Contribution to iterative algorithms for certain optimization problems and fixed point problems in Banach spaces.

by

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As the candidate's supervisor, I have approved this thesis for submission.

Dr. O. T. Mewomo

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Dedication

This thesis is dedicated to my wife Ogechi and my son Kamsiyochukwu.

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I am grateful to God Almighty, my rock, my fortress, my deliverer, my strength, my buckler, the horn of my salvation, and my high tower for His mercies, abundant love and protection over me and over those who have contributed directly or indirectly towards the success of this thesis and my Ph.D. programme.

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May God Almighty bless you all.

Abstract

We study the convergence analysis of the fixed points set of common solution of a one-parameter nonexpansive semigroup, the set of solution of constrained convex minimization problem and the set of solutions of generalized equilibrium problem in a real Hilbert space using the idea of regularized gradient-projection algorithm. Also, we look at the strong convergence of a modified gradient projection algorithm and forward-backward algorithm in Hilbert spaces with numerical computations. We also introduce an iterative algorithm for approximating a common solution of generalized mixed equilibrium problem and fixed point problem in a real reflexive Banach space. Using our algorithm, a strong convergence theorem is proved concerning an element in the intersection of set of solutions of generalized mixed equilibrium problem and the set of solutions of fixed point for a finite family of Bregman strongly nonexpansive mappings.

Moreover, we study and analyze an iterative method for finding a common element of the fixed points set of an infinite family of k-demicontractive mappings which is also a solution to a zero of the sum of two monotone operators, with one operator being maximal monotone and the other inverse-strongly monotone. We further extend our study from the frame work of real Hilbert spaces to more general real smooth and uniformly convex Banach spaces. In this space, we introduce an iterative algorithm with Meir-Keeler contractions for finding zeros of the sum of finite families of m-accretive operators and finite family of inverse strongly accretive operators. We apply our result to the approximation of solution of certain integro-differential equation with generalized p-Laplacian operators.

Furthermore, we study the convergence theorem for a new class of split variational inequality and variational inclusion problem in Hilbert space. We further considered split equality for minimization problem and fixed point sets, split fixed point problem and monotone inclusion problems, split equilibrium problem and fixed point set for multivalued mappings. All these of our algorithms involve a step-size selected in such a way that their implementation does not require the computation or an estimate of the spectral radius.

Again, an iterative algorithm that does not require any knowledge of the operator norm for approximating a solution of split equality equilibrium and fixed point problems in the frame work of p-uniformly convex Banach spaces which are also uniformly smooth is introduced of which we studied the approximation of solution of split equality generalized mixed equilibrium problem and fixed point problem for right Bregman strongly quasi-nonexpansive mappings in q-uniformly convex Banach spaces which are also uniformly smooth. We also study and analyze an iterative algorithm for finding a common element of the set of the split equality for monotone inclusion problem and fixed point of a right Bregman strongly nonexpansive mapping T in the setting of p-uniformly convex uniformly smooth Banach spaces. Finally, we present numerical examples of our theorems and apply our results to study the convex minimization problems and equilibrium problems.

Contents

Declaration

This thesis in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used in the text, proper reference has been made.

Okeke Chibueze Christian

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Contributed papers from the thesis

List of accepted papers

1. C.C. Okeke, A.U. Bello, C. Izuchukwu , O.T. Mewomo, Split equality for monotone inclusion problem and fixed point problem in real Banach spaces, The Australian Journal of Mathematical Analysis and Applications, Volume 14, Issue 2, Article 13, pp. 1-20, 2017
2. C. C. Okeke and O. T. Mewomo, On split equilibrium problem, variational inequality problem and fixed point problem for multi-valued mappings, Ann. Acad. Rom. Sci. Ser. Math. Appl. Vol. 9, No. 2/2017.
3. C. C. Okeke, M. E. Okpala and O. T. Mewomo, Common solution of generalized mixed equilibrium problem and bregman strongly nonexpansive mapping in reflexive banach spaces, Advances in Nonlinear Variational Inequalities Volume 21 (2018), Number 1, 1-16
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Some Articles Under Review

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Chapter 1

Introduction

1.1 Background of study

Fixed point theory is concerned with solutions of the equation

$$x = Tx \tag{1.1.1}$$

where T is (possibly) a nonlinear operator defined on a metric space. Any x that solves (1.1.1) is called a fixed point of T and the collection of all such elements is denoted by $F(T)$. For multi-valued mapping $T : X \to 2^X$, a fixed point of T is any $x \in X$ such that $x \in Tx$.

Fixed point theory is inarguably the most powerful and effective tools used in modern nonlinear analysis. It is an active and fruitful area of research as it has vast applicability in establishing existence and uniqueness of solutions of diverse mathematical models, like solutions to optimization problems, variational inequalities and ordinary differential equations. These models represent various phenomena arising in different fields, such as steady state temperature distribution, neutron transport theory, economic theories, chemical equations, optimal control of systems, models for population, epidemics and flow of fluids.

For example, given an initial value problem

$$\begin{cases} \frac{dx(t)}{dt} = f(t, x(t)), \\ x(t_0) = x_0. \end{cases} \tag{1.1.2}$$

This system is transformed into the functional equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

To establish the existence of solution of (1.1.2), we consider the operator $T : X \to X (X = C([a, b]))$ defined by

$$Tx = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Then, finding a solution to the initial value problem (1.1.2) amounts to finding a fixed point of T.

The existence (and uniqueness) of solution of equation (1.1.1), certainly depends on the geometry of the space and the nature of the mapping T. Existence theorems are concerned with establishing sufficient conditions under which the equation (1.1.1) will have a solution, but does not necessarily show how to find them. There are many existence and uniqueness theorems in the literature (see e.g Kato [92], Kirk [96], Kōmura [99]).

Though existence theorems do not indicate how to construct a process starting from a nonfixed point and convergence of sequence to a fixed point, they nevertheless, enhance the understanding of conditions under which the existence of such fixed points is guaranteed.

On the other hand, iterative methods for fixed points theory are concerned with approximation or computation of sequences which converge to solutions of (1.1.1).

The pivot of the iterative methods of fixed point theory is the Banach contraction mapping principle, which states that a self mapping T on complete metric space (X, d) satisfying

$$d(Tx, Ty) \leq kd(x, y), \quad 0 \leq k < 1, \quad \forall x, y \in X, \tag{1.1.3}$$

necessarily has a unique fixed point and for any starting point x_1, the sequence $\{T_n x_1\}$ converges to that fixed point.

Many authors, see for examples Alber [4], Boyd and Wong [22], have investigated more general conditions under which a mapping will have a unique fixed point and also developed iterative sequences that converges to such fixed points.

If $k = 1$ in inequality (1.1.3), the mapping T is called nonexpansive. There are many examples which show that $x_{n+1} = T^n(x)$ need not to converge to a fixed point of a nonexpansive mapping T, even if it has a unique fixed point. We then need to impose additional conditions on T (and/or the space X) and also modify the sequence $T^n(x)$ to ensure convergence to a fixed point of T.

These notable iterative algorithms were introduced for nonexpansive mappings namely, the Krasnosel'skii sequence presented in [100] as: $x_1 \in X$ and

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n),$$

the Krasnoselskii-Mann algorithm given by: $x_1 \in X$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad \lambda \in (0, 1),$$

the Halpern algorithm given in [84] as: $u \in X$ arbitrary and

$$x_{n+1} = \alpha_n u + (1 - \alpha)Tx_n,$$

and the more general Mann sequence presented in [119] as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n.$$

Diverse convergence theorems have been proved for these sequences, depending on the smoothness of the underlying space and/or the compactness of the mapping T.

Effort to establish convergence theorems for nonexpansive mappings is likely the most rewarding research venture in nonlinear analysis. It has helped in the development of the geometry of Banach spaces and other related class of mappings, namely, monotone and accretive operators.

A mapping $A : X \to X$ is called τ-strongly monotone if

$$\langle x - y, Ax - Ay\rangle \geq \tau\|x - y\|^{,} \ \ \forall\ x, y \in X,$$

and $A : X \to X^*$ is called τ-strongly accretive if

$$\langle Ax - Ay, j(x - y)\rangle \geq \tau\|x - y\|^2, \ \ \forall x, y \in X,$$

where $\langle ., .\rangle$ is the duality pairing between X and X^*, $j(x - y) \in J(x - y)$ where J is the normalized duality mapping. When $\tau = 0$, these mappings are called monotone and accretive respectively. If X is a Hilbert space, these two notions agree and they are simply refereed to as monotone.

Accretive mappings have properties that are similar to those of monotone mappings. However, the use of the strongly nonlinear mapping J makes the study of such mappings difficult. In a sense, the duality mapping on a Banach space has all the properties of Banach space that makes it differ from a Hilbert space and the space can be characterized, almost, exclusively by the mapping.

These two ideas have been proved to be very useful in many areas of interest. The idea of accretive operators vary often in partial differential equations and in the existence theory of nonlinear evolution equations. On the other hand, the idea of monotone operators appears in optimization theory and that, in particular, includes the increasingly important set-valued mapping called the subdifferential. Given a convex, lower semicontinuous function f, the subdifferetial is $\partial f : X \to 2^{X^*}$ given by

$$\partial f(x) := \{x^* \in X^* : f(y) - f(x) \geq \langle y - x, x^*\rangle, \forall y \in \forall y \in X\}.$$

The subdifferential is a monotone mapping and well known that $0 \in \partial f(\bar{x})$ if and only if $f(\bar{x}) = \inf\limits_{x \in X} f(x)$. This motivates the study of the more general problem of finding a zero of monotone operators, i.e $\bar{x}$ such that $0 \in A\bar{x}$, of a monotone operator A.

The question on the existence of zeros is studied under the concept of maximal monotone operators. A monotone mapping A is maximal monotone if the graph $G(A)$ is a maximal element when graphs of monotone operators in $X \times X^*$ are partially ordered by set inclusion. In that case, for any $(x, y) \in X \times X^*$, the inequality

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq 0, \ \forall x_2 \in D(A), \ y_2 \in Ax_2$$

implies $y_1 \in Ax_2$. Maximal accretive mapping are defined accordingly.

The accretive operators are intimately connected with an important generalization of nonexpansive mappings called the pseudocontractive mappings. A mapping is pseudocontractive in terminology of Browder and Petryshyn [26] if for $x, y \in X$, and for all $r > 0$,

$$\|x - y \leq \|(x - y) + r[(x - Tx) - (y - Ty)]\|.$$

By a result of Kato [92], this is equivalent to

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq 0.$$

Thus, a mapping T is pseudocontractive if and only if the complementary operator $A := I-T$ is accretive. Moreover, the zeros of A coincides with the fixed points of T.

Another interesting relationship is that the resolvent of an accretive mapping A always exists (i.e $I+\lambda A$ is invertible) and it is nonexpansive. The resolvent of A is a set valued mapping $J_\lambda : X \to 2^X$ defined

$$J_\lambda(x) = (I+\lambda A)^{-1}x, \quad \lambda > 0.$$

In this case, $A^{-1}(0) = F(J_\lambda)$. More Precisely, the mapping J_λ is in fact firmly nonexpansive, i.e

$$\|J_\lambda(x) - J_\lambda y\|^2 \leq \langle x-y, J_\lambda(x) - J_\lambda(y)\rangle, \quad \forall x, y \in X.$$

The existence and approximation algorithms for zeros of maximal monotone operators are usually formulated in relation with the corresponding problem for fixed points of firmly nonexpansive mappings. This makes the study of firmly nonexpansive and the more general pseudocontractive mappings an important tool for monotone operators and the theory of optimization.

The metric projection operators have become a veritable tool in dealing with variational inequality problems by iterative-projection method in Hilbert spaces. Variational Inequality Problem VIP(A,C) involving an accretive operator A and convex set C can be proved to be equivalent to the fixed point problem involving the nonexpansive mapping

$$T = P_C(I-\lambda A)$$

for arbitrary positive number λ. Conversely, given a differentiable functional f, the VIP(∇f,C) is simply the optimality condition for minimization problem

$$\min_{x\in C} f(x). \tag{1.1.4}$$

Metric projection operators in Hilbert spaces are accretive and nonexpansive and give absolutely best approximations of any element of the closed and convex set. However, in Banach space setting, this operator no longer possess most of those properties that made them so effective in Hilbert spaces.

To study monotone-type mappings and the related pseudocontractive mappings in Banach spaces, some analogues of Hilbert space type projection operators were introduced. These mappings are natural extensions of the classical projection operators to Banach spaces. They have also helped in the approximation of monotone operator in Banach spaces.

Fixed point theory is divided mainly into four branches, namely: set theoretical fixed point theory, topological fixed point theory, fuzzy topological fixed point theory and metric fixed point theory [10, 27, 94, 97]. In this study, we are interested in metric fixed point theory with particular interest on Split Feasibility Problems (SFP) and some of its generalizations. In particular, we study equilibrium problems, monotone inclusion problems, variational inequality problems and minimization problems and obtain some important convergence results which compliment, improve and generalize some existing results in literature.

1.2 Research problems and motivation

In this section, we discuss the research problems and motivation for the study

1.2.1 Research problems

Many authors have studied and introduced different iterative algorithms for fixed point problems, split monotone inclusion problems, convex feasibility problems, split feasibility problems, split variational inequality problems, equilibrium problems and minimization problems see for example [55, 56, 169, 180] and the references therein. These authors have proved weak and strong convergence results in Hilbert spaces and to the best of our knowledge not much has been done on split monotone inclusion problems, split equilibrium problem in a more general setting of Banach spaces other than the Hilbert spaces.

In this work, we study the monotone inclusion problem in uniformly convex and uniformly smooth Banach spaces. Also we extend some results on split variational inequality and equilibrium problem and some of its special cases from Hilbert spaces to higher Banach spaces.
Moreover, most of the results on split monotone inclusion, split equilibrium and split variational inequality problems in Hilbert spaces require a prior knowledge of the operator norm or the the estimation of the spectral radius which are not always easy to compute or estimate. Thus, we introduce some iterative algorithm for solving split equilibrium problems, split variational inequality problems and split monotone inclusion which do not require the knowledge of the operator norm or spectral radius.

We also study the regularized gradient-projection algorithm for solving one-parameter nonexpansive semigroup, constrained minimization and generalized equilibrium problem in Hilbert spaces. Furthermore, we also carry out some studies on fixed point problems and split feasibility problems using Bregman distance techniques in reflexive Banach spaces.

1.2.2 Motivation

1. Fixed Point problems:
   Bruck [30] noted that apart from being an obvious generalization of the contraction mappings, nonexpansive maps are important for the following reasons:
   (a) Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.
   (b) Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusions of the form $0 \in \frac{du}{dt} + T(t)u$ where the operators $T(t)$ are in general set-valued and are accretive or dissipative and minimally continuous.

   (c) Many well-known algorithms in signal processing and image reconstruction are iterative in nature and a wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the Krasnoselskii-Mann iteration procedure, for particular choices of the nonexpansive operator, see [42].

   Despite many existing results for nonexpansive type mapping in the literature, there is still much to be done on other maps which are more general than nonexpansive mappings.

2. Split feasibility problems: Recently, the split feasibility problem have been extended to an infinite-dimensional Hilbert spaces, see [42, 120, 128, 127, 137, 138, 139, 135] and have also been applied in solving problems in areas such as image restoration, computer tomography and radiation therapy treatment planning, see [179, 160, 173, 186, 189, 193] and references therein for existing results on split feasibility problem. Many authors have also studied split equality fixed point problems, split common fixed point problems and split convex feasibility problems, see [42, 7, 82, 106, 161].

   The study of split feasibility problem and split equality problem in Banach spaces outside Hilbert space is rare. Thus, it is necessary to extend study in the frame work of Banach spaces.

3. Split monotone inclusion problems:
   An important and perhaps interesting topic in nonlinear analysis and convex optimization concerns solving inclusions of the form $0 \in A(x)$, where A is a maximal monotone operator on a Banach space X. Its importance in convex optimization is evidenced from the fact that many problems that involve convexity can be formulated as finding zeros of maximal monotone operators. For example, convex minimizations and convex-concave mini-max problems, to mention but a few can be formulated in this way. Furthermore, the variational inclusion problem is important generalization of a variational inequality problem and has been extensively studied and generalized in different directions to study a wide class of problems arising in mechanics, optimization, nonlinear programming, economics, finance and applied sciences. In particular, the subdifferential of a proper, convex and lower semi-continuous (lsc) function f on a Banach space X, ∂f is a maximal monotone operator and a point $p \in X$ minimizes f if and only if $0 \in \partial f(p)$.

   Many authors have studied split monotone variational inclusion problem extensively in Hilbert spaces by introducing different iterative schemes and proving convergence theorems for solving split monotone variational inclusion problems in Hilbert spaces.

   The point of interest here is that as important as the split monotone inclusion problem is, much have not been done on it in Banach spaces more general than Hilbert spaces. Also, most of the existing results in Hilbert spaces involve iterative schemes such as shrinking projection algorithm or require the knowledge of the operator norm which sometimes may be difficult to compute. Thus, there is a need to introduce simpler and much easier iterative algorithms or iterative algorithms that do not require any prior knowledge of the operator norm for solving split monotone variational inclusion problem in certain Banach spaces.

4. Generalized equilibrium problem:
   Solving equilibrium problem represents an important area of mathematical sciences as numerous problems in physics, optimization, operations research, economics, game theory, financial mathematics and mechanics can be formulated as an equilibrium problem. Equilibrium problems include variational inequalities, optimization problems, Nash equilibra problems, saddle point problems, fixed point problems and complementarity problems as special cases, and the generalized mixed equilibrium problems generalize the equilibrium problems.

   In the theory of variational inequalities, variational inclusions and equilibrium problems, the development of an efficient and implementable iterative algorithms is interesting and important. In past years, some iterative methods have been proposed to solve the equilibrium problem and variational inequality problems in Hilbert spaces and Banach spaces, see for instance [14, 126, 48, 144, 143, 195, 110] and the references therein.

   In literature, results on split equilibrium problems are mostly in Hilbert spaces and also depend on a prior knowledge of the operator norm. Hence, it will be important to introduce iterative algorithms for solving split equilibrium problems which do not require knowledge of operator norm. Furthermore, there is also a need to obtain iterative solution of split equilibrium problem or any of its generalizations in Banach spaces more general than Hilbert spaces.

1.3 Objectives of study

The main objectives of this study are:

(a) To introduce iterative algorithms and prove strong convergence theorem for solving split feasibility problems such that our step-size is selected in such a way that its implementation does not involve the computation or estimation of the operator norm.

(b) To extend some existing result on split monotone inclusion problems and split generalized equilibrium problem from the frame work of Hilbert space to q-uniformly smooth Banach spaces which are also uniformly convex.

(c) To propose an iterative method for solving split equality convex minimization problem of the form $\min\{f(x)+g(x)\}$ where f and g are convex functions and give some applications.

(d) To propose a regularized gradient-projection algorithm for solving one-parameter nonexpansive semigroup, constrained convex minimization and generalized equilibrium problems.

(e) To study strong convergence for Meir-Keeler contractions and a countable family of accretive operators in the frame work of uniformly convex and uniformly smooth Banach spaces with applications to certain integro-differential equations with generalized p-Laplacian operators.

(f) To obtain some strong convergence results for a new class of split variational inequality and monotone variational inclusion problems in Hilbert spaces with numerical examples.

1.4 Organization of the thesis

The thesis is divided into seven chapters as follows:

In chapter 1, we give a brief historical background of our study, discuss the research problems and motivation for the study, give the objectives of the study and finally describe the organization of the thesis.

In chapter 2, we introduce some basic concepts and terms and give some existing results and classical inequalities that are needed in establishing our results in this work. Some notable results on metric projection in Banach and the concept of Bregman distance are also discussed.

Our major work begins in Chapter 3 and this comprises of three sections.

In section 3.1, we give the definition of one parameter nonexpansive semi group and a brief introduction of regularized gradient-projection algorithm in Hilbert spaces.
In section 3.2, we study an iterative algorithm for finding a common element of the fixed points set of common solution of a one-parameter nonexpansive semigroup, the set of solution of constrained convex minimization problem and the set of solutions of generalized equilibrium problem in a real Hilbert space using the idea of regularized gradient-projection algorithm under suitable conditions. We also give an application.
In section 3.3, a strong convergence theorem is proved concerning an element in the intersection of set of solutions of generalized mixed equilibrium problem and the set of solutions of fixed point for a finite family of Bregman strongly nonexpansive mappings.

In chapter 4, we study the monotone inclusion, minimization and fixed point problem and this comprises of 4 sections.
In section 4.1, a brief introduction of sum of two monotone operators and Peaceman-Rachford algorithm is given.
In section 4.2, a strong convergence result is obtained for finding a common element of the fixed points set of an infinite family of k-demicontractive mappings which is also a zero of the sum of two monotone operators, with one operator being maximal monotone and the other inverse-strongly monotone.
In section 4.3, we study an iterative algorithm with Meir-Keeler contractions for finding zeros of the sum of finite families of m-accretive operators and finite family of α-inverse strongly accretive operators in a real smooth and uniformly convex Banach spaces.We also discuss application of this method to the approximation of solution to certain integro-differential equation with generalized p-Laplacian operators.
In section 4.4, we propose a new modification of the Gradient Projection Algorithm (GPA) and the Forward-backward Algorithm (FBA). Using our proposed algorithms, we establish two strong convergence theorems. We also apply our result to solve split feasibility

problem and we give numerical example.

Chapter 5 is devoted to the study of split feasibility variational inequality and fixed point problems and this comprises of 3 sections.
In section 5.1, we propose an algorithm involving a step-size selected in such a way that its implementation does not require the computation or an estimate of the operator norm. Using our algorithm we prove strong convergence theorem for finding a common element that solves a class of split variational inequality problems.
In section 5.2, we propose an algorithm involving a step-size selected in such a way that its implementation does not require the computation or an estimate of the spectral radius. Using our algorithm we prove strong convergence theorem for common solution of a split equilibrium problem, a variational inequality problem and fixed point problem for multi-valued quasi-nonexpansive mappings in real Hilbert spaces.
In section 5.3, we propose an iterative algorithm involving a step-size selected in such a way that its implementation does not require the computation or an estimate of the spectral radius. Using our algorithm, we state and prove a strong convergence theorem for approximating a common solution to a monotone inclusion problem and a fixed point problem for multi-valued Lipschitz hemicontractive-type mappings whose image under a bounded linear operator is a fixed point of a demicontractive mapping.

In chapter 6, we study split equality equilibrium, split equality monotone inclusion, split equality minimization and fixed point problems in Banach spaces and this comprises of three sections.
In section 6.1, an iterative algorithm for approximating a solution of a split equality minimization problem and split equality fixed point problem for demi-contractive mappings is introduced. Using our iterative algorithm, we state and prove a strong convergence theorem for approximating an element in the intersection of the solution set of a split equality minimization problem and the solution set of split equality fixed point problem for demicontractive maps. Our result do not require any compactness assumption and do not require the prior knowledge of the operator norm.
In section 6.2, we propose a new iterative algorithm for approximating a common solution of split equality monotone inclusion problem and split equality fixed point problem. Using our algorithm, we state and prove a strong convergence theorem for approximating an element in the intersection of the set of solutions of a split equality monotone inclusion problem and the set of solutions of a split equality fixed point problem for right Bregman strongly nonexpansive mappings in the setting of p-uniformly convex Banach spaces which are also uniformly smooth. We also give an application.
In section 6.3, we study the approximation of solution of split equality generalized mixed equilibrium problem and fixed point problem for right Bregman strongly quasi-nonexpansive mappings in q-uniformly convex Banach spaces which are also uniformly smooth. We introduce a simultaneous algorithm and prove strong convergence without prior information of the operator norms involved.

In the last chapter, our contribution to knowledge is discussed and some areas of future research are also pointed out.

Chapter 2

Literature Review

In this chapter, we give definitions of some basic terms and concepts that will be useful throughout the work. We also present some useful results and give detailed literature review of concepts that are relevant to the work.

2.1 Preliminaries and definitions

Unless otherwise specified, X represents a Banach space with norm $\|.\|$. The dual space X^* of X is the Banach space of all bounded linear functionals on X. It is endowed with the norm

$$\|x^*\|_{X^*} := \sup_{\|x\|=1} \langle x, x^* \rangle,$$

where $\langle ., . \rangle$ represent the pairing between the elements of X and X^*. Given any sequence $\{x_n\}$ in X, we take $x_n \to x^*$ to mean $\{x_n\}$ converges strongly to x^* and $x_n \rightharpoonup x^*$ to mean that $\{x_n\}$ converges weakly to x^*.

2.1.1 Some basic concepts and definitions

Definition 2.1.1. *A mapping $T : X \to X$ is called L-Lipschitzian if there exists $L > 0$ such that*

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in X. \tag{2.1.1}$$

Remark 2.1.2. *If $L = 1$ in the inequality (2.1.1), the mapping is called nonexpansive and if $0 < L < 1$, it is called a strict contraction. It is well known that $F(T)$ is closed and convex whenever T is nonexpansive.*

Definition 2.1.3. *A mapping $T : X \to X$ is pseudocontractive in the terminology of Browder and Pertryshn [26] if*

$$\|x - y\| \leq \|(x - y) + r[(x - Tx) - (y - Ty)]\|, \quad \forall x, y \in X, \ r > 0. \tag{2.1.2}$$

By the result of Kato [92], (2.1.2) is equivalent to

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq 0.$$

Thus, a mapping T is pseudocontractive if and only if the complementary operator $A := I-T$ is accretive.

Definition 2.1.4. *Given a real Hilbert space H and a closed convex subset C of H, let $T : C \to C$ be a mapping. Then T is said to be*

(B1) strictly pseudocontractive if there exists $\mu \in [0,1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + \mu\|(x-Tx)-(y-Ty)\|^2, \quad \forall x, y \in C; \tag{2.1.3}$$

(B2) demi-contractive if $F(T) \neq \emptyset$ and there exists $\mu \in [0,1)$ such that

$$\|Tx-Tp\|^2 \leq \|x-p\|^2 + \mu\|x-Tx\|, \quad \forall\ (x,p) \in C \times F(T); \tag{2.1.4}$$

(B3) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx-Tp\| \leq \|x-p\|, \quad \forall (x,p) \in C \times F(T). \tag{2.1.5}$$

It is known that (2.1.3) is equivalent to

$$\langle Tx-Ty, x-y\rangle \leq \|x-y\|^2 - \frac{1-\mu}{2}\|(x-Tx)-(y-Ty)\|^2, \quad \forall x, y \in C. \tag{2.1.6}$$

If $\mu = 1$ in the inequality (2.1.4), the mapping is called hemicontractive. (2.1.4) is equivalent to

$$\langle Tx-Tp, x-p\rangle \leq \|x-p\|^2 - \frac{1-\mu}{2}\|Tx-x\|^2 \qquad \forall\ \ x \in C, \quad p \in F(T). \tag{2.1.7}$$

Definition 2.1.5. *Given a mapping $f : X \to \mathbb{R} \cup \{+\infty\}$. We say that f is*

- *proper if*

$$D(f) := \{x \in X : f(x) < \infty\} \neq \emptyset;$$

- *convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall \lambda \in (0,1),\ \ x, y \in D(f);$$

- *lower semi-continuous (lsc) at $x_0 \in D(f)$ if*

$$f(x_0) \leq \liminf_{x \to x_0} f(x);$$

- *Gâteaux-differentiable ($G-$differentiable) at $x_0 \in D(f)$, if there exists a bounded linear mapping $f' \in X^*$ such that*

$$\langle h, f'(x)\rangle = \lim_{t \to 0} \frac{f(x_0+th)-f(x_0)}{t};$$

- *Fre'chet-differentiable at* $x_0 \in D(f)$, *if it is* $G-$*differentiable at* $x_0 \in D(f)$, *with derivative* $f'(x_0)$, *and*

$$\lim_{t\to 0} \sup_{\|h\|=1} \left| \frac{f(x_0+th)-f(x_0)}{t} - \langle h, f'(x_0)\rangle \right| = 0;$$

- *subdifferential at* $x_0 \in D(f)$, *if there exists* $g \in X^*$, *called a subgradient element, such that*

$$f(x)-f(x_0) \geq \langle x - x_0, g\rangle, \ \forall x \in X.$$

Remark 2.1.6. *It is known that every convex lower semicontinuous function* f *is subdifferentiable in the interior of its domain, see for example Cioranescu [64]. Moreover,* f *is* $G-$*differentiable if and only if the subdifferential* $\partial f(x)$ *contains only one element, namely* $f'(x) = \nabla f(x)$, *for each* $x \in D(f)$.

Definition 2.1.7. *The subdifferential of a functional* f *at* x_0 *is the set valued mapping* $\partial f : X \to 2^{X^*}$ *defined by*

$$\partial f(x_0) := \{g \in X^* : f(x) - f(x_0) \geq \langle x - x_0, g\rangle\} \ \forall x \in X.$$

The subdifferential is an increasingly important multivalued mapping due to its frequent use in the theory of optimization. Many functions of interest, for example, the absolute value function $f(x) = |x|$ on $\mathbb{R}$ are not differentiable. They may however be subdifferentiable. Therefore $0 \in \partial f(\bar{x})$ if and only if $f(x) \geq f(\bar{x})$ holds for all $x \in X$. Finding a minimizer of f therefore is equivalent to finding an $\bar{x} \in X$ with $0 \in \partial f(\bar{x})$. This technique has been applied successfully for example in game theory and market economy, in the existence theory for equilibra.

2.1.2 Duality mappings and characterization of some Banach spaces

We present some characterization of Banach spaces according to their duality mappings.

It is a common knowledge that the domain of a function f is almost never compact in the infinite dimensional spaces and therefore strong convergence is almost never guaranteed. To enforce a form of convergence of a minimizing sequence, one uses some other properties of the functional. In particular, one assumes that f is weakly lower semicontinuous, i.e "if $x_n \rightharpoonup u$, then $f(u) \leq \liminf f(x_n)$". It is known that every convex lower semicontinuous function is weakly lower semicontinuous.

If the mapping f is differentiable, then the convexity can be characterized exclusively by the derivative as follows:

$$\langle u - v, f'(u) - f'(v)\rangle \geq 0 \ \ \forall u, v \in X, \tag{2.1.8}$$

where $\langle ., . \rangle$ is the paring between the elements of the dual X^* and X. Any mapping $A : X \to X^*$ satisfying the type of inequality (2.1.8), i.e

$$\langle u - v, A(u) - A(v)\rangle \geq 0 \ \ \forall u, v \in X,$$

is called a *monotone* mapping. We have noted that if f is convex and lower semicontinuous but not necessarily differentiable, we may still obtain the subdifferential of f. The multivalued mapping ∂f satisfies the inequality (2.1.8) in the sense that

$$\langle u^* - v^*, u - v\rangle \geq 0,\ \forall u, v \in X,\ u^* \in \partial f(u),\ v^* \in \partial(v). \tag{2.1.9}$$

This suggest that the inequality (2.1.8) is applicable to a wide range of areas including multi-valued mappings.

Definition 2.1.8. *Given a Banach space X with its dual X^*. We recall that*

- *a normed space X is called uniformly smooth if for every $x \in X$, $\|x\| = 1$, there exists a unique $x^* \in x^*$ such that $\|x^*\| = 1$ and $\langle x, x^*\rangle = \|x\|$,*
- *the modulus of convexity of X is a mapping $\delta_X : [0, 2] \to \mathbb{R}$ defined by*

$$\delta_X(t) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x - y\| = t\right\}, \tag{2.1.10}$$

- *and the modulus of smoothness is a mapping $\rho_X : (0, \infty) \to \mathbb{R}$ defined by*

$$\rho_X(t) = \sup\left\{\frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : \|x\| = 1,\ \|y\| = t\right\}. \tag{2.1.11}$$

The space X is said to be uniformly convex whenever $\delta_X(t) > 0$ for each $t \in (0, 2]$ and uniformly smooth if $\lim\limits_{t\to 0^+} \frac{\rho_X(t)}{t} = 0$. Given real numbers $p, q > 1$, the space X is called p-uniformly convex (resp. q-uniformly smooth) if for some constant $c > 0$,

$$\delta_X(t) \geq ct^p \quad (\text{resp}\ \ \rho_X(t) \leq ct^q).$$

Moreover, X is uniformly smooth if and only if X^* is uniformly convex and vice versa. Also, if $\frac{1}{p} + \frac{1}{q} = 1$, then X^* is q-uniformly smooth if X is p-uniformly convex and vice versa.

Common examples of p-uniformly convex spaces are the L_p spaces, $1 < p < \infty$. Given a measure space $(\Omega, \mathbb{A}, \mu)$, we define a real Lebesgue space $(L_p(\Omega))$, $1 < p < \infty$ as

$$L_p(\Omega) := \left\{f, f : \Omega \to \mathbb{R} \cup \{+\infty\},\ \ f \text{ is } \mathbb{A} - \text{measurable},\ \int_\Omega |f|^p d\mu < \infty\right\}.$$

In this case, $(L_p(\Omega), \|.\|_p)$, where

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}},$$

is a normed linear space.

In the special case that $\Omega = \mathbb{N}$ and μ is the counting measure δ, the space

$$L_p(\mathbb{N}) = \left\{f, f : \mathbb{N} \to \mathbb{R}, f(n) = x_n, \int_{\mathbb{N}} |f|^p d\delta < \infty\right\},$$

corresponds to

$$l_p := \left\{ (x_n)_n : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Let q be the Holder conjugate exponent of p, i.e

$$q := \frac{p}{p-1}, \quad 1 < p < \infty.$$

For $u \in L_q(\Omega)$, we define the linear functional F_u on $(L_p(\Omega))^*$ by

$$F_u(f) := \int_\Omega f.u \; d\mu, \quad f \in L_p.$$

The Holder inequality

$$|F_u(f)| = \left| \int_\Omega f.u \; d\mu \right| \leq \|u\|_q \|f\|_p$$

gives $\|F_u\|_* \leq \|u\|_q$. Thus, the mapping

$$I_p : L_q(\Omega) \to (L_p(\Omega))^*, \quad u \mapsto F_u,$$

is a one to one bounded linear operator with $\|I_p\|_{B(L_q,(L_p)^*)} \leq 1$. With the isometry above in mind, we will habitually identify the space $(L_p)^*$ with L_q in the sense that for any $\phi \in (L_p)^*$, there exist $u_\phi \in L_q$, such that $\langle f, \phi \rangle = \int_\Omega f.u_\phi \; d\mu, \;\; \forall \, f \in L_p$, and $\|\phi\|_* = \|u_\phi\|_q$. It is well known, see for example Chidume [60], that

(i) X is $p-$ uniformly convex if and only if X^* is $q-$ uniformly smooth.

(ii) X is $q-$ uniformly smooth if and only if X^* is $p-$ uniformly convex.

Let X be a real $p-$uniformly convex and uniformly smooth Banach spaces with dual X^* which is $q-$uniformly smooth and uniformly convex. We define the functional $f_p : X \to \mathbb{R}$ by

$$f_p(x) := \frac{1}{p}\|x\|^p, \quad x \in X.$$

It is obvious that f_p is strictly convex and lower semicontinuous. Then the subdifferential of f_p, which is actually the Fre'tchet derivative is denoted by J_p, where

$$J_p(x) = \{ j_p(x) \in X^* : \langle j_p(x), x \rangle = \|x\|^p = \|j_p(x)\|^p \}. \tag{2.1.12}$$

This mapping $J_p : X \to 2^{X^*}$ is nonlinear and is called the generalized duality mapping of X.

For $p = 2$, the mapping $J_2 := J : X \to 2^{X^*}$ is called the normalized duality mapping. When it is understood that J is single valued, we may use $J(x)$ and $j(x)$ interchangeably. Some of its very useful properties are :

(a) For any $x \in X$, $J(x) \neq \emptyset$ (due to Hahn Banach theorem).

(b) For any real number α, $J(\alpha x) = \alpha J(x), \quad \forall x \in X$.

(c) If X is a reflexive and smooth Banach space, then J is single-valued and onto.

(d) If X is strictly convex, then J is injective.

(e) If X is reflexive and strictly convex and X^* is strictly convex then $J^* : X^* \to X^{**}$ $(= X)$ is a duality mapping on X^* satisfying $J^{-1} = J^*$.

The normalized duality mapping characterize the reflexivity of a Banach space, as shown in the result below:

Theorem 2.1.9. *(Cioranescu [64]) Let X be a Banach space and J the normalized duality mapping. Then X is reflexive if and only if*

$$\bigcup_{x \in X} J(x) = X^*.$$

Below are other basic relationships that exist between the geometric property of Banach space and generalized duality mappings, which can be found in [60] and [64]:

Proposition 2.1.10. *Let X be a Banach space. Then the following assertions holds:*

(a) X is smooth if and only if the generalized duality mapping J_p is single valued.

(b) X is uniformly smooth if and only if the generalized duality mapping J_p is norm to norm uniformly continuous on bounded subsets of X.

(c) If X has a uniformly $G-$differentiable norm, then J_p is norm to weak uniformly continuous on bounded subsets of X.*

Definition 2.1.11. *(Browder [28]) A Banach space X is said to have a weakly continuous duality mapping if there exists a $p > 1$ such that J_p is singled-valued and weak* sequentially continuous, that is*

$$if \quad x_n \rightharpoonup x, \quad then \quad J_P(x_n) \rightharpoonup J_P(x) \quad in \quad the \quad weak^* \quad topology.$$

Example of spaces with weakly continuous duality mapping are l_p, $1 < p < \infty$. For a Banach space with weakly continuous duality, the following result holds:

Theorem 2.1.12. *(Cioranescu [64] and Riech [152]) Suppose that X has a weakly continuous duality mapping J_p and that the sequence $\{x_n\}$ converges weakly to x. Then*

$$\limsup_{n\to\infty} \|x_n - z\|^p = \limsup_{n\to\infty} \|x_n - x\|^p + \|z - x_n\|^p$$

$\forall z \in X$. In particular, X satisfies the Opial's conditions; that is,

$$if \quad x_n \rightharpoonup x, \quad then \quad \limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - z\|$$

for all $z \in X$, $z \neq x$.

Below is another important property of the normalized duality mapping in Banach space which can be found in [92].

Lemma 2.1.13. *(Kato [92]) Let $x, y \in X$. Then $\|x\| \leq \|x + \alpha y\|$ for every $\alpha > 0$ if and only if there exists $j(x) \in J(x)$ such that $\langle y, j(x) \rangle \geq 0$.*

2.1.3 Metric projection in Hilbert spaces

Given a nonempty closed and convex subset C of a Hilbert space H, the metric projection or the proximal mapping on C is a mapping $P_C : H \to C$ such that for each $x \in H$, the uniquely existing element $P_C x \in C$ satisfies

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

A very important inequality that characterize the metric projection in Hilbert spaces is stated below.

Proposition 2.1.14. *For arbitrary $x \in H$, $z = P_C x$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$$

From this proposition, we derive that

(i) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle$ $\forall x, y \in H$, that is, the metric projection is firmly nonexpansive;

(ii) $\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2$ $\forall x \in H$ and $y \in C$;

(iii) if C is a closed subspace, then P_C coincide with the orthogonal projection from H onto C. Thus, for any $y \in C$, $\langle x - P_C x, y\rangle = 0$.

Remark: The convexity of the set C is very crucial in the existence of the mapping P_C. This can be seen in the example where $C := \{e_1, e_2, ..., e_n, ...\} \subset l_2$, $e_n = \left(0, 0, ..., \frac{n+2}{n}, ...\right)$. Certainly, C is closed but not convex. It is easy to see that $P_C 0 = \emptyset$. In some case when the structure of the convex set C is simple, we can easily calculate the metric projection onto such a set.

Example 2.1.15. *(a) Let $C = \bar{B}(u, r) = \{y \in H : \|y - u\| \leq r\}$. Then*

$$P_C x = \begin{cases} u + r\frac{(x-u)}{\|x-u\|}, & if\ x \notin C, \\ x, & if\ x \in C. \end{cases}$$

(b) Given a nonzero mapping $f : H \to \mathbb{R}$ and $C := \{y \in H : f(y) = \alpha\}$ a hyperplane, then

$$P_C x = x - \frac{f(x) - \alpha}{\|f\|^2} f.$$

(c) Given a nonzero mapping $f : H \to \mathbb{R}$ and $C := \{y \in H : f(y) \leq \alpha\}$ a closed half space, then

$$P_C x = \begin{cases} x - \frac{(f(x)-\alpha)}{\|f\|^2} f, & if\ f(x) > \alpha, \\ x, & if\ f(x) < \alpha. \end{cases}$$

2.1.4 Bregman distance and some related notions

In 1967, Bregman [24] introduced a nice and effective method for using the so called Bregman distance function Δ_f in the process of designing and analyzing feasibility and and optimization algorithms. This opened a growing area of research in which Bregman distance technique is applied in various ways in order to design and analyze iterative algorithms for solving equilibra and for computing fixed points of nonlinear mappings (see, e.g., [2, 1, 3, 31, 34, 24, 36, 37, 39, 40, 41, 77, 98] and the references therein).

Let $f : E \to \mathbb{R}$ be an admissible function, that is, a proper, lower semicontinuous, convex and Gáteaux differentiable function. Under these conditions, we know that f is continuous in the $int(domf)$ (see [33]).

Definition 2.1.16. *(***Bregman** *[24]) Let $f : X \to (-\infty, +\infty]$ be a $G-$differentiable function. The function $\Delta_f : D(f) \times intD(f) \to [0, +\infty)$ defined by*

$$\Delta_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle \tag{2.1.13}$$

is called the Bregman distance with respect to f.

Remark 2.1.17. *[179]. The generalized duality mapping J_p is the derivative of the function $f_p(x) = (\frac{1}{p})||x||^p$. Given that $f = f_p$ in the definition above, the Bregman distance with respect to f_p now becomes*

$$\begin{aligned} \Delta_p(x, y) &= \frac{1}{q}||x||^p - \langle J_p x, y\rangle + \frac{1}{p}||y||^p \\ &= \frac{1}{p}(||y||^p - ||x||^p) + \langle J_p x, x - y\rangle \\ &= \frac{1}{q}(||x||^p - ||y||^p) - \langle J_p x - J_p y, y\rangle. \end{aligned} \tag{2.1.14}$$

We note that the Bregman distance is not symmetric, therefore it is not a metric but it has the following important properties for all $x, y, z \in X$:

(i) $\Delta_p(x, x) = 0$,

(ii) $\Delta_p(x, y) \geq 0$,

(iii) $\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_p x - J_p y\rangle$,

(iv) $\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_p x - J_p y\rangle$.

For any p-uniformly convex Banach space X, the metric and Bregman distance have the following relation:

$$k||x - y||^p \leq \Delta_p(x, y) \leq \langle x - y, J_p x - J_p y\rangle,$$

where $k > 0$ is a fixed number.

The modulus of total convexity of f is the bifunction $\nu_f : int(domf) \times [0, +\infty) \to [0, +\infty]$ which is defined by

$$\nu_f(x, t) := \inf\{\Delta_f(y, x) : y \in domf,\ \|y - x\| = t\}.$$

The function f is said to be totally convex at a point $x \in int(domf)$ if $\nu_f(x,t) > 0$ whenever $t > 0$. The function f is said to be totally convex when it is totally convex at every point $x \in int(domf)$. This property is less stringent than uniform convexity (see [36], Section 2.3, page 92). Examples of totally convex functions can be found, for instance, in [35, 36, 39].

We remark that f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets (see [39], Theorem 2.10, page 9).
The Bregman projection (cf. [24]) with respect to f of $x \in int(domf)$ onto a nonempty, closed and convex set $C \subset int(domf)$ is defined as the necessarily unique vector $Proj_C^f(x) \in C$, which satisfies

$$\Delta_f\left(Proj_C^f(x), x\right) = \inf\{\Delta_f(y,x) : y \in C\}. \tag{2.1.15}$$

Let C be a nonempty, closed and convex subset of X, $f : X \to \mathbb{R}$ be a $G-$ differentiable function and totally convex function and let $x \in X$. It is known from [39] that $z = Proj_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0$ for all $y \in C$. We also have

$$\Delta_f(y, Proj_C^f(x)) + \Delta_f(Proj_C^f(x), x) \leq \Delta_f(x,y), \quad \forall x \in X,\ y \in C. \tag{2.1.16}$$

Similar to metric projection in Hilbert space, the Bregman projection with respect to totally convex and $G-$ differentiable function has a variational characterization (cf. [39], Corollary 4.4, page 2.3).

Proposition 2.1.18. *(see [174])(Characterization of Bregman Projections). Suppose that $f : X \to \mathbb{R}$ is totally convex and $G-$ differentiable in $int(domf)$. Let $x \in int(domf)$ and $C \subset int(domf)$ be a nonempty, closed and convex set. If $\hat{x} \in C$, then the following conditions are equivalent:*

(i) The vector $\hat{x}$ is the Bregman projection of x onto C with respect to f.

(ii) The vector $\hat{x}$ is unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \geq 0, \quad \forall y \in C.$$

(iii) The vector $\hat{x}$ is the unique solution of the inequality

$$\Delta_f(y,z) + \Delta_f(z,x) \leq \Delta_f(y,x), \quad \forall \in C.$$

Definition 2.1.19. *Let C be a nonempty, closed and convex subset of X and $T : C \to C$ be any mapping. A point $p \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set asymptotic fixed points of T is denoted by $\hat{F}(T)$.*

$T : C \to C$ is said to be

(i) right Bregman firmly nonexpansive if

$$\langle J_p(Tx) - J_p(Ty), Tx - Ty\rangle \leq \langle J_p(Tx) - J_p(Ty), x - y\rangle, \ \forall x, y \in C,$$

equivalently,

$$\Delta_p(Tx,Ty) + \Delta_p(Ty,Tx) + \Delta_p(x,Tx) + \Delta_p(y,Ty) \leq \Delta_p(x,Ty) + \Delta_p(y,Tx),$$

(ii) right Bregman strongly nonexpansive (see [118]) with respect to a nonempty $\hat{F}(T)$ if

$$\Delta_p(Tx, y) \leq \Delta_p(x, y), \ \forall x \in C, \ y \in \hat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $y \in \hat{F}(T)$ and

$$\lim_{n\to\infty} (\Delta_p(x_n, y) - \Delta_p(Tx_n, y)) = 0,$$

it follows that

$$\lim_{n\to\infty} \Delta_p(x_n, Tx_n) = 0.$$

Remark 2.1.20. *[118]. Every right Bregman firmly nonexpansive mapping is right Bregman strongly nonexpansive mapping with respect to $F(T) = \hat{F}(T)$.*

Definition 2.1.21. *A mapping $B : X \to 2^{X^*}$ is called monotone if*

$$\langle \xi - \eta, x - y\rangle \geq 0 \ \forall x, y \in E, \ \xi \in B(x), \eta \in B(y). \tag{2.1.17}$$

B is said to be maximal if the graph of B denoted by $G(B)$ is not properly contained in the graph of any other monotone mapping. It is generally known that a monotone mapping B is maximal if and only if $\langle x - y, u - v\rangle \geq 0$, for all $(x, u) \in X \times X, \ (y, v) \in G(B)$ implies $u \in Bx$.

The following are examples of monotone mappings.

Example 2.1.22. *Let X be a real Banach space and $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function. The subdifferential ∂f of f defined by*

$$\partial f(x) = \{\xi \in x^* : \langle \xi, y - x\rangle \leq f(y) - f(x) \ \forall x \in x\},$$

is maximal monotone (see [153]).

Example 2.1.23. *Let X be a real Banach space, then the normalized duality mapping J is monotone. Indeed for any $x, y \in E, \ u \in J(x), \ v \in J(y)$, we have*

$$\begin{aligned} \langle x - y, u - v\rangle &= ||x||^2 + ||y||^2 - \langle x, v\rangle - \langle y, u\rangle \\ &\geq ||x||^2 + ||y||^2 - ||x||||v|| - ||y||||u|| \\ &= ||x||^2 + ||y||^2 - 2||x||||y|| \\ &= (||x|| - ||y||)^2 \geq 0. \end{aligned}$$

Example 2.1.24. *Let A be an $n \times n$ matrix with real entries. Consider the operator $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = Ax$. Then f is maximal monotone if f is a positive linear operator (see [83]).*

Definition 2.1.25. *Let X be a p−uniformly convex Banach space and J_p be the generalized duality mapping of X. The resolvent of a maximal monotone mapping B is the operator $Res_p^{\lambda B} : X \to 2^X$ defined by*

$$Res_p^{\lambda B} := (J_p + \lambda B)^{-1} \circ J_p, \quad \lambda > 0. \tag{2.1.18}$$

Remark 2.1.26. *The resolvent operator* $Res_p^{\lambda B}$ *is a Bregman firmly nonexpansive operator. Furthermore,* $0 \in B(x)$ *if and only if* $x = Res_p^{\lambda B}(x)$ *(see [151], for more details).*

Let C be a nonempty closed and convex subset of X. The metric projection

$$P_C x := \underset{y \in C}{argmin}\ ||x - y||$$

for all $x \in X$ is the unique minimizer of the norm distance which can be characterized by a variational inequality:

$$\langle J_p^X(x - P_C x), z - P_C x \rangle \leq 0,\ \forall\ z \in C. \tag{2.1.19}$$

Similar to the metric projection, we define the Bregman projection as

$$\Pi_C x := \underset{y \in C}{argmin}\ \Delta_p(x, y),$$

$\forall\ x \in X$, which is the unique minimizer of the Bregman distance. The Bregman projection can also be characterized by a variational inequality:

$$\langle J_p^X(x) - J_p^X(\Pi_c x), z - \Pi_c x \rangle \leq 0.\ \forall\ z \in C, \tag{2.1.20}$$

which implies that

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x), \tag{2.1.21}$$

forall $z \in C$.

Definition 2.1.27. *[179]. Let* X *be a* $p-$*uniformly convex Banach space. The function* $V_p : X^* \times X \to [0, +\infty)$ *is defined by*

$$V_p(x, y) := \frac{1}{q}||x||^q - \langle x, y \rangle + \frac{1}{p}||y||^p,\ \forall x \in X^*, y \in X. \tag{2.1.22}$$

V_p is nonnegative and $V_p(x, y) = \Delta_p(J_q^*(x), y)$ for all $x \in X^*$ and $y \in X$. Also, by the subdifferential inequality, we have

$$V_p(x^*, x) + \langle y^*, J_q^*(x^*) - x \rangle \leq V_p(x^* + y^*, x),\ \forall x \in X,\ x^*, y^* \in X^* \text{ (see, [179])}. \tag{2.1.23}$$

Furthermore, V_p is convex in the second variable. Thus for all $z \in X$, we have

$$\Delta_p\left(J_q^*\left(\sum_{i=1}^{N} t_i J_p(x_i)\right), z\right) \leq \sum_{i=1}^{N} t_i \Delta_p(x_i, z), \tag{2.1.24}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$ (see [1, 32, 158, 161, 179] for more details).

2.2 Equilibrium, split feasibility and variational inequality problems

2.2.1 Equilibrium problem

In this section, we give a brief survey of some classes of equilibrium problems. Throughout this section C is a nonempty closed and convex subset of a Hilbert space H.

In 1994, Blum and Oettli [14] introduced the following Equilibrium Problem (in short EP). Given a bifunction $F : C \times C \to \mathbb{R}$, the EP is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{2.2.1}$$

In solving the EP, it is assumed that the bifunction F satisfied the following:

A1. $F(x, x) = 0 \quad \forall\ x \in C$,

A2. F is monotone, i.e $F(x, y) + F(y, x) \leq 0 \quad \forall\ x, y \in C$,

A3. $\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$ for each $x, y, z \in C$,

A4. for each $x \in C$, the function $y \longmapsto F(x, y)$ is convex, lower semicontinuous.

The set of solution to (2.2.1) is denoted by EP(F).

Generalized Equilibrium Problem (GEP)
In 1999, Moudafi and Théra [126] introduced the GEP which is to find $x \in C$ such that :

$$F(x, y) + \langle Ax, y - x\rangle \geq 0, \quad y \in C, \tag{2.2.2}$$

where $A : C \to H$ is a nonlinear mapping. The set of solutions to (2.2.2) is denoted by GEP(F).

The EP and GEP have potential and useful applications in nonlinear analysis and mathematical economics as seen below (see Blum and Oettli [14]):

(1) *Optimization*: Let $\phi : C \to \mathbb{R}$ be a convex and lower semi-continuous function, the minimization problem is to find $\bar{x} \in C$ such that

$$\phi(\bar{x}) \leq \phi(y) \qquad \forall y \in C. \tag{2.2.3}$$

Setting $F(x, y) := \phi(y) - \phi(x)$, Problem (2.2.3) coincides with (2.2.1). The function F is monotone in this case.

(2) *Saddle point problem*: Let $\varphi : C_1 \times C_2 \to \mathbb{R}$. Then $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is called a saddle point of the function φ if and only if for $(\bar{x}_1, \bar{x}_2) \in C_1 \times C_2$,

$$\varphi(\bar{x}_1, y_2) \leq \varphi(y_1, \bar{x}_2) \quad \forall (y_1, y_2) \in C_1 \times C_2. \tag{2.2.4}$$

Setting $C = C_1 \times C_2$ and define $F : C \times C \to \mathbb{R}$ by

$$F\Big((x_1, x_2), (y_1, y_2)\Big) := \varphi(y_1, x_2) - \varphi(x_1, y_2).$$

Then $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is a solution of (2.2.1) if and only if $(\bar{x}_1, \bar{x}_2)$ satisfies (2.2.4). F is monotone in this case.

(3) *Nash equilibria in non-cooperative game:* Let I be a finite index set (the set of players). For every $i \in I$, let there be given a set C_i (the strategy set of the $i-th$ player). Let $C := \prod_{i \in I} C_i$. For every $i \in I$, let there be given a function $f_i : C \to \mathbb{R}$ (the loss function of the i^{th} player depending on the strategies of all players). For $x = (x_i)_{i \in I} \in C$, we define $x^i = (x_j)_{j \in I, j \neq i}$. The point $\bar{x} = (\bar{x}_i)_{i \in I} \in C$ is called Nash equilibrium if and only if for all $i \in I$, there holds

$$f_i(\bar{x}_i) \leq f_i(\bar{x}^i, y_i) \quad \forall y_i \in C_i, \tag{2.2.5}$$

(that is, no player can reduce his loss by varying his strategy alone).
Define $F : C \times C \to \mathbb{R}$ by

$$F(x, y) := \sum_{i \in I} \Big(f_i(x^i, y_i) - f_i(x)\Big).$$

Then $x \in C$ is a Nash equilibrium if and only if x fulfills (2.2.1). Indeed: If (2.2.5) holds for all $i \in I$, then it is obvious that (2.2.1) is fulfilled. If for some $i \in I$, we choose $y \in C$ in such a way that $\bar{x}^i = y^i$. Then

$$F(\bar{x}, y) = f_i(\bar{x}^i, y_i) - f_i(\bar{x}).$$

Hence, (2.2.1) implies (2.2.5) for all $i \in I$. F in this case is not automatically monotone.

(4) *Fixed Point Problem (FPP):* Let $T : C \to C$ be a given mapping. The fixed point problem is to find $x \in C$ such that

$$x = Tx. \tag{2.2.6}$$

Setting $F(x, y) = \langle x - Tx, y - x\rangle$. Then x solves (2.2.6) if and only if x is a solution of (2.2.1). Indeed: (2.2.6) $\Rightarrow$ (2.2.1) is obvious. And if (2.2.1) is satisfied, then choose $\bar{y} = T\bar{x}$ to obtain

$$0 \leq F(\bar{x}, \bar{y}) = -||\bar{x} - T\bar{x}||^2, \tag{2.2.7}$$

hence, $\bar{x} = T\bar{x}$. So (2.2.1) $\Rightarrow$ (2.2.6). In this case F is monotone if and only if

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2 \qquad \forall x, y \in C.$$

Hence in particular T is nonexpansive.

(5) *Convex differentiable optimization:* Besides the straightforward connection between optima and equilibria given in (1), there is a more subtle connection in the convex differentiable case. Let $\Phi : C \to \mathbb{R}$ be convex and Gáteaux differentiable, with Gáteaux differential $D_\Phi(x) \in H^*$ at x. Consider the problem

$$\min\{\Phi(x) : x \in C\}. \tag{2.2.8}$$

It is well known from convex analysis that $\bar{x}$ is a solution of (2.2.8) if and only if $\bar{x}$ satisfy the variational inequality

$$\bar{x} \in C, \quad \langle D_\Phi(\bar{x}), y\bar{x}\rangle \geq 0 \qquad \forall y \in C.$$

Upon setting $F(x, y) := \langle D_\Phi(x), y - x\rangle$, this becomes an example of our equilibrium problem (2.2.1). The function F is monotone in this case, since the mapping $x \longmapsto D_\Phi(x)$ is monotone, i.e

$$\langle D_\Phi(y) - D_\Phi(x), y - x\rangle \geq 0 \qquad \forall x, y \in C.$$

(6) *Variational operator inequalities:* Let $\psi : C \to H^*$ be a given mapping. The variational inequality problem is to find $\bar{x} \in H$ such that

$$\bar{x} \in C, \quad \langle \psi\bar{x}, y - \bar{x}\rangle \geq 0 \quad \forall y \in C. \tag{2.2.9}$$

We set $F(x, y) := \langle \psi x, y - x\rangle$. Then clearly (2.2.9) $\Longleftrightarrow$ (2.2.1).

(7) *Complementarity problem:* This is a special case of the previous problem (6). Let C be a closed convex cone with $C^* := \{x^* \in H^* : \langle x^*, y\rangle \geq 0 \quad \forall y \in C\}$ denoting its polar cone. Let $A : C \to H^*$ be a given mapping. The complementarity problem is to find $\bar{x} \in H$ such that

$$\bar{x} \in C, \quad A\bar{x} \in C^*, \quad \langle A\bar{x}, \bar{x}\rangle \geq 0. \tag{2.2.10}$$

It is easily seen that (2.2.10) is equivalent with (2.2.9). Obviously, (2.2.10) $\Rightarrow$ (2.2.9). If (2.2.9) holds, then setting in turn $y := 2\bar{x}$ and $y := 0$, we obtain from (2.2.9) that $\langle A\bar{x}, \bar{x}\rangle = 0$ and thereby, $\langle A\bar{x}, y\rangle \geq 0, \qquad \forall y \in C$. Hence (2.2.9)$\Longrightarrow$ (2.2.10).

Mixed Equilibrium Problem(MEP)
In 2008, Ceng and Yao [48] studied the MEP which is to find $x \in C$, such that

$$F(x, y) + \phi(y) - \phi(x) \geq 0 \quad \forall y \in C, \tag{2.2.11}$$

where $\phi : C \to \mathbb{R} \cup \{+\infty\}$ is a nonlinear functional. The set of solutions of the MEP is denoted by MEP(F, ϕ).

Generalized Mixed Equilibrium Problem(GEMEP)
Also, Peng and Yao [143] studied the GEMEP which is to find $x \in C$, such that

$$F(x, y) + \langle Ax, y - x\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in C, \tag{2.2.12}$$

where $A : C \to H$ is a nonlinear mapping and $\phi : C \to \mathbb{R} \cup \{+\infty\}$ is a nonlinear functional. The set of solutions of GEMEP is denoted by GEMEP(F, A, ϕ).

2.2.2 Split feasibility problem

The SFP introduced in 1994 by Censor and Elfving [50] is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{2.2.13}$$

where C and Q are nonempty closed convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and A is an $m \times n$ real matrix. The SFP has wide applications in many fields, such as phase retrieval, medical image reconstruction, signal processing, and radiation therapy treatment planning (for example see [42, 49, 50, 51, 71, 194] and the references therein).
The SFP has also been studied by numerous authors in both finite and infinite dimensional Hilbert spaces (for examples see [43, 51, 55, 88, 120, 128, 127, 137, 138, 139, 135, 179, 160, 173, 186, 189, 193]). It has been shown (see [188]) that if the SFP (2.2.13) has a solution, then $x^* \in C$ solves SFP (2.2.13) if and only if it solves the fixed point equation

$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*, \tag{2.2.14}$$

where P_C and P_Q are the metric projections onto C and Q respectively, γ is any positive real number, A is a bounded linear operator and A^* is the adjoint of A.
Byrne [43] applied the forward-backward method, a type of projected gradient method, to introduce the so-called CQ-iterative procedure for approximating a solution of (2.2.13), which is defined by

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \; n \in \mathbb{N}, \tag{2.2.15}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.
In 2009, Censor and Segal [55] introduced an important form of the SFP called Split Common Fixed Point Problem (SCFPP), which is to find a point

$$x^* \in F(T) \text{ such that } Ax^* \in F(S), \tag{2.2.16}$$

where T and S are some nonlinear operators on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, A is a real $m \times n$ matrix. Based on the properties of the operators T and S, called directed operators, they presented the following algorithm for solving the SCFPP:

$$x_{n+1} = T(x_n + \gamma A^T(S - I)Ax_n), \quad \forall n \geq 1, \; x_1 \in \mathbb{R}^n, \tag{2.2.17}$$

where $\gamma \in (0, \frac{2}{||A||^2})$. They also obtained a convergence result for this algorithm.
Motivated by the work of Censor and Segal [55], Moudafi [128] presented the following iterative scheme which does not involve the metric projections P_C and P_Q:

$$x_{n+1} = (1 - \alpha_n)\left(x_n + \gamma A^*(S - I)Ax_n\right) + \alpha_n T\left(x_n + \gamma A^*(S - I)Ax_n\right), \; n \in \mathbb{N}, \tag{2.2.18}$$

for approximating a solution of the SCFPP (2.2.16) and obtained a weak convergence results when T and S are demi-contractive.
The SFP has been extended from the setting of Hilbert spaces to more general Banach spaces by many authors. Schopfer *et al.* [154] introduced and studied the following

algorithm (which is a generalization of algorithm (2.2.15)) for solving the SFP (2.2.13) in p-uniformly smooth Banach spaces: For any $x_0 \in X_1$ and $n \geq 0$,

$$x_{n+1} = \prod\nolimits_C J^* \left[J_p^{X_1}(x_n) - tA^* J_p^{X_2}(Ax_n - P_Q(Ax_n)) \right], \tag{2.2.19}$$

where A^* is the adjoint of a bounded linear operator A, t is any positive real number, P_Q is the metric projection onto Q and C, Q are nonempty, closed and convex subsets of X_1, X_2 respectively. They obtained weak convergence result under the assumption that the duality mapping of X is sequentially weak-to-weak continuous.

Wang [182] modified Algorithm (2.2.19) and obtained strong convergence result for the following Multiple-Sets Split Feasibility Problem (MSSFP): Find

$$x \in \cap_{i=1}^{r} C_i \ \text{ such that } \ Ax \in \cap_{j=1+r}^{r+s} Q_j, \tag{2.2.20}$$

where r, s are two given integers, C_i, $i = 1, 2, 3, \ldots, r$ are closed convex subsets of X_1 and Q_j, $j = r+1, \ldots, r+s$ are closed convex subsets of X_2. He introduced the following algorithm: For any $x_0 \in X_1$, define $\{x_n\}$ by

$$\begin{cases} y_n = T_n x_n; \\ D_n = \{u \in X_1 : \Delta_p(y_n, u) \leq \Delta_p(x_n, u)\}; \\ E_n = \{u \in X_1 : \langle x_n - u, J_p^{X_1} x_0 - J_p^{X_1} x_n \rangle \geq 0\}; \\ x_{n+1} = \prod_{D_n \cap X_n}(x_0); \end{cases} \tag{2.2.21}$$

where T_n is defined for each $n \in \mathbb{N}$ by

$$T_n(x) = \begin{cases} \prod_{C_{\mathrm{i}(n)}}(x); & 1 \leq \mathrm{i}(n) \leq r; \\ J_q^{X_1^*} \left[J_p^{X_1} x - t_n A^* J_p^{X_2}(I - P_{Q_{\mathrm{i}(n)}})Ax \right]; & r+1 \leq \mathrm{i}(n) \leq r+s; \end{cases} \tag{2.2.22}$$

where $\mathrm{i} : \mathbb{N} \to I$ is the cyclic control mapping

$$\mathrm{i}(n) = n \bmod (r+s) + 1$$

and t_n satisfies

$$0 < t \leq t_n \leq \left(\frac{q}{C_q ||A||^p} \right)^{\frac{1}{q-1}}. \tag{2.2.23}$$

Very recently, Shehu *et al.* [179] introduced and studied the following iterative algorithm for approximating a common solution of SFP and fixed point problems for right Bregman strongly nonexpansive mapping in $p-$uniformly convex Banach spaces which are also uniformly smooth: For a fixed $u \in C$, $u_0 \in X_1$, define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ recursively by

$$\begin{cases} x_n = \prod_C J_q^{X_1^*} \left[J_p^{X_1}(u_n) - t_n A^* J_p^{X_2}(I - P_Q)Au_n \right]; \\ u_{n+1} = \prod_C J_q^{X_1^*} \left[\alpha_n J_p^{X_1}(u) + \beta_n J_p^{X_1}(x_n) + \gamma_n J_p^{X_1}(Tx_n) \right]; \end{cases} \tag{2.2.24}$$

where $0 < t \leq t_n \leq k < \left(\frac{q}{C_q ||A||^p} \right)^{\frac{1}{q-1}}$. They established strong convergence of algorithm (2.2.24) under some suitable conditions.

2.2.3 Variational inequality problem

The theory of VIP is known to be very useful in solving diverse mathematical problems which include optimization problems, equilibrium problems, boundary valued problems, among others. It is known that many mathematical problems can be posed as a VIP. In particular, VIPs are known to be natural generalization of the theory of boundary value problems and are considered in optimization theory as natural extension of minimization problems (see [63]). We now give some examples of a VIP in the following subsection. To do this, we first recall the definition of a VIP in a real Hilbert space H. Let $A : H \to H$ be an operator and C be a nonempty, closed and convex subset of H. The VIP is the problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \ \forall x \in C. \tag{2.2.25}$$

We denote by Γ the solution set of the VIP (2.2.25). It is generally known that Γ is a closed and convex subset of C (for example, see [165]).

A set valued mapping $\mathcal{T} : H \to 2^H$ is called monotone if for all $x, y \in H$, $u \in \mathcal{T}x$ and $v \in \mathcal{T}y$ imply $\langle x - y, u - v \rangle \geq 0$. A monotone mapping $\mathcal{T} : H_1 \to 2^{H_1}$ is maximal if the graph $G(\mathcal{T})$ of $\mathcal{T}$ is not properly contained in the graph of any other monotone mapping.

It is well known that a monotone mapping $\mathcal{T}$ is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$, for every $(y, v) \in G(\mathcal{T})$ implies $u \in \mathcal{T}x$. Let $B : C \to H$ be an inverse strongly monotone mapping and let $N_C x$ be the normal cone to C at $x \in C$, i.e., $N_C x := \{z \in H : \langle y - x, z \rangle \geq 0, \ \forall \ y \in C\}$. Define

$$\mathcal{T}x = \begin{cases} Ax + N_C x, & \forall x \in C, \\ \emptyset, & \forall x \notin C. \end{cases}$$

Then $\mathcal{T}$ is maximal monotone and $0 \in \mathcal{T}x$ if and only if $x \in \Gamma$, see [134].

Consider the following problem of finding the minimal value of a differentiable function f over a closed interval $I = [a, b]$. For $x^* \in I$, we have the following three possible cases,

1. if $a < x^* < b$ then $f'(x^*) = 0$,
2. if $x^* = a$ then $f'(x^*) \geq 0$,
3. if $x^* = b$ then $f'(x^*) \leq 0$.

The above cases can be summarized as a VIP of finding $x^* \in I$ such that $f'(x^*)(x - x^*) \geq 0 \ \forall x \in I$ (see [7] for more details).

VIPs have been extensively studied in both finite and infinite dimensional spaces by numerous authors. The break through in the study of VIPs in finite dimensional spaces happened in 1980 when Dafermos [69] identified that a certain traffic network equilibrium conditions had a structure of variational inequalities under the monotonicity assumption.

Dafermos [69] used the techniques of the theory of variational inequalities to establish existence of a traffic equilibrium pattern for which he developed an algorithm for the construction of the pattern and derived estimates on the speed of convergence of the algorithm. He also used his algorithm to estimate the user-optimized equilibrium pattern for a simple network with two-way streets. His work attracted the interest of numerous researchers, as a result of this, a lot of research efforts were devoted to the study of VIPs in finite dimensional spaces (for example, see [70], [75], [101], [133]).

The study of VIPs was further extended to infinite dimensional spaces. There are several monographs on VIPs in infinite dimensional spaces, however, we shall mention here a few. Stampacchia [164] established the existence and uniqueness of the solution of problem (2.2.25) under the assumption that A is a coercive and linear operator from a Hilbert space H to its dual space H^*. Lions and Stampacchia [108] further considered the case where A is positive or semicoercive. Hatman and Stampacchia [86] worked on partial differential equations using the VIP (2.2.25) as a tool, with applications to problems arising from mechanics. They proved the existence and uniqueness theorem of the solution of problem (2.2.25) in a reflexive real Banach space when A is assumed to be a monotone hemicontinuous operator. In fact they proved the following theorem.

Theorem 2.2.1. *Let X be a Banach space and X^* be its dual. Let $A : X \to X^*$ be a monotone hemicontinuous operator and K be a bounded convex subset of X. Then there exists at least one solution of problem* (2.2.25).

As we mentioned earlier, there are different methods or ways of obtaining solutions of problem (2.2.25) in infinite dimensional spaces. The methods used by the authors mentioned above has to do with the existence and uniqueness of solutions of VIPs (see [165] for detailed information on different approaches for solving problem (2.2.25)). Unlike the existence and uniqueness problem which is only concerned with establishing conditions under which problem (2.2.25) has solution, the iterative methods of finding solutions of problem (2.2.25) are concerned with the actual computation or approximation of sequences to a solution of problem (2.2.25).

Let C and Q be nonempty, closed and convex subsets of the real Hilbert spaces H_1 and H_2 respectively. Let $f : H_1 \to H_1$, $g : H_2 \to H_2$ be inverse strongly monotone operators and $A : H_1 \to H_2$ be bounded linear operator. Consider the following problem which is called the Split Variational Inequality Problem (SVIP): Find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \ \forall x \in C \tag{2.2.26}$$

and such that $y^* = Ax^* \in Q$ solves

$$\langle g(y^*), y - y^* \rangle \geq 0 \ \forall y \in Q. \tag{2.2.27}$$

If (2.2.26) and (2.2.27) are considered separately, we have that (2.2.26) is a VIP with its solution set $VIP(C, f)$ and (2.2.27) is a VIP with its solution set $VIP(Q, g)$. The SVIP was introduced and studied by Censor *et al.* [52]. They studied this problem as a pair of VIPs in which they obtained a solution of one VIP in H_1 whose image under a given bounded linear operator A is a solution of the second VIP in the second space H_2. They considered two approaches for establishing the solution of the SVIP (2.2.26)-(2.2.27). In each of these approaches they proposed an iterative algorithm and using their algorithms they obtained strong convergence results of the SVIP (2.2.26)-(2.2.27).

In 2012, Censor *et al.* [53] introduced the general Common Solutions to Variational Inequality Problem (CSVIP), which consist of finding common solutions to unrelated variational inequalities for finite number of sets. That is, find $x^* \in \cap_{i=1}^N C_i$ such that for each $i = 1, 2, \ldots, N$,

$$\langle A_i(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in C_i, \; i = 1, 2, \ldots, N, \tag{2.2.28}$$

where $A_i : H \to H$ is a nonlinear operator for each $i = 1, 2, \ldots, N$ and C_i is a nonempty, closed and convex subset of H. They obtained the solution of problem (2.2.28) by considering first, a case where $i = 1, 2$ and later obtained the result of the problem for $i = 1, 2, \ldots, N$. They proposed the following algorithm and proved the corresponding theorem.

$$\begin{cases} x^0 \in H, \\ x^{k+1} = \prod_{i=1}^N (P_{C_i}(I - \lambda A_i))(x^k). \end{cases} \tag{2.2.29}$$

Theorem 2.2.2. *Let H be a real Hilbert space and C_i be nonempty, closed and convex subsets of H for each $i = 1, 2, \ldots, N$. Let $A_i : H \to H$ be α_i-inverse strongly monotone operators with $\lambda \in (0, 2\alpha)$ and $\alpha := \min_i\{\alpha_i\}$. Assume that $\cap_{i=1}^\infty C_i \neq \emptyset$ and $\Gamma := \cap_{i=1}^N SOL(C_i, A_i) \neq \emptyset$. Then any sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm (2.2.29) converges weakly to a point $x^* \in \Gamma$ and furthermore,*

$$x^* = \lim_{k \to \infty} P_\Gamma(x^k). \tag{2.2.30}$$

Censor *et al.* [54] also considered problem (2.2.28) in the case where the operator A is a multivalued mapping. More precisely, they studied the following problem which they called Common Solutions to Variational Inequalities Problem (CSVIP):
Let H be a real Hilbert space and C_i be a nonempty, closed and convex subsets of H with $\cap_{i=1}^N C_i \neq \emptyset$. Let $A_i : H \to 2^H$ be a multivalued mapping for each $i = 1, 2, \ldots, N$. The CSVIP is the problem of finding a point $x \in \cap_{i=1}^N C_i$ such that for each $i = 1, 2, \ldots, N$, there exists $u_i \in A_i(x)$ satisfying

$$\langle u_i, y - x \rangle \geq 0 \; \forall y \in C_i, \; i = 1, 2, \ldots, N. \tag{2.2.31}$$

Their motivation stems from the observation that if $A_i = 0$, then the CSVIP (2.2.31) reduces to the Convex Feasibility Problem (CFP) of finding a point $x \in \cap_{i=1}^N C_i$. Also observe that if C_i are the fixed point sets of a family of nonlinear operators defined on H, then the CFP becomes the Common Fixed Point Problem (CFPP) (for example, see [53] and [54]).

Recently, Tian *et al* [170] considered a class of generalized split feasibility problem for finding an element that solves a variational inequality problem such that its image under a given bounded linear operator is in the fixed point set of a nonexpansive mapping. They proved the following weak convergence theorem:

Theorem 2.2.3. *Let H_1 and H_2 be real Hilbert spaces. Let K be a nonempty closed convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$, $f : K \to H_1$ be a monotone and k-Lipschitz continuous mapping and $T : H_2 \to H_2$ be a nonexpansive mapping. Setting $\Gamma = \{z \in VI(K, f) : Az \in F(T)\}$, assume that $\Gamma \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ be generated by $x_1 = x \in K$ and*

$$\begin{cases} y_n = P_K(x_n - \gamma_n A^*(I - T)Ax_n), \\ t_n = P_K(y_n - \lambda_n f(y_n)), \\ x_{n+1} = P_K(y_n - \lambda_n f(t_n)), \end{cases} \tag{2.2.32}$$

for each $n \in \mathbb{N}$, where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\|A\|^2})$ and $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, \frac{1}{k})$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in \Gamma$, where $z = \lim_{n\to\infty} P_\Gamma x_n$.

Observe that the operator $P_C(I - \lambda A)$ appears in most of the algorithms stated above. Hence, we present the connection between the fixed points set of the operator $P_C(I - \lambda A)$ and the solution set of VIP:

Proposition 2.2.4. *[53]. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to C$ be an inverse strongly monotone operator. If $VIP(C, A)$ is the solution set of the VIP (2.2.25), then for any $\lambda > 0$, we have that $F(P_C(I - \lambda A)) = VIP(C, A)$.*

Proof. Let $\lambda > 0$ and $y \in C$, then from the characterization of the metric projection (see Proposition 2.1.14), we have that

$$\begin{aligned} x \in F(P_C(I - \lambda A)) &\iff x = P_C(I - \lambda A)x, \\ &\iff \langle x - (I - \lambda A)x, y - x\rangle \geq 0, \\ &\iff \langle \lambda A(x), y - x\rangle \geq 0, \\ &\iff \langle A(x), y - x\rangle \geq 0, \\ &\iff x \in \text{VIP}(C, A). \end{aligned}$$

Hence, $F(P_C(I - \lambda A)) =$VIP(C, A). ☐

2.3 Some important results

In this section, we give some useful and important results that will be needed in establishing our main results.

Definition 2.3.1. *A mapping $T : H \to H$ is said to be firmly nonexpansive if and only if $2T - I$ is nonexpansive or equivalently*

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2 \quad \forall x, y \in H.$$

Alternatively T is firmly nonexpansive if and only if T can be express

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive. For example, the projection mapping is firmly nonexpansive.

Definition 2.3.2. *A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity mapping I and a nonexpansive mapping; that is*

$$T = (1 - \alpha)I + \alpha S, \tag{2.3.1}$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive.

More precisely, when (2.3.1) holds, we say that T is α-averaged. Thus, firmly nonexpansive mapping (in particular, projections) are $\frac{1}{2}$-averaged mappings.

Theorem 2.3.3. *(Banach contraction mapping principle [13]). Let (X, d) be a complete metric space and let f be a contraction on X. Then f has a unique fixed point.*

Theorem 2.3.4. *(Meir and Keeler [121]). Let (X, d) be a complete metric space and let f be a Meir-Keeler contraction (MKC, for short) on X, that is, for every $\epsilon > 0$, the exists $\delta > 0$ such that $d(x, y) < \epsilon + \delta$ implies $d(f(x), f(y)) < \epsilon$ for all $x, y \in X$. Then f has a unique fixed point.*

Remark 2.3.5. *It is well known that Theorem 2.3.4 is a generalization of Theorem 2.3.3 since contractions are proper subclass of Meir-Keeler contractions.*

Lemma 2.3.6. *Let H be a real Hilbert space, then the following well known identities holds:*

(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$;

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(iii) $\|\lambda x + (1 - \lambda)y - z\|^2 = \lambda\|x - z\|^2 + (1 - \lambda)\|y - z\|^2 - \lambda(1 - \lambda)\|x - y\|^2$, *for any* $\lambda \in (0, 1)$, $x, y, z \in H$;

(iv) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2, \quad \forall\ x, y \in H$.

Proposition 2.3.7. *(Characterization of Bregman Projection, see [174]). Suppose that $f : X \to \mathbb{R}$ is totally convex and Gateaux differentiable in int(domf). Let $x \in$ int(domf) and $C \subset$ int(domf) be a nonempty, closed and convex set. If $\hat{x} \in C$, then the following conditions are equivalent:*

(i) The vector $\hat{x}$ is the Bregman projection of x onto C with respect to f.

(ii) The vector $\hat{x}$ is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \geq 0,\ \forall\ y \in C.$$

(iii) *The vector $\hat{x}$ is a unique solution of the inequality $D_f(y,z)+D_f(z,x) \leq D_f(y,x)$ $\forall y,z \in C$.*

Lemma 2.3.8. *(see [175], Lemma 3.1): Let $f : X \to \mathbb{R}$ be Gâteaux differentiable and totally convex function. If $x_0 \in X$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

Lemma 2.3.9. *(see [175], Lemma 3.2): Let $f : X \to \mathbb{R}$ be Gâteaux differentiable and totally convex function at $x_0 \in X$ and let C be a nonempty, closed and convex subset of X. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belong to C. If $D_f(x_n, x_0) \leq D_f(Proj_C^f(x_0), x_0)$ for any n, then $\{x_n\}$ converges strongly to $Proj_C^f(x_0)$.*

Lemma 2.3.10. *[36]. The function $f : X \to \mathbb{R}$ is totally convex on bounded set if and only if it is sequentially consistent.*

Lemma 2.3.11. *[174]. If $f : X \to \mathbb{R}$ is uniformly Fretchet differentiable and bounded on bounded subsets of X, then ∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

Definition 2.3.12. *(Demiclosedness property) Let $T : H \to H$ be a nonlinear mapping. Then T is said to be demiclosed at $y \in H$, if $\{x_n\} \rightharpoonup x \in H$ and $Tx_n \to y$, then $y = Tx$.*

Lemma 2.3.13. *(Demiclosedness principle) Let K be a nonempty, closed and convex subset of a real Hilbert space H and let $T : K \to K$ be μ-strictly pseudocontractive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \rightharpoonup x \in K$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.3.14. *[187] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=0}^{\infty}\gamma_n = \infty$,
(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=0}^{\infty}|\delta_n\gamma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3.15. *[115] Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.3.16. *[65, 185] Let C be a nonempty subset of H. The following statements hold:*

i If $T : C \to H$ is α-averaged, then for any $z \in F(T)$ and for all $x \in C$,

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \frac{1-\alpha}{\alpha}\|Tx - x\|^2.$$

ii If $T_1 : H \to H$ and $T_2 : H \to H$ are α_1 and α_2-averaged, respectively. Then T_1T_2 is $(\alpha_1 + \alpha_2 - \alpha_1\alpha_2)$-averaged.

Lemma 2.3.17. *[114] Let $\{a_n\}$ be a sequence of non-negative numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n r_n,$$

where $\{r_n\}$ is a sequence of real numbers bounded from above and $\{\alpha_n\} \subset [0, 1]$ satisfies $\sum \alpha_n = \infty$. Then

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} r_n.$$

Lemma 2.3.18. *[116] Let H be a Hilbert space, $f : H \to H$ a contraction with coefficient $0 < \alpha < 1$, and A a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \bar{\gamma}/\alpha$,*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad x, y \in H.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\alpha$.

Lemma 2.3.19. *[192] Let H be a real Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0, 1]$ for $i = 1, 2, ..., n$ such that $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$, the following equality holds:*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n\|^2 = \sum_{i=1}^{n} \alpha_i \|x_i\|^2 - \sum_{1\leq i,j\leq n} \alpha_i\alpha_j \|x_i - x_j\|^2.$$

Lemma 2.3.20. *(Lopez et.al, [102]) Let H be a real Hilbert space. Let $M : H \to 2^H$ be a maximal monotone operator and $B : H \to H$ be an α-inverse strongly monotone mapping. Then we have*
(i) for $r > 0$, $F(T_r) = (M + B)^{-1}(0) := \{x \in H : 0 \in (M + B)x\}$;
(ii) for $0 < s \leq r$ and $x \in E$, $||x - T_s x|| \leq 2||x - T_r x||$, where $T_r := (I + rM)^{-1}(I - rB) = J_r^M(I - rB)$.

Lemma 2.3.21. *[195] Let C be a nonempty closed convex subset of a smooth Banach space X. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4), $A : C \to X^*$ be a continuous and monotone mapping, $\phi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $r > 0$ be any given number and $x \in X$, there exists $z \in C$ such that*

$$F(z, y) + \langle Az, y - z\rangle + \phi(y) - \phi(z) + \frac{1}{r}\langle J(z) - J(x), y - z\rangle \geq 0, \ \forall\ y \in C. \tag{2.3.2}$$

Define the resolvent mapping $T_r : E \to 2^C$ as follows:+

$$T_r^F(x) = \{z \in C : F(z, y) + \langle Az, y - z\rangle + \phi(y) - \phi(z) + \frac{1}{r}\langle J(z) - J(x), y - z\rangle \geq 0, \ \forall\ y \in C\}, \tag{2.3.3}$$

then, T_r^F has the following properties:
(1) T_r^F is single-valued,
(2) T_r^F is a firmly nonexpansive mapping, that is,

$$\langle T_r^F z - T_r^F y, JT_r^F z - JT_r^F y\rangle \leq \langle T_r^F z - T_r^F y, Jz - Jy\rangle \ \forall z, y \in X,$$

(3)$F(T_r^F) = GMEP(F, A, \phi)$,
(4) $GMEP(F, A, \phi)$ is closed and convex.

It is easy to see that the resolvent operator satisfies the following inequality: for all $r > 0$, $u \in GMEP(F, A, \phi)$ and $x \in X$, then

$$\Delta_p(x, T_r^F x) + \Delta_p(T_r^F x, u) \leq \Delta_p(x, u). \tag{2.3.4}$$

Lemma 2.3.22. *[110] Assume that $F : C \times C \to \mathbb{R}$ is a bifunction satisfying (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $Q_r : H \to C$ as follows:*

$$Q_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall\, y \in C \right\}.$$

Then the following hold:

- *(1) Q_r is single-valued;*
- *(2) Q_r is firmly nonexpansive, i.e. $\|Q_r - Q_r y\|^2 \leq \langle Q_r x - Q_r y, x - y\rangle$ for any $x, y \in H$;*
- *(3) $F(Q_r) = EP(F)$;*
- *(4) $EP(F)$ is closed and convex.*

Lemma 2.3.23. *[185]. Assume that $\{a_n\}_{n=0}^{\infty}$ is a sequence of non negative real numbers such that*

$$a_{n+1} \leq (1 - \delta_n)a_n + \delta_n \sigma_n + b_n, \quad n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are sequences in $(0, 1)$ and $\{\delta_n\}_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ such that

- *(i) $\sum_{n=0}^{\infty} \delta_n = \infty$;*
- *(ii) either $\limsup_{n\to\infty} \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} \delta_n|\sigma_n| < 0$;*
- *(iii) $\sum_{n=0}^{\infty} b_n < \infty$.*

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3.24. *(see [45]) Assume A is a strongly positive bounded operator with coefficient $\bar{\gamma} > 0$ on a real smooth Banach space X and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq I - \rho\bar{\gamma}$.*

Lemma 2.3.25. *(see [166] Lemma 2.3) Let f be an MKC on a convex subset of a Banach space X. Then for each $\epsilon > 0$, there exists $r_\epsilon \in (0, 1)$ such that*

$$\|x - y\| \geq \epsilon \implies \|f(x) - f(y)\| \leq r_\epsilon \|x - y\| \quad \forall x, y \in C. \tag{2.3.5}$$

Lemma 2.3.26. *(see [11]) Let X be a Banach space and let A be an m-accretive operator. For $\lambda > 0$, $\mu > 0$ and $x \in X$, we have*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x \right),$$

where $J_\lambda^A = (I + \lambda A)^{-1}$ and $J_\mu^A = (I + \mu A)^{-1}$.

Lemma 2.3.27. *(see [167]) Let* $\{x_n\}$, $\{z_n\}$ *be bounded sequences in* X *and* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *which satisfied the following condition:* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$*. Suppose that* $x_{n+1} = (1-\beta_n)x_n + \beta_n z_n$ *for all* $n \ge 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$*. Then,* $\lim_{n\to\infty} \|z_n - x_n\| = 0$*.*

Lemma 2.3.28. *(see [61] Let* X *be a real Banach space with Fréchet differentiable norm. For* $x \in X$*, let* $\beta^*(t)$ *be defined for* $0 < t < \infty$ *by*

$$\beta^*(t) = \sup\left\{\left|\frac{\|x+ty\|^2 - \|x\|^2}{t} - 2\langle y, j(x)\rangle\right| : \|y\| = 1\right\}. \tag{2.3.6}$$

Then, $\lim_{t\to 0^+} \beta^*(t) = 0$ *and*

$$\|x+h\|^2 \le \|x\|^2 + 2\langle h, j(x)\rangle + \|h\|\beta^*(\|h\|)$$

for all $h \in X \setminus \{0\}$*.*

In the result of Cholamjik and Suantai [61], the authors assumed that $\beta^*(t) \le 2t$ for $t > 0$. In our more general setting, throughout this work , we will assume that

$$\beta^*(t) \le ct, \quad t > 0 \quad \text{and} \quad \text{for} \quad \text{some} \quad c > 1,$$

where β^* is the function appearing in (2.3.6).

Chapter 3

Equilibrium and Fixed Point Problems

3.1 Introduction

One parameter family of mappings $\mathcal{T} := \{G(t) : 0 \leq t < \infty\}$ is called a continuous Lipschitzian semigroup on C if the following conditions are satisfied:

(1) $G(0)x = x$ for all $x \in C$;

(2) $G(s+t) = G(s)G(t)$ for all $s, t \geq 0$;

(3) for each $t > 0$, there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that $\|G(t)x - G(t)y\| \leq L_t\|x - y\|,\ x, y \in C$;

(4) for each $x \in C$, the mapping $G(.)x$ from $[0, \infty)$ into C is continuous.

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive if $L_t = 1$ for all $t > 0$ and asymptotically nonexpansive if $\limsup\limits_{t\to\infty} L_t \leq 1$. Let $F(\mathcal{T})$ denote the common fixed point set of the semigroup $\mathcal{T}$ i.e., $F(\mathcal{T}) := \{x \in C : G(t)x = x, \forall t > 0\}$.

Consider the following constrained minimization problem:

$$\min_{x \in C} g(x) \tag{3.1.1}$$

where $g : C \to \mathbb{R}$ is real-valued convex and continuously Fréchet differentiable functional. Assume that the constrained convex minimization problem (3.1.1) has a solution, we denote the set of solutions of (3.1.1) by Γ. The Gradient-Projection Algorithm (GPA) generates a sequence $\{x_n\}$ according to the recursive formula

$$x_{n+1} = P_C(I - \gamma_n \nabla g)x_n, \quad \forall n \geq 0, \tag{3.1.2}$$

where the parameters γ_n are real positive numbers, and P_C is the metric projection from H onto C. It is well known that the convergence of the algorithms (3.1.2) is determined by the gradient ∇g and the metric projection onto C. If the gradient ∇g is only assumed to be inverse strongly monotone, then the sequence $\{x_n\}$ defined by the algorithm (3.1.2) can only converge weakly to a minimizer of (3.1.1). If the gradient ∇g is Lipschitz continuous and strongly monotone, then the sequence generated by (3.1.2) can converge strongly to a unique minimizer of (3.1.1) provided the parameters γ_n satisfy appropriate conditions.

In 2011, Xu [185] proposed average mappings to GPA, and he constructed a counter example which shows that the GPA does not have strong convergence in an infinite-dimensional space. Moreover, he provided two convergent modifications of GPA which are shown to converge in norm.

Also, in 2011 motivated by Xu, Cent *et al.* [47] presented the following iterative algorithm:

$$x_{n+1} = P_C\left[\theta_n \gamma f(x_n) + (I - \theta_n \mu F)T_n(x_n)\right], \quad n \geq 0, \tag{3.1.3}$$

where $f : C \to H$ is and l-Lipschitzian mapping with constant $l > 0$, and $F : C \to H$ is a k-Lipschitzian and η-strongly monotone operator with constants $k, \eta > 0$. Let $0 < \mu < 2\eta/k^2$, $0 \leq \gamma l < \tau$ and $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Let T_n and θ_n satisfy $\theta_n = \frac{2-\lambda_n L}{4}$, $P_C(I - \lambda_n \nabla g) = \theta_n I + (1 - \theta_n T_n)$. Under suitable conditions, it is proved that the sequence $\{x_n\}$ generated by (3.1.3) converges strongly to a minimizer x^* of (3.1.2).

In 2012, Tian and Liu [113] introduced the following iterative method in a Hilbert space: $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C, \\ x_{n+1} = \alpha_n \gamma f(u_n) + (1 - \alpha_n A)T_n(u_n), \quad \forall\, n \in \mathbb{N}, \end{cases} \tag{3.1.4}$$

where $F : C \times C \to \mathbb{R}$, $u_n = Q_{r_n}(x_n)$, $P_C(I - \lambda_n \nabla g) = \theta_n I + (I - \theta_n)T_n \theta_n = \frac{2-\lambda_n L}{4}$, and $\{\lambda_n\} \subset (0, 2/L$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\theta_n\}$ satisfy appropriate conditions. Furthermore, they proved that the sequence $\{x_n\}$ converges strongly to a point $q \in \Gamma \cap EP(F)$, which solves the variational inequality

$$\langle (A - \gamma f)q, q - z\rangle \leq 0, \quad z \in \Gamma \cap EP(F).$$

However, it is known that the minimization problem (3.1.1) has more than one solution, so regularization is needed to find a unique solution. Now, consider the following regularized minimization problem:

$$\min_{x \in C} g_\alpha(x) := \min_{x \in C}\left\{ g(x) + \frac{\alpha}{2}\|x\|^2 \right\},$$

where $\alpha > 0$ is the regularization parameter, g is a convex function with a $1/L$-ism continuous gradient ∇g. Then the Regularized Gradient Projection Algorithm (RGPA) generates a sequence $\{x_n\}$ by the following recursive formula:

$$x_{n+1} = P_C(I - \gamma \nabla g_{\alpha_n})x_n = P_C\left[x_n - \gamma(\nabla g + \alpha_n I)(x_n)\right], \tag{3.1.5}$$

where the parameter $\alpha_n > 0$, γ is a constant with $0 < \gamma < 2/L$, and P_C is the metric projection from H onto C. It is well known that the sequence $\{x_n\}$ generated by algorithm (3.1.5) converges weakly to a minimizer of (3.1.1) in the setting of infinite-dimensional space (see [188]).

In 2010, Tian [169] combined the iterative methods of [116, 190] to propose a general iterative method for approximating a fixed point of a nonexpansive mapping T defined on a real Hilbert space. Let f be a l-contraction on C with $0 < l < 1$, and let S be a η-strongly monotone and k-Lipschitzian. For a constant μ satisfying $0 < \mu < 2\eta/k^2$, a constant t satisfying $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$, then for $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n t f(x_n) + (1 - \alpha_n \mu S)Tx_n, \quad n \geq 0, \tag{3.1.6}$$

where $F(T)$ denote the fixed points of mapping T, i.e, $F(T) = \{x \in H : x = Tx\}$.

Recently, motivated by the work of Tian [169], Tian and Liu [113], Ming Tian and Si-Wen Jiao [168] introduced a new iterative algorithm: $x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\ y \in C, \\ x_{n+1} = \alpha_n t f(x_n) + (I - \alpha_n \mu S)T_{\lambda_n}(u_n), \quad \forall\ n \in \mathbb{N}, \end{cases} \tag{3.1.7}$$

for finding a element of $\Gamma \cap EP(F)$, where $F : C \times C \to \mathbb{R}$, $u_n = Q_{r_n}(x_n)$, $P_C(I - \gamma\nabla g_{\lambda_n}) = T_{\lambda_n}$, $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$, $\gamma \in (0, 2/L)$. Under appropriate conditions, they proved that the sequence $\{x_n\}$ generated by (3.1.7) converges strongly to a point $q \in \Gamma \cap EP(F)$, which is also a solution to the variational inequality

$$\langle(\mu S - tf)q, q - z\rangle \leq 0, \quad \forall\ z \in \Gamma \cap EP(F).$$

In [177], Y. Shehu introduced an iterative scheme for finding a common element of the set of common fixed points of a nonexpansive semigroup, the set of solution to a generalized equilibrium problem and the set of solutions to a variational inclusion problem in a real Hilbert space. In particular, they proved the following theorem:

Theorem 3.1.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), ψ a μ- inverse-strongly monotone mapping of C into H, A an α-inverse strongly monotone mapping of C into H and $M : H \to 2^H$ a maximal monotone mapping. Let $\mathcal{T} : \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\mathcal{T}) \cap I(A, M) \cap GEP(F, \psi) \neq \emptyset$ and suppose $f : H \to H$ is a contraction mapping with a constant $\gamma \in (0, 1)$. Let $\{t_n\} \subset (0, \infty)$ be a real sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}$ and $\{u_n\}$ are generated by $x_1 \in H$,*

$$\begin{cases} F(u_n, y) + \langle\psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\ y \in C \quad \textit{and} \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)\left(\frac{1}{t_n}\int_0^{t_n} G(u)[\alpha_n f(x_n) + (1 - \alpha_n)J_{M,\lambda}(u_n - \lambda A u_n)]du\right), \end{cases} \tag{3.1.8}$$

for all $n \neq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ and $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ satisfying:

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\lambda \in (0, 2\alpha]$;

(iv) $0 < a \leq r_n \leq b < 2\mu$, $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(v) $\lim_{n\to\infty} \frac{t_n - t_{n-1}}{t_n} \frac{1}{\alpha_n(1-\beta_n)} = 0$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* z, *where* $z := P_\Omega f(z)$.

Lemma 3.1.2. *[162]. Let* D *be a nonempty, bounded, closed and convex subset of a real Hilbert space* H *and let* $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ *a nonexpansive semigroup on* D, *then for any* $h \geq 0$,

$$\lim_{t\to\infty} \sup_{x\in D} \left\| G(h)\left(\frac{1}{t}\int_0^t G(u)x du\right) - \left(\frac{1}{t}\int_0^t G(u)x du\right) \right\| = 0.$$

Lemma 3.1.3. *[168] Let* H *be a real Hilbert space and* C *be a nonempty, closed and convex subset of* H. *Let* $f : H \to H$ *be a contraction with constant* $l \in (0, 1)$, *and* $S : C \to H$ *be a k-Lipschitzian and* η*-strongly monotone operator with* $k > 0$, $\eta > 0$. *Suppose that* ∇g *is* $1/L$*-ism continuous. Let* Q_{r_n} *be sequence of mappings defined as in Lemma 2.3.22. Consider the following mapping* X_n *on* H *defined by*

$$X_n(x) = \alpha_n t f(x) + (I - \alpha_n \mu S)T_{\lambda_n} Q_{r_n}(x), \quad \forall x \in H, \ n \in \mathbb{N},$$

where $P_C(I - \gamma\nabla g_{\lambda_n}) = T_{\lambda_n}$, $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$, $\gamma \in (0, 2/L)$, $\{\alpha_n\} \subset (0, 1)$, $\mu \in (0, 2\eta/k^2)$, $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$. *Then* X_n *is a contraction. i.e.*

$$\|X_n(x) - X_n(y)\| \leq (1 - \alpha_n(\tau - tl))\|x - y\|.$$

In this chapter, we study the common solution for one-parameter nonexpansive semigroup, constrained minimization problem and generalized equilibrium problem, we also introduce an iterative algorithm for approximating a common solution of generalized mixed equilibrium problem and fixed point problem in a real reflexive Banach space.

3.2 Regularized gradient-projection algorithm for solving one-parameter nonexpansive semigroup, constrained convex minimization and generalized equilibrium problems

In this section, motivated by the work of Y. Shehu [177], Ming Tian and Si-Wen Jiao [168] and ongoing results, we prove strong convergence theorems for finding a common element of the set of common fixed points of a nonexpansive semigroup, the set of solutions to a generalized equilibrium problem and the set of solution to a constrained convex minimization problem in a real Hilbert space. Our contribution lies in the fact that our iterative method solves fixed point problem for nonexpansive semigroup, generalized equilibrium problem and constrained convex minimization problem at same time.

Lemma 3.2.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi : C \to H$ be a monotone mapping and let $g : C \to \mathbb{R}$ be a real-valued convex function, and assume that the gradient ∇g is $1/L$-ism with a constant $L > 0$. Let $f : H \to H$ be a contraction with the constant $0 < l < 1$ and let $S : C \to H$ be η-strongly monotone and k-Lipschitzian. Fix a constant μ satisfying $0 < \mu < 2\eta/k^2$, a constant t satisfying $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$. Let $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Upsilon := F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi) \neq \emptyset$, and $\{t_n\} \subset (0, \infty)$ be a sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by $x_1 \in H$ as follows:*

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall\ y \in C; \\ T_{\lambda_n}(u_n) = P_C(I - \gamma\nabla g_{\lambda_n})u_n; \\ x_{n+1} = \beta_n x_n + (1-\beta_n)\Big(\frac{1}{t_n}\int_0^{t_n} G(u)[\alpha_n t f(x_n) + (1-\alpha_n\mu S)T_{\lambda_n}(u_n)]du\Big), \end{cases} \tag{3.2.1}$$

where $u_n = Q_{r_n}(x_n)$, $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$, $T_{\lambda_n} = P_C(I - \gamma\nabla g_{\lambda_n})$, $\gamma \in (0, 2/L)$. Let $\{\beta_n\}$, $\{r_n\}$, $\{\alpha_n\}$, $\{\lambda_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(ii) $\alpha_n \subset (0, 1)$ $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\{\lambda_n\} \subset (0, 2/\gamma - L)$, $\lambda_n = o(\alpha_n)$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(iv) $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(v) $\lim_{n\to\infty} \frac{t_n - t_{n-1}}{t_n} \frac{1}{\alpha_n(1-\beta_n)} = 0$.

Then $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}$, $\{u_n\}$ and $\left\{\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right\}$ are bounded.

Proof. First, we show that $(I - \gamma\nabla g_{\lambda_n})$ is nonexpansive. For all $x, y \in C$ and $\gamma \in (0, 2/L)$, we have

$$\begin{aligned} \|(I - \gamma\nabla g_{\lambda_n})x - (I - \gamma\nabla g_{\lambda_n})y\|^2 &= \|(x-y) - \gamma(\nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y)\|^2 \\ &= \|x-y\|^2 - 2\gamma\langle x - y, \nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y\rangle + \gamma^2\|\nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y\|^2 \\ &\leq \|x-y\|^2 - \frac{2\gamma}{L}\|\nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y\|^2 + \gamma^2\|\nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y\|^2 \\ &= \|x-y\|^2 + \gamma\left(\gamma - \frac{2}{L}\right)\|\nabla g_{\lambda_n}x - \nabla g_{\lambda_n}y\|^2 \qquad (3.2.2) \\ &\leq \|x-y\|^2. \end{aligned}$$

Next, we show that $\{x_n\}$ is bounded. Let $p \in F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi) \neq \emptyset$ and by Lemma 2.3.14, we know that

$$\|u_n - p\| = \|Q_{r_n}(x_n) - Q_{r_n}(p)\| \leq \|x_n - p\|. \tag{3.2.3}$$

Now, let $y_n := \alpha_n t f(x_n) + (1 - \alpha_n \mu S) T_{\lambda_n}(u_n)$, $n \geq 1$. So

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n t f(x_n) + (I - \alpha_n \mu S) T_{\lambda_n}(u_n) - p\| \\
&\leq \|(I - \alpha_n \mu S) T_{\lambda_n}(u_n) - (I - \alpha_n \mu S) T_{\lambda_n}(p)\| + \|(I - \alpha_n \mu S) T_{\lambda_n}(p) - (I - \alpha_n \mu S) T(p)\| \\
&\quad + \alpha_n t \|f(x_n) - f(p)\| + \alpha_n \|t f(x_n) - \mu S(p)\| \\
&\leq (1 - \alpha_n \tau)\|u_n - p\| + \|T_{\lambda_n}(p) - T(p)\| \\
&\quad + \|\alpha_n \mu S T_{\lambda_n}(p) - \alpha_n \mu S T(p)\| + \alpha_n t l \|x_n - p\| + \alpha_n \|t f(p) - \mu S(p)\| \\
&\leq (1 - \alpha_n(\tau - tl))\|x_n - p\| + (\alpha_n \mu k + 1)\|T_{\lambda_n}(p) - T(p)\| \\
&\quad + \alpha_n \|t f(p) - \mu S(p)\|.
\end{aligned} \tag{3.2.4}$$

For $x \in C$, note that

$$P_C(I - \gamma \nabla g_{\lambda_n})x = T_{\lambda_n} x$$

and

$$P_C(I - \gamma \nabla g)x = Tx.$$

Then we get

$$\begin{aligned}
\|T_{\lambda_n} x - Tx\| &= \|P_C(I - \gamma \nabla g_{\lambda_n})x - P_C(I - \gamma \nabla g)x\| \\
&\leq \lambda_n \gamma \|x\|.
\end{aligned} \tag{3.2.5}$$

It follows from (3.2.4) and (3.2.5) that

$$\begin{aligned}
\|y_n - p\| &\leq (1 - \alpha_n(\tau - tl))\|x_n - p\| \\
&\quad + \alpha_n(\tau - tl)\left[\frac{\lambda_n}{\alpha_n} . \frac{(\alpha_n \mu k + 1)\gamma}{\tau - tl}\|p\| + \frac{\|t f(p) - \mu S(p)\|}{\tau - tl}\right].
\end{aligned} \tag{3.2.6}$$

From (3.2.1), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\| \beta(x_n - p) + (1 - \beta_n)\left(\frac{1}{t_n}\int_0^{t_n} [G(u)y_n - G(u)p] du\right)\right\| \\
&\leq \beta_n \|x_n - p\| + (1 - \beta_n)\|y_n - p\| \\
&\leq \beta_n \|x_n - p\| + (1 - \beta_n)(1 - \alpha_n(\tau - tl))\|x_n - p\| \\
&\quad + \alpha_n(1 - \beta_n)(\tau - tl)\left[\frac{\lambda_n}{\alpha_n} . \frac{(\alpha_n \mu k + 1)\gamma}{\tau - tl}\|p\| + \frac{\|t f(p) - \mu S(p)\|}{\tau - tl}\right] \\
&= [1 - \alpha_n(\tau - tl)(1 - \beta_n)]\,\|x_n - p\| \\
&\quad + \alpha_n(1 - \beta_n)(\tau - tl)\left[\frac{\lambda_n}{\alpha_n} . \frac{(\alpha_n \mu k + 1)\gamma}{\tau - tl}\|p\| + \frac{\|t f(p) - \mu S(p)\|}{\tau - tl}\right].
\end{aligned}$$

Since $\lambda_n = o(\alpha_n)$, there exists a real number $M_1 > 0$ such that $\frac{\lambda_n}{\alpha_n} \leq M_1$. Thus,

$$\begin{aligned}
\|x_{n+1} - p\| &\leq [1 - \alpha_n(\tau - tl)(1 - \beta_n)]\,\|x_n - p\| \\
&\quad + \alpha_n(1 - \beta_n)(\tau - tl)\frac{M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|}{\tau - tl} \\
&\leq \max\left\{\|x_n - p\|, \frac{1}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right)\right\} \\
&\ \ \vdots \\
&\leq \max\left\{\|x_1 - p\|, \frac{1}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right)\right\}, \quad n \geq 1.
\end{aligned}$$

So, $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{u_n\}$ and $\left\{\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right\}$ are also bounded.

□

Lemma 3.2.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi : C \to H$ be a monotone mapping and let $g : C \to \mathbb{R}$ be a real-valued convex function, and assume that the gradient ∇g is $1/L$-ism with a constant $L > 0$. Let $f : H \to H$ be a contraction with the constant $0 < l < 1$ and let $S : C \to H$ be η-strongly monotone and k-Lipschitzian. Fix a constant μ satisfying $0 < \mu < 2\eta/k^2$, a constant t satisfying $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$. Let $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Upsilon := F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi) \neq \emptyset$, and $\{t_n\} \subset (0, \infty)$ be a sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (3.2.1). Then $\lim\limits_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim\limits_{n\to\infty} \|u_{n+1} - u_n\| = 0$.*

Proof. From (3.2.4), we have

$$\begin{aligned}
\|y_n - p\| &\leq (1 - \alpha_n(\tau - tl))\|x_n - p\| + \alpha_n(\tau - tl)\frac{1}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right) \\
&\leq \|x_n - p\| + \frac{1}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right) \\
&\leq \max\left\{\|x_1 - p\|, \frac{1}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right)\right\} \\
&\quad + \frac{1}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right) \\
&\leq \|x_1 - p\| + \frac{2}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right).
\end{aligned}$$

Put $D = \left\{w \in H : \|w - p\| \leq \|x_1 - p\| + \frac{2}{\tau - tl}\left(M_1(\alpha_n \mu k + 1)\gamma\|p\| + \|tf(p) - \mu S(p)\|\right)\right\}$. Then D is a nonempty, bounded, closed and convex subset of H. Since $G(u)$ is nonexpansive for any $u \in [0, \infty)$, D is $G(u)$-invariant for each $u \in [0, \infty)$ and contains $\{y_n\}$. Without loss of generality, we may assume that $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ is a nonexpansive semigroup on D. By Lemma 3.1.2, we get

$$\lim_{n\to\infty} \left\|\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right) - G(h)\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du\right)\right\| = 0, \tag{3.2.7}$$

for every $h \in [0, \infty)$. Furthermore, observe that

$$\begin{aligned}
\|x_{n+1} - G(h)x_{n+1}\| &\leq \left\| x_{n+1} - \frac{1}{t_n}\int_0^{t_n} G(u)y_n du \right\| \\
&\quad + \left\| \left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du \right) - G(h)\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du \right) \right\| \\
&\quad + \left\| G(h)\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du \right) - G(h)x_{n+1} \right\| \\
&\leq 2\left\| x_{n+1} - \frac{1}{t_n}\int_0^{t_n} G(u)y_n du \right\| \\
&\quad + \left\| \left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du \right) - G(h)\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du \right) \right\| \\
&\leq 2\beta_n\left\| x_n - \frac{1}{t_n}\int_0^{t_n} G(u)y_n du \right\| \\
&\quad + \left\| \left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du \right) - G(h)\left(\frac{1}{t_n}\int_0^{t_n} G(u)y_n du \right) \right\|,
\end{aligned}$$

from $\lim\limits_{n\to\infty} \beta_n = 0$ and (3.2.7), we get $\lim\limits_{n\to\infty} \|x_{n+1} - G(h)x_{n+1}\| = 0$ and hence

$$\lim_{n\to\infty} \|x_n - G(h)x_n\| = 0. \tag{3.2.8}$$

Next, we show that $\|x_{n+1} - x_n\| \to 0,\ n \to \infty$. Since ∇g is $1/L-$ism, $P_C(I - \nabla g_{\lambda_n}) = T_{\lambda_n}$, so we have that

$$\begin{aligned}
\|T_{\lambda_n}(u_{n-1}) - T_{\lambda_{n-1}}(u_{n-1})\| &= \|P_C(I - \gamma\nabla g_{\lambda_n})u_{n-1} - (I - \gamma\nabla g\lambda_{n-1})u_{n-1}\| \\
&\leq \|(I - \gamma\nabla g_{\lambda_n})u_{n-1} - (I - \gamma\nabla g_{\lambda_n})u_{n-1}\| \\
&= \gamma\|\nabla g(u_{n-1}) + \lambda_{n-1}u_{n-1} - \nabla g(u_{n-1}) - \lambda_n u_{n-1}\| \\
&= \gamma|\lambda_n - \lambda_{n-1}|\|u_{n-1}\|.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|(\alpha_n tf(x_n) + (I - \alpha_n\mu S)T_{\lambda_n}(u_n) - (\alpha_{n-1}tf(x_{n-1}) + (I - \alpha_{n-1}\mu S)T_{\lambda_{n-1}}(u_{n-1})\| \\
&\leq \|\alpha_n tf(x_n) - \alpha_n tf(x_{n-1})\| + \|\alpha_n tf(x_{n-1}) - \alpha_{n-1}tf(x_{n-1})\| \\
&\quad + \|(I - \alpha_n\mu S)T_{\lambda_n(u_n)} - (I - \alpha_n\mu S)T_{\lambda_n}(u_{n-1})\| \\
&\quad + \|(I - \alpha_n\mu S)T_{\lambda_n}(u_{n-1}) - (I - \alpha_{n-1}\mu S)T_{\lambda_{n-1}}(u_{n-1})\| \\
&\leq \alpha_n tl\|x_n - x_{n-1}\| + t|\alpha_n - \alpha_{n-1}|.\|f(x_{n-1})\| + (1 - \alpha_n\tau)\|u_n - u_{n-1}\| \\
&\quad + \|T_{\lambda_n}(u_{n-1}) - T_{\lambda_{n-1}}(u_{n-1})\| + \|\alpha_{n-1}\mu ST_{\lambda_{n-1}}(u_{n-1}) - \alpha_n\mu ST_{\lambda_n}(u_{n-1})\| \\
&\leq \alpha_n tl\|x_n - x_{n-1}\| + t|\alpha_n - \alpha_{n-1}|.\|f(x_{n-1})\| + (1 + \alpha_{n-1}\mu k)\|T_{\lambda_n}(u_{n-1}) - T_{\lambda_{n-1}}(u_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}|.\|ST_{\lambda_n}(u_{n-1})\| + (1 - \alpha_n\tau)\|u_n - u_{n-1}\| \\
&\leq \alpha_n tl\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|.\left(t\|f(x_n)\| + \mu\|ST_{\lambda_n}(u_{n-1})\|\right) + (1 - \alpha_n\tau)\|u_n - u_{n-1}\| \\
&\quad + \gamma(1 + \mu k)|\lambda_n - \lambda_{n-1}|.\|u_{n-1}\|.
\end{aligned}$$

Since from Lemma 3.2.1 $\{u_n\}$, $\{f(x_n)\}$ and $\{ST_{\lambda_n}(u_n)\}$ are bounded, then there exists a constant $M_2 > 0$ such that

$$M_2 \geq \max\left\{\gamma(1 + \mu k)\|u_{n-1}\|, t\|f(x_n)\| + \mu\|ST_{\lambda_n}(u_{n-1})\right\}, \quad \forall\ n \geq 1.$$

Hence

$$\begin{aligned}\|y_n - y_{n-1}\| \;&\leq\; \alpha_n tl\|x_n - x_{n-1}\| + (1-\alpha_n\tau)\|u_n - u_{n-1}\| \\ &\quad + M_2(|\alpha_n - \alpha_{n-1}| + |\lambda_n - \lambda_{n-1}|). \qquad (3.2.9)\end{aligned}$$

From $u_{n+1} = Q_{r_{n+1}}(x_{n+1})$ and $u_n = Q_{r_n}(x_n)$, we note that

$$F(u_n, y) + \langle \psi x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C \qquad (3.2.10)$$

and

$$F(u_{n+1}, y) + \langle \psi x_{n+1}, y - u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall y \in C. \qquad (3.2.11)$$

Putting $y = u_{n+1}$ in (3.2.10) and $y = u_n$ in (3.2.11), we have

$$F(u_n, u_{n+1}) + \langle \psi x_n, u_{n+1} - u_n\rangle + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and

$$F(u_{n+1}, u_n) + \langle \psi x_{n+1}, u_n - u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0.$$

So, from (A2), we have

$$\langle \psi x_{n+1} - \psi x_n, u_n - u_{n+1}\rangle + \left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0$$

and hence,

$$\begin{aligned} 0 \;&\leq\; \left\langle u_n - u_{n+1}, r_n(\psi x_{n+1} - \psi x_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) - (u_n - x_n)\right\rangle \\ &= \left\langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)u_{n+1} + (x_{n+1} - r_n\psi x_{n+1}) - (x_n - r_n\psi x_n) \right. \\ &\quad \left. - x_{n+1} + \frac{r_n}{r_{n+1}}x_{n+1}\right\rangle \\ &= \left\langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) + (x_{n+1} - r_n\psi x_{n+1}) - (x_n - r_n\psi x_n)\right\rangle. \end{aligned}$$

It then follows that

$$\|u_{n+1} - u_n\|^2 \leq \|u_{n+1} - u_n\|\left\{\left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|\right\}$$

and so we have

$$\|u_{n+1} - u_n\| \leq \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|.$$

Without loss of generality, we assume that there exists $w \in \mathbb{R}$ such that $r_n > w > 0$, $\forall n \geq 1$. Then

$$\begin{aligned}\|u_{n+1}-u_n\| &\leq \|x_{n+1}-x_n\| + \frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\| \\ &\leq \|x_{n+1}-x_n\| + \frac{1}{w}|r_{n+1}-r_n|M_3, \qquad (3.2.12)\end{aligned}$$

where $M_3 := \sup_{n\geq 1}\|u_n - x_n\|$.
From (3.2.9) and (3.2.12), we get

$$\begin{aligned}\|y_n-y_{n-1}\| &\leq \alpha_n tl\|x_n-x_{n-1}\| + (1-\alpha_n\tau)\|u_n-u_{n-1}\| + M_2(|\lambda_n-\lambda_{n-1}|+|\alpha_n-\alpha_{n-1}|) \\ &\leq \alpha_n tl\|x_n-x_{n-1}\| + (1-\alpha_n\tau)\left(\|x_n-x_{n-1}\| + \frac{1}{w}|r_n-r_{n-1}|M_3\right) \\ &\quad +M_2(|\lambda_n-\lambda_{n-1}|+|\alpha_n-\alpha_{n-1}|) \\ &\leq (1-\alpha_n(\tau-tl))\|x_n-x_{n-1}\| + \frac{M_3}{w}|r_n-r_{n-1}| \\ &\quad +M_2(|\lambda_n-\lambda_{n-1}|+|\alpha_n-\alpha_{n-1}|). \qquad (3.2.13)\end{aligned}$$

Let $z_n := \frac{1}{t_n}\int_0^{t_n} G(u)y_n du$; $\quad n \geq 1$. Then we have

$$\|z_n - z_{n-1}\| = \left\|\frac{1}{t_n}\int_0^{t_n}[G(u)y_n - G(u)y_{n-1}]du + \left(\frac{1}{t_n}-\frac{1}{t_{n-1}}\right)\int_0^{t_{n-1}} du + \frac{1}{t_n}\int_{t_{n-1}}^{t_n} G(u)y_{n-1}du\right\|.$$

Given that
$\left(\frac{1}{a}-\frac{1}{b}\right)b = -\frac{a-b}{b}$, $\quad a, b \neq 0$; if $p \in \Omega$, we can write

$$\begin{aligned}\|z_n - z_{n-1}\| &= \left\|\frac{1}{t_n}\int_0^{t_n}[G(u)y_n - G(u)y_{n-1}]du + \left(\frac{1}{t_n}-\frac{1}{t_{n-1}}\right)\int_0^{t_{n-1}}[G(u)y_{n-1} - G(u)p]du \right. \\ &\quad \left. +\frac{1}{t_n}\int_{t_{n-1}}^{t_n}[G(u)y_{n-1} - G(u)p]du\right\|.\end{aligned}$$

Thus,

$$\|z_n - z_{n-1}\| \leq \|y_n - y_{n-1}\| + \left(\frac{2|t_n - t_{n-1}|}{t_n}\right)\|y_{n-1} - p\|. \qquad (3.2.14)$$

Substituting (3.2.13) into (3.2.14), we obtain

$$\begin{aligned}\|z_n - z_{n-1}\| &\leq (1-\alpha_n(\tau-tl))\|x_n-x_{n-1}\| + \frac{M_3}{w}|r_n-r_{n-1}| \\ &\quad +M_2(|\lambda_n-\lambda_{n-1}|+|\alpha_n-\alpha_{n-1}|) \\ &\quad +\left(\frac{2|t_n - t_{n-1}|}{t_n}\right)\|y_{n-1} - p\|. \qquad (3.2.15)\end{aligned}$$

From (3.2.1), we have $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ and this implies that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\beta_n x_n + (1-\beta_n)z_n - \beta_{n-1}x_{n-1} - (1-\beta_{n-1})z_{n-1}\| \\ &= \|\beta_n x_n - \beta_{n-1}x_{n-1} + \beta_n x_{n-1} - \beta_n x_{n-1}(1-\beta_n)z_n - (1-\beta_{n-1})z_{n-1} \\ &\quad +(1-\beta_n)z_{n-1} - (1-\beta_n)z_{n-1}\| \\ &\leq \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| \\ &\quad +|\beta_n - \beta_{n-1}|(\|x_{n-1}\| + \|z_{n-1}\|). \qquad (3.2.16)\end{aligned}$$

Using (3.2.15) in (3.2.16), we obtain

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq \beta_n\|x_n-x_{n-1}\|+(1-\beta_n)\Big[(1-\alpha_n(\tau-tl))\|x_n-x_{n-1}\|+\frac{M_3}{w}|r_n-r_{n-1}| \\ &\quad +M_2(|\lambda_n-\lambda_{n-1}|+|\alpha_n-\alpha_{n-1}|)+\left(\frac{2|t_n-t_{n-1}|}{t_n}\right)\|y_{n-1}-p\|\Big] \\ &\quad +|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|z_{n-1}\|) \\ &\leq [1-\alpha_n(\tau-tl)(1-\beta_n)]\,\|x_n-x_{n-1}\|+\frac{M_3}{w}|r_n-r_{n-1}|+|\beta_n-\beta_{n-1}|(\|x_{n-1}\|+\|z_{n-1}\| \\ &\quad +M_2(|\lambda_n-\lambda_{n-1}|+|\alpha_n-\alpha_{n-1}|)+\left(\frac{2|t_n-t_{n-1}|}{t_n}\right)\|y_{n-1}-p\| \\ &\leq [1-\alpha_n(\tau-tl)(1-\beta_n)]\,\|x_n-x_{n-1}\| \\ &\quad +D\left[|r_n-r_{n-1}|+|\beta_n-\beta_{n-1}|+(|\lambda_n-\lambda_{n-1}|+|\alpha_n-\alpha_{n-1}|)+\frac{2|t_n-t_{n-1}|}{t_n}\right],\end{aligned}$$

where $D := \max\{\sup_{n\geq 1}(\|x_n\|+\|z_n\|), \sup_{n\geq 1}\|y_n-p\|, \frac{M_3}{w}, M_2\}$. From Lemma 2.3.23 taking
$\delta_n=\alpha_n(\tau-tl)(1-\beta_n)$, $b_n=\frac{2D|t_n-t_{n-1}|}{t_n}$ and $\sigma_n=D(|r_n-r_{n-1}|+|\beta_n-\beta_{n-1}|+|\lambda_n-\lambda_{n-1}|+|\alpha_n-\alpha_{n-1}|)$, by using conditions (i)-(v), it follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.2.17}$$

Since $\lim_{n\to\infty}|r_{n+1}-r_n|=0$, then from (3.2.12) and (3.2.17), we have that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0. \tag{3.2.18}$$

$\square$

Lemma 3.2.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi: C\to H$ be a monotone mapping and let $g: C\to\mathbb{R}$ be a real-valued convex function, and assume that the gradient ∇g is $1/L$-ism with a constant $L>0$. Let $f: H\to H$ be a contraction with the constant $0<l<1$ and let $S: C\to H$ be η-strongly monotone and k-Lipschitzian. Fix a constant μ satisfying $0<\mu<2\eta/k^2$, a constant t satisfying $0<t<\mu(\eta-\frac{\mu k^2}{2})/l=\tau/l$. Let $\mathcal{T}:=\{G(u): 0\leq u<\infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Upsilon := F(\mathcal{T})\cap\Gamma\cap GEP(F,\psi)\neq\emptyset$, and $\{t_n\}\subset(0,\infty)$ be a sequence such that $\lim_{n\to\infty}t_n=\infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (3.2.1). Then $\lim_{n\to\infty}\|u_n-T_{\lambda_n}(u_n)\|=0$ and $\lim_{n\to\infty}\|y_n-u_n\|=0$.*

Proof. Furthermore, from (3.2.1), (3.2.2) and (3.2.3), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|\beta_n(x_n-p)+(1-\beta_n)\left(\frac{1}{t_n}\int_0^{t_n}[G(u)y_n-G(u)p]du\right)\right\|^2 \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2 \\
&= \beta_n\|x_n-p\|^2+(1-\beta_n)\|\alpha_n(tf(x_n)-\mu Sp)+(1-\alpha_n\mu S)(T_{\lambda_n}(u_n)-p)\|^2 \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\left[\alpha_n\|tf(x_n)-\mu Sp\|^2+(1-\alpha_n\mu S)\|T_{\lambda_n}(u_n)-p\|^2\right] \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\Big[\alpha_n\|tf(x_n)-\mu Sp\|^2 \\
&\quad +(1-\alpha_n\mu S)(\|u_n-p\|^2+\gamma\left(\gamma-\frac{2}{L}\right)\|\nabla g_{\lambda_n}(u_n)-\nabla g_{\lambda_n}p\|^2)\Big] \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\Big[\alpha_n\|tf(x_n)-\mu Sp\|^2 \\
&\quad +(1-\alpha_n\mu S)(\|x_n-p\|^2+\gamma\left(\gamma-\frac{2}{L}\right)\|\nabla g_{\lambda_n}(u_n)-\nabla g_{\lambda_n}p\|^2)\Big] \\
&\leq \|x_n-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2+\gamma\left(\gamma-\frac{2}{L}\right)\|\nabla g_{\lambda_n}(u_n)-\nabla g_{\lambda_n}p\|^2.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
-\gamma\left(\gamma-\frac{2}{L}\right)\|\nabla g_{\lambda_n}(u_n)-\nabla g_{\lambda_n}p\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2 \\
&\leq \|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|)+\alpha_n\|tf(x_n)-\mu Sp\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ by Lemma 3.2.2 we obtain

$$\lim_{n\to\infty}\|\nabla g_{\lambda_n}(u_n)-\nabla g_{\lambda_n}p\|=0.$$

From (3.2.1), we obtain (noting that $(I-\gamma\nabla g_{\lambda_n})$ is nonexpansive)

$$\begin{aligned}
\|T_{\lambda_n}(u_n)-p\|^2 &= \|P_C(I-\gamma\nabla g_{\lambda_n})u_n-P_C(I-\gamma\nabla g\lambda_n)p\|^2 \\
&\leq [\langle(u_n-\gamma\nabla g_{\lambda_n}u_n)-(p-\gamma\nabla g_{\lambda_n}p),T_{\lambda_n}(u_n)-p\rangle] \\
&= \frac{1}{2}\Big[\|(u_n-\gamma\nabla g_{\lambda_n}u_n)-(p-\gamma\nabla g_{\lambda_n}p)\|^2+\|T_{\lambda_n}(u_n)-p\|^2 \\
&\quad -\|(u_n-\gamma\nabla g_{\lambda_n})-(p-\gamma\nabla g_{\lambda_n}p)-(T_{\lambda_n}(u_n)-p)\|^2\Big] \\
&\leq \frac{1}{2}\left[\|u_n-p\|^2+\|T_{\lambda_n}(u_n)-p\|^2+\|(u_n-T_{\lambda_n}(u_n))-\gamma(\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p)\|^2\right] \\
&\leq \frac{1}{2}\Big[\|u_n-p\|^2+\|T_{\lambda_n}(u_n)-p\|^2-\|u_n-T_{\lambda_n}(u_n)\|^2 \\
&\quad +2\gamma\langle u_n-T_{\lambda_n}(u_n),\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\rangle-\gamma^2\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|^2\Big].
\end{aligned}$$

So, we have

$$\begin{aligned}
\|T_{\lambda_n}(u_n)-p\|^2 &\leq \|u_n-p\|^2-\|T_{\lambda_n}(u_n)-u_n\|^2+2\gamma\langle u_n-T_{\lambda_n}(u_n),\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\rangle \\
&\quad -\gamma^2\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|^2. \qquad (3.2.19)
\end{aligned}$$

From (3.2.1) and (3.2.19), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|\beta_n(x_n-p)+(1-\beta_n)\left(\frac{1}{t_n}\int_0^{t_n}[G(u)y_n-G(u)p]du\right)\right\|^2\\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2\\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\left(\alpha_n\|tf(x_n)-\mu Sp\|^2+(1-\alpha_n\mu S)\|T_{\lambda_n}(u_n)-p\|^2\right)\\
&\leq \beta_n\|x_n-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2+(1-\beta_n)[\|u_n-p\|^2-\|T_{\lambda_n}(u_n)-u_n\|^2\\
&\quad +2\gamma\langle u_n-T_{\lambda_n}(u_n),\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\rangle-\gamma^2\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|^2]\\
&\leq \beta_n\|x_n-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2+(1-\beta_n)[\|x_n-p\|^2-\|T_{\lambda_n}(u_n)-u_n\|^2\\
&\quad +2\gamma\langle u_n-T_{\lambda_n}(u_n),\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\rangle-\gamma^2\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|^2]\\
&\leq \|x_n-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2-(1-\beta_n)\|u_n-T_{\lambda_n}(u_n)\|^2\\
&\quad +2(1-\beta_n)\gamma\|u_n-T_{\lambda_n}(u_n)\|\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|. \qquad (3.2.20)
\end{aligned}$$

From (3.2.20), we have

$$\begin{aligned}
(1-\beta_n)\|u_n-T_{\lambda_n}(u_n)\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|tf(x_n)-\mu Sp\|^2\\
&\quad +2\gamma\|u_n-T_{\lambda_n}(u_n)\|\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ by Lemma 3.2.2 $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|\nabla g_{\lambda_n}u_n-\nabla g_{\lambda_n}p\|=0$, we obtain

$$\lim_{n\to\infty}(1-\beta_n)\|u_n-T_{\lambda_n}(u_n)\|=0.$$

Since $\lim_{n\to\infty}\beta_n=0$, we obtain

$$\lim_{n\to\infty}\|u_n-T_{\lambda_n}(u_n)\|=0. \qquad (3.2.21)$$

From $y_n=\alpha_n tf(x_n)+(1-\alpha_n\mu S)T_{\lambda_n}(u_n)$, we obtain $y_n-T_{\lambda_n}(u_n)=\alpha_n(tf(x_n)-\mu ST_{\lambda_n}(u_n))$. So,

$$\|y_n-T_{\lambda_n}(u_n)\|=\alpha_n\|(tf(x_n)-\mu ST_{\lambda_n}(u_n))\|\to 0, \quad as \quad n\to\infty. \qquad (3.2.22)$$

Next we show that $\lim_{n\to\infty}\|u_n-x_n\|=0$. Indeed, for any $p\in F(\mathcal{T})\cap\Gamma\cap EP(F)$, by Lemma 2.3.22 we have

$$\begin{aligned}
\|u_n-p\|^2 &= \|Q_{r_n}(x_n)-Q_{r_n}(p)\|^2\\
&\leq \langle x_n-p,u_n-p\rangle\\
&= \frac{1}{2}(\|x_n-p\|^2+\|u_n-p\|^2-\|u_n-x_n\|^2).
\end{aligned}$$

This implies that

$$\|u_n-p\|^2 \leq \|x_n-p\|^2-\|u_n-x_n\|^2. \qquad (3.2.23)$$

Then, from (3.2.5) and (3.2.23), we derive that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n t f(x_n) + (I - \alpha_n \mu S)T_{\lambda_n}(u_n) - p\|^2 \\
&\leq [((1-\alpha_n\tau)\|u_n - p\| + \alpha_n t l\|x_n - p\|) + (\lambda_n\gamma(1+\alpha_n\mu k)\|p\| + \alpha_n\|tf(p) - \mu S(p)\|)]^2 \\
&\leq \|u_n - p\|^2 + (\alpha_n^2 t^2 l^2 + 2\alpha_n t l)\|x_n - p\|^2 + \lambda_n^2\gamma^2(1+\alpha_n\mu k)^2\|p\|^2 + \alpha_n^2\|tf(p) - \mu S(p)\|^2 \\
&\quad +2\lambda_n\gamma(1+\alpha_n\gamma)\|p\|.\|tf(x_n) - \mu S(p)\| + 2\lambda_n\gamma(1+tl)(1+\alpha_n\mu k)\|x_n - p\|.\|p\| \\
&\quad +2\alpha_n(1+tl)\|x_n - p\|.\|tf(p) - \mu S(p)\| \\
&\leq (1+\alpha_n t l)^2\|x_n - p\|^2 - \|u_n - x_n\|^2 + \lambda_n^2\gamma^2(1+\alpha_n\mu k)^2\|p\|^2 + \alpha_n^2\|tf(p) - \mu S(p)\|^2 \\
&\quad +2\lambda_n\gamma(1+\alpha_n\gamma)\|p\|.\|tf(x_n) - \mu S(p)\| + 2\lambda_n\gamma(1+tl)(1+\alpha_n\mu k)\|x_n - p\|.\|p\| \\
&\quad +2\alpha_n(1+tl)\|x_n - p\|.\|tf(p) - \mu S(p)\|. \qquad (3.2.24)
\end{aligned}$$

From (3.2.1) ,(3.2.24) and by the convexity of $\|.\|^2$, we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|\beta_n(x_n - p) + (1-\beta_n)\left(\frac{1}{t_n}\int_0^{t_n}[G(u)y_n - G(u)p]du\right)\right\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\Big[(1+\alpha_n t l)^2\|x_n - p\|^2 - \|u_n - x_n\|^2 + \lambda_n^2\gamma^2(1+\alpha_n\mu k)^2\|p\|^2 \\
&\quad +\alpha_n^2\|tf(p) - \mu S(p)\|^2 + 2\lambda_n\gamma(1+\alpha_n\gamma)\|p\|.\|tf(x_n) - \mu S(p)\| \\
&\quad +2\lambda_n\gamma(1+tl)(1+\alpha_n\mu k)\|x_n - p\|.\|p\| + 2\alpha_n(1+tl)\|x_n - p\|.\|tf(p) - \mu S(p)\|\Big].
\end{aligned}$$

Thus, we get

$$\begin{aligned}
(1-\beta_n)\|u_n - x_n\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\Big[(1+\alpha_n t l)^2\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad +\lambda_n^2\gamma^2(1+\alpha_n\mu k)^2\|p\|^2 + \alpha_n^2\|tf(p) - \mu S(p)\|^2 \\
&\quad +2\lambda_n\gamma(1+\alpha_n\gamma)\|p\|.\|tf(x_n) - \mu S(p)\| \\
&\quad +2\lambda_n\gamma(1+tl)(1+\alpha_n\mu k)\|x_n - p\|.\|p\| \\
&\quad +2\alpha_n(1+tl)\|x_n - p\|.\|tf(p) - \mu S(p)\|\Big].
\end{aligned}$$

Since $\{x_n\}$ is bounded by Lemma 3.2.2 $\alpha_n \to 0$, $\beta_n \to 0$, $\lambda_n \to 0$, $n \to \infty$ and $\|x_{n+1} - x_n\| \to 0$ by Lemma 3.2.2 we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \qquad (3.2.25)$$

Furthermore, from (3.2.21), (3.2.22), (3.2.24) and for every $h \in [0,\infty)$, we have that

$$\begin{aligned}
\|G(h)y_n - G(h)x_n\| &\leq \|y_n - x_n\| \\
&\leq \|u_n - x_n\| + \|u_n - T_{\lambda_n}(u_n)\| \\
&\quad +\|T_{\lambda_n}(u_n) - y_n\| \to 0, \quad n \to \infty. \qquad (3.2.26)
\end{aligned}$$

Hence, from (3.2.8) and (3.2.26), we obtain

$$\|G(h)y_n - x_n\| \leq \|G(h)y_n - G(h)x_n\| + \|G(h)x_n - x_n\| \to 0, \quad n \to \infty.$$

Also, we have that

$$\|x_n - T_{\lambda_n}(u_n)\| \leq \|x_n - u_n\| + \|u_n - T_{\lambda_n}(u_n)\| \to 0, \quad n \to \infty.$$

Hence, for every $h \in [0, \infty)$ we have that

$$\|G(h)y_n - y_n\| \leq \|G(h)y_n - x_n\| + \|x_n - T_{\lambda_n}(u_n)\| + \|T_{\lambda_n}(u_n) - y_n\| \to 0, \quad n \to \infty. \tag{3.2.27}$$

Next, we show that $\|x_n - T_{\lambda_n}(x_n)\| \to 0$, $n \to \infty$.

$$\begin{aligned}
\|x_n - T_{\lambda_n}(x_n)\| &= \|x_n - u_n + u_n - T_{\lambda_n}(u_n) + T_{\lambda_n}(u_n) - T_{\lambda_n}(u_n) - T_{\lambda_n}(x_n)\| \\
&\leq \|x_n - u_n\| + \|u_n - T_{\lambda_n}(u_n)\| + \|T_{\lambda_n}(u_n) - T_{\lambda_n}(x_n)\| \\
&\leq \|x_n - u_n\| + \|u_n - T_{\lambda_n}(u_n)\| + \|u_n - x_n\|.
\end{aligned}$$

From,(3.2.21) and (3.2.25), we have

$$\|x_n - T_{\lambda_n}(x_n)\| \to 0, \quad n \to \infty.$$

From (3.2.21) and (3.2.22), we obtain that

$$\|y_n - u_n\| \leq \|u_n - T_{\lambda_n}(u_n)\| + \|y_n - T_{\lambda_n}(u_n)\| \to 0, \quad n \to \infty. \tag{3.2.28}$$

□

Lemma 3.2.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi : C \to H$ be a monotone mapping and let $g : C \to \mathbb{R}$ be a real-valued convex function, and assume that the gradient ∇g is $1/L$-ism with a constant $L > 0$. Let $f : H \to H$ be a contraction with the constant $0 < l < 1$ and let $S : C \to H$ be η-strongly monotone and k-Lipschitzian. Fix a constant μ satisfying $0 < \mu < 2\eta/k^2$, a constant t satisfying $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$. Let $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Upsilon := F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi) \neq \emptyset$, and $\{t_n\} \subset (0, \infty)$ be a sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (3.2.1). Then*

$$\limsup_{n\to\infty} \langle y_n - z, -(\mu S - tf)z\rangle \leq 0,$$

where $z = P_{\Upsilon}(I - \mu S + tf)z$.

Proof. Now if we take a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle y_n - z, -(\mu S - tf)z\rangle = \limsup_{n\to\infty} \langle y_{n_k} - z, -(\mu S - tf)z\rangle, \tag{3.2.29}$$

by (3.2.28) and $y_{n_k} \rightharpoonup q$, we have that $u_{n_k} \rightharpoonup q$. Note that

$$\begin{aligned}
\|u_n - T(u_n)\| &\leq \|u_n - T_{\lambda_n}(u_n)\| + \|T_{\lambda_n}(u_n) - T(u_n)\| \\
&\leq \|u_n - T_{\lambda_n}(u_n)\| + \lambda_n \gamma \|u_n\|.
\end{aligned}$$

Hence, by using the fact that $\|u_n - T_{\lambda_n}(u_n)\| \to 0$ by Lemma 3.2.3 and $\lambda_n \to 0$, we get $\|u_n - T(u_n)\| \to 0$. From Lemma 2.3.13 we get $q \in F(T) = \Gamma$. Next, we show that $q \in GEP(F, \psi)$. Since $u_n = Q_{r_n} x_n$, for any $y \in C$, we obtain

$$F(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0.$$

Furthermore, replacing n by n_j in the last inequality and using (A2), we obtain

$$\langle \psi x_{n_j}, y - u_{n_j} \rangle + \frac{1}{r_{n_j}} \langle y - u_n, u_n - x_n \rangle \geq F(y, u_{n_j}). \tag{3.2.30}$$

Let $z_t := ty + (1-t)q$ for all $t \in (0,1]$ and $y \in C$. This implies that $z_t \in C$. Then, by (3.2.30), we have

$$\begin{aligned} \langle z_t - u_{n_j}, \psi z_t \rangle \;&\geq\; \langle z_t - u_{n_j}, \psi z_t \rangle - \langle z_t - u_{u_j}, \psi x_{n_j} \rangle - \left\langle z_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}} \right\rangle + F(z_t, u_{n_j}) \\ &=\; \langle z_t - u_{n_j}, \psi z_t - \psi u_{n_j} \rangle + \langle z_t - u_{n_j}, \psi u_{n_j} - \psi x_{n_j} \rangle \\ &\quad - \left\langle z_t - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}} \right\rangle + F(z_t, u_{n_j}). \end{aligned} \tag{3.2.31}$$

Since $\lim\limits_{j\to\infty} \|x_{n_j} - u_{n_j}\| = 0$, we have $\lim\limits_{j\to\infty} \|\psi x_{n_j} - \psi u_{n_j}\| = 0$. Furthermore, by the monotonicity of ψ, we obtain $\langle z_t - u_{n_j}, \psi z_t - \psi u_{n_j} \rangle \geq 0$.
Since $\lim\limits_{j\to\infty} \|y_{n_j} - u_{n_j}\| = 0$ and $\lim\limits_{j\to\infty} y_{n_j} = q$, we obtain that $\lim\limits_{j\to\infty} u_{n_j} = q$. Then, using assumption (A4) in (3.2.31), we obtain

$$\langle z_t - q, \psi z_t \rangle \geq F(z_t, q), \quad j \to \infty, \ in \ (3.2.31). \tag{3.2.32}$$

Using (A1), (A4) and (3.2.32), we also obtain

$$\begin{aligned} 0 \;&=\; F(z_t, z_t) \leq tF(z_t, y) + (1-t)F(z_t, q) \\ &\leq\; tF(z_t, y) + (1-t)\langle z_t - q, \psi \rangle \\ &\leq\; tF(z_t, y) + (1-t)t\langle y - q, \psi z_t \rangle \end{aligned}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle y - q, \psi z_t \rangle.$$

Letting $t \to 0$ and using assumption (A3), we have, for each $y \in C$,

$$0 \;\leq\; F(q, y) + \langle y - q, \psi q \rangle. \tag{3.2.33}$$

Hence $q \in GEP(F, \psi)$.

Next, we show that $q \in F(\mathcal{T})$. Assume that $q \neq G(h)q$ for some $h \in [0, \infty)$. Then by Opial's condition, we obtain from (3.2.27) that

$$\begin{aligned} \liminf_{j\to\infty} \|y_{n_j} - q\| \;&<\; \liminf_{j\to\infty} \|y_{n_j} - G(h)q\| \\ &\leq\; \liminf_{j\to\infty} (\|y_{n_j} - G(h)y_{n_j}\| + \|G(h)y_{n_j} - G(h)q\|) \\ &\leq\; \liminf_{j\to\infty} \|y_{n_j} - q\|. \end{aligned}$$

This is a contradiction. Hence, $q \in F(\mathcal{T})$. Thus $q \in \Upsilon := F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi)$. By (3.2.29) and property of metric projection, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle y_n - z(\mu S - tf)z\rangle &= \lim_{j\to\infty}\langle y_{n_j} - z, (\mu S - tf)z\rangle \\ &= \langle y - z, (\mu S - tf)z\rangle \le 0. \qquad (3.2.34)\end{aligned}$$

□

Theorem 3.2.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi : C \to H$ be a monotone mapping and let $g : C \to \mathbb{R}$ be a real-valued convex function, and assume that the gradient ∇g is $1/L$-ism with a constant $L > 0$. Let $f : H \to H$ be a contraction with the constant $0 < l < 1$ and let $S : C \to H$ be η-strongly monotone and k-Lipschitzian. Fix a constant μ satisfying $0 < \mu < 2\eta/k^2$, a constant t satisfying $0 < t < \mu(\eta - \frac{\mu k^2}{2})/l = \tau/l$. Let $\mathcal{T} := \{G(u) : 0 \le u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Upsilon := F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi) \ne \emptyset$, and $\{t_n\} \subset (0, \infty)$ be a sequence such that $\lim_{n\to\infty} t_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are generated by (3.2.1). Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to z, where $z := P_{\Upsilon}(I - \mu S + tf)z$.*

Proof. Now,

$$\begin{aligned} y_n - z &= \alpha_n t f(x_n) + (I - \alpha_n\mu S)T_{\lambda_n}(u_n) - z \\ &= ((I - \alpha_n\mu S)(T_{\lambda_n}(u_n) - (I - \alpha_n\mu S)T_{\lambda_n}(z)) + ((I - \alpha_n\mu S)T_{\lambda_n}(z) - (I - \alpha_n\mu S)T(z)) \\ &\quad + \alpha_n t(f(x_n) - f(z)) + \alpha_n(tf(q) - \mu S(z)).\end{aligned}$$

So, from (3.2.3) and (3.2.5), we derive

$$\begin{aligned}\|y_n - z\|^2 &= \langle (I - \alpha_n\mu S)(T_{\lambda_n}(u_n) - (I - \alpha_n\mu S)T_{\lambda_n}(z), y_n - z\rangle \\ &\quad + \langle (I - \alpha_n\mu S)T_{\lambda_n}(z) - (I - \alpha_n\mu S)T(z), y_n - z\rangle \\ &\quad + \alpha_n t\langle f(x_n) - f(q), y_n - z\rangle + \alpha_n\langle -(\mu S - tf)z, y_n - z\rangle \\ &\le (1 - \alpha_n\tau)\|u_n - z\|.\|y_n - z\| \\ &\quad + \lambda_n\gamma(1 + \alpha_n\mu k)\|z\|.\|z_n - z\| + \alpha_n t l\|x_n - z\|.\|y_n - z\| \\ &\quad + \langle -(\mu S - tf)z, y_n - z\rangle \\ &\le (1 - \alpha_n(\tau - tl))\|x_n - z\|.\|y_n - z\| \\ &\quad + \lambda_n\gamma(1 + \alpha_n\mu k)\|z\|.\|y_n - z\| + \alpha_n\langle -(\mu S - tf)z, y_n - z\rangle \\ &\le (1 - \alpha_n(\tau - tl))\frac{1}{2}(\|x_n - z\|^2 + \|y_n - z\|^2) \\ &\quad + \alpha_n\left[\langle -(\mu S - tf)z, y_n - z\rangle + \frac{\lambda_n}{\alpha_n}\gamma(1 + \alpha_n\mu S)\|z\|.\|y_n - z\|\right].\end{aligned}$$

This implies that

$$\begin{aligned}\|y_n - z\|^2 &\leq \frac{1-\alpha_n(\tau - tl)}{1+\alpha_n(\tau - tl)}\|x_n - z\|^2 \\ &\quad + \frac{2\alpha_n}{1+\alpha_n(\tau - tl)}\Big[\langle -(\mu S - tf)z, y_n - z\rangle + \frac{\lambda_n}{\alpha_n}\gamma(1+\alpha_n\mu k)\|z\|.\|y_n - z\|\Big] \\ &\leq (1-\alpha_n(\tau - tl))\|x_n - z\|^2 + \frac{2\alpha_n}{1+\alpha_n(\tau - tl)}\Big[\langle -(\mu S - tf)z, y_n - z\rangle \\ &\quad + \frac{\lambda_n}{\alpha_n}\gamma(1+\alpha_n\mu k)\|z\|.\|y_n - z\|\Big]. \end{aligned} \tag{3.2.35}$$

Using (3.2.1) in (3.2.35), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \left\|\beta_n(x_n - z) + (1-\beta_n)\left(\frac{1}{t_n}\int_0^{t_n}[G(u)y_n - G(u)p]du\right)\right\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\Big((1-\alpha_n(\tau - tl))\|x_n - z\|^2 \\ &\quad + \frac{2\alpha_n}{1+\alpha_n(\tau - tl)}\Big[\langle -(\mu S - tf)z, y_n - z\rangle + \frac{\lambda_n}{\alpha_n}\gamma(1+\alpha_n\mu k)\|z\|.\|y_n - z\|\Big]\Big) \\ &\leq [1-(1-\beta_n)\alpha_n(\tau - tl)]\|x_n - z\|^2 \\ &\quad + \frac{2\alpha_n(1-\beta_n)}{1+\alpha_n(\tau - tl)}\Big[\langle -(\mu S - tf)z, y_n - z\rangle + \frac{\lambda_n}{\alpha_n}\gamma(1+\alpha_n\mu k)\|z\|.\|y_n - z\|\Big].\end{aligned}$$

Since $\{y_n\}$ is bounded by Lemma 3.2.1, there exists a constant $M > 0$ such that

$$M \geq \|y_n - z\|, \quad n \geq 1.$$

Then, we have that

$$\|x_{n+1} - z\|^2 \leq (1-\delta_n)\|x_n - z\|^2 + \alpha_n\sigma_n, \tag{3.2.36}$$

where $\delta_n := (1-\beta_n)\alpha_n(\tau - tl)$ and $\sigma_n := \frac{2(1-\beta_n)}{1+\alpha_n(\tau - tl)}\Big[\langle -(\mu S - tf)z, y_n - z\rangle + \frac{\lambda_n}{\alpha_n}\gamma(1+\alpha_n\mu k)\|z\|M\Big]$.

By (3.2.34) and $\lambda_n = o(\alpha_n)$, we get $\limsup_{n\to\infty}\sigma_n \leq 0$. Now applying Lemma 2.3.23 to (3.2.36) we conclude that $x_n \to z$ as $n \to \infty$. This completes the proof. □

Remark 3.2.6. *Examples of sequences $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$, $\{t_n\}_{n=1}^\infty$, $\{r_n\}_{n=1}^\infty$ and $\{\lambda_n\}_{n=1}^\infty$ in Theorem 3.2.5 are*

$$\alpha_n = \frac{1}{n^{\frac{1}{4}}}, \quad \beta_n = \frac{1}{(n+1)^{\frac{1}{4}}} \quad t_n = n \quad r_n = \frac{n}{n+1} \quad \lambda_n = \frac{1}{n+1} \quad n \geq 1.$$

Corollary 3.2.7. *: Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4), $\psi : C \to H$ be a monotone* *mapping and let* $g : C \to \mathbb{R}$ *be a real-valued convex function, and assume that the gradient* ∇g *is* $1/L$*-ism with a constant* $L > 0$*. Let* $f : H \to H$ *be a contraction with the constant* $0 < l < 1$*. Let* $\mathcal{T} := \{G(u) : 0 \leq u < \infty\}$ *be a one-parameter nonexpansive semigroup on* H *such that* $\Upsilon := F(\mathcal{T}) \cap \Gamma \cap GEP(F, \psi) \neq \emptyset$*, and* $\{t_n\} \subset (0, \infty)$ *be a sequence such that* $\lim_{n\to\infty} t_n = \infty$*. Suppose* $\{x_n\}_{n=1}^{\infty}$ *and* $\{u_n\}_{n=1}^{\infty}$ *are generated by* $x_1 \in H$ *as follows:*

$$\begin{cases} F(u_n, y) + \langle \psi x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \forall\ y \in C; \\ T_{\lambda_n}(u_n) = P_C(I - \gamma \nabla g_{\lambda_n}) u_n; \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)\Big(\frac{1}{t_n}\int_0^{t_n} G(u)[\alpha_n f(x_n) + (1 - \alpha_n) T_{\lambda_n}(u_n)] du\Big), \end{cases} \tag{3.2.37}$$

where $u_n = Q_{r_n}(x_n)$*,* $\nabla g_{\lambda_n} = \nabla g + \lambda_n I$*,* $T_{\lambda_n} = P_C(I - \gamma \nabla g_{\lambda_n})$*,* $\gamma \in (0, 2/L)$*. Let* $\{\beta_n\}$*,* $\{r_n\}$*,* $\{\alpha_n\}$*,* $\{\lambda_n\}$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(ii) $\alpha_n \subset (0, 1)$ $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\{\lambda_n\} \subset (0, 2/\gamma - L)$, $\lambda_n = o(\alpha_n)$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(iv) $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(v) $\lim_{n\to\infty} \frac{t_n - t_{n-1}}{t_n} \frac{1}{\alpha_n(1-\beta_n)} = 0$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* z*, where* $z := P_\Upsilon f(z)$*.*

3.2.1 Application

Consider the problem of finding a zero of maximal monotone operator in a Hilbert space H. It is well known (see [25]) that the initial value problem

$$\frac{du(t)}{dt} + Au(t) \ni 0 \ \text{ for } \ \text{every } t \geq 0,\ u(0) = x,$$

for any $x \in \overline{D(A)}$ has a unique solution $u : [0, \infty) \to H$ and $\overline{D(A)}$ is closed and convex. Putting $G(t)x = u(t)$, we have that the family of mapping $\mathcal{T} = \{G(t) : 0 \leq t < \infty\}$ of $\overline{D(A)}$ onto itself is a one-parameter nonexpansive semigroup on $\overline{D(A)}$. Moreover, we know that from [25] that $A^{-1}0 = F(\mathcal{T})$. So, we can apply our Theorem 3.2.5 to find zero of A with $H = \overline{D(A)}$. Then the method (3.2.1) has the form $x_1 \in H$,

$$\begin{cases} y_n = (1 - \alpha_n) x_n, \\ x_{n+1} = (1 - \beta_n) x_n + \beta_n \left(\frac{1}{t_n}\int_0^{t_n} G(s) y_n ds\right), \ n \geq 1. \end{cases}$$

3.3 Common solution of generalized mixed equilibrium problem and Bregman strongly nonexpansive mapping in reflexive Banach spaces

Let C be a nonempty, closed and convex subset of a real reflexive Banach space$X$ and let $\phi : C \to \mathbb{R}$ be a lower semi-continuous and convex function and $\psi : C \to X^*$ be continuous monotone mapping. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1)-(A4)$. Then the mixed resolvent of Θ is the operator $Res^f_{\Theta,\phi,\psi} : X \to 2^C$ defined by

$$\begin{aligned} Res^f_{\Theta,\phi,\psi} &= \{z \in C : \Theta(z,y) + \psi(y) + \langle \phi x, y-z\rangle \\ &\quad + \langle \nabla f(z) - \nabla f(x), y-z\rangle \geq \phi(z)\ \forall\ y \in C\}. \end{aligned}$$

Note that if $f : X \to (-\infty,+\infty]$ is coercive and Gâteaux differentiable function, then the mixed resolvent Θ satisfies dom$Res^f_{\Theta,\phi,\psi} = X$ (see [68] lemma 4.14).

Lemma 3.3.1. *(see [68], Lemma 2.15): Let $f : X \to (-\infty,+\infty]$ be a Legendre function. Let C be a nonempty, closed and convex subset of X. If the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the conditions $(A1)-(A4)$, then*

(i) $Res^f_{\Theta,\phi,\psi}$ is single-valued;

(ii) $Res^f_{\Theta,\phi,\psi}$ is a BFNE operator;

(iii) the set of fixed points of $Res^f_{\Theta,\phi,\psi}$ is a solution set of the corresponding generalized mixed equilibrium problem, i.e $F(Res^f_{\Theta,\phi,\psi}) = GMEP(\Theta,\phi,\psi)$;

(iv) $GMEP(\Theta,\phi,\psi)$ is closed and convex subset of C;

(v) for all $x \in X$ and $u \in F(Res^f_{\Theta,\phi,\psi})$, we have

$$D_f(u, Res^f_{\Theta,\phi,\psi}(x)) + D_f(Res^f_{\Theta,\phi,\psi}(x), x) \leq D_f(u,x).$$

In this section, we study the problem of finding a common solution of generalized mixed equilibrium problem and bregman strongly nonexpansive mapping which is stated as: Let X^* be the dual of a reflexive Banach space X, and C_i, $i = 1,2,...,N$ be nonempty, closed and convex subset of X. Let $T_i : C_i \to C_i$, $i = 1,2,...N$ be N countable family of Bregman strongly nonexpansive mappings such that $F(T_i) = \hat{F}(T_i)$, $i = 1,2,...N$. Let $\Theta_i : C_i \times C_i \to \mathbb{R}$ be a bifunction satisfying $A(1)-(A4)$, $\phi : C_i \to \mathbb{R}$ be lower semi-continuous and convex functions and $\psi_i : C_i \to X^*$ be continuous monotone mappings, for all $i = 1,2,...N$. Find $x^* \in \cap_{i=1}^N F(T_i)$ such that

$$x^* \in \cap_{i=1}^N GMEP(\Theta_i,\phi_i,\psi_i). \tag{3.3.1}$$

We denote the solution set of (3.3.1) by Υ. i.e,

$$\Upsilon := \{x^* \in \cap_{i=1}^N F(T_i) : x^* \in \cap_{i=1}^N GMEP(\Theta_i,\phi_i,\psi_i)\}$$

Furthermore we propose an iterative algorithm and using the algorithm, we state and prove a strong convergence result for approximation of a solution of problem (3.3.1) in a real reflexive Banach space. Our theorem takes into account possible computational errors.

Theorem 3.3.2. *Let X be a reflexive Banach space and $C_i,\ i = 1, 2, ..., N$ be nonempty, closed and convex subsets of X. Let $T_i : C_i \to C_i,\ i = 1, 2, ..., N$ be countable families of Bregman strongly nonexpansive mappings such that $F(T_i) = \hat{F}(T_i),\ i = 1, 2, ..., N$. Let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subsets of X such that $C_i \subset int(domf)$. Let $\Theta_i : C_i \times C_i \to \mathbb{R}$ satisfy conditions $(A1) - (A4)$, let $\phi_i : C_i \to \mathbb{R}$ be lower semi-continuous and convex functions and $\psi_i : C_i \to X^*$ be continuous monotone mappings, for all $i = 1, 2, ..., N$. Assume that $\Upsilon \neq \emptyset$. Let $u, x_0 \in C_i$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n^i = \nabla f^* \left(\alpha_n \nabla f(u) + \beta_n \nabla f(x_n + e_n^i) + \gamma_n \nabla f(T_i(x_n + e_n^i))\right),\ i = 1, 2, ...N; \\ u_n^i = Res^f_{\Theta_i,\phi_i,\psi_i} y_n^i; \\ C_{n+1} = \{z \in C_n : \sup_{i\geq 1} D_f(z, u_n^i) \leq \alpha_n D_f(z, u) + D_f(z, x_n + e_n^i)\}; \\ x_{n+1} = Proj^f_{C_{n+1}}(x_0),\ n \geq 0, \end{cases} \tag{3.3.2}$$

satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0,\ \alpha_n + \beta_n + \gamma_n = 1$ *and* $0 < a < \beta_n, \gamma_n < b < 1$, *for some* $a, b > 0$,

(ii) $\lim_{n\to\infty} e_n^i = 0$.

Then $\{x_n\}$ *converges strongly to* $Proj^f_{\Upsilon} x_0$.

Proof. We first show that $\Upsilon \subset C_n\ \forall n \geq 0$.
Since T_i is Bregman strongly nonexpansive, we have from ([176] Lemma 15.5) that $F(T_i)$ is closed and convex for each $i = 1, 2, ..., N$. Also, from Lemma 2.6 we have that $F(Res^f_{\Theta_i,\phi_i,\psi_i}) = GMEP(\Theta_i, \phi_i, \psi_i)$ is nonempty, closed and convex set. So we have Υ is nonempty closed and convex set. Thus $Proj^f_{\Upsilon}$ is well defined. Also, we know that $\Upsilon \subset C = C_0$. Now, suppose that $\Upsilon \subset C_k$ for some $k \in \mathbb{R}$ and let $p \in \Upsilon$. Then

$$\begin{aligned} D_f(p, u_k^i) &= D_f(p, Res^f_{\Theta_i,\phi_i,\psi_i} y_k^i) \\ &\leq D(p, y_k^i) \\ &= D_f(p, \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(x_k + e_k^i) + \gamma_n \nabla f(T_i(x_k + e_k^i)) \\ &\leq \alpha_n D_f(p, u) + \beta_n D_f(p, x_k + e_k^i) + \gamma_n D_f(p, T_i(x_k + e_k^i)) \\ &\leq \alpha_n D_f(p, u) + \beta_n D_f(p, x_k + e_k^i) + \gamma_n D_f(p, x_k + e_k^i) \\ &= \alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, x_k + e_k^i) \\ &\leq \alpha_n D_f(p, u) + D_f(p, x_k + e_k^i). \end{aligned} \tag{3.3.3}$$

Hence $p \in C_{k+1}$. By induction we can conclude that $\Upsilon \subset C_n\ \forall n \geq 0$.
Next, we show that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists.

Let $p \in \Upsilon$, then from (3.3.2), we have that $x_n = Proj^f_{C_n}(x_0)$ and $x_{n+1} = Proj^f_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$. Also, by proposition 2.3.7(iii), we have that

$$\begin{aligned} D_f(x_n, x_0) &= D_f\left(Proj^f_{C_n}(x_0), x_0\right) \\ &\leq D_f(p, x_0) - D_f(p, Proj^f_{C_n}(x_0)) \leq D_f(p, x_0). \end{aligned} \tag{3.3.4}$$

Therefore, the sequence $\{D_f(x_n, x_0)\}$ is bounded. From Lemma 2.3.8 we conclude that $\{x_n\}$ is bounded too.
We also have from proposition 2.3.7(iii) that

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0). \tag{3.3.5}$$

Hence the sequence $\{D_f(x_n, x_0)\}$ is increasing. Therefore, $\lim\limits_{n\to\infty} D_f(x_n, x_0)$ exists. Therefore from (3.3.5), we have that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.3.6}$$

Next, we show that the sequence $\{x_n\}$ is Cauchy in C.
Since $x_m = Proj^f_{C_m}(x_0) \in C_m \subset C_n$ for $m > n$, we have from proposition 2.3.7(iii) that

$$\begin{aligned} D_f(x_m, x_n) &= D_f\left(x_m, Proj^f_{C_n}(x_0)\right) \\ &\leq D_f(x_m, x_0) - D_f\left(Proj^f_{C_n}(x_0), x_0\right) \\ &= D_f(x_m, x_0) - D_f(x_n, x_0), \end{aligned} \tag{3.3.7}$$

which implies that $D_f(x_m, x_n) \to 0$, as $n, m \to \infty$. Since f is totally convex on a bounded subset of X, we have from Lemma 2.3.10 that f is sequentially consistent. That is $\|x_m - x_n\| \to 0$, as $n, m \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence.
Since X is complete and C is closed, there exists $x^* \in C$ such that $\{x_n\}$ converges to x^*. Thus, we have from condition (ii) that $\{(x_n + e^i_n)\}$ converges to x^*.
Next, we show that $\lim_{n\to\infty} \|\nabla f(u^i_n) - \nabla f(x_n + e^i_n)\| = 0,\ i = 1, 2, ..., N$. From Lemma 2.3.10, we have from (3.3.6) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3.8}$$

For any $i = 1, 2, ..., N$, it follows from the definition of Bregman distance that

$$\begin{aligned} D_f(x_n, x_n + e^i_n) &= f(x_n) - f(x_n + e^i_n) - \langle \nabla f(x_n + e^i_n), x_n - (x_n + e^i_n)\rangle \\ &= f(x_n) - f(x_n + e^i_n) + \langle \nabla f(x_n + e^i_n), e^i_n\rangle. \end{aligned} \tag{3.3.9}$$

Since f is bounded on a bounded subsets of X, we have that ∇f is also bounded on a bounded subsets of X. Also, since f is uniformly Frechet differentiable, we have that f is uniformly continuous on a bounded subsets of X. From condition (ii) we have that

$$\lim_{n\to\infty} D_f(x_n, x_n + e^i_n) = 0. \tag{3.3.10}$$

For each $i = 1, 2, ..., N$, we have from the three point identity that

$$\begin{aligned} D_f(x_{n+1}, x_n + e_n^i) &= D_f(x_{n+1}, x_n) + D_f(x_n, x_n + e_n^i) \\ &\quad + \langle \nabla f(x_n) - \nabla f(x_n + e_n^i), x_{n+1} - x_n \rangle. \end{aligned} \tag{3.3.11}$$

From (3.3.6), (3.3.8) and (3.3.10), we have that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n + e_n^i) = 0. \tag{3.3.12}$$

Since $x_{n+1} = Proj_{C_{n+1}} x_0$, then we have from (3.2.3) and (3.3.12), that

$$D_f(x_{n+1}, u_n^i) \leq \alpha_n D_f(x_{n+1}, u) + D_f(x_{n+1}, x_n + e_n^i) \to 0, \ as\ n \to \infty.$$

That is

$$\lim_{n\to\infty} D_f(x_{n+1}, u_n^i) = 0. \tag{3.3.13}$$

From (3.3.12), (3.3.13) and by using Lemma 2.3.9, we have,

$$\lim_{n\to\infty} \|u_n^i - (x_n + e_n^i)\| = 0, \ i = 1, 2, ..., N. \tag{3.3.14}$$

From (3.2.16) and by Lemma 2.3.11, we have that

$$\lim_{n\to\infty} \|\nabla f(u_n^i) - \nabla f(x_n + e_n^i)\| = 0, \ i = 1, 2, ..., N. \tag{3.3.15}$$

Next, we show that that $x^* \in \cap_{i=1}^N F(T_i)$.
From (3.2.3), we have that

$$\begin{aligned} D_f(p, y_n^i) &= D_f(p, \nabla f^*(\alpha_n \nabla f(u) + \beta_n \nabla f(x_n + e_n^i) + \gamma_n \nabla f(T_i(x_n + e_n^i))) \\ &\leq \alpha_n D_f(p, u) + \beta_n D_f(p, x_n + e_n^i) + \gamma_n D_f(p, T_i(x_n + e_n^i)) \\ &\leq \alpha_n D_f(p, u) + \beta_n D_f(p, x_n + e_n^i) + \gamma_n D_f(p, x_n + e_n^i) \\ &= \alpha_n D_f(p, u) + (1 - \alpha_n) D_f(p, x_n + e_n^i) \\ &\leq \alpha_n D_f(p, u) + D_f(p, x_n + e_n^i). \end{aligned} \tag{3.3.16}$$

From Lemma 3.3.1(v) we have that

$$\begin{aligned} D_f(p, u_n^i) + D_f(u_n^i, y_n^i) &= D_f\left(p, Res^f_{\Theta_i,\phi_i,\psi_i} y_n^i\right) + D_f\left(Res^f_{\Theta_i,\phi_i,\psi_i} y_n^i, y_n^i\right) \\ &\leq D_f(p, y_n^i). \end{aligned} \tag{3.3.17}$$

From (3.3.16) and (3.3.17), we have

$$D_f(u_n^i, y_n^i) \leq D_f(p, y_n^i) - D_f(p, u_n^i) \leq \alpha_n D_f(p, u) + D_f(p, x_n + e_n^i) - D_f(p, u_n^i). \tag{3.3.18}$$

From (3.3.14),(3.3.15) and the three point identity, we have

$$\lim_{n\to\infty} D_f(u_n^i, y_n^i) = 0, \ i = 1, 2, ..., N. \tag{3.3.19}$$

From Lemma 2.3.9, we have

$$\lim_{n\to\infty}\|u_n^i - y_n^i\| = 0,\ \ i = 1, 2, ..., N. \tag{3.3.20}$$

By (3.3.20) and Lemma 2.3.11, we have that

$$\lim_{n\to\infty}\|\nabla f(u_n^i) - \nabla f(y_n^i)\| = 0,\ \ i = 1, 2, ..., N. \tag{3.3.21}$$

From (3.3.15) and (3.3.20), we obtain

$$\lim_{n\to\infty}\|(x_n + e_n^i) - y_n^i\| = 0,\ \ i = 1, 2, ..., N. \tag{3.3.22}$$

From Lemma 2.3.11, we have

$$\lim_{n\to\infty}\|\nabla f(x_n + e_n^i) - \nabla f(y_n^i)\| = 0,\ \ i = 1, 2, ..., N. \tag{3.3.23}$$

Let $w_n^i = \nabla f^*\left(\frac{\beta_n}{1-\alpha_n}\nabla f(x_n + e_n^i) + \frac{\gamma_n}{1-\alpha_n}\nabla f(T_i(x_n + e_n^i))\right)$, then

$$\begin{aligned}\|\nabla f(y_n^i) - \nabla f(w_n^i)\| &= \|\alpha_n \nabla f(u) - \frac{\alpha_n\beta_n}{1-\alpha_n}\nabla f(x_n + e_n^i) - \frac{\alpha_n\gamma_n}{1-\alpha_n}\nabla f(T_i(x_n + e_n^i))\| \\ &= \alpha_n\|\nabla f(u) - \frac{\beta_n}{1-\alpha_n}\nabla f(x_n + e_n^i) \\ &\quad -\frac{\gamma_n}{1-\alpha_n}\nabla f(T_i(x_n + e_n^i))\|. \end{aligned} \tag{3.3.24}$$

Since $\alpha_n \to 0$, as $n \to \infty$, we have from (3.3.24) that

$$\lim_{n\to\infty}\|\nabla f(y_n^i) - \nabla f(w_n^i)\| = 0,\ \ i = 1, 2, ..., N. \tag{3.3.25}$$

From (3.3.12) and (3.3.25), we have that

$$\lim_{n\to\infty}\|\nabla f(w_n^i) - \nabla f(x_n + e_n^i)\| = 0,\ \ i = 1, 2, ..., N. \tag{3.3.26}$$

So we have that

$$\begin{aligned}0 = \lim_{n\to\infty}\|\nabla f(w_n^i) - \nabla f(x_n + e_n^i)\| &= \lim_{n\to\infty}\|\frac{\beta_n}{1-\alpha_n}\nabla f(x_n + e_n^i) \\ &\quad +\frac{\gamma_n}{1-\alpha_n}\nabla f(T_i(x_n + e_n^i)) - \nabla f(x_n + e_n^i)\| \\ &= \lim_{n\to\infty}\left(\frac{\gamma_n}{1-\alpha_n}\|\nabla f(T_i(x_n + e_n^i)) \right. \\ &\quad \left. -\nabla f(x_n + e_n^i)\|\right). \end{aligned} \tag{3.3.27}$$

By condition (i), we have

$$\lim_{n\to\infty}\|\nabla f(T_i(x_n + e_n^i)) - \nabla f(x_n + e_n^i)\| = 0,\ \ i = 1, 2, ..N. \tag{3.3.28}$$

Since f is strongly coercive and uniformly convex and bounded subsets of X, we have that f^* is uniformly Fretchet differentiable on bounded subsets of X^* (see [191] Proposition 3.6.2). Also, f is Legendre, thus by Lemma 2.3.11, we have

$$\lim_{n\to\infty} \|T_i(x_n + e_n^i) - (x_n + e_n^i)\| = 0, \;\; i = 1, 2, ..., N. \tag{3.3.29}$$

From (3.3.29), we have that $x^* \in \hat{F}(T_i) = F(T_i)$, $i = 1, 2, ..., N$. Therefore, $x^* \in \cap_{i=1}^N F(T_i)$. Next, we show that $x^* \in \cap_{i=1}^N GMEP(\Theta_i, \phi_i, \psi_i)$.
For each $p \in \Upsilon$ and for $i = 1, 2, ..., N$, we have

$$\begin{aligned} D_f(p, y_n^i) - D_f(p, u_n^i) &= f(u_n^i) - f(y_n^i) \\ &\quad + \langle \nabla f(u_n^i) - \nabla f(y_n^i), p - y_n^i \rangle + \langle \nabla f(u_n^i), y_n^i - u_n^i \rangle. \end{aligned}$$

Since $\{u_n^i\}$ is bounded, $\{\nabla f(u_n^i)\}$ is bounded too. Therefore, from (3.3.20) and (3.3.21), we get

$$\lim_{n\to\infty} \left(D_f(p, y_n^i) - D_f(p, u_n^i) \right) = 0.$$

That is,

$$\lim_{n\to\infty} \Big(D_f(p, y_n^i) - D_f(p, Res^f_{\Theta_i,\phi_i,\psi_i} y_n^i) \Big) = 0,$$

for any $p \in \Upsilon$ and for $i = 1, 2, ..., N$. Since $Res^f_{\Theta_i,\phi_i,\psi_i}$ is BSNE operator and $F(Res^f_{\Theta_i,\phi_i,\psi_i}) = \hat{F}(Res^f_{\Theta_i,\phi_i,\psi_i})$, we have

$$\lim_{n\to\infty} D_f(Res^f_{\Theta_i,\phi_i,\psi_i} y_n^i, y_n^i) = 0, \;\; i = 1, 2, ..., N. \tag{3.3.30}$$

Also, from (3.2.26) and (3.3.30), we have

$$\lim_{n\to\infty} D_f(Res^f_{\Theta_i,\phi_i,\psi_i} y_n^i, (x_n + e_n^i)) = 0, \;\; i = 1, 2, ..., N. \tag{3.3.31}$$

Thus, if $\{y_{n_k}\}$ is any subsequence of $\{y_n\}$ which converges weakly to x^*, then $x^* \in \hat{F}(Res^f_{\Theta_i,\phi_i,\psi_i}) = F(Res^f_{\Theta_i,\phi_i,\psi_i})$ for $i = 1, 2, ...N$. Hence $x^* \in \cap_{i=1}^N GMEP(\Theta_i, \phi_i, \psi_i)$.
Next, we show that $x^* = Proj^f_{\Upsilon} x_0$.
Since $x_n = Proj^f_{C_n} x_0$, we have from Proposition 2.3.7 that

$$\langle \nabla f(x_0) - \nabla f(x_n), x_n - z \rangle \geq 0 \; \forall z \in C_n. \tag{3.3.32}$$

In particular, we have

$$\langle \nabla f(x_0) - \nabla f(x_n), x_n - \bar{x} \rangle \geq 0 \; \forall \bar{x} \in \Upsilon \subset C_n. \tag{3.3.33}$$

Letting $n \to \infty$ in (3.3.33), we obtain

$$\langle \nabla f(x_0) - \nabla f(x^*), x^* - \bar{x} \rangle \geq 0 \; \forall \; \bar{x} \in \Upsilon. \tag{3.3.34}$$

From Proposition 2.3.7 and (3.3.34), we have that $x^* = Proj^f_{\Upsilon} x_0$.
Finally we show that $\{x_n\}$ converges strongly to $Proj^f_{\Upsilon} x_0$.
Since $x^* = Proj^f_{\Upsilon}(x_0)$, $x_{n+1} = Proj^f_{C_{n+1}}(x_0)$ and $\Upsilon \subset C_n$, we have

$$D_f(x_{n+1}, x_0) \leq D_f(Proj^f_{C_{n+1}}(x_0), x_0) = D_f(x^*, x_0).$$

Hence, by Lemma 2.3.9, we obtain that $\{x_n\}$ converges strongly to $x^* = Proj^f_{\Upsilon} x_0$, as $n \to \infty$.

$\square$

This completes the proof.

Remark 3.3.3. *If X is a smooth, strictly convex and reflexive Banach space and $f(x) = \frac{1}{2}\|x\|^2$, then the algorithm (3.2.3) reduces to*

$$\begin{cases} y_n^i = J^{-1}\left(\alpha_n J(u) + \beta_n J(x_n + e_n^i) + \gamma_n J(T_i(x_n + e_n^i))\right), \ i = 1, 2, ...N; \\ u_n^i = Res^f_{\Theta_i,\phi_i,\psi_i} y_n^i; \\ C_{n+1} = \{z \in C_n : \sup_{i\geq 1} \varphi(z, u_n^i) \leq \alpha_n \varphi(z, u) + \varphi(z, x_n + e_n^i)\}; \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \ n \geq 0, \end{cases} \tag{3.3.35}$$

where J is the normalized duality mapping from X into 2^{X^*}, $\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, for all $x, y \in X$, and the sequence $\{x_n\}$ converges strongly to $\Pi_\Upsilon(x_0)$ which is its generalized projection from X onto Υ.

If in Theorem 3.3.2 $\phi_i = 0$ and $\psi_i = 0$ for all $i = 1, 2, ...N$, then for the operator $Res^f_{\Theta_i,\phi_i,\psi_i}$(denoted by $Res^f_{\Theta_i}$), we have the following corollary:

Corollary 3.3.4. *Let X be a reflexive Banach space and C_i, $i = 1, 2, ..., N$ be nonempty, closed and convex subsets of X. Let $T_i : C_i \to C_i$, $i = 1, 2, ..., N$ be countable families of Bregman strongly nonexpansive mappings such that $F(T_i) = \hat{F}(T_i)$, $i = 1, 2, ..., N$. Let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X with $C_i \subset int(domf)$. Let $\Theta_i : C_i \times C_i \to \mathbb{R}$ satisfy conditions $(A1) - (A4)$, $i = 1, 2, ..., N$. Assume that $\Gamma = \{x^* \in \cap_{i=1}^N F(T_i) : x^* \in \cap_{i=1}^N EP(\Theta_i)\} \neq \emptyset$. Let $u, x_0 \in C_i$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n^i = \nabla f^*\left(\alpha_n \nabla f(u) + \beta_n \nabla f(x_n + e_n^i) + \gamma_n \nabla f(T_i(x_n + e_n^i))\right), \ i = 1, 2, ...N; \\ u_n^i = Res^f_{\Theta_i} y_n^i; \\ C_{n+1} = \{z \in C_n : \sup_{i\geq 1} D_f(z, u_n^i) \leq \alpha_n D_f(z, u) + D_f(z, x_n + e_n^i)\}; \\ x_{n+1} = Proj^f_{C_{n+1}}(x_0), \ n \geq 0, \end{cases} \tag{3.3.36}$$

with $\{\alpha\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ $\{\gamma_n\}_{n=1}^\infty$ and $\{e_n^i\}$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\alpha_n + \beta_n + \gamma_n = 1$ *and* $0 < a < \beta_n, \gamma_n, \alpha_n < b < 1$, *for some* $a, b > 0$,

(ii) $\lim_{n\to\infty} e_n^i = 0$.

Then $\{x_n\}$ converges strongly to $Proj^f_\Gamma x_0$.

3.3.1 Applications to zeroes of Bregman inversely strongly monotone operators

Let the Legendre function f be such that

$$\operatorname{ran}(\nabla f - A) \subseteq \operatorname{ran}(\nabla f). \tag{3.3.37}$$

The operator $A : X \to 2^{X^*}$ is called Bregman inversely strongly monotone (BISM) if

$$(\mathrm{dom}A) \cap (\mathrm{int}(\mathrm{dom})f) \neq \emptyset$$

and for any $x, y \in$ int(dom)f, and each $\xi \in Ax$, $\eta \in Ay$, we have

$$\langle \xi - \eta, (\nabla f(x) - \xi) - \nabla f^*(\nabla f(y) - \eta)\rangle \geq 0.$$

The class of operators was introduced by Butnaru and Kassey (see [38]). For any operator $A : X \to 2^{X^*}$, the anti resolvent $A^f : X \to 2^{X^*}$ of A is defined by

$$A^f := \nabla f^* \circ (\nabla f - A).$$

Observe that dom$A^f \subset (\mathrm{dom}A) \cap$(int(dom)f and ran$A^f \subseteq \mathrm{inf}(\mathrm{dom})f$. The operator A (see [38]) is BISM if and only if the anti-resolvent A^f is single-valued BFNE operator which can be seen in [38]. From the definition of anti-resolvent and ([38] Lemma3.5) we obtain the following proposition.

Proposition 3.3.5. *(see [176]) Let $f : X \to (-\infty, +\infty]$ be Legendre function and let $A : X \to 2^{X^*}$ be a BISM operator such that $A^{-1}(0)^* \neq \emptyset$. Then the following holds:*

(i) $A^{-1}(0)^* = F(A^f)$,

(ii) for any $u \in A^{-1}(0)^*$ and $x \in domA^f$, we have

$$D_f(u, A^f) + D_f(A^f x, x) \leq D_f(u, x).$$

So, if the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subset of X, then the resolvent A^f of A is a single-valued BSNE operator which satisfies

$$F(A^f) = \hat{F}(A^f).$$

In Theorem 3.3.2 if we let $T_i = A_i^f$ and f be the Legendre function such that (3.3.37) is satisfied, then we obtain the strong convergence result for approximating a common solutions of zeroes of a finite family of Bregman inversely strongly monotone operator and a common solution to a finite system of equilibrium problems.

Chapter 4

Monotone Inclusion, Minimization and Fixed Point Problems

4.1 Introduction

Let $T : C \to C$ be a mapping. Then T is said to be:

(i) accretive if for all $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq 0,$$

where J is the normalized duality mapping;

(ii) α- inverse strongly accretive if for all $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \alpha\|Tx - Ty\|^2,$$

for some $\alpha > 0$;

(iii) m-accretive if T is accretive and $R(I + \lambda T) = X$, $\forall\ \lambda > 0$;

(iv) strongly positive if X is a real Banach space and there exists $\bar{\gamma} > 0$ such that

$$\langle Tx, jx\rangle \geq \bar{\gamma}\|x\|^2, \forall\ x \in C.$$

Let C be a convex subset of X, let K be a nonempty subset of C and let p be a retraction from C onto K, i.e, $Px = x$ for each $x \in K$. P is said to be sunny if $P(Px+t(x-Px)) = Px$ for each $x \in C$. and $t \geq 0$ with $Px + t(x - Px) \in C$. If there is a sunny nonexpansive retraction from C onto K, K is said to be a sunny nonexpansive retract of C.

A set valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$,with $u \in M(x)$ and $v \in M(y)$ then

$$\langle x - y, u - v\rangle \geq 0.$$

A monotone mapping M is said to be *maximal* if the graph of M, denoted as $G(M)$, is not properly contained in the graph of any other monotone mapping, where for multivalued mapping M,

$$G(M) = \{(x, y) : y \in M(x)\}.$$

It is well known that M is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in M(x)$. The resolvent operator J_λ^M associated with M and λ is the mapping $J_\lambda^M : H \to H$ defined by

$$J_\lambda^M(x) = (I + \lambda M)^{-1}x, \quad x \in H, \lambda > 0. \tag{4.1.1}$$

It is known that the resolvent operator $J_\lambda^M(x)$ is single valued, nonexpansive and 1-inverse strongly monotone (see, for example, [25]). The inverse-strongly monotone (also referred to as co-coercive) operators have been widely used to solve practical problem in various fields, for instance, in traffic assignment problems (see for example, [15, 85] and the references therein). It can easily be seen that (i) if T is nonexpansive, then $I - T$ is monotone; (ii) the projection mapping P_C is 1-ism.

A fundamental problem is to find a zero of a maximal monotone operator $T : H \to 2^H$ in real Hilbert space H

$$\text{find} \quad x \in H : \quad 0 \in Tx. \tag{4.1.2}$$

This problem includes, as special cases, variational inequality problems, non-smooth convex optimization problems and convex-concave saddle-point problems. Therefore this problem finds many important applications in scientific field such as image processing, computer vision, machine learning and signal processing. It is known that the solution of (4.1.2) is a fixed point of $J_r^T, \ \forall r > 0$ and the set of zeroes $T^{-1}(0) := \{x \in H : 0 \in Tx\}$ is closed and convex (see, for example, [103]).

In case, T is a general monotone operator, the classical algorithm to solve (4.1.2) is the proximal point algorithm which can be traced back to the early works of Minty [122] and Martinet [117]. See also the thesis of Eckstein [78] for detailed treatment of the subject.

The proximal point algorithm generates a sequence x_n according to the recursion

$$x_{n+1} = (I + r_nT)^{-1}(x_n) \tag{4.1.3}$$

where $r_n > 0$ is a regularization parameter. The operator $J_{r_n}^T := (I + r_nT)^{-1}$ is the so-called resolvent operator, that has been introduced by Moreau [124]. In the context of algorithms, the resolvent operator is often referred to as the backward operator. Rockafellar [149] has proved that the sequence x_n generated by the proximal point algorithm (4.1.3) converges weakly to a point x^* satisfying $0 \in Tx^*$.

In many problems, however, the operator T can be written as the sum of two maximal monotone operators, i.e., $T = A + B$, such that the resolvent operators $(I + rA)^{-1}$ and $(I + rB)^{-1}$ are much easier to compute than the full resolvent $(I + \lambda T)^{-1}$. Then, by combining the resolvents with respect to A and B in a certain way, one might be able to mimic the effect of the full proximal step based on T. The two most successful instances that are based on combining forward and backward steps with respect to A and B, are the Peaceman-Rachford splitting algorithm [142],

$$x_{n+1} = (I + rB)^{-1}(I - rA)(I + rA)^{-1}(I - rB)(x_n),$$

and the Douglas-Rachford splitting algorithm [73],

$$x_{n+1} = (I + rB)^{-1}[(I + rA)^{-1}(I - rB) + rB](x_n).$$

These splitting techniques have been originally proposed in the context of linear operators and therefore cannot be applied to general monotone operators. In [107], Lion and Mercier have analysed and further developed these splitting algorithms. Their idea was to perform a change of variables $x_n = (I + rB)^{-1}(v_n)$, such that the Peaceman-Rachford and Douglas-Rachford splitting algorithms have meaning even for A and B being multivalued operators. Regarding convergence of the algorithms, the Peaceman-Rachford algorithm still needs to assume that B is single-valued but the Douglas-Rachford algorithm converges even in the general setting, where $A + B$ is just maximal monotone.

In [76], Eckstein has pointed out that the Douglas-Rachford splitting algorithm can be rewritten in the form of (4.1.3). Hence, it is basically a certain instance of the proximal point algorithm. Moreover, Eckstein has shown that the application of the Douglas-Rachford algorithm to the dual of a certain structured convex optimization problem coincides with the so-called alternating direction method of multipliers. It is remarkable, that the Douglas-Rachford splitting algorithm and its variants have seen a considerable renaissance in the modern convex optimization [23, 81]. The main reason for the renewed interest lies in the fact that it is well suited for distributed convex programming. This is an important aspect for solving large scale convex optimization problems arising in recent image processing and machine learning applications.

Another important line of splitting methods is given by the so-called forward-backward splitting technique [29, 82, 104, 107]. In contrast to the more complicated splitting technique discussed above, the forward-backward scheme is based (as the name suggests) on the recursive application of an explicit forward step with respect to B, followed by an implicit backward step with respect to A. The forward-backward algorithm is written as:

$$x_{n+1} = (I + r_n A)^{-1}(I - r_n B)(x_n). \tag{4.1.4}$$

In the most general setting, where both A and B are general monotone operators, the convergence result is rather weak [141], basically, r_n has to fulfil the same step-size restrictions as unconstrained subgradient descend schemes. However, if in addition B is single valued and Lipschtz, e.g., B is the gradient of a smooth convex function, the situation becomes much more beneficial. In fact, if B is L- Lipschitz, and r_n is chosen such that $r_n < \frac{2}{L}$, the forward- backward algorithm (4.1.4) converges to zero of $T = A + B$ [80, 171]. Similar to the Douglas-Rachford splitting algorithm, the forward-backward algorithm has seen a renewed interest. It has been proposed and further improved in the context of sparse

signal recovery [66, 67], image processing [148], and machine learning [74] applications. Recently, modifications of forward- backward algorithm (4.1.4) have been proposed and studied in [102] and [62] and strong convergence results obtained.
Recently, Wei and Duan [184] presented the following iterative algorithm with errors in a real smooth and uniformly convex Banach space:

$$\begin{cases} x_0 \in C, \\ y_n = Q_C[(1-\alpha_n)(x_n + e_n)], \\ z_n = (1-\beta_n)x_n + \beta_n[a_0 y_n + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(y_n - r_{n,i}B_i y_n)], \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)z_n, \quad n \geq 0, \end{cases} \tag{4.1.5}$$

where C is a nonempty, closed and convex sunny nonexpansive retract of X, Q_C is the sunny nonexpansive retraction of X onto C, $\{e_n\} \subset X$ is the error sequence, $\{A_i\}_{i=1}^N$ is finite family of m-accretive operators and $\{B_i\}_{i=1}^N$ is a finite family of α-inverse strongly accretive operators. $T : X \to X$ is a strongly positive bounded linear operator with coefficient $\bar{\gamma}$ and $f : X \to X$ is a contraction with coefficient $k \in (0,1)$. $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$, for $i = 1, 2, ..., N$, $\sum_{i=0}^N a_i = 1$, $0 < a_i < 1$, for $i = 0, 1, 2, ..., N$. Then $\{x_n\}$ converges strongly to $p_0 \in \cap_{i=1}^N (A_i + B_i)^{-1}0$, which is also a solution of some variational inequality problem.

In this chapter, we study the common fixed point of infinite family of demicontractive mappings which is also a solution to the sum of two monotone inclusion problems. We also study a countable family of accretive operators with Meir-Keeler contraction mappings in Banach space and finally work on modified gradient projection algorithm studied by Shehu and Gang Cai in [46].

4.2 Strong convergent result for monotone inclusions and fixed point problem in Hilbert spaces

In this section, we introduce an iterative algorithm for finding a common fixed point of an infinite family of demicontractive mappings which is also a solution to a monotone inclusion (4.1.2), where $T = A + B$ with A being a maximal monotone operator and B an α-inverse strongly monotone, in a real Hilbert space and prove strong convergence of the sequence generated by our scheme. Our contribution lies in the fact that our iterative method solves monotone inclusion problem of sum of two monotone operators and fixed point problem of infinite family of demicontractive mappings at the same time.

For each $i \geq 1$, let $S_i : H \to H$ be a k-demicontractive mapping. Let $A : H \to 2^H$ be maximal monotone operator and $B : H \to H$ be an α-inverse strongly monotone mapping. Suppose $\Gamma := \cap_{i=1}^{\infty} F(S_i) \cap (A+B)^{-1}(0) \neq \emptyset$. We study the following iteration method in this paper: given $x_1 \in H$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated by

$$\begin{cases} w_n = \alpha_n x_1 + (1-\alpha_n)x_n, \\ z_n = (I + r_n A)^{-1}(w_n - r_n B w_n), \\ x_{n+1} = z_n + \sum_{i=1}^{\infty} \beta_{n,i} \eta \beta (S_i - I) z_n, \end{cases} \tag{4.2.1}$$

where $\{\alpha_n\}, \{\beta_{n,i}\}$ are in $(0,1)$ and $\eta \in (0, \frac{1-k}{\beta})$. We prove that the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (4.2.1) converges strongly to a point in Γ. To this end, we enumerate the main assumptions used for the rest of the paper:

(H1) $\sum_{i=1}^{\infty} \beta_{n,i} = 1$, and $\liminf\limits_{n\to\infty} \beta_{n,i} > 0$ for each $i \in \mathbb{N}$,

(H2) $\sum\limits_{n\geq 1} \alpha_n = +\infty$ and $\alpha_n \to 0$ as $n \to \infty$,

(H3) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1 - k$ where $k \in [0,1)$,

(H4) $0 < \liminf\limits_{n\to\infty} r_n \leq \limsup\limits_{n\to\infty} r_n < 2\alpha$.

Remark 4.2.1. *We observe that if $S_i = I, \ \ \forall i \geq 1$ in (4.2.1) above, then our iterative method (4.2.1) reduces to the method studied in [102] and [62]. Therefore, our iterative method (4.2.1) is more applicable than the methods studied in [102] and [62] since our method solves monotone inclusion problem and fixed point problem.*

Theorem 4.2.2. *For each $i \in \mathbb{N}$, let $S_i : H \to H$ be a k-demicontractive mapping. Let $A : H \to 2^H$ be a maximal monotone operator and $B : H \to H$ be an α-inverse strongly monotone. Suppose that $\Gamma \neq \emptyset$ and $I - T$ is demiclosed at the origin. Assume that conditions (H1)-(H4) hold. Then the sequence $\{x_n\}$ generated by (4.2.1) converges strongly to $q \in \Gamma$, where $q = P_\Gamma x_1$.*

Proof. We first show that $\{x_n\}$ is bounded. Let $q = P_\Gamma x_1$ and define $T_n := J_{r_n}^A(I - r_n B), \ \ \forall n \geq 1$. Then, for all $x, y \in H$, we have

$$\begin{aligned}
||T_n x - T_n y||^2 &\leq ||J_{r_n}^A(x - r_n Bx) - J_{r_n}^A(y - r_n By)||^2 \\
&\leq ||x - y - r_n(Bx - By)||^2 \\
&= ||x - y||^2 - 2r_n\langle Bx - By, x - y\rangle + r_n^2||Bx - By||^2 \\
&\leq ||x - y||^2 - 2r_n\alpha||Bx - By||^2 + r_n^2||Bx - By||^2 \\
&= ||x - y||^2 - (2\alpha - r_n)r_n||Bx - By||^2 \\
&\leq ||x - y||^2.
\end{aligned}$$

Thus, T_n is nonexpansive for all $n \geq 1$. Furthermore,

$$\begin{aligned}
& x = T_n x \Leftrightarrow x = (I + r_n A)^{-1}(x - r_n Bx) \\
\Leftrightarrow \ & x - r_n Bx \in x + r_n Ax \\
\Leftrightarrow \ & 0 \in Ax + Bx.
\end{aligned}$$

Thus, $F(T_n) = (A + B)^{-1}(0), \ \ \forall n \geq 1$. In particular,

$$\|z_n - q\| = \|T_n w_n - T_n q\| \leq \|w_n - q\|.$$

Using the convexity of $\|.\|^2$ and the fact that S_i is k-demicontractive, we have from Lemma 2.3.6 (i) and (4.2.1) that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \Big\|z_n+\sum_{i=1}^{\infty}\beta_{n,i}\eta\beta(S_i-I)z_n-p\Big\|^2\\
&\leq \sum_{i=1}^{\infty}\beta_{n,i}\|z_n+\eta\beta(S_i-I)z_n-q\|^2\\
&= \sum_{i=1}^{\infty}\beta_{n,i}\Big[\|z_n-q\|^2+\eta^2\beta^2\|(S_i-I)z_n\|^2+2\eta\beta\langle z_n-q,(S_i-I)z_n\rangle\Big]\\
&= \sum_{i=1}^{\infty}\beta_{n,i}\Big[\|z_n-q\|^2+\eta^2\beta^2\|(S_i-I)z_n\|^2+2\eta\beta\langle(z_n-q)\\
&\quad +(S_i-I)z_n-(S_i-I)z_n,(S_i-I)z_n\rangle\Big]\\
&= \sum_{i=1}^{\infty}\beta_{n,i}\Big[\|z_n-q\|^2+\eta^2\beta^2\|(S_i-I)z_n\|^2+2\eta\beta\langle S_iz_n-q,S_iz_n-z_n\rangle\\
&\quad -\|(S_i-I)z_n\|^2\Big]\\
&\leq \sum_{i=1}^{\infty}\beta_{n,i}\Big[\|z_n-q\|^2+\eta^2\beta^2\|(S_i-I)z_n\|^2+2\eta\beta\Big(\frac{1+k}{2}\|(S_i-I)z_n\|^2-\|(S_i-I)z_n\|^2\Big)\Big]\\
&= \sum_{i=1}^{\infty}\beta_{n,i}\Big[\|z_n-q\|^2+\eta^2\beta^2\|(S_i-I)z_n\|^2-\eta\beta(1-k)\|(S_i-I)z_n\|^2\Big].
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \|z_n-q\|^2-\sum_{i=1}^{\infty}\beta_{n,i}\eta\beta(1-k-\eta\beta)\|(S_i-I)z_n\|^2\\
&\leq \|z_n-q\|^2\leq\|w_n-q\|^2. \qquad (4.2.2)
\end{aligned}$$

Hence, from (4.2.2) and (4.2.1), we obtain

$$\begin{aligned}
\|x_{n+1}-q\| &\leq \|w_n-q\|=\|\alpha_nx_1+(1-\alpha_n)x_n-q\|\\
&= \|(1-\alpha_n)(x_n-q)+\alpha_n(x_1-q)\|\\
&\leq (1-\alpha_n)\|x_n-q\|+\alpha_n\|x_1-q\|\\
&\leq \max\{\|x_n-q\|,\|x_1-q\|\}\\
&\ \vdots\\
&\leq \max\{\|x_1-q\|,\|x_1-q\|\}.
\end{aligned}$$

Hence $\{x_n\}$ is bounded and consequently $\{z_n\},\{S_iz_n\}$ and $\{w_n\}$ are all bounded.

Using Lemma 2.3.6 (i) in (4.2.1), we obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \left\|z_n+\sum_{i=1}^{\infty}\beta_{n,i}\eta\beta(S_i-I)z_n-q\right\|^2\\
&\leq \|z_n-q\|^2-\sum_{i=1}^{\infty}\beta_{n,i}\eta\beta(1-k-\eta\beta)\|(S_i-I)z_n\|^2\\
&\leq \|w_n-q\|^2-\sum_{i=1}^{\infty}\beta_{n,i}\eta\beta(1-k-\eta\beta)\|(S_i-I)z_n\|^2\\
&= \|\alpha_n u+(1-\alpha_n)x_n-q\|^2-\sum_{i=1}^{\infty}\beta_{n,i}\eta\beta(1-k-\eta\beta)\|(S_i-I)z_n\|^2\\
&= \|x_n-q\|^2+\alpha_n^2\|x_n-u\|^2-2\alpha_n\langle x_n-q,x_n-u\rangle\\
&\quad -\sum_{i=1}^{\infty}\beta_{n,i}\eta\beta(1-k-\eta\beta)\|(S_i-I)z_n\|^2. \qquad (4.2.3)
\end{aligned}$$

Since $\{x_n\}$ and $\{z_n\}$ are bounded, there exists $M>0$ such that $-2\langle x_n-q,x_n-u\rangle\leq M$ for all $n\geq 1$. Therefore

$$\begin{aligned}
\|x_{n+1}-q\|^2-\|x_n-q\|^2 &+ \sum_{i=1}^{\infty}\beta_{n,i}\eta\beta(1-k-\eta\beta)\|(S_i-I)z_n\|^2\\
&\leq \alpha_n^2\|x_n-u\|^2+\alpha_n M. \qquad (4.2.4)
\end{aligned}$$

We now divide the rest of the proof into two cases.
Case 1 : Suppose that there exists $n_0\in\mathbb{N}$ such that $\{\|x_n-q\|\}_{n=n_0}^{\infty}$ is non-increasing. Then $\{\|x_n-q\|\}$ converges and

$$\|x_{n+1}-q\|^2-\|x_n-q\|^2\to 0,\qquad n\to\infty. \qquad (4.2.5)$$

Subsequently from (4.2.4), we get

$$\begin{aligned}
\beta_{n,i}\eta\beta(1-k-\eta\beta)\|(S_i-I)z_n\|^2 &\leq \alpha_n^2\|x_n-u\|^2+\alpha_n M\\
&\quad +\|x_n-q\|^2-\|x_{n+1}-q\|^2,\quad \forall i\geq 1. \qquad (4.2.6)
\end{aligned}$$

Using the fact that $\beta\in(0,1)$, $\eta\in(0,\frac{1-k}{\beta})$, $\liminf_{n\to\infty}\beta_{n,i}>0$ and the condition that $\alpha_n\to 0$ in (4.2.6) we have that

$$\|z_n-S_iz_n\|\to 0\qquad n\to\infty,\quad \forall i\in\mathbb{N}. \qquad (4.2.7)$$

Observe that $(I-r_nB)$ is nonexpansive and $J_{r_n}^A$ is firmly nonexpansive mapping. Then we get

$$\begin{aligned}
\|z_n - q\|^2 &= \|J^A_{r_n}(w_n - r_nBw_n) - J^A_{r_n}(q - r_nBq)\|^2 \\
&\leq \langle (w_n - r_nBw_n) - (q - r_nBq, z_n - q\rangle \\
&= \frac{1}{2}\Big[\|w_n - r_nBw_n - (q - r_nBq)\|^2 + \|z_n - q\|^2 \\
&\quad -\|(w_n - r_nBw_n) - (q - r_nBq) - (z_n - q)\|^2\Big] \\
&\leq \frac{1}{2}\Big[\|w_n - q\|^2 + \|z_n - q\|^2 - \|(w_n - z_n) - r_n(Bw_n - Bq)\|^2\Big] \\
&= \frac{1}{2}\Big[\|w_n - q\|^2 + \|z_n - q\|^2 - \|w_n - z_n\|^2 + 2r_n\langle w_n - z_n, Bw_n - Bq\rangle \\
&\quad -r_n^2\|Bw_n - Bq\|^2\Big].
\end{aligned}$$

Therefore

$$\begin{aligned}
\|z_n - q\|^2 &\leq \|w_n - q\|^2 - \|w_n - z_n\|^2 + 2r_n\langle w_n - z_n, Bw_n - Bq\rangle \\
&\quad -r_n^2\|Bw_n - Bq\|^2. \qquad (4.2.8)
\end{aligned}$$

Furthermore, using Lemma 2.3.6 (i) in (4.2.1), we have

$$\begin{aligned}
\|z_n - q\|^2 &\leq \|(w_n - r_nBw_n) - (q - r_nBq)\|^2 \\
&= \|w_n - q\|^2 - 2r_n\langle w_n - q, Bw_n - Bq\rangle + r_n^2\|Bw_n - Bq\|^2 \\
&\leq \|w_n - q\|^2 - 2\alpha r_n\|Bw_n - Bq\|^2 + r_n^2\|Bw_n - Bq\|^2 \\
&= \|w_n - q\|^2 - r_n(2\alpha - r_n)\|Bw_n - Bq\|^2 \\
&= \|x_n - q + \alpha_n(u - x_n)\|^2 - r_n(2\alpha - r_n)\|Bw_n - Bq\|^2 \\
&= \|x_n - q\|^2 + 2\alpha_n\langle u - x_n, x_n - q\rangle + \alpha_n^2\|x_n - u\|^2 - r_n(2\alpha - r_n)\|Bw_n - Bq\|^2 \\
&\leq \|x_n - q\|^2 + 2\alpha_n\|x_n - u\|\|x_n - q\| + \alpha_n^2\|x_n - u\|^2 - r_n(2\alpha - r_n)\|Bw_n - Bq\|^2.
\end{aligned}$$

This implies that for some $M^* > 0$ and condition (H4), we have

$$\begin{aligned}
a\Big(2\alpha - b\Big)\|Bw_n - Bq\|^2 &\leq r_n(2\alpha - r_n)\|Bw_n - Bq\|^2 \\
&\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
&\quad +\alpha_n^2\|x_n - u\|^2 + \alpha_nM^*. \qquad (4.2.9)
\end{aligned}$$

Using condition $(H2)$ in (4.2.9), we obtain

$$\lim_{n\to\infty}\|Bw_n - Bq\| = 0. \qquad (4.2.10)$$

By (4.2.8), we have

$$\begin{aligned}
\|w_n - z_n\|^2 &\leq \|w_n - q\|^2 - \|z_n - q\|^2 + 2r_n\langle w_n - z_n, Bw_n - Bq\rangle - r_n^2\|Bw_n - Bq\|^2 \\
&\leq \|w_n - q\|^2 - \|x_{n+1} - q\|^2 + 2r_n\langle w_n - z_n, Bw_n - Bq\rangle - r_n^2\|Bw_n - Bq\|^2 \\
&= \|x_n - q - \alpha_n(x_n - u)\|^2 - \|x_{n+1} - q\|^2 + 2r_n\langle w_n - z_n, Bw_n - Bq\rangle \\
&\quad -r_n^2\|Bw_n - Bq\|^2 \\
&\leq \|x_n - q\|^2 + 2\alpha_n\|x_n - q\|\|x_n - u\| + \alpha_n^2\|x_n - u\|^2 - \|x_{n+1} - q\|^2 \\
&\quad +2r_n\langle w_n - z_n, Bw_n - Bq\rangle - r_n^2\|Bw_n - Bq\|^2 \\
&= \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 2\alpha_n\|x_n - u\|\|x_n - q\| + \alpha_n^2\|x_n - u\|^2 \\
&\quad +2r_n\langle w_n - z_n, Bw_n - Bq\rangle - r_n^2\|Bw_n - Bq\|^2. \qquad (4.2.11)
\end{aligned}$$

Using condition (H2) and (4.2.10) in (4.2.11), we get

$$\lim_{n\to\infty}\|T_n w_n - w_n\| = \lim_{n\to\infty}\|z_n - w_n\| = 0.$$

From (3.2.37), we have that

$$\|w_n - x_n\| \to 0, \quad n \to \infty.$$

Since $\{x_n\}$ is bounded, we can extract a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_1 - q, x_n - q\rangle = \lim_{j\to\infty}\langle x_1 - q, x_{n_j} - q\rangle \qquad (4.2.12)$$

and $\{x_{n_j}\}$ converges weakly to some element $p \in H$. Furthermore by the fact that $\|w_n - x_n\| \to 0$, we have that $w_{n_j} \rightharpoonup p$. Similarly, observe that $\lim_{n\to\infty}\|z_n - w_n\| = 0$ implies that $z_{n_j} \rightharpoonup p$. Using (4.2.7) and the demiclosedness of $(I - S_i)$ at the origin, we have that $p \in \cap_{i=1}^{\infty}F(S_i)$.

Since $\liminf_{n\to\infty} r_n > 0$, there exists $\epsilon > 0$ such that $r_n \geq \epsilon, \quad \forall n \geq 1$. Then, by Lemma 2.3.20, we have

$$\lim_{n\to\infty}||T_\epsilon w_n - w_n|| \leq 2\lim_{n\to\infty}||T_n w_n - w_n|| = 0.$$

By Lemma 2.3.13, we have that $p \in F(T_\epsilon) = (A+B)^{-1}(0)$. This implies that $p \in \cap_{i=1}^{\infty}F(S_i) \cap (A+B)^{-1}(0) = \Gamma$.

Next we prove that $\{x_n\}$ converges strongly to q. From (4.2.1), (4.2.2) and Lemma 2.3.6 (ii), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \|z_n - q\|^2 \leq \|w_n - q\|^2 \\
&= \|(1-\alpha_n)(x_n - q) + \alpha_n(x_1 - q)\|^2 \\
&\leq (1-\alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle x_1 - q, w_n - q\rangle. \qquad (4.2.13)
\end{aligned}$$

We observe that

$$\langle x_1 - q, w_n - q\rangle = \langle x_1 - q, w_n - x_n\rangle + \langle x_1 - q, x_n - q\rangle$$

By the fact that $\|w_n - x_n\| \to 0, n \to \infty$ and (4.2.12), we have that

$$\limsup_{n\to\infty} \langle x_1 - q, w_n - q\rangle \leq \lim_{j\to\infty} \langle x_1 - q, x_{n_j} - q\rangle = \langle x_1 - q, p - q\rangle \leq 0. \tag{4.2.14}$$

Using Lemma 2.3.14 and (4.2.14) in (4.2.13), we obtain that $\|x_n - q\| \to 0$. That is $x_n \to 0$ as $n \to \infty$.

Case 2: Assume that there is no $n_0 \in \mathbb{N}$ such that $\{\|x_n - q\|\}_{n=n_0}^{\infty}$ is monotonically decreasing.

Set $\Gamma_n = \|x_n - q\|^2$ for all $n \geq 1$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\},$$

i.e. $\tau(n)$ is the largest number k in $\{1, \ldots, n\}$ such that Γ_k increases at $k = \tau(n)$; note that, in view of Case 2, this $\tau(n)$ is well-defined for all sufficiently large n. Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

Since $\{x_{\tau(n)}\}$ is bounded there exist a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$ which converges weakly to $p \in H$.After a similar conclusion from (4.2.7), it is easy to see that

$$\lim_{n\to\infty} \|z_{\tau(n)} - S_i z_{\tau(n)}\| = 0.$$

By using similar argument as in case 1, we conclude immediately that

$$\lim_{n\to\infty} \|T_{\tau(n)} w_{\tau(n)} - w_{\tau(n)}\| = \lim_{n\to\infty} \|z_{\tau(n)} - w_{\tau(n)}\| = 0$$

and

$$\limsup_{n\to\infty} \langle x_1 - q, w_{\tau(n)} - q\rangle \leq 0.$$

Observe that since $\lim\limits_{n\to\infty} \|x_{\tau(n)} - w_{\tau(n)}\| = 0$, we also have $w_{\tau(n)} \rightharpoonup p$. Since $I - S_i$ is demiclosed and $\|z_{\tau(n)} - S_i z_{\tau(n)}\| \to 0$ as $n \to \infty$, we have that $p \in \cap_{i=1}^{\infty} F(S_i)$. Similarly, we can show that $p \in (A + B)^{-1}(0)$. Therefore, $p \in \Gamma$. At the same time, we note from (4.2.13) that for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - q\|^2 - \|x_{\tau(n)} - q\|^2 \\ &\leq \alpha_{\tau(n)} \Big[2\langle x_1 - q, w_{\tau(n)} - q\rangle - \|x_{\tau(n)} - q\|^2 \Big] \end{aligned}$$

which implies that (since $\alpha_{\tau(n)} > 0$)

$$\|x_{\tau(n)} - q\|^2 \leq 2\langle x_1 - q, w_{\tau(n)} - q\rangle.$$

Thus,

$$\limsup_{n\to\infty} \|x_{\tau(n)} - q\|^2 \leq 2 \limsup_{n\to\infty} \langle x_1 - q, w_{\tau(n)} - q\rangle \leq 0.$$

Hence

$$\lim_{n\to\infty}\|x_{\tau(n)}-q\|=0.$$

Using (4.2.7) in (3.2.37), get

$$\|x_{\tau(n)+1}-z_{\tau(n)}\|\to 0,\quad n\to\infty.$$

Now,

$$\|x_{\tau(n)+1}-x_{\tau(n)}\|\leq\|x_{\tau(n)+1}-z_{\tau(n)}\|+\|z_{\tau(n)}-w_{\tau(n)}\|+\|w_{\tau(n)}-x_{\tau(n)}\|\to 0,\quad n\to\infty.$$

Hence, we deduce that

$$\|x_{\tau(n)+1}-q\|\leq\|x_{\tau(n)+1}-x_{\tau(n)}\|+\|x_{\tau(n)}-q\|\to 0,\quad n\to\infty,$$

and so

$$\lim_{n\to\infty}\Gamma_{\tau(n)}=\lim_{n\to\infty}\Gamma_{\tau(n)+1}=0.$$

Furthermore, for $n\geq n_0$, it is easy to see that $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$ if $n\neq\tau(n)$ (that is $\tau(n)<n$),because $\Gamma_j\geq\Gamma_{j+1}$ for $\tau(n)+1\leq j\leq n$. As a consequence, we obtain for all $n\geq n_0$,

$$0\leq\Gamma_n\leq\max\{\Gamma_{\tau(n)},\Gamma_{\tau(n)+1}\}=\Gamma_{\tau(n)+1}.$$

Hence $\lim\limits_{n\to\infty}\Gamma_n=0$ and so $\{x_n\}$ converges strongly to q. This completes the proof. ☐

Let f and g be two convex, lower semi continuous functions from H to $\mathbb{R}\cup\{+\infty\}$ such that f is differentiable with $L-$Lipschitz continuous gradient, and g is "simple" meaning that its "proximal map"

$$x\to arg\min_{y\in H} g(y)+\frac{||x-y||^2}{2\tau}$$

can easily be computed. Let us consider the following minimization problem

$$\min_{x\in H} F(x):=f(x)+g(x) \tag{4.2.15}$$

and assume that this problem has at least a solution.

In Theorem 4.2.2, take $A:=\nabla f$ and $B:=\partial g$, where ∇f is the gradient of f and ∂g is the subdifferential of g which is defined by $\partial g(x):=\big\{s\mid g(y)\geq g(x)+\langle s,y-x\rangle\ \forall y\big\}$. Therefore, we obtain the following strong convergence result for finding a solution of problem (4.2.15) which is also a common fixed point of an infinite family of k-demicontractive mappings.

Corollary 4.2.3. *Let H be a real Hilbert space and for each $i\in\mathbb{N}$, let $S_i:H\to H$ a k-demicontractive mapping. Let f and g be two convex, lower semicontinuous functions from H to $\mathbb{R}\cup\{+\infty\}$ such that f is differentiable with Lipschitz continuous gradient, and g is simple, be such that $\Gamma:=\cap_{i=1}^{\infty}F(S_i)\cap(\nabla f+\partial g)^{-1}(0)\neq\emptyset$ Let $\{r_n\}$ denote a nonnegative real sequence and L is the Lipschitz constant of ∇f. and $I-S_i$ is demiclosed at the origin. Assume conditions (H1)-(H4) hold. Given $x_1\in H$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} w_n=\alpha_n x_1+(1-\alpha_n)x_n,\\ z_n=\text{prox}_{r_n g}(w_n-r_n\nabla f(w_n)),\\ x_{n+1}=z_n+\sum_{i=1}^{\infty}\beta_{n,i}\eta\beta(S_i-I)z_n,\end{cases} \tag{4.2.16}$$

Then $\{x_n\}$ converges strongly to $q:=P_\Gamma x_1$.

4.3 Strong convergence result for Meir-Keeler contractions and a countable family of accretive operators in Banach spaces with applications

Let $A : X \to X$ be a single-valued nonlinear mapping and $B : X \to 2^X$ be a set-valued mapping where X is real smooth uniform and convex Banach space. We consider the following inclusion problem: find $u \in X$ such that

$$0 \in (A + B)x. \tag{4.3.1}$$

In this section, motivated by the works of Song *et al.* [163], Wei and Duan [184] and Shehu and Cai [180], we study and prove strong convergence results, under some mild conditions, using generalized forward-backward method which involve viscosity approximation method with Meir-Keeler contractions for solving the inclusion problem (4.3.1) for a finite family of m-accretive and α- inverse strongly accretive operators in the framework of uniformly convex and uniformly smooth Banach spaces. Finally we provide some applications of our result to certain integro-differential equation with generalized p-Laplacian operator. Our results is interesting and it also improves and compliments the result of Song *et al.* [163] and Wei and Duan [184].

Lemma 4.3.1. *Let X be a real smooth and uniformly convex Banach space and C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be a nonexpansive mapping and $f : C \to C$ be MKC, $M : X \to X$ be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$. Suppose that the duality mapping $J : X \to X^*$ is weakly sequentially continuous at zero, $0 \leq \eta < \frac{\bar{\gamma}}{2}$ and $F(T) \neq \emptyset$. If for each $t \in (0, 1)$, define $S_t : X \to X$ by*

$$S_t x := t\eta f(x) + (I - tM)Tx, \tag{4.3.2}$$

then S_t has a fixed point x_t, for each $0 < t \leq \|M\|^{-1}$, which converges strongly to the fixed point of T, as $t \to 0$. That is $\lim\limits_{t\to 0} x_t = x_0 \in F(T)$. Moreover, x_0 satisfies the following variational inequality

$$\langle (M - \eta f)x_0, j(x_0 - z)\rangle \leq 0, \quad \forall z \in F(T). \tag{4.3.3}$$

Proof. From the definition of MKC, we can see that MKC is also a nonexpansive mapping. Hence we obtain

$$\begin{aligned} \|S_t x - S_t y\| &\leq t\eta\|f(x) - f(y)\| + \|(1 - tM)(Tx - Ty)\| \\ &\leq t\eta\|f(x) - f(y)\| + (1 - t\bar{\gamma})\|x - y\| \\ &\leq t\eta\|x - y\| + (1 - t\bar{\gamma})\|x - y\| \\ &\leq [1 - t(\bar{\gamma} - k\eta)]\,\|x - y\|, \end{aligned}$$

which implies that S_t is a contraction since $0 < \eta < \frac{\bar{\gamma}}{2}$. Then Theorem 2.3.3 implies that S_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation

$$x_t = t\eta f(x_t) + (I - tM)Tx_t. \tag{4.3.4}$$

Next we show that the solution to the variational inequality (4.3.3) is unique. Suppose both $x_0 \in F(T)$ and $\hat{x}$ are solutions of (4.3.3), without lost of generalities, we may assume that there is a number ϵ such that $\|x_0 - \hat{x}\| \geq \epsilon$. Then by Lemma 2.3.25, there exists a number $k > 0$ such that $\|f(x_0) - f(\hat{x})\| \leq k_\epsilon \|x_0 - \hat{x}\|$. From (4.3.3) we obtain

$$\begin{cases} \langle (M - \eta f)x_0, j(x_0 - \hat{x}) \rangle \leq 0, \\ \langle (M - \eta f)\hat{x}, j(\hat{x} - x_0) \rangle \leq 0. \end{cases} \tag{4.3.5}$$

Adding up (4.3.5), we obtain

$$\begin{aligned} \langle (M - \eta f)\hat{x} - (M - \eta f)x_0, j(\hat{x} - x_0) \rangle &= \langle M(\hat{x} - x_0), j(\hat{x} - x_0) \rangle - \eta \langle f(\hat{x}) - f(x_0), j(\hat{x} - x_0) \rangle \\ &\geq \hat{\gamma} \|\hat{x} - x_0\|^2 - k\eta \|\hat{x} - x_0\|^2 \\ &= (\hat{\gamma} - k\eta) \|\hat{x} - x_0\|^2 \\ &\geq (\hat{\gamma} - k\eta)\epsilon^2 \\ &> 0. \end{aligned}$$

Therefore $x_0 = \bar{x}$ and the uniqueness is proved. Hence x_0 is a unique solution of (4.2.3).

Now we show that $\{x_t\}$ is bounded. Indeed, we may assume with no loss of generality, $t < \|M\|^{-1}$, for all $p \in F(T)$, fixed ϵ_1, for each $t \in (0,1)$.

Case 1 ($\|x_t - p\| < \epsilon_1$): In this case, $\{x_t\}$ is bounded.

Case 2 ($\|x_t - p\| \geq \epsilon_1$): In this case, we obtain by Lemma 2.3.24 and 2.3.25 that there is a number r_1 such that

$$\|f(x_t) - f(p)\| < r_1 \|x_t - p\|. \tag{4.3.6}$$

Hence we obtain

$$\begin{aligned} \|x_t - p\| &= \|t\eta f(x_t) + (I - tM)Tx_t - p\| \\ &= \|t(\eta f(x_t) - Mp) + (I - tM)(Tx_t - p)\| \\ &\leq t\|\eta f(x_t) - Mp)\| + (1 - t\bar{\gamma})\|x_t - p\| \\ &\leq t\|\eta f(x_t) - \eta f(p)\| + t\|\eta f(p) - Mp\| + (1 - t\bar{\gamma})\|x_t - p\| \\ &\leq t\eta r_1 \|x_t - p\| + t\|\eta f(p) - Mp\| + (1 - t\bar{\gamma})\|x_t - p\|. \end{aligned}$$

Therefore

$$\|x_t - p\| \leq \frac{\|\eta f(p) - Mp\|}{\bar{\gamma} - \gamma r_1}. \tag{4.3.7}$$

This implies that $\{x_t\}$ is bounded. Consequently $\{f(x_t)\}$ and $\{Tx_t\}$ are bounded.

Since $\{f(x_t)\}$ and $\{Tx_t\}$ are bounded, we obtain from (4.2.4) that

$$\|x_t - Tx_t\| = t\|\eta f(x_t) - MTx_t\| \to 0, \quad as \quad t \to 0. \tag{4.3.8}$$

To prove that $x_t \to x_0$ ($x_0 \in F(T)$) as $t \to 0$.

Since $\{x_t\}$ is bounded and X uniformly convex by Milman Pettis Theorem we have X is reflexive. Hence there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n} \rightharpoonup x^*$. By (4.3.7) we have that $x_{t_n} - Tx_{t_n} \to 0$, as $t_n \to 0$. Since X satisfies Opial's condition, it follows from Lemma 2.3.13 that $x^* \in F(T)$. Claim

$$\|x_{t_n} - x^*\| \to 0. \tag{4.3.9}$$

Suppose by contradiction, there is a number ϵ_0 and a subsequence $\{x_{t_m}\}$ of $\{x_{t_n}\}$ such that $\|x_{t_m} - x^*\| \geq \epsilon_0$. From Lemma 2.3.25, there is a number $r_{\epsilon_0} > 0$ such that $\|f(x_{t_m}) - f(x^*)\| \leq r_{\epsilon_0}\|x_{t_m} - x^*\|$, we have

$$\begin{aligned}
\|x_{t_m} - x^*\|^2 &= t_m\langle \eta f(x_{t_m}) - Mx^*, j(x_{t_m} - x^*)\rangle + \langle (1-t_m)(Tx_{t_m} - x^*), j(x_{t_m} - x^*)\rangle \\
&\leq t_m\langle \eta f(x_{t_m}) - Ax^*, j(x_{t_m} - x^*)\rangle + (1 - t_m\bar{\gamma})\|x_{t_m} - x^*\|^2.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\|x_{t_m} - x^*\|^2 &\leq \frac{1}{\bar{\gamma}}\langle \eta f(x_{t_m}) - Mx^*, j(x_{t_m} - x^*)\rangle \\
&\leq \frac{1}{\bar{\gamma}}[\langle \eta f(x_{t_m}) - \eta f(x^*), j(x_{t_m} - x^*)\rangle + \langle \eta f(x^*) - Mx^*, j(x_{t_m} - x^*)\rangle] \\
&\leq \frac{1}{\bar{\gamma}}[\eta r_{\epsilon_0}\|x_{t_m} - x^*\|^2 + \langle \eta f(x^*) - Mx^*, j(x_{t_m} - x^*)\rangle].
\end{aligned}$$

Therefore

$$\|x_{t_m} - x^*\|^2 \leq \frac{\langle \eta f(x^*) - Mx^*, j(x_{t_m} - x^*)\rangle}{\bar{\gamma} - \eta r_{\epsilon_0}}. \tag{4.3.10}$$

Using the fact the duality map j is single valued and weakly sequentially continuous at zero by (4.3.10), we get that $x_{t_m} \to x^*$. It is a contradiction. Hence, we have $x_{t_n} \to x^*$.

Finally, we show that x^* solves the variational inequality (4.2.3). Since

$$x_t = t\eta f(x_t) + (I - tM)Tx_t,$$

we obtain

$$(M - \eta f)x_t = -\frac{1}{t}(I - tM)(1 - T)x_t. \tag{4.3.11}$$

Notice

$$\begin{aligned}
\langle (I-T)x_t - (I-T)z, j(x_t - z)\rangle &\geq \|x_t - z\|^2 - \|Tx_t - Tz\|\|x_t - z\| \\
&\geq \|x_t - z\|^2 - \|x_t - z\|^2 \\
&= 0.
\end{aligned}$$

It follows that, for $z \in F(T)$,

$$\begin{aligned}
\langle (M - \eta f)x_t, j(x_t - z)\rangle &= -\frac{1}{t}\langle (I - tM)(I - T)x_t, j(x_t - z)\rangle \\
&= -\frac{1}{t}\langle (I-T)x_t - (I-T)z, j(x_t - z)\rangle + \langle M(I-T)x_t, j(x_t - z)\rangle \\
&\leq \langle M(I-T)x_t, j(x_t - z)\rangle. \qquad (4.3.12)
\end{aligned}$$

Now, replacing t in (4.3.12) with t_n and letting $n \to \infty$, noticing that $(I-T)x_{t_n} \to (I-T)x^* = 0$ for $x^* \in F(T)$, we obtain $\langle (M-\eta f)x_t, j(x_t - z)\rangle \leq 0$. That is $x^* \in F(T)$ is a solution of (4.2.3). Hence $x_0 = x^*$ by uniqueness. Hence, we have show that each cluster point of $\{x_t\}$ as $t \to 0$ equals $\hat{x}$, therefore, $x_t \to \hat{x}$ as $t \to 0$. $\square$

Lemma 4.3.2. *Let X be a real smooth and uniformly convex Banach space. Let C be a nonempty convex and closed subset of E. Let $A_i : X \to 2^X$ $(i = 1,2,...,N)$ be m-accretive operators such that $\overline{D(A_i)} \subseteq C$ and let $B_i : C \to X$ be α_i-inverse strongly accretive operators such that $\cap_{i=1}^N (A_i + B_i)^{-1}0 \neq \emptyset$. Let $a_0, a_1, ..., a_N$ be real numbers in $(0,1)$ such that $\sum_{i=0}^N a_i = 1$ and $P_n = a_0 I + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)$, where $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$ and $0 < r_{n,i} \leq \frac{2\alpha_i}{c}$ $\forall$ $i = 1,2,...,N$ and $n \geq 1$. Then $P_n : C \to C$ is nonexpansive and $F(P_n) = \cap_{i=1}^N (A_i + B_i)^{-1}0$, for all $n \geq 1$.*

Proof. First, we show that P_n is nonexpansive for all $n \geq 1$. Let $x, y \in C$. Then for $i = 1,2,...,N$, it follows that

$$\begin{aligned}
\|(I - r_{n,i}B_i)x - (I - r_{n,i}B_i)y\|^2 &= \|x - y - r_{n,i}(B_i x - B_i y)\|^2 \\
&\leq \|x-y\|^2 - 2r_{n,i}\langle B_i x - B_i y, j(x-y)\rangle + c r_{n,i}^2 \|B_i x - B_i y\|^2 \\
&\leq \|x-y\|^2 - 2r_{n,i}\alpha\|B_i x - B_i y\|^2 + c r_{n,i}^2 \|B_i x - B_i y\|^2 \\
&= \|x-y\|^2 - (2\alpha - c r_{n,i}) r_{n,i}\|B_i x - B_i y\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Thus $(I - r_{n,i}B_i)$ is nonexpansive for all $i = 1,2,...,N$. Since $J_{r_{n,i}}^{A_i}$ and $(1 - r_{n,i}B_i)$ are nonexpansive for all $i = 1,2,...,N$, we get that

$$\begin{aligned}
\|P_n x - P_n y\| &\leq a_0\|x-y\| + \sum_{i=1}^N a_i \left\| J_{r_{n,i}}^{A_i}(1 - r_{n,i}B_i)x - J_{r_{n,i}}^{A_i}(1 - r_{n,i}B_i)y \right\| \\
&\leq a_0\|x-y\| + \sum_{i=1}^N a_i \|(1 - r_{n,i}B_i)x - (1 - r_{n,i}B_i)y\| \\
&\leq a_0\|x-y\| + \sum_{i=1}^N a_i\|x-y\| \\
&= \|x-y\|.
\end{aligned}$$

Thus P_n is nonexpansive for all $n \geq 1$.

Next we show that $F(P_n) = \cap_{i=1}^N (A_i + B_i)^{-1}0$, for all $n \geq 1$. It is obvious that $\cap_{i=1}^N (A_i + B_i)^{-1}0 \subseteq F(P_n)$. So, we are left to show that $F(P_n) \subseteq \cap_{i=1}^N (A_i + B_i)^{-1}0$. Let $u \in F(P_n)$. Then $P_n u = u$ and for all $v \in \cap_{i=1}^N (A_i + B_i)^{-1}0 \subseteq F(P_n)$, we have

$$\begin{aligned}
\|u - v\| &\leq a_0\|u-v\| + a_1\left\| J_{r_{n,1}}^{A_1}(I - r_{n,1}B_1)u - v \right\| + ... + a_N \left\| J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v \right\| \\
&\leq (a_0 + a_1 + ... + a_{N-1})\|u-v\| + a_N\|J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v\| \\
&\leq (1 - a_N)\|u-v\| + a_N\left\| J_{r_{n,N}}^{A_N}(I - r_{n,N}B_N)u - v \right\|.
\end{aligned}$$

Therefore

$$\|u-v\| = (1-a_N)\|u-v\| + a_N \left\| J_{r_{n,N}}^{A_N}(I-r_{n,N}B_N)u - v \right\|,$$

which implies that

$$\|u-v\| = \left\| J_{r_{n,N}}^{A_N}(I-r_{n,N}B_N)u - v \right\|.$$

Similarly,

$$\|u-v\| = \left\| J_{r_{n,1}}^{A_1}(I-r_{n,1}B_1)u - v \right\| = ... = \left\| J_{r_{n,N-1}}^{A_{N-1}}(I-r_{n,N-1}B_{N-1})u - v \right\|.$$

Then

$$\begin{aligned} \|u-v\| &= \frac{a_1}{\sum_{i=1}^N a_i}\| \big(J_{r_{n,1}}(I-r_{n,1}B_1)u - v\big) \| + \frac{a_2}{\sum_{i=1}^N a_i}\| \big(J_{r_{n,2}}(I-r_{n,2}B_2)u - v\big) \| + ... \\ &\quad + \frac{a_N}{\sum_{i=1}^N a_i}\| \big(J_{r_{n,N}}(I-r_{n,N}B_N)u - v\big) \|. \end{aligned}$$

By strict convexity of X, we have that

$$u - v = J_{r_{n,1}}(I-r_{n,1}B_1)u - v = J_{r_{n,2}}(I-r_{n,2}B_2)u - v = ... = J_{r_{n,N}}(I-r_{n,N}B_N)u - v.$$

Therefore, $J_{r_{n,i}}(I-r_{n,i}B_i)u = u$, for $i = 1, 2, ..., N$. Then $u \in \cap_{i=1}^N (A_i+B_i)^{-1}0$.

Thus $F(P_n) \subseteq \cap_{i=1}^N (A_i+B_i)^{-1}0$. □

Theorem 4.3.3. *Let X be a real smooth and uniformly convex Banach space and C be a nonempty, closed and convex subset of X, and let $f : C \to C$ be a MKC. Let $M : C \to C$ be a strong positive bounded linear operator, $\bar{\gamma} > 0$ such that $0 \le \eta < \frac{\bar{\gamma}}{2}$. Suppose that the duality mapping $j : X \to X^*$ is weakly sequentially continuous at zero. Let $A_i : C \to 2^X$ be m-accretive operators and $B_i : C \to X$ be α_i-inverse strongly accretive operators, for $i = 1, 2, ..., N$ such that $\cap_{i=1}^N (A_i+B_i)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be generated by $x_1 \in X$,*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)\left[a_0 x_n + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)x_n\right], \\ x_{n+1} = \alpha_n \eta f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n M)y_n, \ n \ge 1, \end{cases} \tag{4.3.13}$$

for all $n \ge 1$, where $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$ for $i = 1, 2, ..., N$, and $0 < a_i < 1$, for $i = 0, 1, 2, ..., N$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real number sequence in $(0, 1)$ and $\{r_{n,i}\} \subset (0, \infty)$. Suppose that the above sequence satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < r_{n,i} < \frac{2\alpha}{c}$ *and* $\sum_{n=1}^\infty |r_{n+1,i} - r_{n,i}| < \infty$ *for* $n \ge 1$ *and* $i = 1, 2, ...N$, *where c is a constant;*

(iii) $\lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0$;

(iv) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$.

Then $\{x_n\}$ *converges strongly to a point* $x_0 \in \cap_{i=1}^N (A_i+B_i)^{-1}0$, *which is the unique solution of the variational inequality:* $\forall z \in \cap_{i=1}^N (A_i+B_i)^{-1}0$.

$$\langle (M-\eta f)x_0, J(x_0-z)\rangle \leq 0, \tag{4.3.14}$$

where $x_0 = Q_{\cap_{i=1}^N (A_i+B_i)^{-1}0} f(x_0)$, *and* $Q_{\cap_{i=1}^N (A_i+B_i)^{-1}0}$ *is the unique sunny nonexpansive retraction of* X *onto* $\cap_{i=1}^N (A_i+B_i)^{-1}0$.

Proof. Put $P_n = a_0 I + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)$ and $u_{n_{n,i}} = (I - r_{n,i}B_i)x_n$ for $i = 1, 2, 3, ..., N$ and $n \geq 1$. Then we obtain from (4.3.13) and Lemma 4.3.2 that

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)P_n x_n - p\| \\ &\leq \|\beta_n(x_n - p) + (1-\beta_n)(P_n x_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{4.3.15}$$

From the definition of MKC and Lemma 2.3.25, for each $\epsilon > 0$ there is a number $r_\epsilon \in (0,1)$, if $\|x_n - z\| < \epsilon$ then $\|f(x_n) - f(z)\| \leq r_\epsilon \|x_n - z\|$. it follows from (4.3.13) and (4.3.15) that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n \eta f(x_n) + \gamma_n x_n + (1-\gamma_n)I - \alpha_n M)y_n - p\| \\ &= \|\alpha_n(\eta f(x_n) - Mp) + \gamma_n(x_n - p) + ((1-\gamma_n)I - \alpha_n M)(y_n - p)\| \\ &\leq \alpha_n\|\eta f(x_n) - Mp\| + \gamma_n\|x_n - p\| + (1 - \gamma_n - \alpha_n\bar{\gamma})\|x_n - p\| \\ &\leq \alpha_n \eta \max\{r_\epsilon\|x_n - p\|, \epsilon\} + \alpha_n\|\eta f(p) - Mp\| + (1-\alpha_n\bar{\gamma})\|x_n - p\| \\ &= \max\{(1-\alpha_n\bar{\gamma})\|x_n - p\| + \alpha_n\eta r_\epsilon\|x_n - p\| + \alpha_n\|\eta f(p) - Mp\|, \\ &\quad (1-\alpha_n\bar{\gamma})\|x_n - p\| + \alpha_n\eta\epsilon + \alpha_n\|\eta f(p) - Mp\|\} \\ &= \max\{(1-\alpha_n\bar{\gamma} + \alpha_n\eta r_\epsilon)\|x_n - p\| + \alpha_n\|\eta f(p) - Mp\|, (1-\alpha_n\bar{\gamma})\|x_n - p\| \\ &\quad + \alpha_n\eta\epsilon + \alpha_n\|\eta f(p) - Mp\|\} \\ &= \max\{[1 - (\alpha_n\bar{\gamma} - \alpha_n\eta r_\epsilon)]\|x_n - p\| + \alpha_n\|\eta f(p) - Mp\|, (1-\alpha_n\bar{\gamma})\|x_n - p\| \\ &\quad + \alpha_n\eta\epsilon + \alpha_n\|\eta f(p) - Mp\|\}. \end{aligned}$$

Inductively, we obtain

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\eta f(p) - Mp\|}{\bar{\gamma} - \eta r_\epsilon}, \frac{\gamma\epsilon + \|\eta f(p) - Mp\|}{\bar{\gamma}}, \right\} \quad n \geq 1, \tag{4.3.16}$$

which implies that the sequence $\{x_n\}$ is bounded.

Next we show that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$.

First we consider $\|J_{r_{n+1,i}}^{A_i} u_{n+1,i} - J_{r_{n,i}}^{A_i} u_{n,i}\|$, if $r_{n,i} \leq r_{n+1,i}$ then it follows from Lemma

2.3.26 that

$$\begin{aligned}\left\|J^{A_i}_{r_{n+1,i}}u_{n+1,i}-J^{A_i}_{r_{n,i}}u_{n,i}\right\| &= \left\|J^{A_i}_{r_{n,i}}\left(\frac{r_{n,i}}{r_{n+1,i}}u_{n+1,i}+\left(1-\frac{r_{n,i}}{r_{n+1,i}}\right)J^{A_i}_{r_{n+1,i}}u_{n+1,i}\right)-J^{A_i}_{r_{n,i}}u_{n,i}\right\| \\ &\leq \left\|\frac{r_{n,i}}{r_{n+1,i}}u_{n+1,i}+\left(1-\frac{r_{n,i}}{r_{n+1,i}}\right)J^{A_i}_{r_{n+1,i}}u_{n+1,i}-u_{n,i}\right\| \\ &\leq \frac{r_{n,i}}{r_{n+1,i}}\|u_{n+1,i}-u_{n,i}\|+\left(1-\frac{r_{n,i}}{r_{n+1,i}}\right)\left\|J^{A_i}_{n+1,i}u_{n+1,i}-u_{n,i}\right\| \\ &\leq \|u_{n+1,i}-u_{n,i}\|+\frac{r_{n+1,i}-r_{n,i}}{b}2M_1. \qquad (4.3.17)\end{aligned}$$

If $r_{n+1,i}\leq r_{n,i}$, using similar proof as in (4.3.17), we obtain

$$\left\|J^{A_i}_{r_{n+1,i}}u_{n+1,i}-J^{A_i}_{r_{n,i}}u_{n,i}\right\| \leq \|u_{n+1,i}-u_{n,i}\|+\frac{r_{n,i}-r_{n+,i}}{b}2M_1. \qquad (4.3.18)$$

Combining (4.3.17) and (4.3.18), we have, for $n\geq 1$,

$$\begin{aligned}\left\|J^{A_i}_{r_{n+1,i}}u_{n+1,i}-J^{A_i}_{r_{n,i}}u_{n,i}\right\| &\leq \|u_{n+1,i}-u_{n,i}\|+\frac{2|r_{n,i}-r_{n+,i}|}{b}M_1 \\ &\leq \|(I-r_{n+1,i}B_i)(x_{n+1}-x_n)\|+|r_{n+1,i}-r_{n,i}|\|B_ix_n\|+\frac{2|r_{n+1,i}-r_{n,i}|}{b}M_1 \\ &\leq \|x_{n+1}-x_n\|+|r_{n+1,i}-r_{n,i}|\|B_ix_n\|+\frac{2|r_{n+1,i}-r_{n,i}|}{b}M_1.\end{aligned} \qquad (4.3.19)$$

Set $M_2=\left(\frac{2}{b}+M_1\right)$ and using (4.3.19), we obtain

$$\begin{aligned}\|P_{n+1}x_{n+1}-P_nx_n\| &\leq a_0\|x_{n+1}-x_n\|+\sum_{i=1}^{N}\left\|a_i\left(J^{A_i}_{r_{n+1,i}}(I-r_{n+1,i}B_i)x_n-J^{A_i}_{r_{n,i}}(I-r_{n,i}B_i)x_n\right)\right\| \\ &\leq \|x_{n+1}-x_n\|+M_2\sum_{i=1}^{N}|r_{n,i}-r_{n+1,i}|. \qquad (4.3.20)\end{aligned}$$

Next, from (4.3.13), we get that

$$x_{n+1}=\alpha_n\eta f(x_n)+\gamma_nx_n+[(1-\gamma)I-\alpha_nM]Q_nx_n. \qquad (4.3.21)$$

Now, define

$$z_n=\frac{x_{n+1}-\gamma_nx_n}{1-\gamma_n}. \qquad (4.3.22)$$

Hence, we obtain

$$\begin{aligned}z_{n+1}-z_n &= \frac{\alpha_{n+1}\eta f(x_{n+1})+\gamma_{n+1}x_{n+1}+[(1-\gamma_{n+1})I-\alpha_{n+1}M]\,Q_{n+1}x_{n+1}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\ &\quad -\frac{\alpha_nf(x_n)+\gamma_nx_n+[(1-\gamma_n)I-\alpha_nM]\,Q_nx_n-\gamma_nx_n}{1-\gamma_n} \\ &= \frac{\alpha_{n+1}\left[\eta f(x_{n+1})-MQ_{n+1}x_{n+1}\right]}{1-\gamma_{n+1}}-\frac{\alpha_n\left[\eta f(x_n)-MQ_nx_n\right]}{1-\gamma_n}+Q_{n+1}x_{n+1}-Q_nx_n,\end{aligned}$$

which implies that

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}\|\eta f(x_{n+1}) - MQ_{n+1}x_{n+1}\|}{1-\gamma_{n+1}} + \frac{\alpha_n\|\eta f(x_n) - MQ_nx_n\|}{1-\gamma_n} \\ &\quad + \|Q_{n+1}x_{n+1} - Q_nx_n\|.\end{aligned} \tag{4.3.23}$$

Now, we estimate $\|Q_{n+1}x_{n+1} - Q_nx_n\|$.

$$\begin{aligned}\|Q_{n+1}x_{n+1} - Q_nx_n\| &= \|\,[\beta_{n+1}x_{n+1} + (1-\beta_{n+1})P_{n+1}x_{n+1}] - [\beta_nx_n + (1-\beta_n)P_nx_n]\,\| \\ &\leq (1-\beta_{n+1})\|P_{n+1}x_{n+1} - P_{n+1}x_n\| + |\beta_{n+1}-\beta_n|\|P_nx_n\| \\ &\quad +\beta_{n+1}\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_n\| \\ &\leq (1-\beta_{n+1})\|x_{n+1}-x_n\| + M_2(1-\beta_{n+1})\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| + |\beta_{n+1}-\beta_n|\|P_nx_n \\ &\quad +\beta_{n+1}\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_n\| \\ &\leq \|x_{n+1}-x_n\| + M_2(1-\beta_{n+1})\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| + |\beta_{n+1}-\beta_n|\|P_nx_n\| \\ &\quad +|\beta_{n+1}-\beta_n|\|x_n\|.\end{aligned} \tag*{(4.3.24}$$

From (4.3.23) and (4.3.24), we obtain

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \frac{\alpha_{n+1}\|\eta f(x_{n+1}) - MQ_{n+1}x_{n+1}\|}{1-\gamma_{n+1}} + \frac{\alpha_n\|\eta f(x_n) - MQ_nx_n\|}{1-\gamma_n} \\ &\quad +\|x_{n+1}-x_n\| + M_2(1-\beta_{n+1})\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| + |\beta_{n+1}-\beta_n|\|P_nx_n\| \\ &\quad +|\beta_{n+1}-\beta_n|\|x_n\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1}-x_n\| &\leq \frac{\alpha_{n+1}\|\eta f(x_{n+1}) - MQ_{n+1}x_{n+1}\|}{1-\gamma_{n+1}} + \frac{\alpha_n\|\eta f(x_n) - MQ_nx_n\|}{1-\gamma_n} \\ &\quad +M_2(1-\beta_{n+1})\sum_{i=1}^{N}|r_{n,i} - r_{n+1,i}| + |\beta_{n+1}-\beta_n|\|P_nx_n\| \\ &\quad +|\beta_{n+1}-\beta_n|\|x_n\|.\end{aligned} \tag{4.3.25}$$

Since $\{x_n\}$, $\{f(x_n)\}$ and $\{P_nx_n\}$ and $\{Q_nx_n\}$ are bounded by conditions (i), (ii) and (iii), we have that

$$\limsup_{n\to\infty}\{\|z_{n+1} - z_n\| - \|x_{n+1}-x_n\|\} \leq 0. \tag{4.3.26}$$

Thus by Lemma 2.3.27, we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{4.3.27}$$

Hence we obtain from (4.3.23) and (4.3.27) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.3.28}$$

Also from (4.3.13), we obtain

$$\begin{aligned}\|Q_n x_n - x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Q_n x_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n \eta f(x_n) + \gamma_n(x_n - Q_n x_n) - \alpha_n M Q_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n(\|\eta f(x_n)\| + \|M Q_n x_n\|) + \gamma_n \|x_n - Q_n x_n\|,\end{aligned}$$

which implies that

$$\|Q_n x_n - x_n\| \leq \frac{1}{1-\gamma_n}(\|x_n - x_{n+1}\| + \alpha_n(\|\eta f(x_n)\| + \|M Q_n x_n\|)). \tag{4.3.29}$$

Hence from condition (i), (4.3.28) and (4.3.29), we get that

$$\lim_{n\to\infty} \|Q_n x_n - x_n\| = 0. \tag{4.3.30}$$

Next, we estimate $\|P_n x_n - x_n\|$

$$\begin{aligned}\|P_n x_n - x_n\| &\leq \|x_n - Q_n x_n\| + \|Q_n x_n - P_n x_n\| \\ &\leq \|x_n - Q_n x_n\| + \|\beta_n x_n + (1-\beta_n) P_n x_n - P_n x_n\| \\ &\leq \|x_n - Q_n x_n\| + \beta_n \|x_n - P_n x_n\|,\end{aligned}$$

which implies that

$$\|P_n x_n - x_n\| \leq \frac{1}{1-\beta_n}\|x_n - Q_n x_n\| \to 0, \quad n \to \quad \infty. \tag{4.3.31}$$

Also we have

$$\begin{aligned}\|y_n - x_n\| &= \|\beta_n x_n + (1-\beta_n) P_n x_n - x_n\| \\ &= \beta_n \|x_n - P_n x_n\| + \|P_n x_n - x_n\| \to 0, \quad n \to \infty.\end{aligned} \tag{4.3.32}$$

Also we can obtain that

$$\|y_n - P_n x_n\| \leq \|y_n - x_n\| + \|x_n - P_n x_n\| \to 0, \quad n \to \infty.$$

In similar way, we obtain

$$\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \|x_n - y_n\| \to 0, \quad n \to \infty.$$

From (4.2.7) and Lemma 4.3.2, we know that there exists z_t such that $z_t = t\eta f(x_t) + (1 - tM) P_n T x_t$ for $t \in (0,1)$. Moreover, $z_t \to x_0 \in F(P_n) = \cap_{n=1}^{N}(A_i + B_i)^{-1}0, \ as\ t \to 0$, and x_0 is the unique solution of the variational inequality (4.3.2).

Next we show that

$$\limsup_{n\to\infty} \langle \eta f(\eta) - M\hat{x}, j(x_n - \hat{x})\rangle \leq 0, \tag{4.3.33}$$

where $\hat{x} = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \longmapsto t\eta f(x) + (1 - tM)P_nTx. \tag{4.3.34}$$

Now, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \eta f(\hat{x}) - M\hat{x}, j(x_n - \hat{x})\rangle = \lim_{k\to\infty} \langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_k} - \hat{x})\rangle. \tag{4.3.35}$$

We may also assume that $x_{n_k} \rightharpoonup q$. Note that $q \in F(P_n)$ by Lemma 2.3.13 and (4.3.33). Since j is weakly sequentially continuous duality mapping, we obtain from Lemma 4.3.2 that

$$\begin{aligned}\limsup_{n\to\infty} \langle \eta f(\hat{x}) - M\hat{x}, j(x_n - \hat{x})\rangle &= \lim_{k\to\infty} \langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_k} - \hat{x})\rangle \\ &= \langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_k} - \hat{x})\rangle \le 0. \qquad (4.3.36)\end{aligned}$$

Hence, we obtain

$$\limsup_{n\to\infty} \langle \eta f(\hat{x}) - M\hat{x}, j(x_n - \hat{x})\rangle \le 0.$$

Finally, we show that $\|x_n - \hat{x}\| \to 0, \quad n \to \infty$. To do this, we divide the rest of the proof into two cases.
By contradiction, there is number ϵ_0 such that

$$\limsup_{n\to\infty} \|x_n - \hat{x}\| \ge \epsilon_0. \tag{4.3.37}$$

case 1. Fixed ϵ_1 $(\epsilon_1 < \epsilon_0)$, if for some $n \ge N \in \mathbb{N}$ such that $\|x_n - \hat{x}\| \ge \epsilon_0 - \epsilon_1$, and for the other $n \ge N \in \mathbb{N}$ such that $\|x_n - \hat{x}\| < \epsilon_0 - \epsilon_1$. Let

$$M_n = \frac{2\langle \eta f(\hat{x}) - M\hat{x}, j(x_{n+1} - \hat{x})\rangle}{(\epsilon_0 - \epsilon_1)^2}. \tag{4.3.38}$$

From (4.3.33), we know that $\limsup_{n\to\infty} M_n \le 0$. Hence, there is a number N, when $n > N$, we have $M_n \le \bar{\gamma} - \eta$. There exists $n_0 \ge N$ such that $\|x_{n_0} - \hat{x}\| < \epsilon_0 - \epsilon_1$, then we have

$$\begin{aligned}\|x_{n_0+1} - \hat{x}\|^2 &= \|\alpha_{n_0} f(x_{n_0}) + \gamma_{n_0} x_{n_0} + [(1-\gamma_{n_0})I - \alpha_{n_0} M] y_{n_0} - \hat{x}\|^2 \\ &= \|[(1-\gamma_{n_0})I - \alpha_{n_0} M](y_{n_0} - \hat{x}) + \alpha_{n_0}(\eta f(x_{n_0}) - M\hat{x}) + \gamma_{n_0}(x_{n_0} - \hat{x})\|^2 \\ &= \langle [(1-\gamma_{n_0})I - \alpha_{n_0} M] y_{n_0} - \hat{x}) + \alpha_{n_0}(\eta f(x_{n_0}) - M\hat{x}) + \gamma_{n_0}(x_{n_0} - \hat{x}), j(x_{n_0+1} - \hat{x})\rangle \\ &= \langle [(1-\gamma_{n_0})I - \alpha_{n_0} M](y_{n_0} - \hat{x}), j(x_{n_0+1} - \hat{x})\rangle + \langle \alpha_{n_0}(\eta f(x_{n_0}) - M\hat{x}), j(x_{n_0+1} - \hat{x})\rangle \\ &\quad + \langle \gamma_{n_0}(x_{n_0} - \hat{x}), j(x_{n_0+1} - \hat{x})\rangle \\ &= \langle [(1-\gamma_{n_0})I - \alpha_{n_0} M](y_{n_0} - \hat{x}), j(x_{n_0+1} - \hat{x})\rangle + \alpha_{n_0}\eta\langle f(x_{n_0}) - f(\hat{x}), j(x_{n_0+1} - \hat{x})\rangle \\ &\quad + \alpha_{n_0}\langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_0+1} - \hat{x})\rangle + \langle \gamma_{n_0}(x_{n_0} - \hat{x}), j(x_{n_0+1} - \hat{x})\rangle \\ &\le (1 - \gamma_{n_0} - \alpha_{n_0}\bar{\gamma})\|x_{n_0} - \hat{x}\|\|x_{n_0+1} - \hat{x}\| + \alpha_{n_0}\eta\|f(x_{n_0}) - f(\hat{x})\|\|x_{n_0+1} - \hat{x}\| \\ &\quad + \alpha_{n_0}\langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_0+1} - \hat{x})\rangle + \gamma_{n_0}\|x_{n_0} - \hat{x}\|\|x_{n_0+1} - \hat{x}\| \\ &< [1 - \alpha_{n_0}(\bar{\gamma} - \eta)](\epsilon_0 - \epsilon_1)\|x_{n_0+1} - \bar{x}\| + \alpha_{n_0}\langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_0+1} - \hat{x})\rangle \\ &\le \frac{1}{2}[1 - \alpha_{n_0}(\bar{\gamma} - \eta)]^2(\epsilon_0 - \epsilon_1)^2 + \frac{1}{2}\|x_{n_0+1} - \hat{x}\|^2 \\ &\quad + \alpha_{n_0}\langle \eta f(\hat{x}) - M\hat{x}, j(x_{n_0+1} - \hat{x})\rangle, \qquad (4.3.39)\end{aligned}$$

which implies from (4.3.39) that

$$\begin{aligned}
\|x_{n_0+1}-\hat{x}\|^2 &\leq [1-\alpha_{n_0}(\bar{\gamma}-\eta)]^2(\epsilon_0-\epsilon_1)^2+2\alpha_{n_0}\langle \eta f(\hat{x})-M\hat{x}, j(x_{n_0+1}-\hat{x})\rangle \\
&\leq [1-\alpha_{n_0}(\bar{\gamma}-\eta)](\epsilon_0-\epsilon_1)^2+2\alpha_{n_0}\langle \eta f(\hat{x})-M\hat{x}, j(x_{n_0+1}-\hat{x})\rangle \\
&= [1-\alpha_{n_0}(\bar{\gamma}-\eta-M_n)](\epsilon_0-\epsilon_1)^2 \\
&\leq (\epsilon_0-\epsilon_1)^2. \qquad (4.3.40)
\end{aligned}$$

Hence, we have

$$\|x_{n_0+1}-\hat{x}\| < \epsilon_0-\epsilon_1, \quad for\ \epsilon_0 > \epsilon_1.$$

In similar manner, we obtain

$$\|x_n-\hat{x}\| < \epsilon_0-\epsilon_1, \quad \forall\ n \geq n_0,$$

which contradicts the fact that $\limsup_{n\to\infty}\|x_n-\hat{x}\| \geq \epsilon_0$.

Case 2. Fixed ϵ_1 $(\epsilon_1 < \epsilon_0)$, if $\|x_n-\hat{x}\| \geq \epsilon_0-\epsilon_1$ for all $n \geq N \in \mathbb{N}$, from Lemma 2.3.25, there is a number r_ϵ, $(0 < r_\epsilon < 1)$ such that

$$\|f(x_n)-f(\hat{x})\| \leq r\|x_n-\hat{x}\|, \quad n \geq N. \qquad (4.3.41)$$

From (4.3.13) and (4.3.41), we obtain

$$\begin{aligned}
\|x_{n_0+1}-\hat{x}\|^2 &= \|\alpha_n\eta f(x_n)+\gamma_n x_n+[(1-\gamma_n)I-\alpha_n M]y_n-\hat{x}\|^2 \\
&= \|[(1-\gamma_n)I-\alpha_n M](y_n-\hat{x})+\alpha_n(\eta f(x_n)-M\hat{x})+\gamma_n(x_n-\hat{x})\|^2 \\
&= \langle[(1-\gamma_n)I-\alpha_n M](y_n-\hat{x})+\alpha_n(\eta f(x_n)-M\hat{x})+\gamma_n(x_n-\hat{x}), j(x_{n_0+1}-\hat{x})\rangle \\
&= \langle[(1-\gamma_n)I-\alpha_n M](y_n-\hat{x}), j(x_{n+1}-\hat{x})\rangle+\langle\alpha_n(\eta f(x_n)-M\hat{x}), j(x_{n+1}-\hat{x})\rangle \\
&\quad +\langle\gamma_n(x_n-\hat{x}), j(x_{n+1}-\hat{x})\rangle \\
&\leq \langle[(1-\gamma_n)I-\alpha_n M](y_n-\hat{x}), j(x_{n+1}-\hat{x})\rangle+\langle\alpha_n(\eta f(x_n)-f(\hat{x})), j(x_{n+1}-\hat{x})\rangle \\
&\quad +\langle\alpha_n\eta f(\hat{x}-M\hat{x}), j(x_{n+1}-\hat{x})\rangle+\langle\gamma_n(x_n-\hat{x}), j(x_{n+1}-\hat{x})\rangle \\
&\leq (1-\gamma_n-\alpha_n\hat{\gamma})\|x_n-\hat{x}\|\|x_{n+1}-\hat{x}\|+\alpha_n\eta r\|x_n-\hat{x}\|\|x_{n+1}-\hat{x}\| \\
&\quad +\langle\alpha_n\eta f(\hat{x}-M\hat{x}), j(x_{n+1}-\hat{x})\rangle+\gamma_n\|x_n-\hat{x}\|\|x_{n+1}-\hat{x}\| \\
&\leq [1-\alpha_n(\hat{\gamma}-\eta r)]\|x_n-x_{n+1}\|\|x_{n+1}-\hat{x}\|+\langle\alpha_n\eta f(\hat{x}-M\hat{x}), j(x_{n+1}-\hat{x})\rangle \\
&\leq [1-\alpha_n(\hat{\gamma}-\eta r)]\frac{1}{2}\|x_n-\hat{x}\|^2+\frac{1}{2}\|x_{n+1}-\hat{x}\|^2+\langle\alpha_n\eta f(\hat{x}-M\hat{x}), j(x_{n+1}-\hat{x})\rangle,
\end{aligned}$$

which implies that

$$\|x_{n+1}-\hat{x}\|^2 \leq [1-\alpha_n(\bar{\gamma}-\eta r)]\|x_n-\hat{x}\|+2\alpha_n\langle\eta f(\hat{x}-M\hat{x}), j(x_{n+1}-\hat{x})\rangle. \qquad (4.3.42)$$

Hence from Lemma 2.3.14 and (4.3.42), we conclude that $x_n \to \hat{x}$ as $n \to \infty$, which contradict the fact that $\|x_n-\hat{x}\| \geq \epsilon_0-\epsilon_1$. This complete the proof. □

If $i = 1$ and f is a contraction, then from Theorem 4.3.3 we obtain the following:

Corollary 4.3.4. *Let X be a real smooth and uniformly convex Banach space and C be a nonempty, closed and convex subset of X, and let $f : C \to C$ be a contraction mapping with $k \in (0,1)$. Let $M : C \to C$ be a strong positive bounded linear operator $\bar{\gamma} > 0$ such that $0 \leq \eta < \frac{2\bar{\gamma}}{k}$. Suppose that the duality mapping $j : X \to X^*$ is weakly sequentially continuous at zero. Let $A : C \to 2^X$ be m-accretive operator and $B : C \to X$ be α-inversely strongly accretive operator, such that $(A+B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be generated by the following algorithm:*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) J_{r_n}^A (I - r_n B) x_n, \\ x_{n+1} = \alpha_n \eta f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n M) y_n, \ n \geq 1, \end{cases} \tag{4.3.43}$$

for all $n \geq 1$, where $J_{r_n}^A = (I + r_n A)^{-1}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real number sequence in $(0,1)$ and $\{r_n\} \subset (0,\infty)$. Suppose that the above sequence satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < r_n < \frac{2\alpha}{c}$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ *for* $n \geq 1$ *and* c *is a constant;*

(iii) $\lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0$;

(iv) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Then $\{x_n\}$ converges strongly to a point $x_0 \in (A+B)^{-1}0$, which is the unique solution of the variational inequality: $\forall z \in (A+B)^{-1}0$.

$$\langle (M - \eta f)x_0, J(x_0 - z)\rangle \leq 0. \tag{4.3.44}$$

where $x_0 = Q_{(A+B)^{-1}(0)} f(x_0)$, and $Q_{(A+B)^{-1}(0)}$ is the unique sunny nonexpansive retraction of X onto $(A+B)^{-1}(0)$.

4.3.1 Applications

In this section, we give an application of our Corollary 4.3.4 to approximation of solution of certain nonlinear integro-differential equation involving the generalized p-Laplacian. Throughout this section, we shall assume $N \geq 1$, $\frac{2N}{N+1} < r \leq \min\{p, p'\} < +\infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, and $\frac{1}{r} + \frac{1}{r'} = 1$.

Let $V = L^p(0,T; W^{1,p}(\Omega))$ and V^* be the dual space of V. The norm in V will be denoted by $\|.\|_v$, which is defined by

$$\|u(x,t)\|_v := \left(\int_0^T \|u(x,t)\|^p_{W^{1,p}(\Omega)} dt \right)^{\frac{1}{p}}, \quad u(x,t) \in V.$$

Also, let $W = L^{\max\{p,p'\}}(0,T; L^{\max\{p,p'\}}(\Omega))$.

Now, using the result obtained in Corollary 4.3.4, we shall study the existence and uniqueness of the solution and iterative approximation of the unique solution of the following nonlinear integro-differential equation.

$$\begin{cases} \frac{\partial u}{\partial t} - \mathrm{div}\left[a(x)\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right] + b(x)|u|^{q-2}u + c(x)|u|^{r-2}u \\ +g(x,u,\nabla u) + a_1\frac{\partial}{\partial t}\int_\Omega u dx = f(x,t) \quad a.e. \text{ in } \Omega\times(0,T) \\ -\left\langle \vartheta, a(x)\left(1+\frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right\rangle \in \beta_x(u(x)) \quad a.e \text{ on } \Gamma\times(0,T) \\ u(x,0) = u(x,T), \end{cases} \tag{4.3.45}$$

where Ω is a bounded conical domain of the Euclidean space $\mathbb{R}^N$, Γ is the boundary Ω with $\Gamma \in C^1$ and ϑ denotes the exterior normal derivatives to Γ. Also $f(x,t) \in W$, a, b and c are strictly positive bounded and continuous functions on Ω such that

$$\begin{aligned} 0 < a^- &= \inf_{x\in\Omega} a(x) \le a^+ = \sup_{x\in\Omega} a(x) < \infty \\ 0 < b^- &= \inf_{x\in\Omega} b(x) \le b^+ = \sup_{x\in\Omega} b(x) < \infty \\ 0 < c^- &= \inf_{x\in\Omega} c(x) \le c^+ = \sup_{x\in\Omega} c(x) < \infty. \end{aligned}$$

Moreover, a_1 is a positive constant and β_x is the subdifferential of ϑ_x, where $\vartheta_x = \vartheta(x,.) : \mathbb{R} \to \mathbb{R}$ for $x \in \Gamma$ and $\vartheta : \Gamma \times \mathbb{R} \to \mathbb{R}$ is the given function.

Lemma 4.3.5. *[183] Let A be m-accretive operator, then the mapping $A : W \to 2^W$ is m-accretive.*

Lemma 4.3.6. *[183] Define $B : D(B) \subset L^{\max\{p,p'\}}(0,T;W^{1,\max\{p,p'\}}(\Omega)) \subset W \to W$ by*

$$(Bu)(x,t) = g(x,u,\nabla u) - f(x,t),$$

for $u(x,t) \in D(B)$. Then B is inversely strongly accretive.

Recently, Y. Shehu and G. Cai [180] proved the following theorem

Theorem 4.3.7. *[180] $u(x,t) \in W$ is the unique solution of the nonlinear boundary value problem (4.3.45) if and only if $u(x,t) \in (A+B)^{-1}(0)$.*

Now, using Theorem 4.3.7, Lemma 4.3.5 and 4.3.6 we obtain the following result.

Theorem 4.3.8. *Let $2 \le p < \infty$. Suppose A and B are the same as those in Lemma 4.3.5 and 4.3.6 respectively. Let $f : W = L^{\max\{p,p'\}}(0,T;L^{\max\{p,p'\}}(\Omega)) \to L^{\max\{p,p'\}}(0,T;L^{\max\{p,p'\}}(\Omega))$ be a fixed contraction with coefficient $k \in (0,1)$. Let $M : L^{\max\{p,p'\}}(0,T;L^{\max\{p,p'\}}(\Omega)) \to L^{\max\{p,p'\}}(0,T;L^{\max\{p,p'\}}(\Omega))$ be a strong positive bounded linear operator $\bar{\gamma} > 0$ such that $0 \le \eta < \frac{2\bar{\gamma}}{k}$. Suppose that the duality mapping $j_{\max\{p,p'\}} : X \to X^*$ is weakly sequentially continuous at zero such that the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < r_n < \frac{2\alpha}{c}$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ *for* $n \geq 1$ *and* c *is a constant;*

(iii) $\lim_{n\to\infty}(\beta_{n+1} - \beta_n) = 0$;

(iv) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Let the sequence $\{u_n(x,t)\}_{n=1}^{\infty}$ *be generated by* $u_1(x,t) \in W$,

$$\begin{cases} y_n = \beta_n u_n(x,t) + (1-\beta_n) J_{r_n}^A (I - r_n B) u_n(x,t), \\ u_{n+1}(x,t) = \alpha_n \eta f(u_n(x,t)) + \gamma_n u_n(x,t) + ((1-\gamma_n)I - \alpha_n M) y_n, \ n \geq 1. \end{cases} \tag{4.3.46}$$

Then $\{u_n(x,t)\}_{n=1}^{\infty}$ *converges strongly to* $u(x,t) \in (A+B)^{-1}(0)$, *which is the unique solution of the variational inequality:* $\forall z(x,t) \in (A+B)^{-1}0$.

$$\langle (M - \eta f) u(x,t), j_{\max\{p,p'\}}(u(x,t) - z(x,t)) \rangle \leq 0. \tag{4.3.47}$$

where $u(x,t) = Q_{(A+B)^{-1}(0)} f(u(x,t))$, *and* $Q_{(A+B)^{-1}(0)}$ *is the unique sunny nonexpansive retraction of* E *onto* $(A+B)^{-1}(0)$.

4.4 Strong convergence results for convex minimization and monotone variational inclusion problems in Hilbert spaces

In this section, motivated by the above work of Y. Shehu and Gang Cai [46], we will modify the GPA by adopting the idea of algorithm to the GPA. We observe that, to prove strong convergence results for the GPA problem and other related optimization problems, the CQ (modified Haugazeau) algorithms are often used. In some other cases (where algorithms other than the CQ algorithm are used), some compactness conditions are assumed on the operators under consideration, or the proof may be divided into two cases which may result to a very long proof. Moreover, our method for proving strong convergence of our algorithm will be different from that of the existing methods in the literature.

Theorem 4.4.1. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Suppose that the minimization problem* (3.1.1) *is consistent and* Υ *denote its solution set such that* $\Upsilon \neq \emptyset$. *Assume that the gradient* ∇g *is L-Lipschitzian with constant* $L > 0$. *Let the sequence* $\{x_n\}$ *be generated for fixed* $x_1, u \in C$ *by*

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n) x_n; \\ x_{n+1} = (1-\beta_n) y_n + \beta_n P_C (I - \lambda_n \nabla g) y_n, \ n \geq 1, \end{cases} \tag{4.4.1}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *and* $\{\lambda_n\}$ *a sequence in* $(0, \frac{2}{L})$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{L}$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $z \in \Upsilon$*, where* $z = P_\Upsilon u$*.*

Proof. It is well known that $z \in C$ solves the minimization problem (3.1.1) if and only if z solves the fixed point equation.

$$z = P_C(I - \lambda\nabla g)z,$$

where $\lambda > 0$ is any fixed positive number. We may assume that (due to condition (ii))

$$0 < a \leq \lambda_n \leq b < \frac{2}{L}, \quad n \geq 1,$$

where a and b are constant. Furthermore it is well known that the gradient ∇g is $\frac{1}{L}$ -ism, $(I - \lambda_n\nabla g)$ is nonexpansive and that $P_C(I - \lambda\nabla g)$ is $\frac{2+\lambda L}{4}$-averaged for $0 < \lambda < \frac{2}{L}$. Hence, we find that for each n, $P_C(I - \lambda_n\nabla g)$ is $\frac{2+\lambda_n L}{4}$-averaged. Therefore we can write

$$P_C(I - \lambda_n\nabla g) = \frac{2 - \lambda L}{4} + \frac{2 + \lambda_n L}{4}T_n = (1 - \mu_n)I + \mu_n T_n, \tag{4.4.2}$$

where T_n are nonexpansive, $\mu_n = \frac{2+\lambda_n L}{4} \in [a_1, b_1] \subset (0, 1)$, $a_1 = \frac{2+aL}{4}$ and $b_1 = \frac{2+bL}{4} < 1$. Let $z_n = P_C(I - \lambda_n\nabla g)y_n$. Then by (4.4.2), we obtain

$$z_n = P_C(I - \lambda_n\nabla g)y_n = (1 - \mu_n)y_n + \mu_n T_n y_n.$$

Firstly, we show that $\{x_n\}$ is bounded. Let $z \in \Upsilon$, from the above equality, we obtain

$$\begin{aligned}
\|z_n - z\|^2 &= \|(1 - \mu_n)y_n + \mu_n T_n y_n - z\|^2 \\
&= (1 - \mu_n)\|y_n - z\|^2 + \mu_n\|T_n y_n - z\|^2 - \mu_n(1 - \mu_n)\|y_n - T_n y_n\|^2 \\
&\leq \|y_n - z\|^2 - \mu_n(1 - \mu_n)\|y_n - T_n y_n\|^2 \\
&\leq \|y_n - z\|^2. \qquad (4.4.3)
\end{aligned}$$

Now, we obtain from (4.4.1) and and the convexity of norm that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|(y_n - z) - \beta_n(y_n - z_n)\|^2 \\
&= \|y_n - z_n\|^2 + \beta_n^2\|y_n - z_n\|^2 - 2\beta_n\langle y_n - z, y_n - z_n\rangle \\
&\leq \|y_n - z_n\|^2 - \beta_n(1 - \beta_n)\|y_n - z_n\|^2. \qquad (4.4.4)
\end{aligned}$$

But,

$$z_n - y_n = \frac{1}{\beta_n}(x_{n+1} - y_n). \tag{4.4.5}$$

Therefore, from (4.4.4), we obtain

$$\|x_{n+1} - z\|^2 \leq \|y_n - z\|^2 - \frac{1}{\beta_n}(1 - \beta_n)\|x_{n+1} - y_n\|^2. \tag{4.4.6}$$

Using (4.4.1) and (4.4.6), we get

$$
\begin{aligned}
\|x_{n+1}-z\| &\leq \|y_n - z\| \\
&\leq \alpha_n\|u-z\| + (1-\alpha_n)\|x_n - z\| \\
&\leq \max\{\|x_n - z\|, \|u-z\|\} \\
&\vdots \\
&\leq \max\{\|x_1 - z\|, \|u-z\|\},
\end{aligned}
$$

which shows that $\{x_n\}$ is bounded. We have from (4.4.5) that

$$
\begin{aligned}
\|z_n - y_n\|^2 &= \left\|\frac{1}{\beta_n}(x_{n+1}-y_n)\right\|^2 \\
&= \frac{1}{\beta_n^2}\|x_{n+1}-y_n\|^2 \\
&= \frac{\alpha_n}{\beta_n}\left(\frac{\|x_{n+1}-y_n\|^2}{\alpha_n\beta_n}\right).
\end{aligned} \tag{4.4.7}
$$

Using Lemma 2.3.6 (ii) and (4.4.1) (noting that $\alpha_n \in (0,1)$), we have

$$
\begin{aligned}
\|y_n - z\|^2 &= \|\alpha_n(u-z) + (1-\alpha_n)(x_n - z)\|^2 \\
&= \alpha_n^2\|u-z\|^2 + 2\alpha_n(1-\alpha_n)\langle u-z, x_n - z\rangle + (1-\alpha_n)^2\|x_n - z\|^2 \\
&\leq \alpha_n^2\|u-z\|^2 + 2\alpha_n(1-\alpha_n)\langle u-z, x_n - z\rangle + (1-\alpha_n)\|x_n - z\|^2 \\
&\leq \alpha_n^2\|u-z\|^2 - 2\alpha_n(1-\alpha_n)\langle u-z, z - x_n\rangle \\
&\quad +(1-\alpha_n)\|x_n - z\|^2.
\end{aligned} \tag{4.4.8}
$$

Putting (4.4.8) in (4.4.6), we obtain

$$
\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \alpha_n^2\|u-z\|^2 - 2\alpha_n(1-\alpha_n)\langle u-z, z-x_n\rangle \\
&\quad +(1-\alpha_n)\|x_n - z\|^2 - \frac{1}{\beta_n}(1-\beta_n)\|x_{n+1}-y_n\|^2 \\
&= (1-\alpha_n)\|x_n - z\|^2 - \alpha_n\Big(-\alpha_n\|u-z\|^2 + 2(1-\alpha_n)\langle u-z, z-x_n\rangle \\
&\quad +\frac{1}{\alpha_n\beta_n}(1-\beta_n)\|x_{n+1}-y_n\|^2\Big).
\end{aligned} \tag{4.4.9}
$$

Let

$$
\Gamma_n := -\alpha_n\|u-z\|^2 + 2(1-\alpha_n)\langle u-z, z-x_n\rangle + \frac{1}{\alpha_n\beta_n}(1-\beta_n)\|x_{n+1}-y_n\|^2,\ n \geq 1 \tag{4.4.10}
$$

Then, (4.4.9) becomes

$$
\|x_{n+1}-z\|^2 \leq (1-\alpha_n)\|x_n - z\|^2 - \alpha_n\Gamma_n. \tag{4.4.11}
$$

Since $\{x_n\}$ is bounded an so it is bounded below. Hence, Γ_n is bounded below. Furthermore, using Lemma 2.3.17 and condition (i) in (4.4.11), we obtain

$$
\begin{aligned}
\limsup_{n\to\infty}\|x_n - z\|^2 &\leq \limsup_{n\to\infty}(-\Gamma_n) \\
&= -\liminf_{n\to\infty}\Gamma_n.
\end{aligned} \tag{4.4.12}
$$

Therefore, $\liminf_{n\to\infty}\Gamma_n$ is a finite. We have from (4.4.10) that

$$\liminf_{n\to\infty}\Gamma_n = \liminf_{n\to\infty}\left(2\langle u-z, z-x_n\rangle + \frac{1}{\alpha_n\beta_n}(1-\beta_n)\|x_{n+1}-y_n\|^2\right).$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q \in H$ and

$$\liminf_{n\to\infty}\Gamma_n = \lim_{k\to\infty}\left(2\langle u-z, z-x_{n_k}\rangle + \frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})\|x_{n_k+1}-y_{n_k}\|^2\right). \tag{4.4.13}$$

Since $\{x_n\}$ is bounded and $\liminf\limits_{n\to\infty}\Gamma_n$ is finite, we have that $\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})\|x_{n_k+1}-y_{n_k}\|^2$ is bounded. Also, by assumption (iii), we have that there exists $b\in(0,1)$ such that $\beta_n \le b < 1$ and this implies that $\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k}) \ge \frac{1}{\alpha_{n_k}\beta_{n_k}}(1-b) > 0$ and so we have that $\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})\|x_{n_k+1}-y_{n_k}\|^2$ is bounded. Observe from assumptions (i) and (iii) that there exists $a\in(0,1)$ such that

$$0 < \frac{\alpha_{n_k}}{\beta_{n_k}} \le \frac{\alpha_{n_k}}{a} \to 0,\ k\to\infty.$$

This implies that $\frac{\alpha_{n_k}}{\beta_{n_k}} \to 0,\ k\to\infty$. Therefore, we obtain from (4.4.7) and $\frac{\alpha_{n_k}}{\beta_{n_k}} \to 0,\ k\to\infty$ that

$$\|z_{n_k}-y_{n_k}\| = ||P_C(I-\lambda_{n_k}\nabla g)y_{n_k} - y_{n_k}|| \to 0,\ k\to\infty. \tag{4.4.14}$$

From (4.4.5) and (4.4.14), we have that

$$\|x_{n_k+1}-y_{n_k}\| = \beta_{n_k}\|z_{n_k}-y_{n_k}\| \to 0,\ k\to\infty.$$

Hence,

$$\|x_{n_k+1}-x_{n_k}\| \le \|x_{n_k+1}-y_{n_k}\| + \|y_{n_k}-x_{n_k}\| \to 0,\ k\to\infty.$$

Observe that $y_{n_k} \rightharpoonup x^*\in C,\ k\to\infty$ since $y_{n_k}-x_{n_k}\to 0,\ k\to\infty$ and $x_{n_k}\rightharpoonup x^*\in C,\ k\to\infty$. We may assume that $\lambda_{n_k}\rightharpoonup\lambda$; then we have $0<\lambda<\frac{2}{L}$. Set $T := P_C(I-\lambda\nabla g)$, then T is nonexpansive and we get from (4.4.14) that

$$\begin{aligned}
\|P_C(I-\lambda\nabla g)y_{n_k}-y_{n_k}\| &\le \|P_C(I-\lambda\nabla g)y_{n_k} - P_C(I-\lambda_{n_k}\nabla g)y_{n_k}\| + \|P_C(I-\lambda_{n_k}\nabla g)y_{n_k}-y_{n_k}\|\\
&\le \|(I-\lambda\nabla g)y_{n_k} - (I-\lambda_{n_k}\nabla g)y_{n_k}\| + \|P_C(I-\lambda_{n_k}\nabla g)y_{n_k}-y_{n_k}\|\\
&= |\lambda_{n_k}-\lambda|\|\nabla g(y_{n_k})\| + \|P_C(I-\lambda_{n_k}\nabla g)y_{n_k}-y_{n_k}\| \to 0.
\end{aligned}$$

It then follows from Lemma 2.3.13 that $x^*\in F(T)$. But $F(T)=\Upsilon$. Therefore we have that $x^*\in\Upsilon$. Now, we obtain from (4.4.10) and the property of P_Υ that

$$\begin{aligned}
\liminf_{n\to\infty}\Gamma_n &= \lim_{k\to\infty}\left(2\langle u-z, z-x_{n_k}\rangle + \frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})\|x_{n_k+1}-y_{n_k}\|^2\right)\\
&\ge 2\lim_{k\to\infty}\langle u-z, z-x_{n_k}\rangle\\
&= 2\langle u-z, z-x^*\rangle \ge 0.
\end{aligned} \tag{4.4.15}$$

Then from (4.4.12), we have that

$$\limsup_{n\to\infty}\|x_n-z\|^2 \le -\liminf_{n\to\infty}\Gamma_n \le 0.$$

Therefore, $\lim_{n\to\infty}\|x_n-z\|=0$ and this implies that $\{x_n\}$ converges strongly to z. □

We next investigate the problem of finding a zero of the sum of two monotone operators, which is formulated as the following monotone variational inclusion problem: Find $x \in H$ such that

$$0 \in (A+B)x, \tag{4.4.16}$$

where $A : H \to H$ and $B : H \to 2^H$ are two monotone operators in Hilbert space H.

Lemma 4.4.2. *Let C be a nonempty subset of H, $\nu \in \mathbb{R}^+$, $T : C \to H$ be ν-ism and $\gamma \in (0, 2\nu)$. Then $I - \gamma T$ is $\gamma/2\nu$-averaged.*

Proof. Set $N = I - 2\nu T$. Since T is ν-ism, we obtain from Lemma (2.3.6) (i) that

$$\begin{aligned} \|Nx - Ny\|^2 &= \|(I - 2\nu T)x - (I - 2\nu T)y\|^2 \\ &= \|(x-y) - 2\nu(Tx - Ty)\|^2 \\ &= \|x-y\|^2 + 4\nu^2\|Tx - Ty\|^2 - 4\nu\langle x - y, Tx - Ty\rangle \\ &\leq \|x-y\|^2 + 4\nu^2\|Tx - Ty\|^2 - 4\nu^2\|Tx - Ty\| \\ &= \|x-y\|^2. \end{aligned}$$

Hence, N is nonexpansive. Thus, we obtain that

$$I - \gamma T = (1 - \gamma/2\nu)I + (\gamma/2\nu)I - \gamma T = (1 - \gamma/2\nu)I + (\gamma/2\nu)N.$$

Since $\gamma \in (0, 2\nu)$, then $\gamma/2\nu \in (0,1)$, thus we have that $I - \gamma T$ is $\gamma/2\nu$-averaged. □

We shall assume that problem (4.4.16) is consistent, namely its solution set, denoted by Θ is nonempty. We now introduce an iterative algorithm that converges strongly to a solution of (4.4.16). More accurately, our algorithm starts with an arbitrary initial guess $x_0 \in H$, and generates x_{n+1} according to the recursion process

$$\begin{cases} y_n = \alpha_n u + (1 - \alpha_n)x_n; \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n J_{\gamma_n B}(I - \gamma_n A)y_n. \end{cases} \tag{4.4.17}$$

Noting that $\Theta = F(J_{\gamma_n B}(I - \gamma_n A))$ i.e $x \in \Theta$, if and only if

$$\begin{aligned} 0 \in (A+B)x = Ax + Bx &\Leftrightarrow x - \gamma Ax \in x + \gamma Bx \\ &\Leftrightarrow x \in (I + \gamma B)^{-1}(I - \gamma A)x \\ &\Leftrightarrow x = J_{\gamma B}(I - \gamma A)x. \end{aligned}$$

Let $z \in \Theta$, from Lemma 4.4.2, we obtain that $I - \gamma_n A$ is $\gamma_n A/2\nu$-averaged for every $n \in \mathbb{N}$. Since $J_{\gamma_n B}$ is nonexpansive, then it is $\frac{1}{2}$-averaged. It follows from Lemma 2.3 (ii) that $J_{\gamma_n B}(I - \gamma_n A)$ is $(2\nu + \gamma_n)/4\nu$ - averaged. Let $\mu_n = \frac{2\nu + \gamma_n}{4\nu}$, in view of $\gamma_n \in (0, 2\nu)$, we have that $\mu_n \in (0,1)$. So $J_{\gamma_n B}(I - \gamma_n A)$ is μ_n-averaged. Hence it follows from Definition 1.2 that

$$J_{\gamma_n B}(I - \gamma_n A) = (1 - \mu_n)I + \mu_n T_n, \tag{4.4.18}$$

where T_n is nonexpansive for every $n \in \mathbb{N}$. By (4.4.18), we obtain that

$$F(J_{\gamma_n B}(I - \gamma_n A)) = F(T_n).$$

Hence we obtain the following strong convergence theorem for finding a zero of the sum of two monotone operators.

Theorem 4.4.3. *Let $A : H \to H$ be a ν-inverse strongly monotone mapping and $B : H \to 2^H$ be a monotone mapping. Let $\{\gamma_n\}$ be a sequence in $(0, 2\nu)$ and let the following conditions be satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 2\nu$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ generated by (4.4.17) converges strongly to a solution z of the inclusion problem (4.4.16).

Proof. Set $J_{\gamma_n B}(I - \gamma_n A)y_n = P_C(I - \lambda_n \nabla g)y_n$, we obtain the desired result from Theorem 4.4.1 □

4.4.1 Application to split feasibility problem

In this section, we give an application of Theorem 4.4.1 to Split Feasibility Problem (SFP).

The SFP is finding a point x such that

$$x \in C \quad \text{and} \quad Bx \in Q, \tag{4.4.19}$$

where C and Q are nonempty, closed and convex subsets of Hilbert space H_1 and H_2, respectively and $B : H_1 \to H_2$ is a bounded linear operator.

Clearly x^* is a solution to the SFP (4.4.19) if and only if $x^* \in C$ and $Bx^* - P_Q Bx^* = 0$. The proximity function g is defined by

$$g(x) = \frac{1}{2}\|Bx - P_Q Bx\|^2 \tag{4.4.20}$$

and we consider the constrained convex minimization problem

$$\min_{x\in C} g(x) = \min_{x\in C} \frac{1}{2}\|Bx - P_Q Bx\|^2. \tag{4.4.21}$$

Then x^* solves the SFP (4.4.19) if and only if x^* solves the minimization problem (4.4.21). In [9], the following CQ algorithm was introduced to solve the SFP,

$$x_{n+1} = P_C(I - \lambda B^*(I - P_Q)B)x_n, \qquad n \geq 0 \tag{4.4.22}$$

where $0 < \lambda < \frac{2}{\|B\|^2}$ and B^* is the adjoint of B. It was proved that the sequence generated by (4.4.22) converges weakly to a solution of the SFP.

We now propose the following algorithm as an application of Theorem 4.4.1 to obtain a strong convergence iterative sequence to solve the SFP.

Theorem 4.4.4. *Let C and Q be nonempty, closed and convex subset of real Hilbert spaces H_1 and H_2 respectively, and $B : H_1 \to H_2$ be bounded linear operator. Let $g(x) = \frac{1}{2}\|Bx - P_Q Bx\|^2$ and let $\Upsilon = argmin_{c\in C} g(x)$ such that $\Upsilon \neq \emptyset$. Let the sequence $\{x_n\}$ be generated for fixed $x_1, u \in C$ by*

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)x_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n P_C(I - \lambda_n B^*(I - P_Q)B)y_n, \end{cases} \tag{4.4.23}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{\lambda_n\}$ a sequence in $(0, \frac{2}{\|B\|^2})$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{\|B\|^2}$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to a solution z of the split feasibility problem (4.4.19)

Proof. By the definition of proximity function g, we have that $\nabla g = B^*(I - P_Q)B$ and ∇g is $||B||^2$-Lipschitz continuous. Hence, by setting $\nabla g = B^*(I - P_Q)B$ in Algorithm (4.4.1), we obtain the desired result. □

4.4.2 Numerical example

In this section, we present numerical example with $H_1 = H_2 = \mathbb{R}^4$ to illustrate the performance of our algorithm.

Example 4.4.5. *Let $\nabla g(x) = B^*(I - P_Q)Bx$, where $Bx = (3x_1 + x_2 + x_3 - x_4, -2x_1 - x_2 - 3x_3 + 2x_4, -4x_1 - x_2 + 5x_3 - 2x_4, x_1 - x_2 + x_3 - x_4)$, $Q = \{x \in \mathbb{R}^4 : \langle w, x\rangle = b\}$, $w = (-1, 2, 4, 7)^T$, $b = 2$, $P_Q(x) = \max\left\{0, \frac{b - \langle w,x\rangle}{||w||^2}\right\} w + x$. Since B is a bounded linear operator and P_Q is a metric projection onto Q, then ∇g is L-Lipschitz continuous with $L = ||B||^2 = 50$. Let $C = \{x \in \mathbb{R}^4 : \langle y, x\rangle \geq a\}$, $y = (2, -5, -7, 1)^T$, $a = 3$, $P_C(x) = \frac{a - \langle y,x\rangle}{||y||^2} y + x$.*

Now, take $\alpha_n = \frac{1}{3n+1}$, $\beta_n = \frac{n+1}{3n}$ and $\lambda_n = \frac{n}{25n+3}$. Hence, Algorithm (4.4.23) becomes

$$\begin{cases} y_n = \frac{1}{3n+1} u + (1 - \frac{1}{3n+1})x_n; \\ x_{n+1} = (1 - \frac{n+1}{3n})y_n + \frac{n+1}{3n} P_C(I - \lambda_n B^*(I - P_Q)B)y_n, \ n \geq 1. \end{cases} \tag{4.4.24}$$

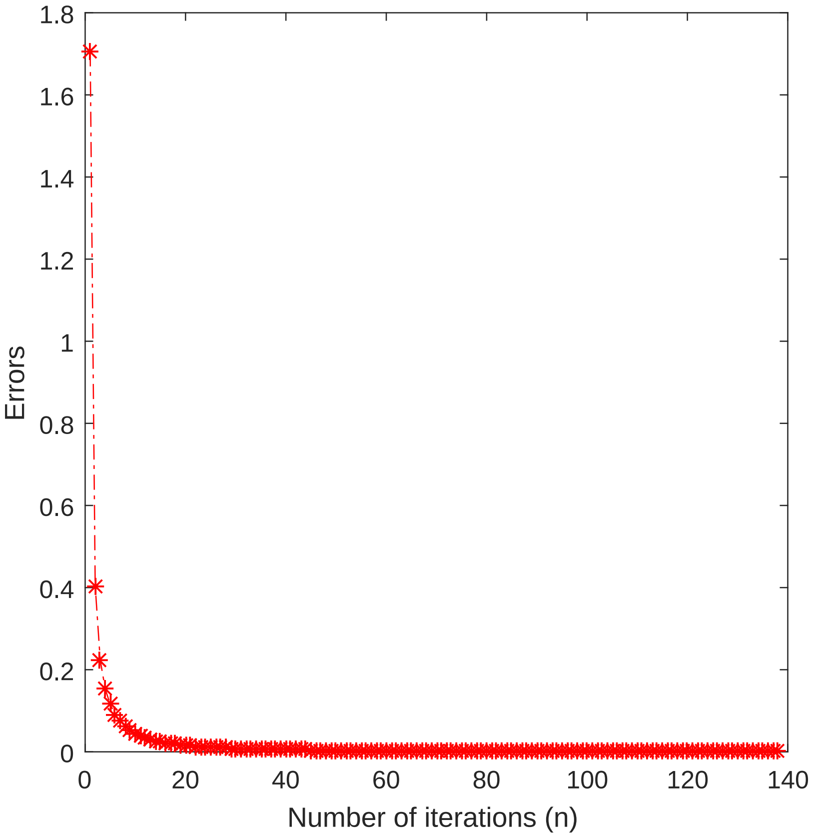

Figure 4.1: Errors vs number of iterations for **Case I.**

We consider the following cases.

Case I *Take* $x_1 = (-1, 2, 1, 0.5)^T$ *and* $u = (-1, 2, 1, -2)^T$.

Case II *Take* $x_1 = (-1, 2, 1, 0.5)^T$ *and* $u = (-1, 7, 1, -5)^T$.

Case II *Take* $x_1 = (1, 7, -5, 3)^T$ *and* $u = (-1, 7, 1, -5)^T$.

Remark 4.4.6. *We remark from this numerical example that different choices of* x_1 *and* u *within the specified spaces and range have no effect on the number of iterations required for convergence and with very insignificant effect on the cpu run time as can be seen from the Table aand corresponding figures. Th representation of the error* ($\|x_{n+1} - x_n\|$) *against the number of iterations in the figure. This is because the values of the error are either thesame or very close to each other and so the curves overlap each other. Also, we have seen from the table and graphs that the closer the values of* x_n *to zero, the less the number of iterations required for the convergence.*

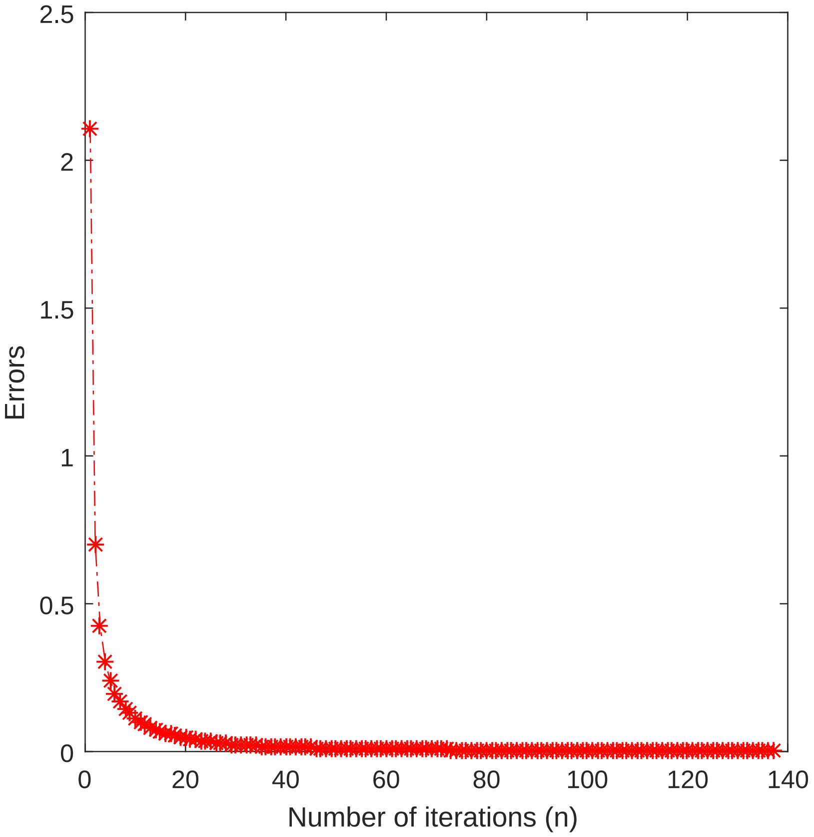

Figure 4.2: Errors vs number of iterations for **Case II**.

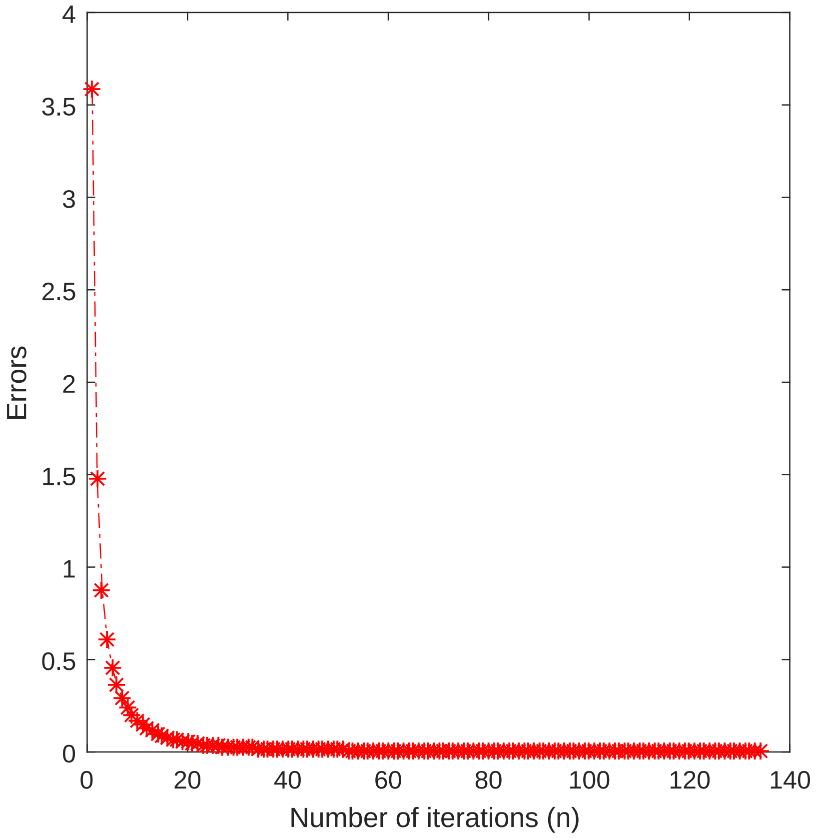

Figure 4.3: Errors vs number of iterations for **Case III**.

Table 4.1: Showing numerical results for **Case I**, **Case II** and **Case III**.

No. of iteration	**Errors for Case I**	**Errors for Case II**	**Errors for Case III**
1	-	-	-
2	1.7062	2.1097	3.5829
3	0.4044	0.7020	1.4746
4	0.2234	0.4231	0.8781
5	0.1531	0.3037	0.6089
6	0.1151	0.2383	0.4571
7	0.0911	0.1967	0.3596
8	0.0744	0.1675	0.2916
9	0.0622	0.1454	0.2415
10	0.0529	0.1280	0.2031
11	0.0456	0.1138	0.1729
12	0.0397	0.1018	0.1486
13	0.0349	0.0917	0.1288
14	0.0309	0.0830	0.1125
15	0.0276	0.0754	0.0989
16	0.0248	0.0687	0.0875
17	0.0224	0.0629	0.0779
18	0.0204	0.0577	0.0697
19	0.0186	0.0531	0.0627
20	0.0171	0.0490	0.0566
21	0.0157	0.0454	0.0514
22	0.0145	0.0421	0.0469
23	0.0135	0.0392	0.0430
24	0.0126	0.0366	0.0395
25	0.0118	0.0342	0.0365
26	0.0110	0.0321	0.0338
27	0.0104	0.0302	0.0315
28	0.0098	0.0284	0.0294
29	0.0092	0.0268	0.0275
30	0.0088	0.0253	0.0258

Chapter 5

Split Feasibility Variational Inequality and Fixed Point Problems

5.1 Introduction

In this chapter, we study a new class of split variational inequality and variational inclusion problem in Hilbert space. Also we studied the common solution of equilibrium, variational inequality and fixed point problems for multivalued mapping with an application to split monotone inclusion problem in Hilbert space. Lastly we proved a strong convergence result for finding fixed point of a multivalued Lipschitz hemicontractive mapping which is also a solution of a monotone variational inclusion problem.

5.2 Strong convergence theorem for a new class of split variational inequality and monotone variational inclusion problem in Hilbert spaces

In this section, motivated by the work of Tian *et al* [169], we propose an iterative algorithm for the class of generalized split feasibility problem for finding an element that solves a class of variational inequality problem and such that its image under a bounded linear operator is in a fixed point set of a pseudocontractive mappings. We also prove that our algorithm converges strongly to the common solutions. In all our results the step-size is selected in such a way that its implementation does not involve the computation or an estimate of the operator norm. Hence Our result improve and extend the result of Tian *et al* [169] and other results in this direction.

Theorem 5.2.1. *Let* H_1 *and* H_2 *be real Hilbert spaces and let* K *be a nonempty closed and convex subset of* H_1. *Let* $A : H_1 \to H_2$ *be a bounded linear operator,* $f : K \to H_1$ *be* $1/\alpha$*-ism mapping and* $T : H_2 \to H_2$ *be* μ*-pseudo contractive mapping for* $0 < \mu < 1$ *such that* $\Upsilon = \{x^* \in VI(K, f) : \ Ax^* \in F(T)\} \neq \emptyset$. *Let the step size* γ_n *be chosen such that for* $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \frac{(1-\mu)\|TAu_n - Au_n\|^2}{\|A^*(T-I)Au_n\|^2} - \epsilon\right)$, *if* $TAu_n \neq Au_n$; *otherwise* $\gamma_n = \gamma(\gamma$ *being any* *nonnegative real number). Suppose the sequence* $\{x_n\}$ *is generated for arbitrary* $x_1, u \in K$ *by*

$$\begin{cases} u_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = P_K(u_n + \gamma_n A^*(T-I)Au_n), \\ x_{n+1} = P_K(y_n - \lambda_n f(y_n)), \end{cases} \tag{5.2.1}$$

where the sequences $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *satisfy the following conditions:*

(i) $\alpha_n \in (0,1),\ \sum\limits_{n\geq 1}\alpha_n = +\infty,$

(ii) $\lim\limits_{n\to\infty}\alpha_n = 0,$

(iii) $\lambda_n \in (0, 2/\alpha), \alpha > 0.$

Then $\{x_n\}$ *converges strongly to* $x^* \in \Upsilon$*, where* $x^* = P_\Upsilon u$.

Proof. First we show that $(I-\lambda_n f)$ is nonexpansive. For all $x, y \in K$ and $\lambda_n \in (0, 2/\alpha)$, we obtain

$$\begin{aligned} \|(I-\lambda_n f)x - (I-\lambda_n f)y\|^2 &= \|(x-y) - \lambda_n(f(x)-f(y)\|^2 \\ &= \|x-y\|^2 - 2\lambda_n\langle x-y, f(x)-f(y)\rangle + \lambda_n^2\|f(x)-f(y)\|^2 \\ &\leq \|x-y\|^2 - \frac{2\lambda_n}{\alpha}\|f(x)-f(y)\|^2 + \lambda_n^2\|f(x)-f(y)\|^2 \\ &\leq \|x-y\|^2 + \lambda_n\left(\lambda_n - \frac{2}{\alpha}\right)\|f(x)-f(y)\|^2 \\ &\leq \|x-y\|^2. \end{aligned}$$

Hence $(I-\lambda_n f)$ is nonexpansive.
Next we show that $\{x_n\}$ is bounded.
From (3.2.1), we have

$$\begin{aligned} \|y_n - p\|^2 &= \|P_K(u_n + \gamma_n A^*(T-I)Au_n - p\|^2 \\ &\leq \|u_n + \gamma_n A^*(T-I)Au_n - p\|^2 \\ &\leq \|u_n - p\|^2 + \gamma_n^2\|A^*(T-I)Au_n\|^2 + 2\gamma_n\langle u_n - p, A^*(T-I)Au_n\rangle \\ &\leq \|u_n - p\|^2 + \gamma_n^2\|A^*(T-I)Au_n\|^2 + 2\gamma_n\langle A(u_n-p), (T-I)Au_n\rangle \\ &= \|u_n - p\|^2 + \gamma_n^2\|A^*(T-I)Au_n\|^2 \\ &\quad +2\gamma_n\left[\langle Au_n - Ap, TAu_n - Ap\rangle - \|Au_n - Ap\|^2\right] \\ &\leq \|u_n - p\|^2 + \gamma_n^2\|A^*(T-I)Au_n\|^2 \\ &\quad +2\gamma_n\left[\|Au_n - Ap\|^2 + \frac{(\mu-1)}{2}\|(Au_n - TAu_n) - (Ap - TAp)\|^2 - \|Au_n - Ap\|^2\right] \\ &= \|u_n - p\|^2 + \gamma_n\left[\gamma_n\|A^*(T-I)Au_n\|^2 + (\mu-1)\|Au_n - TAu_n\|^2\right]. \end{aligned} \tag{5.2.2}$$

From the choice of γ_n, (5.2.2), (5.2.1) and using the fact that $(I-\lambda_n f)$ is nonexpansive, we obtain

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \|P_K(y_n-\lambda_n f(y_n)-p\| \\
&\leq \|y_n-\lambda_n f(y_n)-p\| \\
&\leq \|y_n-p\| \\
&\leq \|u_n-p\| \\
&= \|(1-\alpha_n)(x_n-p)+\alpha_n(u-p)\| \\
&\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\|u-p\| \\
&\leq \max\{\|x_n-p\|,\|u-p\|\} \\
&\vdots \\
&\leq \max\{\|x_1-p\|,\|u-p\|\}.
\end{aligned}$$

Hence $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are bounded. We now divide the rest of the proof into two cases to get strong convergence.

Case 1. Suppose that $\{\|x_n-p\|\}$ monotonically decreasing, we have that $\lim_{n\to\infty}\{\|x_n-p\|\}$ exists. Hence

$$\|x_{n+1}-p\|-\|x_n-p\| \to, \ n\to\infty. \tag{5.2.3}$$

If $TAu_n = Au_n$, then

$$\lim_{n\to\infty}\|A^*(T-I)Au_n\|=0.$$

Suppose that $TAu_n \neq Au_n$, then $\gamma_n \in \left(\epsilon, \frac{(1-\mu)\|TAu_n-Au_n\|^2}{\|A^*(T-I)Au_n\|^2}-\epsilon\right)$ and we have from (5.2.2) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|u_n-p\|^2+\gamma_n\left[\gamma_n\|A^*(T-I)Au_n\|^2+(\mu-1)\|TAu_n-Au_n\|^2\right] \\
&\leq \|x_n-p\|^2+\alpha_n^2\|x_n-u\|^2-2\alpha_n\langle x_n-p,x_n-u\rangle \\
&\quad +\gamma_n\left[\gamma_n\|A^*(T-I)Au_n\|^2+(\mu-1)\|TAu_n-Au_n\|^2\right] \\
&\leq \|x_n-p\|^2+\alpha_n^2\|x_n-u\|^2-2\alpha_n\langle x_n-p,x_n-u\rangle \\
&\quad -\gamma_n\epsilon\|A^*(T-I)Au_n\|^2.
\end{aligned} \tag{5.2.4}$$

Therefore,

$$\begin{aligned}
\gamma_n\epsilon\|A^*(T-I)Au_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2\|x_n-u\|^2 \\
&\quad -\alpha_n\langle x_n-p,x_n-u\rangle.
\end{aligned} \tag{5.2.5}$$

Thus, we obtain

$$\lim_{n\to\infty}\|A^*(T-I)Au_n\|=0. \tag{5.2.6}$$

Also from (5.2.4), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2+\alpha_n^2\|x_n-u\|^2-2\alpha_n\langle x_n-p,x_n-u\rangle \\
&\quad +\gamma_n\left[\gamma_n\|A^*(T-I)Au_n\|^2+(\mu-1)\|TAu_n-Au_n\|^2\right],
\end{aligned}$$

which implies

$$\begin{aligned}\gamma_n(1-\mu)\|TAu_n - Au_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n^2\|x_n - u\|^2\\ &\quad -2\alpha_n\langle x_n - p, x_n - u\rangle + \gamma_n^2\|A^*(T-I)Au_n\|^2.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|TAu_n - Au_n\| = 0. \tag{5.2.7}$$

From (5.2.1), we have

$$\|u_n - x_n\| = \alpha_n\|x_n - u\| \to 0,\ n\to\infty. \tag{5.2.8}$$

Again from (5.2.2), we obtain

$$\begin{aligned}\|y_n - p\|^2 &= \|P_K(u_n + \gamma_n A^*(T-I)Au_n) - p\|^2\\ &\leq \langle y_n - p, u_n + \gamma_n A^*(T-I)Au_n - p\rangle\\ &= \frac{1}{2}[\|y_n - p\|^2 + \|u_n + \gamma_n A^*(T-I)Au_n - p\|^2\\ &\quad -\|y_n - p - (u_n + \gamma_n A^*(T-I)Au_n - p)\|^2]\\ &\leq \frac{1}{2}[\|y_n - p\|^2 + \|u_n - p\|^2 + \gamma_n\left[\gamma_n\|A^*(T-I)Au_n\|^2 + (\mu-1)\|Au_n - TAu_n\|^2\right]\\ &\quad -\|y_n - u_n - \gamma_n A^*(T-I)Au_n\|^2]\\ &\leq \frac{1}{2}[\|y_n - p\|^2 + \|u_n - p\|^2 - (\|y_n - u_n\|^2 + \gamma_n^2\|A^*(T-I)Au_n\|^2\\ &\quad -2\gamma_n\langle y_n - u_n, A^*(T-I)Au_n\rangle)]\\ &\leq \frac{1}{2}[\|y_n - p\|^2 + \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2\gamma_n\|Ay_n - Au_n\|\|(T-I)Au_n\|.\end{aligned}$$

That is,

$$\|y_n - p\|^2 \leq \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2\gamma_n\|Ay_n - Au_n\|\|(T-I)Au_n\|. \tag{5.2.9}$$

Hence, from (5.2.9), we have

$$\begin{aligned}\|y_n - u_n\|^2 &\leq \|u_n - p\|^2 - \|y_n - p\|^2 + 2\gamma_n\|Ay_n - Au_n\|\|(T-I)Au_n\|\\ &\leq \|x_n - p + \alpha_n(u - x_n)\|^2 - \|x_{n+1} - p\|^2 + 2\gamma_n\|Ay_n - Au_n\|\|(T-I)Au_n\|\\ &= \|x_n - p\|^2 + 2\alpha_n\langle u - x_n, x_n - p\rangle + \alpha_n^2\|x_n - u\|^2 - \|x_{n+1} - p\|^2\\ &\quad +2\gamma_n\|Ay_n - Au_n\|\|(T-I)Au_n\|\\ &\leq \|x_n - p\|^2 + 2\alpha_n\|x_n - u\|\|x_n - p\| + \alpha_n^2\|x_n - u\|^2 - \|x_{n+1} - p\|^2\\ &\quad +2\gamma_n\|Ay_n - Au_n\|\|(T-I)Au_n\|.\end{aligned}$$

This with condition (ii) implies that

$$\begin{aligned}\|y_n - u_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\|x_n - u\|^2 + 2\alpha_n\|x_n - u\|\|x_n - p\|\\ &\quad +2\gamma_n\|Ay_n - Au_n\|\|(T-I)Au_n\| \to 0,\ n\to\infty.\end{aligned} \tag{5.2.10}$$

Therefore,

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{5.2.11}$$

Next we show that $\lim\limits_{n\to\infty} \|f(y_n) - f(p)\| = 0$.

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(y_n - \lambda_n f(y_n)) - (p - \lambda_n f(p))\|^2 \\
&= \|y_n - p\|^2 - 2\lambda_n \langle y_n - p, f(y_n) - f(p)\rangle + \lambda_n^2 \|f(y_n) - f(p)\|^2 \\
&\leq \|y_n - p\|^2 - \frac{2\lambda_n}{\alpha}\|f(y_n) - f(p)\|^2 + \lambda_n^2 \|f(y_n) - f(p)\|^2 \\
&\leq \|u_n - p\|^2 - \frac{2\lambda_n}{\alpha}\|f(y_n) - f(p)\|^2 + \lambda_n^2 \|f(y_n) - f(p)\|^2 \\
&\leq \|x_n - p + \alpha_n(u - x_n)\|^2 + \lambda_n \left(\lambda_n - \frac{2}{\alpha}\right) \|f(y_n) - f(p)\|^2 \\
&= \|x_n - p\|^2 + 2\alpha_n \langle u - x_n, x_n - p\rangle + \alpha_n^2 \|x_n - u\|^2 + \lambda_n \left(\lambda_n - \frac{2}{\alpha}\right) \|f(y_n) - f(p)\|^2 \\
&\leq \|x_n - p\|^2 + 2\alpha_n \|x_n - u\| \|x_n - p\| + \alpha_n^2 \|x_n - u\|^2 + \lambda_n \left(\lambda_n - \frac{2}{\alpha}\right) \|f(y_n) - f(p)\|^2.
\end{aligned}$$

This implies

$$\begin{aligned}
a\left(\frac{2}{\alpha} - b\right) \|f(y_n) - f(p)\|^2 &\leq \lambda_n \left(\frac{2}{\alpha} - \lambda_n\right) \|f(y_n) - f(p)\|^2 \\
&\leq 2\alpha_n \|x_n - u\| \|x_n - p\| + \alpha_n^2 \|x_n - u\|^2 \\
&\quad + \|x_n - p\|^2 - \|x_{n+1} - p\|^2.
\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|f(y_n) - f(p)\| = 0. \tag{5.2.12}$$

From (5.2.1) and Proposition 2.1.14 (i),(noting that $(I - \lambda_n f)$ is nonexpansive), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|P_C(y_n - \lambda_n f(y_n) - P_C(p - \lambda_n f(p))\|^2 \\
&\leq \langle y_n - \lambda_n f(y_n) - (p - \lambda_n f(p)), x_{n+1} - p\rangle \\
&\leq \frac{1}{2}\Big[\|(y_n - \lambda_n f(y_n)) - (p - \lambda_n f(p)\|^2 + \|x_{n+1} - p\|^2 \\
&\quad - \|(y_n - \lambda_n f(y_n) - (p - \lambda_n f(p)) - (x_{n+1} - p)\|^2\Big] \\
&\leq \frac{1}{2}\left[\|y_n - p\|^2 + \|x_{n+1} - p\|^2 - \|(x_{n+1} - y_n) - \lambda_n(f(y_n) - f(p))\|^2\right] \\
&\leq \frac{1}{2}\Big[\|y_n - p\|^2 + \|x_{n+1} - p\|^2 - \|x_{n+1} - y_n\|^2 + 2\lambda\langle x_{n+1} - y_n, f(y_n) - f(p)\rangle \\
&\quad - \lambda_n^2 \|f(y_n) - f(p)\|^2\Big].
\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|y_n-p\|^2-\|x_{n+1}-y_n\|^2+2\lambda_n\langle x_{n+1}-y_n, f(y_n)-f(p)\rangle\\ &\quad -\lambda_n^2\|f(y_n)-f(p)\|^2.\end{aligned} \tag{5.2.13}$$

Hence from (5.2.13), we obtain

$$\begin{aligned}\|x_{n+1}-y_n\|^2 &\leq \|y_n-p\|^2-\|x_{n+1}-y_n\|^2+2\lambda_n\langle x_{n+1}-y_n, f(y_n)-f(p)\rangle\\ &\quad -\lambda_n^2\|f(y_n)-f(p)\|^2\\ &\leq \alpha_n\|u-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\lambda_n\langle x_{n+1}-y_n, f(y_n)-f(p)\rangle\\ &\quad -\lambda_n^2\|f(y_n)-f(p)\|^2\\ &\leq \alpha_n\|u-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\lambda_n\|x_{n+1}-y_n\|\|f(y_n)-f(p)\|-\lambda_n^2\|f(y_n)\end{aligned}$$

By using (5.2.12) and (5.2.14), we have

$$\lim_{n\to\infty}\|P_K(y_n-\lambda_n f(y_n))-y_n\|=\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{5.2.15}$$

Also from (5.2.11) and (5.2.15), we obtain

$$\|x_{n+1}-u_n\|=\|x_{n+1}-y_n\|+\|y_n-u_n\|\to 0,\ n\to\infty. \tag{5.2.16}$$

From (5.2.8) and (5.2.15), we get

$$\|x_{n+1}-x_n\|=\|x_{n+1}-u_n\|+\|u_n-x_n\|\to 0,\ n\to\infty. \tag{5.2.17}$$

Since $\{x_n\}$ is bounded and H_1 is a Hilbert space there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\rightharpoonup x^*$, for some $x^*\in H_1$. Furthermore, $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$, implies that $x_{n_j+1}\rightharpoonup x^*$.

We now show that $x^*\in VI(K,f)$, that is x^* satisfies $\langle f(x^*), x-x^*\rangle\geq 0\ \forall x\in K$.
Let $N_K z$ be the normal cone of K at a point $z\in K$, then we defined the following set-valued operator $B: H\to 2^H$ by

$$Bz=\begin{cases} fz+N_K z, & z\in K\\ \emptyset, & z\notin K.\end{cases}$$

Then, B is maximal monotone. Let $(z,w)\in G(M)$, then $w-fz\in N_K z$, and so we have

$$\langle z-x_{n_j+1}, w-fz\rangle\geq 0. \tag{5.2.18}$$

From $x_{n_j+1}=P_K(y_{n_j}-\lambda_{n_j}f(y_{n_j}))$, we have $\langle z-x_{n_j+1}, x_{n_j+1}-(y_{n_j}-\lambda_{n_j}f(y_{n_j})\rangle\geq 0$, which implies

$$\left\langle z-x_{n_j+1}, \frac{x_{n_j+1}-y_{n_j}}{\lambda_{n_j}}+f(y_{n_j})\right\rangle\geq 0.$$

From (5.2.18), we get

$$\begin{aligned}
\langle z - x_{n_j+1}, w\rangle &\geq \langle z - x_{n_j+1}, fz\rangle \\
&\geq \langle z - x_{n_j+1}, fz\rangle - \left\langle z - x_{n_j+1}, \frac{x_{n_j+1} - y_{n_j}}{\lambda_{n_j}} + f(y_{n_j})\right\rangle \\
&= \left\langle z - x_{n_j+1}, fz - f(y_{n_j}) - \frac{x_{n_j+1} - y_{n_j}}{\lambda_{n_j}}\right\rangle \\
&= \langle z - x_{n_j+1}, fz - f(x_{n_j+1})\rangle + \langle z - x_{n_j+1}, f(x_{n_j+1}) - f(y_{n_j})\rangle \\
&\quad - \left\langle z - x_{n_j+1}, \frac{x_{n_j+1} - y_{n_j}}{\lambda_n}\right\rangle \\
&\geq \langle z - x_{n_j+1}, f(x_{n_j+1}) - f(y_{n_j})\rangle \\
&\quad - \left\langle z - x_{n_j+1}, \frac{x_{n_j+1} - y_{n_j}}{\lambda_{n_j}}\right\rangle. \qquad (5.2.19)
\end{aligned}$$

Since f is inverse strongly monotone, hence it is Lipschitz continuous, and from (5.2.19), we have that

$$\lim_{n\to\infty} \|f(x_{n+1}) - f(y_n)\| = 0. \qquad (5.2.20)$$

Using (5.2.20) together with the fact that $\{x_{n_j+1}\}$ converges weakly to x^*, we obtain from (5.2.19) that $\langle z - x^*, w\rangle \geq 0$. Since B is maximal monotone, this gives that $x^* \in B^{-1}(0)$ which implies that $0 \in B(x^*)$. Hence $x^* \in VI(K, f)$ that is $\langle f(x^*), z - x^*\rangle \geq 0,\ \forall z \in K$.

Also, since $\|y_n - u_n\| \to 0$ as $n \to \infty$, we have that Au_{n_j} converges weakly to Ax^* and by (5.2.7) and the fact that $I - T$ is demiclosed at 0, we have

$$Ax^* \in F(T).$$

Hence $x^* \in \Upsilon$. Next we prove that $\{x_n\}$ converges strongly to x^*. From (5.2.1) and Lemma 2.3.6, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|y_n - x^*\|^2 \leq \|u_n - x^*\|^2 \\
&= \|\alpha_n u + (1 - \alpha_n)x_n - x^*\|^2 \\
&= \|(1 - \alpha_n(x_n - x^*) - \alpha_n(x^* - u)\|^2 \\
&\leq (1 - \alpha_n)\|x_n - x^*\|^2 - 2\alpha_n\langle u_n - x^*, x^* - u\rangle. \qquad (5.2.21)
\end{aligned}$$

We observe that $\limsup_{n\to\infty}[-2\langle u_n - x^*, x^* - u\rangle] \leq -2\langle p - x^*, x^* - u\rangle \leq 0$ (since $x^* = P_\Upsilon u$). Therefore from Lemma 2.3.14 $\|x_n - x^*\| \to 0$, as $n \to \infty$.

Case 2: Assume that $\{\|x_n - x^*\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n - x^*\|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k < \Gamma_{k+1}\}.$$

Clearly τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} \geq 0 \ \ \forall n \geq n_0.$$

Again from (5.2.5), we have

$$\begin{aligned}\gamma_{\tau(n)}\epsilon\|A^*(T-I)Au_{\tau(n)}\|^2 \quad &\leq \quad \|x_{\tau(n)}-p\|^2-\|x_{\tau(n)+1}-p\|^2+\alpha_n^2\|x_{\tau(n)}-u\|^2 \\ &\quad\quad -\alpha_{\tau(n)}\langle x_{\tau(n)}-p, x_{\tau(n)}-u\rangle \end{aligned} \tag{5.2.22}$$

Thus, we obtain

$$\lim_{n\to\infty}\|A^*(T-I)Au_{\tau(n)}\| = 0. \tag{5.2.23}$$

Also from (5.2.4), we have

$$\begin{aligned}\|x_{\tau(n)+1}-p\|^2 \quad &\leq \quad \|x_{\tau(n)}-p\|^2+\alpha_{\tau(n)}^2\|x_{\tau(n)}-u\|^2-2\alpha_{\tau(n)}\langle x_{\tau(n)}-p, x_{\tau(n)}-u\rangle \\ &\quad\quad +\gamma_{\tau(n)}\left[\gamma_{\tau(n)}\|A^*(T-I)Au_{\tau(n)}\|^2+(\mu-1)\|TAu_{\tau(n)}-Au_{\tau(n)}\|^2\right],\end{aligned}$$

which implies

$$\begin{aligned}\gamma_{\tau(n)}(1-\mu)\|TAu_{\tau(n)}-Au_{\tau(n)}\|^2 \quad &\leq \quad \|x_{\tau(n)}-p\|^2-\|x_{\tau(n)+1}-p\|^2+\alpha_{\tau(n)}^2\|x_{\tau(n)}-u\|^2 \\ &\quad\quad -2\alpha_{\tau(n)}\langle x_{\tau(n)}-p, x_{\tau(n)}-u\rangle+\gamma_{\tau(n)}^2\|A^*(T-I)Au_{\tau(n)}\|^2.\end{aligned}$$

Hence

$$\lim_{n\to\infty}\|TAu_{\tau(n)}-Au_{\tau(n)}\| = 0. \tag{5.2.24}$$

By using the same argument as in case 1, we get that there is a subsequence $\{x_{\tau(n_j)}\}$ of $\{x_{\tau(n)}\}$ which converges weakly to $x^* \in \Upsilon$ as $\tau(n_j) \to \infty$.

At the same time, we note from (5.2.21) that for all $n \geq n_0$,

$$\begin{aligned}0 &\leq \|x_{\tau(n)+1}-x^*\|^2-\|x_{\tau(n)}-x^*\|^2 \\ &\leq \alpha_{\tau(n)}\left[2\langle u-x^*, u_{\tau(n)}-x^*\rangle-\|x_{\tau(n)}-x^*\|^2\right]\end{aligned}$$

which implies that (since $\alpha_{\tau(n)} > 0$)

$$\|x_{\tau(n)}-x^*\|^2 \leq 2\langle x_1-x^*, u_{\tau(n)}-x^*\rangle.$$

Thus,

$$\limsup_{n\to\infty}\|x_{\tau(n)}-x^*\|^2 \leq 2\limsup_{n\to\infty}\langle x_1-x^*, u_{\tau(n)}-x^*\rangle \leq 0.$$

Hence

$$\lim_{n\to\infty}\|x_{\tau(n)}-x^*\| = 0.$$

Using (5.2.15), we get

$$\|x_{\tau(n)+1}-y_{\tau(n)}\| \to 0, \quad n\to\infty.$$

Now,

$$\|x_{\tau(n)+1}-x_{\tau(n)}\| \leq \|x_{\tau(n)+1}-y_{\tau(n)}\|+\|y_{\tau(n)}-u_{\tau(n)}\|+\|u_{\tau(n)}-x_{\tau(n)}\| \to 0, \quad n\to\infty.$$

Hence, we deduce that

$$\|x_{\tau(n)+1}-x^*\| \leq \|x_{\tau(n)+1}-x_{\tau(n)}\|+\|x_{\tau(n)}-x^*\| \to 0, \quad n\to\infty,$$

and so

$$\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim\limits_{n\to\infty}\Gamma_n = 0$ and so $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Corollary 5.2.2. *Let H_1 and H_2 be real Hilbert spaces. Let K be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator, $f : K \to H_1$ be $1/\alpha$-ism mapping and $T : H_2 \to H_2$ nonexpansive mapping such that $\Upsilon = \{x^* \in VI(K, f) :\ Ax^* \in F(T)\} \neq \emptyset$. Let the step size γ_n be chosen such that for $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \frac{\|TAu_n - Au_n\|^2}{\|A^*(T-I)Au_n\|^2} - \epsilon\right)$, if $TAu_n \neq Au_n$; otherwise $\gamma_n = \gamma$(γ being any nonnegative real number). Let the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in K$ by*

$$\begin{cases} u_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = P_K(u_n + \gamma_n A^*(T-I)Au_n), \\ x_{n+1} = P_K(y_n - \lambda_n f(y_n)), \end{cases} \tag{5.2.25}$$

where the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\alpha_n \in (0,1)$, $\sum\limits_{n\geq 1}\alpha_n = +\infty$,

(ii) $\lim\limits_{n\to\infty}\alpha_n = 0$,

(iii) $\lambda_n \in (0, 2/\alpha), \alpha > 0$.

Then $\{x_n\}$ converges strongly to $x^* \in \Upsilon$, where $x^* = P_\Upsilon u$.

Let $f : H \to H$ be a single valued nonlinear mapping and let $M : H \to 2^H$ be a set valued mapping. The Monotone Variational Inclusion Problem (MVIP) is to find $x \in H$ such that

$$0 \in B(x) + M(x), \tag{5.2.26}$$

where 0 is the zero vector in H. The set of solutions to the MVIP (5.2.26) is denoted by $I(B, M)$. For more information on MVIPs, see for example [105, 106, 144] and the references therein.

It is well known that $M : H \to 2^H$ is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in M(x)$. The resolvent operator J^M_η associated with M and η is the mapping $J^M_\eta : H \to H$ defined by

$$J^M_\eta(x) = (I + \eta M)^{-1}x, \quad x \in H, \eta > 0. \tag{5.2.27}$$

It is a common knowledge that the resolvent operator J_η^M is single valued, nonexpansive and 1-inversely monotone (for example see [25]) and the solution of (5.2.26) is a fixed point of $J_\eta^M(I-\eta B), \quad \forall \eta > 0$ (see for example [125]). If f is ν-inversely strongly monotone mapping with $0 < \eta < 2\nu$, then $J_\eta^M(I-\eta B)$ is nonexpansive and $I(B, M)$ is closed and convex [125].

Lemma 5.2.3. *[25] Let $M : H \to 2^H$ be a maximal monotone mapping and $B : H \to H$ be a Lipschitz continuous mapping. Then the mapping $G = M + B : H \to 2^H$ is a maximal monotone mapping.*

If in Corollary 5.2.2, $T : H_2 \to H_2$ is taken to be the mapping $J_\lambda^M(I-\lambda B)$, we obtain the following convergence result for finding a point $x^* \in VI(K, f)$ such that $0 \in M(Ax^*) + B(Ax^*)$.

Corollary 5.2.4. *Let H_1 and H_2 be real Hilbert spaces. Let K be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator and $f : K \to H_1$ be $1/\alpha$-ism mapping. Let $M : H_2 \to 2^{H_2}$ be multivalued maximal monotone mapping, $B : H_2 \to H_2$ be ν-inverse strongly monotone mapping with $0 < \eta < 2\nu$ and assume that $\Gamma = \{x^* \in VI(K, f) : 0 \in M(Ax^*) + B(Ax^*)\} \neq \emptyset$. Let the step size γ_n be chosen such that for $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \frac{\|J_\eta^M(I-\eta B)Au_n - Au_n\|^2}{\|A^*(J_\eta^M(I-\eta B)-I)Au_n\|^2} - \epsilon\right)$, if $J_\eta^M(I-\eta B)Au_n \neq Au_n$; otherwise $\gamma_n = \gamma$(γ being any nonnegative real number). Let the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in K$ by*

$$\begin{cases} u_n = (1-\alpha_n)x_n + \alpha_n u, \\ y_n = P_K(u_n + \gamma_n A^*(J_\eta^M(I-\eta B) - I)Au_n), \\ x_{n+1} = P_K(y_n - \lambda_n f(y_n)), \end{cases} \tag{5.2.28}$$

where the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\alpha_n \in (0,1), \sum\limits_{n\geq 1} \alpha_n = +\infty,$

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0,$

(iii) $\lambda_n \in (0, 2/\alpha), \alpha > 0.$

Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$, where $x^* = P_\Gamma u$.

5.2.1 Numerical example:

In this section we give numerical example in $\mathbb{R}$ (with the usual metric) to support Theorem 5.2.1 Let $H_1 = H_2 = \mathbb{R}$ and $K = [-2, 2]$ and $P_K : \mathbb{R} \to K$ be defined by

$$P_K(x) = \begin{cases} x, & if \quad x \in K \\ \frac{2x}{|x|}, & if \quad x \notin K \end{cases} \tag{5.2.29}$$

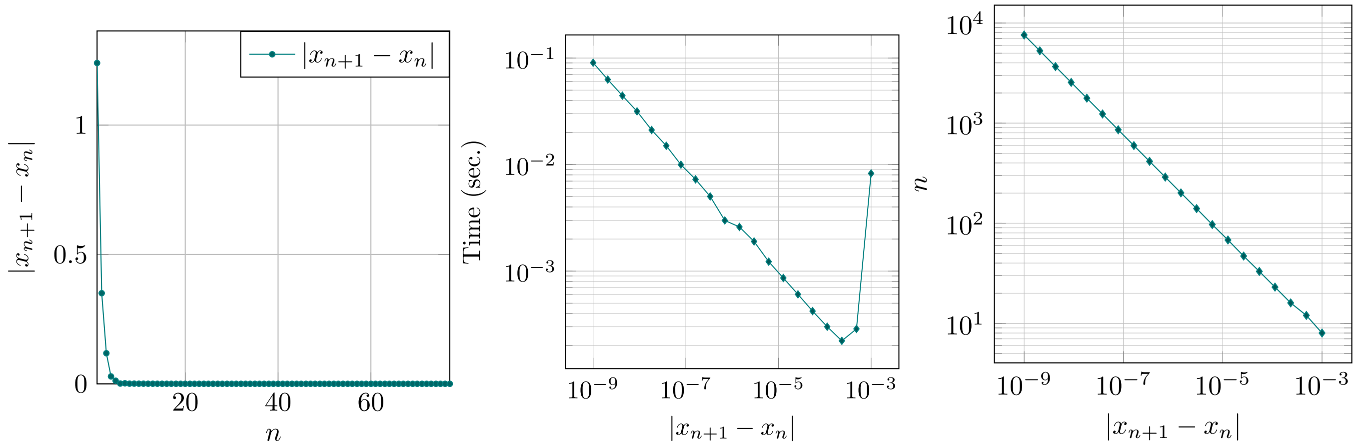

Figure 5.1: 1: errors vs number of iterations (top left and right); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

We define $T:\mathbb{R}\to\mathbb{R}$ by $Tx=-2x$. Also, we define $A:\mathbb{R}\to\mathbb{R}$ by $A(x)=3x$, so that $A^*(x)=3x$. Let $f:K\to K$ be defined by $f(x)=\frac{x}{5}$. Take $\alpha_n=\frac{1}{4n+1}$, $\lambda_n=\frac{2n^2}{n^2+1}\in(0,2/\alpha)$, for $\alpha>0$.
It is easy to see that T is μ-strictly pseudo contraction with $\mu=\frac{1}{3}$, f is $\frac{1}{\alpha}$ inverse strongly monotone operator with $\frac{1}{\alpha}=5$ and A is linear operator, take $\gamma_n\in(\epsilon,\frac{2}{27}-\epsilon)$. Hence for $u,x_1\in K$, our Algorithm (5.2.1) becomes

$$\begin{cases} u_n=\frac{4n}{4n+1}x_n+\frac{1}{4n+1}u,\\ y_n=P_K(u_n-27\gamma_n u_n),\\ x_{n+1}=P_K(y_n-\frac{2n^2}{n^2+1}f(y_n)). \end{cases} \tag{5.2.30}$$

1. Take $u=1$ and $x_1=-1$
2. Take $u=-1$ and $x_1=-1$
3. Take $u=1$ and $x_1=2$.

5.3 On split equilibrium problem, variational inequality problem and fixed point problem for multi-valued mappings

In this section, we assume that H_1 and H_2 are real Hilbert spaces. Let D be a nonempty closed and bounded subset of H_1. Let $f_n:D\to H_1$ be uniformly convergence sequence of contraction mappings. Then there exists a real numbers $\rho_n\in(0,1)$ such that

$$\|f_n(x)-f_n(y)\|\le\rho_n\|x-y\|,\quad\forall x,y\in D.$$

A subset C of H_1 is called proximinal if, for each $x\in H_1$, there exists $c\in C$ such that

$$\|c-x\|=\inf\{\|x-y\|:y\in C\}=d(x,C).$$

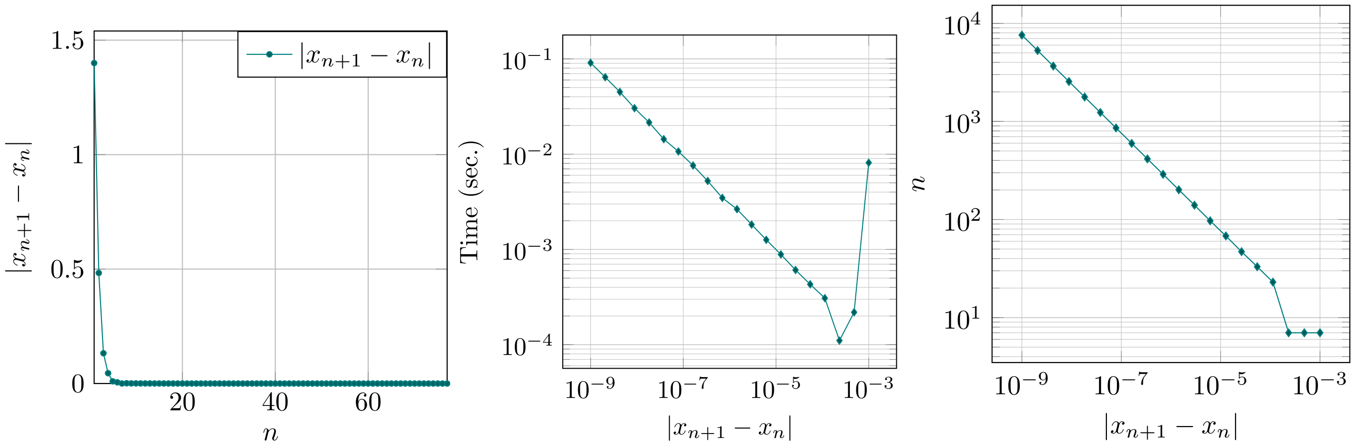

Figure 5.2: 2: errors vs number of iterations (top left and right); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

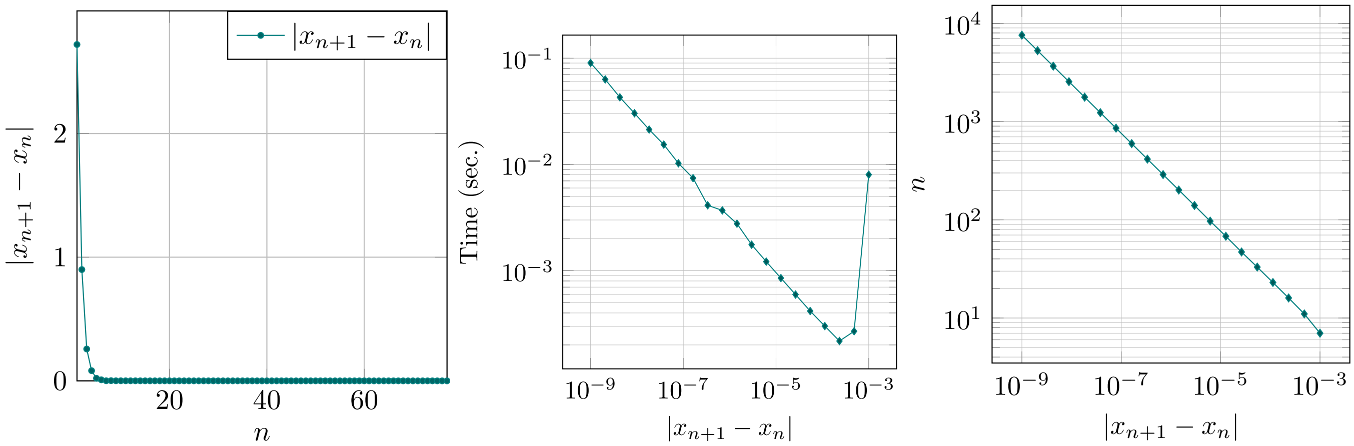

Figure 5.3: 3: errors vs number of iterations (top left and right); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

Let C be a closed convex and nonempty subset of H_1. It is well known that in a Hilbert space, closed and convex sets are Proximinal (see for example, [90, 156]). In the sequel, we denote by $CB(H_1)$ the collection of all nonempty, closed and bounded subsets of H_1. The Hausdorff metric $\mathcal{H}$ on $CB(H_1)$ is defined by

$$\mathcal{H}(A,B) := \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}, \quad \forall A,B \in CB(H),$$

where $d(x,A) := \inf_{y\in A} d(x,y)$.

Definition 5.3.1. *Let $S : H \to CB(H)$ be a multi-valued mapping. An element $p \in H$ is said to be a fixed point of S, if $p \in Sp$. The mapping S is said to be: (i) nonexpansive, if $\mathcal{H}(Sx,Sy) \le \|x-y\|$, $\forall x,y \in H$;*
(ii) quasi-nonexpansive, if $F(S) \neq \emptyset$ and $\mathcal{H}(Sx,Sy) \le \|x-p\|$, $\forall x \in H$, $p \in F(S)$.

Definition 5.3.2. *A mapping $S : C \to CB(C)$ is said to be a multivalued hemicontractive-type mapping if $F(S) \neq \emptyset$ and for all $p \in F(S)$, $x \in C$,*

$$\mathcal{H}^2(Sx,Sp) \le \|x-p\|^2 + \|x-u\|^2, \quad \forall\ u \in Sx. \tag{5.3.1}$$

A Fixed Point Problem (FPP, for short) for multi-valued mapping S is to find $x \in C$ such that

$$x \in Sx. \tag{5.3.2}$$

The solution set of FPP (5.3.2) is denoted by $F(S)$. Fixed point theory for multi-valued mappings has many useful application in various fields, in particular, game theory and mathematical economics. Thus, it is natural to extend the known fixed point results for single-valued mappings to the setting of multi-valued mappings. Several authors have investigated the approximations of fixed point of multi-valued nonexpansive mappings in the literature (see, for example,[90, 95, 140, 155, 156, 177, 178, 181]).

Moudafi [125] introduced an iterative method, an extension of a method by Censor et al. [56] for the following split monotone variational inclusion:

$$\text{Find} \;\; x^* \in H_1 \;\; \text{such} \;\; \text{that} \;\; 0 \in f(x^*) + B_1(x^*),$$

and such that

$$y^* = Ax^* \in H_2 \;\; \text{solves} \;\; 0 \in g(y^*) + B_2(y^*),$$

where $B_i : H_i \to 2^{H_i}$ is a set-valued mapping for $i = 1,2$. Later on Byrne et al. [44] generalize and extend the work of Censor et al. [56] and Moudafi [125].

Lemma 5.3.3. *[132] Let H be a Hilbert space. Let $A,B \in CB(H)$ and $a \in A$. Then, for $\epsilon > 0$, there exists a point $b \in B$ such that $\|a-b\| \le \mathcal{H}(A,B) + \epsilon$. As a consequence of this, taking $\epsilon = \mathcal{H}(A,B)$, we obtain that $\|a-b\| \le 2\mathcal{H}(A,B)$.*

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonnilear bifunctions and $A : H_1 \to H_2$ be bounded linear operator, then the Split Equilibrium Problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{5.3.3}$$

and such that

$$y^* = Ax^* \in Q \;\; \text{solves} \;\; F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{5.3.4}$$

When looked upon separately, we observed that (5.3.3) is the classical Equilibrium Problem (EP) and we denote it's solution set by $EP(F_1)$. The SEP (5.3.3) and (5.3.4) constitute a pair of equilibrium problems which we have to solve so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution the EP (5.3.3) in H_1 is the solution set of EP (5.3.4) in another space H_2. The solution set of SEP (5.3.3) and (5.3.4) is denoted by

$$\Omega = \{p \in EP(F_1) : Ap \in EP(F_2)\}.$$

Motivated by the work of Censor et al.[55, 56], Moudafi [125], Byrne et al [44], Liu et al. [111], Kazmi and Rizvi [93] obtained the following converges result.

Theorem 5.3.4. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D : C \to H_1$ be a τ-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions satisfying hypothesis $(A1) - (A4)$ and F_2 is upper semicontinuous in first argument. Let $S : C \to C$ be a nonexpansive mapping such that $\Theta := F(S) \cap \Omega \cap \Gamma \neq \emptyset$. For a given $x_0 = v \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} u_n = J_{r_n}^{F_1}(x_n + \gamma A^*(J_{r_n}^{F_2} - I)Ax_n); \\ y_n = P_C(u_n - \lambda_n D u_n); \\ x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n S y_n, \end{cases} \tag{5.3.5}$$

where $r_n \subset (0, \infty)$, $\lambda_n \in (0, 2\tau)$ and $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the following

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty}, \alpha_n = \infty$,

(iii) $0 < \liminf\limits_{n\to\infty} \lambda_n \leq \limsup\limits_{n\to\infty} \lambda_n < 2\tau$, and $\lim\limits_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

(iv) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$,

(v) $\lim\limits_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$,

(vi) $\lim\limits_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Theta$, where $z = P_\Theta v$.

Motivated by the work of Kazmi and Rizvi [93], we introduce an iterative scheme for approximating a common solution of SEP(5.3.3) and (5.3.4), VIP (2.2.25) and FPP (5.3.2) for multi-valued quasi-nonexpansive in real Hilbert space. Using our proposed algorithm we prove strong convergence theorem for approximating a common solution of SEP(5.3.3) and (5.3.4), VIP (2.2.25) and FPP (5.3.2) for multi-valued quasi-nonexpansive in real Hilbert. In all our results the variable step-size is selected in such a way that its implementation does not involve the computation or an estimate of the spectral radius.

Theorem 5.3.5. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_n : H_1 \to H_1$ be a sequence of ρ_n-contractive mappings with $0 < \underline{\rho} \leq \rho_n \leq \bar{\rho} < 1$ and $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of H_1. Let $B : C \to H_1$ be τ-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions satisfying hypothesis $(A1) - (A4)$ and F_2 is upper semicontinuous in the first argument. Let $S : H_1 \to CB(H_1)$ be a multi-valued quasi-nonexpansive mapping such that S is demiclosed at the origin, $Sp = \{p\}$ $\forall p \in F(S)$ and $\Upsilon := F(S) \cap \Omega \cap \Gamma \neq \emptyset$. For arbitrary $x_1 \in H_1$, define the iterative sequence $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ by*

$$\begin{cases} u_n = J_{r_n}^{F_1}(x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n(\sigma w_n + (1-\sigma)y_n), \quad w_n \in Sx_n, \quad n \geq 1, \end{cases} \tag{5.3.6}$$

where $\gamma_n := \mu_n \frac{\|(J_{r_n}^{F_2}-I)Ax_n\|^2}{\|A^*(J_{r_n}^{F_2}-I)Ax_n\|^2}$ with $0 < a \leq \mu_n \leq b < 1$,
$r_n \in (0,\infty)$, $\lambda_n \in (0, 2\tau)$, $\sigma, \bar{\rho}, \underline{\rho} \in (0,1)$ and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions

- *(i)* $\alpha_n + \beta_n + \delta_n = 1$;
- *(ii)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty}, \alpha_n = \infty$;
- *(iii)* $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\tau$;
- *(iv)* $\beta_n \geq \epsilon_1 > 0 \quad \delta_n \geq \epsilon_2 > 0$.

Then the sequence $\{x_n\}$ converges strongly to $p \in \Upsilon$ where $p = P_\Upsilon f(p)$.

Proof. Let $p \in \Upsilon := \mathrm{Fix}(S) \cap \Omega \cap \Gamma$, i.e $p \in \Omega$, we have $p = J_{r_n}^{F_1} p$ and $Ap = J_{r_n}^{F_2} Ap$. We then obtain

$$\begin{aligned} \|u_n - p\|^2 &= \|J_{r_n}^{F_1}(x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n - p\|^2 \\ &= \|J_{r_n}^{F_1}(x_n + \gamma_n A_i^*(J_{r_n}^{F_2} - I)Ax_n - J_{r_n}^{F_1}(p)\|^2 \\ &\leq \|x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n - p\|^2 \\ &= \|x_n - p + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\ &\leq \|x_n - p\|^2 + 2\gamma_n \langle x_n - p, A^*(J_{r_n}^{F_2} - I)A_i x_n\rangle + \gamma_n^2 \|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2. \end{aligned}$$

Thus,

$$\begin{aligned}\|u_n - p\|^2 &= \|x_n - p\|^2 + 2\gamma_n\langle x_n - p, A^*(J_{r_n}^{F_2} - I)Ax_n\rangle \\ &\quad + \gamma_n^2\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2. \qquad (5.3.7)\end{aligned}$$

Since $Ap \in F(J_{r_n}^{F_2})$, we have that

$$\begin{aligned}\langle x_n - p, A^*(J_{r_n}^{F_2} - I)Ax_n\rangle &= \langle A(x_n - p), (J_{r_n}^{F_2} - I)Ax_n\rangle \\ &= \langle A(x_n) - A(p) + (J_{r_n}^{F_2} - I)Ax_n \\ &\quad -(J_{r_n}^{F_2} - I)Ax_n, (J_{r_n}^{F_2} - I)Ax_n\rangle \\ &= \langle J_{r_n}^{F_2}Ax_n - Ap - (J_{r_n}^{F_2} - I)Ax_n, (J_{r_n}^{F_2} - I)Ax_n\rangle \\ &\leq \langle J_{r_n}^{F_2}Ax_n - Ap, (J_{r_n}^{F_2} - I)Ax_n\rangle - \|(J_{r_n}^{F_2} - I)Ax_n\|^2 \\ &= \frac{1}{2}\left[\|J_{r_n}^{F_2}Ax_n - Ap\|^2 + \|(J_{r_n}^{F_2} - I)Ax_n\|^2 - \|Ax_n - Ap\|^2\right] \\ &\quad -\|(J_{r_n}^{F_2} - I)Ax_n\|^2 \\ &= \frac{1}{2}\left[\|J_{r_n}^{F_2}Ax_n - Ap\|^2 - \|Ax_n - Ap\|^2\right] - \frac{1}{2}(\|J_{r_n}^{F_2} - I)Ax_n\|^2 \\ &= \frac{1}{2}\left[\|J_{r_n}^{F_2}Ax_n - Ap\|^2 - \|Ax_n - Ap\|^2 - \|(J_{r_n}^{F_2} - I)Ax_n\|^2\right] \\ &\leq \frac{1}{2}\left[\|Ax_n - Ap\|^2 - \|Ax_n - Ap\|^2 - \|(J_{r_n}^{F_2} - I)Ax_n\|^2\right] \\ &= -\frac{1}{2}\|(J_{r_n}^{F_2} - I)Ax_n\|^2.\end{aligned}$$

Therefore,

$$\langle x_n - p, A^*(J_{r_n}^{F_2} - I)Ax_n\rangle \leq -\frac{1}{2}\|(J_{r_n}^{F_2} - I)Ax_n\|^2. \qquad (5.3.8)$$

Substituting (5.3.8) into (5.3.7), we get

$$\begin{aligned}\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \gamma_n\|(J_{r_n}^{F_2} - I)Ax_n\|^2 + \gamma_n^2\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\ &\leq \|x_n - p\|^2 \\ &\quad -\gamma_n^2\left[\|(J_{r_n}^{F_2} - I)Ax_n\|^2 - \gamma_n\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2\right]. \qquad (5.3.9)\end{aligned}$$

Using the definition of γ_n, we have

$$\|u_n - p\| \leq \|x_n - p\|. \qquad (5.3.10)$$

Now, we estimate

$$\begin{aligned}\|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Bu_n) - P_C(p - \lambda_n Bp)\|^2 \\ &\leq \|(u_n - \lambda_n Bu_n) - (p - \lambda_n Bp)\|^2 \\ &\leq \|u_n - p\|^2 - \lambda_n(2\tau - \lambda_n)\|Bu_n - Bp\|^2 \qquad (5.3.11) \\ &\leq \|u_n - p\|^2 \\ &\leq \|x_n - p\|^2.\end{aligned}$$

Let $z_n := \sigma w_n + (1-\sigma)y_n$. Then

$$\begin{aligned}\|z_n - p\| &= \|\sigma(w_n - p) + (1-\sigma)(y_n - p)\| \\ &\leq \sigma\|w_n - p\| + (1-\sigma)\|y_n - p\| \\ &\leq \sigma\mathcal{H}(P_S(x_n), P_S(p)) + (1-\sigma)\|x_n - p\| \\ &\leq \sigma\|x_n - p\| + (1-\sigma)\|x_n - p\| \\ &= \|x_n - p\|.\end{aligned}$$

By using (5.3.6), we have that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n f_n(x_n) + \beta_n x_n + \delta_n z_n - p\| \\ &= \|\alpha_n(f_n(x_n) - f_n(p)) + \alpha_n(f_n(p) - p) + \beta_n(x_n - p) + \delta_n(z_n - p)\| \\ &\leq \alpha_n(\|f_n(x_n) - f_n(p)\| + \|f_n(p) - p\|) + \beta_n\|x_n - p\| + \delta_n\|z_n - p\| \\ &\leq \alpha_n(\|f_n(x_n) - f_n(p)\| + \|f_n(p) - p\|) + (1-\alpha_n)\|x_n - p\|.\end{aligned}$$

Hence,

$$\|x_{n+1} - p\| \leq \alpha_n(\|f_n(x_n) - f_n(p)\| + \|f_n(p) - p\|) + (1-\alpha_n)\|x_n - p\|. \tag{5.3.12}$$

By the uniform convergence of $\{f_n(x)\}$ on D, there exists $M > 0$ such that $\|f_n(p) - p\| \leq M$, $\forall n \geq 1$. Hence, we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n\rho_n\|x_n - p\| + \alpha_n\|f_n(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \alpha_n\bar{\rho}\|x_n - p\| + \alpha_n\|f_n(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &= (\alpha_n\bar{\rho} + (1-\alpha_n)\|x_n - p\| + \alpha_n\|f_n(p) - p\| \\ &= (1 - \alpha_n(1-\bar{\rho}))\|x_n - p\| + \alpha_n\|f_n(p) - p\| \\ &= (1 - \alpha_n(1-\bar{\rho}))\|x_n - p\| + \alpha_n(1-\bar{\rho})\frac{\|f_n(p) - p\|}{1-\bar{\rho}} \\ &\leq \max\left\{\|x_n - p\|, \frac{M}{1-\bar{\rho}}\right\} \\ &\leq \vdots \\ &\leq \max\left\{\|x_1 - p\|, \frac{M}{1-\bar{\rho}}\right\}.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded.

By (5.3.6) and (5.3.10), we have that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n f_n(x_n)+\beta_n x_n+\delta_n z_n-p\|^2 \\
&= \|\beta_n(x_n-p)+\delta_n(z_n-p)+\alpha_n(f_n(x_n)-f_n(p))+\alpha_n(f_n(p)-p)\|^2 \\
&\leq \|\beta_n(x_n-p)+\delta_n(z_n-p)+\alpha_n(f_n(x_n)-f_n(p))\|^2 \\
&\quad +2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle \\
&\leq \|\beta_n(x_n-p)+\delta_n(z_n-p)\|^2+\alpha_n^2\|(f_n(x_n)-f_n(p))\|^2 \\
&\quad +2\alpha_n\langle \beta_n(x_n-p)+\delta_n(z_n-p), f_n(x_n)-f_n(p)\rangle+2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle \\
&= \beta_n(\beta_n+\delta_n)\|x_n-p\|^2+\delta_n(\beta_n+\delta_n)\|z_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2 \\
&\quad +\alpha_n^2\|(f_n(x_n)-f_n(p))\|^2+2\alpha_n\langle \beta_n(x_n-p)+\delta_n(z_n-p), f_n(x_n)-f_n(p)\rangle \\
&\quad +2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle \\
&\leq (\beta_n+\delta_n)^2\|x_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2 \\
&\quad +\alpha_n^2\|f_n(x_n)-f_n(p)\|^2+2\alpha_n\langle \beta_n(x_n-p)+\delta_n(z_n-p), f_n(x_n)-f_n(p)\rangle \\
&\quad +2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle \\
&= (1-\alpha_n)^2\|x_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2 \\
&\quad +\alpha_n^2\|f_n(x_n)-f_n(p)\|^2+2\alpha_n\langle \beta_n(x_n-p)+\delta_n(z_n-p), f_n(x_n)-f_n(p)\rangle \\
&\quad +2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)^2\|x_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2 \\
&\quad +\alpha_n^2\rho_n^2\|x_n-p\|^2+2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle+2\alpha_n\|\beta_n(x_n-p) \\
&\quad +\delta_n(z_n-p)\|\|f_n(x_n)-f_n(p)\| \\
&\leq (1-\alpha_n)^2\|x_n-p\|^2-\beta_n\delta_n\|x_n-z_n\|^2 \\
&\quad +\alpha_n^2\rho_n^2\|x_n-p\|^2+2\alpha_n\langle f_n(p)-p, x_{n+1}-p\rangle \\
&\quad +2\bar{\rho}\alpha_n(1-\alpha_n)\|x_n-p\|^2. \qquad (5.3.13)
\end{aligned}$$

We divide the rest of the proof into two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-p\|\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{\|x_n-p\|\}$ converges and $\|x_n-p\|-\|x_{n+1}-p\| \to 0$ as $n \to \infty$. By (5.3.6), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n^2\|f_n(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\delta_n\|z_n-p\|^2 \\
&\leq \alpha_n\|f_n(x_n)-p\|^2+\beta_n\|x_n-p\|^2+\delta_n(\sigma\|x_n-p\|^2+(1-\sigma)\|y_n-p\|^2) \\
&\leq \alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\sigma\delta_n)\|x_n-p\|^2+(1-\sigma)\|u_n-p\|^2.
\end{aligned}$$

This implies that

$$-\|u_n-p\|^2 \leq \frac{1}{(1-\sigma)\delta_n}\Big[\alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\sigma\delta_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2\Big].$$

From (5.3.8) and (5.3.9), we have that

$$\begin{aligned}
&\gamma_n\Big[\|(J_{r_n}^{F_2}-I)Ax_n\|^2-\gamma_n\|A^*(J_{r_n}^{F_2}-I)Ax_n\|^2\Big]\leq\|x_n-p\|^2-\|u_n-p\|^2\\
&\leq\|x_n-p\|^2-\frac{1}{(1-\sigma)\delta_n}\|x_{n+1}-p\|^2+\frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n)-p\|^2+\frac{\beta_n+\sigma\delta_n}{(1-\sigma)\delta_n}\|x_n-p\|^2\\
&=\frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_n-p\|^2-\frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_{n+1}-p\|^2+\frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n)-p\|^2\\
&=\frac{1}{(1-\sigma)\delta_n}\Big[\|x_n-p\|^2-\|x_{n+1}-p\|^2\Big]+\frac{\alpha_n}{(1-\sigma)\delta_n}\Big[\|f_n(x_n)-p\|^2-\|x_n-p\|^2\Big].
\end{aligned}$$

Since $\alpha_n\to0$ as $n\to\infty$ by condition (i), we have that

$$\gamma_n\Big[\|(J_{r_n}^{F_2}-I)Ax_n\|^2-\gamma_n\|A^*(J_{r_n}^{F_2}-I)Ax_n\|^2\Big]\to0\ \ as\ \ n\to\infty,$$

which by definition of γ_n implies that

$$\frac{\mu_n(1-\mu_n)\|(J_{r_n}^{F_2}-I)Ax_n\|^4}{\|A^*(J_{r_n}^{F_2}-I)Ax_n\|^2}\to0\ \ as\ \ n\to\infty.$$

Since $0<a\leq\mu_n\leq b<1$ and $\|A^*(J_{r_n}^{F_2}-I)Ax_n\|$ is bounded, we have that

$$\|(J_{r_n}^{F_2}-I)Ax_n\|\to0\ \ as\ \ n\to\infty. \tag{5.3.14}$$

Now,

$$\|A^*(J_{r_n}^{F_2}-I)Ax_n\|=\|A\|\|(J_{r_n}^{F_2}-I)Ax_n\|\to0\ \ as\ \ n\to\infty. \tag{5.3.15}$$

By (5.3.13), we have that

$$\beta_n\delta_n\|x_n-z_n\|^2\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_nM_1,$$

for some $M_1>0$ and this implies that

$$\|x_n-z_n\|\to0,\ n\to\infty.$$

Hence,

$$\begin{aligned}
\|x_{n+1}-x_n\|&=\|\alpha_nf_n(x_n)+\beta_nx_n+\delta_nz_n-(\alpha_nx_n+\beta_nx_n+\delta_nx_n)\|\\
&\leq\alpha_n\|f_n(x_n)-x_n\|+\alpha_n\|x_n-z_n\|\to0,n\to\infty.
\end{aligned}$$

Similarly,

$$\|x_{n+1}-z_n\|\leq\|x_{n+1}-x_n\|+\|x_n-z_n\|\to0,n\to\infty.$$

From (5.3.6) and the fact that $J_{r_n}^{F_1}$ is firmly nonexpansive, we have that

$$
\begin{aligned}
\|u_n - p\|^2 &= \|J_{r_n}^{F_1}(x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n) - p\|^2 \\
&= \|J_{r_n}^{F_1}(x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n) - J_{r_n}^{F_1}(p)\|^2 \\
&= \langle u_n - p, x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n - p\rangle \\
&\leq \frac{1}{2}[\|u_n - p\|^2 + \|x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n - p\|^2 \\
&\quad -\|u_n - p - (x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n - p)\|^2] \\
&\leq \frac{1}{2}[\|u_n - p\|^2 + \|x_n - p + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\
&\quad -\|u_n - x_n + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\|^2] \\
&\leq \frac{1}{2}[\|u_n - p\|^2 + \|x_n - p + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\
&\quad -[\|u_n - x_n\|^2 + \gamma_n\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 - 2\langle u_n - x_n, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle]] \\
&= \frac{1}{2}[\|u_n - x_n\|^2 + \|x_n - p + \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 - \|u_n - x_n\|^2 \\
&\quad -\gamma_n\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 + 2\langle u_n - x_n, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle] \\
&= \frac{1}{2}[\|u_n - x_n\|^2 + \|x_n - p\|^2 + \gamma_n\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\
&\quad +2\langle x_n - p, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle - \|u_n - x_n\|^2 - \gamma_n\|A^*(J_{r_n}^{F_2} - I)Ax_n\|^2 \\
&\quad +2\langle u_n - x_n, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle] \\
&\leq \frac{1}{2}[\|x_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\langle x_n - p, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle \\
&\quad +2\langle u_n - x_n, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle] \\
&\leq \|x_n - p\|^2 - \frac{1}{2}\|u_n - x_n\|^2 + \langle u_n - p, \gamma_n A^*(J_{r_n}^{F_2} - I)Ax_n\rangle \\
&\leq \|x_n - p\|^2 - \frac{1}{2}\|u_n - x_n\|^2 + \gamma_n\|u_n - p\|\|A^*(J_{r_n}^{F_2} - I)Ax_n\|.
\end{aligned}
$$

That is

$$
\|u_n - p\|^2 \leq \|x_n - p\|^2 - \frac{1}{2}\|u_n - x_n\|^2 + \gamma_n\|u_n - p\|\|A^*(J_{r_n}^{F_2} - I)Ax_n\|. \tag{5.3.16}
$$

Hence,

$$\begin{aligned}
\frac{1}{2}\|u_n - x_n\|^2 &\leq \|x_n - p\|^2 - \|u_n - p\|^2 + \gamma_n \|u_n - p\| \|A^*(J_{r_n}^{F_2} - I)Ax_n\| \\
&\leq \|x_n - p\|^2 - \frac{1}{(1-\sigma)\delta_n}\|x_{n+1} - p\|^2 + \frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n) - p\|^2 + \frac{\beta_n + \sigma\delta_n}{(1-\sigma)\delta_n}\|x_n - p\|^2 \\
&\quad +\gamma_n \|u_n - p\| \|A^*(J_{r_n}^{F_2} - I)Ax_n\| \\
&= \frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_n - p\|^2 - \frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_{n+1} - p\|^2 + \frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n) - p\|^2 \\
&\quad +\gamma_n \|u_n - p\| \|A^*(J_{r_n}^{F_2} - I)Ax_n\| \\
&= \frac{1}{(1-\sigma)\delta_n}\Big[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big] + \frac{\alpha_n}{(1-\sigma)\delta_n}\Big[\|f_n(x_n) - p\|^2 - \|x_n - p\|^2\Big] \\
&\quad +\gamma_n \|u_n - p\| \|A^*(J_{r_n}^{F_2} - I)Ax_n\|,
\end{aligned}$$

and this implies that

$$\|u_n - x_n\| \to 0, n \to \infty,$$

and

$$\|z_n - u_n\| \leq \|u_n - x_n\| + \|x_n - z_n\| \to 0, n \to \infty.$$

Next, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n \|f_n(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \delta_n \|z_n - p\|^2 \\
&\leq \alpha_n \|f_n(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \delta_n(\sigma\|x_n - p\|^2 + (1-\sigma)\|y_n - p\|^2) \\
&= \alpha_n \|f_n(x_n) - p\|^2 + (\beta_n + \sigma\delta_n)\|x_n - p\|^2 + (1-\sigma)\delta_n\|y_n - p\|^2 \\
&\leq \alpha_n \|f_n(x_n) - p\|^2 + (\beta_n + \sigma\delta_n)\|x_n - p\|^2 + (1-\sigma)\delta_n[\|u_n - p - \lambda_n(Bu_n - Bp)\|^2] \\
&\leq \alpha_n \|f_n(x_n) - p\|^2 + (\beta_n + \sigma\delta_n)\|x_n - p\|^2 \\
&\quad +(1-\sigma)\delta_n\left[\|u_n - p\|^2 - 2\lambda_n\langle u_n - p, Bu_n - Bp\rangle + \lambda_n^2\|Bu_n - Bp\|^2\right] \\
&\leq \alpha_n \|f_n(x_n) - p\|^2 + (\beta_n + \sigma\delta_n)\|x_n - p\|^2 + (1-\sigma)\delta_n\|x_n - p\|^2 \\
&\quad -(1-\sigma)2\delta_n\lambda_n\tau\|Bu_n - Bp\|^2 + (1-\sigma)\delta_n\lambda_n^2\|Bu_n - Bp\|^2 \\
&\leq \alpha_n \|f_n(x_n) - p\|^2 + (\beta_n + \sigma\delta_n)\|x_n - p\|^2 + (1-\sigma)\delta_n\|x_n - p\|^2 \\
&\quad +(1-\sigma)\delta_n\lambda_n(\lambda_n - 2\tau)\|Bu_n - Bp\|^2,
\end{aligned}$$

and so

$$\begin{aligned}
\lambda_n(2\tau-\lambda_n)\|Bu_n-Bp\|^2 &\leq \|x_n-p\|^2-\frac{1}{(1-\sigma)\delta_n}\|x_{n+1}-p\|^2\\
&\quad+\frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n)-p\|^2+\frac{\beta_n+\sigma\delta_n}{(1-\sigma)\delta_n}\|x_n-p\|^2\\
&= \frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_n-p\|^2-\frac{1-\alpha_n}{(1-\sigma)\delta_n}\|x_{n+1}-p\|^2\\
&\quad+\frac{\alpha_n}{(1-\sigma)\delta_n}\|f_n(x_n)-p\|^2\\
&= \frac{1}{(1-\sigma)\delta_n}\Big[\|x_n-p\|^2-\|x_{n+1}-p\|^2\Big]\\
&\quad+\frac{\alpha_n}{(1-\sigma)\delta_n}\Big[\|f_n(x_n)-p\|^2-\|x_n-p\|^2\Big].
\end{aligned}$$

Since $\alpha_n\to 0$ and $0<\lim_{n\to\infty}\lambda_n=\lambda<2\tau$, we have that

$$\lim_{n\to\infty}\|Bu_n-Bp\|=0. \tag{5.3.17}$$

Since P_C is firmly nonexpansive and $(I-\lambda_nB)$ is nonexpansive by (5.3.6), we have

$$\begin{aligned}
\|y_n-p\|^2 &= \|P_C(u_n-\lambda_nBu_n)-P_C(p-\lambda_nBp)\|^2\\
&\leq \langle y_n-p, u_n-\lambda_nBu_n-(p-\lambda_nBp)\rangle\\
&= \frac{1}{2}(\|y_n-p\|^2+\|(I-\lambda_nB)u_n-(I-\lambda_nB)p\|^2-\|y_n-u_n+\lambda_n(Bu_n-Bp)\|^2)\\
&\leq \frac{1}{2}(\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-u_n+\lambda_n(Bu_n-Bp)\|^2)\\
&= \frac{1}{2}(\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-u_n\|^2-\lambda_n^2\|Bu_n-Bp\|^2\\
&\quad-2\lambda_n\langle y_n-u_n, Bu_n-Bp\rangle)\\
&\leq \frac{1}{2}(\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-u_n\|^2-\lambda_n^2\|Bu_n-Bp\|^2\\
&\quad+2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|),
\end{aligned}$$

and so

$$\begin{aligned}
\|y_n-p\|^2 &\leq \|u_n-p\|^2-\|y_n-u_n\|^2-\lambda_n^2\|Bu_n-Bp\|^2\\
&\quad+2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|\\
&\leq \|x_n-p\|^2-\|y_n-u_n\|^2+2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|.
\end{aligned} \tag{5.3.18}$$

From (5.3.6) and (5.3.18), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\sigma\delta_n)\|x_n-p\|^2+(1-\sigma)\delta_n\|y_n-p\|^2\\
&\leq \alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\sigma\delta_n)\|x_n-p\|^2\\
&\quad+(1-\sigma)\delta_n[\|x_n-p\|^2-\|y_n-u_n\|^2+2\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|]\\
&\leq \alpha_n\|f_n(x_n)-p\|^2+(\beta_n+\sigma\delta_n)\|x_n-p\|^2\\
&\quad+(1-\sigma)\delta_n\|x_n-p\|^2-(1-\sigma)\delta_n\|y_n-u_n\|^2+2(1-\sigma)\delta_n\lambda_n\|y_n-u_n\|\|Bu_n-Bp\|.
\end{aligned}$$

Therefore, we have

$$\|y_n - u_n\|^2 \leq \frac{1}{(1-\sigma)\delta_n}\Big[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\Big] + \frac{\alpha_n}{(1-\sigma)\delta_n}\Big[\|f_n(x_n) - p\|^2 - \|x_n - p\|^2\Big]$$
$$+ 2\lambda_n(\|y_n\| + \|u_n\|)\|Bu_n - Bp\|.$$

Since $\lim\limits_{n\to\infty} \alpha_n = 0$ and both $\{y_n\}$ and $\{u_n\}$ are bounded by (5.3.17), we have

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{5.3.19}$$

From $z_n = \sigma w_n + (1-\sigma)y_n$, we get

$$\|w_n - y_n\| = \frac{1}{\sigma}[\|z_n - u_n\| + \|y_n - u_n\|] \to 0, \quad n \to \infty.$$

So,

$$\|w_n - x_n\| \leq \|w_n - y_n\| + \|y_n - u_n\| + \|u_n - x_n\| \to 0, \quad n \to \infty.$$

Therefore,

$$d(x_n, Sx_n) \leq \|x_n - w_n\| \to 0, \quad n \to \infty. \tag{5.3.20}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^* \in H_1$ and

$$\limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle = \limsup_{j\to\infty}\langle f(p) - p, x_{n_j} - p\rangle.$$

By demiclosedness principle for multi-valued map S at zero and (5.3.20), we have that $x^* \in F(S)$.

Next, we show that $x^* \in EP(F_1)$. Since $u_n = J_{r_n}^{F_1}x_n$, we have $F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C$.

It follows from monotonicity of F_1 that $\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_1(y, u_n)$ and hence $\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_n}\right\rangle \geq F_1(y, u_{n_i})$. Since $\|u_n - x_n\| \to 0$, $d(x_n, Sx_n) \leq \|x_n - w_n\| \to 0, \quad n \to \infty$, we get $u_{n_i} \rightharpoonup x^*$ and $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$. It follows by Assumption 2.1(iv) that $0 \geq F_1(y, x^*)$, $\forall x^* \in C$. For $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)x^*$. Since $y \in C$, $x^* \in C$, we get $y_t \in C$ and $F_1(y_1, x^*) \leq 0$. So from A1 and A4 we have

$$0 = F_1(y_t, y_t) \leq tF(y_t, y) + (1-t)F_1(y_t, x^*) \leq tF_1(y_t, y).$$

Therefore, from A3 we have $0 \leq F_1(x^*, y)$. Hence $x^* \in EP(F_1)$.

Next, we show that $Ax^* \in EP(F_2)$. Since $\|u_n - x_n\| \to 0$, $u_n \rightharpoonup x^*$ as $n \to \infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_k\}$ such that $x_{n_k} \rightharpoonup x^*$ and since A is a bounded linear operator so that $Ax_{n_k} \rightharpoonup Ax^*$.

Now setting $v_{n_k} = Ax_{n_k} - J_{r_n}^{F_2}Ax_{n_k}$. It follows from (5.3.14) that $\lim\limits_{k\to\infty} v_{n_k} = 0$ and $Ax_{n_k} - v_{n_k} = J_{r_{n_k}}^{F_2}Ax_{n_k}$.

Therefore from Lemma 2.3.22 we have

$$F_2(Ax_{n_k} - v_{n_k}, z) + \frac{1}{r_{n_k}}\langle z - (Ax_{n_k} - v_{n_k}), (Ax_{n_k} - v_{n_k}) - Ax_{n_k}\rangle \geq 0, \quad \forall z \in Q.$$

Since F_2 is upper semicontinuous in the first argument, taking limsup of the above inequality as $k \to \infty$ and using condition (iv), we obtain

$$F_2(Ax^*, z) \geq 0, \quad \forall z \in Q,$$

which means that $Ax^* \in EP(F_2)$ and hence $x^* \in \Omega$.

Finally, by using the argument as in the proof of Theorem 3.1 of [134], we can show that $x^* \in \Gamma$. Meanwhile, since $\{f_n(x_n)\}$ is uniformly convergent on D, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f_n(p) - p, x_n - p\rangle &= \limsup_{j\to\infty}\langle f_{n_j}(p) - p, x_{n_j} - p\rangle \\ &= \langle f(p) - p, x^* - p\rangle \geq 0.\end{aligned}$$

By (5.3.13), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq\ & (1 - 2\alpha_n(1 - \bar{\rho}(1 - \alpha_n)))\|x_n - p\|^2 - \beta_n\delta_n\|x_n - z_n\|^2 \\ & + \alpha_n^2(1 + \bar{\rho}^2)\|x_n - p\|^2 + 2\alpha_n\langle f_n(p) - p, x_{n+1} - p\rangle \\ \leq\ & (1 - 2\alpha_n(1 - \bar{\rho}))\|x_n - p\|^2 - \beta_n\delta_n\|x_n - z_n\|^2 \\ & + \alpha_n\left[\alpha_n(1 + \bar{\rho}^2)\|x_n - p\|^2 + 2\langle f_n(p) - p, x_{n+1} - p\rangle\right].\end{aligned} \tag{5.3.21}$$

Using Lemma 2.3.23 we have that $x_n \to p$ as $n \to \infty$.

Case 2. Assume that$\{\|x_n - p\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n - p\|^2$ and Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

This implies that $\|x_{\tau(n)} - p\| \leq \|x_{\tau(n)+1} - p\|$, $\forall n \geq n_0$. Thus $\lim\limits_{n\to\infty}\|x_{\tau(n)} - p\|$ exists. In a similar way as in case 1, we can show that

$$\|A^*(J^{F_2}_{r_{\tau(n)}} - I)Ax_{\tau(n)}\| \to 0, \quad n \to \infty. \tag{5.3.22}$$

Similarly,

$$\|x_{\tau(n)} - w_{\tau(n)}\| \to 0, \quad n \to \infty, \tag{5.3.23}$$

so that

$$d(x_{\tau(n)}, Sx_{\tau(n)}) \leq \|x_{\tau(n)} - w_{\tau(n)}\| \to 0, \quad n \to \infty,$$

and

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0, \quad n \to \infty. \tag{5.3.24}$$

We can also show that

$$\begin{aligned}&\|u_{\tau(n)} - x_{\tau(n)}\| \to 0, \quad n \to \infty, \\ &\|w_{\tau(n)} - y_{\tau(n)}\| \to 0, \quad n \to \infty,\end{aligned}$$

and

$$\|(J^{F_2}_{r_{\tau(n)}} - I)A_{x_{\tau(n)}}\| \to 0, \quad n \to \infty.$$

From the fact that $\{x_{\tau(n)}\}$ is bounded, we have that there exists a subsequence of $\{x_{\tau(n)}\}$, denoted as $\{x_{\tau(n)}\}$, that converges weakly to $x^* \in H_1$. Since $\|u_{\tau(n)} - x_{\tau(n)}\| \to 0$, it follows that $u_{\tau(n)} \rightharpoonup x^* \in H_1$. As in Case 1, we can show that $x^* \in \Upsilon$ and

$$\limsup_{n\to\infty} \langle f_{\tau(n)}(p) - p, x_{\tau(n)+1} - p\rangle \geq 0.$$

By (5.3.13), we get

$$\begin{aligned} \|x_{\tau(n)+1} - p\|^2 &\leq (1 - 2\alpha_{\tau(n)}(1 - \bar{\rho}(1 - \alpha_{\tau(n)})))\|x_{\tau(n)} - p\|^2 - \beta_n\delta_n\|x_n - z_n\|^2 \\ &+ \alpha^2_{\tau(n)}(1 + \bar{\rho}^2)\|x_{\tau(n)} - p\|^2 + 2\alpha_{\tau(n)}\langle f_{\tau(n)}(p) - p, x_{\tau(n)+1} - p\rangle \\ &\leq (1 - 2\alpha_{\tau(n)}(1 - \bar{\rho}))\|x_{\tau(n)} - p\|^2 + \alpha^2_{\tau(n)}(1 + \bar{\rho}^2)\|x_{\tau(n)} - p\|^2 \\ &+ 2\alpha_{\tau(n)}\langle f_{\tau(n)}(p) - p, x_{\tau(n)+1} - p\rangle. \end{aligned} \tag{5.3.25}$$

Which implies that (noting that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$)

$$\begin{aligned} 2(1 - \bar{\rho})\|x_{\tau(n)} - p\|^2 \;\; &\leq \;\; \alpha_{\tau(n)}(1 + \bar{\rho}^2)\|x_{\tau(n)} - p\|^2 \\ & \quad +2\langle f_{\tau(n)}(p) - p, x_{\tau(n)+1} - p\rangle. \end{aligned} \tag{5.3.26}$$

This implies that

$$\limsup_{n\to\infty} \|x_{\tau(n)} - p\| \leq 0.$$

Thus

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\| = 0. \tag{5.3.27}$$

Therefore,

$$\|x_{\tau(n)+1} - p\| \leq \|x_{\tau(n)} - p\| + \|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0, \quad n \to \infty.$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j \leq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim \Gamma_n = 0$, that is, $\{x_n\}$ converges to $\bar{x}$. This completes the proof. □

If S is a single-valued quasi-nonexpansive mapping. We obtain the following result.

Corollary 5.3.6. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_n : H_1 \to H_1$ be a sequence of ρ_n-contractive mappings with $0 < \underline{\rho} \leq \rho_n \leq \bar{\rho} < 1$ and $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of H_1. Let $B : C \to H_1$ be τ-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are two bifunctions satisfying hypothesis $(A1) - (A4)$ and F_2 is upper semicontinuous in the first argument. Let $S : H_1 \to H_1$ be a quasi-nonexpansive mapping such that $I - S$ is demiclosed at the origin, $Sp = p\ \forall p \in$ Fix*

(S) and $\Upsilon := Fix\ (S) \cap \Omega \cap \Gamma \neq \emptyset$. For arbitrary $x_1 \in H_1$, define the iterative sequence $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ by

$$\begin{cases} u_n = J^{F_1}_{r_n}(x_n + \gamma_n A^*(J^{F_2}_{r_n} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n Bu_n), \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n(\sigma Sx_n + (1-\sigma)y_n), \quad n \geq 1, \end{cases} \tag{5.3.28}$$

where $\gamma_n := \mu_n \frac{\|(J^{F_2}_{r_n} - I)Ax_n\|^2}{\|A^*(J^{F_2}_{r_n} - I)Ax_n\|^2}$ with $0 < a \leq \mu_n \leq b < 1$,
$r_n \in (0,\infty)$, $\lambda_n \in (0, 2\tau)$, $\sigma, \bar{\rho}, \underline{\rho} \in (0,1)$ and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions

(i) $\alpha_n + \beta_n + \delta_n = 1$;

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty}, \alpha_n = \infty$,

(iii) $0 < \liminf\limits_{n\to\infty} \lambda_n \leq \limsup\limits_{n\to\infty} \lambda_n < 2\tau$,

(iv) $\beta_n \geq \epsilon_1 > 0 \quad \delta_n \geq \epsilon_2 > 0$.

Then the sequence $\{x_n\}$ converges strongly to $p \in \Upsilon$ where $p = P_\Upsilon f(p)$.

5.3.1 Application to split monotone inclusion problem.

Let $G : H_1 \to 2^{H_1}$ be a multivalued mapping. The multi-valued mapping G is said to be monotone if for each $x, y \in H_1$ any $u \in G(x), v \in G(y)$, we have that

$$\langle u - v, x - y \rangle \geq 0.$$

A monotone multi-valued mapping G is said to be a maximal monotone mapping if the Graph$(G) = \{(x,u) \in H_1 \times H_1, u \in Gx\}$ is not properly contained in the graph of any other monotone mapping on H_1, see [12]. To every maximal monotone multi-valued mapping G, there is an associated mapping $J^G_\lambda : H_1 \to H_1$, is called the resolvent of G, defined by

$$J^G_\lambda(x) := (I + \lambda G)^{-1}(x), \quad \forall x \in H_1,$$

for some $\lambda > 0$, where I is the identity mapping on H_1. The resolvent mapping J^G_λ is single valued and firmly nonexpansive (hence nonexpansive) (see for example, [25, 122, 123] for more details).
Let H_1 and H_2 be real Hilbert spaces. Let $G_1 : H_1 \to 2^{H_1}$ and $G_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings. Let $A : H_1 \to H_2$ be a bounded linear mapping. The split monotone inclusion problem (see, for example, [130]) is to find $x^* \in H_1$ such that

$$0 \in G_1(x^*), \qquad x^* \in Sx^* \tag{5.3.29}$$

and

$$0 \in G_2(Ax^*), \tag{5.3.30}$$

where $S : H_1 \to CB(H_1)$ is a multi-valued quasi-nonexpansive mapping. We shall denote by $\mho$, the solution set of (5.3.29) - (5.3.30). That is,

$$\mho = \{x^* \in H_1 : 0 \in G_1(x^*) \text{ and } G_2(Ax^*)\}.$$

Putting $F_1 = G_1$ and $F_2 = G_2$ in Theorem 5.3.5 we obtain the following result

Corollary 5.3.7. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_n : H_1 \to H_1$ be a sequence of ρ_n-contractive mappings with $0 < \underline{\rho} \leq \rho_n \leq \bar{\rho} < 1$ and $\{f_n(x)\}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of H_1. Let $B : C \to H_1$ be τ-inverse strongly monotone mapping. Assume that $G_1 : H_1 \to 2^{H_1}$ and $G_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $S : H_1 \to CB(H_1)$ be a multi-valued quasi-nonexpansive mapping such that S is demiclosed at the origin, $Sp = \{p\}$ $\forall p \in F(S)$ and $\Upsilon := Fix\ (S) \cap \mho \cap \Gamma \neq \emptyset$. For arbitrary $x_1 \in H_1$, define the iterative sequence $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ by*

$$\begin{cases} u_n = J_\lambda^{G_1}(x_n + \gamma_n A^*(J_\lambda^{G_2} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n + \delta_n(\sigma w_n + (1-\sigma)y_n), \quad w_n \in Sx_n, \quad n \geq 1, \end{cases} \tag{5.3.31}$$

where $\gamma_n := \mu_n \frac{\|(J_\lambda^{G_2} - I)Ax_n\|^2}{\|A^*(J_\lambda^{G_2} - I)Ax_n\|^2}$ with $0 < a \leq \mu_n \leq b < 1$, $\lambda_n \in (0, 2\tau)$, $\sigma, \bar{\rho}, \underline{\rho} \in (0,1)$ and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions

(i) $\alpha_n + \beta_n + \delta_n = 1$;

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty}, \alpha_n = \infty$,

(iii) $0 < \liminf\limits_{n\to\infty} \lambda_n \leq \limsup\limits_{n\to\infty} \lambda_n < 2\tau$,

(iv) $\beta_n \geq \epsilon_1 > 0 \quad \delta_n \geq \epsilon_2 > 0$.

Then the sequence $\{x_n\}$ converges strongly to $p \in \Upsilon$ where $p = P_\Upsilon f(p)$.

5.4 Iterative solution of split fixed point problems and monotone inclusion problems in Hilbert space

In this section, we introduce an iterative algorithm and prove a strong convergence theorem for finding a fixed point of a multi-valued Lipschitz hemicontractive-type mapping which is also a solution of a monotone variational inclusion problem (4.1.2), where $T = M + B$ with M being a maximal monotone operator and B an α-inverse strongly monotone mapping and whose image under a bounded linear operator is a fixed point of a demicontractive mapping. In our result the step-size is selected in such a way that its implementation does not involve the computation or an estimate of the operator norm. Hence Our result improves and extends many known results in this direction.

Theorem 5.4.1. *Let H_1 and H_2 be two real Hilbert spaces and let C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $M : H_1 \to 2^{H_1}$ be a maximal monotone operator and $B : C \to H_1$ be τ-inverse strongly monotone mapping. Let $S : H_1 \to CB(H_1)$ be a L-Lipschitz hemicontractive-type mapping and $T : H_2 \to H_2$ be μ-demicontractive mapping such that $\Upsilon := F(S) \cap (M+B)^{-1}(0) \cap A^{-1}F(T) \neq \emptyset$. Let the step size γ_n be chosen such that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \frac{(1-\mu)\|TAx_n - Ax_n\|^2}{\|A^*(T-I)Ax_n\|^2} - \epsilon\right)$, if $TAx_n \neq Ax_n$; otherwise $\gamma_n = \gamma$(γ being any nonnegative real number). Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ and suppose the following conditions are satisfied:*

- *(i)* $\alpha_n + \beta_n + \delta_n = 1$;
- *(ii)* $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty}, \alpha_n = \infty$;
- *(iii)* $0 < \liminf\limits_{n\to\infty} r_n \leq \limsup\limits_{n\to\infty} r_n < 2\tau$;
- *(iv)* $\alpha_n + \beta_n \leq \lambda_n \leq \lambda \leq \frac{1}{\sqrt{1+4L^2}+1}$,
- *(v)* $Sp = \{p\}$ $\forall p \in \Upsilon$, $(I-T)$ *and* $(I-S)$ *are demiclosed at* 0.

Then the sequence $\{x_n\}$ generated for any $x_1, u \in H_1$ by

$$\begin{cases} w_n = P_C(x_n + \gamma_n A^*(T-I)Ax_n), \\ z_n = (I + r_n M)^{-1}(w_n - r_n B w_n), \\ y_n = (1-\lambda_n)z_n + \lambda_n u_n, \\ x_{n+1} = \alpha_n u + \beta_n v_n + \delta_n z_n, \quad n \geq 1, \end{cases} \tag{5.4.1}$$

where $u_n \in Sz_n$, and $v_n \in Sy_n$ converges strongly to $q \in \Upsilon$ where $q = P_{\Upsilon}u$.

Proof. We first show that $\{x_n\}$ is bounded. Let $q = P_{\Upsilon}u$ and define $T_n := J_{r_n}^M(I - r_n B)$, $\forall n \geq 1$. Then, T_n is nonexpansive for all $n \geq 1$ and

$$\|z_n - q\| = \|T_n w_n - T_n q\| \leq \|w_n - q\|.$$

From (2.1.7), (5.4.1) and $q \in \Upsilon$, we have

$$\begin{aligned}
\|w_n - q\|^2 &= \|P_C(x_n + \gamma_n A^*(T-I)Ax_n) - q\|^2 \\
&= \|x_n + \gamma_n A^*(T-I)Ax_n - q\|^2 \\
&= \|x_n - q\|^2 + \gamma_n^2 \|A^*(T-I)Ax_n\|^2 + 2\gamma_n \langle x_n - q, A^*(T-I)Ax_n\rangle \\
&= \|x_n - q\|^2 + \gamma_n^2 \|A^*(T-I)Ax_n\|^2 + 2\gamma_n \langle A(x_n - q), (T-I)Ax_n\rangle \\
&= \|x_n - q\|^2 + \gamma_n^2 \|A^*(T-I)Ax_n\|^2 \\
&\quad + 2\gamma_n[\langle Ax_n - Ap, TAx_n - Aq\rangle + \langle Ax_n - Aq, Aq - Ax_n\rangle] \\
&= \|x_n - q\|^2 + \gamma_n^2 \|A^*(T-I)Ax_n\|^2 \\
&\quad + 2\gamma_n[\langle Ax_n - Aq, TAx_n - Aq\rangle - \|Ax_n - Aq\|^2] \\
&\leq \|x_n - q\|^2 + \gamma_n^2 \|A^*(T-I)Ax_n\|^2 \\
&\quad + 2\gamma_n[\|Ax_n - Aq\|^2 - \frac{(1-\mu)}{2}\|TAx_n - Ax_n\|^2 - \|Ax_n - Aq\|^2] \\
&= \|x_n - q\|^2 + \gamma_n^2 \|A^*(T-I)Ax_n\|^2 + \gamma_n(\mu - 1)\|TAx_n - Ax_n\|^2 \\
&= \|x_n - q\|^2 + \gamma_n[\gamma_n \|A^*(T-I)Ax_n\|^2 + (\mu - 1)\|TAx_n - Ax_n\|^2]
\end{aligned} \tag{5.4.2}$$

From the choice of γ_n and (5.4.2), we get

$$\|w_n - q\|^2 \leq \|x_n - q\|^2. \tag{5.4.3}$$

Since S is hemicontractive-type mapping and $u_n \in Sz_n$, we obtain from (5.4.1) and (5.4.3) that

$$\begin{aligned}
\|y_n - q\|^2 &= \|(1-\lambda_n)(z_n - q) + \lambda_n(u_n - q)\|^2 \\
&= (1-\lambda_n)\|z_n - q\|^2 + \lambda_n\|u_n - q\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - q\|^2 + \lambda_n \mathcal{H}^2(Sz_n, Sq) - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda)\|z_n - q\|^2 + \lambda_n\left(\|z_n - q\|^2 + \|z_n - u_n\|^2\right) - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&= \|z_n - q\|^2 + \lambda_n\|z_n - u_n\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq \|x_n - q\|^2 + \lambda_n^2\|z_n - u_n\|^2.
\end{aligned} \tag{5.4.4}$$

Also, since S is hemicontractive-type mapping and $v_n \in Sy_n$, from (5.4.1) and (5.4.4), we obtain

$$\begin{aligned}
\|v_n - q\|^2 &= (d(v_n, Sq))^2 \leq \mathcal{H}^2(Sy_n, Sq) \\
&\leq \|y_n - q\|^2 + \|y_n - v_n\|^2 \\
&\leq \|x_n - q\|^2 + \lambda_n^2\|z_n - u_n\|^2 + \|y_n - v_n\|^2.
\end{aligned} \tag{5.4.5}$$

Also from (5.4.1), we have

$$\begin{aligned}
\|z_n - y_n\|^2 &= \|z_n - ((1-\lambda_n)z_n + \lambda_n u_n)\|^2 \\
&= \lambda_n^2\|z_n - u_n\|^2.
\end{aligned} \tag{5.4.6}$$

Since S is L-Lipschitzian mapping and $\|u_n - v_n\| \leq 2\mathcal{H}(Sz_n, Sy_n)$, using (5.4.6), we obtain that

$$
\begin{aligned}
\|y_n - v_n\|^2 &= \|(1-\lambda_n)(z_n - v_n) + \lambda_n(u_n - v_n)\|^2 \\
&= (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n\|u_n - v_n\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n\|u_n - v_n\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - v_n\|^2 + 4\lambda_n\mathcal{H}^2(Tz_n, Ty_n) - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&\leq (1-\lambda_n)\|z_n - v_n\|^2 + 4\lambda_n L^2\|z_n - y_n\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \\
&= (1-\lambda_n)\|z_n - v_n\|^2 + 4\lambda_n^3 L^2\|z_n - u_n\|^2 - \lambda_n(1-\lambda_n)\|z_n - u_n\|^2 \tag{5.4.7}
\end{aligned}
$$

Hence, substituting (5.4.7) into (5.4.5), we obtain that

$$
\begin{aligned}
\|v_n - q\|^2 &\leq \|x_n - q\|^2 + \lambda_n^2\|z_n - u_n\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n(4L^2\lambda_n^2 + \lambda_n - 1)\|z_n - u_n\|^2 \\
&= \|x_n - q\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2. \tag{5.4.8}
\end{aligned}
$$

Thus, from (5.4.1), (5.4.8) and Lemma 2.3.19, we have that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n u + \beta_n v_n + \delta_n z_n - q\|^2 \\
&= \alpha_n\|u - q\|^2 + \beta_n\|v_n - q\|^2 + \delta_n\|z_n - q\|^2 - \beta_n\delta_n\|z_n - v_n\|^2 \\
&\leq \|u - q\|^2 + \beta_n\left[\|x_n - q\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2\right] \\
&\quad + \delta_n\|z_n - q\|^2 - \beta_n\delta_n\|z_n - v_n\|^2 \\
&\leq \alpha_n\|u - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 + \beta_n(1-\lambda_n)\|z_n - v_n\|^2 \\
&\quad + \beta_n\lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 - \beta_n\delta_n\|z_n - v_n\|^2 \\
&= \alpha_n\|u - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 + \beta_n(1 - \delta_n - \lambda_n)\|z_n - v_n\|^2 \\
&\quad - \beta_n\lambda_n(1 - 4L^2\lambda_n^2 - 2\lambda_n)\|z_n - u_n\|^2. \tag{5.4.9}
\end{aligned}
$$

From assumption (iv), we have

$$
1 - 4L^2\lambda_n^2 - 2\lambda_n \geq 1 - 4L^2\lambda^2 - 2\lambda > 0 \quad \text{and} \quad (\alpha_n + \beta_n) - \lambda_n \leq 0, \quad \forall n \geq 1. \tag{5.4.10}
$$

Therefore, from (5.4.9) and (5.4.10), we have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \alpha_n\|u - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 \\
&\leq \max\{\|u - q\|^2, \|x_n - q\|^2\} \\
&\vdots \\
&\leq \max\{\|u - q\|^2, \|x_1 - q\|^2\}.
\end{aligned}
$$

Hence $\{x_n\}$ is bounded. From (5.4.1), we have

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \|\alpha_n u + \beta_n v_n + \delta_n z_n - q\|^2 \\
&\leq \|\beta_n(v_n - q) + \delta_n(z_n - q)\|^2 + 2\alpha_n\langle u - q, x_{n+1} - q\rangle \\
&\leq \beta_n\|v_n - q\|^2 + \delta_n\|z_n - q\|^2 - \beta_n\delta_n\|z_n - u_n\|^2 + 2\alpha_n\langle u - q, x_{n+1} - q\rangle \\
&\leq \beta_n\left[\|x_n - q\|^2 + \lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 + (1-\lambda_n)\|z_n - v_n\|^2\rangle\right] \\
&\quad + \delta_n\|x_n - q\|^2 - \beta_n\delta_n\|z_n - u_n\|^2 + 2\alpha_n\langle u - q, x_{n+1} - q\rangle \\
&= (1-\alpha_n)\|x_n - q\|^2 - \beta_n\lambda_n(1 - 4\lambda_n^2 L^2 - 2\lambda_n)\|z_n - u_n\|^2 \\
&\quad + \beta_n(\alpha_n + \beta_n - \lambda_n)\|z_n - v_n\|^2 + 2\alpha_n\langle u - q, x_{n+1} - q\rangle. \tag{5.4.11}
\end{aligned}
$$

Which implies that

$$\|x_{n+1}-q\|^2 \;\leq\; (1-\alpha_n)\|x_n-q\|^2+2\alpha_n\langle u-q, x_{n+1}-q\rangle. \tag{5.4.12}$$

Now, to get strong convergence we divide the proof into two cases

<u>Case 1.</u> Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-q\|\}$ is decreasing for all $n \geq n_0$. Then, we get that $\{\|x_n-q\|\}$ is convergent. Hence we have that

$$\|x_{n+1}-q\|^2-\|x_n-q\|^2 \to 0, \quad n \to \infty. \tag{5.4.13}$$

Thus, from (5.4.10) and (5.4.11), we have that

$$\begin{aligned}\beta_n\lambda_n(1-4L^2\lambda_n^2-2\lambda_n)\|z_n-u_n\|^2 \;&\leq\; (1-\alpha_n)\|x_n-q\|^2\\ &\quad -\|x_{n+1}-q\|^2+2\alpha_n\langle u-q, x_{n+1}-q\rangle.\end{aligned}$$

Hence form (5.4.10), (5.4.13) and the fact that $\alpha_n \to 0$ as $n \to \infty$, we have that

$$\|z_n-u_n\| \to 0 \;\text{ as }\; n \to \infty, \tag{5.4.14}$$

which implies that

$$d(z_n, Sz_n) \leq \|z_n-u_n\| \to 0 \;\; as \;\; n \to \infty. \tag{5.4.15}$$

Also, since S is Lipschitz we obtain from (5.4.6) and (5.4.14) that

$$\begin{aligned}\|z_n-v_n\| \;&\leq\; \|z_n-u_n\|+\|u_n-v_n\|\\ &\leq\; \|z_n-u_n\|+2L\|z_n-y_n\|\\ &=\; \|z_n-u_n\|+2L\lambda_n\|z_n-u_n\| \to 0 \;\; as \;\; n \to \infty.\end{aligned} \tag{5.4.16}$$

Suppose $TAx_n \neq Ax_n$, then $\gamma_n \in \left(\epsilon, \frac{(1-\mu)\|TAx_n-Ax_n\|^2}{\|A^*(T-I)Ax_n\|^2}-\epsilon\right)$ and from (5.4.2), we have

$$\begin{aligned}\|w_n-q\|^2 \;&\leq\; \|x_n-q\|^2+\gamma_n[\gamma_n\|A^*(T-I)Ax_n\|^2+(\mu-1)\|TAx_n-Ax_n\|^2]\\ &\leq\; \|x_n-q\|^2-\gamma_n\epsilon\|A^*(T-I)Ax_n\|^2.\end{aligned} \tag{5.4.17}$$

Also from (5.4.1), (5.4.8), (5.4.17) and Lemma 2.3.19, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 \;&\leq\; \alpha_n\|u-q\|^2+\beta_n\|v_n-q\|^2+\delta_n\|z_n-q\|^2-\beta_n\delta_n\|z_n-v_n\|^2\\ &\leq\; \alpha_n\|u-q\|^2+\beta_n\left[\|x_n-q\|^2+(1-\lambda_n)\|z_n-v_n\|^2+\lambda_n(4L^2\lambda_n^2+2\lambda_n-1)\|z_n-u_n\|^2\right]\\ &\quad +\delta_n\|w_n-q\|^2-\delta_n\beta_n\|z_n-v_n\|^2\\ &\leq\; \alpha_n\|u-q\|^2+\beta_n\|x_n-q\|^2+\beta_n(1-\lambda_n)\|z_n-v_n\|^2-\delta_n\beta_n\|z_n-v_n\|^2\\ &\quad +\beta_n\lambda_n(4L^2\lambda^2+2\lambda_n-1)\|z_n-u_n\|^2+\delta_n\|x_n-q\|^2-\delta_n\gamma_n\epsilon\|A^*(T-I)Ax_n\|^2,\end{aligned} \tag{5.4.18}$$

which implies that

$$\begin{aligned}\delta_n\gamma_n\epsilon\|A^*(T-I)Ax_n\|^2 \;&\leq\; \alpha_n\|u-q\|^2-(1-\alpha_n)\|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\quad +\beta_n(1-\delta_n-\lambda_n)\|z_n-v_n\|^2\\ &\quad +\beta_n\lambda_n(4L^2\lambda_n^2+2\lambda_n-1)\|z_n-u_n\|^2.\end{aligned} \tag{5.4.19}$$

From (5.4.13), (5.4.14), (5.4.16) and the fact that $\alpha_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|A^*(T-I)Ax_n\| = 0. \tag{5.4.20}$$

Also, from (5.4.17) and (5.4.18), we have

$$\begin{aligned} \delta_n\gamma_n\|TAx_n - Ax_n\|^2 \;\; \leq \;\; & \alpha_n\|u-q\|^2 + (1-\alpha_n)\|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \beta_n(1-\delta_n-\lambda_n)\|z_n - v_n\|^2 \\ & +\beta_n\lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 + \delta_n\gamma_n\|A^*(T-I)Ax_n\|^2. \end{aligned} \tag{5.4.21}$$

From (5.4.13), (5.4.14), (5.4.16), (5.4.20) and the fact that $\alpha_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0. \tag{5.4.22}$$

Now, from (5.4.1), we have

$$\begin{aligned} \|w_n - q\|^2 &= \|P_C(x_n + \gamma_n A^*(T-I)Ax_n) - q\|^2 \\ &\leq \langle w_n - q, x_n + \gamma_n A^*(T-I)Ax_n - q\rangle \\ &= \frac{1}{2}[\|w_n - q\|^2 + \|x_n + \gamma_n A^*(T-I)Ax_n - q\|^2 - \|w_n - q - (x_n + \gamma_n A^*(T-I)Ax_n) - q\| \\ &= \frac{1}{2}[\|w_n - q\|^2 + \|x_n - q\|^2 + \gamma_n[\gamma_n\|A^*(T-I)Ax_n\|^2 \\ &\quad +(\mu - 1)\|TAx_n - Ax_n\|^2] - \|w_n - x_n - \gamma_n A^*(T-I)Ax_n - q\|^2] \\ &\leq \frac{1}{2}[\|w_n - q\|^2 + \|x_n - q\|^2 - \|w_n - x_n\|^2 + \gamma_n\|Aw_n - Ax_n\|\|(T-I)Ax_n\|]. \end{aligned}$$

That is,

$$\|w_n - q\|^2 \;\; \leq \;\; \|x_n - q\|^2 - \|w_n - x_n\|^2 + 2\gamma_n\|Aw_n - Ax_n\|\|(T-I)Ax_n\|. \tag{5.4.23}$$

From (5.4.19) and (5.4.23), we obtain

$$\begin{aligned} \|x_{n+1} - q\|^2 \;\; \leq \;\; & \alpha_n\|u-q\|^2 + \beta_n\left[\|x_n - q\|^2 + (1-\lambda_n)\|z_n - v_n\|^2 + \lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2\right] \\ & +\delta_n\|w_n - q\|^2 - \delta_n\beta_n\|z_n - v_n\|^2 \\ \leq \;\; & \alpha_n\|u-q\|^2 + \beta_n\|x_n - q\|^2 + \beta_n(1-\delta_n-\lambda_n)\|z_n - v_n\|^2 \\ & +\beta_n\lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 + \delta_n\|x_n - q\|^2 - \delta_n\|w_n - x_n\|^2 \\ & +\delta_n\gamma_n\|Aw_n - Ax_n\|\|(T-I)Ax_n\|, \end{aligned}$$

which implies that

$$\begin{aligned} \delta_n\|w_n - x_n\|^2 \;\; \leq \;\; & \alpha_n\|u-q\|^2 + (1-\alpha_n)\|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \beta_n(1-\delta_n-\lambda_n)\|z_n - v_n\|^2 \\ & +\beta_n\lambda_n(4L^2\lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 \\ & +\delta_n\gamma_n\|Aw_n - Ax_n\|\|(T-I)Ax_n\|. \end{aligned} \tag{5.4.24}$$

From (5.4.13), (5.4.14), (5.4.16), (5.4.24) and the fact that $\alpha_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{5.4.25}$$

Next, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \alpha_n\|u-q\|^2+\beta_n\|v_n-q\|^2+\delta_n\|z_n-q\|^2-\delta_n\beta_n\|z_n-v_n\|^2\\
&\le \alpha_n\|u-q\|^2+\beta_n\left[\|x_n-q\|^2+(1-\lambda_n)\|z_n-v_n\|^2+\lambda_n(4L^2\lambda_n^2+2\lambda_n-1)\|z_n-u_n\|^2\right]\\
&\quad +\delta_n\left[\|(w_n-r_nBw_n)-(q-r_nBq)\|^2\right]-\beta_n\delta_n\|z_n-v_n\|^2\\
&= \alpha_n\|u-q\|^2+\beta_n(1-\delta_n-\lambda_n)\|z_n-v_n\|^2+\beta_n\lambda_n(4L^2\lambda_n^2+2\lambda_n-1)\|z_n-u_n\|^2\\
&\quad +\beta_n\|x_n-q\|^2+\delta_n\left[\|w_n-q\|^2-2r_n\langle w_n-q,Bw_n-Bq\rangle+r_n^2\|Bw_n-Bq\|^2\right]\\
&\le \alpha_n\|u-q\|^2+(1-\alpha_n)\|x_n-q\|^2+\delta_n r_n(r_n-2\tau)\|Bw_n-Bq\|^2\\
&\quad +\beta_n(1-\delta_n-\lambda_n)\|z_n-v_n\|^2+\beta_n\lambda_n(4L^2\lambda_n^2+2\lambda_n-1)\|z_n-u_n\|^2,
\end{aligned}$$

and so,

$$\begin{aligned}
\delta_n r_n(2\tau-r_n)\|Bw_n-Bq\|^2 &\le \alpha_n\|u-q\|^2+(1-\alpha_n)\|x_n-q\|^2-\|x_{n+1}-q\|^2\\
&\quad +\beta_n(1-\delta_n-\lambda_n)\|z_n-v_n\|^2+\beta_n\lambda_n(4L^2\lambda_n^2+2\lambda_n-1)\|z_n-u_n\|^2.
\end{aligned}$$

Hence from condition (iii), we obtain that

$$\lim_{n\to\infty}\|Bw_n-Bq\|=0. \tag{5.4.26}$$

Observe that $(I-r_nB)$ is nonexpansive and $J_{r_n}^M$ is firmly nonexpansive mapping. Then we get

$$\begin{aligned}
\|z_n-q\|^2 &= \|J_{r_n}^M(w_n-r_nBw_n)-J_{r_n}^M(q-r_nBq)\|^2\\
&\le \langle (w_n-r_nBw_n)-(q-r_nBq,z_n-q\rangle\\
&= \frac{1}{2}\Big[\|w_n-r_nBw_n-(q-r_nBq)\|^2+\|z_n-q\|^2\\
&\quad -\|(w_n-r_nBw_n)-(q-r_nBq)-(z_n-q)\|^2\Big]\\
&\le \frac{1}{2}\Big[\|w_n-q\|^2+\|z_n-q\|^2-\|(w_n-z_n)-r_n(Bw_n-Bq)\|^2\Big]\\
&= \frac{1}{2}\Big[\|w_n-q\|^2+\|z_n-q\|^2-\|w_n-z_n\|^2+2r_n\langle w_n-z_n,Bw_n-Bq\rangle\\
&\quad -r_n^2\|Bw_n-Bq\|^2\Big].
\end{aligned}$$

Therefore

$$\|z_n-q\|^2\le\|w_n-q\|^2-\|w_n-z_n\|^2+2r_n\langle w_n-z_n,Bw_n-Bq\rangle-r_n^2\|Bw_n-Bq\|^2 \tag{5.4.27}$$

By (5.4.1) and (5.4.27), we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \alpha_n\|u_n-q\|^2+(1-\alpha_n)\|x_n-q\|^2+\beta_n(1-\delta_n-\lambda_n)\|z_n-v_n\|^2\\
&\quad +\beta_n\lambda_n(4L^2\lambda_n^2+2\lambda_n-1)\|z_n-u_n\|^2-\delta_n\|w_n-z_n\|^2\\
&\quad +2\delta_n r_n\|w_n-z_n\|\|Bw_n-Bq\|-r_n^2\|Bw_n-Bq\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}\delta_n\|w_n-z_n\|^2 &\leq \alpha_n\|u-q\|^2+(1-\alpha_n)\|x_n-q\|^2-\|x_{n+1}-q\|^2\\ &\quad+\beta_n(1-\delta_n-\lambda_n)\|z_n-v_n\|^2+\beta_n\lambda_n(4L^2\lambda_n^2+2\lambda_n-1)\|z_n-u_n\|^2\\ &\quad+2\delta_n r_n(\|w_n\|+\|z_n\|)\|Bw_n-Bq\|. \end{aligned} \tag{5.4.28}$$

Since $\lim\limits_{n\to\infty}\alpha_n=0$ and both $\{z_n\}$ and $\{w_n\}$ are bounded by (5.4.28), we have

$$\lim_{n\to\infty}\|T_nw_n-w_n\|=\lim_{n\to\infty}\|w_n-z_n\|=0. \tag{5.4.29}$$

Also, from (5.4.25) and (5.4.29), we obtain

$$\|z_n-x_n\|\leq\|z_n-w_n\|+\|w_n-x_n\|\to 0,\quad n\to\infty. \tag{5.4.30}$$

Therefore, from (5.4.1), (5.4.14), (5.4.30) and the fact that $\alpha_n\to 0$ as $n\to\infty$, we get

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq \|x_{n+1}-z_n\|+\|z_n-x_n\|\\ &= \|\alpha_n(u-z_n)+\beta_n(v_n-z_n)\|+\|z_n-x_n\|\\ &\leq \alpha_n\|u-z_n\|+\beta_n\|v_n-z_n\|+\|z_n-x_n\|\to 0,\ \text{as } n\to\infty. \end{aligned} \tag{5.4.31}$$

Now, let $z=P_{\Upsilon}u$. We claim that $\lim\limits_{n\to\infty}\langle u-z,x_{n+1}-z\rangle\leq 0$. Since $\{x_{n+1}\}$ is a bounded sequence in a Hilbert space H, which is a reflexive Banach space, there exists a subsequence $\{x_{n_j+1}\}$ of $\{x_{n+1}\}$ and an element in H, say p, such that

$$x_{n_j+1}\rightharpoonup p \quad\text{and}\quad \limsup_{n\to\infty}\langle u-z,x_{n+1}-z\rangle=\lim_{j\to\infty}\langle u-z,x_{n_j+1}-z\rangle.$$

Since C is weakly closed, we have $p\in C$ and from (5.4.31) it follows that $x_{n_j}\rightharpoonup p$ as $j\to\infty$. From (5.4.30), we obtain that $z_{n_j}\rightharpoonup p$ as $j\to\infty$ and hence using the fact that $(I-S)$ is demiclosed at zero and (5.4.15), we conclude that

$$p\in F(S).$$

Since $\liminf\limits_{n\to\infty}r_n>0$, there exists $\epsilon>0$ such that $r_n\geq\epsilon,\quad\forall n\geq 1$. Then, by Lemma 2.3.20, we have

$$\lim_{n\to\infty}||T_\epsilon w_n-w_n||\leq 2\lim_{n\to\infty}||T_nw_n-w_n||=0.$$

By Lemma 2.3.13, we have that $p\in F(T_\epsilon)=(M+B)^{-1}(0)$.

Moreover, since $\|w_n-x_n\|\to 0$, as $n\to\infty$, we have that Aw_{n_j} converges weakly to Ax^* and by (5.4.22) and the fact that $I-T$ is demiclosed at 0, we get that

$$Ap\in F(T).$$

Hence $p\in\Upsilon$. Since $z=P_Cu$ and $x_{n_j}\rightharpoonup p$, then from Proposition 2.1.14, we get that

$$\begin{aligned}\limsup_{n\to\infty}\langle u-z,x_{n+1}-z\rangle &= \lim_{j\to\infty}\langle u-z,x_{n+1}-z\rangle\\ &= \langle u-z,p-z\rangle\leq 0. \end{aligned} \tag{5.4.32}$$

Thus, since $z \in \Upsilon$, from (5.4.12), (5.4.32), condition (ii) and Lemma 2.3.23, we get

$$\|x_n - z\| \to 0 \text{ as } n \to \infty.$$

Hence, $x_n \to z = P_{\Upsilon}u$.

Case 2. Assume that$\{\|x_n - q\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n - q\|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is non decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

This implies that $\|x_{\tau(n)} - q\| \leq \|x_{\tau(n)+1} - q\|$, $\forall n \geq n_0$. Thus $\lim\limits_{n\to\infty} \|x_{\tau(n)} - q\|$ exists.

Again, from (5.4.19)

$$\begin{aligned} \delta_{\tau(n)}\gamma_{\tau(n)}\epsilon\|A^*(T-I)Ax_{\tau(n)}\|^2 \;\; \leq \;\; & \alpha_{\tau(n)}\|u-q\|^2 - (1-\alpha_{\tau(n)})\|x_{\tau(n)} - q\|^2 - \|x_{\tau(n)+1} - q\|^2 \\ & + \beta_{\tau(n)}(1 - \delta_{\tau(n)} - \lambda_{\tau(n)})\|z_{\tau(n)} - v_{\tau(n)}\|^2 + \beta_{\tau(n)}\lambda_{\tau(n)}(4L^2\lambda^2_{\tau(n)} \\ & + 2\lambda_{\tau(n)} - 1)\|z_{\tau(n)} - u_{\tau(n)}\|^2. \end{aligned} \tag{5.4.33}$$

Thus, we obtain

$$\lim_{n\to\infty} \|A^*(T-I)Ax_{\tau(n)}\| = 0. \tag{5.4.34}$$

Also, from (5.4.21), we have

$$\begin{aligned} \delta_{\tau(n)}\gamma_{\tau(n)}\|TAx_{\tau(n)} - Ax_{\tau(n)}\|^2 \;\; \leq \;\; & \alpha_{\tau(n)}\|u-q\|^2 + (1-\alpha_{\tau(n)})\|x_{\tau(n)} - q\|^2 - \|x_{\tau(n)+1} - q\|^2 \\ & + \beta_{\tau(n)}(1 - \delta_{\tau(n)} - \lambda_{\tau(n)})\|z_{\tau(n)} - v_{\tau(n)}\|^2 + \beta_{\tau(n)}\lambda_{\tau(n)}(4L^2\lambda^2_{\tau(n)} \\ & + 2\lambda_{\tau(n)} - 1)\|z_{\tau(n)} - u_{\tau(n)}\|^2 \\ & + \delta_{\tau(n)}\gamma_{\tau(n)}\|A^*(T-I)Ax_{\tau(n)}\|^2. \end{aligned} \tag{5.4.35}$$

Hence

$$\lim_{n\to\infty} \|TAx_{\tau(n)} - Ax_{\tau(n)}\| = 0. \tag{5.4.36}$$

By using the same argument as in case 1, we get that there is a subsequence $\{x_{\tau(n_j)}\}$ of $\{x_{\tau(n)}\}$ which converges weakly to $x^* \in \Upsilon$ as $\tau(n_j) \to \infty$. At the same time, we note from (5.4.12) that for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \\ &\leq \alpha_{\tau(n)}\left[2\langle u - x^*, x_{\tau(n)+1} - x^*\rangle - \|x_{\tau(n)} - x^*\|^2\right], \end{aligned}$$

which implies that (since $\alpha_{\tau(n)} > 0$)

$$\|x_{\tau(n)} - x^*\|^2 \leq 2\langle u - x^*, x_{\tau(n)+1} - x^*\rangle.$$

Thus,

$$\limsup_{n\to\infty}\|x_{\tau(n)}-x^*\|^2 \leq 2\limsup_{n\to\infty}\langle u-x^*, x_{\tau(n)+1}-x^*\rangle \leq 0.$$

Hence

$$\lim_{n\to\infty}\|x_{\tau(n)}-x^*\| = 0.$$

Also, from (5.4.31), we have

$$\begin{aligned}\|x_{\tau(n)+1}-x_{\tau(n)}\| &\leq \alpha_{\tau(n)}\|u-z_{\tau(n)}\| + \beta_{\tau(n)}\|v_{\tau(n)}-z_{\tau(n)}\| \\ &\quad +\|z_{\tau(n)}-x_{\tau(n)}\| \to 0, \text{ as } n\to\infty. \end{aligned} \tag{5.4.37}$$

Hence, we deduce that

$$\|x_{\tau(n)+1}-x^*\| \leq \|x_{\tau(n)+1}-x_{\tau(n)}\| + \|x_{\tau(n)}-x^*\| \to 0, \quad n\to\infty,$$

and so

$$\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim_{n\to\infty}\Gamma_n = 0$ and so $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Corollary 5.4.2. *Let H_1 and H_2 be two real Hilbert spaces and let C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $M : H_1 \to 2^{H_1}$ be a maximal monotone operator and $B : C \to H_1$ be τ-inverse strongly monotone mapping. Let $S : H_1 \to CB(H_1)$ be a L-Lipschitz hemicontractive-type mapping and $T : H_2 \to H_2$ be nonexpansive mapping such that $\Upsilon := F(S)\cap(M+B)^{-1}(0)\cap A^{-1}F(T) \neq \emptyset$. Let the step size γ_n be chosen such that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \frac{\|TAx_n-Ax_n\|^2}{\|A^*(T-I)Ax_n\|^2}-\epsilon\right)$, if $TAx_n \neq Ax_n$; otherwise $\gamma_n = \gamma$(γ being any nonnegative real number). Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ and suppose the following conditions are satisfied:*

- *(i)* $\alpha_n+\beta_n+\delta_n = 1$;
- *(ii)* $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}, \alpha_n = \infty$;
- *(iii)* $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\tau$;
- *(iv)* $\alpha_n+\beta_n \leq \lambda_n \leq \lambda \leq \frac{1}{\sqrt{1+4L^2}+1}$,
- *(v)* $Sp = \{p\}\ \forall p \in \Upsilon$, $(I-T)$ *and* $(I-S)$ *are demiclosed at* 0.

Then the sequence $\{x_n\}$ generated for any $x_1, u \in H_1$ by

$$\begin{cases} w_n = P_C(x_n+\gamma_n A^*(T-I)Ax_n), \\ z_n = (I+r_nM)^{-1}(w_n-r_nBw_n), \\ y_n = (1-\lambda_n)z_n+\lambda_n u_n, \\ x_{n+1} = \alpha_n u+\beta_n v_n+\delta_n z_n, \quad n \geq 1, \end{cases} \tag{5.4.38}$$

where $u_n \in Sz_n$, and $v_n \in Sy_n$ converges strongly to $q \in \Upsilon$ where $q = P_\Upsilon u$.

5.4.1 Application to minimization problem

Let f and g two convex, lower semi continuous functions from H to $\mathbb{R} \cup \{+\infty\}$ such that f is differentiable with $L-$Lipschitz continuous gradient, and g is "simple" meaning that its "proximal map"

$$x \to arg \min_{y \in H} \left\{ g(y) + \frac{||x-y||^2}{2\tau} \right\}$$

can easily be computed. Let us consider the following minimization problem

$$\min_{x \in H} F(x) := \min_{x \in H} \{ f(x) + g(x) \} \tag{5.4.39}$$

and assume that this problem has at least a solution. Recall that the subdifferential of a function $g : H \to \mathbb{R}$ at x is the set-valued operator on a Hilbert space H defined by

$$\partial f(x) = \{ z \in H : f(y) \geq f(x) + \langle z, y-x \rangle \}; \ \forall y \in H,$$

and $\text{prox}_{\gamma g}(x) = (I + \gamma \partial g)^{-1}(x), \quad \gamma > 0$.

It is well known that a point $x^* \in H$ is a solution to the problem (5.4.39), that is, x^* is a minimizer of $f(x) + g(x)$, if and only if $0 \in \nabla f(x^*) + \partial g(x^*)$, where ∇f is the gradient of f. For any $\gamma > 0$ this optimality condition holds if and only if the following equivalent statements hold:

$$\begin{aligned} 0 &\in \gamma \nabla f(x^*) + \gamma \partial g(x^*) \\ 0 &\in \gamma \nabla f(x^*) - x^* + x^* + \gamma \partial g(x^*) \\ (I + \gamma \partial g)(x^*) &\in (I - \gamma \nabla f)(x^*) \\ x^* &= (I + \gamma \partial g)^{-1}(I - \gamma \nabla f)(x^*) \\ x^* &= \text{prox}_{\gamma g}(x^* - \gamma \nabla f(x^*)). \end{aligned} \tag{5.4.40}$$

The last two expressions hold with equality because the proximal operator is single-valued. The final statement says that x^* minimizes $f + g$ if and only if it is a fixed point of $\text{prox}_{\gamma g}(I - \gamma \nabla f)$.

Definition 5.4.3. *A mapping $T : H \to H$ is said to be averaged if it can be written as $T = (1 - \alpha)I + \alpha S$, where $\alpha \in (0,1)$ and $S : H \to H$ is a nonexpansive mapping.*

The condition $\gamma \in (0, \frac{2}{L}]$, where L is the Lipschitz constant of ∇f guarantees that $\text{prox}_{\gamma g}(I - \gamma \nabla f)$ is averaged and hence nonexpansive.

In Corollary 5.4.2, If we take $T := \text{prox}_{\gamma_n g}(I - \gamma_n \nabla f)$ with $\gamma_n \in (0, \frac{2}{L}]$, then we obtain a strong convergence result for approximating a point x^* which solves a fixed point problem for multivalued Lipschitz hemicontractive mappings and also solves monotone variational inclusion problems and the image under a bounded linear operator is a minimizer of sum of two functions in real Hilbert spaces.

Chapter 6

Split Equality and Fixed Point Problems

6.1 Introduction

The Split Equality Fixed Point Problem (SEFPP) which generalizes the SFP (2.2.13) was introduced by Moudafi and Al-Shemas [129]:

$$\text{Find } \; x \in C := F(T), \;\; y \in Q := F(S) \; \text{ such that } \; Ax = By, \tag{6.1.1}$$

where $A : H_1 \to H_3,\ B : H_2 \to H_3$ are two bounded linear operators, $F(T)$ and $F(S)$ denotes the sets of fixed points of operators T and S defined on H_1 and H_2 respectively. Note that if $H_2 = H_3$ and $B = I$ (where I is the identity map on H_2) in (6.1.1), then problem (6.1.1) reduces to problem (2.2.13).

Further, Moudafi and Al-Shemas presented the following algorithm for solving the SEFPP

$$\begin{cases} x_{n+1} = T(x_n - \gamma_n A^*(Ax_n - By_n)); \\ y_{n+1} = S(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1; \end{cases} \tag{6.1.2}$$

where $T : H_1 \to H_1,\ S : H_2 \to H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \to H_3,\ B : H_2 \to H_3$ are two bounded linear operators, A^* and B^* are the adjoints of A and B respectively, $\{\gamma_n\} \subset \left(\epsilon, \frac{2}{\lambda_{A^*A} + \lambda_{B^*B}} - \epsilon\right)$, λ_{A^*A} and λ_{B^*B} denote the spectral radius of A^*A and B^*B, respectively. Moudafi established the weak convergence result for problem (6.1.1) using algorithm (6.1.2).
Yaun-Fang *et al.* [79] presented the following algorithm for solving problem (6.1.1):

$$\begin{cases} \forall x_1 \in H_1,\ y_1 \in H_2; \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n - \gamma_n A^*(Ax_n - By_n)); \\ y_{n+1} = (1 - \alpha_n)y_n + \alpha_n S(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1; \end{cases} \tag{6.1.3}$$

where $T : H_1 \to H_1, S : H_2 \to H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \to H_3, B : H_2 \to H_3$ are two bounded linear operators, A^* and B^* are the adjoints of A and

B respectively, $\{\gamma_n\} \subset \left(\epsilon, \frac{2}{\lambda_{A^*A}+\lambda_{B^*B}} - \epsilon\right)$ (for ϵ small enough), λ_{A^*A} and λ_{B^*B} denote the spectral radius of A^*A and B^*B respectively and $\alpha_n \in [\alpha, 1]$
(for some $\alpha > 0$) and established a strong and weak convergence results.
Based on the work of Moudafi and Al-Shemas [129], Chidume *et al.* [58] proposed the following algorithm for solving the SEFPP for demi-contractive mappings:

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = (1-\alpha)\left(x_n - \gamma A^*(Ax_n - By_n)\right) + \alpha T\left(x_n - \gamma A^*(Ax_n - By_n)\right); \\ y_{n+1} = (1-\alpha)\left(y_n + \gamma B^*(Ax_n - By_n)\right) + \alpha S\left(y_n + \gamma B^*(Ax_n - By_n)\right), \quad \forall n \geq 1; \end{cases} \tag{6.1.4}$$

where $T : H_1 \to H_1, \quad S : H_2 \to H_2$ are two demi-contractive mappings. Chidume *et al.* [58] proved weak and strong convergence theorems of the iterative scheme (6.1.4) to a solution of the SEFPP in a real Hilbert spaces.

Recently, Rahaman et al [147] considered the split equality generalized mixed equilibrium problem and fixed point problem for certain nonlinear mappings in Hilbert spaces. They obtained the following simultaneous iterative algorithms and proved strong convergence theorems based on the algorithm under some mild conditions:

$$\left\{ \begin{array}{l} F(\lambda_1 u_n + (1-\lambda_1)b, u) + \langle T(u_n), u - u_n\rangle + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ G(\lambda_2 v_n + (1-\lambda_2)c, v) + \langle S(v_n), v - v_n\rangle + \psi(v) - \psi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \\ x_{n+1} = (1-\alpha_n)(u_n - \gamma_n A^*(Au_n - Bv_n)) + \alpha_n P(u_n - \gamma_n A^*(Au_n - Bv_n)), \\ y_{n+1} = (1-\alpha_n)(v_n + \gamma_n B^*(Au_n - Bv_n)) + \alpha_n Q(v_n + \gamma_n B^*(Au_n - Bv_n)), \end{array} \right. \tag{6.1.5}$$

for every $u, b \in C$, $v, c \in Q$ and $n \geq 1$, where λ_A and λ_B denote the spectral radii of A^*A and B^*B respectively, $\{\gamma_n\}$ is a positive sequence such that $\gamma_n \in \left(\epsilon, \frac{2}{\lambda_A+\lambda_B} - \epsilon\right)$, for sufficiently small ϵ, $P : H_1 \to H_1$ and $Q : H_2 \to H_2$ are demicontractive mappings, $\{\alpha_n\}$ is a sequence in (k,1) and $\{r_n\} \subset (0, \infty)$ which satisfies some conditions.

Also, Karahan [91] studied a similar algorithm and proved a strong convergence theorem under some certain conditions:

$$\left\{ \begin{array}{l} F(\lambda_1 u_n + (1-\lambda_1)b, u) + \langle T(u_n), u - u_n\rangle + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \\ G(\lambda_2 v_n + (1-\lambda_2)c, v) + \langle S(v_n), v - v_n\rangle + \psi(v) - \psi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \\ x_{n+1} = (1-\alpha_n)P_1(u_n - \gamma_n A^*(Au_n - Bv_n)) + \alpha_n P_2(u_n - \gamma_n A^*(Au_n - Bv_n)), \\ y_{n+1} = (1-\alpha_n)P_3(v_n + \gamma_n B^*(Au_n - Bv_n)) + \alpha_n P_4(v_n + \gamma_n B^*(Au_n - Bv_n)), \end{array} \right. \tag{6.1.6}$$

where $P_1, P_2 : H_1 \to H_1$ and $P_3, P_4 : H_2 \to H_2$ are nonexpansive mappings.

In this chapter, we study the split equality minimization and fixed point problem for pseudocontractive mapping. Also we extended some results on split equality monotone inclusion and split equality generalized mixed equilibrium problems from the frame work of Hilbert space to p uniformly convex and uniformly smooth Banach space. We give applications and numerical example to illustrate the performance of our algorithms.

6.2 On split equality minimization and fixed point problems

We now consider the following Split Equality Minimization and Fixed Point Problem (SEMFPP).

Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear maps. Let $f_1 : H_1 \to \mathbb{R}$ and $f_2 : H_2 \to \mathbb{R}$ be differentiable maps with L_1 and L_2-Lipschitz continuous gradients respectively. Let $g_i : H_i \to \mathbb{R}$ $(i = 1, 2)$ be "simple" maps. The (SEMFPP) is to find $x^* \in F(T)$ and $y^* \in F(S)$ such that

$$\begin{cases} f_1(x^*) + g_1(x^*) & = \min\limits_{x \in H_1}[f_1(x) + g_1(x)], \\ f_2(y^*) + g_2(y^*) & = \min\limits_{x \in H_2}[f_2(x) + g_2(x)], \end{cases} \tag{6.2.1}$$

and $Ax^* = By^*$. where $T : H_1 \to H_1$ and $S : H_2 \to H_2$ are two nonlinear mappings. Assume that this problem has a solution, let's denote the solution set of (6.2.1) by Υ. Furthermore, we propose an iterative scheme and using the iterative scheme, we state and prove a strong convergence result for the approximation of a solution of problem (6.2.1).

Theorem 6.2.1. *Let H_1, H_2, and H_3 be real Hilbert spaces. Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be demicontractive mappings with constants k_1 and k_2 respectively, such that $I - S$ and $I - T$ are demi-closed at 0. Let f_i and g_i $(i = 1, 2)$ be two convex and lower semicontinuous functions such that $f_1 : H_1 \to \mathbb{R}$ and $f_2 : H_2 \to \mathbb{R}$ are differentiable with L_1 and L_2- Lipschitz continuous gradient, $g_1 : H_1 \to \mathbb{R}$ and $g_2 : H_2 \to \mathbb{R}$ be simple maps and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Upsilon \neq \emptyset$ and let the step-size $\gamma_n \in \left(\epsilon, \frac{2||At_n - Br_n||^2}{||A^*(At_n - Br_n)||^2 + ||B^*(At_n - Br_n)||^2} - \epsilon\right), n \in \Omega$*

otherwise, $\gamma_n = \gamma$(γ being any nonnegative value), where the set of indexes $\Omega = \{n : At_n - Br_n \neq 0\}$.
Let $u, x_0 \in Q_1$ and $v, y_0 \in Q_2$ be arbitrary and the sequences $\{(x_n, y_n)\}$ be generated by

$$\begin{cases} t_n = (1 - \alpha_n)x_n + \alpha_n u; \\ r_n = (1 - \alpha_n)y_n + \alpha_n v; \\ u_n = Prox_{\delta_n g_1}(I - \delta_n \nabla f_1)(t_n - \gamma_n A^*(At_n - Br_n)); \\ v_n = Prox_{\delta_n g_2}(I - \delta_n \nabla f_2)(r_n + \gamma_n B^*(At_n - Br_n)); \\ x_{n+1} = (1 - \zeta_n)u_n + \zeta_n T u_n; \\ y_{n+1} = (1 - \psi_n)v_n + \psi_n S v_n; \end{cases} \tag{6.2.2}$$

where $\{\delta_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$, $\{\zeta_n\}$ and $\{\psi_n\}$ are sequences in $(0, 1)$, with conditions

i $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

ii $\zeta_n \in (a, 1 - k_1) \subseteq (0, 1)$ for some $a > 0,$

iii $\psi_n \in (b, 1-k_2) \subseteq (0,1)$ for some $b > 0$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Υ.

Proof. Clearly γ_n is well defined since for any $(x,y) \in \Upsilon$, we have

$$\langle A^*(At_n - Br_n), t_n - x\rangle = \langle At_n - Br_n, At_n - Ax\rangle \tag{6.2.3}$$

and

$$\langle B^*(At_n - Br_n), y - r_n\rangle = \langle At_n - Br_n, By - Br_n\rangle. \tag{6.2.4}$$

Adding (6.2.3) and (6.2.4) and taking into account the fact that $Ax = By$, we obtain $\forall n \in \Omega$,

$$\begin{aligned} ||At_n - Br_n||^2 &= \langle A^*(At_n - Br_n), t_n - x\rangle + \langle B^*(At_n - Br_n), y - r_n\rangle \\ &\leq ||A^*(At_n - Br_n)||||t_n - x|| + ||B^*(At_n - Br_n)||||y - r_n||. \end{aligned}$$

Therefore, for $n \in \Omega$, that is, $||At_n - Br_n|| > 0$, we have $||A^*(At_n - Br_n)|| \neq 0$ or $||B^*(At_n - Br_n)|| \neq 0$. Thus γ_n is well defined.
Let $(p,q) \in \Upsilon$, we have from (6.2.2) that

$$\begin{aligned} \|u_n - p\|^2 &= \|Prox_{\delta_n g_1}(I - \delta_n \nabla f_1)(t_n - \gamma_n A^*(At_n - Br_n)) - p\|^2 \\ &\leq \|t_n - \gamma_n A^*(At_n - Br_n) - p\|^2 \\ &= \|t_n - p\|^2 - 2\gamma_n\langle t_n - p, A^*(At_n - Br_n)\rangle + \gamma_n^2\|A^*(At_n - Br_n)\|^2. \end{aligned} \tag{6.2.5}$$

From Lemma 2.1 and noting that A^* is adjoint of A, we have

$$\begin{aligned} -2\langle t_n - p, A^*(At_n - Br_n)\rangle &= -2\langle At_n - Ap, At_n - Br_n\rangle \\ &= -\|At_n - Ap\|^2 - \|At_n - Br_n\|^2 + \|Br_n - Ap\|^2. \end{aligned} \tag{6.2.6}$$

From (6.2.5) and (6.2.6), we obtain

$$\begin{aligned} \|u_n - p\|^2 \leq \|t_n - p\|^2 &- \gamma_n\|At_n - Ap\|^2 - \gamma_n\|At_n - Br_n\|^2 \\ &+ \gamma_n\|Br_n - Ap\|^2 + {\gamma_n}^2\|A^*(At_n - Br_n)\|^2. \end{aligned} \tag{6.2.7}$$

Similarly, from (6.2.2), we have

$$\begin{aligned} \|v_n - q\|^2 \leq \|r_n - q\|^2 &- \gamma_n\|Br_n - Bq\|^2 - \gamma_n\|At_n - Br_n\|^2 \\ &+ \gamma_n\|At_n - Bq\|^2 + \gamma_n^2\|B^*(At_n - Br_n)\|^2. \end{aligned} \tag{6.2.8}$$

Adding inequality (6.2.7) and (6.2.8), and using the fact that $Ap = Bq$, we obtain

$$\begin{aligned} \|u_n - p\|^2 + \|v_n - q\|^2 &\leq \|t_n - p\|^2 + \|r_n - q\|^2 - \gamma_n[2\|At_n - Br_n\|^2 \\ &\quad - \gamma_n(\|A^*(At_n - Br_n)\|^2 + \|B^*(At_n - Br_n)\|^2)] \\ &\leq \|t_n - p\|^2 + \|r_n - q\|^2. \end{aligned} \tag{6.2.9}$$

From (6.2.2) and the fact that T is demi-contractive, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\zeta_n)u_n+\zeta_n Tu_n-p\|^2\\
&= \|(1-\zeta_n)(u_n-p)+\zeta_n(Tu_n-p)\|^2\\
&= (1-\zeta_n)^2\|u_n-p\|^2+\zeta_n^2\|Tu_n-p\|^2+2\zeta_n(1-\zeta_n)\langle u_n-p,Tu_n-p\rangle\\
&\leq (1-\zeta_n)^2\|u_n-p\|^2+\zeta_n^2[\|u_n-p\|^2+k_1\|u_n-Tu_n\|^2]\\
&\quad +2\zeta_n(1-\zeta_n)\left[\|u_n-p\|^2-\frac{1-k_1}{2}\|u_n-Tu_n\|^2\right]\\
&= (1-2\zeta_n+\zeta_n^2)\|u_n-p\|^2+\zeta_n^2[\|u_n-p\|^2+k_1\|u_n-Tu_n\|^2]\\
&\quad +2\zeta_n\|u_n-p\|^2-2\zeta_n^2\|u_n-p\|^2-\zeta_n(1-\zeta_n)(1-k_1)\|u_n-Tu_n\|^2\\
&= \|u_n-p\|^2-\zeta_n(1-\zeta_n-k_1)\|u_n-Tu_n\|^2\\
&\leq \|u_n-p\|^2. \qquad (6.2.10)
\end{aligned}$$

Similarly, we have that

$$\|y_{n+1}-q\|^2\leq\|v_n-q\|^2. \tag{6.2.11}$$

Adding (6.2.10) and (6.2.11), and using (6.2.9), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2+\|y_{n+1}-q\|^2 &\leq \|u_n-p\|^2+\|v_n-q\|^2\\
&\leq \|t_n-p\|^2+\|r_n-q\|^2. \qquad (6.2.12)
\end{aligned}$$

From (6.2.2), (6.2.12) and Lemma 2.2, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2+\|y_{n+1}-q\|^2 &\leq \|(1-\alpha_n)(x_n-p)+\alpha_n(u-p)\|^2+\|(1-\alpha_n)(y_n-q)+\alpha_n(v-q)\|^2\\
&\leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|u-p\|^2+(1-\alpha_n)\|y_n-q\|^2+\alpha_n\|v-q\|^2\\
&= (1-\alpha_n)[\|x_n-p\|^2+\|y_n-q\|^2]+\alpha_n[\|u-p\|^2+\|v-q\|^2]\\
&\leq \max\{\|x_n-p\|^2+\|y_n-q\|^2,\|u-p\|^2+\|v-q\|^2\}\\
&\ \ \vdots\\
&\leq \max\{\|x_0-p\|^2+\|y_0-q\|^2,\|u-p\|^2+\|v-q\|^2\}.
\end{aligned}$$

Hence, $\{\|x_n-p\|^2+\|y_n-q\|^2\}$ is bounded. Consequently $\{x_n\}$, $\{y_n\}$, $\{t_n\}$, $\{r_n\}$, $\{u_n\}$, $\{v_n\}$, $\{Ax_n\}$, $\{By_n\}$ are bounded. From (6.2.9), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2+\|y_{n+1}-q\|^2 &\leq \|t_n-p\|^2+\|r_n-q\|^2-\gamma_n[2\|At_n-Br_n\|^2\\
&\quad -\gamma_n(\|A^*(At_n-Br_n)\|^2+\|B^*(At_n-Br_n)\|^2)]\\
&\leq (1-\alpha_n)[\|x_n-p\|^2+\|y_n-q\|^2]+\alpha_n[\|u-p\|^2+\|v-q\|^2]\\
&\quad -\gamma_n[2\|At_n-Br_n\|^2-\gamma_n(\|A^*(At_n-Br_n)\|^2+\|B^*(At_n-Br_n)\|^2)],
\end{aligned}$$

which implies

$$\begin{aligned}
\epsilon^2\left(||A^*(At_n-Br_n)||^2+||B^*(At_n-Br_n)||^2\right) &\leq (1-\alpha_n)\left[||x_n-p||^2+||y_n-q||^2\right]\\
&\quad +\alpha_n\left[||u-p||^2+||v-q||^2\right]\\
&\quad -\left[||x_{n+1}-p||^2+||y_{n+1}-q||^2\right]. \qquad (6.2.13)
\end{aligned}$$

We divided the remaining part of the proof into two cases to establish strong convergence.
Case 1: Assume that $\{||x_n-p||^2+||y_n-q||^2\}$ is monotone decreasing, then $\{||x_n-p||^2+||y_n-q||^2\}$

is convergent, thus

$$\lim_{n\to\infty}\left[\left(||x_{n+1}-p||^2+||y_{n+1}-q||^2\right)-\left(||x_n-p||^2+||y_n-q||^2\right)\right]=0.$$

It follows from (6.2.13) that

$(||A^*(At_n-Br_n)||^2+||B^*(At_n-Br_n)||^2)\to 0$, as $n\to\infty$.
Since $At_n-Br_n=0$, if $n\notin\Omega$, we have

$$\lim_{n\to\infty}||A^*(At_n-Br_n)||^2=\lim_{n\to\infty}||B^*(At_n-Br_n)||^2=0. \tag{6.2.14}$$

From (6.2.2), we have

$$\begin{aligned}||t_n-x_n||^2 &= ||(1-\alpha_n)x_n+\alpha_n u-x_n||^2\\ &= \alpha_n^2||u-x_n||^2\to 0,\ \text{ as } n\to\infty.\end{aligned}$$

$$\implies \lim_{n\to\infty}||t_n-x_n||^2=0. \tag{6.2.15}$$

Similarly, we have

$$\lim_{n\to\infty}||r_n-y_n||^2=0. \tag{6.2.16}$$

Also, from (6.2.2), Lemma 2.3.6 (iii) and Lemma 2.3.6 (iv), we have that

$$\begin{aligned}||u_n-p||^2 &= ||Prox_{\delta_n g_1}(I-\delta_n\nabla f_1)(t_n-\gamma_n A^*(At_n-Br_n))-p||^2\\
&\leq \langle u_n-p,t_n-\gamma_n A^*(At_n-Br_n)-p\rangle\\
&= \frac{1}{2}[||u_n-p||^2+||t_n-\gamma_n A^*(At_n-Br_n)-p||^2\\
&\quad -||u_n-p-(t_n-\gamma_n A^*(At_n-Br_n)-p)||^2]\\
&\leq \frac{1}{2}[||u_n-p||^2+||t_n-p||^2+\gamma_n^2||A^*(At_n-Br_n)||^2\\
&\quad +2\gamma_n||t_n-p||||A^*(At_n-Br_n)||\\
&\quad -(||u_n-t_n||^2+\gamma_n^2||A^*(At_n-Br_n)||^2-2\gamma_n\langle u_n-t_n,A^*(At_n-Br_n)\rangle])\\
&= \frac{1}{2}[||u_n-p||^2+||t_n-p||^2+2\gamma_n||t_n-p||||A^*(At_n-Br_n)||\\
&\quad -||u_n-t_n||^2+2\gamma_n\langle u_n-t_n,A^*(At_n-Br_n)\rangle]\\
&\leq \frac{1}{2}[||u_n-p||^2+||t_n-p||^2+2\gamma_n||t_n-p||||A^*(At_n-Br_n)||\\
&\quad -||u_n-t_n||^2+2\gamma_n||u_n-t_n||||A^*(At_n-Br_n)||]\\
&\leq \frac{1}{2}[||u_n-p||^2+(1-\alpha_n)||x_n-p||^2+\alpha_n||u-p||^2\\
&\quad +2\gamma_n||t_n-p||||A^*(At_n-Br_n)||-||u_n-t_n||^2\\
&\quad +2\gamma_n||u_n-t_n||||A^*(At_n-Br_n)||]\\
&\leq \frac{1}{2}[||u_n-p||^2+||x_n-p||^2+\alpha_n||u-p||^2\\
&\quad +2\gamma_n||t_n-p||||A^*(At_n-Br_n)||\\
&\quad -||u_n-t_n||^2+2\gamma_n||u_n-t_n||||A^*(At_n-Br_n)||],\end{aligned} \tag{6.2.17}$$

which implies

$$\begin{aligned} ||u_n-p||^2 &\leq ||x_n-p||^2+\alpha_n||u-p||^2+2\gamma_n||w_n-p||||A^*(At_n-Br_n)|| \\ &\quad -||u_n-t_n||^2+2\gamma_n||u_n-t_n||||A^*(At_n-Br_n)||. \end{aligned} \tag{6.2.18}$$

From (6.2.10) and (6.2.18), we have

$$\begin{aligned} ||x_{n+1}-p||^2 &\leq ||x_n-p||^2+\alpha_n||u-p||^2+2\gamma_n||t_n-x^*||||A^*(At_n-Br_n)|| \\ &\quad -||u_n-t_n||^2+2\gamma_n||u_n-t_n||||A^*(At_n-Br_n)||. \end{aligned} \tag{6.2.19}$$

Similarly, we have

$$\begin{aligned} ||y_{n+1}-q||^2 &\leq ||y_n-q||^2+\alpha_n||v-q||^2+2\gamma_n||r_n-q||||B^*(At_n-Br_n)|| \\ &\quad -||v_n-r_n||^2+2\gamma_n||v_n-r_n||||B^*(At_n-Br_n)||. \end{aligned} \tag{6.2.20}$$

Adding (6.2.19) and (6.2.20), we have

$$\begin{aligned} ||x_{n+1}-p||^2+||y_{n+1}-q||^2 &\leq ||x_n-p||^2+||y_n-q||^2+\alpha_n[||u-p||^2+||v-q||^2] \\ &\quad +2\gamma_n[||t_n-x^*||||A^*(At_n-Br_n)|| \\ &\quad +||r_n-y^*||||B^*(At_n-Br_n)||] \\ &\quad -[||u_n-t_n||^2+||v_n-r_n||^2] \\ &\quad +2\gamma_n[||u_n-t_n||||A^*(At_n-Br_n)|| \\ &\quad +||v_n-r_n||||B^*(At_n-Br_n)||]. \end{aligned} \tag{6.2.21}$$

Using (6.2.14) together with the fact that $\alpha_n \to 0$, as $n \to \infty$ in (6.2.21), we have

$$\lim_{n\to\infty}\left[||u_n-t_n||^2+||v_n-r_n||^2\right]=0,$$

which implies

$$\lim_{n\to\infty}||u_n-t_n||^2=0 \tag{6.2.22}$$

and

$$\lim_{n\to\infty}||v_n-r_n||^2=0. \tag{6.2.23}$$

Observe that since T is demicontractive and $p \in F(T)$, so we have

$$\begin{aligned} ||Tx-p||^2 &\leq ||x-p||^2+k_1||x-Tx||^2 \\ \Longrightarrow \langle Tx-p,Tx-p\rangle &\leq \langle x-p,x-p\rangle+k_1||x-Tx||^2 \\ \Longrightarrow \langle Tx-p,Tx-x\rangle+\langle Tx-p,x-p\rangle &\leq \langle x-p,x-p\rangle+k_1||x-Tx||^2 \\ \Longrightarrow \langle Tx-p,Tx-x\rangle &\leq \langle x-Tx,x-p\rangle+k_1||x-Tx||^2 \\ \Longrightarrow \langle Tx-x,Tx-x\rangle+\langle x-p,Tx-x\rangle &\leq \langle x-Tx,x-p\rangle+k_1||x-Tx||^2 \\ ||Tx-x||^2 &\leq \langle x-p,x-Tx\rangle-\langle x-p,Tx-x\rangle+k_1||x-Tx||^2 \\ \Longrightarrow (1-k_1)||Tx-x||^2 &\leq 2\langle x-p,x-Tx\rangle. \end{aligned} \tag{6.2.24}$$

From (6.2.2) and (6.2.24), we have

$$\begin{aligned} ||x_{n+1}-p||^2 &= ||(1-\zeta_n)u_n+\zeta_n Tu_n-p||^2 \\ &= ||u_n-p+\zeta_n(Tu_n-u_n)||^2 \\ &= ||u_n-p||^2+\zeta_n^2||Tu_n-u_n||^2-2\zeta_n\langle u_n-p,u_n-Tu_n\rangle \\ &\leq ||u_n-p||^2+\zeta_n^2||Tu_n-u_n||^2-(1-k_1)\zeta_n||Tu_n-u_n||^2 \\ &= ||u_n-p||^2+\zeta_n(\zeta_n-(1-k_1))||u_n-Tu_n||^2. \end{aligned} \tag{6.2.25}$$

Similarly, we have that

$$||y_{n+1}-q||^2 \leq ||v_n-q||^2+\psi_n(\psi_n-(1-k_2))||v_n-Sv_n||^2. \tag{6.2.26}$$

Adding (6.2.25) and (6.2.26), we have

$$\begin{aligned} ||x_{n+1}-p||^2+||y_{n+1}-q||^2 &\leq ||u_n-p||^2+||v_n-q||^2+\zeta_n(\zeta_n-(1-k_1))||Tu_n-u_n||^2 \\ &+ \psi_n(\psi_n(1-k_2))||v_n-Sv_n||^2 \\ &\leq ||t_n-p||+||r_n-q||^2+\zeta_n(\zeta_n-(1-k_1))||Tu_n-u_n||^2 \\ &+ \psi_n(\psi_n-(1-k_2))||v_n-Sv_n||^2 \\ &\leq (1-\alpha_n)\left[||x_n-p||^2+||y_n-q||^2\right]+\alpha_n\left[||u-p||^2+||v-q||^2\right] \\ &+ \zeta_n(\zeta_n-(1-k_1))||u_n-Tu_n||^2 \\ &\quad +\psi_n(\psi_n-(1-k_2))||v_n-Sv_n||^2. \end{aligned} \tag{6.2.27}$$

Let $K_n=\zeta_n((1-k_1)-\zeta_n)||u_n-Tu_n||^2+\psi_n((1-k_2)-\psi_n)||v_n-Sv_n||^2$,

then

$$\begin{aligned} K_n &\leq (1-\alpha_n)\left[||x_n-p||^2+||y_n-q||^2\right]-\left[||x_{n+1}-p||^2+||y_{n+1}-q||^2\right] \\ &\quad +\alpha_n\left[||u-p||^2+||v-q||^2\right]\to 0, \text{ as } n\to\infty, \end{aligned}$$

which implies

$$||u_n-Tu_n||^2+||v_n-Sv_n||^2\to 0,\ n\to\infty.$$

That is

$$\lim_{n\to\infty}||u_n-Tu_n||^2=0, \tag{6.2.28}$$

and

$$\lim_{n\to\infty}||v_n-Sv_n||^2=0. \tag{6.2.29}$$

From (6.2.28), we have

$$\lim_{n\to\infty}||x_{n+1}-u_n|| = \lim_{n\to\infty}\zeta_n||u_n-Tu_n||=0. \tag{6.2.30}$$

Similarly, from (6.2.29), we have

$$\lim_{n\to\infty}||y_{n+1}-v_n|| = \lim_{n\to\infty}\psi_n||v_n-Sv_n||=0. \tag{6.2.31}$$

From (6.2.15) and (6.2.22), we have $||x_n - u_n|| \leq ||x_n - t_n|| + ||t_n - u_n|| \to 0$, which implies that

$$\lim_{n\to\infty} ||x_n - u_n|| = 0. \tag{6.2.32}$$

Similarly, from (6.2.16) and (6.2.23), we have

$$\lim_{n\to\infty} ||y_n - v_n|| = 0. \tag{6.2.33}$$

Also, from (6.2.30) and (6.2.32), we have

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - u_n|| + ||u_n - x_n|| \to 0, \quad as \quad n \to \infty$$

,
which implies that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{6.2.34}$$

Similarly, from (6.2.31) and (6.2.33), we have

$$\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0. \tag{6.2.35}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $\bar{x} \in H_1$. By (6.2.32) and (6.2.15), we have that $\{u_n\}$ and $\{t_n\}$ converges weakly to $\bar{x}$ and by the demi-closeness of $I - T$ at 0 and (6.2.28), we have that $\bar{x} \in F(T)$. Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$) such that $\{y_{n_k}\}$ converge weakly to $\bar{y} \in H_2$. By (6.2.33) and (6.2.16), we have that $\{v_n\}$ and $\{r_n\}$ converge weakly to $\bar{y}$ and by the demi-closeness of $I - S$ at 0 and (6.2.29), we have that $\bar{y} \in F(S)$.
Also, since A and B are bounded linear operators, we have that $\{At_n\}$ converges weakly to $A\bar{x}$ and $\{Br_n\}$ converges weakly to $B\bar{y}$.
Next, we show that $A\bar{x} = B\bar{y}$.

$$\begin{aligned}
||A\bar{x} - B\bar{y}||^2 &= \langle A\bar{x} - B\bar{y}, A\bar{x} - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - B\bar{y} + At_n - At_n + Br_n - Br_n\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - At_n\rangle + \langle A\bar{x} - B\bar{y}, At_n - Br_n\rangle + \langle A\bar{x} - B\bar{y}, Br_n - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - At_n\rangle + \langle A\bar{x}, At_n - Br_n\rangle - \langle B\bar{y}, At_n - Br_n\rangle \\
&\quad + \langle A\bar{x} - B\bar{y}, Br_n - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - At_n\rangle + \langle \bar{x}, A^*(At_n - Br_n)\rangle - \langle \bar{y}, B^*(At_n - Br_n)\rangle \\
&\quad + \langle A\bar{x} - B\bar{y}, Br_n - B\bar{y}\rangle \\
&\leq \langle A\bar{x} - B\bar{y}, A\bar{x} - At_n\rangle + ||\bar{x}||||A^*(At_n - Br_n)|| + ||\bar{y}||||B^*(At_n - Br_n)|| \\
&+ \langle A\bar{x} - B\bar{y}, Br_n - B\bar{y}\rangle \to 0, n \to \infty,
\end{aligned}$$

which implies that $||A\bar{x} - B\bar{y}|| = 0$. Hence $A\bar{x} = B\bar{y}$.

Let $p_n = t_n - \gamma_n A^*(At_n - Br_n)$.

Then $\|p_n - t_n\|^2 = {\gamma_n}^2\|A^*(At_n - Br_n)\|^2 \to 0$ as $n \to \infty$, and

$$\|u_n - p_n\| \leq \|u_n - t_n\| + \|t_n - p_n\| \to 0 \quad as \quad n \to \infty. \tag{6.2.36}$$

We get from (6.2.36) that

$$\|prox_{\delta_n g_1}(p_n - \delta_n \nabla f_1(p_n)) - p_n\| \to 0. \tag{6.2.37}$$

Similarly, if we let $a_n = r_n + \gamma_n B^*(At_n - Br_n)$, we obtain

$$\|prox_{\delta_n g_2}(a_n - \delta_n \nabla f_2(a_n)) - a_n\| \to 0. \tag{6.2.38}$$

Hence by Lemma 2.3.13 (demiclosedness principle) we have $w_\omega(p_n) = w_\omega(x_n) \subset \Upsilon$. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$, for some $\bar{x} \in \Upsilon$. By similar argument, we get $\bar{y} \in \Upsilon$. Hence $(\bar{x}, \bar{y}) \in \Upsilon$.
Next, we show that $(\{x_n\}, \{y_n\})$ converges strongly to $(\bar{x}, \bar{y})$.
From (6.2.12), we have

$$\begin{aligned} ||x_{n+1} - \bar{x}||^2 + ||y_{n+1} - \bar{y}||^2 &\leq ||t_n - \bar{x}||^2 + ||r_n - \bar{y}||^2 \\ &= (1-\alpha_n)^2||x_n - \bar{x}||^2 + \alpha_n^2||u - \bar{x}||^2 + 2(1-\alpha_n)\alpha_n\langle x_n - \bar{x}, u - \bar{x}\rangle \\ &\quad +(1-\alpha_n)^2||y_n - \bar{y}||^2 + \alpha_n^2||v - \bar{y}||^2 + 2(1-\alpha_n)\alpha_n\langle y_n - \bar{y}, v - \bar{y}\rangle \\ &\leq (1-\alpha_n)^2\left[||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2\right] + \alpha_n\big[\alpha_n||u - \bar{x}||^2 \\ &+ 2(1-\alpha_n)\langle x_n - \bar{x}, u - \bar{x}\rangle + \alpha_n||v - \bar{y}||^2 \\ &\quad +2(1-\alpha_n)\langle y_n - \bar{y}, v - \bar{y}\rangle\big]. \end{aligned} \tag{6.2.39}$$

Since $\limsup\limits_{n\to\infty}\langle x_n - \bar{x}, u - \bar{x}\rangle \leq \langle p - \bar{x}, u - \bar{x}\rangle \leq 0$ and $\limsup\limits_{n\to\infty}\langle y_n - \bar{y}, v - \bar{y}\rangle \leq \langle q - \bar{y}, v - \bar{y}\rangle \leq 0$, applying Lemma 2.3.14 to (6.2.39), we have that $(\{x_n\}, \{y_n\})$ converges strongly to $(\bar{x}, \bar{y})$.

Case 2. Assume that $\{||x_n - p||^2 + ||y_n - q||^2\}$ is not monotone decreasing. Set $\Gamma_n = ||x_n - p||^2 + ||y_n - q||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0.$$

From (6.2.13), we have

$$\begin{aligned} \epsilon^2\left(||A^*(At_{\tau(n)} - Br_{\tau(n)})||^2 + ||B^*(At_{\tau(n)} - Br_{\tau(n)})||^2\right) &\leq (1-\alpha_{\tau(n)})[||x_{\tau(n)} - p||^2 + ||y_{\tau(n)} - q||^2] \\ &\quad -[||x_{\tau(n)+1} - p||^2 + ||y_{\tau(n)+1} - q||^2] \\ &\quad +\alpha_{\tau(n)}[||u - p||^2 + ||v - p||^2] \\ &\leq \alpha_{\tau(n)}[||u - p||^2 + ||v - q||^2]. \end{aligned} \tag{6.2.40}$$

Therefore,

$$\left(||A^*(At_{\tau(n)} - Br_{\tau(n)})||^2 + ||B^*(At_{\tau(n)} - Br_{\tau(n)})||^2\right) \to 0, \text{ as } n \to \infty.$$

Note that $At_{\tau(n)} - Br_{\tau(n)} = 0$, if $\tau(n) \notin \Omega$.
Hence,

$$\lim_{n\to\infty} ||A^*(At_{\tau(n)} - Br_{\tau(n)})||^2 = 0, \tag{6.2.41}$$

and

$$\lim_{n\to\infty} ||B^*(At_{\tau(n)} - Br_{\tau(n)})||^2 = 0. \tag{6.2.42}$$

Using the same argument as in case 1, we have $(\{x_{\tau(n)}\}, \{y_{\tau(n)}\})$ converges weakly to $(\bar{x}, \bar{y}) \in \Gamma$.
Now for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq [||x_{\tau(n)+1} - p||^2 + ||y_{\tau(n)+1} - q||^2] - [||x_{\tau(n)} - p||^2 + ||y_{\tau(n)} - q||^2] \\ &\leq (1 - \alpha_{\tau(n)})[||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2] - [||x_{\tau(n)} - p||^2 + ||y_{\tau(n)} - q||^2] \\ &\quad + \alpha_{\tau(n)}[\alpha_{\tau(n)}[||u - \bar{x}||^2 + ||v - \bar{y}||^2] + 2(1 - \alpha_{\tau(n)})(\langle x_{\tau(n)} - \bar{x}, u - \bar{x}\rangle + \langle y_{\tau(n)} - \bar{y}, v - \bar{y}\rangle)], \end{aligned}$$

which implies

$$\begin{aligned} ||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2 &\leq \alpha_{\tau(n)}[||u - \bar{x}||^2 + ||v - \bar{y}||^2] + 2(1 - \alpha_{\tau(n)})(\langle x_{\tau(n)} - \bar{x}, u - \bar{x}\rangle \\ &\quad + \langle y_{\tau(n)} - \bar{y}, v - \bar{y}\rangle) \to 0. \end{aligned}$$

Hence

$$\lim_{n\to\infty} (||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2) = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Moreover, for $n \geq n_0$, it is clear that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$.
Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Thus, $\lim_{n\to\infty} \Gamma_n = 0$. That is $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

□

Corollary 6.2.2. *Let H_1, H_2, and H_3 be real Hilbert spaces Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be two nonexpansive mappings such that $I - S$ and $I - T$ are demi-closed at 0. Let f_i and g_i ($i = 1, 2$) be two convex and lower semicontinuous functions such that $f_1 : H_1 \to \mathbb{R}$ and $f_2 : H_2 \to \mathbb{R}$ are differentiable with L_1- and L_2- Lipschitz continuous gradient, $g_1 : H_1 \to \mathbb{R}$ and $g_2 : H_2 \to \mathbb{R}$ be simple maps and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Upsilon \neq \emptyset$ and let the step-size $\gamma_n \in \left(\epsilon, \frac{2||At_n - Br_n||^2}{||A^*(At_n - Br_n)||^2 + ||B^*(At_n - Br_n)||^2} - \epsilon\right), n \in \Omega$*

otherwise, $\gamma_n = \gamma$(γ being any nonegative value), where the set of indexes $\Omega = \{n : At_n - Br_n \neq 0\}$.

Let $u, x_0 \in Q_1$ and $v, y_0 \in Q_2$ be arbitrary and the sequences $\{(x_n, , y_n)\})$ be generated by

$$\begin{cases} t_n = (1-\alpha_n)x_n + \alpha_n u \\ r_n = (1-\alpha_n)y_n + \alpha_n v \\ u_n = Prox_{\delta_n g_1}(I - \delta_n \nabla f_1)(t_n - \gamma_n A^*(At_n - Br_n)) \\ v_n = Prox_{\delta_n g_2}(I - \delta_n \nabla f_2)(r_n + \gamma_n B^*(At_n - Br_n)) \\ x_{n+1} = (1-\zeta_n)u_n + \zeta_n T u_n \\ y_{n+1} = (1-\psi_n)v_n + \psi_n S v_n \end{cases} \tag{6.2.43}$$

where $\{\delta_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$, $\{\zeta_n\}$ and $\{\psi_n\}$ are sequences in $(0,1)$, with conditions

i $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

ii $\zeta_n \in (a, 1-k_1) \subseteq (0,1)$ for some $a > 0,$

iii $\psi_n \in (b, 1-k_2) \subseteq (0,1)$ for some $b > 0.$

Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ in Υ.

Corollary 6.2.3. *Let H_1, H_2, and H_3 be real Hilbert spaces. Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be k_1-strictly pseudocontractive and k_2-strictly pseudocontractive mappings respectively such that $I - S$ and $I - T$ are demi-closed at 0. Let f_i and g_i $(i = 1, 2)$ be two convex and lower semicontinuous functions such that $f_1 : H_1 \to \mathbb{R}$ and $f_2 : H_2 \to \mathbb{R}$ are differentiable with L_1- and L_2- Lipschitz continuous gradient, $g_1 : H_1 \to \mathbb{R}$ and $g_2 : H_2 \to \mathbb{R}$ be simple maps and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Upsilon \neq \emptyset$ and let the step-size $\gamma_n \in \left(\epsilon, \frac{2||At_n - Br_n||^2}{||A^*(At_n - Br_n)||^2 + ||B^*(At_n - Br_n)||^2} - \epsilon\right), n \in \Omega$*

otherwise, $\gamma_n = \gamma$(γ being any nonegative value), where the set of indexes $\Omega = \{n : At_n - Br_n \neq 0\}$.
Let $u, x_0 \in Q_1$ and $v, y_0 \in Q_2$ be arbitrary and the sequences $(\{x_n\}, \{y_n\})$ be generated by

$$\begin{cases} t_n = (1-\alpha_n)x_n + \alpha_n u \\ r_n = (1-\alpha_n)y_n + \alpha_n v \\ u_n = Prox_{\delta_n g_1}(I - \delta_n \nabla f_1)(t_n - \gamma_n A^*(At_n - Br_n)) \\ v_n = Prox_{\delta_n g_2}(I - \delta_n \nabla f_2)(r_n + \gamma_n B^*(At_n - Br_n)) \\ x_{n+1} = (1-\zeta_n)u_n + \zeta_n T u_n \\ y_{n+1} = (1-\psi_n)v_n + \psi_n S v_n \end{cases} \tag{6.2.44}$$

where $\{\delta_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$, $\{\zeta_n\}$ and $\{\psi_n\}$ are sequences in $(0,1)$, with conditions

i $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

ii $\zeta_n \in (a, 1-k_1) \subseteq (0,1)$ for some $a > 0$,

iii $\psi_n \in (b, 1-k_2) \subseteq (0,1)$ for some $b > 0$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Υ.

6.2.1 Application to the split equality monotone variational inclusion problem

Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Let $f_1 : H_1 \to H_1$, $f_2 : H_2 \to H_2$ be α_1, (respectively, α_2)-inverse strongly monotone mappings and $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings. The split equality monotone variational inclusion problem (SEMVIP) is to find $x^* \in H_1$ and $y^* \in H_2$ such that

$$0 \in f_1(x^*) + M_1(x^*), \tag{6.2.45}$$

$$0 \in f_2(y^*) + M_2(y^*), \quad \text{and} \quad Ax^* = By^*. \tag{6.2.46}$$

Let SOL(f_i, M_i), $(i = 1, 2)$ be the solution set of SEMVIP. The operator $J_\sigma^{M_i}(I - \lambda\phi)$ $(i = 1, 2)$ is an averaged nonexpansive operator and $F(J_\sigma^{M_i}(I - \lambda f_i))$ =SOL(f_i, M_i), $i = 1, 2$, where $\sigma > 0$, $\lambda \in (0, 2\alpha)$ and $J_\sigma^{M_i}(I - \lambda\phi)$ is the resolvent of M_i with parameter σ (see for example [6, 136]).

Averaged nonexpansive mapping with nonempty fixed point set is also quasi-nonexpansive. In Corollary 6.2.2, if we let $T = J_\sigma^{M_1}(I - \lambda f_1)$ and $S = J_\sigma^{M_2}(I - \lambda f_2)$, then we obtain a strong convergence result for approximating a common solution of SEMVIP and SEMFPP.

6.3 Split equality for monotone inclusion problem and fixed point problem in real Banach spaces

In this section, we study the following problem:

Let X_1, X_2 and X_3 be $p-$uniformly convex Banach spaces which are also uniformly smooth and $A : X_1 \to X_3$, $B : X_2 \to X_3$ be bounded linear operators. Let $M_i : X_1 \to 2^{X_1^*}$, $N_i : X_2 \to 2^{X_2^*}$ $i = 1, 2, ..., m$ be multivalued maximal monotone mappings and $T : X_1 \to X_1$, $S : X_2 \to X_2$ be right Bregman strongly nonexpansive mappings: Find $\bar{x} \in F(T)$ and $\bar{y} \in F(S)$ such that

$$0 \in M_i(\bar{x}), \tag{6.3.1}$$

$$0 \in N_i(\bar{y}) \quad \text{and} \quad A\bar{x} = B\bar{y}. \tag{6.3.2}$$

Furthermore, motivated by the recent work of Shehu *et al.* [179] we propose a new iterative algorithm and using the algorithm, we state and prove a strong convergence result for the approximation of a solution of problem (6.3.1)-(6.3.2).

6.3.1 Main result

Theorem 6.3.1. *Let X_1, X_2 and X_3 be three p-uniformly convex Banach space which are also uniformly smooth and $A : X_1 \to X_3$, $B : X_2 \to X_3$ be two bounded linear operators. Let $M_i : X_1 \to 2^{X_1^*}$, $N_i : X_2 \to 2^{X_2^*}$ $i = 1, 2, ..., m$ be multivalued maximal monotone mappings and $T : X_1 \to X_1$, $S : X_2 \to X_2$ be right Bregman strongly nonexpansive mappings such that $F(T) = \hat{F}(T)$ and $F(S) = \hat{F}(S)$. Suppose that $\Gamma \neq \emptyset$ and $\{\alpha_n\}$, $\{\beta_n\}$, are sequences in $(0, 1)$. Let $u, x_0 \in X_1$ and $v, y_0 \in X_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by*

$$\begin{cases} v_n = Res_p^{\lambda N_m} \circ Res_p^{\lambda N_{m-1}} \circ \cdots \circ Res_p^{\lambda N_1} J_q^{X_2}\left[J_q^{X_2}(y_n) + t_n B^* J_p^{X_3}(Ax_n - By_n)\right]; \\ u_n = Res_p^{\lambda M_m} \circ Res_p^{\lambda M_{m-1}} \circ \cdots \circ Res_p^{\lambda M_1} J_q^{X_1^*}\left[J_p^{X_1}(x_n) - t_n A^* J_p^{X_3}(Ax_n - By_n)\right]; \\ w_n = J_q^{X_2^*}\left[(1-\beta_n)J_p^{X_2}(v_n) + \beta_n J_p^{X_2} S v_n\right]; \\ z_n = J_q^{X_1^*}\left[(1-\beta_n)J_p^{X_1}(u_n) + \beta_n J_p^{X_1} T u_n\right]; \\ y_{n+1} = J_q^{X_2^*}\left[\alpha_n J_p^{X_2}(v) + (1-\alpha_n)J_p^{X_2} w_n\right]; \\ x_{n+1} = J_q^{X_1^*}\left[\alpha_n J_p^{X_1}(u) + (1-\alpha_n)J_p^{X_1} z_n\right], \end{cases} \tag{6.3.3}$$

with conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < t \leq t_n \leq k \leq \left(\frac{q}{2C_q||A||^q}\right)^{\frac{1}{q-1}}$, $0 < t \leq t_n \leq k \leq \left(\frac{q}{2D_q||B||^q}\right)^{\frac{1}{q-1}}$,

(iv) $\beta_n \in (a, b)$ *for some* $a, b \in (0, 1)$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Gamma$.

Proof. Let $(x^*, y^*) \in \Gamma$, $\Phi^m = Res_p^{\lambda M_m} \circ Res_p^{\lambda M_{m-1}} \circ \cdots \circ Res_p^{\lambda M_1}$, where $\Phi^0 = I$ and $\Psi^m = Res_p^{\lambda N_m} \circ Res_p^{\lambda N_{m-1}} \circ \cdots \circ Res_p^{\lambda N_1}$, where $\Psi^0 = I$. Let $a_n = J_q^{X_1^*}\left[J_p^{X_1}(x_n) - t_n A^* J_p^{X_3}(Ax_n - By_n)\right]$ and
$b_n = J_q^{X_2^*}\left[J_p^{X_2}(y_n) + t_n B^* J_p^{X_3}(Ax_n - By_n)\right]$. Then, using (2.1.25), (6.3.3) and Lemma

2.1.27, we have

$$\begin{aligned}
\Delta_p(u_n,x^*) &= \Delta_p(\Phi^m J_q^{X_1^*}\left[J_p^{X_1}(x_n)-t_nA^*J_p^{X_3}(Ax_n-By_n)\right],x^*)\\
&\leq \Delta_p(\Phi^{m-1} J_q^{X_1^*}\left[J_p^{X_1}(x_n)-t_nA^*J_p^{X_3}(Ax_n-By_n)\right],x^*)\\
&\ \vdots\\
&\leq \Delta_p(J_q^{X_1^*}\left[J_p^{X_1}(x_n)-t_nA^*J_p^{X_3}(Ax_n-By_n)\right],x^*)\\
&= \frac{1}{q}||J_p^{X_1}(x_n)-t_nA^*J_p^{X_3}(Ax_n-By_n)||^q-\langle J_p^{X_1}(x_n),x^*\rangle\\
&\quad +t_n\langle Ax^*,J_p^{X_3}(Ax_n-By_n)\rangle+\frac{1}{p}||x^*||^p\\
&\leq \frac{1}{q}||J_p^{X_1}(x_n)||^q-t_n\langle Ax_n,J_p^{X_3}(Ax_n-By_n)\rangle+\frac{C_q(t_n||A||)^q}{q}||J_p^{X_3}(Ax_n-By_n)||^q\\
&\quad -\langle J_p^{X_1}(x_n),x^*\rangle+\frac{1}{p}||x^*||^p+t_n\langle Ax^*,J_P^{X_3}(Ax_n-By_n)\rangle\\
&= \frac{1}{q}||x_n||^p-\langle J_p^{X_1}(x_n),x^*\rangle+\frac{1}{p}||x^*||^p+t_n\langle Ax^*-Ax_n,J_p^{X_3}(Ax_n-By_n)\rangle\\
&\quad +\frac{C_q(t_n||A||)^q}{q}||(Ax_n-By_n)||^p\\
&= \Delta_p(x_n,x^*)+t_n\langle J_p^{X_3}(Ax_n-By_n),Ax^*-Ax_n\rangle\\
&\quad +\frac{C_q(t_n||A||)^q}{q}||Ax_n-By_n||^p. \qquad (6.3.4)
\end{aligned}$$

Similarly, from (6.3.3) and Lemma 2.1.27, we have

$$\begin{aligned}
\Delta_p(v_n,y^*) &\leq \Delta_p(y_n,y^*)-t_n\langle J_p^{X_3}(Ax_n-By_n),By^*-By_n\rangle\\
&\quad +\frac{D_q(t_n||B||)^q}{q}||Ax_n-By_n||^p. \qquad (6.3.5)
\end{aligned}$$

Adding (6.3.4) and (6.3.5) and using the fact that $Ax^*=By^*$, we have

$$\begin{aligned}
\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*) &\leq \Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)-t_n\langle J_p^{X_3}(Ax_n-By_n),Ax_n-By_n\rangle\\
&\quad +\frac{C_q(t_n||A||)^q}{q}||Ax_n-By_n||^p+\frac{D_q(t_n||B||)^q}{q}||Ax_n-By_n||^p\\
&= \Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\\
&\quad -\left[t_n-\left(\frac{C_q(t_n||A||)^q}{q}+\frac{D_q(t_n||B||)^q}{q}\right)\right]||Ax_n-By_n|| \qquad (6.3.6)
\end{aligned}$$

Using condition (iii) in (6.3.6), we have

$$\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*) \leq \Delta_p(x_n,x^*)+\Delta_p(y_n,y^*). \qquad (6.3.7)$$

From (6.3.3) and (2.1.24), we have

$$\begin{aligned}
\Delta_p(z_n,x^*) &= \Delta_p\left(J_q^{X_1^*}\left[(1-\beta_n)J_p^{X_1^*}u_n+\beta_nJ_p^{X_1^*}Tu_n\right],x^*\right)\\
&\leq (1-\beta_n)\Delta_p(u_n,x^*)+\beta_n\Delta_p(Tu_n,x^*)\\
&\leq (1-\beta_n)\Delta_p(u_n,x^*)+\beta_n\Delta_p(Tu_n,x^*)\\
&\leq \Delta_p(u_n,x^*). \qquad (6.3.8)
\end{aligned}$$

Similarly, we have

$$\Delta_p(w_n,y^*)\leq \Delta_p(v_n,y^*). \tag{6.3.9}$$

From (6.3.3) and (6.3.8) and (2.1.24), we have

$$\begin{aligned}\Delta_p(x_{n+1},x^*) &= \Delta_p\left(J_q^{X_1^*}\left[\alpha_n J_p^{X_1}(u)+(1-\alpha_n)J_p^{X_1}z_n\right],x^*\right)\\ &\leq \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\Delta_p(z_n,x^*)\\ &\leq \alpha_n\Delta_p(u,x^*)+(1-\alpha_n)\Delta_p(u_n,x^*)\end{aligned} \tag{6.3.10}$$

Similarly, from (6.3.3) and (6.3.9), we have

$$\Delta_p(y_{n+1},y^*) \;\leq\; \alpha_n\Delta_p(v,y^*)+(1-\alpha_n)\Delta_p(v_n,y^*). \tag{6.3.11}$$

Adding (6.3.10) and (6.3.11) and using (6.3.7), we have

$$\begin{aligned}\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*) &\leq \alpha_n\left[\Delta_p(u,x^*)+\Delta_p(v,y^*)\right]+(1-\alpha_n)\left[\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*)\right]\\ &\leq \alpha_n\left[\Delta_p(u,x^*)+\Delta_p(v,y^*)\right]+(1-\alpha_n)\left[\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\right]\\ &\leq \max\{\Delta_p(u,x^*)+\Delta_p(v,y^*),\ \Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\}\\ &\;\vdots\\ &\leq \max\{\Delta_p(u,x^*)+\Delta_p(v,y^*),\ \Delta_p(x_0,x^*)+\Delta_p(y_0,y^*)\}.\end{aligned} \tag{6.3.12}$$

Therefore, $\{\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\}$ is bounded and consequently, $\{\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*)\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$, $\{Ax_n\}$ and $\{By_n\}$ are all bounded.

Also, from (6.3.3) and inequality (2.1.23) with $y^*=-\alpha_n(J_p^{X_1}(u)-J_p^{X_1}(x^*))$, we obtain

$$\begin{aligned}\Delta_p(x_{n+1},x^*) &= \Delta_p\left(J_q^{X_1^*}\left[\alpha_n J_p^{X_1}(u)+(1-\alpha_n)J_p^{X_1}(z_n)\right],x^*\right)\\ &= V_p\left(\alpha_n J_p^{X_1}(u)+(1-\alpha_n)J_p^{X_1}(z_n),x^*\right)\\ &\leq V_p\left(\alpha_n J_p^{X_1}(u)+(1-\alpha_n)J_p^{X_1}(z_n)-\alpha_n(J_p^{X_1}(u)-J_p^{X_1}(x^*)),x^*\right)\\ &\quad -\langle-\alpha_n(J_p^{X_1}(u)-J_p^{X_1}(x^*)),J_q^{X_1^*}\left[\alpha_n J_p^{X_1}(u)+(1-\alpha_n)J_p^{X_1}(z_n)\right]-x^*\rangle\\ &= V_p\left(\alpha_n J_p^{X_1}(x^*)+(1-\alpha_n)J_p^{X_1}(z_n),x^*\right)+\alpha_n\langle J_p^{X_1}(u)-J_p^{X_1}(x^*),x_{n+1}-x^*\rangle\\ &= \Delta_p\left(J_q^{X_1^*}\left[\alpha_n J_p^{X_1}(x^*)+(1-\alpha_n)J_p^{X_1}(z_n)\right],x^*\right)+\alpha_n\langle J_p^{X_1}(u)-J_p^{X_1}(x^*),x_{n+1}-x^*\rangle\\ &\leq \alpha_n\Delta_p(x^*,x^*)+(1-\alpha_n)\Delta_p(z_n,x^*)+\alpha_n\langle J_p^{X_1}(u)-J_p^{X_1}(x^*),x_{n+1}-x^*\rangle\\ &\leq (1-\alpha_n)\Delta_p(z_n,x^*)+\alpha_n\langle J_p^{X_1}(u)-J_p^{X_1}(x^*),x_{n+1}-x^*\rangle\\ &\leq (1-\alpha_n)\Delta_p(u_n,x^*)+\alpha_n\langle J_p^{X_1}(u)-J_p^{X_1}(x^*),x_{n+1}-x^*\rangle.\end{aligned} \tag{6.3.13}$$

Similarly, we have

$$\Delta_p(y_{n+1},y^*) \;\leq\; (1-\alpha_n)\Delta_p(v_n,y^*)+\alpha_n\langle J_p^{X_2}(v)-J_p^{X_2}(y^*),y_{n+1}-y^*\rangle. \tag{6.3.14}$$

Adding (6.3.13) and (6.3.14), we have

$$\begin{aligned}\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*) &\leq (1-\alpha_n)\left[\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*)\right]\\ &\quad +\alpha_n\left[\langle J_p^{X_1}(u)-J_p^{X_1}(x^*),x_{n+1}-x^*\rangle+\langle J_p^{X_2}(v)-J_p^{X_2}(y^*),y_{n+1}-y^*\rangle\right]\\ &\leq (1-\alpha_n)\left[\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\right]\\ &\quad +\alpha_n\Big[\langle J_p^{X_1}(u)-J_p^{X_1}(x^*),x_{n+1}-x^*\rangle\\ &\quad +\langle J_p^{X_2}(v)-J_p^{X_2}(y^*),y_{n+1}-y^*\rangle\Big].\end{aligned} \tag{6.3.15}$$

We now consider two cases to establish the strong convergence of $\{(x_n, y_n)\}$ to $(\bar{x}, \bar{y})$.
Case 1. Suppose that $\{\Delta_p(x_n, x^*)+\Delta_p(y_n, y^*)\}$ is monotone non-increasing, then $\{\Delta_p(x_n, x^*)+\Delta_p(y_n, y^*)\}$ is convergent. Thus,

$$\lim_{n\to\infty} [(\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*))] = 0.$$

From (6.3.8), (6.3.9), (6.3.10) and (6.3.11), we have

$$\begin{aligned} 0 &\leq (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) - (\Delta_p(w_n, x^*) + \Delta_p(z_n, y^*)) \\ &= (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\ &\quad + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(w_n, x^*) + \Delta_p(z_n, y^*)) \\ &\leq (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\ &\quad + \alpha_n (\Delta_p(u, x^*) + \Delta_p(v, y^*)) + (1-\alpha_n)(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\ &\quad - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \to 0, \text{ as } n \to \infty, \end{aligned}$$

which implies

$$\lim_{n\to\infty} (\Delta_p(u_n, x^*) - \Delta_p(z_n, x^*)) = \lim_{n\to\infty} (\Delta_p(v_n, y^*) - \Delta_p(w_n, y^*)) = 0. \tag{6.3.16}$$

Also, from the definition of z_n, we have

$$\begin{aligned} \Delta_p(z_n, x^*) &= \Delta_p\left(J_q^{X_1^*}\left((1-\beta_n)J_p^{X_1}(u_n) + \beta_n J_p^{X_1}(Tu_n)\right), x^*\right) \\ &\leq (1-\beta_n)\Delta_p(u_n, x^*) + \beta_n \Delta_p(Tu_n, x^*) \\ &= \Delta_p(u_n, x^*) + \beta_n [\Delta_p(Tu_n, x^*) - \Delta_p(u_n, x^*)]. \end{aligned} \tag{6.3.17}$$

Also, from (6.3.16), (6.3.17) and condition (iv) we obtain

$$\beta_n (\Delta_p(u_n, x^*) - \Delta_p(Tu_n, x^*)) \leq \Delta_p(u_n, x^*) - \Delta_p(z_n, x^*) \to 0, \text{ as } n \to \infty. \tag{6.3.18}$$

Since $\{\beta_n\}$ is bounded (see condition (iv)), we have

$$\lim_{n\to\infty} (\Delta_p(u_n, x^*) - \Delta_p(Tu_n, x^*)) = 0. \tag{6.3.19}$$

Similarly, we have

$$\lim_{n\to\infty} (\Delta_p(v_n, y^*) - \Delta_p(Sv_n, y^*)) = 0. \tag{6.3.20}$$

Since T and S are right Bregman strongly nonexpansive mappings, then from (6.3.19) and (6.3.20), we have

$$\lim_{n\to\infty} \Delta_p(Tu_n, u_n) = 0$$

and

$$\lim_{n\to\infty} \Delta_p(Sv_n, v_n) = 0$$

respectively, which implies

$$\lim_{n\to\infty} ||Tu_n - u_n|| = 0 \tag{6.3.21}$$

and

$$\lim_{n\to\infty} ||Sv_n - v_n|| = 0. \tag{6.3.22}$$

From (6.3.6) and (6.3.13), we have

$$\begin{array}{rl}
& \left[t_n - \left(\dfrac{C_q(t_n||A||)^q}{q} + \dfrac{D_q(t_n||B||)^q}{q}\right)\right] ||Ax_n - By_n||^p \\
\leq & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
= & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
\leq & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& +(1-\alpha_n)\left(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)\right) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
& +\alpha_n \left[\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle\right] \\
= & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& +\alpha_n(\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle - \Delta_p(u_n, x^*)) \\
& +\alpha_n(\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle - \Delta_p(v_n, y^*)) \to 0, \text{ as } n \to \infty.
\end{array}$$

That is,

$$\lim_{n\to\infty} \left[t_n - \left(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\right)\right] ||Ax_n - By_n||^p = 0.$$

Since $0 < t\left(1 - \left(\frac{C_q k^{q-1}(||A||)^q}{q} + \frac{D_q k^{q-1}(||B||)^q}{q}\right)\right) \leq \left(t_n - \left(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\right)\right)$, we have

$$\lim_{n\to\infty} ||Ax_n - By_n||^p = 0. \tag{6.3.23}$$

From the definitions of a_n and b_n, we have

$$\begin{array}{rcl}
||J_p^{X_1}a_n - J_p^{X_1}x_n|| & = & ||J_p^{X_1}(x_n) - t_n A^* J_p^{X_3}(Ax_n - By_n) - J_p^{X_1}(x_n)|| \\
& \leq & t_n ||A^*|| ||J_p^{X_3}(Ax_n - By_n)|| \\
& \leq & \left(\dfrac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}} ||A^*|| ||Ax_n - By_n|| \to 0, \ n \to \infty.
\end{array}$$

Since $J_p^{X_1^*}$ is norm to norm uniformly continuous on bounded subsets of X_1^*, we have

$$\lim_{n\to\infty} ||a_n - x_n|| = 0. \tag{6.3.24}$$

Similarly, we have

$$\lim_{n\to\infty} ||b_n - y_n|| = 0. \tag{6.3.25}$$

Since $\mathrm{Res}_p^{\lambda M_m}$ is a right Bregman firmly nonexpansive mapping, we have

$$\begin{aligned}
&\Delta_p\left(\mathrm{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n), x^*\right) + \Delta_p\left(x^*, \mathrm{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n)\right) + \Delta_p(\Phi^{m-1}a_n, \mathrm{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n)) + \Delta_p(x^*, x^*)\\
&\qquad \leq \Delta_p(\Phi^{m-1}a_n, x^*) + \Delta_p\left(x^*, \mathrm{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n)\right),
\end{aligned}$$

which implies

$$\Delta_p\left(\mathrm{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n), x^*\right) + \Delta_p\left(\Phi^{m-1}a_n, \mathrm{Res}_p^{\lambda M_m}(\Phi^{m-1}a_n)\right) \leq \Delta_p(\Phi^{m-1}a_n, x^*).$$

That is,

$$\Delta_p(\Phi^{m-1}a_n, \Phi^m a_n) \;\leq\; \Delta_p(\Phi^{m-1}a_n, x^*) - \Delta_p(\Phi^m a_n, x^*). \tag{6.3.26}$$

Similarly, we have

$$\Delta_p(\Psi^N b_n, \Psi^{N-1}b_n) \;\leq\; \Delta_p(\Psi^{N-1}b_n, y^*) - \Delta_p(\Psi^N b_n, y^*). \tag{6.3.27}$$

Adding (6.3.26) and (6.3.27), we have

$$\begin{aligned}
& \Delta_p(\Phi^{m-1}a_n, \Phi^m a_n) + \Delta_p(\Psi^{N-1}b_n, \Psi^N b_n)\\
\leq\;& \Delta_p(\Phi^{m-1}a_n, x^*) + \Delta_p(\Psi^{N-1}b_n, y^*) - \left(\Delta_p(\Phi^m a_n, x^*) + \Delta_p(\Psi^N b_n, y^*)\right)\\
\vdots\;&\\
\leq\;& \Delta_p(a_n, x^*) + \Delta_p(b_n, y^*) - \left(\Delta_p(\Phi^m a_n, x^*) + \Delta_p(\Psi^N b_n, y^*)\right) \qquad (6.3.28)\\
\leq\;& \left(\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)\right) - \left(\Delta_p(\Phi^m a_n, x^*) + \Delta_p(\Psi^N b_n, y^*)\right)\\
=\;& (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*))\\
& + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*))\\
\leq\;& (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*))\\
& +(1-\alpha_n)\left(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)\right) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*))\\
& +\alpha_n\left[\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle\right]\\
=\;& (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*))\\
& +\alpha_n(\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle - \Delta_p(u_n, x^*))\\
& +\alpha_n(\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle - \Delta_p(v_n, y^*)) \to 0, \text{ as } n \to \infty, \qquad (6.3.29)
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \Delta_p(\Phi^{m-1}a_n, \Phi^m a_n) = \lim_{n\to\infty} \Delta_p(\Psi^{N-1}b_n, \Psi^N b_n) = 0. \tag{6.3.30}$$

By the same argument as (6.3.26)-(6.3.28), we have

$$\begin{aligned}
& \Delta_p(\Phi^{m-2}a_n, \Phi^{m-1}a_n) + \Delta_p(\Psi^{N-2}b_n, \Psi^{N-1}b_n) \\
\leq\ & \Delta_p(a_n, x^*) + \Delta_p(b_n, y^*) - \big(\Delta_p(\Phi^{m-1}a_n, x^*) + \Delta_p(\Psi^{N-1}b_n, y^*)\big) \\
\leq\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - \big(\Delta_p(\Phi^{m}a_n, x^*) + \Delta_p(\Psi^{N}b_n, y^*)\big) \\
=\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
\leq\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& +(1-\alpha_n)\,(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
& +\alpha_n\left[\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle\right] \\
=\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& +\alpha_n(\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle - \Delta_p(u_n, x^*)) \\
& +\alpha_n(\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle - \Delta_p(v_n, y^*)) \to 0, \text{ as } n \to \infty,
\end{aligned} \tag{6.3.31}$$

which implies

$$\lim_{n\to\infty} \Delta_p(\Phi^{m-2}a_n, \Phi^{m-1}a_n) = \lim_{n\to\infty} \Delta_p(\Psi^{N-2}b_n, \Psi^{N-1}b_n) = 0. \tag{6.3.32}$$

Continuing in the same manner, we have that

$$\lim_{n\to\infty} \Delta_p(\Phi^{m-3}a_n, \Phi^{m-2}a_n) = \cdots = \lim_{n\to\infty} \Delta_p(\Phi^{1}a_n, \Phi^{2}a_n) = \lim_{n\to\infty} \Delta_p(a_n, \Phi^{1}a_n) = 0, \tag{6.3.33}$$

and

$$\lim_{n\to\infty} \Delta_p(\Psi^{N-3}a_n, \Psi^{N-2}a_n) = \cdots = \lim_{n\to\infty} \Delta_p(\Psi^{1}a_n, \Psi^{2}a_n) = \lim_{n\to\infty} \Delta_p(a_n, \Phi^{1}a_n) = 0. \tag{6.3.34}$$

From (6.3.30), (6.3.32), (6.3.33) and (6.3.34), we can conclude that

$$\lim_{n\to\infty} \Delta_p(\Phi^{l-1}a_n, \Phi^{l}a_n) = 0,\ l = 1, 2, \ldots, m,$$

$$\lim_{n\to\infty} \Delta_p(\Psi^{r-1}b_n, \Psi^{r}b_n) = 0,\ r = 1, 2, \ldots, N.$$

Which implies

$$\lim_{n\to\infty} ||\Phi^{l}a_n - \Phi^{l-1}a_n|| = 0,\ l = 1, 2, \ldots, m, \tag{6.3.35}$$

$$\lim_{n\to\infty} ||\Psi^{r}b_n - \Psi^{r-1}b_n|| = 0,\ r = 1, 2, \ldots, N. \tag{6.3.36}$$

Also, we have that

$$\lim_{n\to\infty} \|a_n - \Phi(a_n)\| = 0.$$

Hence,

$$\lim_{n\to\infty} ||a_n - u_n|| \leq \lim_{n\to\infty} \left[||a_n - \Phi^{1}a_n|| + ||\Phi^{1}a_n - \Phi^{2}a_n|| + \cdots + ||\Phi^{m-1}a_n - u_n||\right] \to 0, \tag{6.3.37}$$

which implies

$$\lim_{n\to\infty} ||a_n - u_n|| = 0. \tag{6.3.38}$$

Similarly, we obtain

$$\lim_{n\to\infty} ||b_n - v_n|| = 0. \tag{6.3.39}$$

From (6.3.24) and (6.3.38), we have

$$\lim_{n\to\infty} ||x_n - u_n|| = 0. \tag{6.3.40}$$

Similarly, we have

$$\lim_{n\to\infty} ||y_n - v_n|| = 0. \tag{6.3.41}$$

Also we have

$$\begin{array}{rcl} \|x_n - Tx_n\| & = & \|x_n - u_n + u_n - Tu_n + Tu_n - Tx_n\| \\ & \leq & \|x_n - u_n\| + \|u_n - Tu_n\| + \|Tu_n - Tx_n\| \\ & \leq & 2\|u_n - x_n\| + \|u_n - Tu_n\|. \end{array}$$

Hence from (6.3.21) and (6.3.40), we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{6.3.42}$$

Similarly, from (6.3.22) and (6.3.41) we obtain

$$\lim_{n\to\infty} \|y_n - Sy_n\| = 0. \tag{6.3.43}$$

Since $J_p^{X_1}$ and $J_p^{X_2}$ are uniformly continuous on bounded subsets of X_1 and X_2 respectively, we have from (6.3.35) and (6.3.36) that

$$\lim_{n\to\infty} ||J_p^{X_1}\Phi^l a_n - J_p^{X_1}\Phi^{l-1}a_n|| = 0, \ l = 1, 2, \ldots, m, \tag{6.3.44}$$

and

$$\lim_{n\to\infty} ||J_p^{X_2}\Psi^r b_n - J_p^{X_2}\Psi^{r-1}b_n|| = 0, \ r = 1, 2, \ldots, N. \tag{6.3.45}$$

Since $\{x_n\}$ is bounded in X_1 and X_1 is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to $\bar{x}$. By (6.3.42), we have that $\bar{x} \in F(T)$ since $F(T) = \hat{F}(T)$. Also since $\{y_n\}$ is bounded in X_2 and X_2 is reflexive, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ that converges weakly to $\bar{y}$. By (6.3.43), we have that $\bar{y} \in F(S)$ since $F(S) = \hat{F}(S)$.

Next, we show that $0 \in M_l(\bar{x})$ and $0 \in M_r(\bar{y})$, for each $l = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, N$. Let $i \in \{1, 2, 3, ...m\}$. Let $(z, \eta) \in G(M_l)$, then $\eta \in M_l z.$, From $\Phi^l a_n = \mathrm{Res}_p^{\lambda M_l}(\Phi^{l-1}a_n)$, we have that

$$J_p^{X_1}\Phi^{l-1}a_n \in \left(J_p^{X_1} + \lambda M_l\right)\Phi^l a_n,$$

which implies

$$\frac{1}{\lambda}\left(J_p^{X_1}\Phi^{l-1}a_n - J_p^{X_1}\Phi^l a_n\right) \in M_l\Phi^l a_n.$$

By the monotonicity of M_l, for each $l = 1, 2, \dots, m$, we have

$$\langle \eta - \frac{1}{\lambda}\left(J_p^{X_1}\Phi^{l-1}a_n - J_p^{X_1}\Phi^l a_n\right), z - \Phi^l a_n\rangle \geq 0.$$

This implies

$$\langle \eta, z - \Phi^l a_n\rangle \geq \left\langle \frac{1}{\lambda}\left(J_p^{X_1}\Phi^{l-1}a_n - J_p^{X_1}\Phi^l a_n\right), z - \Phi^{l-1}a_n \right\rangle.$$

Since $\{x_n\}$ converges weakly to $\bar{x}$, we have from (6.3.44) and (6.3.40) that

$$\langle \eta, z - \bar{x}\rangle \geq 0.$$

Hence, by the maximal monotonicity of M_l, we have that $0 \in M_l(\bar{x})$. Since i was arbitrary, we have $0 \in \cap_{l=1}^m M_l(\bar{x})$.

By similar argument, we obtain that $0 \in \cap_{r=1}^N M_r(\bar{y})$.

We now show that $A\bar{x} = B\bar{y}$.
Since $A : X_1 \to X_3$ and $B : X_2 \to X_3$ are bounded linear operators,and $\{x_n\}$ and $\{y_n\}$ converges weakly to $\bar{x}$ and $\bar{y}$, respectively we have that for arbitrary $f \in X_3^*$,

$$f(Ax_n) = (f \circ A)(x_n) \to (f \circ A)(\bar{x}) = f(A\bar{x}).$$

Similarly

$$f(Bx_n) = (f \circ B)(y_n) \to (f \circ B)(\bar{y}) = f(B\bar{y}).$$

This convergence implies that

$$Ax_n - By_n \rightharpoonup A\bar{x} - B\bar{y}.$$

Also, by weakly semi-continuity of the norm, it follows that

$$||A\bar{x} - B\bar{y}|| \leq \liminf_{n\to\infty} ||Ax_n - By_n|| = 0. \tag{6.3.46}$$

That is, $A\bar{x} = B\bar{y}$. Therefore $(\bar{x}, \bar{y}) \in \Gamma$.

We now show that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

$$\Delta_p(z_n, u_n) \leq (1 - \beta_n)\Delta_p(u_n, u_n) + \beta_n\Delta_p(Tu_n, u_n) \to 0, \text{ as } n \to \infty. \tag{6.3.47}$$

Also, we have

$$\begin{aligned}\Delta_p(x_{n+1}, u_n) &= \Delta_p\left(J_q^{X_1^*}\left[\alpha_n J_p^{X_1}(u) + (1 - \alpha_n)J_p^{X_1}z_n\right], u_n\right) \\ &\leq \alpha_n\Delta_p(u, u_n) + (1 - \alpha_n)\Delta_p(z_n, u_n) \to 0, \text{ as } n \to \infty, \end{aligned} \tag{6.3.48}$$

which implies

$$\lim_{n\to\infty} ||x_{n+1} - u_n|| = 0. \tag{6.3.49}$$

Similarly, we have

$$\lim_{n\to\infty} ||y_{n+1} - v_n|| = 0. \tag{6.3.50}$$

From (6.3.40) and (6.3.49), we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{6.3.51}$$

Similarly, we have

$$\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0. \tag{6.3.52}$$

From (6.3.15), we have

$$\begin{aligned}\Delta_p(x_{n+1}, \bar{x}) + \Delta_p(y_{n+1}, \bar{y}) \ \leq \ & (1 - \alpha_n)\left[\Delta_p(x_n, \bar{x}) + \Delta_p(y_n, \bar{y})\right] \\ & + \alpha_n \Big[\langle J_p^{X_1}(u) - J_p^{X_1}(\bar{x}), x_{n+1} - \bar{x}\rangle \\ & + \langle J_p^{X_2}(v) - J_p^{X_2}(\bar{y}), y_{n+1} - \bar{y}\rangle\Big].\end{aligned} \tag{6.3.53}$$

Using Lemma 2.3.14 in (6.3.53), we conclude that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.
Case 2: Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\Delta_p(x_{n_i}, x^*) + \Delta_p(y_{n_i}, y^*) < \Delta_p(x_{n_i+1}, x^*) + \Delta_p(y_{n_i+1}, y^*) \ \forall \ i \in \mathbb{N}.$$

By Lemma 2.3.15, we can find a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and for all $k \in \mathbb{N}$, we have

$$\Delta_p(x_{m_k}, x^*) + \Delta_p(y_{m_k}, y^*) \leq \Delta_p(x_{m_k+1}, x^*) + \Delta_p(y_{m_k+1}, y^*)$$

and

$$\Delta_p(x_k, x^*) + \Delta_p(y_k, y^*) \leq \Delta_p(x_{m_k+1}, x^*) + \Delta_p(y_{m_k+1}, y^*). \tag{6.3.54}$$

Then, by the same arguments as in (6.3.16), (6.3.17) and (6.3.18), we have that

$$\lim_{k\to\infty} ||Tu_{m_k} - u_{m_k}|| = 0 \tag{6.3.55}$$

and

$$\lim_{k\to\infty} ||Sv_{m_k} - v_{m_k}|| = 0. \tag{6.3.56}$$

From (6.3.15), we have

$$\begin{aligned}\Delta_p(x_{m_k+1}, \bar{x}) + \Delta_p(y_{m_k+1}, \bar{y}) \ \leq \ & (1 - \alpha_{m_k})\left(\Delta_p(x_{m_k}, \bar{x}) + \Delta_p(y_{m_k}, \bar{y})\right) \\ & + \alpha_{m_k}\Big(\langle J_p^{X_1}(u) - J_p^{X_1}(\bar{x}), x_{m_k+1} - \bar{x}\rangle \\ & + \langle J_p^{X_2}(v) - J_p^{X_2}(\bar{y}), y_{m_k+1} - \bar{y}\rangle\Big),\end{aligned} \tag{6.3.57}$$

which implies

$$\begin{aligned}
\alpha_{m_k}\left(\Delta_p(x_{m_k},\bar{x})+\Delta_p(y_{m_k},\bar{y})\right) &\leq (\Delta_p(x_{m_k},\bar{x})+\Delta_p(y_{m_k},\bar{y}))-(\Delta_p(x_{m_k+1},\bar{x})+\Delta_p(y_{m_k+1},\bar{y}))\\
&\quad +\alpha_{m_k}\Big(\langle J_p^{X_1}(u)-J_p^{X_1}(\bar{x}),x_{m_k+1}-\bar{x}\rangle\\
&\quad +\langle J_p^{X_2}(v)-J_p^{X_2}(\bar{y}),y_{m_k+1}-\bar{y}\rangle\Big)\\
&\leq \alpha_{m_k}\Big(\langle J_p^{X_1}(u)-J_p^{X_1}(\bar{x}),x_{m_k+1}-\bar{x}\rangle\\
&\quad +\langle J_p^{X_2}(v)-J_p^{X_2}(\bar{y}),y_{m_k+1}-\bar{y}\rangle\Big). \qquad (6.3.58)
\end{aligned}$$

That is

$$(\Delta_p(x_{m_k},\bar{x})+\Delta_p(y_{m_k},\bar{y}))\leq\left(\langle J_p^{X_1}(u)-J_p^{X_1}(\bar{x}),x_{m_k+1}-\bar{x}\rangle+\langle J_p^{X_2}(v)-J_p^{X_2}(\bar{y}),y_{m_k+1}-\bar{y}\rangle\right).$$

Which implies

$$\lim_{k\to\infty}\left(\Delta_p(x_{m_k},\bar{x})+\Delta_p(y_{m_k},\bar{y})\right)=0. \qquad (6.3.59)$$

From (6.3.54) and (6.3.59), we have

$$\Delta_p(x_k,\bar{x})+\Delta_p(y_k,\bar{y})\leq\Delta_p(x_{m_k+1},\bar{x})+\Delta_p(y_{m_k+1},\bar{y})\to 0,\ \text{as } k\to\infty,$$

which implies that $\{(x_k,y_k)\}$ converges strongly to $(\bar{x},\bar{y})$. Thus, $\{(x_n,y_n)\}$ converges strongly to $(\bar{x},\bar{y})\in\Gamma$. □

Corollary 6.3.2. *Let X_1, X_2 and X_3 be three p-uniformly convex real Banach spaces which are also uniformly smooth and $A:X_1\to X_3,\ B:X_2\to X_3$ be two bounded linear operators. Let $M:X_1\to 2^{X_1^*},\ N:X_2\to 2^{X_2^*}$ be multivalued maximal monotone mappings and $T:X_1\to X_1,\ S:X_2\to X_2$ be right Bregman strongly nonexpansive mappings such that $F(T)=\hat{F}(T)$ and $F(S)=\hat{F}(S)$. Suppose that $\Gamma:=\{(\bar{x},\bar{y})\in F(T)\times F(S)$ such that $0\in M(\bar{x}),\ 0\in N(\bar{y})$ and $A\bar{x}=B\bar{y}\}\neq\emptyset$ and $\{\alpha_n\}$, $\{\beta_n\}$, are sequences in $(0,1)$. Let $u,x_0\in X_1$ and $v,y_0\in X_2$ be arbitrary and the sequence $\{(x_n,y_n)\}$ be generated by*

$$\begin{cases}
v_n=Res_p^{\lambda N}J_q^{X_2}\left[J_q^{X_2}(y_n)+t_nB^*J_p^{X_3}(Ax_n-By_n)\right];\\
u_n=Res_p^{\lambda M}J_q^{X_1^*}\left[J_p^{X_1}(x_n)-t_nA^*J_p^{X_3}(Ax_n-By_n)\right];\\
w_n=J_q^{X_2^*}\left[(1-\beta_n)J_p^{X_2}(v_n)+\beta_nJ_p^{X_2}Sv_n\right];\\
z_n=J_q^{X_1^*}\left[(1-\beta_n)J_p^{X_1}(u_n)+\beta_nJ_p^{X_1}Tu_n\right];\\
y_{n+1}=J_q^{X_2^*}\left[\alpha_nJ_p^{X_2}(v)+(1-\alpha_n)J_p^{X_2}w_n\right];\\
x_{n+1}=J_q^{X_1^*}\left[\alpha_nJ_p^{X_1}(u)+(1-\alpha_n)J_p^{X_1}z_n\right],
\end{cases} \qquad (6.3.60)$$

with conditions

(i) $\lim_{n\to\infty}\alpha_n=0$,

(ii) $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(iii) $0 < t \leq t_n \leq k \leq \left(\frac{q}{2C_q||A||^q}\right)^{\frac{1}{q-1}}$, $0 < t \leq t_n \leq k \leq \left(\frac{q}{2D_q||B||^q}\right)^{\frac{1}{q-1}}$,

(iv) $\beta_n \in (a, b)$ *for some* $a, b \in (0, 1)$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Gamma$.

Recall that in a real Hilbert space, the duality mapping J_p becomes the identity mapping I. Thus, the resolvent of M now becomes:

$$\text{Res}_p^{\lambda M} = (I + \lambda M)^{-1} \circ I = (I + \lambda M)^{-1} =: J_\lambda^M.$$

Also note that if H is a real Hilbert space, then $H = H^*$. Using these facts, we obtain the following corollary in real Hilbert spaces.

Corollary 6.3.3. *Let* H_1*,* H_2 *and* H_3 *be three real Hilbert spaces and* $A : H_1 \to H_3$*,* $B : H_2 \to H_3$ *be two bounded linear operators. Let* $M_i : H_1 \to 2^{H_1}$*,* $N_i : H_2 \to 2^{H_2}$ $i = 1, 2, ..., m$ *be multivalued maximal monotone mappings and* $T : H_1 \to H_1$*,* $S : H_2 \to H_2$ *be strongly nonexpansive mappings. Suppose that* $\Gamma \neq \emptyset$ *and* $\{\alpha_n\}$*,* $\{\beta_n\}$*, are sequences in* $(0, 1)$*. Let* $u, x_0 \in H_1$ *and* $v, y_0 \in H_2$ *be arbitrary and the sequence* $\{(x_n, y_n)\}$ *be generated by*

$$\begin{cases} v_n = J_\lambda^{N_m} \circ J_\lambda^{N_{m-1}} \circ \cdots \circ J_\lambda^{N_1} \left[y_n + t_n B^*(Ax_n - By_n)\right]; \\ u_n = J_\lambda^{M_m} \circ J_\lambda^{M_{m-1}} \circ \cdots \circ J_\lambda^{M_1} \left[x_n - t_n A^*(Ax_n - By_n)\right]; \\ w_n = (1 - \beta_n)v_n + \beta_n S v_n; \\ z_n = (1 - \beta_n)u_n + \beta_n T u_n; \\ y_{n+1} = \alpha_n v + (1 - \alpha_n)w_n; \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \end{cases} \tag{6.3.61}$$

with conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $t_n \in \left(\epsilon, \dfrac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\right), n \in \Omega$ *otherwise,* $t_n = t$ *(*t *being any nonegative value), where the set of indexes* $\Omega = \{n : Ax_n - By_n \neq 0\}$,

(iv) $\beta_n \in (a, b)$ *for some* $a, b \in (0, 1)$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Gamma$.

6.3.2 Application to split equality variational inequality problem and split equality fixed point problem

Let $D : C \subset X \to X^*$ be strongly positive bounded linear mapping with coefficient $\tau > 0$ and $f : C \subset X \to X^*$ be a contraction mapping with coefficient $0 < \alpha < 1$. Then Lemma 2.3.18 also holds in a more general Banach space, i.e., we have that $(D - \gamma f)$ is a monotone operator in X. Indeed, for all $x, y \in C$ and $0 < \gamma < \frac{\tau}{\alpha}$, we obtain

$$\begin{aligned} \langle (D-\gamma f)x - (D-\gamma f)y, x-y\rangle &= \langle Ax - Ay - \gamma(fx - fy), x - y\rangle \\ &= \langle Ax - Ay, x-y\rangle - \gamma\langle fx - fy, x-y\rangle \\ &\geq \tau||x-y||^2 - \gamma||fx - fy||||x-y|| \\ &\geq \tau||x-y||^2 - \gamma\alpha||x-y||^2, \end{aligned}$$

which implies that $\langle (D-\gamma f)x - (D-\gamma f)y, x-y\rangle \geq 0$.

It can also be shown that $(D - \gamma f)$ is a Lipschitzian mapping. Indeed, $\forall x, y \in C$, we have

$$\begin{aligned} \|(D-\tau f)x - (D-\tau f)y\|_* &= \|Dx - \tau fx - Dy + \tau fy\|_* \\ &\leq \|Dx - Dy\|_* + \tau\|fx - fy\|_* \\ &\leq k\|x-y\| + \tau\alpha\|x-y\| \\ &\leq (k + \alpha\tau)\|x-y\|, \end{aligned} \tag{6.3.62}$$

which implies that $(D - \tau f)$ is Lipschitzian with coefficient $L = k + \alpha\tau$. Hence, $(D - \tau f)$ is a monotone and L-Lipschitz mapping if $0 < \gamma < \frac{\tau}{\alpha}$. Therefore, if we define $M : X \to 2^{X^*}$ by

$$Mx = \begin{cases} N_C x + (D-\tau f)x & if \quad x \in C, \\ \emptyset & if \quad x \notin C, \end{cases} \tag{6.3.63}$$

where $N_C x$ is the normal cone of C at x, defined by

$$N_C x = \{w \in X^* : \langle w, x - y\rangle \geq 0, \ \forall y \in C\}.$$

Then, M is maximal monotone and $M^{-1}(0) = VIP(C, D - \tau f)$ (see, for example, [[150], Theorem 3]), where $VIP(C, D - \tau f)$ is the solution set of the variational inequality problem: Find $x^* \in C$ such that

$$\langle (D - \tau f)x^*, y - x^*\rangle \geq 0 \quad \forall y \in C. \tag{6.3.64}$$

Let $\Gamma^* := \{(\bar{x}, \bar{y}) \in F(T) \times F(S)$, such that $(\bar{x}, \bar{y}) \in (\cap_{l=1}^m VIP(C, D_l - \tau f_l)) \times \big(\cap_{r=1}^N VIP(Q, D_r - \tau f_r)\big)$ Then, we state the following theorem for approximating a common solution of split equality variational inequality problem and split equality fixed point problem, whose proof follows from the proof of Theorem 6.3.1.

Theorem 6.3.4. *Let X_1, X_2 and X_3 be three p-uniformly convex real Banach spaces which are also uniformly smooth and C, Q be nonempty, closed and convex subset of X_1, X_2 respectively. For each $l = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, N$, let $D_l : C \to X_1^*$, $D_r : Q \to X_2^*$ be strongly positive bounded linear mappings with coefficient $\tau > 0$ and $f_l : C \to X_1^*$,*

$f_r : Q \to X_2^*$ be contraction mappings with coefficient $0 < \alpha < 1$ such that $0 < \gamma < \frac{\tau}{\alpha}$. Let $T : X_1 \to X_1$, $S : X_2 \to X_2$ be right Bregamn strongly nonexpansive mappings such that $F(T) = \hat{F}(T)$ and $F(S) = \hat{F}(S)$. Let $A : X_1 \to X_3, B : X_2 \to X_3$ be two bounded linear operators and $\Gamma^* \neq \emptyset$. Let the sequence $\{(x_n, y_n)\}$ be generated by Algorithm 6.3.3, then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Gamma^*$.

Remark 6.3.5. *Our work extend results for split equality monotone inclusion problem from the framework of Hilbert spaces to the more general p-uniformly convex Banach spaces which are also uniformly smooth.*

6.4 Solving split equality equilibrium and fixed point problems in Banach spaces without prior knowledge of the operator norms

Let H_1, H_2 and H_3 be real Hilbert spaces, C and Q be nonempty closed convex subsets of H_1 and H_2 respectively. Let $F : C \times C \to \mathbb{R}$, $G : Q \times Q \to \mathbb{R}$ be bifunctions, $T : H_1 \to H_1$, $S : H_2 \to H_2$ be nonlinear operators and $\phi : H_1 \to H_1$, $\psi_2 \to H_2$ be real-valued functions. The Split Equality Generalized Mixed Equilibrium Problem (SEGMEP) is to find $\bar{x} \in C$ and $\bar{y} \in Q$ such that

$$F(\bar{x}, x) + \langle T\bar{x}, x - \bar{x}\rangle + \phi(x) - \phi(\bar{x}) \geq 0, \;\; \forall\, x \in C,$$
$$G(\bar{y}, y) + \langle S\bar{y}, x - \bar{y}\rangle + \psi(y) - \psi(\bar{y}) \geq 0, \;\; \forall\, y \in Q,$$

and

$$A\bar{x} = B\bar{y},$$

where $A : H_1 \to H_3$, and $B : H_2 \to H_3$ are bounded linear operators. We denote the set of solutions for the SEGMEP by $SEGMEP(F, G, T, S, \phi, \psi)$. It is easy to see that the SEGMEP consist of two generalized mixed equilibrium problems with sets of solutions $GMEP(F, T, \phi)$ and $GMEP(G, R, \psi)$.

In this section, we consider the split equality problems in p-uniformly convex Banach spaces which are also uniformly smooth. We propose a simultaneous iterative algorithm for approximating a common solution of split equality fixed point problem for right Bregman strongly quasi nonexpansive mappings and split equality generalized mixed equilibrium problem in p-uniformly convex Banach spaces. We also introduce a choice of stepsize for our algorithm so that the algorithm does not depend on the operator norms. Our results in this paper extend and improve the result of Rahaman et al [147] and Karahan [91] from Hilbert space to more general Banach spaces and many recent results on split equality fixed point problem and common solution of split equality fixed point problem and equilibrium problem obtained by many authors in literature.

6.4.1 Main results

Theorem 6.4.1. *Let X_1, X_2 and X_3 be p-uniformly convex real Banach spaces which are also uniformly smooth, C and Q be nonempty closed convex subsets of X_1 and X_2 respectively and $A : X_1 \to X_3$, $B : X_2 \to X_3$ be bounded linear operators and $A^* : X_3^* \to X_1^*$ and $B^* : X_3^* \to X_2^*$ be the adjoint of A and B. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunctions which satisfy conditions (A1)-(A4), $S : C \to X_1^*$ and $R : Q \to X_2^*$ be continuous monotone mappings and $\phi : C \to \mathbb{R} \cup \{+\infty\}$ and $\psi : Q \to \mathbb{R} \cup \{+\infty\}$ be convex and lower semicontinuous mappings. Let $T_1 : C \to C$ and $T_2 : Q \to Q$ be right Bregman strongly quasi-nonexpansive mappings such that $F(T_1) = \hat{F}(T_1)$ and $F(T_2) = \hat{F}(T_2)$ respectively. Suppose $\Gamma := SEGMEP(F, G, S, R, \phi, \psi) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Choose an initial guess $x_0 \in X_1$ and $y_0 \in X_2$ and let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[a, b] \subset (0, 1)$. For a fixed $u \in C$ and $v \in Q$, assume that the nth iterate $\{(x_n, y_n)\} \subset (X_1 \times X_2)$ has been constructed, then we calculate the $(n + 1)$th iterate (x_{n+1}, y_{n+1}) via the formula:*

$$\begin{cases} u_n = T_{r_n}^F J_q^{X_1^*}(J_p^{X_1}(x_n) - t_n A^* J_p^{X_3}(Ax_n - By_n)), \\ x_{n+1} = J_q^{X_1^*}(\alpha_n J_p^{X_1}(u) + (1-\alpha_n)(\beta_n J_p^{X_1}(u_n) + (1-\beta_n) J_p^{X_1} T_1 u_n)), \\ v_n = T_{r_n}^G J_q^{X_2^*}(J_p^{X_2}(y_n) + t_n B^* J_p^{X_3}(Ax_n - By_n)), \\ y_{n+1} = J_p^{X_2^*}(\alpha_n J_p^{X_2}(v) + (1-\alpha_n)(\beta_n J_p^{X_2}(v_n) + (1-\beta_n) J_p^{X_2}) T_2 v_n)), \end{cases} \tag{6.4.1}$$

where the stepsize t_n is choosen in such a way that

$$t_n \in \Big(\epsilon, \Big(\frac{q||Ax_n - By_n||^p}{C_q||A^* J_p^{X_3}(Ax_n - By_n)||^q + Q_q||B^* J_p^{X_3}(Ax_n - By_n)||^q} - \epsilon\Big)^{\frac{1}{q-1}}\Big), \ n \in \Omega,$$

for small enough ϵ, otherwise $t_n = t$(t being any nonegative value), where the set of indices $\Omega = \{n : Ax_n - By_n \neq 0\}$. Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $(1 - \alpha_n)a < (1 - \beta_n)$, $a \in (0, \frac{1}{2})$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iv) $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{(x_n, y_n)\}$ strongly converges to a solution $(x^*, y^*) \in \Gamma$.

Proof. Let $(x^*, y^*) \in \Gamma$. This means that $T_{r_n}^F x^* = x^*, T_1 x^* = x^*$, $T_{r_n}^G y^* = y^*, T_2 y^* = y^*$ and

$Ax^* = By^*$. Thus, from (6.4.1) and (3.2.3), we have

$$\begin{aligned}
\Delta_p(u_n, x^*) &= \Delta_p(T_{r_n}^F J_q^{X_1^*}(J_p^{X_1}(x_n) - t_n A^* J_p^{X_3}(Ax_n - By_n)), x^*) \\
&\leq \Delta_p(J_q^{X_1^*}(J_p^{X_1}(x_n) - t_n A^* J_p^{X_3}(Ax_n - By_n)), x^*) \\
&= V_p(J_p^{X_1}(x_n) - t_n A^* J_p^{X_3}(Ax_n - By_n)), x^*) \\
&= \frac{1}{q}||J_p^{X_1}(x_n) - t_n A^* J_p^{X_3}(Ax_n - By_n)||^q - \langle J_p^{X_1}(x_n), x^*\rangle \\
&\quad + t_n\langle A^* J_p^{X_3}(Ax_n - By_n), x^*\rangle + \frac{1}{p}||x^*||^p \\
&\leq \frac{1}{q}||J_p^{X_1}(x_n)||^q - t_n\langle J_p^{X_3}(Ax_n - By_n), Ax_n\rangle + \frac{C_q}{q}t_n^q||A^* J_p^{X_3}(Ax_n - By_n)||^q \\
&\quad - \langle J_p^{X_1}(x_n), x^*\rangle \\
&\quad + t_n\langle J_p^{X_3}(Ax_n - By_n), Ax^*\rangle + \frac{1}{p}||x^*||^p \\
&= \frac{1}{q}||x_n||^q - \langle J_p^{X_1}(x_n), x^*\rangle + \frac{1}{p}||x^*||^p - t_n\langle J_p^{X_3}(Ax_n - By_n), Ax_n - Ax^*\rangle \\
&\quad + \frac{C_q}{q}t_n^q||A^* J_p^{X_3}(Ax_n - By_n)||^q \\
&= \Delta_p(x_n, x^*) - t_n\langle J_p^{X_3}(Ax_n - By_n), Ax_n - Ax^*\rangle \\
&\quad + \frac{C_q t_n^q}{q}||A^* J_p^{X_3}(Ax_n - By_n)||^q. \qquad (6.4.2)
\end{aligned}$$

Following similar process, we obtain

$$\begin{aligned}
\Delta_p(v_n, y^*) &\leq \Delta_p(y_n, y^*) - t_n\langle J_p^{X_3}(Ax_n - By_n), By^* - By_n\rangle \\
&\quad + \frac{Q_q t_n^q}{q}||B^* J_p^{X_3}(Ax_n - By_n)||^q. \qquad (6.4.3)
\end{aligned}$$

Adding (6.4.2) and (6.4.3), noting that $Ax^* = By^*$, we have

$$\begin{aligned}
\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*) &\leq \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*) - t_n\Big[||Ax_n - By_n||^p \\
&\quad - \frac{t_n^{q-1}}{q}(C_q||A^* J_p^{X_3}(Ax_n - By_n)||^q \\
&\quad + Q_q||B^* J_p^{X_3}(Ax_n - By_n)||^q)\Big]. \qquad (6.4.4)
\end{aligned}$$

Thus

$$\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*) \leq \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*). \qquad (6.4.5)$$

Also from (6.4.1) and (2.1.24), we have

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) &= \Delta_p(J_q^{X_1^*}(\alpha_n J_p^{X_1}(u) + (1-\alpha_n)(\beta_n J_p^{X_1}(u_n) + (1-\beta_n)J_p^{X_1}T_1u_n)), x^*) \\
&= \Delta_p(J_q^{X_1^*}(\alpha_n J_p^{X_1}(u) + (1-\alpha_n)\beta_n J_p^{X_1}(u_n) + (1-\alpha_n)(1-\beta_n)J_p^{X_1}T_1u_n), x^*) \\
&\leq \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\beta_n\Delta_p(u_n, x^*) + (1-\alpha_n)(1-\beta_n)\Delta_p(T_1u_n, x^*) \\
&\leq \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\beta_n\Delta_p(u_n, x^*) + (1-\alpha_n)(1-\beta_n)\Delta_p(u_n, x^*) \\
&= \alpha_n\Delta_p(u, x^*) + (1-\alpha)\Delta_p(u_n, x^*). \qquad (6.4.6)
\end{aligned}$$

Similarly, we have

$$\Delta_p(y_{n+1}, y^*) \leq \alpha_n \Delta_p(v, y^*) + (1-\alpha_n)\Delta_p(v_n, y^*). \tag{6.4.7}$$

Hence, from (6.4.5),(6.4.6) and (6.4.7), we have

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*) &\leq \alpha_n(\Delta_p(u, x^*) + \Delta_p(v, y^*)) + (1-\alpha_n)(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
&\leq \alpha_n(\Delta_p(u, x^*) + \Delta_p(v, y^*)) + (1-\alpha_n)(\Delta_p(x_n, x^*) \\
&\quad + \Delta_p(y_n, y^*)) \qquad (6.4.8) \\
&\leq max\{\Delta_p(u, x^*) + (\Delta_p(v, y^*), \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)\} \\
&\vdots \\
&\leq max\{\Delta_p(u, x^*) + (\Delta_p(v, y^*), \Delta_p(x_0, x^*) + \Delta_p(y_0, y^*)\}
\end{aligned}$$

. Thus $\{\Delta_p(x_{n+1}, x^*)+\Delta_p(y_{n+1}, y^*)\}$ is bounded. Consequently, $\{\Delta_p(x_n, x^*)\}$ and $\{\Delta_p(y_n, y^*)\}$ are bounded. It therefore, follows that $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ are bounded. Furthermore, from (6.4.1) and inequality (2.1.23), with $\bar{y} = -\alpha_n(J_p(u)^{X_1} - J_p^{X_1}(x^*))$, we have

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) &= \Delta_p(J_q^{X_1^*}(\alpha_n J_p^{X_1}(u) + (1-\alpha_n)(\beta_n J_p^{X_1}(u_n) + (1-\beta_n)J_p^{X_1}T_1 u_n)), x^*) \\
&= V_p(\alpha_n J_p^{X_1}(u) + (1-\alpha_n)(\beta_n J_p^{X_1}(u_n) + (1-\beta_n)J_p^{X_1}T_1 u_n), x^*) \\
&= V_p(\alpha_n J_p^{X_1}(u) + (1-\alpha_n)(\beta_n J_p^{X_1}(u_n) + (1-\beta_n)J_p^{X_1}T_1 u_n) - \alpha_n(J_p^{X_1}(u) - J_p^{X_1}(x^*)), x^* \\
&\quad -\langle \alpha_n(J_p^{X_1}(u) - J_p^{X_1}(x^*)), J_q^{X_1^*}(\alpha_n J_p^{X_1}(u) + (1-\alpha_n)(\beta_n J_p^{X_1}(u_n) + (1-\beta_n)J_p^{X_1}T_1 u_n)) \\
&= V_p(\alpha_n J_p^{X_1}(x^*) + (1-\alpha_n)(\beta_n J_p^{X_1}(u_n) + (1-\beta_n)J_p^{X_1}T_1 u_n), x^*) \\
&\quad +\alpha_n\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle \\
&= \Delta_p(J_q^{X_1^*}(\alpha_n J_p^{X_1}(x^*) + (1-\alpha_n)(\beta_n J_p^{X_1}(u_n) + (1-\beta_n)J_p^{X_1}T_1 u_n)), x^*) \\
&\quad +\alpha_n\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle \\
&= \Delta_p(J_q^{X_1^*}(\alpha_n J_p^{X_1}(x^*) + (1-\alpha_n)\beta_n J_p^{X_1}(u_n) + (1-\alpha_n)(1-\beta_n)J_p^{X_1}T_1 u_n), x^*) \\
&\quad +\alpha_n\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle \\
&\leq \alpha_n \Delta_p(x^*, x^*) + (1-\beta_n)\beta_n \Delta_p(u_n, x^*) + (1-\alpha_n)(1-\beta_n)\Delta_p(T_1 u_n, x^*) \\
&\quad +\alpha_n\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\Delta_p(u_n, x^*) + \alpha_n\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle,
\end{aligned}$$

Similarly, we can obtain

$$\Delta_p(y_{n+1}, y^*) \leq (1-\alpha_n)\Delta_p(v_n, y^*) + \alpha_n\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle.$$

Therefore, from (6.4.6), we have

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*) &\leq (1-\alpha_n)(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
&\quad +\alpha_n(\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle) \\
&\leq (1-\alpha_n)(\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) \\
&\quad +\alpha_n(\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle \\
&\quad +\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle). \qquad (6.4.9)
\end{aligned}$$

Now, by setting $s_n(x^*, y^*) := \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)$, we divide the remaining proof into two cases.

CASE 1: Suppose there exists $n_0 \in \mathbb{N}$ such that $s_n(x^*, y^*)$ is monotonically non-increasing for all $n \geq n_0$. Then $\{s_n(x^*, y^*)\}$ converges as $n \to \infty$ and so

$$s_n(x^*, y^*) - s_{n+1}(x^*, y^*) \to 0 \text{ as } n \to \infty.$$

Moreover, from (6.4.4) we have

$$\begin{aligned} & t_n\Big[||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}(C_q||A^* J_p^{X_3}(Ax_n - By_n)||^q + Q_q||B^* J_p^{X_3}(Ax_n - By_n)||^q)\Big] \\ \leq \quad & \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*) - (\Delta_p(u_n, x^*) + (\Delta_p(v_n, y^*)). \end{aligned} \tag{6.4.10}$$

Putting $\Lambda_n := C_q||A^* J_p^{X_3}(Ax_n - By_n)||^q + Q_q||B^* J_p^{X_3}(Ax_n - By_n)||^q$, we have from (6.4.10)

$$\begin{aligned} t_n\Big[||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}\Lambda_n\Big] \quad & \leq \quad \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*) - (\Delta_p(u_n, x^*) + (\Delta_p(v_n, y^*)) \\ & = \quad s_n(x^*, y^*) - s_{n+1}(x^*, y^*) + s_{n+1}(x^*, y^*) \\ & \qquad -(\Delta_p(u_n, x^*) + (\Delta_p(v_n, y^*)). \end{aligned} \tag{6.4.11}$$

It follows from (6.4.9) and (6.4.11) and the fact that $\alpha_n \to 0$ as $n \to \infty$ that

$$\begin{aligned} t_n\Big[||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}\Lambda_n\Big] \quad & \leq \quad s_n(x^*, y^*) - s_{n+1}(x^*, y^*) + (1-\alpha_n)(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\ & \qquad +\alpha_n(\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle \\ & \qquad -(\Delta_p(u_n, x^*) + (\Delta_p(v_n, y^*)) \to 0, \qquad (6.4.12 \end{aligned}$$

as $n \to \infty$. By the condition on the stepsize t_n, we have that

$$t_n^{q-1} < \frac{q||Ax_n - By_n||^p}{\Lambda_n} - \epsilon,$$

which implies that

$$t_n^{q-1}\Lambda_n < q||Ax_n - By_n||^p - \epsilon\Lambda_n.$$

Thus

$$\frac{\epsilon\Lambda_n}{q} < ||Ax_n - By_n||^p - \frac{t_n^{q-1}}{q}\Lambda_n \to 0, \text{ as } n \to \infty.$$

Therefore,

$$C_q||A^* J_p^{X_3}(Ax_n - By_n)||^q + Q_q||B^* J_p^{X_3}(Ax_n - By_n)||^q \to 0 \text{ as } n \to \infty.$$

It follows that

$$\lim_{n\to\infty} ||A^* J_p^{X^3}(Ax_n - By_n)||^q = 0, \tag{6.4.13}$$

and

$$\lim_{n\to\infty} ||B^* J_p^{X^3}(Ax_n - By_n)||^q = 0. \tag{6.4.14}$$

Also, we have that

$$\begin{aligned} t_n||Ax_n - By_n||^p &\leq s_n(x^*, y^*) - s_{n+1}(x^*, y^*) + (1-\alpha_n)(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\ &\quad + \alpha_n(\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle) \\ &\quad - (\Delta_p(u_n, x^*) + (\Delta_p(v_n, y^*)) + \frac{t_n^q}{q}\Lambda_n \to 0, \end{aligned}$$

as $n \to \infty$. Hence

$$\lim_{n\to\infty} ||Ax_n - By_n||^p = 0. \tag{6.4.15}$$

Putting

$$k_n = \beta_n J_p^{X_1} u_n + (1-\beta_n) J_p^{X_1} T_1 u_n, \tag{6.4.16}$$

and

$$l_n = \beta_n J_p^{X_2} v_n + (1-\beta_n) J_p^{X_2} T_2 v_n, \tag{6.4.17}$$

we have,

$$\begin{aligned} \Delta_p(k_n, x^*) &\leq \beta_n \Delta_p(u_n, x^*) + (1-\beta_n)\Delta_p(T_1 u_n, x^*) \\ &\leq \beta_n \Delta_p(u_n, x^*) + (1-\beta_n)\Delta_p(u_n, x^*) \\ &= \Delta_p(u_n, x^*). \end{aligned} \tag{6.4.18}$$

Similarly, we obtain that

$$\Delta_p(l_n, y^*) \leq \Delta_p(v_n, y^*). \tag{6.4.19}$$

Hence

$$\Delta_p(k_n, x^*) + \Delta_p(l_n, y^*) \leq (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)). \tag{6.4.20}$$

Hence, form (6.4.20), we obtain

$$\begin{aligned} 0 &\leq (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) - (\Delta_p(k_n, x^*) + \Delta_p(l_n, y^*)) \\ &\leq (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\ &\quad + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(k_n, x^*) + \Delta_p(l_n, y^*)) \\ &\leq (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) + \alpha_n(\Delta_p(u, x^*) + \Delta_p(v, y^*)) \\ &\quad + (1-\alpha_n)(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\ &\quad - (\Delta_p(k_n, x^*) + \Delta_p(l_n, y^*)) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{6.4.21}$$

Thus

$$\lim_{n\to\infty} (\Delta_p(u_n, x^*) - \Delta_p(k_n, x^*)) = 0. \tag{6.4.22}$$

Similarly, we obtain

$$\lim_{n\to\infty} (\Delta_p(v_n, y^*) - \Delta_p(l_n, y^*)) = 0. \tag{6.4.23}$$

But

$$\begin{aligned}\Delta_p(k_n,x^*) &\leq \beta_n\Delta_p(u_n,x^*)+(1-\beta_n)\Delta_p(T_1u_n,x^*)\\ &= \beta_n\Delta_p(u_n,x^*)-\beta_n\Delta_p(T_1u_n,x^*)+\Delta_p(T_1u_n,x^*)\\ &\leq \Delta_p(u_n,x^*)+\beta_n\left[\Delta_p(u_n,x^*)-\Delta_p(T_1u_n,x^*)\right].\end{aligned} \tag{6.4.24}$$

Hence, since $\alpha_n \to 0$ as $n \to \infty$ and by (6.4.22), we have

$$\beta_n(\Delta_p(T_1u_n,x^*)-\Delta_p(u_n,x^*)) \leq \Delta_p(u_n,x^*)-\Delta_p(k_n,x^*) \to 0,\ n\to\infty. \tag{6.4.25}$$

Therefore

$$\lim_{n\to\infty}(\Delta_p(T_1u_n,x^*)-\Delta_p(u_n,x^*))=0. \tag{6.4.26}$$

Similarly, we have

$$\lim_{n\to\infty}(\Delta_p(T_2v_n,y^*)-\Delta_p(v_n,y^*))=0. \tag{6.4.27}$$

Since T_1 and T_2 are right Bregman strongly quasi-nonexpansive, we have

$$\lim_{n\to\infty}\Delta_p(T_1u_n,u_n)=0, \tag{6.4.28}$$

and

$$\lim_{n\to\infty}\Delta_p(T_2v_n,v_n)=0. \tag{6.4.29}$$

Therefore

$$\lim_{n\to\infty}||T_1u_n-u_n||=0, \tag{6.4.30}$$

and

$$\lim_{n\to\infty}||T_2v_n-v_n||=0. \tag{6.4.31}$$

Since $\{u_n\}$ and $\{v_n\}$ are bounded, there exist subsequences $\{u_{n_i}\}$ of $\{u_n\}$ and $\{v_{n_i}\}$ of $\{v_n\}$ which converges weakly to $\hat{x}\in C$ and $\hat{y}\in Q$ respectively. Since $F(T_1)=\hat{F}(T_1)$ and $F(T_2)=\hat{F}(T_2)$, it follows from (6.4.30) and (6.4.31) that $\hat{x}\in F(T_1)$ and $\hat{y}\in F(T_2)$.

We now show that $\hat{x}\in GMEP(F,S,\phi)$ and $\hat{y}\in GMEP(G,R,\psi)$. Putting $w_n = J_q^{X_1^*}(J_p^{X_1}(x_n)-t_nA^*J_p^{X_3}(Ax_n-By_n))$ and $z_n = J_q^{X_2}(J_p^{X_2}(y_n)+t_nB^*J_p^{X_3}(Ax_n-By_n))$, then $u_n = T_{r_n}^F w_n$ and $v_n = T_{r_n}^G z_n$. We note from (6.4.2),(6.4.3) and (6.4.4) that

$$\Delta_p(w_n,x^*)+\Delta_p(z_n,y^*) \leq \Delta_p(x_n,x^*)+\Delta_p(y_n,y^*).$$

It follows from (2.3.4) that

$$\begin{aligned}\Delta_p(w_n,u_n)+\Delta_p(z_n,v_n) &= \Delta_p(w_n,T_{r_n}^Fw_n)+\Delta_p(z_n,T_{r_n}^Gz_n)\\ &\leq (\Delta_p(w_n,x^*)+\Delta_p(z_n,y^*))-(\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*))\\ &\leq (\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*))-(\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*))\\ &= (\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*))-(\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*))\\ &\quad +(\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*))-(\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*))\\ &\leq (\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*))-(\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*))\\ &\quad +\alpha_n(\Delta_p(u,x^*)+\Delta_p(v,y^*))+(1-\alpha_n)(\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*))\\ &\quad -(\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*)) \to 0,\ n\to\infty.\end{aligned} \tag{6.4.32}$$

Hence

$$\lim_{n\to\infty} \Delta_p(w_n, u_n) = 0,$$

and

$$\lim_{n\to\infty} \Delta_p(z_n, v_n) = 0.$$

Thus

$$\lim_{n\to\infty} ||w_n - u_n|| = 0, \tag{6.4.33}$$

and

$$\lim_{n\to\infty} ||z_n - v_n|| = 0. \tag{6.4.34}$$

Since X_1 and X_2 are uniformly smooth, $J_p^{X_1}$ and $J_p^{X_2}$ are uniformly continuous on bounded subsets of X_1 and X_2, respectively. Thus

$$\lim_{n\to\infty} ||J_p^{X_1} w_n - J_p^{X_1} u_n|| = 0, \tag{6.4.35}$$

and

$$\lim_{n\to\infty} ||J_p^{X_2} z_n - J_p^{X_2} v_n|| = 0. \tag{6.4.36}$$

From the definition of w_n, we have

$$\begin{array}{rcl} \|J_p^{X_1} w_n - J_p^{X_1} x_n\| &=& \|J_p^{X_1}(x_n) - t_n A^* J_p^{X_3}(Ax_n - By_n) - J_p^{X_1}(x_n)\| \\ &=& t_n \|A^* J_p^{X_3}(Ax_n - By_n)\| \to 0, \quad n \to \infty. \end{array}$$

Since X_1 and X_2 are uniformly smooth, $J_p^{X_1}$ and $J_p^{X_2}$ are uniformly continuous on bounded subsets of X_1 and X_2, respectively. Thus, we have

$$\|w_n - x_n\| \to 0 \quad n \to \infty.$$

In similar way, we obtain

$$\|z_n - y_n\| \to 0, \quad n \to \infty.$$

Also, we obtain that

$$\|x_n - u_n\| \le \|x_n - w_n\| + \|w_n - u_n\| \to 0, \quad n \to \infty,$$

and

$$\|y_n - v_n\| \le \|y_n - z_n\| + \|z_n - v_n\| \to 0, \quad n \to \infty.$$

By the definition of $T_{r_n}^F$, we have for any $u \in C$ and $r > 0$

$$F(u_n, u) + \langle Su_n, u - u_n \rangle + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle J_p^{X_1} u_n - J_p^{X_1} w_n, u - u_n \rangle \ge 0.$$

By (A2), we obtain

$$\begin{array}{rcl} F(u, u_n) &\le& -F(u_n, u) \\ &\le& \langle Su_n, u - u_n \rangle + \phi(u) - \phi(u_n) + \dfrac{1}{r_n}\langle J_p^{X_1} u_n - J_p^{X_2} w_n, u - u_n \rangle. \end{array}$$

For any $t \in (0,1)$ and $u \in C$, define $u_t = tu + (1-t)\hat{x}$. Then $u_t \in C$. It follows by the monotonicity of S that

$$\begin{aligned} F(u_t, u_n) - \phi(u_t) + \phi(u_n) &\leq \langle Su_n - Su_t, u - u_n\rangle + \langle Su_t, u - u_n\rangle + \frac{1}{r_n}\langle J_p^{X_1}u_n - J_p^{X_1}w_n, u - u_n\rangle \\ &\leq \langle Su_t, u - u_n\rangle + \frac{1}{r_n}\langle J_p^{X_1}u_n - J_p^{X_1}w_n, u - u_n\rangle. \end{aligned}$$

It follows from (6.4.35) and the fact that $u_n \rightharpoonup \hat{x}$, as $n \to \infty$, that

$$F(u_t, \hat{x}) - \phi(u_t) + \phi(\hat{x}) \leq \langle Su_t, u - \hat{x}\rangle. \tag{6.4.37}$$

By (A1) and (A4) and the convexity of ϕ, we have

$$\begin{aligned} 0 &= F(u_t, u_t) + \phi(u_t) - \phi(u_t) \\ &\leq tF(u_t, u) + (1-t)F(u_t, \hat{x}) + t\phi(u) + (1-t)\phi(\hat{x}) - \phi(u_t) \\ &= t(F(u_t, u) + \phi(u) - \phi(u_t)) + (1-t)(F(u_t, \hat{x}) + \phi(\hat{x}) - \phi(u_t)) \\ &\leq t(F(u_t, u) + \phi(u) - \phi(u_t)) + (1-t)\langle Su_t, u - \hat{x}\rangle. \end{aligned} \tag{6.4.38}$$

By (A3), the weakly lower semicontinuity of ϕ and the continuity of S, letting $t \to 0$, we obtain

$$F(\hat{x}, u) + \phi(u) - \phi(\hat{x}) + \langle S\hat{x}, u - \hat{x}\rangle \geq 0, \ \forall u \in C.$$

This shows that $\hat{x} \in GMEP(F, S, \phi)$.
Following similar argument, we can show that $\hat{y} \in GMEP(G, R, \psi)$.

We now show that $A\hat{x} = B\hat{y}$.
Since $A : X_1 \to X_3$ and $B : X_2 \to X_3$ are bounded linear operators, and $\{x_n\}$ and $\{y_n\}$ converges weakly to $\hat{x}$ and $\hat{y}$, respectively, we have that for arbitrary $f \in X_3^*$,

$$f(Ax_n) = (f \circ A)(x_n) \to (f \circ A)(\hat{x}) = f(A\hat{x}).$$

Similarly

$$f(Bx_n) = (f \circ B)(y_n) \to (f \circ B)(\hat{y}) = f(B\hat{y}).$$

This convergence implies that

$$Ax_n - By_n \rightharpoonup A\hat{x} - B\hat{y}.$$

Also, by weakly semi-continuity of the norm, it follows that

$$||A\hat{x} - B\hat{y}||^p \leq \liminf_{n\to\infty} ||Ax_n - By_n||^p = 0. \tag{6.4.39}$$

That is, $A\hat{x} = B\hat{y}$. Therefore $(\hat{x}, \hat{y}) \in \Gamma$.

We now show that the sequences $\{(x_n, y_n)\}$ strongly converges to $(x^*, y^*) = \Pi_\Gamma(u, v)$. From (6.4.9), we have

$$\begin{aligned} \Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*) &\leq (1-\alpha_n)(\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) \\ &\quad +\alpha_n(\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle \\ &\quad +\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle). \end{aligned} \tag{6.4.40}$$

Choose subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle = \lim_{j\to\infty}\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n_j+1} - x^*\rangle,$$

and

$$\limsup_{n\to\infty}\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle = \lim_{j\to\infty}\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n_j+1} - y^*\rangle.$$

Since $x_{n_j} \rightharpoonup \bar{x}$ and $y_{n_j} \rightharpoonup \bar{y}$, it follows from (2.1.20) that

$$\begin{aligned}\limsup_{n\to\infty}\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n+1} - x^*\rangle &= \lim_{j\to\infty}\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{n_j+1} - x^*\rangle \\ &= \langle J_p^{X_1}(u) - J_p^{X_1}(x^*), \bar{x} - x^*\rangle \leq 0, \quad (6.4.41)\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n+1} - y^*\rangle &= \lim_{j\to\infty}\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{n_j+1} - y^*\rangle) \\ &= \langle J_p^{X_2}(v) - J_p^{X_2}(y^*), \bar{y} - y^*\rangle \leq 0. \quad (6.4.42)\end{aligned}$$

Using Lemma 2.3.14 in (6.4.40), we conclude that

$$\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*) \to 0, \ n \to \infty. \tag{6.4.43}$$

Thus, $\Delta_p(x_n, x^*) \to 0$ and $\Delta_p(y_n, y^*) \to 0$, $n \to \infty$. Therefore $x_n \to x^*$ and $y_n \to y^*$.

Case 2: Assume that $\{s_n(x^*, y^*)\}$ is not monotonically decreasing. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) defined by

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n, \tau_k \leq \tau_{k+1}\}.$$

Clearly, τ is nondecreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$0 \leq s_{\tau(n)}(x^*, y^*) \leq s_{\tau(n)+1}(x^*, y^*), \quad \forall n \geq n_0.$$

After a similar argument as in Case 1, it is easy to see that

$$\lim_{n\to\infty}||A^* J_p^{X^3}(Ax_{\tau(n)} - By_{\tau(n)})||^q = 0,$$

and

$$\lim_{n\to\infty}||B^* J_p^{X^3}(Ax_{\tau(n)} - By_{\tau(n)})||^q = 0.$$

Following similar analysis as in Case 1, we immediately conclude that $\lim\limits_{n\to\infty}||Ax_{\tau(n)} - By_{\tau(n)}||^p = 0$; $\lim\limits_{n\to\infty}||u_{\tau(n)} - T_1 u_{\tau(n)}|| = 0$ and $\lim\limits_{n\to\infty}||v_{\tau(n)} - T_1 v_v|| = 0$. Also we have that

$$\limsup_{n\to\infty}\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{\tau(n)+1} - x^*\rangle \leq 0 \tag{6.4.44}$$

and

$$\limsup_{n\to\infty}\langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{\tau(n)+1} - y^*\rangle \leq 0. \tag{6.4.45}$$

Now Since $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ are bounded, there exist subsequences of $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ still denoted as $\{x_{\tau(n)}\}$ and $\{y_{\tau(n)}\}$ which converge weakly to $\bar{x} \in X_1$ and $\bar{y} \in X_2$ respectively. From (6.4.9), we have

$$\begin{aligned} s_{\tau(n)+1}(x^*, y^*) &\leq (1-\alpha_{\tau(n)})s_{\tau(n)}(x^*, y^*) + \alpha_{\tau(n)}(\langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{\tau(n)+1} - x^*\rangle \\ &\quad + \langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{\tau(n)+1} - y^*\rangle). \end{aligned} \tag{6.4.46}$$

Since $s_{\tau(n)}(x^*, y^*) \leq s_{\tau(n)+1}(x^*, y^*)$, it follows from (6.4.46) that

$$s_{\tau(n)}(x^*, y^*) \leq \langle J_p^{X_1}(u) - J_p^{X_1}(x^*), x_{\tau(n)+1} - x^*\rangle + \langle J_p^{X_2}(v) - J_p^{X_2}(y^*), y_{\tau(n)+1} - y^*\rangle.$$

Then from (6.4.44), (6.4.45) and (6.4.46), we have that

$$\lim_{n\to\infty} s_{\tau(n)}(x^*, y^*) = \lim_{n\to\infty} (\Delta_p(x_{\tau(n)}, x^*) + \Delta_p(y_{\tau(n)}, y^*)) = 0.$$

Hence, $\lim_{n\to\infty} \Delta_p(x_{\tau(n)}, x^*) = 0$ and $\lim_{n\to\infty} \Delta_p(y_{\tau(n)}, y^*) = 0$. Thus we have

$$\lim_{n\to\infty} ||x_{\tau(n)} - x^*|| = 0 \quad \text{and} \quad \lim_{n\to\infty} ||y_{\tau(n)} - y^*|| = 0.$$

Also, we have

$$\lim_{n\to\infty} ||x_{\tau(n)+1} - x^*|| = 0 \quad \text{and} \quad \lim_{n\to\infty} ||y_{\tau(n)+1} - y^*|| = 0.$$

Furthermore, for $n \geq n_0$, it is easy to see that $s_{\tau(n)}(x^*, y^*) \leq s_{\tau(n)+1}(x^*, y^*)$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $s_k(x^*, y^*) \geq s_{k+1}(x^*, y^*)$ for $\tau(n) + 1 \leq k \leq n$. As a consequence, we obtain for all $n \geq n_0$

$$0 \leq s_n(x^*, y^*) \leq \max\{s_{\tau(n)}(x^*, y^*), s_{\tau(n)+1}(x^*, y^*)\} = s_{\tau(n)+1}(x^*, y^*).$$

Hence, $\lim_{n\to\infty} s_n(x^*, y^*) = \lim_{n\to\infty} (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) = 0$. Thus $\lim_{n\to\infty} \Delta_p(x_n, x^*) = \lim_{n\to\infty} \Delta_p(y_n, y^*)$. Therefore, we have

$$\lim_{n\to\infty} ||x_n - x^*|| = 0 \quad \text{and} \quad \lim_{n\to\infty} ||y_n - y^*|| = 0.$$

This implies that the sequences $\{(x_n, y_n)\}$ strongly converges to $(x^*, y^*) = \Pi_\Gamma(u, v)$. This completes the proof. ☐

6.4.2 Numerical example

We give an example in $(\mathbb{R}^3, ||\cdot||_2)$ of our main Theorem 6.4.1. Let $X_1 = X_2 = X_3 = \mathbb{R}^3$, with $p = 2$, $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$. Let $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, $S : \mathbb{R}^3 \to \mathbb{R}^3$ and $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F(\bar{x}, \bar{y}) = -\frac{1}{2}\bar{x}^2 + \frac{1}{2}\bar{y}^2, \quad S(\bar{x}) = \bar{x} \quad \text{and} \quad \phi(\bar{x}) = \frac{1}{2}\bar{x}^2 \quad \text{respectively.}$$

For any $r > 0$ and $\bar{x} \in \mathbb{R}^3$, Lemma 2.3.21 ensures that ethere exists $\bar{z} \in \mathbb{R}^3$ such that for any $\bar{y} \in \mathbb{R}^3$

$$F(\bar{x}, \bar{y}) + \langle S\bar{x}, \bar{y} - \bar{x}\rangle + \phi(\bar{y}) - \phi(\bar{x}) + \frac{1}{r}\langle \bar{y} - \bar{x}, \bar{x} - \bar{z}\rangle \geq 0.$$

Then, we obtain

$$T_{r_n}^F \bar{x} = \frac{\bar{x}}{3r_n + 1}.$$

Also, let $G : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, $R : \mathbb{R}^3 \to \mathbb{R}^3$ and $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as

$$G(\bar{u}, \bar{v}) = -3\bar{u} + 2\bar{u}\bar{v} + \bar{v}^2, \quad R(\bar{u}) = 2\bar{u} \quad \text{and} \quad \psi(\bar{u}) = \bar{u}^2 \quad \text{respectively.}$$

For any $r > 0$ and $\bar{u} \in \mathbb{R}^3$, Lemma 2.3.21 ensures that there exists $\bar{w} \in \mathbb{R}^3$ such that, for any $\bar{v} \in \mathbb{R}^3$, we have

$$G(\bar{u}, \bar{v}) + \langle R\bar{u}, \bar{v} - \bar{u} \rangle + \psi(\bar{v}) - \psi(\bar{u}) + \frac{1}{r}\langle \bar{v} - \bar{u}, \bar{u} - \bar{w} \rangle \geq 0.$$

Then, we also obtain that

$$T_{r_n}^G \bar{u} = \frac{\bar{u}}{8r_n + 1}.$$

Now take $C := \{\bar{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle \bar{a}, \bar{x} \rangle \geq b\}$, where $\bar{a} = (1, -3, 5)$ and $b = 4$, then

$$\Pi_C(\bar{x}) = P_C(\bar{x}) = \frac{b - \langle \bar{a}, \bar{x} \rangle}{||\bar{a}||^2}\bar{a} + \bar{x}.$$

Also, take $Q := \{\bar{y} = (y_1, y_2, y_3) \in \mathbb{R}^3 : \langle \bar{c}, \bar{y} \rangle = d\}$, where $\bar{c} = (4, 5, 6)$ and $d = 2$, then

$$\Pi_Q(\bar{y}) = P_Q(\bar{y}) = \max\left\{0, \frac{d - \langle \bar{c}, \bar{y} \rangle}{||\bar{c}||^2}\right\}\bar{c} + \bar{y}.$$

Then, let $T_1 = \Pi_C$ and $T_2 = \Pi_Q$ respectively. Furthermore, let $A : \mathbb{R}^3 \to \mathbb{R}^3$ and $B : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$A(\bar{x}) = \begin{pmatrix} 4 & -2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad B(\bar{y}) = \begin{pmatrix} 5 & 6 & 8 \\ 7 & 4 & -1 \\ 8 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Choosen $r_n = \frac{n+1}{n+2}$, $\alpha_n = \frac{1}{9n+2}$ and $\beta_n = \frac{5n-1}{6(n+1)}$, then our algorithm (6.4.1) becomes

$$\begin{cases} \bar{u}_n = \frac{n+2}{5n+5}(\bar{x}_n - t_n A^T(A\bar{x}_n - B\bar{y}_n)), \\ \bar{x}_{n+1} = \frac{1}{9n+2}\bar{u} + \frac{9n+1}{9n+2}\Big[\frac{5n-1}{6(n+1)}\bar{u}_n + \frac{n+7}{6(n+1)}\Pi_C(\bar{u}_n)\Big], \\ \bar{v}_n = \frac{n+2}{9n+10}(\bar{y}_n + t_n B^T(A\bar{x}_n - B\bar{y}_n)), \\ \bar{y}_{n+1} = \frac{1}{9n+2}\bar{v} + \frac{9n+1}{9n+2}\Big[\frac{5n-1}{6(n+1)}\bar{v}_n + \frac{n+7}{6(n+1)}\Pi_Q(\bar{v}_n)\Big], \end{cases} \tag{6.4.47}$$

where A^T and B^T are transpose of A and B respectively and the step size t_n is choosen in such a way that

$$t_n \in \left(\epsilon, \frac{2||A\bar{x}_n - B\bar{y}_n||^2}{||A^T(A\bar{x}_n - B\bar{y}_n)||^2 + ||B^T(A\bar{x}_n - B\bar{y}_n)||^2} - \epsilon\right),$$

otherwise, $t_n = t$ (t being any nonnegative value), where the set of indexes $\Omega = \{n : A\bar{x}_n - B\bar{y}_n \neq 0\}$.

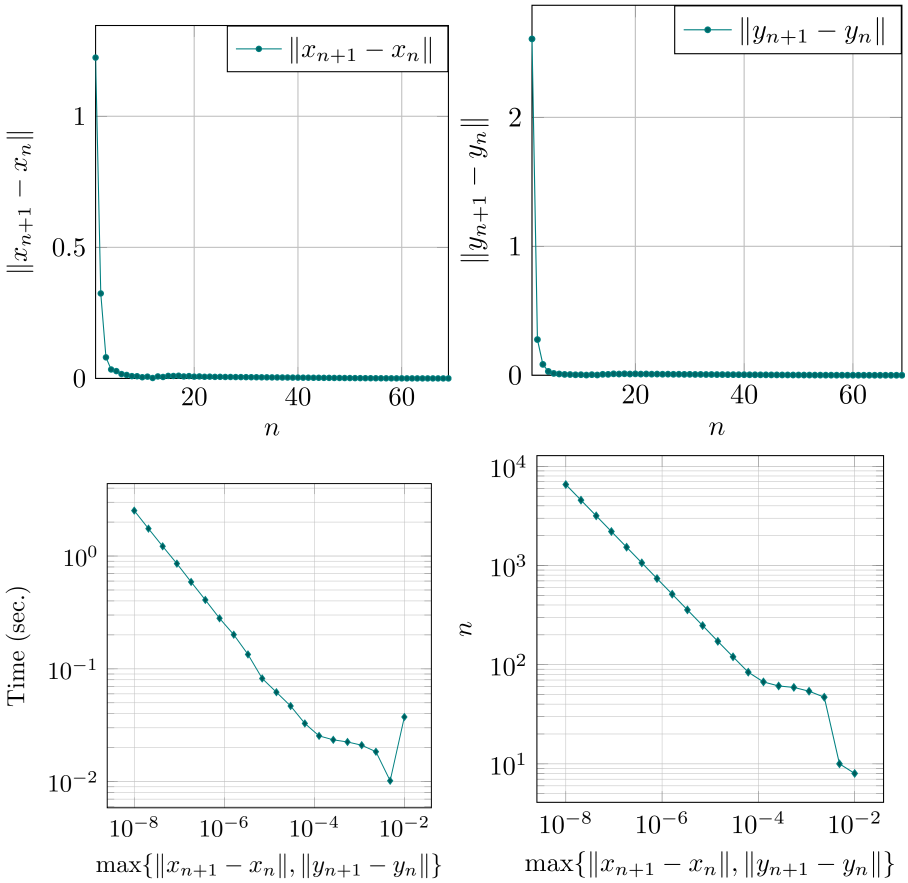

Figure 6.1: Case A(i): errors vs number of iterations (top left and right); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

We make different choose of $\bar{x}_0$, $\bar{y}_0$, $\bar{u}$ and $\bar{v}$ in order to see how change in initial values affect the number of iterations. We note that the choice of t_n, as long as it is in the range, does not have any significant effect on both the number of iterations and cpu time.

Case A

(i) Take $\bar{x}_0 = (1, 1, 0.5)^T$, $\bar{y}_0 = (-0.5, 1, 2.2)^T$, $\bar{u} = (-1, 0, -1)^T$ and $\bar{v} = (2, 1, -2)^T$.

(ii) Take $\bar{x}_0 = (4, 1, 1)^T$, $\bar{y}_0 = (0, 1, 3)^T$, $\bar{u} = (5, 1, -3)^T$ and $\bar{v} = (0.5, 1, 3)^T$.

Case B

(i) Take $\bar{x}_0 = (5, 1, 5)^T$, $\bar{y}_0 = (-0.5, -3, 2)^T$, $\bar{u} = (1, 3, -1)^T$ and $\bar{v} = (3, 4, -2)^T$.

(ii) Take $\bar{x}_0 = (-2, 1, 10)^T$, $\bar{y}_0 = (2, 1, -2)^T$, $\bar{u} = (-1.5, 0, 4)^T$ and $\bar{v} = (0, 5, -2)^T$.

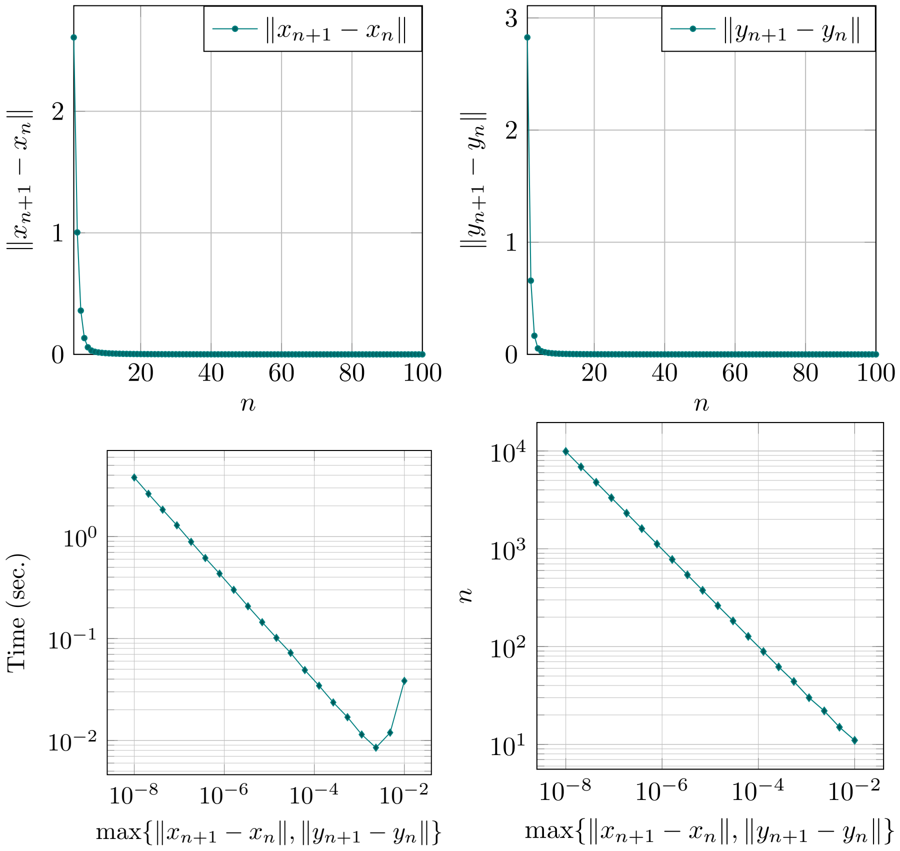

Figure 6.2: Case A(ii): errors vs number of iterations (top left and right); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

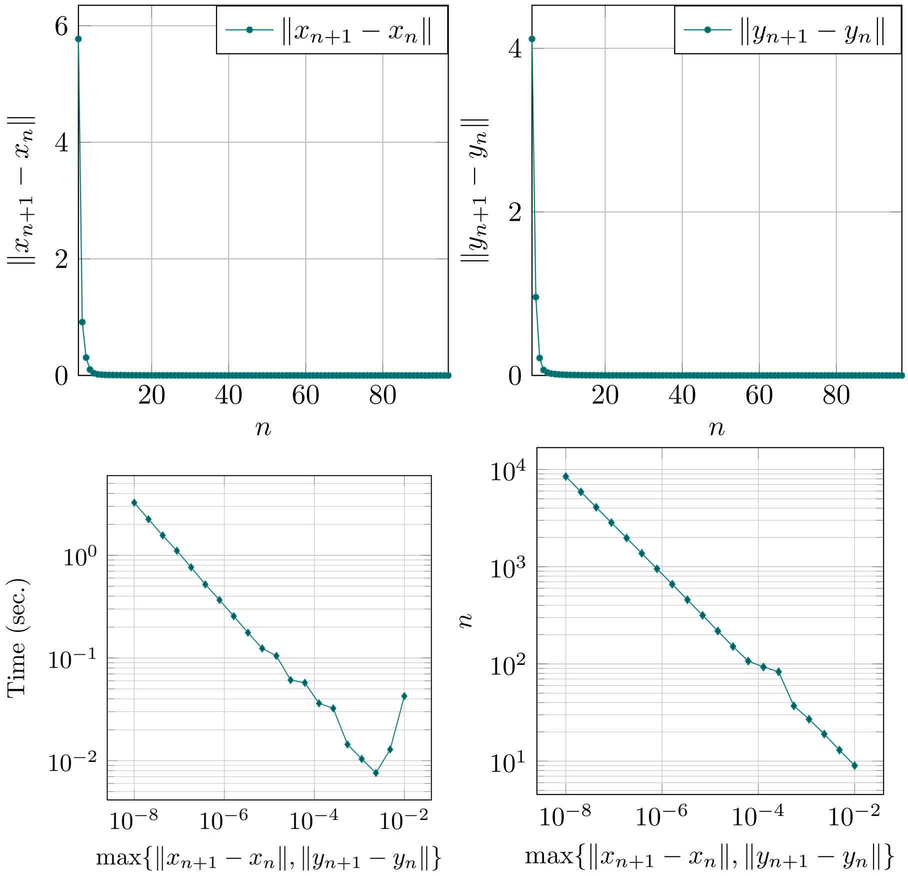

Figure 6.3: Case B(i): errors vs number of iterations (top left and right); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

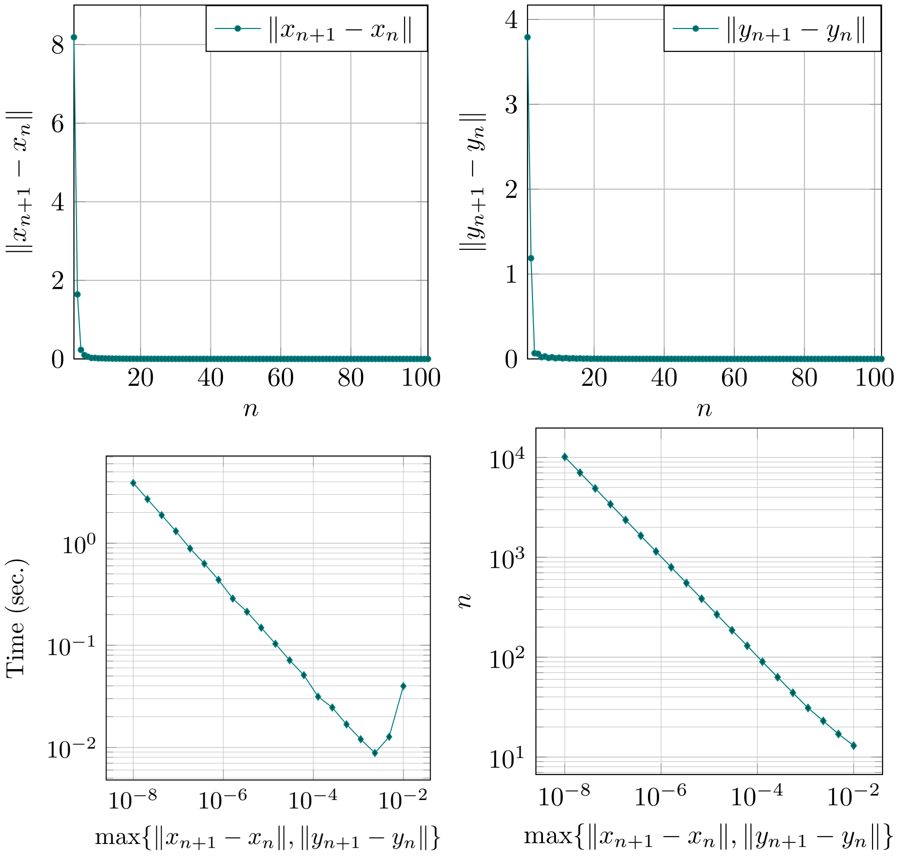

Figure 6.4: Case B(ii): errors vs number of iterations (top left and right); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

Chapter 7

Contribution to Knowledge and Area of Future Research

In this chapter, we give a summary of the results obtained in this work which serve as our contribution to knowledge. Also, possible areas of future research are also identified and discussed.

7.1 Contribution to knowledge

We now highlight our contributions in this work.

1. Tian *et al.* [169] introduced a new iterative algorithm for finding a common element of the set of solutions of the constrained convex minimization problem and equilibrium problem. Motivated by the works of Shehu [177] and Tian *et al.* [169], in Theorem 3.1.1, we propose a new iterative algorithm for finding a common element of the fixed points set of common solution of a one-parameter nonexpansive semigroup, the set of solution of constrained convex minimization problem and the set of solutions of generalized equilibrium problem in a real Hilbert space using the idea of regularized gradient-projection algorithm under suitable conditions. Hence our results here improve, extend and compliment the work of Tian *et al.* [169].

2. In 2010, Riech and Sabach [175] used the CQ algorithm to obtain strong convergence result for finitely many maximal monotone mappings B_i using the Bregman distance for computation. Moreover, in 2017 Truong Minh Tuyen [172] introduced three parallel iterative methods which use techniques of Bregman distance, Bregman projections, Bregman strongly nonexpansive operators and hybrid or shrinking projection methods for solving systems of generalized mixed equilibrium problems in real reflexive Banach space. We know that the CQ algorithm is difficult to compute. Hence, our result in Theorem 3.3.2 extend the results of [172] and [175]. Also our method of proof here is shorter, easier to read and less technical.

3. The following contributions were made in Theorem 4.3.3:

   (i) We know that the Meir-Keeler contraction is a generalization of the contraction mapping. Moreover, the condition

   $$\langle Bx - By, j((I - rB)x - (I - rB)y\rangle \geq 0$$

   for all $x, y \in E$ and for all $r > 0$ assumed in the result of Wei and Duan [184] is dispensed in our result. Hence, our results improve the results of Wei and Duan [184].

   (ii) It is well known that real smooth and uniformly convex Banach space are more general than Hilbert space or q-uniformly smooth Banach space and also our normalized duality mapping j is weakly sequentially continuous in most of the existing related work is weaken to j weakly sequentially continuous at zero. Hence our result extends the results of Song *et al.* [163].

   We also point out some differences between the presentation of our method of proof of Theorem 4.4.1 and that of Theorem 2.4 of [46] viz:

   (i) The major key in proving Theorem 4.4.1 is to show that $\limsup_{n\to\infty}(-\Gamma_n) \leq 0$ as given in (4.4.15) and using Lemma 2.3.17 in (3.3.10).

   (ii) In our convergence analysis, we did not make use of Lemma 2.3 of [46], which was used in the convergence analysis of proof of Theorem 2.4 in [46]; rather we used Lemma 2.3.17 in this result.

4. Our result on Theorem 5.3.5 extends and compliments some recent results in the following ways:

   (i) Our result improves and extend the result of Kazmi and Rizvi [93], from single-valued nonexpansive to multi-valued quasi-nonexpansive mappings.

   (ii) In contrast with other related methods, our algorithm does not require any estimate of some spectral radius. Our iterative scheme is proposed with a way of selecting the step-size γ_n such that its implementation does not need any prior information about the spectral radius of the operator A^*A. The constant step-size γ in the result of Kazmi and Rizvi [93], for example, depends on the spectral radius of the operator A^*A and we know that computing the spectral radius of this operator A^*A can be difficult to find at times. Therefore, our result improve and extend the result of Kazmi and Rizvi [93].

5. Tian *et al* [169] considered a class of generalized split feasibility problems for finding an element that solves a variational inequality problem such that its image under a given bounded linear operator is in the fixed point set of a nonexpansive mapping and obtained weak convergence. In Theorem 5.2.1 we propose an iterative algorithm for the class of generalized split feasibility problem for finding an element that solves a class of variational inequality problem and such that its image under a bounded linear operator is in a fixed point set of a pseudo contractive mapping. We also prove that our algorithm converges strongly to their common solutions. In all our results

the step-size is selected in such a way that its implementation does not involve the computation or an estimate of the operator norm. Hence our result improve and extend the result of Tian *et al* [169] and other results in this direction.

6. In [58], the author imposed the demi-compactness condition on the demicontractive mappings to obtain strong convergence result. Also, in [59], the author imposed the hemi-compactness condition on the multi-valued demicontractive mappings to obtain strong convergence result. However, in Theorem 6.2.1 we obtained strong convergence results without imposing these conditions on the mappings considered in our study. Hence, our results show that these conditions can be dispensed with.

7. Our result in Theorem 6.3.1 extends results for split equality monotone inclusion problem and split equality fixed point problem from the frame work of Hilbert spaces to the more general p-uniformly convex Banach spaces which are also uniformly smooth.

8. It is worthy to note that the assumption of demicompactness of the nonlinear operators in the results of Rahaman et al. [147] and Karahan [91] is too strong. Also the algorithms (6.1.5) and (6.1.6) depend on the operator norms $||A||$ and $||B||$ of the bounded linear operators A and B respectively are in general not easy to compute (or at least, estimate). Even in finite dimensions, computing the norm of bounded linear operator is a difficult task (see [87]). Hence our result in Theorem 6.4.1 extends the results of Rahaman et.al. [147] and Karahan [91] also Theorem 6.4.1 extends the results of Rahaman et.al. [147] and Karahan [91] from Hilbert space to p-uniformly convex Banach spaces which is also uniformly smooth.

7.2 Future research

Let us recall that the inertial term is based upon a discrete version of a second order dissipative dynamical system [8] and can be regarded as a procedure of speeding up the convergence properties (see, e.g.,[5] and some reference therein). Recently, there have been increasing interests in studying inertial type algorithms, see, for example, inertial forward-backward splitting methods [112], inertial Douglas-Rachford splitting method [19], inertial ADMM [20], and inertial forward-backward-forward method [21]. Some inertial algorithms for solving nonsmooth and nonconvex optimization problems have been recently studied in [17, 18]. For example, it is known that acceleration scheme developed by Nesterov improves the theoretical rate of convergence of forward-backward method from the standard $O(k^{-1})$ down to $O(k^{-2})$ and the inertial extrapolation scheme of Nesterov's accelerated forward-backward method is actually $o(k^{-2})$ rather than $O(k^{-2})$ (see [108]). These results and other related ones analyzed the convergence properties of inertial type algorithms and demonstrated their performance numerically on some imaging and data analysis problems.

In [5], Alvarez and Attouch translated the idea of the heavy ball method in [145, 146] to the setting of a general maximal monotone operator using the framework of proximal point algorithm. The resulting algorithm is called the inertial proximal point algorithm and it is written as:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n B)^{-1} y_n, \quad n \geq 1. \end{cases} \tag{7.2.1}$$

Alverez and Attouch [5] proved that under the condition

$$\sum \alpha_n \|x_n - x_{n-1}\|^2 < \infty, \tag{7.2.2}$$

the algorithm (7.2.1) converges weakly to a zero of B.
In [131] Moudafi and Oliny introduced an additional single-valued, coercive and Lipschitz continuous operator A into the inertial proximal point algorithm,

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n B)^{-1}(y_n - r_n A x_n), \quad n \geq 1. \end{cases} \tag{7.2.3}$$

Moudafi and Oliny [131] obtained a weak convergence result using their algorithm (7.2.3) and condition (7.2.2) imposed above in [5]. As remarked in [109], the algorithm (7.2.3) does not take the form of a forward-backward splitting algorithm, since operator A is still evaluated at the point x_n for $\alpha_n > 0$.

Recently, Dong, Jiang, Cholamjik and Shehu [72] studied and proved strong convergence results using a combination of Hangazeau's algorithm and Nesterov's acceleration scheme to solve the inclusion problem (4.3.1) in the frame work of real Hilbert space. They proposed the following algorithm

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = (I + r_n B)^{-1}(y_n - r_n A y_n), \\ C_n = \{u \in H : \|z_n - u\|^2 \leq \|x_n - u\|^2 - 2\alpha_n\langle x_n - u, x_{n-1} - x_n\rangle + \alpha_n^2\|x_{n-1} + x_n\|^2\}, \\ Q_n = \{u \in H : \langle u - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{7.2.4}$$

where $\alpha_n \in (0,1)$ and $r_n \in (0,\infty)$. Under some mild conditions, they prove that $\{x_n\}$ converges strongly to $x \in (A+B)^{-1}(0)$.

7.2.1 Research question

. The following questions will be at the core of future research work:

1. Can we develop iterative method with inertial extrapolation term which does not involve the construction of the sets C_n and Q_n as given in [72] and the sequence of iterates generated by this method converges strongly to a solution of $x \in (A + B)^{1-}(0) \neq \emptyset$.
2. Can we extend the result in [19] to strongly nonexpansive mappings?
3. Can we extend the results of [19] and [72] to reflexive Banach spaces with some appropriate geometric property?

4. Can we obtain strong convergence result of [19] in a real Hilbert space?.
5. Can we develop iterative method with inertial extrapolation term for most of our results in this thesis?

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