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Conformal symmetries and classification in shear-free spherically symmetric spacetimes

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This dissertation is submitted to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science, University of KwaZulu-Natal, Durban, in fulfilment of the requirements for the degree of Master of Science, under the supervision of Professor S. D. Maharaj and Dr S. Moopanar.

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As the candidate's supervisors, we have approved this dissertation for submission.

Signed: Prof S. D. Maharaj January 2014.

Signed: Dr S. Moopanar January 2014.

Abstract

In this thesis we study the conformal geometry of static and non-static spherically symmetric spacetimes. We analyse the general solution of the conformal Killing vector equation subject to integrability conditions which place restrictions on the metric functions. The Weyl tensor is used to characterise the conformal geometry, and we calculate the Weyl tensor components for the spherically symmetric line element. The accuracy of our results is verified using Mathematica (Wolfram 2010) and Maple (2009). We show that the standard result in the conformal motions for static spacetimes is incorrect. This mistake is identified and corrected. Two nonlinear ordinary differential equations are derived in the classification of static spacetimes. Both equations are solved in general. Two nonlinear partial differential equations are derived in the classification of non-static spacetimes. The first equation is solved in general and the second equation admits a particular solution. Our treatment is the first complete classification of conformal motions in static and non-static spherically symmetric spacetimes using the Weyl tensor.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Addial Mackingtosh Manjonjo

Declaration 1 - Plagiarism

I, Addial Mackingtosh Manjonjo, declare that

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Dedication

To

Him who sits on the Throne

Contents

Chapter 1

Introduction

The theory of general relativity provides the best description of the behaviour of the gravitational field. In general relativity we take the spacetime to be a four dimensional differentiable manifold endowed with a symmetric metric tensor field. The matter content of the universe can be treated as a relativistic fluid and is described by the symmetric energy momentum tensor. The Einstein field equations relate the matter content to the curvature of spacetimes. Determining explicit solutions to the Einstein field equations is necessary for astrophysical and cosmological applications.

Spacetimes admitting a symmetry play an important role in general relativity. There are different types of vector symmetries on a manifold, namely those generated by Killing vectors, conformal Killing vectors and homothetic vectors. Without imposing a symmetry condition, solving the Einstein field equations becomes a difficult task. Symmetry properties are also used to put structure on the set of solutions of the Einstein equations by a classification scheme based on the groups of motions or other invariant vectors. In this sense, conditions put on the metric tensor field in spacetime would be of interest in obtaining new solutions to the field equations. There are symmetries which arise from geometric quantities other than the metric tensor field; for instance conditions can be placed on the curvature, e.g. Ricci collineations. Some of these have been extensively studied by Maartens and Maharaj (1986), Duggal and Sharma (1999) and Stephani *et al* (2003)

The traditional approach used to find exact solutions of Einstein field equations is to specify the form of the metric and the type of fluid. However this method often leads to physically unreliable astrophysical or cosmological models. One way around this problem is to assume that the spacetime has a conformal symmetry. Therefore conformal symmetries have been extensively studied in the literature. Conformal symmetries have the geometric property of preserving the structure of the null cone by mapping null geodesics to null geodesics. They are physically significant as they generate constants of the motion along null geodesics for massless particles. Conformal symmetries have been applied to cosmology in different spacetimes. The conformal geometry has been studied in Robertson-Walker spacetimes by Maartens and Maharaj (1986) and Keane and Barrett (2000). A detailed analysis of conformal vectors has been undertaken by Maartens and Maharaj (1991) and Keane and Tupper (2004) in *pp*-wave spacetimes. Maartens *et al* (1986) investigated the kinematic and dynamic properties of conformal Killing vectors in anisotropic fluids. Castejon-Amenedo and Coley (1992) and Hansraj *et al* (2005) have considered the applications of conformal symmetries in conformally related spacetimes. Comprehensive analysis of the conformal geometry, especially in spherically symmetric spacetimes, and their kinematical and dynamical quantities were performed by Coley and Tupper (1990a, 1990b, 1994). The conformal Killing vector was explicitly derived in static spacetimes (Maharaj *et al* (1995) and Maartens *et al* (1995, 1996)) and non-static case (Moopanar and Maharaj (2013)) with spherical symmetry. There are various applications of conformal motions in relativistic astrophysics. Herera *et al* (1984) and Herera and Ponce de Leon (1985) use conformal motions in modelling an anisotropic sphere. Rahaman *et al* (2010) studied the role of pressure anisotropy with conformal symmetry. Mak and Harko (2004) studied charged strange stars with a quark equation of state. Esculpi and Aloma (2010) generated anisotropic relativistic charged fluid spheres with a linear barotropic equation of state. Usmani *et al* (2011) extended the concept of a Bose-Einstein condensate to gravity to construct gravastars. Herrera *et al* (2012) studied irreversible dissipative processes and Landau damping in relativistic stellar systems.

For this thesis we follow the work of Maartens *et al* (1995), who examined the classification of conformal motions in static symmetric spacetimes. We complete their analysis for static spacetimes and attempt to extend this classification to the non-static spacetimes.

We now provide an outline of this thesis:

Chapter 2 gives an overview of the elements of differential geometry and general relativity which are essential for the work covered in later chapters of this thesis. The concept of a manifold is briefly explained. This provides the foundation for looking at other important tensors such as the Riemann, Ricci, Weyl and Einstein tensors. The Einstein field equations satisfy the conservation laws through the Bianchi identity. In particular we study the Weyl tensor and its symmetries. In this chapter we also analyse properties of the Lie derivative and its properties. We conclude the chapter by studying the conformal Killing equation. This equation is useful to later calculations and represent an important part of this thesis.

In Chapter 3 static and non-static spherically symmetric spacetimes are examined. We consider the static and the non-static cases separately. The conformal Killing vector equation is solved and the general solution is subject to a set of integrability conditions. This work on conformal motions for these spacetimes has been extensively studied earlier by Maartens *et al* (1995) and by Moopanar and Maharaj (2013). We have used both Mathematica (Wolfram 2010) and Maple (2009) to verify the calculations presented in this and the subsequent chapters. The conformal geometry may be written more compactly by introducing a new transformation.

In Chapter 4 we derive the non-zero Weyl tensor components for the static spherically symmetric spacetimes. We use these components to calculate the Lie derivatives of the Weyl tensor components. The classification depends on whether the Weyl tensor vanishes. We compare our results with those shown in the work of Maartens *et al* (1995) and identify an error in that paper. We then give a complete classification of conformal motions in static spacetimes.

In Chapter 5 we follow the procedure as in Chapter 4 for non-static spherically symmetric spacetimes. By calculating the Lie derivatives of the Weyl tensor components we present a complete classification of conformal motions subject to integrability conditions. Again the classification depends on whether the Weyl tensor components vanish. We generate two nonlinear partial differential equations; the first equation has a general solution and the second equation has a particular solution.

In the conclusion we review the results obtained. We believe some of the results obtained in this thesis are original. Our analysis provides a complete classification of conformal motions in both static and non-static spherically symmetric spacetimes.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter we consider the basic elements of differential geometry and general relativity. Differential geometry is an indispensable tool in analysing the mathematical aspects of general relativity. At the foundation of the subject is the concept of a manifold. A manifold is an abstraction of the concept of a smooth surface in Euclidean space. This generalisation has proved useful in general relativity as we encounter smooth sets which cannot be presented as subsets of Euclidean space. It is on this differentiable manifold that we define the metric tensor field. We also define a number of tensors which form the fundamental part of our study, namely the Riemann, Weyl, Ricci and Einstein tensors. The Weyl tensor in particular is the crucial quantity that enables us to complete the classification of the conformal geometry in subsequent chapters. We then introduce the Lie derivative, Lie algebras and the conformal Killing vector. These provide the framework for the work covered in the remaining chapters. We also briefly introduce the Einstein field equations in general relativity.

2.2 Differential geometry

2.2.1 Differentiable manifolds

An n-dimensional differentiable manifold M is a topological space which locally resembles the structure of Euclidean space $\mathbb{R}^n$. Formally an n-manifold is defined as follows:

Definition 2.2.1 (**Manifold**). An n-dimensional differential manifold M is a connected Hausdorff topological space such that for every neighbourhood $\mathcal{U}_x$ of a point $x \in M$ the following conditions hold:

i) There exists a one-to-one mapping α_x with

$$\alpha_x : \mathcal{U}_x \mapsto \mathbb{R}^n,$$

where α_x is called the map of $\mathcal{U}_x$.

ii) If the neighbourhoods $\mathcal{U}_x$ and $\mathcal{U}_y$, $x, y \in M$, are such that $\mathcal{U}_x \cap \mathcal{U}_y \neq \emptyset$ and if α_x is a map of $\mathcal{U}_x$ and α_y is map of $\mathcal{U}_y$, then the mappings $\alpha_y \circ {\alpha_x}^{-1}$ and $\alpha_x \circ {\alpha_y}^{-1}$ are mappings of $\mathbb{R}^n$ into itself.

For this thesis we restrict our analysis to a 4-dimensional differentiable manifold where each point in M can be uniquely described by coordinates $x^a = (x^0, x^1, x^2, x^3)$, where x^0 is the timelike coordinate and x^1, x^2 and x^3 are spatial coordinates.

2.2.2 Tensors

Tensors are invariant quantities defined on the manifold M. The analysis of physical phenomena does not depend on our choice of a coordinate system. This necessitates the study of tensors as these are quantities that remain invariant in all coordinate systems. Some of the important tensors relevant to our study are described below.

Metric tensors

The main object of study in general relativity is the metric tensor **g**. It is a symmetric tensor of order two and is a function of the coordinates x^a. The invariant distance between two neighbouring points on the manifold M is represented by the line element

$$ds^2 = g_{ab}dx^a dx^b, \tag{2.1}$$

where **g** is non-degenerate and symmetric. The fundamental theorem of Riemannian geometry implies the existence of the metric connection. The connection coefficients are given by

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d}), \tag{2.2}$$

where commas represent partial differentiation. The connection coefficients do not transform tensorially; the definition (2.2) ensures the covariant derivative transforms as a tensor. The components $\Gamma^a{}_{bc}$ are also known as the Christoffel symbols of the second kind.

Riemann curvature tensor

The Riemann curvature tensor, or simply the Riemann tensor **R**, plays a central role in describing the geometry of curved spacetimes. Apart from measuring the extent to which a manifold is locally equivalent to $\mathbb{R}^n$, it is also used to generate other tensors, e.g. Ricci and Weyl tensors. The components of the Riemann tensor are given by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de}. \tag{2.3}$$

The Riemann tensor possesses important algebraic symmetries which may be used to simplify many calculations. These are given below

$$R_{abcd} = -R_{bacd} = -R_{abdc}, \tag{2.4}$$

$$R_{abcd} = R_{cdab}, \tag{2.5}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0. \tag{2.6}$$

These symmetries reduce the number of independent components of the Riemann tensor from n^4 to $\frac{1}{12}n^2(n^2-1)$ in a manifold of dimension n. For a 4-dimensional manifold, the Riemann tensor has twenty independent components. In addition the Riemann tensor also satisfies the Bianchi identity

$$R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = 0, \tag{2.7}$$

where semi-colons represent covariant differentiation.

Ricci tensor, Ricci scalar and Einstein tensor

The contraction of the Riemann tensor across the first and third indices produces the Ricci tensor. This is defined by

$$R_{bd} = g^{ac}R_{abcd}. \tag{2.8}$$

The Ricci tensor has $\frac{1}{2}n(n+1)$ independent components in an n-dimensional manifold; in a 4-dimensional spacetime the number of independent components is ten. From equation (2.5) it follows that the Ricci tensor is symmetric. Further contracting the Ricci tensor, we obtain the Ricci scalar

$$R = g^{ab} R_{ab}. \tag{2.9}$$

The Einstein tensor is defined by

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}. \tag{2.10}$$

The Einstein tensor is divergence-free so that

$$G^{ab}{}_{;b} = 0.$$

This identity is used to derive the conservation laws from the Einstein field equations. This result makes the Einstein tensor a natural choice in deriving the field equations, since the Ricci tensor itself is not divergence-free.

Weyl tensor

The Weyl tensor plays an important role in this thesis. It represents the tidal forces that a body experiences when moving along a geodesic. It has the same symmetries as the Riemann tensor with the extra condition that it is trace-free, i.e. it is a Riemann tensor with the Ricci terms subtracted out. Hence the Weyl tensor contains information about how the shape of an object is distorted by tidal forces whilst the volume is preserved as there are no Ricci terms. An important property of the Weyl tensor is that it remains invariant under conformal transformations. Hence it is sometimes known as the conformal tensor. For any manifold with dimension $n \geqslant 3$ the Weyl tensor is given by

$$C_{abcd} = R_{abcd} - \frac{1}{2}(g_{ab}R_{cd} - g_{ad}R_{bc} - g_{bc}R_{ad} + g_{bd}R_{ac}) - \frac{1}{6}R(g_{ad}g_{bc} - g_{ac}g_{bd}). \quad (2.11)$$

The Weyl tensor satisfies the following symmetries

$$C_{abcd} = -C_{bacd} = -C_{abdc}, \quad (2.12)$$

$$C_{abcd} = C_{cdab}, \quad (2.13)$$

$$C_{abcd} + C_{adbc} + C_{acdb} = 0. \quad (2.14)$$

A manifold of dimension $n \geqslant 4$ is said to be conformally flat if and only if the Weyl tensor vanishes.

2.2.3 Lie theory

Symmetries that exist in the manifold M play an important role as they simplify the number of unknown functions in many calculations of physical importance in astrophysics and cosmology. Lie derivatives are a natural way of analysing these symmetries as they provide a coordinate independent way in which to describe symmetries on the manifold M.

Lie algebra

Definition 2.2.2 (Lie algebra). A Lie algebra is a vector space V on which a product $[\mathbf{X}, \mathbf{Y}]$ of vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in V$ is defined as follows:

i) $[\mathbf{X}, \mathbf{Y}]$ is bilinear,

ii) $[\mathbf{X}, \mathbf{X}] = 0$,

iii) $[\mathbf{X}, \mathbf{Y}] = -[\mathbf{X}, \mathbf{Y}]$ (Anti-symmetry),

iv) $[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0$ (Jacobi identity).

The product $[\mathbf{X}, \mathbf{Y}]$ is known as the Lie bracket and is defined by

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{X}\mathbf{Y} - \mathbf{Y}\mathbf{X}. \tag{2.15}$$

where products on the right represent composition of functions.

Lie derivative

The Lie derivative $\mathcal{L}_{\mathbf{X}}$ along a vector field $\mathbf{X}$ is a very important concept in differential geometry that tells us how a geometric object changes as it is pushed along the curve of a given tangent vector. It plays a central role in defining symmetries on manifolds and describing conservation laws.

Definition 2.2.3 (**Lie derivative**)**.** The Lie derivative for an arbitrary tensor field $T^{a_1a_2\cdots a_m}{}_{b_1b_2\cdots b_n}$ along a vector $\mathbf{X}$ is given by

$$\begin{aligned}
\mathcal{L}_{\mathbf{X}} T^{a_1a_2\cdots a_m}{}_{b_1b_2\cdots b_n} &= X^c T^{a_1a_2\cdots a_m}{}_{b_1b_2\cdots b_n,c} \\
&\quad - T^{ca_2\cdots a_m}{}_{b_1b_2\cdots b_n} X^{a_1}{}_{,c} - \cdots - T^{a_1a_2\cdots c}{}_{b_1b_2\cdots b_n} X^{a_m}{}_{,c} \\
&\quad + T^{a_1a_2\cdots a_m}{}_{cb_2\cdots b_n} X^c{}_{,b_1} + \cdots + T^{a_1a_2\cdots a_m}{}_{b_1b_2\cdots c} X^c{}_{,b_n}.
\end{aligned} \tag{2.16}$$

The Lie derivative $\mathcal{L}_{\mathbf{X}}$ satisfies the following properties:

i) $\mathcal{L}_{\mathbf{X}}$ preserves tensor type, i.e. $\mathcal{L}_{\mathbf{X}}T$ is a tensor field of the same type as T,

ii) $\mathcal{L}_{\mathbf{X}}$ is linear and satisfies the Leibniz rule for differentiation,

iii) $\mathcal{L}_{\mathbf{X}}$ commutes with contraction,

v) $\mathcal{L}_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}]$ for all vector fields $\mathbf{Y}$,

vi) $\mathcal{L}_{[\mathbf{X},\mathbf{Y}]} = \mathcal{L}_{\mathbf{X}}\mathcal{L}_{\mathbf{Y}} - \mathcal{L}_{\mathbf{X}}\mathcal{L}_{\mathbf{Y}}$ for all vector fields $\mathbf{X}$ and $\mathbf{Y}$.

Lie group

A Lie group is a manifold that also has a group structure, i.e. it is manifold in the sense of Definition 2.2.1, and in addition the set of points of this manifold form an algebraic group. The manifold structure allows us to consider the notion of smoothness which would not be possible in more general groups.

Definition 2.2.4 (**Lie group**)**.** Let M be an n-dimensional differentiable manifold. Then M is a Lie group if the following are satisfied:

i) M is a group in the usual algebraic sense,

ii) the group multiplication

$$m : M \times M \longrightarrow M,$$

and the group inverse

$$i : M \longrightarrow M,$$

are both smooth maps.

Every Lie group defines a unique Lie algebra and conversely every Lie algebra defines a unique Lie group. For this investigation we utilise the symmetries of the metric tensor field on manifolds. These concepts are considered in great detail by Choquet-Bruhet *et al* (1977), Dubrovin *et al* (1984, 1985) and Stephani *et al* (2003).

Conformal Killing vector

A Killing vector leaves the metric unchanged and leads to a conservation law. There are some useful ways to consider symmetries that are not Killing symmetries. We introduce a more general notion of a symmetry by considering the case of a conformal Killing vector. In this thesis we are concerned with groups of conformal symmetries which preserve the metric up to a factor. The conformal Killing vector field is defined by the following relation

$$\mathcal{L}_{\mathbf{X}} g_{ab} = 2\psi g_{ab}, \tag{2.17}$$

where $\psi = \psi(x^a)$ is the conformal factor. The set of all conformal Killing vectors generates a Lie algebra with basis $\{\mathbf{X}_I\}$. The elements of the basis $\{\mathbf{X}_I\}$ are related by

$$[\mathbf{X}_I, \mathbf{X}_J] = C^K{}_{IJ}\mathbf{X}_K$$

where $C^K{}_{IJ}$ are the structure constants of the group. The structure constants satisfy

$$C^K{}_{IJ} = -C^K{}_{JI} \qquad \text{(anti-symmetry)},$$

$$C^K{}_{LM}C^M{}_{IJ} + C^K{}_{IM}C^M{}_{JL} + C^K{}_{JM}C^M{}_{LI} = 0 \qquad \text{(Lie identity)}.$$

The integrability condition for the existence of the conformal vector (2.17) is given by

$$\mathcal{L}_{\mathbf{X}} C^a{}_{bcd} = 0, \tag{2.18}$$

as established by Hall and Steele (1991).

2.3 Field equations

Einstein field equations are the general relativistic generalisation of Newton's gravitational laws. Einstein postulated that the field equations should indicate how energy and matter curve spacetime and that they must obey the conservation laws. The matter content is described by the symmetric divergence-free tensor

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{2.19}$$

called the energy-momentum tensor. In this definition μ is the proper density, p is the isotropic pressure, q_a is the energy flux vector and π_{ab} is the anisotropic stress tensor. The vector $\mathbf{u}$ is timelike and unit ($u^a u_a = -1$). The energy-momentum tensor (2.19) is coupled to the Einstein tensor (2.10) via the Einstein field equations. This generates the system

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^2} T_{ab}, \tag{2.20}$$

where G is the gravitational constant and c is the velocity of light. The system (2.20) is a system of non-linear partial differential equations, and a large part of research in general relativity is concerned with finding exact solutions to this system. The equations (2.20) constitute a system of nonlinear differential equations, which determine the behaviour of the gravitating system in the presence of matter. Exact solutions to (2.20) relevant to astrophysics and cosmology are given by Krasinski (1997) and Stephani *et al* (2003).

Chapter 3

Spherically symmetric spacetimes

3.1 Introduction

Spherically symmetric spacetimes are widely used in applications in cosmology and astrophysics. In this chapter we investigate the conformal geometry of static and non-static spherically symmetric spacetimes. Killing vectors are relatively well known as indicated in the work of Stephani *et al* (2003). Conformal symmetries will be analysed in this chapter as they remain an important area of research. We give the general solution of the conformal Killing equation and the conformal factor for both static and non-static spherical spacetimes. This solution is subject to integrability conditions which place restrictions on the gravitational potentials. In §3.2 we consider the conformal geometry of static spherically symmetric spacetimes, and give the general form of the conformal Killing vector and the conformal factor first obtained by Maharaj *et al* (1995). Maartens *et al* (1995) adopted a notation to express the solution more compactly and we provide this form as well. In §3.3 the general solution of the conformal Killing vector and conformal factor for the non-static spherically symmetric spacetimes is given. This solution was first obtained by Moopanar and Maharaj (2013).

3.2 Static spacetimes

3.2.1 Spacetime geometry

The general form of the line element for static spherically symmetric spacetimes is given by

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu(r)}\mathrm{d}t^2 + \mathrm{e}^{2\lambda(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{3.1}$$

in comoving coordinates. The quantities $\nu(r)$ and $\lambda(r)$ represent the gravitational potentials. The metric (3.1) admits four Killing vectors

$$\xi_{\mathbf{0}} = \frac{\partial}{\partial t}, \tag{3.2a}$$

$$\xi_{\mathbf{1}} = \frac{\partial}{\partial \phi}, \tag{3.2b}$$

$$\xi_{\mathbf{2}} = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}, \tag{3.2c}$$

$$\xi_{\mathbf{3}} = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}, \tag{3.2d}$$

as the spacetime is static and invariant under rotations.

3.2.2 Conformal geometry

To generate the conformal geometry for static spherically symmetric spacetimes we need to find the conformal vector $\mathbf{X} = (X^0, X^1, X^2, X^3)$ and the conformal factor

ψ. The conformal Killing vector equation (2.17) for the line element (3.1) can be decomposed into the system:

$$\nu_r X^1 + X^0{}_t = \psi, \tag{3.3a}$$

$$-\mathrm{e}^{2\nu} X^0{}_r + \mathrm{e}^{2\lambda} X^1{}_t = 0, \tag{3.3b}$$

$$-\mathrm{e}^{2\nu} X^0{}_\theta + r^2 X^2{}_t = 0, \tag{3.3c}$$

$$-\mathrm{e}^{2\nu} X^0{}_\phi + r^2 \sin^2\theta X^3{}_t = 0, \tag{3.3d}$$

$$\lambda_r X^1 + X^1{}_r = \psi, \tag{3.3e}$$

$$\mathrm{e}^{2\lambda} X^1{}_\theta + r^2 X^2{}_r = 0, \tag{3.3f}$$

$$\mathrm{e}^{2\lambda} X^1{}_\phi + r^2 \sin^2\theta X^3{}_r = 0, \tag{3.3g}$$

$$X^1 + r X^2{}_\theta = r\psi, \tag{3.3h}$$

$$X^2{}_\phi + \sin^2\theta X^3{}_\theta = 0, \tag{3.3i}$$

$$X^1 + r\cot\theta X^2 + r X^3{}_\phi = r\psi. \tag{3.3j}$$

Note that (3.3) comprises a system of coupled first order differential equations. The system (3.3) can be integrated to yield $\mathbf{X}$ and ψ subject to integrability conditions.

This was achieved by Maharaj *et al* (1995) and Maartens *et al* (1995, 1996). We have re-derived this solution and verified that it is correct with the help of Mathematica (Wolfram 2010). We do not give the integration procedure for the above system and simply present the general solution below. The components for the conformal vector are

$$X^0 = r^2 e^{-2\nu} \sin\theta(\mathcal{A}_t \sin\phi - \mathcal{B}_t \cos\phi) - r^2 e^{-2\nu}\mathcal{C}_t \cos\theta + \mathcal{E} + a_0, \tag{3.4a}$$

$$X^1 = r^2 e^{-2\lambda} \sin\theta(-\mathcal{A}_r \sin\phi + \mathcal{B}_r \cos\phi) + r^2 e^{-2\lambda}\mathcal{C}_r \cos\theta + \mathcal{D}, \tag{3.4b}$$

$$X^2 = \cos\theta(\mathcal{A}\sin\phi - \mathcal{B}\cos\phi) + \mathcal{C}\sin\theta + a_1 \sin\phi + a_2 \cos\phi, \tag{3.4c}$$

$$X^3 = \csc\theta(\mathcal{A}\cos\phi + \mathcal{B}\sin\phi) + \cot\theta(a_1 \cos\phi - a_2 \sin\phi) + a_3. \tag{3.4d}$$

The conformal factor is given by

$$\begin{aligned}\psi &= r^2 \sin\theta \sin\phi\Big(- \nu_r e^{-2\lambda}\mathcal{A}_r + e^{-2\nu}\mathcal{A}_{tt}\Big) \\ &\quad + r^2 \sin\theta\cos\phi\Big(\nu_r e^{-2\lambda}\mathcal{B}_r - e^{-2\nu}\mathcal{B}_{tt}\Big) \\ &\quad + r^2\cos\theta\Big(\nu_r e^{-2\lambda}\mathcal{C}_r - e^{-2\nu}\mathcal{C}_{rr}\Big) + \nu_r\mathcal{D} + \mathcal{E}_t. \end{aligned}\tag{3.4e}$$

The quantities $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ are arbitrary functions of t and r, and a_0-a_3 are constants. These quantities arise from the integration process. In the integration, the following integrability conditions are generated

$$\mathrm{e}^{2\nu}(r^2\mathrm{e}^{-2\nu}\mathcal{A}_t)_r + r^2\mathcal{A}_{tr} = 0, \tag{3.5a}$$

$$r^2\nu_r\mathrm{e}^{-2\lambda}\mathcal{A}_r - r^2\mathrm{e}^{-2\nu}\mathcal{A}_{tt} - \lambda_r r^2\mathrm{e}^{-2\lambda}\mathcal{A}_r - (r^2\mathrm{e}^{-2\lambda}\mathcal{A}_r)_r = 0, \tag{3.5b}$$

$$r\mathrm{e}^{-2\lambda}\mathcal{A}_r + \mathcal{A} - r^2(\nu_r\mathrm{e}^{-2\lambda}\mathcal{A}_r - \mathrm{e}^{-2\nu}\mathcal{A}_{tt}) = 0, \tag{3.5c}$$

$$\mathrm{e}^{2\nu}(r^2\mathrm{e}^{-2\nu}\mathcal{B}_t)_r + r^2\mathcal{B}_{tr} = 0, \tag{3.5d}$$

$$r^2\nu_r\mathrm{e}^{-2\lambda}\mathcal{B}_r - r^2\mathrm{e}^{-2\nu}\mathcal{B}_{tt} - \lambda_r r^2\mathrm{e}^{-2\lambda}\mathcal{B}_r - (r^2\mathrm{e}^{-2\lambda}\mathcal{B}_r)_r = 0, \tag{3.5e}$$

$$r\mathrm{e}^{-2\lambda}\mathcal{B}_r + \mathcal{B} - r^2(\nu_r\mathrm{e}^{-2\lambda}\mathcal{B}_r - \mathrm{e}^{-2\nu}\mathcal{B}_{tt}) = 0, \tag{3.5f}$$

$$\mathrm{e}^{2\nu}(r^2\mathrm{e}^{-2\nu}\mathcal{C}_t)_r + r^2\mathcal{C}_{tr} = 0, \tag{3.5g}$$

$$r^2\nu_r\mathrm{e}^{-2\lambda}\mathcal{C}_r - r^2\mathrm{e}^{-2\nu}\mathcal{C}_{tt} - \lambda_r r^2\mathrm{e}^{-2\lambda}\mathcal{C}_r - (r^2\mathrm{e}^{-2\lambda}\mathcal{C}_r)_r = 0, \tag{3.5h}$$

$$r\mathrm{e}^{-2\lambda}\mathcal{C}_r + \mathcal{C} - r^2(\nu_r\mathrm{e}^{-2\lambda}\mathcal{C}_r - \mathrm{e}^{-2\nu}\mathcal{C}_{tt}) = 0, \tag{3.5i}$$

$$(r^{-1} - \nu_r)\mathcal{D} - \mathcal{E}_t = 0, \tag{3.5j}$$

$$\mathrm{e}^{2\nu}\mathcal{E}_r - \mathrm{e}^{2\lambda}\mathcal{D}_t = 0, \tag{3.5k}$$

$$\nu_r\mathcal{D} + \mathcal{E}_t - \lambda_r\mathcal{D} - \mathcal{D}_r = 0. \tag{3.5l}$$

The solution (3.4) gives the most general conformal symmetry of the static spherically symmetric spacetime (3.1). The equations (3.5) act as integrability conditions restricting the forms of the gravitational potentials. The metric functions $\nu(r)$ and $\lambda(r)$ depend on the functions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and $\mathcal{E}$.

The above solution can be expressed in a more compact form by making use of the following transformation adopted by Maartens *et al* (1995, 1996). We introduce the new variables

$$A^i = (A^1, A^2, A^3)$$

$$\equiv (\tilde{\mathcal{B}}, -\tilde{\mathcal{A}}, \mathcal{C}),$$

$$\eta_i = (\eta_1, \eta_2, \eta_3)$$

$$= (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta),$$

$$A^0 \equiv \mathcal{E},$$

$$A^4 \equiv \mathcal{D}.$$

Using this notation, the solution (3.4) can be transformed to

$$X^0 = -r^2 \mathrm{e}^{-2\nu} A^i{}_t \eta_i + A^0 + a_0, \tag{3.6a}$$

$$X^1 = r^2 \mathrm{e}^{-2\lambda} A^i{}_r \eta_i + A^4, \tag{3.6b}$$

$$X^2 = -A^i (\eta_i)_\theta + a_1 \sin\phi + a_2 \cos\phi, \tag{3.6c}$$

$$X^3 = -\csc^2\theta A^i (\eta_i)_\phi + \cot\theta (a_1 \cos\phi - a_2 \sin\phi) + a_3. \tag{3.6d}$$

The conformal factor is then given by

$$\psi = r^2 (\nu_r \mathrm{e}^{-2\lambda} A^i{}_r - \mathrm{e}^{-2\nu} A^i{}_{tt}) \eta_i + A^0 + \nu_r A^4. \tag{3.6e}$$

The integrability conditions become

$$(r\mathrm{e}^{-\nu} A^i{}_t)_r = 0, \tag{3.7a}$$

$$\mathrm{e}^{2(\lambda-\nu)} A^i{}_{tt} + A^i{}_{rr} + (2r^{-1} - \lambda_r - \nu_r) A^i{}_r = 0, \tag{3.7b}$$

$$A^i{}_{tt} + \mathrm{e}^{2(\nu-\lambda)} (r^{-1} - \nu_r) A^i{}_r + r^{-2} \mathrm{e}^{2\nu} A^i = 0, \tag{3.7c}$$

$$A^4{}_r + (\lambda_r - r^{-1}) A^4 = 0, \tag{3.7d}$$

$$\mathrm{e}^{2\nu} A^0{}_r - \mathrm{e}^{2\lambda} A^4{}_t = 0, \tag{3.7e}$$

$$A^0{}_t + (\nu_r - r^{-1}) A^4 = 0. \tag{3.7f}$$

Equations (3.6) and (3.7), equivalent to (3.4) and (3.5) respectively, have a compact and transparent form. They comprise the most general conformal structure for the static spherically symmetric spacetimes (3.1). Observe that the angular dependence on the θ and ϕ coordinates has been completely determined in the conformal vector $\mathbf{X}$ and conformal factor ψ. There is freedom only in the t and r coordinates. Also note that the conformal geometry has a dependence in the timelike coordinate t even though the spacetime is static. The integrability conditions restrict the forms of the potentials ν and λ. The Einstein field equations provide further constraints on the behaviour of the gravitational field.

3.3 Non-static spacetimes

3.3.1 Spacetime geometry

The general form of the line element for non-static spherically symmetric spacetimes is given by

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu(t,r)}\mathrm{d}t^2 + \mathrm{e}^{2\lambda(t,r)}\Big[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\Big], \tag{3.8}$$

in suitable coordinates. The quantities $\nu(t,r)$ and $\lambda(t,r)$ represent gravitational potentials. In this line element the coordinate system is simultaneously comoving and isotropic. The non-vanishing kinematical quantities are the acceleration $\dot{u}_a$ and the expansion Θ. The spacetime (3.8) is invariant under the action of rotational Killing vectors. The Lie algebra is spanned by the following three linearly independent Killing vectors

$$\xi_{\mathbf{1}} = \frac{\partial}{\partial\phi}, \tag{3.9a}$$

$$\xi_{\mathbf{2}} = \cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi}, \tag{3.9b}$$

$$\xi_{\mathbf{3}} = \sin\phi \frac{\partial}{\partial\theta} + \cos\phi \cot\theta \frac{\partial}{\partial\phi}, \tag{3.9c}$$

for the spacetime (3.8).

3.3.2 Conformal geometry

The conformal Killing vector equation (2.17) is decomposed into the following system of ten coupled partial differential equations for the metric (3.8):

$$\nu_t X^0 + \nu_r X^1 + X^0_t = \psi, \tag{3.10a}$$

$$\mathrm{e}^{2\lambda} X^1_t - \mathrm{e}^{2\nu} X^0_r = 0, \tag{3.10b}$$

$$r^2 \mathrm{e}^{2\lambda} X^2_t - \mathrm{e}^{2\nu} X^0_\theta = 0, \tag{3.10c}$$

$$r^2 \mathrm{e}^{2\lambda} \sin^2\theta X^3_t - \mathrm{e}^{2\nu} X^0_\phi = 0, \tag{3.10d}$$

$$\lambda_t X^0 + \lambda_r X^1 + X^1_r = \psi, \tag{3.10e}$$

$$r^2 X^2_r + X^1_\theta = 0, \tag{3.10f}$$

$$r^2 \sin^2\theta X^3_r + X^1_\phi = 0, \tag{3.10g}$$

$$\lambda_t X^0 + (r^{-1} + \lambda_r) X^1 + X^2_\theta = \psi, \tag{3.10h}$$

$$\sin^2\theta X^3_\theta + X^2_\phi = 0, \tag{3.10i}$$

$$\lambda_t X^0 + (r^{-1} + \lambda_r) X^1 + \cot\theta X^2 + X^3_\phi = \psi. \tag{3.10j}$$

The system (3.10) maybe integrated to yield $\mathbf{X}$ and ψ subject to integrability conditions. This was done by Moopanar and Maharaj (2013). Here we simply state the results from the integration. The components of the conformal vector are

$$X^0 = r^2 \mathrm{e}^{2(\lambda-\nu)} \sin\theta(\mathcal{C}_t \sin\phi - \mathcal{D}_t \cos\phi) - r^2 \mathrm{e}^{2(\lambda-\nu)} \mathcal{I}_t \cos\theta + \mathcal{J}, \tag{3.11a}$$

$$X^1 = -r^2 \sin\theta(\mathcal{C}_r \sin\phi - \mathcal{D}_r \cos\phi) + r^2 \mathcal{I}_r \cos\theta + \mathcal{K}, \tag{3.11b}$$

$$\begin{aligned} X^2 = {} & \cos\theta(\mathcal{C}\sin\phi - \mathcal{D}\cos\phi) + \cos\theta(a_1 \sin\phi - a_2 \cos\phi) \\ & -a_3 \sin\phi + a_4 \cos\phi + \mathcal{I}\sin\phi, \end{aligned} \tag{3.11c}$$

$$\begin{aligned} X^3 = {} & \csc\theta(\mathcal{C}\cos\phi + \mathcal{D}\sin\phi) + \csc\theta(a_1 \cos\phi + a_2 \sin\phi) \\ & - \cot\theta(a_3 \cos\phi + a_4 \sin\phi) + a_5, \end{aligned} \tag{3.11d}$$

and the conformal factor is given by

$$\begin{aligned}
\psi \;\; = \;\; & r^2 \sin\theta \sin\phi \Big(\mathrm{e}^{2(\lambda-\nu)} \mathcal{C}_{tt} + (2\lambda_t - \nu_t) \mathrm{e}^{2(\lambda-\nu)} \mathcal{C}_t - \nu_r \mathcal{C}_r \Big) \\
& - r^2 \sin\theta \cos\phi \Big(\mathrm{e}^{2(\lambda-\nu)} \mathcal{D}_{tt} + (2\lambda_t - \nu_t) \mathrm{e}^{2(\lambda-\nu)} \mathcal{D}_t - \nu_r \mathcal{D}_r \Big) \\
& - r^2 \cos\theta \Big(\mathrm{e}^{2(\lambda-\nu)} \mathcal{I}_{tt} + (2\lambda_t - \nu_t) \mathrm{e}^{2(\lambda-\nu)} \mathcal{I}_t - \nu_r \mathcal{I}_r \Big) \\
& + \mathcal{J}_t + \nu_t \mathcal{J} + \nu_r \mathcal{K}, \qquad (3.11e)
\end{aligned}$$

where $\mathcal{C}, \mathcal{D}, \mathcal{I}, \mathcal{J}$ and $\mathcal{K}$ are arbitrary functions of t and r, and a_1-a_5 are constants. These quantities arise from the integration process. In the integration the following integrability conditions are generated

$$\mathcal{C}_{tr} + (r^{-1} + \lambda_r - \nu_r)\mathcal{C}_t = 0, \qquad (3.12a)$$

$$\mathcal{D}_{tr} + (r^{-1} + \lambda_r - \nu_r)\mathcal{D}_t = 0, \qquad (3.12b)$$

$$\mathcal{I}_{tr} + (r^{-1} + \lambda_r - \nu_r)\mathcal{I}_t = 0, \qquad (3.12c)$$

$$\mathrm{e}^{2(\lambda-\nu)}\mathcal{C}_{tt} + \mathcal{C}_{rr} + (\lambda_t - \nu_t)\mathrm{e}^{2(\lambda-\nu)}\mathcal{C}_t + (2r^{-1} + \lambda_r - \nu_r)\mathcal{C}_r = 0, \qquad (3.12d)$$

$$\mathrm{e}^{2(\lambda-\nu)}\mathcal{D}_{tt} + \mathcal{D}_{rr} + (\lambda_t - \nu_t)\mathrm{e}^{2(\lambda-\nu)}\mathcal{D}_t + (2r^{-1} + \lambda_r - \nu_r)\mathcal{D}_r = 0, \qquad (3.12e)$$

$$\mathrm{e}^{2(\lambda-\nu)}\mathcal{I}_{tt} + \mathcal{I}_{rr} + (\lambda_t - \nu_t)\mathrm{e}^{2(\lambda-\nu)}\mathcal{I}_t + (2r^{-1} + \lambda_r - \nu_r)\mathcal{I}_r = 0, \qquad (3.12f)$$

$$r^2\mathrm{e}^{2(\lambda-\nu)}\mathcal{C}_{tt} + r^2(\lambda_t - \nu_t)\mathrm{e}^{2(\lambda-\nu)}\mathcal{C}_t + r^2(r^{-1} + \lambda_r - \nu_r)\mathcal{C}_r + \mathcal{C} + a_1 = 0, \qquad (3.12g)$$

$$r^2 e^{2(\lambda-\nu)}\mathcal{D}_{tt} + r^2(\lambda_t - \nu_t)e^{2(\lambda-\nu)}\mathcal{D}_t + r^2(r^{-1} + \lambda_r - \nu_r)\mathcal{D}_r + \mathcal{D} + a_2 = 0, \qquad (3.12h)$$

$$r^2 e^{2(\lambda-\nu)}\mathcal{I}_{tt} + r^2(\lambda_t - \nu_t)e^{2(\lambda-\nu)}\mathcal{I}_t + r^2(r^{-1} + \lambda_r - \nu_r)\mathcal{I}_r + \mathcal{I} = 0, \qquad (3.12i)$$

$$e^{2\lambda}\mathcal{K}_t - e^{2\nu}\mathcal{J}_r = 0, \qquad (3.12j)$$

$$\mathcal{J}_t + (\lambda_t - \nu_t)\mathcal{J} + (r^{-1} + \lambda_r - \nu_r)\mathcal{K} = 0, \qquad (3.12k)$$

$$\mathcal{J}_t + \mathcal{K}_r + (\lambda_t - \nu_t)\mathcal{J} + (\lambda_r - \nu_r)\mathcal{K} = 0. \qquad (3.12l)$$

The result (3.12) places further restrictions on the gravitational potentials ν and λ. These potentials are dependent on $\mathcal{C}, \mathcal{D}, \mathcal{I}, \mathcal{J}$ and $\mathcal{K}$.

We again adopt the procedure used by Maartens *et al* (1995, 1996), to express the above solution in a more compact form. However, note that in this case the potentials also depend on the coordinate t. We first let

$$\tilde{\mathcal{C}} = \mathcal{C} + a \text{ and } \tilde{\mathcal{D}} = \mathcal{D} + a.$$

We then introduce new variables as follows

$$A^i = (A^1, A^2, A^3)$$

$$= (\tilde{\mathcal{C}}, \tilde{\mathcal{D}}, -\mathcal{I}),$$

$$\eta_i = (\eta_1, \eta_2, \eta_3)$$

$$= (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta),$$

$$A^0 = \mathcal{J},$$

$$A^4 = \mathcal{K}.$$

Using the above notation (3.11) can be transformed into the following form

$$X^0 = r^2 \mathrm{e}^{2(\lambda-\nu)} A^i{}_t \eta_i + A^0, \tag{3.13a}$$

$$X^1 = -r^2 A^i{}_r \eta_i + A^4, \tag{3.13b}$$

$$X^2 = A^i(\eta_i)_\theta - a_3 \sin\phi + a_4 \cos\phi, \tag{3.13c}$$

$$X^3 = \csc^2\theta A^i(\eta_i)_\phi - \cot\theta(a_3\cos\phi + a_4\sin\phi) + a_5. \tag{3.13d}$$

The conformal factor is then given by

$$\psi = r^2 \Big[\mathrm{e}^{2(\lambda-\nu)} A^i{}_{tt} + (2\lambda_t - \nu_t)\mathrm{e}^{2(\lambda-\nu)} A^i{}_t - \nu_r A^i{}_r\Big]\eta_i + A^0{}_t + \nu_t A^0 + \nu_r A^4. \tag{3.13e}$$

The integrability conditions (3.12) are then transformed to

$$A^i{}_{tr} + (r^{-1} + \lambda_r - \nu_r)A^i{}_t = 0, \tag{3.14a}$$

$$\mathrm{e}^{2(\lambda-\nu)}A^i{}_{tt} + A^i{}_{rr} + (\lambda_t - \nu_t)\mathrm{e}^{2(\lambda-\nu)}A^i{}_t + (2r^{-1} + \lambda_r - \nu_r)A^i{}_r = 0, \tag{3.14b}$$

$$r^2\mathrm{e}^{2(\lambda-\nu)}A^i{}_{tt} + r^2(\lambda_t - \nu_t)\mathrm{e}^{2(\lambda-\nu)}A^i{}_t + r^2(r^{-1} + \lambda_r - \nu_r)A^i{}_r + A^i = 0, \tag{3.14c}$$

$$\mathrm{e}^{2\lambda}A^4{}_t - \mathrm{e}^{2\nu}A^0{}_r = 0, \tag{3.14d}$$

$$-A^0{}_t + (\lambda_t - \nu_t)A^0 + (r^{-1} + \lambda_r - \nu_r)A^4 = 0, \tag{3.14e}$$

$$-A^0{}_t + A^4{}_r + (\lambda_t - \nu_t)A^0 + (\lambda_r - \nu_r)A^4 = 0. \tag{3.14f}$$

Equations (3.13) and (3.14) are equivalent to (3.11) and (3.12), respectively. They comprise the most general conformal geometry for the non-static spherically symmetric spacetimes (3.8). As in the static case we observe that the angular dependence on the θ and ϕ coordinates has been completely determined in the conformal vector $\mathbf{X}$ and the conformal factor ψ, and there is freedom only in the t and r coordinates. The integrability conditions restrict the forms of the potentials ν and λ. Clearly the Einstein field equations will also further restrict the conformal symmetry.

Chapter 4

Classification of conformal motions in static spacetimes

4.1 Introduction

Classification of symmetries is important as this helps to identify spacetimes of physical interest with a symmetry property. It provides a deeper insight into spacetime geometry and helps to produce new solutions to the Einstein field equations. It can also help to invariantly describe known models. In this chapter we classify conformal symmetries in terms of the Weyl tensor which represents tidal forces. We first consider the Weyl tensor components for the static line element (3.1). In §4.2 we calculate the non-zero components of the Weyl tensor in static spherical spacetimes. The accuracy of these components has been verified using both Mathematica (Wolfram 2010) and Maple (2009). In §4.3 we review the result obtained by Maartens *et al* (1995), and identify an error in their conclusion. In §4.4 we impose the first integrability condition and classify the conformal geometry in terms of conformally flat and non-conformally flat spacetimes. The nonlinear differential equation in both cases can be integrated in general. In so doing, we rectify the error in the Maartens *et al* (1995) paper.

4.2 Weyl tensor

The Weyl tensor plays an important role in our classification scheme. We calculate the Weyl tensor components for the line element (3.1) by using the definition (2.11). The results have been verified with the assistance of Mathematica (Wolfram 2010) and Maple (2009). The non-zero components of the Weyl tensor for the line element (3.1) are

$$C^0{}_{101} = -\frac{1}{3}\Big[\nu_{rr} + {\nu_r}^2 - \lambda_r\nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda})\Big], \tag{4.1a}$$

$$C^0{}_{202} = \frac{1}{6}r^2\mathrm{e}^{-2\lambda}\Big[\nu_{rr} + {\nu_r}^2 - \lambda_r\nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda})\Big], \tag{4.1b}$$

$$C^0{}_{303} = \frac{1}{6}r^2\sin^2\theta\mathrm{e}^{-2\lambda}\Big[\nu_{rr} + {\nu_r}^2 - \lambda_r\nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda})\Big], \tag{4.1c}$$

$$C^1{}_{212} = \frac{1}{6}r^2\mathrm{e}^{-2\lambda}\Big[\nu_{rr} + {\nu_r}^2 - \lambda_r\nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda})\Big], \tag{4.1d}$$

$$C^1{}_{313} = \frac{1}{6}r^2\sin^2\theta\mathrm{e}^{-2\lambda}\Big[\nu_{rr} + {\nu_r}^2 - \lambda_r\nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda})\Big], \tag{4.1e}$$

$$C^2{}_{323} = -\frac{1}{3}r^2\sin^2\theta\mathrm{e}^{-2\lambda}\Big[\nu_{rr} + {\nu_r}^2 - \lambda_r\nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda})\Big]. \tag{4.1f}$$

It is convenient to introduce the quantity

$$\Lambda = \nu_{rr} + {\nu_r}^2 - \lambda_r\nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda}), \tag{4.2}$$

so that we can rewrite (4.1) as

$$C^0{}_{101} = -\frac{1}{3}\Lambda, \tag{4.3a}$$

$$C^0{}_{202} = \frac{1}{6}r^2 \mathrm{e}^{-2\lambda}\Lambda, \tag{4.3b}$$

$$C^0{}_{303} = \frac{1}{6}r^2 \sin^2\theta \mathrm{e}^{-2\lambda}\Lambda, \tag{4.3c}$$

$$C^1{}_{212} = C^0{}_{202}, \tag{4.3d}$$

$$C^1{}_{313} = C^0{}_{303}, \tag{4.3e}$$

$$C^2{}_{323} = -2C^0{}_{303}. \tag{4.3f}$$

The introduction of the quantity Λ assists in the classification scheme for the conformal symmetry. When $\Lambda = 0$ all components of the Weyl tensor vanish and the spacetime is conformally flat.

4.3 Earlier results

In this section we state the results obtained by Maartens *et al* (1995) in their work on the classification of conformal motions in static spacetimes. The motivation for including these results is to point out an error in their calculations which will be corrected in the next section. We use the same notation adopted in that paper to ease comparison with their results. The equivalent of (4.2) is

$$\Gamma = \nu_{rr} + {\nu_r}^2 - \lambda_r\nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda}). \tag{4.4}$$

Then Maartens *et al* (1995) claimed that

$$\mathcal{L}_\xi {C^0}_{101} = 0 \quad \Rightarrow \quad \Gamma({A^1}_r \cos\theta\cos\phi + {A^2}_r \cos\theta\sin\phi - {A^3}_r \sin\theta) = 0, \tag{4.5}$$

where ξ is the conformal Killing vector. (Note that in the above paper the first equality is incorrectly expressed as $\mathcal{L}_\xi {C^0}_{201} = 0$). Hence if $\Gamma \neq 0$, then ${A^i}_r = 0$. From this condition it was observed that

$$\mathcal{L}_\xi {C^0}_{202} = 0$$

$$\Rightarrow (r^2\mathrm{e}^{-2\lambda}\Gamma)_r A^4 + 2r^2\mathrm{e}^{-2\lambda}\Gamma[\sin\theta(A^1\cos\phi + A^2\sin\phi) + A^3\cos\theta] = 0, \tag{4.6}$$

which leads to the conclusion that

$$\Gamma \neq 0 \quad \Rightarrow \quad A^i = 0 \quad \text{and} \quad (r^2\mathrm{e}^{-2\lambda}\Gamma)_r A^4 = 0. \tag{4.7}$$

However (4.5) is not true as we will show in the next section. The restrictions obtained in (4.6) and (4.7) are therefore not correct. Consequently, the results of Maartens *et al* (1995) need to be modified.

4.4 Integrability conditions

In this section we give the accurate version of the results discussed in the previous section. If we apply the general Lie derivative formula (2.16) to the Weyl tensor components given in (4.3) we obtain

$$\mathcal{L}_{\mathbf{X}}C^{0}{}_{101} = -\frac{1}{3}X^{1}\Lambda_{r} - \frac{2}{3}X^{1}{}_{r}\Lambda, \tag{4.8a}$$

$$\mathcal{L}_{\mathbf{X}}C^{0}{}_{202} = \frac{1}{6}(r^{2}\mathrm{e}^{-2\lambda}\Lambda)_{r}X^{1} + \frac{2}{6}r^{2}\mathrm{e}^{-2\lambda}\Lambda X^{2}{}_{\theta}, \tag{4.8b}$$

$$\begin{aligned}\mathcal{L}_{\mathbf{X}}C^{0}{}_{303} &= \frac{1}{6}X^{1}(r^{2}\sin^{2}\theta\mathrm{e}^{-2\lambda}\Lambda)_{r} + \frac{1}{6}X^{2}(r^{2}\sin^{2}\theta\mathrm{e}^{-2\lambda}\Lambda)_{\theta} \\ &\quad + \frac{2}{6}r^{2}\sin^{2}\theta\mathrm{e}^{-2\lambda}\Lambda X^{3}{}_{\phi},\end{aligned} \tag{4.8c}$$

$$\mathcal{L}_{\mathbf{X}}C^{1}{}_{212} = \mathcal{L}_{\mathbf{X}}C^{0}{}_{202}, \tag{4.8d}$$

$$\mathcal{L}_{\mathbf{X}}C^{1}{}_{313} = \mathcal{L}_{\mathbf{X}}C^{0}{}_{303}, \tag{4.8e}$$

$$\mathcal{L}_{\mathbf{X}}C^{2}{}_{323} = -2\mathcal{L}_{\mathbf{X}}C^{0}{}_{303}. \tag{4.8f}$$

For the existence of the conformal symmetry (2.18) holds. Hence (4.8d), (4.8e) and (4.8f) are identically satisfied for vanishing tidal forces. Thus we do not utilise them in further calculations.

Now we consider(4.8c). Simplifying the right hand side we get

$$\begin{aligned}\mathcal{L}_{\mathbf{X}}C^{0}{}_{303} &= \frac{1}{6}(r^{2}\mathrm{e}^{-2\lambda}A^{i}{}_{r}\eta_{i} + A^{4})[2r(1-r\lambda_{r})\Lambda + r^{2}\Lambda_{r}]\sin^{2}\theta\mathrm{e}^{-2\lambda} \\ &\quad + \frac{1}{6}[-A^{i}(\eta_{i})_{\theta} + a_{1}\sin\phi + a_{2}\cos\phi](2r^{2}\sin\theta\cos\theta\mathrm{e}^{-2\lambda}\Lambda)\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{6}r^2\sin^2\theta \mathrm{e}^{-2\lambda}\Lambda[-\csc^2\theta A^i(\eta_i)_{\phi\phi}-\cot\theta(a_1\sin\phi+a_2\cos\phi)]\\
= \quad & \frac{1}{6}\Big\{r^2\mathrm{e}^{-2\lambda}A^i{}_r[2r(1-r\lambda_r)\Lambda+r^2\Lambda_r]+2\Lambda A^i-2\Lambda A^i\Big\}r^2\sin^2\theta \mathrm{e}^{-2\lambda}\eta_i\\
&+A^4[2r(1-r\lambda_r)\Lambda+r^2\Lambda_r]\mathrm{e}^{-2\lambda}\sin^2\theta\\
&-\frac{2}{6}r^2\Lambda A^i\mathrm{e}^{-2\lambda}[\sin\theta\cos\theta(\eta_i)_\theta+(\eta_i)_{\phi\phi}]\\
&+\frac{2}{6}r^2\mathrm{e}^{-2\lambda}\Lambda(a_1\sin\phi+a_2\cos\phi)(\sin\theta\cos\theta-\sin^2\theta\cot\theta)\\
= \quad & \sin^2\theta\mathcal{L}_{\mathbf{X}}C^0{}_{202}-\frac{2}{6}r^2\Lambda A^i\mathrm{e}^{-2\lambda}[\sin^2\theta\eta_i+\sin\theta\cos\theta(\eta_i)_\theta+(\eta_i)_{\phi\phi}]. \quad (4.9)
\end{aligned}
$$

The last trigonometric term vanishes identically, so (4.9) has the simple form

$$
\mathcal{L}_{\mathbf{X}}C^0{}_{303}=\sin^2\theta\mathcal{L}_{\mathbf{X}}C^0{}_{202}. \tag{4.10}
$$

Equation (4.10) is satisfied identically because of the condition (2.18). For this reason (4.8c) is not used in further calculations.

The remaining conditions to be considered are (4.8a) and (4.8b). With the help of (3.6), (4.8a) and (4.8b) yield

$$
\begin{aligned}
\mathcal{L}_{\mathbf{X}}C^0{}_{101} \quad = \quad & -\frac{1}{3}\Big\{r^2A^i{}_r\Lambda_r+2\Lambda[2r(1-r\lambda_r)A^i{}_r+r^2A^i{}_{rr}]\Big\}\mathrm{e}^{-2\lambda}\eta_i\\
&-\frac{1}{3}(A^4\Lambda_r+2A^4{}_r\Lambda), \quad (4.11)
\end{aligned}
$$

$$\begin{aligned}
\mathcal{L}_{\mathbf{X}}C^0{}_{202} &= \frac{1}{6}(r^2 \mathrm{e}^{-2\lambda}A^i{}_r\eta_i + A^4)[2r(1-r\lambda_r)\Lambda + r^2\Lambda_r]\mathrm{e}^{-2\lambda} \\
&\quad + \frac{2}{6}r^2\mathrm{e}^{-2\lambda}\Lambda A^i\eta_i \\
&= \frac{1}{6}\Big\{\mathrm{e}^{-2\lambda}A^i{}_r[2r(1-r\lambda_r)\Lambda + r^2\Lambda_r] + 2\Lambda A^i\Big\}r^2\mathrm{e}^{-2\lambda}\eta_i \\
&\quad + [2r(1-r\lambda_r)\Lambda + r^2\Lambda_r]\mathrm{e}^{-2\lambda}A^4. \qquad (4.12)
\end{aligned}$$

By invoking the integrability condition (2.18) we have $\mathcal{L}_{\mathbf{X}}C^0{}_{101} = 0$ and $\mathcal{L}_{\mathbf{X}}C^0{}_{202} = 0$. Then using the linear independence of trigonometric functions we obtain the following conditions

$$[r^2\Lambda_r + 4r(1-r\lambda_r)\Lambda]A^i{}_r + 2r^2A^i{}_{rr}\Lambda = 0, \qquad (4.13a)$$

$$\Lambda_r A^4 + 2\Lambda A^4{}_r = 0, \qquad (4.13b)$$

$$[2r(1-r\lambda_r)\Lambda + r^2\Lambda_r]\mathrm{e}^{-2\lambda}A^i{}_r + 2\Lambda A^i = 0, \qquad (4.13c)$$

$$[2r(1-r\lambda_r)\Lambda + r^2\Lambda_r]A^4 = 0. \qquad (4.13d)$$

The equations in the system (4.13) are the necessary and sufficient conditions for the integrability conditions (2.18) to be satisfied. The structure of the system (4.13) suggests that there is a natural classification system for the conformal symmetries of the static spherically symmetric spacetimes (3.1). We consider the two cases

i) $\Lambda = 0$,

ii) $\Lambda \neq 0$.

This classification naturally separates the spacetimes into conformally flat ($\Lambda = 0$) and non-conformally flat ($\Lambda \neq 0$) categories.

Case $I : \Lambda = 0$

This case has vanishing Weyl tensor. From (4.2) we obtain

$$\nu_{rr} + {\nu_r}^2 - \lambda_r \nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda}) = 0. \tag{4.14}$$

This second order nonlinear partial differential equation is difficult to solve. However, it can be solved using the procedure adopted by Herrera *et al* (2001) in their work on conformally flat anisotropic spheres. We first write (4.14) in the following form

$$\left[\frac{\mathrm{e}^{-2\lambda}\nu_r}{r}\right]_r + \mathrm{e}^{-2(\nu+\lambda)}\left[\frac{\mathrm{e}^{2\nu}\nu_r}{r}\right]_r - \left[\frac{\mathrm{e}^{-2\lambda} - 1}{r^2}\right]_r = 0. \tag{4.15}$$

Next we introduce new variables

$$y = \mathrm{e}^{-2\lambda}\ ,\ \nu_r = \frac{u_r}{u}. \tag{4.16}$$

This transforms (4.15) into

$$y_r + \frac{2[u_{rr} - \frac{u_r}{r} + \frac{u}{r^2}]}{[u_r - \frac{u}{r}]}y - \frac{2u}{r^2[u_r - \frac{u}{r}]} = 0. \tag{4.17}$$

Equation (4.17) is a first order differential equation in y and its integrating factor is given by

$$\exp\left\{2\int \frac{[u_{rr} - \frac{u_r}{r} + \frac{u}{r^2}]}{[u_r - \frac{u}{r}]}\mathrm{d}r\right\} = \left[u_r - \frac{u}{r}\right]^2. \tag{4.18}$$

Multiplying both sides of (4.17) by the integrating factor and simplifying we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(u_r - \frac{u}{r}\right)^2 y\right] &= \frac{2u}{r^2}\left[u_r - \frac{u}{r}\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{u^2}{r^2}\right]. \end{aligned} \tag{4.19}$$

Integrating (4.19) we obtain

$$\left[u_r - \frac{u}{r}\right]^2 y = \frac{u^2}{r^2} + c_1, \tag{4.20}$$

where c_1 is a constant of integration, which is the same as

$$\left[\frac{u_r}{u} - \frac{1}{r}\right]^2 y = \frac{1}{r^2}\left[1 + c_1 r^2 u^{-2}\right]. \tag{4.21}$$

We can change back to the original variables using $u = \mathrm{e}^{\nu}$ to get

$$\left[\nu_r - \frac{1}{r}\right]^2 \mathrm{e}^{-2\lambda} = \frac{1}{r^2}\left[1 - c^2 r^2 \mathrm{e}^{-2\nu}\right] \tag{4.22}$$

where we have set $c_1 = -c^2$. Equation (4.22) can be written as

$$\nu_r - \frac{1}{r} = \frac{\mathrm{e}^{\lambda}}{r}\sqrt{1 - c^2 r^2 \mathrm{e}^{-2\nu}}. \tag{4.23}$$

Rearranging (4.23) and integrating with respect to r we get

$$\int \frac{\nu_r - \frac{1}{r}}{\sqrt{1 - c^2 r^2 \mathrm{e}^{-2\nu}}}\mathrm{d}r = \int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r. \tag{4.24}$$

Now let

$$\alpha = cr\mathrm{e}^{-\nu}. \tag{4.25}$$

Then

$$\begin{aligned} \mathrm{d}\alpha &= -cr\mathrm{e}^{-\nu}\left[\nu_r - \frac{1}{r}\right]\mathrm{d}r \\ &= -\alpha\left[\nu_r - \frac{1}{r}\right]\mathrm{d}r. \end{aligned} \tag{4.26}$$

Thus (4.24) becomes

$$-\int \frac{\mathrm{d}\alpha}{\alpha\sqrt{1-\alpha^2}} = \int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r. \tag{4.27}$$

Upon integration this gives

$$\mathrm{sech}^{-1}(\alpha) = \int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + \tilde{C}, \tag{4.28}$$

where $\tilde{C}$ is a constant. Simplifying (4.28) we obtain

$$\mathrm{e}^{\nu} = cr\cosh\left[\int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + \tilde{C}\right]. \tag{4.29}$$

The result (4.29) is the general solution of the nonlinear equation (4.14).

Gathering all this information leads to the theorem:

Theorem 4.4.1. *In a static spherically symmetric spacetime which is conformally flat, i.e. $\Lambda = 0$, the gravitational potentials ν and λ are related by the equation*

$$\mathrm{e}^{\nu} = cr\cosh\left[\int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + \tilde{C}\right],$$

where $\tilde{C}$ is a constant. The conformal Killing vector $\boldsymbol{X}$ is given by (3.6) *and the functions $A^0(t,r), A^4(t,r)$ and $A^i(t,r)$ are arbitrary.*

Case II: $\Lambda \neq 0$

This case is not conformally flat. We can write (4.13a) in the form

$$(A^i{}_r)_r + \Big[\frac{\Lambda_r}{2\Lambda} + \frac{2}{r} - 2\lambda_r\Big]A^i{}_r = 0, \tag{4.30}$$

If we consider this as a first order equation in $A^i{}_r$ then we can integrate to get

$$A^i{}_r = \frac{\mathrm{e}^{2\lambda}}{r^2\Lambda^{1/2}}h_1(t). \tag{4.31}$$

where $h_1(t)$ is a function that arises in the integration process. Substituting (4.31) into (4.13c) we obtain

$$A^i = -\frac{1}{2\Lambda^{1/2}}\Big[2(\frac{1}{r} - \lambda_r) + \frac{\Lambda_r}{\Lambda}\Big]h_1(t), \tag{4.32}$$

and we have an explicit form for A^i. From (4.13b), we have

$$A^4 = \frac{g_1(t)}{\Lambda^{1/2}}, \tag{4.33}$$

where $g_1(t)$ is a function of integration. It remains to solve (4.13d). Substitute (4.33) in (4.13d) to get

$$[2r(1 - r\lambda_r)\Lambda + r^2\Lambda_r]g_1(t) = 0.$$

This leads to two subcases:

Case $II(a) : g_1(t) = 0$

For this subcase we have

$$A^4 = 0 \ ,$$

$$A^i = -\frac{1}{2\Lambda^{1/2}}\Big[2\Big(\frac{1}{r} - \lambda_r\Big) + \frac{\Lambda_r}{\Lambda}\Big]h_1(t).$$

The above forms, together with (3.7e) and (3.7f), imply

$$A^0{}_r = 0, \ \ A^0{}_t = 0.$$

Hence

$$A^0 = C,$$

where C is a constant.

Case $II(b) : g_1(t) \neq 0$

This case implies that

$$2r(1 - r\lambda_r)\Lambda + r^2\Lambda_r = 0.$$

We can solve the above equation to obtain

$$\Lambda = \frac{\mathrm{e}^{2\lambda}}{r^2}k. \tag{4.34}$$

where k is a constant. Then (4.2) and (4.34) give

$$\nu_{rr} + {\nu_r}^2 - \lambda_r \nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}[1 - (1+k)\mathrm{e}^{2\lambda}] = 0. \tag{4.35}$$

Equation (4.35) is a highly nonlinear equation. Fortunately the method utilised in **Case** *I* may also be applied to solve (4.35). We believe that (4.35) is a new equation that arises in conformal symmetries of static spacetimes and has not been solved previously. We first write (4.35) in the following form

$$\left[\frac{\mathrm{e}^{-2\lambda}\nu_r}{r}\right]_r + \mathrm{e}^{-2(\nu+\lambda)}\left[\frac{\mathrm{e}^{2\nu}\nu_r}{r}\right]_r - \left[\frac{\mathrm{e}^{-2\lambda} - (1+k)}{r^2}\right]_r = 0. \tag{4.36}$$

We introduce the variables

$$y = \mathrm{e}^{-2\lambda} \ , \ \nu_r = \frac{u_r}{u}.$$

Then (4.36) becomes

$$y_r + \frac{2[u_{rr} - \frac{u_r}{r} + \frac{u}{r^2}]}{[u_r - \frac{u}{r}]} y - \frac{2(1+k)u}{r^2[u_r - \frac{u}{r}]} = 0. \tag{4.37}$$

We multiply (4.37) by the integrating factor.

$$\exp\left\{2\int \frac{[u_{rr} - \frac{u_r}{r} + \frac{u}{r^2}]}{[u_r - \frac{u}{r}]}\mathrm{d}r\right\} = \left[u_r - \frac{u}{r}\right]^2.$$

Then (4.37) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(u_r - \frac{u}{r}\right)^2 y\right] = (1+k)\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{u^2}{r^2}\right], \tag{4.38}$$

which can be integrated to give

$$\left[u_r - \frac{u}{r}\right]^2 y = (1+k)\frac{u^2}{r^2} + c_2, \tag{4.39}$$

where c_2 is a constant of integration. If we transform back to the original variables using $u = \mathrm{e}^{\nu}$ then (4.39) becomes

$$\left[\nu_r - \frac{1}{r}\right]^2 \mathrm{e}^{-2\lambda} = (1+k)\frac{1}{r^2}\left[1 - \tilde{c}^2 r^2 \mathrm{e}^{-2\nu}\right], \tag{4.40}$$

where we have set $\frac{c_2}{1+k} = -\tilde{c}^2$. This equation gives

$$\nu_r - \frac{1}{r} = \sqrt{1+k}\frac{\mathrm{e}^{\lambda}}{r}\sqrt{1 - \tilde{c}^2 r^2 \mathrm{e}^{-2\nu}}. \tag{4.41}$$

Rearranging (4.41) and integrating with respect to r we get

$$\int \frac{\nu_r - \frac{1}{r}}{\sqrt{1 - \tilde{c}^2 r^2 \mathrm{e}^{-2\nu}}}\mathrm{d}r = \sqrt{1+k}\int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r. \tag{4.42}$$

Now let

$$\beta = \tilde{c} r \mathrm{e}^{-\nu}. \tag{4.43}$$

Then

$$\mathrm{d}\beta = -\tilde{c} r \mathrm{e}^{-\nu}\left[\nu_r - \frac{1}{r}\right]\mathrm{d}r$$

$$= -\beta\left[\nu_r - \frac{1}{r}\right]\mathrm{d}r. \tag{4.44}$$

Thus (4.42) becomes

$$-\int \frac{\mathrm{d}\beta}{\beta\sqrt{1-\beta^2}} = \sqrt{1+k}\int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r. \tag{4.45}$$

Upon integrating this gives

$$\mathrm{sech}^{-1}(\beta) = \sqrt{1+k}\int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + \tilde{C}_1, \tag{4.46}$$

where $\tilde{C}_1$ is a constant. Simplifying (4.46) we obtain

$$\mathrm{e}^{\nu} = \tilde{c}r\cosh\left[\sqrt{1+k}\int \frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + \tilde{C}_1\right], \tag{4.47}$$

where $\tilde{C}_1$ is a constant of integration.

We conclude from (4.13c), (4.33) and (4.34) that

$$A^i = 0\ ,\ \ A^4 = r\mathrm{e}^{-\lambda}g_1(t).$$

Using (3.7f) we see that

$$A^0{}_t = (1 - r\nu_r)\mathrm{e}^{-\lambda}g_1(t).$$

Integrating this we obtain

$$A^0 = (1 - r\nu_r)\mathrm{e}^{-\lambda}g_2(t) + g_3(r),$$

where $g_2(t) = \int g_1(t)\mathrm{d}t$, and $g_3(r)$ is a function of integration.

We gather all these results in the following theorem:

Theorem 4.4.2. *For static spherically symmetric spacetimes which are non-conformally flat, i.e. $\Lambda \neq 0$, we have either A^4 or A^i must vanish. The two cases are given by:*

a)

$$A^4 = 0,$$

$$A^0 = C, \ \ \textit{a constant}$$

$$A^i(t,r) = -\frac{1}{2\Lambda^{1/2}}\left[2\left(\frac{1}{r} - \lambda_r\right) + \frac{\Lambda_r}{\Lambda}\right]h_1(t),$$

where $h_1(t)$ is arbitrary.

b)

$$A^4(t,r) = r\mathrm{e}^{-\lambda}g_1(t),$$

$$A^i(t,r) = 0,$$

$$A^0(t,r) = (1 - r\nu_r)\mathrm{e}^{-\lambda}g_2(t) + g_3(r),$$

where $g_1(t)$ and $g_3(r)$ are arbitrary functions and $g_2(t) = \int g_1(t)\mathrm{d}t$.

Furthermore, the gravitational potentials are related by the equation

$$\mathrm{e}^{\nu} = \tilde{c}r\cosh\left[\sqrt{1+k}\int\frac{\mathrm{e}^{\lambda}}{r}\mathrm{d}r + \tilde{C}_1\right],$$

where k, $\tilde{c}$ and $\tilde{C}_1$ are constants.

Theorem 4.4.1 and Theorem 4.4.2 represent the classification of the conformal geometry in static spherically symmetric spacetimes in terms of the Weyl tensor. We emphasize that this classification is the most general in terms of the Weyl tensor. It is remarkable that the nonlinear differential equations relating the gravitational potentials can be solved in general. The potentials ν and λ are related by the hyperbolic function cosh. Also note that the theorems correct the Maartens *et al* (1995, 1996) result. Their analysis is simpler because the nonlinear differential equations (4.14) and (4.35) do not arise due to the mistake in their integrability conditions. As we have demonstrated these nonlinear equations lead to a more rich structure in the conformal geometry directly involving the gravitational potentials. Our analysis completes the general solution and classification of conformal motions in static spherically symmetric spacetimes.

Chapter 5

Classification of Conformal motions in non-static spacetimes

5.1 Introduction

In this chapter we consider the classification of symmetries in terms of conformal motions for non-static spherical spacetimes. We classify the conformal symmetries in terms of the Weyl tensor which represents tidal forces. We first calculate the Weyl tensor components for the non-static line element (3.8) in §5.2. We have checked the correctness of these expressions with Mathematica (Wolfram 2010) and Maple (2009). We then use these Weyl tensor components to calculate the Lie derivatives in §5.3. We apply the integrability condition (2.18) to classify the conformal motions for these spacetimes. Two cases arise which are distinguished by the vanishing or non-vanishing of the Weyl tensor. In the first case we explicitly obtain a relationship relating the metric functions by solving a nonlinear second order partial differential. In the second case we find that the metric functions are related via a nonlinear third order partial differential equation for which particular solutions exist.

5.2 Weyl tensor

As in Chapter 4 the Weyl tensor is crucial for the classification scheme of the conformal symmetries. Below we give all the non-vanishing independent Weyl tensor components of the line element (3.8). The nonzero components for the spacetime (3.8) are

$$C^0{}_{101} = -\frac{1}{3r}\Big[(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr})\Big], \tag{5.1a}$$

$$C^0{}_{202} = \frac{1}{6}r\Big[(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr})\Big], \tag{5.1b}$$

$$C^0{}_{303} = \frac{1}{6}r\sin^2\theta\Big[(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr})\Big], \tag{5.1c}$$

$$C^1{}_{212} = \frac{1}{6}r\Big[(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr})\Big], \tag{5.1d}$$

$$C^1{}_{313} = \frac{1}{6}r\sin^2\theta\Big[(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr})\Big], \tag{5.1e}$$

$$C^2{}_{323} = -\frac{1}{3}r\sin^2\theta\Big[(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr})\Big]. \tag{5.1f}$$

Observe that if we set

$$\Psi = (\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr}), \tag{5.2}$$

we can rewrite (5.1) as

$$C^0{}_{101} = -\frac{1}{3r}\Psi, \tag{5.3a}$$

$$C^0{}_{202} = \frac{1}{6}r\Psi, \tag{5.3b}$$

$$C^0{}_{303} = \frac{1}{6}r\sin^2\theta\Psi, \tag{5.3c}$$

$$C^1{}_{212} = C^0{}_{202}, \tag{5.3d}$$

$$C^1{}_{313} = C^0{}_{303} \tag{5.3e}$$

$$C^2{}_{323} = -2C^0{}_{303}. \tag{5.3f}$$

The accuracy of these components has been verified by use of both Mathematica (Wolfram 2010) and Maple (2009).

5.3 Lie derivative

In this section we apply the general formula given in (2.16) to the Weyl tensor components (5.3) to obtain

$$\begin{aligned}\mathcal{L}_{\mathbf{X}}C^0{}_{101} &= -\frac{1}{3r}\Psi_t\Big[r^2\mathrm{e}^{2(\lambda-\nu)}A^i{}_t\eta_i + A^0\Big]\\ &+\Big[-r^2A^i{}_r\eta_i + A^4\Big]\Big[-\frac{1}{3r}\Psi_r + \frac{1}{3r^2}\Psi\Big]\end{aligned}$$

$$-\frac{2}{3r}\Psi\Big[-(r^2{A^i}_{rr}+2r{A^i}_r)\eta_i+{A^4}_r\Big], \tag{5.4a}$$

$$\mathcal{L}_{\mathbf{X}}{C^0}_{202} = \frac{1}{6}r\Psi_t\Big[r^2\mathrm{e}^{2(\lambda-\nu)}{A^i}_t\eta_i+A^0\Big]+\frac{1}{6}\Big[-r^2{A^i}_r\eta_i+A^4\Big]\Big[r\Psi_r+\Psi\Big]$$

$$+\frac{1}{3}r\Psi\Big[A^i(\eta_i)_{\theta\theta}\Big], \tag{5.4b}$$

$$\mathcal{L}_{\mathbf{X}}{C^0}_{303} = \frac{1}{6}r\sin^2\theta\Psi_t\Big[r^2\mathrm{e}^{2(\lambda-\nu)}{A^i}_t\eta_i+A^0\Big]$$

$$+\frac{1}{6}\sin^2\theta(r\Psi_r+\Psi)\Big[-r^2{A^i}_r\eta_i+A^4\Big]$$

$$+\frac{1}{3}r\sin\theta\cos\theta\Psi\Big[A^i(\eta_i)_\theta-a_3\sin\phi+a_4\cos\phi\Big]$$

$$+\frac{1}{3}r\sin^2\theta\Psi\Big[\csc^2\theta A^i(\eta_i)_{\phi\phi}+\cot\theta(a_3\sin\phi-a_4\cos\phi)\Big], \tag{5.4c}$$

$$\mathcal{L}_{\mathbf{X}}{C^1}_{212} = \mathcal{L}_{\mathbf{X}}{C^0}_{202}, \tag{5.4d}$$

$$\mathcal{L}_{\mathbf{X}}{C^1}_{313} = \mathcal{L}_{\mathbf{X}}{C^0}_{303}, \tag{5.4e}$$

$$\mathcal{L}_{\mathbf{X}}{C^2}_{323} = -2\mathcal{L}_{\mathbf{X}}{C^0}_{303}. \tag{5.4f}$$

We note that (5.4d)-(5.4f) do not provide new information as they are multiples of (5.4b) and (5.4c). Thus we do not consider them in further calculations. If we rearrange (5.4a)-(5.4c) in terms of linearly independent trigonometric functions, we get the following

$$\mathcal{L}_{\mathbf{X}}{C^0}_{101} = \Big[-\frac{1}{3}r\Psi_t \mathrm{e}^{2(\lambda-\nu)}{A^i}_t - \frac{1}{3}(\Psi - r\Psi_r){A^i}_r + \frac{2}{3}\Psi(r{A^i}_{rr} + 2{A^i}_r)\Big]\eta_i$$

$$+\Big[-\frac{1}{3r}\Psi_t A^0 + (-\frac{1}{3r}\Psi_r + \frac{1}{3r^2}\Psi)A^4 - \frac{2}{3r}{A^4}_r\Big], \qquad (5.5a)$$

$$\mathcal{L}_{\mathbf{X}}{C^0}_{202} = \Big[\frac{1}{6}r^3\Psi_t \mathrm{e}^{2(\lambda-\nu)}{A^i}_t - \frac{1}{6}r^2(r\Psi_r + \Psi){A^i}_r - \frac{1}{3}r\Psi A^i\Big]\eta_i$$

$$+\frac{1}{6}\Big[r\Psi_t A^0 + (r\Psi_r + \Psi)A^4\Big], \qquad (5.5b)$$

$$\mathcal{L}_{\mathbf{X}}{C^0}_{303} = \Big[\frac{1}{6}r^3\Psi_t \mathrm{e}^{2(\lambda-\nu)}{A^i}_t - \frac{1}{6}r^2(r\Psi_r + \Psi){A^i}_r\Big]\eta_i \sin^2\theta$$

$$+\frac{1}{6}\Big[r\Psi_t A^0 + (r\Psi_r + \Psi)A^4\Big]\sin^2\theta$$

$$+\frac{1}{3}r\Psi A^i(\eta_i)_\theta \sin\theta\cos\theta + \frac{1}{3}r\Psi A^i(\eta_i)_{\phi\phi}. \qquad (5.5c)$$

We simplify (5.5c) as follows

$$\mathcal{L}_{\mathbf{X}}{C^0}_{303} = \Big[\frac{1}{6}r^3\Psi_t \mathrm{e}^{2(\lambda-\nu)}{A^i}_t - \frac{1}{6}r^2(r\Psi_r + \Psi){A^i}_r - \frac{1}{3}r\Psi A^i + \frac{1}{3}r\Psi A^i\Big]\eta_i \sin^2\theta$$

$$+\frac{1}{6}\Big[r\Psi_t A^0 + (r\Psi_r + \Psi)A^4\Big]\sin^2\theta$$

$$+\frac{1}{3}r\Psi A^i(\eta_i)_\theta \sin\theta\cos\theta + \frac{1}{3}r\Psi A^i(\eta_i)_{\phi\phi}$$

$$= \sin^2\theta\,\mathcal{L}_{\mathbf{X}}{C^0}_{202} + \frac{1}{3}r\Psi A^i\Big[\eta_i\sin^2\theta + (\eta_i)_\theta\sin\theta\cos\theta + (\eta_i)_{\phi\phi}\Big]. \qquad (5.6)$$

We use the condition (2.18) and simplify the trigonometric functions on the right hand side of the above equation. Thus (5.6) has the simple form

$$\mathcal{L}_{\mathbf{X}} C^0{}_{303} = \sin^2\theta \mathcal{L}_{\mathbf{X}} C^0{}_{202}. \tag{5.7}$$

Equation (5.7) is identically satisfied, and as a result, we only need consider (5.5a) and (5.5b) for our analysis. By invoking the integrability condition (2.18) on equations (5.5a) and (5.5b) and using the linear independence of the trigonometric functions we obtain the following conditions

$$-\frac{1}{3}r\Psi_t \mathrm{e}^{2(\lambda-\nu)} A^i{}_t - \frac{1}{3}(\Psi - r\Psi_r)A^i{}_r + \frac{2}{3}\Psi(rA^i{}_{rr} + 2A^i{}_r) = 0, \tag{5.8a}$$

$$-\frac{1}{3r}\Psi_t A^0 + (-\frac{1}{3r}\Psi_r + \frac{1}{3r^2}\Psi)A^4 - \frac{2}{3r}\Psi A^4{}_r = 0, \tag{5.8b}$$

$$\frac{1}{6}r^3\Psi_t \mathrm{e}^{2(\lambda-\nu)} A^i{}_t - \frac{1}{6}r^2(r\Psi_r + \Psi)A^i{}_r - \frac{1}{3}r\Psi A^i = 0, \tag{5.8c}$$

$$r\Psi_t A^0 + (r\Psi_r + \Psi)A^4 = 0. \tag{5.8d}$$

If we multiply (5.8a) by $\frac{1}{2}r^2$ and add the resulting equation to (5.8c) we obtain the following

$$(r^2 A^i{}_{rr} + rA^i{}_r - A^i)\Psi = 0. \tag{5.9}$$

Similarly, if we multiply (5.8b) by $3r^2$ and add the resulting equation to (5.8d) we obtain the following

$$({A^4}_r - r^{-1}A^4)\Psi = 0. \tag{5.10}$$

In order for the above equations to be satisfied we look at the two cases for which this is possible, i.e.

i) $\Psi = 0$,

ii) $\Psi \neq 0$.

Case I : $\Psi = 0$

This case is equivalent to

$$(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr}) = 0, \tag{5.11}$$

were we have used (5.2). This is a nonlinear second order equation with two unknown functions λ and ν. It is possible to reduce this equation in terms of a single unknown function. If we let

$$w(t,r) = \frac{(\lambda_r - \nu_r)}{r}, \tag{5.12}$$

then equation (5.11) is transformed into

$$w_r = rw^2, \tag{5.13}$$

which is a separable first order partial differential equation. Solving (5.13) yields

$$w = -\frac{2}{r^2 + 2a_1(t)}, \tag{5.14}$$

where $a_1(t)$ is a function of integration. Then (5.12) and (5.14), give

$$\lambda_r - \nu_r = -\frac{2r}{r^2 + 2a_1(t)}. \tag{5.15}$$

Integrating (5.15) we obtain

$$\lambda - \nu = -\ln[r^2 + 2a_1(t)] + a_2(t), \tag{5.16}$$

where $a_2(t)$ is another function of integration.

Theorem 5.3.1. *In non-static spherically symmetric spacetimes, which are conformally flat, i.e. $\Psi = 0$, the functions, $A^0(t,r), A^4(t,r)$* and *$A^i(t,r)$ are arbitrary. Furthermore, the gravitational potentials are related by the formula*

$$\lambda - \nu = -\ln[r^2 + 2a_1(t)] + a_2(t),$$

where $a_1(t)$ and $a_2(t)$ are arbitrary functions of integration.

Case $II: \;\; \Psi \neq 0$

For this case (5.9) reduces to

$$r^2 A^i{}_{rr} + rA^i{}_r - A^i = 0. \tag{5.17}$$

Equation (5.17) is an Euler-Cauchy partial differential equation. Solving this equation we obtain

$$A^i = rF^i(t) + r^{-1}G^i(t), \tag{5.18}$$

where F^i and G^i are functions of integration. Substituting (5.18) into (5.8a) or (5.8c), we obtain

$$\begin{aligned} -r\Psi_t \mathrm{e}^{2(\lambda-\nu)}(rF^i{}_t + r^{-1}G^i{}_t) + (r\Psi_r - \Psi)(F^i - r^{-2}G^i) & \\ +2\Psi\Big[r(2r^{-3}G^i) + 2(F^i - r^{-2}G^i)\Big] &= 0. \end{aligned} \tag{5.19}$$

Simplifying this equation we get

$$-\mathrm{e}^{2(\lambda-\nu)}(r^2F^i{}_t + G^i{}_t)\Big[(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr})\Big]_t$$

$$+(rF^i - r^{-1}G^i)\Big[(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr})\Big]_r$$

$$+(3F^i + r^{-2}G^i)\Big[(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr})\Big] \;=\; 0. \qquad (5.20)$$

Equation (5.20) is a third order non-linear partial differential which is given in terms of the difference between the potentials λ and ν. This equation is difficult to solve in general. We can simplify the equation substantially if we let

$$w(t,r) = \frac{\lambda_r - \nu_r}{r}. \qquad (5.21)$$

Then (5.20) becomes

$$\begin{aligned}
-\exp\left\{2\int^r sw(t,s)\mathrm{d}s\right\}\Big[r^2{F^i}_t + {G^i}_t\Big]\Big[r^2(rw - w_r)\Big]_t \\
+(rF^i - r^{-1}G^i)\Big[r^2(rw - w_r)\Big]_r \\
+(3F^i + r^{-2}G^i)\Big[r^2(rw - w_r)\Big] \;&=\; 0. \qquad (5.22)
\end{aligned}$$

The two functions λ and ν have been replaced by the single function w in (5.22). Equation (5.22) is simpler but remains nonlinear and a general solution is not obvious. There may be transformations, other than (5.21), that could lead to a general solution; we will pursue this avenue in future research. Clearly particular solutions to (5.20) are possible. For example

$$\lambda - \nu \equiv f(t)$$

does not depend on the radial coordinate r and (5.20) is satisfied.

Now (5.10) reduces to

$$A^4{}_r - r^{-1}A^4 = 0. \tag{5.23}$$

On solving the differential equation (5.23) we obtain

$$A^4 = rF^4(t), \tag{5.24}$$

where $F^4(t)$ is a function of integration. Thus, from (5.8b) or (5.8d), A^0 is given by

$$A^0 = -\frac{r\Psi_r + \Psi}{\Psi_t}F^4(t). \tag{5.25}$$

Theorem 5.3.2. *In non-static spherically symmetric spacetimes which are non-conformally flat, i.e.* $\Psi \neq 0$, *the functions* $A^0(t,r), A^4(t,r)$ *and* $A^i(t,r)$ *are given by*

$$A^0 = -\frac{r\Psi_r + \Psi}{\Psi_t}F^4(t),$$

$$A^4 = rF^4(t),$$

$$A^i = rF^i(t) + r^{-1}G^i(t).$$

The gravitational potentials ν *and* λ *are related by the following non-linear partial* *differential equation*

$$-\exp\left\{2\int^{r} sw(t,s)\mathrm{d}s\right\}\left[r^2{F^i}_t + {G^i}_t\right]\left[r^2(rw - w_r)\right]_t$$

$$+(rF^i - r^{-1}G^i)\left[r^2(rw - w_r)\right]_r$$

$$+(3F^i + r^{-2}G^i)\left[r^2(rw - w_r)\right] = 0,$$

where $rw = \lambda_r - \nu_r$.

We believe this investigation is the first attempt to classify the conformal geometry in non-static spacetimes in terms of the Weyl tensor. In **Case** I we obtained a partial differential equation which we solved in general to obtain a solution which relates the potentials ν and λ. This is a new result. In **Case** II we obtained a highly nonlinear partial differential equation which we could not solve in general. We transformed this equation into an equivalent form and showed particular solutions are possible. This is also a new result. In Chapter 4 the integrability conditions were ordinary differential equations. Here the integrability conditions are partial differential equations which are more difficult to solve because the potentials ν and λ depend on both r and t. Our analysis completes the classification of conformal motions in non-static spherically symmetric spacetimes subject to finding the general solution of (5.22).

Chapter 6

Conclusion

Our main objective in this thesis was to classify conformal symmetries in static and non-static spherically symmetric spacetimes. We first analysed the classification in static spacetimes following the work of Maartens *et al* (1995). We calculated the Weyl tensor components and the Lie derivatives of these components. From our calculations we identified an error in the work of Maartens *et al* (1995) which we went on to correct. Using the integrability condition for the existence of conformal symmetry we derived two cases for our classification: the conformally flat case and the non-conformally flat case. In both cases we derived nonlinear second order ordinary differential equations which we solved in general. The solution to these nonlinear equations helped us complete the classification of conformal motions in static spacetimes. We then analysed the classification of conformal symmetries in non-static spacetimes. Following the same procedure as in the static case, we calculated the Weyl tensor components and their respective Lie derivatives. The two cases of the conformally flat and the non-conformally flat spacetimes arose in our analysis. In the conformally flat case a nonlinear second order partial differential equation was identified and solved in general to give a complete classification of conformal motions in non-static spacetimes. The non-conformally flat case contains a nonlinear partial differential equation for which only particular solutions have been found.

We now provide an outline of the results obtained in this investigation:

- In Chapter 2 we discussed some aspects of differential geometry that are important in our study. We introduced some important tensors which are widely used in general relativity. In particular we discussed the Weyl tensor which plays a crucial role in the study of conformal symmetries. The conformal Killing equation was also introduced.

- In Chapter 3 the general solution of the conformal Killing vector equation was presented in spherically symmetric spacetimes. We considered both the static and non-static cases. In each case the solution was given subject to integrability conditions that placed further restrictions on the metric functions. We adopted the work of Maartens *et al* (1995,1996) to write the conformal geometry and their integrability conditions in a more compact form.

- In Chapter 4 we presented the non-vanishing Weyl tensor components for static spherically symmetric spacetimes. We proceeded to use these values to calculate the Lie derivatives. We identified an error in the paper by Maartens *et al* (1995) which is corrected in this thesis. Two cases arose in our classification, namely the conformally flat and the non-conformally flat case. In the conformally flat case we derived the following second order partial differential equation

$$\nu_{rr} + {\nu_r}^2 - \lambda_r \nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}(1 - \mathrm{e}^{2\lambda}) = 0. \tag{6.1}$$

  This equation was solved in general. It is interesting to observe that our approach was similar to the method followed by Herera *et al* (2001) in the evolution of anisotropic general relativistic spheres. For the non-conformally flat case we derived the nonlinear second order differential equation

$$\nu_{rr} + {\nu_r}^2 - \lambda_r \nu_r + r^{-1}(\lambda_r - \nu_r) + r^{-2}[1 - (1+k)\mathrm{e}^{2\lambda}] = 0. \tag{6.2}$$

We solved (6.2) in the same manner as (6.1). We presented results from the two cases in the form of theorems. This completed the classification of conformal motions in static spacetimes.

- In Chapter 5 we followed the approach of Chapter 4 for the non-static spherically symmetric spacetimes. Two cases arose in our analysis, namely the conformally flat case and the non-conformally flat case. For the conformally flat case we derived the following nonlinear partial differential equation

$$(\lambda_r - \nu_r) + r(\lambda_r - \nu_r)^2 - r(\lambda_{rr} - \nu_{rr}) = 0. \tag{6.3}$$

Equation (6.3) was integrated in general. For the non-conformally flat case we derived the third order nonlinear partial differential equation

$$\begin{aligned} -\exp\left\{2\int^r sw(t,s)\mathrm{d}s\right\}\left[r^2{F^i}_t + {G^i}_t\right]\left[r^2(rw - w_r)\right]_t & \\ +(rF^i - r^{-1}G^i)\left[r^2(rw - w_r)\right]_r & \\ +(3F^i + r^{-2}G^i)\left[r^2(rw - w_r)\right] &= 0. \end{aligned} \tag{6.4}$$

Equation (6.4) has particular solutions but a general solution remains outstanding. We presented our results in the form of a theorem. This completed our general classification of conformal symmetries in non-static spherically symmetric spacetimes.

In this thesis we have demonstrated the power of using the Weyl tensor in classifying the conformal geometry in spherically symmetric spacetimes. This suggests that it may be worthwhile analysing the role of the Weyl tensor in other spacetimes of physical interest.

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