., x_i, ...), \quad x_i \in \mathbb{R} \quad \text{and} \quad \sum_{i=1}^{\infty}|x_i|^2 < \infty\},$$

with inner product $\langle ., .\rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ and norm $\|.\| : \ell_2 \to \mathbb{R}$ defined by

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i \quad \text{and} \quad \|x\| = \left(\sum_{i=1}^{\infty}|x_i|^2\right)^2, \quad \text{where} \quad x = \{x_i\}_{i=1}^\infty, \quad y = \{y_i\}_{i=1}^\infty \in \ell_2.$$

Let $C = \{x \in E : \|x\| \leq 1\}$. Define the bifuntion $f : C \times C \to \mathbb{R}$ by

$$f(x,y) = (3 - \|x\|)\langle x, y - x\rangle, \quad \forall\ x, y \in C.$$

It can easily be verified that f is pseudo-monotone and satisfies conditions (A1)-(A4) of **Assumption A** *with Lipschitz constant $c_1 = c_2 = \frac{5}{2}$. Also, for $j = 1, 2, ...m$, we define $D_j : \ell_2 \to \ell_2$ by $D_j(x) = \frac{x}{2j} = T_j(x)$. It is easy to verify that D_j is Bregman quasi-nonexpansive for each $j = 1, 2, ..., m$.*

We test Example 3.2.12, Example 3.2.13 and Example 3.2.14 under the following experiments:

Experiment 3.2.15. *In this experiment, we check the behavior of our method by fixing the other parameters and varying θ in Example 3.2.12. We do this to check the effects of this parameter and the sensitivity of our method on it.*

We choose $q_n = \frac{n}{2n+1}u$, $g_n = \frac{n}{2n+1}u$ and $p \in \{5, 10, 15, 20\}$. Using $\|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion, we plot the graphs of $\|x_{n+1} - x_n\|$ against the number of iterations in each case. The numerical results are reported in Fig. 3.1 and Table 3.1.17.

Experiment 3.2.16. *In this experiment, we check the behavior of our method by fixing the other parameters and varying ϵ_n in Example 3.2.13 We do this to check the effects of this parameter and the sensitivity of our method on it.*
We consider $\epsilon_n \in \{\frac{1}{(2n+1)^3}, \frac{2}{(2n+5)^3}, \frac{1}{(2n+1)^4}, \frac{2}{(2n+5)^4}, \frac{1}{(2n+1)^5}\}$ which satisfies Assumption 3.2.1 (5)(C2). We choose $q_n = \frac{n}{2n+1}u$, $g_n = \frac{n}{2n+1}u$ and $p \in \{5, 10, 20, 30\}$. Using $\|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion, we plot the graphs of $\|x_{n+1} - x_n\|$ against the number of iterations in each case. The numerical results are reported in Fig. 3.2 and Table 3.1.18.

Experiment 3.2.17. *In this experiment, we check the behavior of our method by fixing the other parameters and varying ρ in Example 3.2.14. We do this to check the effects of this parameter and the sensitivity of our method on it. We consider $\rho \in \{0.5, 1.0, 1.5, 2.0, 2.5\}$. Using $\|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion, we plot the graphs of $\|x_{n+1} - x_n\|$ against the number of iterations in each case. We choose $u = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$, $q_n = \frac{n}{2n+1}u$, $g_n = \frac{n}{2n+1}u$ and consider different cases of initial values x_0 and x_1 as follows:*

Case 1 : $x_0 = (\frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \cdots)$, $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.

Case 2 : $x_0 = (2, 1, \frac{1}{2}, \cdots)$, $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.

Case 3 : $x_0 = (\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \cdots)$, $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.

Case 4 : $x_0 = (\frac{7}{9}, \frac{7}{18}, \frac{7}{36}, \cdots)$, $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.

The numerical results are reported in Fig. 3.3 and Table 3.1.19.

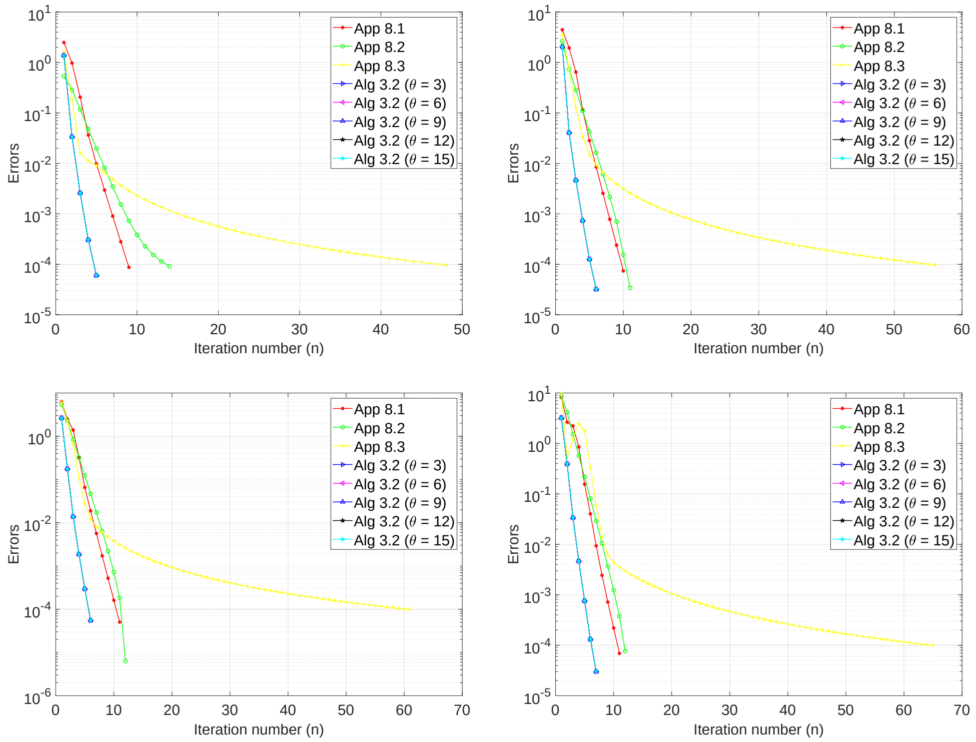


Figure 3.1: Top left: p = 5; Top right: p = 10; Bottom left: p = 15; Bottom right: p = 20.

Table 3.1.17. *Numerical results for Example 3.2.12* (**Experiment 3.2.15**).

Cases		*App. 9.1.1*	*App. 9.1.2*	*App. 9.1.3*	*Alg. 3.2.2* ($\theta = 3$)	*Alg. 3.2.2* ($\theta = 6$)	*Alg. 3.2.2* ($\theta = 9$)
p=5	*CPU*	*0.0819*	*0.0478*	*0.3300*	*0.0356*	*0.0318*	*0.0323*
	Iter.	*9*	*14*	*48*	*5*	*5*	*5*
	CPU	*0.0870*	*0.0462*	*0.4494*	*0.0501*	*0.0481*	*0.0603*
p=10	*Iter.*	*10*	*11*	*56*	*6*	*6*	*6*
	CPU	*0.1107*	*0.0513*	*0.5404*	*0.0574*	*0.0541*	*0.0527*
p=15	*Iter.*	*11*	*12*	*61*	*6*	*6*	*6*
	CPU	*0.1138*	*0.0608*	*0.4254*	*0.0723*	*0.0554*	*0.0575*
p=20	*Iter.*	*11*	*12*	*65*	*7*	*7*	*7*

Table 3.1.18. *Numerical results for Example 3.2.13* (**Experiment 3.2.16**).

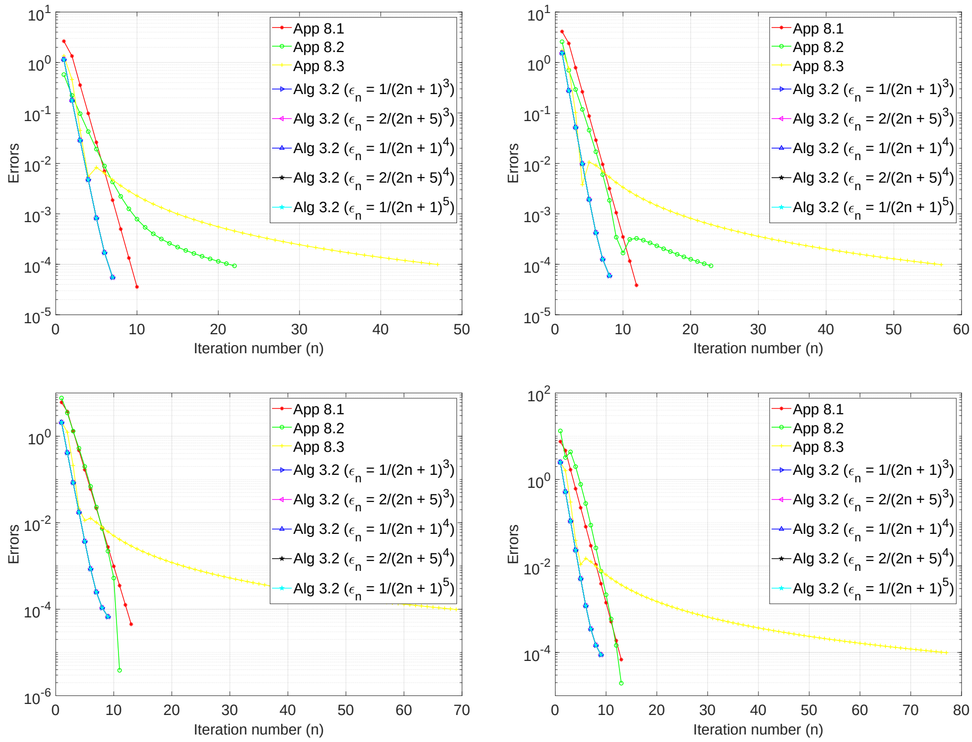


Figure 3.2: Top left: p = 5; Top right: p = 10; Bottom left: p = 20; Bottom right: p = 30.

Cases		*App.* *9.1.1*	*App.* *9.1.2*	*App.* *9.1.3*	*Alg.* *3.2.2* $(\epsilon_n = \frac{1}{(2n+1)^3})$	*Alg.* *3.2.2* $(\epsilon_n = \frac{2}{(2n+5)^3})$	*Alg.* *3.2.2* $(\epsilon_n = \frac{1}{(2n+1)^4})$
p=5	*CPU*	*0.1510*	*0.1280*	*0.6744*	*0.0838*	*0.0852*	*0.0930*
	Iter.	*10*	*22*	*47*	*7*	*7*	*7*
	CPU	*0.0949*	*0.0786*	*0.4007*	*0.0781*	*0.0620*	*0.0580*
p=10	*Iter.*	*12*	*23*	*57*	*8*	*8*	*8*
	CPU	*0.1115*	*0.0480*	*0.4606*	*0.0782*	*0.0848*	*0.0754*
p=20	*Iter.*	*13*	*11*	*69*	*9*	*9*	*9*
	CPU	*0.0892*	*0.0803*	*0.5478*	*0.0551*	*0.0723*	*0.0804*
p=30	*Iter.*	*13*	*13*	*77*	*9*	*9*	*9*

Table 3.1.19 *Numerical results for Example 3.2.14 (***Experiment 3.2.17***).*

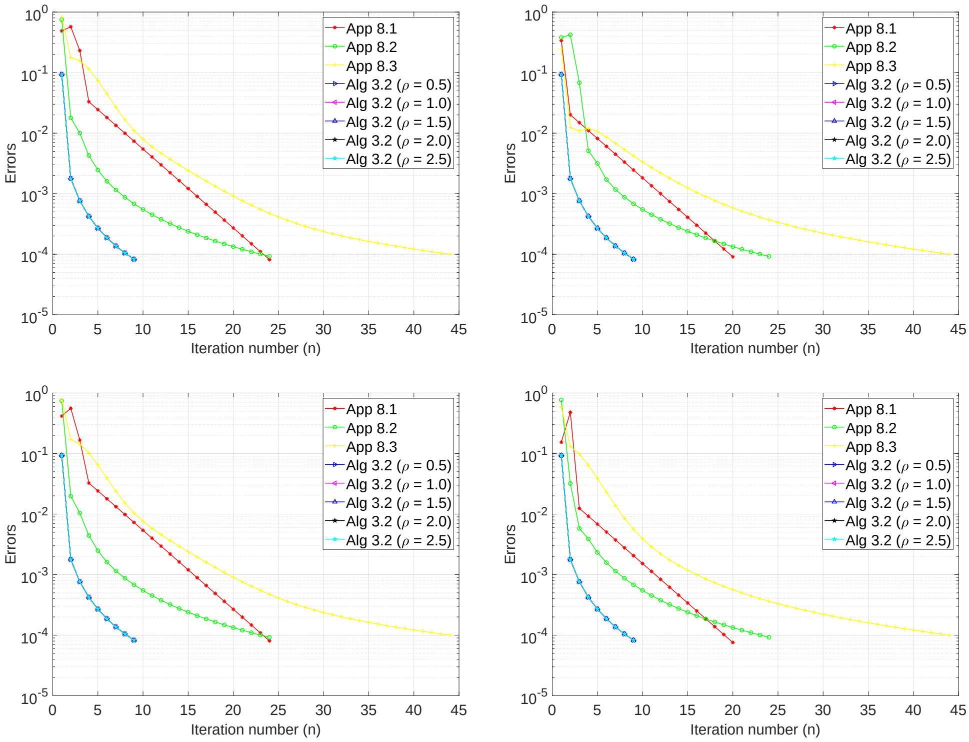


Figure 3.3: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

Cases		*App.* 9.1.1	*App.* 9.1.2	*App.* 9.1.3	*Alg.* 3.2.2 ($\rho = 0.5$)	*Alg.* 3.2.2 ($\rho = 1.0$)	*Alg.* 3.2.2 ($\rho = 1.5$)
1	*CPU*	*0.0054*	*0.0047*	*0.0044*	*0.0152*	*0.0113*	*0.0113*
	Iter.	*24*	*24*	*44*	*9*	*9*	*9*
2	*CPU*	*0.0037*	*0.0067*	*0.0041*	*0.0115*	*0.0098*	*0.0099*
	Iter.	*20*	*24*	*44*	*9*	*9*	*9*
3	*CPU*	*0.0008*	*0.0018*	*0.0012*	*0.0029*	*0.00251*	*0.0031*
	Iter.	*24*	*24*	*44*	*9*	*9*	*9*
4	*CPU*	*0.0026*	*0.0036*	*0.0027*	*0.0027*	*0.0031*	*0.0044*
	Iter.	*20*	*24*	*44*	*9*	*9*	*9*

Remark 3.2.18. *We use distinct starting points, values of p and vary the other parameters in Experiment 3.2.15, Experiment 3.2.16 and Experiment 3.2.17 respectively, we get the numerical results shown in Tables 3.1.17-3.1.19 and Figures 3.1-3.3. We compared our method, Algorithm 3.2.2 with the methods in Appendix 9.1.1, Appendix 9.1.2 and Appendix 9.1.3.*
In addition, we observe the following from our numerical tests and experiments:

- *In the numerical experiments, we randomly selected the parameters and noted that regardless of the choices made, the number of iteration does not change and no significant difference in the CPU time.*

- *In Experiment 3.2.15, we examine the sensitivity of θ for each case of p in order to know if the choices of θ affect the performance of our method. Clearly, from Table 3.1.17 and Fig. 3.1, the number of iterations for our method is well behaved for $\theta \in \{3.0, 6.0, 9.0, 12.0, 15.0\}$. In addition, there is no significant difference in the CPU time as we vary the value of θ.*

- *In Experiment 3.2.16, we examine the sensitivity of ϵ_n for each case of p in order to know if the choices of ϵ_n affect the performance of our method. Clearly, from Table 3.1.18 and Fig. 3.2, the number of iterations for our method is well behaved for $\epsilon_n \in \{\frac{1}{(2n+1)^3}, \frac{2}{(2n+5)^3}, \frac{1}{(2n+1)^4}, \frac{2}{(2n+5)^4}, \frac{1}{(2n+1)^5}\}$. Moreover, there is no significant difference in the CPU time as we vary the value of $\beta_{n,i}$.*

- *In Experiment 3.2.17, we examine the sensitivity of ρ for each starting points in order to know if the choices of ρ affect the performance of our method. Clearly, from Table 3.1.19 and Fig. 3.3, the number of iterations for our method is well behaved for $\rho \in \{0.5, 1.0, 1.5, 2.0, 2.5\}$. In addition, there is no significant difference in the CPU time as we vary the value of θ.*

- *From Tables 3.1.17-3.1.19, Fig. 3.1, Fig. 3.2 and Fig. 3.3, we noted clearly that in terms of number of iterations, our method, Algorithm 3.2.2 performs better than the existing methods in Appendix 9.1.1, Appendix 9.1.2 and Appendix 9.1.3 and no significant difference in the CPU time.*

3.3 Split generalized equilibrium problem with multiple output sets.

In this section, we introduce and study the notion of Split Generalized Equilibrium Problem with Multiple Output Sets (SGEPMOS). We propose a new iterative method which employs viscosity approximation technique for approximating the common solution of the SGEPMOS and common fixed point problem for an infinite family of multivalued demicontractive mappings in real Hilbert spaces. Using our algorithm, we state and prove a strong convergence result of our iteration sequences. Our method uses self-adaptive step size which does not require prior knowledge of the operator norm. An application to Split Variational Inequality with Multiple Output Sets (SVIPWMOS) was considered.

The following results are needed:

Lemma 3.3.1. *[249] Let H be a Hilbert space. Let $x, y, z \in H$ and $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha + \beta + \gamma = 1$. Then, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Assumption 3.3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ be two bifunctions satisfy the following conditions:*

(B1) $F(x, x) \geq 0$ for all $x \in C$;

(B2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0 \ \ \forall x, y \in C$;

(B3) F is upper hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to \infty} F(tz + (1 - t)x, y) \leq F(x, y);$$

(B4) For each $x \in C$ fixed, the function $y \mapsto F(x, y)$ is convex and lower semicontinuous;

(B5) $\phi(x, x) \geq 0$ for all $x \in C$;

(B6) For each $y \in C$ fixed, the function $x \to \phi(x, y)$ is upper semicontinuous;

(B7) For each $x \in C$ fixed, the function $y \to \phi(x, y)$ is convex and lower semicontinuous,

and assume that for fixed $r > 0$ and $z \in C$,. there exits a nonempty compact convex subset K of H and $x \in C \cap K$ such that

$$F(y, x) + \phi(y, x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \ \forall \ y \in C \backslash K.$$

Lemma 3.3.3. *[154] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ be bifunctions satisfying the assumptions*

B1-B7 and ϕ is monotone. For $r > 0$ and for all $x \in H$, define a mapping $T_r^{(F,\phi)} : H \to C$ as follows:

$$T_r^{(F,\phi)}x = \{x^* \in C : F(x^*, y) + \phi(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - x\rangle \geq 0, \quad \forall y \in C\}.$$

Then, the following conclusions hold:

(i) $T_r^{(F,\phi)}$ is single-valued;

(ii) $T_r^{(F,\phi)}$ is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r^{(F,\phi)}x - T_r^{(F,\phi)}y\|^2 \leq \langle T_r^{(F,\phi)}x - T_r^{(F,\phi)}y, x - y\rangle; \tag{3.56}$$

(iii) $Fix(T_r^{(F,\phi)}) = GEP(F, \phi)$;

(iv) $GEP(F, \phi)$ is compact and convex.

Lemma 3.3.4. *[240] Let H be a real Hilbert space. A mapping $T : H \to H$ is firmly nonexpansive if and only if its complement $I - T$ is firmly nonexpansive.*

3.3.1 Main result

In this section, we present a modified viscosity-type algorithm for approximating a common element of the set of solution of the split generalized equilibrium problem with multiple output sets and the common fixed point problem for an infinite family of multivalued demicontractive mappings in real Hilbert spaces.

Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, ..., N$, let C_i be nonempty closed convex subset of Hilbert spaces H_i and let $A_i : H \to H_i$ be bounded linear operators. Let $F, \phi : C \times C \to \mathbb{R}, F_i, \phi_i : C_i \times C_i \to \mathbb{R}$ be bifunctions satisfying assumptions (B1)-(B7) in Assumption 3.3.2 and for each $j \in \mathbb{N}$, let $S_j : H \to CB(H)$ be a family of multivalued demicontractive mappings with constant $k_j \in (0, 1)$ such that $I - S_j$ is demiclosed at zero and $S_j(p) = \{p\}$ for each $j \in \mathbb{N}, p \in \cap_{j=1}^{\infty}\mathrm{Fix}(S_j)$. Suppose the solution set denoted by

$$\Gamma := \bigcap_{j=1}^{\infty} \mathrm{Fix}(S_j) \cap GEP(F, \phi) \cap \big(\bigcap_{i=1}^{N} A_i^{-1}(GEP(F_i, \phi_i))\big) \neq \emptyset. \tag{3.57}$$

Let $g : H \to H$ be a ρ-contraction with constant $\rho \in (0, 1)$. Let $\{\alpha_n\}, \{\delta_n\}, \{\mu_n\}, \{\gamma_{n,j}\}, j \in \mathbb{N}$, be sequences in $(0, 1)$, and $\{\phi_{n,i}\}$ is a sequence of positive real numbers for each $i = 0, 1, 2, \ldots, N$ and $n \geq 0$. Let $\{x_n\}$ be a sequence generated as follows:

Algorithm 3.3.5.

Step 0: *For any $x_0 \in H$, let $H_0 = H, T_0 = I^H, F_0 = F, \phi_0 = \phi$ and set $n = 0$.*

Step 1: *Compute*

$$v_n = \sum_{i=0}^{N} \beta_{i,n} \Big[x_n - \tau_{i,n} A_i^* \big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)} \big) A_i x_n \Big]. \tag{3.58}$$

Step 2: *Compute*

$$y_n = \gamma_{n,0} v_n + \sum_{j=1}^{\infty} \gamma_{n,j} z_n^j, \tag{3.59}$$

where $z_n^j \in S_j v_n, \quad j = 1, 2, ...$

Step 3: *Compute*

$$x_{n+1} = \alpha_n g(x_n) + \delta_n x_n + \mu_n y_n, \quad n \in \mathbb{N}. \tag{3.60}$$

Update:

$$\tau_{i,n} = \theta_{i,n} \frac{\| \big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)} \big) A_i x_n \|^2}{\| A_i^* \big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)} \big) A_i x_n \|^2 + \phi_{i,n}}. \tag{3.61}$$

Set $n = n + 1$ *and go to* **Step 1**.

The following assumptions are needed in order to establish the strong convergence result for Algorithm 3.3.5:

(A1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\alpha_n + \delta_n + \mu_n = 1, \mu_n \subset [a, b] \subset (0, 1)$;

(A2) $\sum_{j=0}^{\infty} \gamma_{n,j} = 1, \liminf_{n\to\infty} \gamma_{n,j}(\gamma_{n,0} - k) > 0$ for each $j = 1, 2, \ldots,$ where $k := \sup_{j\geq 1}\{k_j\} < 1$;

(A3) $\{\beta_{i,n}\} \subset [c, d] \subset (0, 1)$ such that $\sum_{i=0}^{N} \beta_{i,n} = 1, \{\theta_{i,n}\} \subset [e, f] \subset (0, 2)$;

(A4) $r_i > 0$ for each $i = 1, 2, \ldots, N, \max_{i=0,1,\ldots,N}\{\sup_n\{\phi_{i,n}\}\} = K < \infty$.

Lemma 3.3.6. *The sequence* $\{x_n\}$ *generated by Algorithm 3.3.5 is bounded.*

Proof. Let $p \in \Gamma$, we get $A_i p = T_{r_i}^{(F_i,\phi_i)} A_i p$, for each $i = 0, 1, ..., N$. Thus, by the convexity of the function $\| \cdot \|^2$ we obtain

$$\begin{aligned} \|v_n - p\|^2 &= \| \sum_{i=0}^{N} \beta_{i,n} \Big(x_n - \tau_{i,n} A_i^* \big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)} \big) A_i x_n \Big) - p\|^2 \\ &\leq \sum_{i=0}^{N} \beta_{i,n} \| x_n - \tau_{i,n} A_i^* \big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)} \big) A_i x_n - p\|^2. \end{aligned} \tag{3.62}$$

Applying the firmly nonexpansiveness of $I^{H_i} - T_{r_i}^{(F_i,\phi_i)}$ for each $i = 0, 1, ..., N$, we obtain

$$\begin{aligned}
&\|x_n - \tau_{i,n} A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n - p\|^2 \\
&= \|x_n - p\|^2 + \tau_{i,n}^2 \|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^2 \\
&- 2\tau_{i,n}\langle A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n, x_n - p\rangle \\
&= \|x_n - p\|^2 + \tau_{i,n}^2 \|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^2 \\
&- 2\tau_{i,n}\langle \big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n, A_i x_n - A_i p\rangle \\
&= \|x_n - p\|^2 - 2\tau_{i,n}\langle \big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n - \big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i p, A_i x_n - A_i p\rangle \\
&+ \tau_{i,n}^2 \|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^2 \\
&\leq \|x_n - p\|^2 - 2\tau_{i,n}\|\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^2 \\
&+ \tau_{i,n}^2 \big(\|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^2 + \phi_{i,n}\big) \\
&= \|x_n - p\|^2 - \theta_{i,n}(2 - \theta_{i,n})\frac{\|\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^4}{\|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^2 + \phi_{i,n}},
\end{aligned} \tag{3.63}$$

which follows from (3.62) and (3.63) that

$$\|v_n - p\|^2 \leq \|x_n - p\|^2 - \sum_{i=0}^{N} \beta_{i,n}\theta_{i,n}(2 - \theta_{i,n})\frac{\|\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^4}{\|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^2 + \phi_{i,n}}. \tag{3.64}$$

Now, from Algorithm 3.3.5, Lemma 2.5.12 and the fact that S_j is demicontractive for each $j \in \mathbb{N}$ we have that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\gamma_{n,0} v_n + \sum_{j=1}^{\infty} \gamma_{n,j} z_n^j - p\|^2 \\
&= \|\gamma_{n,0}(v_n - p) + \sum_{j=1}^{\infty} \gamma_{n,j}(z_n^j - p)\|^2 \\
&= \gamma_{n,0}\|v_n - p\|^2 + \sum_{j=1}^{\infty} \gamma_{n,j}\|z_n^j - p\|^2 - \sum_{j=1}^{\infty} \gamma_{n,0}\gamma_{n,j}\|v_n - z_n^j\|^2 \\
&\leq \gamma_{n,0}\|v_n - p\|^2 + \sum_{j=1}^{\infty} \gamma_{n,j}\mathcal{H}^2(S_j v_n, S_j p) - \sum_{j=1}^{\infty} \gamma_{n,0}\gamma_{n,j}\|v_n - z_n^j\|^2 \\
&\leq \gamma_{n,0}\|v_n - p\|^2 + \sum_{j=1}^{\infty} \gamma_{n,j}[\|v_n - p\|^2 + k\text{dist}(v_n, S_j v_n)^2] - \sum_{j=1}^{\infty} \gamma_{n,0}\gamma_{n,j}\|v_n - z_n^j\|^2 \\
&\leq \gamma_{n,0}\|v_n - p\|^2 + \sum_{j=1}^{\infty} \gamma_{n,j}[\|v_n - p\|^2 + k\|v_n - z_n^j\|^2] - \sum_{j=1}^{\infty} \gamma_{n,0}\gamma_{n,j}\|v_n - z_n^j\|^2 \\
&= \|v_n - p\|^2 - \sum_{j=1}^{\infty} \gamma_{n,j}(\gamma_{n,0} - k)\|v_n - z_n^j\|^2.
\end{aligned}$$

It follows from (3.64) that

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \sum_{j=1}^{\infty} \gamma_{n,j}(\gamma_{n,0} - k)\|v_n - z_n^j\|^2$$
$$- \sum_{i=0}^{N} \beta_{i,n}\theta_{i,n}(2 - \theta_{i,n}) \frac{\|\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^4}{\|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_n\|^2 + \phi_{i,n}}, \tag{3.65}$$

and it follows from the conditions (A2)-(A4), that

$$\|y_n - p\|^2 \le \|x_n - p\|^2. \tag{3.66}$$

Further, by applying (3.66) we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n g(x_n) + \delta_n x_n + \mu_n y_n - p\| \\
&\le \alpha_n \|g(x_n) - p\| + \delta_n \|x_n - p\| + \mu_n \|y_n - p\| \\
&\le \alpha_n \big(\|g(x_n) - g(p)\| + \|g(p) - p\|\big) + \delta_n \|x_n - p\| + \mu_n \|y_n - p\| \\
&\le \alpha_n \big(\rho\|x_n - p\| + \|g(p) - p\|\big) + \delta_n \|x_n - p\| + \mu_n \|x_n - p\| \\
&= (\rho\alpha_n + \delta_n + \mu_n)\|x_n - p\| + \alpha_n \|g(p) - p\| \\
&= \big(1 - \alpha_n(1 - \rho)\big)\|x_n - p\| + \frac{\alpha_n(1-\rho)\|g(p) - p\|}{1 - \rho} \\
&\le \max\Big\{\|x_n - p\|, \frac{\|g(p) - p\|}{1 - \rho}\Big\}.
\end{aligned}$$

It therefore follows by induction that

$$\|x_{n+1} - p\| \le \max\Big\{\|x_1 - p\|, \frac{\|g(p) - p\|}{1 - \rho}\Big\}.$$

Hence, $\{x_n\}$ is bounded and consequently, $\{v_n\}$, $\{y_n\}$ and $\{z_n^j\}$ are all bounded. $\square$

It is easy to see that the operator $P_\Gamma \circ g$ is a contraction. Thus, by the Banach Contraction Principle, there exists a unique point $x^* \in \Gamma$ such that $x^* = P_\Gamma \circ g(x^*)$. It follows from the characterization of the projection mapping that

$$\langle g(x^*) - x^*, x - x^* \rangle \le 0, \quad \forall\, x \in \Gamma. \tag{3.67}$$

Lemma 3.3.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3.5 and let $p \in \Gamma$. Then, under conditions (A1)-(A4) and Assumption 3.3.2 the following inequality holds for all $n \in \mathbb{N}$:*

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le \left(1-\frac{2\alpha_n[1-\rho]}{(1-\alpha_n\rho)}\right)\|x_n-p\|^2\\
&+\frac{2\alpha_n(1-\rho)}{(1-\alpha\rho)}\left[\frac{\alpha_n M_1}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle g(p)-p, x_{n+1}-p\rangle\right]\\
&-\frac{\mu_n(1-\alpha_n)}{(1-\alpha_n\rho)}\left[\sum_{j=1}^{\infty}\gamma_{n,j}(\gamma_{n,0}-k)\|v_n-z_n^j\|^2\right.\\
&\left.+\sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^4}{\|A_i^*\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^2+\phi_{i,n}}\right]
\end{aligned}
$$

Proof. Let $p \in \Gamma$. By applying Lemma 2.5.18(iii) and (3.65), we have

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 = \|\alpha_n g(x_n)+\delta_n x_n+\mu_n y_n-p\|^2\\
&\le \|\delta_n(x_n-p)+\mu_n(y_n-p)\|^2+2\alpha_n\langle g(x_n)-p, x_{n+1}-p\rangle\\
&\le \delta_n^2\|x_n-p\|^2+\mu_n^2\|y_n-p\|^2+2\delta_n\mu_n\|x_n-p\|\|y_n-p\|+2\alpha_n\langle g(x_n)-p, x_{n+1}-p\rangle\\
&\le \delta_n^2\|x_n-p\|^2+\mu_n^2\|y_n-p\|^2+\delta_n\mu_n\left(\|x_n-p\|^2+\|y_n-p\|^2\right)+2\alpha_n\langle g(x_n)-g(p), x_{n+1}-p\rangle\\
&+2\alpha_n\langle g(p)-p, x_{n+1}-p\rangle\\
&= \delta_n\left(\delta_n+\mu_n\right)\|x_n-p\|^2+\mu_n(\mu_n+\delta_n)\|y_n-p\|^2+2\alpha_n\langle g(x_n)-g(p), x_{n+1}-p\rangle\\
&+2\alpha_n\langle g(p)-p, x_{n+1}-p\rangle\\
&\le \delta_n(1-\alpha_n)\|x_n-p\|^2+\mu_n(1-\alpha_n)\left[\|x_n-p\|^2-\sum_{j=1}^{\infty}\gamma_{n,j}(\gamma_{n,0}-k)\|v_n-z_n^j\|^2\right.\\
&\left.-\sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^4}{\|A_i^*\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^2+\phi_{i,n}}\right]\\
&+2\alpha_n\rho\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n\langle g(p)-p, x_{n+1}-p\rangle\\
&\le (1-\alpha_n)^2\|x_n-p\|^2-\mu_n(1-\alpha_n)\left[\sum_{j=1}^{\infty}\gamma_{n,j}(\gamma_{n,0}-k)\|v_n-z_n^j\|^2\right.\\
&\left.+\sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^4}{\|A_i^*\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^2+\phi_{i,n}}\right]+\alpha_n\rho\left[\|x_n-p\|^2+\|x_{n+1}-p\|^2\right]\\
&+2\alpha_n\langle g(p)-p, x_{n+1}-p\rangle\\
&= \left((1-\alpha_n)^2+\alpha_n\rho\right)\|x_n-p\|^2+\alpha_n\rho\|x_{n+1}-p\|^2+2\alpha_n\langle g(p)-p, x_{n+1}-p\rangle\\
&-\mu_n(1-\alpha_n)\left[\sum_{j=1}^{\infty}\gamma_{n,j}(\gamma_{n,0}-k)\|v_n-z_n^j\|^2\right.\\
&\left.+\sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^4}{\|A_i^*\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^2+\phi_{i,n}}\right].
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \frac{(1-2\alpha_n+\alpha_n^2+\alpha_n\rho)}{(1-\alpha_n\rho)}\|x_n-p\|^2+\frac{2\alpha_n}{(1-\alpha_n\rho)}\langle g(p)-p, x_{n+1}-p\rangle \\
&-\frac{\mu_n(1-\alpha_n)}{(1-\alpha_n\rho)}\Bigg[\sum_{j=1}^{\infty}\gamma_{n,j}(\gamma_{n,0}-k)\|v_n-z_n^j\|^2 \\
&+\sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^4}{\|A_i^*\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^2+\phi_{i,n}}\Bigg] \\
&=\frac{(1-2\alpha_n+\alpha_n\rho)}{(1-\alpha_n\rho)}\|x_n-p\|^2+\frac{\alpha_n^2}{(1-\alpha_n\rho)}\|x_n-p\|^2+\frac{2\alpha_n}{(1-\alpha_n\rho)}\langle g(p)-p, x_{n+1}-p\rangle \\
&-\frac{\mu_n(1-\alpha_n)}{(1-\alpha_n\rho)}\Bigg[\sum_{j=1}^{\infty}\gamma_{n,j}(\gamma_{n,0}-k)\|v_n-z_n^j\|^2 \\
&+\sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^4}{\|A_i^*\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^2+\phi_{i,n}}\Bigg] \\
&\leq\left(1-\frac{2\alpha_n(1-\rho)}{(1-\alpha_n\rho)}\right)\|x_n-p\|^2+\frac{2\alpha_n(1-\rho)}{(1-\alpha\rho)}\left[\frac{\alpha_nM_1}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle g(p)-p, x_{n+1}-p\rangle\right] \\
&-\frac{\mu_n(1-\alpha_n)}{(1-\alpha_n\rho)}\Bigg[\sum_{j=1}^{\infty}\gamma_{n,j}(\gamma_{n,0}-k)\|v_n-z_n^j\|^2 \\
&+\sum_{i=0}^{N}\beta_{i,n}\theta_{i,n}(2-\theta_{i,n})\frac{\|\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^4}{\|A_i^*\left(I^{H_i}-T_{r_i}^{(F_i,\phi_i)}\right)A_ix_n\|^2+\phi_{i,n}}\Bigg],
\end{aligned}
$$

where $M_1 := \sup\{\|x_n-p\|^2 : n\in\mathbb{N}\}$. Hence, the proof is complete. □

Theorem 3.3.8. *Suppose that conditions (A1)-(A4) and Assumption 3.3.2 hold. Then, the sequence $\{x_n\}$ generated by Algorithm 3.3.5 converges strongly to $\hat{x}\in\Gamma$, where $\hat{x} = P_\Gamma\circ g(\hat{x})$.*

Proof. Let $\hat{x} = P_\Omega\circ g(\hat{x})$. From Lemma 3.3.7, we obtain

$$
\begin{aligned}
\|x_{n+1}-\hat{x}\|^2 \leq &\left(1-\frac{2\alpha_n[1-\rho]}{(1-\alpha_n\rho)}\right)\|x_n-\hat{x}\|^2 \\
&+\frac{2\alpha_n(1-\rho)}{(1-\alpha\rho)}\left[\frac{\alpha_nM_1}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle g(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle\right] \qquad (3.68)
\end{aligned}
$$

Next, we show that the sequence $\{\|x_n-\hat{x}\|\}$ converges to zero. In order to establish this, by Lemma 2.5.55, it is enough to show that $\limsup_{k\to\infty}\langle g(\hat{x})-\hat{x}, x_{n_k+1}-\hat{x}\rangle\leq 0$ for every subsequence $\{\|x_{n_k}-\hat{x}\|\}$ of $\{\|x_n-\hat{x}\|\}$ satisfying

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-\hat{x}\|-\|x_{n_k}-\hat{x}\|\right)\geq 0. \qquad (3.69)$$

Now, suppose that $\{\|x_{n_k} - \hat{x}\|\}$ is a subsequence of $\{\|x_n - \hat{x}\|\}$ such that (3.69) holds. Then,

$$\begin{aligned}\liminf_{k\to\infty}\left(\|x_{n_k+1} - \hat{x}\|^2 - \|x_{n_k} - \hat{x}\|^2\right) &= \liminf_{k\to\infty}\Big[(\|x_{n_k+1} - \hat{x}\| - \|x_{n_k} - \hat{x}\|)\\ &\quad (\|x_{n_k+1} - \hat{x}\| + \|x_{n_k} - \hat{x}\|)\Big]\\ &\geq 0. \end{aligned} \tag{3.70}$$

Again, from Lemma 3.3.7 we have

$$\begin{aligned}\frac{\mu_{n_k}(1-\alpha_{n_k})}{(1-\alpha_{n_k}\rho)}\sum_{j=1}^{\infty}\gamma_{n_k,j}(\gamma_{n_k,0}-k)\|v_{n_k} - z_{n_k}^j\|^2 &\leq \left(1 - \frac{2\alpha_{n_k}[1-\rho]}{(1-\alpha_{n_k}\rho)}\right)\|x_{n_k} - \hat{x}\|^2\\ &- \|x_{n_k+1} - \hat{x}\|^2 + \frac{2\alpha_n(1-\rho)}{(1-\alpha\rho)}\left[\frac{\alpha_{n_k}M_1}{2(1-\rho)} + \frac{1}{(1-\rho)}\langle g(\hat{x}) - \hat{x}, x_{n_k+1} - \hat{x}\rangle\right].\end{aligned}$$

By applying (3.70) together with condition (A1), we obtain

$$\frac{\mu_{n_k}(1-\alpha_{n_k})}{(1-\alpha_{n_k}\rho)}\sum_{j=1}^{\infty}\gamma_{n_k,j}(\gamma_{n_k,0}-k)\|v_{n_k} - z_{n_k}^j\|^2 \to 0, \quad k\to\infty.$$

Consequently, we have

$$\lim_{k\to\infty}\|v_{n_k} - z_{n_k}^j\| = 0, \quad j = 1, 2, ... \tag{3.71}$$

It then follows that

$$\lim_{k\to\infty}\operatorname{dist}(v_{n_k}, S_j v_{n_k}) \leq \lim_{k\to\infty}\|v_{n_k} - z_{n_k}^j\| = 0, \quad j = 1, 2, ... \tag{3.72}$$

By similar argument, we obtain from Lemma 3.3.7 that

$$\lim_{k\to\infty}\mu_{n_k}\left[\sum_{i=0}^{N}\beta_{i,n_k}\theta_{i,n_k}(2-\theta_{i,n_k})\frac{\|\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_{n_k}\|^4}{\|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_{n_k}\|^2 + \phi_{i,n_k}}\right] = 0$$
$\forall\ i = 0, 1, 2, ..., N.$

By condition (A3), we have

$$\frac{\|\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_{n_k}\|^4}{\|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_{n_k}\|^2 + \phi_{i,n_k}} \to 0, \quad k\to\infty, \quad \forall\ i = 0, 1, 2, ..., N. \tag{3.73}$$

It follows from the boundedness of the operators A_i, the nonexpansivity of the mappings $T_{r_i}^{(F_i,\phi_i)}$, and the boundedness of the sequence $\{x_{n_k}\}$ that $L := \max_{i=0,1,...,N}\{\sup_n\{\|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_{n_k}\|^2\}\} < \infty$. Therefore, it follows from (A4), that

$$\frac{\|\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_{n_k}\|^4}{\|A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_{n_k}\|^2 + \phi_{i,n_k}} \geq \frac{\|\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_{n_k}\|^4}{L+K}.$$

Using the last inequality together with (3.73), we have

$$\lim_{k\to\infty} \|\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_{n_k}\| = 0, \quad \forall\ i = 0,1,2,...,N. \tag{3.74}$$

Furthermore, we obtain from (3.58) that

$$\lim_{k\to\infty} \|v_{n_k} - x_{n_k}\| = \|\sum_{i=0}^{N} \beta_{i,n_k}\tau_{i,n_k}A_i^*\big(I^{H_i} - T_{r_i}^{(F_i,\phi_i)}\big)A_i x_{n_k}\|. \tag{3.75}$$

Applying (3.74) together with (A3), it follows from the last inequality that

$$\lim_{k\to\infty} \|v_{n_k} - x_{n_k}\| = 0. \tag{3.76}$$

Also, from (3.59) and (3.71), we have that

$$\begin{aligned}\|y_{n_k} - v_{n_k}\| &= \|\gamma_{n_k,0}(v_{n_k} - v_{n_k}) + \sum_{j=1}^{\infty}\gamma_{n_k,j}(z_{n_k}^j - v_{n_k})\| \\ &\leq \sum_{j=1}^{\infty}\gamma_{n_k,j}\|v_{n_k} - z_{n_k}^j\| \to 0, \qquad k\to\infty. \end{aligned} \tag{3.77}$$

Observe that from (3.71) and (3.77), we have

$$\|y_{n_k} - z_{n_k}^j\| \leq \|y_{n_k} - v_{n_k}\| + \|v_{n_k} - z_{n_k}^j\| \to 0, \qquad k\to\infty,\ \ \forall\ j = 1,2,... \tag{3.78}$$

It follows from (3.76) and (3.77) that

$$\|x_{n_k} - y_{n_k}\| \leq \|x_{n_k} - v_{n_k}\| + \|v_{n_k} - y_{n_k}\| \to 0, \qquad k\to\infty. \tag{3.79}$$

Consequently, by applying condition (A1) we have

$$\begin{aligned}\|x_{n_{k+1}} - x_{n_k}\| &= \|\alpha_{n_k}g(x_{n_k}) + \delta_{n_k}x_{n_k} + \mu_{n_k}y_{n_k} - x_{n_k}\| \\ &\leq \alpha_{n_k}\|g(x_{n_k}) - x_{n_k}\| + \delta_{n_k}\|x_{n_k} - x_{n_k}\| + \mu_{n_k}\|y_{n_k} - x_{n_k}\| \to 0, \quad k\to\infty.\end{aligned} \tag{3.80}$$

To complete the proof, we need to show that $w_\omega(x_n) \subset \Gamma$. Since the sequence $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $\bar{x} \in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. From (3.76), we have that $v_{n_k} \rightharpoonup \bar{x}$. Now, from the fact that $I - S_j$ is demiclosed at zero for each $j = 1, 2, ...$, and since from (3.71) $\lim_{k\to\infty} \|v_{n_k} - z^j_{n_k}\| \to 0$ as $k \to \infty$ for each $j = 1, 2, ...$, we have that $\bar{x} \in \text{Fix}(S_j)$ for all $j = 1, 2, ...$ Hence, $\bar{x} \in \cap_{j=1}^{\infty}\text{Fix}(S_j)$. Also, since for each $i = 0, 1, ..., N$, A_i is a bounded linear operator, it follows that $A_i x_{n_k} \rightharpoonup A_i\bar{x}$. Thus, by the demiclosedness principle it follows from (3.74) that $A_i\bar{x} \in \text{Fix}(T^{F_i,\phi_i}_{r_i})$ for all $i = 0, 1, ..., N$. Hence, $A_i\bar{x} \in \bigcap_{i=0}^{N}(GEP(F_i, \phi_i))$. Consequently, we have $\bar{x} \in \Gamma$, which implies that $w_\omega(x_n) \subset \Gamma$. By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\lim_{j\to\infty} \langle g(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\rangle = \limsup_{k\to\infty}\langle g(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle.$$

Since $\hat{x} = P_\Omega \circ g(\hat{x})$, then from (3.80) and (3.67), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle g(\hat{x}) - \hat{x}, x_{n_{k+1}} - \hat{x}\rangle &= \limsup_{k\to\infty}\langle g(\hat{x}) - \hat{x}, x_{n_{k+1}} - x_{n_k}\rangle \\ &\quad + \limsup_{k\to\infty}\langle g(\hat{x}) - \hat{x}, x_{n_k} - \hat{x}\rangle \\ &= \limsup_{j\to\infty}\langle g(\hat{x}) - \hat{x}, x_{n_{k_j}} - \hat{x}\rangle \\ &= \langle g(\hat{x}) - \hat{x}, x^\dagger - \hat{x}\rangle \leq 0. \end{aligned} \tag{3.81}$$

Applying Lemma 2.5.55 to (3.68), and using (3.81) together with the fact that $\lim_{n\to\infty} \alpha_n = 0$, we deduce that $\lim_{n\to\infty} \|x_n - \hat{x}\| = 0$ as required.

□

If we take $\phi_i = 0, i = 0, 1, 2, \ldots, N$ in Theorem 3.3.8, we have the following consequent result for approximating a common solution of the set of solution of split equilibrium problem with multiple output sets and the common fixed point problem for an infinite family of multivalued demicontractive mappings in real Hilbert spaces.

Corollary 3.3.9. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, ..., N$, let C_i be nonempty closed convex subset of Hilbert spaces H_i and let F, F_i, and A_i be as defined in Theorem 3.3.8. For each $j \in \mathbb{N}$, let $S_j : H \to CB(H)$ be a family of multivalued demicontractive mappings with constant $k_j \in (0, 1)$ such that $I - S_j$ is demiclosed at zero for each $j \in \mathbb{N}$. Suppose the solution set denoted by $\Gamma := \bigcap_{j=1}^{\infty} Fix(S_j) \cap EP(F) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(EP(F_i))\right) \neq \emptyset$, and conditions (A1)-(A4) and Assumption 3.3.2 hold. Then, the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $\hat{x} \in \Gamma$, where $\hat{x} = P_\Gamma \circ g(\hat{x})$.*

Algorithm 3.3.10.

Step 0: *For any $x_0 \in H$, let $H_0 = H, T_0 = I^H, F_0 = F$ and set $n = 0$.*

Step 1: *Compute*

$$v_n = \sum_{i=0}^{N} \beta_{i,n}\Big[x_n - \tau_{i,n} A_i^*\big(I^{H_i} - T_{r_i}^{F_i}\big) A_i x_n\Big]. \tag{3.82}$$

Step 2: *Compute*

$$y_n = \gamma_{n,0} v_n + \sum_{j=1}^{\infty} \gamma_{n,j} z_n^j, \tag{3.83}$$

where $z_n^j \in S_j v_n, \quad j = 1, 2, ...$

Step 3: *Compute*

$$x_{n+1} = \alpha_n g(x_n) + \delta_n x_n + \mu_n y_n, \quad n \in \mathbb{N}. \tag{3.84}$$

Update:

$$\tau_{i,n} = \theta_{i,n} \frac{\|\big(I^{H_i} - T_{r_i}^{F_i}\big) A_i x_n\|^2}{\|A_i^*\big(I^{H_i} - T_{r_i}^{F_i}\big) A_i x_n\|^2 + \phi_{i,n}}.$$

Set $n = n + 1$ *and go to* **Step 1**.

3.3.2 Application

Let C be a nonempty closed convex subset of a real Hilbert space H, and $B : H \to H$ be a single-valued mapping. The *Variational Inequality Problem* (shortly, VIP) is defined as follows:

$$\text{Find } \ x^* \in C \ \text{ such that } \ \langle y - x^*, Bx^* \rangle \geq 0, \quad \forall\, y \in C. \tag{3.85}$$

The solution set of the VIP is denoted by $VI(C, B)$. Variational inequality was first introduced independently by Fichera [87] and Stampacchia [215]. The VIP is a useful mathematical model, which unifies many important concepts in applied mathematics, such as necessary complementarity problems, network equilibrium problems, optimality conditions, and systems of nonlinear equations (see [90, 130]). Several methods have been proposed and analyzed for approximating the solution of VIP (3.85) (see [5, 6, 49, 125] and references therein).

Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, ..., N$, let C_i be nonempty closed convex subset of Hilbert spaces H_i and let $A_i : H \to H_i$ be bounded linear operators. Let $B : C \to H, B_i : C_i \to H_i$ be monotone mappings. The *Split Variational Inequality Problem with Multiple Output Sets* (shortly, SVIPMOS) is formulated as finding a point $x^* \in C$ such that

$$x^* \in VI(C, B) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}\big(VI(C_i, B_i)\big) \right) \neq \emptyset. \tag{3.86}$$

We denote the solution set of problem (3.86) by $\mathcal{F}$. By taking $F_i(x,y) := \langle y-x, B_i x\rangle, i = 0,1,2,\dots,N$, where $F_0 = F, B_0 = B$, then the SVIPMOS (3.86) becomes the problem of finding a solution of split equilibrium problem with multiple output sets. Consequently, Corollary 3.3.9 can be used to approximate the common solution of SVIPMOS (3.86) and the common fixed point problem for an infinite family of multivalued demicontractive mappings $S_j : H \to CB(H), j \in \mathbb{N}$ in real Hilbert spaces, where the solution set denoted by $\Gamma := \bigcap_{j=1}^{\infty} \mathrm{Fix}(S_j) \cap \mathcal{F}$ is assumed to be nonempty.

3.3.3 Numerical examples

This section provides some examples to illustrate the implementation of our proposed methods, Algorithm 3.3.5. In our experiment, we let $g(x) = \frac{x}{3}, \alpha_n = \frac{1}{140n+1}, \quad \delta_n = \frac{1}{3n+14}, \quad \mu_n = 1-\alpha_n-\delta_n$, for $i = 0,1,2$, let $\quad r_i = r = 0.5, , \beta_{i,n} = \frac{1}{3}, \quad \gamma_{n,0} = \frac{1}{2}, \gamma_{n,j} = \frac{1}{2^{j+1}}, j = 1,2,\dots$. Moreover, we consider the effect of varying values of the following parameters $\theta_{i,n} = \frac{1}{2+i}, 1.0, \frac{2}{3+i}, 1.5, \frac{3}{4+i}, 1.9, \phi_{i,n} = 0.5, \frac{3}{4+i}, 1.0, \frac{5}{2+i}, 2.0, \frac{7}{3+i}$ on our method. All numerical computations were carried out using Matlab version R2019(b). We plot the graphs of errors against the number of iterations in each case. The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-9}$. The numerical results are reported in Figures 3.4, 3.5, 3.6 and 3.7 and Tables 3.2.11, 3.2.12, 3.2.13 and 3.2.14. In Tables 3.2.11-3.2.14, “Iter.” means the number of iterations while “CPU” means the CPU time in seconds.

Example 3.3.11. *Let H, $H_i = \mathbb{R}^2$ for $i = 0,1,2$, with $H = H_0$. We define the mappings $F = F_0 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $\quad F_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $F_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ respectively by $F(x,y) = -3x^2 + xy + 2y^2, \quad F_1(x,y) = -4x^2 + xy + 3y^2$ and $F_2(x,y) = -5y^2 + 2y + 5xy - 5xy^2$ for each $x = (x_1,x_2) \in \mathbb{R}^2$ and $y = (y_1,y_2) \in \mathbb{R}^2$. Also, for $i = 0,1,2$, let $\phi_0 = \phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad \phi_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $\phi_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\phi(x,y) = x^2 - xy, \quad \phi_1(x,y) = 2x(x-y)$ and $\phi_2(x,y) = 5y^2 - 2x$ respectively for each $x = (x_1,x_2) \in \mathbb{R}^2$ and $y = (y_1,y_2) \in \mathbb{R}^2$. For some $r > 0$, we obtain by some simple calculation that*

$$v = T_r^{F,\phi} u = \frac{1}{4r+1} u, \quad y = T_r^{F_1,\phi_1} x = \frac{1}{1+5r} x \quad \textit{and} \quad w = T_r^{F_2,\phi_2} z = \frac{z-2r}{1+5r}.$$

Let $A_i : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $A_i(x) = \frac{x}{i+1}$ where $x = (x_1,x_2) \in \mathbb{R}^2$. Let $S_j : C \to C(B)$ be defined by

$$S_j x := \frac{-3}{2j} x, \quad j = 1,2,...$$

It is easy to see that S_j is demicontractive for each $j = 1,2,\dots$.

Table 3.2.11. Numerical results for Example 3.3.11 (Experiment 1).

Cases		$\theta_{i,n} = \frac{1}{2+i}$	$\theta_{i,n} = 1$	$\theta_{i,n} = \frac{2}{3+i}$	$\theta_{i,n} = 1.5$	$\theta_{i,n} = \frac{3}{4+i}$
1	*CPU time(sec)* *No of Iter.*	*0.0155* *14*	*0.0128* *14*	*0.0096* *14*	*0.0140* *14*	*0.0150* *14*
2	*CPU time (sec)* *No of Iter.*	*0.0136* *14*	*0.0088* *14*	*0.0128* *14*	*0.0139* *14*	*0.0150* *14*
3	*CPU time (sec)* *No of Iter.*	*0.0137* *14*	*0.0089* *14*	*0.0145* *14*	*0.0162* *14*	*0.0161* *14*
4	*CPU time (sec)* *No of Iter.*	*0.0135* *14*	*0.0123* *14*	*0.0097* *14*	*0.0146* *14*	*0.0151* *14*

Table 3.2.12. Numerical results for Example 3.3.11 (Experiment 2).

Cases		$\phi_{i,n} = 0.5$	$\phi_{i,n} = \frac{3}{4+i}$	$\phi_{i,n} = 1.0$	$\theta_{i,n} = \frac{5}{2+i}$	$\theta_{i,n} = 2.0$
1	*CPU time(sec)* *No of Iter.*	*0.0138* *14*	*0.0090* *14*	*0.0132* *14*	*0.0156* *14*	*0.0092* *14*
2	*CPU time(sec)* *No of Iter.*	*0.0139* *14*	*0.0089* *14*	*0.0128* *14*	*0.0144* *14*	*0.0140* *14*
3	*CPU time(sec)* *No of Iter.*	*0.0139* *14*	*0.0091* *14*	*0.0148* *14*	*0.0151* *14*	*0.0093* *14*
4	*CPU time(sec)* *No of Iter.*	*0.0136* *14*	*0.0088* *14*	*0.0140* *14*	*0.0152* *14*	*0.0092* *14*

The next example is in the framework of an infinite dimensional Hilbert spaces.

Example 3.3.12. *Let $H,\ H_i = \ell_2$ for $i = 0, 1, 2, \ldots$ be the linear spaces whose elements consists of 2-summable sequences $(x_1, x_2, \ldots, x_i, \ldots)$ of scalars, i.e.,*

$$\ell_2 = \{x : x = (x_1, x_2, \ldots, x_i, \ldots) \ \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty\},$$

with an inner product $\langle ., . \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i \quad \text{where} \quad x = \{x_i\}_{i=1}^{\infty}, \quad y = \{y_i\}_{i=1}^{\infty} \in \ell_2.$$

For $i = 0, 1, 2,$ let the mapping $A_i : \ell_2 \to \ell_2$ be defined by $A_i x = \left(\frac{x_1}{3}, \frac{x_2}{3}, \ldots, \frac{x_m}{3}, \ldots\right)$ for all $x = \{x_m\}_{m=1}^{\infty} \in \ell_2$ and $A_i^* : \ell_2 \to \ell_2$ be defined by $A_i^* z = \left(\frac{z_1}{3}, \frac{z_2}{3}, \ldots, \frac{z_m}{3}, \ldots\right)$ for

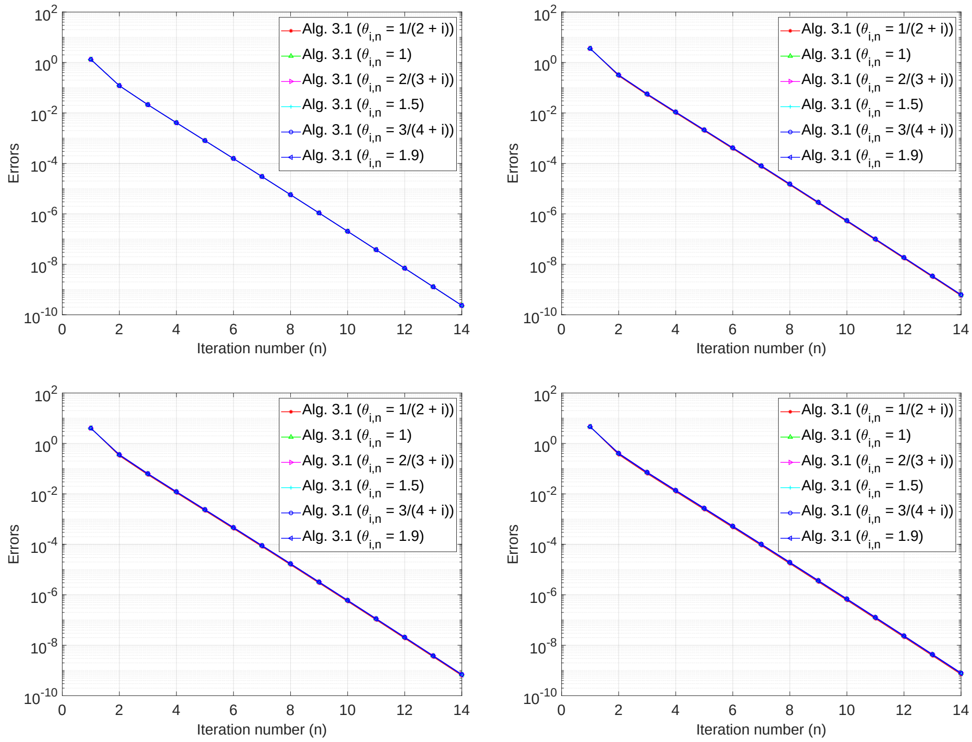


Figure 3.4: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

all $z = \{z_m\}_{m=1}^{\infty} \in \ell_2$. *Define the mapping* $F_i : \ell_2 \times \ell_2 \to \mathbb{R}$ *by* $F_i, = F$ *such that* $F(x, y) = -x^2 + y^2, \quad \forall \quad x = \{x_i\}_{i=1}^{\infty}, \quad y = \{y_i\}_{i=1}^{\infty}$. *and* $\phi_i = 0$, *for each* $i = 0, 1, 2$. *It is easy to see that*

$$T_r^{(F,\phi)} x = \frac{1-r}{5r+1} x.$$

Also, for $j = 1, 2, ...,$ *we define* $S_j : C \to CB(\ell_2)$ *by*

$$S_j x = \Big[0, \frac{x}{5j}\Big] \quad \forall j = 1, 2, ...$$

It is easy to see that S_j *is* 0*-demicontractive for each* $j = 1, 2, ...$ *and* $Fix(S_j) = \{0\}$.

Table 3.2.13. Numerical results for Example 3.3.12 (Experiment 1).

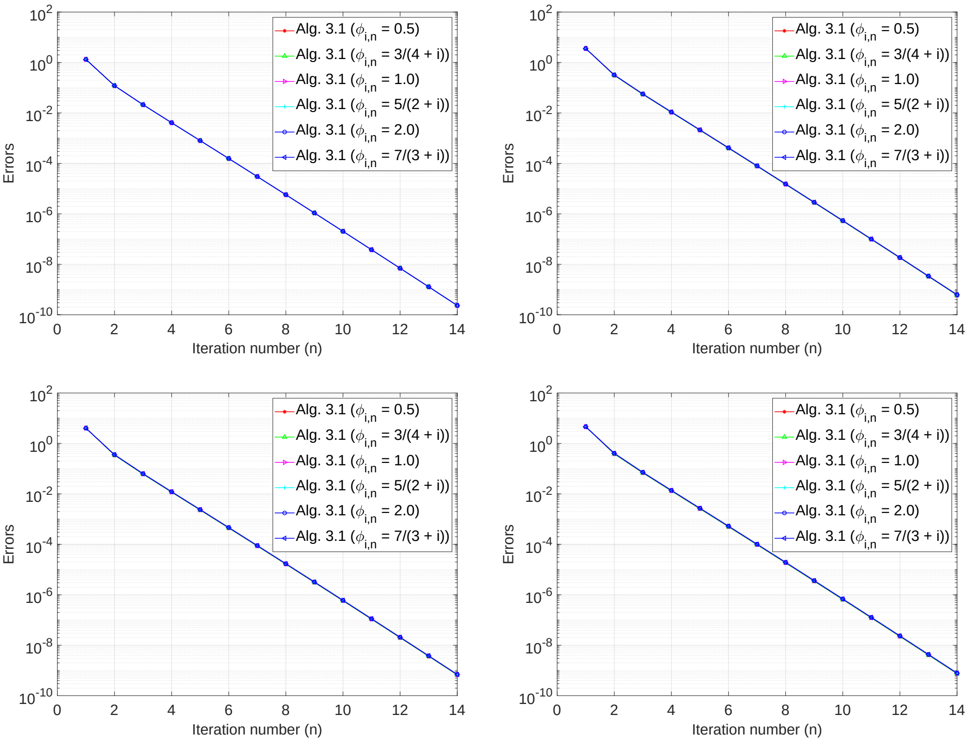


Figure 3.5: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

Cases		$\theta_{i,n} = \frac{1}{2+i}$	$\theta_{i,n} = 1$	$\theta_{i,n} = \frac{2}{3+i}$	$\theta_{i,n} = 1.5$	$\theta_{i,n} = \frac{3}{4+i}$
1	*CPU time(sec)*	*0.0544*	*0.0112*	*0.0144*	*0.0130*	*0.0158*
	No of Iter.	*33*	*33*	*33*	*33*	*33*
2	*CPU time(sec)*	*0.0129*	*0.0093*	*0.0133*	*0.0133*	*0.0138*
	No of Iter.	*34*	*33*	*34*	*33*	*34*
3	*CPU time(sec)*	*0.0134*	*0.0124*	*0.0096*	*0.0142*	*0.0137*
	No of Iter.	*33*	*33*	*33*	*33*	*33*
4	*CPU time(sec)*	*0.0161*	*0.0174*	*0.0164*	*0.0166*	*0.0133*
	No of Iter.	*33*	*33*	*33*	*34*	*34*

Table 3.2.14. Numerical results for Example 3.3.12 (Experiment 2).

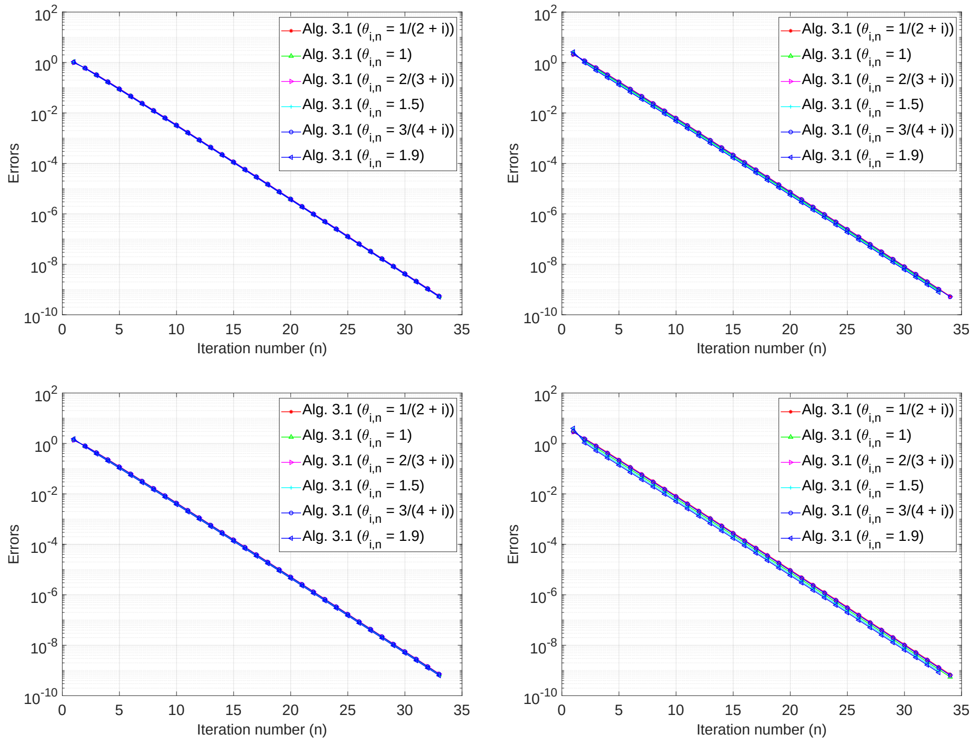


Figure 3.6: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

Cases		$\phi_{i,n} = 0.5$	$\phi_{i,n} = \frac{3}{4+i}$	$\phi_{i,n} = 1.0$	$\theta_{i,n} = \frac{5}{2+i}$	$\theta_{i,n} = 2.0$
1	*CPU time(sec)*	*0.0132*	*0.0100*	*0.0112*	*0.0116*	*0.0118*
	No of Iter.	*33*	*33*	*33*	*33*	*33*
2	*CPU time(sec)*	*0.0154*	*0.0122*	*0.0102*	*0.0146*	*0.0138*
	No of Iter.	*33*	*33*	*33*	*33*	*33*
3	*CPU time(sec)*	*0.0141*	*0.0100*	*0.0133*	*0.0141*	*0.0140*
	No of Iter.	*33*	*33*	*33*	*33*	*33*
4	*CPU time(sec)*	*0.0161*	*0.0169*	*0.0142*	*0.0169*	*0.0160*
	No of Iter.	*33*	*33*	*33*	*34*	*34*

We test these examples under the following experiments:

Experiment 1:
In this experiment, we check the behavior of our method by fixing the other parameters and varying $\theta_{i,n}$. We do this to check the effects of the parameter $\theta_{i,n}$ on our method.

For **Example 3.3.11** We consider the following cases for the initial value of x_0 :

Case 1 : $x_0 = (0.78, 1.25)$;

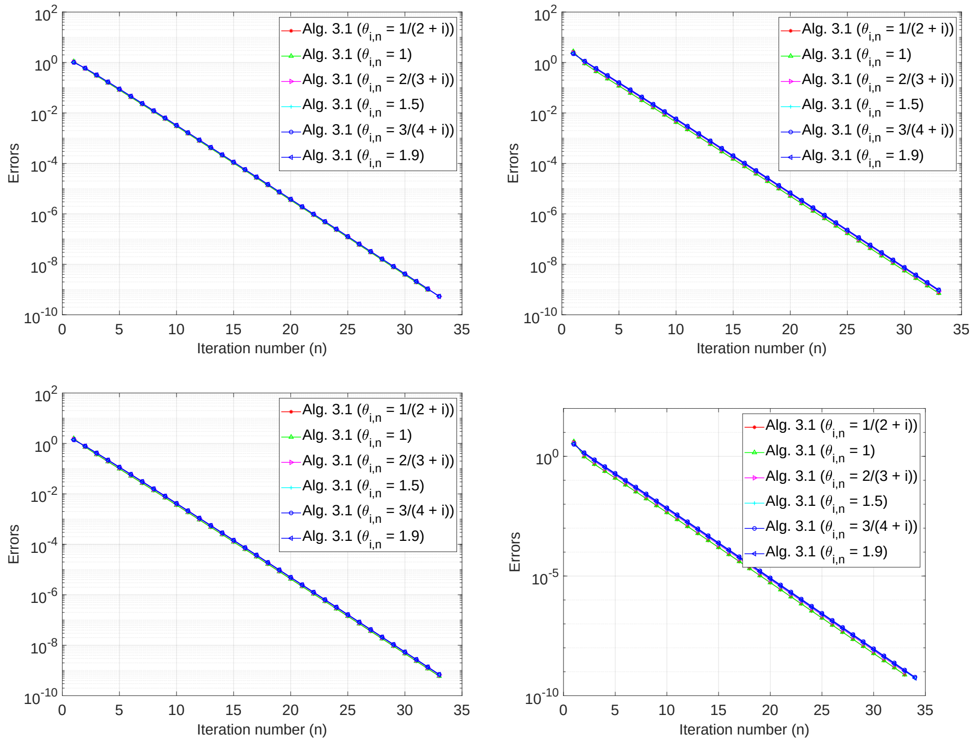


Figure 3.7: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

Case 2 : $x_0 = (3.78, 1.25)$;

Case 3 : $x_0 = (4, 2)$;

Case 4 : $x_0 = (-1, -5)$.

Also, we consider $\theta_{i,n} \in \{\frac{1}{2+i}, 1.0, \frac{2}{3+i}, 1.5, \frac{3}{4+i}, 1.9\}$, which satisfies Assumption (A3). We use Algorithm 3.3.5 for the experiment and report the numerical results in Table 3.2.11 and Figure 3.4.

For Example 3.3.12 We consider the following cases for the initial value of x_0 :

Case 1 : $x_0 = (2, 1, \frac{1}{2}, \cdots)$;

Case 2 : $x_0 = (4, -2, 1, \cdots)$;

Case 3 : $x_0 = (-3, \frac{3}{5}, -\frac{3}{25}, \cdots)$;

Case 4 : $x_0 = (6, 1, \frac{1}{6}, \cdots)$.

Also, we consider $\theta_{i,n} \in \{\frac{1}{2+i}, 1.0, \frac{2}{3+i}, 1.5, \frac{3}{4+i}, 1.9\}$, which satisfies Assumption (A3). We use Algorithm 3.3.5 for the experiment and report the numerical results in Table 3.2.13 and Figure 3.6.

Experiment 2:
In this experiment, we check the behavior of our method by fixing the other parameters and varying $\phi_{i,n}$. We do this to check the effects of the parameter $\phi_{i,n}$ on our method.

For Example 3.3.11: We consider the following cases for the initial value of x_0 :

Case 1 : $x_0 = (0.78, 1.25)$;

Case 2 : $x_0 = (3.78, 1.25)$;

Case 3 : $x_0 = (4, 2)$;

Case 4 : $x_0 = (-1, -5)$.

Also, we consider $\phi_{i,n} \in \{0.5, \frac{3}{4+i}, 1.0, \frac{5}{2+i}, 2.0, \frac{7}{3+i}\}$, which satisfies Assumption (A4). We use Algorithm 3.3.5 for the experiment and report the numerical results in Table 3.2.12 and Figure 3.5.

For Example 3.3.12: We consider the following cases for the initial value of x_0 :

Case 1 : $x_0 = (2, 1, \frac{1}{2}, \cdots)$;

Case 2 : $x_0 = (4, -2, 1, \cdots)$;

Case 3 : $x_0 = (-3, \frac{3}{5}, -\frac{3}{25}, \cdots)$;

Case 4 : $x_0 = (6, 1, \frac{1}{6}, \cdots)$.

Also, we consider $\phi_{i,n} \in \{0.5, \frac{3}{4+i}, 1.0, \frac{5}{2+i}, 2.0, \frac{7}{3+i}\}$, which satisfies Assumption (A4). We use Algorithm 3.3.5 for the experiment and report the numerical results in Table 3.2.14 and Figure 3.7.

Chapter 4

Split Monotone Variational Inclusion Problems and Fixed Point Problems

4.0.1 Introduction

In this chapter, we present our result on relaxed double inertial Tseng's extragradient method with self-adaptive step sizes for solving split monotone variational inclusion problem (SMVIP) involving non-Lipschitz operators and fixed point problem of strict pseudo-contractive mappings. Furthermore, results on generalized split feasibility problem over a solution set of monotone variational inclusion problem was studied. For each of these problems, we propose iterative algorithm with self-adaptive step size for approximating the solution and prove strong convergence theorem. We also give applications of our results and illustrate our algorithms with numerical examples.

4.1 Non-Lipschitz split monotone variational inclusion problem with fixed points constraints

In this section, we prove that our proposed scheme converges strongly to a minimum-norm solution of the aforementioned problem in real Hilbert spaces. We point out that while the operators are non-Lipschitz, our method does not involve linesearch procedure which is known to be time-consuming, but we employ a more efficient self-adaptive step size technique that generates non-monotonic sequence of step sizes at each iteration. Results of the numerical experiments demonstrate the comparative advantage of our method over existing methods in the literature. Our result extends and complements several existing results in the direction of this research in a unified way.

4.1.1 Main result

In this section, we present our algorithm. Firstly, we begin with the following assumptions under which the strong convergence theorem for the algorithm is established.

Assumption 4.1.1. *Let H_1 and H_2 be real Hilbert spaces. We make the following assumptions:*

(1) (a)$A : H_1 \to H_1$ and $D : H_2 \to H_2$ are monotone and uniformly continuous.
(b) $B : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ are maximal monotone operators.
(c) $T : H_1 \to H_2$ is a bounded linear operator such that $T \neq 0$ and $T^* : H_2 \to H_1$ is the adjoint of T.
(d) $S : H_1 \to H_1$ is a k-strict pseudocontractive mapping.
(e) The solution set $\Upsilon := Fix(S) \cap \Omega \neq \emptyset$.

(2) $\{\alpha_n\}_{n=1}^{\infty}$, $\{\phi_n\}_{n=1}^{\infty}$, $\{\varepsilon_n\}_{n=1}^{\infty}$, $\{\tau_n\}_{n=1}^{\infty}$, $\{\kappa_n\}_{n=1}^{\infty}$, $\{\vartheta_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are all positive sequences which satisfy the following conditions:
(a)$\{\alpha_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim\limits_{n\to\infty} \dfrac{\varepsilon_n}{\alpha_n} = 0$, $\lim\limits_{n\to\infty} \dfrac{\tau_n}{\alpha_n} = 0, 0 < a \leq \phi_n \leq b < 1$.
(b) $0 < \kappa < \kappa' < 1$, $0 < \sigma < \sigma' < 1, 0 < \vartheta < \vartheta' < 1$, $\lim\limits_{n\to\infty} \kappa_n = 0, \lim\limits_{n\to\infty} \sigma_n = 0, \lim\limits_{n\to\infty} \vartheta_n = 0, \sum_{n=1}^{\infty} s_n < +\infty$ and $\sum_{n=1}^{\infty} t_n < +\infty$, $\alpha \in [k, 1)$.

Below is our proposed scheme for this paper.

Algorithm 4.1.2.

Initialization: *Suppose $\theta, \varrho, \gamma_1, \lambda_1 > 0$, $x_0, x_1 \in H_1$ be arbitrary and $x_0 = b_0$.*

Iterative Steps: *Set $n := 1$. Calculate x_{n+1} as follows:*

Step 1. *Given the iterates x_{n-1} and x_n $(n \geq 1)$, we choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases} \tag{4.1}$$

Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Compute

$$y_n = (I + \gamma_n G)^{-1}(I - \gamma_n D)Tw_n, \tag{4.2}$$

where

$$\gamma_{n+1} = \begin{cases} \min\left\{\frac{(\kappa_n+\kappa)\|Tw_n - y_n\|}{\|DTw_n - Dy_n\|}, \ \gamma_n + s_n\right\}, & \text{if } DTw_n \neq Dy_n, \\ \gamma_n + s_n, & \text{otherwise,} \end{cases} \tag{4.3}$$

and

$$z_n = y_n - \gamma_n(Dy_n - DTw_n).$$

Step 3. *Compute*

$$b_n = w_n + \eta_n T^*(z_n - Tw_n),$$

where

$$\eta_n = \begin{cases} \frac{(\vartheta_n+\vartheta)\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2}, & \text{if } \|T^*(Tw_n - z_n)\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Step 4. *Choose* ϱ_n *such that* $0 \leq \varrho_n \leq \bar{\varrho}_n$, *where*

$$\bar{\varrho}_n := \begin{cases} \min\left\{\varrho, \frac{\tau_n}{\|b_n - b_{n-1}\|}\right\}, & \text{if } b_n \neq b_{n-1} \\ \\ \varrho, & \text{otherwise.} \end{cases} \tag{4.5}$$

Step 5 : *Compute*

$$l_n = (1 - \alpha_n)\left(b_n + \varrho_n(b_n - b_{n-1})\right)$$

$$u_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)l_n,$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{(\sigma_n+\sigma)\|u_n - l_n\|}{\|Au_n - Al_n\|}, \ \lambda_n + t_n\right\}, & \text{if } Au_n \neq Al_n, \\ \lambda_n + t_n, & \text{otherwise,} \end{cases} \tag{4.6}$$

and

$$v_n = u_n - \lambda_n(Au_n - Al_n).$$

Step 6. *Compute*

$$x_{n+1} = (1 - \phi_n)l_n + \phi_n S_\alpha v_n, \tag{4.7}$$

where

$$S_\alpha = \alpha I + (1 - \alpha)S.$$

Set $n := n + 1$ *and go back to* **Step 1.**

Remark 4.1.3. *Observe from* (4.1) *and Assumption 4.1.1 (2)(a) that*

$$\lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$$

Similarly, we obtain from (4.5) *and Assumption 4.1.1 (2)(a) that*

$$\lim_{n\to\infty} \varrho_n\|b_n - b_{n-1}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\varrho_n}{\alpha_n}\|b_n - b_{n-1}\| = 0.$$

Lemma 4.1.4. *The sequences $\{\gamma_n\}$ and $\{\lambda_n\}$ of step sizes generated by Algorithm 4.1.2 under Assumption 4.1.1 are well-defined and bounded.*

Proof. Since D is uniformly continuous, using (2.55), we have that for any given $\epsilon > 0$, there exists $K < +\infty$ such that $\|DTw_n - Dy_n\| \le K\|Tw_n - y_n\| + \epsilon$. Furthermore, whenever $DTw_n - Dy_n \neq 0$ for all $n \ge 1$, we obtain

$$\frac{(\kappa_n+\kappa)\|Tw_n-y_n\|}{\|DTw_n-Dy_n\|} \ge \frac{(\kappa_n+\kappa)\|Tw_n-y_n\|}{K\|Tw_n-y_n\|+\epsilon} = \frac{(\kappa_n+\kappa)\|Tw_n-y_n\|}{(K+\epsilon_1)\|Tw_n-y_n\|} = \frac{\kappa_n+\kappa}{N} \ge \frac{\kappa}{N},$$

where $\epsilon = \epsilon_1\|Tw_n - y_n\|$ for some $\epsilon_1 \in (0,1)$ and $N = K + \epsilon_1$. It follows from the definition of γ_{n+1} that the sequence $\{\gamma_n\}$ has lower bound $\min\{\frac{\kappa}{N}, \gamma_1\}$ and has upper bound $\gamma_1 + \Psi$. By Lemma 2.5.23, the limit $\lim_{n\to\infty}\gamma_n$ exists and $\lim_{n\to\infty}\gamma_n = \gamma$. Obviously, we have that $\gamma \in \left[\min\{\frac{\kappa}{N}, \gamma_1\},\ \gamma_1 + \Psi\right]$; where $\Psi = \sum_{n=1}^{\infty} s_n$.
Similarly, we obtain $\lim_{n\to\infty}\lambda_n = \lambda \in \left[\min\{\frac{\sigma}{M}, \lambda_1\},\ \lambda_1 + \Phi\right]$; for some $M > 0$ where $\Phi = \sum_{n=1}^{\infty} t_n$. □

Lemma 4.1.5. *Suppose $\|T^*(Tw_n - z_n)\| \neq 0$, then the sequence $\{\eta_n\}$ generated by Algorithm 4.1.2 has a positive lower bound.*

Proof. Let $\|T^*(Tw_n - z_n)\| \neq 0$, it follows that

$$\eta_n = \frac{(\vartheta_n + \vartheta)\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2}.$$

Since T is a bounded linear operator, $\vartheta \in (0,1)$ and $\lim_{n\to\infty}\vartheta_n = 0$, we have

$$\frac{(\vartheta_n+\vartheta)\|Tw_n-z_n\|^2}{\|T^*(Tw_n-z_n)\|^2} \ge \frac{(\vartheta_n+\vartheta)\|Tw_n-z_n\|^2}{\|T\|^2\|(Tw_n-z_n)\|^2} = \frac{(\vartheta_n+\vartheta)}{\|T\|^2} \ge \frac{\vartheta}{\|T\|^2}.$$

Thus, $\frac{\vartheta}{\|T\|^2}$ is a lower bound of $\{\eta_n\}$. □

Lemma 4.1.6. *Let Assumption 4.1.1 (2) hold. Then, there exists a positive integer N such that $\vartheta_n + \vartheta, \frac{\gamma_n(\kappa_n+\kappa)}{\gamma_{n+1}}, \frac{\lambda_n(\sigma_n+\sigma)}{\lambda_{n+1}} \in (0,1)\ \forall\ n \ge N$.*

Proof. Since $0 < \vartheta < \vartheta' < 1$ and $\lim_{n\to\infty}\vartheta_n = 0$, there exists a positive integer N_1 such that

$$0 < \vartheta + \vartheta_n \le \vartheta' < 1, \quad \forall n \ge N_1.$$

Also, since $0 < \kappa < \kappa' < 1$, $\lim_{n\to\infty}\kappa_n = 0$ and $\lim_{n\to\infty}\gamma_n = \gamma$, we obtain

$$\lim_{n\to\infty}\left(1 - \frac{\gamma_n(\kappa_n+\kappa)}{\gamma_{n+1}}\right) = 1 - \kappa > 1 - \kappa' > 0.$$

Thus, there exists a positive integer N_2 such that

$$1 - \frac{\gamma_n(\kappa_n+\kappa)}{\gamma_{n+1}} > 0,\ \forall\ n \ge N_2.$$

In the same vein, there exists a positive integer N_3 such that

$$1 - \frac{\lambda_n(\sigma_n + \sigma)}{\lambda_{n+1}} > 0, \ \forall \ n \geq N_3.$$

Now, setting $N = \max\{N_1, N_2, N_3\}$, we have $\vartheta_n + \vartheta, \frac{\gamma_n(\kappa_n+\kappa)}{\gamma_{n+1}}, \quad \frac{\lambda_n(\sigma_n+\sigma)}{\lambda_{n+1}} \in (0,1) \ \forall \ n \geq N$. $\square$

Lemma 4.1.7. *Suppose Assumptions 4.1.1 holds and let $\{z_n\}$ be a sequence generated by Algorithm 4.1.2. Then, for all $\bar{x} \in \Upsilon$, we have*

$$\|z_n - T\bar{x}\|^2 \leq \|Tw_n - T\bar{x}\|^2 - \left(1 - \frac{\gamma_n^2(\kappa_n + \kappa)^2}{\gamma_{n+1}^2}\right)\|Tw_n - y_n\|^2, \tag{4.8}$$

and

$$\|z_n - y_n\| \leq \frac{\gamma_n(\kappa_n + \kappa)}{\gamma_{n+1}}\|Tw_n - y_n\|. \tag{4.9}$$

Proof. From (4.3), we obtain

$$\|DTw_n - Dy_n\| \leq \frac{(\kappa_n + \kappa)}{\gamma_{n+1}}\|Tw_n - y_n\|, \quad \forall n \in \mathbb{N}. \tag{4.10}$$

Observe that (4.10) holds for both $DTw_n = Dy_n$ and $DTw_n \neq Dy_n$. Let $\bar{x} \in \Upsilon$, then we have $T\bar{x} \in (D+G)^{-1}(0)$. From **Step 2** and by applying (4.10) and Lemma 2.5.18 (i), we obtain

$$\begin{aligned}
\|z_n - T\bar{x}\|^2 =& \|y_n - \gamma_n(Dy_n - DTw_n) - T\bar{x}\|^2 \\
&= \|y_n - T\bar{x}\|^2 + \gamma_n^2\|Dy_n - DTw_n\|^2 - 2\gamma_n\langle y_n - T\bar{x}, Dy_n - DTw_n\rangle \\
&= \|y_n - Tw_n\|^2 + \|Tw_n - T\bar{x}\|^2 + 2\langle y_n - Tw_n, Tw_n - T\bar{x}\rangle \\
&+ \gamma_n^2\|Dy_n - DTw_n\|^2 - 2\gamma_n\langle y_n - T\bar{x}, Dy_n - DTw_n\rangle \\
&= \|Tw_n - T\bar{x}\|^2 + \|y_n - Tw_n\|^2 - 2\langle y_n - Tw_n, y_n - Tw_n\rangle \\
&+ 2\langle y_n - Tw_n, y_n - T\bar{x}\rangle + \gamma_n^2\|Dy_n - DTw_n\|^2 - 2\gamma_n\langle y_n - T\bar{x}, Dy_n - DTw_n\rangle \\
&= \|Tw_n - T\bar{x}\|^2 - \|y_n - Tw_n\|^2 + 2\langle y_n - Tw_n, y_n - T\bar{x}\rangle \\
&+ \gamma_n^2\|Dy_n - DTw_n\|^2 - 2\gamma_n\langle y_n - T\bar{x}, Dy_n - DTw_n\rangle \\
&= \|Tw_n - T\bar{x}\|^2 - \|y_n - Tw_n\|^2 - 2\langle Tw_n - y_n - \gamma_n(DTw_n - Dy_n), y_n - T\bar{x}\rangle \\
&+ \gamma_n^2\|Dy_n - DTw_n\|^2 \\
&\leq \|Tw_n - T\bar{x}\|^2 - \left(1 - \frac{\gamma_n^2(\kappa_n + \kappa)^2}{\gamma_{n+1}^2}\right)\|Tw_n - y_n\|^2 \\
&- 2\langle Tw_n - y_n - \gamma_n(DTw_n - Dy_n), y_n - T\bar{x}\rangle.
\end{aligned} \tag{4.11}$$

Now, recall from **Step 2** that $y_n = (I + \gamma_n G)^{-1}(I - \gamma_n D)Tw_n$, which implies that $(I - \gamma_n D)Tw_n \in (I + \gamma_n G)y_n$. Since G is maximal monotone, there exists $h_n \in Gy_n$ such that

$$Tw_n - \gamma_n DTw_n = y_n + \gamma_n h_n,$$

which implies that

$$h_n = \frac{1}{\gamma_n}\left(Tw_n - \gamma_n DTw_n - y_n\right). \tag{4.12}$$

Furthermore, since $T\bar{x} \in (D+G)^{-1}(0)$, we have $0 \in (D+G)T(\bar{x})$ and $Dy_n + h_n \in (D+G)y_n$. In addition, since $D+G$ is maximal monotone, we obtain

$$\langle Dy_n + h_n, y_n - T\bar{x}\rangle \geq 0. \tag{4.13}$$

Using (4.12) and (4.13), we have

$$\frac{1}{\gamma_n}\langle Tw_n - \gamma_n DTw_n - y_n + \gamma_n Dy_n, y_n - T\bar{x}\rangle \geq 0,$$

which implies that

$$\langle Tw_n - y_n - \gamma_n(DTw_n - Dy_n), y_n - T\bar{x}\rangle \geq 0. \tag{4.14}$$

Therefore, applying (4.14) in (4.11), we obtain

$$\|z_n - T\bar{x}\|^2 \leq \|Tw_n - T\bar{x}\|^2 - \left(1 - \frac{\gamma_n^2(\kappa_n + \kappa)^2}{\gamma_{n+1}^2}\right)\|Tw_n - y_n\|^2. \tag{4.15}$$

Additionally, from the definition of z_n and inequality (4.10), we have

$$\|z_n - y_n\| \leq \frac{\gamma_n(\kappa_n + \kappa)}{\gamma_{n+1}}\|Tw_n - y_n\|.$$

This completes the proof. □

Lemma 4.1.8. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.1.2 under Assumption 4.1.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $\bar{x} \in \Upsilon$. From **Step 1** and Assumption 4.1.1 (2), we obtain that $\theta_n\|x_n - x_{n-1}\| \leq \varepsilon_n \ \forall \ n \in \mathbb{N}$, and by implication we have

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \frac{\varepsilon_n}{\alpha_n} \to 0, \quad \text{as } n \to \infty.$$

Thus, there exists $M_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1 \ \forall \ n \in \mathbb{N}.$$

Thus, from **Step 2**, we have

$$\begin{aligned}\|w_n - \bar{x}\| &\leq \|x_n - \bar{x}\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - \bar{x}\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \qquad (4.16)\\ &\leq \|x_n - \bar{x}\| + \alpha_n M_1. \qquad (4.17)\end{aligned}$$

Furthermore, by Lemma 4.1.6 there exists $N \in \mathbb{N}$ such that $\left(1 - \frac{\gamma_n^2(\kappa_n+\kappa)^2}{\gamma_{n+1}^2}\right) > 0 \ \forall\ n \geq N$. Therefore, from (4.8), we have

$$\|z_n - T\bar{x}\|^2 \leq \|Tw_n - \bar{x}\|^2, \quad \forall\ n \geq N. \tag{4.18}$$

Using Lemma 2.5.18, (4.18) and the definition of b_n, we have

$$\begin{aligned}
\|b_n - \bar{x}\|^2 &= \|w_n + \eta_n T^*(z_n - Tw_n) - \bar{x}\|^2 \\
&= \|w_n - \bar{x}\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 + 2\eta_n\langle w_n - \bar{x}, T^*(z_n - Tw_n)\rangle \qquad (4.19)\\
&= \|w_n - \bar{x}\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 + 2\eta_n\langle Tw_n - T\bar{x}, z_n - Tw_n\rangle \\
&= \|w_n - \bar{x}\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 \\
&+ \eta_n\left[\|z_n - T\bar{x}\|^2 - \|Tw_n - T\bar{x}\|^2 - \|z_n - Tw_n\|^2\right] \\
&\leq \|w_n - \bar{x}\|^2 + \eta_n^2\|T^*(z_n - Tw_n)\|^2 - \eta_n\|z_n - Tw_n\|^2 \\
&= \|w_n - \bar{x}\|^2 - \eta_n[\|z_n - Tw_n\|^2 - \eta_n\|T^*(z_n - Tw_n)\|^2]. \qquad (4.20)
\end{aligned}$$

If $\|T^*(z_n - Tw_n)\|^2\| \neq 0$, we obtain from (4.4) that

$$\|z_n - Tw_n\|^2 - \eta_n\|T^*(z_n - Tw_n)\|^2 = (1 - (\vartheta_n + \vartheta))\,\|Tw_n - z_n\|^2 \geq 0. \tag{4.21}$$

It follows from (4.17) and (4.20) that

$$\|b_n - \bar{x}\| \leq \|w_n - \bar{x}\| \leq \|x_n - \bar{x}\| + \alpha_n M_1, \ \forall\ n \geq \mathbb{N}. \tag{4.22}$$

On the other hand, if $\|T^*(z_n - Tw_n)\|^2 = 0$, then from (4.19), $\|b_n - \bar{x}\| \leq \|w_n - \bar{x}\|$ clearly holds.

Next, observe that from (4.6) we have

$$\|Au_n - Al_n\| \leq \frac{(\sigma_n + \sigma)}{\lambda_{n+1}}\|u_n - l_n\|, \quad \forall n \in \mathbb{N}. \tag{4.23}$$

Observe that (4.23) holds for both $Au_n = Al_n$ and $Au_n \neq Al_n$. Now, from the definition of v_n and by applying (4.23) we have

$$\begin{aligned}
\|v_n - \bar{x}\|^2 &= \|u_n - \lambda_n(Au_n - Al_n) - \bar{x}\|^2 \\
&= \|u_n - \bar{x}\|^2 + \lambda_n^2\|Au_n - Al_n\|^2 - 2\lambda_n\langle u_n - \bar{x}, Au_n - Al_n\rangle \\
&= \|u_n - l_n\|^2 + \|l_n - \bar{x}\|^2 + 2\langle u_n - l_n, l_n - \bar{x}\rangle + \lambda_n^2\|Au_n - Al_n\|^2 \\
&- 2\lambda_n\langle u_n - \bar{x}, Au_n - Al_n\rangle \\
&= \|l_n - \bar{x}\|^2 + \|u_n - l_n\|^2 - 2\langle u_n - l_n, u_n - l_n\rangle + 2\langle u_n - l_n, u_n - \bar{x}\rangle \\
&+ \lambda_n^2\|Au_n - Al_n\|^2 - 2\lambda_n\langle u_n - \bar{x}, Au_n - Al_n\rangle \\
&= \|l_n - \bar{x}\|^2 - \|u_n - l_n\|^2 + 2\langle u_n - l_n, u_n - \bar{x}\rangle + \lambda_n^2\|Au_n - Al_n\|^2 \\
&- 2\lambda_n\langle u_n - \bar{x}, Au_n - Al_n\rangle \\
&= \|l_n - \bar{x}\|^2 - \|u_n - l_n\|^2 - 2\langle l_n - u_n - \lambda_n(Al_n - Au_n), u_n - \bar{x}\rangle \\
&+ \lambda_n^2\|Au_n - Al_n\|^2 \\
&\leq \|l_n - \bar{x}\|^2 - \left(1 - \frac{\lambda_n^2(\sigma_n + \sigma)^2}{\lambda_{n+1}^2}\right)\|u_n - l_n\|^2 \\
&- 2\langle l_n - u_n - \lambda_n(Al_n - Au_n), u_n - \bar{x}\rangle. \qquad (4.24)
\end{aligned}$$

From $u_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)l_n$, we have $(I - \lambda_n A)l_n \in (I + \lambda_n B)u_n$. Since B is maximal monotone, there exists $d_n \in Bu_n$ such that

$$(I - \lambda_n A)l_n = u_n + \lambda_n d_n,$$

which implies that

$$d_n = \frac{1}{\lambda_n}(l_n - u_n - \lambda_n A l_n). \tag{4.25}$$

Since $\bar{x} \in \Upsilon$, we have $0 \in (A + B)\bar{x}$ and $Au_n + d_n \in (A + B)u_n$. Since $A + B$ is maximal monotone, we obtain

$$\langle Au_n + d_n, u_n - \bar{x}\rangle \geq 0. \tag{4.26}$$

Substituting (4.25) into (4.26), we obtain

$$\frac{1}{\lambda_n}\langle l_n - u_n - \lambda_n A l_n + \lambda_n A u_n, u_n - \bar{x}\rangle \geq 0,$$

which gives

$$\langle l_n - u_n - \lambda_n(Al_n - Au_n), u_n - \bar{x}\rangle \geq 0. \tag{4.27}$$

Applying (4.27) in (4.24), we have

$$\|v_n - \bar{x}\|^2 \leq \|l_n - \bar{x}\|^2 - \left(1 - \frac{\lambda_n^2(\sigma_n + \sigma)^2}{\lambda_{n+1}^2}\right)\|u_n - l_n\|^2. \tag{4.28}$$

By the definition of v_n and applying (4.23), we obtain

$$\|v_n - u_n\| \leq \frac{\lambda_n(\sigma_n + \sigma)}{\lambda_{n+1}}\|u_n - l_n\|. \tag{4.29}$$

Now, from **Step 4** and Assumption 4.1.1 (2), we have

$$\frac{\varrho_n}{\alpha_n}\|b_n - b_{n-1}\| \leq \frac{\tau_n}{\alpha_n} \to 0, \quad \text{as } n \to \infty. \tag{4.30}$$

By the definition of l_n, we have

$$\begin{aligned}
\|l_n - \bar{x}\| &= \|(1 - \alpha_n)(b_n + \varrho_n(b_n - b_{n-1})) - \bar{x}\| \\
&= \|(1 - \alpha_n)(b_n - \bar{x}) + (1 - \alpha_n)\varrho_n(b_n - b_{n-1}) - \alpha_n\bar{x}\| \\
&\leq (1 - \alpha_n)\|b_n - \bar{x}\| + (1 - \alpha_n)\varrho_n\|b_n - b_{n-1}\| + \alpha_n\|\bar{x}\| \\
&= (1 - \alpha_n)\|b_n - \bar{x}\| + \alpha_n\left[(1 - \alpha_n)\frac{\varrho_n}{\alpha_n}\|b_n - b_{n-1}\| + \|\bar{x}\|\right],
\end{aligned} \tag{4.31}$$

By (4.30), we obtain

$$\lim_{n\to\infty}\left[(1 - \alpha_n)\frac{\varrho_n}{\alpha_n}\|b_n - b_{n-1}\| + \|\bar{x}\|\right] = \|\bar{x}\|.$$

Thus, there exists $M_2 > 0$ such that

$$(1-\alpha_n)\frac{\varrho_n}{\alpha_n}\|b_n - b_{n-1}\| + \|\bar{x}\| \leq M_2, \quad \forall n \in \mathbb{N}. \tag{4.32}$$

Combining (4.31) and (4.32), we have

$$\|l_n - \bar{x}\| \leq (1-\alpha_n)\|b_n - \bar{x}\| + \alpha_n M_2, \ \forall\ n \geq \mathbb{N}. \tag{4.33}$$

By Lemma 4.1.6 there exists $N \in \mathbb{N}$ such that $\left(1 - \frac{\lambda_n^2(\sigma_n+\sigma)^2}{\lambda_{n+1}^2}\right) > 0\ \forall\ n \geq N$. Thus, from (4.28) we obtain

$$\|v_n - \bar{x}\|^2 \leq \|l_n - \bar{x}\|^2, \quad \forall\ n \geq N. \tag{4.34}$$

Using the last inequality together with (4.22) and (4.33) we obtain

$$\begin{aligned}\|v_n - \bar{x}\| \leq \|l_n - \bar{x}\| &\leq (1-\alpha_n)\|w_n - \bar{x}\| + \alpha_n M_2 \\ &\leq (1-\alpha_n)(\|x_n - \bar{x}\| + +\alpha_n M_1) + \alpha_n M_2 \\ &\leq (1-\alpha_n)\|x_n - \bar{x}\| + \alpha_n(M_1 + M_2).\end{aligned} \tag{4.35}$$

Now, from **Step 6**, (4.34),Lemma 2.5.13 and Lemma 2.5.18(iii), we obtain

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &= \|(1-\phi_n)l_n + \phi_n S_\alpha v_n - \bar{x}\|^2 \\ &= \|(1-\phi_n)(l_n - \bar{x}) + \phi_n(S_\alpha v_n - \bar{x})\|^2 \\ &= (1-\phi_n)\|l_n - \bar{x}\|^2 + \phi_n\|S_\alpha v_n - \bar{x}\|^2 - (1-\phi_n)\phi_n\|S_\alpha v_n - l_n\|^2 \\ &\leq (1-\phi_n)\|l_n - \bar{x}\|^2 + \phi_n\|v_n - \bar{x}\|^2 - (1-\phi_n)\phi_n\|S_\alpha v_n - l_n\|^2 \\ &\leq \|l_n - \bar{x}\|^2 - (1-\phi_n)\phi_n\|S_\alpha v_n - l_n\|^2,\end{aligned} \tag{4.36}$$

which follows from (4.35) that

$$\begin{aligned}\|x_{n+1} - \bar{x}\| &\leq \|l_n - \bar{x}\| \\ &\leq (1-\alpha_n)\|x_n - \bar{x}\| + \alpha_n(M_1 + M_2) \\ &= (1-\alpha_n)\|x_n - \bar{x}\| + \alpha_n K \\ &\leq \max\{\|x_n - \bar{x}\|, K\} \\ &\vdots \\ &\leq \max\{\|x_N - \bar{x}\|, K\}.\end{aligned}$$

where $K = M_1 + M_2$. Therefore, the sequence $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{b_n\}$, $\{l_n\}$, $\{u_n\}$, $\{v_n\}$ are all bounded. □

Lemma 4.1.9. *Suppose Assumption 4.1.1 hold;* $\lim\limits_{j\to\infty}\|b_{n_j} - w_{n_j}\| = 0 = \lim\limits_{j\to\infty}\|l_{n_j} - v_{n_j}\|$ *and* $\{x_n\}$ *is a sequence generated by Algorithm 4.1.2. If there exists a subsequence* $\{x_{n_j}\}$ *of* $\{x_n\}$ *which converges weakly to an element* $\hat{x} \in H_1$*, then* $\hat{x} \in \Omega$.

Proof. Let $\bar{x} \in \Omega$. Then, we obtain from (4.20) and (4.21) that

$$\|b_{n_j} - \bar{x}\|^2 \leq \|w_{n_j} - \bar{x}\|^2 - \eta_{n_j}\left[1 - (\vartheta_{n_j} + \vartheta)\right]\|Tw_{n_j} - z_{n_j}\|^2, \tag{4.37}$$

which implies that

$$\begin{aligned}\eta_{n_j}\left[1 - (\vartheta_{n_j} + \vartheta)\right]\|Tw_{n_j} - z_{n_j}\|^2 &\leq \|w_{n_j} - \bar{x}\|^2 - \|b_{n_j} - \bar{x}\|^2 \\ &\leq \|w_{n_j} - b_{n_j}\|^2 + 2\|w_{n_j} - b_{n_j}\|\|b_{n_j} - \bar{x}\|. \end{aligned}\tag{4.38}$$

Since $\{b_{n_j}\}$ is bounded, then by the hypothesis that $\lim_{n\to\infty}\|b_{n_j} - w_{n_j}\| = 0$, from (4.38) we obtain

$$\lim_{j\to\infty}\eta_{n_j}\left[1 - (\vartheta_{n_j} + \vartheta)\right]\|Tw_{n_j} - z_{n_j}\|^2 = 0,$$

which by the conditions on the control parameters implies that

$$\lim_{j\to\infty}\|Tw_{n_j} - z_{n_j}\| = 0. \tag{4.39}$$

From (4.8), we obtain

$$\|z_{n_j} - T\bar{x}\|^2 \leq \|Tw_{n_j} - T\bar{x}\|^2 - \left(1 - \frac{\gamma_{n_j}^2(\kappa_{n_j} + \kappa)^2}{\gamma_{n_{j+1}}^2}\right)\|Tw_{n_j} - y_{n_j}\|^2,$$

which implies that

$$\begin{aligned}\left(1 - \frac{\gamma_{n_j}^2(\kappa_{n_j} + \kappa)^2}{\gamma_{n_{j+1}}^2}\right)\|Tw_{n_j} - y_{n_j}\|^2 &\leq \|Tw_{n_j} - T\bar{x}\|^2 - \|z_{n_j} - T\bar{x}\|^2 \\ &= \|Tw_{n_j} - z_{n_j}\|^2 + \|z_{n_j} - T\bar{x}\|^2 \\ &\quad + 2\langle Tw_{n_j} - z_{n_j}, z_{n_j} - T\bar{x}\rangle - \|z_{n_j} - T\bar{x}\|^2 \\ &\leq \|Tw_{n_j} - z_{n_j}\|^2 + 2\|Tw_{n_j} - z_{n_j}\|\|z_{n_j} - T\bar{x}\|.\end{aligned}$$

It follows from (4.39) and Assumption 4.1.1 (2) that

$$\lim_{j\to\infty}\|Tw_{n_j} - y_{n_j}\| = 0. \tag{4.40}$$

Now, let $(v, w) \in \text{Graph}(D + G)$, then, $w - Dv \in Gv$. Since $y_{n_j} = (I + \gamma_{n_j}G)^{-1}(I - \gamma_{n_j}D)Tw_{n_j}$, we have

$$Tw_{n_j} - \gamma_{n_j}DTw_{n_j} \in y_{n_j} + \gamma_{n_j}Gy_{n_j},$$

which implies that

$$\frac{1}{\gamma_{n_j}}(Tw_{n_j} - y_{n_j} - \gamma_{n_j}DTw_{n_j}) \in Gy_{n_j}.$$

Thanks to the maximal monotonicity of G, which guarantees that

$$\langle v - y_{n_j}, w - Dv - \frac{1}{\gamma_{n_j}}(Tw_{n_j} - y_{n_j} - \gamma_{n_j}DTw_{n_j})\rangle \geq 0. \tag{4.41}$$

Using (4.41) and the monotonicity of D, we have

$$\begin{aligned}\langle v - y_{n_j}, w\rangle &\geq \langle v - y_{n_j}, Dv + \frac{1}{\gamma_{n_j}}(Tw_{n_j} - y_{n_j} - \gamma_{n_j} DTw_{n_j})\rangle\\ &= \langle v - y_{n_j}, Dv - Dy_{n_j}\rangle + \langle v - y_{n_j}, Dy_{n_j} - DTw_{n_j}\rangle\\ &+ \langle v - y_{n_j}, \frac{1}{\gamma_{n_j}}(Tw_{n_j} - y_{n_j})\rangle\\ &\geq \langle v - y_{n_j}, Dy_{n_j} - DTw_{n_j}\rangle + \langle v - y_{n_j}, \frac{1}{\gamma_{n_j}}(Tw_{n_j} - y_{n_j})\rangle. \end{aligned} \tag{4.42}$$

Moreover, we obtain from the definition of w_{n_j} and θ_{n_j} and by Remark 5.2.4 that

$$\lim_{j\to\infty} \|w_{n_j} - x_{n_j}\| = \lim_{j\to\infty} \theta_{n_j}\|x_{n_j} - x_{n_j-1}\| = 0. \tag{4.43}$$

Furthermore, since $x_{n_j} \rightharpoonup \hat{x} \in H_1$, then it follows that there exists a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $w_{n_j} \rightharpoonup \hat{x}$. Also, since T is a bounded linear operator, we have $Tw_{n_j} \rightharpoonup T\hat{x}$. It follows from (4.40) and the continuity of D that

$$y_{n_j} \rightharpoonup T\hat{x} \text{ and } \lim_{j\to\infty} \|DTw_{n_j} - Dy_{n_j}\| = 0. \tag{4.44}$$

Thus, passing limit as $j \to \infty$ in (4.42), we obtain

$$\langle v - T\hat{x}, w\rangle \geq 0, \quad \forall (v, w) \in \text{Graph}(D + G). \tag{4.45}$$

Thus, by the maximal monotonicity of $D + G$, we conclude that $T\hat{x} \in (D + G)^{-1}(0)$.

Also, by applying similar argument used in obtaining (4.40), (4.42), (4.43), we have

$$\lim_{j\to\infty} \|l_{n_j} - u_{n_j}\| = 0, \tag{4.46}$$

$$\lim_{j\to\infty} \|l_{n_j} - b_{n_j}\| = 0, \tag{4.47}$$

$$\langle \hat{v} - u_{n_j}, \hat{w}\rangle \geq 0, \quad \forall\ (\hat{v}, \hat{w}) \in \text{Graph}(A + B). \tag{4.48}$$

In the same vein, since $x_{n_j} \rightharpoonup \hat{x} \in H_1$, we obtain using (4.43) and our hypothesis $\lim_{j\to\infty} \|b_{n_j} - w_{n_j}\| = 0$ together with (4.46)-(4.48) that $\{w_{n_j}\}, \{b_{n_j}\}, \{l_{n_j}\}$ and $\{u_{n_j}\}$ all converge weakly to $\hat{x}$. Hence, we have

$$\langle \hat{v} - \hat{x}, \hat{w}\rangle = \lim_{j\to\infty} \langle \hat{v} - u_{n_j}, \hat{w}\rangle \geq 0, \quad \forall\ (\hat{w}, \hat{v}) \in \text{Graph}(A + B).$$

By the maximal monotonicity of $A + B$, we conclude that $\hat{x} \in (A + B)^{-1}(0)$. Therefore, $\hat{x} \in \Omega$.

□

Theorem 4.1.10. *Let $\{x_n\}$ be an iterative sequence generated by Algorithm 4.1.2 such that Assumptions 4.1.1 are satisfied. Then, $\{x_n\}$ converges strongly to an element $\bar{x} \in \Upsilon$, where $\|\bar{x}\| = \min\{\|p\| : p \in \Upsilon\}$.*

Proof. Let $\bar{x} \in \Upsilon$. Since $\bar{x} = \min\{\|p\| : p \in \Upsilon\}$, we have $\bar{x} = P_{\Upsilon}(0)$. We obtain using Cauchy-Schwartz inequality, **Step 2**, Lemma 2.5.18 and (4.16) that

$$\begin{aligned}
\|w_n - \bar{x}\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - \bar{x}\|^2 \\
&= \|x_n - \bar{x}\|^2 + 2\theta_n\langle x_n - \bar{x},\ x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - \bar{x}\|^2 + 2\theta_n\|x_n - \bar{x}\|.\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - \bar{x}\|^2 + \theta_n\|x_n - x_{n-1}\|(2\|x_n - \bar{x}\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|) \\
&\leq \|x_n - \bar{x}\|^2 + \theta_n\|x_n - x_{n-1}\|[\alpha_n M_1 + 2\|x_n - \bar{x}\|]. \qquad (4.49)
\end{aligned}$$

Now, since $\{x_n\}$ is bounded, then, there exists $M_3 > 0$ such that $\alpha_n M_1 + 2\|x_n - \bar{x}\| \leq M_3,\ \forall\ n \geq 1$. It follows from (4.49) that

$$\|w_n - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2 + \theta_n\|x_n - x_{n-1}\|M_3. \qquad (4.50)$$

In the same way, we obtain from Lemma 2.5.18 and (4.30) that

$$\begin{aligned}
\|l_n - \bar{x}\|^2 &= \|(1-\alpha_n)\left(b_n + \varrho_n(b_n - b_{n-1})\right) - \bar{x}\|^2 \\
&= \|(1-\alpha_n)(b_n - \bar{x}) + \varrho_n(1-\alpha_n)(b_n - b_{n-1}) - \alpha_n\bar{x}\|^2 \\
&\leq \|(1-\alpha_n)(b_n - \bar{x}) + \varrho_n(1-\alpha_n)(b_n - b_{n-1})\|^2 + \alpha_n^2\|\bar{x}\|^2. \qquad (4.51)
\end{aligned}$$

But

$$\begin{aligned}
&\|(1-\alpha_n)(b_n - \bar{x}) + \varrho_n(1-\alpha_n)(b_n - b_{n-1})\|^2 \\
&\quad \leq (1-\alpha_n)^2\|b_n - \bar{x}\|^2 + 2(1-\alpha_n)\varrho_n\|b_n - \bar{x}\|\|b_n - b_{n-1}\| + \varrho_n^2\|b_n - b_{n-1}\|^2 \\
&\quad \leq (1-\alpha_n)^2\|b_n - \bar{x}\|^2 + \varrho_n\|b_n - b_{n-1}\|\left(2(1-\alpha_n)\|b_n - \bar{x}\| + \alpha_n\frac{\varrho_n}{\alpha_n}\|b_n - b_{n-1}\|\right) \\
&\quad \leq (1-\alpha_n)\|b_n - \bar{x}\|^2 + \varrho_n\|b_n - b_{n-1}\|\left(2\|b_n - \bar{x}\| + \alpha_n M_2^*\right),
\end{aligned}$$

for some $M_2^* > 0$. Thus, there exists $M_4 > 0$ such that $2\|b_n - \bar{x}\| + \alpha_n M_2^* \leq M_4,\ \ \forall\ n \geq 1$. Using (4.51), we get

$$\begin{aligned}
\|l_n - \bar{x}\|^2 &\leq (1-\alpha_n)\|b_n - \bar{x}\|^2 + \varrho_n\|b_n - b_{n-1}\|M_4 + \alpha_n^2\|\bar{x}\|^2 \\
&= (1-\alpha_n)\|b_n - \bar{x}\|^2 + \alpha_n\left(\frac{\varrho_n}{\alpha_n}\|b_n - b_{n-1}\|M_4 + \alpha_n\|\bar{x}\|^2\right). \qquad (4.52)
\end{aligned}$$

Since $\{l_n\}$ is bounded, then there exists $M_5 > 0$ such that $\frac{\varrho_n}{\alpha_n}\|b_n - b_{n-1}\|M_4 + \alpha_n\|\bar{x}\|^2 \leq M_5,\ \forall\ n \geq 1$. It follows from (4.52) that

$$\|l_n - \bar{x}\|^2 \leq (1-\alpha_n)\|b_n - \bar{x}\|^2 + \alpha_n M_5. \qquad (4.53)$$

Using Lemma 2.5.18, Lemma 2.5.13, (4.20), (4.28), (4.50) and (4.53), we obtain from **Step 6** that

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &= \|(1-\phi_n)l_n+\phi_n S_\alpha v_n-\bar{x}\|^2\\
&= \|(1-\phi_n)(l_n-\bar{x})+\phi_n(S_\alpha v_n-\bar{x})\|^2\\
&= (1-\phi_n)\|l_n-\bar{x}\|^2+\phi_n\|S_\alpha v_n-\bar{x}\|^2-(1-\phi_n)\phi_n\|S_\alpha v_n-l_n\|^2\\
&\leq (1-\phi_n)\|l_n-\bar{x}\|^2+\phi_n\|v_n-\bar{x}\|^2-(1-\phi_n)\phi_n\|S_\alpha v_n-l_n\|^2\\
&\leq (1-\phi_n)\|l_n-\bar{x}\|^2+\phi_n\left[\|l_n-\bar{x}\|^2-\left(1-\frac{\lambda_n^2(\sigma_n+\sigma)^2}{\lambda_{n+1}^2}\right)\|u_n-l_n\|^2\right]\\
&\quad-(1-\phi_n)\phi_n\|S_\alpha v_n-l_n\|^2\\
&= \|l_n-\bar{x}\|^2-\phi_n\left(1-\frac{\lambda_n^2(\sigma_n+\sigma)^2}{\lambda_{n+1}^2}\right)\|u_n-l_n\|^2\\
&\quad-(1-\phi_n)\phi_n\|S_\alpha v_n-l_n\|^2 \qquad (4.54)\\
&\leq (1-\alpha_n)\|b_n-\bar{x}\|^2+\alpha_n M_5-\phi_n\left(1-\frac{\lambda_n^2(\sigma_n+\sigma)^2}{\lambda_{n+1}^2}\right)\|u_n-l_n\|^2\\
&\quad-(1-\phi_n)\phi_n\|S_\alpha v_n-l_n\|^2 \qquad (4.55)\\
&\leq (1-\alpha_n)\|w_n-\bar{x}\|^2+\alpha_n M_5-\phi_n\left(1-\frac{\lambda_n^2(\sigma_n+\sigma)^2}{\lambda_{n+1}^2}\right)\|u_n-l_n\|^2\\
&\quad-(1-\phi_n)\phi_n\|S_\alpha v_n-l_n\|^2\\
&\leq (1-\alpha_n)\|x_n-\bar{x}\|^2+\theta_n(1-\alpha_n)\|x_n-x_{n-1}\|M_3+\alpha_n M_5\\
&\quad-\phi_n\left(1-\frac{\lambda_n^2(\sigma_n+\sigma)^2}{\lambda_{n+1}^2}\right)\|u_n-l_n\|^2-(1-\phi_n)\phi_n\|S_\alpha v_n-l_n\|^2. \qquad (4.56)
\end{aligned}$$

Furthermore, we obtain from (4.50), (4.54), and Lemma 2.5.18(ii) that

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\leq \|l_n-\bar{x}\|^2\\
&= \|(1-\alpha_n)(b_n-\bar{x})+(1-\alpha_n)\varrho_n(b_n-b_{n-1})-\alpha_n\bar{x}\|^2\\
&\leq \|(1-\alpha_n)(b_n-\bar{x})+(1-\alpha_n)\varrho_n(b_n-b_{n-1})\|^2+2\alpha_n\langle-\bar{x},l_n-\bar{x}\rangle\\
&\leq (1-\alpha_n)^2\|b_n-\bar{x}\|^2+2(1-\alpha_n)\varrho_n\|b_n-\bar{x}\|\|b_n-b_{n-1}\|+\varrho_n^2\|b_n-b_{n-1}\|^2\\
&\quad+2\alpha_n\langle-\bar{x},l_n-x_{n+1}\rangle+2\alpha_n\langle-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\leq (1-\alpha_n)\|w_n-\bar{x}\|^2+2(1-\alpha_n)\varrho_n\|b_n-\bar{x}\|\|b_n-b_{n-1}\|+\varrho_n^2\|b_n-b_{n-1}\|^2\\
&\quad+2\alpha_n\|\bar{x}\|\|l_n-x_{n+1}\|+2\alpha_n\langle-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\leq (1-\alpha_n)[\|x_n-\bar{x}\|^2+\theta_n\|x_n-x_{n-1}\|M_3]+2(1-\alpha_n)\varrho_n\|b_n-\bar{x}\|\|b_n-b_{n-1}\|\\
&\quad+\varrho_n^2\|b_n-b_{n-1}\|^2+2\alpha_n\|\bar{x}\|\|l_n-x_{n+1}\|+2\alpha_n\langle-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\leq (1-\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n\Big[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|M_3\\
&\quad+2(1-\alpha_n)\frac{\varrho_n}{\alpha_n}\|b_n-b_{n-1}\|\|b_n-\bar{x}\|\\
&\quad+\varrho_n\|b_n-b_{n-1}\|\frac{\varrho_n}{\alpha_n}\|b_n-b_{n-1}\|+2\|\bar{x}\|\|l_n-x_{n+1}\|+2\langle-\bar{x},x_{n+1}-\bar{x}\rangle\Big]\\
&\leq (1-\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n\Phi_n, \qquad (4.57)
\end{aligned}$$

where
$\Phi_n = (1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|M_3+2(1-\alpha_n)\frac{\varrho_n}{\alpha_n}\|b_n-b_{n-1}\|\|b_n-\bar{x}\|+\varrho_n\|b_n-b_{n-1}\|\frac{\varrho_n}{\alpha_n}\|b_n-b_{n-1}\| + 2\|\bar{x}\|\|l_n - x_{n+1}\| + 2\langle -\bar{x}, x_{n+1} - \bar{x}\rangle$.

Next, we claim that the sequence $\Big\{\|x_n-\bar{x}\|\Big\}$ converges to zero. Indeed, in view of Lemma 2.5.55 and by Remark 4.1.3, it is enough to establish that $\limsup_{k\to\infty}\langle -\bar{x}, x_{n_{k+1}} - \bar{x}\rangle \le 0$ and $\lim_{k\to\infty}\|l_{n_k} - x_{n_k+1}\| = 0$, for every subsequence $\Big\{\|x_{n_k} - \bar{x}\|\Big\}$ of $\Big\{\|x_n - \bar{x}\|\Big\}$ satisfying

$$\liminf_{k\to\infty}\Big(\|x_{n_k+1} - \bar{x}\| - \|x_{n_k} - \bar{x}\|\Big) \ge 0. \tag{4.58}$$

Now, let $\Big\{\|x_{n_k}-\bar{x}\|\Big\}$ be a subsequence of $\Big\{\|x_n-\bar{x}\|\Big\}$ such that (4.58) holds. From (4.56), we get

$$\begin{aligned}
&\limsup_{k\to\infty}\left[\phi_{n_k}\left(1-\frac{\lambda_{n_k}^2(\sigma_{n_k}+\sigma)^2}{\lambda_{n_{k+1}}^2}\right)\|u_{n_k}-l_{n_k}\|^2\right]\\
&\le \limsup_{k\to\infty}\Bigg((1-\alpha_{n_k})\|x_{n_k}-\bar{x}\|^2 - \|x_{n_{k+1}}-\bar{x}\|^2\\
&+\alpha_{n_k}\left(\frac{\theta_{n_k}}{\alpha_{n_k}}(1-\alpha_{n_k})\|x_{n_k}-x_{n_{k-1}}\|M_3+M_5\right)\Bigg)\\
&= -\liminf_{k\to\infty}\left(\|x_{n_{k+1}}-\bar{x}\|^2 - \|x_{n_k}-\bar{x}\|^2\right)\\
&\le 0.
\end{aligned}$$

Since $\lim_{k\to\infty}\alpha_{n_k} = 0$, it follows from Assumption 4.1.1 (2) that

$$\lim_{k\to\infty}\|u_{n_k} - l_{n_k}\| = 0. \tag{4.59}$$

In like manner, we obtain from (4.56) that

$$\begin{aligned}
&\limsup_{k\to\infty}(1-\phi_{n_k})\phi_{n_k}\|S_\alpha v_{n_k} - l_{n_k}\|^2\\
&\le \limsup_{k\to\infty}\Bigg((1-\alpha_{n_k})\|x_{n_k}-\bar{x}\|^2 - \|x_{n_{k+1}}-\bar{x}\|^2\\
&+\alpha_{n_k}\left(\frac{\theta_{n_k}}{\alpha_{n_k}}(1-\alpha_{n_k})\|x_{n_k}-x_{n_{k-1}}\|M_3+M_5\right)\Bigg)\\
&= -\liminf_{k\to\infty}\left(\|x_{n_{k+1}}-\bar{x}\|^2 - \|x_{n_k}-\bar{x}\|^2\right)\\
&\le 0.
\end{aligned}$$

Consequently, by the conditions of the control parameters, we obtain

$$\lim_{k\to\infty}\|S_\alpha v_{n_k} - l_{n_k}\| = 0. \tag{4.60}$$

In addition, we obtain from (4.55) and (4.37) that

$$\begin{aligned}\|x_{n_{k+1}}-\bar{x}\|^2 &\leq (1-\alpha_{n_k})\|b_{n_k}-\bar{x}\|^2+\alpha_{n_k}M_5\\ &\leq (1-\alpha_{n_k})\|w_{n_k}-\bar{x}\|^2-\eta_{n_k}[1-(\vartheta_{n_k}+\vartheta)]\|Tw_{n_k}-z_{n_k}\|^2+\alpha_{n_k}M_5\\ &\leq (1-\alpha_{n_k})\|x_{n_k}-\bar{x}\|^2-\eta_{n_k}[1-(\vartheta_{n_k}+\vartheta)]\|Tw_{n_k}-z_{n_k}\|^2\\ &+\theta_{n_k}(1-\alpha_{n_k})\|x_{n_k}-x_{n_k-1}\|M_3+\alpha_{n_k}M_5.\end{aligned}$$

By applying (4.58), Assumption 4.1.1 (2), Lemma 4.1.5 and the fact that $\lim_{k\to\infty}\alpha_{n_k}=0$ we have

$$\begin{aligned}&\limsup_{k\to\infty}\left(\eta_{n_k}[1-(\vartheta_{n_k}+\vartheta)]\|Tw_{n_k}-z_{n_k}\|^2\right)\\ &\leq \limsup_{k\to\infty}\Bigg((1-\alpha_{n_k})\|x_{n_k}-\bar{x}\|^2-\|x_{n_{k+1}}-\bar{x}\|^2\\ &+\alpha_{n_k}\left(\frac{\theta_{n_k}}{\alpha_{n_k}}(1-\alpha_{n_k})\|x_{n_k}-x_{n_{k-1}}\|M_3+M_5\right)\Bigg)\\ &=-\liminf_{k\to\infty}\left(\|x_{n_{k+1}}-\bar{x}\|^2-\|x_{n_k}-\bar{x}\|^2\right)\\ &\leq 0.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\|Tw_{n_k}-z_{n_k}\|=0.$$

It is easy to see from **Step 3** that

$$\|b_{n_k}-w_{n_k}\|=\eta_{n_k}\|T^*(z_{n_k}-Tw_{n_k})\|=(\vartheta_{n_k}+\vartheta)\|Tw_{n_k}-z_{n_k}\|\to 0 \text{ as } k\to\infty. \tag{4.61}$$

Observe that when $\|T^*(z_{n_k}-Tw_{n_k})\|=0$, clearly, $\|b_{n_k}-w_{n_k}\|=0$.

Also, from **Step 5**, we obtain

$$\begin{aligned}\|l_{n_k}-b_{n_k}\| &\leq (1-\alpha_{n_k})\|b_{n_k}-b_{n_k}\|+(1-\alpha_{n_k})\varrho_{n_k}\|b_{n_k}-b_{n_{k-1}}\|+\alpha_{n_k}\|b_{n_k}\|\\ &=\alpha_{n_k}\left((1-\alpha_{n_k})\frac{\varrho_{n_k}}{\alpha_{n_k}}\|b_{n_k}-b_{n_{k-1}}\|+\|b_{n_k}\|\right).\end{aligned}$$

Since $\lim_{k\to\infty}\alpha_{n_k}=0$, we get

$$\lim_{k\to\infty}\|l_{n_k}-b_{n_k}\|=0. \tag{4.62}$$

Using (4.29) and (4.59), we obtain

$$\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\|=0. \tag{4.63}$$

Combining (4.59) and (4.63), we obtain

$$\lim_{k\to\infty}\|v_{n_k}-l_{n_k}\|\leq \lim_{k\to\infty}\left(\|v_{n_k}-u_{n_k}\|+\|u_{n_k}-l_{n_k}\|\right)=0. \tag{4.64}$$

It is easy to see from (4.60) and (4.64) that

$$\lim_{k\to\infty} \|S_\alpha v_{n_k} - v_{n_k}\| \le \lim_{k\to\infty} (\|S_\alpha v_{n_k} - l_{n_k}\| + \|l_{n_k} - v_{n_k}\|) = 0. \tag{4.65}$$

Using (4.62) and (4.64), we obtain

$$\lim_{k\to\infty} \|b_{n_k} - v_{n_k}\| \le \lim_{k\to\infty} (\|l_{n_k} - b_{n_k}\| + \|v_{n_k} - l_{n_k}\|) = 0. \tag{4.66}$$

Now, by Remark 4.1.3, we obtain

$$\|w_{n_k} - x_{n_k}\| = \theta_{n_k}\|x_{n_k} - x_{n_{k-1}}\| \to 0,\ k \to \infty. \tag{4.67}$$

Since $\{b_n\}$ is bounded, it follows from (4.62), (4.65) and (4.66) and **Step 6** that

$$\|x_{n_{k+1}} - b_{n_k}\| \le (1-\phi_{n_k})\|l_{n_k} - b_{n_k}\| + \phi_{n_k}\|S_\alpha v_{n_k} - b_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{4.68}$$

Also, using (4.61) and (4.68), we obtain

$$\lim_{k\to\infty} \|x_{n_{k+1}} - w_{n_k}\| \le \lim_{k\to\infty} (\|x_{n_{k+1}} - b_{n_k}\| + \|b_{n_k} - w_{n_k}\|) = 0. \tag{4.69}$$

Moreover, we obtain using (4.62) and (4.68) that

$$\lim_{k\to\infty} \|l_{n_k} - x_{n_{k+1}}\| \le \lim_{k\to\infty} (\|l_{n_k} - b_{n_k}\| + \|x_{n_{k+1}} - b_{n_k}\|) = 0. \tag{4.70}$$

It follows from (4.67) and (4.69) that

$$\lim_{k\to\infty} \|x_{n_{k+1}} - x_{n_k}\| = 0. \tag{4.71}$$

Since $\{x_{n_k}\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Now, suppose $x^* \in w_\omega(x_n)$ be arbitrarily selected. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Using (4.67), we get $w_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Now, employing Lemma 4.1.9 together with (4.61) and (4.64), we have

$$x^* \in \Omega. \tag{4.72}$$

From (4.64), (4.70) and (4.71), we have that $v_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Furthermore, since $I - S_\alpha$ is demiclosed at zero, then by Lemma 2.5.13 and the fact that $\|S_\alpha v_{n_k} - v_{n_k}\| \to 0$, we have

$$x^* \in \mathrm{Fix}(S_\alpha) = \mathrm{Fix}(S). \tag{4.73}$$

Since $x^* \in w_\omega(x_n)$ is arbitrary, we obtain using (4.72) and (4.73) that

$$w_\omega(x_n) \subset \Upsilon.$$

Additionally, since $\{x_{n_k}\}$ is bounded, then there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that weakly converges to $x^\dagger \in H_1$ and

$$\lim_{j\to\infty} \langle -\bar{x},\ x_{n_{k_j}} - \bar{x}\rangle = \limsup_{k\to\infty} \langle -\bar{x},\ x_{n_k} - \bar{x}\rangle.$$

Since $\bar{x} = \mathrm{P}_{\Upsilon}(0)$, we have

$$\limsup_{k\to\infty}\langle -\bar{x},\ x_{n_k} - \bar{x}\rangle = \lim_{j\to\infty}\langle -\bar{x},\ x_{n_{k_j}} - \bar{x}\rangle = \langle -\bar{x},\ x^{\dagger} - \bar{x}\rangle \leq 0. \tag{4.74}$$

From (4.71) and (4.74), we obtain

$$\limsup_{k\to\infty}\langle -\bar{x},\ x_{n_{k+1}} - \bar{x}\rangle = \limsup_{k\to\infty}\langle -\bar{x},\ x_{n_k} - \bar{x}\rangle = \langle -\bar{x},\ x^{\dagger} - \bar{x}\rangle \leq 0. \tag{4.75}$$

Applying Lemma 2.5.55 to (4.57) and using (4.75) combined with (4.70), and the fact that $\lim\limits_{k\to\infty} \frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_{k-1}}\| = 0$, $\lim\limits_{k\to\infty} \frac{\varrho_{n_k}}{\alpha_{n_k}}\|b_{n_k} - b_{n_{k-1}}\| = 0$ and $\lim\limits_{k\to\infty} \alpha_{n_k} = 0$, we obtain that

$$\lim_{n\to\infty}\|x_n - x^*\| = 0.$$

Consequently, we conclude that $\{x_n\}$ converges strongly to the point $\bar{x} = P_{\Upsilon}(0)$. □

4.1.2 Applications

In this subsection, we apply our result to study some optimization problems and fixed point problem.

Split Equilibrium Problem and Fixed Point Problem

Let C and Q be two nonempty, closed and convex subsets of a real Hilbert spaces H_1 and H_2, respectively. Let $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ be two bifunctions. The Split Equilibrium Problem (SEP) is defined as follows:

$$\text{Locate } x^* \in C \text{ that solves } f(x^*, x) \geq 0\ \forall\ x \in C \tag{4.76}$$

such that

$$y^* = Tx^* \in Q \text{ solves } g(y^*, y) \geq 0\ \forall\ y \in Q. \tag{4.77}$$

Observe that whenever $H_1 = H_2$, $g = 0$ and $T = I_H$, then problem (4.76) and (4.77) reduces to the well-known classical Equilibrium Problem (EP) (4.76).

For solving equilibrium problem, we assume that the bifunction f satisfies the following conditions:

(F1) $f(x, x) = 0\ \forall\ x \in C$;

(F2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0\ \forall\ x, y \in C$;

(F3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;

(F4) for each $x \in C$, the mapping defined by $y \mapsto f(x, y)$ is convex and lower semicontinuous.

The following lemma is needed to solve equilibrium problems (4.76) and (4.77).

Lemma 4.1.11. *Let f be a bifunction satisfying $(F1)-(F4)$, and let B_f be a multi-valued mapping from H into itself defined by*

$$B_f x = \begin{cases} \{x^* \in H : f(x,y) \geq \langle y - x, x^* \rangle \ \forall\ y \in C\} & \text{if } x \in C \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then B_f is maximal monotone with domain $D(B_f) \subset C$ and $B_f^{-1}(0) = EP(f)$, where $EP(f)$ denotes the solution set of EP (4.76). On the other hand, the resolvent $J_\gamma^f := (I_H + \gamma B_f)^{-1}$ of B_f is defined by

$$J_\gamma^f x = \{x^* \in C : f(x^*, y) + \frac{1}{\gamma}\langle y - x^*, x^* - x\rangle \geq 0, \ \forall\ y \in C\} \ \forall\ x \in H.$$

With Lemma 4.1.11 in mind and setting $D = 0, A = 0$ in Theorem 4.1.10, we obtain the following relaxed double inertial method from Algorithm 4.1.2 for approximating the common solution of split equilibrium problem (4.76) and (4.77) and fixed point problem of strict pseudocontractive mappings in the framework of real Hilbert spaces.

Algorithm 4.1.12.

Initialization: *Suppose $\theta, \varrho, \gamma_1, \lambda_1 > 0$, $x_0, x_1 \in H_1$ be arbitrary and $x_0 = b_0$.*

Iterative Steps: *Set $n := 1$. Calculate x_{n+1} as follows:*

Step 1. *Given the iterates x_{n-1} and x_n $(n \geq 1)$, set*

$$\begin{aligned} w_n &= x_n + \theta(x_n - x_{n-1}) \\ y_n &= J_{\gamma_n}^g T w_n \\ \gamma_{n+1} &= \gamma_n + s_n. \end{aligned}$$

Step 2. *Compute*

$$b_n = w_n + \eta_n T^*(y_n - T w_n),$$

where

$$\eta_n = \begin{cases} \frac{(\vartheta_n + \vartheta)\|T w_n - y_n\|^2}{\|T^*(T w_n - y_n)\|^2}, & \text{if } \|T^*(T w_n - y_n)\|^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.78}$$

Step 3. *Compute*

$$\begin{aligned} l_n &= (1 - \alpha_n)\left(b_n + \varrho_n(b_n - b_{n-1})\right) \\ u_n &= J_{\lambda_n}^f l_n \\ \lambda_{n+1} &= \lambda_n + t_n \\ x_{n+1} &= (1 - \phi_n) l_n + \phi_n S_\alpha u_n, \end{aligned}$$

where

$$S_\alpha = \alpha I + (1 - \alpha) S.$$

Set $n := n + 1$ and go back to **Step 1.**

Theorem 4.1.13. *Let $f : H_1 \times H_1 \to \mathbb{R}$ and $g : H_2 \times H_2 \to \mathbb{R}$ be two bifunctions which satisfy conditions $(F1)-(F4)$ and assume $\Upsilon := \{x^* \in Fix(S_\alpha) \cap EP(f) : Tx^* \in EP(g)\} \neq \emptyset$. Suppose Assumption 4.1.1 holds, then the iterative sequence $\{x_n\}$ generated by Algorithm 4.1.12 converges strongly to an element of Υ.*

Split convex minimization problem and fixed point problem

Let $f_1 : H_1 \to \mathbb{R}$ and $f_2 : H_2 \to \mathbb{R}$ be two convex and continuously differentiable functions, $g_1 : H_1 \to \mathbb{R}$ and $g_2 : H_2 \to \mathbb{R}$ be two convex and lower semi-continuous functions, and let $T : H_1 \to H_2$ be a bounded linear mapping. We define the Split Convex Minimization Problem (for short, SCMP) as follows:

$$\text{Locate } \bar{x} \in H_1 \text{ that solves } f_1(\bar{x}) + g_1(\bar{x}) = \min_{x \in H_1}[f_1(x) + g_1(x)], \tag{4.79}$$

$$\text{such that } \bar{y} = T\bar{x} \in H_2 \text{ solves } f_2(\bar{y}) + g_2(\bar{y}) = \min_{y \in H_2}[f_2(y) + g_2(y)]. \tag{4.80}$$

Observe that when $H_1 = H_2 = H,\ f_2 = 0,\ g_2 = 0$ and $T = I_H$, then the SCMP (4.79) and (4.80) becomes the convex minimization problem (see [26]). We denote by Υ the solution set of SCMP.

It is a known fact that the gradients ∇f_1 and ∇f_2 of f_1 and f_2, respectively, are monotone and continuous (see [26]). In addition, the subdifferentials ∂g_1 of g_1 and ∂g_2 of g_2 are maximal monotone (see for instance [201]). Moreover, it is known that

$$f_1(\bar{x}) + g_1(\bar{x}) = \min_{x \in H_1}[f_1(x) + g_1(x)] \iff 0 \in [\nabla f_1(\bar{x}) + \partial g_1(\bar{x})].$$

Therefore, if we let $A = \nabla f_1,\ D = \nabla f_2,\ B = \partial g_1$ and $G = \partial g_2$ in Theorem 4.1.10, we obtain the following consequent relaxed double inertial scheme for approximating the common solution of SCMP (4.79) and (4.80) and fixed point of strict pseudocontractive mappings.

Algorithm 4.1.14.

Initialization: *Suppose $\theta, \varrho, \gamma_1, \lambda_1 > 0$, $x_0, x_1 \in H_1$ be arbitrary and $x_0 = b_0$.*

Iterative Steps: *Set $n := 1$. Calculate x_{n+1} as follows:*

Step 1. *Given the iterates x_{n-1} and x_n $(n \geq 1)$, set*

$$w_n = x_n + \theta(x_n - x_{n-1}).$$

Then compute

$$y_n = prox_{\gamma_n}^{g_2}(Tw_n - \gamma_n \nabla f_2 Tw_n) = (I + \gamma_n \partial g_2)^{-1}(Tw_n - \gamma_n \nabla f_2 Tw_n).$$

where

$$\gamma_{n+1} = \begin{cases} \min\left\{\frac{(\kappa_n+\kappa)\|Tw_n - y_n\|}{\|\nabla f_2 Tw_n - \nabla f_2 y_n\|},\ \gamma_n + s_n\right\}, & \text{if } \nabla f_2 Tw_n \neq \nabla f_2 y_n, \\ \gamma_n + s_n, & \text{otherwise}, \end{cases} \tag{4.81}$$

$$z_n = y_n - \gamma_n(\nabla f_2 y_n - \nabla f_2 Tw_n).$$

Step 2. *Compute*

$$b_n = w_n + \eta_n T^*(z_n - Tw_n),$$

where

$$\eta_n = \begin{cases} \frac{(\vartheta_n+\vartheta)\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2}, & \text{if } \|T^*(Tw_n - z_n)\|^2 \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.82}$$

$$\begin{aligned} l_n &= (1-\alpha_n)\left(b_n + \varrho_n(b_n - b_{n-1})\right) \\ u_n &= prox_{\lambda_n}^{g_1}(l_n - \lambda_n \nabla f_1 l_n) = (I + \lambda_n \partial g_1)^{-1}(l_n - \lambda_n \nabla f_1 l_n), \end{aligned}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{(\sigma_n+\sigma)\|u_n - l_n\|}{\|\nabla f_1 u_n - \nabla f_1 l_n\|}, \ \lambda_n + t_n\right\}, & \text{if } \nabla f_1 u_n \neq \nabla f_1 l_n, \\ \lambda_n + t_n, & \text{otherwise,} \end{cases} \tag{4.83}$$

Step 3. *Compute*

$$\begin{aligned} v_n &= u_n - \lambda_n(\nabla f_1 u_n - \nabla f_1 l_n) \\ x_{n+1} &= (1-\phi_n) l_n + \phi_n S_\alpha v_n, \end{aligned}$$

where

$$S_\alpha = \alpha I + (1-\alpha)S.$$

Set $n := n+1$ *and go back to* **Step 1.**

Theorem 4.1.15. *Let* f_1, f_2, g_1, g_2 *and* T *be as defined above and assume that* $\Upsilon := \{x^* \in Fix(S_\alpha) \cap prox(f_1, g_1) : Tx^* \in prox(f_2, g_2)\} \neq \emptyset$. *Suppose Assumption 4.1.1 holds, then the iterative sequence* $\{x_n\}$ *generated by Algorithm 4.1.14 converges strongly to an element of* Υ.

4.1.3 Numerical example

In this section, we demonstrate the numerical behavior of our method, (Pro Alg. 4.1.2) as well as comparing it with some related methods. In particular, we compare its performance with Algorithm 9.1.4 proposed by Kazmi & Rizvi (Kaz & Riz Alg.), Appendix 9.1.5 proposed by Izuchukwu *et al.* (Iz *et al.* Alg.), Appendix 9.1.6 proposed by Thong *et al.* (Th *et al.* Alg.), Appendix 9.1.7 by Shehu & Ogbuisi (Sh & Ogb Alg.), Appendix 9.1.8 by Izuchukwu, Riech *et al.*(Iz, Re *et al.* Alg.) and Appendix 9.1.9 by Wang *et al.* (Wa *et al.* Alg.).

The codes for the numerical analysis are implemented in Matlab 2021 (b) and performed on a personal computer. For our numerical simulations, we arbitrarily select the starting points $x_0, x_1 \in H_1$ (see the cases below). We also choose the parameters (randomly) $\gamma_1 = 0.03$, $\lambda_1 = 0.1, \theta = 0.85$, $\varrho = 0.82$, $\alpha = 0.125$, $\kappa = 0.97$, $\sigma = 0.81, \vartheta = 0.9$, $Sx = \frac{2x}{3}$ and the control sequences $\alpha_n = \frac{1}{2n+1}$, $\phi_n = \frac{1}{2}(1-\alpha_n)$, $\varepsilon_n = \frac{1}{(2n+1)^3}, \tau_n =$

$\frac{1}{(2n+1)^4}, \kappa_n = \frac{50}{n^{0.1}}, \vartheta_n = \frac{1}{n^{0.01}}, \sigma_n = \frac{200}{n^{0.01}}, s_n = \frac{20}{n^2}, t_n = \frac{50}{n^3}$ in Algorithm 4.1.2 and we take $f_n(x_n) = 0.5x_n, \lambda_n = 0.05, \gamma = \gamma_n = 0.03,\ \nu = 0.5,\ \mu = 0.01,\ \delta_n = \frac{1}{2}(1-\alpha_n)$ in Algorithm 9.1.5. In addition, take $\omega_n = \frac{1}{3n+1},\ \theta_n = 0.4$ in Algorithm 9.1.6 and Algorithm 9.1.8. Also, take $b_1 = 1,\quad b_2 = 1,\quad \xi = 1,\quad \hat{\xi} = 1,\quad \bar{\epsilon} = 0.01$ in Algorithm 9.1.8. Let $\chi_n = \frac{1}{6n+5},\quad \pi_n = \frac{1}{5n+6}, a_1 = 0.4,\ a_2 = 0.6$ in Algorithm 9.1.9.

Example 4.1.16. *Let $H_1 = H_2 = \mathbb{R}$, be the set of all real numbers equipped with the inner product $\langle x, y\rangle = xy\ \forall\ x, y \in \mathbb{R}$ and the corresponding norm $|\cdot|$. Define $A : H_1 \to H_1$ by $Ax := x + \sin x$, $B : H_1 \to 2^{H_1}$ by $Bx = 3x$, $D : H_1 \to H_1$ by $Dx = 3x$ and $G : H_2 \to 2^{H_2}$ by $Gx = 2x$ for all $x \in \mathbb{R}$. Clearly, A is $\frac{1}{2}$-inverse strongly monotone and B is maximal monotone. Similarly, it is easy to see that D is monotone and G is maximal monotone. Let $Tx := \frac{x}{5},\ \ \forall\ x \in \mathbb{R}$. Then T is a bounded linear operator on $\mathbb{R}$ with adjoint $T^*y = \frac{y}{5}\ \ \forall\ y \in \mathbb{R}$.*

We consider different initial starting points as follows:
Case 1: *Take $x_0 = 137$ and $x_1 = 18$*
Case 2: *Take $x_0 = 15$ and $x_1 = -15$*
Case 3: *Take $x_0 = -39$ and $x_1 = -16$*
Case 4: *Take $x_0 = -128$ and $x_1 = 15$.*

We compare the performance of our Algorithm 4.1.2 with other related methods. We plot the graph of errors against the number of iterations in each case using $|x_{n+1} - x_n| < 10^{-4}$ as the stopping criterion. The numerical computations are reported in Figure 4.1 and Table 4.1.16.

Table 4.1.16. Numerical Results for Example 4.1.16

	Case 1		*Case 2*		*Case 3*		*Case 4*	
	Iter.	*CPU Time*	*Iter.*	*CPU Time*	*Iter.*	*CPU Time*	*Iter.*	*CPU Time*
Kaz Alg.	*31*	0.0047	*26*	0.0046	*28*	0.0041	*31*	0.0041
Iz Alg.	*58*	0.0080	*52*	0.0083	*68*	0.0087	*75*	0.0077
Th Alg.	*540*	0.0111	*424*	0.0129	*489*	0.0112	*475*	0.0108
Sh Alg.	*56*	0.0037	*46*	0.0036	*50*	0.0036	*56*	0.0032
IzR Alg.	*84*	0.0194	*67*	0.0074	*77*	0.0073	*78*	0.0069
Wa Alg.	*38*	0.0099	*37*	0.0091	*37*	0.0086	*37*	0.0091
Pro Alg.	*19*	0.0090	*19*	0.0074	*19*	0.0078	*19*	0.0081

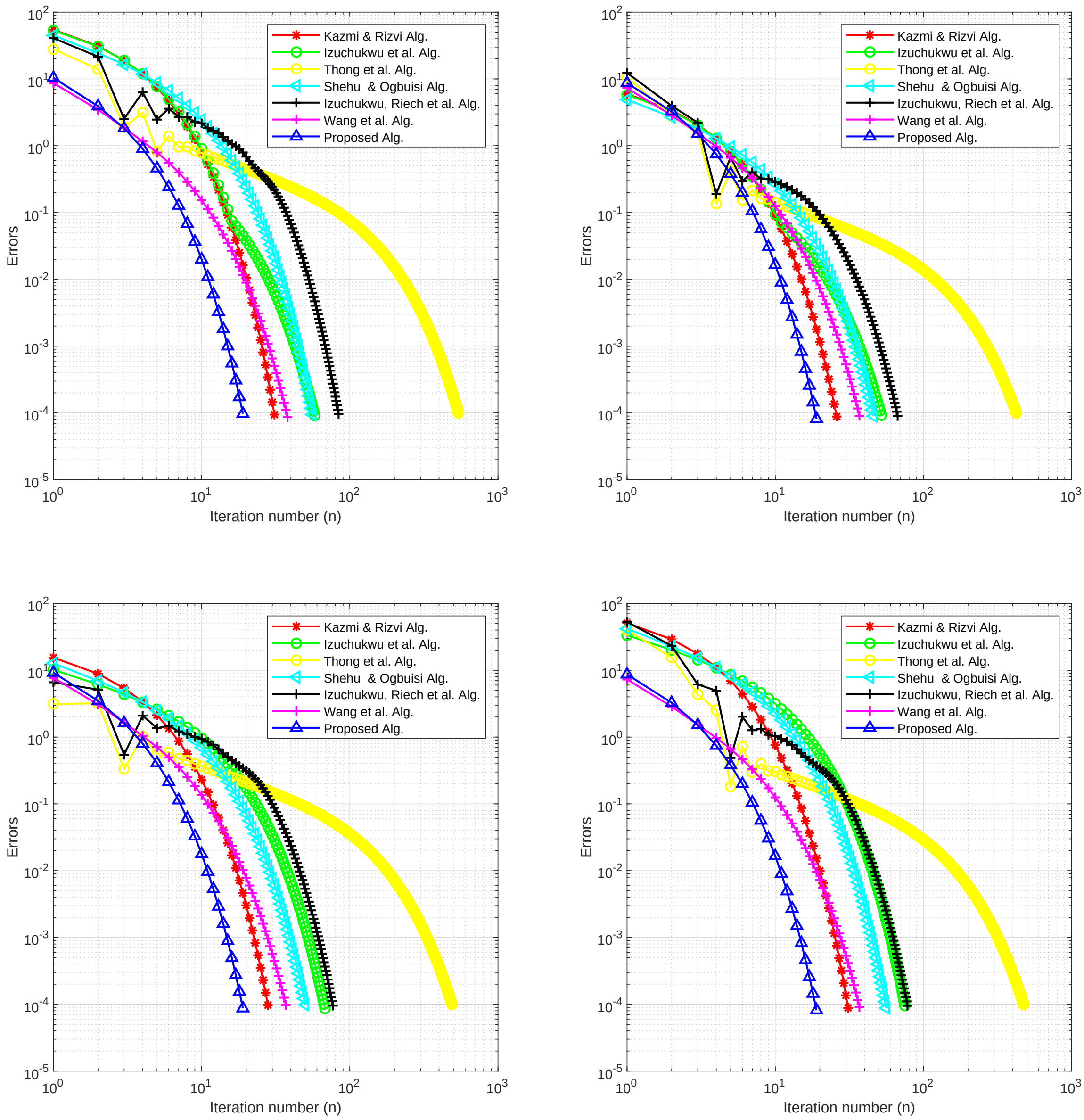


Figure 4.1. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

In the next example, we compare our proposed Algorithm 4.1.2 with Algorithm 9.1.4, Appendix 9.1.5, Appendix 9.1.6, Appendix 9.1.7 and Appendix 9.1.9 which are all strong convergence result.

Example 4.1.17. *Let $H_1 = H_2 = (\ell_2(\mathbb{R}, \|.\|_{\ell_2})$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, ...),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$ endowed with the norm $\|x\|_{\ell_2} := (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}},\ \forall\ x \in \ell_2(\mathbb{R})$. Now, consider the Split Variational Inequality Problem (SVI_qP) with nonempty closed and convex subsets C and Q of H_1 and H_2, respectively, where C and Q are defined by*

$$C := \{x \in \ell_2(\mathbb{R}) : \|x - b\|_{\ell_2} \leq 3\} \text{ and } Q := \{x \in \ell_2(\mathbb{R}) : \|x - b\|_{\ell_2} \leq 1\},$$

respectively. We set $J_{\gamma_n}^B = P_C$ and $J_{\lambda_n}^G = P_Q$. Then, the projection operators P_C and P_Q

have the following explicit formulas

$$P_C(x) := \begin{cases} 3\frac{x-b}{\|x-b\|_{\ell_2}} + b, & \|x-b\|_{\ell_2} > 3, \\ x, & \|x-b\|_{\ell_2} \leq 3, \end{cases} \quad \text{and} \quad P_Q(x) := \begin{cases} \frac{x-b}{\|x-b\|_{\ell_2}} + b, & \|x-b\|_{\ell_2} > 1, \\ x, & \|x-b\|_{\ell_2} \leq 1, \end{cases}$$

where $b = \left(1, \frac{1}{2}, \frac{1}{3}, ...\right)$. *Also, let* $T : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ *be defined by*

$$Tx = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...\right), \ \forall \ x \in \ell_2(\mathbb{R}).$$

Then, T *is a bounded linear operator with adjoint* T^*, *defined by* $T^*y = \left(y_2, \frac{y_3}{2}, \frac{y_4}{3}, ...\right), \ \forall \ y \in \ell_2(\mathbb{R})$.
Define the mapping $A, D : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ *by*

$$A(x) = \left(x_1 e^{-x_1^2}, 0, 0, ...\right) \text{ and } D(x) = \left(5x_1 e^{-x_1^2}, 0, 0, ...\right), \quad \text{respectively.}$$

Now, let $B = \partial\delta_C$ *and* $G = \partial\delta_Q$ *in Theorem 4.1.10 (where* δ_C *and* δ_Q *are the indicator functions) and use*

$$Tol_n := \frac{1}{2}\left(\|x_n - P_C(x_n - \gamma_n A x_n)\|^2 + \|Tx_n - P_Q(Tx_n - \lambda_n DTx_n)\|^2\right) < \epsilon$$

for stopping criterion, where $\epsilon = 10^{-4}$.

We consider four cases for our numerical experiments:
Case 1: *Take* $x_0 = \left(-7, 1, -\frac{1}{7}, ...\right)$ *and* $x_1 = \left(2, 1, \frac{1}{2}, ...\right)$
Case 2: *Take* $x_0 = \left(8, 1, \frac{1}{8}, ...\right)$ *and* $x_1 = (1, 0.1, 0.01, ...)$
Case 3: *Take* $x_0 = \left(9, 1, \frac{1}{9}, ...\right)$ *and* $x_1 = (-2, 0.2, -0.02, ...)$
Case 4: *Take* $x_0 = \left(-6, 1, -\frac{1}{6}, ...\right)$ *and* $x_1 = \left(-3, 1, -\frac{1}{3}, ...\right)$.
The numerical results are displayed in Figure 4.2 and Table 4.1.17.

Table 4.1.17. Numerical Results for Example 4.1.17

	Case 1		*Case 2*		*Case 3*		*Case 4*	
	Iter.	*CPU Time*	*Iter.*	*CPU Time*	*Iter.*	*CPU Time*	*Iter.*	*CPU Time*
KR	*26*	0.0117	*26*	0.0098	*26*	0.0093	*25*	0.0085
Izu	*57*	0.0075	*57*	0.0074	*58*	0.0076	*56*	0.0071
Thn	*620*	0.0153	*1248*	0.0173	*478*	0.0150	*1759*	0.0189
She	*42*	0.0061	*43*	0.0070	*43*	0.0065	*42*	0.0069
IzR	*169*	0.0129	*194*	0.0121	*168*	0.0034	*203*	0.0126
Prop	*41*	0.0117	*41*	0.0132	*41*	0.0132	*41*	0.0132

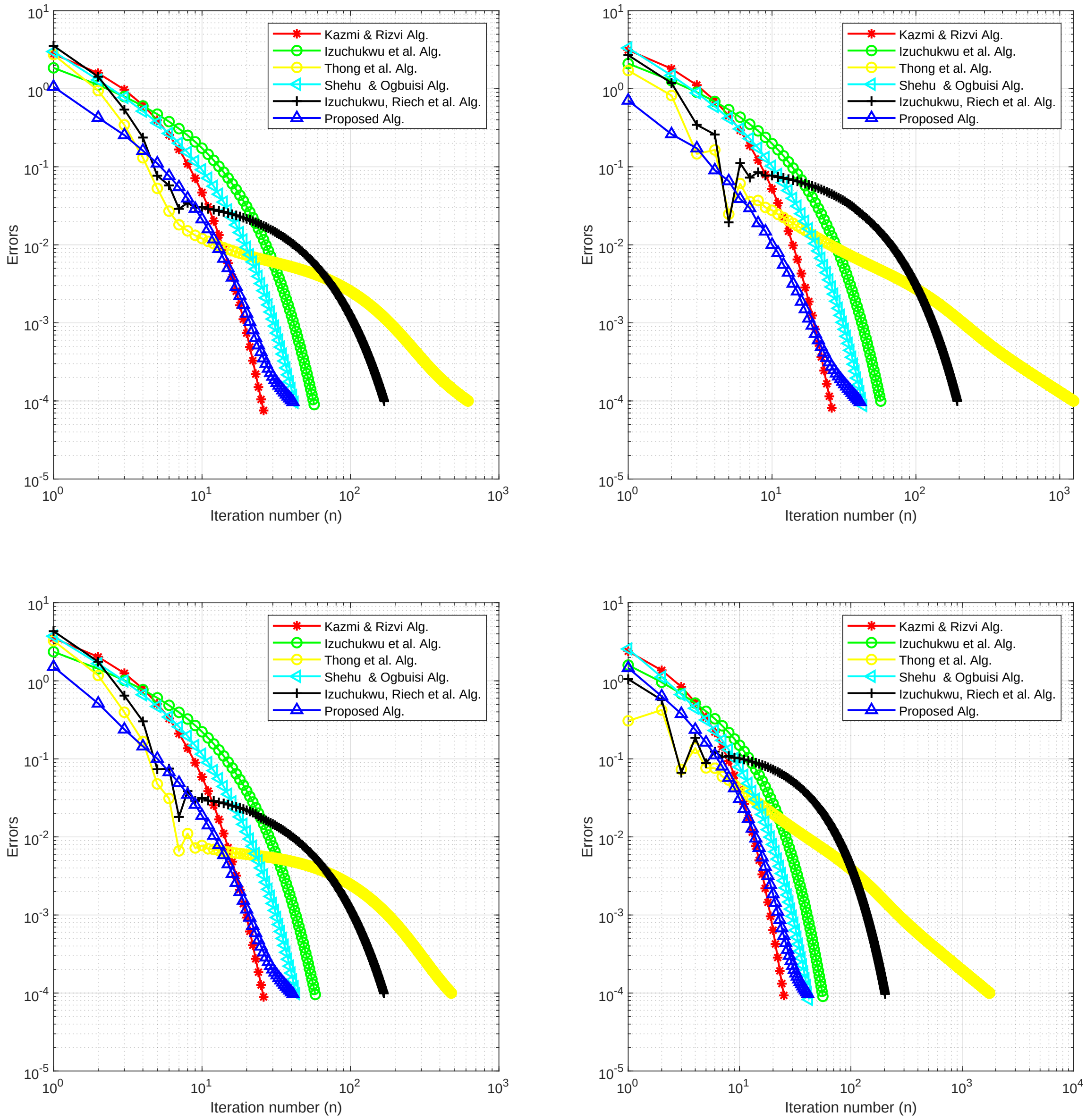


Figure 4.2. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

4.2 Generalized split feasibility problems.

In this section, we propose to solve Generalized Split Feasibility Problem (GSFP) over the solution set of Monotone Variational Inclusion Problem (MVIP) which is defined as:

$$\text{Find } \ x^* \in H_1 \ \text{ such that } \ x^* \in (A+B)^{-1}(0) \ \text{ and } \ Tx^* \in F(S), \tag{4.84}$$

where $B : H_1 \to 2^{H_1}$ is a maximal monotone operator, $T : H_1 \to H_2$ is a bounded linear operator, $A : H_1 \to H_1$ is a monotone and Lipschitz continuous operator and $S : H_2 \to H_2$ is a nonexpansive mapping.

Lemma 4.2.1. *[244] Let H be a real Hilbert space and $S : H \to H$ be a nonexpansive mapping with $F(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in H converging weakly to x and if $\{(I - S)x_n\}$ converges strongly to y, then $(I - S)x = y$.*

4.2.1 Main result

In this section, we present our method and discuss some of its features. We begin with the following assumptions under which our strong convergent result is obtained.

Assumption 4.2.2. *Let H_1 and H_2 be two real Hilbert spaces. We assume that the following conditions hold:*

(1) (a) $B : H_1 \to 2^{H_1}$ is a maximal monotone operator.
(b) $A : H_1 \to H_1$ is a monotone and Lipschitz continuous operator (but the Lipschitz constant need not to be known).
(c) $S : H_2 \to H_2$ is a nonexpansive mapping.
(d) $T : H_1 \to H_2$ is a bounded linear operator such that $T \neq 0$.
(e) The solution set $\Gamma := \{z \in (A+B)^{-1}(0) : Tz \in F(S)\}$ is nonempty.

(2) $\{\theta_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\epsilon_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$ are positive sequences satisfying the following conditions:
(a)$\{\theta_n\} \subset (0,1)$, $\lim_{n\to\infty} \theta_n = 0$, $\sum_{n=1}^{\infty} \theta_n = \infty$.
(b) $\lim_{n\to\infty} \frac{\epsilon_n}{\theta_n} = 0$.
(c) $\beta_n \subset (a, 1-\theta_n)$ for some $a > 0$.
(d) $\{\psi_n\} \subset [b, 1]$, $\lim_{n\to\infty} \psi_n$ exists for some $b > 0$.

Henceforth, we define $Gx := T^*(I - S)Tx$ for all $x \in H_1$. Thus, it follows from Lemma 2.5.25 that G is $2\|T\|^2$-Lipschitz continuous.
We now present the proposed method.

Algorithm 4.2.3. *Initialization: Choose sequences $\{\theta_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\epsilon_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$ such that the conditions from Assumption 4.2.2 (2) hold and let $\gamma_1, \lambda_1 > 0$, $\mu, \eta \in (0,1)$, $\alpha \geq 3$ and $x_0, x_1 \in H_1$ be given arbitrarily.*

Iterative Steps: *Set $n := 1$.*

Step 1. *Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where*

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases} \tag{4.85}$$

Step 2. *Set*

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = w_n - \gamma_n G w_n \quad \text{and} \quad y_n = J^B_{\lambda_n}(I - \lambda_n A)u_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)u_n, \tag{4.86}$$

where

$$\gamma_{n+1} = \begin{cases} \min\left\{\frac{\mu\|u_n - w_n\|}{\|Gu_n - Gw_n\|}, \gamma_n\right\}, & \text{if } Gu_n \neq Gw_n, \\ \gamma_n, & \text{otherwise} \end{cases} \tag{4.87}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\eta\|u_n - y_n\|}{\|Au_n - Ay_n\|}, \lambda_n\right\}, & \text{if } Au_n \neq Ay_n, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{4.88}$$

Step 3. *Compute*

$$x_{n+1} = (1 - \beta_n - \theta_n)u_n + \beta_n z_n, \tag{4.89}$$

where $z_n := (1-\psi_n)u_n + \psi_n\Big(y_n + \lambda_n\big(A(u_n) - A(y_n)\big)\Big)$.

Stopping criterion: *If $y_n = u_n = w_n = x_n$, then stop, otherwise, set $n := n+1$ and go back to* **Step 1.**

Remark 4.2.4. $\lambda_{n+1} \leq \lambda_n \quad \forall n \in \mathbb{N}$ *and* $\lim\limits_{n\to\infty} \lambda_n \geq \min\{\lambda_1, \frac{\eta}{L}\}$.
Indeed, from (4.88), it is obvious that $\lambda_{n+1} \leq \lambda_n \quad \forall n \in \mathbb{N}$. Furthermore, since A is L-Lipschitz continuous, we get in the case of $Au_n \neq Ay_n$, that

$$\lambda_{n+1} = \min\left\{\frac{\eta\|u_n - y_n\|}{\|Au_n - Ay_n\|}, \lambda_n\right\} \geq \min\left\{\frac{\eta}{L}, \lambda_n\right\},$$

which by induction implies that $\lambda_n \geq \min\{\lambda_1, \frac{\eta}{L}\}$. This gives that the limit of $\{\lambda_n\}$ exists and $\lim\limits_{n\to\infty} \lambda_n \geq \min\{\lambda_1, \frac{\eta}{L}\}$.

Since G is $2\|T\|^2$-Lipschitz continuous, then in a similar manner, we have from (4.87) that $\gamma_{n+1} \leq \gamma_n \quad \forall n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \gamma_n \geq \min\left\{\gamma_1, \frac{\mu}{2\|T\|^2}\right\}$.

We now show that the stopping criterion of Algorithm 4.2.3 is valid.

Lemma 4.2.5. *If $y_n = u_n = w_n = x_n$ in Algorithm 4.2.3, then $x_n \in \Gamma$.*

Proof. Let $y_n = u_n = w_n = x_n$. Then, from (4.86), it is clear that $x_n = J_{\lambda_n}^B(x_n - \lambda_n A(x_n))$. This implies that $x_n \in (A+B)^{-1}(0)$. From (4.86), we also obtain that $x_n = x_n - \gamma_n G x_n = x_n - \gamma_n T^*(I-S)Tx_n$, which implies that $T^*(I-S)Tx_n = 0$. That is,

$$STx_n = Tx_n + \bar{z},$$

where $T^*\bar{z} = 0$. Now, let $z \in \Gamma$, then we obtain that

$$\begin{aligned}
\|Tx_n - Tz\|^2 &= \|Tx_n - Tz\|^2 + 2\langle x_n - z, T^*\bar{z}\rangle \\
&= \|Tx_n - Tz\|^2 + 2\langle Tx_n - Tz, \bar{z}\rangle \\
&= \|Tx_n - Tz + \bar{z}\|^2 - \|\bar{z}\|^2 \\
&= \|STx_n - Tz\|^2 - \|\bar{z}\|^2 \\
&\leq \|Tx_n - Tz\|^2 - \|\bar{z}\|^2,
\end{aligned}$$

which implies that $\|\bar{z}\| = 0$. That is, $\bar{z} = 0$. Hence $STx_n = Tx_n$, which gives that $Tx_n \in F(S)$. Therefore, $x_n \in \Gamma$.

□

Lemma 4.2.6. *Let Assumption 4.2.2 hold and let $\{x_n\}$ be a sequence generated by Algorithm 4.2.3. Then $\{x_n\}$ is bounded.*

Proof. From (4.88), we have that

$$\|Au_n - Ay_n\| \leq \frac{\eta}{\lambda_{n+1}}\|u_n - y_n\| \quad \forall n \in \mathbb{N} \tag{4.90}$$

holds for both $Au_n = Ay_n$ and $Au_n \neq Ay_n$. Now, let $x^* \in \Gamma$. Then, by Step 3, we obtain

$$\begin{aligned}
&\|z_n - x^*\|^2 = \|(1-\psi_n)u_n + \psi_n y_n + \psi_n\lambda_n\big(A(u_n) - A(y_n)\big) - x^*\|^2 \\
&= \|(1-\psi_n)(u_n - x^*) + \psi_n(y_n - x^*) + \psi_n\lambda_n\big(A(u_n) - A(y_n)\big)\|^2 \\
&= (1-\psi_n)^2\|u_n - x^*\|^2 + \psi_n^2\|y_n - x^*\|^2 + \psi_n^2\lambda_n^2\|A(u_n) - A(y_n)\|^2 \\
&+ 2\psi_n(1-\psi_n)\langle u_n - x^*, y_n - x^*\rangle \\
&+ 2\psi_n(1-\psi_n)\lambda_n\langle u_n - x^*, A(u_n) - A(y_n)\rangle + 2\psi_n^2\lambda_n\langle y_n - x^*, A(u_n) - A(y_n)\rangle.
\end{aligned} \tag{4.91}$$

Using Lemma 2.5.18 (i), we obtain

$$2\langle u_n - x^*, y_n - x^*\rangle = \|u_n - x^*\|^2 + \|y_n - x^*\|^2 - \|u_n - y_n\|^2. \tag{4.92}$$

Combining (4.91) and (4.92), we get

$$\begin{aligned}
\|z_n - x^*\|^2 = (1-\psi_n)\|u_n - x^*\|^2 &+ \psi_n\|y_n - x^*\|^2 + \psi_n^2\lambda_n^2\|A(u_n) - A(y_n)\|^2 \\
&- \psi_n(1-\psi_n)\|u_n - y_n\|^2 + 2\psi_n(1-\psi_n)\lambda_n\langle u_n - x^*, A(u_n) - A(y_n)\rangle \\
&+ 2\psi_n^2\lambda_n\langle y_n - x^*, A(u_n) - A(y_n)\rangle.
\end{aligned} \tag{4.93}$$

Now, from (4.86), we obtain

$$Ay_n + \frac{1}{\lambda_n}(u_n - \lambda_n Au_n - y_n) \in (A+B)y_n.$$

Since $x^* \in \Gamma$, we have that $0 \in (A+B)(x^*)$. Hence, we obtain from Lemma 2.5.15 that

$$\langle y_n - u_n - \lambda_n\big(A(y_n) - A(u_n)\big), y_n - x^*\rangle \leq 0,$$

which implies that

$$\langle y_n - u_n, y_n - x^*\rangle \leq \lambda_n\langle A(y_n) - A(u_n), y_n - x^*\rangle. \tag{4.94}$$

Now, observe that

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \|y_n - u_n\|^2 + \|u_n - x^*\|^2 + 2\langle y_n - u_n, u_n - x^*\rangle \\
&\leq \|y_n - u_n\|^2 + \|u_n - x^*\|^2 - 2\langle y_n - u_n, y_n - u_n\rangle + 2\langle y_n - u_n, y_n - x^*\rangle \\
&\leq \|u_n - x^*\|^2 - \|y_n - u_n\|^2 - 2\lambda_n\langle A(u_n) - A(y_n), y_n - x^*\rangle, \qquad (4.95)
\end{aligned}$$

where the last inequality follows from (4.94). Substituting (4.95) into (4.93), and using (4.90), we obtain

$$\begin{aligned}
&\|z_n - x^*\|^2 \leq (1-\psi_n)\|u_n - x^*\|^2 + \psi_n\big[\|u_n - x^*\|^2 - \|u_n - y_n\|^2 \\
&- 2\lambda_n\langle A(u_n) - A(y_n), y_n - x^*\rangle\big] - \psi_n(1-\psi_n)\|u_n - y_n\|^2 + \psi_n^2\lambda_n^2\|A(u_n) - A(y_n)\|^2 \\
&+ 2\psi_n(1-\psi_n)\lambda_n\langle u_n - x^*, A(u_n) - A(y_n)\rangle + 2\psi_n^2\lambda_n\langle y_n - x^*, A(u_n) - A(y_n)\rangle \\
&= \|u_n - x^*\|^2 - \psi_n(2-\psi_n)\|u_n - y_n\|^2 + \psi_n^2\lambda_n^2\|A(u_n) - A(y_n)\|^2 \\
&- 2\psi_n\lambda_n\langle A(u_n) - A(y_n), y_n - x^*\rangle + 2\psi_n^2\lambda_n\langle y_n - x^*, A(u_n) - A(y_n)\rangle \\
&+ 2\psi_n(1-\psi_n)\lambda_n\langle u_n - x^*, A(u_n) - A(y_n)\rangle \\
&= \|u_n - x^*\|^2 - \psi_n(2-\psi_n)\|u_n - y_n\|^2 + \psi_n^2\lambda_n^2\|A(u_n) - A(y_n)\|^2 \\
&+ 2\psi_n(1-\psi_n)\lambda_n\langle A(u_n) - A(y_n), u_n - y_n\rangle \\
&\leq \|u_n - x^*\|^2 - \psi_n(2-\psi_n)\|u_n - y_n\|^2 + \psi_n^2\lambda_n^2\frac{\eta^2}{\lambda_{n+1}^2}\|u_n - y_n\|^2 \\
&+ 2\psi_n(1-\psi_n)\lambda_n\frac{\eta}{\lambda_{n+1}}\|u_n - y_n\|^2 \\
&= \|u_n - x^*\|^2 - \psi_n\left[2 - \psi_n - 2(1-\psi_n)\eta\frac{\lambda_n}{\lambda_{n+1}} - \psi_n\eta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right]\|u_n - y_n\|^2. \qquad (4.96)
\end{aligned}$$

Since the limit of $\{\lambda_n\}$ exists (see Remark 4.2.4 (d)), then $\lim_{n\to\infty}\lambda_n = \lim_{n\to\infty}\lambda_{n+1}$. Also, by Assumption 4.2.2 (2)(d), there exists $\psi \in (0,1]$ such that $\psi = \lim_{n\to\infty}\psi_n$. Thus, we obtain that

$$\lim_{n\to\infty}\left[2 - \psi_n - 2(1-\psi_n)\eta\frac{\lambda_n}{\lambda_{n+1}} - \psi_n\eta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right] = (1-\eta)(2-\psi+\psi\eta) > 0. \tag{4.97}$$

Hence, there exists $n_0 \in \mathbb{N}$ such that $2 - \psi_n - 2(1-\psi_n)\eta\frac{\lambda_n}{\lambda_{n+1}} - \psi_n\eta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} > 0 \ \ \forall n \geq n_0$. Thus, we obtain from (4.96) that

$$\|z_n - x^*\| \leq \|u_n - x^*\| \ \ \forall n \geq n_0. \tag{4.98}$$

Observe from Lemma 2.5.25 and Lemma 2.5.16 (ii), (iii), that $(I-\gamma_n G)$ is $\gamma_n\|T\|^2$-averaged. That is, $(I-\gamma_n G)=(1-\beta_n)I+\beta_n V_n, \quad \forall \;\; n\in\mathbb{N}$, where $\beta_n=\gamma_n\|T\|^2\in(0,1)$ and V_n is nonexpansive for all $n\in\mathbb{N}$. Therefore, we can rewrite u_n from (4.86) as:

$$u_n=(1-\beta_n)w_n+\beta_n V_n w_n, \quad n\geq 1. \tag{4.99}$$

Thus, we obtain that

$$\begin{aligned}\|u_n-x^*\|^2&=(1-\beta_n)\|w_n-x^*\|^2+\beta_n\|V_nw_n-x^*\|^2-\beta_n(1-\beta_n)\|V_nw_n-w_n\|^2\\&\leq\|w_n-x^*\|^2-\beta_n(1-\beta_n)\|V_nw_n-w_n\|^2\\&\leq\|w_n-x^*\|^2.\end{aligned} \tag{4.100}$$

Now, from Step 1 and Assumption 4.2.2 (2), we have that $\alpha_n\|x_n-x_{n-1}\|\leq\epsilon_n \;\; \forall n\in\mathbb{N}$, which implies that

$$\frac{\alpha_n}{\theta_n}\|x_n-x_{n-1}\|\leq\frac{\epsilon_n}{\theta_n}\to 0, \quad \text{as} \quad n\to\infty. \tag{4.101}$$

Hence, there exists $M_1>0$ such that

$$\frac{\alpha_n}{\theta_n}\|x_n-x_{n-1}\|\leq M_1 \quad \forall\; n\in\mathbb{N}. \tag{4.102}$$

Thus, from Step 2, we obtain

$$\begin{aligned}\|w_n-x^*\|&\leq\|x_n-x^*\|+\alpha_n\|x_n-x_{n-1}\|\\&=\|x_n-x^*\|+\theta_n\frac{\alpha_n}{\theta_n}\|x_n-x_{n-1}\|\\&\leq\|x_n-x^*\|+\theta_nM_1,\end{aligned} \tag{4.103}$$

and so,

$$\|u_n-x^*\|\leq\|w_n-x^*\|\leq\|x_n-x^*\|+\theta_nM_1. \tag{4.104}$$

From (4.89), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|&=\|(1-\beta_n-\theta_n)u_n+\beta_nz_n-x^*\|\\&=\|(1-\beta_n-\theta_n)(u_n-x^*)+\beta_n(z_n-x^*)-\theta_nx^*\|\\&\leq\|(1-\beta_n-\theta_n)(u_n-x^*)+\beta_n(z_n-x^*)\|+\theta_n\|x^*\|.\end{aligned} \tag{4.105}$$

Note that

$$\begin{aligned}&\|(1-\beta_n-\theta_n)(u_n-x^*)+\beta_n(z_n-x^*)\|^2\\&=(1-\beta_n-\theta_n)^2\|u_n-x^*\|^2+2(1-\beta_n-\theta_n)\beta_n\langle u_n-x^*,z_n-x^*\rangle\\&+\beta_n^2\|z_n-x^*\|^2\\&\leq(1-\beta_n-\theta_n)^2\|u_n-x^*\|^2+2(1-\beta_n-\theta_n)\beta_n\|u_n-x^*\|.\|z_n-x^*\|\\&+\beta_n^2\|z_n-x^*\|^2\\&\leq(1-\beta_n-\theta_n)^2\|u_n-x^*\|^2+(1-\beta_n-\theta_n)\beta_n\|u_n-x^*\|^2\\&+(1-\beta_n-\theta_n)\beta_n\|z_n-x^*\|^2+\beta_n^2\|z_n-x^*\|^2\\&\leq(1-\beta_n-\theta_n)(1-\theta_n)\|u_n-x^*\|^2+(1-\theta_n)\beta_n\|z_n-x^*\|^2\\&\leq(1-\beta_n-\theta_n)(1-\theta_n)\|u_n-x^*\|^2+(1-\theta_n)\beta_n\|u_n-x^*\|^2\\&=(1-\theta_n)^2\|u_n-x^*\|^2,\end{aligned} \tag{4.106}$$

which implies from (4.104) and (4.105) that

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq (1-\theta_n)\|x_n-x^*\|+\theta_n(1-\theta_n)M_1+\theta_n\|x^*\|\\ &\leq (1-\theta_n)\|x_n-x^*\|+\theta_n(\|x^*\|+M_1).\end{aligned}$$

It follows from Lemma 2.5.24 that $\{x_n\}$ is bounded. $\square$

Lemma 4.2.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.2.3 such that Assumption 4.2.2 holds. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to a point $z \in H_1$ and $\lim\limits_{n\to\infty}\|u_{n_k}-y_{n_k}\| = 0 = \lim\limits_{n\to\infty}\|u_{n_k}-x_{n_k}\|$. Then, $z \in \Gamma$.*

Proof. From Step 2 and (4.101), we have

$$\|w_n-x_n\| = \theta_n\frac{\alpha_n}{\theta_n}\|x_n-x_{n-1}\| \to 0, \quad \text{as} \quad n\to\infty. \tag{4.107}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ which is weakly convergent to a point $z \in H_1$. Then, the subsequences $\{u_{n_k}\}, \{y_{n_k}\}$ and $\{w_{n_k}\}$ are weakly convergent to $z \in H_1$.

We first show that $Tz \in F(S)$. To do this, recall from Remark 4.2.4 that $\lim\limits_{n\to\infty}\gamma_n \geq \min\left\{\gamma_1, \frac{\mu}{2\|T\|^2}\right\} > 0$. Let $\lim\limits_{n\to\infty}\gamma_n = \gamma > 0$, since $\{T^*(I-S)Tw_{n_k}\}$ is bounded, we get that

$$\|(I-\gamma_{n_k}T^*(I-S)T)w_{n_k}-(I-\gamma T^*(I-S)T)w_{n_k}\| = |\gamma_{n_k}-\gamma|\|T^*(I-S)Tw_{n_k}\| \to 0, \text{ as } k\to\infty.$$

That is, $\lim\limits_{k\to\infty}\|u_{n_k}-(I-\gamma T^*(I-S)T)w_{n_k}\| = 0$, which implies from (4.107) and our hypothesis that

$$\lim_{k\to\infty}\|w_{n_k}-\big(I-\gamma T^*(I-S)T\big)w_{n_k}\| = 0. \tag{4.108}$$

Thus, we obtain from Lemma 2.5.55, that $z \in F\big(I-\gamma T^*(I-S)T\big)$. Hence, using the same line of argument as in the proof of Lemma 4.2.5, we obtain that $Tz \in F(S)$.

Now, let $(v,w) \in G(A+B)$, then $w-Av \in B(v)$. Also, we obtain from (4.86) that

$$\frac{1}{\lambda_{n_k}}(u_{n_k}-\lambda_{n_k}Au_{n_k}-y_{n_k}) \in B(y_{n_k}).$$

Thus, by the monotonicity of B, we obtain that

$$\langle v-y_{n_k}, w-Av-\frac{1}{\lambda_{n_k}}\big(u_{n_k}-\lambda_{n_k}Au_{n_k}-y_{n_k}\big)\rangle \geq 0. \tag{4.109}$$

Using (4.109) and the monotonicity of A, we obtain

$$\begin{aligned}\langle v-y_{n_k}, w\rangle &\geq \langle v-y_{n_k}, Av+\frac{1}{\lambda_{n_k}}(u_{n_k}-y_{n_k})-Au_{n_k}\rangle\\ &= \langle v-y_{n_k}, Av-A(y_{n_k})\rangle + \langle v-y_{n_k}, A(y_{n_k})-A(u_{n_k})\rangle\\ &+ \langle v-y_{n_k}, \frac{1}{\lambda_{n_k}}(u_{n_k}-y_{n_k})\rangle\\ &\geq \langle v-y_{n_k}, A(y_{n_k})-A(u_{n_k})\rangle + \langle v-y_{n_k}, \frac{1}{\lambda_{n_k}}(u_{n_k}-y_{n_k})\rangle.\end{aligned} \tag{4.110}$$

Recall that $\lim_{k\to\infty} \lambda_{n_k} \geq \min\{\lambda_1, \frac{\eta}{L}\} > 0$, $\lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| = 0$ and by the Lipschitz continuity of A, $\lim_{k\to\infty} \|A(u_{n_k}) - A(y_{n_k})\| = 0$. Hence, passing limit as $k \to \infty$ in (4.110), we obtain

$$\langle v - z, w\rangle \geq 0.$$

Also, by Lemma 2.5.15, $A + B$ is maximal monotone, thus, $0 \in (A + B)z$. This together with the fact that $Tz \in F(S)$ gives that $z \in \Gamma$. □

Theorem 4.2.8. *Suppose that Assumption 4.2.2 holds. Then, the sequence $\{x_n\}$ generated by Algorithm 4.2.3 converges strongly to $x^* \in \Gamma$, where*

$$\|x^*\| = \min\{\|z\| : z \in \Gamma\}.$$

Proof. Let $x^* \in \Gamma$. Then, we obtain from (4.98), (4.100) and Step 2 that

$$\begin{aligned}
\|(1-\beta_n)u_n + \beta_n z_n - x^*\|^2 &= \|(1-\beta_n)(u_n - x^*) + \beta_n(z_n - x^*)\|^2 \\
&= (1-\beta_n)^2\|u_n - x^*\|^2 + \beta_n^2\|z_n - x^*\|^2 + 2(1-\beta_n)\beta_n\langle u_n - x^*, \quad z_n - x^*\rangle \\
&\leq (1-\beta_n)^2\|u_n - x^*\|^2 + \beta_n^2\|z_n - x^*\|^2 + 2(1-\beta_n)\beta_n\|u_n - x^*\|.\|z_n - x^*\| \\
&\leq (1-\beta_n)^2\|u_n - x^*\|^2 + \beta_n^2\|z_n - x^*\|^2 \\
&+ (1-\beta_n)\beta_n\left(\|u_n - x^*\|^2 + \|z_n - x^*\|^2\right) \\
&= (1-\beta_n)\|u_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 \\
&\leq \|w_n - x^*\|^2. \qquad (4.111)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|x_n - x^*\|^2 + 2\alpha_n\langle x_n - x^*, \quad x_n - x_{n-1}\rangle + \alpha_n^2\|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - x^*\|^2 + 2\alpha_n\|x_n - x^*\|.\|x_n - x_{n-1}\| + \alpha_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - x^*\|^2 + \alpha_n\|x_n - x_{n-1}\|(2\|x_n - x^*\| + \alpha_n\|x_n - x_{n-1}\|) \\
&\leq \|x_n - x^*\|^2 + 3\alpha_n\|x_n - x_{n-1}\|M_2, \qquad (4.112)
\end{aligned}$$

for some $M_2 > 0$. Combining (4.111) and (4.112), we obtain

$$\|(1-\beta_n)u_n + \beta_n z_n - x^*\|^2 \leq \|x_n - x^*\|^2 + 3\alpha_n\|x_n - x_{n-1}\|M_2. \qquad (4.113)$$

Thus, from Step 3 and Lemma 2.5.18 (ii), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 = \|(1-\theta_n)\big[(1-\beta_n)u_n + \beta_n z_n - x^*\big] - \big[\theta_n\beta_n(u_n - z_n) + \theta_n x^*\big] \\
&\leq (1-\theta_n)^2\|(1-\beta_n)u_n + \beta_n z_n - x^*\|^2 - 2\langle\theta_n\beta_n(u_n - z_n) + \theta_n x^*, \quad x_{n+1} - x^*\rangle \\
&\leq (1-\theta_n)\|(1-\beta_n)u_n + \beta_n z_n - x^*\|^2 + 2\langle\theta_n\beta_n(u_n - z_n), \quad x^* - x_{n+1}\rangle \\
&+ 2\theta_n\langle x^*, \quad x^* - x_{n+1}\rangle \\
&\leq (1-\theta_n)\|(1-\beta_n)u_n + \beta_n z_n - x^*\|^2 + 2\theta_n\beta_n\|u_n - z_n\|.\|x^* - x_{n+1}\| \\
&+ 2\theta_n\langle x^*, \quad x^* - x_{n+1}\rangle. \qquad (4.114)
\end{aligned}$$

Therefore, using (4.113) and (4.114), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le (1-\theta_n)\|x_n-x^*\|^2+3(1-\theta_n)\alpha_n\|x_n-x_{n-1}\|M_2\\
&+2\theta_n\beta_n\|u_n-z_n\|.\|x^*-x_{n+1}\|+2\theta_n\langle x^*,\quad x^*-x_{n+1}\rangle\\
&=(1-\theta_n)\|x_n-x^*\|^2\\
&+\theta_n\Big[3\frac{\alpha_n}{\theta_n}(1-\theta_n)\|x_n-x_{n-1}\|M_2+2\beta_n\|u_n-z_n\|.\|x^*-x_{n+1}\|\\
&+2\langle x^*,\quad x^*-x_{n+1}\rangle\Big]\\
&=(1-\theta_n)\|x_n-x^*\|^2+\theta_n d_n,\quad \forall n\ge n_0,
\end{aligned}\tag{4.115}$$

where $d_n := \Big[3\frac{\alpha_n}{\theta_n}(1-\theta_n)\|x_n-x_{n-1}\|M_2+2\beta_n\|u_n-z_n\|.\|x^*-x_{n+1}\|+2\langle x^*,\quad x^*-x_{n+1}\rangle\Big]$.

To show that the sequence $\Big\{\|x_n-x^*\|\Big\}$ converges to zero, by Lemma 2.5.55, it is enough to show that $\limsup_{k\to\infty} d_{n_k}\le 0$ (where $\{d_{n_k}\}$ is a subsequence of $\{d_n\}$), for every subsequence $\Big\{\|x_{n_k}-x^*\|\Big\}$ of $\Big\{\|x_n-x^*\|\Big\}$ satisfying

$$\liminf_{k\to\infty}\Big(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|\Big)\ge 0.\tag{4.116}$$

Now, suppose that $\Big\{\|x_{n_k}-x^*\|\Big\}$ is a subsequence of $\Big\{\|x_n-x^*\|\Big\}$ such that (4.116) holds. Then,

$$\begin{aligned}
&\liminf_{k\to\infty}\Big(\|x_{n_k+1}-x^*\|^2-\|x_{n_k}-x^*\|^2\Big)\\
&\quad=\liminf_{k\to\infty}\Big((\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|)(\|x_{n_k+1}-x^*\|+\|x_{n_k}-x^*\|)\Big)\ge 0.
\end{aligned}\tag{4.117}$$

Now, by Step 3, we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|(1-\beta_n-\theta_n)(u_n-x^*)+\beta_n(z_n-x^*)-\theta_n x^*\|^2\\
&=\|(1-\beta_n-\theta_n)(u_n-x^*)+\beta_n(z_n-x^*)\|^2-2\theta_n\langle(1-\beta_n-\theta_n)(u_n-x^*)\\
&+\beta_n(z_n-x^*),\quad x^*\rangle+\theta_n^2\|x^*\|^2\\
&\le\|(1-\beta_n-\theta_n)(u_n-x^*)+\beta_n(z_n-x^*)\|^2+\theta_n M_3,
\end{aligned}\tag{4.118}$$

for some $M_3>0$. Substituting (4.106) into (4.118), we have

$$\|x_{n+1}-x^*\|^2\le(1-\beta_n-\theta_n)(1-\theta_n)\|u_n-x^*\|^2+(1-\alpha_n)\beta_n\|z_n-x^*\|^2+\theta_n M_3.\tag{4.119}$$

Using (4.96), (4.112) and (4.119), we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\leq (1-\beta_n-\theta_n)(1-\theta_n)\|u_n-x^*\|^2+(1-\theta_n)\beta_n\Bigg(\|u_n-x^*\|^2\\
&-\psi_n\left[2-\psi_n-2(1-\psi_n)\eta\frac{\lambda_n}{\lambda_{n+1}}-\psi_n\eta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right]\|u_n-y_n\|^2\Bigg)+\theta_nM_3\\
&=(1-\theta_n)^2\|u_n-x^*\|^2-(1-\theta_n)\beta_n\psi_n\left[2-\psi_n-2(1-\psi_n)\eta\frac{\lambda_n}{\lambda_{n+1}}-\psi_n\eta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right]\|u_n-y_n\|^2\\
&+\theta_nM_3\\
&\leq (1-\theta_n)^2\left[\|x_n-x^*\|^2+3\alpha_n\|x_n-x_{n-1}\|M_2\right]-(1-\theta_n)\beta_n\psi_n\left[2-\psi_n-2(1-\psi_n)\eta\frac{\lambda_n}{\lambda_{n+1}}\right.\\
&\left.-\psi_n\eta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right]\|u_n-y_n\|^2+\theta_nM_3\\
&\leq \|x_n-x^*\|^2+\theta_n\left[3\frac{\alpha_n}{\theta_n}\|x_n-x_{n-1}\|M_2+M_3\right]-(1-\theta_n)\beta_n\psi_n\left[2-\psi_n-2(1-\psi_n)\eta\frac{\lambda_n}{\lambda_{n+1}}\right.\\
&\left.-\psi_n\eta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right]\|u_n-y_n\|^2.
\end{aligned}$$

Hence, by (4.102) and (4.117), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}(1-\theta_{n_k})\beta_{n_k}\psi_{n_k}\left[2-\psi_{n_k}-2(1-\psi_{n_k})\eta\frac{\lambda_{n_k}}{\lambda_{n_k+1}}-\psi_{n_k}\eta^2\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}\right]\|u_{n_k}-y_{n_k}\|^2\\
&\leq \limsup_{k\to\infty}\left\{\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2+\theta_{n_k}\left[3M_1M_2+M_3\right]\right\}\\
&=\limsup_{k\to\infty}\left\{\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2\right\}\\
&=-\liminf_{k\to\infty}\left\{\|x_{n_k+1}-x^*\|^2-\|x_{n_k}-x^*\|^2\right\}\leq 0,
\end{aligned}$$

which implies from (4.97) that

$$\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\|=0. \tag{4.120}$$

Using (4.119), (4.98) and (4.100), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2&\leq (1-\beta_n-\theta_n)(1-\theta_n)\|u_n-x^*\|^2+(1-\theta_n)\beta_n\|z_n-x^*\|^2+\theta_nM_3\\
&\leq (1-\theta_n)^2\|u_n-x^*\|^2+\theta_nM_3\\
&\leq \|w_n-x^*\|^2-\beta_n(1-\beta_n)\|V_nw_n-w_n\|^2+\theta_nM_3\\
&\leq \|x_n-x^*\|^2+3\alpha_n\|x_n-x_{n-1}\|M_2-\beta_n(1-\beta_n)\|V_nw_n-w_n\|^2+\theta_nM_3.
\end{aligned}$$

Thus, by (4.117) we obtain

$$\begin{aligned}\limsup_{k\to\infty}\left[\beta_{n_k}(1-\beta_{n_k})\|V_{n_k}w_{n_k}-w_{n_k}\|^2\right] &\le \limsup_{k\to\infty}\Big[\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2\\ &\quad+\theta_{n_k}\left[3M_1M_2+M_3\right]\Big]\\ &= -\liminf_{k\to\infty}\left[\|x_{n_k+1}-x^*\|^2-\|x_{n_k}-x^*\|^2\right]\le 0,\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|V_{n_k}w_{n_k}-w_{n_k}\|=0. \tag{4.121}$$

Thus, we obtain from (4.99) that

$$\lim_{k\to\infty}\|u_{n_k}-w_{n_k}\|=\lim_{k\to\infty}\beta_{n_k}\|V_{n_k}w_{n_k}-w_{n_k}\|=0. \tag{4.122}$$

Using (4.122) and (4.107), we obtain

$$\lim_{k\to\infty}\|u_{n_k}-x_{n_k}\|=0. \tag{4.123}$$

Observe from Step 3 and (4.120) that

$$\begin{aligned}\|z_{n_k}-y_{n_k}\| &= (1-\psi_{n_k})\|u_{n_k}-y_{n_k}\|+\psi_{n_k}\|\lambda_{n_k}(Au_{n_k}-Ay_{n_k})\|\\ &\le (1-\psi_{n_k})\|u_{n_k}-y_{n_k}\|+\psi_{n_k}\lambda_{n_k}L\|u_{n_k}-y_{n_k}\|\to 0 \text{ as } k\to\infty.\end{aligned} \tag{4.124}$$

Thus, from (4.120), we obtain

$$\|z_{n_k}-u_{n_k}\|=0. \tag{4.125}$$

Also, from Step 3, we obtain

$$\|x_{n_k+1}-u_{n_k}\|\le\beta_{n_k}\|z_{n_k}-u_{n_k}\|+\theta_{n_k}\|u_{n_k}\|\to 0 \text{ as } k\to\infty. \tag{4.126}$$

Thus, we obtain from (4.123) and (4.126) that

$$\lim_{k\to\infty}\|x_{n_{k+1}}-x_{n_k}\|=0. \tag{4.127}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to some $z\in H_1$, such that

$$\limsup_{k\to\infty}\langle x^*,\ \ x^*-x_{n_k}\rangle=\lim_{j\to\infty}\langle x^*,\ \ x^*-x_{n_{k_j}}\rangle=\langle x^*,\ \ x^*-z\rangle. \tag{4.128}$$

Also, we obtain from (4.120), (4.123) and Lemma 4.2.7 that $z\in\Gamma$.

Thus, since $x^*=P_\Gamma 0$, we obtain from (4.128) that

$$\limsup_{k\to\infty}\langle x^*,\ \ x^*-x_{n_k}\rangle=\langle x^*,\ \ x^*-z\rangle\le 0,$$

which implies from (4.127) that

$$\limsup_{k\to\infty}\langle x^*, \quad x^* - x_{n_k+1}\rangle \leq 0. \tag{4.129}$$

Now, recall that $d_{n_k} := \Big[3\dfrac{\alpha_{n_k}}{\theta_{n_k}}(1-\theta_{n_k})\|x_{n_k} - x_{n_k-1}\|M_2 + 2\beta_{n_k}\|u_{n_k} - z_{n_k}\|.\|x^* - x_{n_k+1}\| + 2\langle x^*, \quad x^* - x_{n_k+1}\rangle\Big]$.

Thus, by (4.129), (4.101) and (4.125), we obtain $\limsup_{k\to\infty} d_{n_k} \leq 0$. Hence, we get that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Therefore, we conclude that $\{x_n\}$ converges strongly to $x^* = P_\Gamma 0$.

□

Some Corollaries

If we set $B = N_C$ in Algorithm 4.2.3, then we obtain from the definition of normal cone (see (2.5.44)) that $J_\lambda^{N_C}(I-\lambda A) = P_C(I-\lambda A)$. In this case, we know that $(A+N_C)^{-1}(0) = VIP(C,A)$. Thus, we obtain the following consequent result.

Algorithm 4.2.9. *Initialization: Choose sequences $\{\theta_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\epsilon_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ such that the conditions from Assumption 4.2.2 (2) hold and let $\gamma_1, \lambda_1 > 0$, $\mu, \eta \in (0,1)$, $\alpha \geq 3$ and $x_0, x_1 \in H_1$ be given arbitrarily.*

Iterative Steps: *Set $n := 1$.*

Step 1. *Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha}_n$, where*

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases} \tag{4.130}$$

Step 2. *Set*

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = w_n - \gamma_n G w_n \quad \text{and} \quad y_n = P_C(I - \lambda_n A)u_n, \tag{4.131}$$

where

$$\gamma_{n+1} = \begin{cases} \min\left\{\frac{\mu\|u_n - w_n\|}{\|Gu_n - Gw_n\|}, \ \gamma_n\right\}, & \text{if } Gu_n \neq Gw_n, \\ \gamma_n, & \text{otherwise} \end{cases} \tag{4.132}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\eta\|u_n-y_n\|}{\|Au_n-Ay_n\|},\ \lambda_n\right\}, & \text{if } Au_n \neq Ay_n, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{4.133}$$

Step 3. *Compute*

$$x_{n+1} = (1-\beta_n-\theta_n)u_n + \beta_n z_n, \tag{4.134}$$

where $z_n := (1-\psi_n)u_n + \psi_n\Big(y_n + \lambda_n\big(A(u_n) - A(y_n)\big)\Big)$.

Stopping criterion: *If* $y_n = u_n = w_n = x_n$, *then stop, otherwise, set* $n := n+1$ *and go back to* **Step 1.**

Corollary 4.2.10. *Suppose that Assumption 4.2.2 holds with* $B = N_C$ *and* $\Gamma := \{z \in VIP(C, A) : Tz \in F(S)\}$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 4.2.9 converges strongly to* $x^* \in \Gamma$, *where* $\|x^*\| = \min\{\|z\| : z \in \Gamma\}$.

If $A \equiv 0$, then we obtain the following consequent result.

Algorithm 4.2.11. *Initialization: Choose sequences* $\{\theta_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{\epsilon_n\}_{n=1}^\infty$, $\{\psi_n\}_{n=1}^\infty$ *such that the conditions from Assumption 4.2.2 (2) hold and let* $\gamma_1 > 0$, $\mu \in (0,1)$, $\alpha \geq 3$ *and* $x_0, x_1 \in H_1$ *be given arbitrarily.*

Iterative Steps: *Set* $n := 1$.

Step 1. *Given the iterates* x_{n-1} *and* x_n $(n \geq 1)$, *choose* α_n *such that* $0 \leq \alpha_n \leq \bar{\alpha}_n$, *where*

$$\bar{\alpha}_n := \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\epsilon_n}{\|x_n-x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases} \tag{4.135}$$

Step 2. *Set*

$$w_n = x_n + \alpha_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = w_n - \gamma_n G w_n \quad \text{and} \quad y_n = J_{\lambda_n}^B u_n, \tag{4.136}$$

where $\inf\limits_{n\geq 1} \lambda_n \geq \lambda > 0$ *and*

$$\gamma_{n+1} = \begin{cases} \min\left\{\frac{\mu\|u_n-w_n\|}{\|Gu_n-Gw_n\|},\ \gamma_n\right\}, & \text{if } Gu_n \neq Gw_n. \\ \gamma_n, & \text{otherwise} \end{cases} \tag{4.137}$$

Step 3. *Compute*

$$x_{n+1} = (1 - \beta_n - \theta_n)u_n + \beta_n z_n, \tag{4.138}$$

where $z_n := (1 - \psi_n)u_n + \psi_n y_n$.

Stopping criterion: *If* $y_n = u_n = w_n = x_n$*, then stop, otherwise, set* $n := n+1$ *and go back to* **Step 1.**

Corollary 4.2.12. *Suppose that Assumption 4.2.2 holds with* $A \equiv 0$ *and* $\Gamma := \{z \in B^{-1}(0) : Tz \in F(S)\}$*. Then, the sequence* $\{x_n\}$ *generated by Algorithm 4.2.11 converges strongly to* $x^* \in \Gamma$*, where* $\|x^*\| = \min\{\|z\| : z \in \Gamma\}$.

Remark 4.2.13. *If we set* $H_1 = H_2 = H$ *and* $T = I = S$ *in Algorithms 4.2.3, 4.2.9 and 4.2.11, we recover relaxed inertial methods for solving the classical MVIP (see [17]), VIP (see [14]) and NPP (see [126]), respectively. Furthermore, under appropriate settings, Algorithms 4.2.3, 4.2.9 and 4.2.11 reduce to relaxed inertial methods for solving the classical SMVIP of Moudafi [171], SVIP of Censor et al. [54] and SCNPP, respectively.*

4.2.2 Application to image restoration problems

First, consider the following Split Linear Inverse Problem (SLIP), which is modified from [26]:

$$\text{Find } \ x^* \in H_1 \text{ such that } P(x^*) + G(x^*) = \min_{x \in H_1} [P(x) + G(x)], \text{ and } Tx^* \in F(S), \tag{4.139}$$

where $P : H_1 \to \mathbb{R}$ is convex and continuously differentiable, $G : H_1 \to \mathbb{R}$ is convex and lower semi-continuous, $T : H_1 \to H_2$ is a bounded linear operator and $S : H_2 \to H_2$ is any nonlinear mapping.

Now, recall that problem (4.139) contains as a special case, the following *image restoration problem* (which is our main interest for this application):

$$\min_{x \in \mathbb{R}^N} \ \{||Dx - b||_2^2 + \lambda ||x||_1\}, \tag{4.140}$$

where $\lambda > 0$, $x \in \mathbb{R}^N$ is the original image to be recovered, $b \in \mathbb{R}^M$ is the observed image and $D : \mathbb{R}^N \to \mathbb{R}^M$ is the blurring operator. In this case, $P(x) = ||Dx - b||_2^2$ and $G(x) = \lambda ||x||_1$.

To apply Algorithm 4.2.3 to solve the image restoration problem (4.140) via numerical computations, we recall that since P is convex and continuously differentiable, then the gradient ∇P of P is monotone and continuous. Also, since G is convex and lower semi-continuous, then the subdifferential ∂G of G is maximal monotone. Moreover,

$$P(x^*) + G(x^*) = \min_{x \in H_1} [P(x) + G(x)] \Leftrightarrow 0 \in (\nabla P(x^*) + \partial G(x^*)).$$

Thus, we can set $A = \nabla P$ and $B = \partial G$ in our algorithm, where ∇P is $\|D\|^2$-Lipschitz continuous and monotone. We then consider the (256×256) Cameraman Image and

128×128 Medical Resonance Imaging (MRI) from MATLAB Image Processing Toolbox. Moreover, we use the Gaussian blur of size 9×9 and standard deviation $\sigma = 4$ to create the blurred and noisy image (observed image). Also, we measure the quality of the restored image using the signal-to-noise ratio defined as

$$\text{SNR} = 20 \times \log_{10}\left(\frac{\|x\|_2}{\|x - x^*\|_2}\right),$$

where x is the original image and x^* is the restored image. Note that, the larger the SNR, the better the quality of the restored image. We also choose the initial values as $x_0 = \mathbf{0} \in \mathbb{R}^{N\times N}$ and $x_1 = \mathbf{1} \in \mathbb{R}^{N\times M}$, and take $\psi_n = \frac{n}{2n+5}$.
Then, the results are reported in Table 1 which shows the CPU time and SNR values for each algorithm, and Figure 1 which shows the original, blurred and restored images. The two major advantages of our Algorithm over the other two algorithms are the higher SNR value and lower CPU time for generating the recovered images.

Table 4.2.2. Numerical results for image restoration problem 4.140

Algorithms	Cameraman		MRI	
	CPU	SNR	CPU	SNR
Algorithm 4.2.3	17.4408	34.2025	3.1058	27.0411
Tian & Jiang Alg.	21.2987	27.9775	6.7426	22.3356
Chidume & Nnakwe Alg.	21.6154	18.5223	6.9751	18.2135

4.2.3 Numerical experiments

In this section, we discuss the numerical behavior of our proposed methods; Algorithm 4.2.3, Algorithm 4.2.9 and Algorithm 4.2.11. Furthemore, we compare them with *related strong convergent methods* in the literature; namely, the method of Tian and Jiang [229] and the method of Chidume and Nnakwe [64, Algorithm (3.1)] (other related methods in the literature are mostly weak convergent methods).

The codes for the numerical analysis are written in Matlab 2016 (b) and performed on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM. In Tables 1-4, "Iter." means the number of iterations while "CPU" means the CPU time in seconds. In our numerical computation, we randomly choose the relaxation stepsize ψ_n and the starting points $x_0, x_1 \in H_1$ (see the cases below). We also choose the parameters (randomly) $\gamma_1 > 0$, $\mu, \eta \in (0,1)$, $\alpha \geq 3$ (the choices of these parameters will be discussed in Remark 4.2.17), and the control sequences $\theta_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2} - \theta_n$, $\alpha_n = \bar{\alpha}_n$ and $\epsilon_n = \frac{\theta_n}{n^{0.01}}$ for Algorithm 4.2.3, Algorithm 4.2.9, Algorithm 4.2.11; $\lambda_n = \frac{1}{2L}$ for Algorithm Tian and Jiang [229], Chidume and Nnakwe [64, Algorithm (3.1)]; $\lambda_n = \frac{2n}{3n+7}$ for Algorithm 4.2.11; $\gamma_n = \frac{1}{2\|T\|^2}$ for Algorithm Tian and Jiang [229], Chidume and Nnakwe [64, Algorithm (3.1)]; and $\alpha_n = \frac{1}{n+1}$ for Algorithm Tian and Jiang [229].
Furthermore, we define $\text{TOL}_n := \frac{1}{2}\left(||x_n - J_\lambda^B(x_n - \lambda A x_n)||^2 + ||Tx_n - STx_n||^2\right)$ for Algorithm 4.2.3; $\text{TOL}_n := \frac{1}{2}\left(||x_n - P_C(x_n - \lambda A x_n)||^2 + ||Tx_n - STx_n||^2\right)$ for Algorithm 4.2.9, Algorithm Tian and Jiang [229], Chidume and Nnakwe [64, Algorithm (3.1)];

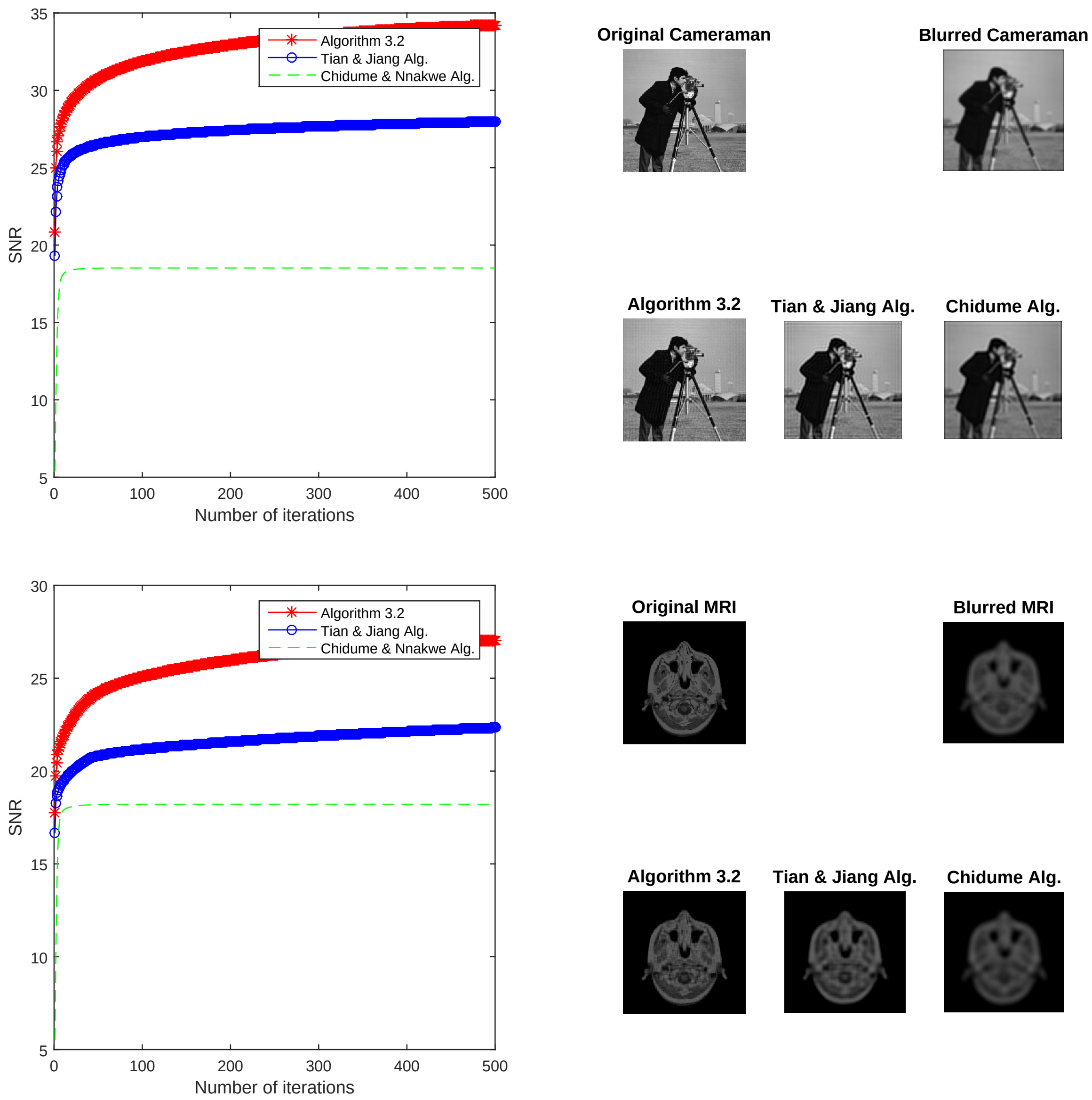


Figure 4.3. Comparison for Numerical results for image restoration problem 4.140: Top: **Cameraman**; Bottom: **MRI**.

$\text{TOL}_n := \frac{1}{2}\left(||x_n - J_\lambda^B x_n||^2 + ||Tx_n - STx_n||^2\right)$ for Algorithm 4.2.11; and use the stopping criterion $\text{TOL}_n < \varepsilon$ for the iterative processes, where ε is the predetermined error. Note that if $\text{TOL}_n = 0$, then x_n is a solution of the problem under consideration.

Example 4.2.14. *Let $H_1 = H_2 = L_2([0,1])$ be equipped with the inner product*

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \quad \forall \quad x, y \in L_2([0,1]) \quad \text{and} \quad \|x\| := \sqrt{\int_0^1 |x(t)|^2 dt} \quad \forall x, y, \in L_2([0,1]).$$

Now, define the operators $A, B : L_2([0,1]) \to L_2([0,1])$ by

$$Ax(t) = \int_0^1 \left(x(t) - \left(\frac{2tse^{t+s}}{e\sqrt{e^2-1}}\right)\cos x(s)\right)ds + \frac{2te^t}{e\sqrt{e^2-1}}, x \in L_2([0,1]),$$
$$Bx(t) = \max\{0, x(t)\}, t \in [0,1].$$

Then A is Lipschitz continuous and monotone, and B is maximal monotone on $L_2([0,1])$. Let $T : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Tx(s) = \int_0^1 \kappa(s,t)x(t)dt \quad \forall \quad x \in L_2([0,1]),$$

where κ is a continuous real-valued function defined on $[0,1]\times[0,1]$. Then, T is a bounded linear operator with adjoint

$$T^*x(s) = \int_0^1 \kappa(t,s)x(t)dt \quad \forall \quad x \in L_2([0,1]).$$

Let $S : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Sx(t) = \int_0^1 tx(s)ds, \quad t \in [0,1].$$

Then, S is nonexpansive. Indeed, we have

$$|Sx(t) - Sy(t)|^2 = |\int_0^1 t\big(x(s) - y(s)\big)|^2 \le \left(\int_0^1 t|x(s) - y(s)|ds\right)^2$$
$$\le \int_0^1 |x(s) - y(s)|^2 ds = \|x - y\|^2.$$

Now, let $C = \{x \in L_2([0,1]) : \langle y, x\rangle \le b\}$, where $y = e^t + 1$ and $b = 10$, then C is a nonempty closed and convex subset of $L_2([0,1])$. Thus, we define the metric projection P_C as:

$$P_C(x) = \begin{cases} \frac{b - \langle y, x\rangle}{||y||^2} y + x, & \text{if } \langle y, x\rangle > b, \\ x, & \text{if } \langle y, x\rangle \le b. \end{cases}$$

For Algorithm Tian and Jiang [229], we define $h : L_2([0,1]) \to L_2([0,1])$ by

$$hx(t) = \int_0^1 \frac{t}{2} x(s)ds, \quad t \in [0,1].$$

Then, h is a contraction.
We now consider the following cases for the relaxation stepsize ψ_n and the starting points x_0, x_1:
Case 1: *Take $x_1(t) = t^3 - 3$, $x_0(t) = t$ and $\psi_n = \frac{n}{2n+5}$.*
Case 2: *Take $x_1(t) = \sin(t)$, $x_0(t) = t + 1$ and $\psi_n = \frac{n}{n+10}$.*
Case 3: *Take $x_1(t) = \cot(t)$, $x_0(t) = \sin(t)$ and $\psi_n = \frac{n}{n+10}$.*
Case 4: *Take $x_1(t) = e^t$, $x_0(t) = t^2 + t + 1$ and $\psi_n = \frac{2n}{18n+1}$.*
The numerical results are displayed in Table 4.2.14 and Figure 4.4.

Table 4.2.14. Numerical results for Example 4.2.14 with $\varepsilon = 10^{-7}$.

Cases		*Algorithm 4.2.9*	*Algorithm 4.2.11*	*Tian Jiang*	*Chidume Nnakwe*
1	*CPU*	*15.8066*	*7.2564*	*21.7214*	*35.9767*
	Iter.	*64*	*51*	*114*	*158*
2	*CPU*	*17.8827*	*7.6196*	*61.6392*	*137.3976*
	Iter.	*39*	*32*	*96*	*140*
3	*CPU*	*20.4864*	*9.6206*	*55.5797*	*144.6658*
	Iter.	*42*	*35*	*102*	*145*
4	*CPU*	*22.8433*	*10.4064*	*46.0686*	*56.3887*
	Iter.	*82*	*63*	*109*	*153*

Example 4.2.15. *Let $H_1 = H_2 = \ell_2$, where $\ell_2 = \ell_2(\mathbb{R}) := \{x = (x_1, x_2, ..., x_i, ...), \quad x_i \in \mathbb{R} : \sum\limits_{i=1}^{\infty} |x_i|^2 < \infty\}$, with inner product $\langle ., . \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle := \sum_{i=1}^{\infty} x_i y_i$ and the norm $\|.\| : \ell_2 \to \mathbb{R}$ by $\|x\| := \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$, where $x = \{x_i\}_{i=1}^{\infty}$ and $y = \{y_i\}_{i=1}^{\infty}$. Define the mapping $A : \ell_2 \to \ell_2$ by $Ax = \left(\frac{x_1 + |x_1|}{2}, \frac{x_2 + |x_2|}{2}, ..., \frac{x_i + |x_i|}{2}, ...\right)$ $\forall x \in \ell_2$. Then, A is Lipschitz continuous and monotone (see [14]).*
Let $B : \ell_2 \to \ell_2$ be defined by $B(x) = (2x_1, 2x_2, ..., 2x_i, ...)$ $\forall x \in \ell_2$. Then, B is maximal monotone.
Let $T : \ell_2 \to \ell_2$ be defined by $Tx = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, ...\right)$, for all $x \in \ell_2$. Then, T is a bounded linear operator on ℓ_2 with adjoint $T^*y = \left(y_2, \frac{y_3}{2}, \frac{y_4}{3}, ...\right)$ for all $y \in \ell_2$.
Now, $C = \{x \in \ell_2 : ||x - a||_{\ell_2} \leq r\}$, where $a = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and $r = 3$. Then C is a

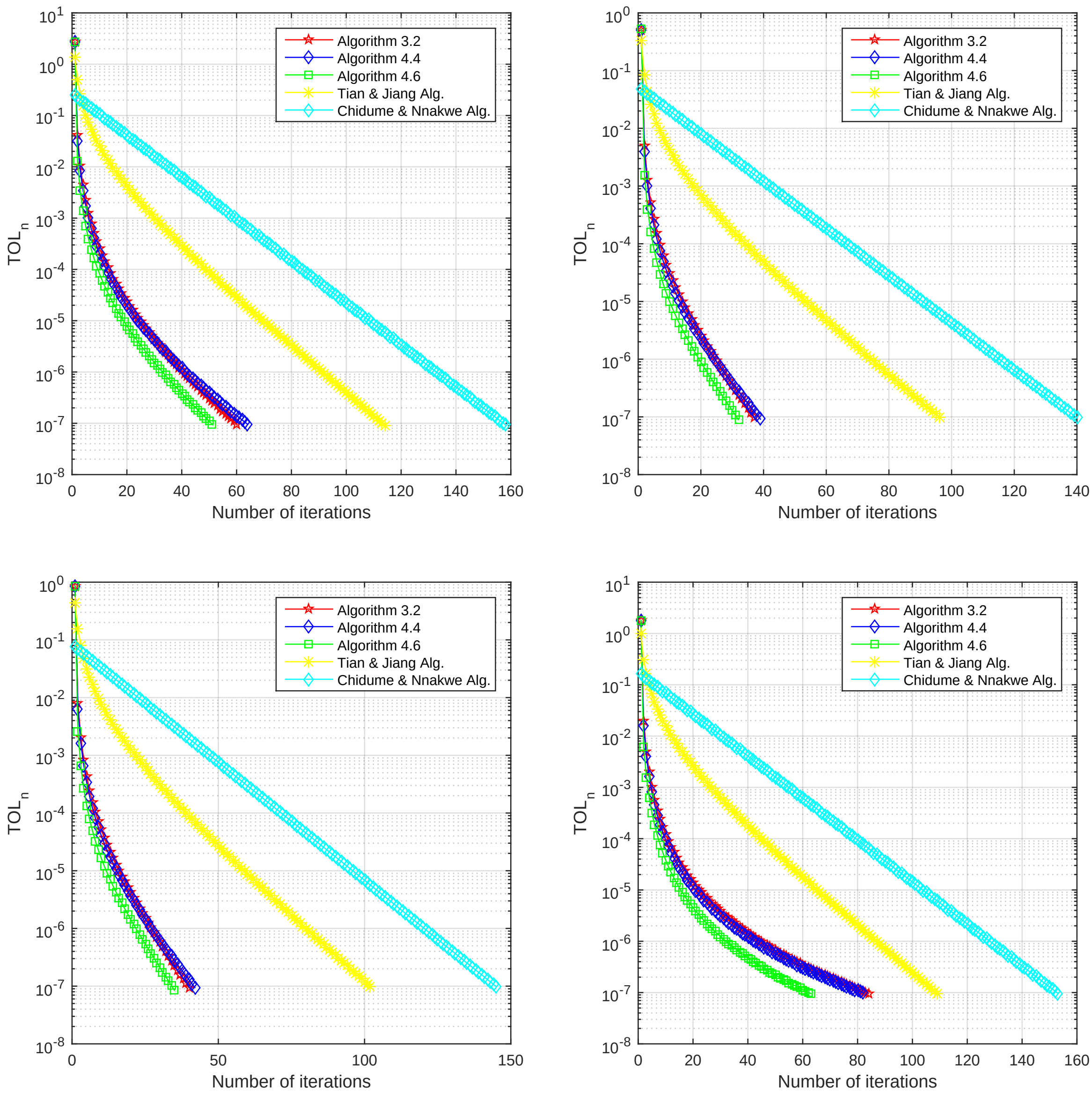


Figure 4.4. The behavior of TOL_n with $\varepsilon = 10^{-7}$ for Example 4.2.14: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

nonempty closed and convex subset of ℓ_2. *Thus,*

$$P_C(x) = \begin{cases} x, & \text{if } ||x-a||_{\ell_2} \leq r, \\ \frac{x-a}{||x-a||_{\ell_2}} r + a, & \text{otherwise.} \end{cases}$$

Furthermore, we define the mappings $S, h : \ell_2 \to \ell_2$ *by* $Sx = (0, x_1, x_2, \dots)$ *and* $hx = \left(0, \frac{x_1}{2}, \frac{x_2}{2}, \cdots\right)$ *for all* $x \in \ell_2$, *and consider the following cases for the relaxation stepsize* ψ_n *and the starting points* x_0, x_1:

Case 1: *Take* $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$, $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$ *and* $\psi_n = \frac{n}{2n+5}$.

Case 2: *Take* $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$, $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ *and* $\psi_n = \frac{n}{n+10}$.

Case 3: *Take* $x_1 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$, $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$ *and* $\psi_n = \frac{n}{n+10}$.

Case 4: *Take* $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$, $x_0 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$ *and* $\psi_n = \frac{2n}{18n+1}$.

The numerical results are displayed in Table 4.2.15 and Figure 4.5.

Table 4.2.15. Numerical results for Example 4.2.15 with $\varepsilon = 10^{-8}$**.**

Cases		*Algorithm 4.2.9*	*Algorithm 4.2.11*	*Tian Jiang*	*Chidume Nnakwe*
1	*CPU*	*0.0566*	*0.0325*	*1.0297*	*1.0889*
	Iter.	*76*	*56*	*449*	*509*
2	*CPU*	*0.0449*	*0.0208*	*1.0290*	*1.0679*
	Iter.	*67*	*51*	*449*	*506*
3	*CPU*	*0.0394*	*0.0223*	*1.0268*	*1.0719*
	Iter.	*74*	*55*	*449*	*510*
4	*CPU*	*0.0389*	*0.0199*	*1.0290*	*1.0676*
	Iter.	*83*	*60*	*449*	*510*

Example 4.2.16. *In many practical problems, it is important if the minimum-norm solutions of such problems can be found. Such problems can be formulated mathematically as (see, for instance [206, Example 3.4]):*

$$\text{Find } x^* \in H \text{ such that } \|x^*\| = \min\{\|x\| : x \in H\}, \tag{4.141}$$

where H *is a real Hilbert space. Note that (4.141) can be reformulated as the following particular variational inequality problem:*

$$\text{Find } x^* \in H \text{ such that } \langle x^*, x^* - x \rangle \leq 0, \ \forall x \in H. \tag{4.142}$$

Now, let $H_1 = H_2 = L_2([a,b])$, $C = \{x \in L_2([a,b]) : \langle a, x \rangle = b\}$ *and* $Q = \{x \in L_2([a,b]) : \langle a, x \rangle \geq b\}$, *for some* $a \in L_2([a,b]) - \{0\}$ *and* $b \in \mathbb{R}$.

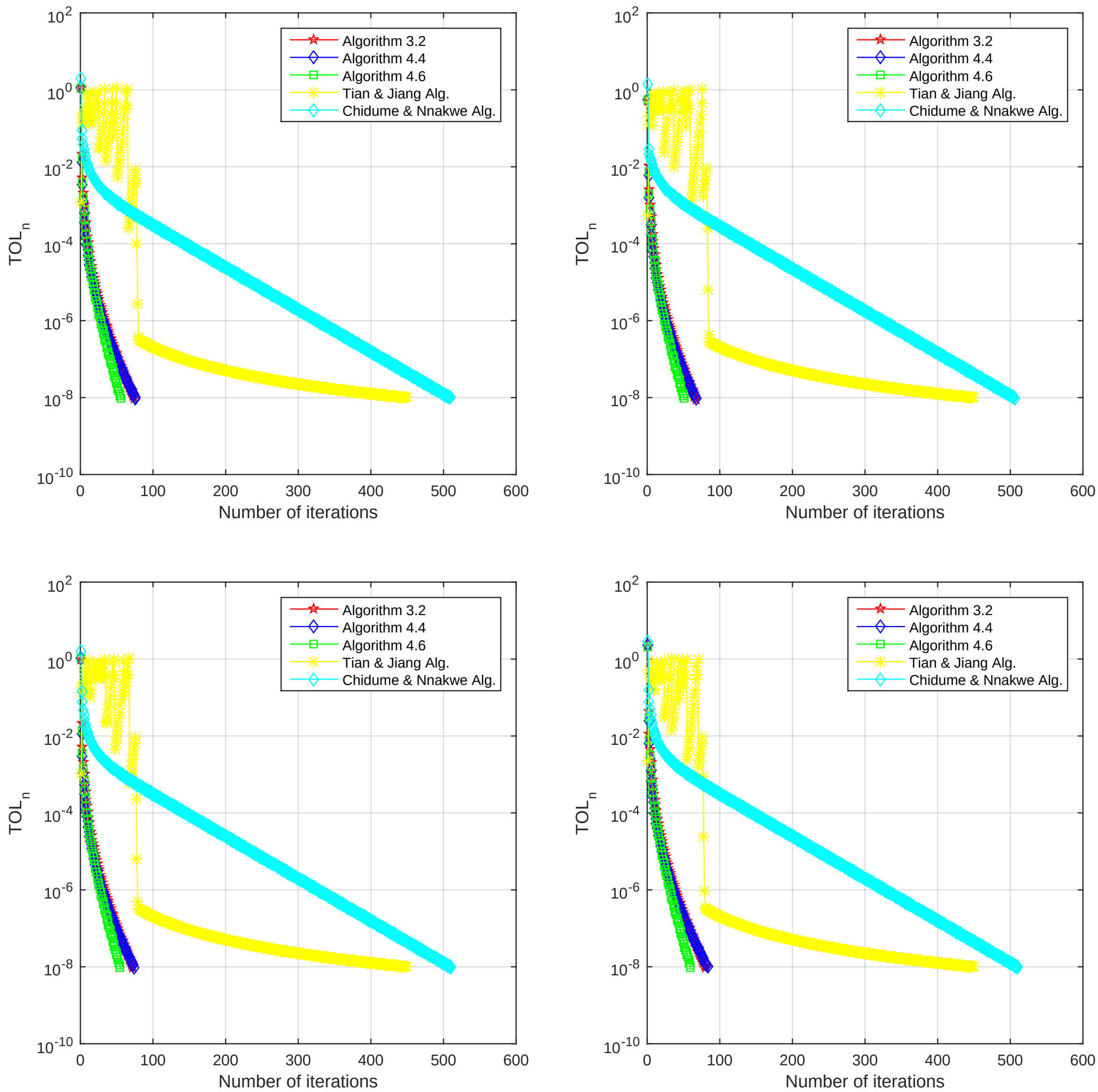


Figure 4.5. The behavior of TOL_n with $\varepsilon = 10^{-8}$ for Example 4.2.15: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

Then, x^* minimizes $||.||+\delta_C$ if and only if $0 \in \partial\left(||.||+\delta_C\right)(x^*)$ and Tx^* minimizes $||.||+\delta_Q$ if and only if $0 \in \partial\left(||.||+\delta_Q\right)(Tx^*)$, where δ_C (defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$, otherwise) and δ_Q denote the indicator functions of C and Q, respectively. Now, if we set in (4.84), $A = 0$, $B = (||.|| + \delta_C)$ and $F(S) = \partial\left(||.|| + \delta_Q\right)$, then problem (4.84) becomes the following problem:

$$\text{Find } x^* \in C \text{ such that } x^* = \text{argmin}\{\|x\| : x \in C\}, \tag{4.143}$$

and such that

$$Tx^* \in Q \quad \text{solves} \quad Tx^* = \text{argmin}\{\|y\| : y \in Q\}. \tag{4.144}$$

It is known that the solution to problem (4.143)-(4.144) is a minimum-norm solution (see [206, Example 3.4]).
Now, for the numerical computation, we define the bounded linear operator T as in Example 4.2.14. We also choose the starting points and relaxation stepsize as in **Case 1-Case 4** *of Example 4.2.14. The numerical results are reported in Table 4.2.16 and Figure 4.6.*

Table 4.2.16. Numerical results for Example 4.2.16 with $\varepsilon = 10^{-8}$.

Cases		Algorithm 4.2.9	Algorithm 4.2.11	Tian Jiang	Chidume Nnakwe
1	CPU	19.8207	15.4169	29.0280	50.9212
	Iter.	121	101	136	182
2	CPU	26.5295	25.3138	87.9523	187.6199
	Iter.	92	83	119	165
3	CPU	41.3551	34.5019	82.0133	204.0348
	Iter.	95	93	125	170
4	CPU	25.2010	22.7278	77.0343	96.5969
	Iter.	110	99	132	178

Remark 4.2.17. *We now summarize this section by highlighting some of the observations from the numerical results.*

- *During the numerical computations, we observed that irrespective of the choices of the parameters $\gamma_1, \lambda_1 > 0$, $\mu, \eta \in (0, 1)$ and $\alpha \geq 3$, the number of iteration does not change and no significant difference in the CPU time. Therefore, we randomly choose these parameters.*

- *Throughout the experiments, we see clearly that both in CPU time and number of iterations, Algorithm 4.2.11 outperforms Algorithm 4.2.3 and Algorithm 4.2.9. This is expected since Algorithm 4.2.11 does not require any evaluation of A and the stepsize λ_n in Algorithm 4.2.11 does not involve any form of "inner loop" during implementation, which are required in Algorithm 4.2.3 and Algorithm 4.2.9.*

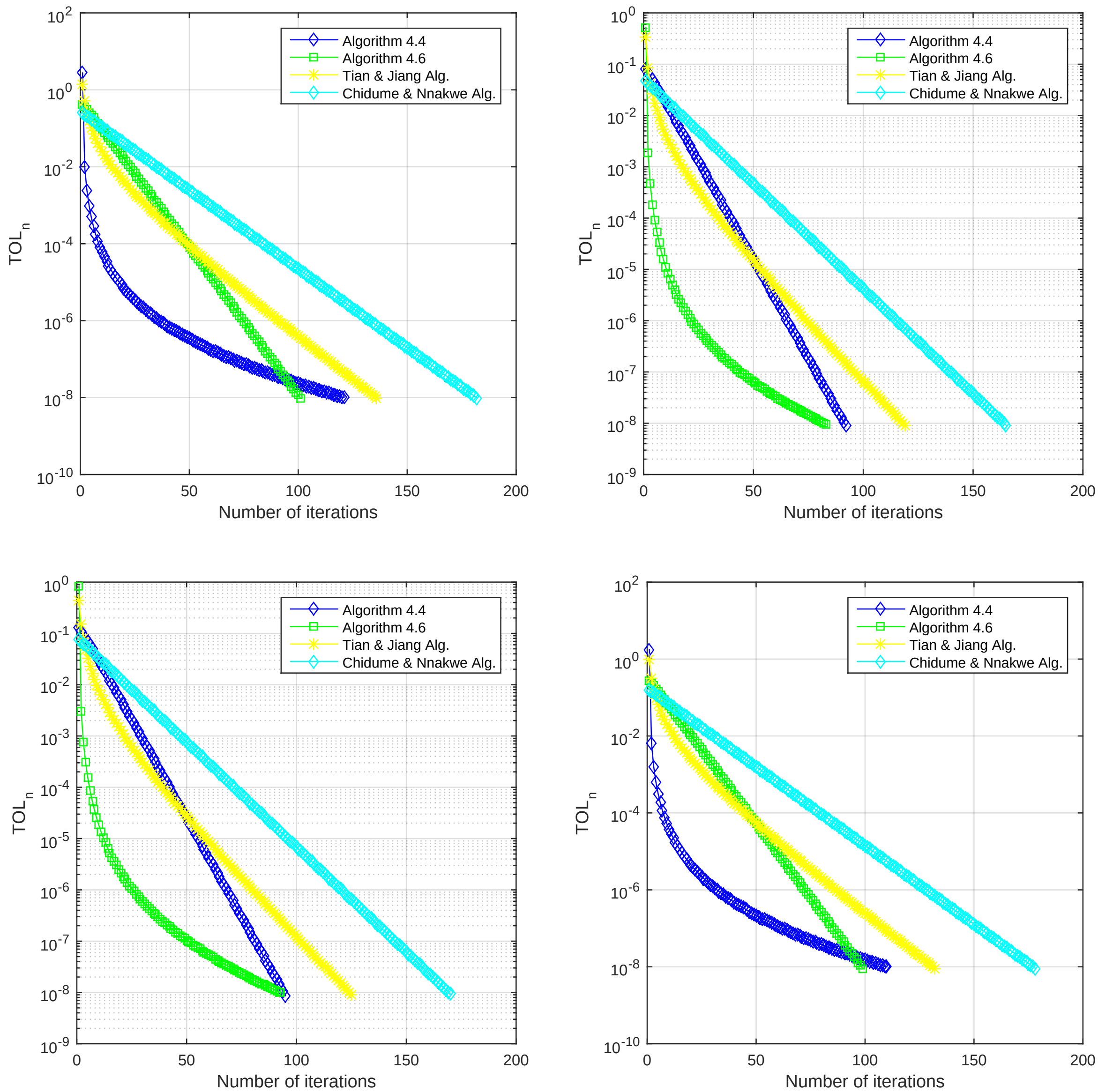


Figure 4.6. The behavior of TOL_n with $\varepsilon = 10^{-8}$ for Example 4.2.16: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

- *It can also be seen from the tables and figures that Algorithm 4.2.3 and Algorithm 4.2.9 perform better than Algorithm Tian and Jian [229] and Algorithm (3.1) of Chidume and Nnakwe [64]. This is also expected due to the presence of both relaxation and inertial steps in our methods. We can also see from the cases that irrespective of the choices of the relaxation stepsize, our proposed methods converge more than twice as fast as the other methods.*

Chapter 5

Inertial Type Algorithms for Solving Variational Inequality Problems and Fixed Point Problems

5.1 Introduction

In this chapter, we propose some inertial-type algorithms for solving some optimization and fixed point problems. Strong convergence of the algorithms are established under mild assumptions. We demonstrate the performance of our algorithms with numerical examples.

5.2 Variational inequality and fixed point problems

In this section, we introduce a new relaxed inertial Tseng extragradient method with self-adaptive step size for approximating common solutions of monotone variational inequality and fixed point problems of quasi-pseudo-contraction mappings in real Hilbert spaces. We prove strong convergence result for the proposed algorithm without the knowledge of the Lipschitz constant of the cost operator. Moreover, we apply our results to approximate solution of convex minimization problem and we present some numerical experiments to show the efficiency and applicability of our method in comparison with some existing methods in the literature. Our proposed method is easy to implement. It requires only one projection onto a constructible half-space.

5.2.1 Main result

In this section, we present our proposed method and discuss some of its important features. We begin with the following assumptions under which our strong convergence result is obtained.

Assumption 5.2.1. *Suppose that the following conditions hold:*

1. *The set C is defined by*

$$C = \{x \in H \mid c(x) \leq 0\};$$

where $c : H \to \mathbb{R}$ satisfies the following conditions:

   (a) $c : H \to \mathbb{R}$ is a continuously differentiable convex function such that $c'(.)$ is M-Lipschitz continuous;

   (b) $c(x)$ is weakly lower semicontinuous (w-lsc) on H.

2. *(a) $A : H \to H$ is monotone and L-Lipschitz continuous with constant $L > 0$. However, prior knowledge of the Lipschitz constant is not required for the implementation of the proposed algorithm.*

   (b) $T : H \to H$ is a K-Lipschitz continuous quasi-pseudo-contractive mapping, which is demiclosed at zero and with $K \geq 1$.

   (c) The solution set $\Omega = F(T) \cap VI(C, A)$ is nonempty.

   (d) $D : H \to H$ is a strongly positive bounded linear operator with coefficient $\bar{\gamma}$ and $f : H \to H$ is a contraction with coefficient $\rho \in (0, 1)$ such that $0 < \gamma < \frac{\bar{\gamma}}{\rho}$.

3. *$\{\alpha_n\}_{n=1}^{+\infty}$, $\{\beta_n\}_{n=1}^{+\infty}$ and $\{\gamma_n\}_{n=1}^{+\infty}$ are positive sequences satisfying the following conditions:*

   (a) $\alpha_n \in (0, 1)$, $\lim\limits_{n\to+\infty} \alpha_n = 0$, $\sum_{n=1}^{+\infty} \alpha_n = +\infty$, $\lim\limits_{n\to+\infty} \dfrac{\gamma_n}{\alpha_n} = 0, 0 < c_1 \leq \beta_n \leq c_2 < 1$.

   (b) $\{\phi_n\} \subset (0, 1]$ such that $\lim\limits_{n\to+\infty} \phi_n = \phi \in (0, 1], 0 < \xi < \mu < \frac{1}{1+\sqrt{1+K^2}}$.

   (c) $0 < \delta < \sqrt{(1-\phi)(2-\phi) + (1-\kappa^2)} - (1-\phi)$ for some $\kappa > 0$.

Algorithm 5.2.2.

Initialization: *Choose sequences $\{\alpha_n\}_{n=1}^{+\infty}$, $\{\beta_n\}_{n=1}^{+\infty}$ and $\{\gamma_n\}_{n=1}^{+\infty}$ such that the conditions from Assumption 5.2.1(3) hold and let $\lambda_1 > 0$, $\alpha \geq 3$ and $x_0, x_1 \in H$ be given arbitrarily. Set $n = 1$.*

Iterative Steps:

Step 1. *Given the iterates x_{n-1} and $x_n (n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\gamma_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases} \tag{5.1}$$

Step 2 *Compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}),$$

Step 3 *Construct the half-space*

$$C_n = \{x \in H : c(w_n) + \langle c'(w_n), x - w_n\rangle \le 0\}.$$

and compute

$$y_n = P_{C_n}(w_n - \lambda_n A w_n),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\delta\|w_n - y_n\|}{\|Aw_n - Ay_n\| + \|c'(w_n) - c'(y_n)\|},\ \lambda_n\right\}, & \text{if } \|Aw_n - Ay_n\| \\ & +\|c'(w_n) - c'(y_n)\| \ne 0, \\ \lambda_n, & \text{otherwise.} \end{cases} \quad (5.2)$$

Step 4. *Compute*

$$t_n := (1-\phi_n)w_n + \phi_n\Big(y_n + \lambda_n\big(Aw_n - Ay_n\big)\Big).$$

Step 5. *Compute*

$$x_{n+1} = \alpha_n \gamma f(w_n) + (I - \alpha_n D)\big[(1-\beta_n)t_n + \beta_n \mathbb{V} t_n\big],$$

where

$$\mathbb{V} = (1-\xi)I + \xi T\big((1-\mu)I + \mu T\big).$$

Set $n = n+1$ *and go back to* **Step 1**.

Remark 5.2.3. *(a) Our method only requires one projection onto a half-space* C_n*, which can be easily computed. Hence, our method is less computationally expensive than several other methods in the literature (see for example, [44, 104]), for solving variational inequality problems. By the definition of the subgradient, we observe that* $C \subset C_n$*.*
(b) The step size $\{\lambda_n\}$ *given by* (5.2) *is generated at each iteration by some simple computations. Thus,* $\{\lambda_n\}$ *is easily implemented and does not depend on the Lipschitz constant of the cost operator.*
(c) **Step 1** *of Algorithm 5.2.2 is also easily implemented since the value of* $\|x_n - x_{n-1}\|$ *is known before choosing* θ_n*.*

Remark 5.2.4. *Observe that from* (5.2) *in Algorithm 5.2.2, we have* $\lambda_{n+1} \le \lambda_n\ \forall n \ge 1$*. Moreover, since* A *is* L*-Lipschitz continuous and* c' *is* K*-Lipschitz continuous, we obtain in the case of* $\|Aw_n - Ay_n\| + \|c'(w_n) - c'(y_n)\| \ne 0$ *in Algorithm 5.2.2, that*

$$\lambda_{n+1} = \min\left\{\frac{\delta\|w_n - y_n\|}{\|Aw_n - Ay_n\| + \|c'(w_n) - c'(y_n)\|},\ \lambda_n\right\} \ge \min\left\{\frac{\delta}{L+K},\ \lambda_n\right\},$$

which by induction, implies that $\{\lambda_n\}$ *is bounded below by* $\min\left\{\frac{\delta}{L+K},\ \lambda_1\right\}$*. Since* $\{\lambda_n\}$ *is monotone nonincreasing, it follows that the limit exists, and* $\lim\limits_{n\to+\infty} \lambda_n \ge \min\{\frac{\delta}{L+K}, \lambda_1\} > 0$.

Remark 5.2.5. *By Assumption (5.2.1) 3(a), one can easily verify from (5.1) that*

$$\lim_{n\to+\infty} \theta_n||x_n - x_{n-1}|| = 0 \quad \textit{and} \quad \lim_{n\to+\infty} \frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0. \tag{5.3}$$

Lemma 5.2.6. *Let $p \in \Omega$ and $\{x_n\}$ be a sequence generated by Algorithm 5.2.2 under Assumption 5.2.1. Then, we have the following inequality:*

$$\|t_n - p\|^2 \le \|w_n - p\|^2 - \phi_n\big[2 - \phi_n - \delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}} - \rho^2\big]\|w_n - y_n\|^2. \tag{5.4}$$

Proof. Let $p \in \Omega$. Then, $p \in C \subset C_n$. Since $y_n = P_{C_n}(w_n - \lambda_n Aw_n)$ and $p \in C_n$, then by the characterization of P_{C_n} we have

$$\langle w_n - \lambda_n Aw_n - y_n, y_n - p\rangle \ge 0,$$

or equivalently

$$2\langle w_n - y_n, y_n - p\rangle - 2\lambda_n\langle Aw_n - Ay_n, y_n - p\rangle - 2\lambda_n\langle Ay_n, y_n - p\rangle \ge 0. \tag{5.5}$$

Next, applying Lemma 2.5.18(i), we get

$$2\langle w_n - y_n, y_n - p\rangle = \|w_n - p\|^2 - \|w_n - y_n\|^2 - \|y_n - p\|^2. \tag{5.6}$$

By the monotonicity of A, we obtain

$$\begin{aligned}\langle Ay_n, y_n - p\rangle &= \langle Ay_n - Ap, y_n - p\rangle + \langle Ap, y_n - p\rangle \\ &\ge \langle Ap, y_n - p\rangle.\end{aligned} \tag{5.7}$$

Substituting (5.6) and (5.7) in (5.5), we have

$$\|y_n - p\|^2 \le \|w_n - p\|^2 - \|w_n - y_n\|^2 - 2\lambda_n\langle Aw_n - Ay_n, y_n - p\rangle + 2\lambda_n\langle Ap, p - y_n\rangle. \tag{5.8}$$

Next, using the definition of t_n and Lemma 2.5.18, we obtain

$$\begin{aligned}\|t_n - p\|^2 &= \|(1-\phi_n)w_n + \phi_n y_n + \phi_n\lambda_n(Aw_n - Ay_n) - p\|^2 \\ &= \|(1-\phi_n)(w_n - p) + \phi_n(y_n - p) + \phi_n\lambda_n(Aw_n - Ay_n)\|^2 \\ &= (1-\phi_n)^2\|w_n - p\|^2 + \phi_n^2\|y_n - p\|^2 + \phi_n^2\lambda_n^2\|Aw_n - Ay_n\|^2 \\ &+ 2\phi_n(1-\phi_n)\langle w_n - p, y_n - p\rangle + 2\lambda_n\phi_n(1-\phi_n)\langle w_n - p, Aw_n - Ay_n\rangle \\ &+ 2\lambda_n\phi_n^2\langle y_n - p, Aw_n - Ay_n\rangle \\ &= (1-\phi_n)^2\|w_n - p\|^2 + \phi_n^2\|y_n - p\|^2 + \phi_n^2\lambda_n^2\|Aw_n - Ay_n\|^2 \\ &+ \phi_n(1-\phi_n)\big[\|w_n - p\|^2 + \|y_n - p\|^2 - \|w_n - y_n\|^2\big] \\ &+ 2\lambda_n\phi_n(1-\phi_n)\langle w_n - p, Aw_n - Ay_n\rangle + 2\lambda_n\phi_n^2\langle y_n - p, Aw_n - Ay_n\rangle \\ &= (1-\phi_n)\|w_n - p\|^2 + \phi_n\|y_n - p\|^2 - \phi_n(1-\phi_n)\|w_n - y_n\|^2 \\ &+ \phi_n^2\lambda_n^2\|Aw_n - Ay_n\|^2 \\ &+ 2\lambda_n\phi_n(1-\phi_n)\langle w_n - p, Aw_n - Ay_n\rangle + 2\lambda_n\phi_n^2\langle y_n - p, Aw_n - Ay_n\rangle.\end{aligned} \tag{5.9}$$

Using (5.8) in (5.9), we get

$$\begin{aligned}
\|t_n - p\|^2 &\le (1-\phi_n)\|w_n - p\|^2 + \phi_n\big[\|w_n - p\|^2 - \|w_n - y_n\|^2 - 2\lambda_n\langle Aw_n - Ay_n, y_n - p\rangle \\
&\quad + 2\lambda_n\langle Ap, p - y_n\rangle\big] - \phi_n(1-\phi_n)\|w_n - y_n\|^2 + \phi_n^2\lambda_n^2\|Aw_n - Ay_n\|^2 \\
&\quad + 2\phi_n(1-\phi_n)\lambda_n\langle w_n - p, Aw_n - Ay_n\rangle + 2\phi_n^2\lambda_n\langle y_n - p, Aw_n - Ay_n\rangle \\
&= \|w_n - p\|^2 - \phi_n(2-\phi_n)\|w_n - y_n\|^2 + \phi_n^2\lambda_n^2\|Aw_n - Ay_n\|^2 \\
&\quad + 2\lambda_n\phi_n(1-\phi_n)\langle Aw_n - Ay_n, w_n - y_n\rangle + 2\lambda_n\phi_n\langle Ap, p - y_n\rangle. \qquad (5.10)
\end{aligned}$$

Now, we consider the following two cases:

Case 1: $Ap = 0$. If $Ap = 0$, then from (5.10), we have

$$\begin{aligned}
\|t_n - p\|^2 &\le \|w_n - p\|^2 - \phi_n(2-\phi_n)\|w_n - y_n\|^2 + \phi_n^2\lambda_n^2\|Aw_n - Ay_n\|^2 \\
&\quad + 2\lambda_n\phi_n(1-\phi_n)\langle Aw_n - Ay_n, w_n - y_n\rangle \\
&\le \|w_n - p\|^2 - \phi_n(2-\phi_n)\|w_n - y_n\|^2 + \phi_n\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\|w_n - y_n\|^2 \\
&\quad + 2\phi_n(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 \\
&= \|w_n - p\|^2 - \phi_n\Big[2 - \phi_n - \delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}}\Big]\|w_n - y_n\|^2 \\
&\le \|w_n - p\|^2 - \phi_n\Big[2 - \phi_n - \delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}} - \kappa^2\Big]\|w_n - y_n\|^2,
\end{aligned}$$

for some $\kappa > 0$, which is the required inequality.

Case 2: $Ap \neq 0$.
By Lemma 2.5.11, it follows that $p \in \partial C$ and there exists $\kappa > 0$ such that $Ap = -\kappa c'(p)$. Observe that $c(p) = 0$ since $p \in \partial C$. By the subdifferential inequality (2.54), we get

$$\begin{aligned}
c(y_n) &\ge c(p) + \langle c'(p), y_n - p\rangle \\
&= \frac{-1}{\kappa}\langle Ap, y_n - p\rangle,
\end{aligned}$$

which implies that

$$\langle Ap, p - y_n\rangle \le \kappa c(y_n). \qquad (5.11)$$

Since $y_n \in C_n$, we get

$$c(w_n) + \langle c'(w_n), y_n - w_n\rangle \le 0. \qquad (5.12)$$

Again, by applying the subdifferential inequality (2.54), we have

$$c(y_n) + \langle c'(y_n), w_n - y_n\rangle \le c(w_n). \qquad (5.13)$$

Adding (5.12) and (5.13), we have

$$c(y_n) \le \langle c'(y_n) - c'(w_n), y_n - w_n\rangle. \qquad (5.14)$$

From (5.11) and (5.14), we obtain

$$\langle Ap, p - y_n\rangle \le \kappa\langle c'(y_n) - c'(w_n), y_n - w_n\rangle. \tag{5.15}$$

Also, observe that

$$2\lambda_n\kappa\langle c'(y_n) - c'(w_n), y_n - w_n\rangle \le \lambda_n^2\|c'(y_n) - c'(w_n)\|^2 + \kappa^2\|y_n - w_n\|^2. \tag{5.16}$$

Applying (5.15) and (5.16) in (5.10) and using the definition of the step size, we get

$$\begin{aligned}
&\|t_n - p\|^2 \le \|w_n - p\|^2 - \phi_n(2-\phi_n)\|w_n - y_n\|^2 + \phi_n\lambda_n^2\|Aw_n - Ay_n\|^2\\
&\quad + 2\lambda_n\phi_n(1-\phi_n)\langle w_n - y_n, Aw_n - Ay_n\rangle + \phi_n\lambda_n^2\|c'(y_n) - c'(w_n)\|^2 + \phi_n\kappa^2\|y_n - w_n\|^2\\
&\quad = \|w_n - p\|^2 - \phi_n(2-\phi_n)\|w_n - y_n\|^2 + \phi_n\lambda_n^2\big[\|Aw_n - Ay_n\|^2 + \|c'(y_n) - c'(w_n)\|^2\big]\\
&\quad + 2\lambda_n\phi_n(1-\phi_n)\langle w_n - y_n, Aw_n - Ay_n\rangle + \phi_n\kappa^2\|y_n - w_n\|^2\\
&\quad \le \|w_n - p\|^2 - \phi_n(2-\phi_n)\|w_n - y_n\|^2 + \phi_n\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\|w_n - y_n\|^2\\
&\quad + 2\phi_n(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 + \phi_n\kappa^2\|w_n - y_n\|^2\\
&\quad = \|w_n - p\|^2 - \phi_n\big[2 - \phi_n - \delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}} - \kappa^2\big]\|w_n - y_n\|^2,
\end{aligned}$$

which is the required inequality. Hence, we have the desired result.

□

Since the limit of $\{\lambda_n\}$ exists, $\lim_{n\to+\infty}\lambda_n = \lim_{n\to+\infty}\lambda_{n+1}$. Thus, by Assumption (5.2.1) 3(b) and 3(c), we have

$$\lim_{n\to+\infty}\phi_n\big[2 - \phi_n - \delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}} - \kappa^2\big] = \phi\big[2 - \phi - \delta^2 - 2(1-\phi)\delta - \kappa^2\big] > 0.$$

Hence, there exists $n_0 \ge 1$ such that for all $n \ge n_0$ we have

$$\lim_{n\to+\infty}\phi_n\big[2 - \phi_n - \delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}} - \kappa^2\big] > 0.$$

Consequently, from (5.4) we have that for all $n \ge n_0$,

$$\|t_n - p\| \le \|w_n - p\|. \tag{5.17}$$

Lemma 5.2.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 5.2.2. Then, under Assumption 5.2.1, $\{x_n\}$ is bounded.*

Proof. First, we claim that $P_\Omega(I - D + \gamma f)$ is a contraction on H. For all $x, y \in H$, we have

$$\begin{aligned}
\|P_\Omega(I - D + \gamma f)x - P_\Omega(I - D + \gamma f)y\| &\le \|(I - D + \gamma f)x - (I - D + \gamma f)y\|\\
&\le \|(I - D)x - (I - D)y\| + \gamma\|f(x) - fy\|\\
&\le (1 - \bar{\gamma})\|x - y\| + \gamma\rho\|x - y\|\\
&= (1 - (\bar{\gamma} - \gamma\rho))\|x - y\|.
\end{aligned}$$

Since $0 < (1 - (\bar{\gamma} - \gamma\rho)) < 1$, $P_\Omega(I - D + \gamma f)$ is a contraction. Thus, there exists $p \in \Omega$ such that $p = P_\Omega(I - D + \gamma f)(p)$. Since $p \in \Omega$, then we have that $\mathbb{V}p = p$. Next, from the definition of w_n, we obtain

$$\begin{aligned}
\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\
&\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\| + \alpha_n \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|. \qquad (5.18)
\end{aligned}$$

By (5.3), there exists $M_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1, \quad \forall\, n \geq 1, \qquad (5.19)$$

and so, it follows from (5.18) that

$$\|w_n - p\| \leq \|x_n - p\| + \alpha_n M_1, \quad \forall n \geq 1. \qquad (5.20)$$

Let $T_n = (1 - \beta_n)I + \beta_n\mathbb{V}$ for each $n \geq 1$. By the conditions on ξ and μ, and by Lemma 2.5.3, we know that $\mathbb{V}$ is quasi-nonexpansive. Consequently, we have that

$$\begin{aligned}
\|T_n t_n - p\| &= \|(1 - \beta_n)t_n + \beta_n\mathbb{V}t_n - p\| \\
&\leq (1 - \beta_n)\|t_n - p\| + \beta_n\|\mathbb{V}t_n - p\| \\
&\leq (1 - \beta_n)\|t_n - p\| + \beta_n\|t_n - p\| \\
&= \|t_n - p\| \qquad (5.21)
\end{aligned}$$

Now, by applying (5.17), (5.20) and (5.21) we obtain for all $n \geq n_0$,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n\gamma f(w_n) + (I - \alpha_n D)T_n t_n - p\| \\
&= \|\alpha_n(\gamma f(w_n) - Dp) + (I - \alpha_n D)(T_n t_n - p)\| \\
&\leq \alpha_n\|\gamma f(w_n) - Dp\| + (1 - \alpha_n\bar{\gamma})\|T_n t_n - p\| \\
&\leq \alpha_n\|\gamma(f(w_n) - f(p)) + (\gamma f(p) - Dp)\| + (1 - \alpha_n\bar{\gamma})\|t_n - p\| \\
&\leq \alpha_n\gamma\rho\|w_n - p\| + \alpha_n\|\gamma f(p) - Dp\| + (1 - \alpha_n\bar{\gamma})(\|x_n - p\| + \alpha_n M_1) \\
&\leq \alpha_n\gamma\rho\big(\|x_n - p\| + \alpha_n M_1\big) + \alpha_n\|\gamma f(p) - Dp\| + (1 - \alpha_n\bar{\gamma})(\|x_n - p\| + \alpha_n M_1) \\
&= \big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\big)\|x_n - p\| + \alpha_n\|\gamma f(p) - Dp\| + \big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\big)\alpha_n M_1 \\
&\leq \big(1 - \alpha_n(\bar{\gamma} - \gamma\rho)\big)\|x_n - p\| + \alpha_n(\bar{\gamma} - \gamma\rho)\left[\frac{\|\gamma f(p) - Dp\|}{\bar{\gamma} - \gamma\rho} + \frac{M_1}{\bar{\gamma} - \gamma\rho}\right] \\
&\leq \max\left\{\|x_n - p\|, \quad \frac{\|\gamma f(p) - Dp\|}{\bar{\gamma} - \gamma\rho} + \frac{M_1}{\bar{\gamma} - \gamma\rho}\right\} \\
&\;\;\vdots \\
&\leq \max\left\{\|x_{n_0} - p\|, \quad \frac{\|\gamma f(p) - Dp\|}{\bar{\gamma} - \gamma\rho} + \frac{M_1}{\bar{\gamma} - \gamma\rho}\right\}.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded. Consequently, $\{w_n\}, \{y_n\}$ and $\{t_n\}$ are also bounded.

$\square$

Lemma 5.2.8. *Suppose $\{w_n\}$ and $\{y_n\}$ are sequences generated by Algorithm 5.2.2 such that $\lim_{n\to+\infty} ||w_n - y_n|| = 0$. If $\{w_{n_j}\}$ converges weakly to some $\hat{x} \in H$ as $j \to +\infty$, then $\hat{x} \in VI(C, A)$.*

Proof. Since $w_{n_j} \rightharpoonup \hat{x}$, by the hypothesis of the lemma it follows that $y_{n_j} \rightharpoonup \hat{x}$ as $j \to +\infty$. Also, since $y_{n_j} \in C_{n_j}$, then by the definition of C_n we get

$$c(w_{n_j}) + \langle c'(w_{n_j}), y_{n_j} - w_{n_j}\rangle \leq 0.$$

Applying the Cauchy-Schwartz Inequality, we have

$$c(w_{n_j}) \leq \|c'(w_{n_j})\|\|y_{n_j} - w_{n_j}\|. \tag{5.22}$$

Since $c'(\cdot)$ is Lipschitz continuous, it follows that it is bounded on any bounded subsets of H. Hence, by the boundedness of $\{w_n\}$ there exists a constant $M > 0$ such that $\|c'(w_{n_j})\| \leq M \quad$ for all $\quad j \geq 0$. Then, from (5.22) we obtain

$$c(w_{n_j}) \leq M\|y_{n_j} - w_{n_j}\| \to 0 \quad \text{as} \quad j \to +\infty, \tag{5.23}$$

By condition 1(b) in Assumption 5.2.1, we have $c(\hat{x}) \leq \liminf_{j\to+\infty} c(w_{n_j}) \leq 0$. This implies that $\hat{x} \in C$. By the characterization of P_{C_n}, we obtain

$$\langle y_{n_j} - w_{n_j} + \lambda_{n_j} A w_{n_j},\ z - y_{n_j}\rangle \geq 0, \quad \forall\ z \in C \subseteq C_{n_j}.$$

Since A is monotone, we have

$$\begin{aligned}
0 &\leq \langle y_{n_j} - w_{n_j},\ z - y_{n_j}\rangle + \lambda_{n_j}\langle A w_{n_j},\ z - y_{n_j}\rangle \\
&= \langle y_{n_j} - w_{n_j},\ z - y_{n_j}\rangle + \lambda_{n_j}\langle A w_{n_j},\ z - w_{n_j}\rangle + \lambda_{n_j}\langle A w_{n_j},\ w_{n_j} - y_{n_j}\rangle \\
&\leq \langle y_{n_j} - w_{n_j},\ z - y_{n_j}\rangle + \lambda_{n_j}\langle A z,\ z - w_{n_j}\rangle + \lambda_{n_j}\langle A w_{n_j},\ w_{n_j} - y_{n_j}\rangle.
\end{aligned}$$

Letting $j \to +\infty$, and since $\lim_{j\to+\infty} ||y_{n_j} - w_{n_j}|| = 0$, we have

$$\langle Az,\ z - \hat{x}\rangle \geq 0, \quad \forall\ z \in C.$$

Applying Lemma 2.5.9, we have that $\hat{x} \in VI(C, A)$. $\square$

Lemma 5.2.9. *Let $\{x_n\}$ be a sequence generated by Algorithm 5.2.2. Then, under Assumption 5.2.1, we have the following inequality for all $p \in \Omega \quad$ and $n \in \mathbb{N}$:*

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le & \Big(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big)||x_n-p||^2+\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\rho)}M_3 \\
&+3M_2\frac{\big((1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho\big)}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{1}{(\bar{\gamma}-\gamma\rho)}\langle\gamma f(p)-Dp, x_{n+1}-p\rangle\Big\} \\
&-\frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\phi_n\big[2-\phi_n-\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}-2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}}-\kappa^2\big]\|w_n-y_n\|^2 \\
&+\beta_n(1-\beta_n)\|\mathbb{V}t_n-t_n\|^2\Big\}.
\end{aligned}$$

Proof. Using the Cauchy-Schwartz Inequality and Lemma 2.5.18, we get

$$\begin{aligned}
||w_n-p||^2 &= ||x_n+\theta_n(x_n-x_{n-1})-p||^2 \\
&= ||x_n-p||^2+\theta_n^2||x_n-x_{n-1}||^2+2\theta_n\langle x_n-p, x_n-x_{n-1}\rangle \\
&\le ||x_n-p||^2+\theta_n^2||x_n-x_{n-1}||^2+2\theta_n||x_n-x_{n-1}||||x_n-p|| \\
&= ||x_n-p||^2+\theta_n||x_n-x_{n-1}||(\theta_n||x_n-x_{n-1}||+2||x_n-p||) \\
&\le ||x_n-p||^2+3M_2\theta_n||x_n-x_{n-1}|| \\
&= ||x_n-p||^2+3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||, \qquad (5.24)
\end{aligned}$$

where $M_2 := \sup_{n\in\mathbb{N}}\{||x_n-p||, \theta_n||x_n-x_{n-1}||\} > 0$.

Now, applying Lemma 2.5.18, (5.4) and (5.24), we have

$$\begin{aligned}
\|T_nt_n-p\|^2 &= \|(1-\beta_n)(t_n-p)+\beta_n(\mathbb{V}t_n-p)\|^2 \\
&\le (1-\beta_n)\|t_n-p\|^2+\beta_n\|\mathbb{V}t_n-p\|^2-\beta_n(1-\beta_n)\|\mathbb{V}t_n-t_n\|^2 \\
&\le (1-\beta_n)\|t_n-p\|^2+\beta_n\|t_n-p\|^2-\beta_n(1-\beta_n)\|\mathbb{V}t_n-t_n\|^2 \\
&= \|t_n-p\|^2-\beta_n(1-\beta_n)\|\mathbb{V}t_n-t_n\|^2 \\
&\le \|w_n-p\|^2-\phi_n\big[2-\phi_n-\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}-2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}}-\kappa^2\big]\|w_n-y_n\|^2 \\
&\quad -\beta_n(1-\beta_n)\|\mathbb{V}t_n-t_n\|^2. \qquad (5.25)
\end{aligned}$$

Next, by applying Lemma 2.5.18, (5.24) and (5.25) we obtain

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 = \|\alpha_n\gamma f(w_n) + (I-\alpha_n D)T_n t_n - p\|^2\\
&= ||\alpha_n(\gamma f(w_n) - Dp) + (I-\alpha_n D)(T_n t_n - p)||^2\\
&\leq (1-\alpha_n\bar{\gamma})^2||T_n t_n - p||^2 + 2\alpha_n\langle \gamma f(w_n) - Dp, x_{n+1}-p\rangle\\
&\leq (1-\alpha_n\bar{\gamma})^2\Big(\|w_n-p\|^2 - \phi_n\big[2-\phi_n-\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}} - \kappa^2\big]\|w_n-y_n\|^2\\
&-\beta_n(1-\beta_n)\|\mathbb{V}t_n - t_n\|^2\Big) + 2\alpha_n\gamma\langle f(w_n)-f(p), x_{n+1}-p\rangle + 2\alpha_n\langle\gamma f(p) - Dp, x_{n+1}-p\rangle\\
&\leq (1-\alpha_n\bar{\gamma})^2\Big(\|w_n-p\|^2 - \phi_n\big[2-\phi_n-\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}} - \kappa^2\big]\|w_n-y_n\|^2\\
&-\beta_n(1-\beta_n)\|\mathbb{V}t_n - t_n\|^2\Big) + \alpha_n\gamma\rho\Big(\|w_n-p\|^2 + \|x_{n+1}-p\|^2\Big) + 2\alpha_n\langle\gamma f(p) - Dp, x_{n+1}-p\rangle\\
&\leq (1-\alpha_n\bar{\gamma})^2\Big(||x_n-p||^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\\
&-\phi_n\big[2-\phi_n-\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}} - \kappa^2\big]\|w_n-y_n\|^2 - \beta_n(1-\beta_n)\|\mathbb{V}t_n - t_n\|^2\Big)\\
&+\alpha_n\gamma\rho\Big(||x_n-p||^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}|| + \|x_{n+1}-p\|^2\Big) + 2\alpha_n\langle\gamma f(p) - Dp, x_{n+1}-p\rangle.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \leq \frac{(1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}||x_n-p||^2\\
&+3M_2\frac{\big((1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho\big)}{(1-\alpha_n\gamma\rho)}\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{2\alpha_n}{(1-\alpha_n\gamma\rho)}\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\\
&-\frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\phi_n\big[2-\phi_n-\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}-2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}}-\kappa^2\big]\|w_n-y_n\|^2\\
&+\beta_n(1-\beta_n)\|\mathbb{V}t_n-t_n\|^2\Big\}\\
&=\frac{(1-2\alpha_n\bar{\gamma}+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}||x_n-p||^2+\frac{(\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}||x_n-p||^2\\
&\quad+3M_2\frac{\big((1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho\big)}{(1-\alpha_n\gamma\rho)}\alpha_n\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{2\alpha_n}{(1-\alpha_n\gamma\rho)}\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\\
&\quad-\frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\phi_n\big[2-\phi_n-\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}-2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}}-\kappa^2\big]\|w_n-y_n\|^2\\
&+\beta_n(1-\beta_n)\|\mathbb{V}t_n-t_n\|^2\Big\}\\
&\leq\Big(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big)||x_n-p||^2+\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\rho)}M_3\\
&\quad+3M_2\frac{\big((1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho\big)}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{1}{(\bar{\gamma}-\gamma\rho)}\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\Big\}\\
&\quad-\frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\Big\{\phi_n\big[2-\phi_n-\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}-2(1-\phi_n)\delta\frac{\lambda_n}{\lambda_{n+1}}-\kappa^2\big]\|w_n-y_n\|^2\\
&+\beta_n(1-\beta_n)\|\mathbb{V}t_n-t_n\|^2\Big\},
\end{aligned}
$$

where $M_3 := \sup\{||x_n-p||^2 : n \in \mathbb{N}\}$. This gives the required inequality. $\square$

Theorem 5.2.10. *Let $\{x_n\}$ be a sequence generated by Algorithm 5.2.2 under Assumption 5.2.1. Then, the sequence $\{x_n\}$ converges strongly to a point $p \in \Omega$, where $p = P_\Omega(I - D + \gamma f)p$ is a solution of the variational inequality*

$$\langle (D-\gamma f)p, p-x\rangle \leq 0,\ \forall x \in \Omega.$$

Proof. From Lemma 5.2.9, we obtain

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \leq \Big(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big)||x_n-p||^2+\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\Big\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\rho)}M_3\\
&+3M_2\frac{\big((1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho\big)}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||+\frac{1}{(\bar{\gamma}-\gamma\rho)}\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\Big\}. \quad (5.26)
\end{aligned}
$$

Now, we claim that the sequence $\{\|x_n - p\|\}$ converges to zero. To establish this, by Lemma 2.5.55, it suffices to show that $\limsup_{k\to+\infty}\langle \gamma f(p) - Dp, x_{n_k+1} - p\rangle \le 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying

$$\liminf_{k\to+\infty} \left(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\right) \ge 0. \tag{5.27}$$

Suppose that $\{\|x_{n_k} - p\|\}$ is a subsequence of $\{\|x_n - p\|\}$ such that (5.27) holds. Again, from Lemma 5.2.9 we get

$$\begin{aligned}
&\frac{(1-\alpha_{n_k}\bar{\gamma})^2}{(1-\alpha_{n_k}\gamma\rho)}\phi_{n_k}\left[2-\phi_{n_k}-\delta^2\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}-2(1-\phi_{n_k})\delta\frac{\lambda_{n_k}}{\lambda_{n_k+1}}-\kappa^2\right]\|w_{n_k}-y_{n_k}\|^2\\
&\le \left(1-\frac{2\alpha_{n_k}(\bar{\gamma}-\gamma\rho)}{(1-\alpha_{n_k}\gamma\rho)}\right)||x_{n_k}-p||^2-||x_{n_k+1}-p||^2+\frac{2\alpha_{n_k}(\bar{\gamma}-\gamma\rho)}{(1-\alpha_{n_k}\gamma\rho)}\Big\{\frac{\alpha_{n_k}\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\rho)}M_3\\
&\quad+3M_2\frac{\left((1-\alpha_{n_k}\bar{\gamma})^2+\alpha_{n_k}\gamma\rho\right)}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||+\frac{1}{(\bar{\gamma}-\gamma\rho)}\langle\gamma f(p)-Dp,x_{n_k+1}-p\rangle\Big\}.
\end{aligned}$$

Applying (5.27) and the fact that $\lim_{k\to+\infty}\alpha_{n_k}=0$, we have

$$\frac{(1-\alpha_{n_k}\bar{\gamma})^2}{(1-\alpha_{n_k}\gamma\rho)}\phi_{n_k}\left[2-\phi_{n_k}-\delta^2\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}-2(1-\phi_{n_k})\delta\frac{\lambda_{n_k}}{\lambda_{n_k+1}}-\kappa^2\right]\|w_{n_k}-y_{n_k}\|^2\to 0,\quad k\to+\infty.$$

By the conditions on the control parameters, we get

$$\|w_{n_k}-y_{n_k}\|\to 0,\quad k\to+\infty. \tag{5.28}$$

Following similar argument, from Lemma 5.2.9 we obtain

$$\|\mathbb{V}t_{n_k}-t_{n_k}\|\to 0,\quad k\to+\infty. \tag{5.29}$$

From the definition of t_{n_k} in **Step 4** and (5.28), we have

$$\begin{aligned}
\|t_{n_k}-w_{n_k}\| &= \|(1-\phi_{n_k})w_{n_k}+\phi_{n_k}(y_{n_k}+\lambda_{n_k}(Aw_{n_k}-Ay_{n_k}))-w_{n_k}\|\\
&\le (1-\phi_{n_k})\|w_{n_k}-w_{n_k}\|+\phi_{n_k}(\|y_{n_k}-w_{n_k}\|+\lambda_{n_k}\|Ay_{n_k}-Aw_{n_k}\|)\\
&\le \phi_{n_k}(\|y_{n_k}-w_{n_k}\|+\delta\frac{\lambda_{n_k}}{\lambda_{n_k+1}}\|w_{n_k}-y_{n_k}\|)\\
&= \phi_{n_k}\left(1+\frac{\lambda_{n_k}\delta}{\lambda_{n_k+1}}\right)\|y_{n_k}-w_{n_k}\|\to 0.
\end{aligned}$$

Consequently, by (5.28) and the conditions on the control parameters we obtain

$$\lim_{k\to+\infty}\|t_{n_k}-w_{n_k}\|=0. \tag{5.30}$$

From (5.28) and (5.30) we have

$$\lim_{k\to+\infty} \|y_{n_k} - t_{n_k}\| = 0. \tag{5.31}$$

Now, from **Step 2** and by (5.3), we have

$$\lim_{k\to+\infty} \|w_{n_k} - x_{n_k}\| = \lim_{k\to+\infty} \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\| = 0. \tag{5.32}$$

Next, by applying (5.29), we get

$$\begin{aligned} \|T_{n_k}t_{n_k} - t_{n_k}\| &= \|(1-\beta_{n_k})t_{n_k} + \beta_{n_k}\mathbb{V}_{n_k}t_{n_k} - t_{n_k}\| \\ &\leq (1-\beta_{n_k})\|t_{n_k} - t_{n_k}\| + \beta_{n_k}\|\mathbb{V}_{n_k}t_{n_k} - t_{n_k}\| \to 0, \quad k\to+\infty. \end{aligned} \tag{5.33}$$

Now, by using (5.30), (5.31), (5.32) and (5.33) we obtain

$$\begin{gathered} \lim_{k\to+\infty} \|t_{n_k} - x_{n_k}\| = 0, \quad \lim_{k\to+\infty} \|y_{n_k} - x_{n_k}\| = 0, \quad \lim_{k\to+\infty} ||T_{n_k}t_{n_k} - w_{n_k}|| = 0, \\ \lim_{k\to+\infty} ||T_{n_k}t_{n_k} - x_{n_k}|| = 0. \end{gathered} \tag{5.34}$$

Consequently, by applying the fact that $\lim\limits_{k\to+\infty} \alpha_{n_k} = 0$ we get

$$\begin{aligned} \|x_{n_k+1} - x_{n_k}\| &= \|\alpha_{n_k}\gamma f(w_{n_k}) + (I - \alpha_{n_k}D)T_{n_k}t_{n_k} - x_{n_k}\| \\ &\leq \alpha_{n_k}\|\gamma f(w_{n_k}) - Dx_{n_k}\| + (1-\alpha_{n_k}\bar{\gamma})\|T_{n_k}t_{n_k} - x_{n_k}\| \to 0, \quad k\to+\infty. \end{aligned} \tag{5.35}$$

To complete the proof, we need to show that $w_\omega(x_n) \subset \Omega$. Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $x^* \in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k\to+\infty$. By Lemma 5.2.8 and (5.28), we obtain $x^* \in VI(C,A)$. Consequently, we have $w_\omega(x_n) \subset VI(C,A)$. From (5.34), we have that $t_{n_k} \rightharpoonup x^*$ as $k\to+\infty$. Since $I - \mathbb{V}$ is demiclosed at zero, it follows from (5.29) and Lemma 2.5.3 that $x^* \in F(\mathbb{V}) = F(T)$. That is, $w_\omega(x_n) \subset F(T)$. Thus, we have $w_\omega(x_n) \subset \Omega$.

Moreover, from (5.34) it follows that $w_\omega\{x_n\} = w_\omega\{t_n\}$. By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\lim_{j\to+\infty}\langle\gamma f(p) - Dp, x_{n_{k_j}} - p\rangle = \limsup_{k\to+\infty}\langle\gamma f(p) - Dp, x_{n_k} - p\rangle = \limsup_{k\to+\infty}\langle\gamma f(p) - Dp, t_{n_k} - p\rangle. \tag{5.36}$$

Since $p = P_\Omega(I - D + \gamma f)(p)$, it follows from (5.36) that

$$\limsup_{k\to+\infty}\langle\gamma f(p) - Dp, x_{n_k} - p\rangle = \lim_{j\to+\infty}\langle\gamma f(p) - Dp, x_{n_{k_j}} - p\rangle = \langle\gamma f(p) - Dp, x^\dagger - p\rangle \leq 0. \tag{5.37}$$

Hence, from (5.35) and (5.37), we get

$$\begin{aligned}\limsup_{k\to+\infty}\langle \gamma f(p)-Dp, x_{n_k+1}-p\rangle &= \limsup_{k\to+\infty}\langle \gamma f(p)-Dp, x_{n_k+1}-x_{n_k}\rangle \\ &\quad+\limsup_{k\to+\infty}\langle \gamma f(p)-Dp, x_{n_k}-p\rangle \\ &= \langle \gamma f(p)-Dp, x^{\dagger}-p\rangle \le 0. \end{aligned} \tag{5.38}$$

Applying Lemma 2.5.55 to (5.26), and using (5.38) together with the fact that $\lim_{n\to+\infty}\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||=0$ and $\lim_{n\to+\infty}\alpha_n=0$, we deduce that $\lim_{n\to+\infty}||x_n-p||=0$ as required.

□

5.2.2 Applications

In this subsection, we apply our results to study convex minimization problem.

Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $\phi : C \to \mathbb{R}$ be a convex function. The convex minimization problem is formulated as finding an element $x^* \in C$, such that

$$\phi(x^*) = \min_{x\in C}\phi(x). \tag{5.39}$$

The solution set for the convex minimization problem (5.39) is denoted by :

$$\Gamma = \{x^* \in C : \phi(x^*) = \min_{x\in C}\phi(x)\}.$$

We need the following lemma in establishing our next result.

Lemma 5.2.11. *[230] Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let ϕ be a convex function of H into $\mathbb{R}$. If ϕ is differentiable, then z is a solution of the problem (5.39) if and only if $z \in VI(C, \nabla\phi)$, where $\nabla\phi$ is the gradient of ϕ.*

By applying Theorem 5.2.10 and Lemma 5.2.11, we have the following consequent result for approximating a common solution of convex minimization problem and fixed point of quasi-pseudo-contractions in Hilbert spaces.

Theorem 5.2.12. *Let H be a real Hilbert space and $T : H \to H$ be a K-Lipschitz continuous quasi-pseudo-contractive mapping, which is demiclosed at zero and with $K \geq 1$. Let $\phi : H \to \mathbb{R}$ be a differentiable convex function such that $\nabla\phi$ is $\alpha-$ism. Let $\{x_n\}$ be a sequence generated as follows:*

Algorithm 5.2.13.

Initialization: *Choose sequences $\{\alpha_n\}_{n=1}^{+\infty}$, $\{\beta_n\}_{n=1}^{+\infty}$ and $\{\gamma_n\}_{n=1}^{+\infty}$ such that the conditions from Assumption 5.2.1(3) hold and let $\lambda_1 > 0$, $\alpha \geq 3$ and $x_0, x_1 \in H$ be given arbitrarily. Set $n = 1$.*

Iterative Steps:

Step 1. *Given the iterates* x_{n-1} *and* $x_n (n \geq 1)$, *choose* θ_n *such that* $0 \leq \theta_n \leq \bar{\theta}_n$, *where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\alpha-1}, \frac{\gamma_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \frac{n-1}{n+\alpha-1}, & \text{otherwise.} \end{cases} \tag{5.40}$$

Step 2 *Compute*

$$w_n = x_n + \theta_n(x_n - x_{n-1}),$$

Step 3 *Construct the half-space*

$$C_n = \{x \in H : c(w_n) + \langle c'(w_n), x - w_n\rangle \leq 0\}.$$

and compute

$$y_n = P_{C_n}(w_n - \lambda_n \nabla\phi w_n),$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\delta\|w_n - y_n\|}{\|\nabla\phi w_n - \nabla\phi y_n\| + \|c'(w_n) - c'(y_n)\|}, \lambda_n\right\}, & \text{if } \|\nabla\phi w_n - \nabla\phi y_n\| \\ & +\|c'(w_n) - c'(y_n)\| \neq 0, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{5.41}$$

Step 4. *Compute*

$$t_n := (1 - \phi_n)w_n + \phi_n\Big(y_n + \lambda_n\big(\nabla\phi w_n - \nabla\phi y_n\big)\Big).$$

Step 5. *Compute*

$$x_{n+1} = \alpha_n\gamma f(w_n) + (I - \alpha_n D)\big[(1 - \beta_n)t_n + \beta_n \mathbb{V} t_n\big],$$

where

$$\mathbb{V} = (1 - \xi)I + \xi T\big((1 - \mu)I + \mu T\big).$$

Set $n = n + 1$ *and go back to* **Step 1**.

Suppose that the solution set $\Omega = F(T) \cap \Gamma$ *is nonempty and other conditions of Assumption 5.2.1 are satisfied. Then, the sequence* $\{x_n\}$ *converges strongly to a point* $p \in \Omega$, *where* $p = P_\Omega(I - D + \gamma f)p$ *is a solution of the variational inequality*

$$\langle (D - \gamma f)p, p - x\rangle \leq 0, \ \forall x \in \Omega.$$

Proof. Taking $A = \nabla\phi$ in Theorem 5.2.10, the desired result follows by applying Lemma 5.2.11. □

5.2.3 Numerical example

In this section, we proceed to perform two numerical experiments to show the computational efficiency of our Algorithm 5.2.2 in comparison with Appendix 9.1.10, Appendix 9.1.11, Appendix 9.1.12 and Appendix 9.1.13. The graph of errors is plotted against the number of iterations in each case. All numerical computations were carried out using Matlab 2019(b). The parameters are chosen as follows:

- Let $f(x) = \frac{1}{3}x$, then $\rho = \frac{1}{3}$ is the Lipschitz constant for f. Let $D(x) = \frac{x}{2}$ with constant $\bar{\gamma} = \frac{1}{2}$, then we take $\gamma = 1$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Choose $\alpha = 3, \lambda_1 = 0.8, \alpha_n = \frac{2}{3n+1}, \gamma_n = (\frac{2}{3n+1})^2, \beta_n = \frac{2n}{3n+2}, \phi_n = \frac{2n+1}{5n+2}$ in our Algorithm 5.2.2.
- Take $Sx = \frac{1}{2}x$ in Appendix 9.1.10.
- Let $Ux = -\frac{3}{2}x, \delta_n = \frac{2n}{31n+15}$ in Appendix 9.1.11, Appendix 9.1.12 and Appendix 9.1.13.

Example 5.2.14. *Let $C = \{x \in \mathbb{R}^2 : c(x) := x_1^2 - x_2 \leq 0\}$, where $x = (x_1, x_2)$. Define the operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$A(x) = (3h(x_1), 2x_1 + x_2)$$

where

$$h(s) = \begin{cases} e(s-1) + e, & \text{if } \ s > 1, \\ e^s, & \text{if } \ -1 \leq s \leq 1, \\ e^{-1}(s+1) + e^{-1}, & \text{if } \ s < -1. \end{cases}$$

Then, it can easily be verified that A is monotone and L-Lipschitz continuous with $L = \sqrt{9e^2 + 5}$. Moreover, c is a continuously differentiable convex functions with Lipschitz constant $M = 2$. Also, we have that $\kappa = 3\sqrt{e^2 + 1}$ (see [107]). Define $T : H \to H$ by $Tx = -2x$. Then T is 2-Lipschitzian quasi-pseudo-contractive. We take $\xi = 0.2, \mu = 0.25$. We test the algorithms for four different starting points as follows:

Case I: $x_0 = (-1, 2), x_1 = (-2, 7)$;

Case II: $x_0 = (1, 2), x_1 = (-2, 5)$;

Case III: $x_0 = (2, 5), x_1 = (-\frac{1}{5}, 4)$;

Case IV: $x_0 = (4, 21), x_1 = (\frac{2}{3}, 6)$.

The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-2}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 5.1 and Table 5.1.14.

Table 5.1.14 Numerical results for Example 5.2.14

		App. 9.1.10	App. 9.1.11	App. 9.1.12	App. 9.1.13	Alg. 5.2.2
Case I	CPU time (sec)	0.0152	0.0114	0.0212	0.0186	0.0199
	No of Iter.	5	12	15	20	7
Case II	CPU time (sec)	0.0152	0.0102	0.0148	0.0138	0.0180
	No. of Iter.	5	12	15	20	8
Case III	CPU time (sec)	0.0150	0.0102	0.0168	0.0245	0.0227
	No of Iter.	5	12	15	20	7
Case IV	CPU time (sec)	0.0147	0.0124	0.0099	0.0105	0.0210
	No of Iter.	5	12	15	20	8

Example 5.2.15. *Consider the infinite dimensional real Hilbert space $H = L^2([0,1])$ with the inner product defined by*

$$\langle u, v\rangle = \int_0^1 u(t)v(t)dt, \quad \forall\ u,\ v \in H$$

and the induced norm

$$\|u\| = \int_0^1 |u(t)|^2 dt, \quad \forall u \in H.$$

Let $C[0,1]$ denote the continuous function space defined on the interval $[0,1]$ and $\varphi \in C[0,1]$ be an arbitrary fixed continuous function. Let $C := \{x \in H : ||\varphi x|| \leq 1\}$. It can easily be verified that C is a nonempty closed convex subset of H. The operator $A : H \to H$ is given by

$$(Au)(t) = \max\{0, u(t)\}, \qquad \forall t \in [0,1],\ u \in H.$$

It is easy to see that A is monotone and Lipschitz continuous on H with the Lipschitz constant $L = 2$. Define $c : H \to \mathbb{R}$ by

$$c(u) = \frac{1}{2}(||\varphi u||^2 - 1), \qquad \forall\ u \in H,$$

then c is a convex function and C is a level set of c, i.e., $C = \{u \in H : c(u) \leq 0\}$. Moreover, c is differentiable on H and $c'(u) = \varphi^2 u, \quad \forall\ u \in H$. Moreover, we have that $\kappa = e^2$ (see [104]). Here, we choose $\varphi(t) = e^{-t}, \quad \forall\ t \in [0,1]$. Let $T : H \to H$ be defined by $Tx = -\frac{5}{4}x$. Then T is $\frac{5}{4}$-Lipschitzian quasi-pseudo-contractive. We take $\xi = 0.3, \mu = 0.35$ and $\delta_n = \frac{2n}{3n+5}$ in Appendix 9.1.11, Appendix 9.1.12 and Appendix 9.1.13.

We test the algorithms for four different starting points as follows:

Case I: $x_0 = 2t^5 + t^3 + 1, x_1 = t^3 + 3t^2 - 2;$

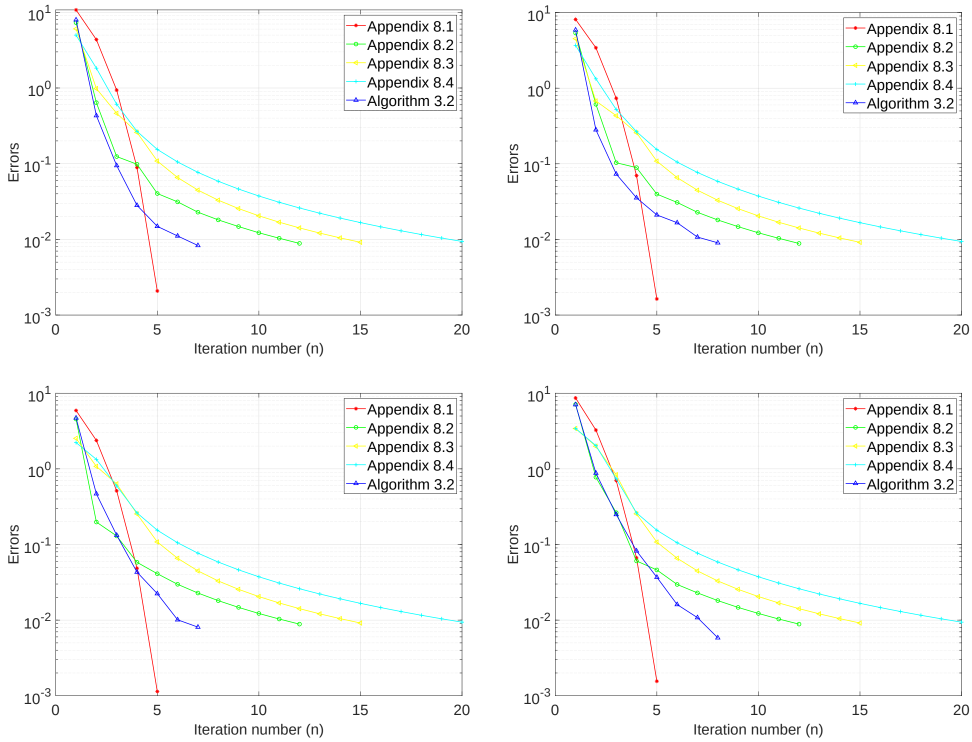


Figure 5.1: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

Case II: $x_0 = t^3 + 3t + 1, x_1 = \exp(-t)$;

Case III: $x_0 = 3t^4 - 2, x_1 = \sin 2t$;

Case IV: $x_0 = 4t^3 + t + 1, x_1 = \cos(t)$.

The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-2}$. *We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 5.2 and Table 5.1.15.*

Table 5.1.15 Numerical results for Example 5.2.15

		App. 9.1.10	App. 9.1.11	App. 9.1.12	App. 9.1.13	Alg. 5.2.2
Case I	No of Iter.	13	6	8	11	5
Case II	No. of Iter.	15	7	9	12	5
Case III	No of Iter.	13	6	8	11	5
Case IV	No of Iter.	15	7	9	12	5

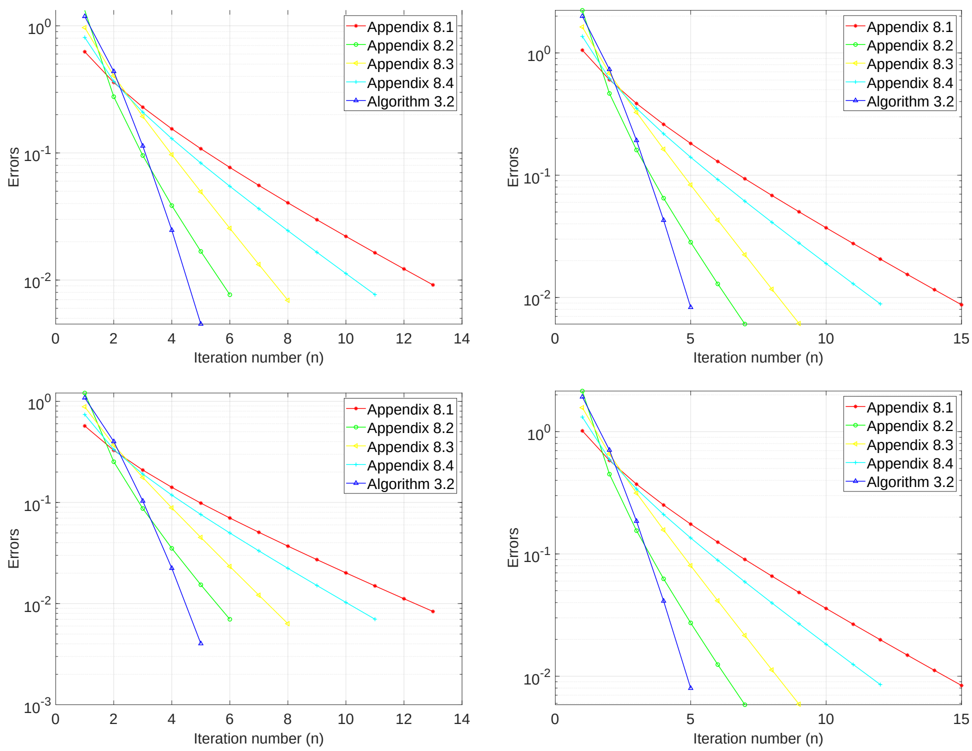


Figure 5.2: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

5.3 Two-level variational inequality and fixed point problems involving pseudomonotone and $\varrho-$ demimetric mapping

In this section, we introduce an iterative algorithm that approximates the solution of two-level variational inequality and fixed point problem in a real Hilbert space where the underlying operators are pseudo-monotone and $\varrho-$demimetric. An iterative algorithm was developed and shown to converge strongly to the solution set of two-level variational inequality and fixed point problem. Four numerical examples are presented to further demonstrate the usefulness and applicability of our method. The result obtained extends, generalizes and compliments several existing results in this direction of research.

5.3.1 Main result

Assumptions

(S_1) The feasible set C is a nonempty closed convex subset of real Hilbert space H.

(S_2) The operator $A : H \to H$ is L_1-Lipschitz continuous, pseudomonotone on H and sequentially weakly continuous on C.

(S_3) Given I as the identity operator on H, let $T : H \to H$ be a uniformly continuous $\varrho-$demimetric operator with $(I - T)$ demiclosed on zero. Also, let $0 < h \leq t_n \leq (1 - \varrho)$ for all $n \geq 1$.

(S_4) The operator $f : H \to H$ is an $L-$contraction with $L \in (0, \frac{1}{2})$.

(S_5) The solution set of the variational inequality is nonempty. That is, $VI(C, A) \neq \emptyset$. Also, $\Omega \neq \emptyset$ where $\Omega := \{x \in VI(C, A) \ : \ T(x) = x\}$.

(S_6) Let $\{\alpha_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ be positive sequences such that $\alpha_n \in (0, 1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n\geq 1} \alpha_n = \infty$, and $\lim\limits_{n\to\infty} \frac{\gamma_n}{\alpha_n} = 0$.

We introduce and study the following algorithm:

Algorithm 5.3.1. *Calculation of the sequence* $\{x_n\}$.

Initialization: *Let* $\tau_0 > 0$, $\mu \in (0, 1)$, $\beta \in (0, 2)$

Iterative Steps: *Given the current iterate* x_n*, calculate* x_{n+1} *as follows:*

Step 1: *Given* x_{n-1}, x_n *with* $n \geq 1$*, choose* θ_n *such that* $0 \leq \theta_n \leq \bar{\theta}_n$ *where*

$$\bar{\theta}_n = \begin{cases} \min\{\frac{n-1}{n-1+\epsilon}, \frac{\gamma_n}{\|x_n - x_{n-1}\|}\}, & \textit{if } x_n \neq x_{n-1} \\ \frac{n-1}{n-1+\epsilon}, & \textit{otherwise} \end{cases}$$

Step 2: *Compute*

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}) \\ y_n &= P_C(w_n - \tau_n A w_n). \end{aligned}$$

Step 3: *Compute* $z_n = w_n - \beta\eta_n d_n$ *where*

$$d_n := w_n - y_n - \tau_n(Aw_n - Ay_n)$$

and

$$\eta_n := \begin{cases} \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}, & \textit{if } d_n \neq 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Step 4: *Compute*

$$\begin{aligned} v_n &= z_n - t_n(z_n - Tz_n), \\ x_{n+1} &= \alpha_n f(v_n) + (1 - \alpha_n)v_n. \end{aligned} \tag{5.42}$$

Update

$$\tau_{n+1} = \begin{cases} \min\{\mu\frac{\|w_n-y_n\|}{\|Aw_n-Ay_n\|}, \tau_n\}, \textit{if } Aw_n \neq Ay_n, \\ \tau_n, \qquad\qquad\qquad\qquad\qquad \textit{otherwise.} \end{cases} \tag{5.43}$$

Step 5: *Set $n := n+1$ and go to* **Step 1**.

Remark 5.3.2. *The sequence $\{\tau_n\}$ generated by (5.43) is a non-increasing sequence and*

$$\lim_{n\to\infty} \tau_n = \tau \geq \min\{\tau_0, \frac{\mu}{L}\}.$$

Moreover, we have that

$$\|Aw_n - Ay_n\| \leq \frac{\mu}{\tau_{n+1}}\|w_n - y_n\|, \quad \forall n \geq 0.$$

We begin with the following lemmas which are critical in obtaining our strong convergence result.

Lemma 5.3.3. *Let $\{x_n\}_{n\geq 0}$ be a sequence in H iteratively generated by Algorithm 5.3.1. Assume that the conditions $(S_1) - (S_6)$ hold, then $\{x_n\}$ is bounded.*

Proof. Let $x^* \in \mathrm{VI}(C, A)$. We obtain from the definition of w_n that

$$\begin{aligned} \|w_n - x^*\| &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\| \\ &\leq \|x_n - x^*\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - x^*\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|. \end{aligned}$$

Now from **Step 1**, observe that $\theta_n\|x_n - x_{n-1}\| \leq \gamma_n, \quad \forall\ n \geq 1$, which implies that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \frac{\gamma_n}{\alpha_n} \to 0, \quad \text{as} \quad n \to \infty. \tag{5.44}$$

Thus, there exists $K_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq K_1, \quad \forall\ n \geq 1, \tag{5.45}$$

and so,

$$\|w_n - x^*\| \leq \|x_n - x^*\| + \alpha_n K_1, \quad \forall n \geq 1. \tag{5.46}$$

Since $y_n = P_C(w_n - \tau_n Aw_n)$, we obtain from properties of projection that,

$$\langle w_n - \tau_n Aw_n - y_n, y_n - x^*\rangle \geq 0. \tag{5.47}$$

Also, since $x^* \in VI(C, A)$ and $y_n \in C$, we have that $\langle Ax^*, y_n - x^*\rangle \geq 0$. By the pseudomonotonicity of A, we get that,

$$\langle Ay_n, y_n - x^*\rangle \geq 0. \tag{5.48}$$

Since $\tau_n > 0$, we have that

$$\langle \tau_n A(y_n), y_n - x^* \rangle \geq 0. \tag{5.49}$$

Using (5.47) and (5.49), we obtain that;

$$\langle y_n - x^*, w_n - y_n - \tau_n (A w_n - A y_n) \rangle \geq 0. \tag{5.50}$$

Equivalently,

$$\langle y_n - x^*, d_n \rangle \geq 0. \tag{5.51}$$

Observe from **Step 3** and (5.51) that

$$\begin{aligned} \langle w_n - x^*, d_n \rangle &= \langle w_n - y_n, d_n \rangle + \langle y_n - x^*, d_n \rangle \\ &\geq \langle w_n - y_n, d_n \rangle. \end{aligned} \tag{5.52}$$

Using Algorithm 5.3.1, (5.46) and (5.52), we get that (using the fact that $\|d_n\| \neq 0$),

$$\begin{aligned} \|z_n - x^*\|^2 &= \|w_n - \beta \eta_n d_n - x^*\|^2 \\ &= \|w_n - x^*\|^2 + \beta^2 \eta_n^2 \|d_n\|^2 - 2\beta \eta_n \langle w_n - x^*, d_n \rangle \\ &\leq \|w_n - x^*\|^2 + \beta^2 \eta_n \langle w_n - y_n, d_n \rangle - 2\beta \eta_n \langle w_n - y_n, d_n \rangle \\ &= \|w_n - x^*\|^2 - (2 - \beta) \beta \eta_n \langle w_n - y_n, d_n \rangle \\ &= \|w_n - x^*\|^2 - (2 - \beta) \beta \|\eta_n d_n\|^2 \\ &\leq \|w_n - x^*\|^2 - \frac{1}{\beta}(2 - \beta) \|w_n - z_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \frac{1}{\beta}(2 - \beta) \|w_n - z_n\|^2 + \alpha_n K_1. \end{aligned} \tag{5.53}$$

Using Algorithm 5.3.1, (5.53) and the definition of T, we obtain that,

$$\begin{aligned} \|v_n - x^*\|^2 &= \|z_n - t_n(z_n - T z_n) - x^*\|^2 \\ &= \|z_n - x^*\|^2 + t_n^2 \|z_n - T z_n\|^2 - 2t_n \langle z_n - T z_n, z_n - x^* \rangle \\ &\leq \|z_n - x^*\|^2 - t_n(1 - \varrho - t_n) \|z_n - T z_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \frac{1}{\beta}(2 - \beta) \|w_n - z_n\|^2 - t_n(1 - \varrho - t_n) \|z_n - T z_n\|^2 + \alpha_n K_1. \end{aligned} \tag{5.54}$$

Again, using Algorithm 7.2.3, (5.54), and the fact that f is a contraction, we obtain that;

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(v_n)+(1-\alpha_n)v_n-x^*\|^2\\
&\le \alpha_n\|f(v_n)-f(x^*)+f(x^*)-x^*\|^2+(1-\alpha_n)\|v_n-x^*\|^2\\
&\le 2\alpha_n[L\|v_n-x^*\|^2+\|f(x^*)-x^*\|^2]+(1-\alpha_n)\|v_n-x^*\|^2\\
&= [1-\alpha_n(1-2L)]\|v_n-x^*\|^2+2\alpha_n\|f(x^*)-x^*\|^2\\
&\le [1-\alpha_n(1-2L)]\|x_n-x^*\|^2+\alpha_n(1-2L)\frac{2}{1-2L}\|f(x^*)-x^*\|^2\\
&- t_n[1-\alpha_n(1-2L)](1-\varrho-t_n)\|z_n-Tz_n\|^2\\
&- \frac{1}{\beta}(2-\beta)[1-\alpha_n(1-2L)]\|w_n-z_n\|^2\\
&+ \alpha_n[1-\alpha_n(1-2L)]\alpha_n K_1
\end{aligned}
\tag{5.55}
$$

Using the assumptions S_3 and S_6, and for some $K_2>0$, we have that;

$$
\|x_{n+1}-x^*\|^2 \le [1-\alpha_n(1-2L)]\big(\|x_n-x^*\|^2+K_2\big)+\alpha_n(1-2L)\frac{2}{1-2L}\|f(x^*)-x^*\|^2.
\tag{5.56}
$$

If we define $M:=\max\left\{\|x_0-x^*\|^2+K_2,\frac{2}{1-2L}\|f(x^*)-x^*\|^2\right\}$, then it is easy to see that

$$
\|x_{n+1}-x^*\|^2 \le M \quad \text{for all } n\ge 0. \tag{5.57}
$$

Thus, $\{x_n\}_{n=1}^{\infty}$ is bounded. Consequently, $\{w_n\},\{Aw_n\},\{y_n\},\{Ay_n\},\{z_n\},\{Tz_n\}$, and $\{v_n\}$ are all bounded. ■

Lemma 5.3.4. *Suppose that conditions S_1 and S_2 hold, and $\{x_n\}_{n=1}^{\infty}$ is a sequence generated by Algorithm 5.3.1, then, there exists $n_0\in\mathbb{N}$ such that*

$$
\|w_n-y_n\|^2 \le \frac{\left(1+\mu\frac{\tau_n}{\tau_{n+1}}\right)^2}{\left[\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\beta\right]^2}\|z_n-w_n\|^2 \qquad \forall\ n\ge n_0. \tag{5.58}
$$

Proof. From Algorithm 3.1, we easily see that,

$$
\begin{aligned}
\|d_n\| &= \|w_n-y_n-\tau_n(Aw_n-Ay_n)\| \ge \|w_n-y_n\|-\tau_n\|Aw_n-Ay_n\|\\
&\ge \|w_n-y_n\|-\mu\frac{\tau_n}{\tau_{n+1}}\|w_n-y_n\|\\
&= \left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n-y_n\|.
\end{aligned}
\tag{5.59}
$$

But $\lim_{n\to\infty}\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)=1-\mu>0$. So, there exists $n_0\in\mathbb{N}$ such that

$$
1-\mu\frac{\tau_n}{\tau_{n+1}} > \frac{1-\mu}{2} \qquad \forall\ n\ge n_0.
$$

Therefore, for each $n \geq n_0$, we have that $\|d_n\| > \frac{1-\mu}{2}\|w_n - y_n\| > 0$.

Moreover,

$$\begin{aligned}\|d_n\| &\leq \|w_n - y_n\| + \tau_n\|Aw_n - Ay_n\| \\ &\leq \|w_n - y_n\| + \mu\frac{\tau_n}{\tau_{n+1}}\|w_n - y_n\| \\ &= \left(1 + \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - y_n\| \qquad \forall\ n \geq n_0. \end{aligned} \tag{5.60}$$

Thus, for each $n \geq n_0$, we obtain that,

$$\|d_n\|^2 \leq \left(1 + \mu\frac{\tau_n}{\tau_{n+1}}\right)^2\|w_n - y_n\|^2. \tag{5.61}$$

Also, from Algorithm 5.3.1, we get that,

$$\begin{aligned}\langle w_n - y_n, d_n\rangle &= \langle w_n - y_n, w_n - y_n - \tau_n(Aw_n - Ay_n)\rangle \\ &= \|w_n - y_n\|^2 - \tau_n\langle w_n - y_n, Aw_n - Ay_n\rangle \\ &\geq \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - y_n\|^2. \end{aligned} \tag{5.62}$$

So, for each $n \geq n_0$, we obtain from (5.61) and (5.62);

$$\eta_n = \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2} \geq \frac{\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)}{\left(1 + \mu\frac{\tau_n}{\tau_{n+1}}\right)^2}. \tag{5.63}$$

Also, from Algorithm 5.3.1 and (5.62), we obtain that;

$$\eta_n\|d_n\|^2 = \langle w_n - y_n, d_n\rangle \geq \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - y_n\|^2. \tag{5.64}$$

Thus, from (5.64) and Algorithm 5.3.1,

$$\begin{aligned}\|w_n - y_n\|^2 &\leq \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)^{-1}\eta_n\|d_n\|^2 \\ &= \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)^{-1}\|\beta\eta_n d_n\|^2\frac{1}{\beta^2}\frac{1}{\eta_n} \\ &= \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)^{-1}\|z_n - w_n\|^2\frac{1}{\beta^2}\frac{1}{\eta_n}. \end{aligned} \tag{5.65}$$

Thus, using (5.63) in (5.65), we obtain that,

$$\|w_n - y_n\|^2 \leq \frac{\left(1 + \mu\frac{\tau_n}{\tau_{n+1}}\right)^2}{\left[\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\beta\right]^2}\|z_n - w_n\|^2 \qquad \forall\ n \geq n_0.$$

■

Lemma 5.3.5. *Let $\{x_n\}$ be sequence generated by Algorithm 5.3.1 under Assumptions S. Suppose that there exists a subsequence $\{x_{n_b}\}$ of $\{x_n\}$ which converges weakly to $\bar{x} \in H$ and $\lim_{b\to\infty} \|w_{n_b} - y_{n_b}\| = 0$, then $\bar{x} \in VI(C, A)$.*

Proof. Clearly, from Algorithm 5.3.1 and Assumption S_6, we obtain that,

$$\|w_n - x_n\| = \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq \frac{\gamma_n}{\alpha_n} \to 0 \text{ as } n \to \infty. \tag{5.66}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_b}\}$ of $\{x_n\}$ which converges weakly to $\bar{x} \in H$. From (2.9), we have that;

$$\langle w_{n_b} - \tau_{n_b} A w_{n_b} - y_{n_b}, x - y_{n_b} \rangle \leq 0, \qquad \forall\, x \in C.$$

Rearranging, we obtain that,

$$\frac{1}{\tau_{n_b}} \langle w_{n_b} - y_{n_b}, x - y_{n_b} \rangle \leq \langle A w_{n_b}, x - y_{n_b} \rangle, \qquad \forall\, x \in C.$$

Thus,

$$\frac{1}{\tau_{n_b}} \langle w_{n_b} - y_{n_b}, x - y_{n_b} \rangle + \langle A w_{n_b}, y_{n_b} - w_{n_b} \rangle \leq \langle A w_{n_b}, x - w_{n_b} \rangle, \quad \forall\, x \in C. \tag{5.67}$$

Fix $x \in C$. Since $\{w_{n_b}\}$ is bounded and A is Lipschitz continuous, then $\{Aw_{n_b}\}$ is bounded. Also, using the assumption that $\lim_{b\to\infty} \|w_{n_b} - y_{n_b}\| = 0$ and, taking $\liminf$ of (5.67) as $b \to \infty$, we get

$$\liminf_{b\to\infty} \langle A w_{n_b}, x - w_{n_b} \rangle \geq 0. \tag{5.68}$$

By assumptions that $\lim_{b\to\infty} \|w_{n_b} - y_{n_b}\| = 0$, and A is Lipschitz continuous, we get that

$$\lim_{b\to\infty} \|A w_{n_b} - A y_{n_b}\| = 0. \tag{5.69}$$

It is easy to see that,

$$\langle A y_{n_b}, x - y_{n_b} \rangle = \langle A y_{n_b} - A w_{n_b}, x - w_{n_b} \rangle + \langle A w_{n_b}, x - w_{n_b} \rangle + \langle A y_{n_b}, w_{n_b} - y_{n_b} \rangle \tag{5.70}$$

Using (5.68) and (5.69) in (5.70), we get that;

$$\liminf_{b\to\infty} \langle A y_{n_b}, x - y_{n_b} \rangle \geq 0. \tag{5.71}$$

Let us choose a decreasing sequence of positive numbers $\{\beta_b\}$ with $\lim_{b\to\infty} \beta_b = 0$. For each $b \in \mathbb{N}$, we denote by N_b the smallest positive number such that

$$\langle A y_{n_j}, x - y_{n_j} \rangle + \beta_b \geq 0 \quad \forall\, j \geq N_b. \tag{5.72}$$

Since $\{\beta_b\}$ is decreasing, then the sequence $\{N_b\}$ is increasing. Furthermore, for each $b \in \mathbb{N}$, since $\{y_{n_b}\} \subset C$, we assume that $A y_{N_b} \neq 0$ (else, y_{N_b} is a solution of of VI(C,A)). Set

$$m_{N_b} = \frac{A y_{N_b}}{\|A y_{N_b}\|^2},$$

where $\langle Ay_{N_b}, m_{N_b}\rangle = 1$ for each $b \in \mathbb{N}$. From (5.72), for each $b \in \mathbb{N}$, we obtain that,

$$\langle Ay_{N_b}, x + \beta_b m_{N_b} - y_{N_b}\rangle \geq 0. \tag{5.73}$$

Since A is pseudomonotone, we obtain that,

$$\langle A(x + \beta_b m_{N_b}), x + \beta_b m_{N_b} - y_{N_b}\rangle \geq 0.$$

Thus,

$$\langle Ax, x - y_{N_b}\rangle \geq \langle Ax - A(x + \beta_b m_{N_b}), x + \beta_b m_{N_b} - y_{N_b}\rangle - \beta_b\langle Ax, m_{N_b}\rangle. \tag{5.74}$$

Since $\{x_{n_b}\}$ converges weakly to $\bar{x}$ as $b \to \infty$, then, using (5.66) and the assumption that $\lim\limits_{b\to\infty} \|w_{n_b} - y_{n_b}\| = 0$, we easily obtain that $y_{n_b} \rightharpoonup \bar{x}$ as $b \to \infty$. By sequentially weakly continuity of A, we have that $Ay_{n_b} \rightharpoonup A\bar{x}$ as $b \to \infty$. Assume that $A\bar{x} \neq 0$ (otherwise, $\bar{x}$ is a solution), by the sequentially weakly lower semicontinuity of norm, we have that

$$0 < \|A\bar{x}\| \leq \liminf_{b\to\infty} \|Ay_{n_b}\|.$$

Since $\{y_{N_b}\} \subset \{y_{n_b}\}$ and $\beta_b \to 0$ as $b \to \infty$, we have that,

$$0 \leq \limsup_{b\to\infty} \|\beta_b m_{N_b}\| = \limsup_{b\to\infty} \left(\frac{\beta_b}{\|Ay_{n_b}\|}\right) \leq \frac{\limsup_{b\to\infty} \beta_b}{\liminf_{b\to\infty} \|Ay_{n_b}\|} \leq \frac{0}{\|A\bar{x}\|} = 0.$$

Thus, $\|\beta_b m_{N_b}\| \to 0$ as $b \to \infty$. Hence, taking the limit as $b \to \infty$ in (5.74), we obtain that,

$$\langle Ax, x - \bar{x}\rangle \geq 0.$$

Since $x \in H$ is arbitrary, we obtain from Lemma 2.5.10 that $\bar{x} \in VI(C, A)$ ■

Lemma 5.3.6. *Let $\{x_n\}_{n\geq 0}$ be a sequence in H iteratively generated by Algorithm 5.3.1. Assume that $\zeta_n = (1-L)\alpha_n$, and assumptions $S_1 - S_6$ hold, then*

$$\|x_{n+1} - x^*\|^2 \leq (1 - \zeta_n)\Big[\|x_n - x^*\|^2 + \alpha_n K_1\Big] + \zeta_n \frac{2}{1-L}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle. \tag{5.75}$$

Proof. From Algorithm 5.3.1, (5.54), Lemma 2.5.18 and Assumption S_6, we obtain that,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(v_n) + (1-\alpha_n)v_n - x^*\|^2 \\
&= \|\alpha_n[f(v_n) - f(x^*)] + (1-\alpha_n)[v_n - x^*] + \alpha_n[f(x^*) - x^*]\|^2 \\
&\leq \|\alpha_n[f(v_n) - f(x^*)] + (1-\alpha_n)[v_n - x^*]\|^2 + 2\alpha_n\langle x_{n+1} - x^*, f(x^*) - x^*\rangle \\
&\leq [1 - (1-L)\alpha_n]\|v_n - x^*\|^2 + 2\alpha_n\langle x_{n+1} - x^*, f(x^*) - x^*\rangle \\
&\leq [1 - (1-L)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\langle x_{n+1} - x^*, f(x^*) - x^*\rangle \\
&\quad + [1 - (1-L)\alpha_n]\alpha_n K_1 - \frac{1}{\beta}(2-\beta)[1 - (1-L)\alpha_n]\|w_n - z_n\|^2 \\
&\quad - t_n(1 - \varrho - t_n)[1 - (1-L)\alpha_n]\|z_n - Tz_n\|^2.
\end{aligned}$$

So, using assumption S_4, we obtain that,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \le & [1 - (1-L)\alpha_n]\|x_n - x^*\|^2 + [1 - (1-L)\alpha_n]\alpha_n K_1 \\ & + (1-L)\alpha_n \frac{2}{1-L}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \end{aligned} \tag{5.76}$$

If we let $\zeta_n = (1-L)\alpha_n$, then we obtain from (5.76) that,

$$\|x_{n+1} - x^*\|^2 \le (1-\zeta_n)\Big[\|x_n - x^*\|^2 + \alpha_n K_1\Big] + \zeta_n \frac{2}{1-L}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle. \tag{5.77}$$

■

Theorem 5.3.7. *Let $\{x_n\}_{n\ge 0}$ be a sequence in H iteratively generated by Algorithm 5.3.1. Assume that the conditions $(S_1) - (S_6)$ hold, then $\{x_n\}$ converges strongly to $x^* = P_\Omega(f(x^*))$.*

Proof.
From Lemma 5.3.3, $\{x_n\}_{n=1}^\infty$ is bounded. Let $x^* = P_\Omega(f(x^*))$. Now, from Algorithm 7.2.3, (5.54), and Lemma 2.5.18, we obtain that,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n f(v_n) + (1-\alpha_n)v_n - x^*\|^2 \\ &\le \|\alpha_n f(v_n) + (1-\alpha_n)v_n - x^* - \alpha_n(f(v_n) - x^*)\|^2 \\ &+ 2\alpha_n\langle f(v_n) - x^*, x_{n+1} - x^*\rangle \\ &\le (1-\alpha_n)\|v_n - x^*\|^2 + 2\alpha_n\langle f(v_n) - x^*, x_{n+1} - x^*\rangle \\ &\le (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle f(v_n) - x^*, x_{n+1} - x^*\rangle \\ &- \frac{1}{\beta}(2-\beta)(1-\alpha_n)\|w_n - z_n\|^2 + (1-\alpha_n)\alpha_n K_1 \\ &- (1-\alpha_n)t_n(1-\varrho - t_n)\|z_n - Tz_n\|^2 \\ &\le (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle f(v_n) - x^*, x_{n+1} - x^*\rangle \\ &- \frac{1}{\beta}(2-\beta)(1-\alpha_n)\|w_n - z_n\|^2 + \alpha_n K_1 \\ &- (1-\alpha_n)t_n(1-\varrho - t_n)\|z_n - Tz_n\|^2. \end{aligned} \tag{5.78}$$

To show that the sequence $\{\|x_{n+1} - x^*\|\}$ converges to zero, we consider two possible cases:

Case 1: Suppose there exists $n_0 \in \mathbb{N}$ such that the real sequence $\|x_n - x^*\|$ is decreasing for all $n \ge n_0$. It then follows that $\|x_n - x^*\|$ is convergent. Since $\{x_n\}$ is bounded, then from (5.78), assumption S_3, and using the fact that $\alpha_n \to 0$ as $n \to \infty$, we obtain that

$$\lim_{n\to\infty}\|z_n - Tz_n\| = 0, \lim_{n\to\infty}\|w_n - z_n\| = 0. \tag{5.79}$$

From Algorithm 5.3.1, we get that;

$$\begin{aligned} \|v_n - z_n\| = t_n\|z_n - Tz_n\| \to 0 \text{ as } n \to \infty, \\ \|x_{n+1} - v_n\| = \alpha_n\|f(v_n) - v_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{5.80}$$

Thus, using (5.66), (5.79), and (5.80), we obtain that,

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|x_{n+1} - v_n\| + \|v_n - z_n\| + \|z_n - w_n\| + \|w_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{5.81}$$

From the results above, we can easily deduce that,

$$\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty. \tag{5.82}$$

Claim:

$$\limsup_{n\to\infty} \Big\langle f(x^*) - x^*, x_n - x^* \Big\rangle \leq 0. \tag{5.83}$$

Proof of Claim: Let $\{x_{n_q}\}_{q\geq 0}$ be a subsequence of $\{x_n\}_{n\geq 0}$ such that

$$\limsup_{n\to\infty} \Big\langle f(x^*) - x^*, x_n - x^* \Big\rangle = \lim_{q\to\infty} \Big\langle f(x^*) - x^*, x_{n_q} - x^* \Big\rangle \tag{5.84}$$

Since $\{x_{n_q}\}_{q\geq 0}$ is a bounded sequence in H, there exists a subsequence $\{x_{n_{q_b}}\}_{b\geq 0}$ of $\{x_{n_q}\}_{q\geq 0}$ which converges weakly to $\bar{x}$ in H. Hence, $x_{n_{q_b}} \rightharpoonup \bar{x}$ as $b \to \infty$. For convenience, and WLOG, we will represent $\{x_{n_{q_b}}\}_{b\geq 0}$ by $\{x_{n_b}\}_{b\geq 0}$.

Recall from Lemma 5.3.4,

$$\|w_{n_b} - y_{n_b}\|^2 \leq \frac{\left(1 + \mu \frac{\tau_{n_b}}{\tau_{n_b+1}}\right)^2}{\left[\left(1 - \mu \frac{\tau_{n_b}}{\tau_{n_b+1}}\right)\alpha\right]^2} \|z_{n_b} - w_{n_b}\|^2 \qquad \forall\ n_b \geq n_0.$$

Using (5.79), we have that $\|w_{n_b} - y_{n_b}\| \to 0$ as $n \to \infty$. Thus, using Lemma 5.3.5, and the fact that $x_{n_b} \rightharpoonup \bar{x}$ as $b \to \infty$, we have that $\bar{x} \in VI(C, A)$.

Next, we show that $\bar{x} \in \Omega$. From (5.79) and the fact that $(I - T)$ is demiclosed at zero, we can deduce that $\bar{x} \in F(T)$.

Therefore, it follows from our argument above that $\bar{x} \in \Omega$. Thus, we obtain from Lemma 2.5.26 and (5.83) that

$$\limsup_{n\to\infty} \Big\langle f(x^*) - x^*, x_n - x^* \Big\rangle = \lim_{q\to\infty} \Big\langle f(x^*) - x^*, x_{n_q} - x^* \Big\rangle \leq 0. \tag{5.85}$$

Also, using the fact that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we get from (5.85) that,

$$\limsup_{n\to\infty} \Big\langle f(x^*) - x^*, x_{n+1} - x^* \Big\rangle \leq 0. \tag{5.86}$$

Recall that from Lemma 5.3.6, we obtain that,

$$\|x_{n+1} - x^*\|^2 \leq (1 - \zeta_n)\Big[\|x_n - x^*\|^2 + \alpha_n K_1\Big] + \zeta_n \frac{2}{1 - L} \langle f(x^*) - x^*, x_{n+1} - x^* \rangle. \tag{5.87}$$

Thus, by Lemma 2.5.26, it follows that $\|x_n - x^*\|$ converges strongly to zero as $n \to \infty$. Hence, $\{x_n\}$ converges to $x^* \in \Omega$. This completes the proof for the first case.

Case 2: Suppose that there exists a subsequence $\{\|x_{n_j} - x^*\|\}_{j=0}^{\infty}$ of $\{\|x_n - x^*\|\}_{n\geq 0}$ such that $\|x_{n_j} - x^*\| < \|x_{n_j+1} - x^*\|$ for all $j \geq 0$, then we obtain by Lemma 2.5.41 that there exists a non-decreasing sequence $\{m_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$ and $\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\|$ for all $k \in \mathbb{N}$. Since the sequences $\{x_{m_k}\}_{k=1}^{\infty}$ is bounded, we obtain from (5.66), (5.79), (5.80) (and using the arguments displayed in Case 1) that as $k \to \infty$;

$$\begin{aligned} \|w_{m_k} - x_{m_k}\|,\ \|w_{m_k} - z_{m_k}\| \to 0 \\ \|v_{m_k} - z_{m_k}\|,\ \|x_{m_k+1} - v_{m_k}\| \to 0. \end{aligned} \tag{5.88}$$

Applying (5.88), we obtain that,

$$\|x_{m_k+1} - x_{m_k}\| \to 0 \text{ as } k \to \infty. \tag{5.89}$$

Furthermore, following the arguments used in Case 1, we obtain that

$$\limsup_{k\to\infty} \left\langle f(x^*) - x^*, x_{m_k} - x^* \right\rangle \leq 0. \tag{5.90}$$

Again, using (5.89) and the same arguments as case 1, we have that,

$$\limsup_{k\to\infty} \left\langle f(x^*) - x^*, x_{m_k+1} - x^* \right\rangle \leq 0. \tag{5.91}$$

Again, from Lemma 5.3.6 and using the fact that $\|x_{m_k} - x^*\|^2 \leq \|x_{m_k+1} - x^*\|^2$ for any $k \in \mathbb{N}$ we obtain by rearranging that,

$$\begin{aligned} \zeta_{m_k}\|x_{m_k} - x^*\|^2 \leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + \zeta_{m_k}\frac{2}{1-L}\langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle \\ + (1-\zeta_{m_k})\alpha_{m_k}K_1. \end{aligned} \tag{5.92}$$

Applying Lemma 2.5.41, we obtain that,

$$\begin{aligned} \zeta_{m_k}\|x_k - x^*\|^2 \leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + \zeta_{m_k}\frac{2}{1-L}\langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle \\ +(1-\zeta_{m_k})\alpha_{m_k}K_1. \end{aligned} \tag{5.93}$$

Thus, it follows that $\limsup_{k\to\infty}\|x_k - x^*\| = 0$. Hence, $\lim_{k\to\infty}\|x_k - x^*\| = 0$.

Thus, $x_n \to x^*$ as $n \to \infty$. This completes the proof. ■

Remark 5.3.8. *Theorem 5.3.7 provides an iterative scheme for approximation of the solution of two-level variational inequality and fixed point problem where the A is Lipschitz continuous and pseudo-monotone and T is $\varrho-$demimetric. The result extends to solution of variational inequality problem of monotone operators since every pseudo-monotone operator is monotone.*

5.3.2 Numerical example

In this section, we give some numerical examples to show the applicability and convergence of Algorithm 5.3.1. We will also present some graphical representations of the convergence of our scheme. All numerical computations were carried out using Matlab version R2021(b).

In our computations, we choose $\tau_0 = \mu = 0.1$, $\alpha_n = \frac{1}{n+1}$ and $\gamma_n = \frac{1}{(n+1)^2}$ for all $n \geq 1$ and we select $\beta = \{0.1, 0.5, 1.0, 1.9\}$ in order to find the value of β that gives the best approximate solution.

In the first two examples, we consider the quadratic programming problem in the form below

$$\begin{cases} \min \aleph = x^T \Theta x + Y^T x \\ \text{subject to } x \in \mathbb{R}^n \\ x_i \geq 0, \ i = 1, \cdots, n, \end{cases} \tag{5.94}$$

in an $n-$dimensional Euclidean space. When Θ is symmetric and positive-definite, then, $A = \Theta x + Y$ is pseudo-monotone and Lipschitz continuous with the Lipschitz constant as $L = \|\Theta\|$.

Example 5.3.9. *For the first example, we consider the choices of Θ and Y for $H = \mathbb{R}^4$ as seen in [93]:*

$$\Theta = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

We now define the operators T and f as;

$$Tx = \begin{pmatrix} x_1 \\ \sin x_2 \\ x_3 \cos x_3 \\ \sin x_4 \end{pmatrix}, \quad fx = \frac{1}{7}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad x \in H.$$

The functions defined, sequences and constants satisfies conditions of Theorem 5.3.7 such that $x^* \in \Omega$ where $x^* = (1, 0, 0, 0)$. To approximate the solution of (5.94) using Algorithm 5.3.1, we take $\|x_{n+1} - x_n\| = 10^{-6}$ as the stopping criterion and choose the starting points as follows:

Case 1: *Take $x_1 = (5, 5, 3, 4)$ and $x_0 = (0, 0.5, 0, 0.5)$*

Case 2: *Take $x_1 = (4, 5, 4, 2)$ and $x_0 = (1, 0.5, 1, 0.5)$.*
We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 5.3 and Table 5.2.9.

Table 5.2.9. Numerical results for Example 5.3.9

		$\beta = 0.1$	$\beta = 0.5$	$\beta = 1.0$	$\beta = 1.9$
Case 1	No. of Iter.	61	54	35	105
	CPU time (sec)	0.0162	0.0103	0.0075	0.0133
Case 2	No. of Iter.	62	53	32	104
	CPU time (sec)	0.0148	0.0127	0.0081	0.0182

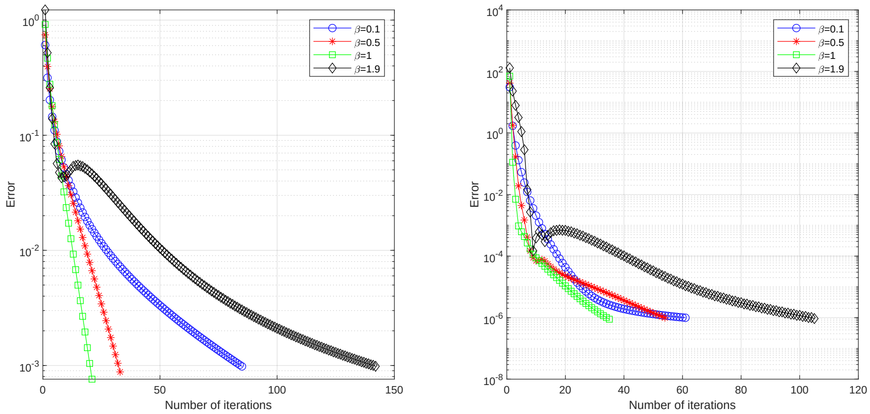


Figure 5.3: Left: Case 1 with various β; Right: Case 2 with various β.

Example 5.3.10. *Here, we consider the choices of* Θ *and* Y *for* $H = \mathbb{R}^5$ *as seen in [86]:*

$$\Theta = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 5 & 6 & 6 & 6 \\ 2 & 6 & 9 & 10 & 10 \\ 2 & 6 & 10 & 13 & 14 \\ 2 & 6 & 10 & 14 & 17 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

The operators T *and* f *are given as;*

$$Tx = \begin{pmatrix} x_1 \\ \sin x_2 \\ x_3 \\ \sin x_4 \\ x_5 \end{pmatrix}, \quad fx = \frac{1}{10}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5, \end{pmatrix}, \quad x \in H\ .$$

It is obvious that the functions, sequences and constants defined satisfies conditions of Theorem 5.3.7 such that $x^* \in \Omega$ *where* $x^* = (1, 0, 0, 0, 0)$*. To approximate the solution of (5.94) using Algorithm 7.2.3, we take* $\|x_{n+1} - x_n\| = 10^{-6}$ *as the stopping criterion and choose the starting points as follows:*

Case 1: *Take* $x_1 = (0.5, 5, 0.5, 5, 0.5)$ *and* $x_0 = (1, 0.5, 1, 0.5, 1)$

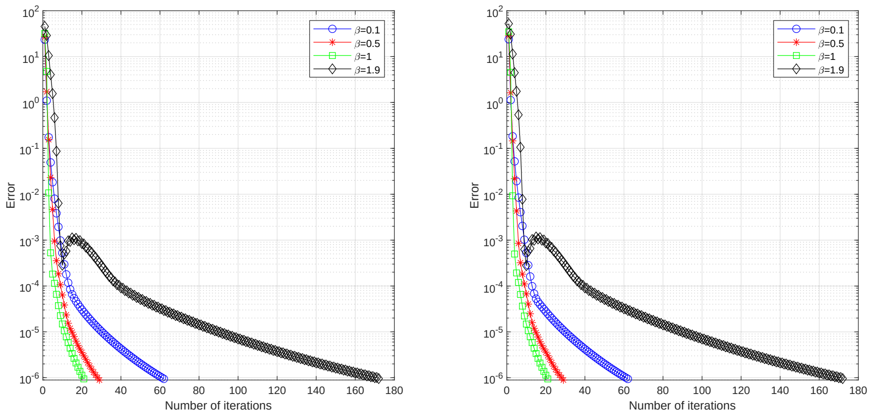


Figure 5.4: Left: Case 1 with various β; Right: Case 2 with various β.

Case 2: *Take* $x_1 = (2, 1, 0.5, 5, 1)$ *and* $x_0 = (2, 1, 1, 0.5, 2)$.

We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 5.4 and Table 5.2.10.

Table 5.2.10. Numerical results for Example 5.3.10

		$\beta = 0.1$	$\beta = 0.5$	$\beta = 1.0$	$\beta = 1.9$
Case 1	No. of Iter.	62	29	21	172
	CPU time (sec)	0.0480	0.0183	0.0162	0.0331
Case 2	No. of Iter.	62	29	21	172
	CPU time (sec)	0.0175	0.0105	0.0084	0.0359

The last two examples are in infinite dimensional spaces.

Example 5.3.11. *Let* $H = (\ell_2(\mathbb{R}), \|.\|_{l_2})$, *where* $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, ...), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} < \infty\}$ *and* $\|x\|_{\ell_2} := (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$, $\forall\ x \in \ell_2(\mathbb{R})$. *Let* $C = \{x \in \ell_2(\mathbb{R}) : |x_i| \leq \frac{1}{i}, i = 1, 2, 3, ...\}$. *Thus, we have explicit formula for* P_C. *Now, define the operator* $A : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ *by*

$$Ax = \left(\|x\| + \frac{1}{\|x\| + \alpha} \right) x,$$

for some $\alpha > 0$. *Then, it is easy to see that* A *is pseudomonotone on* $\ell_2(\mathbb{R})$ *(see [226]). Additionally, define the mapping* $T : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ *by* $Tx = (0, x_1, x_2, ...)$. *Then,* T *is demimetric on* $\ell_2(\mathbb{R})$.
We take $\|x_{n+1} - x_n\| = 10^{-3}$ *as the stopping criterion and choose the starting points as follows:*

Case 1: *Take* $x_1 = (2, 1, \frac{1}{2}, ...)$ *and* $x_0 = (3, 1, \frac{1}{3}, ...)$

Case 2: *Take* $x_1 = (-2, 1, -\frac{1}{2}, ...)$ *and* $x_0 = (4, 1, \frac{1}{4}, ...)$

Case 3: *Take* $x_1 = (-2, 1, -\frac{1}{2}, ...)$ *and* $x_0 = (-3, 1, -\frac{1}{3}, ...)$

Case 4: *Take* $x_1 = (2, 1, \frac{1}{2}, ...)$ *and* $x_0 = (-4, 1, -\frac{1}{4}, ...)$.

We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 5.5 and Table 5.2.11.

Table 5.2.11. Numerical results for Example 5.3.11

		$\beta = 0.1$	$\beta = 0.5$	$\beta = 1.0$	$\beta = 1.9$
Case 1	No. of Iter.	85	35	21	142
	CPU time (sec)	0.0836	0.0220	0.0271	0.0725
Case 2	No. of Iter.	84	33	21	142
	CPU time (sec)	0.0309	0.0146	0.0101	0.0392
Case 3	No. of Iter.	84	33	21	142
	CPU time (sec)	0.0168	0.0063	0.0047	0.0266
Case 4	No. of Iter.	85	33	21	142
	CPU time (sec)	0.0167	0.0048	0.0028	0.0221

Example 5.3.12. *Consider the infinite dimensional Hilbert space* $H = L^2[0, 1]$ *with the inner product (see [111])*

$$\langle u, v\rangle = \int_0^1 u(t)v(t)dt, \quad \forall\ u,\ v \in H$$

and the corresponding norm

$$\|u\| = \left(\int_0^1 u(t)^2 dt\right)^{\frac{1}{2}}, \quad \forall u \in H.$$

Let $p,\ \ R$ *be two real numbers such that* $R > p > \left(\frac{k}{k+1}\right) R > 0$ *for some* $k > 1$*. Take the feasible set* $C = \{u \in H : \|u\| \leq p\}$ *and the operator* $A : H \to H$ *given by*

$$A(x) = (R - \|x\|)\, x, \quad \forall x \in H.$$

Note that the operator A *is not monotone. Since* $\frac{R}{k+1} < \frac{p}{k} < p$*, we can choose* $\bar{x} \in C$ *such that* $\frac{R}{k+1} < \|\bar{x}\| < \frac{p}{k}$ *and then set* $\bar{y} = k\bar{x}$*. Also, since* $\|\bar{y}\| = k\|\bar{x}\| < k.(\frac{p}{k}) = p$*, it is easy to see that* $\bar{y} \in C$*. Moreover, since* $\|\bar{x}\| > \frac{R}{k+1} > 0$ *and* $k > 1$*, we obtain*

$$\langle A(\bar{x}) - A(\bar{y}), \bar{x} - \bar{y}\rangle = (1 - k)^2\|\bar{x}\|^2\,(R - (1 + k)\|\bar{x}\|) < 0.$$

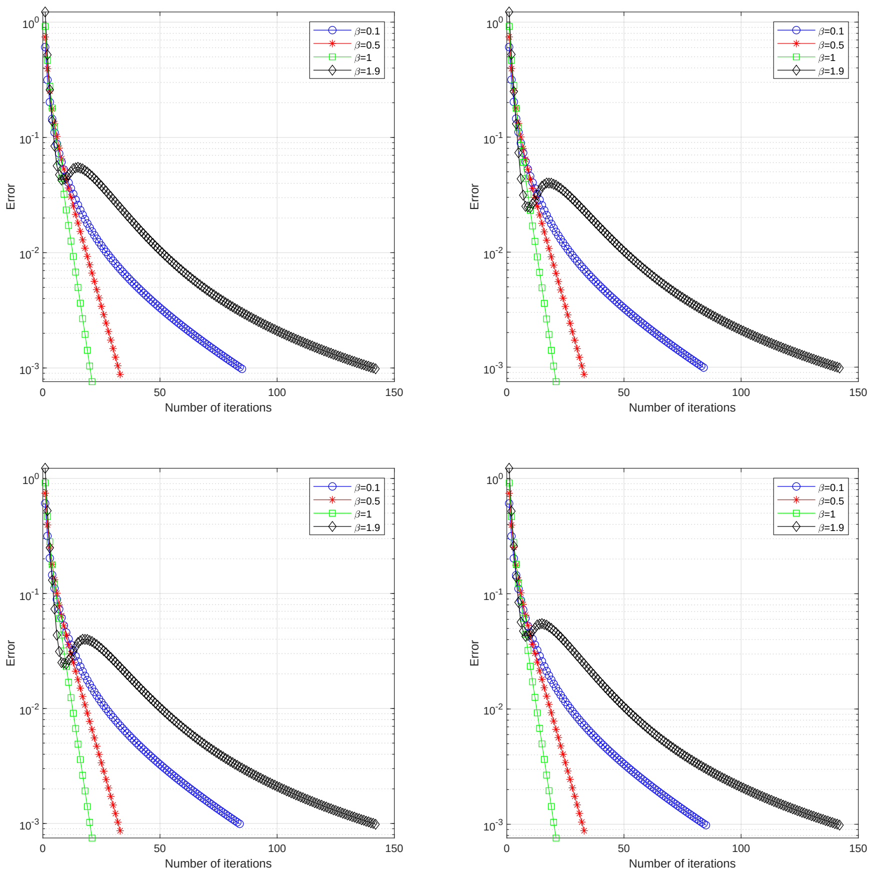


Figure 5.5: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

It is easy to see that the operator A is pseudomonotone on C. Indeed, if $\langle A(x), y-x\rangle \geq 0$ for all $x, y \in C$, i.e., $\langle (R-\|x\|)\, x, y-x\rangle \geq 0$, then we have $\langle x, y-x\rangle \geq 0$ since $\|x\| \leq p < R$. Thus, we obtain

$$\begin{aligned}\langle A(y), y-x\rangle &= \langle (R-\|y\|)\, y, y-x\rangle \\ &\geq (R-\|y\|)\, y\, (\langle y, y-x\rangle - \langle x, y-x\rangle) \\ & (R-\|y\|)\, \|y-x\|^2 \geq 0,\end{aligned}$$

since $\|y\| < p < R$. Also, define the mapping $T : H \to H$ by $Tx = \frac{x}{2}$. Then, it is easy to show that T is demimetric on H.

For our numerical result, we choose $R = 1.5, \quad p = 1, \quad k = 1.1$. The exact solution of the problem is $x^*(t) = 0$. Since the solution is known, we use the sequence $E(x_n) = \|x_n - x^*\|^2$ for each $n \geq 0$ to illustrate the numerical behaviour of our algorithm. The projection on C is computed by the explicit formula $P_C(x) = x$ if $\|x\| \leq p$ and$P_C(x) = \frac{px}{\|x\|}$ if $\|x\| > p$. We take $\|x_{n+1} - x_n\| = 10^{-3}$ as the stopping criterion and choose the starting points as follows:

Case 1: *Take $x_1 = t^2$ and $x_0 = t^3 \cos t$*

Case 2: *Take $x_1 = t^3$ and $x_0 = exp(-2t)$*

Case 3: *Take $x_1 = t^4$ and $x_0 = \sin 2t$*

Case 4: *Take $x_1 = t^5$ and $x_0 = t^2 \exp(-t)$.*

We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 5.6 and Table 5.2.12.

Table 5.2.12. Numerical results for Example 5.3.12

		$\beta = 0.1$	$\beta = 0.5$	$\beta = 1.0$	$\beta = 1.9$
Case 1	No. of Iter.	9	8	7	6
	CPU time (sec)	2.2148	0.8072	0.6564	0.6117
Case 2	No. of Iter.	9	8	7	6
	CPU time (sec)	1.0137	0.6741	0.6032	0.5348
Case 3	No. of Iter.	9	8	7	6
	CPU time (sec)	1.1558	0.7063	0.6036	0.5523
Case 4	No. of Iter.	9	8	7	6
	CPU time (sec)	1.2158	0.8600	0.6336	0.5204

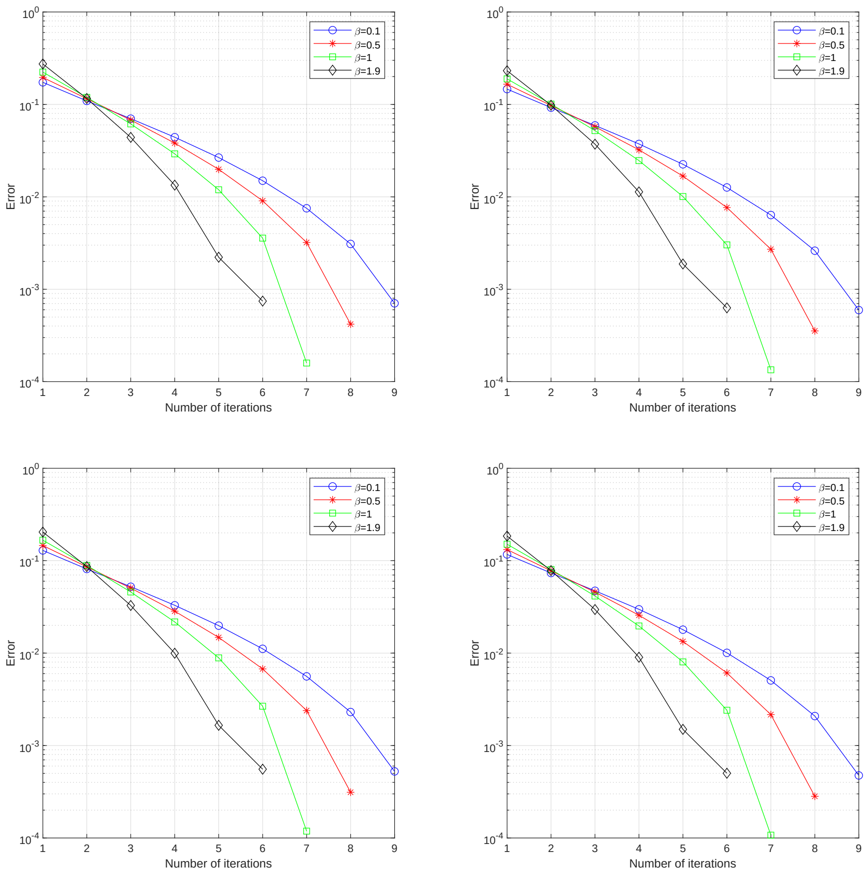


Figure 5.6: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

Remark 5.3.13. *Examples 5.3.9, 5.3.10, 5.3.11, 5.3.12 shows that even though our assumption is that $\beta \in (0,2)$, our proposed method perform best in terms of number of iterations and CPU time when $\beta \approx 1$.*

5.4 An inertial extrapolation method for solving generalized split feasibility problems

In this section, we propose a new inertial extrapolation method for solving a certain class of generalized split feasibility problems in two real Hilbert spaces. We prove that the sequence generated by our algorithm is strongly convergent to a minimum norm. Furthermore, some examples and numerical experiments to show the efficiency and implementation of our method were also discussed in the framework of infinite dimensional Hilbert spaces.

5.4.1 Main result

In this section, we first state the assumptions under which our strong convergence result is obtained.

Assumption 5.4.1. *Suppose that the following conditions hold:*

(1) *(a) The set C is a nonempty closed and convex subset of the real Hilbert space H_1.*
(b) $A : H_1 \to H_1$ is pseudomonotone, sequentially weakly continuous and uniformly continuous on bounded subsets of C.
(c) $S : H_2 \to H_2$ is a nonexpansive mapping, where H_2 is a real Hilbert space.
(d) $T : H_1 \to H_2$ is a bounded linear operator such that $T \neq 0$.
(e) The solution set $\Gamma := \{z \in VI(C, A) : Tz \in F(S)\}$ is nonempty.

(2) *$\{\alpha_n\}_{n=1}^{\infty}$ and $\{\epsilon_n\}_{n=1}^{\infty}$ are positive sequences satisfying the following conditions:*
(a) $\alpha_n \in (0, 1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$.
(b) $\lim\limits_{n\to\infty} \dfrac{\epsilon_n}{\alpha_n} = 0$.
(c) $\beta_n \subset (a, 1 - \alpha_n)$ for some $a > 0$.

Henceforth, we define $Gx := T^*(I - S)Tx$ for all $x \in H_1$. Thus, it follows from Lemma 2.5.25 that G is $2\|T\|^2$-Lipschitz continuous.
We now present the proposed method of this paper.

Algorithm 5.4.2. *Initialization: Choose sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\epsilon_n\}_{n=1}^{\infty}$ such that the conditions from Assumption 3.1 (2) hold and let $\tau_1 > 0$, $\delta, \alpha, \sigma \in (0, 1)$, $\theta \in [0, 1)$ and $x_0, x_1 \in H_1$ be given arbitrarily. Set $n = 1$.*

Iterative Steps:

Step 1. *Given the iterates x_{n-1} and $x_n (n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise.} \end{cases} \tag{5.95}$$

Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = w_n - \tau_n G w_n \quad \text{and} \quad z_n = P_C(u_n - A(u_n)). \tag{5.96}$$

Step 3. *Compute* $y_n := u_n - \eta_n(u_n - z_n)$, *where* $\eta_n := \alpha^{m_n}$ *and* m_n *is the smallest nonnegative integer satisfying*

$$\langle A(y_n), u_n - z_n\rangle \geq \frac{\sigma}{2}||u_n - z_n||^2. \tag{5.97}$$

Step 4. *Compute*

$$x_{n+1} = (1 - \beta_n - \alpha_n)u_n + \beta_n P_{C_n}(u_n), \tag{5.98}$$

where $C_n := \{x \in H_1 : F_n(x) \leq 0\}$ *and*

$$F_n(x) := \langle A(y_n), x - y_n\rangle. \tag{5.99}$$

Update:

$$\tau_{n+1} = \begin{cases} \min\left\{\frac{\delta||w_n - u_n||}{||Gw_n - Gu_n||},\ \tau_n\right\}, & \text{if } Gw_n \neq Gu_n, \\ \tau_n, & \text{otherwise.} \end{cases} \tag{5.100}$$

Stopping criterion: *If* $z_n = w_n = u_n = x_n$, *then stop, otherwise, set* $n := n + 1$ *and go back to* **Step 1.**

Remark 5.4.3.

(a) Assumption 5.4.1 (1) (b) requires that the operator A *is pseudomonotone and uniformly continuous which is much more, a weaker assumption than the monotonicity and Lipschitz continuity assumption used in [228, 229], as well as the inverse strongly monotonicity assumptions used in other papers for solving split variational inequality problems (see for example [54, 119, 136, 137, 178]). Thus, our method is applicable for a much more general class of Pseudomonotone and uniformly continuous operators.*

(b) In addition, our method requires at each iteration, only one projection onto C *and another projection onto the half-space* C_n, *which can be easily computed. In fact,*

$$P_{C_n}(u_n) := \begin{cases} u_n - \frac{\langle A(y_n), u_n - y_n\rangle}{||A(y_n)||^2}A(y_n), & \text{if } \langle A(y_n), u_n - y_n\rangle > 0, \\ u_n, & \text{if } \langle A(y_n), u_n - y_n\rangle \leq 0. \end{cases}$$

Hence, our method is less computationally expensive than other methods in the literature [54, 228, 229], for solving the split variational inequality problems.

Remark 5.4.4. *If $\{y_n\}$ is bounded, then F_n is Lipschitz continuous.*
Indeed, if $\{y_n\}$ is bounded, then by Lemma 2.5.50, there exists $\bar{M} > 0$ such that $\|A(y_n)\| \leq \bar{M}$ for all $n \geq 1$. Thus, for each $x, y \in C$, we obtain

$$\begin{aligned}
\|F_n(x) - F_n(y)\| &= \|\langle A(y_n), x - y_n\rangle - \langle A(y_n), y - y_n\rangle\| \\
&= \|\langle A(y_n), x - y\rangle\| \\
&\leq \|A(y_n)\|\|x - y\| \\
&\leq \bar{M}\|x - y\|.
\end{aligned}$$

Therefore, F_n is Lipschitz continuous.

We now show that the stopping criterion of Algorithm 5.4.2 is valid.

Lemma 5.4.5. *If $z_n = w_n = u_n = x_n$ in Algorithm 5.4.2, then $x_n \in \Gamma$.*

Proof. If $z_n = w_n = u_n = x_n$, then from (5.96), it is clear that $x_n = P_C(x_n - A(x_n))$. Also, we obtain that $x_n = x_n - \tau_n G x_n = x_n - \tau_n T^*(I - S)Tx_n$, which implies that $T^*(I - S)Tx_n = 0$. That is,

$$STx_n = Tx_n + \bar{z},$$

where $T^*\bar{z} = 0$. Now, let $z \in \Gamma$, then we obtain that

$$\begin{aligned}
\|Tx_n - Tz\|^2 &= \|Tx_n - Tz\|^2 + 2\langle x_n - z, T^*\bar{z}\rangle \\
&= \|Tx_n - Tz\|^2 + 2\langle Tx_n - Tz, \bar{z}\rangle \\
&= \|Tx_n - Tz + \bar{z}\|^2 - \|\bar{z}\|^2 \\
&= \|STx_n - Tz\|^2 - \|\bar{z}\|^2 \\
&\leq \|Tx_n - Tz\|^2 - \|\bar{z}\|^2,
\end{aligned}$$

which implies that $\|\bar{z}\| = 0$. That is, $\bar{z} = 0$. Hence $STx_n = Tx_n$, which gives that $Tx_n \in F(S)$. Therefore, $x_n \in \Gamma$.

□

Next, we show that the limit of the stepsize $\{\tau_n\}$ generated by (5.100) exists.

Lemma 5.4.6. *The limit of the stepsize $\{\tau_n\}$ exists and $\lim\limits_{n\to\infty} \tau_n > 0$.*

Proof. From (5.100), it is obvious that $\tau_{n+1} \leq \tau_n \quad \forall n \in \mathbb{N}$.
Also, we know by Lemma 2.5.25 that G is $2\|T\|^2$-Lipschitz continuous. Thus, we get in the case of $Gu_n \neq Gw_n$ that

$$\tau_{n+1} = \min\left\{\frac{\delta\|w_n - u_n\|}{\|Gw_n - Gu_n\|}, \tau_n\right\} \geq \min\left\{\frac{\delta}{2\|T\|^2}, \tau_n\right\}.$$

Hence, by induction, we obtain that $\{\tau_n\}$ is bounded below by $\min\left\{\frac{\delta}{2\|T\|^2}, \tau_1\right\}$. Hence, the limit of $\{\tau_n\}$ exists and $\lim\limits_{n\to\infty} \tau_n \geq \min\left\{\tau_1, \frac{\delta}{2\|T\|^2}\right\} > 0$.

□

Lemma 5.4.7. *Let Assumption 5.4.1 hold and let $\{x_n\}$ be a sequence generated by Algorithm 5.4.2. Assume that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to a point x^*, and $\lim_{k\to\infty}\|x_{n_k}-u_{n_k}\|=0$, $\lim_{k\to\infty}\|u_{n_k}-w_{n_k}\|=0$ and $\lim_{k\to\infty}\|u_{n_k}-z_{n_k}||=0$, then $x^*\in\Gamma$.*

Proof. By Lemma 2.5.48 we obtain

$$\langle u_{n_k}-A(u_{n_k})-z_{n_k},\quad x-z_{n_k}\rangle\leq 0,\ \forall\ x\in C,$$

which implies that

$$\langle u_{n_k}-z_{n_k},\quad x-z_{n_k}\rangle\leq\langle A(u_{n_k}),\quad x-z_{n_k}\rangle,\ \forall\ x\in C.$$

Hence

$$\langle u_{n_k}-z_{n_k},\quad x-z_{n_k}\rangle+\langle A(u_{n_k}),\quad z_{n_k}-u_{n_k}\rangle\leq\langle A(u_{n_k}),\quad x-u_{n_k}\rangle,\ \forall x\in C. \tag{5.101}$$

Fix $x\in C$ and let $k\to\infty$ in (5.101). Since $\lim_{k\to\infty}\|u_{n_k}-z_{n_k}||=0$, we have

$$0\leq\liminf_{k\to\infty}\langle A(u_{n_k}),x-u_{n_k}\rangle\quad\forall x\in C. \tag{5.102}$$

Now, choose a sequence $\{\delta_k\}$ of positive numbers such that $\delta_{k+1}\leq\delta_k,\ \forall\ k\geq 1$ and $\delta_k\to 0$ as $k\to\infty$. Then, for each δ_k, we denote by N_k (which exists as a result of (5.102)) the smallest positive integer such that

$$\langle A(u_{n_j}),x-u_{n_j}\rangle+\delta_k\geq 0\quad\forall j\geq N_k. \tag{5.103}$$

Since $\{\delta_k\}$ is decreasing, we have that $\{N_k\}$ is increasing. Furthermore, we set for each $k\geq 1$, $m_{N_k}=\frac{A(u_{N_k})}{\|A(u_{N_k})\|^2}$, provided $A(u_{N_k})\neq 0$. Then it is easy to see that $\langle A(u_{N_k}),m_{N_k}\rangle=1$ for each $k\geq 1$. Thus, by (5.103), we have that

$$\langle A(u_{N_k}),x+\delta_k m_{N_k}-u_{N_k}\rangle\geq 0,$$

which implies by the pseudo-monotonicity of A that

$$\langle A(x+\delta_k m_{N_k}),x+\delta_k m_{N_k}-u_{N_k}\rangle\geq 0. \tag{5.104}$$

Since $\{x_{n_k}\}$ converges weakly to x^*, we obtain by our hypothesis that $\{u_{n_k}\}$, $\{z_{n_k}\}$ and $\{w_{n_k}\}$ also converge weakly to x^*. Thus, by the sequentially weakly continuity of A, we have that $\{A(u_{n_k})\}$ converges weakly to $A(x^*)$. If $A(x^*)=0$, then $x^*\in VI(C,A)$. On the other hand, if we suppose that $A(x^*)\neq 0$, then by the weakly lower semicontinuity of $\|\cdot\|$, we obtain that

$$0<\|A(x^*)\|\leq\liminf_{k\to\infty}\|A(u_{n_k})\|.$$

Since $\{u_{N_k}\} \subset \{u_{n_k}\}$, we obtain that

$$
\begin{aligned}
0 \leq \limsup_{k\to\infty} \|\delta_k m_{N_k}\| &= \limsup_{k\to\infty} \left(\frac{\delta_k}{\|A(u_{n_k})\|}\right) \\
&\leq \frac{\limsup_{k\to\infty} \delta_k}{\liminf_{k\to\infty} \|A(u_{n_k})\|} \\
&\leq \frac{0}{\|A(x^*)\|} = 0.
\end{aligned}
$$

Therefore, $\lim_{k\to\infty} \|\delta_k m_{N_k}\| = 0$. Thus, letting $k \to \infty$ in (5.104) yields

$$\langle A(x), x - x^*\rangle \geq 0 \quad \forall x \in C, \tag{5.105}$$

which implies by Lemma 2.5.9 that $x^* \in VI(C, A)$.

Now, to show that $Tx^* \in F(S)$, recall that Lemma 5.4.6 gives that $\lim_{n\to\infty} \tau_n = \tau > 0$. Furthermore, since $G \equiv T^*(I - S)T$ is Lipschitz continuous, $\{T^*(I - S)Tw_{n_k}\}$ is bounded. Hence, we get that

$$\|(I-\tau_{n_k}T^*(I-S)T)w_{n_k}-(I-\tau T^*(I-S)T)w_{n_k}\| = |\tau_{n_k}-\tau|\|T^*(I-S)Tw_{n_k}\| \to 0, \text{ as } k \to \infty.$$

That is, $\lim_{k\to\infty} \|u_{n_k} - (I - \tau T^*(I - S)T)w_{n_k}\| = 0$, which implies by our hypothesis that

$$\lim_{k\to\infty} ||w_{n_k} - \big(I - \tau T^*(I - S)T\big)w_{n_k}|| = 0. \tag{5.106}$$

Thus, we obtain from Lemma 2.5.55, that $x^* \in F\big(I-\tau T^*(I-S)T\big)$. Hence, using the same line of argument as in the proof of Lemma 5.4.5, we obtain that $Tx^* \in F(S)$. Therefore $x^* \in \Gamma$.

□

Lemma 5.4.8. *Suppose Assumption 5.4.1 holds. Then, the sequence $\{x_n\}$ generated by Algorithm 5.4.2 is bounded.*

Proof. From Lemma 2.5.16 (ii), (iii), and (iv), u_n can be written in the form

$$u_n = (1 - \beta_n)w_n + \beta_n V_n w_n, \tag{5.107}$$

where $\beta_n = \tau_n||T||^2$ and V_n is a nonexpansive mapping for each $n \in \mathbb{N}$. That is, $(I - \tau_n G)$ is $\tau_n||T||^2$-averaged.
Now, let $x^* \in \Gamma$, we obtain from (5.107) that

$$
\begin{aligned}
||u_n - x^*||^2 &= ||(1 - \beta_n)w_n + \beta_n V_n w_n - x^*||^2 \\
&= (1 - \beta_n)||w_n - x^*||^2 + \beta_n||V_n w_n - x^*||^2 - \beta_n(1 - \beta_n)||V_n w_n - w_n||^2 \\
&\leq ||w_n - x^*||^2 - \beta_n(1 - \beta_n)||V_n w_n - w_n||^2 \qquad (5.108)\\
&\leq ||w_n - x^*||^2.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}||w_n - x^*|| &= ||x_n + \theta_n(x_n - x_{n-1}) - x^*|| \\ &\le ||x_n - x^*|| + \theta_n||x_n - x_{n-1}|| \\ &= ||x_n - x^*|| + \alpha_n\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||.\end{aligned}$$

Now from (5.95), we observe that $\dfrac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \le \dfrac{\epsilon_n}{\alpha_n} \to 0$. Thus there exists $M_1 > 0$ such that $\dfrac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| \le M_1 \quad \forall\ n \ge 1$,
and so,

$$||u_n - x^*|| \le ||w_n - x^*|| \le ||x_n - x^*|| + \alpha_n M_1. \tag{5.109}$$

From (5.98), we obtain

$$\begin{aligned}||x_{n+1} - x^*|| &= ||(1-\beta_n-\alpha_n)u_n + \beta_n P_{C_n}u_n - x^*|| \\ &= ||(1-\beta_n-\alpha_n)(u_n - x^*) + \beta_n(P_{C_n}u_n - x^*) - \alpha_n x^*|| \\ &\le ||(1-\beta_n-\alpha_n)(u_n - x^*) + \beta_n(P_{C_n}u_n - x^*)|| + \alpha_n||x^*||. \qquad (5.110)\end{aligned}$$

Note that

$$\begin{aligned}&||(1-\beta_n-\alpha_n)(u_n - x^*) + \beta_n(P_{C_n}u_n - x^*)||^2 \\ &= (1-\beta_n-\alpha_n)^2||u_n - x^*||^2 + 2(1-\beta_n-\alpha_n)\beta_n\langle u_n - x^*, \quad P_{C_n}u_n - x^*\rangle \\ &+ \beta_n^2||P_{C_n}u_n - x^*||^2 \\ &\le (1-\beta_n-\alpha_n)^2||u_n - x^*||^2 + 2(1-\beta_n-\alpha_n)\beta_n||u_n - x^*||||P_{C_n}u_n - x^*|| \\ &+ \beta_n^2||P_{C_n}u_n - x^*||^2 \\ &\le (1-\beta_n-\alpha_n)^2||u_n - x^*||^2 + (1-\beta_n-\alpha_n)\beta_n||u_n - x^*||^2 \\ &+ (1-\beta_n-\alpha_n)\beta_n||P_{C_n}u_n - x^*||^2 + \beta_n^2||P_{C_n}u_n - x^*||^2 \\ &\le (1-\beta_n-\alpha_n)(1-\alpha_n)||u_n - x^*||^2 + (1-\alpha_n)\beta_n||P_{C_n}u_n - x^*||^2 \qquad (5.111) \\ &\le (1-\beta_n-\alpha_n)(1-\alpha_n)||u_n - x^*||^2 + (1-\alpha_n)\beta_n||u_n - x^*||^2 \\ &= (1-\alpha_n)^2||u_n - x^*||^2,\end{aligned}$$

which implies from (5.109) that

$$\begin{aligned}||(1-\beta_n-\alpha_n)(u_n - x^*) + \beta_n(P_{C_n}u_n - x^*)|| &= (1-\alpha_n)||x_n - x^*|| + (1-\alpha_n)\alpha_n M_1 \\ &\le (1-\alpha_n)||x_n - x^*|| + \alpha_n M_1. \qquad (5.112)\end{aligned}$$

Now, combining (5.110) and (5.112) gives

$$\begin{aligned}||x_{n+1} - x^*|| &\le (1-\alpha_n)||x_n - x^*|| + \alpha_n||x^*|| + \alpha_n M_1 \\ &= (1-\alpha_n)||x_n - x^*|| + \alpha_n(||x^*|| + M_1).\end{aligned}$$

It follows from Lemma 2.5.24 that $\{x_n\}$ is bounded. Consequently, $\{A(u_n)\}, \{u_n\}, \{w_n\}$, $\{z_n\}$ and $\{A(y_n)\}$ are also bounded. $\square$

Lemma 5.4.9. *Let $\{x_n\}$ be generated by Algorithm 5.4.2 such that Assumption 5.4.1 holds. Assume that $\lim_{k\to\infty} ||P_{C_{n_k}} u_{n_k} - u_{n_k}|| = 0 = \lim_{k\to\infty} \eta_{n_k} ||u_{n_k} - z_{n_k}||^2$, then $\lim_{k\to\infty} \|u_{n_k} - z_{n_k}\| = 0$.*

Proof. From **Step 3**, we have that $\eta_n = \alpha^{m_n}$ with $\alpha \in (0,1)$. Hence, $\{\eta_n\} \subset (0,1)$ is bounded. Thus, there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\liminf_{k\to\infty} \eta_{n_k} \geq 0$.

Case 1. Suppose that $\liminf_{k\to\infty} \eta_{n_k} > 0$. Then

$$0 \leq \|u_{n_k} - z_{n_k}\|^2 = \frac{\eta_{n_k} \|u_{n_k} - z_{n_k}\|^2}{\eta_{n_k}},$$

which implies that

$$\begin{aligned}
\limsup_{k\to\infty} \|u_{n_k} - z_{n_k}\|^2 &\leq \limsup_{k\to\infty} \Big(\eta_{n_k} \|u_{n_k} - z_{n_k}\|^2\Big) \Big(\limsup_{k\to\infty} \frac{1}{\eta_{n_k}}\Big) \\
&= \Big(\limsup_{k\to\infty} \eta_{n_k} \|u_{n_k} - z_{n_k}\|^2\Big) \frac{1}{\liminf_{k\to\infty} \eta_{n_k}} \\
&= 0.
\end{aligned}$$

Hence, $\limsup_{k\to\infty} \|u_{n_k} - z_{n_k}\| = 0$. Therefore, $\lim_{k\to\infty} \|u_{n_k} - z_{n_k}\| = 0$.

Case 2 . Suppose that $\liminf_{k\to\infty} \eta_{n_k} = 0$. Then, we may assume without loss of generality that $\lim_{k\to\infty} \eta_{n_k} = 0$ and $\lim_{k\to\infty} \|u_{n_k} - z_{n_k}\| = a \geq 0$.

Define $\bar{y}_{n_k} - u_{n_k} := \dfrac{1}{\alpha} \eta_{n_k} (z_{n_k} - u_{n_k})$. Since $\{z_{n_k} - u_{n_k}\}$ is bounded and $\lim_{k\to\infty} \eta_{n_k} = 0$, then

$$\lim_{k\to\infty} \|\bar{y}_{n_k} - u_{n_k}\| = 0. \tag{5.113}$$

From the step-size rule and definition of $\bar{y}_k$, we have

$$\langle A(\bar{y}_{n_k}), \quad u_{n_k} - z_{n_k} \rangle < \frac{\sigma}{2} \|u_{n_k} - z_{n_k}\|^2, \ \forall k \in \mathbb{N}.$$

This implies that,

$$2\langle A(u_{n_k}), \quad u_{n_k} - z_{n_k} \rangle + 2\langle A(\bar{y}_{n_k}) - A(u_{n_k}), \quad u_{n_k} - z_{n_k} \rangle < \sigma \|u_{n_k} - z_{n_k}\|^2, \ \forall k \in \mathbb{N}.$$

Setting $\mu_{n_k} := u_{n_k} - A(u_{n_k})$, we obtain

$$2\langle u_{n_k} - \mu_{n_k}, \quad u_{n_k} - z_{n_k} \rangle + 2\langle A(\bar{y}_{n_k}) - A(u_{n_k}), \quad u_{n_k} - z_{n_k} \rangle < \sigma \|u_{n_k} - z_{n_k}\|^2, \ \forall k \in \mathbb{N}.$$

Using Lemma 2.5.18(ii) we have

$$2\langle u_{n_k} - \mu_{n_k}, \quad u_{n_k} - z_{n_k} \rangle = \|u_{n_k} - z_{n_k}\|^2 + \|u_{n_k} - \mu_{n_k}\|^2 - \|z_{n_k} - \mu_{n_k}\|^2.$$

Therefore,

$$\|u_{n_k} - \mu_{n_k}\|^2 - \|z_{n_k} - \mu_{n_k}\|^2 < (\sigma - 1)\|u_{n_k} - z_{n_k}\|^2 - 2\langle A(\bar{y}_{n_k}) - A(u_{n_k}), \quad u_{n_k} - z_{n_k}\rangle, \quad \forall k \in \mathbb{N}.$$

Since A is uniformly continuous on bounded subsets of C and (5.113), if $a > 0$ then the right hand side of the inequality above converges to $(\sigma - 1)a < 0$ as $k \to \infty$. From the last inequality, we obtain

$$\limsup_{k\to\infty} \left(\|u_{n_k} - \mu_{n_k}\|^2 - \|z_{n_k} - \mu_{n_k}\|^2\right) \le (\sigma - 1)a < 0.$$

For $\epsilon = \dfrac{-(\sigma - 1)a}{2} > 0$, there exists $N \in \mathbb{N}$ such that

$$\|u_{n_k} - \mu_{n_k}\|^2 - \|z_{n_k} - \mu_{n_k}\|^2 \le (\sigma - 1)a + \epsilon = \frac{(\sigma - 1)a}{2} \le 0 \quad \forall k \in \mathbb{N}, k \ge N;$$

Thus

$$\|u_{n_k} - \mu_{n_k}\| < \|z_{n_k} - \mu_{n_k}\| \quad \forall k \in N, k \ge N.$$

which is a contradiction to the definition of $z_{n_k} = P_C\big(u_{n_k} - A(u_{n_k})\big)$. Hence $a = 0$, the proof is complete. $\square$

Theorem 5.4.10. *Suppose that Assumption 5.4.1 holds. Then, the sequence $\{x_n\}$ generated by Algorithm 5.4.2 converges strongly to the point $x^* \in \Gamma$, where*

$$\|x^*\| = \min\{\|\bar{x}\| : \bar{x} \in \Gamma\}.$$

Proof. Using Lemma 2.5.8 and Lemma 2.5.49, we obtain

$$\begin{aligned} ||P_{C_n}(u_n) - x^*|| &\le ||u_n - x^*||^2 - ||P_{C_n}(u_n) - u_n||^2 \\ &= ||u_n - x^*||^2 - dist^2(u_n, C_n) \\ &\le ||u_n - x^*||^2 - \left(\frac{1}{m}F_n(u_n)\right)^2 \\ &= ||u_n - x^*||^2 - \left(\frac{1}{m}\langle A(y_n), u_n - y_n\rangle\right)^2 \\ &= ||u_n - x^*||^2 - \left(\frac{\eta_n}{m}\langle A(y_n), u_n - z_n\rangle\right)^2 \\ &\le ||u_n - x^*||^2 - \left(\frac{\sigma\eta_n}{2m}||u_n - z_n||^2\right)^2. \end{aligned} \tag{5.114}$$

Furthermore, we have

$$x_{n+1} = (1 - \beta_n - \alpha_n)u_n + \beta_n P_{C_n}(u_n) = (1 - \beta_n)u_n + \beta_n P_{C_n}(u_n) - \alpha_n u_n.$$

Let $\phi_n = (1-\beta_n)u_n + \beta_n P_{C_n}(u_n)$, then we obtain

$$\begin{aligned}
||\phi_n - x^*||^2 &= ||(1-\beta_n)u_n + \beta_n P_{C_n}(u_n) - x^*||^2 \\
&= ||(1-\beta_n)(u_n - x^*) + \beta_n(P_{C_n}(u_n) - x^*)||^2 \\
&= (1-\beta_n)^2||u_n - x^*||^2 + \beta_n^2||P_{C_n}(u_n) - x^*||^2 + 2(1-\beta_n)\beta_n\langle u_n - x^*, \quad P_{C_n}(u_n) - x^*\rangle \\
&\leq (1-\beta_n)^2||u_n - x^*||^2 + \beta_n^2||P_{C_n}(u_n) - x^*||^2 + 2(1-\beta_n)\beta_n||u_n - x^*||||P_{C_n}(u_n) - x^*|| \\
&\leq (1-\beta_n)^2||u_n - x^*||^2 + \beta_n^2||P_{C_n}(u_n) - x^*||^2 + (1-\beta_n)\beta_n||u_n - x^*||^2 \qquad (5.115) \\
&+ (1-\beta_n)\beta_n||P_{C_n}(u_n) - x^*||^2 \\
&= (1-\beta_n)||u_n - x^*||^2 + \beta_n||P_{C_n}(u_n) - x^*||^2 \\
&\leq ||w_n - x^*||^2. \qquad (5.116)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
||w_n - x^*||^2 &= ||x_n + \theta_n(x_n - x_{n-1}) - x^*||^2 \\
&\leq ||x_n - x^*||^2 + 2\theta_n\langle x_n - x^*, \quad x_n - x_{n-1}\rangle + \theta_n^2||x_n - x_{n-1}||^2 \\
&\leq ||x_n - x^*||^2 + 2\theta_n||x_n - x^*||||x_n - x_{n-1}|| + \theta_n^2||x_n - x_{n-1}||^2 \\
&= ||x_n - x^*||^2 + \theta_n||x_n - x_{n-1}||(2||x_n - x^*|| + \theta_n||x_n - x_{n-1}||) \\
&\leq ||x_n - x^*||^2 + \theta_n||x_n - x_{n-1}||M_2, \qquad (5.117)
\end{aligned}$$

for some $M_2 > 0$. Combining (5.116) and (5.117) we obtain

$$||\phi_n - x^*||^2 \leq ||x_n - x^*||^2 + \theta_n||x_n - x_{n-1}||M_2. \qquad (5.118)$$

Since $\phi_n = (1-\beta_n)u_n + \beta_n P_{C_n}(u_n)$, we have $u_n - \phi_n = \beta_n(u_n - P_{C_n}(u_n))$. Therefore it follows that

$$x_{n+1} = \phi_n - \alpha_n u_n = (1-\alpha_n)\phi_n - \alpha_n(u_n - \phi_n) = (1-\alpha_n)\phi_n - \alpha_n\beta_n(u_n - P_{C_n}(u_n)).$$

Thus, from Lemma 2.5.18 (ii), we obtain

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||(1-\alpha_n)\phi_n - \alpha_n\beta_n(u_n - P_{C_n}(u_n)) - x^*||^2 \\
&= ||(1-\alpha_n)(\phi_n - x^*) - \big(\alpha_n\beta_n(u_n - P_{C_n}(u_n)) + \alpha_n x^*\big)||^2 \\
&\leq (1-\alpha_n)^2||\phi_n - x^*||^2 - 2\langle\alpha_n\beta_n(u_n - P_{C_n}(u_n)) + \alpha_n x^*, \quad x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)||\phi_n - x^*||^2 + 2\langle\alpha_n\beta_n(u_n - P_{C_n}(u_n)), \quad x^* - x_{n+1}\rangle \\
&+ 2\alpha_n\langle x^*, \quad x^* - x_{n+1}\rangle \\
&\leq (1-\alpha_n)||\phi_n - x^*||^2 + 2\alpha_n\beta_n||u_n - P_{C_n}(u_n)||||x^* - x_{n+1}|| \\
&+ 2\alpha_n\langle x^*, \quad x^* - x_{n+1}\rangle. \qquad (5.119)
\end{aligned}$$

Therefore, using (5.118) and (5.119), we have

$$\begin{aligned}
&||x_{n+1} - x^*||^2 \leq (1-\alpha_n)||x_n - x^*||^2 + (1-\alpha_n)\theta_n||x_n - x_{n-1}||M_2 \\
&+ 2\alpha_n\beta_n||u_n - P_{C_n}(u_n)||||x^* - x_{n+1}|| + 2\alpha_n\langle x^*, \quad x^* - x_{n+1}\rangle \\
&= (1-\alpha_n)||x_n - x^*||^2 + \alpha_n\Big[\frac{\theta_n}{\alpha_n}(1-\alpha_n)||x_n - x_{n-1}||M_2 + 2\beta_n||u_n - P_{C_n}(u_n)||||x^* - x_{n+1}|| \\
&+ 2\langle x^*, \quad x^* - x_{n+1}\rangle\Big] \\
&= (1-\alpha_n)||x_n - x^*||^2 + \alpha_n d_n, \qquad (5.120)
\end{aligned}$$

where $d_n := \left[\frac{\theta_n}{\alpha_n}(1-\alpha_n)||x_n-x_{n-1}||M_2+2\beta_n||u_n-P_{C_n}(u_n)||||x^*-x_{n+1}||+2\langle x^*,\ \ x^*-x_{n+1}\rangle\right]$.

To show that the sequence $\Big\{||x_n - x^*||\Big\}$ converges to zero, by Lemma 2.5.55, it is enough to show that $\limsup\limits_{k\to\infty} d_{n_k} \leq 0$ (where $\{d_{n_k}\}$ is a subsequence of $\{d_n\}$), for every subsequence $\Big\{||x_{n_k} - x^*||\Big\}$ of $\Big\{||x_n - x^*||\Big\}$ satisfying

$$\liminf_{k\to\infty}\Big(||x_{n_k+1} - x^*|| - ||x_{n_k} - x^*||\Big) \geq 0.$$

Now, suppose that $\Big\{||x_{n_k} - x^*||\Big\}$ is a subsequence of $\Big\{||x_n - x^*||\Big\}$ such that

$$\liminf_{k\to\infty}\Big(||x_{n_k+1} - x^*|| - ||x_{n_k} - x^*||\Big) \geq 0,$$

then, $\liminf\limits_{k\to\infty}\Big(||x_{n_k+1} - x^*||^2 - ||x_{n_k} - x^*||^2\Big)$

$$= \liminf_{k\to\infty}\Big((||x_{n_k+1} - x^*|| - ||x_{n_k} - x^*||)(||x_{n_k+1} - x^*|| + ||x_{n_k} - x^*||)\Big) \geq 0. \tag{5.121}$$

Now,

$$\begin{aligned}
||x_{n+1} - x^*||^2 &= ||(1-\beta_n-\alpha_n)u_n + \beta_n P_{C_n}(u_n) - x^*||^2\\
&= ||(1-\beta_n-\alpha_n)(u_n - x^*) + \beta_n(P_{C_n}(u_n) - x^*) - \alpha_n x^*||^2\\
&= ||(1-\beta_n-\alpha_n)(u_n - x^*) + \beta_n(P_{C_n}(u_n) - x^*)||^2\\
&\quad - 2\alpha_n\langle(1-\beta_n-\alpha_n)(u_n - x^*) + \beta_n(P_{C_n}(u_n) - x^*),\ \ x^*\rangle + \alpha_n^2||x^*||^2\\
&\leq ||(1-\beta_n-\alpha_n)(u_n - x^*) + \beta_n(P_{C_n}(u_n) - x^*)||^2 + \alpha_n M_3,
\end{aligned} \tag{5.122}$$

for some $M_3 > 0$. Substituting (5.111) into (5.122), we have

$$||x_{n+1} - x^*||^2 \leq (1-\beta_n-\alpha_n)(1-\alpha_n)||u_n - x^*||^2 + (1-\alpha_n)\beta_n||P_{C_n}(u_n) - x^*||^2 + \alpha_n M_3. \tag{5.123}$$

Using (5.114), (5.117), and (5.123), we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq (1-\beta_n-\alpha_n)(1-\alpha_n)||u_n - x^*||^2 + (1-\alpha_n)\beta_n||u_n - x^*||^2\\
&\quad - (1-\alpha_n)\beta_n\left(\frac{\sigma\eta_n}{2m}||u_n - z_n||^2\right)^2 + \alpha_n M_3\\
&= (1-\alpha_n)^2||u_n - x^*||^2 - (1-\alpha_n)\beta_n\left(\frac{\sigma\eta_n}{2m}||u_n - z_n||^2\right)^2 + \alpha_n M_3\\
&\leq ||w_n - x^*||^2 - (1-\alpha_n)\beta_n\left(\frac{\sigma\eta_n}{2m}||u_n - z_n||^2\right)^2 + \alpha_n M_3\\
&\leq ||x_n - x^*||^2 + \theta_n||x_n - x_{n-1}||M_2 - (1-\alpha_n)\beta_n\left(\frac{\sigma\eta_n}{2m}||u_n - z_n||^2\right)^2 + \alpha_n M_3\\
&= ||x_n - x^*||^2 + \alpha_n\left[\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}||M_2 + M_3\right]\\
&\quad - (1-\alpha_n)\beta_n\left(\frac{\sigma\eta_n}{2m}||u_n - z_n||^2\right)^2.
\end{aligned}$$

Hence

$$(1-\alpha_n)\beta_n\left(\frac{\sigma\eta_n}{2m}||u_n-z_n||^2\right)^2 \leq ||x_n-x^*||^2-||x_{n+1}-x^*||^2 \\ +\alpha_n\left[\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|M_2+M_3\right]. \tag{5.124}$$

By (5.121) and (5.124), we obtain

$$\limsup_{k\to\infty}\left((1-\alpha_{n_k})\beta_{n_k}\left(\frac{\sigma\eta_{n_k}}{2m}||u_{n_k}-z_{n_k}||^2\right)^2\right)$$

$$\begin{aligned}
&\leq \limsup_{k\to\infty}\left\{||x_{n_k}-x^*||^2-||x_{n_k+1}-x^*||^2\right\}+\limsup_{k\to\infty}\alpha_{n_k}\left[M_1M_2+M_3\right]\\
&=\limsup_{k\to\infty}\left\{||x_{n_k}-x^*||^2-||x_{n_k+1}-x^*||^2\right\}\\
&=-\liminf_{k\to\infty}\left\{||x_{n_k+1}-x^*||^2-||x_{n_k}-x^*||^2\right\}\leq 0,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\eta_{n_k}||u_{n_k}-z_{n_k}||^2=0. \tag{5.125}$$

Now, from Step 3, we have

$$y_{n_k}=u_{n_k}-\alpha^{m_n}\left(u_{n_k}-z_{n_k}\right)=u_{n_k}-\eta_{n_k}\left(u_{n_k}-z_{n_k}\right).$$

Thus, by (5.125), we obtain that

$$\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\|=0. \tag{5.126}$$

Also, observe that equation (5.111) and (5.122), gives

$$\|x_{n+1}-x^*\|^2\leq(1-\alpha_n)\|u_n-x^*\|^2+\alpha_nM_3.$$

Substituting (5.108) into the last inequality above, we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq (1-\alpha_n)\left[\|w_n-x^*\|^2-\beta_n(1-\beta_n)\|V_nw_n-w_n\|^2\right]+\alpha_nM_3\\
&\leq (1-\alpha_n)\left[\|x_n-x^*\|^2+\theta_n\|x_n-x_{n-1}\|M_2\right]-\beta_n(1-\beta_n)(1-\alpha_n)\|V_nw_n-w_n\|^2\\
&+\alpha_nM_3.
\end{aligned}$$

Hence

$$\beta_n(1-\beta_n)(1-\alpha_n)\|V_nw_n-w_n\|^2\leq\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\ +\alpha_n\left[\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|M_2+M_3\right]. \tag{5.127}$$

By (5.121) and (5.127), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\Big[\beta_{n_k}(1-\beta_{n_k})\|V_{n_k}w_{n_k}-w_{n_k}\|^2\Big] &\leq \limsup_{k\to\infty}\Big[\|x_{n_k}-x^*\|^2-\|x_{n_k+1}-x^*\|^2\Big]\\ &\quad+\limsup_{k\to\infty}\alpha_{n_k}\Big[M_1M_2+M_3\Big]\\ &=-\liminf_{k\to\infty}\Big[\|x_{n_k+1}-x^*\|^2-\|x_{n_k}-x^*\|^2\Big]\leq 0,\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|V_{n_k}w_{n_k}-w_{n_k}\|=0. \tag{5.128}$$

Thus, we obtain from (5.107) that

$$\lim_{k\to\infty}\|u_{n_k}-w_{n_k}\|=\lim_{k\to\infty}\beta_{n_k}\|V_{n_k}w_{n_k}-w_{n_k}\|=0. \tag{5.129}$$

Also, combining (5.117) and (5.123) we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq (1-\beta_n-\alpha_n)(1-\alpha_n)\|u_n-x^*\|^2+(1-\alpha_n)\beta_n\|u_n-x^*\|^2\\ &\quad-(1-\alpha_n)\beta_n\|P_{C_n}u_n-u_n\|^2+\alpha_nM_3\\ &=(1-\alpha_n)^2\|u_n-x^*\|^2-(1-\alpha_n)\beta_n\|P_{C_n}u_n-u_n\|^2+\alpha_nM_3\\ &\leq\|x_n-x^*\|^2+\theta_n\|x_n-x_{n-1}\|M_2-(1-\alpha_n)\beta_n\|P_{C_n}u_n-u_n\|^2+\alpha_nM_3\\ &\leq\|x_n-x^*\|^2+\alpha_n\left[\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|M_2+M_3\right]-(1-\alpha_n)\beta_n\|P_{C_n}u_n-u_n\|^2,\end{aligned}$$

which implies that

$$(1-\alpha_n)\beta_n\|P_{C_n}u_n-u_n\|^2\leq\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\Big[M_1M_2+M_3\Big]. \tag{5.130}$$

Thus, we obtain by (5.121) that

$$\lim_{k\to\infty}\|P_{C_{n_k}}u_{n_k}-u_{n_k}\|=0. \tag{5.131}$$

Combining this with (5.125), we obtain from Lemma 5.4.9 that

$$\lim_{k\to\infty}\|u_{n_k}-z_{n_k}\|=0. \tag{5.132}$$

Furthermore, from Step 2 we obtain

$$\|w_{n_k}-x_{n_k}\|=\alpha_{n_k}\left[\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|\right]\to 0 \quad\text{as}\quad k\to\infty. \tag{5.133}$$

Using (5.129) and (5.133), we obtain

$$\lim_{k\to\infty}\|u_{n_k}-x_{n_k}\|=0. \tag{5.134}$$

From (5.98) and (5.131), we have

$$\|x_{n_{k+1}} - u_{n_k}\| \leq \beta_{n_k}\|P_{C_{n_k}}u_{n_k} - u_{n_k}\| + \alpha_{n_k}||u_{n_k}|| \to 0.$$

Thus, we obtain from (5.134) that

$$\lim_{k\to\infty} \|x_{n_{k+1}} - x_{n_k}\| = 0.$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to some $\bar{x} \in H_1$, such that

$$\limsup_{k\to\infty}\langle x^*, \quad x^* - x_{n_k}\rangle = \lim_{j\to\infty}\langle x^*, \quad x^* - x_{n_j}\rangle = \langle x^*, \quad x^* - \bar{x}\rangle.$$

Also, we obtain from (5.129), (5.132), (5.134) and Lemma 5.4.7 that $\bar{x} \in \Gamma$.

From $x^* = P_\Gamma 0$, we obtain

$$\limsup_{k\to\infty}\langle x^*, \quad x^* - x_{n_k}\rangle = \langle x^*, \quad x^* - \bar{x}\rangle \leq 0.$$

Since $\lim_{k\to\infty} \|x_{n_{k+1}} - x_{n_k}\| \to 0$, we obtain

$$\limsup_{k\to\infty}\langle x^*, \quad x^* - x_{n_{k+1}}\rangle \leq 0. \tag{5.135}$$

Now, recall that $d_{n_k} := \Big[\frac{\theta_{n_k}}{\alpha_{n_k}}(1-\alpha_{n_k})\|x_{n_k} - x_{n_k-1}\|M_2 + 2\beta_{n_k}\|u_{n_k} - P_{C_{n_k}}(u_{n_k})\|$
$\|x^* - x_{n_{k+1}}\| + 2\langle x^*, \quad x^* - x_{n_{k+1}}\rangle\Big].$

Hence, by (5.135), $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, (5.131) and Lemma 2.5.55, we obtain

$$\lim_{n\to\infty}\|x_n - x^*\| = 0.$$

Therefore, $\{x_n\}$ converges strongly to $x^* = P_\Gamma 0$. □

5.4.2 Numerical example

In this section, we provide numerical example of the algorithm considered in Section 3 of this paper and compare it with Algorithms (9.1.14), (9.1.15) of Takahashi *et al.* [220] and Algorithm (9.1.16) of Tian and Jiang [229].
The codes are implemented in Matlab 2016 (b). We perform all computations on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM.

Example 5.4.11. *Let $H_1 = H_2 = L^2([0,1])$ with norm $\|x\| := \sqrt{\int_0^1 |x(t)|^2 dt}$ and inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)dt,\ \forall x, y \in L^2([0,1])$. Define the operator $A : L^2([0,1]) \to L^2([0,1])$ by*

$$A(x)(t) := e^{-\|x\|}\int_0^t x(s)ds, \quad \forall x \in L^2([0,1]), \quad t \in [0,1].$$

Then, A is uniformly continuous and pseudomonotone but not monotone on H_1 (see [232]). Let $C := \{x \in L^2([0,1]) : \langle a, x\rangle \leq b\}$ where $a = t^2 + 1$ and $b = 1$, then

$$P_C(x) = \begin{cases} \frac{b-\langle a,x\rangle}{\|a\|^2_{L^2}} a + x, & \langle a, x\rangle > b, \\ \\ x, & \langle a, x\rangle \leq b. \end{cases}$$

We also define the mapping $S : L^2([0,1]) \to L^2([0,1])$ be defined by

$$Sx(t) = \int_0^1 tx(s)ds, \quad t \in [0,1].$$

Then, S is nonexpansive. Indeed, we have

$$\begin{aligned} |Sx(t) - Sy(t)|^2 = |\int_0^1 t(x(s) - y(s))ds|^2 &\leq \left(\int_0^1 t|x(s) - y(s)|ds\right)^2 \\ &\leq \int_0^1 |x(s) - y(s)|^2 ds = ||x - y||^2. \end{aligned}$$

Thus, we obtain that

$$||Sx - Sy||^2 = \int_0^1 |Sx(t) - Sy(t)|^2 \leq ||x - y||^2.$$

Also, let $T : L^2([0,1]) \to L^2([0,1])$ be defined by $T(x(t)) = \int_0^1 x(t)dt \quad \forall\ x \in H_1$.

For Algorithm (9.1.16), we define $h : L_2([0,1]) \to L_2([0,1])$ by

$$hx(t) = \int_0^1 \frac{t}{2} x(s)ds, \quad t \in [0,1].$$

Then, h is a contraction mapping.

For the parameters, we choose $\alpha_n = \frac{1}{5n+2}$, $\beta_n = \frac{1}{2} - \alpha_n$, $\theta_n = \bar{\theta}_n$ and $\varepsilon_n = \frac{\alpha_n}{n^{0.01}}$. Furthermore, we randomly choose $x_0, x_1 \in L_2([0,1])$, $\tau_1 > 0$ and $\delta, \alpha, \sigma, \theta \in (0,1)$ (see the cases below). Note that in our method, the stepsize $\{\tau_n\}$ is generated in each iteration. But for Algorithms (9.1.14), (9.1.15) and (9.1.16), we shall take $\tau_n = \frac{1}{2||T||^2}$.

We now consider the following cases for our numerical results shown in Figure 5.7 and Table 5.3.14 below.

Case 1: *Take $x_1(t) = t^3 - 3$, $x_0(t) = t$, $\theta = 0.1$, $\tau_1 = 100$.*

Case 2: *Take $x_1(t) = t$, $x_0(t) = t^3 - 3$, $\theta = 0.1$, $\tau_1 = 100$.*

Case 3: *Take $x_1(t) = 2t^2$, $x_0(t) = e^t$, $\theta = 0.9$, $\tau_1 = 10$.*

Case 4: *Take $x_1(t) = t^2 + t + 1$, $x_0(t) = e^t + t$, $\theta = 0.9$, $\tau_1 = 10$.*

Table 5.3.14. Numerical results.

		Alg. 5.4.2	*Alg. (9.1.14)*	*Alg. (9.1.15)*	*Alg. (9.1.16)*
Case 1	*No. of Iter.* *CPU time (sec)*	*31* *2.235*	*67* *8.767*	*63* *4.679*	*56* *4.276*
Case 2	*No. of Iter.* *CPU time (sec)*	*26* *1.910*	*53* *7.178*	*51* *4.088*	*45* *3.726*
Case 3	*No. of Iter.* *CPU time (sec)*	*26* *3.377*	*54* *8.413*	*52* *4.395*	*46* *0.5523*
Case 4	*No. of Iter.* *CPU time (sec)*	*29* *1.2158*	*62* *0.8600*	*58* *0.6336*	*52* *4.358*

Remark 5.4.12. *The stopping criterion used for our numerical computations is* $\|x_{n+1} - x_n\| \leq 10^{-4}$. *We plot the graphs of errors against the number of iterations in each case. The figures and numerical results are shown in Figure 5.7 and Table 5.3.14, respectively. From the figure and table, it can be inferred that our algorithm (Algorithm 5.4.2) outperforms Algorithms* (9.1.14), (9.1.15) *and* (9.1.16), *in both number of iterations and time taken for the computation.*
Therefore, our method is more efficient than Algorithms (9.1.14), (9.1.15) *and* (9.1.16).

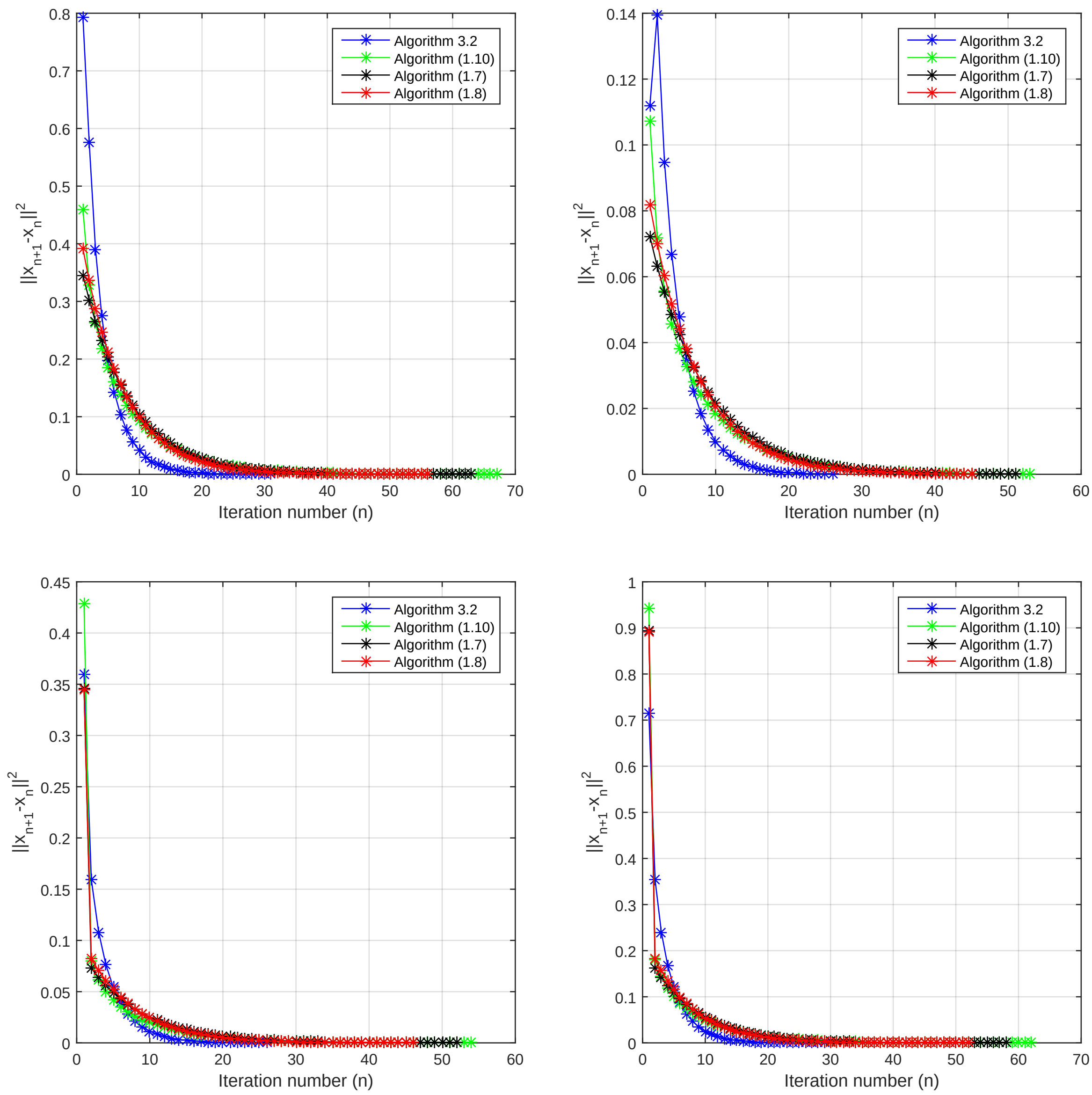


Figure 5.7: Errors vs Iteration numbers(n): **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

Chapter 6

Split-Type Problems Minimization and Hierarchical Fixed Point Problems

6.1 Introduction

In this chapter, we present an algorithm for solving split common fixed point problem for asymptotically demicontractive mapping in two real Hilbert spaces. Under some mild conditions, we prove that the proposed method converges strongly to a solution of the problem. We give examples to illustrate that the class of asymptotically demicontractive mappings and the class of demicontractive mappings are independent. In addition, we introduce and study the notion of Split Minimization Problem with Multiple Output Sets. We propose a new iterative method, which employs an inertial Halpern approximation technique for approximating the common solution of Split Minimization Problem with Multiple Output Sets and fixed point problem for a finite family of Bregman relatively nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. Moreover, we present several numerical experiments to show the efficiency and applicability of our method in comparison with a related methods in the literature.

6.2 Split common fixed point problem of asymptotically demicontractive mappings

Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two mappings with nonempty fixed point sets and $A : H_1 \to H_2$ be a bounded linear operator. The split common fixed point problem (in short, SCFPP) is defined as follows:

$$\text{Find } x^* \in H_1 \text{ such that } x^* \in F(U) \text{ and } Ax^* \in F(T). \tag{6.1}$$

We shall denote the solution set of the SCFPP (6.1) by

$$\Omega := \{x^* \in F(U) \text{ such that } \ Ax^* \in F(T)\} = F(U) \cap A^{-1}(F(T)),$$

where U and F are κ-strictly asymptotically demicontractive mapping and β-strictly asymptotically demicontractive mapping respectively.

6.2.1 Main result

Lemma 6.2.1. *Let $U : H \to H$ be a κ-strictly asymptotically demicontractive mapping. Define ${U_\lambda}^n := (1-\lambda_n)I + \lambda_n U^n$ for any $\lambda_n \in (0, 1-\kappa)$. Then, for any $x \in H$ and $x^* \in F(U)$,*

$$\|U_\lambda^n x - x^*\|^2 \leq \left[1 - \lambda_n\left(1-\kappa_n^2\right)\right]\|x-x^*\|^2 - \lambda_n(1-\kappa-\lambda_n)\|U^n x - x\|^2.$$

Proof. Firstly,

$$\begin{aligned}\|U_\lambda^n x - x^*\|^2 &= \|(x-x^*) + \lambda_n\left(U^n x - x\right)\|^2 \\ &= \|x-x^*\|^2 + 2\lambda_n\langle U^n x - x, x - x^*\rangle + \lambda_n^2\|U^n x - x\|^2, \end{aligned} \tag{6.2}$$

applying inequality (2.7), we have

$$2\lambda_n\langle U^n x - x, x-x^*\rangle \leq \lambda_n(\kappa-1)\|x-U^n x\|^2 - \lambda_n(1-\kappa_n^2)\|x-x^*\|^2, \tag{6.3}$$

adding (6.2) and (6.3), we get

$$\|U_\lambda^n x - x^*\|^2 \leq \left[1-\lambda_n\left(1-\kappa_n^2\right)\right]\|x-x^*\|^2 - \lambda_n\left[1-\kappa-\lambda_n\right]\|U^n x - x\|^2,$$

which is the desired result. □

Lemma 6.2.2. *Let $A : H_1 \to H_2$ be a bounded linear operator and $T : H_2 \to H_2$ a β-strictly asymptotically demicontractive operator with $\beta < 1$. If $A^{-1}\left(F(T)\right) \neq \emptyset$, then*

$$\begin{aligned}\|x_n - \rho A^*(I-T^n)Ax_n - x^*\|^2 \leq &\left[1+\rho\|A\|^2(\beta_n^2-1)\right]\|x_n-x^*\|^2 \\ &- \rho((1-\beta) - \rho\|A\|^2)\|(I-T^n)Ax_n\|^2, \end{aligned}\tag{6.4}$$

where $x_n \in H$, $Ax_n \neq T(Ax_n)$, $x^* \in A^{-1}\left(F(T)\right)$ and $\rho \in (0, \frac{1-\beta}{\|A\|^2})$.

Proof. Since T is β-strictly asymptotically demicontractive, we have

$$\begin{aligned}\|x_n - \rho A^*(I-T^n)Ax_n - x^*\|^2 &= \|x_n - x^*\|^2 - 2\rho\langle x_n - x^*, A^*(I-T^n)Ax_n\rangle \\ &+ \rho^2\|A^*(I-T^n)Ax_n\|^2 \\ &= \|x_n-x^*\|^2 - 2\rho\langle Ax_n - Ax^*, (I-T^n)Ax_n\rangle + \rho^2\|A^*(I-T^n)Ax_n\|^2 \\ &\leq \|x_n-x^*\|^2 - \rho\left[(1-\beta_n^2)\|Ax_n - Ax^*\|^2 + (1-\beta)\|Ax_n - T^nAx_n\|^2\right] \\ &+ \rho^2\|A^*(I-T^n)Ax_n\|^2 \\ &\leq \|x_n-x^*\|^2 + \rho\|A\|^2(\beta_n^2-1)\|x_n-x^*\|^2 - \rho(1-\beta)\|Ax_n - T^nAx_n\|^2 \\ &+ \rho^2\|A^*(I-T^n)Ax_n\|^2 \\ &= \left[1+\rho\|A\|^2(\beta_n^2-1)\right]\|x_n-x^*\|^2 - \rho((1-\beta)-\rho\|A\|^2)\|(I-T^n)Ax_n\|^2,\end{aligned}$$

where the inequality follows from (2.7). □

Lemma 6.2.3. *Let H_1 and H_2 be two real Hilbert spaces. $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be κ-strictly asymptotically demicontractive mapping with sequence $\{\kappa_n\} \in [1, \infty)$ and β-strictly asymptotically demicontractive mapping with sequence $\{\beta_n\} \in [1, \infty)$, respectively. Assuming $F(U) \neq \emptyset$, $F(T) \neq \emptyset$ and $\Omega := \{x^* \in H_1 : x^* \in F(U)$ and $Ax^* \in F(T)\} \neq \emptyset$. Let $f : H_1 \to H_1$ be a contraction mapping with constant $v \in [0, \frac{1}{\sqrt{2}})$ and $\{x_n\}$ be a sequence generated by $x_1 \in H_1$ and*

$$x_{n+1} = U_\lambda^n \left(\alpha_n x_n + \gamma_n f(x_n) + \delta_n \big(x_n - \rho A^*(I - T^n)Ax_n\big)\right), \quad n \geq 1, \tag{6.5}$$

where

$$U_\lambda^n = (1 - \lambda_n)I + \lambda_n U^n$$

and the following conditions are satisfied:

(A1) $0 < a \leq \lambda_n \leq b < 1 - \kappa$;

(A2) $\alpha_n \in [0, 1), \delta_n, \gamma_n \in (0, 1)$ *such that* $\alpha_n + \gamma_n + \delta_n = 1, \liminf\limits_{n\to\infty} \delta_n > 0, \lim\limits_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(A3) $\rho \in (0, \frac{1-\beta}{\|A\|^2})$;

(A4) $\sum_{n=1}^{\infty}(\kappa_n^2 - 1) < \infty$ *and* $\sum_{n=1}^{\infty}(\beta_n^2 - 1) < \infty$;

(A5) $(\kappa_n^2 - 1) = o(\gamma_n)$ *and* $(\beta_n^2 - 1) = o(\gamma_n)$.

Then, $\{x_n\}$ is bounded.

Proof. Let $p \in \Omega$. Then,

$$U_\lambda^n p = (1 - \lambda_n)p + \lambda_n U_\lambda^n p = p.$$

Set $t_n = x_n - \rho A^*(I - T^n)Ax_n$ and $w_n = \alpha_n x_n + \gamma_n f(x_n) + \delta_n t_n$.
It therefore follows from Lemmas 6.2.1 and 6.2.2 that

$$\|U_\lambda^n w_n - p\| \leq \left[1 - \lambda_n \left(1 - \kappa_n^2\right)\right] \|w_n - p\| \tag{6.6}$$

and

$$\|t_n - p\| \leq \left[1 + \rho\|A\|^2(\beta_n^2 - 1)\right] \|x_n - p\|. \tag{6.7}$$

Thus, it follows from (6.5), (6.6) and (6.7), that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|U_\lambda^n w_n - p\| \\
&\leq \left[1 - \lambda_n\left(1 - \kappa_n^2\right)\right]\|w_n - p\| \\
&\leq \left[1 - \lambda_n\left(1 - \kappa_n^2\right)\right]\Bigg(\alpha_n\|x_n - p\| + \gamma_n\|f(x_n) - p\| + \delta_n\|t_n - p\|\Bigg) \\
&\leq \left[1 - \lambda_n\left(1 - \kappa_n^2\right)\right]\Bigg(\gamma_n\|f(x_n) - p\| + \alpha_n\|x_n - p\| \\
&+ \delta_n\left[1 + \rho\|A\|^2(\beta_n^2 - 1)\right]\|x_n - p\|\Bigg) \\
&\leq \left[1 - \lambda_n\left(1 - \kappa_n^2\right)\right]\left[1 + \rho\|A\|^2(\beta_n^2 - 1)\right]\Bigg(\gamma_n\|f(x_n) - p\| + (1 - \gamma_n)\|x_n - p\|\Bigg) \\
&\leq (1 + \sigma_n)(1 + \phi_n)\Bigg(\gamma_n[\|f(x_n) - f(p)\| + \|f(p) - p\|] + (1 - \gamma_n)\|x_n - p\|\Bigg) \\
&\leq (1 + \omega_n)\Bigg(\gamma_n v\|x_n - p\| + \gamma_n\|f(p) - p\| + (1 - \gamma_n)\|x_n - p\|\Bigg) \\
&= (1 + \omega_n)\Bigg(\left(1 - \gamma_n(1 - v)\right)\|x_n - p\| + \frac{\gamma_n(1 - v)\|f(p) - p\|}{1 - v}\Bigg) \\
&\leq (1 + \omega_n)\max\{\|x_n - p\|, \ \frac{\|f(p) - p\|}{1 - v}\} \\
&\vdots \\
&\leq \sum_{j=1}^{n}(1 + \omega_j)\max\{\|x_1 - p\|, \ \frac{\|f(p) - p\|}{1 - v}\},
\end{aligned}
$$

where $\sigma_n = \lambda_n(\kappa_n^2 - 1)$ and $\phi_n = \rho\|A\|^2(\beta_n^2 - 1)$. Also, since $\omega_n = \sigma_n + \phi_n + \sigma_n\phi_n$ and $\sum\omega_n < \infty$, it follows that $\{\|x_{n+1} - p\|\}$ is bounded. Consequently, $\{x_n\}$, $\{t_n\}$, $\{w_n\}$ are all bounded. $\square$

We now present the following convergence theorem.

Theorem 6.2.4. *Let H_1 and H_2 be two real Hilbert spaces. $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be uniformly L_1-Lipschitzian κ-strictly asymptotically demicontractive mapping with sequence $\{\kappa_n\} \in [1, \infty)$ and uniformly L_2-Lipschitzian β-strictly asymptotically demicontractive mapping with sequence $\{\beta_n\} \in [1, \infty)$, respectively. Assuming $F(U) \neq \emptyset$, $F(T) \neq \emptyset$ and $I - U$ and $I - T$ are demiclosed at 0. Let $f : H_1 \to H_1$ be a contraction mapping with constant $v \in [0, \frac{1}{\sqrt{2}})$ and suppose*

$$\Omega := \{x^* \in H_1 : x^* \in F(U) \ \text{ and } \ Ax^* \in F(T)\} \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by $x_1 \in H_1$ and

$$x_{n+1} = U_\lambda^n\left(\alpha_n x_n + \gamma_n f(x_n) + \delta_n\big(x_n - \rho A^*(I - T^n)Ax_n\big)\right), \quad n \geq 1, \tag{6.8}$$

where

$$U_\lambda^n = (1-\lambda_n)I + \lambda_n U^n$$

and the conditions (A1) - (A5) are satisfied. Then, the sequence $\{x_n\}$ generated by (6.8) converges strongly to $p \in \Omega$, where p is the unique solution of the variational inequality problem: Find $p \in \Omega$ such that

$$\langle (I-f)p, x-p\rangle \geq 0 \quad \forall\ x \in \Omega. \tag{6.9}$$

Proof. Since f is a v-contraction mapping, then for all $x, y \in H_1$,

$$\begin{aligned}\|(I-f)x-(I-f)y\| &= \|(x-y)+(f(y)-f(x))\| \\ &\leq \|x-y\| + \|f(y)-f(x)\| \\ &\leq (1+v)\|x-y\| \end{aligned} \tag{6.10}$$

on the other hand,

$$\begin{aligned}\langle (I-f)x-(I-f)y, x-y\rangle &= \langle x-y, x-y\rangle - \langle f(x)-f(y), x-y\rangle \\ &\geq \|x-y\|^2 - v\|x-y\|^2 \\ &= (1-v)\|x-y\|^2. \end{aligned} \tag{6.11}$$

Thus, we have from (6.10) and (6.11) that $(I-f)$ is $(1+v)$-Lipschitz continuous and $(1-v)$-strongly monotone, respectively. Therefore, it follows from Lemma 2.5.9 that there exists a unique element $p \in \Omega$ such that (6.9) is satisfied.
We now prove that the sequence $\{x_n\}$ is strongly convergent to p. Set $t_n = x_n - \rho A^*(I - T^n)Ax_n$ and $w_n = \alpha_n x_n + \gamma_n f(x_n) + \delta_n t_n$.

Firstly,

$$\begin{aligned}\langle f(x_n)-p, t_n-p\rangle &= \langle f(x_n)-f(p), t_n-p\rangle + \langle f(p)-p, t_n-p\rangle \\ &\leq \|f(x_n)-f(p)\|.\|t_n-p\| + \langle f(p)-p, t_n-p\rangle \\ &\leq \frac{1}{2}\left(\|f(x_n)-f(p)\|^2 + \|t_n-p\|^2\right) + \langle f(p)-p, t_n-p\rangle \\ &\leq \frac{1}{2}v^2\|x_n-p\|^2 + \frac{1}{2}\|t_n-p\|^2 + \langle f(p)-p, t_n-p\rangle. \end{aligned} \tag{6.12}$$

Also,

$$\begin{aligned}\|f(x_n)-p\|^2 &\leq (\|f(x_n)-f(p)\| + \|f(p)-p\|)^2 \\ &\leq (v\|x_n-p\| + \|f(p)-p\|)^2 \\ &\leq 2v^2\|x_n-p\|^2 + 2\|f(p)-p\|^2. \end{aligned} \tag{6.13}$$

Now, from Lemma 2.5.18, (6.12), (6.13) and Lemma 6.2.2, we have

$$
\begin{aligned}
&\|w_n - p\|^2 = \|\alpha_n x_n + \gamma_n f(x_n) + \delta_n t_n - p\|^2 \\
&\le \|\alpha_n(x_n - p) + (1-\alpha_n)\big(\frac{\gamma_n}{1-\alpha_n} f(x_n) + \frac{\delta_n}{1-\alpha_n} t_n - p\big)\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\|\frac{\gamma_n}{1-\alpha_n} f(x_n) + \frac{\delta_n}{1-\alpha_n} t_n - p\Big\|^2 \\
&= \alpha_n\|x_n - p\|^2 \\
&+ (1-\alpha_n)\left(\frac{\gamma_n^2}{(1-\alpha)^2}\|f(x_n) - p\|^2 + 2\frac{\gamma_n\delta_n}{(1-\alpha_n)^2}\langle f(x_n) - p, t_n - p\rangle + \frac{\delta_n^2}{(1-\alpha_n)^2}\|t_n - p\|^2\right) \\
&\le \alpha_n\|x_n - p\|^2 + \left(\frac{2v^2\gamma_n^2 + v^2\gamma_n\delta_n}{(1-\alpha_n)}\|x_n - p\|^2 + \frac{\delta_n^2 + \gamma_n\delta_n}{(1-\alpha_n)}\|t_n - p\|^2\right) \\
&+ \left(\frac{2\gamma_n\delta_n}{(1-\alpha_n)}\langle f(p) - p, t_n - p\rangle + \frac{2\gamma_n^2}{(1-\alpha_n)}\|f(p) - p\|^2\right) \\
&\le \alpha_n\|x_n - p\|^2 \\
&+ \left(\frac{2v^2\gamma_n^2 + v^2\gamma_n\delta_n}{(1-\alpha_n)}\|x_n - p\|^2\right. \\
&\left.+ \delta_n\left[(1+\phi_n)\|x_n - p\|^2 - \rho((1-\beta) - \rho\|A||^2)\|(I - T^n)Ax_n\|^2\right]\right) \\
&+ \left(\frac{2\gamma_n\delta_n}{(1-\alpha_n)}\langle f(p) - p, t_n - p\rangle + \frac{2\gamma_n^2}{(1-\alpha_n)}\|f(p) - p\|^2\right) \\
&\le \big(1 - \gamma_n\big[1 - \big(1 + \frac{\gamma_n}{1-\alpha_n}\big)v^2\big]\big)\|x_n - p\|^2 \\
&+ \gamma_n\left(\frac{2\delta_n}{(1-\alpha_n)}\langle f(p) - p, t_n - p\rangle + \frac{2\gamma_n}{(1-\alpha_n)}\|f(p) - p\|^2\right) \\
&+ \phi_n M - \delta_n\rho((1-\beta) - \rho\|A||^2)\|(I - T^n)Ax_n\|^2.
\end{aligned} \tag{6.14}
$$

Moreover, from Lemma 6.2.1 and (6.14), we obtain

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 = \|U_\lambda^n w_n - p\|^2 \\
&\le \tau_n\|w_n - p\|^2 - \lambda_n(1-\kappa-\lambda_n)\|w_n - U_\lambda^n w_n\|^2 \\
&\le \tau_n\Bigg(\big(1 - \gamma_n\big[1 - \big(1 + \frac{\gamma_n}{1-\alpha_n}\big)v^2\big]\big)\|x_n - p\|^2 \\
&\quad + \gamma_n\left(\frac{2\delta_n}{(1-\alpha_n)}\langle f(p) - p, t_n - p\rangle + \frac{2\gamma_n}{(1-\alpha_n)}\|f(p) - p\|^2\right) \\
&\quad + \phi_n M - \delta_n\rho((1-\beta) - \rho\|A||^2)\|(I - T^n)Ax_n\|^2\Bigg) - \lambda_n(1-\kappa-\lambda_n)\|w_n - U_\lambda^n w_n\|^2 \\
&\le \tau_n(1-\psi_n)\|x_n - p\|^2 + \psi_n b_n + \tau_n\phi_n M - \tau_n\delta_n\rho((1-\beta) - \rho\|A||^2)\|(I - T^n)Ax_n\|^2 \\
&\quad - \lambda_n(1-\kappa-\lambda_n)\|w_n - U_\lambda^n w_n\|^2 \\
&\le (1-\psi_n)\|x_n - p\|^2 + \psi_n b_n + (\tau_n\phi_n + \sigma_n)M - \tau_n\delta_n\rho((1-\beta) - \rho\|A||^2)\|(I - T^n)Ax_n\|^2 \\
&\quad - \lambda_n(1-\kappa-\lambda_n)\|w_n - U_\lambda^n w_n\|^2 \qquad (6.15) \\
&\le (1-\psi_n)\|x_n - p\|^2 + \psi_n b_n + (\tau_n\phi_n + \sigma_n)M, \qquad (6.16)
\end{aligned}
$$

where $\tau_n = 1-\lambda_n(1-\kappa_n^2)$, $\psi_n = \gamma_n\big[1-\big(1+\frac{\gamma_n}{1-\alpha_n}\big)v^2\big]$, $b_n = \frac{\tau_n(2\gamma_n\|f(p)-p\|^2+2\delta_n\langle f(p)-p,t_n-p\rangle)}{(1-\alpha_n)(1-(1+\frac{\gamma_n}{1-\alpha_n})v^2)}$, $\sigma_n = \lambda_n(\kappa_n^2-1)$, $\phi_n = \rho\|A\|^2(\beta_n^2-1)$ and $M := \sup_{n\in\mathbb{N}}\{\|x_n-p\|^2\}$.

To complete the proof, we now consider the following two cases:

Case 1. Suppose there exists some $n_0 \in \mathbb{N}$ such that $\{\|x_n-p\|\}$ is monotone decreasing for $n > n_0$. Then by the boundedness of $\{\|x_n-p\|\}$, it implies that $\lim\limits_{n\to\infty}(\|x_{n+1}-p\|-\|x_n-p\|) = 0$. Then we obtain from (6.15) that

$$\begin{aligned}\tau_n\delta_n\rho((1 \quad - \quad \beta) - \rho\|A\|^2)\|(I-T^n)Ax_n\|^2 + \lambda_n(1-\kappa-\lambda_n)\|w_n - U_\lambda^n w_n\|^2 \\ \leq \quad (1-\psi_n)\|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \psi_n b_n + (\tau_n\phi_n+\sigma_n)M.\end{aligned} \tag{6.17}$$

Since $\lim\limits_{n\to\infty}\|x_n-p\|$ exists, $\psi_n \to 0$, $\sigma_n \to 0$ and $\phi_n \to 0$, we have

$$\lim_{n\to\infty}\|(I-T^n)Ax_n\| = 0 \tag{6.18}$$

and

$$\lim_{n\to\infty}\|w_n - U_\lambda^n w_n\| = 0. \tag{6.19}$$

Observe from (6.18) that

$$\|A^*(I-T^n)Ax_n\| \leq \|A^*\|\|(I-T^n)Ax_n\|^2 \to 0 \quad \text{as} \quad n\to\infty. \tag{6.20}$$

Thus, using (6.20), we derive that

$$\begin{aligned}\|w_n - x_n\| &\leq \gamma_n\|f(x_n)-x_n\| + (1-\gamma_n)\rho\|A^*(I-T^n)Ax_n\| \\ &\to 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{6.21}$$

Also, we have that

$$\|t_n - x_n\| = \rho\|A^*(I-T^n)Ax_n\| \to 0, \quad \text{as} \quad n\to\infty. \tag{6.22}$$

Using (6.21) and (6.22), we obtain

$$\lim_{n\to\infty}\|w_n - t_n\| = 0. \tag{6.23}$$

Next, we prove that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

Indeed, from (6.19) and (6.21), we have that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|(1-\lambda_n)w_n + \lambda_n U_\lambda^n w_n - x_n\| \\ &\leq \|w_n - x_n\| + \lambda_n\|w_n - U_\lambda^n w_n\| \\ &\to 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{6.24}$$

It then follows from (6.21) and (6.24) that

$$\|w_n - w_{n-1}\| \to 0, \text{ as } n \to \infty. \tag{6.25}$$

Moreover, since U is uniformly L_1-Lipschitzian, we have that

$$\begin{aligned}\|w_n - Uw_n\| &\leq \|w_n - U^n w_n\| + \|U^n w_n - Uw_n\| \\ &\leq \|w_n - U^n w_n\| + L_1\|U^{n-1}w_n - w_n\| \\ &\leq \|w_n - U^n w_n\| + L_1\big(\|U^{n-1}w_n - U^{n-1}w_{n-1}\| + \|U^{n-1}w_{n-1} - w_{n-1}\| \\ &\quad +\|w_{n-1} - w_n\|\big) \\ &\leq \|w_n - U^n w_n\| + L_1\big(L_1\|w_n - w_{n-1}\| + \|U^{n-1}w_{n-1} - w_{n-1}\| \\ &\quad +\|w_{n-1} - w_n\|\big).\end{aligned} \tag{6.26}$$

Therefore taking the limit of (6.26) and using (6.19) and (6.25), we get

$$\|w_n - Uw_n\| \to 0, \text{ as } n \to \infty \tag{6.27}$$

and by similar argument we obtain that

$$\|Ax_n - TAx_n\| \to 0, \text{ as } n \to \infty. \tag{6.28}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $x^* \in H$ as $k \to \infty$. By linearity, it follows that $Ax_{n_k} \rightharpoonup Ax^*$ as $k \to \infty$. Thus, it follows from (6.28) and the fact that $(I-T)$ is demiclosed at 0 that $Ax^* \in F(T)$. Also, we conclude from (6.21) that $w_{n_k} \rightharpoonup x^*$. Moreover, since $(I-U)$ is demiclosed at 0, it follows from (6.27) that $x^* \in F(U)$. Therefore, $x^* \in \Omega$. we conclude by showing that $x_n \to p$ as $n \to \infty$. Since $x^* \in \Omega$, we have from (6.9) that

$$\langle (I-f)p, x^* - p\rangle \geq 0. \tag{6.29}$$

Also, from the choices of $\{\gamma_n\}$ and v, it is easy to see that $\sum_{n=1}^{\infty}\psi_n = \sum_{n=1}^{\infty}\gamma_n(1-(1+\gamma_n)v^2) = \infty$ and $\psi_n \to 0$ as $n \to \infty$. Using (6.22) and (6.29), we obtain

$$\limsup_{n\to\infty}\langle f(p) - p, t_n - p\rangle = \lim_{k\to\infty}\langle f(p) - p, t_{n_k} - p\rangle = \lim_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle. \tag{6.30}$$

Thus, we obtain from (6.29) and (6.30) that

$$\limsup_{n\to\infty}\langle f(p) - p, t_n - p\rangle = \langle f(p) - p, x^* - p\rangle \leq 0.$$

So that

$$\limsup_{n\to\infty} b_n \leq 0. \tag{6.31}$$

Hence, by (6.16), (6.31) and Lemma 2.5.51, we obtain

$$\lim_{n\to\infty}\|x_n - p\| = 0.$$

Case 2: Suppose that $\{\|x_n - p\|\}$ is not a monotone decreasing sequence. Then for any n_0, there exists an integer $n \geq n_0$ such that $\|x_n - p\| \leq \|x_{n+1} - p\|$. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ for some n_0 sufficiently large enough by

$$\tau(n) := \max\{k \in \mathbb{N} : n_0 \leq k \leq n, \ \phi_k \leq \phi_{k+1}\}.$$

Then $\{\tau(n)\}$ is a non-decreasing sequence such that $\lim_{n\to\infty} \tau(n) = \infty$ and

$$0 \leq \|x_{\tau(n)} - p\| \leq \|x_{\tau(n)+1} - p\| \ \text{ for all } \ n \geq n_0. \tag{6.32}$$

Following similar argument as in **Case 1**, we get

$$\begin{aligned}
\lim_{n\to\infty} \|w_{\tau(n)} - U_\lambda w_{\tau(n)}\| = 0, \quad \lim_{n\to\infty} \|(I-T)Ax_{\tau(n)}\| = 0,\\
\lim_{n\to\infty} \|w_{\tau(n)} - x_{\tau(n)}\| = 0, \quad \lim_{n\to\infty} \|t_{\tau(n)} - x_{\tau(n)}\| = 0,\\
\lim_{n\to\infty} \|w_{\tau(n)} - t_{\tau(n)}\| = 0, \quad \lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0,
\end{aligned}$$

and

$$\limsup_{n\to\infty} \langle f(p) - p, t_{\tau(n)} - p\rangle \leq 0. \tag{6.33}$$

Thus, it follows from (6.16) and (6.32) that

$$\begin{aligned}
&0 \leq (1-\psi_{\tau(n)})\|x_{\tau(n)} - p\| - \|x_{\tau(n)+1} - p\| + \psi_{\tau(n)} b_{\tau(n)} + (\tau_{\tau(n)}\phi_{\tau(n)} + \sigma_{\tau(n)})M\\
&0 \leq (1-\psi_{\tau(n)})\|x_{\tau(n)+1} - p\| - \|x_{\tau(n)+1} - p\| + \psi_{\tau(n)} b_{\tau(n)} + (\tau_{\tau(n)}\phi_{\tau(n)} + \sigma_{\tau(n)})M\\
&= -\psi_{\tau(n)}\|x_{\tau(n)+1} - p\| + \psi_{\tau(n)} b_{\tau(n)} + (\tau_{\tau(n)}\phi_{\tau(n)} + \sigma_{\tau(n)})M,
\end{aligned}$$

which implies that

$$\|x_{\tau(n)+1} - p\| \leq b_{\tau(n)} + \frac{(\tau_{\tau(n)}\phi_{\tau(n)} + \sigma_{\tau(n)})M}{\psi_{\tau(n)}}. \tag{6.34}$$

Hence by (6.33) and conditions (A2) and (A5), taking the limit of (6.34), we get

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - p\| = 0.$$

Consequently, we have that

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\| = 0.$$

It therefore follows by Lemma 2.5.52 that

$$0 \leq \|x_n - p\| \leq \max\{\|x_n - p\|, \ \|x_{\tau(n)} - p\|\} \leq \|x_{\tau(n)+1} - p\|.$$

Thus $\lim_{n\to\infty} \|x_n - p\| = 0$ and therefore, we conclude that $\{x_n\}$ converges strongly to p. We have proved that in both cases, the sequence $\{x_n\}$ converges strongly to $p \in \Omega$, where p is the unique solution of the variational inequality problem (6.9). $\square$

We now present the consequences of our main result.

Corollary 6.2.5. *Let H_1 and H_2 be two real Hilbert spaces. $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be uniformly L_1-Lipschitzian κ-strictly asymptotically demicontractive mapping with sequence $\{\kappa_n\} \in [1, \infty)$ and uniformly L_2-Lipschitzian β-strictly asymptotically demicontractive mapping with sequence $\{\beta_n\} \in [1, \infty)$, respectively. Assuming $F(U) \neq \emptyset$, $F(T) \neq \emptyset$ and $I - U$ and $I - T$ are demiclosed at 0. Let $u \in H_1$ be arbitrary and suppose*

$$\Omega := \{x^* \in H_1 : x^* \in F(U) \ \ \text{and} \ \ Ax^* \in F(T)\} \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by $x_1 \in H_1$ and

$$x_{n+1} = U_\lambda^n \left(\alpha_n x_n + \gamma_n u + \delta_n \big(x_n - \rho A^*(I - T^n)Ax_n\big)\right), \quad n \geq 1, \tag{6.35}$$

where

$$U_\lambda^n = (1 - \lambda_n)I + \lambda_n U^n$$

and the conditions (A1) - (A5) are satisfied. Then, the sequence $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = P_\Omega u$.

Corollary 6.2.6. *Let H_1 and H_2 be two real Hilbert spaces. $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be uniformly L_1-Lipschitzian κ-strictly asymptotically demicontractive mapping with sequence $\{\kappa_n\} \in [1, \infty)$ and uniformly L_2-Lipschitzian β-strictly asymptotically demicontractive mapping with sequence $\{\beta_n\} \in [1, \infty)$, respectively. Assuming $F(U) \neq \emptyset$, $F(T) \neq \emptyset$ and $I - U$ and $I - T$ are demiclosed at 0. Let $f : H_1 \to H_1$ be a contraction mapping with constant $v \in [0, \frac{1}{\sqrt{2}})$ and suppose*

$$\Omega := \{x^* \in H_1 : x^* \in F(U) \ \ \text{and} \ \ Ax^* \in F(T)\} \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by $x_1 \in H_1$ and

$$x_{n+1} = U_\lambda^n \left(\gamma_n f(x_n) + (1 - \gamma_n)\big(x_n - \rho A^*(I - T^n)Ax_n\big)\right), \quad n \geq 1, \tag{6.36}$$

where

$$U_\lambda^n = (1 - \lambda_n)I + \lambda_n U^n$$

and the conditions (A1) - (A5) are satisfied. Then, the sequence $\{x_n\}$ generated by (6.36) converges strongly to $p \in \Omega$, where p is the unique solution of the variational inequality problem: Find $p \in \Omega$ such that

$$\langle (I - f)p, x - p\rangle \geq 0 \ \ \forall \ x \in \Omega.$$

Proof. The proof follows from the proof of Theorem 6.2.4 by taking $\alpha_n = 0$ for all $n \in \mathbb{N}$. □

Following similar procedure as in the proof of Theorem (6.2.4), we obtain the following corollaries:

Corollary 6.2.7. *Let H_1 and H_2 be two real Hilbert spaces. $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ κ-demicontractive mapping β- demicontractive mapping, respectively. Assuming $F(U) \neq \emptyset$, $F(T) \neq \emptyset$ and $I - U$ and $I - T$ are demiclosed at 0. Let $f : H_1 \to H_1$ be a contraction mapping with constant $v \in [0, \frac{1}{\sqrt{2}})$ and suppose*

$$\Omega_1 := \{x^* \in H_1 : x^* \in F(U) \ \text{ and } \ Ax^* \in F(T)\} \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by $x_1 \in H_1$ and

$$x_{n+1} = U_\lambda \left(\alpha_n x_n + \gamma_n f(x_n) + \delta_n\big(x_n - \rho A^*(I - T)Ax_n\big)\right), \quad n \geq 1, \tag{6.37}$$

where

$$U_\lambda = (1 - \lambda_n)I + \lambda_n U$$

and the conditions (A1) - (A5) are satisfied. Then, the sequence $\{x_n\}$ generated by (6.37) converges strongly to $p \in \Omega_1$, where p is the unique solution of the variational inequality problem: Find $p \in \Omega_1$ such that

$$\langle (I - f)p, x - p\rangle \geq 0 \quad \forall\ x \in \Omega_1.$$

6.2.2 Numerical examples

In this section, we demonstrate the efficiency and applicability of our proposed method with numerical example in the framework of an infinite dimensional Hilbert spaces and give numerical comparison with Algorithm (9.1.17).

Example 6.2.8. *Let $H_1 = \{x \in \ell_2 : \|x\| \leq 1\}$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \dots, x_n, \dots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$, $\|x\| = (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}\ \forall\ x \in \ell_2(\mathbb{R})$ and $E_2 = (\ell_2(\mathbb{R}), \|\cdot\|_2)$. We define $U : H_1 \to H_1$ by*

$$Ux = (0, x_1^2, a_2x_2, a_3x_3, \dots),\ \forall x \in H_1$$

where $\{a_j\}_{j=1}^{\infty}$ is a real sequence satisfying: $a_2 > 0, 0 < a_j < 1, j \neq 2$, and $\prod_{j=2}^{\infty} a_j = \frac{1}{2}$, $x \in \ell_2(\mathbb{R})$, $T : H_2 \to H_2$ by $Tx = \frac{1}{2}x, \forall x \in H_2$ and $A : H_1 \to H_2$ by $Ax = 2x$. Then T is 0- asymptotically demicontrative with sequence $\{1 + \frac{1}{2^n}\}$ and U is 0-asymptotically demicontractive. In the case $F(U) = F(T) = \{(0, 0, 0, \dots)\}$ and $\Omega = \{(0, 0, 0, \dots)\}$. We choose $a_j = \frac{1}{4}(\frac{(1+j)^2}{j^2+2j})$, for $j \geq 2$, $\delta_n = \alpha_n = \frac{n}{2(n+1)}$, $\gamma_n = \frac{1}{n+1}$, $\rho = 0.05$, $\lambda_n = 0.85$ and we define $f : E_1 \to E_1$ by $f(x) = \frac{1}{3}x$, for all $x \in E_1$. We make different choices of initial values x_1 as follows:
Case I: $x_1 = (\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots)$;
Case II: $x_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, \dots)$;
Case III: $x_1 = (\frac{1}{5}, -\frac{1}{10}, \frac{1}{20}, \dots)$;
Case IV: $x_1 = (-\frac{1}{2}, \frac{1}{6}, -\frac{1}{18}, \dots)$.

Using MATLAB 2017(b), we use the various choices above to test the efficacy and computability of our iterative scheme and compare with Algorithm (9.1.17). The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-7}$. We plot the graphs of errors against the number of iterations in each case. The figures and numerical results are shown in Figure 6.1 and Table 6.1.15, respectively.

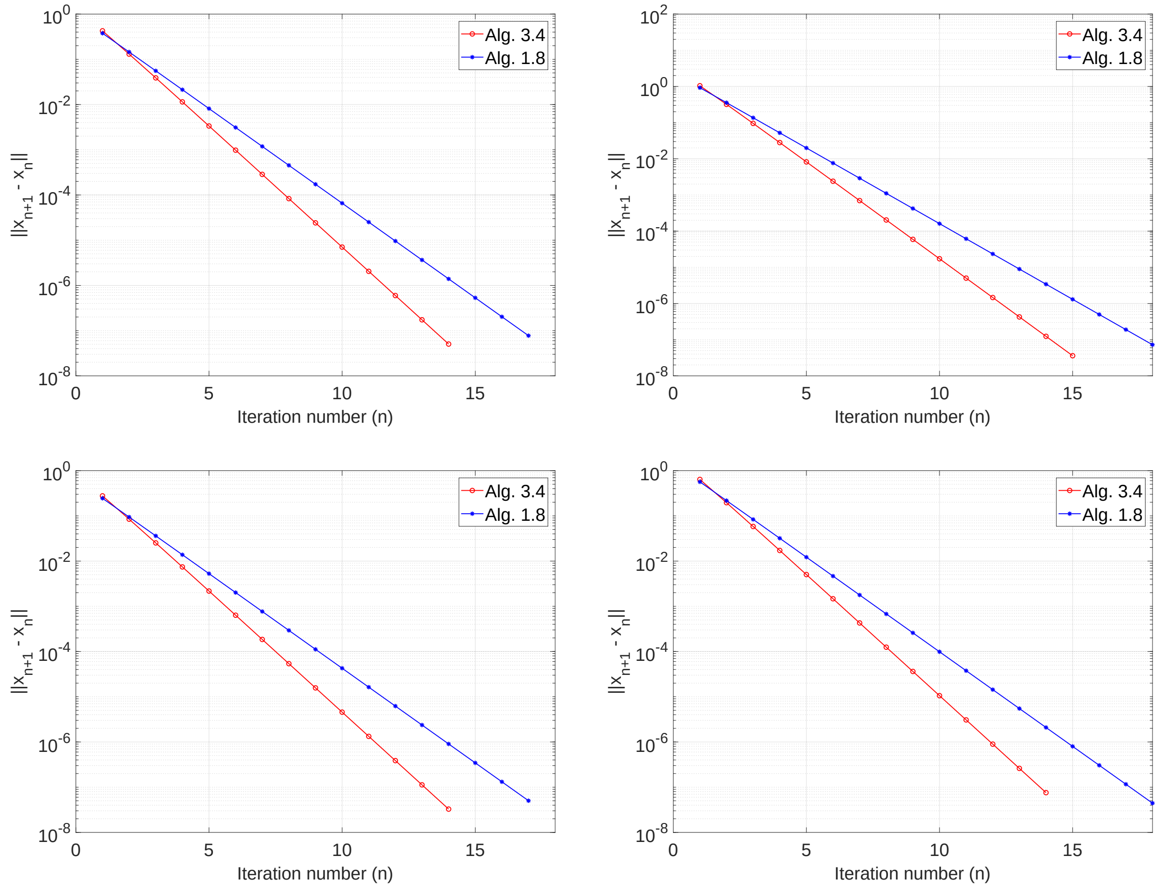


Figure 6.1: Example 6.2.8: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

Table 6.1.15. Numerical results for Example 6.2.8.

		Algorithm (6.36)	Algorithm (9.1.17)
Case I	CPU time (sec)	0.0017	0.0018
	No. of Iter.	14	17
Case II	CPU time (sec)	0.0026	0.0027
	No. of Iter.	15	18
Case III	CPU time (sec)	0.0025	0.0026
	No. of Iter.	14	17
Case IV	CPU time (sec)	0.0016	0.0017
	No. of Iter.	14	18

6.3 Split minimization problem with multiple output sets and common fixed point problem.

We consider the problem of approximating a common solution of the Split Convex Minimization Problems With Multiple Output Sets (SCMPWMOS) and fixed point problem for a finite family of Bregman relatively nonexpansive mappings $S_j, j = 1, 2, ...m$, in the framework of p-uniformly convex Banach spaces which are also uniformly smooth. That is, find an element x^* such that

$$x^* \in \bigcap_{j=1}^{m} \mathrm{Fix}(S_j) \cap \arg\min f \cap \left(\bigcap_{i=1}^{N} T_i^{-1}\big(\arg\min f_i\big) \right), \tag{6.38}$$

We denote by Γ the solution set of problem (6.38).
Let E be a p-uniformly convex uniformly smooth Banach space and $f : E \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. The proximal operator $\mathrm{Prox}_{\lambda f} : E \to E$ associated with f with respect to the Bregman distance is defined as

$$\mathrm{Prox}_{\lambda} f(x) := \arg\min_{u \in E} \left[f(u) + \frac{1}{\lambda} \Delta_p(u, x) \right], \quad \forall\, u \in E.$$

Proximal operators have very interesting properties which are suitable for solving minimization problems. Take for example, $\mathrm{Prox}_{\lambda} f$ is firmly nonexpansive; that is,

$$\|\mathrm{Prox}_{\lambda} f(x) - \mathrm{Prox}_{\lambda} f(y)\|^2 \le \|x - y\|^2 - \|(x - \mathrm{Prox}_{\lambda} f(x)) - (y - \mathrm{Prox}_{\lambda} f(x))\|^2, \quad \forall\, x, y \in E.$$

On the other hand, the set of fixed points of $\mathrm{Prox}_{\lambda} f(x)$ is the set of minimizers of f, (see [25]) for more properties of the proximal operators and the references contained therein. Furthermore, we note from Bauschke et al. [25] that $\mathrm{dom}\, \mathrm{Prox}_{\lambda} f \subset \mathrm{int}\, \mathrm{dom} g$ and $\mathrm{ran}\, \mathrm{Prox}_{\lambda} f \subset \mathrm{dom} g \cap \mathrm{dom} f$, where $g(x) = \frac{1}{p}\|x\|^p$ and it is convex and Gâteaux differentiable. Also, if $\mathrm{ran}\, \mathrm{Prox}_{\lambda} f \subset \mathrm{int}\, \mathrm{dom} g$, then $\mathrm{Prox}_{\lambda} f$ is Bregman firmly nonexpansive and single-valued on its domain if int dom$g$ is strictly convex.

We have the following result from the work of Aoyoma et al. [16]:

$$\left\langle \mathrm{Prox}_{\lambda}^{f}(x) - x^*, J\left(x - \mathrm{Prox}_{\lambda}^{f}(x)\right) \right\rangle \geq 0, \quad \forall x \in E, \quad x^* \in \arg\min f. \tag{6.39}$$

Lemma 6.3.1. *[39] Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Let $f : E \to (-\infty, +\infty]$ be lower semi-continuous and convex function such that $dom(g) \cap dom(f) \neq \emptyset$ and $ran(Prox_{\lambda}^{f}) \subset int(domg)$. For all $x \in E$, $u \in F(Prox_{\lambda}^{g})$ and $\lambda > 0$, then, we have the following*

$$\Delta_g(u, Prox_{\lambda}^{f}(x)) + \Delta_g(Prox_{\lambda}^{f}(x), x) \leq \Delta_g(u, x).$$

6.3.1 Main result

In this section, we present our proposed method and highlight its features. We begin with the following assumptions under which our strong convergence result is obtained.

Assumption 6.3.2.

1. *E_i, $i = 0, 1, 2, ..., N$ (where $E_0 = E$) are p-uniformly convex real Banach spaces which are also uniformly smooth.*
2. *$T_i : E \to E_i$, $i = 0, 1, 2, .., N$ (where $T_0 = I^E$) are bounded linear operators.*
3. *$f : E \to (-\infty, +\infty]$ and $f_i : E_i \to (-\infty, +\infty]$, $i = 0, 1, 2, ...N$ (where $f_0 = f$) are proper convex and lower semi-continuous functions.*
4. *$S_j : E \to E$, for $j = 1, 2, \ldots, m$ are Bregman relatively nonexpansive mappings.*
5. $\Gamma := \{x^* \in \bigcap_{j=1}^{m} Fix(S_j) \cap \arg\min f \cap \bigcap_{i=1}^{N} T_i^{-1} (\arg\min f_i)\} \neq \emptyset$.

   Let $\{\alpha_n\}$ and $\{\beta_{i,n}\}$ be positive sequences satisfying the following conditions:

   (i) $\{\alpha_n\} \subset (0, 1)$ *such that* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim\limits_{n\to\infty} \alpha_n = 0$, $\beta_{i,n} \subset [a, b] \subset (0, 1)$ *such that* $\sum\limits_{i=0}^{N} \beta_{i,n} = 1$;

   (ii) Let $\{\epsilon_n\}$ be a positive sequence such that $\epsilon_n = \circ(\alpha_n)$, that is, $\lim\limits_{n\to\infty} \dfrac{\epsilon_n}{\alpha_n} = 0$,

   (iii) $\theta > 0$ and for $i = 0, 1, \cdots, N$, let λ_n^i be such that $\min\limits_{i=0,1,\cdots,N} \{\inf\limits_{n} \lambda_n^i\} = \lambda > 0$,

   (iv) $\{\phi_{n,j}\} \in (0, 1)$, $\sum_{j=0}^{m} \phi_{n,j} = 1$ *and* $\liminf_{n\to\infty} \phi_{n,0}\phi_{n,j} > 0$ *for each j.*

Algorithm 6.3.3. *Calculation of the sequence* $\{x_n\}$.

Initialization: *Let* $x_0, x_1 \in E$, *choose* θ_n *such that* $\theta_n \in [0, \bar{\theta}_n]$ *where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \dfrac{\epsilon_n}{\|J_E^P(x_n) - J_E^P(x_{n-1})\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{6.40}$$

Step 0: *For* $x_0 \in E$, *let* $E_0 = E, T_0 = I^E, Prox_{\lambda_n^0}^{f_0} = Prox^f$ *and set* $n = 1$.

Iterative steps: *Given iterates* $x_{n-1},\ x_n$, *compute* $\{x_n\}$ *as follows:*

$$\begin{cases} w_n = J_{E^*}^q\Big[J_E^p(x_n) + \theta_n\big(J_E^p(x_{n-1}) - J_E^p(x_n)\big)\Big] \\ y_n = J_{E^*}^q\Bigg[\sum_{i=0}^N \beta_{i,n}\Bigg(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p\big(I^{E_i} - Prox_{\lambda_n^i}^{f_i}\big)T_i(w_n)\Bigg)\Bigg] \\ z_n = J_{E^*}^q\Big(\phi_{n,0}J_E^p(y_n) + \sum_{j=1}^m \phi_{n,j}J_E^p(S_j y_n)\Big) \\ x_{n+1} = J_{E^*}^q\Bigg(\alpha_n J_E^p(u) + (1-\alpha_n)J_E^p(z_n)\Bigg), \end{cases} \tag{6.41}$$

Suppose that for $\epsilon > 0$, *we choose the stepsize* $\tau_{i,n}$ *in such a way that*

$$\tau_{i,n} \in \left(\epsilon, \left(\frac{q\|T_i(w_n) - \big(Prox_{\lambda_n^i}^{f_i}\big)T_i(w_n)\|^p}{C_q\|T_i^* J_{E_i}^p\big(I^{E_i} - Prox_{\lambda_n^i}^{f_i}\big)T_i(w_n)\|^q} - \epsilon\right)^{\frac{1}{q-1}}\right), \quad \forall \ \ n \in \Omega, \tag{6.42}$$

for small enough ϵ; *where the index set* $\Omega := \{n \in \mathbb{N} : T_i(w_n) - \big(Prox_{\lambda_n^i}^{f_i} T_i(w_n) \neq 0\big)\}$, *otherwise,* $\tau_{i,n} = \tau_i$, τ_i *is any nonnegative real number for each* $i = 0, 1, \ldots, N$.

We now highlight some of the features of our proposed algorithm.

Remark 6.3.4.

- *The step size* $\{\tau_{i,n}\}$ *given by* (6.42) *is generated at each iteration by some simple computation. Thus, Algorithm* 6.3.3 *is easily implemented without prior knowledge of the operators norm.*
- *Also, the inertial technique employed is easily implemented since the value of* $\|x_n - x_{n-1}\|$ *is a priori known before choosing* α_n.

Remark 6.3.5. *By conditions (i) and (ii), it can be verified from* (6.40) *that*

$$\lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \theta_n\|J_E^P(x_n) - J_E^P(x_{n-1})\| = 0. \tag{6.43}$$

Lemma 6.3.6. *Let* $\{x_n\}$ *be a sequence generated by Algorithm* 6.3.3 *such that* $\{x_n\}$ *is bounded. Suppose conditions (i) and (ii) hold, then for all* $x^* \in \Gamma$, *we have:*

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\Big(\triangle_p(x_{n-1}, x^*) - \triangle_p(x_n, x^*)\Big) = 0.$$

Proof. Let $x^* \in \Gamma$. Observe that

$$\begin{aligned}
\Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*) &= \frac{1}{q}\|x_{n-1}\|^p - \langle J_E^P(x_{n-1}), x^*\rangle + \frac{1}{p}\|x^*\|^p \\
&\quad - \Big(\frac{1}{q}\|x_n\|^p - \langle J_E^P(x_n), x^*\rangle + \frac{1}{p}\|x^*\|^p\Big) \\
&= \frac{1}{q}\Big(\|x_{n-1}\|^p - \|x_n\|^p\Big) + \langle J_E^P(x_n) - J_E^P(x_{n-1}), x^*\rangle \\
&\leq \frac{1}{q}M\|x_{n-1} - x_n\| + \|J_E^P(x_n) - J_E^P(x_{n-1})\|\|x^*\|, \qquad (6.44)
\end{aligned}$$

for some constant $M > 0$.

Since $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, then it follows from Remark 6.3.5 that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0. \qquad (6.45)$$

Similarly, we have that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|J_E^P(x_n) - J_E^P(x_{n-1})\| = 0. \qquad (6.46)$$

Applying (6.45) and (6.46), it follows from (6.44) that

$$\begin{aligned}
\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\Big(\Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*)\Big) &\leq \lim_{n\to\infty} \Big(\frac{M}{q} \cdot \frac{\theta_n}{\alpha_n}\|x_{n-1} - x_n\| \\
&\quad + \|x^*\|\frac{\theta_n}{\alpha_n}\|J_E^P(x_n) - J_E^P(x_{n-1})\|\Big) \\
&= 0,
\end{aligned}$$

which is the required result.

□

Lemma 6.3.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 6.3.3 such that Assumption 6.3.2 holds. Then, $\{x_n\}$ is bounded.*

Proof. Let $x^* \in \Gamma$. Then we have from (6.3.3)

$$\begin{aligned}
\Delta_p(w_n, x^*) &= \Delta_p(J_{E^*}^q(J_E^p x_n + \theta_n(J_E^p x_{n-1} - J_E^p x_n), x^*) \\
&\leq (1 - \theta_n)\Delta_p(x_n, x^*) + \theta_n\Delta_p(x_{n-1}, x^*). \qquad (6.47)
\end{aligned}$$

Also, we obtain from (6.41) that

$$\Delta_p(y_n, x^*) = \Delta_p\left(J_{E^*}^q\left(\sum_{i=0}^N \beta_{i,n}(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n))\right), x^*\right)$$
$$= V_p\left(\sum_{i=0}^N \beta_{i,n}(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)), x^*\right)$$
$$= \frac{1}{p}\|x^*\|^p - \langle\sum_{i=0}^N \beta_{i,n}(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)), x^*\rangle$$
$$+ \frac{1}{q}\|\sum_{i=0}^N \beta_{i,n}(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n))\|^q$$
$$= \frac{1}{p}\|x^*\|^p - \langle J_E^p(w_n), x^*\rangle + \sum_{i=0}^N \beta_{i,n}\tau_{i,n}\langle T_i x^*, J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle$$
$$+ \frac{1}{q}\|\sum_{i=0}^N \beta_{i,n}(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n))\|^q. \tag{6.48}$$

By the convexity property of Δ_p, we have that

$$\|\sum_{i=0}^N \beta_{i,n}(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n))\|^q \leq \sum_{i=0}^N \beta_{i,n}\|J_E^p(w_n)$$
$$- \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q.$$

Thus, by applying Lemma 2.55, we obtain

$$\|J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q \leq \|J_E^p w_n\|^q$$
$$- q\tau_{i,n}\langle w_n, T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle$$
$$+ C_q\tau_{i,n}^q\|T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q$$
$$= \|w_n\|^q - q\tau_{i,n}\langle T_i w_n, J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle$$
$$+ C_q\tau_{i,n}^q\|T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q.$$

Substituting this into (6.48), we obtain

$$\Delta_p(y_n, x^*) = \frac{1}{p}\|x^*\|^p - \langle J_E^p(w_n), x^*\rangle + \sum_{i=0}^N \beta_{i,n}\tau_{i,n}\langle T_i(x^*), J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle + \frac{1}{q}\|w_n\|^q$$
$$- \sum_{i=0}^N \beta_{i,n}\tau_{i,n}\langle T_i(w_n), J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle + \sum_{i=0}^N \beta_{i,n}\frac{C_q\tau_{i,n}^q}{q}\|T_i^*(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q$$
$$= \Delta_p(w_n, x^*) + \sum_{i=0}^N \beta_{i,n}\tau_{i,n}\langle T_i(x^*) - T_i(w_n), J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle$$
$$+ \sum_{i=0}^N \beta_{i,n}\frac{C_q\tau_{i,n}^q}{q}\|T_i^*(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q. \tag{6.49}$$

By applying (6.39), we obtain

$$\begin{aligned}\langle T_i(x^*) - T_i(w_n), J^p_{E_i}(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\rangle &= \langle T_i(x^*) - T_i(w_n) - \mathrm{Prox}^{f_i}_{\lambda^i_n} T_i(w_n) \\ &\quad + \mathrm{Prox}^{f_i}_{\lambda^i_n} T_i(w_n), J^p_{E_i}(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\rangle \\ &= -\|(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\|^p \\ &\quad + \langle T_i(x^*) - \mathrm{Prox}^{f_i}_{\lambda^i_n} T_i(w_n), J^p_{E_i}(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\rangle \\ &\leq -\|(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\|^p. \qquad (6.50)\end{aligned}$$

Substituting (6.50) into (6.49) and applying (6.42) , we obtain

$$\begin{aligned}\Delta_p(y_n, x^*) \leq \Delta_p(w_n, x^*) - \sum_{i=0}^{N} \beta_{i,n}\tau_{i,n}\Bigg(&\|(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\|^p \\ - \frac{C_q \tau^{q-1}_{i,n}}{q}&\|T_i^* J^p_{E_i}(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\|^q\Bigg) \qquad (6.51) \\ \leq \Delta_p(w_n, x^*). & \qquad (6.52)\end{aligned}$$

Also, from Algorithm 6.3.3 and by applying Lemma 2.5.43 together with the Bregman relative nonexpansivity of S_j for each $j = 1, 2, \dots, m$, we obtain

$$\begin{aligned}\Delta_p(z_n, x^*) &= \Delta_p\left(J^q_{E^*}\left(\phi_{n,0} J^p_E(y_n) + \sum_{j=1}^{m} \phi_{n,j} J^p_E(S_j y_n)\right), x^*\right) \\ &\leq \phi_{n,0}\Delta_p(y_n, x^*) + \sum_{j=1}^{m} \phi_{n,j}\Delta_p(S_j y_n, x^*) - \phi_{n,0}\sum_{j=1}^{m} \phi_{n,j} g^*_r(\|J^p_E(y_n) - J^p_E(S_j y_n)\|) \\ &\leq \phi_{n,0}\Delta_p(y_n, x^*) + \sum_{j=1}^{m} \phi_{n,j}\Delta_p(y_n, x^*) - \phi_{n,0}\sum_{j=1}^{m} \phi_{n,j} g^*_r(\|J^p_E(y_n) - J^p_E(S_j y_n)\|) \\ &= \Delta_p(y_n, x^*) - \phi_{n,0}\sum_{j=1}^{m} \phi_{n,j} g^*_r(\|J^p_E(y_n) - J^p_E(S_j y_n)\|) \qquad (6.53) \\ &\leq \Delta_p(y_n, x^*). \qquad (6.54)\end{aligned}$$

Futhermore, from Algorithm 6.3.3, (6.47), (6.51), (6.52) and (6.54), we have

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) &= \Delta_p\Big(J_{E^*}^q\big(\alpha_n J_E^p(u) + (1-\alpha_n)J_E^p(z_n)\big), x^*\Big) \\
&\leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(z_n, x^*) \\
&\leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(y_n, x^*) \\
&\leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(w_n, x^*) \\
&\leq \alpha_n \Delta_p(u, x^*) + (1-\alpha_n)[(1-\theta_n)\Delta_p(x_n, x^*) + \theta_n \Delta_p(x_{n-1}, x^*)] \\
&\leq \max\{\Delta_p(u, x^*), \max\{\Delta_p(x_n, x^*), \Delta_p(x_{n-1}, x^*)\}\} \\
&\ \ \vdots \\
&\leq \max\{\Delta_p(u, x^*), \max\{\Delta_p(x_1, x^*), \Delta_p(x_0, x^*)\}\} < \infty.
\end{aligned}$$

Thus, $\{\Delta_p(x_n, x^*)\}$ is bounded. Consequently, $\{\Delta_p(w_n, x^*)\}$, $\{\Delta_p(y_n, x^*)\}$ and $\{\Delta_p(z_n, y^*)\}$ are bounded. Therefore, it follows from Lemma 2.5.31 that $\{w_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are all bounded. □

We now prove the strong convergence theorem for the proposed algorithm.

Theorem 6.3.8. *Let $\{x_n\}$ be a sequence generated by Algorithm 6.3.3 under Assumption 6.3.2. Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Gamma$, where $x^* = \Pi_\Gamma u$.*

Proof. Let $x^* \in \Gamma$. Then from Algorithm 6.3.3, (6.51), (6.53) and Lemma 2.5.45(iii), we have

$$
\begin{aligned}
\Delta_p(x_{n+1}, x^*) &= \Delta_p(J^q_{E^*}(\alpha_n J^p_E(u) + (1-\alpha_n)J^p_E(z_n)), x^*) \\
&= V_p(\alpha_n J^p_E(u) + (1-\alpha_n)J^p_E(z_n), x^*) \\
&\leq V_p(\alpha_n J^p_E(u) + (1-\alpha_n)J^p_E(z_n) - \alpha_n(J^p_E(u) - J^p_E(x^*)), x^*) \\
&+ \alpha_n\langle J^p_E(u) - J^p_E(x^*), x_{n+1} - x^*\rangle \\
&= V_p(\alpha_n J^p_E(x^*) + (1-\alpha_n)J^p_E(z_n), x^*) + \alpha_n\langle J^p_E(u) - J^p_E(x^*), x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)V_p(J^p_E(z_n), x^*) + \alpha_n\langle J^p_E(u) - J^p_E(x^*), x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)\Delta_p(z_n, x^*) + \alpha_n\langle J^p_E(u) - J^p_E(x^*), x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\left[\Delta_p(y_n, x^*) - \phi_{n,0}\sum_{j=1}^m \phi_{n,j} g^*_r(\|J^p_E(y_n) - J^p_E(S_j y_n)\|)\right] \\
&+ \alpha_n\langle J^p_E(u) - J^p_E(x^*), x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\Delta_p(w_n, x^*) - (1-\alpha_n)\sum_{i=0}^N \beta_{i,n}\tau_{i,n}\Bigg(\|(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\|^p \\
&- \frac{C_q\tau^{q-1}_{i,n}}{q}\|T^*_i J^p_{E_i}(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\|^q\Bigg) \\
&- (1-\alpha_n)\phi_{n,0}\sum_{j=1}^m \phi_{n,j} g^*_r(\|J^p_E(y_n) - J^p_E(S_j y_n)\|) \\
&+ \alpha_n\langle J^p_E(u) - J^p_E(x^*), x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)[(1-\theta_n)\Delta_p(x_n, x^*) + \theta_n\Delta_p(x_{n-1}, x^*)] \\
&+ \alpha_n\langle J^p_E(u) - J^p_E(x^*), x_{n+1} - x^*\rangle \\
&- (1-\alpha_n)\sum_{i=0}^N \beta_{i,n}\tau_{i,n}\Bigg(\|(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\|^p \\
&- \frac{C_q\tau^{q-1}_{i,n}}{q}\|T^*_i J^p_{E_i}(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\|^q\Bigg) \\
&- (1-\alpha_n)\phi_{n,0}\sum_{j=1}^m \phi_{n,j} g^*_r(\|J^p_E(y_n) - J^p_E(S_j y_n)\|) \\
&\leq (1-\alpha_n)\Delta_p(x_n, x^*) - (1-\alpha_n)\sum_{i=0}^N \beta_{i,n}\tau_{i,n}\Bigg(\|(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\|^p \\
&- \frac{C_q\tau^{q-1}_{i,n}}{q}\|T^*_i J^p_{E_i}(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_n})T_i(w_n)\|^q\Bigg) \\
&- (1-\alpha_n)\phi_{n,0}\sum_{j=1}^m \phi_{n,j} g^*_r(\|J^p_E(y_n) - J^p_E(S_j y_n)\|) + \alpha_n\psi_n, \qquad (6.55)
\end{aligned}
$$

where

$$
\psi_n := \left(\frac{\theta_n}{\alpha_n}(\Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*)) + \langle J^p_E(u) - J^p_E(x^*), x_{n+1} - x^*\rangle\right).
$$

Thus, it follows from (6.55), that

$$(1-\alpha_n)\sum_{i=0}^{N}\beta_{i,n}\tau_{i,n}\Bigg(\|(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^p-\frac{C_q\tau_{i,n}^{q-1}}{q}\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q\Bigg)$$
$$\leq \Delta_p(x_n,x^*)-\Delta_p(x_{n+1},x^*)+\alpha_n\psi_n. \tag{6.56}$$

Similarly, from (6.55) we obtain

$$(1-\alpha_n)\phi_{n,0}\sum_{j=1}^{m}\phi_{n,j}g_r^*(\|J_E^p(y_n)-J_E^p(S_jy_n)\|)\leq \Delta_p(x_n,x^*)-\Delta_p(x_{n+1},x^*)+\alpha_n\psi_n. \tag{6.57}$$

Furthermore, from (6.55) we obtain

$$\Delta_p(x_{n+1},x^*)\leq(1-\alpha_n)\Delta_p(x_n,x^*)+\alpha_n\psi_n. \tag{6.58}$$

We now show that $\{x_n\}$ converges strongly to x^*. Let $a_n:=\Delta_p(x_n,x^*)$. It is easy to see that (6.58) satisfies the inequality

$$a_{n+1}\leq(1-\alpha_n)a_n+\alpha_n\psi_n.$$

Using Lemma 2.5.55, it suffices to show that $\limsup_{k\to\infty}\psi_{n_k}\leq 0$ (where $\{\psi_{n_k}\}$ is a subsequence of $\{\psi_n\}$), for every subsequence $\{\Delta_p(x_{n_k},x^*)\}$ of $\{\Delta_p(x_n,x^*)\}$ satisfying the condition

$$\liminf_{k\to\infty}\Bigg(\Delta_p(x_{n_k+1},x^*)-\Delta_p(x_{n_k},x^*)\Bigg)\geq 0. \tag{6.59}$$

Now, suppose that $\{\Delta_p(x_{n_k},x^*)\}$ is a subsequence of $\{\Delta_p(x_n,x^*)\}$ such that (6.59) holds. Then using (6.56), (6.59) and condition (i), we have that

$$\begin{aligned}\limsup_{k\to\infty}(1-\alpha_{n_k})\sum_{i=0}^{N}\beta_{i,n_k}\tau_{i,n_k}&\Bigg(\|(I^{E_i}-\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^p\\
&-\frac{C_q\tau_{i,n_k}^{q-1}}{q}\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^q\Bigg)\\
&\leq\limsup_{k\to\infty}\Bigg(\Delta_p(x_{n_k},x^*)-\Delta_p(x_{n_k+1},x^*)+\alpha_{n_k}\psi_{n_k}\Bigg)\\
&=\limsup_{k\to\infty}\Bigg(\Delta_p(x_{n_k},x^*)-\Delta_p(x_{n_k+1},x^*)\Bigg)\\
&\leq-\liminf_{k\to\infty}\Bigg(\Delta_p(x_{n_k+1},x^*)-\Delta_p(x_{n_k},x^*)\Bigg)\\
&\leq 0.\end{aligned} \tag{6.60}$$

Observing that by the choice of our stepsize τ_{i,n_k}, it follows that

$$\tau_{i,n_k}^{q-1} < \frac{q\|T_i(w_{n_k}) - (\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^p}{C_q\|T_i^* J_{E_i}^p(I^{E_i} - (\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k}))\|^q} - \epsilon, \tag{6.61}$$

which implies from (6.61) that

$$\begin{aligned}\frac{\epsilon C_q}{q}\|T_i^* J_{E_i}^p(I^{E_i} - (\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k}))\|^q < \Bigg(&\|T_i(w_{n_k}) - \mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^p \\ &- \frac{C_q\tau_{i,n_k}^{q-1}}{q}\|T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^q\Bigg).\end{aligned} \tag{6.62}$$

Thus by passing the limit as $k \to \infty$, from (6.60) and (6.62), we obtain

$$\lim_{k\to\infty} \|T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^q = 0, \quad \forall\ i = 0, 1, 2, ..., N. \tag{6.63}$$

Similarly, from (6.60), (6.61) and condition (i) of Assumption 6.3.2, we obtain that

$$\lim_{k\to\infty} \|T_i(w_{n_k}) - (\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^p = 0, \quad \forall\ i = 0, 1, 2, ..., N. \tag{6.64}$$

Also, we obtain from (6.57) and (6.59) that

$$\begin{aligned}(1-\alpha_{n_k})\phi_{n_k,0} \limsup_{k\to\infty} &\sum_{j=1}^{m} \phi_{n_k,j} g_r^*(\|J_E^p(y_{n_k}) - J_E^p(S_j y_{n_k})\|) \\ &\le \limsup_{k\to\infty}\Big(\Delta_p(x_{n_k}, x^*) - \Delta_p(x_{n_k+1}, x^*) + \alpha_{n_k}\psi_{n_k}\Big) \\ &= \limsup_{k\to\infty}\Big(\Delta_p(x_{n_k}, x^*) - \Delta_p(x_{n_k+1}, x^*)\Big) \\ &= -\liminf_{k\to\infty}\Big(\Delta_p(x_{n_k+1}, x^*) - \Delta_p(x_{n_k}, x^*)\Big) \\ &\le 0.\end{aligned}$$

Thus, by conditions (i) and (iv), we have

$$(1-\alpha_{n_k})\phi_{n_k,0} \limsup_{k\to\infty} \sum_{j=1}^{m} \phi_{n_k,j} g_r^*(\|J_E^p(y_{n_k}) - J_E^p(S_j y_{n_k})\|) = 0.$$

Consequently, we obtain

$$\lim_{k\to\infty} g_r^*(\|J_E^p(y_{n_k}) - J_E^p(S_j y_{n_k})\|) = 0, \quad j = 1, 2, \dots, m.$$

By the properties of g_r^*, it follows that

$$\lim_{k\to\infty} \|J_E^p(y_{n_k}) - J_E^p(S_j y_{n_k})\| = 0, \quad j = 1, 2, \dots, m.$$

Since J_E^p is norm-to-norm uniformly continuous on bounded subsets of E, we obtain

$$\lim_{k\to\infty} \|y_{n_k} - S_j y_{n_k}\| = 0, \quad j = 1, 2, \dots, m. \tag{6.65}$$

Recall that $w_{n_k} = J_{E^*}^q\Big[J_E^p(x_{n_k}) + \theta_{n_k}\big(J_E^p(x_{n_k-1}) - J_E^p(x_{n_k})\big)\Big]$. It then follows from Remark 6.3.5 that

$$\lim_{k\to\infty} \|J_E^p(w_{n_k}) - J_E^p(x_{n_k})\| = \lim_{k\to\infty} \theta_{n_k}\|J_E^p(x_{n_k-1}) - J_E^p(x_{n_k})\| = 0. \tag{6.66}$$

By the uniform continuity of $J_{E^*}^q$ on bounded subsets of E^* and (6.66), we obtain that

$$\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0. \tag{6.67}$$

Also, from (6.41) and (6.63), we have

$$\|J_E^p(y_{n_k}) - J_E^p(w_{n_k})\| \leq \sum_{i=0}^{N} \beta_{i,n_k}\tau_{i,n_k}\|T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\| \to 0 \ \text{ as } \ k \to \infty, \tag{6.68}$$

for all $i = 0, 1, 2, ..., N$. By the uniform continuity of $J_{E^*}^q$ on bounded subsets of E^*, we obtain that

$$\lim_{k\to\infty} \|y_{n_k} - w_{n_k}\| = 0. \tag{6.69}$$

Thus, it is easy to see from (6.67) and (6.69) that

$$\lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| = 0. \tag{6.70}$$

Moreover, we get from condition (iv) and (6.65) that

$$\begin{aligned}\lim_{k\to\infty} \|J_E^p z_{n_k} - J_E^p y_{n_k}\| &\leq \lim_{k\to\infty} \Big(\phi_{n_k,0}\|J_E^p(y_{n_k}) - J_E^p(y_{n_k})\| + \sum_{j=1}^{m} \phi_{n_k,j}\|J_E^p(S_j y_{n_k}) - J_E^p(y_{n_k})\|\Big)\\ &= 0.\end{aligned}$$

It follows from the uniform continuity of $J_{E^*}^q$ on bounded subsets of E^* that

$$\lim_{k\to\infty} \|z_{n_k} - y_{n_k}\| = 0. \tag{6.71}$$

It is not difficult to see from (6.69) and (6.71) that

$$\lim_{k\to\infty} \|z_{n_k} - w_{n_k}\| \leq \lim_{k\to\infty} (\|z_{n_k} - y_{n_k}\| + \|y_{n_k} - w_{n_k}\|) = 0.$$

Similarly, from (6.70) and (6.71), we have

$$\lim_{k\to\infty} \|z_{n_k} - x_{n_k}\| \leq \lim_{k\to\infty} (\|z_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\|) = 0. \tag{6.72}$$

Also, from (6.3.3) we obtain

$$\begin{aligned}\lim_{k\to\infty} \|J_E^p(x_{n_k+1}) - J_E^p(z_{n_k})\| &= \lim_{k\to\infty} \|\alpha_{n_k} J_E^p(u) + (1-\alpha_{n_k}) J_E^p(z_{n_k}) - J_E^p(z_{n_k})\| \\ &\leq \alpha_{n_k} \|J_E^p(u) - J_E^p(z_{n_k})\| + (1-\alpha_{n_k}) \|J_E^p(z_{n_k}) - J_E^p(z_{n_k})\| \to 0,\ k \to \infty. \end{aligned} \tag{6.73}$$

Since $J_{E^*}^q$ is uniformly continuous on bounded subsets of E^*, we obtain

$$\lim_{k\to\infty} \|x_{n_k+1} - z_{n_k}\| = 0.$$

Applying this together with (6.72), we get

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{6.74}$$

We now show that $\lim\limits_{k\to\infty} \psi_{n_k} \leq 0$. To do this, we need to show that

$$\limsup_{k\to\infty} \langle J_E^p(u) - J_E^p(x^*), x_{n_k+1} - x^* \rangle \leq 0.$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ which converges weakly to $\bar{x} \in E$ such that

$$\limsup_{k\to\infty} \langle J_E^p(u) - J_E^p(x^*), x_{n_k+1} - x^* \rangle = \lim_{j\to\infty} \langle J_E^p(u) - J_E^p(x^*), x_{n_{k_j}+1} - x^* \rangle. \tag{6.75}$$

To complete the proof, we need to show that $w_\omega(x_n) \subset \Gamma$. Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $\bar{x} \in w_\omega(x_n)$ be an arbitrarily chosen element. Then, from (6.67) and (6.70), there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to $\bar{x} \in E$ and subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ which converges weakly to $\bar{x} \in E$. Also, $w_{n_{k_j}} \rightharpoonup \bar{x} \in E$. Since for every $i = 0, 1, 2, \cdots, N$, T_i is a bounded linear operator, we obtain $T_i w_{n_{j_k}} \rightharpoonup T_i\bar{x} \in E_i$ as $k \to \infty$. From (6.64), we obtain $T_i\bar{x} \in Fix(\mathrm{Prox}_{\lambda^i_{n_k}}^{f_i})$ for $i = 0, 1, 2, \cdots, N$, which implies that $\bar{x} \in arg\min f \cap \bigcap_{i=1}^N T_i^{-1} (\arg\min f_i)$. Also, from (6.65) and the fact that $\hat{Fix}(S_j) = Fix(S_j)$, for all $j = 1, 2, ..., m$, we have that $\bar{x} \in Fix(S_j)$ for all $j = 1, 2, \ldots, m$. This implies that $\bar{x} \in \bigcap_{j=1}^m Fix(S_j)$. Thus, $\bar{x} \in \Gamma$. Since $\bar{x} \in w_\omega(x_n)$ is an arbitrary element, then we have $w_\omega(x_n) \subset \Gamma$.

Since $x^* = \Pi_\Gamma u$, then from (2.19) and (6.74) we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\langle J_E^p(u) - J_E^p(x^*), x_{n_k+1} - x^*\rangle &\leq \limsup_{k\to\infty}\langle J_E^p(u) - J_E^p(x^*), x_{n_k} - x^*\rangle \\
&\quad + \limsup_{k\to\infty}\langle J_E^p(u) - J_E^p(x^*), x_{n_k+1} - x_{n_k}\rangle \\
&= \lim_{j\to\infty}\langle J_E^p(u) - J_E^p(x^*), x_{n_{k_j}} - x^*\rangle \\
&= \langle J_E^p(u) - J_E^p(x^*), \bar{x} - x^*\rangle \leq 0. \qquad (6.76)
\end{aligned}$$

By applying Lemma 2.5.55, Lemma 6.3.6 and (6.76) to (6.58), we conclude that $\Delta_p(x_n, x^*) \to 0$ as $n \to \infty$. Therefore, we conclude by Lemma 2.5.31 that $\{x_n\}$ converges strongly to x^*. □

We have the following corollary as a consequent result of our main result.

Let $E_i = H_i, i = 0, 1, \cdots, N$ be real Hilbert spaces. We obtain the following result for approximating a common solution of split minimization problem with multiple output sets and common fixed point problem for a finite family of quasi-nonexpansive mappings in real Hilbert spaces.

Corollary 6.3.9. *Let $E_i = H_i, i = 0, 1, \cdots, N$, with $H_0 = H$ be real Hilbert spaces, and T_i be bounded linear operators such that $T_i \neq 0$ with $T_0 = I^H$. Let $f : H \to (-\infty, +\infty]$, $f_i : H_i \to (-\infty, +\infty]$ be proper, convex and lower semi-continuous functions. Let $S_j : H \to H$, for $j = 1, 2, \ldots, m$ be quasi-nonexpansive mappings which are demiclosed at zero and such that $\Gamma := \{x^* \in \bigcap_{j=1}^m Fix(S_j) \cap \arg\min f \cap \bigcap_{i=1}^N T_i^{-1}(\ arg\min f_i)\} \neq \emptyset$. Suppose that other conditions of Assumption 6.3.2 are satisfied. Let $x_0,\ x_1 \in H$ and $\{x_n\}$ be a sequence generated as follows:*

$$\begin{cases}
w_n = x_n + \theta_n(x_n - x_{n-1}) \\
y_n = \sum\limits_{i=0}^{N} \beta_{i,n}(w_n - \tau_{i,n}T_i^*(I^{H_i} - Prox_{\lambda_n^i}^{f_i})T_i(w_n)) \\
z_n = \phi_{n,0}y_n + \sum_{j=1}^m \phi_{n,j}S_jy_n \\
x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \quad n \geq 1.
\end{cases}$$

Choose θ_n such that $\theta_n \in [0, \bar{\theta}_n]$ where

$$\bar{\theta}_n = \begin{cases}
\min\left\{\theta, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\
\theta, & \text{otherwise},
\end{cases} \qquad (6.77)$$

Suppose that for $\epsilon > 0$, the step size $\tau_{i,n}$ is chosen such that

$$\tau_{i,n} \in \left(\epsilon, \frac{2\|T_i(w_n) - \big(Prox_{\lambda_n^i}^{f_i}\big)T_i(w_n)\|^2}{\|T_i^*\big(I^{H_i} - Prox_{\lambda_n^i}^{f_i}\big)T_i(w_n)\|^2} - \epsilon\right) \quad \forall \ \ n \in \Omega, \qquad (6.78)$$

for small enough ϵ; where the index set $\Omega := \{n \in \mathbb{N} : T_i(w_n) - \big(Prox_{\lambda_n^i}^{f_i}T_i(w_n) \neq 0\big)\}$, otherwise, $\tau_{i,n} = \tau_i$, τ_i is any nonnegative real number for each $i = 0, 1, \ldots, N$. Then, the sequence $\{x_n\}$ converges strongly to $x^* = P_\Gamma u$, where $P_\Gamma : H \to \Gamma$ is the metric projection of H onto Γ.

6.3.2 Applications

Equilibrium Problem

Let C be a nonempty closed and convex subset of the Banach space E and $g : C \times C \to \mathbb{R}$ be a bifunction. We recall that the equilibrium problem (EP) consists of finding the point $x \in C$ such that

$$g(x, y) \geq 0 \quad \forall \quad y \in C. \tag{6.79}$$

Let $\bar{x} \in C$. Setting

$$g(\bar{x}, \bar{y}) := \phi(\bar{y}) - \phi(\bar{x}) \quad \forall \, \bar{y} \in C,$$

the equilibrium problem (6.79) coincides with the convex minimization problem (2.20) and the function g satisfies the following conditions:

(i) $g(x, x) = 0 \quad \forall \, x \in C$,

(ii) g is monotone, i.e., $g(x, y) + g(y, x) \leq 0, \quad \forall \quad x, y \in C$,

(iii) $\limsup_{t \to 0} g(tz + (1 - t)x, y) \leq g(x, y) \quad \forall \quad x, y, z \in C$.

(iv) for each $x \in C$, the function $\rightarrowtail g(x, y)$ is convex lower semi-continuous.

The solution set to (6.79) is denoted by $\mathrm{EP}(g)$. The resolvent of the bifunction g is the function $\mathrm{Res}_g^\phi : E \to 2^C$ defined by (see, [7, 183, 194])

$$\mathrm{Res}_g^\phi(x) = \{z \in C : g(z, y) + \langle \Delta\phi(z) - \Delta\phi(x), y - z \rangle \geq 0, \quad \forall \quad y \in C\}. \tag{6.80}$$

Proposition 6.3.10. *(see [195]) Let $\phi : E \to (-\infty, +\infty]$ be a coercive and Legendre function. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies the conditions (i)-(iv), then*

- *Res_g^ϕ is single-valued,*
- *Res_g^ϕ is Bregman firmly nonexpansive,*
- *$F(Res_g^\phi) = EP(g)$,*
- *$EP(g)$ is a closed and convex subset of C,*
- *for all $x \in E$ and $q \in F(Res_g^\phi)$*

$$\Delta_\phi(q, Res_g^\phi(x)) + \Delta_\phi(Res_g^\phi(x), x) \leq \Delta_\phi(q, x).$$

Observe that every Bregman firmly nonexpansive mapping is Bregman relatively nonexpansive mapping. Hence, by setting $S_j = \mathrm{Res}_{g_j}^{\phi_j}, \quad j = 1, 2, ..., m$ in Theorem 7.2.6, then we have a strong convergence theorem for approximating a common solution of split minimization problem with multiple output sets which is also a common solution of a finite family of equilibrium problems in p-uniformly convex Banach spaces which are also uniformly smooth.

Zeroes of Bregman Inverse Strongly Monotone Operators
Let the Legendre function g be such that

$$\text{ran}(\Delta g - B) \subseteq \text{ran}(\Delta g), \tag{6.81}$$

the operator $B : E \to 2^{E^*}$ is called Bregman Inverse Strongly Monotone (BISM) if $(\text{dom}B) \cap (\text{int}(\text{dom}g)) \neq \emptyset$ and for any $x, y \in (\text{int}(\text{dom}g))$ and each $\psi \in Bx, \quad \eta \in By$, we have

$$\langle \psi - \eta, (\Delta g(x) - \psi) - \Delta g^*(\Delta g(y) - \eta) \rangle \geq 0. \tag{6.82}$$

This class of operators was introduced by Butnariu and Kassay (see [42]). For any operator $B : E \to 2^{E^*}$, the anti-resolvent $B^g : E \to 2^E$ of B is defined by

$$B^g := \Delta g^* \circ (\Delta g - B). \tag{6.83}$$

Observe that $\text{dom}B^g \subseteq (\text{dom}B) \cap (\text{int}(\text{dom}g))$ and $\text{ran}B^g \subseteq \text{int}(\text{dom}g)$. The operator B is BISM if and only if the anti-resolvent B^g is a single-valued Bregman Firmly Nonexpansive Mapping (BFNM) (see [42]). Some examples of BISM operators can be seen in [42]. From the definition of anti-resolvent and ([42], Lemma 3.5), we obtain the following proposition.

Proposition 6.3.11. *Let $g : E \to (-\infty, +\infty]$ be a Legendre function and let $B : E \to 2^{E^*}$ be a BISM operator such that $B^{-1}(0)^* \neq \emptyset$. Then the following statements hold:*

(i) $B^{-1}(0)^* = F(B^g)$

(ii) for any $u \in B^{-1}(0)^*$ and $x \in domB^g$, we have

$$D_g(u, B_x^g) + D_g(B_x^g, x) \leq D_g(u, x).$$

So, if the Legendre function g is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the anti-resolvent B^g of B is a single-valued Bregman Firmly Nonexpansive Mapping (BFNM) which satisfies

$$F(B^g) = \hat{F}(B^g).$$

(see [194], Lemma 1.3.2).

In Theorem 6.3.8, if we let $S_j = B_j^g$ and let g be the Legendre function such that (6.81) is satisfied, then we obtain a strong convergence theorem for approximating a common zero of a countable family of Bregman inverse strongly monotone operators which is also a solution of split minimization problem with multiple output sets in p-uniformly convex and uniformly smooth Banach spaces.

6.3.3 Numerical Examples

This section provides some numerical experiments to illustrate our iterative Algorithm 6.3.3 and check the effects of the key parameters on our method.

Example 6.3.12. *Let $E = \mathbb{R}^2 = E_i,\ i = 0, 1, 2, 3$ and $f(x) = \|x\|_2$ for all $x \in \mathbb{R}^2$, be the Euclidean norm. For a unit ball B, the projection onto B is given by*

$$P_B(x) = \begin{cases} \frac{x}{\|x\|_2}, & \text{if } \|x\|_2 > 1 \\ \\ x, & \text{if } \|x\|_2 \leq 1. \end{cases}$$

Then, the proximal operator $Prox^{f_i}(x)$ is given by

$$Prox^{\ f_i}(x) := \begin{cases} \left(1 - \frac{i+1}{\|x\|_2}\right)x, & \text{if } \|x\|_2 \geq 1 \\ \\ 0, & \text{if } \|x\|_2 < 1. \end{cases} \tag{6.84}$$

The proximal operator (6.84) is called the block soft thresholding (see [3]).
Let $T_i x := (i+1)x$, where $x : (x_1, x_2) \in \mathbb{R}^2$. We now consider the following problem:

$$x^* \in \arg\min f \cap (\bigcap_{i=1}^{N} T_i^{-1}(\arg\min f_i)) \neq \emptyset.$$

Also, let $S_j : C \times C \to \mathbb{R}$ be defined by

$$S_j x := \frac{2}{3j}x, \qquad j = 1, 2, \ldots, 7.$$

It is easy to see that S_j is relatively nonexpansive for each $j = 1, 2, \ldots, 7$. Take $\alpha_n = \frac{1}{n+1}$ for all $n \geq 1$, $u = (1, 1)$, $\beta_{i,n} = \frac{1}{4}$, $\phi_{n,0} = \frac{1}{2}$, $\phi_{n,j} = \frac{1}{2^{j+1}}, j = 1, 2, \cdots, 7$, then the iterative Algorithm 6.3.3 becomes

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = \sum_{i=0}^{3} \frac{1}{4}\left(w_n - \tau_{i,n}T_i^*\left(T_i(w_n) - Prox_{\lambda_n^i}^{f_i} T_i(w_n)\right)\right) \\ z_n = \left(\frac{1}{2}(y_n) + \sum_{j=1}^{7} \frac{1}{2^{j+1}}(S_j y_n)\right) \\ x_{n+1} = \frac{1}{n+1}u + \frac{n}{n+1}z_n, \end{cases}$$

for $n \geq 1$. Suppose that for $\epsilon > 0$, the step size $\tau_{i,n}$ is chosen such that

$$\tau_{i,n} \in \left(\epsilon, \frac{2\|T_i(w_n) - \left(Prox_{\lambda_n^i}^{f_i}\right)T_i(w_n)\|^2}{\|T_i^*\left(I^{H_i} - Prox_{\lambda_n^i}^{f_i}\right)T_i(w_n)\|^2} - \epsilon\right) \quad \forall \ \ n \in \Omega, \tag{6.85}$$

for small enough ϵ; where the index set $\Omega := \{n \in \mathbb{N} : T_i(w_n) - \big(Prox^{f_i}_{\lambda^i_n} T_i(w_n) \neq 0\big)\}$, *otherwise,* $\tau_{i,n} = \tau_i$, τ_i *is any nonnegative real number for each* $i = 0, 1, 2, 3$.

Using $\|x_{n+1} - x_n\| < 10^{-3}$ *as the stopping criterion, we plot the graphs of* $\|x_{n+1} - x_n\|$ *against the number of iterations in each case. The numerical results can be found in Fig. 6.2, Fig. 6.3, Table 6.2.12 and Table 6.2.13.*

Table 6.2.12. Numerical results for Example 6.3.12 (Experiment 1).

Cases		$\theta = 1.5$	$\theta = 2.0$	$\theta = 2.5$	$\theta = 3.0$	$\theta = 3.5$
1	*CPU time(sec)*	*0.0125*	*0.0114*	*0.0115*	*0.0108*	*0.0154*
	No of Iter.	*40*	*40*	*40*	*40*	*40*
2	*CPU time(sec)*	*0.0207*	*0.0181*	*0.0131*	*0.0121*	*0.0181*
	No of Iter.	*40*	*40*	*40*	*40*	*40*
3	*CPU time(sec)*	*0.0121*	*0.0141*	*0.0137*	*0.0118*	*0.0157*
	No of Iter.	*40*	*40*	*40*	*40*	*40*
4	*CPU time(sec)*	*0.0128*	*0.0116*	*0.0122*	*0.0107*	*0.0152*
	No of Iter.	*40*	*40*	*40*	*40*	*40*

Table 6.2.13. Numerical results for Example 6.3.12 (Experiment 2).

Cases		$\epsilon = \frac{1}{(n+2)^2}$	$\epsilon = \frac{2}{(n+5)^2}$	$\epsilon = \frac{1}{(n+3)^3}$	$\epsilon = \frac{1}{(n+1)^4}$	$\epsilon = \frac{2}{(n+4)^4}$
1	*CPU time(sec)*	*0.0166*	*0.0127*	*0.0157*	*0.0164*	*0.0157*
	No of Iter.	*40*	*40*	*40*	*40*	*40*
2	*CPU time(sec)*	*0.0122*	*0.0121*	*0.0125*	*0.0114*	*0.0159*
	No of Iter.	*40*	*40*	*40*	*40*	*40*
3	*CPU time(sec)*	*0.0128*	*0.0125*	*0.0120*	*0.0112*	*0.0160*
	No of Iter.	*40*	*40*	*40*	*40*	*40*
4	*CPU time(sec)*	*0.0134*	*0.0119*	*0.0123*	*0.0113*	*0.0159*
	No of Iter.	*40*	*40*	*40*	*40*	*40*

The next example is in infinite dimensional Hilbert space.

Example 6.3.13. *Let* $E = L^2([0, 2\pi]) = E_i,\ i = 0, 1, 2, 3$ *with norm* $\|x\| = (\int_0^{2\pi} |x(t)|^2 dt)^{\frac{1}{2}}$ *and the corresponding inner product* $\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)dt, \quad \forall \quad x, y \in L^2([0, 2\pi])$. *Suppose* $C := \{x \in L^2([0, 2\pi]) : \int_0^{2\pi}(t^2+1)x(t)dt \leq 1\}$ *and* $C_i := \{x \in L^2([0, 2\pi]) : \int_0^{2\pi} |x(t) - sin(t)|^2 \leq 16\}$ *are subsets of* E *and* E_i*, respectively. Define* $T : L^2([0, 2\pi]) \to L^2([0, 2\pi])$ *by* $T(x)(t) = \int_0^{2\pi} e^{-st}x(t)dt$ *for all* $x \in L^2([0, 2\pi])$ *and* $T_ix(t) = \int_0^{2\pi} \frac{1}{10}(x(t))dt$. *It is easy*

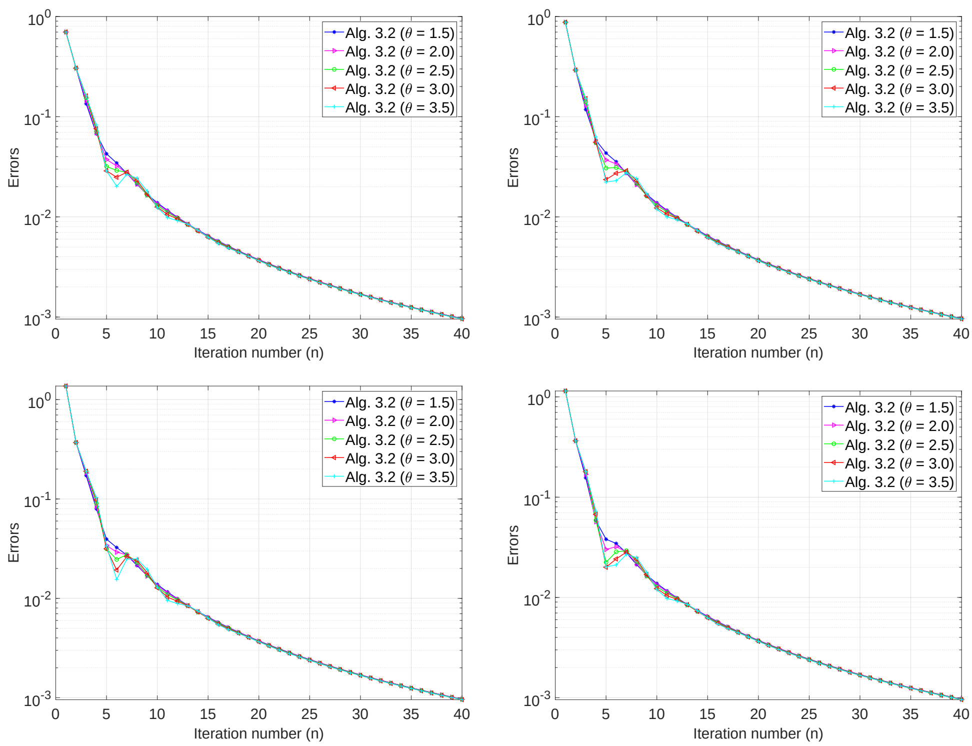


Figure 6.2: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

to see that T and T_i for $i = 0, 1, 2, 3$ are bounded linear operators. Also, for $j = 1, 2, \ldots, 7$, let $S_j : L^2([0, 2\pi]) \to L^2([0, 2\pi])$ by

$$S_j x = \frac{1}{2j} x, \ \forall j = 1, 2, \ldots, 7.$$

It is easy to see that S_j is relatively nonexpansive for each $j = 1, 2, \ldots, 7$. Let $f = i_C$ $f_i = i_{C_i}$ be the indicator functions on C and C_i, respectively, then $Prox_\lambda f = \Pi_C$ and $Prox_\lambda f_i = \Pi_{C_i}$. Also, we choose $u = t + 1$, $\alpha_n = \frac{1}{(n+1)}, \beta_{i,n} = \frac{1}{4}$, $\phi_{n,0} = \frac{1}{2}$, $\phi_{n,j} = \frac{1}{2^{j+1}}$. Then Algorithm 7.2.3 now becomes

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = \sum_{i=0}^{3} \frac{1}{4}\Big(w_n - \tau_{i,n} T_i^* \big(T_i(w_n) - Prox_{\lambda_n^i}^{f_i} T_i(w_n)\big)\Big) \\ z_n = \Big(\frac{1}{2}(y_n) + \sum_{j=1}^{7} \frac{1}{2^{j+1}}(S_j y_n)\Big) \\ x_{n+1} = \frac{1}{(n+2)}(t+1) + \frac{n+1}{(n+2)} z_n, \end{cases} \tag{6.86}$$

for $n \geq 1$. Suppose that for $\epsilon > 0$, the step size $\tau_{i,n}$ is chosen such that

$$\tau_{i,n} \in \left(\epsilon, \frac{2\|T_i(w_n) - \big(Prox_{\lambda_n^i}^{f_i}\big) T_i(w_n)\|^2}{\|T_i^* \big(I^{H_i} - Prox_{\lambda_n^i}^{f_i}\big) T_i(w_n)\|^2} - \epsilon\right) \quad \forall \ \ n \in \Omega, \tag{6.87}$$

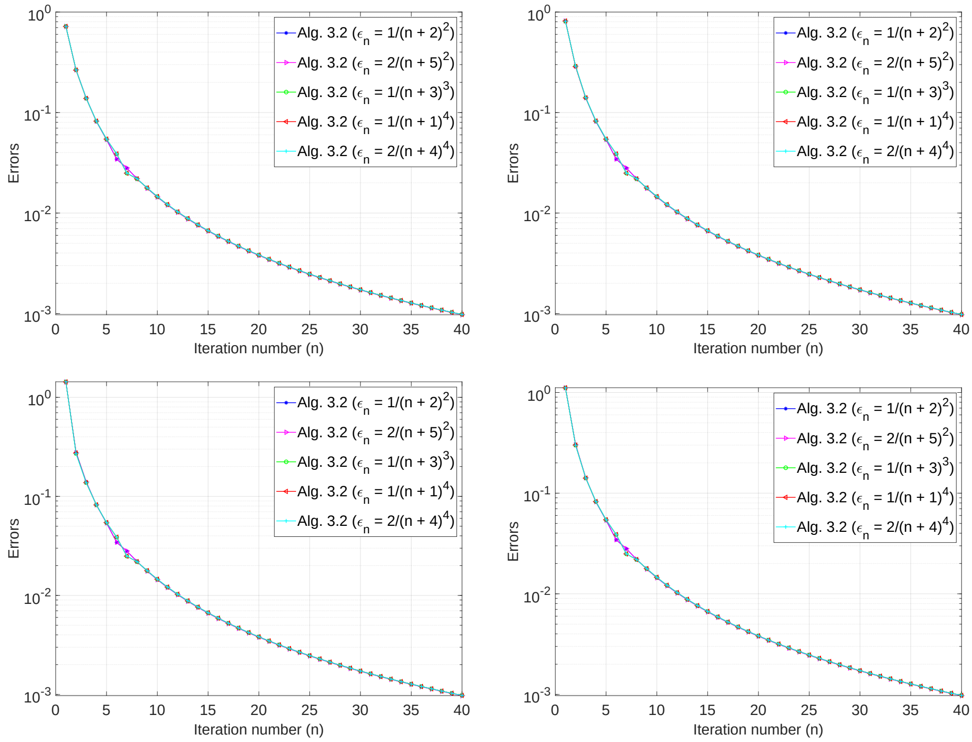


Figure 6.3: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

for small enough ϵ*; where the index set* $\Omega := \{n \in \mathbb{N} : T_i(w_n) - \left(Prox^{f_i}_{\lambda^i_n} T_i(w_n) \neq 0\right)\}$*, otherwise,* $\tau_{i,n} = \tau_i$*,* τ_i *is any nonnegative real number for each* $i = 0, 1, 2, 3$*. Using* $\|x_{n+1} - x_n\| < 10^{-3}$ *as the stopping criterion, we plot the graphs of* $\|x_{n+1} - x_n\|$ *against the number of iterations in each cases. The numerical results are reported in Fig. 6.4, Fig. 6.5, Table 6.2.14 and Table 6.2.15.*

Table 6.2.14. Numerical results for Example 6.3.13 (Experiment 1).

Cases		$\theta = 0.2$	$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$	$\theta = 1.0$
1	*CPU time(sec)*	*18.2522*	*9.5605*	*9.6427*	*9.9466*	*6.4868*
	No of Iter.	*42*	*42*	*42*	*42*	*42*
2	*CPU time(sec)*	*76.3470*	*78.2310*	*71.6249*	*73.9699*	*22.4244*
	No of Iter.	*42*	*42*	*42*	*42*	*42*
3	*CPU time(sec)*	*55.4661*	*54.4661*	*50.8791*	*51.8060*	*0.0161*
	No of Iter.	*42*	*42*	*42*	*42*	*14*
4	*CPU time(sec)*	*17.936*	*18.2225*	*10.8839*	*6.2995*	*7.1470*
	No of Iter.	*42*	*42*	*42*	*42*	*42*

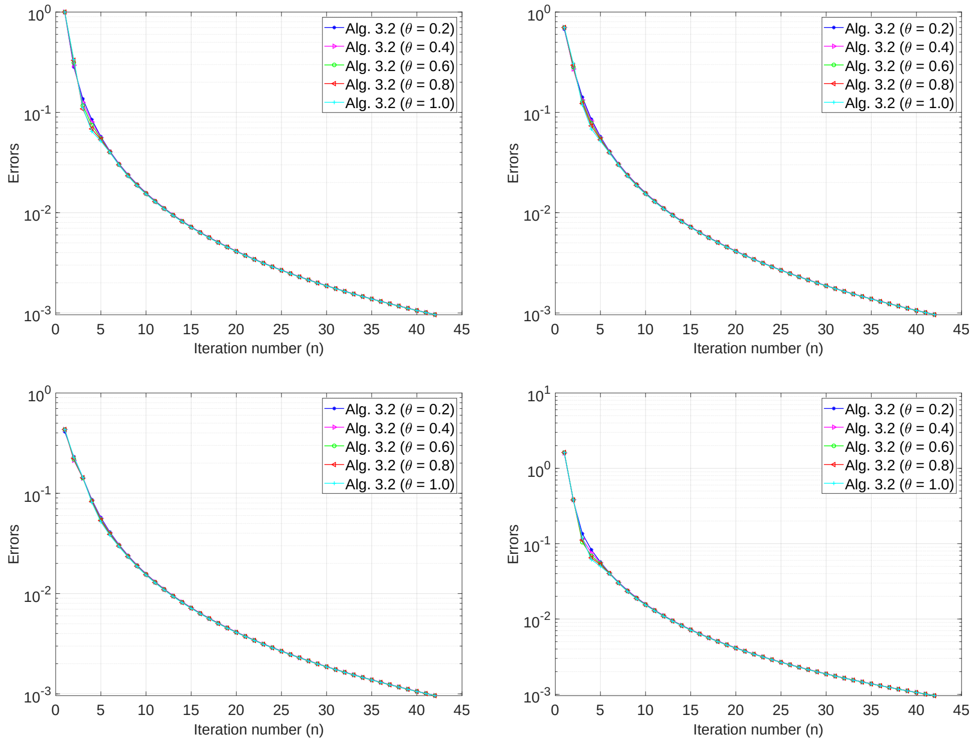


Figure 6.4: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

Table 6.2.15. Numerical results for Example 6.3.13 (Experiment 2).

Cases		$\epsilon = \frac{1}{(n+2)^2}$	$\epsilon = \frac{1}{(n+3)^2}$	$\epsilon = \frac{2}{(n+3)^3}$	$\epsilon = \frac{2}{(n+3)^3}$	$\epsilon = \frac{2}{(n+1)^4}$
1	*CPU time(sec)*	*7.1705*	*7.0260*	*7.0490*	*7.1383*	*7.6806*
	No of Iter.	*42*	*42*	*42*	*42*	*42*
2	*CPU time(sec)*	*5.6297*	*5.5916*	*5.6409*	*6.2183*	*27.9355*
	No of Iter.	*42*	*42*	*42*	*42*	*42*
3	*CPU time(sec)*	*6.1491*	*5.6183*	*5.5601*	*6.2239*	*27.7076*
	No of Iter.	*42*	*42*	*42*	*42*	*42*
4	*CPU time(sec)*	*6.9287*	*6.7615*	*6.7168*	*6.7157*	*7.7159*
	No of Iter.	*42*	*42*	*42*	*42*	*42*

We test these examples under the following experiments:

Experiment 1:
In this experiment, we check the behavior of our method by fixing the other parameters and varying θ and ϵ in Example 6.3.12. We do this to check the effects of these parameters

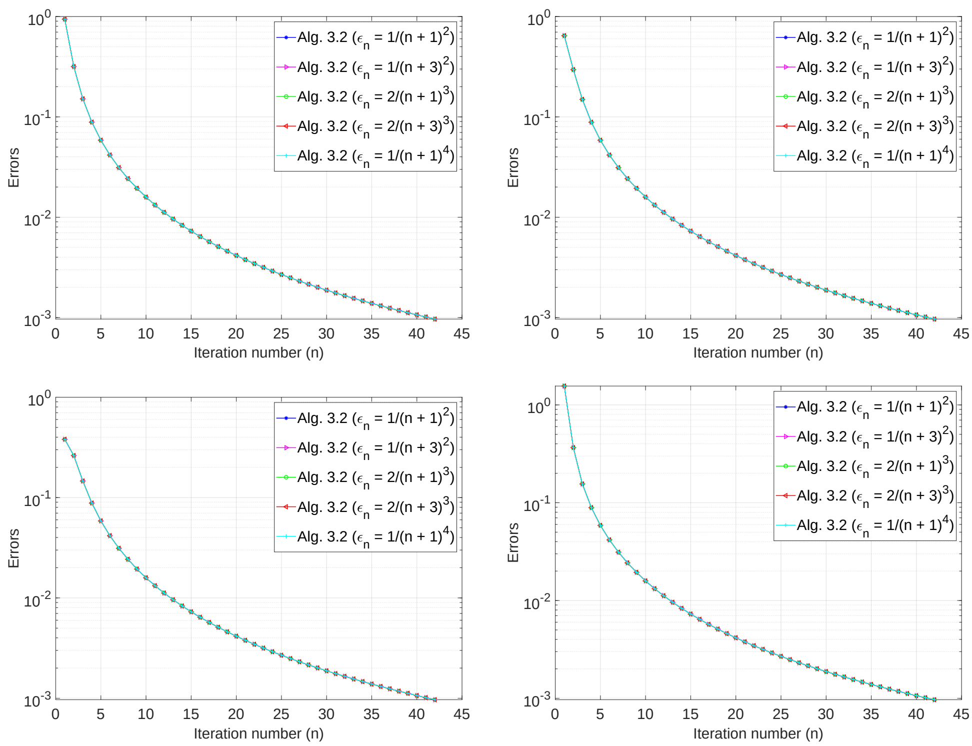


Figure 6.5: Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

on our method. We consider the following cases for the initial values of x_0, x_1 :

Case 1 : $x_0 = (1.00, 1.25)$; $x_1 = (0.34, 1.30)$

Case 2 : $x_0 = (1.50, 2.60)$; $x_1 = (0.75, 1.40)$

Case 3 : $x_0 = (-2.50, 1.30)$; $x_1 = (1.75, -0.45)$

Case 4 : $x_0 = (2.50, -1.30)$ $x_1 = (1.75, 0.45)$.

Also, we consider $\theta \in \{1.5, 2.0, 2.5, 3.0, 3.5\}$ and $\epsilon_n \in \{\frac{1}{(n+2)^2}, \frac{2}{(n+5)^2}, \frac{1}{(n+3)^3}, \frac{1}{(n+1)^4}, \frac{2}{(n+4)^4}\}$ which satisfies Assumption 3.1 (5)(ii-iii). We use Algorithm 6.3.3 for the experiment and report the numerical results in Fig. 6.2, Fig. 6.3, Table 6.2.12 and Table 6.2.13.

Experiment 2:
In this experiment, we check the behavior of our method by fixing the other parameters and varying θ and ϵ in Example 6.3.12. We do this to check the effects of these parameters on our method. We consider the following cases for the initial values of x_0, x_1 :

Case 1 : $x_0 = t + 5$; $x_1 = t^3 + t + 1$, Case 2 : $x_0 = e^{2t}; \quad x_1 = \frac{3}{10}e^{2t}$,

Case 3 : $x_0 = e^{2t}; \quad x_1 = t + 1$,

Case 4 : $x_0 = t^2 + t + 3; \quad x_1 = t + 2$.

We consider $\theta \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ and $\epsilon_n \in \{\frac{1}{(n+1)^2}, \frac{1}{(n+3)^2}, \frac{2}{(n+1)^3}, \frac{2}{(n+3)^3}, \frac{1}{(n+1)^4}\}$ which satisfies Assumption 3.1 (5)(ii-iii). We use Algorithm 6.3.3 for the experiment and report the numerical results in Fig. 6.4, Fig. 6.5, Table 6.2.14 and Table 6.2.15.

Chapter 7

Split Equality Equilibrium Fixed Point and Monotone Variational Inclusion Problems

7.1 Introduction

In this chapter, we devote time to study new algorithms for finding a common element of the sets of solutions of split equality pseudomonotone equilibrium, split equality monotone variational inclusion and fixed point problems for Bregman relatively nonexpansive mappings in p-uniformly convex and uniformly smooth Banach spaces. In addition, we study multiple sets split equality equilibrium problems and set of fixed points for two finite families of Bregman quasi-nonexpansive mappings in the framework of uniformly convex and uniformly smooth Banach spaces.

7.2 On split equality equilibrium, monotone variational inclusion and fixed point problems.

In this section, we propose and study a modified Halpern method for approximating the common solution of split equality equilibrium, monotone variational inclusion and fixed point problem of Bregman relatively nonexpansive mappings in p-uniformly convex Banach spaces which are uniformly smooth. We apply our result to solve split equality variational inequality and split equality convex minimization problems and present several numerical examples to back our algorithm.

7.2.1 Main results

In this section, we first establish the following lemma needed in the convergence analysis of our main theorem.

Lemma 7.2.1. *Let E be a reflexive Banach space, $T : E \to E$ be a Bregman relatively nonexpansive mapping and $B : E \to 2^{E^*}$ be a maximal monotone operator. Suppose $g : E \to [-\infty, \infty]$ is a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subsets of E and $A : E \to E^*$ be a BISM mapping such that $(A + B)^{-1}(0) \neq \emptyset$. Then,*

$$Fix\big(T(Res^g_{\sigma B} \circ A^g_\sigma)\big) = Fix(T) \cap Fix(Res^g_{\sigma B} \circ A^g_\sigma).$$

Proof. Clearly, $\mathrm{Fix}(T) \cap \mathrm{Fix}(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma) \subseteq \mathrm{Fix}\big(T(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)\big)$. We only need to prove that $\mathrm{Fix}\big(T(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)\big) \subseteq \mathrm{Fix}(T) \cap \mathrm{Fix}(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)$. Let $x \in \mathrm{Fix}\big(T(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)\big)$ and $y \in \mathrm{Fix}(T) \cap \mathrm{Fix}(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)$, then

$$\begin{aligned} \Delta_p(y, x) &= \Delta_p(y, T(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)x) \\ &\leq \Delta_p(y, (\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)x). \end{aligned} \tag{7.1}$$

Now, by applying Lemma 6.3.1 and (7.1), we have

$$\begin{aligned} \Delta_p\big(x, (\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)x\big) &\leq \Delta_p(y, x) - \Delta_p\big(y, (\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)x\big) \\ &\leq \Delta_p(y, x) - \Delta_p(y, x) \\ &= 0. \end{aligned}$$

Hence, $x \in \mathrm{Fix}(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)$.
Next, we show that $x \in \mathrm{Fix}(T)$ since $x \in \mathrm{Fix}\big(T(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)\big)$ and $x \in \mathrm{Fix}(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)$, we have

$$\begin{aligned} \Delta_p(x, Tx) &= \Delta_p(x, \big(T(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)\big)x) \\ &= \Delta_p(x, x) \\ &= 0. \end{aligned}$$

Hence, $x \in \mathrm{Fix}(T)$. This implies that $x \in \mathrm{Fix}(T) \cap \mathrm{Fix}(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)$. Therefore, we conclude that $\mathrm{Fix}\big(T(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)\big) = \mathrm{Fix}(T) \cap \mathrm{Fix}(\mathrm{Res}^g_{\sigma B} \circ A^g_\sigma)$. $\square$

Throughout this section, we assume that

Assumption 7.2.2.

1. *E_1, E_2 and E_3 are three p-uniformly convex real Banach spaces which are also uniformly smooth.*
2. *C and Q are two nonempty closed and convex subsets of E_1 and E_2 respectively.*
3. *$A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators.*

4. $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$ *are bifunctions satisfying* $A_1 - A_4$ *of* **Assumption A**.

5. $T : E_1 \to E_1$ *and* $S : E_2 \to E_2$ *are Bregman relatively nonexpansive mappings.*

6. $A_1 : E_1 \to E_1^*$ *and* $A_2 : E_2 \to E_2^*$ *are BISM operators.* $B_1 : E_1 \to 2^{E_1^*}$ *and* $B_2 : E_2 \to 2^{E_2^*}$ *are maximal monotone operators.*

7. *Assume that* $\Gamma := \{x^* \in Fix(T) \cap (A_1 + B_1)^{-1}(0) \cap EP(C, f_1), \quad y^* \in Fix(S) \cap (A_2 + B_2)^{-1}(0) \cap EP(Q, f_2) \quad \text{such that} \quad Ax^* = By^*\} \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *be positive sequences satisfying the following conditions:*

   (i) $\alpha_n \in (0, 1), \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$

   (ii) $\beta_n \in (0, 1)$ *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$

   (iii) $\{\lambda_n\} \subset (0, h)$, *where* $h = \min\{\frac{1}{c_1}, \frac{1}{c_2}\}$ *and* c_1, c_2 *are the Bregman-Lipschitz coefficients of* f_i *for* $1 \leq i \leq 2$.

Algorithm 7.2.3. *For fixed* $u \in E_1$ *and* $v \in E_2$, *choose an initial guess* $(x_1, y_1) \in E_1 \times E_2$. *Suppose that the nth iterate* $(x_n, y_n) \subset E_1 \times E_2$ *has been constructed; then we compute the* $(n + 1)$*th iterate* (x_{n+1}, y_{n+1}) *via the iteration*

$$\begin{cases} u_n = J_q^{E_1^*}\Big(J_p^{E_1}(x_n) - \gamma_n A^* J_p^{E_3}(Ax_n - By_n)\Big), \\ z_n = \arg\min\{\lambda_n f_1(u_n, w) + \Delta_p(w, u_n) : \quad w \in C\}, \\ t_n = \arg\min\{\lambda_n f_1(z_n, w) + \Delta_p(w, u_n) : \quad w \in C\}, \\ x_{n+1} = J_q^{E_1^*}\Bigg(\alpha_n J_p^{E_1}(u) + (1 - \alpha_n)\Big[\beta_n J_p^{E_1}(t_n) + (1 - \beta_n) J_p^{E_1}(T(Res_{\sigma B_1}^g \circ A_{1\sigma}^g) t_n)\Big]\Bigg), \\ v_n = J_q^{E_2^*}\Big(J_p^{E_2}(y_n) + \gamma_n B^* J_p^{E_3}(Ax_n - By_n)\Big), \\ g_n = \arg\min\{\lambda_n f_2(v_n, d) + \Delta_p(d, v_n) : \quad d \in Q\}, \\ h_n = \arg\min\{\lambda_n f_2(g_n, d) + \Delta_p(d, v_n) : \quad d \in Q\}, \\ y_{n+1} = J_q^{E_2^*}\Bigg(\alpha_n J_p^{E_2}(v) + (1 - \alpha_n)\Big[\beta_n J_p^{E_2}(h_n) + (1 - \beta_n) J_p^{E_2}(S(Res_{\sigma B_2}^g \circ A_{2\sigma}^g) h_n)\Big]\Bigg), \end{cases} \tag{7.2}$$

where $\gamma_n \in \left(\epsilon, \left(\frac{q\|Ax_n - By_n\|^p}{C_q\|A^* J_p^{E_3}(Ax_n - By_n)\|^q + Q_q\|B^* J_p^{E_3}(Ax_n - By_n)\|^q} - \epsilon\right)^{\frac{1}{q-1}}\right)$, $n \in \Omega$, *for small enough* ϵ; C_q *and* Q_q *are constants of smoothness of* E_1 *and* E_2, *respectively. Otherwise,* $\gamma_n = \gamma$ *(*γ *being any nonnegative value), where the set of indexes* $\Omega = \{n : Ax_n - By_n \neq 0\}$.

Remark 7.2.4. *From Algorithm 7.2.3,* $g : E \to \mathbb{R}$ *is a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subsets of* E *such that* $C \subset intdomg$.

Lemma 7.2.5. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by Algorithm 7.2.3 such that Assumption 7.2.2 hold. Then,* $\{x_n\}$ *and* $\{y_n\}$ *are bounded.*

Proof. Let $(x^*, y^*) \in \Gamma$. Then, from (7.2), Lemma 2.55 and Lemma 2.5.45, we have

$$\begin{aligned}
\Delta_p(u_n, x^*) &= \Delta_p\left(J_q^{E_1^*}\left(J_p^{E_1}(x_n) - \gamma_n A^* J_p^{E_3}(Ax_n - By_n)\right), x^*\right) \\
&= V_p\left(J_p^{E_1}(x_n) - \gamma_n A^* J_p^{E_3}(Ax_n - By_n), x^*\right) \\
&= \frac{1}{q}\|J_p^{E_1}(x_n) - \gamma_n A^* J_p^{E_3}(Ax_n - By_n)\|^q - \langle J_p^{E_1}(x_n), x^*\rangle \\
&+ \gamma_n\langle A^* J_p^{E_3}(Ax_n - By_n), x^*\rangle + \frac{1}{p}\|x^*\|^p \\
&\leq \frac{1}{q}\|J_p^{E_1}(x_n)\|^q - \gamma_n\langle J_p^{E_3}(Ax_n - By_n), Ax_n\rangle + \frac{C_q}{q}\gamma_n^q\|A^* J_p^{E_3}(Ax_n - By_n)\|^q \\
&- \langle J_p^{E_1}(x_n), x^*\rangle + \gamma_n\langle J_p^{E_3}(Ax_n - By_n), Ax^*\rangle + \frac{1}{p}\|x^*\|^p \\
&= \frac{1}{q}\|x_n\|^p - \langle J_p^{E_1}(x_n), x^*\rangle + \frac{1}{p}\|x^*\|^p - \gamma_n\langle J_p^{E_3}(Ax_n - By_n), Ax_n - Ax^*\rangle \\
&+ \frac{C_q}{q}\gamma_n^q\|A^* J_p^{E_3}(Ax_n - By_n)\|^q \\
&= \Delta_p(x_n, x^*) - \gamma_n\langle J_p^{E_3}(Ax_n - By_n), Ax_n - Ax^*\rangle + \frac{C_q}{q}\gamma_n^q\|A^* J_p^{E_3}(Ax_n - By_n)\|^q.
\end{aligned} \tag{7.3}$$

Similarly, we obtain

$$\Delta_p(v_n, y^*) \leq \Delta_p(y_n, y^*) - \gamma_n\langle J_p^{E_3}(Ax_n - By_n), By^* - By_n\rangle + \frac{Q_q}{q}\gamma_n^q\|B^* J_p^{E_3}(Ax_n - By_n)\|^q. \tag{7.4}$$

Adding (7.3) and (7.4), with $Ax^* = By^*$, we obtain

$$\begin{aligned}
&\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*) \leq \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*) \\
&- \gamma_n\left[\|Ax_n - By_n\|^p - \frac{\gamma_n^{q-1}}{q}\left(C_q\|A^* J_p^{E_3}(Ax_n - By_n)\|^q + Q_q\|B^* J_p^{E_3}(Ax_n - By_n)\|^q\right)\right].
\end{aligned} \tag{7.5}$$

Thus,

$$\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*) \leq \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*). \tag{7.6}$$

Also, from Algorithm 7.2.3 and Lemma 2.5.43, we have

$$\begin{aligned}
\Delta_p(x_{n+1},x^*) &= \Delta_p\Bigg(J_q^{E_1^*}\Big(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)\Big[\beta_n J_p^{E_1}(t_n)\\
&\quad +(1-\beta_n)J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1}\circ A^g_{1\sigma})t_n)\Big]\Big),x^*\Bigg)\\
&= \Delta_p\Bigg(J_q^{E_1^*}\Big(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)\beta_n J_p^{E_1}(t_n)\\
&\quad +(1-\alpha_n)(1-\beta_n)J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1}\circ A^g_{1\sigma})t_n)\Big),x^*\Bigg)\\
&\le \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\beta_n\Delta_p(t_n,x^*)+(1-\alpha_n)(1-\beta_n)\Delta_p(T(\mathrm{Res}^g_{\sigma B_1}\circ A^g_{1\sigma})t_n,x^*)\\
&\le \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\beta_n\Delta_p(t_n,x^*)+(1-\alpha_n)(1-\beta_n)\Delta_p((\mathrm{Res}^g_{\sigma B_1}\circ A^g_{1\sigma})t_n,x^*)\\
&\le \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\beta_n\Delta_p(t_n,x^*)+(1-\alpha_n)(1-\beta_n)\Delta_p(t_n,x^*)\\
&\le \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\beta_n\Delta_p(u_n,x^*)+(1-\alpha_n)(1-\beta_n)\Delta_p(u_n,x^*)\\
&= \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\Delta_p(u_n,x^*). \qquad (7.7)
\end{aligned}$$

Similarly, we have

$$\Delta_p(y_{n+1},y^*)\le \alpha_n\Delta_p(v,y^*)+(1-\alpha_n)\Delta_p(v_n,y^*). \qquad (7.8)$$

Hence, from (7.6), (7.7) and (7.8), we have

$$\begin{aligned}
\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*) &\le \alpha_n\big(\Delta_p(u,x^*)+\Delta_p(v,y^*)\big)\\
&\quad +(1-\alpha_n)\big(\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\big)\\
&\le \max\{\Delta_p(u,x^*)+\Delta_p(v,y^*),\ \Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\}\\
&\ \vdots\\
&\le \max\{\Delta_p(u,x^*)+\Delta_p(v,y^*),\ \Delta_p(x_1,x^*)+\Delta_p(y_1,y^*)\}, n\ge 1. \qquad (7.9)
\end{aligned}$$

Thus, $\{\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\}$ is bounded. Consequently, $\{\Delta_p(x_n,x^*)\}$ and $\{\Delta_p(y_n,y^*)\}$ are bounded. Therefore, it follows from Lemma 2.5.34 that $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ are all bounded. □

Theorem 7.2.6. *Let $\{(x_n,y_n)\}$ be a sequence generated by Algorithm 7.2.3 under Assumption 7.2.2. Then the sequence $\{(x_n,y_n)\}$ converges strongly to $(x^*,y^*)\in\Gamma$.*

Proof. Let $(x^*,y^*)\in\Gamma$. Then from Algorithm 7.2.3 and Lemma 2.5.45(iii), with $\bar{y}_n =$

$-\alpha_n\big(J_p^{E_1}(u) - J_p^{E_1}(x^*)\big)$, we have

$$
\begin{aligned}
\Delta_p(x_{n+1}, x^*) &= \Delta_p\Bigg(J_q^{E_1^*}\Bigg(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big[\beta_n J_p^{E_1}(t_n) \\
&\quad + (1-\beta_n) J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1} \circ A^g_{1\sigma})t_n)\Big]\Bigg), x^*\Bigg) \\
&= V_p\Bigg(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big[\beta_n J_p^{E_1}(t_n) + (1-\beta_n) J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1} \circ A^g_{1\sigma})t_n)\Big], x^*\Bigg) \\
&= V_p\Bigg(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big[\beta_n J_p^{E_1}(t_n) + (1-\beta_n) J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1} \circ A^g_{1\sigma})t_n)\Big] \\
&\quad - \alpha_n\big(J_p^{E_1}(u) - J_p^{E_1}(x^*)\big), x^*\Bigg) - \langle \alpha_n\big(J_p^{E_1}(u) - J_p^{E_1}(x^*)\big), \\
&\quad J_q^{E_1^*}\Bigg(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big[\beta_n J_p^{E_1}(t_n) + (1-\beta_n) J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1} \circ A^g_{1\sigma})t_n)\Big]\Bigg) - x^*\rangle \\
&= V_p\Bigg(\alpha_n J_p^{E_1}(x^*) + (1-\alpha_n)\Big[\beta_n J_p^{E_1}(t_n) + (1-\beta_n) J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1} \circ A^g_{1\sigma})t_n)\Big], x^*\Bigg) \\
&\quad + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle \\
&= \Delta_p\Bigg(J_q^{E_1^*}\Bigg(\alpha_n J_p^{E_1}(x^*) + (1-\alpha_n)\Big[\beta_n J_p^{E_1}(t_n) + (1-\beta_n) J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1} \circ A^g_{1\sigma})t_n)\Big]\Bigg), x^*\Bigg) \\
&\quad + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle \\
&\le \alpha_n \Delta_p(x^*, x^*) + (1-\alpha_n)\beta_n \Delta_p(t_n, x^*) + (1-\alpha_n)(1-\beta_n)\Delta_p(T(\mathrm{Res}^g_{\sigma B_1} \circ A^g_{1\sigma})t_n, x^*) \\
&\quad + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)\Delta_p(t_n, x^*) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)\Delta_p(u_n, x^*) + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle.
\end{aligned}
$$

Similarly, we obtain

$$\Delta_p(y_{n+1}, y^*) \le (1-\alpha_n)\Delta_p(v_n, y^*) + \alpha_n\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle.$$

Therefore, we obtain from (7.6) that

$$\begin{aligned}\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*) &\le (1-\alpha_n)\Bigg(\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*)\Bigg)\\ &\quad+\alpha_n\Bigg(\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle+\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle\Bigg)\\ &\le (1-\alpha_n)\Bigg(\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\Bigg)\\ &\quad+\alpha_n\Bigg(\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle+\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle\Bigg)\\ &=(1-\alpha_n)\Bigg(\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\Bigg)+\alpha_n d_n,\quad \forall n\ge n_1,\end{aligned}\tag{7.10}$$

where $d_n := \Bigg(\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle+\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle\Bigg)$.
To show that the sequence $\{(x_n,y_n)\}$ converges strongly to (x^*,y^*), using Lemma 2.5.55, it is enough to show that $\limsup_{k\to\infty} d_{n_k}\le 0$ (where $\{d_{n_k}\}$ is a subsequence of $\{d_n\}$), for every subsequence $\{\Delta_p(x_{n_k},x^*)\}$ of $\{\Delta_p(x_n,x^*)\}$ and $\{\Delta_p(y_{n_k},y^*)\}$ of $\{\Delta_p(y_n,y^*)\}$ satisfying the condition

$$\liminf_{k\to\infty}\Bigg(\big[\Delta_p(x_{n_k+1},x^*)+\Delta_p(y_{n_k+1},y^*)\big]-\big[\Delta_p(x_{n_k},x^*)+\Delta_p(y_{n_k},y^*)\big]\Bigg)\ge 0.\tag{7.11}$$

Now, using (7.2) and Lemma 2.5.47, we have

$$\begin{aligned}\Delta_p(x_{n+1},x^*) &= \Delta_p\Bigg(J_q^{E_1^*}\Bigg(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)\Big[\beta_n J_p^{E_1}(t_n)\\ &\quad+(1-\beta_n)J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1}\circ A^g_{1\sigma})t_n)\Big]\Bigg),x^*\Bigg)\\ &\le \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\beta_n\Delta_p(t_n,x^*)+(1-\alpha_n)(1-\beta_n)\Delta_p(t_n,x^*)\\ &=\alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\Delta_p(t_n,x^*)\\ &\le \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\Delta_p(u_n,x^*)-(1-\alpha_n)(1-\lambda_n c_1)\Delta_p(z_n,u_n)\\ &\quad-(1-\alpha_n)(1-\lambda_n c_2)\Delta_p(t_n,z_n).\end{aligned}\tag{7.12}$$

Similarly,

$$\begin{aligned}\Delta_p(y_{n+1},y^*) &\le \alpha_n\Delta_p(v,y^*)+(1-\alpha_n)\Delta_p(v_n,y^*)-(1-\alpha_n)(1-\lambda_n c_1)\Delta_p(g_n,v_n)\\ &\quad-(1-\alpha_n)(1-\lambda_n c_2)\Delta_p(h_n,g_n).\end{aligned}\tag{7.13}$$

Adding (7.12) and (7.13), we get

$$\begin{aligned}\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*)\leq{}&\alpha_n\big[\Delta_p(u,x^*)+\Delta_p(v,y^*)\big]\\&+(1-\alpha_n)\big[\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*)\big]\\&-(1-\alpha_n)(1-\lambda_nc_1)\Delta_p(z_n,u_n)\\&-(1-\alpha_n)(1-\lambda_nc_2)\Delta_p(t_n,z_n)\\&-(1-\alpha_n)(1-\lambda_nc_1)\Delta_p(g_n,v_n)\\&-(1-\alpha_n)(1-\lambda_nc_2)\Delta_p(h_n,g_n).\end{aligned}$$

Therefore, using (7.5) and the last inequality above, we have

$$\begin{aligned}\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*)\leq{}&\alpha_n\big[\Delta_p(u,x^*)+\Delta_p(v,y^*)\big]+(1-\alpha_n)\big[\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\big]\\&-(1-\alpha_n)\gamma_n\bigg[\|Ax_n-By_n\|^p-\frac{\gamma_n^{q-1}}{q}\bigg(C_q\|A^*J_p^{E_3}(Ax_n-By_n)\|^q\\&+Q_q\|B^*J_p^{E_3}(Ax_n-By_n)\|^q\bigg)\bigg]\\&-(1-\alpha_n)(1-\lambda_nc_1)\Delta_p(z_n,u_n)-(1-\alpha_n)(1-\lambda_nc_2)\Delta_p(t_n,z_n)\\&-(1-\alpha_n)(1-\lambda_nc_1)\Delta_p(g_n,v_n)-(1-\alpha_n)(1-\lambda_nc_2)\Delta_p(h_n,g_n).\end{aligned}\tag{7.14}$$

From (7.11), Assumption 7.2.2 (7)i and (7.14), we have that

$$\begin{aligned}\limsup_{k\to\infty}&\bigg((1-\alpha_{n_k})\gamma_{n_k}\bigg[\|Ax_{n_k}-By_{n_k}\|^p-\frac{\gamma_{n_k}^{q-1}}{q}\bigg(C_q\|A^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|^q\\&+Q_q\|B^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|^q\bigg)\bigg]\bigg)\\&\leq\limsup_{k\to\infty}\bigg(\alpha_{n_k}\big[\Delta_p(u,x^*)+\Delta_p(v,y^*)\big]+(1-\alpha_{n_k})\big[\Delta_p(x_{n_k},x^*)+\Delta_p(y_{n_k},y^*)\big]\\&-\big[\Delta_p(x_{n_k+1},x^*)+\Delta_p(y_{n_k+1},y^*)\big]\bigg)\\&\leq-\limsup_{k\to\infty}\bigg(\big[\Delta_p(x_{n_k+1},x^*)+\Delta_p(y_{n_k+1},y^*)\big]-\big[\Delta_p(x_{n_k},x^*)+\Delta_p(y_{n_k},y^*)\big]\bigg)\\&\leq 0.\end{aligned}\tag{7.15}$$

Following the same process as in (7.15), we obtain from (7.11), Assumption 7.2.2 (7)i and

(7.14) that

$$\limsup_{k\to\infty}\Bigg((1-\alpha_{n_k})\Bigg((1-\lambda_{n_k}c_1)\Delta_p(z_{n_k},u_{n_k})+(1-\lambda_{n_k}c_2)\Delta_p(t_{n_k},z_{n_k})$$

$$+(1-\lambda_{n_k}c_1)\Delta_p(g_{n_k},v_{n_k})+(1-\lambda_{n_k}c_2)\Delta_p(h_{n_k},g_{n_k})\Bigg)\Bigg)$$

$$\le\limsup_{k\to\infty}\Bigg(\alpha_{n_k}\big[\Delta_p(u,x^*)+\Delta_p(v,y^*)\big]+(1-\alpha_{n_k})\big[\Delta_p(x_{n_k},x^*)+\Delta_p(y_{n_k},y^*)\big]$$

$$-\big[\Delta_p(x_{n_k+1},x^*)+\Delta_p(y_{n_k+1},y^*)\big]\Bigg)$$

$$\le-\limsup_{k\to\infty}\Bigg(\big[\Delta_p(x_{n_k+1},x^*)+\Delta_p(y_{n_k+1},y^*)\big]-\big[\Delta_p(x_{n_k},x^*)+\Delta_p(y_{n_k},y^*)\big]\Bigg)$$

$$\le 0. \tag{7.16}$$

Let $\psi_{n_k}=C_q\|A^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|^q+Q_q\|B^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|^q$. By the condition on the stepsize γ_{n_k}, we have that

$$\gamma_{n_k}^{q-1}<\frac{q\|Ax_{n_k}-By_{n_k}\|^p}{\psi_{n_k}}-\epsilon,$$

which implies that

$$\gamma_{n_k}^{q-1}\psi_{n_k}<q\|Ax_{n_k}-By_{n_k}\|^p-\epsilon\psi_{n_k}.$$

Thus by (7.15), we obtain that $\frac{\epsilon\psi_{n_k}}{q}<\|Ax_{n_k}-By_{n_k}\|^p-\frac{\gamma_{n_k}^{q-1}}{q}\psi_{n_k}\to 0,\ \text{ as }\ k\to\infty$. Therefore $C_q\|A^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|^q+Q_q\|B^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|^q\to 0\ \text{ as }\ k\to\infty$. It follows that

$$\lim_{k\to\infty}\|A^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|^q=0, \tag{7.17}$$

and

$$\lim_{k\to\infty}\|B^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|^q=0. \tag{7.18}$$

Also, we have from (7.15) that

$$\begin{aligned}
&\limsup_{k\to\infty}\left((1-\alpha_{n_k})\gamma_{n_k}\Big[\|Ax_{n_k}-By_{n_k}\|^p\Big]\right)\\
&\le \limsup_{k\to\infty}\Bigg(\alpha_{n_k}\big[\Delta_p(u,x^*)+\Delta_p(v,y^*)\big]+(1-\alpha_{n_k})\big[\Delta_p(x_{n_k},x^*)+\Delta_p(y_{n_k},y^*)\big]\\
&-\big[\Delta_p(x_{n_k+1},x^*)+\Delta_p(y_{n_k+1},y^*)\big]\Bigg)+\limsup_{k\to\infty}\frac{\gamma_{n_k}^q}{q}\psi_{n_k}\\
&\le -\limsup_{k\to\infty}\Bigg(\big[\Delta_p(x_{n_k+1},x^*)+\Delta_p(y_{n_k+1},y^*)\big]-\big[\Delta_p(x_{n_k},x^*)+\Delta_p(y_{n_k},y^*)\big]\Bigg)\\
&+\limsup_{k\to\infty}\frac{\gamma_{n_k}^q}{q}\psi_{n_k}\\
&\le 0.
\end{aligned}\tag{7.19}$$

Therefore, we conclude from (7.15), (7.16) and (7.19) that

$$\lim_{k\to\infty}\|Ax_{n_k}-By_{n_k}\|^p=0,\tag{7.20}$$

$$\begin{cases}\lim_{k\to\infty}\Delta_p(z_{n_k},u_{n_k})=0,\\ \lim_{k\to\infty}\Delta_p(t_{n_k},z_{n_k})=0,\\ \lim_{k\to\infty}\Delta_p(g_{n_k},v_{n_k})=0,\\ \lim_{k\to\infty}\Delta_p(h_{n_k},g_{n_k})=0.\end{cases}\tag{7.21}$$

Therefore, by Lemma 2.5.31, we obtain

$$\begin{cases}\lim_{k\to\infty}\|z_{n_k}-u_{n_k}\|=0,\\ \lim_{k\to\infty}\|t_{n_k}-z_{n_k}\|=0,\\ \lim_{k\to\infty}\|g_{n_k}-v_{n_k}\|=0,\\ \lim_{k\to\infty}\|h_{n_k}-g_{n_k}\|=0.\end{cases}\tag{7.22}$$

Also, by Lemma 2.5.47, (7.22) and Lemma 2.5.31, we have

$$\lim_{k\to\infty}\|t_{n_k}-u_{n_k}\|=0\tag{7.23}$$

Similarly, we have

$$\lim_{k\to\infty} \|v_{n_k} - h_{n_k}\| = 0. \tag{7.24}$$

Furthermore, let

$\tau_n = J_q^{E_1^*}\big(\beta_n J_p^{E_1}(t_n) + (1-\beta_n)J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n)\big)$ and $\eta_n = J_q^{E_2^*}\big(\beta_n J_p^{E_2}(h_n) + (1-\beta_n)J_p^{E_2}(S(\mathrm{Res}_{\sigma B_2}^g \circ A_{2\sigma}^g)h_n)\big)$. Then by (2.14) and Lemma 7.3.1, we have

$$\begin{aligned}
\Delta_p(\tau_n, x^*) &= \Delta_p\Big(J_q^{E_1^*}\big(\beta_n J_p^{E_1}(t_n) + (1-\beta_n)J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n)\big), x^*\Big)\\
&= \frac{1}{p}\|x^*\|^p - \beta_n\langle J_p^{E_1}t_n, x^*\rangle - (1-\beta_n)\langle J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n), x^*\rangle\\
&+ \frac{1}{q}\|\beta_n J_p^{E_1}(t_n) + (1-\beta_n)J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n)\|^q\\
&\le \frac{1}{p}\beta_n\|x^*\|^p + (1-\beta_n)\frac{1}{p}\|x^*\|^p - \beta_n\langle J_p^{E_1}t_n, x^*\rangle - (1-\beta_n)\langle J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n), x^*\rangle\\
&+ \frac{1}{q}\beta_n\|t_n\|^p + \frac{(1-\beta_n)}{q}\|T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n\|^P\\
&- \frac{W_q(\beta_n)}{q}g\big(\|J_p^{E_1}t_n - J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n)\|\big)\\
&= \beta_n\Delta_p(t_n, x^*) + (1-\beta_n)\Delta_p((T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n), x^*)\\
&- \frac{W_q(\beta_n)}{q}g\big(\|J_p^{E_1}t_n - J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n)\|\big)\\
&\le \Delta_p(t_n, x^*) - \frac{W_q(\beta_n)}{q}g\big(\|J_p^{E_1}t_n - J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n)\|\big)\\
&\le \Delta_p(u_n, x^*) - \frac{W_q(\beta_n)}{q}g\big(\|J_p^{E_1}t_n - J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n)\|\big).
\end{aligned} \tag{7.25}$$

Combining (7.25) and (7.2), we have,

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) &= \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\Big[\Delta_p(u_n, x^*)\\
&- \frac{W_q(\beta_n)}{q}g\big(\|J_p^{E_1}t_n - J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n)\|\big)\Big]\\
&\le \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(x_n, x^*)\\
&- (1-\alpha_n)\frac{W_q(\beta_n)}{q}g\big(\|J_p^{E_1}(t_n) - J_p^{E_1}(T(\mathrm{Res}_{\sigma B_1}^g \circ A_{1\sigma}^g)t_n)\|\big).
\end{aligned} \tag{7.26}$$

Similarly, we obtain

$$\begin{aligned}
\Delta_p(y_{n+1}, y^*) \le \alpha_n\Delta_p(v, y^*) + (1-\alpha_n)\Delta_p(y_n, y^*) - (1-\alpha_n)\frac{W_q(\beta_n)}{q}g\big(\|J_p^{E_2}(h_n)\\
- J_p^{E_2}(S(\mathrm{Res}_{\sigma B_2}^g \circ A_{2\sigma}^g)h_n)\|\big).
\end{aligned} \tag{7.27}$$

Adding (7.26) and (7.27), we have

$$\begin{aligned}
\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*) &\le \alpha_n[\Delta_p(u,x^*)+\Delta_p(v,y^*)]+(1-\alpha_n)[\Delta_p(x_n,x^*)\\
&\quad+\Delta_p(y_n,y^*)]-(1-\alpha_n)\frac{W_q(\beta_n)}{q}g\big(\|J_p^{E_1}(t_n)-J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1}\circ A^g_{1\sigma})t_n)\|\\
&\quad+\|J_p^{E_2}(h_n)-J_p^{E_2}(S(\mathrm{Res}^g_{\sigma B_2}\circ A^g_{2\sigma})h_n)\|\big). \qquad (7.28)
\end{aligned}$$

Using (7.11) and (7.28), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\Bigg((1-\alpha_{n_k})\frac{W_q(\beta_{n_k})}{q}g\big(\|J_p^{E_1}(t_{n_k})-J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1}\circ A^g_{1\sigma})t_{n_k})\|\\
&\quad+\|J_p^{E_2}(h_{n_k})-J_p^{E_2}(S(\mathrm{Res}^g_{\sigma B_2}\circ A^g_{2\sigma})h_{n_k})\|\big)\Bigg)\\
&\quad\le \limsup_{k\to\infty}\Bigg(\alpha_{n_k}[\Delta_p(u,x^*)+\Delta_p(v,y^*)]+(1-\alpha_{n_k})[\Delta_p(x_{n_k},x^*)+\Delta_p(y_{n_k},y^*)]\\
&\quad-[\Delta_p(x_{n_{k+1}},x^*)+\Delta_p(y_{n_{k+1}},y^*)]\Bigg)\\
&\quad\le -\limsup_{k\to\infty}\Big([\Delta_p(x_{n_{k+1}},x^*)+\Delta_p(y_{n_{k+1}},y^*)]-\Delta_p[(x_{n_k},x^*)+(y_{n_k},y^*)]\Big)\\
&\quad\le 0. \qquad (7.29)
\end{aligned}$$

Hence,

$$\lim_{k\to\infty}\frac{W_q(\beta_{n_k})}{q}g\big(\|J_p^{E_1}(t_{n_k})-J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1}\circ A^g_{1\sigma})t_{n_k})\|\big)=0.$$

Therefore, we obtain that

$$\lim_{k\to\infty}g\big(\|J_p^{E_1}(t_{n_k})-J_p^{E_1}(T(\mathrm{Res}^g_{\sigma B_1}\circ A^g_{1\sigma})t_{n_k})\|\big)=0.$$

Since g is continuous and the fact that $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets, we have

$$\lim_{k\to\infty}\|T(\mathrm{Res}^g_{\sigma B_1}\circ A^g_{1\sigma})t_{n_k}-t_{n_k}\|=0. \qquad (7.30)$$

Similarly, we obtain

$$\lim_{k\to\infty}\|S(\mathrm{Res}^g_{\sigma B_2}\circ A^g_{2\sigma})h_{n_k}-h_{n_k}\|=0. \qquad (7.31)$$

From the definition of u_n, we have

$$\begin{aligned}
\|J_p^{E_1}(u_{n_k})-J_p^{E_1}(x_{n_k})\| &= \|J_p^{E_1}(x_{n_k})-\gamma_{n_k}A^*J_p^{E_3}(Ax_{n_k}-By_{n_k})-J_p^{E_1}(x_{n_k})\|\\
&=\gamma_{n_k}\|A^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|\to 0 \quad \text{as} \quad k\to\infty. \qquad (7.32)
\end{aligned}$$

Furthermore, since E_1 is uniformly smooth, $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1, then we have

$$\lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0. \tag{7.33}$$

Similarly, we have

$$\lim_{k\to\infty} \|v_{n_k} - y_{n_k}\| = 0. \tag{7.34}$$

Observe that from (7.23) and (7.33), we have

$$\|t_{n_k} - x_{n_k}\| \leq \|t_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0 \quad \text{as} \quad n \to \infty. \tag{7.35}$$

Similarly, we obtain from (7.24) and (7.34) that

$$\|h_{n_k} - y_{n_k}\| \leq \|h_{n_k} - v_{n_k}\| + \|v_{n_k} - y_{n_k}\| \to 0 \quad \text{as} \quad n \to \infty. \tag{7.36}$$

Now, from (7.2) and Lemma 2.5.31, we have

$$\lim_{k\to\infty} \|J_p^{E_1} x_{n_{k+1}} - J_p^{E_1} t_{n_k}\| = 0. \tag{7.37}$$

Similarly, from (7.2) and Lemma 2.5.31, we have

$$\lim_{k\to\infty} \|J_p^{E_2} y_{n_{k+1}} - J_p^{E_2} h_{n_k}\| = 0. \tag{7.38}$$

Now, applying the fact that $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1, we get

$$\lim_{k\to\infty} \|x_{n_{k+1}} - t_{n_k}\| = 0. \tag{7.39}$$

Similarly, we obtain

$$\lim_{k\to\infty} \|y_{n_{k+1}} - h_{n_k}\| = 0. \tag{7.40}$$

We can therefore conclude from (7.35) and (7.39) that

$$\lim_{k\to\infty} \|x_{n_{k+1}} - x_{n_k}\| = 0. \tag{7.41}$$

Similarly, from (7.36) and (7.40), we have

$$\lim_{k\to\infty} \|y_{n_{k+1}} - y_{n_k}\| = 0. \tag{7.42}$$

From (7.35) and (7.36), there exist subsequences $\{t_{n_{k_j}}\}$ of $\{t_{n_k}\}$ and $\{h_{n_{k_j}}\}$ of $\{h_{n_k}\}$ that converge weakly to $\bar{x} \in C$ and $\bar{y} \in Q$ respectively. Using (7.30) and (7.31) together with Lemma 2.5.54 and Lemma 7.2.1, we have that $\bar{x} \in \mathrm{Fix}(T) \cap \mathrm{Fix}(\mathrm{Res}^g_{\sigma B_1} \circ A^g_{1\sigma}) = \hat{\mathrm{Fix}}(T) \cap \hat{\mathrm{Fix}}(\mathrm{Res}^g_{\sigma B_1} \circ A^g_{1\sigma})$ and $\bar{y} \in \mathrm{Fix}(S) \cap \mathrm{Fix}(\mathrm{Res}^g_{\sigma B_2} \circ A^g_{2\sigma}) = \hat{\mathrm{Fix}}(S) \cap \hat{\mathrm{Fix}}(\mathrm{Res}^g_{\sigma B_2} \circ A^g_{2\sigma})$.

Next, we show that $\bar{x} \in \text{EP}(f_1, C)$ and $\bar{y} \in \text{EP}(f_2, Q)$. We know from Lemma 2.5.47(i) that

$$\begin{aligned}\lambda_{n_k}[f_1(u_{n_k}, z_{n_k}) - f_1(u_{n_k}, w)] &\leq \langle J_p^{E_1} u_{n_k} - J_p^{E_1} z_{n_k}, z_{n_k} - w\rangle \\ &\leq \|J_p^{E_1} u_{n_k} - J_p^{E_1} z_{n_k}\| \|z_{n_k} - w\|. \end{aligned} \tag{7.43}$$

By the uniform continuity of $J_p^{E_1}$ and using (7.22) in (7.43), we have

$$\lambda_{n_k}[f_1(\bar{x}, x) - f_1(\bar{x}, w)] \leq 0. \tag{7.44}$$

From (7.44), we obtain that $f_1(\bar{x}, w) \geq 0$. Similarly, we have that $f_2(\bar{y}, w) \geq 0$. Hence, $\bar{x} \in \text{EP}(f_1, C)$ and $\bar{y} \in \text{EP}(f_2, Q)$.

Also, we show that $\|A\bar{x} - B\bar{y}\| = 0$. Since $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators, $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converges weakly to $\bar{x}$ and $\bar{y}$ respectively. We let $d \in E_3^*$ be arbitrary such that

$$d(Ax_{n_k}) = (d \circ A)(x_{n_k}) \to (d \circ A)(\bar{x}) = d(A\bar{x}).$$

Similarly, we have

$$d(By_{n_k}) = (d \circ B)(y_{n_k}) \to (d \circ A)(\bar{y}) = d(B\bar{y}).$$

This implies that $Ax_{n_k} - By_{n_k} \rightharpoonup A\bar{x} - B\bar{y}$. Also, by weakly semi-continuity of the norm, it follows that

$$\|A\bar{x} - B\bar{y}\|^p \leq \liminf_{k\to\infty} \|A\bar{x}_{n_k} - B\bar{y}_{n_k}\|^p = 0. \tag{7.45}$$

Thus, $A\bar{x} = B\bar{y}$. Therefore $(\bar{x}, \bar{y}) \in \Gamma$.

We now show that the sequence $\{(x_n, y_n)\}$ converges strongly to (x^*, y^*). Recall that since $\{x_{n_k}\}$, $\{y_{n_k}\}$ are bounded, there exist subsequences $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ and $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ which converges weakly to $\bar{x}$, $\bar{y}$ in E_1 and E_2 respectively such that

$$\begin{aligned}&\limsup_{k\to\infty}\Big(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_k+1} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n_k+1} - y^*\rangle\Big) \\ &\quad\leq \lim_{j\to\infty}\Big(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_{k_j}+1} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n_{k_j}+1} - y^*\rangle\Big) \\ &\quad= \Big(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), \bar{x} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), \bar{y} - y^*\rangle\Big). \end{aligned} \tag{7.46}$$

Hence, we obtain that

$$\begin{aligned}&\limsup_{k\to\infty}\Big(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_k+1} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n_k+1} - y^*\rangle\Big) \\ &\qquad= \Big(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), \bar{x} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), \bar{y} - y^*\rangle\Big) \\ &\qquad\leq 0. \end{aligned} \tag{7.47}$$

On substituting (7.47), and Lemma 2.5.47 into (7.10), we conclude that $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) \in \Gamma$. $\square$

7.2.2 Application

Convex minimization problem

Let E_1, E_2 and E_3 be three p-uniformly convex real Banach spaces which are also uniformly smooth and $g_1 : E_1 \to (-\infty, +\infty]$, $g_2 : E_2 \to (-\infty, +\infty]$ be proper, convex and lower semi-continuous functions which attains their minimum over E_1 and E_2 respectively. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be two bounded linear operators and $T : E_1 \to E_1$, $S : E_2 \to E_2$ be Bregman relatively nonexpansive mappings such that $\mathrm{Fix}(T) = \hat{\mathrm{Fix}}(T)$ and $\mathrm{Fix}(S) = \hat{\mathrm{Fix}}(S)$.

Now, consider the following Split Equality Fixed Point Convex Minimization (SEFPCMP) [119]: Find $x^* \in \mathrm{Fix}(T)$ and $y^* \in \mathrm{Fix}(S)$ such that

$$g_1(x^*) = \min_{x \in E_1} g_1(x), \tag{7.48}$$

$$g_2(y^*) = \min_{y \in E_2} g_2(y), \quad \text{and} \quad Ax^* = By^*. \tag{7.49}$$

It is well known that the above SEFPCMP can be formulated as follows: Find $x^* \in \mathrm{Fix}(T)$ and $y^* \in \mathrm{Fix}(S)$ such that

$$0 \in \partial g_1(x^*), \tag{7.50}$$

$$0 \in \partial g_2(y^*), \quad \text{and} \quad Ax^* = By^*. \tag{7.51}$$

where ∂g_1 and ∂g_2 are the subdifferentials of g_1 and g_2 respectively. It is a known fact that the subdifferentials ∂g_1 and ∂g_2 are maximal monotone whenever g_1 and g_2 are proper, convex and lower semi-continuous functions. Hence, by applying Algorithm (7.2), we obtain the solution of the SCFPCMP (7.48)-(7.49).

Split equality variational inequality problem

Consider the particular split equality equilibrium problem corresponding to the functions f_1 and f_2 defined by:

$$f_1(x, y) = \langle Mx, y - x \rangle, \quad \forall\, x, y \in C,$$
$$f_2(x, y) = \langle Nx, y - x \rangle, \quad \forall x, y \in Q,$$

with $M : C \subset E_1 \to E_3^*$ and $N : Q \subset E_2 \to E_3^*$. The classical split equality variational inequality problem is of the form:

$$\begin{aligned} &\text{find} \quad x^* \in C \quad \text{such that} \quad \langle Mx^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \\ &\text{and} \quad y^* \in Q \quad \text{such that} \quad \langle Ny^*, z - y^* \rangle \geq 0, \quad \forall z \in Q, \\ &\qquad\qquad\qquad\qquad \text{such that} \quad Mx^* = Ny^*. \end{aligned} \tag{7.52}$$

When considered separately, (7.139) is the classical Variational Inequality Problem (VIP) whose solution set is $\mathrm{VI}(M,C)$ and $\mathrm{VI}(N,Q)$, respectively. Variational inequalities have been found very useful in several real-world problems such as optimization problems, minimax theorems, differential equations and in certain applications to economic theory and mechanics. For more study of systems of variational inequalities (see, [54, 130] and the references contained therein).

Lemma 7.2.7. *[82] Let C be a nonempty, closed convex subset of a reflexive, smooth and strictly convex Banach space E, $A : C \to E^*$ be a mapping. Then*

$$\Pi_C\Big(J_q^{E^*}[J_p^E(x) - \lambda M(y)]\Big) = \arg\min_{w\in C}\{\lambda\langle w-y, M(y)\rangle + \Delta_p(w,x)\}, \tag{7.53}$$

for all $x \in E$, $y \in C$ and $\lambda \in (0,+\infty)$.

Setting $f_1(x,y) = \langle Mx, y-x\rangle, \ \forall\, x,y \in C$, and $f_2(x,y) = \langle Nx, y-x\rangle, \ \forall x,y \in Q$ in Algorithm (7.2), we obtain the following important result.
Suppose that

$$\begin{aligned} \Omega := \{x^* \in Fix(T) \cap (A_1+B_1)^{-1}(0) \cap \mathrm{VI}(M,C),\\ y^* \in Fix(S) \cap (A_2+B_2)^{-1}(0) \cap \mathrm{VI}(N,Q) \ \text{ such that } \ Mx^* = Ny^*\} \neq \emptyset. \end{aligned} \tag{7.54}$$

Algorithm 7.2.8. *For fixed $u \in E_1$ and $v \in E_2$, choose an initial guess $(x_1,y_1) \in E_1\times E_2$. Suppose that the nth iterate $(x_n,y_n) \subset E_1\times E_2$ has been constructed; then we compute the $(n+1)$th iterate (x_{n+1},y_{n+1}) via the iteration*

$$\begin{cases} u_n = J_q^{E_1^*}\Big(J_p^{E_1}(x_n) - \gamma_n A^* J_p^{E_3}(Ax_n - By_n)\Big),\\ z_n = \Pi_C[J_p^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n M(u_n))],\\ t_n = \Pi_C[J_p^{E_1^*}(J_p^{E_1}(z_n) - \lambda_n N(u_n))],\\ x_{n+1} = J_q^{E_1^*}\Bigg(\alpha_n J_p^{E_1}(u) + (1-\alpha_n)\Big[\beta_n J_p^{E_1}(t_n) + (1-\beta_n)J_p^{E_1}(T(Res^g_{\sigma B_1}\circ A^g_{1\sigma})t_n)\Big]\Bigg),\\ v_n = J_q^{E_2^*}\Big(J_p^{E_2}(y_n) + \gamma_n B^* J_p^{E_3}(Ax_n - By_n)\Big),\\ g_n = \Pi_C[J_p^{E_2^*}(J_p^{E_2}(v_n) - \lambda_n N(v_n))]\\ h_n = \Pi_C[J_p^{E_2^*}(J_p^{E_2}(u_n) - \lambda_n N(g_n))]\\ y_{n+1} = J_q^{E_2^*}\Bigg(\alpha_n J_p^{E_2}(v) + (1-\alpha_n)\Big[\beta_n J_p^{E_2}(h_n) + (1-\beta_n)J_p^{E_2}(S(Res^g_{\sigma B_2}\circ A^g_{2\sigma})h_n)\Big]\Bigg), \end{cases} \tag{7.55}$$

Further, we choose the stepsize γ_n such that
$\gamma_n^{q-1} \in \left(\epsilon, \left(\frac{q\|Ax_n - By_n\|^p}{C_q\|A^*J_p^{E3}(Ax_n-By_n)\|^q + Q_q\|B^*J_p^{E3}(Ax_n-By_n)\|^q} - \epsilon\right)^{\frac{1}{q-1}}\right)$ *$n \in \Gamma$, for small enough ϵ, where C_q and Q_q are constants of smoothness of E_1 and E_2, respectively. Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value).*

Suppose the following conditions are satisfied:

(i) $\alpha_n \in (0,1), \quad \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$

(ii) $\beta_n \in (0,1)$ and $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$

(iii) $\{\lambda_n\} \subset (0,p)$, where $p = \frac{2\tau}{L}$ and τ is given by (2.18).

Corollary 7.2.9. *Let $f_1(x,y) = \langle Mx, y-x\rangle, \quad \forall\, x,y \in C$, and $f_2(x,y) = \langle Nx, y-x\rangle, \quad \forall x,y \in Q$ and (7.54) holds, then the sequences $\{x_n, y_n\}$ generated by Algorithm (7.55) converges strongly $(x^*, y^*) \in \Omega$.*

Proof. The proof is similar to the proof in [81]. □

7.3 Multiple set split equality equilibrium and fixed point problems.

In this section, using an Halpern extragradient method, we study a new iterative scheme for finding a common element of the set of solutions of multiple set split equality equilibrium problems consisting of pseudomonotone bifunctions and the set of fixed points for two finite families of Bregman quasi-nonexpansive mappings in the framework of p-uniformly convex Banach spaces which are also uniformly smooth. For this purpose, we design an algorithm so that it does not depend on prior estimates of the Lipschitz-type constants for the pseudomonotone bifunctions. Furthermore, we present an application of our study to investigate a common element of the set of solutions of multiple set split equality variational inequality problems and fixed points set for two finite families of Bregman quasi-nonexpansive mappings. Finally, we conclude with two numerical experiments to support our proposed algorithm.
We need the following already established result in the sequel.

Lemma 7.3.1. *[245] Let $q \ge 1$ and $r > 0$ be two fixed real numbers. Then, a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^*$, $g(0) = 0$ such that for all $x, y \in B_r$ and $0 \le \alpha < 1$,*

$$\|\alpha x + (1-\alpha)y\|^q \le \alpha\|x\|^q + (1-\alpha)\|y\|^q - W_q(\alpha)g(\|x-y\|),$$

where $W_q(\alpha) := \alpha^q(1-\alpha) + \alpha(1-\alpha)^q$ and $B_r := \{x \in E : \|x\| \le r\}$.

7.3.1 Main result

In this section, we present our method and discuss some of its features. We begin with the following assumptions under which our strong convergent result is obtained.

Assumption 7.3.2. *We assume that the following conditions hold:*

(1) (a) E_1, E_2 and E_3 are three p-uniformly convex and uniformly smooth real Banach spaces.
(b) C_i and Q_j are nonempty closed and convex subsets of E_1 and E_2, respectively, for $i = 1, 2, ..., N$ and $j = 1, 2, ..., M$.
(c) $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators.
(d) $f_i : C_i \times C_i \to \mathbb{R}$ and $g_j : Q_j \times Q_j \to \mathbb{R}$ are bifunctions satisfying conditions $C_1 - C_4$ of **Assumption A.**
(e) $D_s : E_1 \to E_1$ and $G_t : E_2 \to E_2$ are Bregman quasi-nonexpansive mappings such that $I - D_s$ and $I - G_t$ are demiclosed at zero for each $s = 1, 2, ..., l$ and $t = 1, 2, ..., m$.
(f) Assume that the solution set
$\Upsilon := \{\bar{x} \in \bigcap_{s=1}^{l} F(D_s) \cap \bigcap_{i=1}^{N} EP(C_i, f_i), \quad \bar{y} \in \bigcap_{t=1}^{m} F(G_t) \cap \bigcap_{j=1}^{M} EP(Q_j, g_j) : A\bar{x} = B\bar{y}\} \neq 0.$

(2) $\{\beta_n\}_{n=1}^{\infty}$, $\{\alpha_{n,s}\}_{s=0}^{l}$, $\{\eta_{n,t}\}_{t=0}^{m}$ are positive sequences satisfying the following conditions:
(a)$\{\beta_n\} \subset (0, 1), \quad \lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty, \tau_0 > 0, \lambda_0 > 0, \kappa \in (0, 1), \epsilon \in (0, 1).$
(b) $\{\alpha_{n,s}\} \subset (0, 1), \quad \sum_{s=0}^{l} \alpha_{n,s} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,s} > 0.$
(c) $\{\eta_{n,t}\} \subset (0, 1), \quad \sum_{t=0}^{m} \eta_{n,t} = 1$ and $\liminf_{n\to\infty} \eta_{n,0}\eta_{n,t} > 0.$

We now present the proposed method of this paper.

Algorithm 7.3.3. *For fixed $\mu \in E_1$ and $\vartheta \in E_2$, choose an initial guess $(x_0, y_0) \in E_1 \times E_2$. Suppose that the nth iterate $(x_n, y_n) \subset E_1 \times E_2$ has been constructed; then we compute the*

$(n+1)$th iterate (x_{n+1}, y_{n+1}) via the iteration

$$\begin{cases} s_n = J^q_{E_1^*}\Big(J^p_{E_1}(x_n) - \rho_n A^* J^p_{E_3}(Ax_n - By_n)\Big), \\ a^i_n = \arg\min\{f_i(s_n,\sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n): \quad \sigma \in C_i\}, \\ z^i_n = \arg\min\{f_i(a^i_n,\sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n): \quad \sigma \in C_i\}. \\ \textit{Obtain the farthest element of } z^i_n \textit{ from } s_n, \textit{ i.e.,} \\ i_n \in \arg\max\{\Delta_p(s_n, z^i_n) : i = 1, ..., N\}. \\ \textit{Set} \;\; z^{i_n}_n = \bar{z}_n \\ u_n = J^{E_1^*}_q\left(\alpha_{n,0} J^p_{E_1}(\bar{z}_n) + \sum_{s=1}^l \alpha_{n,s} J^p_{E_1}(D_s \bar{z}_n)\right) \\ x_{n+1} = J^q_{E_1^*}\left(\beta_n J^p_{E_1}(\mu) + (1-\beta_n) J^p_{E_1}(u_n)\right), \\ t_n = J^q_{E_2^*}\Big(J^p_{E_2}(y_n) + \rho_n B^* J^p_{E_3}(Ax_n - By_n)\Big), \\ b^j_n = \arg\min\{g_j(t_n,\varphi) + \frac{1}{\lambda_n}\Delta_p(\varphi, t_n): \quad \varphi \in Q_j\}, \\ h^j_n = \arg\min\{g_j(b^j_n,\varphi) + \frac{1}{\lambda_n}\Delta_p(\varphi, t_n): \quad \varphi \in Q_j\}. \\ \textit{Obtain the farthest element of } h^j_n \textit{ from } t_n, \textit{ i.e.,} \\ j_n \in \arg\max\{\Delta_p(t_n, h^j_n) : j = 1, ..., M\}. \\ \textit{Set} \;\; h^{j_n}_n = \bar{\theta}_n \\ v_n = J^q_{E_1^*}\left(\eta_{n,0} J^p_{E_2}(\bar{\theta}_n) + \sum_{t=1}^m \eta_{n,t} J^p_{E_2}(G_t \bar{\theta}_n)\right) \\ y_{n+1} = J^q_{E_2^*}\left(\beta_n J^p_{E_2}(\vartheta) + (1-\beta_n) J^{E_2}_p(v_n)\right), \end{cases} \tag{7.56}$$

where

$$\rho_n \in \left(\zeta, \left(\frac{q\|Ax_n - By_n\|^p}{C_q\|A^* J^p_{E_3}(Ax_n - By_n)\|^q + Q_q\|B^* J^p_{E_3}(Ax_n - By_n)\|^q} - \zeta\right)^{\frac{1}{q-1}}\right), \quad n \in \Omega, \tag{7.57}$$

for small enough ζ; C_q and Q_q are constants of smoothness of E_1 and E_2, respectively. Otherwise, $\rho_n = \rho$ (ρ being any nonnegative value), where the set of indexes $\Omega = \{n : Ax_n - By_n \neq 0\}$.

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \min\limits_{1\le i\le N}\left\{\frac{\kappa(\Delta_p(a^i_n, s_n) + \Delta_p(z^i_n, a^i_n))}{f_i(s_n, z^i_n) - f_i(s_n, a^i_n) - f_i(a^i_n, z^i_n)}\right\}\right\}, & \textit{if} \;\; f_i(s_n, z^i_n) - f_i(s_n, a^i_n) - f_i(a^i_n, z^i_n) > 0, \\ \tau_n, & \textit{otherwise.} \end{cases} \tag{7.58}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \min\limits_{1\le j\le M}\left\{\frac{\epsilon(\Delta_p(b^j_n, t_n) + \Delta_p(h^j_n, b^j_n))}{g_j(t_n, h^j_n) - g_j(t_n, b^j_n) - g_j(b^j_n, h^j_n)}\right\}\right\}, & \textit{if } g_j(t_n, h^j_n) - g_j(t_n, b^j_n) - g_j(b^j_n, h^j_n) > 0, \\ \lambda_n, & \textit{otherwise.} \end{cases} \tag{7.59}$$

Remark 7.3.4.

(a) Algorithm 7.3.3 solves split equality equilibrium problem consisting of two strongly convex optimization problems in parallel for $i = 1, 2, ..., N$ as well as another two strongly convex optimization problems in parallel for $j = 1, 2, ..., M$ under bounded linear operators.

(b) The step size $\{\rho_n\}$ given by (7.57) is generated at each iteration by some simple computations. Thus, $\{\rho_n\}$ is easily implemented without the prior knowledge of the operator norms $\|A\|$ and $\|B\|$. Similarly, the step size $\{\tau_n\}$ given by (7.58) and step size $\{\lambda_n\}$ given by (7.59) does not depend on the prior estimates of the Lipschitz-like constants of the pseudomonotone bifunctions f_i, $\quad i = 1, 2, ..., N$ and g_j, $\quad j = 1, 2, ..., M$, unlike the step sizes used in [82, 112] which require finding the prior estimates of the Lipschitz-like constants of the pseudomonotone bifunctions, which is known to be computationally expensive.

(c) Moreover, our result in this study extends the results in [112, 133, 139] from the framework of Hilbert spaces to Banach spaces.

7.3.2 Convergence analysis

Lemma 7.3.5. *The sequences $\{\tau_n\}$ and $\{\lambda_n\}$ of step sizes generated by Algorithm 7.3.3 are well-defined and bounded.*

Proof. Clearly, from (7.58) and (7.59) we have $\tau_{n+1} \leq \tau_n \ \ \forall\, n \in \mathbb{N}$ and $\lambda_{n+1} \leq \lambda_n \ \ \forall\, n \in \mathbb{N}$. This implies that $\{\tau_n\}$ and $\{\lambda_n\}$ are monotonically decreasing sequences. Moreover, it follows from condition C_2 of **Assumption A** that

$$f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f_i(a_n^i, z_n^i) \leq k_{1,i}\Delta_p(a_n^i, s_n) + k_{2,i}\Delta_p(z_n^i, a_n^i), \quad \forall\, i = 1, 2, ..., N.$$

Hence, we obtain for all $i = 1, 2, ..., N$

$$\begin{aligned}
\frac{\kappa\,(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i))}{f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f_i(a_n^i, z_n^i)} &\geq \frac{\kappa\,(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i))}{k_{1,i}\Delta_p(a_n^i, s_n) + k_{2,i}\Delta_p(z_n^i, a_n^i)} \\
&\geq \frac{\kappa\,(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i))}{\max\{k_{1,i}, k_{2,i}\}(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i))} \\
&\geq \frac{\kappa}{\max\{k_{1,i}, k_{2,i}\}}.
\end{aligned}$$

Similarly, we obtain

$$\frac{\epsilon\,(\Delta_p(b_n^j, t_n) + \Delta_p(h_n^j, b_n^j))}{g_j(t_n, h_n^j) - g_j(t_n, b_n^j) - g_j(b_n^j, h_n^j)} \geq \frac{\epsilon}{\max\{c_{1,j}, c_{2,j}\}}, \quad \forall\, j = 1, 2, ..., M.$$

Hence, we conclude that $\{\tau_n\}$ has lower bound $\min\{\tau_0, \frac{\kappa}{\max\limits_{1\leq i\leq N}\{k_{1,i}, k_{2,i}\}}\} > 0$ and $\{\lambda_n\}$ has lower bound
$\min\{\lambda_0, \frac{\epsilon}{\max\limits_{1\leq j\leq M}\{c_{1,j}, c_{2,j}\}}\} > 0$. It then follows that $\lim\limits_{n\to\infty} \tau_n = \tau > 0$ and $\lim\limits_{n\to\infty} \lambda_n = \lambda > 0$. $\square$

Lemma 7.3.6. *Let $C_i, i = 1, 2, ..., N$ and $Q_j, j = 1, 2, ..., M$ be nonempty, closed and convex subsets of E_1 and E_2, respectively. Suppose that $f_i : C_i \times C_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$ and $g_j : Q_j \times Q_j \to \mathbb{R}$, $j = 1, 2, \ldots, M$ are bifunctions satisfying conditions $C_1 - C_4$. Then, for all $(\bar{x}, \bar{y}) \in \Upsilon$ we have*

$$\Delta_p(\bar{x}, z_n^i) \leq \Delta_p(\bar{x}, s_n) - \left(1 - \kappa \frac{\tau_n}{\tau_{n+1}}\right) \left(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i)\right), \quad \forall\, i = 1, 2, ..., N \tag{7.60}$$

and

$$\Delta_p(\bar{y}, h_n^j) \leq \Delta_p(\bar{y}, t_n) - \left(1 - \epsilon \frac{\lambda_n}{\lambda_{n+1}}\right) \left(\Delta_p(b_n^j, t_n) + \Delta_p(h_n^j, b_n^j)\right), \quad \forall\, j = 1, 2, ..., M. \tag{7.61}$$

Proof. Since $z_n^i = \arg\min\{f_i(a_n^i, \sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n) : \ \sigma \in C_i\}$, then from Lemma 2.5.44 we get

$$0 \in \partial_2(\tau_n f_i(a_n^i, z_n^i) + \Delta_p(z_n^i, s_n)) + N_{C_i}(z_n^i).$$

Then, there exists $\xi \in \partial_2 f_i(a_n^i, z_n^i)$, $\bar{\xi} \in N_{C_i}(z_n^i)$, such that

$$\tau_n \xi + J_{E_1}^p(z_n^i) - J_{E_1}^p(s_n) + \bar{\xi} = 0. \tag{7.62}$$

Also, by the definition of $\partial_2 f_i(a_n^i, z_n^i)$, we obtain

$$f_i(a_n^i, \sigma) - f_i(a_n^i, z_n^i) \geq \langle \sigma - z_n^i, \xi \rangle, \quad \forall\, \sigma \in C_i.$$

If we replace σ with $\bar{x}$ in the inequality above, we have

$$f_i(a_n^i, \bar{x}) - f_i(a_n^i, z_n^i) \geq \langle \bar{x} - z_n^i, \xi \rangle, \quad \forall\, \bar{x} \in \Upsilon. \tag{7.63}$$

Using the definition of $N_{C_i}(z_n^i)$ together with (7.62), we have

$$\langle \sigma - z_n^i, J_{E_1}^p(z_n^i) - J_{E_1}^p(s_n) \rangle \geq \tau_n \langle z_n^i - \sigma, \xi \rangle, \quad \forall\, \sigma \in C_i. \tag{7.64}$$

Again, if we let $\sigma = \bar{x}$ in (7.64), we get

$$\langle \bar{x} - z_n^i, J_{E_1}^p(z_n^i) - J_{E_1}^p(s_n) \rangle \geq \tau_n \langle z_n^i - \bar{x}, \xi \rangle, \quad \forall\, \bar{x} \in \Upsilon. \tag{7.65}$$

The combination of (7.63) and (7.65) gives

$$\begin{aligned} \langle \bar{x} - z_n^i, J_{E_1}^p(z_n^i) - J_{E_1}^p(s_n) \rangle &\geq \tau_n \langle f_i(a_n^i, z_n^i) - f_i(a_n^i, \bar{x}) \rangle \\ &\geq \tau_n f_i(a_n^i, z_n^i), \end{aligned} \tag{7.66}$$

because $f_i(\bar{x}, a_n^i) \geq 0$ and f_i is pseudomonotone on $C_i \quad \forall \quad i = 1, 2, ..., N$. Similarly, since $a_n^i = \arg\min\{f_i(s_n, \sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n) : \quad \sigma \in C_i\}$, we obtain

$$\langle a_n^i - z_n^i, J_{E_1}^p(a_n^i) - J_{E_1}^p(s_n)\rangle \geq \tau_n \left[f_i(s_n, z_n^i) - f_i(s_n, a_n^i)\right]. \tag{7.67}$$

Using (7.66) and (7.67) together, we get

$$\begin{aligned} &\langle \bar{x} - z_n^i, J_{E_1}^p(z_n^i) - J_{E_1}^p(s_n)\rangle + \langle a_n^i - z_n^i, J_{E_1}^p(a_n^i) - J_{E_1}^p(s_n)\rangle \\ &\qquad\qquad\qquad\qquad \geq \tau_n \left[f_i(s_n, z_n^i) - f_i(s_n, a_n^i) + f_i(a_n^i, z_n^i)\right]. \end{aligned}$$

Applying Bregman three point identity (2.13), we obtain

$$\Delta_p(\bar{x}, z_n^i) \leq \Delta_p(\bar{x}, s_n) - \Delta_p(a_n^i, s_n) - \Delta_p(z_n^i, a_n^i) + \tau_n\{f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f(a_n^i, z_n^i)\}.$$

Furthermore, by the definition of τ_n, we obtain

$$\begin{aligned} \Delta_p(\bar{x}, z_n^i) &\leq \Delta_p(\bar{x}, s_n) - \Delta_p(a_n^i, s_n) - \Delta_p(z_n^i, a_n^i) \\ &\quad + \frac{\tau_n}{\tau_{n+1}}\tau_{n+1}\{f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f_i(a_n^i, z_n^i)\} \\ &\leq \Delta_p(\bar{x}, s_n) - \Delta_p(a_n^i, s_n) - \Delta_p(z_n^i, a_n^i) + \frac{\tau_n}{\tau_{n+1}}\kappa\left(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i)\right) \\ &= \Delta_p(\bar{x}, s_n) - \left(1 - \frac{\tau_n}{\tau_{n+1}}\kappa\right)\left(\Delta_p(a_n^i, s_n) + \Delta_p(z_n^i, a_n^i)\right). \end{aligned} \tag{7.68}$$

Following similar procedure, we obtain

$$\Delta_p(\bar{y}, h_n^j) \leq \Delta_p(\bar{y}, t_n) - \left(1 - \epsilon\frac{\lambda_n}{\lambda_{n+1}}\right)\left(\Delta_p(b_n^j, t_n) + \Delta_p(h_n^j, b_n^j)\right). \tag{7.69}$$

□

Observe that since $\lim\limits_{n\to\infty}\left(1 - \frac{\tau_n}{\tau_{n+1}}\kappa\right) = 1 - \kappa > 0$, then there exists $K \in \mathbb{N}$ such that

$$\left(1 - \frac{\tau_n}{\tau_{n+1}}\kappa\right) > 0, \quad \forall n \geq K.$$

Hence, from (7.68) we get

$$\Delta_p(\bar{x}, z_n^i) \leq \Delta_p(\bar{x}, s_n), \quad \forall i = 1, 2 \ldots, N, \quad n \geq K. \tag{7.70}$$

Similarly, from (7.69) we obtain

$$\Delta_p(\bar{y}, h_n^j) \leq \Delta_p(\bar{y}, t_n), \quad \forall n \geq L \in \mathbb{N}. \tag{7.71}$$

Lemma 7.3.7. *Suppose $\{x_n\}$ and $\{y_n\}$ are iterative sequences generated by Algorithm 7.3.3 under Assumption 7.3.2. Then, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.*

Proof. Let $(\bar{x}, \bar{y}) \in \Upsilon$. Since D_s is Bregman quasi-nonexpansive for each $s = 1, 2 \ldots, l$, we obtain from (7.56) that

$$\begin{aligned}
\Delta_p(\bar{x}, u_n) &= \Delta_p\left(\bar{x}, J_q^{E_1^*}\left(\alpha_{n,0} J_{E_1}^p(\bar{z}_n) + \sum_{s=1}^{l} \alpha_{n,s} J_{E_1}^p(D_s \bar{z}_n)\right)\right) \\
&\leq \alpha_{n,0} \Delta_p(\bar{x}, \bar{z}_n) + \sum_{s=1}^{l} \alpha_{n,s} \Delta_p(\bar{x}, D_s \bar{z}_n) \\
&\leq \alpha_{n,0} \Delta_p(\bar{x}, \bar{z}_n) + \sum_{s=1}^{l} \alpha_{n,s} \Delta_p(\bar{x}, \bar{z}_n) \\
&= \Delta_p(\bar{x}, \bar{z}_n). \qquad (7.72)
\end{aligned}$$

Similarly, we obtain

$$\Delta_p(\bar{y}, v_n) \leq \Delta_p(\bar{y}, \bar{\theta}_n). \qquad (7.73)$$

Furthermore, from (7.56), Lemma 2.55 and Lemma 2.5.45, we obtain

$$\begin{aligned}
\Delta_p(\bar{x}, s_n) &= \Delta_p\left(\bar{x}, J_{E_1^*}^q\left(J_{E_1}^p(x_n) - \rho_n A^* J_{E_3}^p(Ax_n - By_n)\right)\right) \\
&= V_p\left(\bar{x}, J_{E_1}^p(x_n) - \rho_n A^* J_{E_3}^p(Ax_n - By_n)\right) \\
&= \frac{1}{p}\|\bar{x}\|^p - \langle \bar{x}, J_{E_1}^p(x_n)\rangle + \rho_n \langle \bar{x}, A^* J_{E_3}^p(Ax_n - By_n)\rangle \\
&+ \frac{1}{q}\|J_{E_1}^p(x_n) - \rho_n A^* J_{E_3}^p(Ax_n - By_n)\|^q \\
&\leq \frac{1}{p}\|\bar{x}\|^p - \langle \bar{x}, J_{E_1}^p(x_n)\rangle + \rho_n \langle A\bar{x}, J_{E_3}^p(Ax_n - By_n)\rangle + \\
&+ \frac{1}{q}\|J_{E_1}^p(x_n)\|^q - \rho_n \langle J_{E_3}^p(Ax_n - By_n), Ax_n\rangle + \frac{C_q}{q}\rho_n^q \|A^* J_{E_3}^p(Ax_n - By_n)\|^q \\
&= \frac{1}{p}\|\bar{x}\|^p - \langle \bar{x}, J_{E_1}^p(x_n)\rangle + \frac{1}{q}\|J_{E_1}^p(x_n)\|^q - \rho_n \langle J_{E_3}^p(Ax_n - By_n), Ax_n - A\bar{x}\rangle \\
&+ \frac{C_q}{q}\rho_n^q \|A^* J_p^{E_3}(Ax_n - By_n)\|^q \\
&= \Delta_p(\bar{x}, x_n) - \rho_n \langle J_{E_3}^p(Ax_n - By_n), Ax_n - A\bar{x}\rangle + \frac{C_q}{q}\rho_n^q \|A^* J_p^{E_3}(Ax_n - By_n)\|^q. \qquad (7.74)
\end{aligned}$$

Similarly, we have

$$\Delta_p(\bar{y}, t_n) \le \Delta_p(\bar{y}, y_n) - \rho_n\langle J_{E_3}^p(Ax_n - By_n), B\bar{y} - By_n\rangle + \frac{Q_q}{q}\rho_n^q\|B^*J_{E_3}^p(Ax_n - By_n)\|^q. \tag{7.75}$$

Combining (7.74) and (7.75) and noting that $A\bar{x} = B\bar{y}$, we have

$$\begin{aligned}&\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n) \le \Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)\\ &\quad - \rho_n\left[\|Ax_n - By_n\|^p - \frac{\rho_n^{q-1}}{q}\Big(C_q\|A^*J_{E_3}^p(Ax_n - By_n)\|^q + Q_q\|B^*J_{E_3}^p(Ax_n - By_n)\|^q\Big)\right].\end{aligned} \tag{7.76}$$

Hence,

$$\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n) \le \Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n). \tag{7.77}$$

Also, from (7.56) and applying (7.70) we obtain

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) &= \Delta_p\left(\bar{x}, J_{E_1^*}^q\Big(\beta_n J_{E_1}^p(\mu) + (1-\beta_n)J_{E_1}^p(u_n)\Big)\right)\\ &\le \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\Delta_p(\bar{x}, u_n)\\ &\le \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\Delta_p(\bar{x}, \bar{z}_n)\\ &\le \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\Delta_p(\bar{x}, s_n).\end{aligned} \tag{7.78}$$

In like manner, we have

$$\Delta_p(\bar{y}, y_{n+1}) \le \beta_n\Delta_p(\bar{y}, \vartheta) + (1-\beta_n)\Delta_p(\bar{y}, t_n). \tag{7.79}$$

It follows from (7.77), (7.78) and (7.79) that

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) &\le \beta_n\big(\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta)\big) + (1-\beta_n)\big(\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n)\big)\\ &\le \beta_n\big(\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta)\big) + (1-\beta_n)\big(\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)\big)\\ &\le \max\{\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta),\ \Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)\}\\ &\ \vdots\\ &\le \max\{\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta),\ \Delta_p(\bar{x}, x_{\bar{N}}) + \Delta_p(\bar{y}, y_{\bar{N}})\},\\ &\quad \bar{N} = \max\{K, L\}.\end{aligned} \tag{7.80}$$

Therefore, $\{\Delta_p(\bar{x}, x_n)+\Delta_p(\bar{y}, y_n)\}$ is bounded and consequently $\{\Delta_p(\bar{x}, x_n)\}$ and $\{\Delta_p(\bar{y}, y_n)\}$ are bounded. Hence, by Lemma 2.5.34, the sequences $\{x_n\}, \{y_n\}$ are bounded. Therefore, $\{s_n\}$, $\{a_n^i\}$, $\{z_n^i\}$, $\{u_n\}$, $\{t_n\}$, $\{b_n^j\}$, $\{h_n^j\}$ and $\{v_n\}$ are all bounded. □

Lemma 7.3.8. *Assume that* $r = \sup\{\|J_{E_1}^p(\bar{z}_n)\|,\ \|J_{E_1}^p(D_s\bar{z}_n)\|\}$ *and let* $(\bar{x}, \bar{y}) \in \Upsilon$. *Then, the following inequality holds:*

$$\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) \leq \beta_n\left[\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta)\right] + (1-\beta_n)[\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)]$$
$$- (1-\beta_n)\Big(\frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big) + \frac{W_q(\eta_{n,t})}{q} g\big(\|J_{E_2}^p(\bar{\theta}_n) - J_{E_2}^p(G_t\bar{\theta}_n)\|\big)\Big), \tag{7.81}$$

where $W_q(\alpha_{n,s}) = (\alpha_{n,0})^q \sum_{s=1}^l \alpha_{n,s} + \alpha_{n,0}(\sum_{s=1}^l \alpha_{n,s})^q$ *and* $W_q(\eta_{n,t}) = (\eta_{n,0})^q \sum_{t=1}^m \eta_{n,t} + \eta_{n,0}(\sum_{t=1}^m \eta_{n,t})^q$.

Proof. Let $(\bar{x}, \bar{y}) \in \Upsilon$. Then, from (7.56), Lemma 2.5.45 and Lemma 7.3.1, we obtain

$$\begin{aligned}
\Delta_p(\bar{x}, u_n) &= \Delta_p\left(\bar{x}, J_{E_1^*}^q\left(\alpha_{n,0}J_{E_1}^p(\bar{z}_n) + \sum_{s=1}^l \alpha_{n,s}J_{E_1}^p(D_s\bar{z}_n)\right)\right) \\
&= V_p\left(\bar{x}, \alpha_{n,0}J_{E_1}^p(\bar{z}_n) + \sum_{s=1}^l \alpha_{n,s}J_{E_1}^p(D_s\bar{z}_n)\right) \\
&= \frac{1}{p}\|\bar{x}\|^p - \alpha_{n,0}\langle \bar{x}, J_{E_1}^p(\bar{z}_n)\rangle - \sum_{s=1}^l \alpha_{n,s}\langle \bar{x}, J_{E_1}^p(D_s\bar{z}_n)\rangle \\
&+ \frac{1}{q}\|\alpha_{n,0}J_{E_1}^p(\bar{z}_n) + \sum_{s=1}^l \alpha_{n,s}J_{E_1}^p(D_s\bar{z}_n)\|^q \\
&\leq \frac{1}{p}\|\bar{x}\|^p - \alpha_{n,0}\langle \bar{x}, J_{E_1}^p(\bar{z}_n)\rangle - \sum_{s=1}^l \alpha_{n,s}\langle \bar{x}, J_{E_1}^p(D_s\bar{z}_n)\rangle \\
&+ \frac{1}{q}\alpha_{n,0}\|J_{E_1}^p(\bar{z}_n)\|^p + \frac{1}{q}\sum_{s=1}^l \alpha_{n,s}\|J_{E_1}^p(D_s\bar{z}_n)\|^P - \frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big) \\
&= \frac{1}{p}\alpha_{n,0}\|\bar{x}\|^p + \sum_{s=1}^l \alpha_{n,s}\frac{1}{p}\|\bar{x}\|^p - \alpha_{n,0}\langle \bar{x}, J_{E_1}^p(\bar{z}_n)\rangle - \sum_{s=1}^l \alpha_{n,s}\langle \bar{x}, J_{E_1}^p(D_s\bar{z}_n)\rangle \\
&+ \frac{1}{q}\alpha_{n,0}\|J_{E_1}^p(\bar{z}_n)\|^p + \frac{1}{q}\sum_{s=1}^l \alpha_{n,s}\|J_{E_1}^p(D_s\bar{z}_n)\|^P - \frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big) \\
&= \alpha_{n,0}\left\{\frac{1}{p}\|\bar{x}\|^p - \langle \bar{x}, J_{E_1}^p(\bar{z}_n)\rangle + \frac{1}{q}\|J_{E_1}^p(\bar{z}_n)\|^p\right\} \\
&+ \sum_{s=1}^l \alpha_{n,s}\left\{\frac{1}{p}\|\bar{x}\|^p - \langle \bar{x}, J_{E_1}^p(D_s\bar{z}_n)\rangle + \frac{1}{q}\|J_{E_1}^p(D_s\bar{z}_n)\|^p\right\} \\
&- \frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big) \\
&= \alpha_{n,0}\Delta_p(\bar{x}, \bar{z}_n) + \sum_{s=1}^l \alpha_{n,s}\Delta_p(\bar{x}, D_s\bar{z}_n) - \frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big).
\end{aligned}$$

By the Bregman quasi-nonexpansivity of D_s for $s = 1, 2, ..., l$, we get

$$\begin{aligned}\Delta_p(\bar{x}, u_n) &\leq \alpha_{n,0}\Delta_p(\bar{x}, \bar{z}_n) + \sum_{s=1}^{l} \alpha_{n,s}\Delta_p(\bar{x}, \bar{z}_n) - \frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big) \\ &= \Delta_p(\bar{x}, \bar{z}_n) - \frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big). \end{aligned} \tag{7.82}$$

By the definition of x_{n+1} and applying (7.70) and (7.72), from (7.82) we obtain

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) &= \Delta_p\left(\bar{x}, J_{E_1^*}^q\Big(\beta_n J_{E_1}^p(\mu) + (1-\beta_n)J_{E_1}^p(u_n)\Big)\right) \\ &\leq \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\Delta_p(\bar{x}, u_n) \\ &\leq \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\left(\Delta_p(\bar{x}, \bar{z}_n) - \frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big)\right) \\ &\leq \beta_n\Delta_p(\bar{x}, \mu) + (1-\beta_n)\Delta_p(\bar{x}, s_n) \\ &- (1-\beta_n)\frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big). \end{aligned} \tag{7.83}$$

Following similar argument, we have

$$\Delta_p(\bar{y}, y_{n+1}) \leq \beta_n\Delta_p(\bar{y}, \vartheta) + (1-\beta_n)\Delta_p(\bar{y}, t_n) - (1-\beta_n)\frac{W_q(\eta_{n,t})}{q} g\big(\|J_{E_2}^p(\bar{\theta}_n) - J_{E_2}^p(G_t\bar{\theta}_n)\|\big). \tag{7.84}$$

By adding (7.83) and (7.84), and applying (7.77), we get

$$\begin{aligned}&\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) \leq \beta_n\left[\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta)\right] + (1-\beta_n)[\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n)] \\ &\quad - (1-\beta_n)\left(\frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big) + \frac{W_q(\eta_{n,t})}{q} g\big(\|J_{E_2}^p(\bar{\theta}_n) - J_{E_2}^p(G_t\bar{\theta}_n)\|\big)\right) \\ &\quad \leq \beta_n\left[\Delta_p(\bar{x}, \mu) + \Delta_p(\bar{y}, \vartheta)\right] + (1-\beta_n)[\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)] \\ &\quad - (1-\beta_n)\left(\frac{W_q(\alpha_{n,s})}{q} g\big(\|J_{E_1}^p(\bar{z}_n) - J_{E_1}^p(D_s\bar{z}_n)\|\big) + \frac{W_q(\eta_{n,t})}{q} g\big(\|J_{E_2}^p(\bar{\theta}_n) - J_{E_2}^p(G_t\bar{\theta}_n)\|\big)\right),\end{aligned}$$

which is the required inequality.

□

We now present the main theorem for our proposed algorithm as follows.

Theorem 7.3.9. *Suppose $\{(x_n, y_n)\}$ is a sequence generated by Algorithm 7.3.3 under Assumption 7.3.2. Then, $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Upsilon$, where $\bar{x} = \Pi_\Upsilon(\mu)$ and $\bar{y} = \Pi_\Upsilon(\vartheta)$.*

Proof. Let $(\bar{x}, \bar{y}) = (\Pi_{\Upsilon}(\mu), \Pi_{\Upsilon}(\vartheta))$. It follows from Algorithm 7.3.3 and by applying Lemma 2.5.45 (iii) that

$$
\begin{aligned}
\Delta_p(\bar{x}, x_{n+1}) &= \Delta_p\left(\bar{x}, J^q_{E_1^*}\left(\beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n)\right)\right) \\
&= V_p\left(\bar{x}, \beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n)\right) \\
&\leq V_p\Bigg(\bar{x}, \beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n) - \beta_n\left(J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x})\right)\Bigg) \\
&+ \beta_n\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle \\
&= V_p\Bigg(\bar{x}, \beta_n J^p_{E_1}(\bar{x}) + (1-\beta_n)J^p_{E_1}(u_n)\Bigg) + \beta_n\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle \\
&\leq \beta_n\Delta_p(\bar{x}, \bar{x}) + (1-\beta_n)\Delta_p(\bar{x}, u_n) + \beta_n\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle \\
&\leq (1-\beta_n)\Delta_p(\bar{x}, s_n) + \beta_n\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle. \qquad (7.85)
\end{aligned}
$$

In the same vein, we have

$$
\Delta_p(\bar{y}, y_{n+1}) \leq (1-\beta_n)\Delta_p(\bar{y}, t_n) + \beta_n\langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle. \qquad (7.86)
$$

Hence, by adding (7.85) and (7.86) and applying (7.77), we get

$$
\begin{aligned}
\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) &\leq (1-\beta_n)[\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n)] \\
&+ \beta_n\left(\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle\right) \\
&\leq (1-\beta_n)[\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)] \\
&+ \beta_n\left(\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle\right) \\
&= (1-\beta_n)[\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)] + \beta_n\chi_n, \quad \forall\ n \geq 1, \qquad (7.87)
\end{aligned}
$$

where $\chi_n := \left(\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle\right)$.

In order to show that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$, by Lemma 2.5.55, we only need to show that
$\limsup_{k\to\infty}\chi_{n_k} \leq 0$ for every subsequence $\{\Delta_p(x_{n_k}, \bar{x})\}$ of $\{\Delta_p(x_n, \bar{x})\}$ and $\{\Delta_p(y_{n_k}, \bar{y})\}$ of $\{\Delta_p(y_n, \bar{y})\}$ satisfy the inequality

$$
\liminf_{k\to\infty}\left(\left[\Delta_p(\bar{x}, x_{n_k+1}) + \Delta_p(\bar{y}, y_{n_k+1})\right] - \left[\Delta_p(\bar{x}, x_{n_k}) + \Delta_p(\bar{y}, y_{n_k})\right]\right) \geq 0. \qquad (7.88)
$$

Now, from Algorithm 7.3.3 and Lemma 7.3.6, we obtain

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) &= \Delta_p\left(\bar{x}, J^q_{E_1^*}(\beta_n J^p_{E_1}(\mu) + (1-\beta_n)J^p_{E_1}(u_n))\right)\\ &\leq \beta_n \Delta_p(\mu, \bar{x}) + (1-\beta_n)\Delta_p(\bar{x}, u_n)\\ &\leq \beta_n \Delta_p(\mu, \bar{x}) + (1-\beta_n)\Delta_p(\bar{x}, \bar{z}_n)\\ &\leq \beta_n \Delta_p(\mu, \bar{x}) + (1-\beta_n)\Delta_p(\bar{x}, s_n)\\ &- (1-\beta_n)\left(1 - \frac{\tau_n}{\tau_{n+1}}\kappa\right)\left(\Delta_p(a_n^{i_n}, s_n) + \Delta_p(z_n^{i_n}, a_n^{i_n})\right). \end{aligned} \tag{7.89}$$

In the same vein, we obtain

$$\begin{aligned}\Delta_p(\bar{y}, y_{n+1}) &\leq \beta_n \Delta_p(\vartheta, \bar{y}) + (1-\beta_n)\Delta_p(\bar{y}, t_n)\\ &- (1-\beta_n)\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\eta\right)\left(\Delta_p(b_n^{j_n}, t_n) + \Delta_p(h_n^{j_n}, b_n^{j_n})\right). \end{aligned} \tag{7.90}$$

Adding (7.89) and (7.90) together, we obtain

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) &\leq \beta_n[\Delta_p(\mu, \bar{x}) + \Delta_p(\vartheta, \bar{y})] + (1-\beta_n)[\Delta_p(\bar{x}, s_n) + \Delta_p(\bar{y}, t_n)]\\ &- (1-\beta_n)\left(1 - \frac{\tau_n}{\tau_{n+1}}\kappa\right)\left(\Delta_p(a_n^{i_n}, s_n) + \Delta_p(z_n^{i_n}, a_n^{i_n})\right)\\ &- (1-\beta_n)\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\eta\right)\left(\Delta_p(b_n^{j_n}, t_n) + \Delta_p(h_n^{j_n}, b_n^{j_n})\right). \end{aligned} \tag{7.91}$$

Applying (7.76) in (7.91), we obtain

$$\begin{aligned}\Delta_p(\bar{x}, x_{n+1}) + \Delta_p(\bar{y}, y_{n+1}) &\leq \beta_n[\Delta_p(\mu, \bar{x}) + \Delta_p(\vartheta, \bar{y})] + (1-\beta_n)[\Delta_p(\bar{x}, x_n) + \Delta_p(\bar{y}, y_n)]\\ &- (1-\beta_n)\rho_n\Bigg[\|Ax_n - By_n\|^p - \frac{\rho_n^{q-1}}{q}\Big(C_q\|A^*J^p_{E_3}(Ax_n - By_n)\|^q\\ &+ Q_q\|B^*J^p_{E_3}(Ax_n - By_n)\|^q\Big)\Bigg]\\ &- (1-\beta_n)\left(1 - \frac{\tau_n}{\tau_{n+1}}\kappa\right)\left(\Delta_p(a_n^{i_n}, s_n) + \Delta_p(z_n^{i_n}, a_n^{i_n})\right)\\ &- (1-\beta_n)\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\eta\right)\left(\Delta_p(b_n^{j_n}, t_n) + \Delta_p(h_n^{j_n}, b_n^{j_n})\right). \end{aligned} \tag{7.92}$$

By (7.88), Assumption 7.3.2 (2)(a) and (7.92), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\Bigg((1-\beta_{n_k})\rho_{n_k}\Bigg[\|Ax_{n_k}-By_{n_k}\|^p-\frac{\rho_{n_k}^{q-1}}{q}\Bigg(C_q\|A^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|^q\\
&\qquad+Q_q\|B^*J_p^{E_3}(Ax_{n_k}-By_{n_k})\|^q\Bigg)\Bigg]\Bigg)\\
&\qquad\leq\limsup_{k\to\infty}\Bigg(\beta_{n_k}\big[\Delta_p(\mu,\bar{x})+\Delta_p(\vartheta,\bar{y})\big]+(1-\beta_{n_k})\big[\Delta_p(\bar{x},x_{n_k})+\Delta_p(\bar{y},y_{n_k})\big]\\
&\qquad-\big[\Delta_p(\bar{x},x_{n_k+1})+\Delta_p(\bar{y},y_{n_k+1})\big]\Bigg)\\
&\qquad=-\liminf_{k\to\infty}\Bigg(\big[\Delta_p(\bar{x},x_{n_k+1})+\Delta_p(\bar{y},y_{n_k+1})\big]-\big[\Delta_p(\bar{x},x_{n_k})+\Delta_p(\bar{y},y_{n_k})\big]\Bigg)\\
&\qquad\leq 0.
\end{aligned}\tag{7.93}$$

In the same vein as in (7.93), we get using (7.88), Assumption 7.3.2 (2)(a) and (7.92) that

$$\begin{aligned}
&\limsup_{k\to\infty}\Bigg((1-\beta_{n_k})\Big(1-\frac{\tau_{n_k}}{\tau_{n_k+1}}\kappa\Big)\Big(\Delta_p(a_{n_k}^{i_{n_k}},s_{n_k})+\Delta_p(z_{n_k}^{i_{n_k}},a_{n_k}^{i_{n_k}})\Big)\\
&\qquad+\Big(1-\frac{\lambda_{n_k}}{\lambda_{n_k+1}}\eta\Big)\Big(\Delta_p(b_{n_k}^{j_{n_k}},t_{n_k})+\Delta_p(h_{n_k}^{j_{n_k}},b_{n_k}^{j_{n_k}})\Big)\Bigg)\\
&\qquad\leq\limsup_{k\to\infty}\Bigg(\beta_{n_k}\big[\Delta_p(\mu,\bar{x})+\Delta_p(\vartheta,\bar{y})\big]+(1-\beta_{n_k})\big[\Delta_p(\bar{x},x_{n_k})+\Delta_p(\bar{y},y_{n_k})\big]\\
&\qquad-\big[\Delta_p(\bar{x},x_{n_k+1})+\Delta_p(\bar{y},y_{n_k+1})\big]\Bigg)\\
&\qquad=-\liminf_{k\to\infty}\Bigg(\big[\Delta_p(\bar{x},x_{n_k+1})+\Delta_p(\bar{y},y_{n_k+1})\big]-\big[\Delta_p(\bar{x},x_{n_k})+\Delta_p(\bar{y},y_{n_k})\big]\Bigg)\\
&\qquad\leq 0.
\end{aligned}\tag{7.94}$$

Now, suppose we let $\varrho_{n_k}=C_q\|A^*J_{E_3}^p(Ax_{n_k}-By_{n_k})\|^q+Q_q\|B^*J_{E_3}^p(Ax_{n_k}-By_{n_k})\|^q$. Using the condition we placed on our step size ρ_{n_k}, we have that

$$\rho_{n_k}^{q-1}<\frac{q\|Ax_{n_k}-By_{n_k}\|^p}{\varrho_{n_k}}-\zeta,$$

it follows that

$$\rho_{n_k}^{q-1}\varrho_{n_k}<q\|Ax_{n_k}-By_{n_k}\|^p-\zeta\varrho_{n_k},\tag{7.95}$$

Hence, by (7.93) and (7.95) we have

$$\frac{\zeta \varrho_{n_k}}{q} < \left(\|Ax_{n_k} - By_{n_k}\|^p - \frac{\rho_{n_k}^{q-1}}{q}\varrho_{n_k}\right) \to 0, \quad \text{as} \quad k \to \infty.$$

Thus, $C_q\|A^* J_{E_3}^p(Ax_{n_k} - By_{n_k})\|^q + Q_q\|B^* J_{E_3}^p(Ax_{n_k} - By_{n_k})\|^q \to 0$ as $k \to \infty$, which implies that

$$\lim_{k\to\infty} \|A^* J_{E_3}^p(Ax_{n_k} - By_{n_k})\|^q = 0, \tag{7.96}$$

and

$$\lim_{k\to\infty} \|B^* J_{E_3}^p(Ax_{n_k} - By_{n_k})\|^q = 0. \tag{7.97}$$

Additionally, we obtain from (7.93) that

$$\begin{aligned}
&\limsup_{k\to\infty}\left((1-\beta_{n_k})\rho_{n_k}\Big[\|Ax_{n_k} - By_{n_k}\|^p\Big]\right) \\
&\quad \le \limsup_{k\to\infty}\Bigg(\beta_{n_k}\big[\Delta_p(\mu,\bar{x}) + \Delta_p(\vartheta,\bar{y})\big] + (1-\beta_{n_k})\big[\Delta_p(\bar{x},x_{n_k}) + \Delta_p(\bar{y},y_{n_k})\big] \\
&\quad - \big[\Delta_p(\bar{x},x_{n_k+1}) + \Delta_p(\bar{y},y_{n_k+1})\big]\Bigg) + \limsup_{k\to\infty}(1-\beta_{n_k})\frac{\rho_{n_k}^q}{q}\varrho_{n_k} \\
&\quad = -\liminf_{k\to\infty}\Bigg(\big[\Delta_p(\bar{x},x_{n_k+1}) + \Delta_p(\bar{y},y_{n_k+1})\big] - \big[\Delta_p(\bar{x},x_{n_k}) + \Delta_p(\bar{y},y_{n_k})\big]\Bigg) \\
&\quad \le 0.
\end{aligned} \tag{7.98}$$

Thus, we conclude from (7.93), (7.94) and (7.98) that

$$\lim_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0, \tag{7.99}$$

$$\begin{cases}
\lim_{k\to\infty} \Delta_p(a_{n_k}^{i_{n_k}}, s_{n_k}) = 0, \\
\lim_{k\to\infty} \Delta_p(z_{n_k}^{i_{n_k}}, a_{n_k}^{i_{n_k}}) = 0, \\
\lim_{k\to\infty} \Delta_p(b_{n_k}^{j_{n_k}}, t_{n_k}) = 0, \\
\lim_{k\to\infty} \Delta_p(h_{n_k}^{j_{n_k}}, b_{n_k}^{j_{n_k}}) = 0.
\end{cases} \tag{7.100}$$

Therefore, by Lemma 2.5.31, we obtain

$$\begin{cases} \lim_{k\to\infty} \|a_{n_k}^{i_{n_k}} - s_{n_k}\| = 0, \\ \lim_{k\to\infty} \|z_{n_k}^{i_{n_k}} - a_{n_k}^{i_{n_k}}\| = 0, \\ \lim_{k\to\infty} \|b_{n_k}^{j_{n_k}} - t_{n_k}\| = 0, \\ \lim_{k\to\infty} \|h_{n_k}^{j_{n_k}} - b_{n_k}^{j_{n_k}}\| = 0. \end{cases} \tag{7.101}$$

Observe that by (7.101) and Lemma 2.5.31, we have

$$\lim_{k\to\infty} \|s_{n_k} - z_{n_k}^{i_{n_k}}\| = 0, \quad \lim_{k\to\infty} \Delta_p(s_{n_k}, z_{n_k}^{i_{n_k}}) = 0. \tag{7.102}$$

In like manner, we have

$$\lim_{k\to\infty} \|t_{n_k} - h_{n_k}^{j_{n_k}}\| = 0, \quad \lim_{k\to\infty} \Delta_p(t_{n_k}, h_{n_k}^{j_{n_k}}) = 0. \tag{7.103}$$

By the definitions of i_n and j_n, it follows that

$$\lim_{k\to\infty} \Delta_p(s_{n_k}, z_{n_k}^{i}) = 0, \quad i = 1, 2, \ldots, N, \quad \text{and} \quad \lim_{k\to\infty} \Delta_p(t_{n_k}, h_{n_k}^{j}) = 0, \quad j = 1, 2, \ldots, M. \tag{7.104}$$

Consequently, we have

$$\lim_{k\to\infty} \|s_{n_k} - z_{n_k}^{i}\| = 0, \quad i = 1, 2, \ldots, N, \quad \text{and} \quad \lim_{k\to\infty} \|t_{n_k}, h_{n_k}^{j}\| = 0, \quad j = 1, 2, \ldots, M. \tag{7.105}$$

From (7.60) and by applying the three-point identity (2.13) and (7.105), we have

$$\begin{aligned} \Big(1 - \kappa\frac{\tau_{n_k}}{\tau_{n_k+1}}\Big)\Delta_p(a_{n_k}^{i}, s_{n_k}) &\le \Delta_p(\bar{x}, s_n) - \Delta_p(\bar{x}, z_{n_k}^{i}) \\ &\le \Delta_p(\bar{x}, s_{n_k}) - \Delta_p(\bar{x}, z_{n_k}^{i}) + \Delta_p(s_{n_k}, z_{n_k}^{i}) \\ &= \langle \bar{x} - s_{n_k}, J_{E_1}^{p}(z_{n_k}^{i}) - J_{E_1}^{p}(s_{n_k})\rangle \to 0, \quad k \to \infty. \end{aligned}$$

Hence, we have

$$\Delta_p(a_{n_k}^{i}, s_{n_k}) \to 0, \quad k \to \infty, \quad i = 1, 2, \ldots, N.$$

Consequently, we obtain

$$\|a^i_{n_k} - s_{n_k}\| \to 0, \quad k \to \infty, \quad i = 1, 2, \dots, N. \tag{7.106}$$

Following similar procedure, we have

$$\|b^j_{n_k} - t_{n_k}\| \to 0, \quad k \to \infty, \quad j = 1, 2, \dots, M. \tag{7.107}$$

Furthermore, using (7.81) and (7.88), we have

$$\begin{aligned}
&\limsup_{k\to\infty}(1-\beta_{n_k})\Big(\frac{W_q(\alpha_{n_k,s})}{q}g\big(\|J^p_{E_1}(\bar z_{n_k}) - J^p_{E_1}(D_s\bar z_{n_k})\|\big)\\
&\quad + \frac{W_q(\eta_{n_k,t})}{q}g\big(\|J^p_{E_2}(\bar\theta_{n_k}) - J^p_{E_2}(G_t\bar\theta_{n_k})\|\big)\Big)\\
&\quad \le \limsup_{k\to\infty}\Bigg(\beta_{n_k}\big[\Delta_p(\mu,\bar x) + \Delta_p(\vartheta,\bar y)\big] + (1-\beta_{n_k})\big[\Delta_p(\bar x, x_{n_k}) + \Delta_p(\bar y, y_{n_k})\big]\\
&\quad - \big[\Delta_p(\bar x, x_{n_k+1}) + \Delta_p(\bar y, y_{n_k+1})\big]\Bigg)\\
&\quad = -\liminf_{k\to\infty}\Bigg(\big[\Delta_p(\bar x, x_{n_k+1}) + \Delta_p(\bar y, y_{n_k+1})\big] - \big[\Delta_p(\bar x, x_{n_k}) + \Delta_p(\bar y, y_{n_k})\big]\Bigg)\\
&\quad \le 0.
\end{aligned} \tag{7.108}$$

Thus,

$$\lim_{k\to\infty}\Big(\frac{W_q(\alpha_{n_k,s})}{q}g\big(\|J^p_{E_1}(\bar z_{n_k}) - J^p_{E_1}(D_s\bar z_{n_k})\|\big) + \frac{W_q(\eta_{n_k,t})}{q}g\big(\|J^p_{E_2}(\bar\theta_{n_k}) - J^p_{E_2}(G_t\bar\theta_{n_k})\|\big)\Big) = 0.$$

Hence, we have

$$\begin{aligned}
\lim_{k\to\infty} g\big(\|J^p_{E_1}(\bar z_{n_k}) - J^p_{E_1}(D_s\bar z_{n_k})\|\big) &= 0,\ s = 1, 2, \dots, l,\\
\lim_{k\to\infty} g\big(\|J^p_{E_2}(\bar\theta_{n_k}) - J^p_{E_2}(G_t\bar\theta_{n_k})\|\big) &= 0,\ t = 1, 2, \dots, m.
\end{aligned}$$

By the property of g, and sine $J^q_{E_1^*}$ and $J^q_{E_2^*}$ are norm-to-norm uniformly continuous on bounded subsets of E_1 and E_2 respectively, then we obtain

$$\lim_{k\to\infty}\|D_s\bar z_{n_k} - \bar z_{n_k}\| = 0, \quad \forall\ s = 1, 2, ...l. \tag{7.109}$$

and

$$\lim_{k\to\infty}\|G_t\bar\theta_{n_k} - \bar\theta_{n_k}\| = 0, \quad \forall\ \ t = 1, 2, ..., m. \tag{7.110}$$

Observe that from (7.56) and by (7.99) we obtain

$$\begin{aligned}
\|J^p_{E_1}(s_{n_k}) - J^p_{E_1}(x_{n_k})\| &= \|J^p_{E_1}(x_{n_k}) - \rho_{n_k}A^*J^p_{E_3}(Ax_{n_k} - By_{n_k}) - J^p_{E_1}(x_{n_k})\|\\
&= \rho_{n_k}\|A^*J^p_{E_3}(Ax_{n_k} - By_{n_k})\| \to 0 \quad \text{as} \quad k\to\infty.
\end{aligned} \tag{7.111}$$

Also, because E_1 is uniformly smooth, $J^q_{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1, then we have

$$\lim_{k\to\infty} \|s_{n_k} - x_{n_k}\| = 0. \tag{7.112}$$

In the same vein, we get

$$\lim_{k\to\infty} \|t_{n_k} - y_{n_k}\| = 0. \tag{7.113}$$

Moreover, it is easy to see from (7.105) and (7.112)

$$\lim_{k\to\infty} \|z^i_{n_k} - x_{n_k}\| \leq \lim_{k\to\infty} \|z^i_{n_k} - s_{n_k}\| + \lim_{k\to\infty} \|s_{n_k} - x_{n_k}\| = 0, \quad \forall \ \ i = 1, 2, ..., N. \tag{7.114}$$

In the same way, we obtain from (7.105) and (7.113) that

$$\|h^j_{n_k} - y_{n_k}\| \leq \|h^j_{n_k} - t_{n_k}\| + \|t_{n_k} - y_{n_k}\| \to 0 \ \text{ as } \ k \to \infty, \quad \forall \ \ j = 1, 2, ..., M. \tag{7.115}$$

Moreover, we obtain from (7.56) and (7.109) that

$$\begin{aligned}\lim_{k\to\infty} \|J^p_{E_1}(u_{n_k}) - J^p_{E_1}(\bar{z}_{n_k})\| &= \|\alpha_{n_k,0} J^p_{E_1}(\bar{z}_{n_k}) + \sum_{s=1}^{l} \alpha_{n_k,s} J^p_{E_1}(D_s \bar{z}_{n_k}) - J^p_{E_1}(\bar{z}_{n_k})\| \\ &\leq \alpha_{n_k,0} \|J^p_{E_1}(\bar{z}_{n_k}) - J(\bar{z}_{n_k})\| + \sum_{s=1}^{l} \alpha_{n_k,s} \|J^p_{E_1}(D_s \bar{z}_{n_k}) - J^p_{E_1}(\bar{z}_{n_k})\|.\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \|J^p_{E_1}(u_{n_k}) - J^p_{E_1} \bar{z}_{n_k}\| = 0.$$

By the uniform continuity of $J^q_{E_1^*}$ on bounded subsets of E_1^* we have

$$\lim_{k\to\infty} \|u_{n_k} - \bar{z}_{n_k}\| = 0. \tag{7.116}$$

Hence, from (7.114) and (7.116) we obtain

$$\lim_{k\to\infty} \|u_{n_k} - z^i_{n_k}\| = 0, \quad \forall \ \ i = 1, 2, ..., N. \tag{7.117}$$

Similarly, we obtain

$$\lim_{k\to\infty} \|v_{n_k} - h^j_{n_k}\| = 0, \quad \forall \ \ j = 1, 2, ..., M. \tag{7.118}$$

It is easy to see from (7.114) and (7.117) that

$$\|u_{n_k} - x_{n_k}\| \leq \|u_{n_k} - z^i_{n_k}\| + \|z^i_{n_k} - x_{n_k}\| \to 0 \ \text{ as } \ k \to \infty. \tag{7.119}$$

Similarly, we obtain from (7.115) and (7.118) that

$$\|v_{n_k} - y_{n_k}\| \leq \|v_{n_k} - h^j_{n_k}\| + \|h^j_{n_k} - y_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty. \tag{7.120}$$

Furthermore, from (7.56) and the the fact that $\lim\limits_{k\to\infty} \beta_{n_k} = 0$, we obtain

$$\lim_{k\to\infty} \|J^p_{E_1}(x_{n_k+1}) - J^p_{E_1}(u_{n_k})\| = 0.$$

In the same way, we get

$$\lim_{k\to\infty} \|J^p_{E_2}(y_{n_k+1}) - J^p_{E_2}(v_{n_k})\| = 0.$$

Since $J^p_{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1, we obtain

$$\lim_{k\to\infty} \|x_{n_k+1} - u_{n_k}\| = 0. \tag{7.121}$$

Similarly, we get

$$\lim_{k\to\infty} \|y_{n_k+1} - v_{n_k}\| = 0. \tag{7.122}$$

Hence, from (7.119) and (7.121), we obtain

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{7.123}$$

In the same vein, from (7.120) and (7.122), we get

$$\lim_{k\to\infty} \|y_{n_k+1} - y_{n_k}\| = 0. \tag{7.124}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then $w_\omega(x_n)$ and $w_\omega(y_n)$ are nonempty. Now, let $(x^*, y^*) \in w_\omega(x_n, y_n)$ be arbitrary elements. Then, there exists subsequences $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ that converge weakly to $x^* \in E_1$ and $y^* \in E_2$, respectively. Also, from (7.114) and (7.115), $\{z^i_{n_k}\}$ converges weakly to $x^* \in C_i$ for each $i = 1, 2, ..., N$ and $\{h^j_{n_k}\}$ converges weakly to $y^* \in Q_j$ for each $j = 1, 2, ..., M$. Using (7.109) and (7.110), and by the demiclosedness of $I - D_s$ and $I - G_t$, we obtain

$$x^* \in F(D_s), \quad \forall \;\; s = 1, 2, ..., l \quad \text{and} \quad y^* \in F(G_t), \quad \forall \;\; t = 1, 2, ..., m, \tag{7.125}$$

which implies that

$$x^* \in \bigcap_{s=1}^{l} F(D_s) \quad \text{and} \quad y^* \in \bigcap_{t=1}^{m} F(G_t). \tag{7.126}$$

Next, recall that

$$a_{n_k}^i = \arg\min_{\sigma \in C_i}\{f_i(s_{n_k}, \sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_{n_k})\}.$$

Using Lemma 2.5.44 and applying condition (C4), we get

$$0 \in \partial_2(\tau_{n_k} f_i(s_{n_k}, a_{n_k}^i) + \Delta_p(a_{n_k}^i, s_{n_k})) + N_{C_i}(a_{n_k}^i).$$

Hence, there exists $\varsigma_{n_k}^i \in \partial_2 f_i(s_{n_k}, a_{n_k}^i)$ and $\bar{\varsigma}_{n_k}^i \in N_{C_i}(a_{n_k}^i)$ such that

$$\tau_{n_k}\varsigma_{n_k}^i + J_{E_1}^p(a_{n_k}^i) - J_{E_1}^p(s_{n_k}) + \bar{\varsigma}_{n_k}^i = 0. \tag{7.127}$$

Since $\bar{\varsigma}_{n_k}^i \in N_{C_i}(a_{n_k}^i)$, $\langle \omega - a_{n_k}^i, \bar{\varsigma}_{n_k}^i\rangle \leq 0$ for all $\omega \in C_i$, then this together with (7.127) gives

$$\tau_{n_k}\langle \omega - a_{n_k}^i, \varsigma_{n_k}^i\rangle \geq \langle a_{n_k}^i - \omega, J_{E_1}^p(a_{n_k}^i) - J_{E_1}^p(s_{n_k})\rangle, \quad \forall\, \omega \in C_i. \tag{7.128}$$

Again, since $\varsigma_{n_k}^i \in \partial_2 f_i(s_{n_k}, a_{n_k}^i)$, we obtain

$$f_i(s_{n_k}, \omega) - f_i(s_{n_k}, a_{n_k}^i) \geq \langle w - a_{n_k}^i, \varsigma_{n_k}^i\rangle \quad \forall\, \omega \in C_i. \tag{7.129}$$

Combining (7.128) and (7.129), we obtain

$$\tau_{n_k}\left[f_i(s_{n_k}, \omega) - f_i(s_{n_k}, a_{n_k}^i)\right] \geq \langle a_{n_k}^i - \omega, J_{E_1}^p(a_{n_k}^i) - J_{E_1}^p(s_{n_k})\rangle, \quad \forall\, \omega \in C_i,$$

which implies that

$$\begin{aligned}\tau_{n_k}\left[f_i(s_{n_k}, a_{n_k}^i) - f_i(s_{n_k}, \omega)\right] &\leq \langle J_{E_1}^p(s_{n_k}) - J_{E_1}^p(a_{n_k}^i), a_{n_k}^i - \omega\rangle \\ &\leq \|J_{E_1}^p(s_{n_k}) - J_{E_1}^p(a_{n_k}^i)\|\|a_{n_k}^i - \omega\|. \end{aligned} \tag{7.130}$$

Since $J_{E_1}^p$ is uniformly continuous, applying (7.106) to (7.130) and using (7.112) together with the fact that $x_{n_k} \rightharpoonup x^*$, we get

$$-f_i(x^*, \omega) \leq 0, \quad \forall\, \omega \in C_i, \quad i = 1, 2, \ldots, N,$$

which implies that

$$f_i(x^*, \omega) \geq 0, \quad \forall\, \omega \in C_i, \quad i = 1, 2, \ldots, N.$$

Hence, we have

$$x^* \in \bigcap_{i=1}^{N} EP(C_i, f_i).$$

Similarly, we obtain

$$g_j(y^*, z) \geq 0, \quad \forall\, z \in Q_j, \quad j = 1, 2, \ldots, M,$$

which implies that

$$y^* \in \bigcap_{j=1}^{M} EP(Q_j, g_j).$$

Next, recall that $\{x_{n_k}\}$ and $\{y_{n_k}\}$ converges to x^* and y^*, respectively, where $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are bounded linear operators. Then, by (7.99) and the weakly lower semi-continuity of the norm we have

$$\|Ax^* - By^*\| \leq \liminf_{k\to\infty} \|Ax_{n_k} - By_{n_k}\| = 0,$$

which implies that

$$Ax^* = By^*$$

Since $(x^*, y^*) \in w_\omega(x_n, y_n)$ is an arbitrary element, then it follows that

$$w_\omega(x_n, y_n) \subset \Upsilon.$$

Next, by the boundedness of $\{x_{n_k}\}$ and $\{y_{n_k}\}$, there exist subsequences $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ and $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup \hat{x} \in E_1$ and $y_{n_{k_j}} \rightharpoonup \hat{y} \in E_2$ and

$$\begin{aligned}
&\lim_{j\to\infty}\Big(\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n_{k_j}} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n_{k_j}} - \bar{y}\rangle\Big)\\
&\quad = \limsup_{k\to\infty}\Big(\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n_k} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n_k} - \bar{y}\rangle\Big)
\end{aligned}$$

Since $\bar{x} = \Pi_\Upsilon(\mu)$ and $\bar{y} = \Pi_\Upsilon(\vartheta)$, then by (2.19), (7.123) and (7.124) we have

$$\begin{aligned}
&\limsup_{k\to\infty}\Big(\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n_{k+1}} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n_{k+1}} - \bar{y}\rangle\Big)\\
&\quad = \limsup_{k\to\infty}\Big(\langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n_k} - \bar{x}\rangle + \langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), x_{n_{k+1} - x_{n_k}}\rangle\Big)\\
&\quad + \limsup_{k\to\infty}\Big(\langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n_k} - \bar{y}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n_{k+1} - y_{n_k}}\rangle\Big)\\
&\quad = \lim_{j\to\infty}\Big(\langle J^p_{E_1}(u) - J^p_{E_1}(\bar{x}), x_{n_{k_j}} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), y_{n_{k_j}} - \bar{y}\rangle\Big)\\
&\quad = \langle J^p_{E_1}(\mu) - J^p_{E_1}(\bar{x}), \hat{x} - \bar{x}\rangle + \langle J^p_{E_2}(\vartheta) - J^p_{E_2}(\bar{y}), \hat{y} - \bar{y}\rangle\\
&\quad \leq 0. \qquad (7.131)
\end{aligned}$$

Hence, by (7.131), we have $\limsup_{k\to\infty} \chi_{n_k} \leq 0$. Therefore, by applying Lemma 2.5.55 to (7.87), it follows that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Upsilon$ as required.

□

Some Corollaries

The following consequent result can easily be obtained from Theorem 7.3.9 by setting $l = m = N = M = 1$.

Corollary 7.3.10. *Let E_1, E_2 and E_3 be three p-uniformly convex Banach space, and C, Q be nonempty, closed and convex subsets of E_1 and E_2, respectively. Suppose $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ be bifunctions satisfying (C1)-(C4) of* **Assumption A**. *Let $A : E_1 \to E_3$ and $B : E_2 \to E_3$ be bounded linear operators, and let $D : E_1 \to E_1$ and $G : E_2 \to E_2$ be Bregman quasi-nonexpansive mappings such that $I - D$ and $I - G$ are demiclosed at zero, and $\Upsilon := \{\bar{x} \in F(D) \cap EP(C, f),\ \ \bar{y} \in F(G) \cap EP(Q, g) : A\bar{x} = B\bar{y}\} \neq \emptyset$. Suppose other conditions of Theorem 7.3.9 hold. For fixed $\mu \in E_1$ and $\vartheta \in E_2$ and initial point $(x_0, y_0) \in E_1 \times E_2$, let $\{(x_n, y_n)\}$ be a sequence generated as follows:*

$$\begin{cases} s_n = J^q_{E_1^*}\Big(J^p_{E_1}(x_n) - \rho_n A^* J^p_{E_3}(Ax_n - By_n)\Big), \\ y_n = \arg\min\{f(s_n, \sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n) : \quad \sigma \in C\}, \\ z_n = \arg\min\{f(y_n, \sigma) + \frac{1}{\tau_n}\Delta_p(\sigma, s_n) : \quad \sigma \in C\}. \\ u_n = J_q^{E_1^*}\left(\alpha_{n,0} J^p_{E_1}(z_n) + \alpha_{n,1} J^p_{E_1}(Dz_n)\right) \\ x_{n+1} = J^q_{E_1^*}\Bigg(\beta_n J^p_{E_1}(\mu) + (1 - \beta_n) J^p_{E_1}(u_n)\Bigg), \\ t_n = J^q_{E_2^*}\Big(J^p_{E_2}(y_n) + \rho_n B^* J^p_{E_3}(Ax_n - By_n)\Big), \\ b_n = \arg\min\{g(t_n, \varphi) + \frac{1}{\lambda_n}\Delta_p(\varphi, t_n) : \quad \varphi \in Q\}, \\ h_n = \arg\min\{g(b_n, \varphi) + \frac{1}{\lambda_n}\Delta_p(\varphi, t_n) : \quad \varphi \in Q\}. \\ v_n = J^q_{E_1^*}\left(\eta_{n,0} J^p_{E_2}(h_n) + \eta_{n,1} J^p_{E_2}(Gh_n)\right) \\ y_{n+1} = J^q_{E_2^*}\Bigg(\beta_n J^p_{E_2}(\vartheta) + (1 - \beta_n) J_p^{E_2}(v_n)\Bigg), \end{cases} \tag{7.132}$$

where $\rho_n \in \left(\zeta, \left(\frac{q\|Ax_n - By_n\|^p}{C_q\|A^* J^p_{E_3}(Ax_n - By_n)\|^q + Q_q\|B^* J^p_{E_3}(Ax_n - By_n)\|^q} - \zeta\right)^{\frac{1}{q-1}}\right)$, $\quad n \in \Omega$, for small enough ζ; C_q and Q_q are constants of smoothness of E_1 and E_2, respectively. Otherwise, $\rho_n = \rho$ (ρ being any nonnegative value), where the set of indexes $\Omega = \{n : Ax_n - By_n \neq 0\}$.

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \min\{\frac{\kappa(\Delta_p(y_n, s_n) + \Delta_p(z_n, y_n))}{f(s_n, z_n) - f(s_n, y_n) - f(y_n, z_n)}\}\right\}, & \textit{if } \ f(s_n, z_n) - f(s_n, y_n) - f(y_n, z_n) > 0, \\ \tau_n, & \textit{otherwise.} \end{cases} \tag{7.133}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \min\{\frac{\epsilon(\Delta_p(b_n, t_n) + \Delta_p(h_n, b_n))}{g(t_n, h_n) - g(t_n, b_n) - g(b_n, h_n)}\}\right\}, & \textit{if } \ g(t_n, h_n) - g(t_n, b_n) - g(b_n, h_n) > 0, \\ \lambda_n, & \textit{otherwise.} \end{cases} \tag{7.134}$$

Then, the sequence $\{(x_n, y_n)\}$ generated by (7.132) converges strongly to $(\bar{x}, \bar{y}) \in \Upsilon$.

Let $E_r = H_r, \ \ r = 1, 2, 3$ be real Hilbert spaces, then we obtain the following consequent result for approximating a common solution of multiple sets split equality pseudo-monotone equilibrium problem and common fixed point problems of quasi-nonexpansive mappings in real Hilbert spaces.

Corollary 7.3.11. *Let $H_1, \ \ H_2$ and H_3 be three real Hilbert spaces, and let C_i and Q_j be nonempty closed and convex subsets of H_1 and H_2, respectively, for $i = 1, 2, ..., N$ and $j = 1, 2, ..., M$. Suppose $f_i : C_i \times C_i \to \mathbb{R}$ and $g_j : Q_j \times Q_j \to \mathbb{R}$ are bifunctions satisfying (C1)-(C4) of* **Assumption A**. *Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $D_s : H_1 \to H_1$ and $G_t : H_2 \to H_2$ be quasi-nonexpansive mappings such that $\Upsilon := \{\bar{x} \in \bigcap_{s=1}^{l} F(D_s) \cap \bigcap_{i=1}^{N} EP(C_i, f_i), \ \ \bar{y} \in \bigcap_{t=1}^{m} F(G_t) \cap \bigcap_{j=1}^{M} EP(Q_j, g_j) : A\bar{x} = B\bar{y}\} \neq \emptyset$. Suppose other conditions of Theorem 7.3.9 hold. For fixed $\mu \in H_1$ and $\vartheta \in H_2$ and initial point $(x_0, y_0) \in H_1 \times H_2$, let $\{(x_n, y_n)\}$ be a sequence generated as follows:*

$$\begin{cases} s_n = \Big(x_n - \rho_n A^*(Ax_n - By_n)\Big), \\ a_n^i = \arg\min\{\tau_n f_i(s_n, \sigma) + \frac{1}{2}\|\sigma - s_n\|^2 : \quad \sigma \in C_i\}, \\ z_n^i = \arg\min\{\tau_n f_i(a_n^i, \sigma) + \frac{1}{2}\|\sigma - s_n\|^2 : \quad \sigma \in C_i\}. \\ \textit{Obtain the farthest element of } z_n^i \textit{ from } s_n, \textit{ i.e.,} \\ i_n \in \arg\max\{\frac{1}{2}\|s_n - z_n^i\|^2 : i = 1, ..., N\}. \\ \textit{Set } \ z_n^{i_n} = \bar{z}_n \\ u_n = \alpha_{n,0}\bar{z}_n + \sum_{s=1}^{l} \alpha_{n,s}(D_s\bar{z}_n) \\ x_{n+1} = \beta_n(\mu) + (1 - \beta_n)(u_n), \\ t_n = \Big(y_n + \rho_n B^*(Ax_n - By_n)\Big), \\ b_n^j = \arg\min\{\lambda_n g_j(t_n, \varphi) + \frac{1}{2}\|\varphi - t_n\|^2 : \quad \varphi \in Q_j\}, \\ h_n^j = \arg\min\{\lambda_n g_j(b_n^j, \varphi) + \frac{1}{2}\|\varphi - t_n\|^2 : \quad \varphi \in Q_j\}. \\ \textit{Obtain the farthest element of } h_n^j \textit{ from } t_n, \textit{ i.e.,} \\ j_n \in \arg\max\{\frac{1}{2}\|t_n, h_n^j\|^2 : j = 1, ..., M\}. \\ \textit{Set } \ h_n^{j_n} = \bar{\theta}_n \\ v_n = \eta_{n,0}\bar{\theta}_n + \sum_{t=1}^{m} \eta_{n,t}(G_t\bar{\theta}_n) \\ y_{n+1} = \beta_n(\vartheta) + (1 - \beta_n)(v_n), \end{cases} \tag{7.135}$$

where $\rho_n \in \left(\zeta, \left(\frac{2\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2} - \zeta\right)\right), \quad n \in \Omega$, *for small enough* ζ. *Otherwise,* $\rho_n = \rho$ *(ρ being any nonnegative value), where the set of indexes* $\Omega = \{n : Ax_n - By_n \neq 0\}$.

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \min\limits_{1 \le i \le N}\{\frac{\kappa}{2} \frac{\|a_n^i - s_n\|^2 + \|z_n^i - a_n^i\|^2}{f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f_i(a_n^i, z_n^i)}\}\right\}, & \textit{if } \ f_i(s_n, z_n^i) - f_i(s_n, a_n^i) - f_i(a_n^i, z_n^i) > 0, \\ \tau_n, & \textit{otherwise.} \end{cases} \tag{7.136}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \min\limits_{1\le j\le M}\left\{\frac{\epsilon}{2}\frac{\|b_n^j - t_n\|^2 + \|h_n^j - b_n^j\|^2}{g_j(t_n,h_n^j) - g_j(t_n,b_n^j) - g_j(b_n^j,h_n^j)}\right\}\right\}, & \text{if } g_j(t_n,h_n^j) - g_j(t_n,b_n^j) - g_j(b_n^j,h_n^j) > 0, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{7.137}$$

Then, the sequence $\{(x_n, y_n)\}$ *generated by* (7.135) *converges strongly to* $(\bar{x}, \bar{y}) \in \Upsilon$.

7.3.3 Application

Multiple Set Split Equality Variational Inequality Problem

In this subsection, we apply our result to study the Multiple Set Split Equality Variational Inequality Problem (MSSEVIP).

Let $U : C \to E^*$ be a nonlinear mapping. The classical variational inequality problem (VIP) is formulated as locating a point

$$x^* \in C \ \text{ such that } \ \langle \bar{x} - x^*, U(x^*)\rangle \ge 0, \ \ \forall\, \bar{x} \in C. \tag{7.138}$$

The solution set of VIP (7.138) is denoted by $\mathrm{VI}(C, U)$. Variational inequalities have been found very applicable in several real-world problems such as optimization problems, minimax theorems, differential equations and in certain applications to economic theory and mechanics. For more details on variational inequalities (see, [4, 180, 233] and the references therein).

Now, we consider the multiple sets split equality variational inequality problem defined as follows:

$$\begin{aligned} &\text{find } \ \bar{x} \in C_i \ \text{ such that } \ \langle U_i\bar{x}, x - \bar{x}\rangle \ge 0, \ \ \forall x \in C_i, \ \ i = 1,2,...,N \\ &\text{and } \ \bar{y} \in Q_j \ \text{ such that } \ \langle V_j\bar{y}, y - \bar{y}\rangle \ge 0, \ \ \forall y \in Q_j, \ \ j = 1,2,...,M \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{such that } \ A\bar{x} = B\bar{y}, \end{aligned} \tag{7.139}$$

where $U_i : E_1 \to E_1$, $V_j : E_2 \to E_2$, are two nonlinear mappings, and $A : E_1 \to E_3$ and $B : E_2 \to E_3$ are two bounded linear operators. When viewed separately, (7.139) consists of two classical multiple sets variational inequality problem (MSVIP) whose solution sets are denoted by $\mathrm{VI}(C_i, U_i)$ and $\mathrm{VI}(Q_j, V_j)$, respectively.

Let $U : C \to E^*$ be a nonlinear mapping. Then, U is said to be

(D1) pseudo-monotone; if for any $x, \ \ y \in C$, we have

$$\langle Ux, y - x\rangle \ge 0 \Longrightarrow \langle Uy, y - x\rangle \ge 0,$$

(D2) K-Lipschitz continuous, if there exists a constant $K > 0$ such that

$$\|Ux - Uy\| \le K\|x - y\|, \ \ \forall x, y \in C,$$

(D3) sequentially weakly continuous, if for any sequence $\{x_n\} \subset C$, we have $x_n \rightharpoonup x \in C$ implies that $Ux_n \rightharpoonup Ux \in E^*$.

We need the following lemma to establish our next result.

Lemma 7.3.12. *[82] Let C be a nonempty, closed convex subset of a reflexive, smooth and strictly convex Banach space E, $U : C \to E^*$ be a nonlinear mapping. Then*

$$\Pi_C\Big(J_{E^*}^q[J_E^p(x) - \lambda U(y)]\Big) = \arg\min_{\omega\in C}\{\lambda\langle \omega - y, U(y)\rangle + \Delta_p(\omega, x)\}, \tag{7.140}$$

for all $x \in E$, $y \in C$ and $\lambda \in (0, +\infty)$.

Setting $f_i(x,y) = \langle U_i x, y - x\rangle, \quad \forall\ x, y \in C_i, \quad i = 1, 2, ..., N$ and $g_j(x, y) = \langle V_j x, y - x\rangle, \ \ \forall x, y \in Q_j, \ \ j = 1, 2, ..., M$ in Algorithm (7.3.3), then the bifunctions f_i and g_j satisfy conditions $(C_1 - C_4)$ of **Assumption A** (see [82]).
Hence, by applying Theorem 7.3.9 and Lemma 7.3.12, we obtain the following consequent result for approximating a common solution of multiple sets split equality variational inequality problem and common fixed point problem for finite families of Bregman quasi-nonexpansive mappings in p-uniformly convex real Banach spaces which are also uniformly smooth.

Theorem 7.3.13. *Let E_1, E_2 and E_3 be three p-uniformly convex and uniformly smooth real Banach spaces. Let C_i, $i = 1, 2, ..., N$ and $Q_j, j = 1, 2, ..., M$ be nonempty, closed and convex subsets of E_1 and E_2, respectively. Let $U_i : C_i \to E^*$ and $V_j : Q_j \to E^*$ be two nonlinear mappings satisfying conditions (D1)-(D3) above. Let $D_s : E_1 \to E_1$, $s = 1, 2, \ldots, l$ and $G_t : E_2 \to E_2$, $t = 1, 2, ..., m$ be two finite families of Bregman quasi-nonexpansive mappings such that $I - D_s$ and $I - G_t$ are demiclosed at zero for each s and t, respectively. Suppose that Assumption 7.3.2 2(a)-2(c) holds and the solution set $\Upsilon := \{\bar{x} \in F(D_s) \cap VI(C_i, U_i), \quad \bar{y} \in F(G_t) \cap VI(Q_j, V_j) : A\bar{x} = B\bar{y}\} \neq \emptyset$. Then, the sequence $\{x_n, y_n\}$ generated by Algorithm (7.3.14) below converges strongly to $(\bar{x}, \bar{y}) \in \Upsilon$, where $\bar{x} = \Pi_\Upsilon(\mu)$ and $\bar{y} = \Pi_\Upsilon(\vartheta)$.*

Algorithm 7.3.14.

For fixed $\mu \in E_1$ and $\vartheta \in E_2$, choose an initial guess $(x_0, y_0) \in E_1 \times E_2$. Suppose that the nth iterate $(x_n, y_n) \subset E_1 \times E_2$ has been constructed; then we compute the $(n+1)$th iterate (x_{n+1}, y_{n+1}) via the iteration

$$
\begin{cases}
s_n = J^q_{E_1^*}\Big(J^p_{E_1}(x_n) - \rho_n A^* J^p_{E_3}(Ax_n - By_n)\Big),\\
a^i_n = \Pi_{C_i}[J^p_{E_1^*}(J^p_{E_1}(s_n) - \tau_n U_i(s_n))], \quad i = 1,2,...,N\\
z^i_n = \Pi_{C_i}[J^p_{E_1^*}(J^p_{E_1}(a^i_n) - \tau_n U_i(s_n))], \quad i = 1,2,...,N\\
i_n \in \arg\max\{\Delta_p(s_n, z^i_n) : i = 1,...,N\}, \quad z^{i_n}_n = \bar{z}_n\\
u_n = J^{E_1^*}_q\left(\alpha_{n,0} J^p_{E_1}(\bar{z}_n) + \sum_{i=1}^N \alpha_{n,i} J^p_{E_1}(D_s \bar{z}_n)\right)\\
x_{n+1} = J^q_{E_1^*}\left(\beta_n J^p_{E_1}(\mu) + (1-\beta_n) J^p_{E_1}(u_n)\right),\\
t_n = J^q_{E_2^*}\Big(J^p_{E_2}(y_n) + \rho_n B^* J^p_{E_3}(Ax_n - By_n)\Big),\\
b^j_n = \Pi_{Q_j}[J^{E_2^*}_p(J^{E_2}_p(t_n) - \lambda_n V_j(t_n))] \quad j = 1,2,...,M\\
h^j_n = \Pi_{Q_j}[J^{E_2^*}_p(J^{E_2}_p(t_n) - \lambda_n V_j(b^j_n))] \quad j = 1,2,...M\\
j_n \in \arg\max\{\Delta_p(t_n, h^j_n) : j = 1,...,M\}, \quad h^{j_n}_n = \bar{\theta}_n\\
v_n = J^q_{E_1^*}\left(\eta_{n,0} J^p_{E_2}(\bar{\theta}_n) + \sum_{t=1}^m \eta_{n,t} J^p_{E_2}(G_t \bar{\theta}_n)\right)\\
y_{n+1} = J^q_{E_2^*}\left(\beta_n J^p_{E_2}(\vartheta) + (1-\beta_n) J^{E_2}_p(v_n)\right).
\end{cases}
$$

where $\rho_n \in \left(\zeta, \left(\frac{q\|Ax_n - By_n\|^p}{C_q\|A^* J^{E_3}_p(Ax_n - By_n)\|^q + Q_q\|B^* J^{E_3}_p(Ax_n - By_n)\|^q} - \zeta\right)^{\frac{1}{q-1}}\right)$ $n \in \Omega$, *for small enough* ζ, C_q *and* Q_q *are constants of smoothness of* E_1 *and* E_2*, respectively. Otherwise,* $\rho_n = \rho$ *(*ρ *being any nonnegative value), where the set of indexes* $\Omega = \{n : Ax_n - By_n \neq 0\}$*.*

$$
\tau_{n+1} = \begin{cases}
\min\left\{\tau_n, \min\limits_{1\le i\le N}\{\frac{\kappa(\Delta_p(a^i_n, s_n) + \Delta_p(z^i_n, a^i_n))}{\langle U_i s_n - U_i a^i_n, z^i_n - a^i_n\rangle}\}\right\}, & \text{if } \langle U_i s_n - U_i a^i_n, z^i_n - a^i_n\rangle > 0,\\
\tau_n, & \text{otherwise.}
\end{cases}
$$

and

$$
\lambda_{n+1} = \begin{cases}
\min\left\{\lambda_n, \min\limits_{1\le j\le M}\{\frac{\epsilon(\Delta_p(b^j_n, t_n) + \Delta_p(h^j_n, b^j_n))}{\langle V_j t_n - V_j b^j_n, h^j_n - b^j_n\rangle}\}\right\}, & \text{if } \langle V_j t_n - V_j b^j_n, h^j_n - b^j_n\rangle > 0,\\
\lambda_n, & \text{otherwise.}
\end{cases}
$$

7.3.4 Numerical example

In this section, we demonstrate the efficiency and applicability of our proposed method with two numerical examples. In all the experiments, we consider the case when $l = m = N = M = 5$.

Example 7.3.15. *Let* $E_r = \mathbb{R}^m$ $r = 1,2,3$ *equipped with induced norm* $\|x\| = \sqrt{\sum_{i=1}^m |x_i|}$ *and the inner product* $\langle x, y\rangle = \sum_{i=1}^m x_i y_i$*, for all* $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m$ *and* $y = (y_1, y_2, ..., y_m) \in \mathbb{R}^m$*. Let* $C_i = Q_j = C$*, where the feasible set* C *has the form*

$$
C = \{(x_1, x_2, ..., x_m) \in \mathbb{R}^m_+ : |x_k| \le 1, \quad k = 1,2,...,m\}.
$$

Consider the following problem:

$$\text{Find} \quad (\bar{x}, \bar{y}) \in \Upsilon := \{\bar{x} \in F(D_s) \cap \bigcap_{i=1}^{N} EP(C_i, f_i), \quad \bar{y} \in F(G_t) \cap \bigcap_{j=1}^{M} EP(Q_j, g_j) : A\bar{x} = B\bar{y}\},$$

where $f_i : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ *is given by*

$$f_i(x, y) = \sum_{k=1}^{n} (q_{ik} y_k^2 - q_{ik} x_k^2), \quad i = 1, 2, ..., N,$$

where $q_{ik} \in (0, 1)$ *is randomly selected* $\forall\ i = 1, 2, ..., N, \quad k = 1, 2, ..., m$ *and* $D_s : \mathbb{R}^m \to \mathbb{R}^m$ *is defined by:*

$$D_s(x) = \frac{x}{s+1}, \quad \forall\ s = 1, 2, ..., l.$$

In the same vein, let $g_j : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ *is given by*

$$g_j(x, y) = \sum_{k=1}^{m} (q_{jk} y_k^2 - q_{jk} x_k^2), \quad j = 1, 2, ..., M,$$

where $q_{jk} \in (0, 1)$ *is randomly selected* $\forall\ j = 1, 2, ..., M, \quad k = 1, 2, ..., 5$ *and* $G_t : \mathbb{R}^m \to \mathbb{R}^m$ *is defined by:*

$$G_t(x) = \frac{x}{t+1}, \quad \forall\ t = 1, 2, ..., 5.$$

It is easy to see that conditions (C1)-(C4) of **Assumption A** *are satisfied and* D_s, G_t *are Bregman quasi-nonexpansive mappings for* $s = 1, 2, ..., l$ *and* $t = 1, 2, ..., m$ *respectively,* $I - D_s$ *and* $I - G_t$ *are demiclosed at zero. Moreover, we define* $A(x) = \frac{x}{2}$ *and* $B(x) = \frac{x}{3}$*, then* A *and* B *are bounded linear operators. Furthermore,* $\Upsilon = \{0\}$*. In this example, we choose* $\beta_n = \frac{3}{2n+3}$, $\kappa = 0.36$, $\tau_0 = 0.24$, $\epsilon = 0.5$, $\lambda_0 = 0.4$, $\alpha_{n,0} = \frac{3n}{8n+11}$, $\alpha_{n,s} = \frac{1}{5}(1 - \frac{3n}{8n+11})$, $s = 1, 2, \ldots, 5$, $\eta_{n,0} = \frac{2n}{4n+7}$, $\eta_{n,t} = \frac{1}{5}(1 - \frac{2n}{4n+7})$ $t = 1, 2, \ldots, 5$*. Using* $\frac{\|x_{n+1} - x_n\|}{\|x_2 - x_1\|} < 10^{-4}$ *as our stopping criterion, we generate randomly different starting points* $(\mu, \vartheta), (x_0, y_0) \in E_1 \times E_2$*, for different cases of* $m = 20, 50, 100, 500$*.*

We plot the graphs $\|x_{n+1} - x_n\|$ *against the number of iterations. The numerical results can be seen in Figure 7.1.*

The next example is presented in an infinite dimensional space setting.

Example 7.3.16. *Let* $E_r = L^2([0, 1])$, $r = 1, 2, 3$ *with the induced norm given by* $\|x\|_L = \int_0^1 |x(s)|^2 ds$ *and the corresponding inner product* $\langle x, y \rangle = \int_0^1 x(s) y(s) ds$*. Let the feasible sets* C_i *and* Q_j *be defined as follows:*

$$C_i := \{x \in H : \|x\|_L \leq 1\} \quad i = 1, 2, ..., 5 \quad \text{and} \quad Q_j := \{x \in H : \|x\|_L \leq 1\} \quad j = 1, 2, ..., 5.$$

Let $f_i(x, y) = \langle S_i x, y - x \rangle$ *and* $g_j(x, y) = \langle T_j x, y - x \rangle$ *with the operators* $(S_i x)(t) = \max\{0, \frac{x(t)}{i}\}$ *for* $i = 1, 2, ..., 5$ *and* $(T_j x)(t) = \max\{0, \frac{x(t)}{j}\}$ *for* $j = 1, 2, ..., 5$*. Then, it*

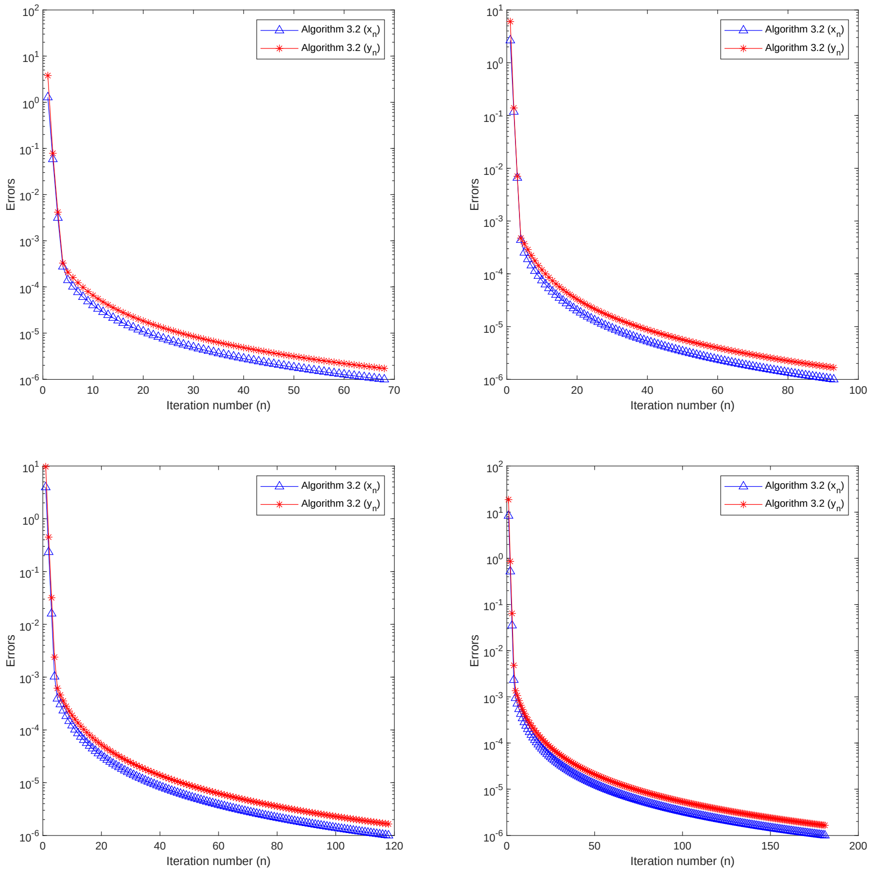


Figure 7.1: Top left: m = 20; Top right: m = 50; Bottom left: m = 100; Bottom right: m = 500.

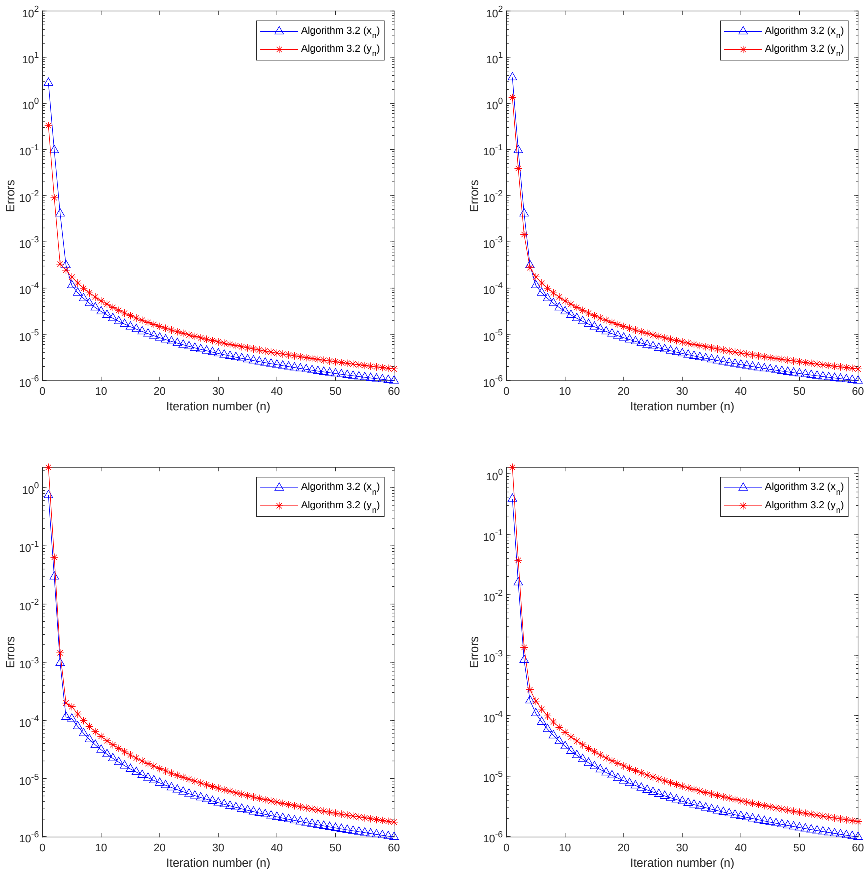


Figure 7.2: Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

is easy to see that each f_i is monotone (and by implication, pseudomonotone) on C_i. Similarly, g_j is pseudomonotone on Q_j. Furthermore, let $D_s : H \to H$ and $G_t : H \to H$ be defined by $D_s(x)(t) = \frac{x(t)}{2s}$ and $G_t(x)(t) = \frac{x(t)}{2t}$, then the mappings D_s and G_t are quasi-nonexpansive $\forall$ $s = 1, 2, ..., 5$ and $t = 1, 2, ..., 5$. Moreover, we define $A(x)(t) = \frac{x(t)}{3}$ and $B(x)(t) = \frac{x(t)}{5}$, then A and B are bounded linear operators. The solution set $\Upsilon = \{0\}$. We choose $\beta_n = \frac{1}{n+2}$, $\alpha_{n,0} = \frac{n+1}{2n+3}$ $\alpha_{n,s} = \frac{1}{5}(1 - \frac{n+1}{2n+3})$, $s = 1, 2, \ldots, 5$, $\eta_{n,0} = \frac{n+2}{2n+5}$ $\eta_{n,t} = \frac{1}{5}(1 - \frac{n+2}{2n+5})$, $t = 1, 2, \ldots, 5$, $\kappa = 0.54$, $\tau_0 = 0.63$, $\epsilon = 0.75$, $\lambda_0 = 0.83$, and using $\frac{\|x_{n+1}-x_n\|}{\|x_2-x_1\|} < 10^{-4}$ as stopping criterion.

We choose fixed points $\mu = t^2 + 2, \vartheta = 4t^3 + 3$ and different starting points as follows:

Case I: $x_0 = t^2 + 4$, $y_0 = t^3 + 2t + 1$
Case II: $x_0 = t^4 + 5$, $y_0 = t^2 + t + 3$
Case III: $x_0 = \sin(2t)$, $y_0 = cos(5t)$
Case IV: $x_0 = exp(t)$, $y_0 = exp(2t)$.

We plot the graph of errors against the number of iterations in each case. The numerical results can be found in Figure 7.2.

Chapter 8

Conclusion, Contributions to Knowledge and Future Research

8.1 Conclusion

In this thesis, we present several explicit iterative algorithms for approximating solutions of certain finite and infinite families of fixed point problems of different nonlinear mappings combined with certain optimization problems. The results obtained in this thesis were obtained in both Hilbert and Banach spaces, most especially the p-uniformly convex Banach spaces where we employed the notion of Bregman distance. In chapters 1 and 2, we briefly explained some definitions and examples of certain nonlinear mappings, state some existing results and gave some geometric properties of our spaces of interest. Our main results are found in chapters 3 to 7, where we discussed our contributions to Equilibrium Problem (EP), Monotone Variational Inclusion Problem (MVIP), Variational Inequality Problem (VIP), Minimization Problem (MP) and some other important optimization problems. In all the chapters, we stated and proved strong convergence theorems. In most cases, we used the concept of inertial and relaxation terms in our algorithms to speed up the rate of convergence of our iterative schemes. In addition, in most cases, we employed a unique approach (which does not involve two case approach) different from the methods common in literature to prove our convergence results. All proposed methods are provided with accurate mathematical formulations and corresponding convergence theorems. We discussed some applications and usefulness of our results and put forward some numerical examples to show the behaviour and performance of our iterative algorithms over existing ones.

8.2 Contributions to knowledge

Below are the highlights of our contributions to knowledge in this thesis which we present in the form of remarks.

Remark 8.2.1. *In chapter three of this thesis, we studied Halpern and viscosity type iterative algorithms to approximate solutions of certain optimization problems. We developed* *an iterative scheme to approximate a common solution of pseudomonotone equilibrium problem which is more general than the monotone-type equilibrium problems studied by most researchers. Moreover, we obtain a strong convergent result by employing a suitable self-adaptive step size technique devoid of the prior information of the Lipschitz constants of the pseudo-monotone bifunction nor any linesearch technique for implementation. We apply our result to solve the classical variational inequality problem. Also, we introduce and study the notion of Split Generalized Equilibrium Problem with Multiple Output Sets (SGEPMOS). We propose a new iterative method which employs viscosity approximation technique for approximating the common solution of the SGEPMOS and common fixed point problem for an infinite family of multivalued demicontractive mappings in real Hilbert spaces.*

Remark 8.2.2. *Most method of the existing methods for solving Split Monotone Variational Inclusion Problem (SMVIP) require that the single-valued mapping be Lipschitz continuous. In this thesis, we propose a new relaxed double inertial Tseng's extragradient method with self-adaptive step sizes for solving split monotone variational inclusion problem (SMVIP) involving non-Lipschitz operators and fixed point problem of strict pseudocontractive mappings. Under more relaxed assumptions, we prove that our proposed scheme converges strongly to a minimum-norm solution of the aforementioned problem in real Hilbert spaces. We point out that while the operators are non-Lipschitz, our method does not involve linesearch procedure which is known to be time-consuming, but we employ a more efficient self-adaptive step size technique that generates non-monotonic sequence of step sizes at each iteration.*

Remark 8.2.3. *The work considered in Section 6.3 of this thesis extends the work of Ma et. al. [155], Kazmi and Rizvi [133] in the sense that they studied the common solution problem of split feasibility and fixed point problems, whereas in our result, we considered a common solution problem of pseudomonotone equilibrium, split feasibility and fixed point problems of relatively nonexpansive mapping. Also, the work in this article extends the work of [213] in the sense that we considered an inertial type algorithm to approximate solutions of split feasibility problem, fixed point problem and pseudomonotone equilibrium problem in the framework of Banach spaces, whereby they only considered a split feasibility problem and pseudomonotone equilibrium problem in the framework of Hilbert spaces.*

Remark 8.2.4. *Furthermore, in this thesis, we proposed a new self-adaptive method and proved that it converges strongly to a minimum-norm solution of generalized split feasibility problem in real Hilbert spaces. The proposed method follows from an explicit discretization of a dynamical system, which combines both the relaxation and inertial techniques for the purpose of increasing the convergence rate of the scheme. The method requires that the underlying single-valued operator is monotone and Lipschitz continuous, and it uses some simple self-adaptive step sizes that are generated at each iteration by some simple computations. As a by-product, we obtained methods for solving other classes of generalized split feasibility problems in real Hilbert spaces. The two major advantages of our algorithm in solving image restoration problems over related algorithms are the higher signal-to-noise ratio value and lower CPU time for generating recovered images.*

8.3 Future research

Computations in fixed point theory have been carried out by researchers working in this direction in linear and nonlinear spaces. Different optimization problems have been introduced and studied in both linear and nonlinear spaces. For instance, the EP also known as Ky Fan's inequality was introduced by Blum and Oettli [29] and it plays a very important role in many fields such as variational inequalities, game theory, mathematical economics, to mention a few. One of the most popular methods for approximating solutions of EP is the proximal point method which was introduced by Martinet [163] for convex minimization and further generalized by Rockafellar [200], and usually imposes the monotone assumption on the bifunctions. Since then, several other methods have been introduced to solve this problem. Korpelevich [142] introduced an extragradient method for solving EP. The advantage of extragradient algorithm is that it can be applied to solve the class of pseudomonotone bifunctions and can be computed easily than the PPA. This method has been considered by many authors in linear spaces. Just recently, extragradient method was introduced to nonlinear spaces. The main reason for extensions are the fact that non-convex problems in linear spaces can be transformed into convex problems non-linear spaces, and constrained optimization problems can also be transformed into unconstrained ones. Thus, nonlinear spaces are more suitable frameworks for the study of optimization problems. One of the recent results of EP in Hadamard maniford is the result of Cruz Neto [177]. He introduced the following iterative algorithm and proved that their sequence converges to the solution of EP. From an initial point $x_0 \in \Omega$, compute x_n, y_n for each $n \in \mathbb{N}$ as follows:

$$\begin{cases} y_n = \arg\min_{t\in\Omega}\{S(x_n,t) + \frac{1}{2\lambda_n} d^2(x_n,t)\}; \\ x_{n+1} = \arg\min_{t\in\Omega}\{S(t,y_n) + \frac{1}{2\lambda_n} d^2(x_n,t)\}, \end{cases}$$

where $\lambda_n \subset (0,+\infty)$. We observe that very few authors have considered this method in Hadamard manifolds.

In our future research, we hope to consider the approximation of solutions of pseudomonotone EP without the Lipschitz constants and prove some strong convergence results in Hadamard manifolds. Also, we hope to introduce the inertial term to this method in Hadamard manifold as this will increase the rate of convergence of iterative algorithms in this space. Lastly, we hope to extend, adapt and utilize the notion of Bregman distance in iterative methods to approximate the solutions of pseudomonotone EP in Hadamard manifolds.

Chapter 9

Appendices

9.1 Lists of Appendices

The following are the lists of all appendices used in this thesis:

9.1.1 Appendix

Algorithm 9.1.1. *[110]*

Initialization: *Choose* $x_0,\ u \in C$ *and the sequences* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\lambda_n\}$ *such that*

$$
\begin{aligned}
y_n &= \arg\min\{\lambda_n f(x_n, y) + \frac{1}{2}\phi(y, x_n) : y \in C\},\\
z_n &= \arg\min\{\lambda_n f(y_n, y) + \frac{1}{2}\phi(y, x_n) : y \in C\}\\
x_{n+1} &= \Pi_C J^{-1}\left(\gamma_n J(u) + (1-\gamma_n)(\beta_n J z_n + (1-\beta_n) J D z_n)\right),
\end{aligned}
$$

where $0 < a \leq \lambda_n \leq b < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}, D : E \to E$ *is a quasi-*ϕ*-nonexpansive mapping.*

Algorithm 9.1.2. *[110]*

Initialization: *Choose* $x_0,\ u \in C$ *and the sequences* $\{\alpha_n\},\ \ \{\beta_n\}$ *and* $\{\lambda_n\}$ *such that*

$$
\begin{aligned}
&y_n = \arg\min\{\lambda_n f(x_n, y) + \frac{1}{2}\phi(y, x_n) : y \in C\},\\
&\textit{Find } m \textit{ the smallest nonnegative integer such that}\\
&\begin{cases} z_{m,n} = (1-\xi^m)x_n + \xi^m y_n,\\ f(z_{m,n}, x_n) - f(z_{m,n}, y_n) \geq \frac{\alpha}{2\lambda_n}\phi(y_n, x_n).\end{cases}\\
&\textit{Set } \rho_n = \xi^m \quad \textit{and } z_n = z_{m,n}\\
&\textit{choose } g_n \in \partial_2 f(z_n, x_n) \quad \textit{and compute } w_n = \Pi_C J^{-1}(Jx_n - \sigma_n g_n)\\
&\textit{where } \sigma_n = \frac{vf(z_n, x_n)}{\|g_n\|^2} \quad \textit{if } y_n \neq x_n \quad \textit{and} \quad \sigma_n = 0 \quad \textit{otherwise.}\\
&x_{n+1} = \Pi_C J^{-1}\left(\gamma_n J(u) + (1-\gamma_n)(\beta_n J w_n + (1-\beta_n) JDw_n)\right),
\end{aligned}
$$

where $\{\lambda_n\} \subset [\lambda, 1], \lambda \in (0, 1], D : E \to E$ is a quasi-ϕ-nonexpansive mapping.

Algorithm 9.1.3. *[82]*

Initialization: **Iterative Step**: *Suppose $\{x_n\}$ is a sequece generated by $x_1 \in C$. Calculate x_{n+1} as follows:*

$$
\begin{aligned}
&w_n^i = \arg\min\{\lambda_n g_i(x_n, w) + D_f(w, x_n) : w \in C\} \quad i = 1, ..., N,\\
&z_n^i = \arg\min\{\lambda_n g_i(w_n^i, z) + D_f(z, x_n) : z \in C\} \quad i = 1, ..., N,\\
&i_n \in \arg\max\{D_f(z_n^i, x_n),\ i = 1, 2, ..., N\},\ \bar{z}_n = z_n^{i_n},\\
&y_n = \nabla f^*\left(\eta_{n,0}\nabla f(\bar{z}_n) + \sum_{j=1}^{m}\eta_{n,j}\nabla f(z_{n,j})\right), \quad z_{n,j} \in T_j\bar{z}_n,\\
&x_{n+1} = Proj_C^f\left(\nabla f^*(\gamma_n \nabla f(q_n)) + (1-\gamma_n)\nabla f(y_n))\right)
\end{aligned}
$$

where $T_j : C \to CB(C)$ is a multivalued Bregman relatively nonexpansive mapping, $\{\lambda_n\} \subset [a, b] \subset (0, p)$, where $p = \min\{\frac{1}{c_1}, \frac{1}{c_2}\}$, $\lim_{n\to\infty} q_n = q \in E$.

9.1.2 Appendix

Algorithm 9.1.4. *[132]*

$$
\begin{aligned}
&u_n = J_\lambda^B\left(x_n + \gamma T^*(J_\lambda^G - I)Tx_n\right)\\
&x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n,
\end{aligned}
$$

Algorithm 9.1.5. *[121]*

Initialization: *Choose $\mu > 0,\ \nu \in (0, 1),\ 0 < \lambda \leq \lambda_n < 2\alpha$ and the sequences $\{\alpha_n\}, \{\phi_n\}$ and $\{\delta_n\}$ such that*

$$
\begin{aligned}
w_n &= P_C\left(x_n - \gamma_n T^* J_E(Tx_n - J_\mu^G T x_n)\right), \\
y_n &= J_{\lambda_n}^B (I - \lambda_n A) w_n \\
x_{n+1} &= \alpha_n f_n(x_n) + \phi_n x_n + \delta_n(\nu S x_n + (1-\nu) y_n), \quad \forall\ n \geq 1,
\end{aligned}
$$

where B and G are maximal monotone operators, $A : H \to H$ is α-inverse strongly monotone, $S : H \to CB(H)$ is a set-valued quasi-nonexpansive mapping, $f_n : H \to H$ is ρ_n-contractive mappings. $\{\alpha_n\}, \{\phi_n\}$ and $\{\delta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n + \phi_n + \delta_n = 1$;

(ii) $\sum_n^\infty \alpha_n = \infty$;

(iii) $0 < \epsilon_1 \leq \phi_n$ *and* $0 < \epsilon_1 \leq \delta_n$.

Algorithm 9.1.6. *[224]*

Initialization: *Choose $\mu > 0$, $x_0, x_1 \in H$ and the sequences $\{\omega_n\}$, $\{\alpha_n\}$ and $\{\lambda_n\}$ such that*

$$
\begin{aligned}
& w_n = x_n + \theta_n(x_n - x_{n-1}) \\
& y_n = J_\mu^B \left[w_n - \lambda_n T^*(I - J_\mu^G) T w_n\right] \\
& z_n = w_n - \omega_n d(w_n, y_n), \\
& \textit{where} \ \ d(w_n, y_n) = w_n - y_n - \lambda_n \left[T^*(I - J_\mu^G) T w_n - T^*(I - J_\mu^G) T y_n\right] \\
& \omega_n := \frac{\langle w_n - y_n, d(w_n, y_n)\rangle}{\|d(w_n, y_n)\|^2} \\
& x_{n+1} = (1 - \alpha_n) z_n + \alpha_n f(x_n),
\end{aligned}
$$

where $\{\lambda_n\} \subset [a, b] \subset \left(0, \frac{1}{L}\right)$ with $L = \|T\|^2$.

Algorithm 9.1.7. *[206]*

Initialization: **Iterative Step**: *Suppose $\{x_n\}$ is a sequece generated by $x_1 \in C$. Calculate x_{n+1} as follows:*

$$
\begin{aligned}
w_n &= (1 - \alpha_n) x_n \\
y_n &= J_\mu^B (I - \mu A)(w_n + \gamma T^*(J_\mu^G(I - \mu D) - I) T w_n) \\
x_{n+1} &= (1 - \phi_n) y_n + \phi_n S y_n, \quad \forall\ n \geq 0,
\end{aligned}
$$

where $0 < \mu < 2\mu$, 2μ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator T^*T. The sequences $\{\alpha_n\}$ and $\{\phi_n\}$ are real sequences in $(0,1)$ with the conditions *(i)* $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_n^{\infty} \alpha_n = \infty;$

(ii) $0 < \liminf \phi_n \leq \limsup \phi_n < 1 - \kappa.$

Algorithm 9.1.8. *[120]*

Initialization: *Choose* $\eta > 0$, $b_i \in (0,1)$, $i = 1,2$, $x_0, x_1 \in H$ *and the sequences* $\{\omega_n\}$, $\{\alpha_n\}$, $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *such that*

$$
\begin{aligned}
&w_n = x_n + \theta_n(x_n - x_{n-1})\\
&y_n = J^G_{\gamma_n}\left[Tw_n - \gamma_n DTw_n\right]\\
&\textit{where } \gamma_{n+1} = \begin{cases} \min\left\{\frac{b_1\|Tw_n - y_n\|}{\|DTw_n - Dy_n\|},\ \gamma_n\right\}, & \textit{if } DTw_n \neq Dy_n,\\ \gamma_n, & \textit{otherwise,}\end{cases}\\
&\hat{t}_n = Tw_n - \hat{\xi}\zeta_n v_n,\\
&\textit{where} \quad v_n = Tw_n - y_n - \gamma_n\left[DTw_n - Dy_n\right],\\
&\zeta_n := \frac{\langle Tw_n - y_n, v_n\rangle}{\|v_n\|^2}, \quad \textit{if} \quad v_n \neq 0; \quad \textit{otherwise,} \quad \zeta_n = 0;\\
&b_n = w_n + \eta_n T^*(\hat{t}_n - Tw_n),\\
&\textit{where} \quad \eta_n \in \left[\bar{\epsilon}, \quad \frac{\|Tw_n - \hat{t}_n\|^2}{\|T^*(Tw_n - \hat{t}_n)\|^2} - \bar{\epsilon}\right] \quad \textit{if} \quad \hat{t}_n \neq Tw_n, \quad \textit{otherwise} \quad \eta_n = \eta,\\
&u_n = J^B_{\lambda_n}(b_n - \lambda_n A b_n),\\
&\textit{where } \lambda_{n+1} = \begin{cases} \min\left\{\frac{b_2\|b_n - u_n\|}{\|Au_n - Ab_n\|},\ \lambda_n\right\}, & \textit{if } Ab_n \neq Au_n,\\ \lambda_n, & \textit{otherwise,}\end{cases}\\
&z_n = b_n - \xi\omega_n r_n\\
&\textit{where} \quad r_n = b_n - u_n - \lambda_n\left[Ab_n - Au_n\right],\\
&\omega_n := \frac{\langle b_n - u_n, r_n\rangle}{\|r_n\|^2}, \quad \textit{if} \quad r_n \neq 0; \quad \textit{otherwise,} \quad \omega_n = 0;\\
&x_{n+1} = (1 - \phi_n)w_n + \phi_n z_n.
\end{aligned}
$$

Algorithm 9.1.9. *[235]*

Initialization: *Choose* $\eta > 0$, $x_0, x_1 \in H$ *and the sequences* $\{\omega_n\}$, $\{\alpha_n\}$, $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *such that such that*

$$
\begin{aligned}
&w_n = x_n + \theta_n(x_n - x_{n-1})\\
&y_n = J^G_{\gamma_n}\left[Tw_n - \gamma_n DTw_n\right]\\
&\textit{where } \gamma_{n+1} = \begin{cases} \min\left\{\frac{(\kappa_n+\kappa)\|Tw_n-y_n\|}{|\langle DTw_n-Dy_n,Tw_n-y_n\rangle|},\ \gamma_n+s_n\right\}, & \textit{if } \langle DTw_n - Dy_n, Tw_n - y_n\rangle,\\ \gamma_n + s_n, & \textit{otherwise,}\end{cases}\\
&\hat{t}_n = Tw_n - \zeta_n r_n,\\
&\textit{where}\ \ r_n = Tw_n - y_n - \gamma_n\left[DTw_n - Dy_n\right],\\
&\zeta_n := \frac{(a_1+\chi_n)\langle Tw_n - y_n, r_n\rangle}{\|r_n\|^2}, \quad \textit{if}\ \ r_n \neq 0; \quad \textit{otherwise,} \quad \zeta_n = 0;\\
&b_n = w_n + \eta_n T^*(\hat{t}_n - Tw_n)\\
&\textit{where}\ \ \eta_n = \begin{cases} \frac{\vartheta\|Tw_n-\hat{t}_n\|^2}{\|T^*(Tw_n-\hat{t}_n)\|^2}, & \textit{if } \|T^*(Tw_n - \hat{t}_n)\| \neq 0,\\ 0, & \textit{otherwise.}\end{cases}\\
&q_n = b_n + \varrho_n(b_n - b_{n-1})\\
&u_n = J^B_{\lambda_n}(q_n - \lambda_n Aq_n),\\
&\textit{where } \lambda_{n+1} = \begin{cases} \min\left\{\frac{(\sigma_n+\sigma)\|u_n-q_n\|}{|\langle Au_n-Aq_n,u_n-q_n\rangle|},\ \lambda_n+t_n\right\}, & \textit{if } \langle Au_n - Aq_n, u_n - q_n\rangle,\\ \lambda_n + t_n, & \textit{otherwise,}\end{cases}\\
&z_n = b_n - \omega_n v_n\\
&\textit{where}\quad v_n = q_n - u_n - \lambda_n\left[Aq_n - Au_n\right],\\
&\omega_n := \frac{(a_2+\pi_n)\langle b_n - u_n, v_n\rangle}{\|v_n\|^2}, \quad \textit{if}\ \ v_n \neq 0; \quad \textit{otherwise,} \quad \omega_n = 0;\\
&x_{n+1} = (1 - \delta_n - \phi_n - \alpha_n)b_n + \delta_n z_n + \phi_n q_n.
\end{aligned}
$$

where θ_n and ϱ_n are as defined in Algorithm 4.1.2.

9.1.3 Appendix

Algorithm 9.1.10. *[144]*

$$
\begin{aligned}
&x_0 \in H,\\
&y_n = P_C(x_n - \mu A x_n),\\
&T_n = \{x \in H : \langle x_n - \mu A x_n - y_n, x - y_n\rangle \leq 0\},\\
&z_n = \alpha_n x_0 + (1-\alpha_n)P_{T_n}(x_n - \mu A y_n),\\
&x_{n+1} = \beta_n x_n + (1-\beta_n)Sz_n,
\end{aligned}
$$

where $\mu \in \left(0, \frac{1}{L}\right), S : H \to H$ *is a quasi-nonexpansive mapping.*

Algorithm 9.1.11. *[224]*

$$
\begin{aligned}
&x_0 \in H,\\
&y_n = P_C(x_n - \mu A x_n),\\
&T_n = \{x \in H : \langle x_n - \mu A x_n - y_n, x - y_n\rangle \le 0\},\\
&z_n = P_{T_n}(x_n - \mu A y_n),\\
&x_{n+1} = (1 - \alpha_n - \delta_n) z_n + \delta_n U z_n,
\end{aligned}
$$

where $\mu \in \left(0, \frac{1}{L}\right), U : H \to H$ *is a* λ*-demicontractive mapping with* $0 \le \lambda < 1$.

Algorithm 9.1.12. *[225]*

Initialization: *Given* $\tau_0 > 0, \quad \mu \in (0,1)$. *Let* $x_0 \in H$ *be arbitrary.* **Iterative Step***: Calculate* x_{n+1} *as follows:*

$$
\begin{aligned}
&y_n = P_C(x_n - \tau_n A x_n),\\
&z_n = P_{T_n}(x_n - \tau_n A y_n),\\
&T_n = \{x \in H : \langle x_n - \tau_n A x_n - y_n, x - y_n\rangle \le 0\},\\
&x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[(1 - \delta_n) z_n + \delta_n U z_n,]
\end{aligned}
$$

where

$$
\tau_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|},\ \tau_n\right\}, & \text{if } Ax_n - Ay_n \neq 0,\\ \tau_n, & \text{otherwise} \end{cases}
$$

where $U : H \to H$ *is a* λ*-demicontractive mapping with* $0 \le \lambda < 1$.

Algorithm 9.1.13. *[225]*

Initialization: *Given* $\tau_0 > 0, \quad \mu \in (0,1)$. *Let* $x_0 \in H$ *be arbitrary.* **Iterative Step***: Calculate* x_{n+1} *as follows:*

$$
\begin{aligned}
&y_n = P_C(x_n - \tau_n A x_n),\\
&x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[(1 - \delta_n) z_n + \delta_n U z_n,]
\end{aligned}
$$

where $z_n = y_n - \tau_n(Ay_n - Ax_n)$, *and*

$$
\tau_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|},\ \tau_n\right\}, & \text{if } Ax_n - Ay_n \neq 0,\\ \tau_n, & \text{otherwise} \end{cases}
$$

Set $n := n + 1$ *and go to* **Step 1**.

where $U : H \to H$ *is a* λ*-demicontractive mapping with* $0 \le \lambda < 1$.

9.1.4 Appendix

Algorithm 9.1.14. *[220]*

$$x_{n+1} = J^B_{\lambda_n}(x_n + \tau_n T^*(S - I)Tx_n), \ n \geq 1, \tag{9.1}$$

where $J^B_{\lambda_n} := (I + \lambda_n B)^{-1}$, $0 < \liminf\limits_{n\to\infty} \lambda_n \leq \limsup\limits_{n\to\infty} \lambda_n < \infty$ *and* $0 < \liminf\limits_{n\to\infty} \tau_n \leq \limsup\limits_{n\to\infty} \tau_n < \frac{1}{||T||^2}$.

Algorithm 9.1.15. *[220]*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)J^B_{\lambda_n}(x_n + \lambda_n T^*(S - I)Tx_n), \ n \geq 1, \tag{9.2}$$

where $\sum\limits_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$, $0 < a \leq \lambda_n \leq b < \frac{1}{||T||^2}$ *and* $\sum\limits_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.

Algorithm 9.1.16. *[229]*

Let $x_1 \in C$, *define the sequence* $\{x_n\}$, $\{y_n\}$ *and* $\{t_n\}$ *by*

$$\begin{cases} y_n = P_C(x_n - \tau_n T^*(I - S)Tx_n), \\ t_n = P_C(y_n - \lambda_n A(y_n)), \\ w_n = P_C(y_n - \lambda_n A(t_n)), \\ x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n)w_n, \end{cases} \tag{9.3}$$

for each $n \in \mathbb{N}$, *where* $\{\tau_n\} \subset [a, b]$ *for some* $a, b \in \left(0, \dfrac{1}{\|T\|^2}\right)$ *and* $\{\lambda_n\} \subset [c, d]$ *for some* $c, d \in \left(0, \dfrac{1}{k}\right)$, $S : H_2 \to H_2$ *is a nonexpansive mapping,* $A : C \to H_1$ *is a monotone and* k*-Lipschitz continuous operator and* h *is a contraction on* C.

9.1.5 Appendix

Algorithm 9.1.17. *[227]*

$$\begin{cases} x_1 \in H_1 \\ x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n U(x_n - \rho_n A^*(I - T)Ax_n), \quad n \geq 1, \end{cases} \tag{9.4}$$

where f *is a contraction,* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0, 1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, $\{\rho_n\} \subset (0, 4)$, *where* U *and* T *are firmly nonexpansive mappings.*

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