COMPLETION OF UNIFORM AND METRIC FRAMES

U G Murugan

"Does the pursuit of truth give you as much pleasure as before? Surely it is not the knowing, but the learning, not the possessing but the acquiring, not the being there but the getting there, that afford the greatest satisfaction. If I have clarified and exhausted something, I leave it in order to go again into the dark. Thus is that insatiable man so strange; when he has completed a structure it is not in order to dwell in it comfortable, but to start another."

Karl Friedrich Gauss

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BY

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DECLARATION

I hereby declare that the contents of this dissertation entitled COMPLETION OF UNIFORM AND METRIC FRAMES is the result of my own investigation and research except where due reference has been made. It has not been submitted before in part or in full for any other degree or to any other institution.

U G MURUGAN
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In conclusion, I wish to dedicate this thesis to the memory of my late mother, Mrs Patchamah Murugan, who though illiterate and lacking in financial means, had the foresight to insist on my pursuing an academic career.

Contents

Chapter 0

Introduction and Summary

The term "frame" was introduced by C H Dowker, who studied them in a long series of joint papers with D Papert Strauss. J R Isbell, in a path - breaking paper [1972] pointed out the need to introduce separate terminology for the opposite of the category of Frames and coined the term "locale". He was the progenitor of the idea that the category of Locales is actually more convenient in many ways than the category of Frames. In fact, this proves to be the case in one of the approaches adopted in this thesis.

Sublocales (quotient frames) have been studied by several authors, notably Dowker and Papert [1966] and Isbell [1972]. The term "sublocale" is due to Isbell, who also used "part" to mean approximately the same thing. The use of nuclei as a tool for studying sublocales (as is used in this thesis) and the term "nucleus" itself was initiated by H Simmons [1978] and his student D

Macnab [1981].

Uniform spaces were introduced by Weil [1937]. Isbell [1958] studied algebras of uniformly continuous functions on uniform spaces. In this thesis, we introduce the concept of a uniform frame (locale) which has attracted much interest recently and here too Isbell [1972] has some results of interest. The notion of a metric frame was introduced by A Pultr [1984]. The main aim of his paper [11] was to prove metrization theorems for pointless uniformities.

This thesis focuses on the construction of completions in Uniform Frames and Metric Frames. Isbell [6] showed the existence of completions using a frame of certain filters. We describe the completion of a frame L as a quotient of the uniformly regular ideals of L, as expounded by Banaschewski and Pultr[3]. Then we give a substantially more elegant construction of the completion of a uniform frame (locale) as a suitable quotient of the frame of all downsets of L. This approach is attributable to Kříž[9]. Finally, we show that every metric frame has a unique completion, as outlined by Banaschewski and Pultr[4]. In the main, this thesis is a standard exposition of known, but scattered material.

Throughout the thesis, choice principles such as C.D.C (Countable Dependent Choice) are used and generally without mention. The treatment of category theory (which is used freely throughout this thesis) is not self-contained. Numbers in brackets refer to the bibliography at the end of the thesis. We will use □ to indicate the end of proofs of lemmas, theorems and propositions.

Chapter 1 covers some basic definitions on frames, which will be utilized in subsequent chapters. We will verify whatever we need in an endeavour to enhance clarity. We define the categories, $\mathbb{F}rm$ of frames and frame homomorphisms, and $\mathbb{L}oc$ the category of locales and frame morphisms. Then we explicate the adjoint situation that exists between $\mathbb{F}rm$ and $\mathbb{T}op$, the category of topological spaces and continuous functions. This is followed by an introduction to the categories, $\mathbb{R}eg\mathbb{F}rm$ of all regular frames and frame homomorphisms, and $\mathbb{K}\mathbb{R}eg\mathbb{F}rm$ the category of compact regular frames and their homomorphisms. We then present the proofs of two very important lemmas in these categories. Finally, we define the compactification of and a congruence on a frame.

In Chapter 2 we recall some basic definitions of covers, refinements and star refinements of covers. We introduce the notion of a uniform frame and define certain mappings (morphisms) between uniform frames (locales). In the terminology of Banaschewski[1] and Křiž [9] we define a complete uniform frame and the completion of a uniform frame.

The aim of Chapter 3 is twofold : first, to construct the compact regular coreflection of uniform frames, that is, the frame counterpart of the Samuel Compactification of uniform spaces [12], and then to use it for a description of the completion of a uniform frame as an alternative to that previously given by Isbell[6].

The main purpose of Chapter 4 is to provide another description of uniform completion in frames (locales), which is in fact even more straightforward than the original topological construction. It simply consists of writing down generators and defining relations. We provide a detailed examination of the main result in this section, that is, a uniform frame L is complete iff each uniform embedding $f : (M,\mathcal{U}M) \to (L,\mathcal{U}L)$ is closed, where $\mathcal{U}M$ and $\mathcal{U}L$ denote the uniformities on the frames M and L respectively.

Finally, in Chapter 5, we introduce the notions of a metric diameter and a metric frame. Using the fact that every metric frame is a uniform frame and hence has a uniform completion, we show that every metric frame L has a unique completion $\gamma : CL \to L$.

Chapter 1

The Theory of Frames and Locales

In this chapter we define frame and frame homomorphisms and explicate the adjoint situation that exists between the categories $\mathbb{F}rm$ and $\mathbb{T}op$. We then introduce the category $\mathbb{L}oc$ as the opposite category of $\mathbb{F}rm$ and adopt the 'locale-theoretic' view that what matters about a space is its lattice of open sets and not its points. This entitles us to use the names of familiar concepts in topology for their natural generalizations in $\mathbb{L}oc$. After recalling some basic terminology, we introduce the categories of $\mathbb{R}eg\mathbb{F}rm$ and $\mathbb{KR}eg\mathbb{F}rm$. The proofs of two important lemmas in these categories are then presented. Next we define the compactification of and a congruence on a frame. Most of the material presented in this chapter is known and can be found in Banascheskwi [1] and Johnstone[7]. Nevertheless, it is included for the sake of completeness and has a bearing on some of the results which are presented later in the thesis.

1.1 Frames and Frame Homomorphisms

1.1.1 Definition : *A frame is a complete lattice L in which the infinite distributive law*

$$a \wedge \bigvee S = \bigvee\{a \wedge s | s \in S\}$$

holds for any $a \in L$ and $S \subseteq L$.

The term 'complete' as used above means that all subsets of L have a meet and hence also a join.

1.1.2 Definition *A frame homomorphism is a map $h : L \to M$ of frames L, M preserving all finite meets, including the unit e, and arbitrary joins, including the zero, 0.*

Thus we have the category $\mathbb{F}rm$ of frames and their homomorphisms.

1.1.3 Definitions :

1.1.3.1 *A frame homomorphism $h : L \to M$ is called an embedding iff h is injective.*

1.1.3.2 *An embedding h is called closed if we have for $a, b \in L, h(a) = h(b)$ implies*

$$a \vee \bigvee_{h(u)=0} u = b \vee \bigvee_{h(u)=0} u.$$

1.1.3.3 *A frame homomorphism h is called dense if $h(x) = 0$ implies $x = 0$.*

1.1.3.4 *A frame homorphism h is called codense if $h(x) = e$ implies $x = e$.*

1.1.4 Remark: Any frame homomorphism $h : L \to M$ has a right adjoint $h_* : M \to L$ defined by the condition

$$h(x) \leq y \text{ iff } x \leq h_*(y) \text{ for all } x \in L \text{ and } y \in M,$$

where $h_*(y) = \bigvee\{x \in L/h(x) \leq y\}$. In particular, if h is onto then $h_*(y)$ is the largest element which h maps to y. We have that h_* preserves arbitrary meets.

1.2 Frames and Topological Spaces

A principal example of the notions in 1.1 is the frame $\mathfrak{D}X$ of open sets of a topological space X, and the frame homomorphism $\mathfrak{D}f : \mathfrak{D}Y \to \mathfrak{D}X$, induced by any continuous map $f : X \to Y$ between topological spaces, taking $U \in \mathfrak{D}Y$ to $f^{-1}(U) \in \mathfrak{D}X$. Indeed, the resulting correspondence constitutes a contravariant functor $\mathfrak{D} : \mathbb{T}op \to \mathbb{F}rm$ from the category $\mathbb{T}op$ of topological spaces and continuous maps to the category $\mathbb{F}rm$. On the other hand, there is a contravariant functor $\sum : \mathbb{F}rm \to \mathbb{T}op$ such that for any frame L, $\sum L$ is the space of all frame homomorphisms $\xi : L \to \underline{2}$ ($\underline{2}$ is the two-element frame$\{0,1\}$), with open sets $\sum_a = \{\xi \in \sum L | \xi(a) = 1\}$ for $a \in L$. For any frame homorphism $h : L \to M, \sum h : \sum M \to \sum L$ acts by composition with h, that is, $(\sum h)(\xi) = \xi h$. Moreover, $\mathfrak{D}$ and $\sum$ are adjoint on the right, with adjunction maps

$$\eta_L : L \to \mathfrak{D}\textstyle\sum L \text{ given by } \eta_L(a) = \sum_a$$

and

$$\varepsilon_X : X \to \sum \mathfrak{D}X \text{ given by } \varepsilon_X(x) = \tilde{x}$$

where $\tilde{x} : \mathfrak{D}X \to \underline{2}$ such that $\tilde{x}(U) = 1$ iff $x \in U$ and $\tilde{x}(U) = 0$ iff $x \notin U$ for all $U \in \mathfrak{D}X$.

1.3 Locales

1.3.1 Definition : : *Given a category K, the opposite (dual) category K^{op} of K has*

$$Ob(K) = Ob(K^{op})$$

and

$$K^{op}(M, L) = K(L, M)$$

that is, both categories have the same objects and if $L \to M$ is a morphism in K then $M \to L$ is the corresponding morphism in K^{op}.

In general, if a statement p is true for a category K then there is a dual statement p^{op}(obtained by changing systematically the directions of all arrows) that will be true for K^{op}.

1.3.2 Category of Locales

We shall write $\mathbb{L}oc$ for the opposite category $\mathbb{F}rm^{op}$ and call its objects *locales.* As long as we are concerned only with the objects, the terms 'frame' and 'locale' are entirely synonymous; it is only when we refer to morphisms that they become different. For example, a *subframe* of a frame L is simply a subset of a frame L, which is closed under finite meets and arbitrary joins but a *sublocale* is something different, corresponding to a quotient frame (the precise definition of a sublocale appears later in 4.1.1). The reason for this dual terminology is that, by making $\mathfrak{O} : \mathbb{T}op \to \mathbb{L}oc$ into a covariant functor, we are entitled to use the names of familiar concepts in topology for their natural generalizations in $\mathbb{L}oc$. For instance, we can talk about closed and dense sublocales of a given locale, whereas we should have had to refer to quotient frames. Thus, there is a natural parallelism between the concepts in $\mathbb{T}op$ and $\mathbb{L}oc$.

We now prove the following lemma, which is attributable to Kříž[9]

1.3.3 Lemma : : *In $\mathbb{L}oc$, a dense closed embedding is an isomorphism.*

Proof. :

Let $h : M \to L$ be a dense closed embedding. Since h is an embedding , h is surjective. It remains to show that h is one-to-one. So, for $a, b \in M$, let $h(a) = h(b)$. Since h is closed we have

$$a \vee \bigvee_{h(u)=0} u = b \vee \bigvee_{h(u)=0} u$$

But, by the density of $h, h(u) = 0$ implies $u = 0$. It follows that $a \vee \bigvee 0 = b \vee \bigvee 0$ implies $a = b$. $\square$

1.4 Compact Regular Frames

1.4.1 Definitions: : *Let L be a frame. Then*

1.4.1.1 *For elements x and y in L, x is said to be rather below y, written $x \prec y$ if there exists an element $t \in L$ such that $x \wedge t = 0$ and $y \vee t = e$.*

1.4.1.2 *L is called a regular frame whenever $a = \bigvee\{x \in L | x \prec a\}$ for all $a \in L$.*

Let $\mathbb{R}eg\mathbb{F}rm$ denote the full subcategory of all regular frames and frame homomorphisms.

We now prove a useful result (Banaschewski [1]).

1.4.2 Lemma : *For $h : L \longrightarrow M$ in $\mathbb{R}eg\mathbb{F}rm$, if h is dense then it is monic.*

Proof. :

Let $h : L \longrightarrow M$ be dense in $\mathbb{R}eg\mathbb{F}rm$ and let $u, v : N \longrightarrow L$ with $hu = hv$. For any $a \in N, a = \bigvee\{x | x \prec a\}$ since $N \in \mathbb{R}eg\mathbb{F}rm$. Since $x \prec a$, there exists $y \in N$ such that $x \wedge y = 0$ and $a \vee y = e$. Now $x \wedge y = 0$ implies $hu(x \wedge y) = hu(0)$, that is, $h(u(x)) \wedge h(u(y)) = 0$. Since $hu = hv$, we have $h(u(x)) \wedge h(v(y)) = 0$ which implies $h(u(x) \wedge v(y)) = 0$. By the density of h, $u(x) \wedge v(y) = 0$. Also $y \vee a = e$ implies $v(y \vee a) = v(e)$, that is, $v(y) \vee v(a) = e$.

Now

$$\begin{aligned}
u(x) &= u(x) \wedge e \\
&= u(x) \wedge (v(y) \vee v(a)) \\
&= (u(x) \wedge v(y)) \vee (u(x) \wedge v(a)) \\
&= 0 \vee (u(x) \wedge v(a)) \\
&= u(x) \wedge v(a) \\
&\leq v(a)
\end{aligned}$$

Therefore $u(a) = \bigvee_{x \prec a} u(x) \leq v(a)$.

By symmetry $v(a) \leq u(a)$ which implies $u = v$ $\square$.

1.4.3 Definition : : *Let L be a meet semilattice with unit e. For any subset $X \subseteq L, X$ is called a downset if $c \in X$ implies $\downarrow c = \{x \in L | x \leq c\} \subseteq X$.*

1.4.4 Definition : : *Let L be any distributive lattice with the bottom element 0. Any subset $J \subseteq L$ is an ideal of L if :*

1. $0 \in J$
2. $a, b \in J$ implies $a \vee b \in J$
3. J is a downset.

We denote by $\mathcal{J}L$, the set of all ideals of L and note that $\mathcal{J}L$ is a frame. The operations $\bigwedge$ and $\bigvee$ which make $\mathcal{J}L$ a frame are defined by : $J \bigwedge K = J \cap K$ and $J \bigvee K = \{a \vee b | a \in J, b \in K\}$.

1.4.5 Definition : : *In a frame L, an element $c \in L$ is called compact if for any $X \subseteq L$ with $c \leq \bigvee X$, there exists finite $E \subseteq X$ with $c \leq \bigvee E$. L is compact if e is compact.*

A useful criterion for compactness (not difficult to establish) is in terms of ideals : A frame L is compact iff $\bigvee : \mathcal{J}L \to L$ is codense, that is, $\bigvee J = e$ implies $e \in J$ for each $J \in \mathcal{J}L$.

1.4.6 Remarks :

1.4.6.1 *Any subframe of a compact frame is compact, but the homomorphic image of a compact frame need not be compact.*

1.4.6.2 *We now have the full subcategory $\mathbb{KR}eg\mathbb{F}rm$ of compact regular frames and their homomorphisms.*

The following is a familiar characterization of embeddings in $\mathbb{KR}eg\mathbb{F}rm$ which is attributed to Banaschewski[2].

1.4.7 Lemma : *For any $h : L \to M$ in $\mathbb{KR}eg\mathbb{F}rm$, the following are equivalent :*

(1) *h is an embedding, that is, h is one-to-one.*

(2) *For all $x \in L, h(x) = 0$ implies $x = 0$, that is, h is dense.*

(3) *For all $x \in L, h(x) = e$ implies $x = e$, that is, h is codense.*

Proof. :

(1) implies (2) : Let $h(x) = 0$. Since h is a homomorphism we have $h(x) = h(0)$. Given that h is one-to-one, we have $x = 0$.

(2) implies (3) : Let $h(x) = e$. Since L is regular, $x = \bigvee_{z_i \prec x} z_i$ which implies $h(x) = \bigvee_{h(z_i) \prec h(x)} h(z_i) = e$.
By compactness $h(x) = h(z_1) \vee \ldots \vee h(z_n) = e$ where $z_i \prec x$ for all $i = 1, \ldots n$. It follows that $h(x) = h(z) = e$ for some $z \prec x$. Then, for $t \in L$ such that $z \wedge t = 0$ and $x \vee t = e$, we have

$$\begin{aligned}
h(t) &= h(t) \wedge e \\
&= h(t) \wedge h(z), \quad h(z) = e \\
&= h(t \wedge z), \qquad h \text{ is a homomorphism} \\
&= h(0), \qquad\quad t \wedge z = 0 \\
&= 0
\end{aligned}$$

Since h is dense, $t = 0$. But $x \vee t = e$ which implies $x = e$.

(3) implies (1) : If $h(x) = h(y)$, take any $z \prec x$ and $t \in L$ such that $z \wedge t = 0$ and $x \vee t = e$. Then $e = h(x \vee t) = h(x) \vee h(t) = h(y) \vee h(t) = h(y \vee t)$. Hence, $y \vee t = e$ by hypothesis. Therefore $z \prec y$ which implies $z \leq y$. Now, by regularity $x = \bigvee_{z_i \prec x} z_i$ and this shows $x \leq y$. By symmetry $y \leq x$ which implies $x = y$. □

1.4.8 Definition : : *A compactification of a frame L is a dense onto homomorphism $h : M \to L$ where M is a compact regular frame.*

1.5 Congruences

1.5.1 Definition : : *A congruence on a frame L is an equivalence relation which is a subframe of $L \times L$*

1.5.2.1 : *Let $\mathfrak{L}L = \{\theta | \theta$ is a congruence on $L\}$. $\mathfrak{L}L$ is closed under arbitrary intersections, so it is a complete lattice with bottom element $\triangle = \{(x, x) | x \in L\}$ and top element, nabla, $\nabla = L \times L$. In fact, $\mathfrak{L}L$ is a frame.*

1.5.2.2 : *If $h : L \rightarrow M$ is a frame homomorphism then $Ker(h) = \{(x, y) | h(x) = h(y)\} \in \mathfrak{L}L$. Conversely, if $\theta \in \mathfrak{L}L$ then $\theta = Ker(h)$ for some h.*

1.5.2.3 : *For any $a \in L$, the following are congruences :*

$$\nabla_a = \{(x, y) | x \vee a = y \vee a\}$$

and

$$\triangle_a = \{(x, y) | x \wedge a = y \wedge a\}.$$

Chapter 2

Uniform Frames (Locales)

This chapter is intended to be a versatile introduction to uniform frames. Using the covering definition, we develop the notion of a uniformity on a frame (locale). We then introduce the relation $\lhd$ (uniformly below) on uniform frames and use it to prove that every uniform frame (locale) is regular. We also define a uniform frame homomorphism. The definitions and lemmas presented here are attributable to Banaschewski and Pultr[3] and foreshadow later material. In the terminology of Křiž[9], we define a complete uniform frame and the completion of a uniform frame. For basic results on uniform frames we refer to Isbell[6] and Křiž[9].

2.1 Uniformities on Frames

2.1.1.1 *Given a frame L, a cover of L is any subset $U \subseteq L$ whose join is the unit e, that is, $\bigvee U = e$.*

2.1.1.2 *For covers U, V of L, U is a refinement of V (and write $U \leq V$) iff for all $a \in U$ there are exists $b \in V$ such that $a \leq b$.*

2.1.1.3 *For a cover U of L and $x \in L$, let*

$$Ux = \bigvee\{a \in U | a \wedge x \neq 0\}.$$

For covers, U, V of L, define $U \leq^* V(U$ star refines $V)$ to mean that the cover $\{Ux | x \in U\}$ refines V.

2.1.2 Definition : *A uniformity on a frame L is a system of covers $\mathcal{U}$ of L satisfying :*

(1) If $U \in \mathcal{U}$ and $U \leq V$ then $V \in \mathcal{U}$.

(2) If $U \in \mathcal{U}$ and $V \in \mathcal{U}$. then $U \bigwedge V \in \mathcal{U}$,

where $U \bigwedge V = \{a \wedge b | a \in U, b \in V\}$.

(3) For all $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \leq^* U$.

(4) For all $a \in L, a = \bigvee\{x \in L | Ux \leq a$, for some $U \in \mathcal{U}\}$.

Thus, a uniformity on L is a *filter* $\mathcal{U}$ of covers of L satisfying conditions (3) and (4) above. We call a system $\mathcal{U}$ of covers of L satisfying conditions (3) and (4) only, a *uniform basis* of L.

In the following L will be a *uniform frame,* that is, a frame together with a specified uniformity. The latter will be denoted $\mathcal{U}L$; we call its members $A, B, C...., U, V, ...$ the *uniform covers* of L.Further, we allow the notational confusion between L and its underlying frame.

2.2 Uniform Frames and Uniform Frame Homomorphisms

2.2.1 Definition : *For any $x, y \in L$, we define $x \lhd y$ (x is* **uniformly below** *y) to mean $Ax \leq y$ for some $A \in \mathcal{U}L$.*

We now note the following familiar and simple consequences of the definition of uniform frames.

2.2.2 Properties of $\lhd$:

(1) If $s \leq x \lhd y \leq t$ then $s \lhd t$.

(2) If $x \lhd y$ and $u \lhd v$ then $x \wedge u \lhd y \wedge v$ and $x \vee u \lhd y \vee v$.

(3) If $x \lhd y$ then there exists z such that $x \lhd z \lhd y$.

(4) For all $a \in L, a = \bigvee\{x \in L | x \lhd a\}$.

2.2.3 Lemma : In a uniform frame, $x \lhd y$ implies that $x \prec y$.(Pultr[10])

Proof. :

If $x \lhd y$ then $Ax \leq y$ for some uniform cover A, where

$Ax = \bigvee\{a \in A | a \wedge x \neq 0\}$.

Take $z = \bigvee\{a \in A | a \wedge x = 0\}$.Then $x \wedge z = 0$ and $(Ax) \vee z = \bigvee A = e$. Hence also $y \vee z = e$. Since $x \wedge z = 0$ and $y \vee z = e$, we have $x \prec y$. □

2.2.4 Corollary : *Every uniform frame (locale) is regular.*

Proof. :

Let L be a uniform frame (locale). For all $a \in L$,

$$a = \bigvee\{x \in L | x \lhd a\}.$$

By Lemma 2.2.3 $a = \bigvee\{x \in L | x \prec a\}$, which implies that L is a regular frame. □

2.2.5 Definition : *For uniform frames L, M, a uniform frame homomorphism $h : L \longrightarrow M$ is a frame homomorphism such that*

$h[A] = \{h(a) | a \in A\}$ belongs to $\mathcal{U}M$ for each as $A \in \mathcal{U}L$.

2.2.6 Lemma : *Let $h : L \longrightarrow M$ be a uniform frame homomorphism. If $x \lhd y$ in L then $h(x) \lhd h(y)$ in M.*

Proof. :

If $x \lhd y$ then $Ax \leq y$ for some uniform cover A. Take any $h(a), a \in A$. For any $x \in L, h(a) \wedge h(x) \neq 0$ implies that $a \wedge x \neq 0$. So $a \leq Ax$ which implies that $h(a) \leq h(Ax)$ for all $a \in A$. Hence,

$$\bigvee\{h(a) \in h[A] | h(a) \wedge h(x) \neq 0\} \leq h(Ax)$$

which implies that $h[A]h(x) \leq h(Ax) \leq h(y)$ since Ax$\leq y$. Therefore $h(x) \lhd h(y)$. $\square$

2.2.7 Definition : *An ideal $J \subseteq L$ will be called uniformly regular whenever $x \in J$ there exists $y \in J$ such that $x \lhd y$.*

2.3 Completion of a Uniform Frame

2.3.1.1 *We call a map $h : L \to M$ of uniform frames a surjection whenever h is a uniform homomorphism mapping L onto M such that the covers $h[A]$ with $A \in \mathcal{U}L$ generate $\mathcal{U}M$.*

2.3.1.2 *A uniform frame (locale) L is complete whenever any dense surjection $M \to L$ is an isomorphism.*

2.3.1.3 *A complete uniform frame (locale) M together with a dense surjection $M \to L$ is called a completion of a uniform frame (locale)L.*

The next chapter provides a detailed examination of the *completion* of a uniform frame.

Chapter 3

Completion of Uniform Frames

Completion of uniform frames have been previously studied by Isbell. We recall that Isbell[6] showed the existence of completions using a frame of certain filters. In this chapter, we firstly construct the compact regular coreflection of uniform frames, that is, the frame counterpart of the Samuel Compactification of uniform spaces. By compactification we imply the operation whereby, given a uniform space X, we construct a compact space having a dense subspace homeomorphic with X. Samuel [12] showed that for any uniformizable space X, with the uniform structure $\mathcal{U}$ we can construct a compactification $\tilde{X}$, which contains as topological subspace the completion $\hat{X}$ of X with respect to $\mathcal{U}$. Samuel describes the compact uniform reflection of an arbitrary uniform space X as a certain quotient of the space of ultrafilters on the underlying set of X.

We then turn to a new description of the completion of a uniform frame L as a quotient of $\mathcal{R}L$, the subframe of all uniformly regular ideals of the frame $\mathcal{J}L$ of all ideals of L. This approach, as expounded by Banaschewski and Pultr [3], is conceptually related to locating the underlying topological space of the completion of a uniform space X inside the Samuel compactification Y of X as the subspace of all those points of Y whose trace filters on X are Cauchy filters in X. Here, a trace filter is the filter given by the intersections with X of all neighbourhoods in Y of some point in Y.

3.1 Compact Regular Coreflection of Uniform Frames

We commence this chapter with some basic results on uniform frames, which are attributable to Banachewski and Pultr[3].

3.1.1 Lemma *The uniformly regular ideals of a frame L form a subframe $\mathcal{R}L$ of the frame $\mathcal{J}L$ of all ideals of L.*

Proof. :

The zero ideal $0 = \{0\}$ and the unit ideal are clearly regular since $0 \lhd 0$ and $x \lhd e$ for all $x \in L$. For regular ideals I and J,
if $x \in I \cap J = \{x \wedge y | x \in I, y \in J\}$ then $x \in I$ and $x \in J$. This implies that $x \lhd y$ and $x \lhd z$ for some $y \in I$ and $z \in J$, and hence by the properties of $\lhd$ (2.2.2), $x \lhd y \wedge z$ where $y \wedge z \in I \cap J$. Thus $I \cap J$ is again regular.

Regarding joins, we note first that a directed union of regular ideals is obviously regular; so it is sufficient to consider binary joins. Now, for any ideals I and J, their join is given by

$$I \bigvee J = \{x \vee y | x \in I \text{ and } y \in J\}$$

by the distributivity of L.
If I and J are regular and $x \in I$ and $y \in J$, then there exist $s \in I$ and $t \in J$ such that $x \lhd s$ and $y \lhd t$. By the properties of $\lhd$ (2.2.2)$x \vee y \lhd s \vee t$

where $s \vee t \in I \bigvee J$, showing that $I \bigvee J$ is regular. This proves that $\mathcal{R}L$ is a subframe. $\square$

A familiar result (not hard to establish) is that for any bounded distributive lattice A, $\mathcal{J}A$ is compact. Since any frame L is a bounded distributive lattice, $\mathcal{J}L$ is compact. It follows from 1.4.6.1 that $\mathcal{R}L$ is compact. We want to show that it is also regular. As a tool for this we introduce the map $k : L \to \mathcal{R}L$ given by

$$k(a) = \{x \in L | x \lhd a\} \text{ for all } a \in L.$$

Now, each $k(a)$(with $a \in L$) is a regular ideal : If $x \in k(a)$ then $x \lhd a$. By the property of $\lhd$ (2.2.2) there exists a $z \in L$ such that $x \lhd z \lhd a$, which implies that $z \in k(a)$.

Moreover, for each $a \in L, a = \bigvee\{x \in L | x \lhd a\}$ since L is a uniform frame, that is, $a = \bigvee k(a)$. Further, for any $a, b \in L \quad k(a) \cap k(b) = k(a \wedge b)$:

$$\begin{aligned} x \in k(a) \cap k(b) \quad &\text{iff} \quad x \in k(a) \text{ and } x \in k(b) \\ &\text{iff} \quad x \lhd a \text{ and } x \lhd b \qquad\qquad (3.1.1) \\ &\text{iff} \quad x \lhd a \wedge b \\ &\text{iff} \quad x \in k(a \wedge b). \end{aligned}$$

Now, we can prove the following lemma.

3.1.2 Lemma : *$\mathcal{R}L$ is a regular frame.*

Proof. :

As a first step towards this, we show that in $\mathcal{R}L$, $a \lhd c$

implies $k(a) \prec k(c)$: (3.1.2)

We prove this by exhibiting $b \in L$ such that $k(a) \cap k(b) = 0$ and $k(c) \vee k(b) = L$.

Now

$$k(a) \cap k(b) = 0 \text{ iff } a \wedge b = 0 :$$

$$\begin{aligned} k(a) \cap k(b) = 0 &\text{ iff } k(a \wedge b) = 0 \qquad (3.1.1) \\ &\text{ iff } \bigvee k(a \wedge b) = \bigvee 0 = 0 \\ &\text{ iff } \bigvee \{x | x \lhd a \wedge b\} = 0 \\ &\text{ iff } \quad a \wedge b = 0 \end{aligned}$$

Since $z \in k(Az)$ for any $z \in L$ and $A \in \mathcal{U}L$, we show there exist $x, z \in L$ such that

$$x \vee z = e, \quad Ax \leq c, \quad a \wedge Az = 0 :$$

We claim $b = Az$ will be the element of the desired kind. For this, note that

$$a \wedge Az = 0 \text{ iff } Aa \wedge z = 0 :$$

Suppose $a \wedge Az = 0$. Take $s \in A$ such that $s \wedge a \neq 0$. If $s \wedge z \neq 0$ then $a \wedge Az \neq 0$. Hence, we have a contradiction and so $s \wedge z = 0$. It follows that $Aa \wedge z = 0$.

In a similar manner one shows that $a \wedge Az = 0$.

Now, given $a \lhd c$, there exist $w, x \in L$ such that $a \lhd w \lhd x \lhd c$, by the properties of $\lhd$. Take $A \in \mathcal{U}L$ such that $Aa \leq w, Aw \leq x, Ax \leq c$ and put $z = \bigvee\{t \in A | t \wedge w = 0\}$. Then $x \vee z = e$ since $Aw \vee z = \bigvee A = e$ and $Aa \wedge z = 0$ since $w \wedge z = 0$. This shows that x, z and A satisfy the stated requirement.

Thus $a \lhd c$ implies $k(a) \prec k(c) \subseteq J \in \mathcal{R}L$, which in turn implies $k(a) \prec J$.

Finally, we show that for each $J \in \mathcal{R}L$,

$$J = \bigvee\{k(a) | a \in J\} :$$

Since $J \in \mathcal{R}L$, for each $a \in J$, there exists $b \in J$ such that $a \lhd b$. It follows that $k(a) \prec k(b)$ from (3.1.2).
Now, $k(b) \subseteq J$: If $y \in k(b)$ then $y \lhd b$ which implies that $y \leq b$. Since J is an ideal, $y \in \downarrow b \subseteq J$. Therefore $y \in J$.
Now k$(a) \prec k(b) \subseteq J$ implies $k(a) \prec J$, which in turn implies $k(a) \subseteq J$. Hence for all $a \in J$, $k(a) \subseteq J$. So

$$\bigcup_{a \in J} k(a) \subseteq J.$$

Conversely, if $a \in J$ then there exists $b \in J$ such that $a \lhd b$.
Since $k(b) = \{y | y \lhd b\}, a \in k(b)$. Therefore,

$$a \in \bigcup_{a \in J} k(a) \text{ for all } a \in J,$$

which implies

$$J \subseteq \bigcup_{a \in J} k(a).$$

Hence, we can conclude that $J = \bigvee\{k(a)|k(a) \prec J\}$.

Therefore $\mathcal{R}L$ is a regular frame. $\square$

3.1.3 Corollary : *$\mathcal{R}L$ is a compact regular frame.*

Proof. :

The proof follows from lemmas 3.1.1 and 3.1.2. $\square$

In the following $\mathbb{KR}eg\mathbb{F}rm$ is the full subcategory of the category $\mathbb{F}rm$ of all frames, given by the compact regular frames, and $\mathbb{U}ni\mathbb{F}rm$ is the category of all uniform frames and uniform frame homomorphisms.

3.1.4 Lemma : *The correspondence $L \rightsquigarrow \mathcal{R}L$ determines a functor*

$\mathcal{R} : \mathbb{U}ni\mathbb{F}rm \longrightarrow \mathbb{KR}eg\mathbb{F}rm.$

Proof. :

Consider any uniform homomorphism $h : L \longrightarrow M$ between uniform frames. Then, for any regular ideal $J \subseteq L$, the ideal generated by its image $h[J]$ is again regular because $x \triangleleft y$ in L implies $h(x) \triangleleft h(y)$ in M. (Lemma 2.2.6)

This says that the map $\mathcal{J}h : \mathcal{J}L \longrightarrow \mathcal{J}M$, induced by h, known to be a frame homomorphism, takes $\mathcal{R}L$ into $\mathcal{R}M$: Let $J \in \mathcal{R}L$. We must show that $\mathcal{J}h(J)$ is again regular. Since J is regular, for each $a \in J$ there exists $b \in J$ such that $a \triangleleft b$. By Lemma 2.2.6 $h(a) \triangleleft h(b)$ and since $h(b) \in \mathcal{J}h(J)$, it follows that $\mathcal{J}h(J) \in \mathcal{R}M$.

This ensures that the correspondence $L \rightsquigarrow \mathcal{R}L$ is functorial because in $\mathbb{F}rm$ the correspondence from frames to their ideal lattices is functorial. $\square$

Next, we need some general facts concerning compact regular frames. First, a basic result which is well known; we present a more direct proof,which is attributable to Banaschewski and Pultr [3].

3.1.5 Lemma : *Any compact regular frame has a unique uniformity, generated by all its finite covers.*

Proof. :

The crucial point here is that any finite cover in a compact regular frame M is star-refined by a finite cover. First, we reduce this to the consideration of two covers as follows : Given any finite cover A , for each $a \in A$

$$a = \bigvee\{x|x \prec a\}$$

by regularity.

Now

$$\bigvee_{a\in A} a = e$$

since A is a cover. This implies that

$$\bigvee_{a\in A}\bigvee_{x\prec a} x = e.$$

By compactness we obtain for each $a \in A$ an $x_a \prec a$. Let z_a be such that $x_a \wedge z_a = 0$ and $a \vee z_a = e$.

Then, since $\{x_a | a \in A\}$ forms a cover, we have $\bigwedge\{z_a | a \in A\} = 0$ and hence the cover $B = \bigwedge\{a, z_a\}$, consisting of all meets formed from elements taken out of each $\{a, z_a\}$ is a refinement of A. Hence, if one obtains a finite cover $C_a \leq^* \{a, z_a\}$ for each $a \in A$ then

$$C = \bigwedge_{a \in A} C_a \leq^* B \leq A$$

so that C star-refines A.

Thus let $\{a, b\}$ be any two-cover. Then take $u \prec a$ and $v \prec b$ such that $u \vee v = e$ (which is possible by regularity and compactness) and let $s, t \in M$ such that

$$u \wedge s = 0, \ a \vee s = e, \ v \wedge t = 0, \ b \vee t = e.$$

Now the cover

$$\begin{aligned} D &= \{a, s\} \wedge \{b, t\} \wedge \{u, v\} \\ &= \{a \wedge b \wedge u, \ a \wedge b \wedge v, \ a \wedge t \wedge u, \ a \wedge t \wedge v, \\ &\quad\ s \wedge b \wedge u, \ s \wedge b \wedge v, \ s \wedge t \wedge u, \ s \wedge t \wedge v\} \\ &= \{a \wedge b \wedge u, \ a \wedge b \wedge v, \ a \wedge t \wedge u, \ s \wedge b \wedge v\} \end{aligned}$$

is the desired star-refinement of $\{a, b\}$ as one sees by direct calculation using the conditions $u \wedge s = 0$ and $v \wedge t = 0$.

This proves that the finite covers of M generate a uniformity. In order to see its uniqueness it has to be shown that the finite covers belong to each uniformity on M. Now, for any such $\mathcal{U}$, if A is any finite cover there exists x_a for each $a \in A$ such that $x_a \lhd a$ relative to $\mathcal{U}$ and there being only finitely many $a \in A$ it follows that there exists $B \in \mathcal{U}$ such that $Bx_a \leq a$ for all $a \in A$. Since $b \wedge x_a \neq 0$ for some $a \in A$, for any $b \in B$, it follows that B refines A and therefore $A \in \mathcal{U}$,by the definition of a uniformity. $\square$

Lemma 3.1.5 means that any compact regular frame may unambiguously be regarded as a uniform frame, and that any frame homomorphism between such frames is automatically uniform. Thus the category $\mathbb{KR}eg\mathbb{F}rm$. appears as a (full)subcategory of the category $\mathbb{U}ni\mathbb{F}rm$ of all uniform frames and uniform frame homomorphisms.

We recall that, for any frame, the join map taking each ideal J to its join $\bigvee J$ is a frame homomorphism, and hence the same holds for its restriction to the regular ideals of a uniform frame. Moreover, we have

3.1.6 Lemma : *The join map $\mathcal{R}L \to L$ is uniform.*

Proof. :

We must show that the image by $\bigvee$ of any finite cover $J_1, \ldots J_n$ of $\mathcal{R}L$ is a uniform cover of L. Take $a_i \in J_i$ such that $a_1 \vee \ldots \vee a_n = e$, given by the fact that $J_1 \vee \ldots \vee J_n = L$. Now, by the regularity of the J_i and the properties of uniformities, there exists $A \in \mathcal{U}L$ for which $Aa_i \in J_i$ and therefore also $Aa_i \leq c_i$, where $c_i = \bigvee J_i$. Now, for any $s \in A$, we have $s \wedge a_i \neq 0$ for some i and therefore $s \leq c_i$, showing that A refines the cover $\{c_1, \ldots, c_n\}$, so that this cover is uniform. $\square$

Now we are able to establish

3.1.7 Theorem : *$\mathbb{KR}eg\mathbb{F}rm$ is coreflective in $\mathbb{U}ni\mathbb{F}rm$, with coreflection functor $\mathcal{R}$ and coreflection maps $\rho_L : \mathcal{R}L \rightarrow L$ given by join.*

Proof. :

By Lemma 3.1.6 the join map $\rho_L : \mathcal{R}L \rightarrow L$ is uniform. It remains to be shown that it is the universal uniform homomorphism from compact regular frames to L, that is,

given any uniform homomorphism $h : M \rightarrow L$ let $\rho_L : \mathcal{R}L \rightarrow L$ be a uniform homomorphism given by $\bigvee$. We must show that there exists a unique uniform homomorphism $t : M \rightarrow \mathcal{R}L$ such that $h = \rho_L \circ t$.

$$\begin{tikzcd}
\mathcal{R}L \arrow[r, "\rho_L"] & L \\
 & M \arrow[u, "h"'] \arrow[ul, dashed, "!t"]
\end{tikzcd}$$

Let $h : M \to L$ be any such homomorphism and consider the diagram

$$\begin{tikzcd} \mathcal{R}L \arrow[r, "\rho_L"] & L \\ \mathcal{R}M \arrow[u, "\mathcal{R}h"] \arrow[r, "\rho_M"] & M \arrow[u, "h"'] \end{tikzcd}$$

which clearly commutes : Let $J \in \mathcal{R}M$. Then

$$\begin{aligned} h \circ \rho_M(J) &= h(\rho_M(J)) \\ &= h(\bigvee J) \\ &= \bigvee h(J) \qquad\qquad h \text{ preserves joins} \end{aligned}$$

and

$$\begin{aligned} \rho_L \circ \mathcal{R}h(J) &= \rho_L(h(J)) \\ &= \bigvee h(J) \end{aligned}$$

Now by compactness ρ_M is one-to-one because it is dense (Lemma 1.4.7), and since it is always onto ($\bigvee k(a) = a$), it is an isomorphism. So it follows that $h \circ \rho_M = \rho_L \circ \mathcal{R}h$ implies that $h = \rho_L \circ \mathcal{R}h \circ \rho_M^{-1}$. Let $t = \mathcal{R}h \circ \rho_M^{-1}$. Therefore $h = \rho_L \circ t$.

For uniqueness, suppose there exists $t' : M \to \mathcal{R}L$ such that $h = \rho_L \circ t'$. Therefore $\rho_L \circ t = \rho_L \circ t'$. Since ρ_L is dense, it follows from Lemma 1.4.2 that ρ_L is monic and therefore $t = t'$. $\square$

3.1.8 Remark : *Samuel [12] describes the compact uniform reflection of an arbitrary uniform space X as a certain quotient of the space of ultrafilters on the underlying set of X. It is conceptually obvious that this quotient must factor through the Stone-Čech compactification (of the underlying space) of X. In the present dual situation, the Stone-Čech compactification of L is the largest regular subframe $\widetilde{RL}$ of the ideal lattice $\mathcal{J}L$ of L, and thus $\mathcal{R}L$ as a regular subframe of $\mathcal{J}L$, is a subframe of $\widetilde{RL}$.*

Hence our description of the reflection from $\mathbb{U}ni\mathbb{F}rm$ to $\mathbb{KR}eg\mathbb{F}rm$ is of the same type as the original description of the Samuel compactification for uniform spaces. Its existence is a consequence of Theorem 3.1.7. It arises here as the space of maximal (uniformly) regular ideals in the lattice of open sets of the uniform space X. This is dual to the more familiar presentation as the space of maximal (uniformly) regular filters.

We now turn to a new description of the completion of L as a quotient of $\mathcal{R}L$.

3.2 The Completion of a Uniform Frame

3.2.1 For each uniform cover A of L, put $K_A = \bigvee\{k(a)|a \in A\}$ in $\mathcal{R}L$, where k is the map $L \to \mathcal{R}L$ given by $k(a) = \{x \in L/x \lhd a\}$. Note that if A is finite and hence also has a finite star refinement $B \in \mathcal{U}L$, then

$K_A = L$. Conversely, if this holds, A has a finite uniform refinement. In general then, K_A will be different from L.

Now, $A \leq B$ in $\mathcal{U}L$ implies $K_A \subseteq K_B$: Let $x \in K_A$.
Then $x \lhd a_1 \vee \ldots \vee a_m$, for some $a_1, \ldots, a_m \in A$. Since $A \leq B$, there exist $b_1, \ldots, b_m \in B$ such that $a_1 \leq b_1, \ldots, a_m \leq b_m$.
So $x \lhd b_1 \vee \ldots \vee b_m$, which implies that $x \in K_B$. Therefore $K_A \subseteq K_B$.

Hence, the K_A with $A \in \mathcal{U}L$ form a filter basis in $\mathcal{R}L$ which, in a sense, measures the deviation of L from being totally bounded, that is, $\mathcal{U}L$ being generated by its finite members. The filter $\mathcal{F}_L$ generated by the filter basis in $\mathcal{R}L$ determines a congruence θ_L on $\mathcal{R}L$, the smallest congruence making each $K_A, A \in \mathcal{U}L$, equivalent to the unit of $\mathcal{R}L$.We put $CL = \mathcal{R}L/\theta_L$, with quotient homomorphism $\nu_L : RL \to CL$ and the corresponding homomorphism $\gamma_L : CL \to L$ such that $\gamma_L \nu_L = \rho_L$, the join map $\mathcal{R}L \to L$. We further take CL as a uniform frame, its uniformity generated by the covers

$$\{\nu_L k(a)/a \in A\} \qquad (A \in \mathcal{U}L)$$

which are covers precisely because, for each $A \in \mathcal{U}L$,

$$\bigvee_{a \in A} \nu_L k(a) = \nu_L \bigvee_{a \in A} k(a) = \nu_L(K_A) = e_{CL},$$

by the very definition of CL.

To check that these covers indeed have the properties of a uniformity basis, consider the collection of all subsets of $\mathcal{R}L$ of the type

$$k[A] = \{k(a) | a \in A\} \qquad (A \in \mathcal{U}L).$$

Although these need not be covers of $\mathcal{R}L$ they have the following properties:

1. If $A \leq B$ then $k[A] \leq k[B]$: Since $A \leq B$ for each $a \in A$ there exists $b \in B$ such that $a \leq b$ which implies that $k(a) \subseteq k(b)$. Hence $k[A] \leq k[B]$.

2. If $A \leq^* B$ then $k[A] \leq^* k[B]$: For any $a \in A$, take any $b \in B$ such that $Aa \leq b$. Then, for any $c \in A, k(a) \cap k(c) \neq 0$ implies $k(a \wedge c) \neq 0$ (by 3.1.1)which in turn implies $a \wedge c \neq 0$. Hence $c \leq b$ and therefore $k(c) \subseteq k(b)$. Thus

$$\bigvee \{k(c) | c \in A, k(c) \cap k(a) \neq 0\} \subseteq k(b).$$

   which proves the assertion.

3. For each $J \in \mathcal{R}L, J = \bigvee \{k(c) | k[A]k(c) \subseteq J$,for some $A \in \mathcal{U}L\}$:

   Given any $x \in J$, take $a \in J$ and $B \in \mathcal{U}L$ such that $Bx \leq a$, and then some $A \leq^* B$ in $\mathcal{U}L$. Now, for $c = Ax$ we have $x \in k(c)$ and $Ac \leq a$. Further, if $k(s) \cap k(c) \neq 0$ for any $s \in A$, then $s \wedge c \neq 0$. Hence $s \leq a$ and therefore $k(s) \subseteq k(a) \subseteq J$. This shows that $x \in k(c)$ and $k[A]k(c) \subseteq J$, proving the assertion.

Now, mapping by $\nu_L : \mathcal{R}L \to CL$ preserves these conditions and turns each $k[A]$ with $A \in \mathcal{U}L$ into the cover $\nu_L k[A]$. It follows that these covers form the basis of a uniformity, as claimed. Note that the homomorphism $\gamma_L : CL \to L$ is uniform since $\gamma_L(\nu_L k(a)) = \bigvee k(a) = a$ for any $a \in L$. Further, by the definition of CL and the fact that $\gamma_L(\nu_L k(a)) = a$ for each $a \in L$, $\gamma_L : CL \to L$ is a dense surjection.

3.2.2 Lemma : *For any dense surjection $h : L \to M$, the induced homomorphism $\mathcal{R}h : \mathcal{R}L \to \mathcal{R}M$ is an isomorphism taking θ_L to θ_M.*

Proof. :

$\mathcal{R}h$ is obviously dense since h is, and since we are dealing with compact regular frames, $\mathcal{R}h$ is one-to-one (Lemma 1.4.7). Thus, it only has to be shown that it is onto. For this, it is enough to see that each $k(c)$ with $c \in M$ is an image. To this end, let $c = h(a)$ with the largest possible $a \in L$. Because $x \lhd a$ in L implies that $h(x) \lhd c$ in M (Lemma 2.2.6),

$$\mathcal{R}h(k(a)) = \bigvee_{x \in k(a)} \downarrow h(x) = \bigvee_{x \lhd a} \downarrow h(x) \subseteq k(h(a)) = k(c),$$

where $\downarrow$ signifies the principal ideal generated by an element.

On the other hand, if $y \in k(c)$ then $y \lhd c$ which means that $Ay \leq c$ for some $A \in \mathcal{U}M$. Take $x \in L$ such that $h(x) = y$ and $B \in \mathcal{U}L$ for which $h[B] \leq A$, using that h is a surjection. Then

$$\begin{aligned} h(Bx) &= h(\bigvee\{b \in B | b \wedge x \neq 0\}) \\ &= \bigvee\{h(b) | b \in B, h(b) \wedge h(x) \neq 0\} \\ &= h[B]y \\ &\leq Ay \\ &\leq c \end{aligned}$$

Now $h(Bx) \leq c$ implies $Bx \leq a$ by the choice of a. Hence $x \in k(a)$ and therefore $y = h(x) \in \mathcal{R}h(k(a))$. In all, this shows that $k(c) = \mathcal{R}h(k(a))$, and hence we have proved the first part of the lemma.

Now, for any $A \in \mathcal{U}L$, let $\tilde{A}$ be the cover of L consisting of all

$$\tilde{a} = \bigvee\{x | h(x) = h(a)\} \quad (a \in A).$$

Since $A \leq \tilde{A}, \tilde{A} \in \mathcal{U}L$ by the properties of uniformities. The previous part of the proof showed that $k(h(a)) = \mathcal{R}h(k(\tilde{a}))$ and hence $Rh(K_{\tilde{A}}) = K_{h[A]}$ by taking joins. Hence, the isomorphism $\mathcal{R}h : \mathcal{R}L \to \mathcal{R}M$ maps $\mathcal{F}_L$ to a filter containing $\mathcal{F}_M$. In order to obtain equality we show that any $A \in \mathcal{U}L$ is refined by some $\tilde{B}$ with $B \in \mathcal{U}L$. This follows directly from the fact that $x \prec a$ implies that $\tilde{x} \prec a$ (since $x \wedge y = 0$ implies $\tilde{x} \wedge y = 0$ by the density of h): if $B \leq^* A$ and for any $x \in B$ we take $a \in A$ such that $Bx \leq a$ then $x \prec a$ and hence $\tilde{x} \leq a$, showing that $\tilde{B} \leq A$. $\square$

3.2.3 Corollary : *For any dense surjection $h : L \to M$, there is a commuting square*

$$\begin{tikzcd} CL \arrow[r, "\gamma_L"] \arrow[d, "Ch"'] & L \arrow[d, "h"] \\ CM \arrow[r, "\gamma_M"'] & M \end{tikzcd}$$

where Ch is an isomorphism.

Proof. :

Since

$$
\begin{aligned}
(h \circ \gamma_L)(\nu_L k(a)) &= h(\gamma_L \circ \nu_L(k(a))) \\
&= h(\rho_L(k(a))) \\
&= h(\bigvee k(a)) \\
&= h(a)
\end{aligned}
$$

and

$$
\begin{aligned}
(\gamma_M \circ Ch)(v_L k(a)) &= \gamma_M(Ch(\nu_L k(a))) \\
&= \gamma_M(\nu_M k(h(a))) \\
&= \bigvee k(h(a)) \\
&= h(a)
\end{aligned}
$$

it follows that the above diagram is a commuting square. The fact that Ch is an isomorphism follows from Lemma 3.2.2. □

3.2.4 Theorem : *CL with $\gamma_L : CL \to L$ is a completion of L, unique up to isomorphism.*

Proof. :

First we show that CL is complete. For this, let $f : M \to CL$ be any dense surjection and consider the commuting square

$$\begin{tikzcd}
CM \arrow[r, "\gamma_M"] \arrow[dd, "l"'] & M \arrow[d, "f"] \\
& CL \arrow[d, "\gamma_L"] \\
CL \arrow[r, "\gamma_L"'] \arrow[uur, "g"] & L
\end{tikzcd}$$

where $l = C(\gamma_L \circ f)$ is the isomorphism corresponding to the dense surjection $\gamma_L \circ f : M \to L$, as in the corollary of Lemma 3.2.2. Now $\gamma_L \circ f \circ \gamma_M = \gamma_L \circ l$ implies that $\gamma_L \circ f \circ \gamma_M \circ l^{-1} = \gamma_L \circ l \circ l^{-1}$. Then for $g = \gamma_M \circ l^{-1}$ we have $\gamma_L \circ f \circ g = \gamma_L \circ id_{CL}$. Hence $f \circ g = id_{CL}$ because γ_L is dense and therefore monic by Lemma 1.4.2. But then $f \circ g = id_{CL}$ implies that

$$f = id_{CL} \circ g^{-1} = id_{CL}(\gamma_M \circ l^{-1})^{-1} = l \circ \gamma_M^{-1}.$$

Hence $g \circ f = (\gamma_M \circ l^{-1})(l \circ \gamma_M^{-1}) = id_{CM}$, the map γ_M and therefore the map g being onto. This shows that f is an isomorphism, as desired. Therefore, by definition 2.3.1.2, CL is complete.

Now consider any dense surjection $f : M \to L$ with M a complete uniform frame. Then in the commuting square

$$\begin{tikzcd} CM \arrow[r, "\gamma_M"] \arrow[d, "Cf"'] & M \arrow[d, "f"] \arrow[dl, "g"'] \\ CL \arrow[r, "\gamma_L"'] & L \end{tikzcd}$$

Cf is an isomorphism by corollary 3.2.3. But so is γ_M, being a dense surjection to the complete uniform frame M. Hence $f \circ \gamma_M = \gamma_L \circ Cf$ implies $f = \gamma_L \circ Cf \circ \gamma_M^{-1}$, which is the desired factorization of f through γ_L. It remains to show the uniqueness of $Cf \circ \gamma_M^{-1}$: Suppose there exists $g : M \to CL$ such that $f = \gamma_L \circ g$.Then $\gamma_L \circ Cf \circ \gamma_M^{-1} = \gamma_L \circ g$. But γ_L is dense and thus monic by Lemma 1.4.2. Therefore it follows that $Cf \circ \gamma_M^{-1} = g$. $\square$

Chapter 4

Another Description of Uniform Completion of Frames (Locales)

In this chapter we define a sublocale and introduce the term 'nucleus' as a tool for studying sublocales. We then give a general description of factorization in frames. The main idea of the approach adopted here belongs to Johnstone [8], who really proved the full strength of Theorem 4.1.4 on factor frames, but restricted the statement by unnecessary assumptions. As outlined by Křiž [9], the theorem is presented in the full generality. It is this important generalization which enables us to give explicit descriptions of frames specified by "generators and relations". We shall assume that our generators form a meet-semilattice.

There is a natural parallelism between the concepts in $\mathbb{T}op$ and $\mathbb{L}oc$. It is, however, often not of much use in technical parts of the proofs presented. For this reason, it is of an advantage to write morphisms as in frames, although we think in locales. Finally, we give a description of uniform completion in frames (locales) which consists of writing down generators and defining relations. This approach was expounded by Kříž[9].

4.1 Sublocales and Factor Frames

Monomorphisms in $\mathbb{L}oc$ (epimorphisms in $\mathbb{F}rm$) are badly behaved. However, regular monomorphisms, which correspond to surjective frame homomorphisms, are more manageable. If $f : L \to M$ is a surjective frame homomorphism, then the composite $j = f_* \circ f : L \to L$ ($f_* : M \to L$ denotes the right adjoint of f, where $f_*(x) = \bigvee_{f(y) \leq x} y$) satisfies the following conditions :

1. $j(a) \geq a$
2. $j(j(a)) = j(a)$
3. $j(a \wedge b) = j(a) \wedge j(b)$

for all $a, b \in L$.

We call such a map a *nucleus* on L. If j is a nucleus on L, then one can put

$$L_j = \{a \in L / j(a) = a\}.$$

Since $j(j(a)) = j(a)$, the image of j is precisely L_j.

4.1.1 Lemma : *L_j is a frame, and $j : L \longrightarrow L_j$ is a frame homomorphism, whose right adjoint is the inclusion $L_j \to L$.*

Proof. :

Since j preserves finite meets and $j(e) = e, L_j$ has finite meets which are the same as those in L. If $S \subseteq L_j$, then $\bigvee\limits_L S$ need not be in L_j, but S clearly has a least upper bound in L_j, namely $j(\bigvee\limits_L S)$. It is clear from this description that $j : L \longrightarrow L_j$ preserves finite meets and all joins, and it is easy to see that it is left adjoint to the inclusion map. So, it remains to verify the infinite distributive law in L_j.

Let $a \in L_j$ and $S \subseteq L_j$. Then

$$
\begin{aligned}
a \wedge \bigvee_{Lj} S &= a \wedge j(\bigvee_L S) \\
&= j(a) \wedge j(\bigvee_L S) && \text{since } a \in L_j \\
&= j(a \wedge \bigvee_L S) && \text{since } j \text{ preserves } \wedge \\
&= j(\bigvee_L \{a \wedge s | s \in S\}) && \text{since } L \text{ is a frame} \\
&= \bigvee_{L_j} \{a \wedge s | s \in S\}
\end{aligned}
$$

□

Thus we obtain a natural one-to-one correspondence between the embeddings and the nuclei. In fact, for any locale (frame) L, there is a bijection between nuclei on L and regular subobjects of L in $\mathbb{L}oc$ (that is, isomorphism classes of regular monomorphisms $M \longrightarrow L$). In view of this, we define a *sublocale* of a locale L to be a subset of the form L_j, for some nucleus j.

It is easily verified that for a T_o topological space X, the continuous maps $X \to Y$, which induce regular monomorphisms $\mathfrak{D}Y \to \mathfrak{D}X$, are precisely the inclusions of subspaces of X, and that in this context the words 'closed' and 'dense' restrict to their usual meanings in topology. However, one respect in which locales differ sharply from spaces is that every locale has a smallest dense sublocale.

4.1.2 Definitions :

4.1.2.1 A *join basis* of a frame L is any subset $L' \subseteq L$ which satisfies:

for all $c \in L$, there exist $S \subseteq L'$ such that $c = \bigvee S$.

4.1.2.2 We call a subset $R \subseteq L \times L$ a *precongruence relation* if for any $a, b \in L$, with $a\,R\,b$, the set $\{s \in L | (a \wedge s) R (b \wedge s)\}$ is a join basis of L.

4.1.2.3 Let $S \subseteq L \times L$ and let L' be a join basis of L, which is closed under finite meets. Then

$$R(S, L') = \{(a \wedge c, b \wedge c) | (a, b) \in S \text{ and } c \in L'\}$$

is a precongruence relation on L.

4.1.2.4 Let $R \subseteq L \times L$. An element $u \in L$ will be called *R-coherent* if for any two $a, b \in L$ with $a\,R\,b$ we have

$$a \leq u \text{ iff } b \leq u.$$

4.1.2.5 We will denote by '$\Rightarrow$' the operation of *implication* in L given by

$$c \leq a \Rightarrow b \text{ iff } c \wedge a \leq b.$$

4.1.3 Lemma : *Let R be a precongruence on L and let $b \in L$ be $R-$coherent. Then the element $a \Rightarrow b$ is $R-$coherent for any $a \in L$.*

Proof. :

Let $x\,R\,y$. Since R is a precongruence relation on L, we can choose $Q \subseteq L$ such that $L = \bigvee Q$ (each $a \in L$ is a join of elements from Q) and for all $t \in Q, (x \wedge t)R(y \wedge t)$.

Now

$$\begin{array}{rlll} x \leq a \Rightarrow b & \text{iff} & x \wedge a \leq b & \\ & \text{iff} & x \wedge t \leq b & \text{for all } t \in Q \\ & \text{iff} & y \wedge t \leq b & (x \wedge t)R(y \wedge t) \\ & \text{iff} & y \wedge a \leq b & \\ & \text{iff} & y \leq a \Rightarrow b & \end{array}$$

□

4.1.4 Theorem : *Let R be a precongruence on L. Then the set $L(R)$ of all the $R-$coherent elements together with the induced ordering is a frame and there exists a nucleus $j : L \to L_j$ such that $L_j = L(R)$. Moreover, $j : L \to L(R)$ is universal among the join-preserving mappings f from L to complete lattices M satisfying*

$$\text{If } a\,R\,b \text{ then } f(a) = f(b)$$

Proof. :

First we show $j : L \to L(R)$ is a nucleus :

Define $j : L \to L(R)$ by

$$j(a) = \bigwedge\{u \in L | u \geq a, u \text{ is } R-\text{coherent}\}.$$

As a first step we show that if $a\,R\,b$ then $j(a) = j(b)$: If $a\,R\,b$ then

$$\begin{aligned} j(a) &= \bigwedge\{u \geq a | u \text{ is } R-\text{coherent}\} \text{ and} \\ j(b) &= \bigwedge\{v \geq b | v \text{ is } R-\text{coherent}\} \end{aligned}$$

Since u is R–coherent $a \leq u$ iff $b \leq u$. It follows that

$$\bigwedge\{u | u \geq a\} = \bigwedge\{u | u \geq b\}.$$

Therefore $j(a) = j(b)$.

We now prove that an arbitrary meet of R–coherent elements is again R–coherent :

Let $a\,R\,b$. Now

$$\begin{aligned} a \leq \bigwedge u_i \ \text{ iff } \ & a \leq u_i && \text{for all } i \\ \text{ iff } \ & b \leq u_i && \text{for all } i \\ \text{ iff } \ & b \leq \bigwedge u_i. \end{aligned}$$

It follows that $a \leq j(a) = j(j(a))$.

Let us show that j preserves finite meets :

Obviously $j(a \wedge b) \leq j(a) \wedge j(b)$ since the right hand element is R–coherent.

Conversely, it is true that

$b \leq (a \Rightarrow j(a \wedge b))$ since $b \wedge a \leq j(a \wedge b)$, and

$a \wedge a = a \leq j(a \wedge b) \Rightarrow j(a \wedge b)$ which implies that $a \wedge j(a \wedge b) \leq j(a \wedge b)$. From this it follows that $a \leq ((a \Rightarrow j(a \wedge b)) \Rightarrow j(a \wedge b))$.
The right hand elements are $R-$coherent by Lemma 4.1.3. Now $a \leq j(a)$ and $b \leq j(b)$ imply that

$$\begin{aligned} a \wedge b &\leq j(a) \wedge j(b) \\ &\leq (a \Rightarrow j(a \wedge b)) \wedge ((a \Rightarrow j(a \wedge b)) \Rightarrow j(a \wedge b)) \\ &\leq j(a \wedge b). \end{aligned}$$

Therefore $j(a \wedge b) = j(a) \wedge j(b)$.
Hence, $j : L \longrightarrow L(R)$ is a nucleus, as required.

Now, we show that $L_j = \{a \in L | j(a) = a\} = L(R)$:
If $x \in L_j$ then $j(x) = x$. Now $j(x)$ is $R-$coherent which implies that $x \in L(R)$. Conversely, if $u \in L(R)$ then u is $R-$coherent. This implies that $j(u) = u$ and thus $u \in L_j$. Therefore, $L_j = L(R)$.

Since j is a nucleus and $L(R) = L_j \subseteq L, L(R)$ is a frame. It remains to prove the universality of $j : L \longrightarrow L(R)$:
Let $f : L \longrightarrow M$ preserve joins and let $a\,R\,b$ imply that $f(a) = f(b)$. Put for $a \in L$
$s(a) = \bigvee\{x \in L | f(x) \leq f(a)\}$
Now $s(a)$ is $R-$coherent : Let xRy. Then

$$
\begin{aligned}
x \leq s(a) \text{ iff } f(x) &\leq f(s(a)) \\
&= f\left(\bigvee_{f(z)\leq f(a)} z \right) \\
&= \bigvee_{f(z)\leq f(a)} f(z) \qquad f \text{ preserves joins} \\
&\leq f(a) \\
\text{iff } f(y) &\leq f(a) \qquad x\,R\,y \text{ implies } f(x) = f(y) \\
\text{iff } y &\leq s(a)
\end{aligned}
$$

Since $s(a)$ is R–coherent and $s(a) \geq a$, we conclude that $s(a) \leq j(a)$:
If $a \leq s(a)$ then $j(a) \leq j(s(a))$ which implies that $j(a) \leq s(a)$.
Consequently $a \leq j(a) \leq s(a)$ implies that f(a)$\leq f(j(a)) \leq f(s(a))$.
On the other hand, since f preserves joins, we have

$$
\begin{aligned}
f(s(a)) &= f\left(\bigvee_{f(x)\leq f(a)} x \right) \\
&= \bigvee_{f(x)\leq f(a)} f(x) \\
&\leq f(a)
\end{aligned}
$$

Therefore $f(a) = f(s(a))$.
To conclude the proof of the theorem, define $g : L(R) \to M$ by $g(a) = f(a)$ for all $a \in L$. Thus, we have

$$
\begin{tikzcd}
L \arrow[r, "j"] \arrow[d, "f"'] & L(R) = Lj \arrow[dl, dashed, "g"] \\
M &
\end{tikzcd}
$$

Now $g(j(a)) = f(j(a))$ by the very definition of g.
Since $f(a) \leq f(j(a)) \leq f(s(a)) \leq f(a)$ for all $a \in L$,$f(j(a)) = f(a)$.
Therefore $f = g \circ j$. □

4.1.5 Corollary : *Preserve the notation of 4.1.4 and assume that $R = R(S, L')$ for a join basis L' and a relation $S \subseteq L \times L$. Then for any frame homomorphism $f : L \to M$ which satisfies*

$$\text{If } a\,S\,b \text{ then } f(a) = f(b)$$

there exists a unique frame homomorphism $g : L(R) \to M$ satisfying $f = g \circ j$.

Proof. :
Let $f : L \to M$ be a frame homomorphism such that $a\,S\,b$ implies that $f(a) = f(b)$. Now $j : L \to L(R)$ satisfies the condition that if $a\,R\,b$ then $j(a) = j(b)$.We must show that there exists a unique frame homomorphism $g : L(R) \to M$ such that $f = g \circ j$, that is, we must show that the following diagram commutes :

$$\begin{tikzcd} L \arrow[r, "j"] \arrow[d, "f"'] & L(R) \arrow[dl, dashed, "!g"] \\ M & \end{tikzcd}$$

First we show that if $a\,R\,b$ then $f(a) = f(b)$:
If $a\,R\,b$ then $(a, b) \in R$ which implies that
$(a, b) = \{(x \wedge c, y \wedge c) | (x, y) \in S, c \in L'\}$.
Now

$$\begin{aligned} f(a) &= f(x \wedge c) \\ &= f(x) \wedge f(c) \quad f \text{ is a frame homomorphism} \\ &= f(y) \wedge f(c) \quad (x, y) \in S \\ &= f(y \wedge c) \\ &= f(b) \end{aligned}$$

By Theorem 4.1.4 there exists a unique join preserving map $g : L(R) \to M$ such that $f = g \circ j$. It remains to show that g preserves finite meets:
Let $a, b \in L$. Since f is a frame homomorphism $f(a \wedge b) = f(a) \wedge f(b)$.
Since $f = g \circ j$, we have $g(j(a \wedge b)) = g(j(a)) \wedge g(j(b))$.
But $j(a \wedge b), j(a)$ and $j(b)$ are $R-$ coherent. Therefore it follows that
$g(a \wedge b) = g(a) \wedge g(b)$. □

4.1.6 Theorem : *The mapping $\downarrow$ gives rise to a reflection from the category $\mathbb{M}SL$ of meet semilattices to $\mathbb{F}rm$.*

Proof. :
Let A be a meet semilattice. Denote by $\downarrow A$ the set of all downsets of A, and let $\downarrow : A \to \downarrow A$ send $a \in A$ to $\downarrow a = \{x \in A | x \leq a\}$. If we order $\downarrow A$ as a subset of $\mathcal{P}(A)$, it is clearly a sub-complete lattice of $\mathcal{P}(A)$; so the infinite distributive law holds (so does its dual, although this is a mere coincidence).

Moreover, by the defining property of meets, we have

$\downarrow(a) \cap \downarrow(b) = \downarrow(a \wedge b)$ and

$\downarrow(e_A) = A$

So $\downarrow : A \to \downarrow A$ is a semi-lattice homomorphism.

Let $f : A \to L$ be a meet semilattice homomorphism, where L is a frame. Define $g : \downarrow A \to L$ by

$$g(S) = \bigvee_L \{f(s) | s \in S\} \qquad (S \in \downarrow A).$$

We must show g is a unique frame homomorphism such that $f = g \downarrow$, that is, we must show that the following diagram commutes :

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "\downarrow"] & \\ L & & \downarrow A \arrow[ll, dashed, "g"] \end{tikzcd}$$

It is immediately clear that g is order preserving, and it extends f since $f(a) \in \{f(s)|s \in \downarrow a\} \subseteq \downarrow f(a)$.

Now

$$\begin{aligned} g(\downarrow a) &= \bigvee\{f(s)|s \in \downarrow a\} \\ &= \bigvee\{f(s)|s \leq a\} \\ &= f(a) \end{aligned}$$

Therefore $f = g \downarrow$ as required.

It is also clear from the form of the definition that g preserves arbitrary joins; so it remains to show that g preserves finite meets :

For $S, T \subseteq \downarrow A$

$$\begin{aligned} g(S) \wedge g(T) &= (\bigvee\{f(s)|s \in S\}) \wedge (\bigvee\{f(t)|t \in T\}) & \\ &= \bigvee\{f(s) \wedge f(t)|s \in S, t \in T\} & \text{by the infinite distributive law in } L \\ &= \bigvee\{f(s \wedge t)|s \in S, t \in T\} & f \text{ preserves finite meets} \\ &= \bigvee\{f(u)|u \in S \cap T\} & S \text{ and } T \text{ are downsets} \\ &= g(S \cap T). & \end{aligned}$$

The uniqueness of g is obvious from the fact that for any $S \subseteq \downarrow A$ we have

$$S = \bigvee_{\downarrow A}\{\downarrow a|a \in S\}$$

and this join must be preserved by g, that is,

$$g(S) = g(\bigvee_{\downarrow A}\{\downarrow a|a \in S\}) = \bigvee_{L}\{g(\downarrow a)|a \in S\}.$$

□

Considering Theorem 4.1.6 one sees that Proposition 1.1 in the paper by Johnstone [8] is a special case of Theorem 4.1.4 for certain type of precongruence relations on $\downarrow A$.

4.2 Completion of Uniform Frames (Locales)

In this section we give a description of uniform completion in terms of generators and defining relations.

4.2.1 Let $(L, \mathcal{U}L)$ be uniform locale. Denote by $S \subseteq \downarrow L \times \downarrow L$ the system of all pairs

$$(\downarrow a, k(a), (\downarrow e, c(U)) \text{ and } (\downarrow 0, \phi)$$

with $a \in L, U \in \mathcal{U}L$, where k and c are given by

$$k(a) = \bigcup_{b \lhd a} \downarrow b \text{ and } c(U) = \bigcup_{a \in U} \downarrow a.$$

Put $\downarrow L' = \{\downarrow a | a \in L\}, R = R(S, \downarrow L')$ where R consists of all pairs

$$\begin{array}{ll} \{(\downarrow a \wedge \downarrow c, k(a) \wedge \downarrow c) & | \ (\downarrow a, k(a)) \in S, \downarrow c \in \downarrow L'\}, \\ \{(\downarrow e \wedge \downarrow c, c(U) \wedge \downarrow c) & | \ (\downarrow e, c(U)) \in S, \downarrow c \in \downarrow L'\} \\ \text{and } \{(\downarrow 0 \wedge \downarrow c, \phi \cap \downarrow c) & | \ (\downarrow 0, \phi) \in S, \downarrow c \in \downarrow L'\} \end{array}$$

Note that $\downarrow L'$ is a join basis for $\downarrow L$ and also $\downarrow L'$ is meet-closed. Recalling the notation of Theorem 4.1.4 write $\bar{L} = \downarrow L(R), j : \downarrow L \to \bar{L}$. Now both $\downarrow L$ and $\bar{L}$ are locales.

Since $\bar{L} \subseteq \downarrow L$, we should be careful when indicating their localic operations. There is no problem with the meets, which are preserved by the nucleus and hence coincide. On the other hand, there is a natural way to distinguish joins : we reserve the symbol '$\bigvee$' for $\bar{L}$, while in $\downarrow L$ we simply use the set-theoretical symbol '$\bigcup$'.

Define $j : \downarrow L \rightarrow \bar{L}$ by

$$j(W) = \bigwedge\{v | v \geq w, v \text{ is } R\text{-coherent}, w \in W\}$$

and let $f : \downarrow L \rightarrow L$ be given by

$$\begin{aligned} f(W) &= f(\bigcup_{a \in W} \downarrow a) \\ &= \bigvee_{a \in W} \downarrow a. \end{aligned}$$

Now, $(\downarrow a, k(a)) \in S$ implies that $\bigvee \downarrow a = a = \bigvee k(a), (\downarrow e, c(U)) \in S$ implies that $\bigvee \downarrow e = e = \bigvee c(U)$ and $(\downarrow 0, \phi) \in S$ implies that $\bigvee \downarrow 0 = 0 = \bigvee \phi$.

Therefore, by Theorem 4.1.4 there exists a unique frame homomorphism $p : \bar{L} \rightarrow L$ such that $pj = \bigvee$.

Thus, we have the following commutative diagram

$$\begin{tikzcd} \downarrow L \arrow[r, "j"] \arrow[d, "\bigvee"'] & \bar{L} \arrow[dl, dashed, "!p"] \\ L & \end{tikzcd}$$

It follows immediately from the definition of S that $\downarrow L' \subseteq \bar{L}$. Hence, we have a mapping $p : \bar{L} \rightarrow L$ given by :

For $u \in \bar{L}$,

$$\begin{array}{rcll} p(u) & = & p(j(u)) & u \text{ is } R\text{-coherent} \\ & = & pj(u) & \\ & = & \bigvee u & pj = \bigvee \\ & = & \bigvee(\bigcup \downarrow a) & \\ & = & \bigvee\{a | \downarrow a \leq u\}. & \end{array}$$

Now, $p : \bar{L} \to L$ is dense:

Let $p(u) = 0, u \in \bar{L}$. Since $p(u) = \bigvee\{a| \downarrow a \leq u\} = 0, a = 0$ which implies that $u = 0$.

Further p is surjective, since for $a \in L, \downarrow a \in \bar{L}$, we have $p(\downarrow a) = p(j(\downarrow a)) = \bigvee(\downarrow a) = a$. Hence p is dense embedding.

Take $U \in \mathcal{U}L$. Then $\overline{\mathcal{U}_0 L} = \{\downarrow a | a \in U, U \in \mathcal{U}L\}$ is a cover of $\bar{L}$ since

$$\begin{array}{rcll} \bigvee\{\downarrow a | a \in U\} & = & j(\bigcup\{\downarrow a | a \in U\}) & \\ & = & j(c(U)) & \\ & = & j(\downarrow e) & (\downarrow e, c(U)) \in S \\ & = & e & \end{array}$$

In fact, we have a uniform basis:

1. Let $U \in \overline{\mathcal{U}_0 L}$. We must show there exists a $V \in \overline{\mathcal{U}_0 L}$ such that $V \leq^* U$, that is,

$$\{\downarrow v | v \in V\} \leq^* \{\downarrow u | u \in U\}.$$

Now

$$\begin{aligned} st(\downarrow v, v \in V) &= \bigvee\{\downarrow w | w \in V \text{ and } \downarrow w \wedge \downarrow v \neq 0\} \\ &= \bigvee\{\downarrow w | w \in V \text{ and } w \wedge v \neq 0\}. \end{aligned}$$

So $st(V, v) = \bigvee\{w \in V | w \wedge v \neq 0\} \leq u$ for some $u \in U$. Hence $st(\downarrow v, v \in V) \leq \downarrow u$ since $w \leq u$ implies that $\bigvee \downarrow w \leq \downarrow u$.

2. Take $W \in \bar{L}$. We must show $W = \bigvee\{\downarrow x | \downarrow x \triangleleft W\}$:

   First we claim that if $x \triangleleft y$ then $\downarrow x \triangleleft \downarrow y$ in $\bar{L}$: (2)

   If $x \triangleleft y$ then there exists $V \in \mathcal{U}L$ such that $Vx \leq y$. Let $\overline{V} \in \overline{\mathcal{U}_0 L}$ where $\overline{V} = \{\downarrow v | v \in V\}$.

   Now $\overline{V} \downarrow x \leq \downarrow y$ since

$$\begin{aligned} \overline{V} \downarrow x &= \bigvee\{\downarrow v | v \in V \text{ and } \downarrow v \wedge \downarrow x \neq 0\} \\ &= \bigvee\{\downarrow v | v \in V \text{ and } \downarrow (v \wedge x) \neq 0\} \\ &= \bigvee\{\downarrow v | v \wedge x \neq 0\} \end{aligned}$$

So $v \in V, v \wedge x \neq 0$ implies that $v \leq y$ which in turn implies $\downarrow v \subseteq \downarrow y$.

Therefore $\bigvee_{v \wedge x \neq 0} \downarrow v \subseteq \downarrow y$, thereby proving the claim.

We now show that $\downarrow x = \bigvee_{y \triangleleft x} \downarrow y$:
Since $\downarrow x \, S \, k(x), j(\downarrow x) = j(k(x))$. Thus we have

$$\downarrow x = j(\downarrow x) = j(k(x)) = j(\bigcup_{y \triangleleft x} \downarrow y) = \bigvee\{\downarrow y | y \triangleleft x\}.$$

It follows that

$$\downarrow x = \bigvee \{\downarrow y / \downarrow y \lhd \downarrow x\}$$

from (2)

Therefore

$$W = \bigvee_{x \in W} \downarrow x = \bigvee \{\bigvee \downarrow y | \downarrow y \lhd \downarrow x\}.$$

Denote by $\overline{\mathcal{U}L}$ the corresponding uniformity on $\bar{L}$. We now show that $p : (\bar{L}, \overline{\mathcal{U}L}) \to (L, \mathcal{U}L)$ is a uniform embedding :

Take $\overline{V}, V \in \mathcal{U}L$. Since $\overline{V} = \{\downarrow v | v \in V\}, p(\overline{V}) = \{p(\downarrow v) | v \in V\}$. Now

$$\begin{aligned} p(\downarrow v) &= \bigvee \{a | \downarrow a \leq \downarrow v\} \\ &= \bigvee \{a | a \leq v\} \end{aligned}$$

So $p(\overline{V}) = \{v | v \in V\} = V \in \mathcal{U}L$ implies that $p(\overline{\mathcal{U}L}) \subseteq \mathcal{U}L$.

Conversely, take $V \in \mathcal{U}L$. Then $\overline{V} \in \overline{\mathcal{U}L}$ and $p(\overline{V}) = V$ implies that $V \in p(\overline{\mathcal{U}L})$. Therefore $p(\overline{\mathcal{U}L}) = \mathcal{U}L$.

The pair $(\bar{L}, \overline{\mathcal{U}L})$ together with the mapping $p : \bar{L} \to L$ will be called the *completion* of the uniform locale $(L, \mathcal{U}L)$.

4.2.2 Let $f : (L, \mathcal{U}L) \to (M, \mathcal{U}M)$ be a uniformly continuous frame morphism. Define $g : L \to \overline{M}$ by putting $g(a) = \downarrow f(a)$. Recall that elements of the form $\downarrow x$ are R-coherent. Using Theorem 4.1.6 we obtain a frame morphism $f_0 : \downarrow L \to \overline{M}$ satisfying

$$f_0(\downarrow a) = g(a) = \downarrow f(a).$$

Thus, we have the following commutative diagram.

$$\begin{tikzcd} & L \arrow[dl, "g"'] \arrow[dr, "\downarrow"] & \\ \overline{M} & & \downarrow L \arrow[ll, dashed, "f_0"] \end{tikzcd}$$

Now the mapping $f_0 \circ k : L \to \overline{M}$ preserves also finite meets :

For $a, b \in L$

$$\begin{aligned} (f_0 \circ k)(a \wedge b) &= f_0(k(a \wedge b)) \\ &= f_0(\bigcup_{c \lhd a \wedge b} \downarrow c) \\ &= f_0(\bigcup_{c \lhd a} \downarrow c \wedge \bigcup_{c \lhd b} \downarrow c) \\ &= f_0(\bigcup_{c \lhd a} \downarrow c) \wedge f_0(\bigcup_{c \lhd b} \downarrow c) \\ &= (f_0 \circ k)(a) \wedge (f_0 \circ k)(b). \end{aligned}$$

So $f_0 \circ k$ is a meet semilattice morphism. Using Theorem 4.1.6 once again we obtain a unique frame morphism $f_1 : \downarrow L \to \overline{M}$.

$$\begin{tikzcd} & L \arrow[dl, "f_0 \circ k"'] \arrow[dr, "\downarrow"] & \\ \overline{M} & & \downarrow L \arrow[ll, dashed, "f_1"] \end{tikzcd}$$

satisfying $f_1(\downarrow a) = f_0 \circ k(a)$.

4.2.3 Lemma : *If* $a\,S\,b$ *then* $f_1(a) = f_1(b)$.

Proof. :

Using corollary 4.1.5 we have

$$f_1(\downarrow 0) = f_0(k(0)) = f_0(\bigcup_{b \lhd 0} \downarrow b) = f_0(\downarrow 0) = \downarrow 0 = f_1(\phi).$$

Now compute

$$\begin{aligned}
f_1(\downarrow a) &= f_0(k(a)) \\
&= f_0(\bigcup_{b \lhd a} \downarrow b) \\
&= f_0(\bigcup_{c \lhd a} \bigcup_{b \lhd c} \downarrow b) \text{ by interpolation} \\
&= \bigvee_{c \lhd a} f_0(\bigcup_{b \lhd c} \downarrow b) \\
&= \bigvee_{c \lhd a} f_0(k(c)) \\
&= \bigvee_{c \lhd a} f_1(\downarrow c) \\
&= f_1(\bigcup_{c \lhd a} \downarrow c) \\
&= f_1(k(a)).
\end{aligned}$$

Take $U \in \mathcal{U}L$. Choose a $W \in \mathcal{U}L$ such that $W \leq^* U$. We conclude the proof by another calculation.

$$
\begin{aligned}
f_1(\downarrow e) &= f_0(k(e)) \\
&= f_0(\bigcup_{b \lhd e} \downarrow b) \\
&= \downarrow e \\
&= j(c(f(W))) \\
&= j(\bigcup_{y \in f(W)} \downarrow y) \\
&= \bigvee_{x \in W} \downarrow f(x) \\
&= \bigvee_{x \in W} f_0(\downarrow x) && f_0(\downarrow x) \;= g(x) = \downarrow f(x) \\
&\leq \bigvee_{y \in U} \bigvee_{x \lhd y} f_0(\downarrow x) && W \leq^* U \\
&= \bigvee_{y \in U} f_0(\bigcup_{x \lhd y} \downarrow x) \\
&= \bigvee_{y \in U} f_0(k(y)) \\
&= \bigvee_{y \in U} f_1(\downarrow y) \\
&= f_1(\bigcup_{y \in U} \downarrow y) \\
&= f_1(c(U))
\end{aligned}
$$

$\square$

4.2.4 Theorem : *For a uniformly continuous morphism* $f : (L,\mathcal{U}L) \to (M,\mathcal{U}M)$, *there exists a unique frame morphism* $\bar{f} : (\bar{L},\overline{\mathcal{U}L}) \to (\overline{M},\overline{\mathcal{U}M})$ *completing the diagram*

$$\begin{tikzcd} \bar{L} \arrow[r, dashed, "\bar{f}"] \arrow[d, "p_L"'] & \overline{M} \arrow[d, "p_M"] \\ L \arrow[r, "f"'] & M \end{tikzcd}$$

Moreover, this morphism is uniformly continuous.

Proof. :

By Lemma 4.2.3 we have $a\,S\,b$ implies that $f_1(a) = f_1(b)$. Let j be the map from $\downarrow L$ to $\bar{L}$. Further $a\,S\,b$ also implies that $j(a) = j(b)$. Using Corollary 4.1.5 there exists a unique frame morphism $\bar{f} : \bar{L} \to \overline{M}$ such that $\overline{f} \circ j = f_1$. Thus we have the following commutative diagram :

$$\begin{tikzcd} & \downarrow L \arrow[dl, "j"'] \arrow[dr, "f_1"] & \\ \bar{L} \arrow[rr, dashed, "\bar{f}"'] & & \overline{M} \end{tikzcd}$$

Completing the diagram we obtain

$$\begin{tikzcd}
& \downarrow L \arrow[dl, "j"'] \arrow[dr, "f_1"] & \\
\bar{L} \arrow[rr, dashed, "\bar{f}"] \arrow[d, "p_L"'] & & \bar{M} \arrow[d, "p_M"] \\
L \arrow[rr, "f"'] & & M
\end{tikzcd}$$

Now the frames $L, M, \overline{L}$ and $\overline{M}$ are uniform and hence regular by Corollary 2.2.4. We now show that $f \circ p_L = p_M \circ \bar{f}$, that is, $(f \circ p_L)(\downarrow x) = (p_M \circ \bar{f})(\downarrow x)$: It is easily seen that

$$(f \circ p_L)(\downarrow x) = f(p_L(\downarrow x)) = f(\bigvee\{a \in L | \downarrow a \leq \downarrow x\}) = f(x).$$

Now

$$\begin{aligned}
(p_M \circ \bar{f})(\downarrow x) &= p_L \circ \bar{f}(j(\downarrow x) \\
&= p_L \circ f_1(\downarrow x) && \bar{f} \circ j = f_1 \\
&= p_L \circ f_0 \circ k(x) && f_1(\downarrow x) = f_0 \circ k(x) \\
&= p_L \circ f_0(\bigcup_{y \triangleleft x} \downarrow y) \\
&= p_L \left(\bigvee_{y \triangleleft x} f_0(\downarrow y) \right) && f_0 \text{ is a frame homomorphism} \\
&= p_L \left(\bigvee_{y \triangleleft x} \downarrow f(y) \right) && f_0(\downarrow y) = g(y) = \downarrow f(y)
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{y \lhd x} p_L(\downarrow (f(y)) \\
&= \bigvee_{y \lhd x} f(y) \\
&= f(\bigvee_{y \lhd x} y) \\
&= f(x)
\end{aligned}
$$

For the uniqueness part of the theorem, suppose there exists $\bar{g} : \bar{L} \longrightarrow \overline{M}$ such that $f \circ p_L = p_M \circ \bar{g}$. Then we have $p_M \circ \bar{f} = p_M \circ \bar{g}$. By Lemma 1.4.2 p_M is monic since it is dense. Therefore $\bar{f} = \bar{g}$.

It remains to prove the uniform continuity of $\bar{f}$:

Let $U_1 = \{\downarrow x | x \in U\}, U \in \mathcal{U}L$. Choose a $W \in \mathcal{U}L$ such that $W \leq^* U$. We compute

$$
\begin{aligned}
\bar{f}(U_1) &= \{\bar{f}(\downarrow x) | x \in U\} \\
&= \{\bar{f}(j(\downarrow x)) | x \in U\} \\
&= \{f_1(\downarrow x)) | x \in U\} && \bar{f} \circ j = f_1 \\
&= \{f_0 \circ k(x) | x \in U\} && f_1(\downarrow x) = f_0 \circ k(x) \\
&\geq \{f_0(\downarrow y) | y \in W\} && W \leq^* U \\
&= \{\downarrow f(y) | y \in W\} && f_0(\downarrow y) = \downarrow f(y) \\
&= \{\downarrow z) | z \in f(W)\} \in && \overline{\mathcal{U}_0 L}.
\end{aligned}
$$

□

We will call a uniform locale $(L, \mathcal{U}L)$ *complete* if the completion $p : \bar{L} \longrightarrow L$ is an isomorphism.

4.2.5 Theorem : *A frame $(L, \mathcal{U}L)$ is complete iff each uniform embedding $f : (M, \mathcal{U}M) \rightarrow (L, \mathcal{U}L)$ is closed.*

Proof. :

If we assume, that each uniform embedding $f : M \to L$ is closed, so is in particular the dense $p : \bar{L} \to L$.

Using Lemma 1.3.3, p is an isomorphism. Therefore $(L, \mathcal{U}L)$ is complete.

We prove the converse. Assume that $(L, \mathcal{U}L)$ is complete. Let $f : (M, \mathcal{U}M) \to (L, \mathcal{U}L)$ be any uniform embedding. We must show that f is closed. It suffices to prove that any dense $f : (M, \mathcal{U}M) \to (L, \mathcal{U}L)$ is an isomorphism (in other cases we simply consider the restriction of f to the closure of L in M).

Thus let f be dense. Put $j_M = f_* \circ f$ where $f_*(x) = \bigvee_{f(y) \leq x} y$ for all $x \in L$.

As a first step towards this we prove a few important Lemmas.

4.2.5.1 Lemma : *If $a \triangleleft b$ then $j_M(a) \triangleleft b$.*

Proof. :

Take $U \in \mathcal{U}M$ with $Ua \leq b$ and put

$$c = \bigvee \{x \in U | x \wedge a = 0\}$$

Then $c \wedge a = 0$ and $Ua \vee c = \bigvee U = e$ implies that $b \vee c = e$. Hence

$$\begin{aligned} f(j_M(a) \wedge c) &= f(j_M(a)) \wedge f(c) \\ &= f(a) \wedge f(c) \\ &= f(a \wedge c) \\ &= 0 \end{aligned}$$

Since f is dense $j_M(a) \wedge c = 0$ which together with $b \vee c = e$ implies that $j_M(a) \triangleleft b$. $\square$

4.2.5.2 Lemma : *For $a \in L$, we have $\bigvee_{x \in k(a)} f_*(x) = f_*(a)$.*

Proof. :

$f_*(a) \geq \bigvee_{x \in k(a)} f_*(x) = \bigvee_{x \lhd a} f_*(x)$ since f is onto and $f(Ub) \geq f(U)f(b)$.

Now $x \lhd a$ implies that $f_*(x) \lhd f_*(a)$ which in turn implies that $f_*(f(y)) \lhd f_*(a)$. Since $j_M = f_* \circ f$ we have $j_M(y) \lhd f_*(a)$, that is, $y \lhd f_*(a)$. Therefore

$$\begin{aligned} f_*(a) &\geq \bigvee_{y \lhd f_*(a)} f_* \circ f(y) \\ &= \bigvee_{y \lhd f_*(a)} j_M(y) \\ &\geq \bigvee_{y \lhd f_*(a)} y \\ &= f_*(a). \end{aligned}$$

It follows that $\bigvee_{x \in k(a)} f_*(x) = f_*(a)$ since

$$f_*(a) \geq \bigvee_{x \in k(a)} f_*(x) \geq f_*(a).$$

$\square$

4.2.5.3 Lemma : *For $U \in \mathcal{U}L$ we have $\bigvee_{x \in c(U)} f_*(x) = \downarrow e$.*

Proof. :

Let $U = f(U_1)$,where $U_1 = \{\downarrow x | x \in U\}$.

We have $f(c(U_1)) = f(\bigcup_{a \in U_1} \downarrow a) = c(U)$.

Now

$$
\begin{aligned}
\bigvee_{x\in c(U)} f_*(x) &= \bigvee_{x\in f(c(U_1))} f_*(x) \\
&= \bigvee_{y\in c(U_1)} f_*\circ f(y) \\
&= \bigvee_{y\in c(U_1)} j_M(y) \\
&= \bigvee_{y\in c(U_1)} y \\
&\geq e
\end{aligned}
$$

$\square$

Take $a, b \in M$. We have the following commutative diagram :

$$
\begin{tikzcd}
\downarrow b \arrow[r, mapsto, "\bar{f}"] \arrow[d, mapsto, "p_M"'] & f(\downarrow b) \arrow[d, mapsto, "p_L"] \\
b \arrow[r, mapsto, "f"'] & f(b)
\end{tikzcd}
$$

Since p_L is an isomorphism there exists $p_L^{-1} : L \to \bar{L}$ given by $f(b) = \downarrow f(b)$ and we deduce that

$$
\begin{aligned}
\downarrow f(b) &= \bar{f}(\downarrow b) \\
&= f_1(\downarrow b) \\
&= f_0\circ k(b) \qquad f_1(b) = f_0\circ k(b) \\
&= f_0(\bigcup_{x\lhd b} \downarrow x) \\
&= j(\bigcup_{x\lhd b} \downarrow f(x)) \qquad (4.2)
\end{aligned}
$$

Put $v = \bigcup\{\downarrow z | f_*(z) \leq b\}$.

We now show that v is $S-$coherent :

1. $\downarrow a \leq v$ iff $k(a) \leq v$:

   Suppose $\downarrow a \leq v$. Since $k(a) = \bigcup\{\downarrow w | w \lhd a\}$ and if $w \lhd a$ then $w \leq a$, we have $\downarrow w \leq \downarrow a \leq v$. Therefore $\bigcup\{\downarrow w | w \lhd a\} \leq v$, that is, $k(a) \leq v$. Conversely, suppose $k(a) \leq v$. If $x \in k(a)$ then $x \in v$ which implies that $x \in \downarrow z$ for some z such that $f_*(z) \leq b$. Hence $x \leq z$ which implies that $f_*(x) \leq f_*(z) \leq b$ for all $x \in k(a)$. Therefore $\bigvee\limits_{x \in k(a)} f_*(x) \leq b$ which implies that $f_*(a) \leq b$ by using Lemma 4.2.5.2. Finally this implies that $\downarrow a \subseteq v$.

2. $\downarrow e \leq v$ iff $c(U) \leq v$:

   Suppose $\downarrow e \leq v$. Now $a \in U$ implies that $\downarrow a \leq \downarrow e \leq v$.

   Hence $\bigcup\limits_{a \in U} \downarrow a \leq \bigcup \downarrow e = \downarrow e \leq v$, that is, $c(U) \leq v$.

   For the converse suppose $c(U) = \bigcup\{\downarrow a | a \in U\} \leq v$.

   Now $e = \bigvee\{a | a \in U\}$.

   If $x \in c(U)$ then $x \in \downarrow a$ for some $a \in U$ which implies that $x \leq a$. Since $c(U) \leq v$, we have $x \leq a$ for some $a \in v$. Hence $x \leq a$ where $a \leq z$ for some z such that $f_*(z) \leq b$. This implies that

   $$f_*(x) \leq f_*(a) \leq f_*(z) \leq b \text{ for all } x \in c(U).$$

   Therefore $\bigvee\limits_{x \in c(U)} f_*(x) \leq b$. Using Lemma 4.2.5.3 we have $\downarrow e \leq b$. Also $f_*(e) = e \leq b$. Therefore we conclude that $\downarrow e \leq v$.

3. $\downarrow 0 \leq v$ iff $\phi \leq v$:

If $\downarrow 0 \leq v$ then the proof that $\phi \leq v$ is trivial. For the converse we need only show, from the definition of v, that $f_*(0) \leq b$:

Since $f_*(0) = \bigvee_{f(y) \leq 0} y$, we have $f(y) = 0$ implies that $y = 0$ by the density of f. Hence $f_*(0) = \bigvee 0 = 0 \leq b$.

By (1), (2), and (3) above we conclude that v is $S-$coherent.

But v is also R-coherent :

1. $\downarrow a \wedge \downarrow c \leq v$ iff $k(a) \wedge \downarrow c \leq v$:

Let $\downarrow a \wedge \downarrow c = \downarrow (a \wedge c) \leq v$. Since $(\downarrow a, k(a)) \in S$ we have $k(a \wedge c) \leq v$, which implies that

$$\bigvee_{x \in k(a \wedge c)} f_*(x) \leq f_*(v).$$

Therefore $f_*(a \wedge c) \leq v$. Since f_* preserves finite meets, $f_*(a) \wedge f_*(c) \leq v$. Now if $\downarrow (a \wedge c) \leq v$ then $a \wedge c \in v$ and $x \in k(a) \cap \downarrow c$ implies that $x \leq a \wedge c$. Therefore $x \in \downarrow (a \wedge c)$ which implies that $x \in v$. For the converse we must show that $a \wedge c \in v$, that is, $f_*(a \wedge c) \leq b$. Now for any $x \in k(a) \wedge \downarrow c, x \leq z$ for some z such that $f_*(z) \leq b$ which implies that $f_*(x) \leq f_*(z) \leq b$. Also $\bigvee_{x \in k(a)} f_*(x) = f_*(a)$ (by Lemma 4.2.5.2) implies that

$$\bigvee_{x \in k(a)} f_*(x) \wedge f_*(c) = f_*(a) \wedge f_*(c).$$

It follows that $\bigvee_{x \in k(a)} f_*(x \wedge c) = f_*(a \wedge c)$.

Now $x \in k(a)$ implies $x \wedge c \in k(a)$ and $x \wedge c \leq c$ implies that $x \wedge c \in \downarrow c$. So $x \wedge c \in k(a) \cap \downarrow c$. Therefore $f_*(a \wedge c) \leq b$, that is, $a \wedge c \in v$ which implies that $\downarrow a \wedge \downarrow c \leq v$.

2. $\downarrow e \wedge \downarrow c = \downarrow c \leq v$ iff $c(U) \wedge \downarrow c \leq v$:

   Let $\downarrow c \leq v$. Then $c(U) \wedge \downarrow c \leq \downarrow c \leq v$. Conversely, suppose

   $c(U) \wedge \downarrow c \leq v$. We must show that $c \in v$, that is, $f_*(c) \leq b$. If $x \in c(U)$ then $x \in \downarrow a$ for some $a \in U$, that is , $x \leq a$. Hence $c \wedge x \leq c \wedge a$ which implies that $c \wedge x \in v$ since $c \wedge x \in c(U) \wedge \downarrow c \leq v$. Therefore $f_*(c \wedge x) \leq b$. So $f_*(c) \wedge f_*(x) \leq b$ for all $x \in c(U)$, since f_* preserves finite meets. It follows that

$$\bigvee_{x \in c(U)} (f_*(c) \wedge f_*(x)) \leq b.$$

   This implies that $f_*(c) \wedge \bigvee_{x \in c(U)} f_*(x) \leq b$.

   By Lemma 4.2.5.3 we have $f_*(c) \wedge \downarrow e \leq b$, that is, $f_*(c) \leq b$ as required.

3. $\downarrow 0 \leq v$ iff $\phi \leq v$: proved earlier.

   Therefore v is $R-$coherent.

On the other hand we have

$$\downarrow f(b) = j(\bigcup_{x \lhd b} \downarrow f(x)) \tag{4.2}$$

By Lemma 4.2.5.1 $x \lhd b$ implies that $j_M(x) \lhd b$. Since $j_M = f_* \circ f$ we have $f_*(f(x)) \lhd b$. So $f_*(z) \lhd b$ for some $z = f(x)$. Hence $f_*(z) \leq b$. It follows that

$$\bigcup_{x \lhd b} \downarrow f(x) \leq \bigcup_{f_*(z) \leq b} \downarrow z = v.$$

Hence $j(\bigcup_{x \lhd b} \downarrow f(x)) \leq j(v) = v$ since v is $R-$ coherent. Thus it follows from (4.2) that $\downarrow f(b) \leq v$, that is, $f(b) \in v$. In consequence

$$j_M(b) = f_*(f(b)) \leq b,$$

which together with the fact that $b \leq j_M(b)$ (properties of nuclei), gives us $j_M(b) = b$. Thus j_M is the identity and hence f is an isomorphism.
It remains to show that the uniform embedding $f : M \to L$ is closed.
Let $\theta = Ker\ f = \{(x,y) | f(x) = f(y)\}$ be a congruence on M. Define $\bar{\theta} = \nabla_t$ where t is the largest element such that $(0,t) \in \theta$. Now $\bar{\theta}$ is a closed congruence. Put $v : M \to M/\bar{\theta}$ where v is the quotient morphism. Further v is uniform. Therefore there exists a unique uniform embedding $\bar{f} : M/\bar{\theta} \to L$ such that $f = \bar{f} \circ v$.
Also $\bar{f}$ is dense. Since $\bar{f}$ is a dense embedding we have $\bar{f}$ is an isomorphism. Hence $\bar{f}$ is closed as isomorphisms are always closed. Now $v : M \to M/\bar{\theta}$ is closed as a frame map. Since compositions of closed maps are closed it follows that $f = \bar{f} \circ v$ is closed. $\square$

Chapter 5

Completion of Metric Frames

In this chapter we introduce the notions of a metric diameter and a metric frame. Then we show that, in any metrizable space X, each metric ρ on X induces a metric diameter d on the frame $\mathfrak{D}X$ of open subsets of X. Metric frames are considered in general, dealing with the dual adjointness to metric spaces. Using the fact that every metric frame is a uniform frame and hence has a uniform completion, we prove that every metric frame has an essentially unique completion. While there are other ways of obtaining this result which avoid the use of the completion of uniform frames, the advantage of the present approach, which is attributable to Banascheswki and Pultr[4], is that it is choice free and that it provides a method applicable in a constructive treatment of this subject.

5.1 Pointfree Metric Topology

We recall that *pointfree topology* deals with frames and their homomorphisms. Now *pointfree metric topology* is concerned with frames equipped with a suitable kind of metric structure (Pultr [11]) as described by the following :

5.1.1 Definition : *A metric diameter on a frame L is a map $d : L \longrightarrow \mathbb{R}^+$ such that*

$(M1)$: $d(0) = 0$.

$(M2)$: If $a \leq b$ then $d(a) \leq d(b)$.

$(M3)$: If $a \wedge b \neq 0$ then $d(a \vee b) \leq d(a) + d(b)$.

$(M4)$: For any $\alpha < d(a)$ and $\varepsilon > 0$, there exists $b, c \leq a$ such that $d(b), d(c) < \varepsilon$ and $\alpha < d(b \vee c)$.

$(M5)$: Each $D_\varepsilon = \{x \in L | d(x) < \varepsilon\}$ is a cover, that is, $\bigvee D_\varepsilon = e$.

$(M6)$: Each $a \in L$ is a join of all $x \in L$ such that there exists $\varepsilon > 0$ for which

$$D_\varepsilon x = \bigvee \{s \in D_\varepsilon | s \wedge x \neq 0\} \leq a.$$

5.1.2 Definition : *A metric frame is a frame equipped with a specified metric diameter.*

We write $L, M \ldots$ for metric frames, $d_L, d_M, \ldots$ for their metric diameters, and $D_\varepsilon^L, D_\varepsilon^M, \ldots$ for the covers consisting of the elements of diameter less than ε.

5.1.3 Lemma : *In any (metrizable) space X, each metric $\rho : X \times X \to \mathbb{R}^+$ determines a metric diameter $d : \mathfrak{D}X \to \mathbb{R}^+$ on the frame $\mathfrak{D}X$ of open subsets of X, given by*

$$d(U) = \sup\{\rho(x,y)|x,y \in U, U \in \mathfrak{D}X\}.$$

Proof. :

$(M1)$: $d(\phi) = \sup\{\rho(x,y)|x,y \in \phi\}$. Suppose $d(\phi) > 0$. Then for $\varepsilon = d(\phi)$ there exists $x_1, y_1 \in \phi$ such that

$$\sup \phi - \varepsilon < \rho(x_1, y_1) \leq \sup \phi.$$

But $x_1, y_1 \in \phi$ and this contradicts the supremum property.

Therefore $d(\phi) = 0$.

$(M2)$: For $A, B \in \mathfrak{D}X$ let $A \subseteq B$.

Then $\{\rho(x,y)|x,y \in A\} \leq \{\rho(x,y)|x,y \in B\}$. This implies that $d(A) \leq d(B)$.

$(M3)$: Let $A, B \in \mathfrak{D}X$ such that $A \cap B \neq \phi$. If$A \subseteq B$ or $B \subseteq A$, the result is clear. So suppose that $x_1 \in A$ and $y_1 \in B$. Then

$$\begin{aligned} \rho(x_1, y_1) &\leq \rho(x_1, z) + \rho(z, y_1) \quad \text{for some } z \in A \cap B \\ &\leq d(A) + d(B) \end{aligned}$$

This implies that $d(A \cup B) \leq d(A) + d(B)$ since $d(A) + d(B)$ is an upper bound for all such $\rho(x_1, y_1), x_1, y_1 \in A \cup B$.

$(M4)$: Let $A \in \mathfrak{D}X$ and $\alpha \in \mathbb{R}^+$ such that $\alpha < d(A)$, that is, $\alpha < \sup\{\rho(x,y)|x,y \in A\}$. Now there exist $x_1, y_1 \in A$ and $\varepsilon > 0$ such that

$$d(A) - \varepsilon < \rho(x_1, y_1) \leq d(A).$$

This implies that $d(A) < \rho(x_1, y_1) + \varepsilon$. Since the sets $S_\rho(x, r)$ are open, we can take for $x_1, x_2 \in A$ suitable $B = S(x_1, r_1)$ and $C = S(x_2, r_2)$ where $r_1 = \frac{\varepsilon}{4} = r_2$. Clearly $d(B), d(C) < \varepsilon$.

Now consider $\varepsilon' = \min\{d(A) - \alpha, \frac{\varepsilon}{2}\}, \varepsilon' > 0$.

Then $d(A) - \varepsilon' < \rho(x_1, y_1)$ for some $y_1 \in A$.

Now $\alpha < d(A) < \rho(x_1, y_1) + \varepsilon' \leq \rho(x_1, y_1) + d(A) - \alpha$, that is,

$d(A) < \rho(x_1, y_1) + d(A) - \alpha$ which implies that $\alpha < \rho(x_1, y_1) \leq d(B \cup C)$.

$(M5)$: Let $D\varepsilon = \{A_i | d(A_i) < \varepsilon\}$ for $\varepsilon > 0$. We claim $\bigcup_i A_i = e_{\mathfrak{D}X} = X$:

Clearly $\bigcup_i A_i \subseteq X$. Now $x \in X$ implies $x \in A$ for some $A \in \mathfrak{D}X$. Since A is open there exist $S(x, \varepsilon) \subseteq A, \varepsilon > 0$. Let $A_i = S(x_i, \varepsilon)$ and so $X = \bigcup_i A_i$. Therefore D_ε is a cover.

$(M6)$: Let $U \in \mathfrak{D}X$. We must show that

$U = \bigcup\{V \in \mathfrak{D}X|$ there exist $\varepsilon > 0$ for which $D_\varepsilon V \subseteq U\}$, where

$D_\varepsilon = \{G \in \mathfrak{D}X | d(G) < \varepsilon\}$ and $D_\varepsilon V = \bigcup\{G \in D_\varepsilon | G \cap V \neq \phi\}$:

For any $x \in U$ there exists $\varepsilon > 0$ such that $S(x, \varepsilon) \subseteq U$. Let $V = S(x, \frac{\varepsilon}{3})$. Now $x \in V$ implies that $D_{\frac{\varepsilon}{3}} V \subseteq U$. We claim that if $S(x, \frac{\varepsilon}{3}) \subseteq U$ then $D_{\frac{\varepsilon}{3}} V \subseteq U$:

If $y \in D_{\frac{\varepsilon}{3}} V$ then there exist $G \in D_{\frac{\varepsilon}{3}}$ such that $y \in G$ and $G \cap V \neq \phi$. This implies that $d(G) < \frac{\varepsilon}{3}, y \in G$ and $G \cap S(x, \frac{\varepsilon}{3}) \neq \phi$.

Now

$$\begin{aligned} d(y,x) &\leq d(y,z) + d(z,x) \quad \text{for some } z \in S(x, \tfrac{\varepsilon}{3}) \cap G \\ &< d(G) + \tfrac{\varepsilon}{3} \\ &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \tfrac{2\varepsilon}{3} < \varepsilon. \end{aligned}$$

Therefore $y \in S(x, \varepsilon) \subseteq U$.

For the converse take any $V \in \mathfrak{D}X$. Then it follows that $V \subseteq D_\varepsilon V \subseteq U$. $\square$

In particular, this means that, with each metric space X, one has the associated metric frame $\mathfrak{D}X$, the frame of open subsets of X equipped with the metric diameter d resulting from the metric on X.

5.1.4 Definition : *A frame homomorphism $h : L \to M$ between metric frames is called uniform, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that D_δ^M refines $h[D_\varepsilon^L]$, that is, for any $x \in D_\delta^M$ there exists $z \in D_\varepsilon^L$ with $x \leq h(z)$.*

5.1.5 Remarks :

5.1.5.1 : The crucial result then is that a continuous map $f : X \to Y$ between metric spaces is uniform in the usual sense iff the associated frame homomorphism $\mathfrak{D}f : \mathfrak{D}Y \to \mathfrak{D}X$ satisfies the condition above. Moreover, for any uniform homomorphism, the induced map between the corresponding metric spectra will be of the analogous type. Thus, if $\mathbb{MF}rm$ and $\mathbb{MS}p$ are the categories of metric frames and metric spaces respectively, each with corresponding uniform maps, then $\mathfrak{D}$ and $\sum$ determine contravariant functors between these categories. Moreover, these functors are again adjoint on the right with adjunctions as in the case of mere frames, while $\mathfrak{D}$ actually provides a full dual embedding of $\mathbb{MS}p$ into $\mathbb{MF}rm$.

5.1.5.2 In any metric frame L, the covers $D_\varepsilon^L, \varepsilon > 0$, define a uniformity, obviously countably generated (take $\varepsilon = \frac{1}{n}$), while any uniform frame of the stated kind has a metric diameter d such that its given uniformity is the same as the uniformity determined by d (Pultr[11]). Thus we can accept that every metric frame is uniform, and hence has a uniform completion.

5.2 Completion of Metric Frames

The fact that a metric space X is complete iff any dense embedding $X \to Y$ is an isomorphism, points to an obvious notion of completeness for metric frames, as follows :

5.2.1 Definitions :

5.2.1.1 A homomorphism $h : L \to M$ between metric frames is called a *dense surjection* if $h(x) = 0$ implies that $x = 0, d_M(h(a)) = d_L(a)$, and h is onto.

5.2.1.2 A metric frame M is called *complete* if any dense surjection $L \to M$ is an isomorphism.

5.2.1.3 For any metric frame L , a complete metric frame M together with a dense surjection $M \to L$ is called a *completion* of L.

5.2.2 Theorem : *Any metric frame L has an essentially unique completion $\gamma_L : CL \to L$, where $CL = \mathcal{R}L\,/\theta_L$.*

Proof. :

Since any metric frame is uniform, L is uniform. We have shown in theorem 3.2.4 that CL with $\gamma_L : CL \to L$ is a uniform completion of L, unique up to a unique isomorphism. Denote γ_L by h and CL by M. A convenient approach to proving that this is in fact a metric completion is to make use of the uniform completion $h : M \to L$ relative to its metric uniformity by showing that $\delta : M \to \mathbb{R}^+$, given by

$$\delta(x) = d_L(h(x)) \text{ for all } x \in M$$

defines a metric diameter on M. Certainly δ is well defined since d_L and h are. It remains to check that the six conditions for a metric diameter are satisfied.

$(M1)$: $\delta(0_M) = d_L(h(0_M)) = d_L(0_L) = 0$.

$(M2)$: If $a \leq b, a, b \in M$, then $h(a) \leq h(b)$ since h is order preserving. By $(M2)$ for d_L,

$$\begin{aligned} d_L(h(a)) &\leq d_L(h(b)), \text{ that is,} \\ \delta(a) &\leq \delta(b). \end{aligned}$$

$(M3)$: Let $a, b \in M$ such that $a \wedge b \neq 0_M$. Now $h(a \vee b) = h(a) \vee h(b)$. So

$$\begin{aligned} d_L(h(a \vee b)) &= d_L(h(a) \vee h(b)) \\ &\leq d_L(h(a)) + d_L(h(b)) \text{ by } (M3) \text{ for } d_L. \end{aligned}$$

Therefore $\delta(a \vee b) \leq \delta(a) + \delta(b)$.

$(M4)$: If $\alpha < \delta(a)$ for some $a \in M$ and $\varepsilon > 0$ then $\alpha < d_L\,(h(a))$. Hence there exists $s, t \leq h(a)$ in D^L_ε for which $d_L(s), d_L(t) < \varepsilon$ and $\alpha < d_L(s \vee t)$ by $(M4)$ for d_L. Now consider $b = a \wedge r(s)$ and $c = a \wedge r(t)$ where $r : L \longrightarrow M$ is the right adjoint of h, defined by $r(a) = \bigvee\{x \in M/h(x) \leq a\}$.
Certainly $b = a \wedge r(s) \leq a$ and $c = a \wedge r(t) \leq a$.

Now

$$\begin{aligned}
\delta(b \vee c) &= d_L(h(b \vee c)) \\
&= d_L(h(a \wedge r(s)) \vee (a \wedge r(t))) \\
&= d_L((h(a) \wedge hr(s)) \vee (h(a) \wedge hr(t))) \\
&= d_L(h(a) \wedge (s \vee t)) \\
&= d_L(s \vee t) \qquad\qquad s, t \leq h(a) \text{ implies } s \vee t \leq h(a)
\end{aligned}$$

Therefore $\alpha < d_L(s \vee t) = \delta(b \vee c)$.

$(M5)$: Let $D^M_\varepsilon = \{b \in M | \delta(b) < \varepsilon\}$. Now for each $b \in M$ there exists $a \in L$ such that $b = r(a)$. So

$$\begin{aligned}
D^M_\varepsilon &= \{r(a) | \delta(r(a)) < \varepsilon\} \\
&= \{r(a) | d_L(a) < \varepsilon\} \\
&= r[D^L_\varepsilon]
\end{aligned}$$

Since D^L_ε is a cover of L, $r[D^L_\varepsilon] = D^M_\varepsilon$ is a cover of M.

$(M6)$: Since M is a uniform frame, for each $a \in M$ we have

$$a = \bigvee\{x \in M/Ax \leq a\} \text{ for some } A \in \mathcal{U}M.$$

We have shown that $D^M_\varepsilon = r[D^L_\varepsilon]$ for $\varepsilon > 0$. Therefore

$a = \bigvee\{x \in M / r[D_\varepsilon^L] \leq a\}$ for $\varepsilon > 0$.

Since $D_\varepsilon^M = h^{-1}[D_\varepsilon^L]$ is a refinement of $r[D_\varepsilon^L]$ because $s \leq rh(s)$ for all $s \in M$, this immediately implies that $a = \bigvee\{x \in M / D_\varepsilon^M x \leq a\}$.

That the resulting metric frame provides a metric completion is obvious, and the uniqueness is an immediate consequence of the familiar uniqueness of uniform completions. $\square$

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