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An Exploration into Grade 11 Learners' Use of Diagrams When Solving Probability Problems in Mathematics in a School in Umlazi District in Durban

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TITLE PAGE

An Exploration into Grade 11 Learners' Use of Diagrams When Solving Probability Problems in Mathematics in a School in Umlazi District in Durban

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DEDICATION

I would like to dedicate this thesis to my entire family (my wife, my children, my parents, my brothers, my sisters and all my relatives) in appreciation of their valuable support and the understanding they showed during the hectic period of compiling this research study.

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I would like to thank my research supervisor, Prof. Vimolan Mudaly, for his outstanding supervision, guidance and motivation throughout the mammoth task of compiling this dissertation.

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Lastly, but not least, I also wish to thank my Creator, God the Almighty, for being there for me throughout the study, from beginning to the end, and most of all for giving me life. To God be the glory!

DECLARATION

I, Mfanimpela Patrick Shoba, hereby declare that the work presented in this thesis entitled An Exploration into Grade 11 Learners' Use of Diagrams When Solving Probability Problems in Mathematics in a School in Umlazi District in Durban is a representation of my own effort and is original. This thesis has never been submitted in part or full for this degree or any other degree to any other University. Where work from another source has been used, proper referencing has been given.

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SHOBA M P		
(The Researcher)	**(Signature)**	**(Date)**

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LIST OF ACRONYMS

APOS: Action, Process, Object, Schema

CAPS: Curriculum Assessment Policy Statement

DBE: Department of Basic Education

DES: Department of Education and Science

DoE: Department of Education

FET: Further Education and Training

IFI: Interpreting Figural Information

N(E): Number of Elements that constitute the Event

N(S): Total Number of elements that constitute the sample space

NCTM: National Council of Teachers of Mathematics

P(E): Probability of an Event to occur

PISA: Programme for International Student Assessment

PME: Psychology of Mathematics Education

S.A: South Africa

VP: Visual Processing

ZDP: Zone of Proximal Development

ABSTRACT

The research study was conducted in a school in Umlazi district of education in Durban in the province of KwaZulu-Natal. Grade 11 mathematics learners in this school provided data for this study. The qualitative research design was chosen for this study because the investigator wanted to capture the experiences of the participants with the aim of responding to the research questions.

Data for this study was collected using two task-based activities, two semi structured interviews, and an observation schedule. Six grade 11 mathematics learners were purposively sampled from the population of grade 11 mathematics learners. The responses of the participants were audio-taped and later transcribed by the investigator to generate the data for the study.

The data gathered from the participants through the task-based activities, the interviews, observation, and document analysis revealed that diagrams as visual representations aided grade 11 learners during probability problem solving. It was also discovered that the English language in which the probability problems were asked posed a challenge for the learners as they struggled to understand the problems. Participants understood problems that had diagrams associated with them better than probability problems that were without diagrams.

Based on the findings of the study, the use of diagrams by both educators, during teaching and setting of questions, and by learners during probability problem solving, play a critical role towards finding the correct solution. It was, therefore, recommended that diagrams as visual representations should not be separated from probability problem solving. It was also recommended that further study is required to investigate the effects of using a language that is foreign to the problem solver in probability problem solving.

CHAPTER 1

1.0 INTRODUCTION

1.1 INTRODUCTION

In South Africa, the performance of a school is judged by the performance of its grade 12 learners in the matric examinations. The performance of learners in the exit point (grade 12) is also dependent on the general performance of learners in the FET phase. Schools that perform below the 60% pass rate are labelled as under-performing schools (Shabalala, 2005). Mpela high school (not its real name) in Umlazi is one of the schools labelled as an underperforming school as it only attained a 43% pass rate in the 2016 matric examinations. One of the contributing factors to the school slow pass rate is the poor pass rate of its learners in mathematics. In 2016 the school attained a 13 % pass rate in mathematics which affected the overall pass rate for the school. The study by Diaz and De la Fuente (2007) informs us of the difficulties and the low success rate students have in solving tasks or problems involving Bayes" rule, or generally in solving conditional probability problems. The South African DBE Examination Analysis (2017) also raised the concern about most learners not attempting questions based on probability problem solving in the 2016 mathematics examinations.

In the DBE CAPS Document (2016: 55) it is noted that problem solving constitutes about 15% of the mathematics examination. Considering that probability problem solving forms the most part of mathematical problem solving in the FET phase and the fact that most learners do not attempt the probability question in the final examinations, indicates a need for a study in that area. Also considering that most of the probability concepts and diagrams used to solve probability problems are learnt in grade 11, the research was conducted with grade 11 learners to find out what diagrams they use, how they use them and why they use them during probability problem solving in a mathematics class. The purpose of this study was to help grade 11 learners to improve their probability solving skills using diagrams and hence improve their performance in mathematics with the aim of making the general pass rate of Mpela high school and other schools better.

1.2 BACKGROUND TO THE STUDY

Grittoni (2007) defines problem solving as a process one goes through when he or she pursues a task-based or goal-based activity. He describes problem solving as a skill that learners need to develop to succeed in an academic setting and that people encounter it in countless different situations. He goes on to emphasize that problem solving has a particular relevance in the mathematics classroom. There is a need for learners to have experience with different kinds of problems as well as specific knowledge to solve those problems. If learners lack that experience and knowledge this may result in the problem becoming difficult for them to solve.

Visualization is a foundation in the learning of mathematics, in that mathematics depends on what is being visualized and the spatial abilities embedded in such visualizations, (Nakin, 2003). Naidoo (2011) defines visualization as the ability to form and transfer a mental image necessary for problem solving in mathematics. As much as many researchers state that visualization is so important in the mathematics classroom, it seems that it receives less attention in the South African curriculum. According to Healey and Holeys (1996) mathematicians know what to look for in a diagram and know what can be generalized from a particular figure and so are able to employ a particular case to stand for a more general observation. This assertion also emphasizes the importance of using diagrams to solve problems. Similarly, Mudaly and Rampersad (2010) argue that the current trend in mathematical inquiry is to explore the role of visual representations which are depicted through data representations, pictures, diagrams, graphs, symbols, words and patterns in the development of mathematical thought. This study explored grade 11 learners' use of diagrams when solving problems in probability in mathematics. This study was motivated by the fact that previous examination analysis has revealed that learners do not perform well in probability problem solving (DBE Examination Analyses, 2017).

1.3 PROBLEM STATEMENT

Stylianon (2001) advocated for the need of educational research on the role of visual images in mathematical problem solving. This, therefore, shows the need for further research in this area. Presmeg (2007) also emphasizes that the need for research on visualization remains strong, not only in mathematics problem solving, but also in the interaction sphere of classroom teaching and learning of mathematics at all levels.

Probability problem-solving, in grade 11 mathematics, contributes about 10% of the entire mathematics curriculum (DoE Mathematics CAPS document, 2016). Grade 11 learners have in the past years shown signs of having challenges with mathematics problem solving in general and probability problem solving. This claim was supported by their performance in these specific mathematics problem areas in their matric examinations as captured by the S.A. examination board in their analysis (DBE Examination Analysis, 2017).

1.4 THE AIM OF THE STUDY

The aim of this study was to explore grade 11 learners' use of diagrams when solving problems in probability in mathematics.

1.5 RESEARCH OBJECTIVES

The objectives of this study are:

1. To determine the types of diagrams that grade 11 learners use when solving probability problems in mathematics.

2. To determine how grade 11 learners use these diagrams in solving probability problems in mathematics.

3. To determine the reasons for grade 11 learners using these diagrams when solving probability problems in mathematics.

1.6 RESEARCH QUESTIONS

The research will answer the following questions:

1. What diagrams do grade 11 learners use when solving probability problems in mathematics?

2. How do grade 11 learners use these diagrams when solving probability problems in mathematics?

3. Why do grade 11 learners use these diagrams when solving probability problems in mathematics?

1.7 SIGNIFICANCE OF THE STUDY

The study was aimed at exploring grade 11 learners' use of diagrams when solving problems in probability in mathematics. Stylianon (2001) advocated for educational research on the role of visual images in mathematical problem solving. This, therefore, showed the need for further research in this area. Presmeg (2007) also emphasizes that the need for research on visualization remains strong, not only in mathematics problem solving, but also in the interaction sphere of classroom teaching and learning of mathematics at all levels. The study will be useful to all educators and learners in the teaching and learning of mathematics in schools and will be of specific benefit to the teaching of probability problem solving. Educators will be encouraged to use diagrams when teaching probability problem solving in mathematics. Learners will also be encouraged to use these diagrams when solving probability problems. The use of diagrams when solving probability problems in mathematics will simplify the understanding of these problems which will eventually help learners to better their performance in mathematics.

1.8 THEORETICAL AND CONCEPTUAL FRAMEWORK

This study was influenced by the interpretivist paradigm which is supported by constructivist learning. It followed the APOS (Action, Process, Object and Schema) theory which is fundamentally a constructivist learning approach. APOS theory is a framework for the process of learning mathematics that pertains specifically to learning more mathematical concepts (Weyer, 2010) as opposed to Piaget's theory of constructivism, Vygotsky‟s theory of scaffolding and Skinner‟s theory of behavioural learning which focuses only on learning. It was an effective theory for the study since it is based on mathematics learning and the study will focus on how diagrams enhance solving probability problems in a mathematics classroom.

APOS theory is grounded in Piaget‟s notion of reflective abstraction. Dubinsky (1991) defines reflective abstraction as a construction of logico-mathematical structures by which an individual during cognitive development is able to build new knowledge. There are five kinds of reflective abstractions; interiorisation, encapsulation, coordination, reversal and generalization (Dubinsky, 1991) which can be linked to the four stages of APOS theory as illustrated by Figure 1 below. It is hypothesized that mathematical knowledge consists of an individual's tendency to deal with perceived mathematical problem situations by constructing mental actions and solving the problems (Dubinsky & McDonald, 2008). This helped to detect the level on which the learners were able to solve the mathematical problem after being enhanced by visualization.

1.9 LOCATION OF THE STUDY

The study was conducted in a school on the outskirts of Umlazi, south-west of Durban in South Africa. The target group for the study was grade 11 mathematics learners in the school. The research was carried out during the second quarter of the 2016 academic year. The school had a 100 % black African enrolment that come from rural and semirural homes around the school. Most of the learners come from relatively poor families.

1.10 RESEARCH METHODS / APPROACH TO THE STUDY

The research followed the qualitative approach using two task-based activities, and an interview with six learners chosen from grade 11 mathematics classes in the same school. Learners in grade 11 were given a task-based activity on probability to write. After completing the task-based activity, six learners were sampled by taking the two with highest score, two with the median score, and two with the lowest score from grade 11. The six learners were each interviewed using a semi-structured interview schedule to answer the research questions.

1.11 VALIDITY AND RELIABILITY AND RIGOUR

Terreblanche and Durrheim (2002) asserted that the credibility of the study is established during the research. To ensure validity of the research, a pilot study was conducted to ensure that the instruments measure what they are supposed to measure. The triangulation method was employed to ensure the reliability of the study. This was achieved using the task-based activities, the semi-structured interviews and the use of the observation schedule.

1.12 CHAPTER ORGANISATION

The dissertation was composed of six chapters. A brief outline of what constituted the six chapters is presented in the discussion that follows.

1.12.1 CHAPTER ONE: INTRODUCTION

This section set the scene for the study and began with an introduction of the study. It included information on the research problem, background to the problem, problem statement, aim of the study, research objectives, research questions, location of the study, the purpose of the study, significance of the study, outline of the research chapters and a conclusion.

1.12.2 CHAPTER TWO: LITERATURE REVIEW

In this chapter a review of the core literature on the research topic was given. It outlined and critically evaluated various works in the areas of mathematics problem solving, probability problem-solving, and the use of visual representations in mathematics education both by educators and learners.

1.12.3 CHAPTER THREE: RESEARCH METHODOLOGY

This chapter described, in detail, how the study was carried out in a way that someone else can study the methodology and replicate the study to test the validity of the original findings. The chapter included information presented under the subheadings: design of the study, sampling strategy, research data collection instruments, data analysis, the pilot study, limitations of the study, and ethical considerations. The chapter also presented the research instruments used to collect data within the study.

1.12.4 CHAPTER FOUR: THEORETICAL FRAMEWORK FOR THE STUDY

The theoretical framework (APOS theory) adapted to guide this study was discussed in this chapter. Other concepts related to the main theoretical framework were also discussed. The theoretical framework for the study and the conceptual frameworks related to the study were discussed in a manner that explained how they fitted to this research.

1.12.5 CHAPTER FIVE: RESULTS PRESENTATION, DISCUSSIONS AND INTERPRETATION OF FINDINGS

This chapter was a straightforward reporting of the findings using verbal transcriptions from the semi-structured interviews. The chapter also presented information about the variables that were used in analysing the data.

In this chapter the investigator explored important relationships among what was done in the past, the purpose of the study, and the results of the current study as depicted by the data collected. The investigator then interpreted the results and discussed their implications in relation to the research. This chapter basically married together the information collected during the literature review, the findings of the primary research that then led to the conclusions and recommendations from the findings in chapter six. The findings, analysis of results, and discussion of findings were presented under the headings of the theoretical framework. The data was presented in a way that relates to the research questions, and the literature related to the study. This discussion of findings was used as evidence for the conclusions and recommendations reached in chapter six.

1.12.6 CHAPTER SIX: CONCLUSIONS AND RECOMMENDATIONS

This chapter discussed the conclusions as drawn logically from the findings of the study. The chapter fully accommodated all the research questions and objectives. The chapter ended with a conclusion to the whole study. This is a chapter in which the reader becomes familiar with the conclusions drawn from data collected during this study. In this chapter the data analysed was discussed based on the research questions and objectives of the study. The theoretical and the conceptual framework guiding the study were also used to guide the conclusions reached in this study. Based on the findings and conclusions of the study, this chapter ended with feasible recommendations suggested for the reader.

1.13 CONCLUSION

This chapter began with a discussion of the research background, the statement of the problem, and the aim of the study. The research objectives and questions to be answered by the study were also stated to guide the focus of the study. The significance and relevance of the study are also briefly discussed. The chapter then presented a short discussion of the theoretical and conceptual framework guiding the study and some highlights of how the data was collected. The chapter then ended with discussion of the how the chapters are organized in this study. In the next chapter a discussion of the relevant related literature to the study is presented.

CHAPTER 2

2.0 REVIEW OF RELATED LITERATURE

2.1 INTRODUCTION

Grade 11 probability problem solving is very much centred on the use of Venn diagrams, contingency tables, and tree diagrams (DBE CAPS Document, 2011). According to the National Curriculum Statement (DBE CAPS Document, 2011), the topic of probability in the FET Further Education and Training Phase) starts in grade 10. The topic starts by a comparison between the relative frequencies of an experimental outcome with the theoretical probability of the outcome. This sub-topic involves using models to compare the relative frequency of events with theoretical probability. The use of Venn diagrams as an aid to solving probability problems then follows. This sub-topic deals with the use of Venn diagrams to solve probability problems, deriving and applying the following relationships for any two events A and B in a sample space S as adapted from the CAPS Document, (DBE CAPS Document, 2011: 5):

$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$

A and B are mutually exclusive if $P(A\ and\ B) = 0$; and A and B are complementary

If they are mutually exclusive; and

$P(A) + P(B) = 1$. This will imply that $P(B) = P(not\ A) = 1 - P(A)$.

In grade 11, the area of probability expands on what was covered in grade 10 probabilities by introducing independent and dependent events. The product rule for independent events $P(A\ and\ B) = P(A) \times P(B)$, is introduced. Venn diagrams, contingency tables, and tree diagrams are then used as aids in solving probability problems. This sub-topic involves using Venn diagrams, contingency tables, and tree diagrams to solve probability problems by deriving and applying formulae for any three events A, B, and C in the sample space S. Tree diagrams are also used at this level to solve probability problems dealing with consecutive or simultaneous events. The fundamental counting principle is only introduced in grade 12. From the presentation above it is clear that content coverage for grade 11 puts more emphasis on the use of diagrams to solve probability problems. The specified diagrams are the Venn diagrams, contingency tables, and tree diagrams.

Various studies have been carried out to establish the importance of using external visual representations such as diagrams to solve mathematical problems in general (Lehrer & Schauble, 1998; Hall, Bailey, & Tillman, 1997; Tversky, 2001; Mayer, Mautone, & Prothero, 2002). Prior research studies also exist in the study of using external visual representations to solve specifically probability problems. This section begins by discussing the available literature under mathematical problem solving in general. Probability problem solving and the use of diagrams in solving probability problems are also discussed. The types of diagrams specified and used in basic school probability are also reviewed. Literature on how learners learn probability is then reviewed by considering some learning theories that are associated with the learning of probability. This section ends with a review of the sub-topics that are specified for basic school probability up to the level of grade 11.

2.2 VISUALIZATION

Visualization is described as the creation of a mental image of a particular concept (Kosslyn, 1996). Based on the teaching point of view, visualization can be viewed as a powerful method to utilize for helping learners to understand a variety of concepts in many disciplines such as computer science, chemistry, physics, biology, engineering, applied statistics and mathematics. Specifically, there are many reasons that substantiate the use of visualization for learning and teaching of mathematics at all levels of schooling, from elementary to university passing through the middle and high school levels. Whiteley, (2004: 3) suggests that we learn to see; we create what we see; visual reasoning or „seeing to think‟ is learned, it can also be taught, and it is important to teach it. Arcavi (2003) defined visualization as a skill that helps learners to recognize shapes, to create new ones and to reveal relationships between them. He went on to add that visualization, imaging, or mental imagery is the ability to create, use, interpret, and reflect upon a pictorial representation, an image, or diagram in the problem solver’s head. De Koning and van der Schoot (2013) noted that these visual representations aid the problem solver’s comprehension of the problem.

Various researchers concur that visualization is a skill that can be improved through training (Werthessen, 1999; Derov & Kinsey, 2009; Yildiz, 2009). Arici (2012) further added that visualization and reasoning skills can be improved through instructional methods. This reveals the importance of teaching problem solving using visual representations to improve the skill of visualization. Visualization, mathematical problem solving, and mathematical achievement have been shown to be interrelated (Van Garderen & Montague, 2003; Jakubowski & Corey, 2009). Visualization helps the problem solver to connect the text with prior knowledge and experiences to create meaningful mind images that aid comprehension and text understanding (De Koning & van der Schoot, 2013). Educators and learners who have learned and become skilful in the use of visualization and „seeing to think" would be able to reinforce mathematical concepts and improve the teaching and learning processes in the mathematics classroom. Whiteley (2004) raised a concern about future and in-service educators of mathematics (elementary, secondary and post-secondary) that they seem surprised to learn that modern abstract and applied mathematics can be intensively visual, combining a very high level of reasoning with a solid grounding in the senses.

When solving a mathematical problem, the problem solver must begin by constructing an internal representation of the problem and then construct a mental model of the problem situation (Mayer, 1992). Visualization will play a very important role in this stage, and in the later stage that requires more pure mathematical steps (Corter & Zehner, 2007). In the area of probability, there are certain schematic diagrams that are conventionally used to visually represent specific concepts and situations. These diagrams include Venn diagrams, for compound events, tree diagrams for sequential experiments, contingency tables, possibility space diagrams, and position holders, for the counting principle (Curriculum and Assessment Policy Statement, 2011). Corter and Zahner (2007) and Russell (2000) mention the fact that learners do exhibit spontaneous use of these standard diagrams in solving probability problems.

De Koning and van der Schoot (2013) alluded to the fact that visual representations especially when supported have a positive impact towards aiding comprehension. They went on to add that external representation lessens the demand on the problem solvers" working memory and shares the demand over verbal and visual working memory. This assertion is in line with the Dual Coding Theory (Sadoski & Paivio, 2013). Leapold and Leutrier (2012) conducted a study to compare the use of visual representations to other comprehension strategies. They produced results that are in favour of the use of visual representation as a comprehension strategy. The quality of the visual representation was also discovered in this study to affect the level of immediate and long-term comprehension with richer and more accurate drawings and diagrams that lead to better comprehension.

Visualization in mathematics refers to the understanding and application of mathematical concepts using visual representations and processes presented in diagrams, computer graphics programs and physical models. Whiteley, (2004: 3) and Gooding, (2009) highlight the following characteristics of visualization in many disciplines:

- Visualization in solving problems is central to numerous fields beside mathematics such as statistics, engineering, computer science, biology and chemistry.
- Visualization is not restricted to geometry or spatially represented mathematics. All fields of mathematics can contain processes and properties that provide visual patterns and visual reasoning. Algebra and symbolic logic rely on visual form and appearance to evoke appropriate steps and comparisons.
- Visualization can simplify mathematics to students who would otherwise not understand it.

In mathematical word problem solving the type of visual representation can affect the problem solver's success. Problem solvers using schematic visual representation were discovered to be more successful than those who were using pictorial visual representations (Eden & Potter, 2010). The process of generating a visual representation is believed by many researchers to be more effective than simply viewing an already produced visual representation towards helping problem solvers to comprehend text and solving mathematical word problems (Dewolf, Van Dooren, Cimrn & Verschaffel, 2014; Berends & Van Lieshout, 2009). This implies that learners need to be taught how to produce their own visual representations as opposed to being provided with readymade visual representations in mathematics education.

The importance of visual representations is described as a strategy that helps to develop learners" literacy: "The research literature can be summarized conclusively, as a spontaneously occurring mental imagery and a natural and important part of literacy, and a mental imagery as a successful practice for comprehension" (Sadoski & Paivio, 2013: 115). They added to this assertion by stating that, "without the activation of mental representations, there can be no meaningful understanding" (Sodoski & Paivio, 2013: 50). Visualization is, therefore, one way of aiding understanding and creating meaningful understanding during mathematical problem solving and it can also be applied in several other fields such as engineering, chemistry, biology, and others. In probability, some visual representations such as Venn diagrams, tree diagrams, contingency tables and others are conventionally used to solve very challenging problems which would otherwise be difficult to solve.

2.3 TEACHING USING VISUALIZATION

Visualization has long been thought to be very useful in the process of mathematical problem solving (Hadamard, 1945). Visualizations have been of great help when the mathematics problems are especially difficult, or when the solutions need to be communicated to others (Cortner & Zahner, 2007; Schreiber, 2004; Russell, 2000). Visualization within mathematics education dates to the beginning of the 1980s when mathematics educators showed an interest in the practical challenges of teaching visualization and visualization of mathematics (David & Tomaz, 2012). Mancosu (2005) alluded to the point that at the end of the 20th century visualization in mathematics education fell into disrepute since it can be deceptive. This meant that though useful, visualization must be used with care when solving problems in mathematics.

The development of visualization techniques in the middle of the 20th century rehabilitated the epistemology of mathematics (Mancosu, 2005). Piaget and Inhelder (1971) advocated that when a person creates a spatial arrangement (like a mathematical inscription or drawing) there is a visual image in the person's mind, guiding this creation. Visualization is thus taken to involve processes of constructing and transforming both visual mental imagery and all the inscriptions of a spatial nature that may be associated within the mathematics topic being done (Presmeg, 1997b). This definition highlights that visualization involves two major aspects of spatial thinking, namely, interpreting figural information (IFI) and visual processing (VP) (Bishop, 1983 in Roth, (2004: 2)).

Visualization for solving problems in mathematics can be in different forms. These include isomorphic visualization which depicts objects and preserves the relationships between them, homomorphic visualization which involves assigning amongst elements but does not exist among their mutual relationships, analogic visualization which uses the relation of analogy between objects, and diagram visualization which intersects with all the mentioned types of visualization which includes graphs and diagrams (Fulier, 2011). Van Garderen (2006) distinguishes between two types of visualizations that are used by learners with different learning abilities in mathematics. He argues that learners with high spatial visualization ability, those who can mentally manipulate, rotate, or twist, or invert a pictorial presented stimulus object, tended to produce schematic visual presentations. Those learners with low spatial visualization ability tended to produce images that were pictorial. This view was also raised by Ryu, Chong, and Song (2007).

On the other hand, Duval (2006) argued that learners' way of seeing, observing, and noticing patterns may be informed by their everyday approach, and not by a mathematical approach. This view is also supported by Revera (2007) when he says that if algebraic generalizations are promoted through visualization, then patterns must be seen as mathematics objects and not just as everyday objects. Deliyianni, Monoyiou, Elia, Georgiou, and Zannettou (2009) had the view that the type of visualization that a problem-solver uses is dependent on his/her age and mental capability. Amongst all the debates about the importance of visualization, it is quite evident that spontaneous visual representations are very helpful in solving mathematics problems (Deliyianni et al, 2009).

The significance of visualization in mathematics education was not well emphasized in mathematical research until the 90s when qualitative research was gaining momentum (Hitt, 2002: 68). It was in 1991, at PME-15 in Assisi, Italy, that visualization in mathematics education was given significant attention as a research field. This was the first time that Imagery and Visualization was presented as a separate category in the list of topics in the proceedings, with the following ten research papers listed under this category (Antonietti & Angelini; Bakar & Tall; Bodner & Goldin; Hershkowitz, Friedlander, & Dreyfus; Lopez-Real; Mariotti; O‟Brien; Presmeg; Shama & Dreyfus; Yerushalmy & Gafni (1991), as well as three other posters. Further, two of the three plenary addresses were directed specifically to this topic (Dorfler, 1991; Dreyfus, 1991). His theory included mental image schemata with their "concrete carriers" such as diagrams as well as protocols of action. Of relevance for visualization were the four kinds of image schemata that he propounded. The four image schemata pronounced by Dorfler are Figurative (purely perceptive), Operative (operates on/with the carrier), Relational (transformation of concrete carrier), and Symbolic image schemata (e.g., formulae with symbols and spatial relations).

Research on visualization has also revealed that learners encounter difficulties with the use of visual representations. This difficulty has been associated with the fact that pattern and dynamic imagery are more apt to be coupled with rigorous analytical thought processes, and this may mean that students are likely to generate visual images but fail to use them for analytical reasoning (Presmeg, 1985). The way the teacher facilities and encourages the learners to use visualization in the mathematics class is also critical in mathematics education. These classroom practices include a nonessential pictorial presentation by the teacher. This means the use of the teacher's own imagery as indicated by gesture or by spatial inscriptions such as arrows in algebraic work, conscious attempts by the teacher to facilitate students‟ construction and use of imagery (either stationary or dynamic), teacher's requesting students to use the motor component of imagery in arm, finger, or body movements, teaching with manipulatives, teacher's use of colour, and finally, teaching that is not guided by rule. These include the use of pattern-seeking procedures, encouragement of students‟ use of intuition, delayed use of symbolism, and deliberate creation of cognitive conflict in learners in the mathematics class (Presmeg, 1991).

The way the curriculum is designed also plays a part in the use of visualization in mathematics. With the growing awareness of the advantages of using visualization in mathematics, technology and computer packages such as Dynamic Geometers Sketch Pad and GeoGebra have also emerged as a dynamic form of visualization that is handy in the teaching of mathematics (Kidman, 2002). The use of dynamic computer packages has made visualization not limited to topics such as geometry and statistics but has also made visualization applicable even in areas such as algebra (Friedlander & Arcavi, 2005). There is some lacuna though in some aspects of visualization such as the extent to which it helps or hinders mathematical learning (White & Mitchelmore, 2003; Pitta Pantazi, Gray, & Christou, 2004). Visualization in mathematics problem solving helps by transforming the mathematical problem into a formal image. This created image helps the learner to understand some aspects of the problem that would have been inaccessible using other approaches.

2.4 MATHEMATICAL PROBLEM SOLVING

Problem solving is defined by Grittoni (2007) as the process through which one goes as he or she pursues a particular task. Polya (1948) defines mathematical problem solving as a practical art like swimming which can be learnt and perfected only by going into the water. This, therefore, suggests that to perfect the art of problem solving one has to solve problems. According to Polya (1948) this art of problem solving involves four steps which are namely: understanding the problem, devising a plan for solving the problem, implementing the plan, and looking back and reflecting on the solution. These four stages which are not necessarily linear but interactive are outlined in the diagram presented in Figure 1 that follows below.

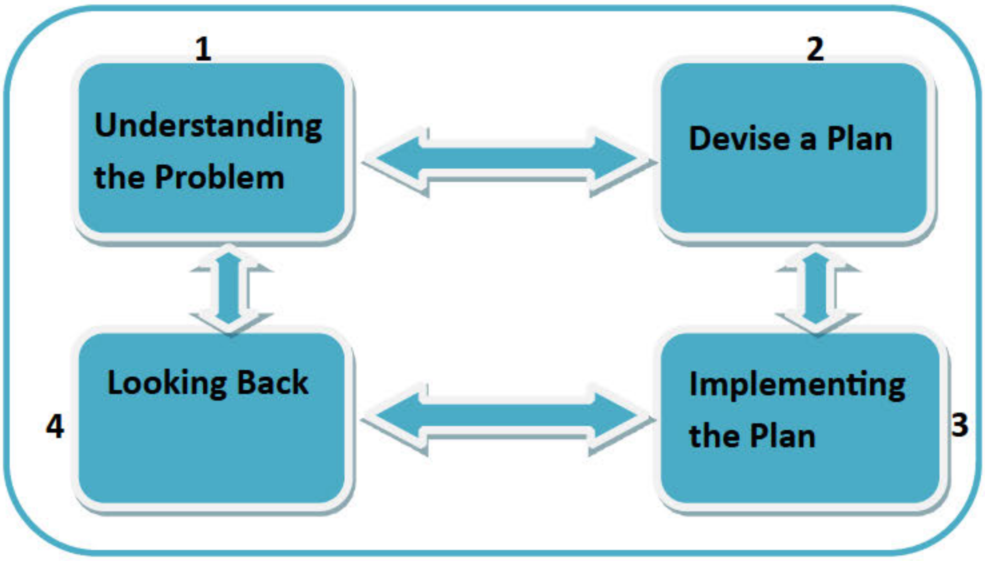


Figure 1: Pólya's Problem Techniques (Adapted from Polya, 1948: 13)

The literature on mathematical problem solving is covered by several different authors who studied and proposed cognitive theories or frameworks that explain the process of mathematical problem solving (Hanich, Jordan, Kaplan, & Dick, (2001); Corter & Zahner, (2007); Jitendra, Griffin, Deatline-Buchman, & Sczesniak, (2007); Montague & Applegate, (1993); Prayitno, Subanji, & Muksar (2016); Prayitno, Sutawidjaja, Subanji, & Muksar (2014)). Mayer"s (1992: 462) model of mathematical problem-solving outlines five different types of knowledge that a problem solver needs to solve a mathematics word problem. These types include:

- linguistic knowledge, which is the student's knowledge of language,
- semantic knowledge, a student's general knowledge of facts about the world (including knowledge about mathematics),
- schematic knowledge, a student's knowledge of the problem topic and the ability to categorize the problem (either correctly or incorrectly) into a particular problem type,
- strategic knowledge, which is a student's knowledge of how to use the various types of available knowledge in generating, planning, and monitoring the solution of problems, such as setting sub-goals.
- and procedural knowledge, or the knowledge of how to perform a sequence of mathematical operations.

The Stepwise processing model of mathematics problem solving was also proposed by Reusser (1996) that involved five consecutive stages. These stages include, constructing a propositional representation of the problem, creating a situational model, transforming the situation model into a formal mathematical representation, applying the operations to calculate the solution, and interpreting the solution in a meaningful way. All the models of probability problem solving can be linked to Pólya's problem-solving model of understanding the problem, devising a strategy, implementing the strategy, and reflecting after obtaining the solution. What can be noted about these models is that they all emphasize the first step as understanding the problem. This stage highlights the challenge posed by language in mathematics problem solving in general and probability problem solving to be specific.

Corter and Zahner (2007) also provided evidence that students follow a sequence of problem activities as they to try solving mathematics word problems. Initially, the problem solver would try to understand the problem. This step would involve understanding the language used in the problem statement and building a mental model of the problem. The problem solver would then try to model the problem in mathematical terms. This would involve relating the problem to previous and familiar mathematical problems and related formulae. After that, the process would proceed by the problem solver developing a plan for solving the problem. The process of problem solving would then end by the execution of the devised plan. This final step involves a sub-step of checking the solution for plausibility. This process of problem solving as described above is represented in the diagram that follows in Figure 2.

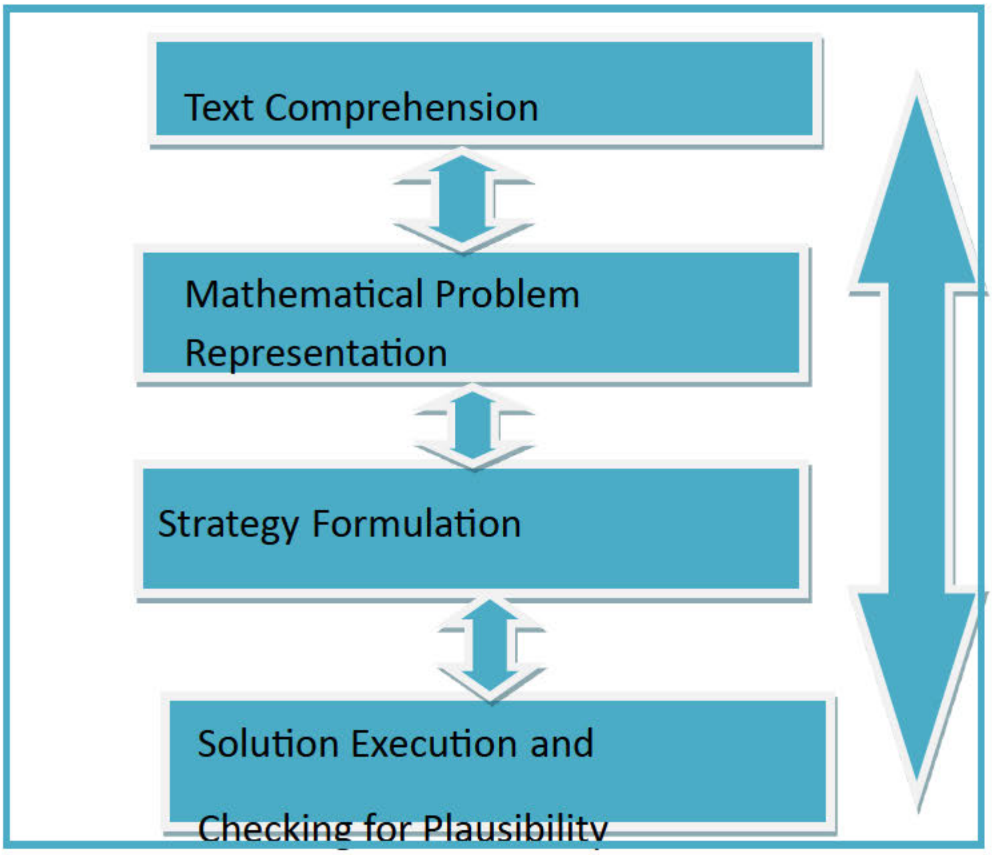


Figure 2: Mathematical Problem-Solving Process (Adapted from Corter & Zahner, 2007: 15)

Maccini and Gagnon (2002) noted that traditional methods of teaching mathematics focused more on operational calculations than on thinking and problem-solving activities. Newer approaches emphasized the importance of conceptual knowledge. Operational knowledge according to Maccini and Gagnon is concerned with how to use the knowledge rather than the rationale behind using the knowledge. Conceptual knowledge on the other hand is about comprehension. Mathematical problem solving is one example of a scientific method that focuses on conceptual learning. In mathematical problem-solving individuals have to follow cognitive processes such as understanding the problem, devising a plan for solving the problem, implementing the plan, and deciding whether the problem has been successfully solved or not.

Noting the difficulty associated with predicting the challenges that learners are going to meet in their professional and general lives, it therefore, becomes imperative that students should be equipped with problem solving skills to cope with any challenge they may encounter. Problem solving will enable the student to think in an advanced way to be able to solve any problem or challenge with which they are faced (Van Garderen, 2007). Garderen went further to highlight that visual representation of information is considered an effective process in mathematics education and explained by the contribution it must developing understanding and institutional perspectives. Students tend to face difficulties when trying to solve mathematical word problems if they cannot relate the known to the unknown. This gap can be bridged using visualization to simulate the student's thinking as compared to focusing on the symbols and the language (Lavy, 2007; Boonen, Van Wessel, Jolles, & Van der Schoot, 2014). Visualization, therefore, makes communication of mathematical ideas easy as it simplifies the language hence making the problem understandable. Countries that do well in mathematics such as Japan, Singapore, and Malaysia have curricula that promote problem solving using visualization techniques (Ho & Lowries, 2014). The success of these countries in mathematics clearly shows the benefits of using visualization techniques in mathematical problem solving.

It is, however, worth noting that mathematics problem solving is a very hectic task to students. Students do not feel at ease solving mathematics problems and, as such, lose interest in mathematics (Hamidreza, Nor, Mohamed, Osman & Suhaimi, 2015). The challenges with solving mathematics problems arise because of the student lacking in the basic skills required to solve the problems (Tambabychik & Meerah, 2010). Solving mathematics problems can be simplified by using visualization. Using visual representations allows the problem solver to link the problem with real life situations (Uesak, Mnalo, & Ichikawa, 2007). Mathematics problem solving becomes difficult for students if they cannot relate the known to the unknown (Boonen, Van der Schoot, Wesel, deVies, & Jolles, 2013). Visualization comes as a powerful tool that can be used to facilitate the comprehension of the mathematics language in problem solving (Lavy, 2007; Van Garderen, 2006). In countries that perform better in mathematics such as Singapore and Japan, visualization is placed at the centre of mathematics problem solving in order to encourage creativity and critical thinking (Beckmann, 2004; Murata, 2008). Students in these countries are trained and encouraged using visual representation to connect ideas in mathematical problem solving (Ho & Lowrie, 2014).

2.5 LANGUAGE AND MATHEMATICAL PROBLEM SOLVING

Communication is generally an important factor in building understanding. The language used in teaching and learning of mathematical problem solving is, therefore, an important factor to be considered. Radford (2003: 124) stresses that language is very important for learning as human beings are homo dialogicus. He continues to say we are currently living in a paradigm of language. Lakatos (1976) echoes the same sentiments by identifying language as the heart of mathematical activity. All these views highlight the importance of language in the teaching and learning of mathematics and mathematics problem solving. These views make it worthwhile to study the impact that the language of teaching and learning and the language used in asking question has on mathematical problem solving in general and probability problem solving in particular.

Non-native learners are more likely to have academic difficulties in general and more particularly with mathematical word problems which could be caused partially by their low command of the language of instruction (Gross, Hudson & Price, 2009; Kempert, Saalbach, & Hardy, 2011; PISA Report, 2012). According to Bernardo and Calleja (2005) learners‟ proficiency in language of instruction can be the most determining factor of the learners‟ mathematical understanding. Language of instruction is, therefore, thought to play a very important and crucial role in mathematical learning. As the instructional language improves, the performance of a learner in mathematics also improves (Van Rinsveld, Brunner, Landerl, Schiltz, & Ugen, 2015).

The language of instruction is very likely to be a contributing factor to the performance of the learner in mathematics, especially if the learner has a poor command of the language (Hudson & Price, 2009; Saalbach & Hardy, 2011; PISA Report, 2012). Other authors have also come out to support the view that proficiency in the language of instruction determines the performance of the learner in mathematics (Bernardo &

Calleja, 2005). This may be even more evident in the case of solving mathematical word problems. Probability problem solving by its nature is classified under mathematical word problems and, therefore, also influenced by the proficiency of the learner in the language of instruction.

Marian and Kanshanskaya (2007) advocated that bilingual learner responded differently to knowledge questions relating to language. In probability the content Knowledge of the learner may also differ depending on the language proficiency. In the South African context, the learners may refer to the club in a pack of playing cards as a fly which tends to confuse them when dealing with probability problems relating to playing cards. Such and similar language differences and variations relating to content knowledge tends to influence the learner's performance in solving probability word problems. Learners may then have to transfer knowledge from one language context to the other before solving the problem which becomes a disadvantage to them as second language users in mathematics (Saalbach, Eckstein, Andri, Hobi, & Grabner, 2013). A learner's mathematical performance may also improve as the learner's proficiency in the language improves (Van Risveld, Brunner, Landerl, Shiltz, Ugen, 2015).

Kyttala and Bjorn (2014) noted that it is not only mathematical knowledge or computational skills that affect mathematical problem solving but the language used in the problem and the ability to comprehend the problem are also contributing factors. Durkin (1991) explains that language and mathematics cannot be separated by saying:
"Mathematics education begins and proceeds in language, it advances and stumbles because of language, and its outcomes are often assessed in language" (Durkin, 1991: 1). Sousa (2008) also noted that human brains process numbers in different parts of the brain when they are in symbols and when they are in words. This implies that mathematical word problems are first converted to number form before being processed. This implies that an extra level of processing is demanded for solving mathematical word problems. Hipwell and Klenowski (2011) also alluded to the point that it is very important that educators recognize, understand, and acknowledge the language demand of all assessments as it affects the achievement of learners.

Saalbach et al. (2013) emphasized that mathematical word problem solving belongs to learned contents that are not transferable from one language to the other without incurring cognitive costs. Learners may end up dropping in their mathematical word problem solving abilities due to problems presented in their non-native language which they have not l mastered well (Kempert, Saalbach, & Hardy, 2011; Gross, Hudson, & Price, 2009). Landerl, Bevan, and Butterworth (2004) suggested that it is vital that any education professional faced with learners with language difficulties such as non-native learners understand and assists them. They further emphasized that many learners with language difficulties will have an increased risk of poor acquisition in mathematics word problem solving.

Problem solvers often try to solve word problems by using a direct translation approach of picking the numbers used and performing operations on them without considering the context of the problem (Kajimes, Vauras & Kinnunen, 2010). This strategy can only work for less complex word problem problems. The way the words are organized and phrased to create meaning (semantic structure) has a big impact on the problem solvers‟ ability to solve word problems and on the strategy, they choose for solving the problem (Thevonot, 2010). The link between reading comprehension and reading ability has also been identified as being problematic regarding mathematical word problem solving (Kyttala & Bjorn, 2014). The use of words in different contexts such as unknown vocabulary, unfamiliar context, and common words used differently may lead to new challenges for mathematical word problem solvers (Benjamin, 2011). Language in mathematical word problem solving can be a challenge for problem solvers as outlined in this discussion. Other challenges emanating from language in mathematical word problem solving are listed here as noted by various researchers:

- Comprehension of the problem (Fuch & Fuch, 2007)
- Linguistic challenge (Abedi & Lord, 2010)
- Wording, length, and grammatic complexity (Fuch & Fuch, 2007)
- Words used in unfamiliar context (Sousa, 2011)
- Synonymous word (Sousa, 2011)
- Complex problem layout involving word comprehension, diagrams, and tables (Gee, 2008).

It is, however, worth noting that in spite of all these challenges caused by language outline here, various studies have been able to identify ways of making mathematical word problem solving accessible to learners. Some of these recommendations are listed here as noted by various researchers:

- Using the learner's first language when setting questions (Sousa, 2011)
- Simplifying the language demand of mathematics problem (Abeedi & Lord, 2010)
- Having specific interventions for word problem solving skills (Swanson, Bocian, Lussier, and Zhang, 2012).

It is, however, worth noting that the recommendations listed here are not solutions to all language related challenges in mathematical word problem solving. Learners need to improve their mathematical word problem solving skills to be able to solve different problems in formal assessments.

Jimenez (2013) highlights the issue of inner speech that bilingual learners undergo when solving mathematical word problems. They translate the given problem into their dominant native language for better understanding and back to the less mastered nonnative language of instruction. Their dominant native language plays an important self-regulating role in thinking out the problem while the other non-dominant second language provides an extra set of cognitive resources and strategies that are visited when needed. Switching from the non-native language to the native language does not only help the learner solve the mathematical problem but also helps the learner to improve in language proficiency (Jimenez, 2013).

There is an obvious link between text comprehension and mathematical word problem solving skills because they both stem from a set of shared cognitive problem-solving abilities (Passolunghi & Pazzaglia, 2005). There is also a clear link between reading comprehension and mathematical skills (Harlaar, Kovas, Dale, Petrill, & Plomin, 2012). Text comprehension through text reading skills in language helps learners to master mathematical word problems skills (Kyttala & Bjorn, 2014). This general view that the level of proficiency in the language of instruction in mathematics affects the learner's mathematical problem-solving ability cannot, therefore, be underestimated in the case of probability problem solving.

2.6 VISUALIZATION IN TEACHING PROBABILITY PROBLEM SOLVING

Mathematics problem solving can be quite difficult for learners although the mathematics involved is quite simple (Garfield & Ahlgren, 1988, O'Connell, 1993, Konold, 1989). Garfield and Ahlgren (1988) and Konold (1988) suggest that probability problem solving

may be especially difficult because people have natural misconceptions about the concept of probability. Mosteller (1980) and Konold (1989) assert that probability word problems may present a unique challenge to the problem solver due to the difficulties experienced in probabilistic reasoning and the abstract nature of the material. Several other authors have come up with recommendations on how to teach the concept of probability (Bantanero, Godino, & Roa, 2004; Keeler & Steinhorst, 2001; Gelman & Nolan, 2002). In all the recommendations external visualizations seem to be possible solutions as they simplify probability problems. Corter and Zahner (2007) assert that learners can spontaneously create external visual presentations during the process of solving probability problems. They further add that external visualization, and external inscriptions may play a role during problem solving by aiding text comprehension. Due to its abstractness, Russell (2000) emphasizes the need to use external visual representations such as diagrams to solve the probability problems.
O'Connell and Corter (1993) and Corter and Zahner (2007) also provided evidence that students follow a sequence of problem-solving activities as they solve probability word problems.

External visual presentations are categorized into schematic (diagrams) and iconic (pictures) presentations. Cortner and Zahner (2007) distinguish between schematic and iconic presentations by highlighting that schematic presentation are those that depict the relationship described in the problem, while iconic presentations are those that depict the physical appearance of elements described in the problem. The use of schematic presentations has been found to lead to a higher success rate in mathematics problem solving than the use of iconic presentations (Hargarty & Kozhhevnikov, 1999). In the area of probability, the schematic presentations that are conventionally used to represent important concepts are Venn diagrams for compound events, outcome trees for sequential experiments, and contingency tables.

Presmeg (2006), Polya (1957) and English (1997) also pointed out that some areas of mathematics cannot be separated from the use of pictures and diagrams to solve them. Probability is one such area where the problem solver may rely on both internal visualization and external sketches and diagrams to solve problems. The external visualizations help the problem solver to understand the problem (Corter & Zahmar, 2007). In probability, the external visualizations can help the problem solver at a later stage when the mathematical and probabilistic concepts are required. Other external inscriptions such as symbols, formulae, and mathematical calculations may be employed to work-out the required probability. Kaufmann (1990), Schwartz and Martin (2004), Sytlianon and Silver (2004) highlighted the vast evidence that exists to emphasize the importance of external visualizations, especially diagrams, in facilitating the process of solving problems in both mathematics and science. Other recent studies on the spontaneous use of external visual presentations in mathematics problem solving have also emphasized the importance of external presentations in facilitating the problem-solving process (deHevia & Spelke, 2009, Eden & Porter, 2008).

Generally, in the probability problem solving, the problem solver's initial efforts are directed at making efforts to understand the problem text and then build a mental model of the problem. Problem solvers then attempt to represent the problem in mathematical terms and possibly relate the current problem to familiar mathematical formulae and/or previously encountered problems. After that, the problem solver proceeds to develop a plan for solving the problem. Finally, the problem solver executes the chosen strategy.
A final sub-step that sometimes does and sometimes does not occur is to check the solution for plausibility. The problem-solving models discussed above can all be aligned to Polya‟s problem solving model in that they all highlight the importance of understanding the problem, devising a plan, implementing the plan, and reflecting on the problem and solution (Passolunghi & Pazzaglia, 2005).

Sedlmeier (2000) advocated that common fallacies in reasoning about conditional probabilities may be simplified by using visual representations. Visualization in probability problem solving involves internal visual representation and external visual representation. Although problem solvers normally spontaneously create external visual representation during problem solving, representations provided by the instructor also aid in simplifying the problem (Chance & Garfield, 2002). The importance of visualization in probability problem solving cannot, therefore be doubted as it is key to how the problem solver understands the probability problem whether the visual representation is provided by the instructor, or it is created by the problem solver.

2.7 ROLE OF VISUAL REPRESENTATIONS IN PROBABILITY PROBLEM SOLVING.

Visualization has been singled out as one of the techniques that assist problem solvers in making decisions under uncertainty in mathematics education (Speigelhalter & Gage, 2014). One reason for this special role of visualization in mathematics education is that it can relate the theoretical problem to real life situations. Ellis, Cokely, Ghazal, and Garcia-Retamero (2014) however, raised the point that though visualization is so important in relating theory to real life situations, it is susceptible to errors and misunderstandings. The problem solver may misunderstand the problem or may make errors when translating the problem from theory to a visual model. These errors according to Binder, Krauss, and Bruckmaier (2015) can be minimized by representing the problem using different forms of visualization models.

Existing forms of visual representations cover a very wide spectrum of applications in mathematics problem-solving. These visualization forms are applied as a powerful and even artistic means of expressing information by many problem-solvers (Vande Moere & Purchase, 2011). Klerk, Verbert and Duval (2014: 4) investigated how visualization forms are used to aid learning. They structured their investigation around the following five basic activities of the learning process:

- To find: How can visualization add value when learners or teachers are searching for relevant information about a problem or topic?
- To understand: How visualization can facilitate better understanding of the content matter?
- To collaborate: How visualization can support collaboration amongst learners and with teachers?
- To reflect: How visualization can help learners to reflect on how they are doing in a particular course or topic?
- To design: How visualization can facilitate the design of learning experiences by teachers for learners to understand?

The role played by visualization in making sense of information across all fields cannot be questioned (Hear & Shneiderman, 2012). The most important intent of visualization is to represent abstract information in a dynamic way that facilitates human interaction for exploration and understanding (Fekete, Van Wyk, Stasko & North, 2008). In mathematics visualization also plays an important role in that it enables learners to see the unseen in data. Arcavi (3002: 223) argues that:

> "Visualization has a powerful complementary role for mathematics students in three aspects: (a) support and illustration of essentially symbolic results, (b) a positive way of solving conflict between correct symbolic solutions and incorrect intuition and (c) as a way of helping us re-engage with and recover conceptual underpinnings which may be easily bypassed by formal solutions".

Visualization is believed to be aiding discovery in mathematics problem solving (Polya, 1957). Pantziara, Gagatsis, and Elias (2009) assert that to the expert mathematician, visual representations may become the most useful tool for exploring non-routine problems. They also add that inventing novel visual representations is part of being creative in mathematics. As indicated previously, probability problem solving can also be classified under non-routine problems. This, therefore, implies that visualization can be of great value during probability problem solving in mathematics. Other studies on the spontaneous use of external visual representations in mathematics problem solving in general suggests that these visual representations are facilitative in the problem-solving process (de Hevia & Spelke, 2009; Edens & Potter, 2008; Uesaka, Manolo, & Ichikawa, 2007). Though this suggestion was based on mathematics problem solving in general, it can also be associated with probability since probability also falls in the problem-solving domain.

Presmeg (2016: 69) illustrates that visualization can play a productive role in mathematical problem solving, both in earlier and later stages of the process. She considered three purposes used by a pair of participants in her study to discover, generalize, and to communicate a rule. She concluded that the mere presence of visual representation and manipulatives does not necessarily guarantee the correct solutions. She suggested that the visual representation that students make should be one that enables them to visualize, test, and examine their mathematical approaches. Her suggestion highlights the importance of reflective thinking in general mathematics problem solving and in probability problem solving to be specific. Arzarello, Paola, Robutti, and Sabena (2009) suggest that different forms of visualization can be used for different roles in mathematical problem solving. This, therefore, explains why some problems in probability can be solved using say a tree diagram while other cannot but require the use of other forms of diagrams to solve them. Presmeg (2014) echoes the same sentiments by pointing out the different types of visualization that are effective for different purposes in mathematical problem solving, which may also include generalization.

The learning process according to Hamidreza et al. (2015) is divided into three levels which are inactive, iconic and symbolic. The inactive level is the most crucial since it is where visualization can perform the role of connecting the practices and the formal levels of understanding and acts as a mediator of communication (Deliyianni et al, 2009). This statement clearly underlines the crucial role played by visual presentations in aiding understanding in the learning process. The diagrams, pictures, or graphs that the student constructs to facilitate comprehension of the problem will automatically generate a big picture in the student's mind that can enhance the solution to the problem (Deliyianni et al., 2009). According to Brown and Wheatley (1997), visual representations also help the student to establish an effective way of solving any problem syntactically, semantically, and pragmatically. A well-constructed visual representation makes a good perspective of meaningful use of visuals to show semantic perspective, while the pictorial signs are used to think, communicate, learn and give a pragmatic perspective (Schnotzi, 2002). Carney and Levin (2002) outlined five functions of pictures regarding text processing in the pragmatic perspective. These are briefly outlined in the table that follows in Figure 3 as transformational pictures, organizational pictures, decorative pictures, representational pictures and interpretational pictures.

Pragmatic Representation	Brief Meaning
Transformational Pictures	A form of mnemonic system that helps the student to recall text information
Organizational Pictures	A structural framework that is usable for the content information
Decorative Pictures	It can be used for designing pages that are related to content information
Representational Pictures	It is used for illustrating part of or the whole content information
Interpretational Pictures	It is used for comprehending of the difficult information in the text

Figure 3: Adapted from Hamidreza et al. (2015: 23)

The information in the table explains the importance of using visual representations in mathematics problem-solving in general and in probability problem-solving. Schnotz (2002) highlighted that students tend to remember information that is presented visually more than information that is presented in words. It is for this reason that expert mathematicians tend to use visual representations more than novice mathematicians to solve mathematical problems (Styliannou & Slver, 2004). The different pragmatic presentations also explain the use of different visual representations when solving different probability problems. Some questions would be better simplified using a Venn diagram, others by using a probability tree diagram or by using a contingency table, while others would be simplified by using position holders.

2.8 FORMS OF VISUAL REPRESENTATIONS

Decision making under uncertainty is an important subject in mathematics education (Spiegelhalter & Gage, 2014). The reason for this is that decision making under uncertainty requires accurate interpretation of the problem. Elliis, Cokely, Ghazal, and Garcia-Retamero (2014) highlighted that the task of making decisions under uncertainty is always prone to errors and misunderstandings. In order to mitigate this challenge different approaches such as using different kinds of visualization forms become a necessity (Blinder, Krauss, & Bruckmaier, 2015). These visualization forms can be used for different purposes in mathematical problems and are suited for simplifying different problems.

Different authors raise various but converging views about what constitute visual representations in mathematics. Besides diagrams, pictures, tables, and graphs, other authors argue that mathematical and arithmetic inscriptions also have visual-spatial aspects as well as symbolic content (Kirshner & Awtry, 2004; Landy & Goldstone, 2007; Presmeg, 1986). This assertion, therefore, suggests that though the common visual representations in probability problem solving are the probability tree diagram, Venn diagrams, contingency tables, the fundamental probability formula and all the symbols associated with probability can also be viewed as visual representations. Diagrams such as the contingency table which is some form of statistical presentation, and the probability tree diagram can be used to simplify mathematical problems with the purpose of minimizing error and misunderstanding (Bocherer-Linders, Eichler, & Vogel, 2015).

Corter & Zahner (2007) classify the forms of visual representations used in probability problem solving into three categories. Under the category of schematic visual representations are Venn diagrams, trees, contingency tables, and novel schematic representations. Spatial reorganization of given information and outcome listings are considered as forms of tabulation. Finally, they classify pictures under the iconic representations of problem elements. Students solving mathematics problems tend to create other types of external inscriptions as well as such equations and computations. The formal language of mathematics used in the question also becomes a form of visualization that affects the comprehension of the question.

2.9 TEACHING OF PROBABILITY PROBLEM SOLVING INTERNATIONALLY

Internationally, it is more than fifteen years since probability became a strand in the curriculum of mainstream in elementary school (Australian Education Council [AEC], 1991; Department of Education and Science and the Welsh Office [DES], 1989; National Council of Teachers of Mathematics [NCTM], 1989). Although research and classroom teaching experience during all these years have produced a prodigious knowledge base on the teaching and learning of probability, there are still plenty of challenges related to what probability should be taught and how probability should be taught to optimize its understanding to learners. During the period before probability became a mainstream strand, probability was only taught, if at all, at the high school level.

The high school curriculum in probability was part of the mathematics program and mainly comprised concepts that had a major emphasis on counting and combinations. With the coming of the curriculum reforms in the late eighties and early nineties, probability did not only commence in the early elementary school, it also focused on key themes that were intended to spiral across the grade levels. In analysing emerging probability curricula of this period, Jones, Langrall and Mooney (2007) identified key themes that percolated through the elementary, middle, and high school curricula.

In the elementary school curriculum, key concepts included the exploration and description of random phenomena arising socially and from random generators, and the representation and ordering of random outcomes. This ordering process involved both experimental estimation and symmetry-based measures of probability. In the middle school, the key ideas of the elementary school curriculum were treated more formally with more precise representations of the sample space, empirical estimations and theoretical measures of the probabilities of events, and some inclusion of compound events and independent events. At the high school level, the coverage of probability was extended to cover random variables, discrete and continuous distributions, and the use of these distributions in modelling random phenomena. Probability distributions were also used to support work in statistical inference.

Researchers and educators of probability have through the years highlighted some of the challenges associated with the teaching and learning of probability (Pratt, 2005; Polaki, 2005). Amongst other things, they highlight the language used and the context of the questions in relation to the learners as key to the understanding of probability questions. Learners and teachers of probability may, because of this challenge, be compelled to devise ways of simplifying probability problems to make them understandable. One such way could be to use visual representations such as diagrams and tables.

Visualization as a form of spatial ability has an indispensable advantage in the learning of probability problem solving. Seker, and Yilmaz (2011) alluded to the fact that human beings are likely to have more than one learning style and the learning styles vary from one person to the other. This implies that there is not one teaching and learning style that can effectively fit all situations. Visualization, therefore, comes as a technique that is meant to aid learners‟‟ understanding in probability problem solving. Spatial ability has been identified to play a crucial role in developing creativity in learners (Kell, Lubinski, Benbow, & Steiger (2013). This role of visualization implies that students with high spatial ability are more likely to be more creative than those with low spatial ability.

Okamoto, Weckbacher, and Hallowell (2014) also held the view when they reached the conclusion that visualization ability is linked to performance in mathematics. This implies that learners‟‟ performance in mathematics can be boosted by teaching them to use visual tools. This can help them to develop their spatial ability and hence boost their general performance in mathematics. According to Aineamani (2011) learners' way of communicating their mathematical reasoning is informed by their learning experiences in class and the activities they did in class with the teacher. The activities and the questions that the teacher uses in class influence the mathematical thinking of the learners. In another perspective male learners are said to possess a higher spatial ability than their female counterparts (Yazici, 2014). This view highlights that there is a need for teachers to provide more assistance to female learners to develop their spatial abilities to improve their general mathematical performance.

2.10 TEACHING OF PROBABILITY PROBLEM SOLVING NATIONALLY

The concept of probability is very important in making learners understand their social environment and to be able to calculate and determine various chances for specific events to occur or not. It is also a topic that is prescribed by the Department of Education in the Learning Programme Guidelines of the subject Mathematical Literacy. The learner is required to be able to collect, summarize, display and analyse data and apply knowledge of statistics and probability to communicate, justify, predict and critically interrogate findings and draw conclusion (DoE 2008: 11).

In South Africa, probability was introduced as a topic of study in mathematics for the first time in the senior phase (grades 7-9) in 1992 (Laridon, 1995). Since its inception, it has been possible for teachers in these middle years to omit the topic, and focus instead on number, algebra and geometry. It is in the more recent years, after the introduction of Curriculum 2005, that data handling has been included as a required outcome. All schools now face Common Tasks of Assessment for Grade 9, and these include assessment of probability concepts and skills. According to research, the formal tuition that was offered to learners in the implementation of the South African curriculum had little effect on the learners' probabilistic thinking. This could indicate that the teaching was not all that effective in developing understanding of probability concepts above that generally were attained by these subjects through their everyday experience (Laridon, 1995: 26).

Mathematics problem solving is enhanced by visualization. A problem in mathematics is better understood and conceptualized based on what is being visualized and the spatial abilities embedded in such visualizations (Nakin, 2003). Naidoo (2011) defines visualization as the ability to form and transfer a mental image into visual representation necessary for solving a mathematics problem. Polya also emphasizes the importance of visual representation for problem solving. He states that solving a problem involves first to understand the problem, and secondly, to come up with a plan to solve the problem. The planning stage may involve drawing a diagram, a picture, a graph, a table or any other form of visual representation (Polya, 2014). The last step in Pólya's problem solving model involves comparing the solution to the original problem and may help the problem solver to decide on whether to revise the visualization model or to maintain the solutions.

2.12 CONCLUSION

As much as many researchers state that visualization is so important in mathematics classrooms, it seems that it receives less attention in the South African curriculum. According to Healey and Holeys (1996) mathematicians know what to look for in a diagram know what can be generalized from a particular figure and so are able to employ a particular case to stand for a more general observation. This assertion also emphasizes the importance of using visual representations to solve problems in mathematics. Similarly, Mudaly and Rampersad (2010) argue that the current trend in mathematical inquiry is to explore the role of visual representations which are depicted through data representations, pictures, diagrams, graphs, symbols, words and patterns in the development of mathematical thought. Although the common visual representations in probability problem solving are the probability tree diagram, Venn diagram, and the contingency table, the fundamental probability formula and all the symbols associated with probability can also be viewed as visual representations (Kirshner & Awtry, 2004; Landy & Goldstone, 2007; Presmeg, 1986).

CHAPTER 3

3.0 THEORETICAL FRAMEWORK

3.1 INTRODUCTION

In this chapter the theoretical framework adopted for the study was discussed. The previous chapter looked at the literature related to the use of visual representations in probability problem solving and highlighted the theory informing this research. The theoretical framework for this research stemmed from the ideas of Piaget concerning reflective abstraction and the theory of constructivism. The chapter began with a discussion of the constructivist paradigm of trying to understand how mathematics is conceptualized (the construction of knowledge in mathematics). The APOS theory was then discussed as emanating from the constructivist paradigm.

The APOS theory as the theoretical framework for this research was then discussed together with examples of previous research conducted using the same theory. The concept of reflective abstraction as advocated by Dubinsky (1991a) for studying mathematical thinking was also discussed next. The APOS theory was also discussed and linked with probability problem solving. Finally, the APOS theory was explained as a mechanism that provided the theoretical framework for the presentation and analysis of data in the subsequent chapters.

3.2 CONSTRUCTION OF MATHEMATICAL KNOWLEDGE

The search for understanding how learners learn mathematics and how the teacher promotes a creative learning environment are central to the constructivist debate (Simon, 2013). Simon (2013:96) went on to note that:

> *Traditionally teaching and curriculum development were based upon showing and telling students what they were to learn (spoon feeding). The assumption was that motivated students would take in the knowledge shared by the teacher and incorporate it into their mathematical knowledge. Although this approach is still the most frequently used, it has been discredited as a primary approach to mathematics instruction because the results have not been good. With the loss of*

confidence in direct instruction came a loss in clarity in the teacher's role in promoting student learning.

This assertion clearly acknowledges the importance of both conceptual and structural knowledge in the quest to understand what gives meaning to one's actions and understanding of mathematics. Researchers such as Prabhu (2016) and Czanorcha, Prabhu, Dias, and Baker (2016) also highlighted the importance of inspiring creativity with both the gifted and the non-gifted and underserved students in the quest to understand how they learn mathematics. Shriki (2011: 73) suggested that creativity enriched teaching should be made part of all students learning without segregation in terms of ability:

It is widely agreed that mathematics students of all levels of learning abilities should be exposed to thinking creatively and flexibly about mathematical concepts and ideas. In consideration of this, teachers must be able to design and implement learning environments that support the development of mathematical creativity.

Prabhu and Czanocha (2014) and Czarnocha et al. (2016) assert that students get stuck when solving mathematics problems not because they are not able to do it but because they have had limited exposure that can allow then to access their own creativity. This view, therefore, suggests that with the appropriate positive exposure learners of all learning abilities can solve mathematical problems.

The main reason for the existence of mathematics is to solve problems (Halmos, 1980). Mathematics is, therefore, concerned with problems and solutions. According to Zimmermann (2003) mathematics is beyond memorizing algorithmic procedure but is about problem solving. According to Mayer (1985) mathematical problem solving involves two major phases: problem representations and problem solutions. Mayer further asserts that problem representation is subdivided into two sub-stages: problem translation, which is about the use of linguistic skills to comprehend what the problem is about, and problem integration, which is about the use of mathematical understanding to interpret the relationships among the problem parts to form a structured presentation. This second sub-stage is synonymous with visualization of the problem. The mathematical problem at this stage can either be represented using mathematical symbols or using diagrams.

Problem solution is a more general phase, in that it is composed of solution planning, strategy selection and carrying out the strategy to find the solution to the problem. The complexity of the stages of mathematical problem solving explains why learners of all ages struggle when it comes to mathematical problem solving. The correctness of the solution normally depends on the accuracy of the stages of problem solving (Jitendra, Griffin, Deatline-Buchman, and Sczesniak, 2007).

Through research on cognitive strategy instruction, Montague, Enders, and Dietz, have validated seven cognitive processes which support successful problem solving and can be placed sequentially into Mayer's model (Montague, 1992; Montague, 1997; Montague & Applegate, 1993; Montague & Applegate, 2000; Montague & Bos, 1986; Montague, Enders, & Dietz, 2009). Montague has developed these processes into a program for solving mathematics word problems called Solve It! (Montague, 2003). To further conceptualize the steps of the problem-solving process and build from this basic framework, Montague's model has been integrated with Mayer's model to illustrate both the phases of the process and the functions required to carry them out. Specifically, problem translation consists of students reading the problem for understanding and then paraphrasing the problem in their own words. Problem integration is when the student visualizes the problem by making a schematic representation. The student then hypothesizes or decides to solve the problem and estimates a reasonable answer during the solution planning stage. The final stage, solution execution, is when students compute or do the arithmetic, and then check to make sure everything is correct.

In general, available literature provides several cognitive theories or frameworks to explain the process of mathematical problem solving. An example is Mayer's (1992) model of mathematical problem solving which specifies five different types of knowledge that a problem solver goes through in the process of problem solving. These types of knowledge are:

- linguistic knowledge, which deals with the solver's knowledge of the language used,
- semantic knowledge, which is about the problem solver's knowledge of the world and mathematics,

- schematic knowledge, which is concerned with the problem solver's knowledge of the problem topic and the ability to categorize the problem,
- strategic knowledge, which is about the problem solver's knowledge of using various types of available knowledge to generate, plan, and monitor the solution of the problem,
- and procedural knowledge, which deals with the problem solver's knowledge of how to perform a sequence of mathematical operations.

3.3 THEORETICAL FRAMEWORK

Lesser (2010: 72) uses Eisenhart's (1992) words in defining conceptual framework as: *An argument that the concepts chosen for investigation, and any anticipated relationships between the concepts will be appropriate and useful given the research problem under investigation. Like theoretical frameworks, conceptual frameworks are based on previous research, but conceptual frameworks are built from an array of current and possibly far-ranging sources.*

In this study the investigator studied how learners make meaning for themselves in mathematics probability solving. The constructivist paradigm was used as a framework for understanding how learners learn probability problem solving. The constructivist paradigm was linked to Vygotsky's (1997) theory of instruction and concept development. Vygosky strongly focused on the role of education in developing a concept in mathematics. He observed that some students could solve certain problems with some assistance while others could not (Vygosky, 1997). Vygosky labelled this phenomenon the learner's zone of proximal development (ZDP).

Piaget and Campbell (2001) also created learning theories about the way cognitive structures are developed and used by learners during mathematical problem solving and they noted the following:

> *Knowledge is basically operative; it is knowledge of what to do with something under certain circumstances. For Piaget, operative knowledge consists of cognitive structures. If knowledge consists of cognitive structures, then development comes down to what structures need to be developed (p. 2).*

Glasersfeld (1995: 65) in his description of an action schema highlighted three parts which are recognition of the situation, association of specific activity with the problem to be solved, and expectation of a certain result. In his learning theory, Piaget uses the term assimilation to describe how a learner recognizes a situation and associates it with an existing schema. He goes on to say that the learner then accommodates the new schema to his or her cognitive structure. When the learner finally reaches equilibrium between the old schema and the new knowledge, it would then lead to the development of a new schema and a development of the cognitive structure (Gallaher & Reid, 2002: 2). Piaget and Garcia (1989) described the mechanism of processing knowledge and generalizing actions to modify the original schema as reflective abstraction: *First, projection essentially establishes correspondences at the next higher level, associating the old contents that may be integrated within the original structure but permitting it to be generalized. Second, these first organizations also lead to the discovery of related contents, which may not be directly assimilated into the original structure. This makes it necessary to transform that structure by means of a constructive process until it becomes integrated within a larger, and therefore partially novel, structure. This mode of construction, by reflective abstraction and constructive generalization, repeats itself indefinitely, at each successive level so that cognitive development is the result of the interaction of a single mechanism.*

Other authors such as Simon et al. (2004: 314) do not fully support Piaget's notion of reflective abstraction and argue that it is not clear about its applicability in the classroom environment:

> *The postulation of reflective abstraction was a significant contribution... describing the kind of process that can derive more advanced structures from those at a lower level... Recent efforts to ground pedagogy in Piagetian theory have been hampered, in our estimation, by the lack of a sufficiently elaborated mechanism for explaining mathematics conceptual development.*

The theoretical framework that is used in this study that is discussed above can also be better summarised in the diagram that is presented in Figure 4 as adopted from Mudaly (2012) and, Mudaly and Rampersad (2010).

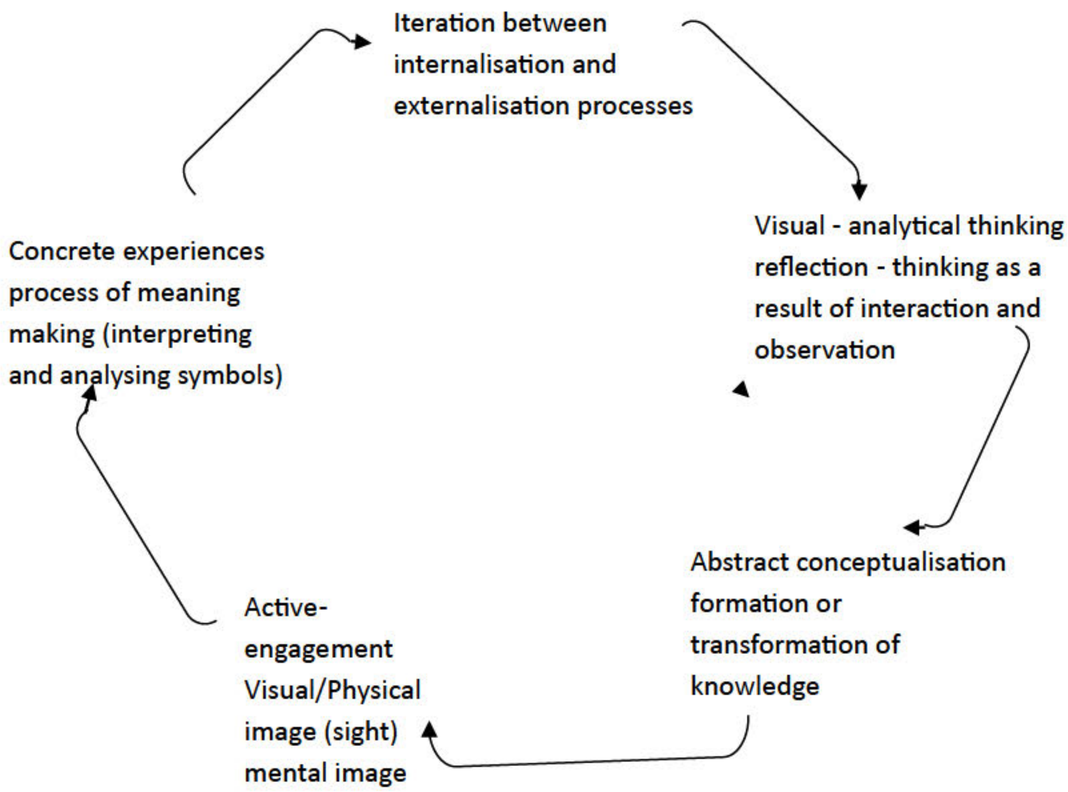


Figure 4: *The process of acquiring new or transforming old knowledge (Adapted from Mudaly (2012) &, Mudaly and Rampersad (2010)).*

For this study the APOS theory by Dudinsky (1991) was employed. Dubinsky studied Piaget's theory of reflective abstraction and made some improvements on it. He further described the theory of reflective abstraction by adding other processes such as interiorization of knowledge, coordination, generalization of processes or actions, reversibility and encapsulation. This model of concept development is referred to as APOS theory or Action-Process-Object-Schema theory and is briefly described by Arnon, Cottrill, Dubinsky, Okta, Fuentes, Trigueros, Weller (2014:19):

> *According to Piaget and adopted by APOS theory, a concept is first conceived as an action, that is, as an externally directed transformation of a previously conceived object or objects. An action is external in the sense that each step of the transformation needs to be performed explicitly and guided by external instruction additionally, each step prompts the next, that is the steps of the action cannot be imagined and none can be skipped.*

3.4 THE APOS THEORY

As noted previously, this study was influenced by the interpretivist paradigm which is supported by constructivist learning. It followed the APOS (Action, Process, Object and Schema) theory which is fundamentally a constructivist learning approach. APOS theory is a framework for the process of learning mathematics that pertains specifically to learning more mathematical concepts (Weyer, 2010) as opposed to Piaget's theory of constructivism, Vygotsky's theory of scaffolding and Skinner's theory of behavioural learning which focuses only on learning. It was an effective theory for the study since it is based on mathematics learning and the study will focus on how diagrams enhance solving probability problems in a mathematics classroom.

APOS theory is grounded on Piaget's notion of reflective abstraction. Dubinsky (1991) and Arnon et al. (2014) define reflective abstraction as construction of logic mathematical structures by which an individual during cognitive development can build new knowledge. There are five kinds of reflective abstractions; exteriorisation, encapsulation, coordination, reversal and generalization (Dubinsky, 1991; Arnon et al., 2014) which can be linked to the four stages of APOS theory as illustrated in Figure 1 above. It is hypothesized that mathematical knowledge consists of an individual"s tendency to deal with perceived mathematical problem situations by constructing mental actions and solving the problems (Dubinsky & McDonald, 2008). Thus, this will help detect the level on which the learners are able to solve the mathematical problem after being enhanced by visualization.

According to Weller, Arnon and Dubinsky (2009) a problem solver goes through four stages of cognitive development in the process of problem solving. The stages are action stage, which is about the individual being able to react to external stimuli, process stage, which is about the individual being able to make a mental model of the problem, object stage, which is about the problem solver being aware of the whole process of solving the problem, and schema stage, which is about the problem solver being able to perform all the steps of working-out the solution to the problem. Dubinsky and McDonald (2001) propose that a learner must have the appropriate mental structures to make sense of a given mathematical concept. The APOS theory is about actions, processes, objects, and schemas that are required by a learner to learn a mathematical concept. This theory advocates that the existing mental structures need to be established and then the necessary appropriate learning activity be designed to support the development of the learner to the next mental structure.

The APOS theory and its applications in teaching and learning are based on the general hypothesis developed to understand Jean Piaget's ideas. Piaget's investigations were conducted with both adolescents and adults to study their thinking. He discovered common central structures and mechanisms that guide concept acquisition (Piaget, 1970; Arnon et al., 2014). The APOS theory is based on two major assumptions which are briefly described below:

- **Assumption on mathematical knowledge**: APOS theory assumes that a learner's mathematical knowledge is determined by his/ her response to mathematical problems and their solutions. This involves how the learner reflects on the mathematical knowledge in a social context. It also involves how the learner constructs and reconstructs mental structures to be used in solving similar mathematical problems.

- **Hypothesis on learning**: This assumption posits that mathematics is not learnt directly but using mental structures to make sense of the intended concept (Piaget, 1964). This implies that learning only occurs if the learner possesses the right mental structures to assist him/her to understand the mathematical concept being taught. This assumption suggests that without the required mental structures for learning a particular concept, learning it cannot be possible.

The basis of the APOS theory described in the preceding paragraph implies that teaching should begin by establishing what mental structures the learner possesses. The process of teaching should then consist of establishing strategies for helping learners to build the relevant mental structures and guiding them to use these mental structures to construct and reconstruct knowledge of the mathematical concept. According to Dubinsky (2010), the mental structures are developed through the processes of exteriorization and encapsulation. APOS theory involves the mental structures, actions, processes, objects, and schemas. The mental structures that build up the APOS theory are briefly described in the discussion that follows as adapted from Weller, Arnon, and Dubinsky (2009) and Arnon et al. (2014).

- **Action**: This mental structure advocates that learning is first conceived as an action. Learning is viewed as a response to stimuli that an individual perceives as external, requires specific teaching and the need to perform each step correctly. A learner who possesses this mental structure can perform each step of the transformation explicitly after specific instructions. In the case of probability problem solving, the learner who has been specifically instructed to complete a contingency table can do so by using the available information and the totals in each row or column. In the case of a tree diagram, the learner would be able to use the available information on the tree diagram to complete it by following steps explicitly. The action mental structure may also be evident when a learner is able to substitute values into the probability formula, $n(E) = \frac{n(E)}{n(S)}$, to work-out the required probability. The learner may then fail to go further than this to using the completed diagram to solve further probability problems such as condition probability and independency of events.

- **Process**: Process as a mental structure develops as a learner repeat and reflects on the actions performed in the first stage. In the process mental structure, the learner basically performs the same operations as in the action mental structure, but the operations are performed wholly in the mind of the learner. The learner at this stage can imagine the procedures and steps involved in the mind without having to execute each step explicitly. A learner who has developed the process mental structure would be able to calculate simple probability without having to substitute values in the probability formula but by doing the substitution in the mind and then writing only the answer. The learner at this stage of cognitive development can work-out some missing values in a contingency table, a Venn diagram, or a tree diagram mentally without writing the steps explicitly.

- **Object**: A learner who has developed the object mental structure becomes aware of the processes as a totality. The learner realizes that the transformations can act on that totality and can construct and reconstruct the transformations explicitly in one's imagination. A learner at this stage is said have encapsulated the process into a cognitive object. A learner who has developed the object mental structure would be able to decide on the probability diagram that would be most suitable to simplify a given probability problem. The learner would be able to draw and complete the diagram and use it to solve that probability problem.

- **Schema**: Probability as a mathematical topic involves actions, processes, and objects that need to be organized and linked into a coherent framework called a schema. A learner who has developed a schema for probability in mathematics is able to decide when presented with a particular probability problem whether probability as a schema is applicable to solve it. For example, the coherence in probability problem solving might lie in understanding that probability can be applied to make predictions about the future. A learner who has developed the probability schema is able to use a given trend about the weather condition (sunny, rainy, or windy) and the mode of transport used by a worker (drive, cycle) to predict the number of days the worker would use a certain mode of transport, say a car, to work in a particular year.

In this example, the learner could begin by deciding to draw a tree diagram to represent the given information, then use the tree diagram to work-out the probability that the worker would, for example, drive to work if it is sunny. The learner may then be able to use the total number of days in that particular year together with the probability that the worker would drive on a sunny day to predict the number of days the worker would drive to work during that particular year.

The descriptions of the mental structures that constitute the APOS theory discussed above can be influenced by other external factors. A learner, for example, may be in the process stage of cognitive development but due to lack of managerial strategies, flexibility, prompts, emotions and other factors fail to use that capability (Dubinsky & McDonald, 2001). The use of the APOS theory in this study would, therefore, be to provide the basis on which data will be categorized and analysed.

3.5 CONCLUSION

The APOS theory as a learning framework was discussed based on the constructivist paradigm. Its relevance to the study was also explained. The APOS learning theory was linked to the learning of probability problem solving as applied to the probability problems of the task-based activity used to gather data for this study. In the next chapter the methodology and the procedure for conducting the study were discussed. This included the discussion of the research design, research philosophy and the research strategy. The population and the selection of the sample were also discussed. The next chapter further outlined the research instruments and outlined the issues related to their management such as piloting and administration of the research instruments. In addition, the limitations associated with the study were recognized and then the chapter was concluded with a brief discussion of the ethical considerations for the study.

CHAPTER 4

4.0 RESEARCH METHODOLOGY AND PROCEDURE

4.1 INTRODUCTION

The previous chapter provided an overview of the APOS theory as the theoretical framework guiding the study. This chapter began by a reintroduction of the critical research questions. The research design, research philosophy and the research strategy were also discussed as were the population and the selection of the sample for the study. This chapter further presented the research instruments and outlined the issues related to their management such as piloting and administration of the research instruments. In addition, the chapter then outlined the limitations governing the study and then concluded with a brief discussion of the ethical considerations for the study.

4.2 THE RESEARCH DESIGN

McMillan and Schumacher (2010) state that qualitative research is the kind of approach that produces findings arrived at from real- world settings where the phenomenon of interest unfolds naturally. In support of that, De Gialdino (2009) defines qualitative research as one that constitutes different orientations and approaches, various intellectual and disciplinary traditions grounded, often in different philosophical assumptions. In a broader spectrum, Denzin and Lincoln (2004) define qualitative research as a multi-method which is about encompassing an interpretive and naturalistic approach to its subject matter. The authors further emphasize that in a quantitative research, things are studied in their natural settings, attempting to make sense of, or interpret phenomena, in terms of the meanings people perceive. Creswell (2008) describes qualitative research as one that addresses a research problem that requires exploration, of a problem about which little is known. He adds that qualitative research is appropriate for a detailed understanding of the central phenomenon and seeks to understand the experiences of the participants from their original setting.

McMillan and Schumacher (2010) on the other hand also point out the advantages of using the quantitative research design. One of this approach's advantages is that it uses a larger sample which makes it more representative of the population. Both approaches are effective depending on the nature of the study. These two approaches are not a substitute for each other but rather complement each other (Creswell, 2008). To understand the central phenomenon of this study, the study followed the qualitative research approach. This approach was chosen to be able to get an in-depth understanding of phenomenon from the respondents in their natural set-up.

The qualitative descriptive approach involves identifying the characteristics of the phenomena or exploring possible correlations among two or more variables. Qualitative methodology was appropriate for this study because, as Rossman and Rallis (1998: 29) have noted, "there are few truths that constitute universal knowledge; rather, there are multiple perspectives about the world". By interviewing the learners personally using a semi-structured interview schedule, better strategies of using diagrams to solve probability problem were unearthed.

The case study research strategy was employed for this research. According to Patton, (2002) and Gomm, Hammersley and Forster (2000) case studies are widely used because they may offer insights that might not be achieved with other approaches and that they are advantageous as they allow for the identification of a group of individuals with diverse experiences. Hancock and Algzzine (2006) suggest that the case study design should be considered if the focus is to answer "how and why" questions. This design was, therefore, relevant as it was able to reveal to the investigator the participants' experiences and it was able to respond to the research questions.

The research followed the qualitative approach using two task-based activities, and two semi-structured interviews with six learners chosen from grade eleven mathematics classes in Mpela high school (not its real name). Learners in grade eleven were given the first task-based activity (Appendix A) comprising of three probability questions to answer for the purpose of selecting a sample. The three questions were such that each question required a different probability diagram to solve. I (the investigator) hand delivered the questions to the 80 grade 11 learners and supervised then while they were answering the questions. While the learners were busy writing, I was observing how they responded to the probability problems using an observation schedule (Appendix E).

After completing the three question of task-based activity 1, I sampled six learners by taking two with highest scores, two with the median scores, and two with the lowest scores from all the 80 learners to sit for the second task-based activity. I then began by interviewing the six learners about what diagrams they used, how they used the diagrams, and why the used those diagrams to solve the probability problems given in task-based activity 1. The interview was conducted using a semi-structured interview schedule (Appendix C) by the researcher. The participants (the grade 11 learners) were audio taped using an electronic sound recorder and their responses were later transcribed by the researcher (Appendix J). The purpose of the interview was to establish their views and approaches in solving the probability problems.

A second task-based activity (Appendix B) was issued by the researcher to each of the six grade 11 learners to answer. This activity consisted of 10 probability questions constructed to cater for the four levels of the APOS theory. While the participants were responding to the probability problem, I was supervising them and observing and recording their activities during the session using the observation schedule. On finishing the second task-based activity, I personally interviewed the participants about how they responded to the probability problems using a second interview schedule (Appendix D). I again audio taped their responses using an electronic sound recorder and later transcribed interview responses as shown in Appendix J. All the papers that were used by the respondents were collected by the researcher for analysis and safe keeping.

4.3 TARGET POPULATION AND SAMPLE SELECTION

There are basically two types of sampling: probability sampling and non-probability sampling. In probability sampling the sample is selected such that the probability of a subject being selected is known. All members of the population have equal chances of being selected and hence bias is greatly minimized. The subjects in probability sampling are representative of the whole population hence the findings from such a sample can be generalized to the whole population (Creswell, 2008). Non-probability sampling, on the hand, is not about selecting participants randomly, but they are rather selected based on specific characteristics, accessibility, and relevance to the study. Non-probability sampling can be categorized into convenience sampling, purposeful sampling, and quota sampling. This type of sampling is most suitable for qualitative research (McMillan& Schumacher, 2010). It was for this reason that the purposeful non-probability sampling method was chosen for this study.

As noted in McMillan and Schumacher (2010) that purposeful non-probability sampling is appropriate for collecting qualitative data it was chosen for this study since the study was of the qualitative research design. This sampling method also allowed for the selection of participants based on characteristics, relevance, and accessibility to the study. The learners selected for the study were those relevant to the study in that they were doing grade 11 at the time and they had done probability problem solving.

In McMillan and Schumacher (2010) the population is described as a group of elements or cases that are in line with criteria, to which we intend to generalize the results of the study. The population for this study was made up of the grade 11 mathematics learners of Mpela high school. The school has three grade 11 mathematics classes with a total of 80 learners. They were appropriate for the study because probability problem solving is done mostly in grade 11 and they were available for this study. A sample is described as a group of subjects or participants from whom the data are collected (McMillan & Schumacher, 2010).

4.4 THE RESEARCH INSTRUMENTS

The research instruments for this study included task-based activity 1 (Appendix A) which was made up of three probability questions, task-based activity 2 (Appendix B) which comprised of 10 probability problems, a semi-structured interview schedule for activity 1 (Appendix C), a semi-structured interview schedule for activity 2 (Appendix D), and an observation Schedule (Appendix E).

4.5 PILOT STUDY

Terreblanche and Durrheim (2002) asserted that the credibility of the study is established during the research. Altrichter, Feldman, Posch, and Somekh (2008) advocated that the credibility of the study is established during the research. To ensure the validity of the research, a pilot study was conducted by the investigator to ensure that the instruments measured what they were intended to measure. According to McMillan and Schumacher, (2010) a pilot study is about testing the research instruments with the participants and is important in ensuring that the research instrument contains items that are clear to the respondent and are related to the focus of study. Creswell (2008) defines a pilot study as the procedure in which the researcher makes changes in an instrument based on feedback from a small number of individuals who completed and evaluated the instrument.

The pilot study was conducted with three learners, from the three grade 11 mathematics classes, to ensure the reliability of the instruments. One learner was chosen from each class since he or she was able to give reliable feedback as he or she was from the same settings of the classes that took part in the study. The investigator personally delivered the papers for the two task-based activities and supervised the three learners as they responded to the probability problems. The investigator also personally interviewed the three learners using the semi-structured interview schedule and audio taped their responses using an electronic sound recorder.

4.6 DATA COLLECTION

The triangulation method was employed to ensure the reliability of the study. This was achieved using more than one research instrument. Two task-based activities, two interview schedules, document analysis and observation schedule were used for this study. The use of more than one method to collect data helped to cross-check the results obtained. Cohen and Manion (2000: 37) defined triangulation as the attempt to outline or explain fully, the richness and complexity of human behaviour by capturing data from more than one standpoint. The different data sources helped to represent the views of the participants from different stand points.

Altrichter, Feldman, Posch, and Somekh (2008: 46) defined triangulation as a means of giving a more detailed and balanced picture of the phenomenon. Denzil (2006: 31) highlighted four types of triangulation: (1) Data triangulation which is about time, space, and persons, (2) Investigator triangulation which advocates for more than one researcher in an investigation, (3) Theory triangulation which is about the use of more than one theory in interpreting the results, and (4) Methodological triangulation which is about using more than one method to gather data.

The investigator personally administered the first task-based activities to all the grade 11 mathematics learners. Their responses were marked and then a total of six learners were chosen by selecting two with low scores, two with middle scores, and two with high scores. The six sampled learners were personally interviewed using a semi-structured interview schedule by the investigator. Their responses were audio taped and later transcribed. The investigator then administered the second task-based activity to the six learners. The investigator then interviewed the six participants using another semi-structured interview schedule. Their responses were audio taped using a sound recorder. The participants" responses for both interviews were later transcribed. During the answering of both task-based activities the activities of the respondents were observed using an observation schedule to generate a different perspective of the data for this research. Data was also generated by analysing documents and through studying related literature.

4.7 DATA ANALYSIS

The data that was collected using an audio recorder from the two interviews were personally transcribed into written form by the investigator. The transcribed data from both interviews and from the observation schedule were analysed by the investigator using inductive analysis. According to McMillan and Schumacher (2010: 367) inductive analysis is the process of making meaning from the raw data by synthesizing it, starting with specific data and ending with categories and patterns. This process involved coding, categorizing, and interpreting the data to provide explanations to the fundamental phenomenon of the study. The investigator was using the data from the transcriptions of the audio taped responses and casual observations to synthesize and make meaning related to the research phenomenon without the assistance of any statistical software.

4.8 LIMITATIONS OF THE STUDY

Since the study followed the interpretive paradigm, the small size of the sample limited the scope of generalizing the research findings to other grade 11 mathematics learners and to other schools. Another limitation to the study was the fact that the questions were in the English language which is foreign to the learners who all Zulu speaking Africans were. This as such limited their understanding of the questions. This was even evident in some of their responses to the interview questions. Another limitation to the study was that the learners were from different backgrounds in terms of their former schools and different teachers. This meant that they have different experiences and varying conceptions of the concept of solving probability problems.

4.9 ETHICAL CONSIDERATIONS

A letter in Appendix A for requesting permission from the head-teacher at the school (school principal) to conduct the research was personally delivered to Mr W. M. Mbhele by the investigator. The school principal granted permission for this study and the permission letter is included as Appendix B. Informed consent forms, as shown in appendix E, were also issued by the researcher to all learners and their parents to read and sign prior to taking part in the research. The investigator used pseudonyms for all participants and places in the study to protect their identity. This was done to ensure confidentiality and anonymity for the protection of all parties involved in the study. At the end of the study participants will be invited to an interactive discussion workshop where the findings will be disseminated and discussed.

4.10 CONCLUSION

This chapter basically discussed all the details related to the methodology by which the data was generated for this research. It began by a discussion of the qualitative research design that was used for this study. The target population and the sample selection procedure were also discussed. The research instruments used to collect the data for this study were then mentioned and briefly discussed. This is followed by a discussion of how the pilot study was conducted to test the instruments for reliability and validity. A detailed discussion of how the data was generated and collected was then presented. This was followed by a discussion of how the data was analysed to address the research phenomenon. The limitations associated with the study are also acknowledged to avoid issues of bias. Lastly, the chapter ended with a discussion of the steps taken towards ensuring that the study was ethically compliant.

CHAPTER 5

5.0 DATA PRESENTATION, DISCUSSION, AND ANALYSIS

5.1 INTRODUCTION

In this chapter all the data that were collected using task-based activities, semi structured interviews, and observation schedule are presented and analysed to understand the perspectives of the participants regarding grade 11 learners' use of diagrams when solving problems in probability in mathematics. The presentation of the data begins with the questions from the two task-based activities and then continues with a discussion of and analysis of how the participants responded to the probability questions. After the discussion and analysis of the data collected from the two task-based activities, the responses of the participants to two semi-structured interviews are presented, discussed and analysed. The data generated from the observation schedule is presented last with its discussion and analysis in relation to the research questions. The questions that the participants were responding to, and their responses are discussed below.

5.2 TASK-BASED ACTIVITY ONE

This task-based activity contained three probability problems. The first problem (refer to Figure 5) required respondents to begin by drawing a probability tree diagram, filling-in the probabilities and then using it to compute the probabilities. Filling-in the probabilities required the respondents to decide whether the items were replaced or not after each draw. This problem was about a person picking up items from a lunch box and eating them. Eating the item meant that there was no replacement after the first draw. Respondents were required to deduce that there was no replacement by understanding and analysing the language. Computing the probability for each case required the respondents to use the formula for probability, $n(E) = \frac{n(E)}{n(S)}$. The respondents also had to substitute in the formula using correct values for each problem.

Question 1

[redacted]

item of food and eats it. He then chooses another item at random and eats it. He then chooses a third item and eats it. Find the probability that he will:

1.1 first choose a banana, then a sandwich and then another sandwich.

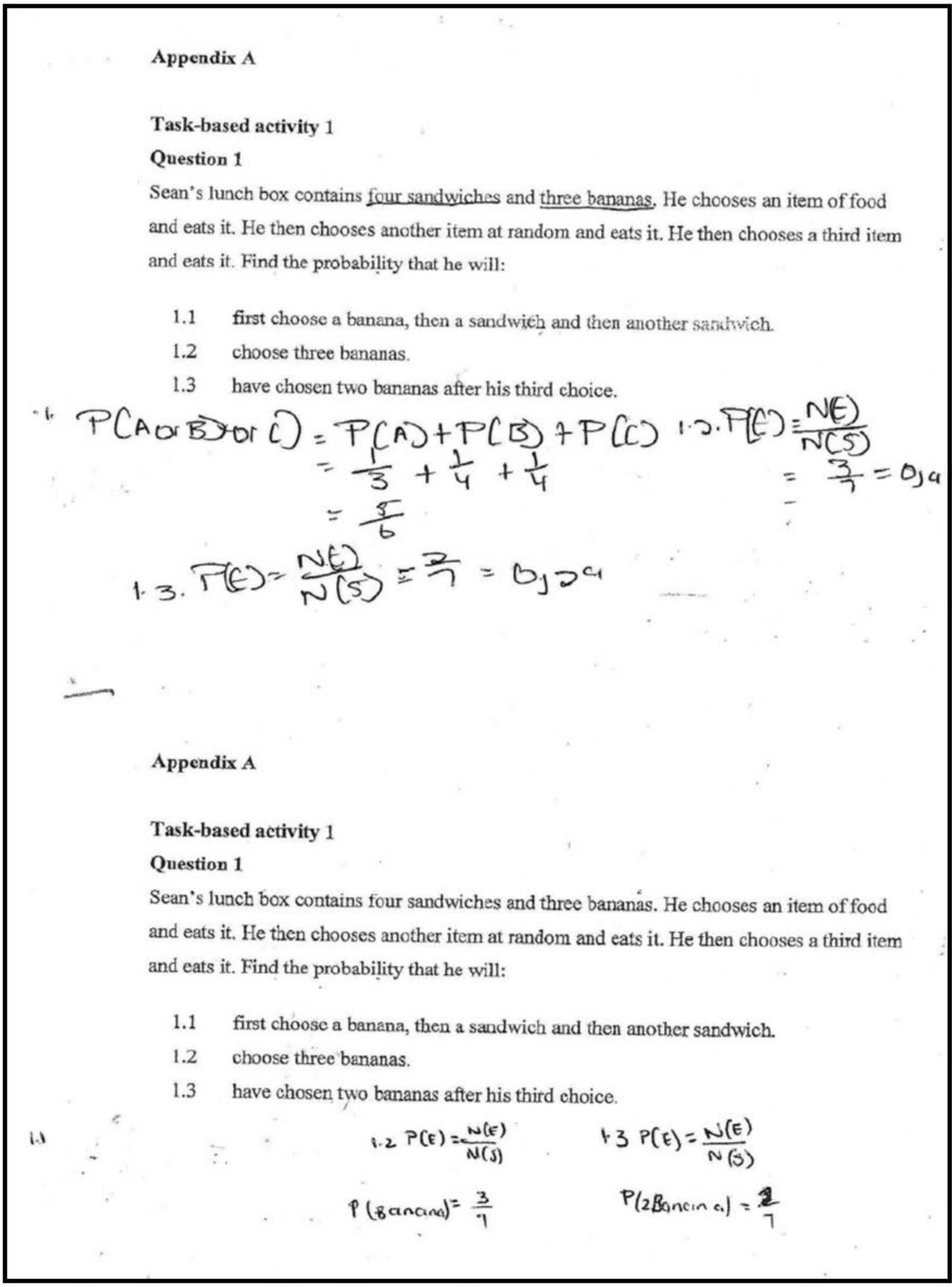
1.2 choose three bananas.

1.3 have chosen two bananas after his third choice.

Figure 5: Question 1 of Task-Based Activity 1.

As stated above, this question required the respondents to draw a probability tree diagram to represent the given information. They then had to complete the probability diagram before they could use it to determine the required probabilities by substituting values in the probability formula. Completing the tree diagram required the respondents to understand the English language so that they could use the fact that the items were eaten to deduce that there was no replacement. The respondents did not realize that there was no replacement. This challenge can be linked to the issue of English being a second language. All five respondents did not use a diagram to determine the probabilities. They claimed that they did not realize that the problem required the use of a tree diagram.

Three of the respondents also did not state or use the probability formula but only wrote wrong probabilities without basing their solutions on any calculation. This was an indication that the respondents had not fully developed the action mental structure of the APOS mental framework in probability. Learners who have fully developed the action mental structure in probability are expected to be able to fill-in the correct values in the tree diagram and substitute the correct values in the probability formula. The other two respondents did state the formula but failed to substitute using the correct values; hence they also got wrong solutions. These respondents appeared to have partly developed the action mental structure as they were able to write the probability formula but substituted using wrong values. Some examples of the learners" responses to the probability problem are presented in Figure 6.



Appendix A

Task-based activity 1

Question 1

Sean's lunch box contains four sandwiches and three bananas. He chooses an item of food and eats it. He then chooses another item at random and eats it. He then chooses a third item and eats it. Find the probability that he will:

1.1 first choose a banana, then a sandwich and then another sandwich.

1.2 choose three bananas.

1.3 have chosen two bananas after his third choice.

1.1 $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{5}{6}$

1.2 $P(E) = \frac{N(E)}{N(S)} = \frac{3}{7} = 0{,}43$

1.3 $P(E) = \frac{N(E)}{N(S)} = \frac{2}{7} = 0{,}29$

Appendix A

Task-based activity 1

Question 1

Sean's lunch box contains four sandwiches and three bananas. He chooses an item of food and eats it. He then chooses another item at random and eats it. He then chooses a third item and eats it. Find the probability that he will:

1.1 first choose a banana, then a sandwich and then another sandwich.

1.2 choose three bananas.

1.3 have chosen two bananas after his third choice.

1.1

1.2 $P(E) = \frac{N(E)}{N(S)}$

$P(3 \text{ banana}) = \frac{3}{7}$

1.3 $P(E) = \frac{N(E)}{N(S)}$

$P(2 \text{ Banana}) = \frac{2}{7}$

Figure 6: Some responses to Question 1 of Task-Based Activity 1.

Question 2

[redacted]

monitor. Find the probability that the class mo nitor is:

2.1 **a girl.**

2.2 **a boy with blue eyes.**

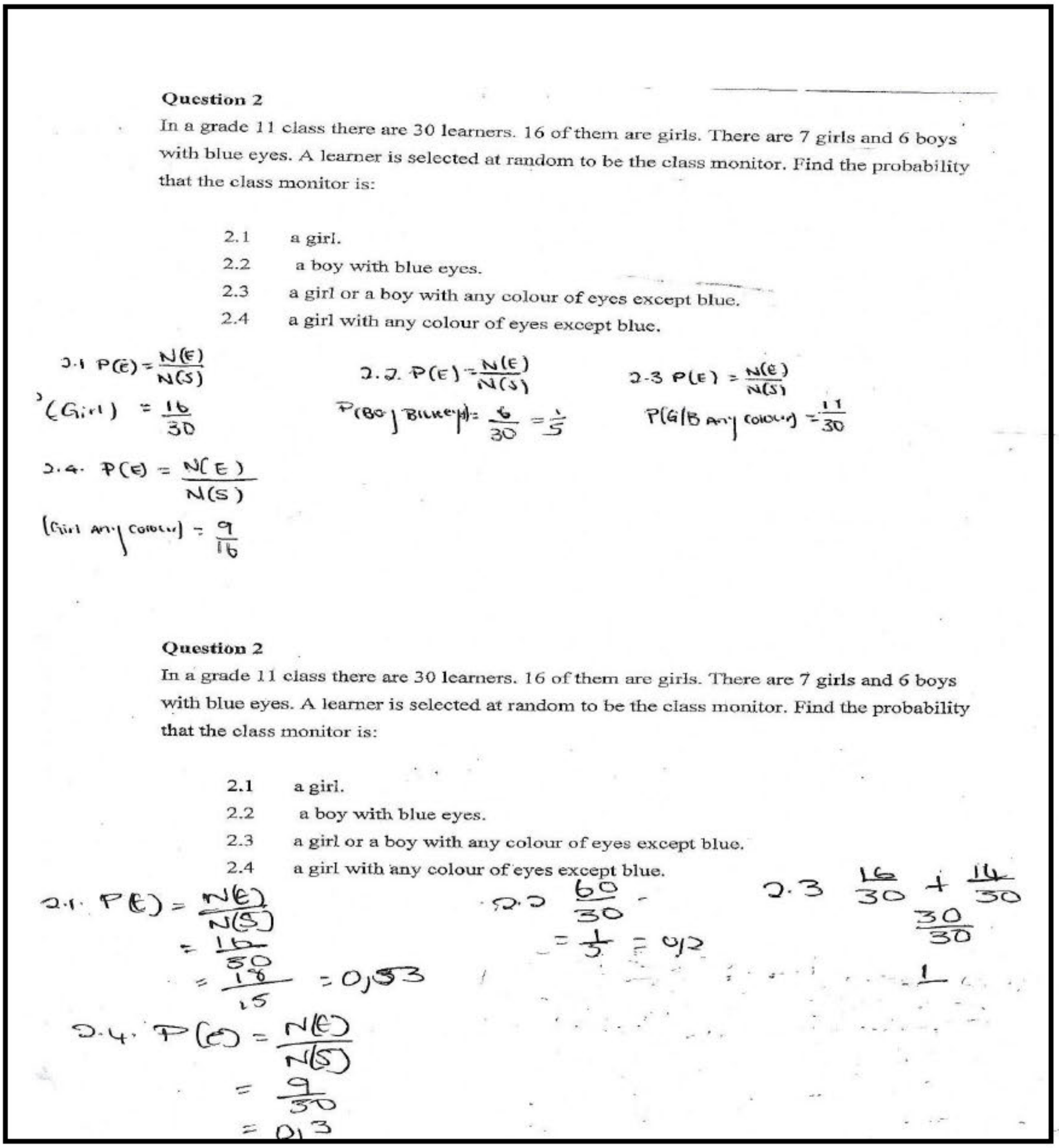
2.3 **a girl or a boy with any colour of eyes except blue.**

2.4 **a girl with any colour of eyes except blue.**

Figure 7: Question 2 of Task-Based Activity 1.

The second probability problem required the respondents to begin by visually representing the given information using a contingency table or a probability tree diagram, completing it, and then using the values to compute the required probabilities by substituting in the probability formula. Being able to realise that a contingency table or tree diagram could be used to simplify the probability problem was the first thing that the respondent was required to do. Deciding on the diagram that could be used to solve the problem could be classified under the first two steps of Polya's problem heuristic of understanding the problem and devising a plan for solving the problem. Using the chosen diagram to work-out the solutions could be classified under the implementation stage of problem solving. Writing the final solution could then require the respondent to reflect on the problem which is the last step of Pólya's problem-solving steps.

None of the five respondents were able to draw a visual representation of the problem. This could mean that they did not understand the problem, or they could not think of a plan to use in solving the problem. It could also mean that the respondents were not exposed to probability problem solving, or if they did, they did not understand it. Two of the respondents started by stating the probability formula and then substituted to get the probabilities but did not simplify their fractions. This meant that these respondents were operating in the action mental stage of the APOS theoretical framework in probability. Two of the respondents only wrote probabilities as fractions without starting by stating the probability formula and did not simplify their final answers. These respondents could be classified as operating below the action mental stage of the APOS framework as they could not even write the probability formula. This could also mean that these respondents were operating in the process mental stage of the APOS framework as they were able to perform some of the steps wholly in their minds and only wrote the final answer to the problem. The fact that all the participants did not simplify their final answers or convert them to decimals could have meant that they have not yet developed their probability mental structures to the object mental stage where they could relate probability answers to equivalent fractions, decimal fractions, and percentages. Some of the respondents‟ solutions to the probability problem are presented in Figure 8.



Question 2

In a grade 11 class there are 30 learners. 16 of them are girls. There are 7 girls and 6 boys with blue eyes. A learner is selected at random to be the class monitor. Find the probability that the class monitor is:

2.1 a girl.

2.2 a boy with blue eyes.

2.3 a girl or a boy with any colour of eyes except blue.

2.4 a girl with any colour of eyes except blue.

2.1 $P(E) = \frac{N(E)}{N(S)}$

P(Girl) $= \frac{16}{30}$

2.2 $P(E) = \frac{N(E)}{N(S)}$

P(Boy | Blue eye) $= \frac{6}{30} = \frac{1}{5}$

2.3 $P(E) = \frac{N(E)}{N(S)}$

P(G/B Any colour) $= \frac{11}{30}$

2.4 $P(E) = \frac{N(E)}{N(S)}$

(Girl Any colour) $= \frac{9}{16}$

Question 2

In a grade 11 class there are 30 learners. 16 of them are girls. There are 7 girls and 6 boys with blue eyes. A learner is selected at random to be the class monitor. Find the probability that the class monitor is:

2.1 a girl.

2.2 a boy with blue eyes.

2.3 a girl or a boy with any colour of eyes except blue.

2.4 a girl with any colour of eyes except blue.

2.1 $P(E) = \frac{N(E)}{N(S)}$

$= \frac{16}{30}$

$= \frac{18}{15} = 0{,}53$

2.2 $\frac{60}{30}$

$= \frac{1}{5} = 0{,}2$

2.3 $\frac{16}{30} + \frac{14}{30}$

$\frac{30}{30}$

1

2.4 $P(E) = \frac{N(E)}{N(S)}$

$= \frac{9}{30}$

$= 0{,}3$

Figure 8: Some responses to Question 2 of Task-Based Activity 1.

Question 3

A school investigated 100 learners to find out the number of learners who arrived at school on time, arrived late, or were absent during a particular week. The results are recorded in the following table.

	Boys	Girls	Total
On time	40	25	
Late	18		25
Absent		3	
Total	65		100

3.1 Copy and complete the contingency table by putting in the missing numbers

3.2 Calculate the probability of a learner:

3.2.1 arriving late for school.

3.2.2 arriving on time.

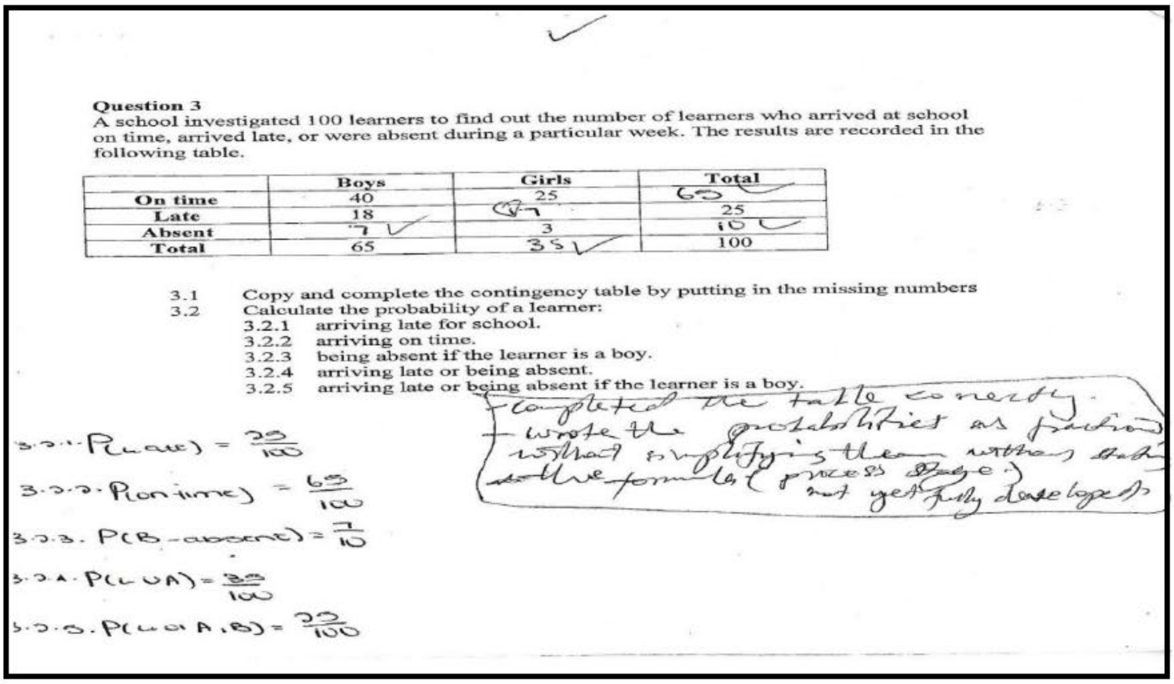
3.2.3 being absent if the learner is a boy.

3.2.4 arriving late or being absent.

3.2.5 arriving late or being absent if the learner is a boy.

Figure 9: Question 3 of Task-Based Activity 1.

This question required that the respondents begin by using the given quantities to complete the contingency table. Being able to complete the contingency table by filling in the missing values could mean that the respondent was operating in the action mental stage of the APOS framework. The respondents then had to use the numbers from the contingency table to work-out the probabilities. All the respondents were generally able to correctly complete the contingency table which meant that they had developed the action mental structure in probability problem solving and in the use of a contingency table. One respondent started by stating the probability formula and then substituted to get the probabilities which implied that this respondent was still operating in the action mental stage of cognitive development. The other respondents only wrote the probabilities without substituting the probability formula which meant that they were operating in the process mental stage of the APOS framework. All the other respondents did not simplify their final answers except one who converted the final answers to decimal fractions which was an indication of operating in the object mental structure. Some examples of the responses written by the respondents and some notes written by the investigator about how they responded are presented and are included with some of the responses in Figure 10.



Question 3
A school investigated 100 learners to find out the number of learners who arrived at school on time, arrived late, or were absent during a particular week. The results are recorded in the following table.

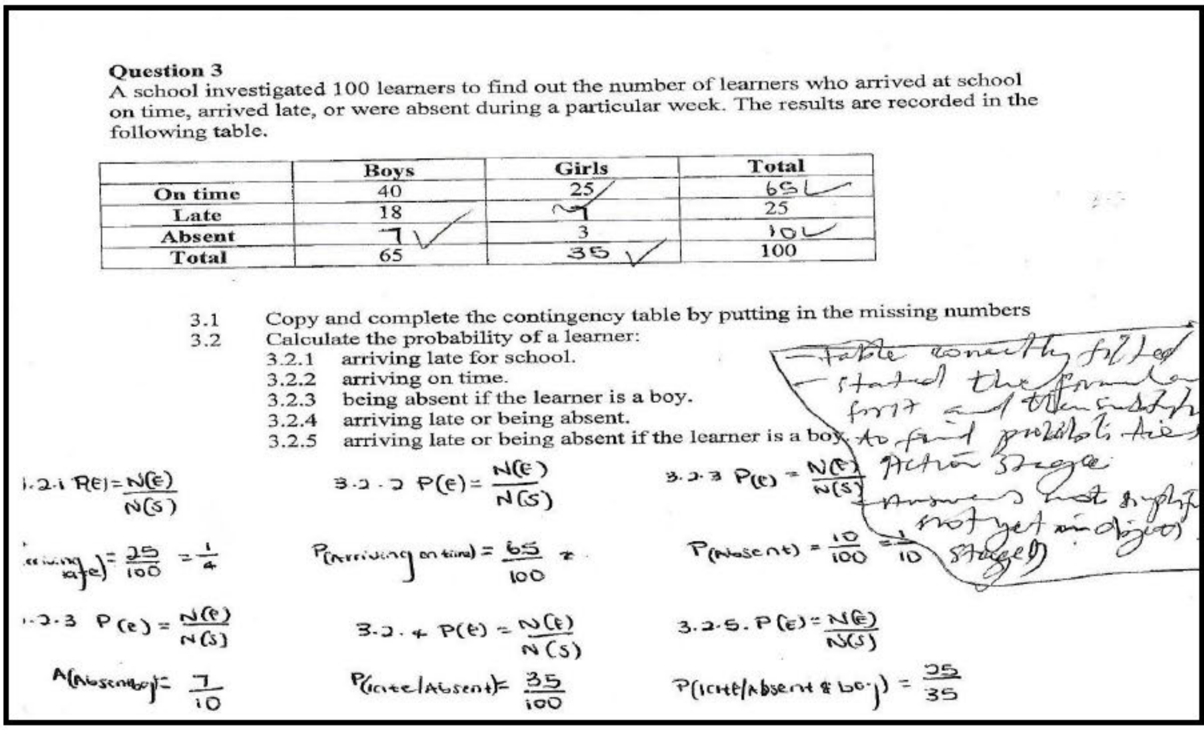
	Boys	Girls	Total
On time	40	25	65
Late	18	7	25
Absent	7	3	10
Total	65	35	100

3.1 Copy and complete the contingency table by putting in the missing numbers
3.2 Calculate the probability of a learner:
3.2.1 arriving late for school.
3.2.2 arriving on time.
3.2.3 being absent if the learner is a boy.
3.2.4 arriving late or being absent.
3.2.5 arriving late or being absent if the learner is a boy.

3.2.1. P(Late) = 25/100
3.2.2. P(on time) = 65/100
3.2.3. P(B-absent) = 7/10
3.2.4. P(L ∪ A) = 35/100
3.2.5. P(L or A, B) = 25/100

- completed the table correctly.
- wrote the probabilities as fractions without simplifying them without stating the formula (process stage.) not yet fully developed

Figure 10a: Some responses to Question 3 of Task-Based Activity 1.



Question 3
A school investigated 100 learners to find out the number of learners who arrived at school on time, arrived late, or were absent during a particular week. The results are recorded in the following table.

	Boys	Girls	Total
On time	40	25	65
Late	18	7	25
Absent	7	3	10
Total	65	35	100

3.1 Copy and complete the contingency table by putting in the missing numbers
3.2 Calculate the probability of a learner:
3.2.1 arriving late for school.
3.2.2 arriving on time.
3.2.3 being absent if the learner is a boy.
3.2.4 arriving late or being absent.
3.2.5 arriving late or being absent if the learner is a boy.

Figure 10b: Some responses to Question 3 of Task-Based Activity 1.

The responses presented here show that the respondents were able to understand the scenario discussed in this question. This is highlighted by the fact that all of them were able to complete the contingency table correctly. This could mean that diagrams in probability problem solving help the problem solver to understand the problem. The respondents were also able to state the probability formula correctly. This could be an indication that they all have passed the action stage of the APOS theoretical framework. Some of the respondents could not, however, substitute the formula with the correct values. This could mean that they have not yet fully developed the object stage of the APOS theoretical framework. This claim is also supported by the fact that even those who were able to substitute with the correct values in the probability formula did not simplify their final answers. Failing to apply the concept of fractions in probability could mean that the respondents are not yet in the stage where they can link the probability schema with the fraction schema. The solutions written by some of the respondents as shown in figure 10 show most of their solutions left without being simplified. Some examples of these solutions as shown in figure 6 include

$\frac{35}{100}, \frac{25}{100}, and\ \frac{65}{100}.$

However, it is worth mentioning that some of the respondents were able to simplify the final answers. Some of the respondents even converted their solutions from common fractions to decimal fractions. This does not necessarily mean that they had developed a schema that linked probability to common fractions and decimal fraction. It might be that the respondents were using a calculator to simplify those answers since they simplify some and did not simplify others.

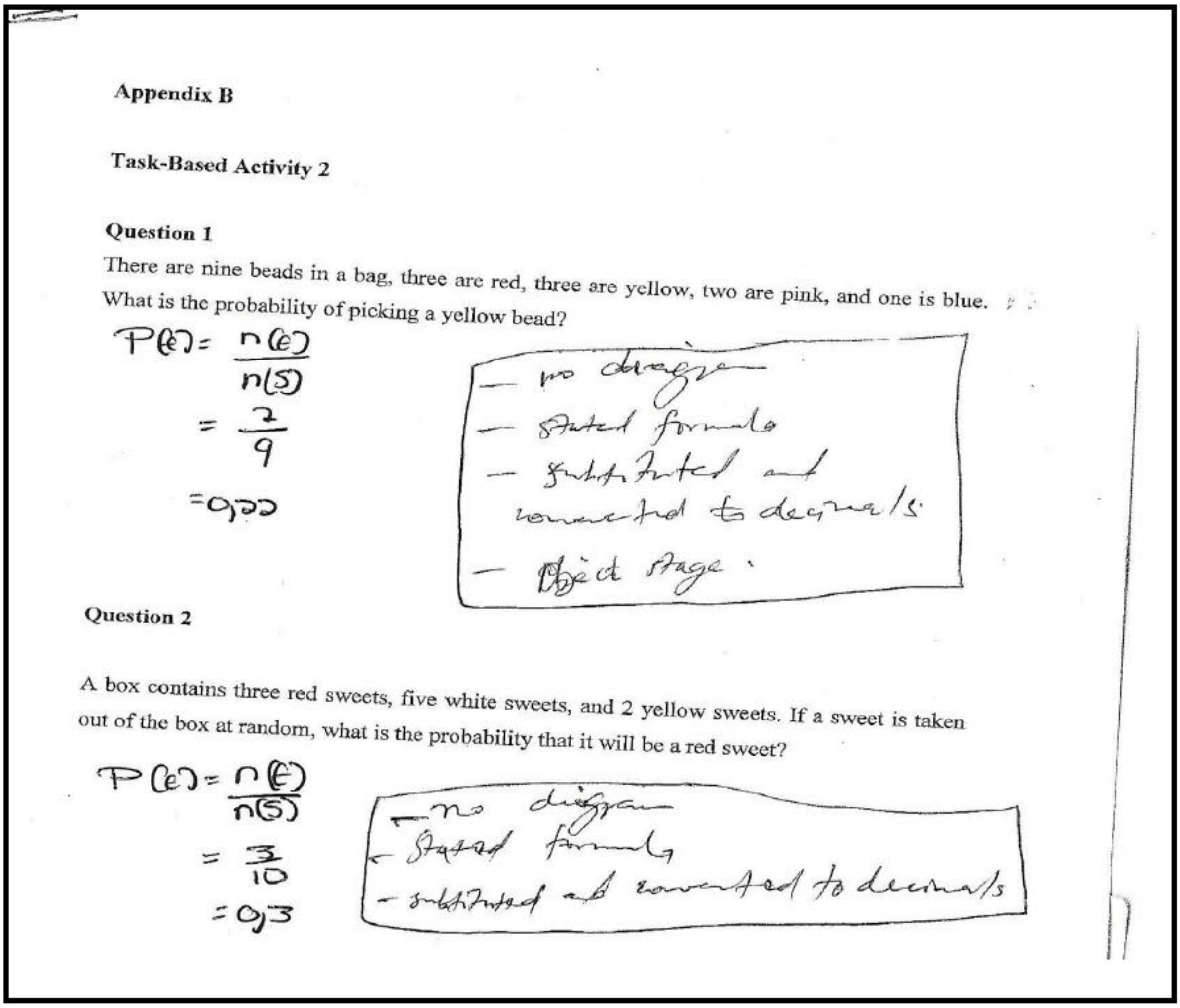
5.3 TASK-BASED ACTIVITY TWO

This task-based activity was made up of ten probability questions taken from the grade 11 mathematics in the South African curriculum context. All the other nine questions had no diagrams except question ten which had an incomplete contingency table. Questions 1 and 2 were on simple probability and could be solved without drawing a probability diagram. Three respondents drew a normal table to visualize the two questions. The other two respondents simply used the probability formula to work-out the probabilities. All the respondents were able to substitute correctly into the probability formula but did not simplify their final answers. This was an indication that they have not yet developed to the object mental structures in probability.

Question 1
There are nine beads in a bag, three are red, three are yellow, two are pink, and one is blue. What is the probability of picking a yellow bead?
Question 2
A box contains three red sweets, five white sweets, and 2 yellow sweets. If a sweet is taken out of the box at random, what is the probability that it will be a red sweet?

Figure 11: Question 1 and 2 of Task-Based Activity 2.

In Questions 1 and 2 the respondents were not given the diagram to use, and they could not decide on which diagram was appropriate for the given probability problem. This indicated that they had not developed to the object mental stage of the APOS framework. In Question 3 the contingency table was given for respondents to complete, and the respondents were able to complete it. This could also emphasize the importance of the first and second steps of Pólya's problem solving heuristic which is to understand the problem and devising a plan for finding the solution to the problem.



Appendix B

Task-Based Activity 2

Question 1

There are nine beads in a bag, three are red, three are yellow, two are pink, and one is blue. What is the probability of picking a yellow bead?

$P(E) = \frac{n(E)}{n(S)}$

$= \frac{3}{9}$

$= 0,33$

- no diagram
- stated formula
- substituted and converted to decimals.
- Object stage.

Question 2

A box contains three red sweets, five white sweets, and 2 yellow sweets. If a sweet is taken out of the box at random, what is the probability that it will be a red sweet?

$P(E) = \frac{n(E)}{n(S)}$

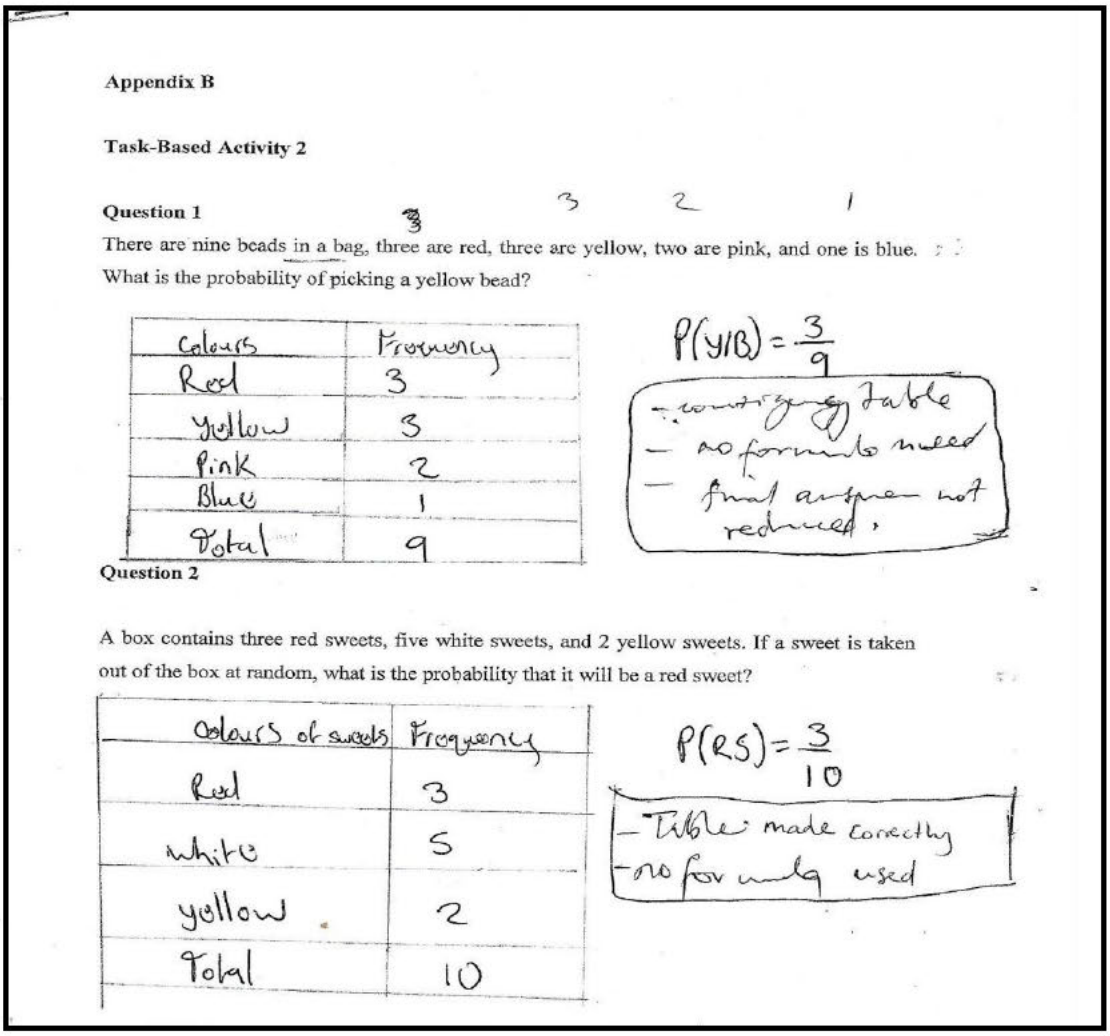
$= \frac{3}{10}$

$= 0,3$

- no diagram
- stated formula
- substituted and converted to decimals

Figure 12a: Some responses to Questions 1 and 2 of Task-Based Activity 2 with the investigator's notes.

The extract of responses presented in Figure 12 reveals that the respondents had a challenge with problems. They seem to have not understood the problems. Respondents could not draw any diagram to represent or model the probability problems. This could mean that they have not developed the object stage of the APOS theoretical framework. It can also be inferred from these extracts that the respondents had difficulty understanding the probability problems without diagrams. This could be interpreted to emphasize the importance of visual representations in the form of diagrams in simplifying probability problems. The respondents were able to state the probability formula correctly, but it was meaningless as they did not know what values needed to be substituted into the formula.



Appendix B

Task-Based Activity 2

Question 1

There are nine beads in a bag, three are red, three are yellow, two are pink, and one is blue.
What is the probability of picking a yellow bead?

Colours	Frequency
Red	3
Yellow	3
Pink	2
Blue	1
Total	9

$P(Y/B) = \frac{3}{9}$

- contingency table
- no formula used
- final answer not reduced.

Question 2

A box contains three red sweets, five white sweets, and 2 yellow sweets. If a sweet is taken out of the box at random, what is the probability that it will be a red sweet?

Colours of sweets	Frequency
Red	3
White	5
yellow	2
Total	10

$P(RS) = \frac{3}{10}$

- Table made correctly
- no formula used

Figure 12b: Some responses to Questions 1 and 2 of Task-Based Activity 2 with the investigator's notes (continued).

Question 3

A blue (B) and green (G) bucket are filled with balls. The blue bucket contains 5 white (W) and 3 red (R) balls. The green bucket contains 2 white and 7 red balls. A bucket is randomly selected, and one ball is thereafter randomly drawn from the bucket.

Determine the probability that a red ball is taken out.

Figure 13a: Questions 3, 4, and 5 of Task-Based Activity 2

Question 4

[redacted]

replaced. A second cube is then taken at random.

[redacted]

cube is taken.

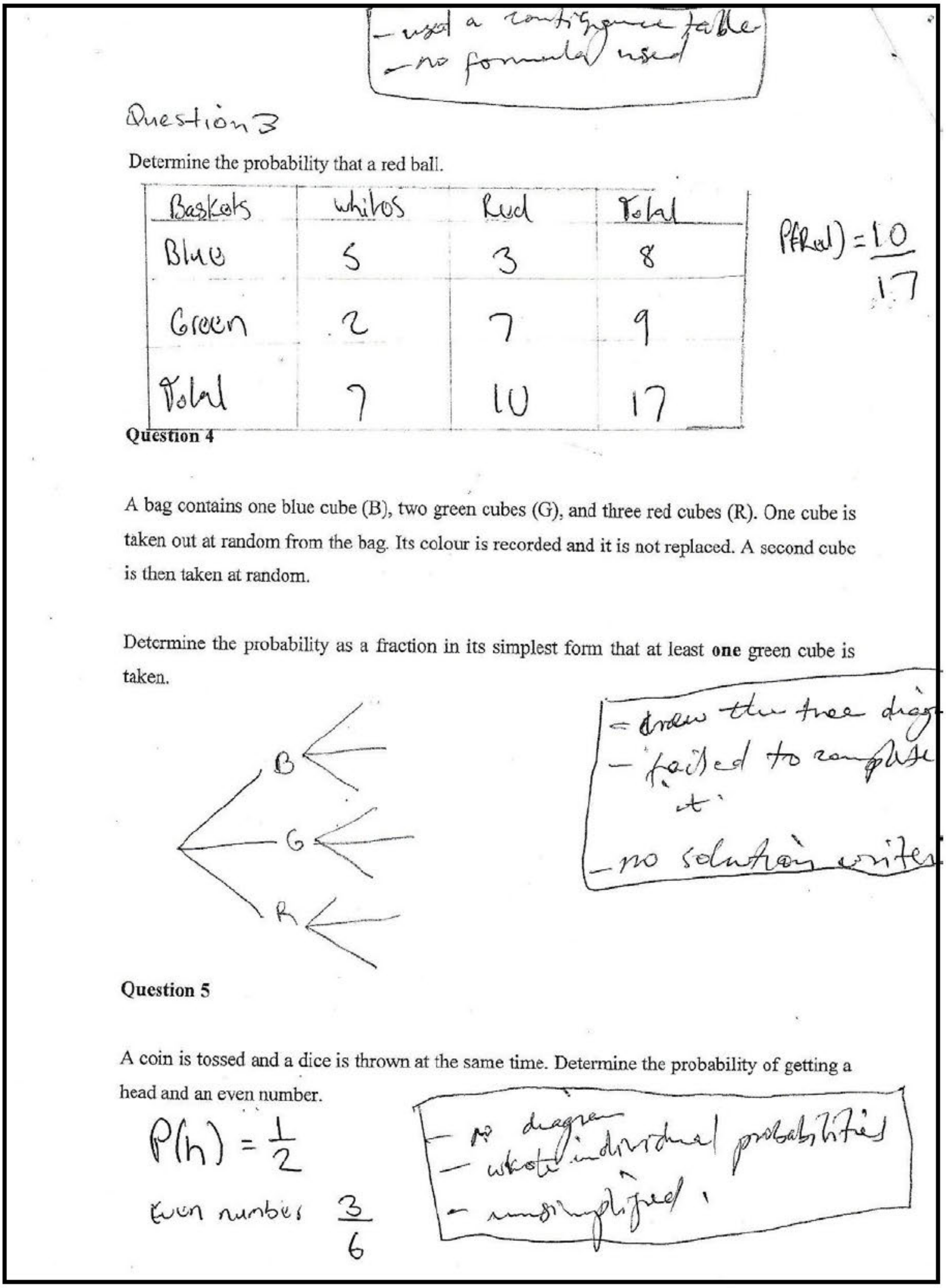
Question 5

[redacted]

getting a head and an even number.

Figure 13b: Questions 3, 4, and 5 of Task-Based Activity 2.

Questions 3, 4, and 5 required the respondents to use either a probability tree diagram or a contingency table to visualize the probability problem. In Questions 3 all four of the respondents used a contingency table to visually represent the information given in the probability problem. Only one used a formula alone. In Questions 4 and 5 the respondents could not draw any probability diagram to visually represent the probability problem. Only one respondent tried to draw a probability tree diagram but also failed to complete it. Their failure to decide on the suitable diagram to use for these questions suggest that they have not yet developed to the object mental structure in probability. In Question 5 they presented the individual probabilities of the two events separately which indicated that they could either be in the action stage or the process stage. They could not link the probabilities of the two events to work-out a single solution. Some of the respondents" solutions to the probability problem are presented in Figure 13 with the investigators note about their responses.


- used a contingency table
- no formula used

Question 3

Determine the probability that a red ball.

Baskets	whites	Red	Total
Blue	5	3	8
Green	2	7	9
Total	7	10	17

$P(Red) = \frac{10}{17}$

Question 4

A bag contains one blue cube (B), two green cubes (G), and three red cubes (R). One cube is taken out at random from the bag. Its colour is recorded and it is not replaced. A second cube is then taken at random.

Determine the probability as a fraction in its simplest form that at least **one** green cube is taken.

B
G
R

- drew the tree diag
- failed to complete it
- no solution writen

Question 5

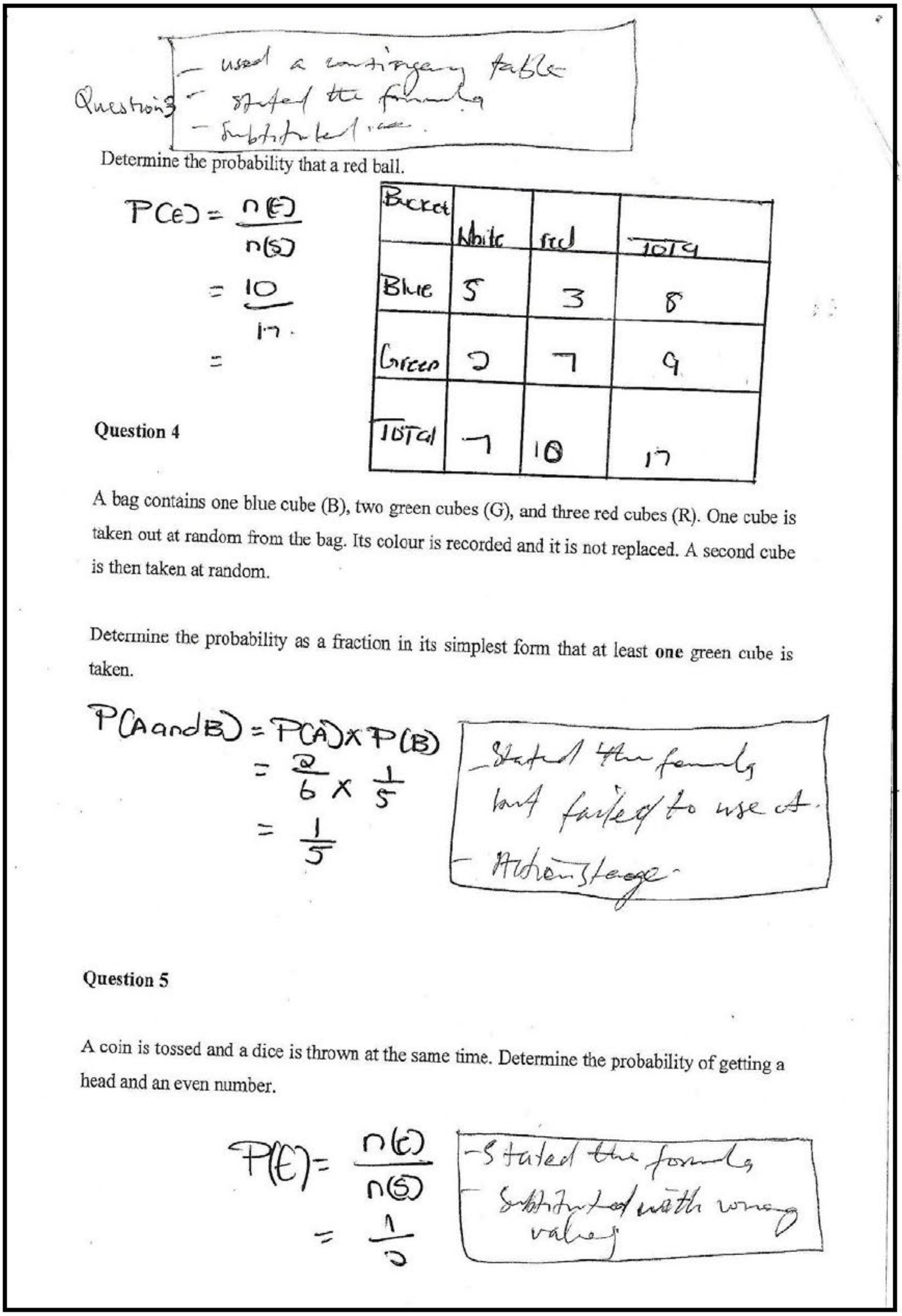
A coin is tossed and a dice is thrown at the same time. Determine the probability of getting a head and an even number.

$P(h) = \frac{1}{2}$

Even numbers $\frac{3}{6}$

- no diagram
- wrote individual probabilities
- unsimplified

Figure 14a: Some of the responses to Questions 3, 4, and 5 of Task-Based Activity 2.


Question 3 – used a contingency table
– stated the formula
– substituted values.

Determine the probability that a red ball.

$P(E) = \frac{n(E)}{n(S)}$

$= \frac{10}{17}$

$=$

Bucket	White	red	Total
Blue	5	3	8
Green	2	7	9
Total	7	10	17

Question 4

A bag contains one blue cube (B), two green cubes (G), and three red cubes (R). One cube is taken out at random from the bag. Its colour is recorded and it is not replaced. A second cube is then taken at random.

Determine the probability as a fraction in its simplest form that at least **one** green cube is taken.

$P(A \text{ and } B) = P(A) \times P(B)$

$= \frac{2}{6} \times \frac{1}{5}$

$= \frac{1}{5}$

– Stated the formula but failed to use it.
– Action stage.

Question 5

A coin is tossed and a dice is thrown at the same time. Determine the probability of getting a head and an even number.

$P(E) = \frac{n(E)}{n(S)}$

$= \frac{1}{2}$

– Stated the formula
– Substituted with wrong values

Figure 14b: Some of the responses to Questions 3, 4, and 5 of Task-Based Activity 2.

Learners" responses as depicted in Figure 14 suggest that some of the learners were aware that they could use diagrams to solve the probability problem. They drew a probability tree diagram but could not complete it. This could mean that they had limited knowledge of how to use a probability tree diagram to solve probability problems or it could also mean that they had forgotten how to use it. On a positive note, it is worth noting that the probability tree diagram shown in Figure 14 has three branches, which indicates that the respondent was able to realize that the problem had three outcomes on the branches. Figure 14 also presents a response whereby the learner tried to use a contingency table to solve Question 3. This also confirms that this respondent was aware that a contingency table could be used to solve probability problems but was not sure where and how to use it. Figure 14 also shows some evidence of the respondents using the probability formula to find the solutions. Though they were able to write the correct formula, they could not find the correct solution as they were substituting using wrong information. It is also worth noting that though the answers they got were incorrect, they were however, reasonable solutions as they were within the range of 0 to 1which is expected for probability.

Some of the respondents used a frequency table as a visual tool to help understand the problem (Figure 14: Question 3). This could be interpreted to mean that learners understand visual representations more than words. In all, the respondents used a diagram in all their workings toward the solutions. Even in cases where they could not finally workout the correct solution, they still had a visual representation accompanying their attempts. Trying to represent the information visually could mean that the respondents were not comfortable with the language as they preferred the diagrams. Some of them even misunderstood the given information especially with the contingency table in Figure 14 as they failed to fill-in the correct numbers. This was also evident in other diagrams and visual presentations that the respondents used in Figure 14 such as the probability tree diagram and probability formula. They were able to draw correct diagrams but failed to fill in the correct probabilities on the branches. They also stated the probability correctly but could not find the correct numbers to substitute with in the formula. All these examples stated here highlight the fact that some of these learners had a challenge with understanding the language used in these problems.

Question 6

In a class of 40 learners the following information is TRUE:

- **7 learners are left-handed**
- **18 learners play soccer**
- **4 learners play soccer and are left-handed**
- **All 40 learners are either right-handed or left-handed**

[redacted] **handed and plays soccer.**

Question 7

The probability that a tennis player has no injuries (NI) is 0.7. The probability that [redacted] **injuries (I), the probability of her winning becomes 0.45.**
Calculate the probability of her winning her next tennis game.

Question 8

[redacted] **on exactly one of the dice is — and the probability that you will get a six shown on both dice is —.**

[redacted] **dice at the same time?**

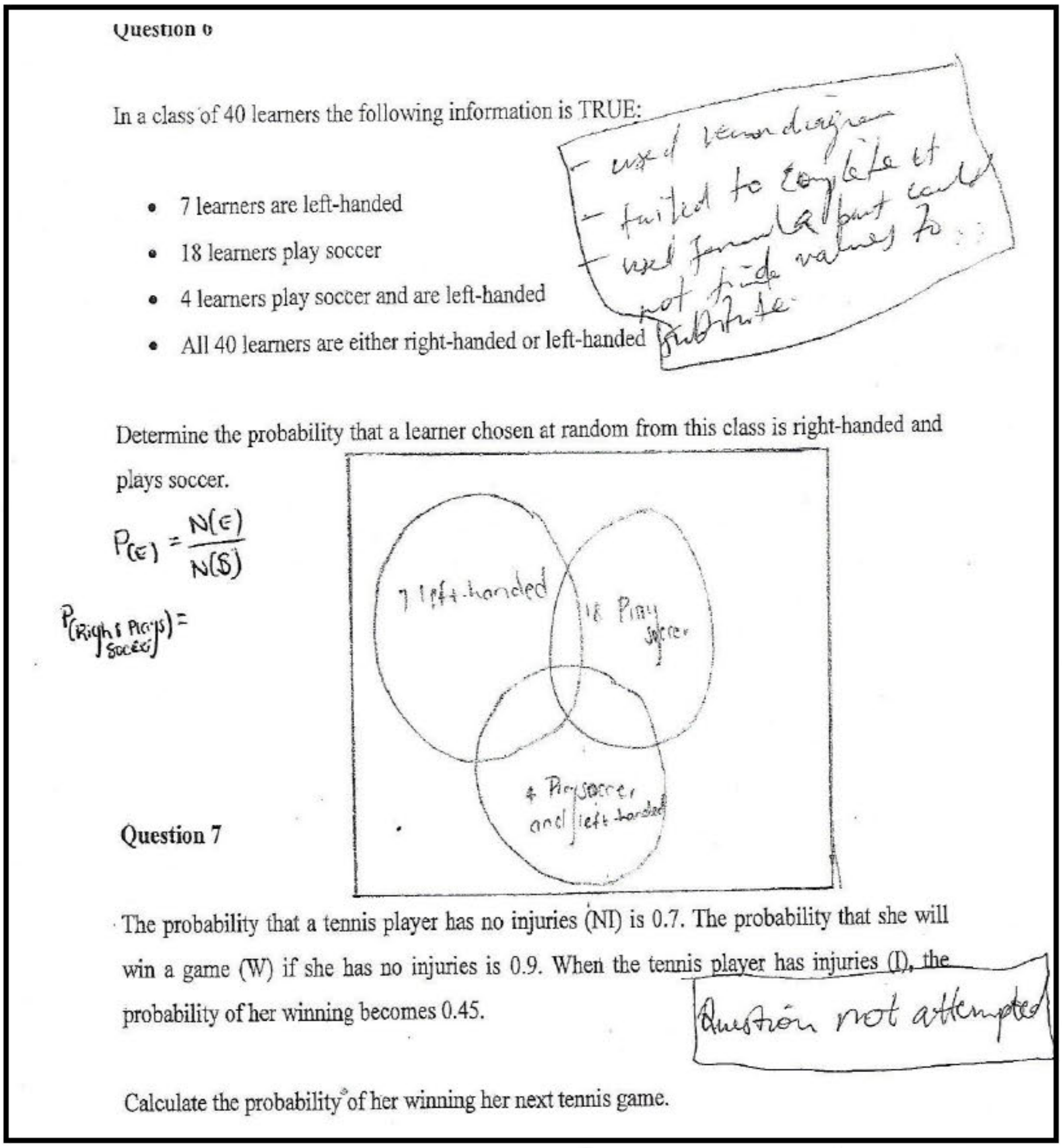
Figure 15: Questions 6, 7, and 8 of Task-Based Activity 2.

Questions 6, 7, and 8 required the respondents to use a Venn diagram to visually represent each of the given probability problems. Drawing the Venn diagram alone was not enough but respondents also had to be able to complete the Venn diagram by putting the given quantities in the correct regions of the diagram and calculating the missing ones before calculating probabilities. Respondents were able to draw a Venn diagram in Question 6, though they could not complete it. In Questions 7 and 8 the respondents could not decide on a suitable diagram to be used to visually represent the probability problems which also suggested that they had not yet developed to the object mental structure in probability. Three of the five respondents did not attempt Questions 7 and 8. The other two respondents used a pencil for these two questions which was also an indication that they doubted what they were writing. Some of the responses presented by some of the respondents are presented with some notes written by the investigator about how they responded in Figure 15.

Some of the respondents used a Venn diagram and the probability formula to try to work-out the solutions to the probability problem in Question 6. This was the most suitable diagram to use in this question, but the respondents were not able to find the correct values to complete the Venn diagram. Some were able to notice that the Venn diagram had to contain three sets. One of the respondents seemed to have not understood the problem as he/she drew two sets instead of three. This could be because of misinterpreting the problem which could be emanating from the language barrier as the problems were in a foreign language (English).

The evidence presented in Figure 15 also indicates that all the respondents did not attempt Question 7. This could mean that they did not understand it, or they could not think of a solution procedure to the problem. This question was posed in words with a few figures which were in decimal format as the probabilities for different events. The English language used in this question could be associated with the difficult of understanding the foreign language. The use of decimal fractions for the probabilities of the events in this problem could also be the source of difficulties if the respondents were familiar with the use of decimal fractions in probability. This could be an indication that the respondents were not yet at the schema stage of the APOS theoretical framework. It was a similar situation with Question 8 as most of the respondents did not attempt it.

This could also mean that the respondents did not understand the problem.



Question 6

In a class of 40 learners the following information is TRUE:

- 7 learners are left-handed
- 18 learners play soccer
- 4 learners play soccer and are left-handed
- All 40 learners are either right-handed or left-handed

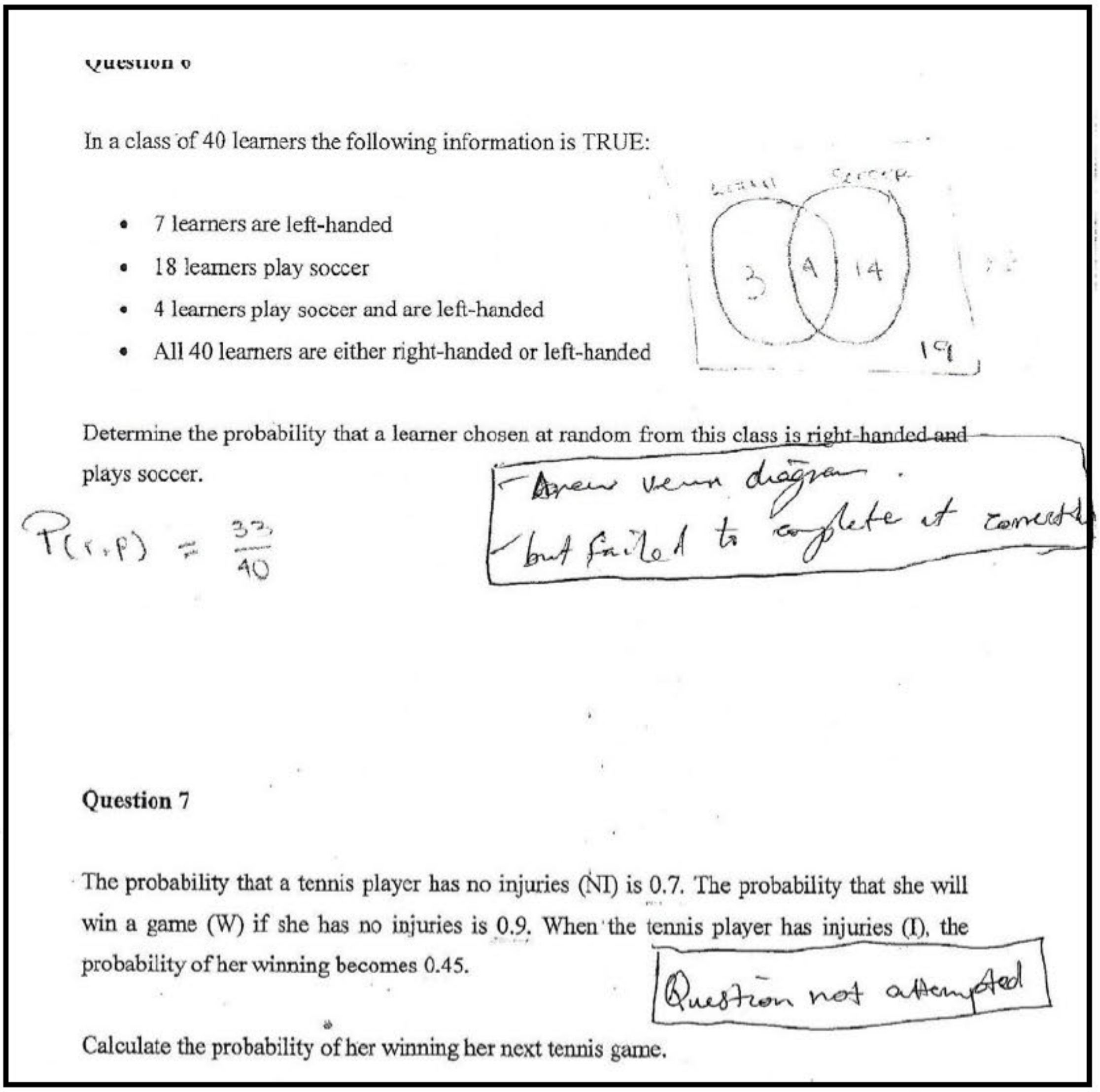
Determine the probability that a learner chosen at random from this class is right-handed and plays soccer.

Question 7

The probability that a tennis player has no injuries (NI) is 0.7. The probability that she will win a game (W) if she has no injuries is 0.9. When the tennis player has injuries (I), the probability of her winning becomes 0.45.

Calculate the probability of her winning her next tennis game.

Figure 16a: Some responses to questions 6, and 7 of Task-Based Activity 2.



Question 6

In a class of 40 learners the following information is TRUE:

- 7 learners are left-handed
- 18 learners play soccer
- 4 learners play soccer and are left-handed
- All 40 learners are either right-handed or left-handed

Determine the probability that a learner chosen at random from this class is right-handed and plays soccer.

$P(r,p) = \frac{33}{40}$

- Drew venn diagram.
- but failed to complete it correctly

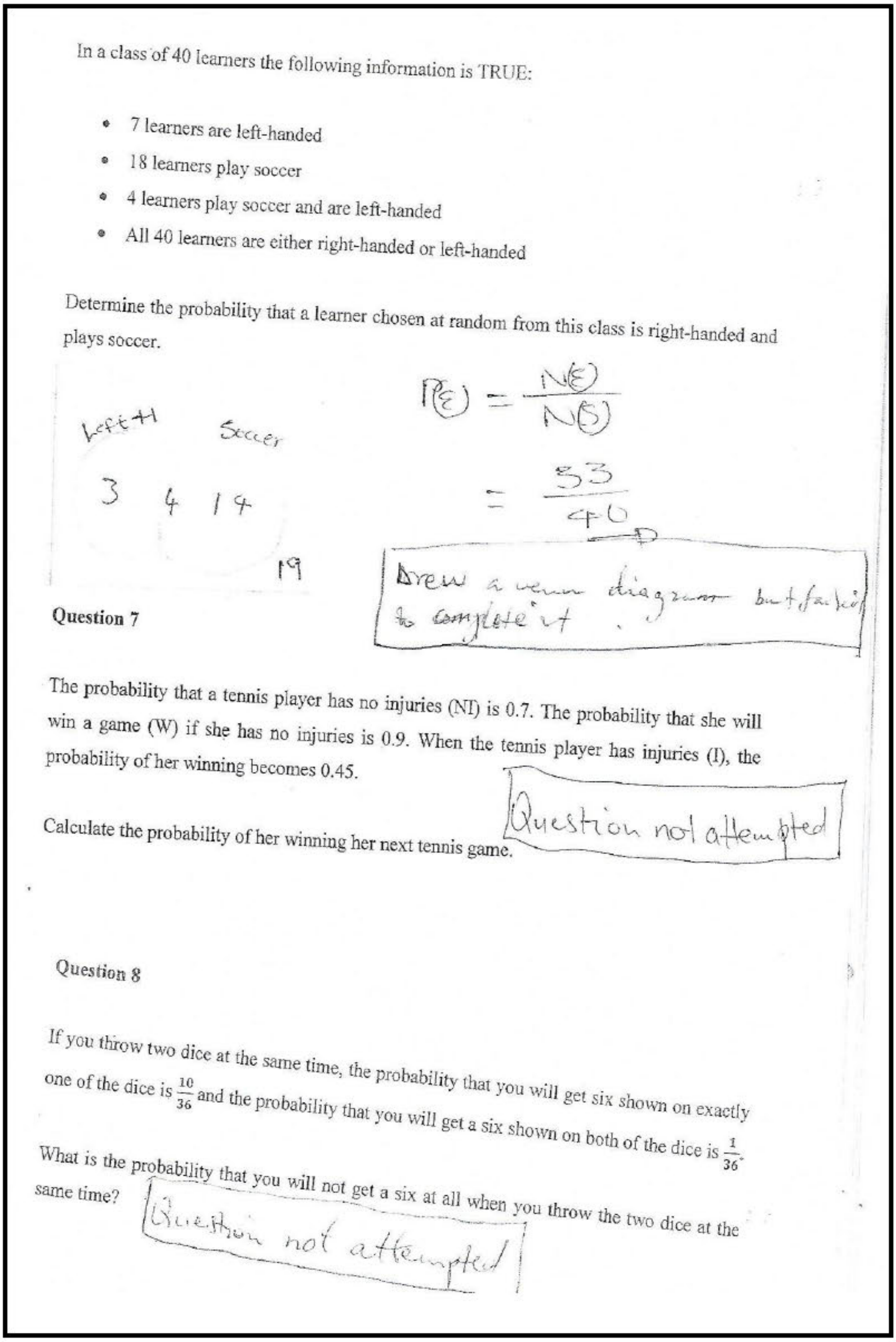
Question 7

The probability that a tennis player has no injuries (NI) is 0.7. The probability that she will win a game (W) if she has no injuries is 0.9. When the tennis player has injuries (I), the probability of her winning becomes 0.45.

Question not attempted

Calculate the probability of her winning her next tennis game.

Figure 16b: Some responses to Questions 6 and 7 of Task-Based Activity 2 (continued).



In a class of 40 learners the following information is TRUE:

- 7 learners are left-handed
- 18 learners play soccer
- 4 learners play soccer and are left-handed
- All 40 learners are either right-handed or left-handed

Determine the probability that a learner chosen at random from this class is right-handed and plays soccer.

$P(E) = \frac{N(E)}{N(S)}$

$= \frac{33}{40}$

Drew a venn diagram but failed to complete it

Question 7

The probability that a tennis player has no injuries (NI) is 0.7. The probability that she will win a game (W) if she has no injuries is 0.9. When the tennis player has injuries (I), the probability of her winning becomes 0.45.

Calculate the probability of her winning her next tennis game.

Question not attempted

Question 8

If you throw two dice at the same time, the probability that you will get six shown on exactly one of the dice is $\frac{10}{36}$ and the probability that you will get a six shown on both of the dice is $\frac{1}{36}$.

What is the probability that you will not get a six at all when you throw the two dice at the same time?

Question not attempted

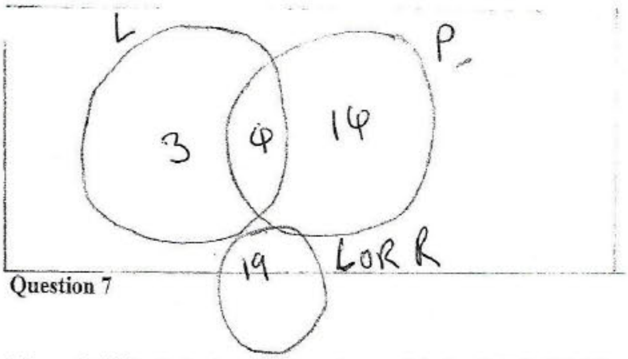
Figure 16c: Some responses to Questions 6 and 7 of Task-Based Activity 2 (continued).

Question 6

In a class of 40 learners the following information is TRUE:

- 7 learners are left-handed
- 18 learners play soccer
- 4 learners play soccer and are left-handed
- All 40 learners are either right-handed or left-handed

Determine the probability that a learner chosen at random from this class is right-handed and plays soccer.



$\frac{14}{40}$

- Drew the venn diagram.
- failed to complete it.
- wrote an unsimplified answer.

Question 7

The probability that a tennis player has no injuries (NI) is 0.7. The probability that she will win a game (W) if she has no injuries is 0.9. When the tennis player has injuries (I), the probability of her winning becomes 0.45.

Calculate the probability of her winning her next tennis game.

- not attempted

Question 8

If you throw two dice at the same time, the probability that you will get six shown on exactly one of the dice is $\frac{10}{36}$ and the probability that you will get a six shown on both of the dice is $\frac{1}{36}$.

What is the probability that you will not get a six at all when you throw the two dice at the same time?.

- Question not attempted.

Figure 16d: Some responses to Questions 6 and 7 of Task-Based Activity 2 (continued).

Question 9

Given that the probability that it is sunny (S) on any day in Durban is

0.8 [redacted]

school is 0.2. Given also that John is expected to be at school for 250 days in a year.

[redacted]

year.

Question 10

[redacted]

colour hair they have. The results are shown in the table below.

		HAND USED TO WRITE WITH		
		Right	**Left**	**Total**
HAIR COLOUR	**Light**	**A**	**b**	**20**
	Dark	**C**	**d**	**40**
	Total	**48**	**12**	**60**

[redacted]

independent events. Calculate the values of a, b, and c.

Figure 17: Questions 9 and 10 of Task-Based Activity 2.

Questions 9 and 10 were on the application of probability in solving general mathematics problems. The respondents needed to have developed the actions, processes, and objects understanding of probability that they needed and be able to organize and linked them into a coherent framework called a schema. This required the respondents to have developed a schema for probability in mathematics to be able to decide when presented with a particular probability problem whether probability as a schema is applicable to solve it. Only one respondent attempted Question 9 by using the formula for probability which could not help. The other respondents did not even attempt Question 9. This was an indication that the respondents had not yet encapsulated the actions, processes and objects into a schema they can use to predict the number of days John would be absent from school in a year. In Question 10 the respondents were given a contingency table which they had to complete before working out the probabilities. The respondents completed the table by using trial and error instead of applying the concept of independent events, hence they got wrong values of a, b, c, and d. Some responses presented by some of the respondents with the investigator's notes are presented in Figure 18.

Question 9

0,9

Given that the probability that it is sunny (S) on any particular day in Durban is **0.8.** Given also that the probability that John will be absent (A) from school if it is sunny is **0.1** and if it is rainy (R), the probability that John will be absent from school is **0.2.** Given also that John is expected to be at school for 250 days in a year; 110

Predict the total number of days John will be absent from school in the whole year.

110

not attempted

Question 10

A survey was conducted asking 60 people which hand they write with and what colour hair they have. The results are shown in the table below.

		HAND USED TO WRITE WITH		
		Right	**Left**	**Total**
HAIR COLOUR	**Light**	**a** 10	**b** 10	**20**
	Dark	**c** 38	**d** 2	**40**
	Total	**48**	**12**	**60**

The survey concluded that the 'hand used for writing' and the 'hair colour' are independent events.

Calculate the values of **a**, **b**, and **c**.

a= 10
b= 10
c= 38/38
d= 2

wrote wrong values of a, b, c, and d.

Figure 18a: Some responses to Questions 9 and 10 of Task-Based Activity 2.

Question 9

Given that the probability that it is sunny (S) on any particular day in Durban is **0.8**. Given also that the probability that John will be absent (A) from school if it is sunny is **0.1** and if it is rainy (R), the probability that John will be absent from school is **0.2**. Given also that John is expected to be at school for 250 days in a year;

Predict the total number of days John will be absent from school in the whole year.

$$P(E) = \frac{n(E)}{n(S)} = \frac{0{,}1}{250} = \frac{1}{2500} = 0{,}0004$$

Stated formula but failed to substitute correctly.

Question 10

		HAND USED TO WRITE WITH		
		Right	Left	Total
HAIR COLOUR	Light	a 10	b 10	20
	Dark	c 38	d 2	40
	Total	48	12	60

The survey concluded that the 'hand used for writing' and the 'hair colour' are independent events.

Calculate the values of **a**, **b**, and **c**.

$$P(A \text{ and } B \text{ and } C) = P(A) \times P(B) \times P(C) = \frac{10}{60} \times \frac{10}{60} \times \frac{38}{60} = \frac{1}{6} \times \frac{1}{6} \times \frac{2}{15}$$

Figure 18b: Some responses to Questions 9 and 10 of Task-Based Activity 2 (continued).

Most of the respondents did not answer Question 9. This could be caused by the fact that the question was in words with a few decimal figures as probabilities of events. Those who attempted the question did not draw the correct diagram required for the problem solution (probability tree diagram). They opted to use the probability formula to work-out the solution to the problem which could not help. All the respondents attempted Question 10. This could be because this question had a diagram drawn to simplify it. Respondents had to complete the diagram and then write the probabilities from the diagram. Even though the respondents could not find the correct solutions to the problem, they were able to write something as the solution to the problem. This could imply that the diagram motivated them to follow the problem. In that case, it would mean that diagrams in probability problem solving are motivating to the problem solver. All the respondents were hindered by the statement: The survey concluded that the hand used for writing and the hair colour were independent events. This confusion could be associated with failing to understand the foreign language well or it could be linked to failing to interpret the meaning of the words independent events in probability problem solving.

5.4 A COMPARISON OF THE RESPONSES TO THE TWO TASKS-BASED ACTIVITIES

The way the respondents responded to the two task-based activities was different. In the first task the respondents did not make use of probability diagrams to visually represent the problems. In the second task diagrams were used. This helped them to get most of the solutions correct. Most of the solutions to the second task were reduced to simplest form which was not done in the first task. This was an indication of an improvement in the development of the mental structures related to probability. The responses to the second task also revealed that not all the participants have developed the probability schema and, as such, could not link it to other topics or cases such as making predictions. In both activities the respondents attempted all the questions that had a diagram associated with them such as those questions where the contingency table was given. Respondents seemed to have problems understanding probability problems which were only presented in words. This could be associated with the fact that the respondents did not attempt most of such probability problems.

5.5 INTERVIEW ONE

This interview was based on the three questions written by the respondents in the first task-based activity. The interview questions were meant to find out the mental interactions that the learners go through during their solving of probability problems. It was also meant to establish if learners find probability diagrams helpful in solving probability problems or not and how they use them. The interview was also meant to find out the types of diagrams that learners use during probability problem solving. The responses of the participants are presented in Figure 19.

Did the questions make sense to you? Were you able to understand the English used and the phasing of the question?

Thabo: *They did make sense kancane…kancane (meaning a little bit).*
Researcher (Probing): Were you able to understand the English used and the phrasing of the questions?
Thabo: *I tried to understand but it was difficult. I only understood a little bit.*
Researcher (Probing): Were all the questions equally difficult to understand?
Thabo: *Some of them I understood like question two, but I failed to draw the diagram. Question three was easy to understand because I know it.*
Minky: *Yes. The English was simple but question 1,… I had a problem since we were not told which diagram to draw. The English was understandable though.* **Pem**: *Question 1 was a little bit tricky. Question 2 and 3 were understandable.* **Sim**: *I was able to understand the questions, especially question 2 and 3. It is only question 1 I failed to understand.*
Mbali: *I was able to answer question three because the table was given, and it made it easy for me to solve the problem. Question 1 and question 2 gave me a hustle because I had forgotten that I must use the diagrams to solve the problems, like the tree diagram, I had forgotten how to draw it.*

Figure 19: Question 1 of Interview 1 and some responses by respondents.

Most of the participants were challenged by the English used in phrasing the questions. Most of them noted that they did not understand the questions very well. Even those who understood some of the questions to some extent they raised the issue of some of these questions being tricky. This concern highlighted the issue of language as a barrier to understanding problems in mathematics.

Did you use any diagrams to work -out the answers to the problems?

Thabo: *I did not use diagrams. I did not know which diagrams to use and how to use it.*
Minky: *Yes. I used diagrams. Like in question 1 I draw a tree diagram.*
Pem: *I did not use diagrams.*
Sim: *I only used the table given in question 3.*
Mbali: *I did not use any diagram except the one given in question 3.*

Figure 20: Question 2 of Interview 1 and some responses by respondents.

Not all the respondents drew diagrams to find the solutions to the problems. They relied on using the probability formula to work-out the solutions. They highlighted that they did not know which diagram they needed to draw for each of the given problems.

Can you name any three types of diagrams you used in solving the probability problems **and state why you chose to use them.**

Thabo: *Venn diagram,… contingency table, and tree diagram.*
Minky: *Yes. I used diagrams. Like in question 1 I draw a tree diagram.*
Pem: *I did not use diagrams.*
Researcher (Probing): What made you not to use diagrams?
Pem: *In question 3 the diagram was already drawn for use while in questions 1 and 2 I did not know which diagrams to draw.*
Sim: *I only used the table given in question 3.*
Mbali: *I did not use any diagram except the one given in question 3.*

Figure 21a: Question 3 of Interview 1 and some of the responses by respondents.

The participants only relied on the contingency table that was already drawn in Question 3. Only one participant drew a sketchy tree diagram that had no probabilities.

What other diagrams did you use in class to solve similar probability problems?

Mention other diagrams that can be used to solve probability problems.

Thabo: *Venn diagram,… contingency table, and tree diagram.*

Minky: *Venn diagram, tree diagram, and conti…gency table.* **Pem**: *Tree diagram, Venn diagram and contingency table.*

Sim: *Venn diagram and tree diagram.*

Mbali: *A Venn diagram, tree diagram, and the…table.*

Figure 22b: Question 4 of Interview 1 and some of the responses by respondents.

The respondents showed that they were aware of some diagrams that could be used to visually represent probability problems. They mentioned the Venn diagram, the probability tree diagram, and the contingency table, though they seemed not to remember its correct name. Some of the participants were even forgetting the names of the diagrams (Minky, in Figure 22). This showed that though they learned these diagrams, they did not practice using them to solve probability problems.

Do you think these probability problems could have been solved in a better way without using diagrams? If yes, how can it be done?

Thabo: *No. It would not be easy because some questions are not easy to understand.*

Minky: *No. I don't think there is a better way. The diagrams are the best.*

Pem: *Yes, I don't think there is a way besides using diagrams.*

Sim: *It is better when one uses the diagrams because they make the question easy to understand.*

Mbali: *No. Diagrams make the probability problems easy to understand. The diagrams also make it easy to get the answers.*

Figure 23: Question 5 of Interview 1 and some of the responses by respondents.

[redacted]

Thabo: *It is necessary because it makes the question easy to understand.*

Minky: *I think it is necessary.*

Pem: *No. They are not important. In question 3 the diagram was drawn and in questions 1 and 2 I could not draw any diagram.*

Sim: *I think it is a necessary to use diagrams in solving the probability problems.*

Mbali: *It is necessary. It makes probability to be understandable and very clear.*

Figure 24: Question 6 of Interview 1 and some of the responses by respondents.

Though the participants did not use diagrams to visually represent the probability problems, they all agreed that the diagrams were necessary in the solving of probability problems. They mentioned that diagrams make the probability problems easy to understand and that diagrams also make it easy to get the solutions to the probability problems.

Was there any of the questions that you could not solve? If yes, what could be the cause? Mention the question and the challenges you encountered with the question.

Thabo: *It was the first and the second questions because I did not know which diagrams to draw. Then in question three, it was easy to understand because the diagram was given.*

Minky: *It is question 1. It confused me because with a tree diagram there is a part that I do not understand about it. I know how to draw it but I cannot put the probabilities.*

Pem: *Question1. I answered but I was not sure how to answer it. The problem was that the question was not specific. I had also forgotten the formula for probability that could be the reason why I struggled with the questions. Another challenge was that I failed to understand the questions.*

Sim: *It was only question 1 in which I could not understand what and how to solve it.*

Mbali: *I failed to answer question 1 and question 2 because I did not practice probability, and I could not use any diagrams.*

Figure 25: Question 7 of Interview 1 and some of the responses by respondents.

All the participants mentioned Questions 1 and 2 as questions that challenged them or as questions that they could not solve. They raised the issue of language as one of the contributing factors. They could not understand what each question required. They also mentioned that they were confused as to what diagram to draw for each probability problem. One of the participants noted that they had challenges with the probability tree diagram, especially when it came to putting the correct probabilities on the branches.

What suggestion can you give to a person who sets the probability questions to make **them understandable to learners?**

Thabo: *He or she can make the questions in simple English because most of us do not understand the English.*
Minky: *They must make sure the questions are clear. They must not try to be tricky when asking questions. If the question requires a diagram to solve, it must be mentioned diagram to use.*
Pem: *When the person sets the questions, he/she must give examples of what the question requires because one ends up not knowing what the question requires. The formula must also be given.*
Sim: *It was only question 1 in which I could not understand what and how to solve it.*
Mbali: *I failed to answer question 1 and question 2 because I did not practice probability, and I could not use any diagrams.*

Figure 26: Question 8 of Interview 1 and some of the responses by respondents.

As language was mentioned earlier as a barrier to understanding the probability problem, the participants noted that for probability problems to be understandable to learners, they must be written using simple understandable English. They also noted that the questions should be set such that they are clear and not tricky to the learners. The participants also said that mentioning the specific diagram to be drawn for each probability problem can help to simplify the problem. Pem also mentioned that the probability diagram must also be given with the probability problem. This indicated that Pem was still operating within the action mental structure in probability.

What advice can you give to fellow learners on how they can successfully solve probability problems?

Thabo: *I would advise them to improve the understanding of English and to be able to draw probability diagrams such as the Venn diagram, and the contingency table.* **Minky**: *They must take more time practising how to solve probability problems. If they encounter problems, they must ask.*

Pem: *The first thing is that they need to ensure that they understand the statement of the question. They must also know the formulae to use in the questions.*

Sim: *They need to practice solving probability problems frequently so that they can master Probability.*

Mbali: *They should just learn to understand the statement first and learn to use diagrams because they make it much easy to answer the questions.*

Figure 27: Question 9 of Interview 1 and some of the responses by respondents.

The participants suggested that fellow learners need to improve their understanding of English as it becomes a barrier to the understanding of probability problems. They mentioned that an improvement in the English language will assist them to understand what each question requires. The participants also suggested that learners need to learn how to draw probability diagrams to visually represent probability problems as it simplifies and clarifies them.

What do you think teachers need to emphasize when teaching probability problem solving?

Thabo: *They must emphasize that learners should make sure they understand the question and then decide on the diagram to draw.*

Minky: *Emmhh…. They need to emphasize on the tree diagram since it is difficult.* **Pem**: *The teacher needs to emphasize that learners understand the statement of the question before they answer.*

Sim: *Teachers need to emphasize the use of diagrams to solve probability problems. If there are certain words that important for the question, the teacher needs to emphasize those words.*

Mbali: *I think they should emphasize the importance of using diagrams because they make probability to be understandable and easier.*

Figure 28: Question 10 of the Interview and some of the responses by respondents.

Thabo mentioned that teachers must emphasize that learners should make sure they understand the question and then decide on the diagram to draw. This was, in fact, the general feeling of all the participants. They felt that emphases by teachers should be on ensuring that the learners understand the probability problem before trying to work-out its solution. They also felt that the use of diagrams to visually represent probability problems also needed to be emphasised.

5.6 INTERVIEW TWO

Did the questions make sense to you? Were you able to understand the English used and the phasing of the question? Were the instructions clear on what the questions require you to do?

Thabo: *Yes. They were understandable though some of them were difficult.*

Minky: *Yes I was able to understand the questions.*

Pem: *Yes, they were understandable.*

Sim: *Ehhh, some questions were clear, some were difficult to understand.*

Mbali: *Some of the question I did not understand such as questions three, four, and five. Others I understood but they were difficult for me to solve.*

Figure 29: Question 1 of Interview 2 and some of the responses by respondents.

The participants highlighted that some of the questions were understandable while others were difficult to understand. They mentioned Questions 3, 4 and 5 as questions that were difficult to understand. They went on to note that there were some of the questions beside the three mentioned here that they understood but failed to solve. This implied that failure to find the solution to the probability problems was not only due to failure to understand the problem but also due to lacking the development of the necessary mental structures required to solve the probability problem.

How did you approach the first three questions? Did you use a formula or a diagram to find the solution?

Thabo: *I used diagrams to make the questions understandable.*

Researcher (Probing): What type of diagrams did you use?

Thabo: *I used the contingency table.*

Minky: *No, I did not use any diagrams in questions one, two, and three. I used the formulae for probability.*

Pem: *In questions one and two, I used formulae but in question three I used a table.* **Sim**: *For the first question I used a formula, for the second and the third questions I used diagrams.*

Mbali: *In questions number one, two, and three I used the table to make the statements understandable. That helped me to understand the questions.*

Figure 30: Question 2 of Interview 2 and some of the responses by respondents.

Some of the participants used a simple table to represent the given information to make it understandable and then wrote the probabilities using the values taken from the table without substituting the probability formula. Being able to write the probabilities without starting by writing the formula meant that these learners had developed the process mental structure of probability. Other respondents only substituted in the probability formula to work-out the solutions to the problem. These participants showed characteristics of operating in the action mental structure of probability understanding.

Did you use a diagram to solve questions four, five, and six? What diagram did you use and how did you use the diagram to help you work-out the required solution? Why did you choose that diagram?

Thabo: *Yes. I used a tree diagram in question four, in question five I did not use any diagram, and in question six I tried to use a Venn diagram.*

Minky: *In question four I tried to use the tree diagram but…it did not work. In question five I did not use any diagram, but I used my general knowledge (the coin has two sides, tail and the tail and that some dice has six sides).* **Pem**: *No. I did not realise they could be used.*

Mbali: *I did not answer questions four and five. In question six I used a Venn diagram to work-out the solutions.*

Researcher (Probing): Did the Venn diagram help you to find the required solutions in question six?

Mbali: *No. I failed to complete the diagram and hence I did not find the solution.*

Figure 31: Question 3 of Interview 2 and some of the responses by respondents.

Did the diagrams you chose help you to find the solution to the questions four, five, and six?

Thabo: *Esh… its only question six where the diagram helped me. In questions four and five the information given was limited. I could not complete the diagrams.*

Pem: *It helped me to know which numbers I can substitute in the formula for probability for the event and the sample space.*

Researcher: Did you use any diagram to solve question seven, eight, nine, and ten?

Pem: *I did not use any diagram except the one that was given with the question. For the other questions, I used the probability formula.*

Sim: *I only used a diagram in question six (a Venn diagram). I tried to use a formula in question three and questions four and five, I failed to find the solutions.*

Mbali: *No. I just found those questions difficult for me.*

Figure 32: Question 4 of Interview 2 and some of the responses by respondents.

In Question 4, 5, and 6 the participants generally struggled to find the correct solutions. Some tried to draw the probability tree diagram in Question 4 and a Venn diagram in question 5 but failed to complete both diagrams using the given information in the probability problems. These participants could be classified under the object mental structure of development in probability understanding. Failure to complete the diagrams they decided to draw could mean they had not yet fully developed the object mental stage of the APOS framework of cognitive development. Other participants did not even attempt to write the solutions to these questions. This could be an indication of underdeveloped mental structures. It could imply that these participants were either at the action stage or even below it.

Did you use diagrams to solve questions eight, nine, and ten? What diagrams did you use and how did you use them?

Thabo: *I did not use diagrams. I did not know which diagrams to use. In fact, I failed to understand the questions.*
Minky: *No, I did not. I only used the one given in question ten.*
Researcher (Probing): Why did you not use a diagram in answering questions eight, and nine?
Minky: *I did not have any idea on what diagram to draw.*
Pem: *I did not use any diagram except the one that was given with the question. For the other questions, I used the probability formula.*
Sim: *I did not use any diagrams.*
Mbali: *I did not answer questions eight, nine, and ten. The questions were difficult.*

Figure 33: Question 5 of Interview 2 and some of the responses by respondents.

In question nine you were required to predict the number of days John will be absent at school in a year. How did you approach that question?

Thabo : *I just used the fact that there are 360 days in a year and that he will be at school for 250 days. I subtracted 250 from 360 to get 110, which is the number of days he will be absent from school (which was totally out of point).*

Minky: *No. I did not know where to begin.*

$(E) = \frac{(\ )}{(\ )}$

Pem: *I first wrote the formula for probability, then I took the number of absent from… school and the number of days in a year. In fact, I failed to understand the question.*

Sim: *I did not answer question nine. I skipped it because I did not know how to solve it.*

Mbali: *I did not answer the question because I did not know how to approach it.*

Figure 34: Question 6 of Interview 2 and some of the responses by respondents.

Were you able to complete the contingency table in question ten? If yes, explain how you solved it. If you could not solve, what do you think was the challenge?

Thabo: *Yes. I just divided the twenty into two to find two numbers that will add-up to 20 and found that in a I have to put 10and also put 10 for b.* For c *I took 48 and subtracted 10 and got 38. In part „d" I just said 12 minus 10 and got 2.*

Researcher (Probing): Why did you think you need to divide 20 into two 10s in you first step?

Thabo: *It is just how I thought of it. I just decided to use 10 and 10 to get the twenty.*

Minky: *I tried but I am not sure if I did it correctly.*

Researcher (Probing): Can you explain how you got the numbers starting from „a" up to the last one?

Minky: *…Esh…in „a" I got 9, in „b", I got 11, in „c" I got 39, and in „d" I got 1.*

Probing: Can you explain how you were getting these numbers?

Minky: *I was just guessing by picking any numbers that add-up to the given number like 20, I used 9 and 11.*

Pem: *Yes, I was able. Hmmm… I added 10 and 10 to find the numbers for „a" and „b" because when you add them they give 20. I then I subtracted 10 from 48 to get 38 for „c", they I subtracted 10 from 12 to get 2 for „d".*

Researcher (Probing): How did you decide that the first number should be 10?

Pem: *Hmmm…I just decided to use it.*

Sim: *For „a" and „b" I just took 14 and 6 which are numbers that add-up to 20. For „c" and „d" I used the 6 and the 14 to work the out.* **Mbali**: *I tried.*

Researcher (Probing): Can you explain how you got the missing numbers for the contingency table given in question 10?

Mbali: *I used the given totals to think of numbers that add-up to the given total. For example, I used 14 and 6 as numbers that add-up to 20. I just chose numbers which were easy for me to add-up to get to the total.*

Figure 35: Question 7 of Interview 2 and some of the responses by respondents.

The participants were greatly challenged by Questions 8, 9, and 10. Some did not even attempt to write the solutions to these problems. During the interviews they mentioned that they did not know where to begin answering the questions. Others tried to solve the problems using the probability formula which did not help. This might be caused by the fact that the participants had a challenge with the language. The challenge might also be emanating from the fact that the participants did not know which diagram they could use to solve these problems. This suggested that these learners were not familiar with these probability problems.

5.7 OBSERVATION SCHEDULE

The following activities by the respondents in Figure 36 were observed by the investigator during their writing of the two task-based activities.

1. Do learners take time to read and understand the probability problem before writing anything?

___

2. Do learners use any diagrams to help them solve the probability problems?

___

3. What type of diagrams do learners used to solve the probability problems?

4. Do learners take time to decide on the type of diagram to use to solve each probability problem?

5

Do learners draw formal diagrams or rough sketches to solve probability problems?

6. Are the diagrams completed correctly as per the information provided?

7. Are the solutions given by learners reasonable and in the context of probability?

Figure 36: The aspects observed by the investigator from respondents‟ activities.

5.8 DATA FROM OBSERVATION SCHEDULE

The participants did not take time to read the questions carefully before attempting to answer the question in the first task-based activity. They rushed into trying to work-out the solutions. This was evident in the time they took reading before starting to write the solutions. Most of the participants would start by trying to write something. If they fail to come-up with a working solution, they will then go back and start reading and analysing the question. Three of the five participants also finished the activities well before the set time yet most of their solutions were incorrect. The first and most important step of problem solving by Polya's (1976) which is to understand the problem, seemed to be the missing link with most of the respondents.

The participants could not use probability diagrams to solve the problems in the first task-based activity. This could be linked with failure to understand the problem which could be caused by the fact that the questions were set in their second language with which they were not comfortable. This raised the issue of language as a contributing factor in solving probability problems. With the second task-based activity they tried to use diagrams though most of them could not complete the diagrams especially the probability tree diagram. They drew the tree diagram but failed to put the correct probabilities on the branches. Others were not sure which diagram to draw for the different questions. They ended up drawing wrong diagrams. Some of their diagrams were only sketches and not complete probability diagrams.

In the second task-based activity the diagram (contingency table) was given and the participants only had to complete it by putting the missing values first before attempting to answer the questions. The participants could not correctly complete the contingency table hence they could not find the correct solution to the question. The respondents needed to have developed the actions, processes, and objects understanding of probability that they needed and be able to organize and link them into a coherent framework called a schema. This required the respondents to have developed a schema for probability in mathematics to be able to decide when presented with a particular probability problem whether probability as a schema is applicable to solve it. Failure by the participants to use the concept of probability to make predictions and failure to use the concept of independent events in probability to complete the contingency table in question 10 highlighted that all the participants did not possess the right action, processes, and objects mental structures to be able to form a schema for probability. This, therefore, made them to fail to relate probability to making predictions and relating the term independent to specific probabilities and the contingency table.

5.9 CONCLUSION

This chapter presented all the data that was collected using task-based activities, semi structured interviews, and observation schedule. The data was presented and analysed to understand the perspectives of the participants with regard to grade 11 learners' use of diagrams when solving problems in probability in mathematics. The data was analysed using the APOS framework of mental development with specific reference to grade 11 probability in the context of South Africa. The participants showed an improvement in their responses to the second task-based activity as compared to the first one. They also showed an improvement in the use of diagrams to visually represent the problems. Most of the responses of the participants suggested that they operated in the action and the process mental levels of cognitive development in probability. It was also noted, though, that one participant could show characteristics of being in one level of mental development with one problem and the same participant could show signs of possessing characteristics of another different level of mental development in the APOS framework of cognitive development with another probability problem.

The next chapter will present the findings of this study based on the analysis of the data collected in this study. The conclusion drawn from this study will then follow from the research findings. The next chapter will, therefore, be specifically guided by the research questions, the research objectives, the literature review, and the findings from the primary data of the study. The research recommendations will then follow the research conclusions.

CHAPTER 6

6.0 CONCLUSIONS AND RECOMMENDATIONS

6.1 INTRODUCTION

In chapter 1 of this research study the general introduction to the study was presented. This introduction included explaining the background to the study and the problem leading to it. The aim, objectives, and the research questions guiding the study were specified together with the significance of the study. In this chapter the literature related to the study was reviewed. The review looked at visualization as a creation of a mental image. The characteristics of visualization in different disciplines and their use in teaching were discussed. The review also covered mathematical problem solving as an art that can only be mastered through practice by problem solvers. The language used by educators when posing questions and by problem solvers when responding to questions was also discussed as a crucial factor in mathematical problem solving in general and in probability problem solving in particular.

Probability problem solving was highlighted as a unique challenge to problem solvers. External visual representations were specially suggested as a possible solution to the challenge by authors such as Corter and Zahner (2007), Bantanero et al. (2004), and Verbert and Duval (2014). They all concurred that visualization could aid comprehension during problem solving. It was, however, noted in the same discussion that Presmeg (2016) highlighted that the mere use of visual representations and manipulatives do not guarantee a correct solution. From all these deliberations in chapter 2 it was clear that the important role played by visual representations in probability problem solving cannot be questioned.

The theoretical and conceptual framework for the study was discussed in chapter 3. The APOS theory as guided by the constructivist paradigm was discussed to explain how it fitted to the study. The APOS theory was defined and discussed using amongst other authors Vygosky (1997), Simon et al. (2004), Mudaly (2012), Arnon et al. (2014), to name a few. The action, process, object, and schema mental structures were discussed and linked with probability problem solving.

Chapter 4 gave a detailed account of how the study was conducted. It began with a discussion of the qualitative research design as the design that was followed in this study. The target population and the sampling strategy employed in this research were explained. The purposeful non-probability sampling was used as it suited the qualitative design. The instruments used for collecting data for the study were also presented in this chapter. This was followed by a discussion of the how the instruments were tested for validity and reliability in the pilot study. A detailed account of how the data were collected and analysed in this study was discussed. The chapter ended by discussing the limitations of the study and the ethical considerations to disclose possible biases and to protect the participants respectively.

The data that were collected in this study was presented, discussed, and analysed in chapter 5. The presentation included responses of the participants to the two task-based activities and the two semi-structured interviews. The data collected using the observation schedule were also presented, discussed, and analysed in this chapter. The discussions and analysis presented in chapter 5 were based on the literature reviewed in chapter 2 and the APOS theoretical framework discussed in chapter 3 with the aim of understanding the perspectives of the participants regarding grade 11 learners' use of diagrams when solving problems in probability in mathematics.

In this chapter, the research findings and conclusions were presented based on the literature review and the primary research. The chapter then presented the research findings in response to the critical questions of the study. The research findings addressed the types of diagrams that grade 11 learners use when solving probability problem in mathematics, how they use diagrams to solve probability problems, and the reasons why they use these diagrams when solving probability problems in mathematics. The conclusions drawn from the research findings were also discussed in this chapter. The research recommendations were then drawn from the conclusions of this study. This chapter was, therefore, specifically guided by the research questions, the objectives of the study, literature reviewed in chapter 2, and the primary data collected during the study.

6.2.0 RESEARCH FINDINGS AND CONCLUSIONS

The research findings from the primary data will be presented based on the research questions and the research objectives. It is for this reason that the research questions and research objectives are revisited before these findings.

6.2.1 RESEARCH QUESTIONS

The research will answer the following questions:

1. What diagrams do grade 11 learners use when solving probability problems in mathematics?

2. How do grade 11 learners use these diagrams when solving probability problems in mathematics?

3. Why do grade 11 learners use these diagrams when solving probability problems in mathematics?

6.2.2 RESEARCH OBJECTIVES

The objectives of this study are:

1. To determine the types of diagrams that grade 11 learners use when solving probability problems in mathematics.

2. To determine how grade 11 learners use these diagrams in solving probability problems in mathematics.

3. To determine the reasons for grade 11 learners to use these diagrams when solving probability problems in mathematics.

6.2.3 THE DIAGRAMS USED BY GRADE 11 LEARNERS WHEN SOLVING PROBABILITY PROBLEMS IN MATHEMATICS

The literature reviewed in this study revealed that in addition to diagrams, pictures, tables, and graphs, mathematical and arithmetic inscriptions also have visual-spatial aspects as well as symbolic content (Kirshner & Awtry, 2004; Landy & Goldstone, 2007; Presmeg, 1986). The probability tree diagram, Venn diagram, contingency table, the fundamental probability formula and all the symbols associated with probability were revealed by the literature for this study as the visual presentations associated with probability problem solving (Bocherer-Linders, Eichler, & Vogel, 2015; Corter & Zahner, 2007).

The types of diagrams that grade 11 learners use when solving probability problems in mathematics included simple tables of values, probability tree diagrams, Venn diagrams, and contingency tables (chapter 5: Figure 14). Grade 11 learners registered their concern about the difficulty of deciding on the suitable diagram to use for a given probability problem (chapter 5: Figure 26). They also mentioned that the probability tree diagram is difficult to complete. Learners showed particular interest in the contingency table though it challenged them when it required them to apply probability concepts such as independent event to complete it.

It was gathered from this study that grade 11 learners lack practice in the use of diagrams to solve probability problems. When interviewed, they mentioned that guiding them on which diagram suits a particular probability problem can assist them in finding the solutions. This was also evident in the way they responded to questions which had the diagram already drawn (chapter 5: Figures 9 & 16). All participants attempted such questions. It was also revealed by this study that all the participants were below the schema level of understanding as they all could not relate or apply probability concepts in solving real life problems such as making predictions.

6.2.4 GRADE 11 LEARNERS' USE OF DIAGRAMS WHEN SOLVING PROBABILITY PROBLEMS IN MATHEMATICS

It was revealed by the literature that visualization forms are applied in mathematics as a powerful and even artistic means of expressing information by many problem-solvers (Vande Moere & Purchase, 2011). This study revealed that grade 11 learners always use diagrams when solving probability problems. It was discovered though that grade 11 learners still had difficulty with making the right decision on the right diagram to use in different probability problems (chapter 5: Figure 18). The study also revealed that probability problems where the diagram was already drawn were easier for learners than those probability problems where the diagram was not given or suggested. It was, however, noted that grade 11 learners had a challenge with completing the Venn diagram and the probability tree diagram. They were able to complete a contingency table that did not have conditions attached to it but could not complete the contingency table that had conditions attached (chapter 5: Figure 21). Grade 11 learners also raised the issue of language as a barrier to them in solving probability problems. They emphasized that the use of the foreign language hindered them from making the right choice of diagram to use for solving the probability problem (chapter 5: Figures 22 & 25). The literature studied in this study also emphasizes the importance of the language used in teaching and learning as an important factor in mathematical problem solving (Radford, 2003; Lakatos, 1976).

6.2.5 THE REASONS WHY GRADE 11 LEARNERS USE DIAGRAMS WHEN SOLVING PROBABILITY PROBLEMS IN MATHEMATICS

Visualization has been singled out as one of the techniques that assist problem solvers in making decisions under uncertainty in mathematics education (Speigelhalter & Gage, 2014). One reason for this special role of visualization in mathematics education is that it can relate the theoretical problem to real life situations. Errors according to Binder, Krauss, and Bruckmaier (2015) can be minimized by representing the problem using different forms of visualization models. The use of diagrams in probability problem solving was revealed by the literature studied as one way of minimizing error. Due to the abstractness of probability problem solving, Russell (2000) emphasizes the need to use external visual representations such as diagrams to solve the probability problems. The use of diagrams helps the problem solver to understand the problem (Corter & Zahmar, 2007). Literature also revealed that the use of diagrams in mathematical problem solving facilitates the process of problem solving. Other recent studies on the spontaneous use of external visual presentations in mathematics problem solving have also emphasized the importance of external presentations in facilitating the problem-solving process (deHevia & Spelke, 2009; Eden & Porter, 2008; Hear & Shneiderman, 2012). Grade 11 learners used diagrams to facilitate their understanding of the problems. They also used diagrams as an alternative way of representing the probability problem in a way they could understand as they noted that they had a challenge with the language used (chapter 5: Figures 26 & 28).

6.3 CONCLUSIONS

The chapter reviewed the critical questions and the objectives of the study. The findings and conclusions of this research were also presented under the three research questions based on the literature studied and the findings from the participants. The recommendations derived from the findings and conclusions of this research were also outlined. Lastly, the issue of language as a challenge to understand probability problem solving as highlighted by the grade 11 learners was revisited. The use of diagrams in probability problem solving was, therefore, revealed by this study as one way of making the problems understandable to learners.

6.4 LIMITATIONS OF THE STUDY

This study was conducted with only six grade 11 mathematics learners in the same school. It is therefore not possible to generalise the findings of this study to other situations such as a full mathematics class. The fact that the participants were also from one school also compromises the generalizability of the research findings to other schools which may have different conditions. The six learners sampled were removed from the rest of their colleagues for the second task-based activity and interviews of this study which could pose a limitation to implementing the finding to a normal class scenario. Another limitation to the study is the mathematical backgrounds of the participants as they were selected from three grade 11 mathematics classes which were not taught by the same educator and were also not taught by the same educators in their lower grades. Lastly the use of audio recording devices to record the responses of the participants during the semi-structured interviews can also be viewed as a limitation to the study as it was the first time for some of the participants to be audio recorded and they showed signs of being less comfortable during the interview sessions.

6.5 RECOMMENDATIONS

Based on the findings of this study and the review of related literature, the following recommendations can be made:

- Educators need to emphasize the use of and use probability diagrams during the teaching of probability problem solving.
- Grade 11 learners must be encouraged to use diagrams to solve probability problems.
- Grade 11 probability problems should be set in simple language that learners can understand.
- The issue of language as a contributing factor to solving of probability problems and mathematical problem solving in general requires further study.
- Further study that will cover a wider scope is recommended to curb the issue of generalizability.
- It would also be worth trying the approach in other classes and other grades. This would help to ascertain if the approach is manageable and realistic for teachers and students to use as part of everyday mathematics teaching and learning.
- The use of diagrams in other topics in mathematics and in other subjects is worth a further investigation to ascertain its applicability in other topics and in different subjects.

6.6 CONCLUSION

The study was able to achieve its aim and objectives in that it was able to establish the type of diagrams that grade 11 learners use when solving probability problems. The research findings together with the reviewed literature related to this study also revealed how and why grade 11 learners use these diagrams during probability problem solving to help them to discover the solutions. These diagrams included amongst others the probability tree diagrams, Venn diagrams, and the contingency table. The study also succeeded in determining the challenges that grade 11 learners are faced with during probability problem solving. These challenges included amongst others the issue of language. Learners raised the point that their first challenge when solving probability problems was to understand the foreign language (English). They also mentioned that they are also challenged by the decision of the type of diagram to use for a particular probability problem. It was established in this study that the success of using these diagrams to aid probability problem solving lay in using simple understandable language when setting probability questions. It also lay in both educators and learners practising and getting used to using these diagrams during teaching and solving probability solving respectively.

The research was conducted with one school in one circuit in Durban in KwaZulu-Natal province. The size of this study, therefore, poses a limitation on the generalization of its findings. Further study can, therefore, be done with other schools in the circuit and beyond to check if they yield similar findings regarding the use of diagrams by grade11 learners when solving probability problems. This study also focused on the grade 11 learners. Future study may be conducted to explore the use of diagrams during probability problem solving by other grades in the school set-up. The use of diagrams in general mathematical problem solving may also be studied to check if it yields similar findings.

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APPENDIXES

APPENDIX A (TASK-BASED ACTIVITY 1)

Task-based activity 1

Question 1

Sean's lunch box contains four sandwiches and three bananas. He chooses an item of food and eats it. He then chooses another item at random and eats it. He then chooses a third item and eats it. Find the probability that he will:

1.4 first choose a banana, then a sandwich and then another sandwich.

1.5 choose three bananas.

1.6 have chosen two bananas after his third choice.

Question 2

In a grade 11 class there are 30 learners. 16 of them are girls. There are 7 girls and 6 boys with blue eyes. A learner is selected at random to be the class monitor. Find the probability that the class monitor is:

2.1 a girl.

2.2 a boy with blue eyes.

2.3 a boy or a girl with any colour of eyes except blue.

2.4 a girl with any colour of eyes except blue.

Question 3

A school investigated 100 learners to find out the number of learners who arrived at school on time, arrived late, or were absent during a particular week. The results are recorded in the following table.

	Boys	Girls	Total
On time	40	25	
Late	18		25
Absent		3	
Total	65		100

3.1 Copy and complete the contingency table by putting in the missing numbers.

3.2 Calculate the probability of a learner:

- 3.2.1 arriving late for school.
- 3.2.2 arriving on time.
- 3.2.3 being absent if the learner is a boy.
- 3.2.4 arriving late or being absent.
- 3.2.5 arriving late or being absent if the learner is a boy.

Demographic Information

Learner's Name:____

Learner's Age:_____________________

Learner's Gender:__________________

Learner's Grade:___________________

APPENDIX B (TASK-BASED ACTIVITY 2)

Task-Based Activity 2

Question 1

There are nine beads in a bag, three are red, three are yellow, two are pink, and one is blue. What is the probability of picking a yellow bead?

Question 2

A box contains three red sweets, five white sweets, and 2 yellow sweets. If a sweet is taken out of the box at random, what is the probability that it will be a red sweet?

Question 3

A blue (B) and green (G) bucket are filled with balls. The blue bucket contains 5 white (W) and 3 red (R) balls. The green bucket contains 2 white and 7 red balls. A bucket is randomly selected, and one ball is thereafter randomly drawn from the bucket. Determine the probability that a red ball is selected.

Question 4

A bag contains one blue cube (B), two green cubes (G), and three red cubes (R). One cube is taken out at random from the bag. Its colour is recorded, and it is not replaced. A second cube is then taken at random. Determine the probability as a fraction in its simplest form that at least **one** green cube is taken.

Question 5

A coin is tossed and a dice is thrown at the same time. Determine the probability of getting a head and an even number.

Question 6

In a class of 40 learners the following information is TRUE:

- 7 learners are left-handed
- 18 learners play soccer
- 4 learners play soccer and are left-handed
- All 40 learners are either right-handed or left-handed

Determine the probability that a learner chosen at random from this class is right-handed and plays soccer.

Question 7

The probability that a tennis player has no injuries (NI) is 0.7. The probability that she will win a game (W) if she has no injuries is 0.9. When the tennis player has injuries (I), the probability of her winning becomes 0.45.

Calculate the probability of her winning her next tennis game.

Question 8

If you throw two dice at the same time, the probability that you will get six shown on exactly one of the dice is $\frac{10}{36}$ and the probability that you will get a six shown on both dice is $\frac{1}{36}$.

What is the probability that you will not get a six at all when you throw the two dice at the same time?

Question 9

Given that the probability that it is sunny (S) on any day in Durban is **0.8**. Given also that the probability that John will be absent (A) from school if it is sunny is **0.1** and if it is rainy (R), the probability that John will be absent from school is **0.2**. Given also that John is expected to be at school for 250 days in a year, predict the total number of days John will be absent from school in the whole year.

Question 10

A survey was conducted in which 60 people were asked about which hand they write with and what colour hair they have. The results of the survey are shown in the table that follows.

		HAND USED TO WRITE WITH		
		Right	**Left**	**Total**
HAIR COLOUR	**Light**	**a**	**b**	**20**
	Dark	**c**	**d**	**40**
	Total	**48**	**12**	**60**

The survey concluded that the „hand used for writing" and the „hair colour" are independent events.

Calculate the values of **a**, **b**, and **c**.

APPENDIX C (INTERVIEW SCHEDULE 1)

Interview Schedule 1

1. Did the questions make sense to you? Were you able to understand the English used and the phrasing of the question?

2. Did you use any diagrams to work-out the answers to the problems?

3. Can you name any three types of diagrams you used in solving the probability problems and state why you chose to use them.

4. What other diagrams did you use in class to solve similar probability problems? Mention other diagrams that can be used to solve probability problems.

5. Do you think these probability problems could have been solved in a better way without using diagrams? If yes, how?

6. In your opinion, is the use of diagrams to solve probability problems a necessity or a waste of time? Motivate your option.

7. Were there any of the questions that you could not solve? If yes, what could be the cause? Mention the question and the challenges you encountered with the question.

8. What suggestion can you give to a person who sets the probability questions to make them understandable to learners?

9. What advice can you give to fellow learners on how they can successfully solve probability problems?

10. What do you think teachers need to emphasize when teaching probability problem solving?

APPENDIX D (INTERVIEW SCHEDULE 2)

Interview schedule 2

1. Did the questions make sense to you? Were you able to understand the English used and the phrasing of the question? Were the instructions clear on what the questions require you to do?

2. How did you approach the first three questions? Did you a formula or a diagram to find the solution?

3. Did you use a diagram to solve questions four, five, and six? What diagram did you use and how did you use the diagram to help you work-out the required solution? Why did you choose that diagram?

4. Did the diagrams you chose help you to find the solution to the questions four, five, and six?

5. Do you think you could have used an alternative method to work-out the solutions to questions four, five, and six?

6. Did you use diagrams to solve questions eight, nine, and ten? What diagrams did you use and how did you use them?

7. In question nine you were required to predict the number of days John will be absent at school in a year. How did you approach that question?

8. Were you able to complete the contingency table in question ten? If yes, explain how you solved it. If you could not solve it, what do you think was the challenge?

APPENDIX E (OBSERVATION SCHEDULE)

Observation Schedule

4. Do learners take time to read and understand the probability problem before writing anything?

____
____
____

5. Do learners use any diagrams to help them solve the probability problems?

____
____
____

6. What type of diagrams do learners used to solve the probability problems?

____
____
____

7. Do learners take time to decide on the type of diagram to use to solve each probability problem?

____
____
____

8. Do learners draw formal diagrams or rough sketches to solve probability problems?

____
____
____

9. Are the diagrams completed correctly as per the information provided?

____
____
____

10. Are the solutions given by learners reasonable and in the context of probability?

APPENDIX F (RESEARCH CLEARANCE LETTER)

UNIVERSITY OF KWAZULU-NATAL

INYUVESI YAKWAZULU-NATALI

22 June 2016

Mr Mfanimpela Patrick Shoba 215081317
School of Education
Edgewood Campus

Dear Mr Shoba

Protocol reference number: HSS/0656/016M
Project Title: Exploring Grade 11 learners' use of diagrams when solving problems in probability in mathematics

Full Approval – Expedited Application

In response to your application received 26 May 2016, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and the protocol has been granted **FULL APPROVAL.**

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment /modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

[redacted]

Dr Shamila Naidoo (Deputy Chair)
Humanities & Social Sciences Research Ethics Committee

/pm

Cc Supervisor: Dr V Mudaly
Cc Academic Leader Research: Dr SB Khoza
Cc School Administrator: Ms Tyzer Khumalo & Ms B Bhengu

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APPENDIX G (GATE-KEEPER'S PERMISSION)

GATE-KEEPER'S PERMISSION TO CONDUCT THE STUDY

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Umlazi Circuit
Durban

Re: Application to Conduct a Research: Mr M.P. SHOBA

Your application to conduct a research was received on 25 June 2016. The title of study reads:"**Exploring Grade 11 learners' use of diagrams when solving problems in probability in mathematics**". I trust that the aim and objectives of the study will benefit the education department. Your request is approved subject to you observing the provisions of the departmental research policy available in the department website. You are also requested to adhere to your University's research ethics as spelt out in your research ethics document.

In terms of the attached research policy, data or any research activity can be conducted after hours as per appointment with affected participants. You are also requested to share your findings with the relevant sections of the department so that we may consider implementing your findings if that will be in the best interest of the department and the school.

The school wishes you well in this important project and pledges to give you the necessary support you may need.

[redacted]

(School Principal)

Date: 26-06-2016

APPENDIX H (INFORMED CONSENT FORM)

INFORMED CONSENT FORM

School of Education, College of Humanities, University of KwaZulu-Natal,
Edgewood Campus,

Dear Participant

INFORMED CONSENT LETTER

My name is Shoba Patrick Mfanimpela. I am a student studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am interested in collecting information on solving probability problems. I am interested in asking you some questions. Please note that:

- Your confidentiality is guaranteed as your inputs will not be attributed to you in person but reported only as a population member opinion.
- The interview may last for about 45 minutes to 1 hour.
- Any information given by you cannot be used against you, and the collected data will be used for purposes of this research only.
- Data will be stored in secure storage and destroyed after 5 years.
- You have a choice to participate, not participate or stop participating in the research. You will not be penalized for taking such an action.
- Your involvement is purely for academic purposes only, and there are no financial benefits involved.
- If you are willing to be interviewed, please indicate (by ticking as applicable) whether you are willing to allow the interview to be recorded by the following equipment:

Equipment	Willing	Not willing
Audio equipment		
Photographic equipment		
Video equipment		

I can be contacted at:

Email: …………………………………………… Cell:
……………………………………………..

My supervisor is ……………………………….. who is located at the School of Education, Edgewood campus of the University of KwaZulu-Natal.

Contact details: Email:…………………………………………………..

Phone Number:………………………………………..

You may also contact the Research Office through:

Ms P Ximba (HSSREC Research Office)

Tel: 031 260 3587

Email: ximbap@ukzn.ac.za)

Thank you for your contribution to this research.

DECLARATION

I…………………………………………………(full names of participant) hereby confirm that I understand the contents of this document and the nature of the research project, and I consent to participating in the research project.

I understand that I am at liberty to withdraw from the project at any time, should I so desire.

SIGNATURE OF PARTICIPANT DATE

……………………………………… ……………………..

SIGNATURE OF PARENT (If participant is a minor) DATE

……………………………………… ……………………..

APPENDIX I (INTERVIEW ONE TRANSCRIPTIONS)

INTERVIEW ONE TRANSCRIPTIONS

1. Interview Schedule One (Thabo)

Researcher: Did the questions make sense to you? Were you able to understand the phrasing of the questions and the English used?

Thabo: *They did make sense kancane…kancane (meaning a little bit).*

Probing: Were you able to understand the English used and the phrasing of the questions?

Thabo: *I tried to understand but it was difficult. I only understood a little bit.*

Probing: Were all the questions equally difficult to understand?

Thabo: *Some of them I understood like question two, but I failed to draw the diagram. Question three was easy to understand because I know it.*

Researcher: Did you use any diagrams to work out the solutions to the questions?

Thabo: *I did not use diagrams. I did not know which diagrams to use and how to use it.*

Researcher: Can you mention any three diagrams that are used to solve probability problems.

Thabo: *Venn diagram,… contingency table, and tree diagram.*

Researcher: Do you think probability problems can be solved in a better way without using diagrams?

Thabo: *No. It would not be easy because some questions are not easy to understand.*

Researcher: Is the use of diagrams in probability problem solving necessary or a waste of time?

Thabo: *It is necessary because it makes the question easy to understand.*

Researcher: Were there any of the questions that challenged you? Mention the question and the challenges that you encountered.

Thabo: *It was the first and the second questions because I did not know which diagrams to draw. Then in question three, it was easy to understand because the diagram was given.*

Researcher: What suggestions can you give to the person who sets the probability questions to make them understandable to learners?

Thabo: *He or she can make the questions in simple English because most of us do not understand the English.*

Researcher: What advice can you give to your fellow learners to help them to be able to solve probability problems?

Thabo: *I would advise them to improve the understanding of English and to be able to draw probability diagrams such as the Venn diagram, and the contingency table.*

Researcher: What do you think teachers need to emphasize when teaching solving of probability problems?

Thabo: *They must emphasize that learners should make sure they understand the question and then decide on the diagram to draw.*

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2. Interview Schedule One (Minky)

Researcher: The first question is a general one in that it pertains to all the questions. Were the questions clear to you? Did you understand what each question required you to do? Were you able to understand the English used and the phrasing of the questions?

Minky: *Yes. The English was simple but question 1,… I had a problem since we were not told which diagram to draw. The English was understandable though.*

Researcher: Did you use diagrams in answering these probability problems?

Minky: *Yes. I used diagrams. Like in question 1 I draw a tree diagram.*

Researcher: Mention any types of diagrams used and why you chose to use them.

Minky: *Venn diagram, tree diagram, and conti…gency table.*

Researcher: What other diagrams did you use in class to solve probability problems?

Minky: *It is just only these that I know of.*

Researcher: Do you think these probability problems could have been solved in a better way without using diagrams?

Minky: *No. I don"t think there is a better way. The diagrams are the best.*

Researcher: In your opinion, is the use of diagrams to solve probability problems necessary or a waste of time? Minky: *I think it is necessary.*

Researcher: Were there any of these questions that you failed to solve? You can mention the question and the challenge you had.

Minky: *It is question 1. It confused me because with a tree diagram there is a part that I do not understand about it. I know how to draw it but I cannot put the probabilities.*

Researcher: What suggestions can you give to a person who sets probability questions to make them understandable to learners?

Minky: *They must make sure the questions are clear. They must not try to be tricky when asking questions. If the question requires a diagram to solve, it must be mentioned diagram to use.*

Researcher: What advice can you give to your fellow learners on how they can effectively solve probability [problems?

Minky: *They must take more time practising how to solve probability problems. If they encounter problems, they must ask.*

Researcher: What do you think the teacher needs to emphasize when teaching probability problem solving to make it understandable to learners?

Minky: *Emmhh…. They need to emphasize on the tree diagram since it is difficult.*

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3. Interview Schedule One (Pem)

Researcher: Did the questions make sense to you? Was the English used understandable in all the three questions?

Pem: *Question 1 was a little bit tricky. Question 2 and 3 were understandable.*

Researcher: Did you use diagrams to answer these questions?

Pem: *I did not use diagrams.*

Prob: What made you not to use diagrams?

Pem: *In question 3 the diagram was already drawn for use while in questions 1 and 2 I did not know which diagrams to draw.*

Researcher: Can you name any diagrams you know in probability.

Pem: *Tree diagram, Venn diagram and contingency table.*

Researcher: Do you think these probability problems could have been solved in a better way without diagrams?

Pem: *Yes, I don"t think there is a way besides using diagrams.*

Researcher: Do you think diagrams are important in solving probability problems.

Pem: *No. They are not important. In question 3 the diagram was drawn and in questions 1 and 2 I could not draw any diagram.*

Researcher: Was there a question you could not solve? If yes, what was the reason? You can mention the question and the difficulty you had.

Pem: *Question1. I answered but I was not sure how to answer it. The problem was that the question was not specific. I had also forgotten the formula for probability that could be the reason why I struggled with the questions. Another challenge was that I failed to understand the questions.*

Researcher: What suggestions can you give to a person who sets probability questions to make them understandable to learners?

Pem: *When the person sets the questions, he/she must give examples of what the question requires because one ends up not knowing what the question requires. The formula must also be given.*

Researcher: What advice would you give to other learners to help them to be able to solve probability problems effectively and efficiently?

Pem: *The first thing is that they need to ensure that they understand the statement of the question. They must also know the formulae to use in the questions.*

Researcher: What do you think teachers need to emphasize when teaching probability problem solving to learners for them to be able to solve probability problems? Pem: *The teacher needs to emphasize that learners understand the statement of the question before they answer.*

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4. Interview One (Sim)

Researcher: The first question pertains to the three questions in general. Did the questions make sense to you? Were you able to understand the questions and the English used in the questions?

Sim: *I was able to understand the questions, especially question 2 and 3. It is only question 1 I failed to understand.*

Researcher: Did you use diagrams to work out the solutions to the questions?

Sim: *I only used the table given in question 3.*

Researcher: Can you name any other diagrams you can use when solving probability problems.

Sim: *Venn diagram and tree diagram.*

Researcher: Do you think the probability problems could have been solved in a better way without using diagrams?

Sim: *It is better when one uses the diagrams because they make the question easy to understand.*

Researcher: In your opinion, is the use of diagrams to solve probability problems a necessity or a waste of time?

Sim: *I think it is a necessary to use diagrams in solving the probability problems.*

Researcher: What makes you to think so?

Sim: *The diagrams make it easy to understand the question.*

Researcher: Was there any question in this task which you could not solve? What could be the reasons? You can mention the question and the challenges you faced in that question.

Sim: *It was only question 1 in which I could not understand what and how to solve it.*

Researcher: What do you think made question 1 to be difficult?

Sim: *I think it was difficult for me to understand what the whole question required.*

Researcher: What suggestions can you make to the person who sets the probability problems to make them understandable to learners?

Sim: *The person who sets the probability problems needs use understandable English language so that it can be easy to understand.*

Researcher: What advice can you give to other learners on how they can master solving probability problems?

Sim: *They need to practice solving probability problems frequently so that they can master Probability.*

Researcher: What do you think teachers need to emphasize when teaching probability problem solving?

Sim: *Teachers need to emphasize the use of diagrams to solve probability problems. If there are certain words that important for the question, the teacher needs to emphasize those words.*

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5. Interview One (Mbali)

Researcher: The whole task had three questions. Did the three questions make sense to you? Were you able to understand the English used in the questions?

Mbali: *I was able to answer question three because the table was given, and it made it easy for me to solve the problem. Question 1 and question 2 gave me a hustle because I had forgotten that I must use the diagrams to solve the problems, like the tree diagram, I had forgotten how to draw it.*

Researcher: Did you use diagrams to solve any of the questions?

Mbali: *I did not use any diagram except the one given in question 3.*

Researcher: Can you name any diagrams that can be used to solve probability problems.

Mbali: *A Venn diagram, tree diagram, and the…table.*

Researcher: Do you think probability problems can be solved in a better way without using diagrams?

Mbali: *No. Diagrams make the probability problems easy to understand. The diagrams also make it easy to get the answers.*

Researcher: In your opinion, is the use of diagrams to solve probability problems a necessity or a waste of time?

Mbali: *It is necessary. It makes probability to be understandable and very clear.*

Researcher: Was there a question in the whole task that you could not solve? If yes, what could be the cause? Mention the question and the challenges you encountered. Mbali: *I failed to answer question 1 and question 2 because I did not practice probability, and I could not use any diagrams.*

Researcher: What suggestions can you make to a person who sets probability questions to make them understandable to learners?

Mbali: *Maybe it should be mentioned that we should use a diagram within the question and that can make learners to remember and be able to answer questions.*

Researcher: What advice would you give to your fellow learners on how they can master probability problem solving?

Mbali: *They should just learn to understand the statement first and learn to use diagrams because they make it much easy to answer the questions.*

Researcher: What do you think teachers need to emphasize when teaching probability problem solving?

Mbali: *I think they should emphasize the importance of using diagrams because they make probability to be understandable and easier.*

APPENDIX J (INTERVIEW TWO TRANSCRIPTIONS)

INTERVIEW TWO TRANSCRIPTIONS

1. Interview Schedule Two (Thabo)

Researcher: Were the instructions in all the questions clear for you to understand what each question required of you?

Thabo: *Yes. They were understandable though some of them were difficult.*

Researcher: How did you solve questions one, two, and three? Thabo:

I used diagrams to make the questions understandable.

Probing: What type of diagrams did you use?

Thabo: *I used the contingency table.*

Researcher: Did you use any diagrams for questions four, five, and six?

Thabo: *Yes. I used a tree diagram in question four, in question five I did not use any diagram, and in question six I tried to use a Venn diagram.*

Researcher: Did the diagrams you used help you to solve the problems?

Thabo: *Esh… its only question six where the diagram helped me. In questions four and five the information given was limited. I could not complete the diagrams.*

Researcher: Did you use diagrams to solve questions six, seven, and eight?

Thabo: *I did not use diagrams. I did not know which diagrams to use. In fact, I failed to understand the questions.*

Researcher: In question number nine, you were required to predict the number of days John would be absent from school in a year, how did you work out that question? Thabo: *I just used the fact that there are 360 days in a year and that he will be at school for 250 days. I subtracted 250 from 360 to get 110, which is the number of days he will be absent from school (which was totally out of point).*

Researcher: Were you able to complete the contingency table in question ten?

Thabo: *Yes. I just divided the twenty into two to find two numbers that will add-up to 20 and found that in a I must put 10and also put 10 for b.* For c *I took 48 and subtracted 10 and got 38. For, d I just said 12 minus 10 and got 2.*

Probing: Why did you think you need to divide 20 into two 10s in you first step?

Thabo: *It is just how I thought of it. I just decided to use 10 and 10 to get the twenty.*

Researcher: Were the instructions in all the questions clear for you to understand what

2. Interview Schedule Two (Minky)

each question required of you?

Minky: *Yes, I was able to understand the questions.*

Researcher: How did you approach questions one, two, and three? Did you use any formula or diagrams?

Minky: *No, I did not use any diagrams in questions one, two, and three. I used the formulae for probability.*

Researcher: In questions four, five, and six, what diagrams did you use to help you work out the solutions?

Minky: *In question four I tried to use the tree diagram but…it did not work. In question five I did not use any diagram, but I used my general knowledge (the coin has two sides, tail and the tail and that a dice has six sides).*

Researcher: Did you use diagrams to work out solutions to questions eight, nine, and ten?

Minky: *No, I did not. I only used the one given in question ten.*

Probing: Why did you not use a diagram in answering questions eight, and nine?

Minky: *I did not have any idea on what diagram to draw.*

Researcher: In question nine, the question wants you to predict the number of days John will be absent from school, were you able to work out the solution?

Minky: *No. I did not know where to begin.*

Researcher: Were you able to complete the contingency table in question ten?

Minky: *I tried but I am not sure if I did it correctly.*

Probing: Can you explain how you got the numbers starting from „a" up to the last one.

Minky: *…Esh…in „a" I got 9, in „b", I got 11, in „c" I got 39, and in, d" I got 1.* Probing: Can you explain how you were getting these numbers?

Minky: *I was just guessing by picking any numbers that add-up to the given number like 20, I used 9 and 11.*

Researcher: Were the instructions in all the questions clear for you to understand what

3. Interview Two (Pem)

each question required of you? Pem:

Yes, they were understandable.

Researcher: How did you approach questions one, two, and three? Did you use any formula or diagrams?

Pem: *In questions one and two, I used formulae but in question three I used a table.*

Researcher: Did you use a diagram to solve questions four, five, and six?

Pem: *No. I did not realise they could be used.*

Researcher: Did the diagram (table) you used in question three help you to work out the solutions to the probability problem?

Pem: *It helped me to know which numbers I can substitute in the formula for probability for the event and the sample space.*

Pem: *It helped me to know which numbers I can substitute in the formula for probability for the event and the sample space.*

Researcher: Did you use any diagram to solve question seven, eight, nine, and ten. Pem: *I did not use any diagram except the one that was given with the question. For the other questions, I used the probability formula.*

Researcher: In question nine, the question wants you to predict the number of days John will be absent from school, were you able to work out the solution?

Pem: *I first wrote the formula for probability,* $P(E)=\frac{n(E)}{n(S)}$, *then I took the number of absent from… school and the number of days in a year. In fact, I failed to understand the question.*

Researcher: Were you able to complete the contingency table in question ten?

Pem: *Yes, I was able. Hmmm… I added 10 and 10 to find the numbers for „a" and „b" because when you add them, they give 20. I then I subtracted 10 from 48 to get 38 for*

Researcher: Were the instructions in all the questions clear for you to understand what
„c", they I subtracted 10 from 12 to get 2 for „d".

Probing: How did you decide that the first number should be 10?

Pem: *Hmmm…I just decided to use it.*

---

4. Interview Schedule Two (Sim)

each question required of you?

Sim: *Ehhh, some questions were clear, some were difficult to understand.*

Researcher: How did you approach questions one, two, and three? Did you use any formula or diagrams?

Sim: *For the first question I used a formula, for the second and the third questions I used diagrams.*

Researcher: Did you use a diagram to solve questions four, five, and six?

Sim: *I only used a diagram in question six (a Venn diagram).*

Researcher: In questions four and five what other options did you have to solve the problems as you say you did not use diagrams?

Sim: *I tried to use a formula in question three and questions four and five, I failed to find the solutions.*

Researcher: Did you use diagrams to work-out solutions to questions seven, eight, and nine?

Sim: *I did not use any diagrams.*

Researcher: Question nine required you to predict the number of days John would be absent from school. How did you approach that question?

Sim: *I did not answer question nine. I skipped it because I did not know how to solve it.*

Researcher: Were you able to complete the contingency table given in question ten? Sim: *For „a" and „b" I just took 14 and 6 which are numbers that add-up to 20. For „c" and „d" I used the 6 and the 14 to work the out.*

Researcher: Were the instructions in all the questions clear for you to understand what

5. Interview Schedule Two (Mbali)

Researcher: Were the instructions in all the questions clear for you to understand what each question required of you?
Mbali: *Some of the question I did not understand such as questions three, four, and five. Others I understood but they were difficult for me to solve.*
Researcher: How did you approach questions one, two, and three? Did you use any formula or diagrams?

Mbali: *In questions number one, two, and three I used the table to make the statements understandable. That helped me to understand the questions.*

Researcher: Did you use a diagram to solve questions four, five, and six?

Mbali: *I did not answer questions four and five. In question six I used a Venn diagram to work-out the solutions.*

Probing: Did the Venn diagram helps you to find the required solutions in question six?

Mbali: *No. I failed to complete the diagram and hence I did not find the solution.*

Researcher: Do you think you could have used any other alternative methods to find the solutions to questions four, five, and six besides using diagrams?

Mbali: *No. I just found those questions difficult for me.*

Researcher: Did you use any diagrams to work out solutions to questions eight, nine, and ten?

Mbali: *I did not answer questions eight, nine, and ten. The questions were difficult.*

Researcher: Question nine required you to predict the number of days John would be absent from school. How did you approach that question?

Mbali: *I did not answer the question because I did not know how to approach it.*

Researcher: Were you able to complete the contingency table given in question ten?

Mbali: *I tried.*

Probing: Can you explain how you got the missing numbers for the contingency table given in question ten.

Mbali: *I used the given totals to think of numbers that add-up to the given total. For example, I used 14 and 6 as numbers that add-up to 20. I just chose numbers which were easy for me to add-up to get to the total.*

APPENDIX K (EDITOR'S LETTER)

Angela Bryan & Associates

[redacted]

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27 November 2017

To whom it may concern

This is to certify that the Master's Dissertation: An Exploration into Grade 11 Learners' Use of Diagrams When Solving Probability Problems in Mathematics in a School in Umlazi District in Durban, South Africa written by Mfanimpela Patrick Shoba has been edited by me for language.

Please contact me should you require any further information.

Kind Regards

Angela Bryan

[redacted]

APPENDIX L (TURN-IT-IN REPORT)

An Exploration into Grade 11 Learners' Use of Diagrams When Solving Probability Problems in Mathematics in a School in Umlazi District in Durban

ORIGINALITY REPORT

8%	%	8%	%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	Doris Zahner. "The Process of Probability Problem Solving: Use of External Visual Representations", Mathematical Thinking and Learning, 04/2010 Publication	3%
2	(HTW Dresden, Informatik, Mathematik). "Proceedings of the tenth international conference Models in developing mathematics education", Saechsische Landesbibliothek-Staats- und Universitaetsbibliothek Dresden, 2012. Publication	1%
3	Kazima, Mercy, and Jill Adler. "Mathematical knowledge for teaching: Adding to the description through a study of probability in practice", Pythagoras, 2011. Publication	<1%
4	Van Rinsveld, Amandine, Christine Schiltz, Martin Brunner, Karin Landerl, and Sonja Ugen.	<1%