THE PRICING THEORY OF ASIAN OPTIONS

By

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Abstract

An Asian option is an example of exotic options. Its payoff depends on the average of the underlying asset prices. The average may be over the entire time period between initiation and expiration or may be over some period of time that begins later than the initiation of the option and ends with the options expiration. The average may be from continuous sampling or may be from discrete sampling. The primary reason to base an option payoff on an average asset price is to make it more difficult for anyone to significantly affect the payoff by manipulation of the underlying asset price. The price of Asian options is not known in closed form, in general, if the arithmetic average is taken into effect. In this dissertation, we shall investigate the pricing theory for Asian options. After a brief introduction to the Black-Scholes theory, we derive the partial differential equations for the value process of an Asian option to satisfy. We do this in several approaches, including the usual extension to Asian options of the Black-Scholes, and the sophisticated martingale approach. Both fixed and floating strike are considered. In the case of the geometric average, we derive a closed form solution for the Asian option. Moreover, we investigate the Asian option price theory under stochastic volatility which is a recent trend in the study of path-dependent option theory

Keywords

Asian Option, Fixed strike Asian option, Floating strike Asian option, Stochastic volatility, Arithmetic averages, Geometric averages.

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Preface

I declare that this thesis is my own, unaided work. It was carried out in the School of Science and Agriculture, in the University of KwaZulu-Natal under the supervision of Prof Hong-Kun Xu. It is being submitted for Master's degree in Mathematics.

This dissertation has not been submitted for any degree or examination to any other University or tertiary institution. Where the work of the others has been used, it is referenced to the text.

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N.G.Z Mkhize

————- day of ————— 2007

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Chapter 1

Black-Scholes

The Black Scholes model is a tool for pricing equity options. Prior to its development there was no standard way to price option. It was easy to work out that for a call option, say the premium should be higher the lower the strike price, the longer the time to maturity the higher the interest cost and the greater the volatility of the underlying stock. But how could these be combined to give an explicit equation that could be used to quickly give the correct or fair price for the option. It was the work of Fischer Black, Myron Scholes and Robert Merton that finally solved the option pricing problem in the early seventees. In 1973, Fischer Black and Myron Scholes published their groundbreaking paper the pricing of options and corporate liabilities see [4] . This first successful options pricing formula also described a general framework for pricing other derivative instruments. Most of the modern option pricing is derived from the ideas behind the Black-Scholes theory. Phenomenal growth of the over the counter options, exotic and swaps markets is intrinsically tied to this powerful model. But this theory is even more fundamental in that it can be applied to virtually any economic or financial activity where some aspect of contingency is inherent. In a very real sense Black-Scholes model marks the beginning of the modern era of financial derivatives.

1.1 Derivative

Financial market instruments are divided into two types, the underlying stock and their derivatives. The underlying stocks are shares, bonds, commodities and foreign currencies and their derivatives are claims that promise some payment or delivery in future contingent on an underlying stock behavior. Derivatives securities are assets whose values depend on the value of some other (underlying) asset. They are also known as the contract (tool) that transfer risks. Its ultimate payoff depends on future events. Their values are derived from the value of the underlying assets. The term derivatives security is broad and lays claim to many different type of financial instrument (security) such as stock, bond and foreign exchange instruments. There are three main types of derivatives securities, namely futures (forward), options and swaps.

1.1.1 Futures/Forward

A futures contract is similar to a forward contract, but the way these contracts are traded differs in some respect. Forward contract (called future contract if traded on exchanges) is an agreement between two parties to buy or sell an asset in the future for a fixed price. The buyer is said to hold the long position, the seller is said to hold the short position. Forward contract is an over the counter (OTC) instrument and trades takes place directly over the telephone for a specific amount and specific delivery dates as negotiated between two parties. In construct, futures contracts are standardized (in terms of contract size), trades takes place on an organized exchange and the contracts are revalued (marked to market) daily.

Forward and Future Contracts

Forwards	*Futures*
Private (non-marketable) contract between two parties.	Traded on an exchange.
Large traders are not communicated to other markets participants.	Traders are immediately known by the other markets participants.
Delivery or cash settlement at expiry.	Contract is usually closed out prior to maturity.
Usually one delivery date.	Range of delivery dates.
No cash paid until expiry.	Cash payments into (out of) margin account daily.
Negotiable choice of delivery dates, size of contract.	Standardized contract.

1.1.2 Options

Options are classic examples of derivatives that can be used to increase or reduce risk exposure. An option is a contract that gives its owner the right but not the obligation to buy or sell an underlying stock or commodity at a future point in time at an agreed upon price (fixed price) called the exercise price or the strike price. The act of buying and selling the asset is known as exercising the option. Options can either be of European or American. European options can only be exercised at an expiration date whereas American options can be exercised at any time prior to the expiring date. There are two main types of options namely call and put option. Call option gives the holder the right to buy an asset at a fixed price. Put option gives the holder the right to sell an asset at a fixed price. The payoff as the first step, we need to know what the contract will be worth at the expiry date. If at the time when the option expires (three months hence) the actual price of the underlying stock is S_T and $S_T > K$ then the option will be exercised. The option is said to be in the money: asset worth S_T can be purchased for just K. The value of the option is then $(S_T - K)$. If on the other hand, $S_T < K$, then it will be cheaper to buy the underlying stock on the open market and so the option will not be exercised. It is this freedom that distinguishes the options from futures. The option is then worthless and is said to be out of the money. If $S_T = K$ the option is said to be at the money. The payoff of the option at time T is thus $(S_T - K) = \max(S_T - K, 0)$.

1.1.3 Swap

This is an agreement between two parties to exchange a periodic stream of benefits or payment over a pre-arranged period. The payment could be based on the market value of an underlying asset. Two types of swaps, interest rate swap and currency swap in its simple form involve two parties exchanging debt denominated in different currencies.

Part of the reason the success of both futures and options is that they provide opportunities for hedging, speculation and arbitrage. The practice of reducing price risks using derivatives is known as hedging. The opposite of hedging is speculation. Speculation involves taking more risk. To illustrate how the Black-Scholes formula is used to provide an explicit equation to determine call and put option.

The Black-Scholes option pricing formulas It is derived by using the stochastic calculus, by assuming a continuous stochastic process for the asset price at each point in time with a given probability distribution. There are several important economic assumptions underlying the Black Schools model:

- It is possible to short sell the underlying stock.
- There are no arbitrage opportunities.
- Trading in the stock is continuous.
- There are no transaction costs/taxes.
- All securities are perfectly divisibly and no dividend is paid on the stock.
- The risk-free interest rate exists and is constant and the same for all maturity dates.
- The price of the underlying instrument e.g. stock follows a Geometric Brownian Motion in particular constant drift μ and volatility σ, $\mathrm{d}S = \mu S dt + \sigma S \mathrm{d}W$.

By combining these assumptions with the idea that the cost of an option should provide no immediate gain to either seller or buyer, a set of equations can be formulated to calculate the price of any option.

1.1.4 Ito's Lemma

Ito's Lemma is essential in the derivation of the Black and Scholes equation. It is an important result in the theory of stochastic processes. It relates the small change in function of random variable to a small change in the random variable itself.
An heuristic approach to Ito's Lemma:

Suppose that a variable X follows a stochastic process of the form

$$\mathrm{d}X = a(X,t)\mathrm{d}t + b(X,t)\mathrm{d}W_t \tag{1.1}$$

where W_t is a white noise, a function $V(X,t)$ a differential function of X and t. Ito's Lemma states that V is another Ito process, and

$$\mathrm{d}V = (\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial X} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial X^2})\mathrm{d}t + b\frac{\partial V}{\partial X}\mathrm{d}W_t. \tag{1.2}$$

Let $a = \mu S$ and $b = \sigma S$ then

$$\mathrm{d}V = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial X^2})\mathrm{d}t + (\sigma S\frac{\partial V}{\partial S})\mathrm{d}W_t.$$

Proof We expand V with respect to a change δV. In the expansion we keep terms up to first order in δt and second order δX. We have

$$V + \delta V = V(X,t) + \frac{\partial V}{\partial X}\delta X + \frac{\partial V}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(\delta X)^2$$

$$= V(X,t) + \frac{\partial V}{\partial X}\delta X + \frac{\partial V}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}(a\delta t + b\delta W_t)^2$$

$$= V(X,t) + \frac{\partial V}{\partial X}\delta X + \frac{\partial V}{\partial t}\delta t + \frac{1}{2}b^2\frac{\partial^2 V}{\partial X^2}b^2\delta W^2$$

$$= V(X,t) + (a\frac{\partial V}{\partial X} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial X^2} + \frac{\partial V}{\partial t})\delta t + b\frac{\partial V}{\partial X}\delta W_t$$

Taking the limits gives (1.2). When V is a function of S and t, and S follows a stochastic equation

$\mathrm{d}S = \mu S\mathrm{d}t + \sigma S\mathrm{d}W_t$, then Ito's lemma gives

$$\mathrm{d}V = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})\mathrm{d}t + \sigma S\frac{\partial V}{\partial S}\mathrm{d}W_t$$

1.2 Option pricing on stock paying no dividend

By assuming a lognormal process for the stock price and creating a riskless hedge portfolio consisting of the option and the underlying assets, Black and Scholes were able to obtain an exact solution for the premium or European call option or put. Black-Scholes option pricing formula prices European put or call on a stock that does not pay a dividend or make other distribution. The formula assumes the underlying stock price follows a Geometric Brownian motion with constant volatility.

1.2.1 Pricing formula for option that pays no dividend

The closed form for a call option is

$$\begin{aligned} C &= SN(d_1) - Ke^{-rT}N(d_2), \qquad (2.3)\\ d_1 &= \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},\\ d_2 = d_1 - \sigma\sqrt{T} &= \frac{\ln(S/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}. \end{aligned}$$

The price of a put option may be computed from the price of the call by the put-call parity

$$P = Ke^{-rT}N(-d_2) - SN(-d_1),$$

where N is the standard normal cumulative distribution function, C is the price of a call option (call premuim), P is the price of a put option, S is a current share price, K is a strike price, T expiry time, σ is the annual standard deviate of the compounded return on the stock.

The terms in the Black-Scholes equation are often referred to as the Greeks. They provide an approximation for the change in price of an option and can also be used to hedge a position either in the underlying asset (e.g. portfolio of stocks) or some portfolio consisting options. Delta measures how much an option price will move relative to the underlying asset. Black-Scholes formula can be used to determine the combination of option and the underlying asset to create a risk-free portfolio over a short time interval of time. This is known as delta hedging. Vega is the sensitivity of the option price to changes in volatility, and theta is the change in option price due to the passage of time.

The Greeks under the Black-Scholes model are easy to calculate. The Greeks-delta (Δ), gamma (Γ), vega (V), theta (θ) and rho (ρ) for call option are given by:

$$\begin{aligned}
\text{Delta} &= N(d_1) \\
\text{Gamma} &= N(d_1)/S\sigma\sqrt{T} \\
\text{Vega} &= SN(d_1)\sqrt{T} \\
\text{Theta} &= -SN(d_1)\sigma/2\sqrt{T} - rXe^{-rT}N(d_2) \\
\text{Rho} &= KTe^{-rT}N(d_2),
\end{aligned}$$

The Greeks-delta, gamma, vega, theta and rho for put are given by:

$$\begin{aligned}
\text{Delta} &= N(d_1) - 1 \\
\text{Gamma} &= N(d_1)/S\sigma\sqrt{T} \\
\text{Vega} &= SN(d_1)\sqrt{T} \\
\text{Theta} &= -SN(d_1)\sigma/2\sqrt{T} + rXe^{-rT}N(-d_2) \\
\text{Rho} &= -KTe^{-rT}N(-d_2),
\end{aligned}$$

Risk-neutral pricing The method of risk-neutral pricing is a mixture of economics and mathematics that allows one to produce a pricing option formula. It states that the value of a derivative is its expected future value discounted at the risk-free interest rate. This is exactly the same result that we would obtain if we assumed that the world was risk-neutral.

Replicating the portfolio This is a technique of mimicking the payoff of a particular instruments using a package of other securities. It is a self financing and replicates the terminal payoff of the option at expiration, by the usual no-arbitrage argument, the initial cost of setting up this replicating portfolio of assets and bonds must be equal to the value of the option replicated. Thus the fair price of an option is the value of its self-replicating portfolio. The concept of replicating portfolio finds wide applications in deriving new formulations of option models. The stock model

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}W_t$$

called the geometric Brownian motion is used to construct an analytic model for an option price.

1.2.2 Black-Scholes Partial Differential Equation (PDE) for options that pay no dividend

We derive the Black-Scholes's PDE.

Let the price S of an underlying security be governed by a geometric Brownian motion process over a time interval $[0, T]$, where W is a standard Weiner process. Let $V(S,t)$ denote the price of a derivative contingent on S. Suppose also that there is a risk-free asset, for example, a bond with interest rate r. The value B_t of the bond satisfies $dB_t = rB_t dt$.

Suppose that the price of a security S is governed by the equation

$$dS = \mu S \mathrm{d}t + \sigma S \mathrm{d}W.$$

A derivative of this security has a price $V(S,t)$ which satisfies the partial differential equation.

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S^2} - rV = 0 \tag{2.4}$$

Proof By Ito's lemma, we have

$$\mathrm{d}V = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})\mathrm{d}t + \sigma S\frac{\partial V}{\partial S}\mathrm{d}W$$

The portfolio S and B are formed to replicate the behaviour of the derivative

a_t=no of shares of stock

b_t= no of bonds

So the total portfolio is

$$G_t = a_t S_t + b_t B_t \tag{2.5}$$

holds for $0 \leq t \leq T$.

$$dG_t = a_t\mathrm{d}S_t + b_t\mathrm{d}B_t$$

$$\mathrm{d}B = rB\mathrm{d}t$$

then

$$dV = (\mu a_t S + rb_t B)\mathrm{d}t + a_t\sigma S\mathrm{d}W$$

equate the dV with Ito's equation

$$\mathrm{d}V = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})\mathrm{d}t + \sigma S\frac{\partial V}{\partial S}\mathrm{d}W$$

$$(\mu a_t S + rb_t B)\mathrm{d}t + a_t\sigma S\mathrm{d}W = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})\mathrm{d}t + \sigma S\frac{\partial V}{\partial S}\mathrm{d}W$$

matching the dW terms gives

$$a_t = \frac{\partial V}{\partial S}$$

$$b_t B = V - S\frac{\partial V}{\partial S}.$$

Thus

$$r(V - S\frac{\partial V}{\partial S})\mathrm{d}t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2})\mathrm{d}t$$

gives us the option price PDE (2.4). That is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial V}{\partial S^2}\mathrm{d}t + rS\frac{\partial V}{\partial S} - rV = 0$$

In the above equation we can see that the principle of risk-neutral valuation is clearly satisfied since the Black Scholes is independent of μ, the expected rate of growth of the underlying security price. It is important to note that the portfolio represents a self-financing, replicating, hedging strategy. It replicates a risk-free investment and it is hedged since it has no stochastic component.

Extending Black-Scholes We extend the Black-Scholes model to cover the option on a stock paying a continuous dividend. Then broaden this approach to include options on stock indices, currency and futures contract.

1.3 Option pricing on stock paying dividend

Clearly, if a stock pays a dividend, the price of the stock will drop by a commensurate amount that will reduce the stock price from S_t to $S_t e^{-q(T-t)}$. From there we could be able to calculate the value of the option as though the stock would pay no dividend. Lower bound/ PCP

$$\begin{aligned} C &> \max(S_t e^{-q(T-t)} - Ke^{-rt}) \\ P &> \max(Ke^{(-rt)} - S_t e^{-q(T-t)}) \\ PCP &\implies C + Ke^{-rt} = P + S_t e^{-q(T-t)}. \end{aligned}$$

1.3.1 Pricing formula for option on stock paying dividend

Replacing S by $S_t e^{-q(T-t)}$ in the Black-Scholes equation, gives the value for call option and put on stock paying dividend as

$$C = S_t e^{-q(T-t)} N(d_1) - Ke^{-rt} N(d_2),$$

$$P = Ke^{rt}N(-d_2) - S_t e^{-q(T-t)}N(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln(S/K) + (r - q - \sigma^2/2)T}{\sigma\sqrt{T}}, \tag{3.6}$$

C, P, S, T, σ and N are defined as in the stock paying no dividend.

1.3.2 A Partial Differential Equation (PDE) for option paying dividend

Let V be the price of a derivative dependent on a stock paying a continuous dividend yield at a rate q. Suppose the stock price follows the process.

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

($\mathrm{d}W_t$ Weiner Process, μ is the expected proportional growth, σ volatility of S_t). Using Ito's lemma

$$\mathrm{d}V = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial X^2})\mathrm{d}t + \sigma S\frac{\partial V}{\partial X}\mathrm{d}W_t.$$

Construct a portfolio consisting of -1 derivative and $\frac{\partial V}{\partial S_t}$ of stock. Then the value of the portfolio

$$\Pi = -V + \frac{\partial V}{\partial S_t}S_t$$

$$\Delta\Pi = -\Delta V + \frac{\partial V}{\partial S_t}\Delta S_t$$

$$\Delta\Pi = -(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2})\Delta t.$$

In the time tick Δt the holder of the portfolio earns capital gains equal to the $\Delta\Pi$ and dividend equal to $qS\frac{\partial V}{\partial S}\Delta t$, and letting ΔQ be the change in the wealth of the portfolio, we have

$$\Delta Q = (-\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + qS_t\frac{\partial V}{\partial S_t})\Delta t.$$

Since this expression is independent of the Weiner process the portfolio is instantaneously risk-free, we have

$$-\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + qS_t\frac{\partial V}{\partial S_t} = r(-V + \frac{\partial V}{\partial S_t}S_t).$$

Therefore the PDE is

$$\frac{\partial V}{\partial t} + (r-q)S_t\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} = rV. \tag{3.7}$$

1.4 Currency option

Foreign currency is analogous to a stock paying a known dividend yield. It pays a dividend equal to the foreign risk-free rate r_f. Since we assume the same stochastic process for the exchange rate S as for a stock on an earlier formula for pricing the call and put, q (the dividend yield) is replaced by r_f.

1.4.1 Pricing formula for a foreign option

$$C = S_t e^{-r_f(T-t)}N(d_1) - Ke^{-rt}N(d_2),$$

$$P = Ke^{-rt}N(-d_2) - S_t e^{-r_f(T-t)}N(-d_1)$$

where

$$d_1 = \frac{\ln(\frac{S_t}{X}) + (r - r_f + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln(\frac{S_t}{K}) + (r - r_f - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \tag{4.8}$$

both r and r_f are assumed to be constant and are continuously compounded for all maturities. As the formula is an extension of Black-Scholes, they apply only to European style option. American style options have the possibility of early exercise and hence are worth more than their European counterparts and must be valued using numerical methods. S_t is the value of unit of the foreign currency in US dollars, σ volatility of the exchange rate and r_f risk free-rate of interest in foreign country.

The above equation can be simplified by noting that the forward rate (for the same maturity date as the option) is given by

$$F = Se^{(r-r_f)(T-t)},$$

where F and S are measured as US Dollar per unit of foreign currency. This is also referred to as S being the currency price of one unit of the foreign currency. The formula assumes that we know the forward price or forward exchange rate F for a maturity T which is given by rearranging equation

$$S = Fe^{-(r-r_f)(T-t)}$$

Thus

$$C = e^{-r(T-t)}(FN(d_1) - KN(d_2)),$$

$$P = e^{-r(T-t)}(KN(-d_2) - FN(-d_1))$$

where

$$d_1 = \frac{\ln(F/K) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(F/K) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}. \tag{4.9}$$

PCP of foreign currency can be established in a similar fashion to that for an option of a dividend paying stock.

$$P + S\sigma e^{-r_f T} = C + e^{-rT}.$$

1.5 Futures options

Options on future contracts require the delivery of an underlying future contract when exercised, the holder acquires a long position in the underlying future contract plus a cash amount equal to the current future minus the strike price. Future contracts are standardized in terms of contract size, trades take place on an organized exchange and the contracts are revalued (marked to market) daily. Future contracts are traded between market makers in a pit on the floor of an exchange of which the largest are CBOT, CME and PSE. Most future contracts are closed out prior to maturity.

1.5.1 Pricing formula for option on Futures

$$C = e^{-r(T-t)}(FN(d_1) - KN(d_2)),$$
$$P = e^{-r(T-t)}(KN(-d_2) - FN(-d_1))$$

where

$$d_1 = \frac{\ln(F/K) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\ln(F/K) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

1.5.2 A PDE for Futures option

A replicating portfolio approach.

Suppose F (future price) follows the process

$$\mathrm{d}F = \mu F \mathrm{d}t + \sigma F \mathrm{d}W_t$$

($\mathrm{d}W_t$ Weiner Process, μ is the expected proportional growth, σ volatility of S_t). Using Ito's lemma we have

$$\mathrm{d}V = (\frac{\partial V}{\partial t} + \mu F \frac{\partial V}{\partial F} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2})\mathrm{d}t + (\sigma F \frac{\partial V}{\partial F})\mathrm{d}W_t.$$

Construct a portfolio consisting of -1 derivative and $\frac{\mathrm{d}V}{\mathrm{d}F}$ future contract. Consider the value of the portfolio Π and let $\Delta\Pi$, Δf, ΔF, be the change in Π , V, F in Δt. Since it cost nothing to enter into a future contract.

$$\Pi = -V.$$

In Δt, the holder of the portfolio earns capital gains equal to the $-\Delta V$ from the derivative and income of $(\partial V/\partial F)\Delta F$ from the future contract. Define ΔQ as the total change in wealth of the portfolio in time Δt. It follows that

$$\Delta Q = \frac{\partial V}{\partial F}\Delta F - \Delta V$$

$$\Delta F = \mu F \Delta t + \sigma F \Delta W_t$$

$$\Delta V = (\frac{\partial V}{\partial t} + \mu F \frac{\partial V}{\partial F} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\mathrm{d}F^2})\Delta t + \sigma F \frac{\partial V}{\partial F}\Delta W_t,$$

where $\Delta W_t = \epsilon\sqrt{\Delta t}$ and ϵ is a random sample from a standardized normal distribution.

It follows that

$$\Delta Q = (-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2})\Delta W_t.$$

This is riskless. Hence $\Delta Q = -r\Pi\Delta t$

$$rV\Delta t = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2})\Delta t.$$

Therefore

$$rV = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2}. \tag{5.10}$$

1.6 Martingale

The Black-Scholes Formula via Martingale The Martingale theory and stochastic analysis is another framework for characterizing arbitrage-free market and for pricing contingent claims. It is also called the risk-neutral valuation. Assume that the stock price follows the linear stochastic differential equation

$$\mathrm{d}S_t = \mu S_t + \sigma S_t \mathrm{d}W_t$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ are the constant drift rate and volatility, respectively. $S_0 > 0$ is the initial stock price. $W_t \in [0, T]$ is a one-dimensional standard Brownian motion on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$. The stochastic differential follows an Ito's integral equation

$$S_t = S_0 + \int_0^t \mu S_u \mathrm{d}u + \int_0^t \sigma S_u \mathrm{d}W_u, \forall t \in [0, T^*]$$

The process S which equals to $S_t = S_0 e^{\sigma W_t + (u - \frac{1}{2}\sigma^2)t}$, $\forall t \in [0, T^*]$ is a solution to the above equation. Assume that the underlying filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$ is the standard augmentation of the natural filtration $\mathcal{F}^W$ of the underlying Brownian motion, that is $\mathcal{F}_t = \mathcal{F}_t^W$, $\forall t \in [0, T^*]$. The risk-free security is assumed to continuously compound in value at the rate r, i.e

$$\mathrm{d}B_t = rB_t\mathrm{d}t$$

or equivalently

$$B_t = B_0 e^{rt}, \forall t \in [0, T^*].$$

The Martingale approach to option pricing

Self-financing strategy A trading strategy $\{a_t, b_t\}$ on the underlying probability space over the time interval $[0, T]$ is self-financing if its wealth process

$$V_t(a_t, b_t) = a_t S_t + b_t W_t, \forall t \in [0, T] \tag{6.11}$$

satisfying the condition

$$V_t(a_t, b_t) = V_0(a_t, b_t) + \int_0^t a_u \mathrm{d}S_u + \int_0^t b_u \mathrm{d}W_u, \forall t \in [0, T]$$

and

$$\mathbb{P}[\int_0^T (a_u)^2 \mathrm{d}u \leq \infty] = 1 \text{ and } \mathbb{P}[\int_0^T \mid a_u \mid \mathrm{d}u \leq \infty] = 1.$$

Martingale measure As in the discrete setting, a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_{T^*})$ equivalent to $\mathbb{P}$, is called a martingale-measure for the process S^*, if S^* is a local martingale under $\mathbb{Q}$. Also a probability measure $\mathbb{P}^*$ is said to be a martingale measure for the spot market, if the discounted wealth of any self-financing trading strategy follows a local martingale under $\mathbb{P}^*$.

Lemma 1 A trading strategy $\{a_t, b_t\}$ is self-financing if and only if its wealth process $V^*(a_t, b_t)$ satisfies

$$\mathrm{d}V_t^* = a_t \mathrm{d}S_t.$$

Proof From Ito's formula

$$\begin{aligned} V_t^*(a_t, b_t) &= e^{-rt} V_t(a_t, b_t) \\ &= V_0^*(a_t, b_t) + \textstyle\int_0^t a_u \mathrm{d}S_u^*, \forall t \in [0, T^*] \end{aligned}$$

where

$$V_t^*(a_t, b_t) = \frac{V_t(a_t, b_t)}{B_t}$$

and $\{a_t, b_t\}$ is self-financing strategy.

$$\begin{aligned}
\mathrm{d}V_t^* &= -re^{-rt}V_t\mathrm{d}t + e{-}rt\mathrm{d}V_t \\
&= -re^{-rt}(b_te^{-rt} + a_tS_t)\mathrm{d}t + e^{-rt}a_t\mathrm{d}(e^{-rt}) + e^{-rt}b_tS_t \\
&= a_t(-re^{-rt}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t \\
&= a_t\mathrm{d}S_t
\end{aligned}$$

The if part

$$\begin{aligned}
a_t\mathrm{d}S_t &= a_t(-re^{-rt}S_t\mathrm{d}t + e^{-rt}\mathrm{d}S_t \\
&= -re^{-rt}(b_te^{-rt} + a_tS_t)\mathrm{d}t + e^{-rt}a_t\mathrm{d}(e^{-rt}) + e^{-rt}b_tS_t \\
&= -re^{-rt}V_t\mathrm{d}t + e{-}rt\mathrm{d}V_t \\
&= \mathrm{d}V_t^*
\end{aligned} \tag{6.12}$$

Lemma 2 A unique martingale measure $\mathbb{Q}$ for the discounted stock price process S^* is given by the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = e^{(\theta W_T^* - \frac{1}{2}\theta^2 T^*)}$$

where

$$\theta = \frac{(r-\mu)}{\sigma}.$$

Under the martingale measure $\mathbb{Q}$, the discounted stock price S^* satisfies

$$\mathrm{d}S_t^* = S_t^*\sigma\mathrm{d}W_t^*. \tag{6.13}$$

By the Girsanov's Theorem the process $W_t^* = W_t - \sigma t$ follows a Brownian motion on a probability space $(\Omega, \mathbb{F}, \mathbb{Q})$.

Definition 1 A trading strategy (a_t, b_t) in a class of all self-financing trading strategies, is called a $\mathbb{P}^*$-admissable if the discounted wealth process

$$V_t^*(a_t, b_t) = B_t^{-1} V_t(a_t, b_t), \forall t \in [0, T]$$

follows a martingale under $\mathbb{P}^*$.

Corollary 1 Let X be a $\mathbb{P}^*$-attainable contingent claim which settles at time T. Then the arbitrage value which settles at time $t \in [0, T]$ in the Black Scholes market is given by the risk-neutral valuation formula

$$\pi_t(X) = B_t \mathbb{E}^{\mathbb{P}^*}(B_T^{-1} X | F_t), \forall t \in [0, T] \tag{6.14}$$

at $t = 0$ the value of X is equal to $\pi_0(X) = E^{\mathbb{P}^*}(B_T^{-1} X)$.

Pricing the Black-Scholes

1.6.1 Option on stock paying no dividend

Theorem 1 In the Black-Scholes market the arbitrage price of a European call option in the Black-Scholes market at $t \in [0, T]$ and a strike price K, is given by the formula

$$C_t = c(S_t, T - t), \forall t \in [0, T],$$

where the function $C : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ is given by

$$c(s, t) = sN(d_1(s, t)) - Ke^{-r(T-t)}(d_2(s, t)) \tag{6.15}$$

$$d_1(s, t) = \frac{\ln(s/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \tag{6.16}$$

$$d_2(s,t) = d_1(s,t) - \sigma\sqrt{T-t} \tag{6.17}$$

and $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-z^2}{2}} \mathrm{d}z, \forall x \in \mathbb{R}$. The unique $\mathbb{P}^*$-admissable replicating strategy (a_t, b_t) of the call option satisfies

$$a_t = \frac{\partial c}{\partial s}(S_t, T-t), b_t = e^{-rt}(c(S_t, T-t) - a_t S_t)$$

$\forall t \in [0, T]$.

Proof (Two alternative proofs)

First proof

It relies on the direct determination of the replicating strategy. It gives the valuation formula, this requires solving the Black-Scholes PDE and explicit formulas for replicating the portfolio.

Assume that the option price C_t satisfies $C_t = v(S_t, t)$ for some function $V : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$. We may thus assume that the replicating strategy $\{a_t, b_t\}$ we are looking for has the following form

$$(a_t, b_t) = (a(S_t, t), b(S_t, t)), \tag{6.18}$$

for $t \in [o, T]$, where $a, b : \Re_+ \times [0, T] \to \Re$ are unknown functions. Since (a_t, b_t) is assumed to be self-financing, the wealth process $V(a_t, b_t)$ which equals

$$V_t(a_t, b_t) = a(S_t, t)S_t + b(S_t, t)B_t = v(S_t, t) \tag{6.19}$$

need to satisfy the following: $\mathrm{d}V_t(a_t, b_t) = a(S_t, t)\mathrm{d}S_t + b(S_t, t)\mathrm{d}B_t$

Under the present assumption, the last equality can be given by the following form

$$\mathrm{d}V_t(a_t, b_t) = (u - r)S_t a(S_t, t)\mathrm{d}t + \sigma S_t a(S_t, t)\mathrm{d}W_t + rv(S_t, t)\mathrm{d}t. \tag{6.20}$$

From the equality (6.19) we obtain

$$b_t = b(S_t, t) = B_t^{-1}(v(S_t, t) - a(S_t, t)S_t).$$

We shall search for the wealth function v in the class of smooth functions on an open domain $\mathcal{D} = (0, +\infty) \times (0, T)$, we assume that $v \in C^{2,1}(\mathcal{D})$.
Applying Ito's Lemma gives

$$\mathrm{d}v(S_t, t) = (\frac{\partial v}{\partial t}(S_t, t) + \mu S_t \frac{\partial v}{\partial s}(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial s^2}(S_t, t))\mathrm{d}t + \sigma S_t \frac{\partial v}{\partial s}(S_t, t)\mathrm{d}W_t.$$

If we combine the above expression with (6.20) we obtain the Ito's different of the process Y, which equals $Y_t = v(S_t, t) - V(a_t, b_t)$

$$\begin{aligned}
&\mathrm{d}Y_t = (\frac{\partial v}{\partial t}(S_t, t) + \mu S_t \frac{\partial v}{\partial s}(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial s^2}(S_t, t))\mathrm{d}t + \sigma S_t \frac{\partial v}{\partial s}(S_t, t)\mathrm{d}W_t \\
&+(r - \mu)S_t a(S_t, t)\mathrm{d}t - \sigma S_t a(S_t, t)\mathrm{d}W_t - rv(S_t, t)\mathrm{d}t.
\end{aligned}$$

On the other hand in view of (6.19), Y vanishes identically, thus $\mathrm{d}Y_t = 0$. The diffusion term in the above decomposition of Y vanishes. In this case, this means that $\forall t \in [0, T]$ we have

$$\int_0^t \sigma S_u(a(S_u, u) - \frac{\partial v}{\partial s}(S_u, u))\mathrm{d}W_u = 0,$$

or equivalently

$$\int_0^T S_u^2(a(S_u, u) - \frac{\partial v}{\partial s}(S_u, u))^2 \mathrm{d}u = 0. \tag{6.21}$$

For (6.21) to hold, it sufficient and necessary that the function a satisfies

$$a(s, t) = \frac{\partial v}{\partial s}(s, t), \forall (s, t) \in \mathbb{R}_+ \times [0, T]. \tag{6.22}$$

We shall now assume that (6.22) holds. By (6.22) we get

$$Y_t = \int_0^t (\frac{\partial v}{\partial t}(S_u, u) + rS_u \frac{\partial v}{\partial s}(S_u, u) + \frac{1}{2}\sigma^2 S_u^2 \frac{\partial^2 v}{\partial s^2}(S_u, u)) - rv(S_u, u))\mathrm{d}u.$$

Y vanishes whenever v satisfies the Black-Scholes PDE equation

$$\frac{\partial v}{\partial t}(s, t) + rs\frac{\partial v}{\partial s}(s, t) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2}(s, t)) - rv(s, t) = 0. \tag{6.23}$$

Since $C_T = v(S_T, T) = (S_T - K)^+$, it is necessary to impose also the terminal condition $v(s,T) = (s-K)^+$, $\forall s \in \mathbb{R}_+$. It easy to check by direct computation that $v(s,t) = c(s, T-t)$, where c is given by (6.15)-(6.17), and check if the replicating strategy $\{a_t, b_t\}$ which equals to $a_t = a(S_t, t) = \frac{\partial v}{\partial s}(S_t, t)$, $b_t = b(S_t, t) = B_t^{-1}(v(S_t,t) - a(S_t,t)S_t)$, is $\mathbb{P}^*$-admissible. We must first check if $\{a_t, b_t\}$ is indeed self-financing. We must check

$$dV_t(a_t, b_t) = a_t \mathrm{d}S_t + b_t \mathrm{d}B_t$$

Since $V_t(a_t, b_t) = a_t S_t + b_t B_t = v(S_t, t)$, by applying Ito's formula, we obtain

$$\mathrm{d}V_t(a_t, b_t) = \frac{\partial v}{\partial t}(S_t, t)\mathrm{d}t + \frac{\partial v}{\partial s}(S_t, t)\mathrm{d}S_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial s^2}(S_t, t)\mathrm{d}t.$$

In view of (6.23), the last equality can also be given the following form

$$\mathrm{d}V_t(a_t, b_t) = \frac{\partial v}{\partial s}(S_t, t)\mathrm{d}S_t + rv(S_t, t)\mathrm{d}t - rS_t \frac{\partial v}{\partial s}(S_t, t)\mathrm{d}t$$

and thus

$$\begin{aligned} \mathrm{d}V_t(a_t, b_t) &= a_t \mathrm{d}S_t + rB_t \tfrac{v(S_t,t) - S_t a_t}{B_t}\mathrm{d}t \\ &= a_t \mathrm{d}S_t + b_t \mathrm{d}B_t \end{aligned}$$

We have now verified that $\{a_t, b_t\}$ is a self-financing strategy. We now have to verify that discounted wealth process $V^*(a_t, b_t)$, which satisfies

$$V_t^*(a_t, b_t) = V_0^*(a_t, b_t) + \int_0^t \frac{\partial v}{\partial s}(S_u, u)\mathrm{d}S_u^*$$

follows a martingale under the martingale measure $\mathbb{P}^*$. By direct computation we obtain $\frac{\partial v}{\partial s}(s,t) = N(d_1(s, T-t))\forall (s,t) \in \mathbb{R}_+ \times [0,T]$ and also, by (6.13) we find that

$$\begin{aligned} V_t^*(a_t, b_t) &= V_0^*(a_t, b_t) + \textstyle\int_0^t \sigma S_u N(d_1(S_u, T-u))\mathrm{d}W_u^* \\ &= V_0^*(a_t, b_t) + \textstyle\int_0^t \beta_u \mathrm{d}W_u^* \end{aligned}$$

where $\beta_u = \sigma S_u N(d_1(S_u, T-u))$. From the general properties of Ito stochastic integral, it is thus clear that $V^*(a_t, b_t)$ follows a martingale follows a local martingale under $\mathbb{P}^*$. To show that $V^*(a_t, b_t)$ is a genuine martingale it is enough to observe that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^*}(\int_0^T \beta_u \mathrm{d}W_u^*)^2 &= \mathbb{E}^{\mathbb{P}^*}(\int_0^T \beta_u^2 \mathrm{d}u) \\ &\leq \sigma^2 \int_0^T \mathbb{E}^{\mathbb{P}^*} \beta_u^2 \mathrm{d}u \\ &< \infty.\end{aligned}$$

For the first inequality see the appendix in [24] and the second inequality follows from the existence of the exponential moments of Gaussian random variable.

Second method

It makes a direct use of the risk-neutral valuation formula of (6.14). It focuses on the explicit computation of the arbitrage price of the hedging strategy. Before we apply the corollary we first check the contigent claim $X = (S_T - K)^+$ if it is attainable in the Black-Scholes market model.

$$V_t^* = V_0^* + \int_0^t \theta_u \mathrm{d}W_u^*, \forall t \in [0, T]$$

follows a continuous martingale under $\mathbb{P}^*$ and

$$\begin{aligned}X^* &= B_T^{-1}(S_T - K)^+ \\ &= \mathbb{E}^{\mathbb{P}^*} X^* + \int_0^T \theta_u \mathrm{d}W_u^* \\ &= \mathbb{E}^{\mathbb{P}^*} X^* + \int_0^T h_u \mathrm{d}S_u^*\end{aligned}$$

where $h_t = \frac{\theta_t}{\sigma S_t^*}$. Consider a trading strategy $\{a_t, b_t\}$ that is given by

$$a_t = h_t, b_t = V_t^* - h_t S_t^* = B_t^{-1}(V_t - h_t S_t),$$

where $V_t = B_t V_t^*$. First we check if trading strategy is self-financing. Observe that the wealth process $V(a_t, b_t)$ agrees with V, and

$$\begin{aligned}
\mathrm{d}V_t(a_t, b_t) &= \mathrm{d}(B_t V_t^*) \\
&= B_t \mathrm{d}V_t^* + r V_t^* B_t \mathrm{d}t \\
&= B_t h_t \mathrm{d}S_t^* + r V_t \mathrm{d}t \\
&= B_t h_t (B_t^{-1} \mathrm{d}S_t - r B_t^{-1} S_t \mathrm{d}t) + r V_t \mathrm{d}t \\
&= h_t \mathrm{d}S_t + r(V_t - h_t S_t)\mathrm{d}t \\
&= a_t \mathrm{d}S_t + b_t \mathrm{d}B_t.
\end{aligned}$$

It is clear that $V_T(a_t, b_t) = V_T = (S_T - K)^+$ so that $\{a_t, b_t\}$ is in fact a $\mathbb{P}^*$-admissable replicating strategy for X. We have to evaluate the arbitrage price of X using the risk-neutral valuation formula. Since $\mathcal{F}_t^W = \mathcal{F}_t^S, \forall t \in [0, T]$, we can write the risk-neutral valuation formula as

$$\begin{aligned}
C_t &= B_t \mathbb{E}^{\mathbb{P}^*}((S_T - K)^+ B_T^{-1} \mid \mathcal{F}_t^S) \\
&= c(S_t, T - t)
\end{aligned}$$

for some function $c : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ the increment $W_T^* - W_t^*$ of the Brownian motion is independent of σ-field $\mathcal{F}_t^W = \mathcal{F}_t^{W^*}$. S_t is manifestly F_t^W-measurable. By virtue of well known properties of conditional expectation we get

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^*}((S_T - K)^+ \mid F_t^S) &= H(S_t, T - t) \\
&= \mathbb{E}^{\mathbb{P}^*}[(s e^{\sigma(W_T^* - W_t^*) + (r - \frac{\sigma^2}{2})(T-t)} - K)^+]
\end{aligned}$$

$\forall (s, t) \in \mathbb{R}_+ \times [0, T]$. Therefore it is enough to find the unconditional expectation

$$\mathbb{E}^{\mathbb{P}^*}((S_T - K)^+ B_T^{-1}) = \mathbb{E}^{\mathbb{P}^*}(S_T B_T^{-1} I_D) - \mathbb{E}^{\mathbb{P}^*}(K B_T^{-1} I_D)$$

$$= J_1 - J_2,$$

where $D = \{S_T > K\}$.

For J_2 we have

$$
\begin{aligned}
J_2 &= e^{-rT} K \mathbb{P}^* \{S_T > K\} \\
&= e^{-rT} K \mathbb{P}^* \{S_t \exp(\sigma W_T^* + (r - \tfrac{1}{2}\sigma^2)T) > K\} \\
&= e^{-rT} K \mathbb{P}^* \{-\sigma W_T^* < \ln(\tfrac{S_t}{K}) + (r - \tfrac{\sigma^2}{2})T\} \\
&= e^{-rT} K \mathbb{P}^* \{\xi < \tfrac{\ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})}{\sigma\sqrt{T}}\} \\
&= e^{-rT} K N(d_2(S_t, T)).
\end{aligned}
$$

Since the random var $\xi = -W_T/\sqrt{t}$ has a standard Gaussian law $N(0,1)$ under the martingale measure $\mathbb{P}^*$ Note that

$$
\begin{aligned}
J_1 &= \mathbb{E}^{\mathbb{P}^*}(S_T B_T^{-1} I_D) && (6.24)\\
&= \mathbb{E}^{\mathbb{P}^*}(S_T^* I_D). && (6.25)
\end{aligned}
$$

We introduce an auxillary probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_T)$ by setting

$$\frac{\mathrm{d}\bar{\mathbb{P}}}{\mathrm{d}\mathbb{P}^*} = e^{\sigma W_T^* - \frac{1}{2}\sigma^2 T}.$$

The process $\bar{W}_t = \tilde{W}_t^* - \sigma t$ follows a standard Brownian motion on the space $(\Omega, \mathbb{F}, \bar{\mathbb{P}})$ we obtain

$$S_T^* = S_t e^{\sigma \bar{W}_T + \frac{1}{2}\sigma^2 T}. \tag{6.26}$$

Combining (6.24) with (6.26) we find that

$$
\begin{aligned}
J_1 &= S\bar{\mathbb{P}}\{S_T^* > K B_T^{-1}\} \\
&= S\bar{\mathbb{P}}\{S_t e^{\sigma \bar{W}_T + \frac{\sigma^2}{2}T} > K e^{-rT}\} \\
&= S\bar{\mathbb{P}}\{-\sigma \bar{W}_T < \ln(\tfrac{S_t}{K}) + (r + \tfrac{1}{2}\sigma^2)T\} \\
&= S N(d_1(S_t, T)).
\end{aligned}
$$

At time t the price of a call option is

$$C = c(S_t, T) = S_t N(d_1(S_t, T)) - K e^{-rT} N(d_2(S_t, T))$$

$$d_1(S_t, T) = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2(S_t, T) = \frac{\ln(S_t/K) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

The above method can also be used to derive the pricing formulas for options paying dividend, foreign options and future options.

The Put-Call parity

Denote the put value by $p : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$. The put-call parity can be used to derive a closed expression for the arbitrage price of a European put option

$$P(S, T) = Ke^{-rT}N(-d_2(S, T)) - SN(-d_1(S, T))$$

Black-Scholes's PDE

Lemma 3 Let W be the one dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. For a Borel measurable $h : \mathbb{R} \to \mathbb{R}$, we define the function $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ by setting

$$u(x, t) = \mathbb{E}^{\mathbb{P}}(e^{-r(T-t)})h(W_T) \mid W_T = x), \forall (x, t) \in \mathbb{R} \times [0, T]. \tag{6.27}$$

Suppose that

$$\int_{\infty}^{+\infty} e^{-ax^2} \mid h(x) \mid \mathrm{d}x < \infty$$

for some $a > 0$. Then the funcion u is defined for $0 < T - t < \frac{1}{2a}$ and $x \in \mathbb{R}$, and has derivative of all orders. In particular, it belongs to the class

$$-\frac{\partial u}{\partial t}(x, t) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x, t) - ru(x, t), \forall (x, t) \in \mathbb{R} \times [0, T]$$

with the teminal condition $u(x, T) = h(x), \forall (x, t) \in \mathbb{R}$.

Proof From the fundamental properties of the Brownian motion, it is clear that u is given by the expression

$$u(x,t) = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} e^{-r(T-t)} h(y) e^{-\frac{(y-x)^2}{2(T-t)}} \mathrm{d}y.$$

Corollary 2 If $g : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function, such that the random variable $X = g(S_T)$ is intergrable under $\mathbb{P}^*$. Then the arbitrage price in the Black-Scholes market of the claim X which settles at time T is given by the equality $\pi_t(X) = v(S_t, t)$, where the function $v : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ solves the Black-Scholes partial differential equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0, \forall (s,t) \in (0,\infty) \times (0,T) \tag{6.28}$$

subject to the terminal condition $v(s,T) = g(s)$.

Proof Here we derive (6.28) from the risk-neutral valuation formula. As in the proof of theorem 1, we find that the price $\pi_t(X)$ satisfies

$$\pi_t(X) = \mathbb{E}^{\mathbb{P}^*}(e^{-r(T-t)} g(S_T) \mid \mathcal{F}_t^S) = v(S_t, t), \text{ for some function } v : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$$

and

$$\pi_t(X) = \mathbb{E}^{\mathbb{P}^*}(e^{-r(T-t)} g(f(W_T^*, T)) \mid \mathcal{F}_t^{W^*}) = v(S_t, t), \tag{6.29}$$

where a strictly positive function $f : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ is given by the formula

$$f(x,t) = S_0 e^{\sigma x + (r - \frac{\sigma^2}{2} t)} \forall x \in \mathbb{R}. \tag{6.30}$$

Denote

$$u(x,t) = \mathbb{E}^{\mathbb{P}^*}(e^{-r(T-t)} g(f(W_T^*, T)) \mid \tilde{W}_t^* = x).$$

By lemma 3, the function $u(x,t)$ satisfies

$$-\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x,t) - ru(x,t), \forall (x,t) \in \mathbb{R} \times [0,T] \tag{6.31}$$

subject to the terminal function $u(x,T) = g(f(x,T)), \forall (x,t) \in \mathbb{R} \times [0,T]$ we obtain the following relationship between $u(x,t)$ and $v(s,t)$ if we compare (6.27) and (6.29) we have

$$u(x,t) = v(f(x,t),t), \forall (x,t) \in \mathbb{R} \times [0,T].$$

Therefore

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial v}{\partial s}(s,t) f_t(x,t) + \frac{\partial v}{\partial t}(s,t)$$

where $s = f(x,t)$ so that $s \in (0, +\infty)$.

$$\frac{\partial u}{\partial x}(x,t) = \frac{\partial v}{\partial s}(s,t) \frac{\partial f}{\partial x}(x,t)$$

and thus

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 v}{\partial s^2}(s,t)(\frac{\partial f}{\partial x})^2(x,t) + \frac{\partial v}{\partial s}(s,t) \frac{\partial^2 f}{\partial x^2}(x,t).$$

It follows from (6.30), that

$$\frac{\partial f}{\partial x}(x,t) = \sigma f(t,x), \frac{\partial^2 f}{\partial x^2}(x,t) = \sigma^2 f(t,x) \tag{6.32}$$

and

$$\frac{\partial f}{\partial t}(x,t) = (r - \frac{1}{2}\sigma^2) f(x,t). \tag{6.33}$$

We conclude that

$$\frac{\partial u}{\partial t}(x,t) = s(r - \frac{1}{2}\sigma^2)\frac{\partial v}{\partial s}(s,t) + \frac{\partial v}{\partial t}(s,t)$$

and

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \sigma^2 s^2 \frac{\partial^2 v}{\partial x^2}(s,t) + \sigma^2 s \frac{\partial v}{\partial s}(s,t)$$

substituting into (6.31) gives

$$s(-r + \frac{1}{2}\sigma^2)\frac{\partial v}{\partial s}(s,t) - \frac{\partial v}{\partial t}(s,t) = \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2}(s,t) + \frac{1}{2}\sigma^2 s \frac{\partial v}{\partial s}(s,t) - rv(s,t)$$

then we get the Black-Scholes partial differential equation (6.28).

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0.$$

1.7 Vanilla option under stochastic volatility

1.7.1 Single-factor stochastic volatility

We introduce the class of the stochastic volatility models, in which the volatility $\sigma_t = f(Y_t)$ is driven by an ergodic process Y_t approaching its unique invariant distribution at exponential rate α. The size of the exponential rate captures clustering effects. The function f is assumed to be sufficiently regular, positive, bounded and bounded away from zero. In particular, we are interested in asymptotic approximation of the price when α is large, which describes bursty volatility. In the family of mean-reverting stochastic volatility model (S_t, Y_t) where S_t is the underlying price, consider for an example Y_t evolves as a one-factor (single-factor) Ornstein-Uhlenbeck (OU) process, as a prototype of an ergodic diffusion. Under the physical probability measure $\mathbb{P}$, the model can be written as

$$\begin{aligned} \mathrm{d}S_t &= rS_t\mathrm{d}t + \sigma_t S_t \mathrm{d}W_t, \\ \sigma_t &= f(Y_t) \\ \mathrm{d}Y_t &= [\alpha(m - Y_t) - \tfrac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(Y_t)]\mathrm{d}t + \beta(\rho \mathrm{d}W_t^* + \sqrt{1-\rho^2}\mathrm{d}Z_t). \end{aligned} \tag{7.34}$$

The stock price S_t evolves as a diffusion with a constant μ in the drift and the random process σ_t in the volatility. The driving volatility Y_t evolves with a mean m, a rate of mean reversion $\alpha > 0$ and the volatility of the volatility β, and independent standard Brownian motions W_t^* and Z_t. Where $\rho \in (-1, 1)$ the instant correlation which captures the leverage effect. Moreover take Y_t to be a diffusion process that allows one to model the asymmetry of return distributions by incorporating a negative correlations ρ.

We introduce a small parameter ϵ such that the rate of mean reversion define by $1/\epsilon$ become large. In the OU case the variance ν^2 is given by $\frac{\beta^2}{(2\alpha)}$ to be a fixed $\mathcal{O}(1)$

constant, which in terms of ϵ implies

$$\beta = \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \tag{7.35}$$
$$\alpha = \frac{1}{\epsilon}.$$

The stochastic volatility OU can be written, under the risk-neutral probability P^*, in terms of the small parameter ϵ. Using the Girsanov's theorem the model under P^* can be written as

$$\mathrm{d}S_t^\epsilon = rS_t^\epsilon \mathrm{d}t + f(Y_t^\epsilon)S_t^\epsilon \mathrm{d}W_t^*,$$

Since the mean reversion of the volatility depends on ϵ, we denote S_t and Y_t by S_t^ϵ and Y_t^ϵ respectively.

$$\mathrm{d}Y_t^\epsilon = [\frac{1}{\epsilon}(m - Y_t^\epsilon) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(Y_t^\epsilon)]\mathrm{d}t + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}(\rho \mathrm{d}W_t^* + \sqrt{1-\rho^2}\mathrm{d}Z_t^*) \tag{7.36}$$

where $(S_t^\epsilon, Y_t^\epsilon)$ indicate explicitly the dependence upon ϵ, and the function Λ is given by

$$\Lambda(y) = \frac{\rho(u-r)}{f(y)} + \gamma(y)\sqrt{1-\rho^2}.$$

Assume that the market price of volatility risk γ is a bounded function of y alone . Also

$$\hat{Z}_t^* = \rho W_t^* + \sqrt{1-\rho^2}Z_t^*$$

with $\rho < 1$, where W^* and Z^* are two independent standard Brownian motion under P^*

Consider a non-negative payoff function $h(s)$ of a European derivative at maturity time T. We denote the price by $P^\epsilon(t,s,y)$ at time $t < T$ of the derivative is a function of the stock's present value $S_t^\epsilon = s$ and the present value $Y_t^\epsilon = y$ of the process driving the volatility.

$$P^\epsilon(t,s,y) = \mathbb{E}^{*(\gamma)}(e^{-r(T-t)}h(X_T^\epsilon) \mid S_t^\epsilon = s, Y_t^\epsilon = y)$$

If we substitute the rescaled parameters α and β in (7.35) we obtain the following partial differential equation

$$\frac{\partial P^{\epsilon}}{\partial t} + \frac{1}{2}f(y)^2 s^2 \frac{\partial^2 P^{\epsilon}}{\partial s^2} + \frac{\rho\nu\sqrt{2}}{\sqrt{\epsilon}} s f(y) \frac{\partial^2 P^{\epsilon}}{\partial s \partial y} + \frac{\nu^2}{\epsilon}\frac{\partial^2 P^{\epsilon}}{\partial y^2}$$
$$+ r(s\frac{\partial P^{\epsilon}}{\partial s} - P^{\epsilon}) + (\frac{1}{\epsilon}(m-y) - \beta\Lambda(y))\frac{\partial P^{\epsilon}}{\partial y} = 0 \tag{7.37}$$

$t < T$ with the terminal condition

$$P^{\epsilon}(T, s, y) = h(s).$$

Note that the price of the derivative depends on the current level of volatility, which is not directly observable, and on the market price of volatility risk which does not appear in the history of the stock price.

The above partial differential equation involves terms of order, $\frac{1}{\epsilon}$, $\frac{1}{\sqrt{\epsilon}}$ and 1. In order to account for these three different orders, we introduce the following convenient operator notation.

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}$$
$$\mathcal{L}_1 = \rho\nu\sqrt{2} s f(y) \frac{\partial^2}{\partial s \partial y} + \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}$$
$$\mathcal{L}_2 = L_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2 s^2 \frac{\partial^2}{\partial s^2} + r(s\frac{\partial}{\partial s} - \cdot) \tag{7.38}$$

where

αL_0 is the infinitesimal generator of OU process Y.

$\mathcal{L}_1$ contains the mixed partial derivative due to the correlation ρ between the two Brownian motions W^* and $\hat{Z}^*$ (it also contains the first order derivative with respect to γ due to the market prices of risk); and

$\mathcal{L}_2$ is the Black-Scholes operator at the volatility level $f(y)$, also denoted by $L_{BS}(f(y))$

Then the pricing partial differential equation (7.37) with the notation becomes

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2)P^{\epsilon} = 0.$$

The Vanilla European call option Take
$H(s) = (s-K)^+$

$$C_{BS}(t, s, \bar{\sigma}(z)) = sN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln(s/K) + (r + \frac{1}{2}\bar{\sigma}^2)(T-t)}{\bar{\sigma}\sqrt{T-t}}$$
$$d_2 = d_1 - \bar{\sigma}\sqrt{T-t} \tag{7.39}$$

$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{\frac{-u^2}{2}}\mathrm{d}u.$
The accuracy of the corrected Black-Scholes price

$$|P^{\epsilon}(t, s, y) - (P_0(t, s) + \tilde{P}_1(t, s))| = \mathcal{O}(\epsilon|\ln\epsilon|)$$

the first order term $P_0(t, x)$ solves the homogenized Black-Scholes PDE with the terminal condition $P_0(T, s) = h(s-K)$. The correction $\tilde{P}_1(t, s)$ satisfies

$$\frac{\partial\tilde{P}_1}{\partial t} + \frac{1}{2}\bar{\sigma}^2 s^2\frac{\partial^2\tilde{P}_1}{\partial s^2} + r(s\frac{\partial\tilde{P}_1}{\partial s} - \tilde{P}_1) = (V_2 s^2\frac{\partial^2 P_0}{\partial s^2} + V_3 s^3\frac{\partial^3 P_0}{\partial s^3}) \tag{7.40}$$

with the zero terminal condition $\tilde{P}_1(T, s) = 0$.

At this stage it is convenient to introduce notation for the first (small) correction,

$$\tilde{P}_1(t, s) = \sqrt{\epsilon}P_1(t, s) \tag{7.41}$$

which is the solution of

$$\mathcal{L}_2(\bar{\sigma})\tilde{P}_1(t, s) \tag{7.42}$$

The correction price is given explicitly by

$$P_0 - (T-t)(V_2 s^2\frac{\partial^2 P_0}{\partial s^2} + V_3 s^3\frac{\partial^3 P_0}{\partial s^3}) \tag{7.43}$$

where P_0 is the Black-Scholes price with constant volatility $\bar{\sigma}$. The parameters V_2 and V_3 are small quantities of order $\sqrt{\epsilon}$, given by

$$V_2 = \frac{\nu\sqrt{\epsilon}}{\sqrt{2}}(2\rho\langle f(y)\phi'(y)\rangle - (\langle\Lambda(y)\phi'(y)\rangle\rangle)$$

$$V_3 = \frac{\rho\nu\sqrt{\epsilon}}{\sqrt{2}}\langle f(y)\phi'(y)\rangle$$

$\langle\cdot\rangle$ denotes the averaging with respect to the invariant distribution $N(m,\nu^2)$ of the OU process Y_t

$$\langle g\rangle = \frac{1}{\nu\sqrt{2\pi}}\int g(y)e^{\frac{-(m-y)^2}{2\nu^2}}\mathrm{d}y.$$

The effective constant volatility $\bar{\sigma}$ is defined by

$$\bar{\sigma}^2 = \langle f^2\rangle. \tag{7.44}$$

and the function $\phi(y)$ is a solution of the Poisson equation

$$\nu^2\frac{\partial^2\phi}{\partial y^2} + (m-y)\frac{\partial\phi}{\partial y} = f(y)^2 - \langle f^2\rangle.$$

The implied volatility I^ϵ of a European call option with mean-reverting stochastic volatility can be written as

$$I^\epsilon = a\frac{\ln(K/s)}{T-t} + b + \mathcal{O}(\sqrt{\epsilon}) \tag{7.45}$$

$$a = \frac{V_3}{\bar{\sigma}^3}$$

$$b = \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}(r + \frac{3}{2}\bar{\sigma}^2) - \frac{V_2}{\bar{\sigma}}$$

a and b parameters are estimated as the slope and intercept of the line fit of the observed implied volatilities plotted as the function of logmoneyness-to-maturity-ratio (LMMR). The parameters V_2 and V_3 calibrated from a and b term structure of the implied volatility surface, are given by

$$\begin{aligned} V_2 &= \bar{\sigma}((\bar{\sigma}-b) - a(r+3/2\bar{\sigma}^2)) \\ V_3 &= -a\bar{\sigma}^3. \end{aligned}$$

1.7.2 Multiscale Stochastic Volatility Models

Consider a family of two-factor stochastic volatility models (S_t, Y_t, Z_t), where S_t is the underlying price, Y_t evolves as an OU process, as a prototype of an ergodic diffusion, and Z_t follows another diffusion process. Then the model can be written as the stochastic volatility OU, under the risk-neutral probability $\mathbb{P}^*$, as follows:

$$\begin{aligned} \mathrm{d}S_t &= rS_t\mathrm{d}t + \sigma_t S_t \mathrm{d}W_t^*, \\ \sigma_t &= f(Y_t, Z_t) \end{aligned}$$

$$\begin{aligned} \mathrm{d}Y_t &= [\alpha(m_f - Y_t) - \nu_f\sqrt{2\alpha}\Lambda(Y_t, Z_t)]\mathrm{d}t + \nu_f\sqrt{2\alpha}(\rho \mathrm{d}W_t^{(0)} + \sqrt{1-\rho_1^2}\mathrm{d}W_t^{(1)}) \\ \mathrm{d}Z_t &= [\delta(m_s - Z_t) - \nu_s\sqrt{2\delta}\Gamma(Y_t, Z_t)]\mathrm{d}t + \nu_s\sqrt{2\delta}(\rho_2 \mathrm{d}W_t^{(0)} \\ &\quad +\rho_{12}\mathrm{d}W_t^{(1)} + \sqrt{1-\rho_2^2-\rho_{12}^2}\mathrm{d}W_t^{(2)}) \end{aligned}$$

where $(W_t^{(0)}, W_t^{(1)}, W_t^{(2)})$ are independent standard Brownian motions, and the instant correlation coefficients ρ_1, ρ_2 and ρ_{12}, satisfies $\rho_1^2 < 1, \rho_1^2 + \rho_{12}^2 < 1$ respectively. Under risk-neutral, the stock price S_t has a constant rate of return equal to constant risk-free interest rate r. The random volatility σ_t depends on the two volatility factors Y_t and Z_t, and the function Λ and Γ are given by

$$\begin{aligned} \Lambda(t, s, y) &= \frac{\rho_1(u-r)}{f(y,z)} + \gamma(t,s,y)\sqrt{1-\rho_1^2} \\ \Gamma(t, s, y) &= \frac{\rho_2(u-r)}{f(y,z)} + \gamma(t,s,y)\rho_{12} + \epsilon(s,y,z)\sqrt{1-\rho_2^2-\rho_{12}}. \end{aligned}$$

The risk neutral probability measure $\mathbb{P}^*$ is determined by the combined market prices of volatility risk Λ_f and Λ_s which we assume to be bounded and independent of the stock price S. The joint process (S_t, Y_t, Z_t) is Markovian. Without Λ_f and Λ_s the driving volatility process Y_t and Z_t is mean reverting around its long run mean m_f and m_s, with a rate of mean reversion $\alpha > 0$ and $\delta > 0$ or at a time scale $\frac{1}{\alpha}$ and $\frac{1}{\delta}$, and 'volatilities of volatilities ' $\nu_f\sqrt{2\alpha}$ and $\nu_s\sqrt{2\delta}$, corresponding to a long run standard deviation ν_f and ν_s. Write OU processes with long run distributions $N(m_f, \nu_f^2)$ and $N(m_s, \nu_s^2)$ as prototype of more general ergodic diffusions. The volatility function $f(y,z)$ is assumed to be smooth in z, bounded and bounded away from 0 , $(0 \leq c_1 \leq f \leq c_2)$. The two stochastic volatility factors Y_t and Z_t are differentiated by their intrinsic time scales. Y_t is fast mean reverting on short time scale $\frac{1}{\alpha}$ and Z_t is slowly varying on long time scale $\frac{1}{\delta}$, that is $\frac{1}{\alpha} < 1 < \frac{1}{\delta}, \alpha \to \infty, \delta \to 0$.

Assume that the risk-free interest rate r is constant and that the market price of volatility risks $\gamma(y,z)$ and $\epsilon(y,z)$ are bounded function of y and z.

The payoff of a European option is a function $H(S_T)$ of the stock price at maturity T. By Markov property, the no-arbitrage price of this option is obtained as the conditional expectation of the discounted payoff given the current stock price and driving volatility levels

$$P(t,x,y,z) = \mathbb{E}^*\{e^{-r(T-t)}H(S_T)|S_t = s, Y_t = y, Z_t = z\}. \tag{7.46}$$

Let $P^{\epsilon,\delta}$ be the price of a European option which solves a three-dimensional PDE equation

$$\mathcal{L}^{\epsilon,\delta}P^{\epsilon,\delta} = 0$$
$$P^{\epsilon,\delta}(T,s,y,z) = h(s)$$

$\epsilon = \frac{1}{\alpha}$, ϵ and α are relative small $0 < \epsilon, \alpha << 1$.
where the partial differential operator $L^{\epsilon,\delta}$ is defined by

$$\mathcal{L}^{\epsilon,\delta} = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3$$

where the component operator are given by

$$\mathcal{L}_0 = \nu_f^2\frac{\partial^2}{\partial y^2} + (m_f - y)\frac{\partial}{\partial y}$$

$$\mathcal{L}_1 = \nu_f\sqrt{2}(\rho_1 s f(y,z)\frac{\partial^2}{\partial x \partial y} - \Lambda_f(y,z))\frac{\partial}{\partial y}$$

$$
\begin{aligned}
\mathcal{L}_2(f(y,z)) &= \frac{\partial}{\partial t} + \frac{1}{2} f(y,z)^2 s^2 \frac{\partial^2}{\partial s^2} + r(s\frac{\partial}{\partial s} - .) \\
\mathcal{M}_1 &= \nu_s \sqrt{2}(-\Gamma_s(y,z)\frac{\partial}{\partial z} + \rho_2 f(y,z) s \frac{\partial^2}{\partial s \partial z}) \\
\mathcal{M}_2 &= (m_s - z)\frac{\partial}{\partial z} + \nu_s^2 \frac{\partial^2}{\partial z^2} \\
\mathcal{M}_3 &= 2\nu_s \nu_f (\rho_1 \rho_2 + \rho_{12}\sqrt{1-\rho_1{}^2})\frac{\partial^2}{\partial y \partial z}.
\end{aligned}
\tag{7.47}
$$

A singular-regular perturbation technique was used by Fouque et al in [14], to derive an explicit formula for the price approximation

$$P^{\epsilon,\delta}(t,s,y,z) \approx \tilde{P}(t,s,z)$$

where

$$P^{\epsilon,\delta}(t,s,y,z) \approx P_0(t,s,z) - \frac{2}{\bar{\sigma}(z)}[V_0^\delta \frac{\partial}{\partial \sigma} + V_1^\delta s \frac{\partial^2}{\partial s \partial \sigma}] + [V_2^\varepsilon s^2 \frac{\partial^2}{\partial s^2} + V_3^\epsilon s^3 \frac{\partial^3}{\partial s^3}] \tag{7.48}$$

and the relevant parameters are defined by

$$
\begin{aligned}
V_2^\epsilon &= \tfrac{\nu\sqrt{\epsilon}}{\sqrt{2}}\langle \Lambda(y,z)\tfrac{\partial \phi(y,z)}{\partial y}\rangle) \\
V_3^\epsilon &= \tfrac{\rho_1 \nu \sqrt{\epsilon}}{\sqrt{2}}\langle f(y,z)\tfrac{\partial \phi(y,z)}{\partial y}\rangle \\
V_0^\delta &= -\tfrac{\nu_s \sqrt{\delta}}{\sqrt{2}}\langle \Gamma \tfrac{\partial \phi(y,z)}{\partial y}\rangle \\
V_1^\delta &= \tfrac{\rho_2 \sqrt{\delta}}{\sqrt{2}}\langle f \tfrac{\partial \phi(y,z)}{\partial y}\rangle.
\end{aligned}
$$

The effective volatility $\bar{\sigma}$ which is a function of the slow factor, is defined by

$$\bar{\sigma}(z) = \langle f^2(\cdot, z)\rangle.$$

The function $\phi(y,z)$ is a solution of the Poisson equation

$$\mathcal{L}_0 \phi(y,z) = f^2(y,z) - \bar{\sigma}^2(z).$$

The leading order price $P_0(t,s,z)$ solves

$$\mathcal{L}_2(\bar{\sigma}(z))P_0(t,s,z) = 0$$

with terminal condition

$$P_0(T,s,z) = h(s).$$

Also with the multi-factor stochastic volatility, the parameters can be calibrated from the implied volatility surface.

The implied volatility $I^{\epsilon,\delta}$ of a European option price is approximated by

$$I^{\epsilon,\delta} \approx \bar{\sigma} + [a^{\epsilon} + a^{\delta}(T-t)]\frac{\ln(K/s)}{T-t} + [b^{\epsilon} + b^{\delta}(T-t)] \tag{7.49}$$

where the z-dependent parameters are defined by

$$\begin{aligned} a^{\epsilon} &= \tfrac{-V_3^{\epsilon}}{\bar{\sigma}^3} \\ a^{\delta} &= \tfrac{-V_1^{\delta}}{\bar{\sigma}^3} \\ b^{\epsilon} &= \bar{\sigma} + \tfrac{V_3^{\epsilon}}{\bar{\sigma}^3}(r + \tfrac{3}{2}\bar{\sigma}^2) - \tfrac{V_3^{\epsilon}}{\bar{\sigma}} \\ b^{\delta} &= -\tfrac{V_0^{\delta}}{\bar{\sigma}} + \tfrac{V_0^{\delta}}{\bar{\sigma}^3}(r - \tfrac{\bar{\delta}^2}{2}). \end{aligned}$$

Therefore the calibration formulas deduced are

$$\begin{aligned} V_0^{\delta}/\bar{\sigma} &= (-b^{\delta} + a^{\delta}(r - \tfrac{\bar{\sigma}^2}{2})) \\ V_1^{\delta}/\bar{\sigma} &= -a^{\delta}\bar{\sigma}^2 \\ V_2^{\epsilon} &= -\bar{\sigma}(b^{\epsilon} + a^{\epsilon}(r - \tfrac{\bar{\sigma}^2}{2})) \\ V_3^{\epsilon} &= -a^{\epsilon}\bar{\sigma}^3. \end{aligned}$$

[18] have shown that the two-factor models and the perturbation method give an excellent fit across strikes and over a wide range of maturities.

Chapter 2

Asian option

An Asian option, also known as average option, is an example of the exotic option. There are two types of options, vanilla and exotic option. Vanilla option is a simple or well understood option, whereas an exotic option is more complex, or less easily understood, inherent risks are more difficult to identify and hence hedging strategy may be complicated. They are largely the over the counter (OTC) instruments or off-exchange market for example, in the interbank transactions and therefore there is no much public data available on details of specific contracts and prices. Asian option are averaging option where the terminal payoffs depend on some form of averaging of the price of the underlying asset over a part or the whole of the life of the option. The pressure and the absence of patent for financial products push financial institutions to design and develop more innovative risk management financial derivatives, many of which are tended towards a specific needs of customers. There has been a growing popularity for path dependent options, so called since their payouts are related to movement in the price of the underlying asset during the whole or part of the life of the option. Asian option can either be of European-type and of

American-type. But most Asian options are of European-type since Asian option of American-type may be redeemed as early as the start of the averaging period and lose the intent of protection from averaging. They were originally used in 1987 when Banker's Trust Tokyo office used them for pricing average options on crude oil contracts, and hence named them Asian because they were in Asia. The reason why Asian options are popular in the market place, is that a company's exposure to future price movement is sometimes naturally expressed as exposure to the average of prices in the future. They are less sensitive to the movements in the underlying assets price when the option's life is close to maturity. Some accounting standards may require the translation of foreign currency assets or liabilities at an average of exchange rates over the accounting period. Furthermore, average option are common place into the currency and energy markets, where a firm that is susceptible to asset price fluctuations could use average options to hedge or speculate on the average of asset prices over a specific time interval, rather than say, the price at the end of period. They are a cheap way to hedge, since they have low volatility. They are commonly traded on currencies and commodity products which have low trading volumes. Average strike options can guarantee that the average price paid for an asset in frequent trading over a period of time is not greater than the final price. Alternatively guarantee that the average price paid for an asset in frequent trading over a period of time is not less than the final price. Methods of Asian options which will be used to derive the Asian option in this chapter

- The PDE approach which usually involves a finite difference scheme.
- Numerical methods, binomial method and fractional step method.
- Monte Carlo simulations.
- Binomial Trees and lattice with various efficiency enhancements.
- Analytical approximation that produce close form-expression at the cost, however of making some over-simplifying assumptions, for example Levy, Turnbull and

Wakeman for arithmetic averages and Kemna and Vorst for geometric averages.

Here we are going show how to derive the governing differential equation that governs the value of an Asian option if the average is a continuous geometric and continuous arithmetic. But, arithmetic Asian options are difficult to price and hedge, since at present there are no closed form for arithmetic averages due to the inappropriate use of lognormal assumption under this form of averaging, a number of approximations have emerged in literature due to the property of these options under which the lognormal assumptions collapse, numerical solution is needed to price them. There are two main classes of Asian options, fixed strike options and the floating strike options. These classes can be either of a Geometric or Arithmetic averaging. The analytical formula for Geometric averages exist, but there are are no analytical formula for Arithmetic averages. This is unfortunate since, almost all over the counter path-dependent contracts are based on Arithmetic rather than Geometric. Another example of numerical techniques for average options employs the idea that an arithmetic average can be approximated by a geometric average with an appropriately adjusted mean and variance.

If Geometric Brownian motion is assumed for the underlying asset price, the analytical derivation of the price formula of a European Asian option with geometric averaging is feasible since the product of lognormal prices remains lognormal. The geometric average is less or equal to the arithmetic average on a set of numbers. These technique tend to overprice the value of an arithmetic Asian option. Using the put-call can underprice the value of arithmetic averages.

Geometric averaging options can be priced via a closed form analytical solution because of the reason that the geometric average of the underlying prices follows a lognormal distribution as well, whereas under average rate options, this condition collapses.

The common averaging procedures are the discrete arithmetic averaging defined by

$$A = \frac{1}{n}\sum_{i=1}^{n} S(t_i)$$

and the discrete geometric averaging is

$$A = \left[\prod_{i=1}^{n} S(t_i)\right]^{\frac{1}{n}}$$

$S(t_i)$ is the asset price at discrete time t_i. In the limit n, the discrete sampled becomes the continuous sampled averages. The continuous arithmetic average is given by

$$A = \frac{1}{T_2 - T_1}\int_0^T S(t_i)\mathrm{d}t$$

while the continuous geometric average is defined to be

$$A = \exp\left(\frac{1}{T_2 - T_1}\int_0^T \ln S(t_i)\mathrm{d}t\right)$$

where $[T_1, T_2]$ is the interval within which the averaging is taken. The terminal payoff functions of an average value are:
fixed strike call option

$$\max(0, A - K)$$

floating strike put option

$$\max(0, K - A)$$

fixed strike call option

$$\max(0, S_T - A)$$

floating strike put option

$$\max(0, A - S_T)$$

where S_T is the asset price at expiry, K is the strike price. The fixed strike option is also called the rate option or the value option or a price option. And the floating strike option is called the strike option.

2.1 Differential equations

The derivation of the governing difference equation for arithmetic and geometric averages of Asian options. Consider the average of the asset price as

$$A = \int_0^t f(S,\tau)\mathrm{d}\tau \tag{1.1}$$

here $f = S$ if an average is of a continuous arithmetic, and $f = \ln S$ if an average is of continuous geometric. The Asian option is a function of time to expiry and two state variables, namely, S and A. By the Mean Value Theorem (MVT)

$$\mathrm{d}A = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} f(S,\tau)\mathrm{d}\tau = \lim_{\Delta t \to 0} f(S,\tilde{\tau})\mathrm{d}t = f(S,t)\mathrm{d}t \tag{1.2}$$

$t < \tilde{\tau} < t + \Delta t$.

$\mathrm{d}A$ is deterministic, hence a riskless hedge for the Asian option requires only eliminating the asset-induced risk. Consider a portfolio that contains one unit of Asian option and $-\Delta$ units of the underlying asset. We then choose Δ such that the stochastic components associated with the option and the underlying asset cancel off each other. Let $\mathrm{d}Z$ be the standard Weiner process, $q(S,t)$ be the dividend yield on the asset, μ be the expected rate of return and σ is the volatility of the asset price. Let the asset price be

$$\mathrm{d}S = (\mu S - q(S,t))\mathrm{d}t + \sigma S \mathrm{d}Z.$$

Let $V(S,A,t)$ be the value of the of the option and let Π denote the value of the portfolio. Then

$$\Pi = V(S,A,t) - \Delta S$$

and Ito's lemma gives

$$\mathrm{d}\Pi = \frac{\partial V}{\partial t}dt + f(S,t)\frac{\partial V}{\partial A}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\mathrm{d}t - \Delta\mathrm{d}S - \Delta q(S,t)\mathrm{d}t.$$

Choosing

$$\Delta = \frac{\partial V}{\partial S}.$$

In the absence of arbitrage

$$\mathrm{d}\Pi = r\Pi\mathrm{d}t$$

r is the riskless interest rate. By equating the two equation we get

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial V^2}{\partial S^2} + (rS - q(S,t))\frac{\partial V}{\partial S} + f(S,t)\frac{\partial V}{\partial A} - rV = 0. \tag{1.3}$$

This is the governing differential equation for $V(S, A, t)$. The specification of the auxillary conditions depend on the specific details of the Asian contract. The partial differential equation for a arithmetic option is

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial V^2}{\partial S^2} + (rS - q(S,t))\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial A} - rV = 0. \tag{1.4}$$

The partial differential equation for a geometric option is

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial V^2}{\partial S^2} + (rS - q(S,t))\frac{\partial V}{\partial S} + \ln S\frac{\partial V}{\partial A} - rV = 0. \tag{1.5}$$

The value of an Asian option is given by the following PDE in two-dimensions in terms of the running sum (I), the solution to this equation is represented as

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial V^2}{\partial S^2} + (rS - q(S,t))\frac{\partial V}{\partial S} + f(S,t)\frac{\partial V}{\partial I} - rV = 0 \tag{1.6}$$

where

$$V(S, A, t) = V(S, I, t).$$

Firstly we will look at the derivation of the fixed strike options. Note that we consider only the case of continuous average value.

2.2 Fixed strike Asian option of a continuous arithmetic averaging

Consider a European call option with terminal payoff function given by

$$V(S, A(T), T) = \max(A(T) - K, 0). \tag{2.7}$$

The average price is given by

$$A = \frac{1}{T_2 - T_1} \int_{T_0}^{t} S(\tau) \mathrm{d}\tau, 0 < T_0 < T, t > T_0$$

$t \in [T_0, T]$, $A(t)$ is a true average when $t = T$. Assume that there are dividend payment on the asset. The governing equation for the Asian option within (T, T_0) is

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} + \frac{1}{T - T_0} S \frac{\partial V}{\partial A} - rV = 0 \tag{2.8}$$

$S > 0, A > 0, t \in (T_0, T)$.

The terminal payoff function is given by equation (2.7). When $A(t) \geq K$ the exact analytic solution exists, and the terminal payoff is guaranteed to be positive, since $A(t)$ is an increasing function of time. The terminal payoff can be expressed as

$$\begin{aligned} \frac{1}{T - T_0} \int_{T_0}^{T} S(\tau) \mathrm{d}\tau &= \frac{1}{T - T_0} \int_{T_0}^{t} S(\tau) \mathrm{d}\tau - K + \frac{1}{T - T_0} \int_{t}^{T} S(\tau) \mathrm{d}\tau \\ &= A(t) - K + \frac{1}{T - T_0} \int_{T_0}^{T} S(\tau) \mathrm{d}\tau. \end{aligned} \tag{2.9}$$

We can obtain the above payoff by using the self-financing replicating the portfolio strategy. Initially an investor can invest $(A(t) - K)e^{-r(T-t)}$ dollars into a riskless bonds so that the amount of $A(t) - K$ is secured at $t = T$. To secure the returns promised by the second term, the investor must transfer $\frac{1}{T - T_0} \int_t^T e^{-r(T-t)} \Delta\tau$ asset to a riskless bond everytime interval $(\tau, \tau + \Delta\tau)$ lapses. The total number of units of asset required from t to T for such transfer is

$$\frac{1}{T - T_0} \int_t^T e^{-r(T-t)} \mathrm{d}\tau = \frac{1 - e^{-r(T-t)}}{r(T - T_0)}. \tag{2.10}$$

By the no-arbitrage principle, the value of the option is equal to the value of the replicating portfolio. The option value for $A(t) \geq K$ is given by (Kemma & Vorst 1990, German and Yor 1993) is

$$V(S, A, t) = (A(t) - K)e^{-r(T-t)} + \frac{1 - e^{-r(T-t)}}{r(T - T_0)} S, A(t) \geq K. \tag{2.11}$$

Note that σ does not appear explicitly in the formula, it appears implicity in S and A and gamma is zero and delta is a function of time only. For $A(t) \leq K$, the option value is governed by equation (2.8), but there is no closed form analytical solution available. The option value can be obtain by the numerical scheme called the finite difference method using the boundary condition.

$$V(0, A, t) = \max(e^{-r(T-t)}(A(T) - K), 0) \tag{2.12}$$

$$\lim_{S \to \infty} \frac{\partial V}{\partial S}(S, A, t) = \frac{T - t}{T - T_0} e^{-r(T-t)} \tag{2.13}$$

$$V(S, K, t) = \frac{1}{T - T_0} \frac{1 - e^{-r(T-t)}}{r} S.$$

The last boundary is obtained by setting $A = K$ in (2.11). Since equation (2.8) resembles a two-dimensional convection-diffussion equation but with the diffusion term missing in one of the spatial dimensions, we may encounter severe oscillations in the finite difference solution. Monte Carlo simulations can also be used as an alternative numerical approach. We will now look at the finite difference method and at the approximation methods by Levy, Turnbull & Wakeman and Monte Carlo method.

The finite difference method Consider a PDE equation (1.6) for arithmetic average

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial V^2}{\partial S^2} + (rS - q(S,t)) \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial I} - rV = 0. \tag{2.14}$$

To price Asian options, we must solve the above PDE, with appropriate final and boundary conditions. We are looking for a numerical solution of the form

$$V_{i,j}^k \approx V(k\Delta t, i\Delta S, j\Delta I).$$

The PDE can be solved backward using the finite difference scheme

$$\begin{aligned}
&\frac{V_{i,j}^{k+1} - V_{i,j}^k}{\Delta t} + (1-\theta)(\frac{\sigma^2}{2} S_i \frac{V_{i+1,j}^{k+1} - 2V_{i,j}^{k+1} + V_{i-1,j}^{k+1}}{\Delta S^2} + rS_i \frac{V_{i+1,j}^{k+1} - V_{i,j}^{k+1}}{\Delta S}) \\
&\quad +(1-\theta)(S_i \frac{-3V_{i,j}^{k+1} + 4V_{i,j+1}^{k+1} - V_{i,j+2}^{k+1}}{2\Delta I} - rV_{i,j}^{k+1}) \\
&\quad +\theta(\frac{\sigma^2}{2} S_i \frac{V_{i+1,j}^k - 2V_{i,j}^k + V_{i-1,j}^k}{\Delta S^2} + rS_i \frac{V_{i+1,j}^k - V_{i,j}^k}{\Delta S}) \\
&\quad +\theta(S_i \frac{-3V_{i,j}^k + 4V_{i,j+1}^k - V_{i,j+2}^k}{2\Delta I} - rV_{i,j}^k) = 0.
\end{aligned} \tag{2.15}$$

At each time level the 2-Dimensional problem can be decoupled into a series of 1D problems, by approximating V line by line, in the direction of a decreasing j index.

A reduced one-dimensional PDE derived by Rogers and Shi:

$$\frac{\partial H}{\partial t} + \frac{\sigma^2}{2} R^2 \frac{\partial^2 H}{\partial R^2} - (\frac{1}{T} + rR) \frac{\partial H}{\partial R} = 0 \tag{2.16}$$

$$\begin{aligned}
H(T,R) &= \max(-R,0), \text{for a fixed call option} \\
&= \max(R,0), \text{for a put option}
\end{aligned}$$

where

$$\begin{aligned}
R &= I/S \\
V(t,I,S) &= SH(t, K - \frac{tI}{T}/S).
\end{aligned}$$

Since this PDE depends on the maturity T, this cannot be applied to an American options. Only two-dimensional can be used to solve American fixed strike options.

2.2.1 Arithmetic Rate Approximation (Turnbull and Wakeman)

Turnbull & Wakeman (T&W)'s approximation makes use of the fact that the distribution under arithmetic averaging is approximately lognormal. It adjust the mean and the variance in order to be consistent with exact moments of the arithmetic averages. They put forward the first and second moments of the average in order to price the option. The mean and variance are used as inputs in the general Black Scholes formula. The value of a call option is given as

$$C = Se^{(b-r)T_2}N(d_1) - Ke^{-rT_2}N(d_2)$$

and the put option is

$$P = Ke^{-rT_2}N(-d_2) - Se^{(b-r)T_2}N(-d_1)$$

where $N(x)$ is the cumulative

$$d_1 = \frac{\ln(S/K) + (b + \frac{\sigma_A^2}{2})T_2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S/K) + (b - \frac{\sigma_A^2}{2})T_2}{\sigma\sqrt{T}} = d_1 - \sigma_A\sqrt{T} \tag{2.17}$$

where T_2 is the time remaining until the maturity. For averaging options which have already begun their averaging period, then T_2 is simply T the original time to maturity, if the averaging period has not yet begun, then T_2 is $T - \tau$.

$$\sigma_A = \sqrt{\frac{\ln(M_2)}{T} - 2b}.$$

The adjusted volatility and dividend yield are given as

$$b = \frac{\ln(M_1)}{T}.$$

Assume that the averaging period has not yet begun, the first and the second moments are given as

$$M_1 = \frac{e^{(r-q)}T - e^{(r-q)\tau}}{(r-q)(T-\tau)},$$

$$M_2 = \frac{2e^{(2(r-q)+\sigma^2)T}S^2}{(r-q+\sigma^2)(2r-2q+\sigma^2)T^2} + \frac{2S^2}{(r-q)T^2}\left[\frac{1}{2(r-q)+\sigma^2} - \frac{e^{(r-q)T}}{r-q+\sigma^2}\right].$$

We adjust the strike price if the averaging period has already begun as

$$K_A = \frac{T}{T_2}K - \frac{(T-T_2)}{T_2}S_A$$

where T is reiterated as the original time to maturity, T_2 as the remaining time to maturity, K as the original strike price and S_A is the average asset price. By Haug (1998) if $r = q$, the formula will not generate a solution.

2.2.2 Arithmetic Rate Approximation (Levy)

This analytical approximation is suggested to give more accurate result than Turnbull and Wakeman approximation. The analytical approximation to a call option is

$$C \approx S_z N(d_1) - K_z e^{-rT_2} N(d_2)$$

and the put option is

$$P \approx C - S_z + K_z e^{-rT_2}$$

where

$$d_1 = \frac{1}{\sqrt{K}}[\frac{\ln(L)}{2} - \ln(K_z)]$$

$$d_2 = d_1 - \sqrt{K} \tag{2.18}$$

and

$$S_z = \frac{S}{(r-q)T}(e^{-qT_2} - e^{-rT_2})$$

$$K_z = K - S_A \frac{T - T_2}{T}$$
$$K = \ln(L) - 2[rT_2 + \ln(S_Z)]$$
$$L = \frac{M}{T^2}$$
$$M = \frac{2S^2}{r - q - \sigma^2} \left[\frac{e^{(2(r-q)+\sigma^2)T_2 - 1}}{2(r-q)+\sigma^2} \right] - \frac{e^{(r-q)T_2} - 1}{r - q}.$$

The price of an Asian call under Turnbull and Wakeman was compared to that of Levy's approximation. Given the following input:

Asset Price= 100, Average Price= 95, q=5%, r=10%, V=15%, T=0, $T_1 = 1$, $T_2 = 0.5$ and K is the strike price.

TABLE 2.1

K	TW	Levy	Absolute Error
95	3.202859	3.199390	0.0034690
96	2.444752	2.440545	0.0042066
97	1.787605	1.782873	0.0047318
98	1.246971	1.242086	0.0048849
99	0.827130	0.822518	0.0046122
100	0.520494	0.516509	0.0039841
101	0.310270	0.307114	0.0031558
102	0.175088	0.172788	0.0022995
103	0.093529	0.091982	0.0015470
104	0.047316	0.046352	0.0009645
105	0.022689	0.022130	0.0005593

The values are said to be similar since, the absolute differences between these two approximations are very small.

2.2.3 Monte Carlo Simulation

Monte Carlo Simulation is convenient and flexible and is very useful to Asian options which are highly path dependent. It is applicable as long as the underlying follows a Markovian-diffusion. Various methods using Monte Carlo simulation have been developed to price Arithmetic Asian option. For example Levy, (Turnball and Wakeman) and Curran. In the next section we will see an approximation analytical solution for geometric closed form by Kemma and Vorst (1990). This solution has been shown by authors that it can be used as control variate within a Monte Carlo simulation framework. The control variate technique can be used to find more accurate analytical solution to a derivative price if there is a similar derivative with a known analytic solution. The arithmetic Asian option can be price using Monte Carlo simulation. It can give a relatively accurate prices for option values. If given the price of a geometric Asian, then one can price the arithmetic Asian option by the equation below.

$$V_A = E(\tilde{V}_A - \tilde{V}_B) + V_B \tag{2.19}$$

where $\tilde{V}_A$ is the estimated value of the arithmetic Asian through simulation, $\tilde{V}_B$ is the simulated value of the geometric Asian, and V_B is the exact value of the geometric Asian.

2.3 Floating strike Asian option with continuous arithmetic averaging

Consider a European call option with strike price K equal to the average asset price during its life $[0, T]$, that is

$$K = \frac{1}{T}\int_0^T S(\tau)\mathrm{d}\tau = \frac{A(T)}{T}$$

where

$$A(T) = \int_0^T S(\tau)\mathrm{d}\tau$$

the governing equation becomes

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial A} - rV = 0$$

when there are no dividend payments,

$$V(0, A, t) = 0$$

$$\lim_{S\to\infty}, \frac{\partial V}{\partial S} = 1$$

$$\lim_{A\to\infty} V(S, A, T) = 0, V(S, A, T) = \max\left\{S - \frac{A(T)}{T}, 0\right\},$$

$V(S, A, t) = AG(y, t)$, where $y = S/A$

$$\frac{\partial V}{\partial S} = \frac{\partial G}{\partial y}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{A}\frac{\partial^2 G}{\partial y^2}$$

$$\frac{\partial V}{\partial A} = G - y\frac{\partial G}{\partial y}$$

$$\frac{\partial V}{\partial t} = A\frac{\partial G}{\partial t}.$$

Then we have

$$\frac{\partial G}{\partial t} + \frac{\sigma^2}{2} y^2 \frac{\partial^2 G}{\partial y^2} + (ry - y^2)\frac{\partial G}{\partial y} + (y - r)G = 0 \tag{3.20}$$

$$G(0, t) = 0,$$

$$\frac{\partial G}{\partial y}(y, t) = 1,$$

as $y \to \infty$, $G(y, T) = \max(y - \frac{1}{T}, 0)$.

The above equation cannot be transformed into a constant co-efficient equation by lognormal transformation of the independent variable. A reduced one-dimensional equation by Rogers and Shi see [9], with the payoff at the maturity

$$V(I, S) = g(I, S),$$

where

$$R = \frac{I}{S}$$

$$V(t, I, S) = SH(t, R).$$

is

$$\frac{\partial H}{\partial t} + \frac{\sigma^2}{2} R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0. \tag{3.21}$$

Note that this PDE can also be applied to American option. German and Yor (1993) in [6] derived a quite complex closed form solution in terms of Bessel functions. Using the fact that a Geometric Brownian motion is a time-changed square Bessel process and the stability by additivity of this process they provide the Laplace transform of the Asian call option. Define

$$C = \frac{e^{-rt}}{t} \frac{4S}{\sigma^2} c(h, q) \tag{3.22}$$

the Laplace transform with respect to $h = \sigma^2 t/4$ of the function $c(h, q)$ is given by

$$C(\lambda, q) = \int_0^{+\infty} e^{-\lambda h} c(h, q) \mathrm{d}h = \frac{\int_0^{1/2q} e^{-x} x^{\frac{1}{2}(\mu - v) - 2} (1 - 2qx)^{\frac{1}{2}(\mu + v) + 1} \mathrm{d}x}{\lambda(\lambda - 2 - 2v)\Gamma(\frac{1}{2}(\mu - v) - 1)} \tag{3.23}$$

where

$$v = \frac{2r}{\sigma^2} - 1;\ q = \frac{\sigma^2 t}{4}\frac{K}{S};\ \mu = \sqrt{2\lambda + v^2}.$$

Bouaziz et al (1996) gave an analytical approximation formula for the option value in more general setting, where the averaging period may cover only part of the life span of the option close to the expiration date.

2.4 Fixed strike Asian option with continuous geometric averaging

Consider a European average value call option with continuous geometric averaging whose terminal payoff function at expiration time T is given by

$$\max\left(e^{\frac{I(T)}{T}} - K, 0\right)$$

where $I(t)$ is defined by

$$I(t) = \int_0^t \ln S(\tau) \mathrm{d}\tau, \quad 0 \le t \le T,$$

and K is the strike price. And the governing equation for the price of the above European call option, $C(S, I, t)$ is given by

$$\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + (r - q) S \frac{\partial C}{\partial S} + \ln S \frac{\partial C}{\partial I} + rC = 0.$$

σ and r are the volatility and interest rate, which may be time dependent. Using the following transformation of variable: with final condition

$$C(S, I, T) = \exp\left[\max\left(0, \frac{I}{\int_0^T p(\tau)\mathrm{d}\tau}\right) - K\right]$$

$$y = \frac{I + (T - t)\ln S}{T}$$

$$V(y,t) = C(S,I,t)$$

the governing equation is reduced to

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \left(\frac{T-t}{T}\right)^2 \frac{\partial^2 V}{\partial y^2} + \left(r - q - \frac{\sigma^2}{2}\right) \frac{T-t}{T}\frac{\partial V}{\partial y} - rV = 0 \qquad (4.24)$$

using the above equation

$$\frac{\partial y}{\partial S} = \frac{T-t}{T}\frac{1}{S}$$

and

$$\frac{\partial y}{\partial t} = \frac{-\ln S}{T}$$

$$\frac{\partial y}{\partial I} = \frac{1}{T}$$

Suppose we write the asset price dynamics as

$$W(S,I,t) = V(Y(S,I,t),t)$$

$$Y(S,I,t) = (y(S,I,t),t)$$

$$\begin{pmatrix} \frac{\partial C}{\partial S} & \frac{\partial C}{\partial I} & \frac{\partial C}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial V}{\partial y} & \frac{\partial V}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial y}{\partial S} & \frac{\partial y}{\partial I} & \frac{\partial y}{\partial t} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial V}{\partial y}\frac{\partial y}{\partial S} & \frac{\partial V}{\partial y}\frac{\partial y}{\partial I} & \frac{\partial V}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial V}{\partial t} \end{pmatrix}$$

$$\frac{\partial C}{\partial t} = \frac{\partial V}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial V}{\partial t} = \frac{-\ln S}{T}\frac{\partial V}{\partial y} + \frac{\partial V}{\partial t}$$

$$\frac{\partial C}{\partial S} = \frac{\partial V}{\partial y}\frac{\partial y}{\partial S} = \frac{T-t}{t}\frac{1}{S}\frac{\partial V}{\partial y}$$

$$\frac{\partial C}{\partial I} = \frac{\partial V}{\partial y}\frac{\partial y}{\partial I} = \frac{1}{T}\frac{\partial V}{\partial y}$$

$$\frac{\partial^2 C}{\partial S^2} = -\frac{T-t}{t}\frac{1}{S^2}\frac{\partial V}{\partial y} + \frac{T-t}{t}\frac{1}{S}\frac{\partial}{\partial S}\frac{\partial V}{\partial y}\bigg|_{y(S,I,t)}$$

$$\frac{\partial^2 C}{\partial S^2} = -\frac{T-t}{t}\frac{1}{S^2}\frac{\partial V}{\partial y} + \frac{T-t}{t}\frac{1}{S}\left(\frac{\partial^2 V}{\partial t \partial y}\frac{\partial t}{\partial S} + \frac{\partial^2 V}{\partial y^2}\frac{\partial y}{\partial S}\right)$$

since

$$\frac{\partial t}{\partial S} = 0 \quad \text{and} \quad \frac{\partial t}{\partial S} = \frac{T-t}{t}\frac{1}{S},$$

$$\frac{\partial^2 C}{\partial S^2} = -\frac{T-t}{t}\frac{1}{S^2}\frac{\partial V}{\partial y} + \frac{T-t}{t}\frac{1}{S}\left(\frac{T-t}{t}\frac{1}{S}\frac{\partial^2 V}{\partial y^2}\right)$$

$$\frac{\partial^2 C}{\partial S^2} = -\frac{T-t}{t}\frac{1}{S^2}\frac{\partial V}{\partial y} + \left(\frac{T-t}{t}\right)^2\frac{1}{S^2}\frac{\partial^2 V}{\partial y^2},$$

substituting this to the above PDE, we get

$$\begin{aligned}
&\frac{-\ln S}{T}\frac{\partial V}{\partial y} + \frac{\partial V}{\partial t} + \frac{-\ln S}{T}\frac{\partial V}{\partial y} \\
&+\frac{\sigma^2}{2}S^2\left[-\frac{T-t}{t}\frac{1}{S^2}\frac{\partial V}{\partial y} + \left(\frac{T-t}{t}\right)^2\frac{1}{S^2}\frac{\partial^2 V}{\partial y^2}\right] \\
&+(r-q)S\left(\frac{T-t}{t}\frac{1}{S}\frac{\partial V}{\partial y}\right) - rV = 0.
\end{aligned}$$

Then the governing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2[\frac{(T-t)}{T}]^2\frac{\partial^2 V}{\partial y^2} + (r-q-\frac{\sigma^2}{2})\frac{(T-t)}{T}\frac{\partial V}{\partial y} - rV = 0$$

for $0 \leq t \leq T$, $-\infty < y < \infty$

$$V(y,T) = \left(\exp\left(\frac{y}{\int_0^T \rho(\tau)\mathrm{d}\tau}\right) - K\right)^+.$$

2.4.1 Geometric Closed Form (Kemna and Vorst)

In 1990 Kemna and Vorst put forward a closed form pricing solution to geometric averaging options by altering the volatility, and cost of carry term. The solution to the geometric call option is

$$C = Se^{(b-r)(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

and the put option is

$$P = Ke^{-r(T-t)}N(-d_2) - Se^{(b-r)(T-t)}N(-d_1)$$

where $N(x)$ is the cumulative of $N(0,1)$,

$$d_1 = \frac{\ln(S/K) + (b + \frac{\sigma_A^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S/K) + (b - \frac{\sigma_A^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma_A\sqrt{T}. \tag{4.25}$$

The adjusted volatility and dividend yield are given as

$$\sigma_A = \frac{\sigma}{\sqrt{3}}$$

$$b = \frac{1}{2}(r - D - \frac{\sigma^2}{6})$$

where σ is the observed volatility, r is the risk free rate of interest.

As we have said before, that in the limit n, the discrete sample averages become the continuous sampled averages. We can now see how Yue-Kuen Kwok in [34] derived the analytic price formula for a continuous geometric averaging Asian option using the discrete averaging geometric averaging Asian option formula.

2.4.2 Analytical value of a continuous fixed strike geometric option

It has been said that the analytic price formula for geometric averaging Asian options exist, provided that the Geometric Brownain motion is assumed for the stochastic movement of the underlying asset price. Consider the discrete geometric averaging of the asset prices at evenly distributed discrete times $t_i = T_0 + i\Delta t, i = 1, 2, \cdots, n$, where Δt is the uniform time interval between fixings and $t_n = T$ is the time of expiration. Define the running geometric averaging by

$$A_G = \left(\prod_{i=1}^{n} S(t_i)\right)^{\frac{1}{n}}.$$

The terminal payoff of a European average value call option with discrete geometric averaging is given by $\max(A_{Gn} - K, 0)$, where K is is the strike price. Assume that the asset price follows a lognormal distribution with variance rate σ^2. Henceforth the price ratio $R_i = \frac{S(t_i)}{S(t_{i-1})}$, $i = 1, 2, \cdots, n$ is also lognormally distributed. More specifically, in the risk neutral world, we have

$$\ln R_i \sim ((r - \frac{\sigma^2}{2})\Delta t, \sigma^2 \Delta t), \ i = 1, 2, \cdots, n,$$

where r is the risk-free interest rate and $N(\mu, \sigma^2)$ represents a normal distribution with mean μ and variance σ^2.

The price formula of the above European average value call depends on whether the current time t is prior to or within the averaging period. Consider the current time before the averaging period that is $t < T_0$. We write

$$\frac{A_{Gn}}{S(t)} = \frac{S(t_0)}{S(t)} \left[\frac{S(t_n)}{S(t_{n-1})} (\frac{S(t_{n-1})}{S(t_{n-2})})^2 (\frac{S(t_{n-2})}{S(t_{n-3})})^3 \cdots (\frac{S(t_1)}{S(t_0)})^n \right]^{\frac{1}{n}}$$

$$\ln \frac{A_{Gn}}{S(t)} = \ln \frac{S(t_0)}{S(t)} + \frac{1}{n} \left(\ln R_n + 2 \ln R_{n-1} + \cdots + n \ln R_1 \right), \ t < T_0$$

since $\ln R_i$ and $\ln \frac{S(t_0)}{S(t)}$ are normally distributed and independent, we deduce that $\ln \frac{A_{Gn}}{S(t)}$ is normally distributed with mean

$$(r - \frac{\sigma^2}{2})(T_0 - t) + \frac{1}{n}(r - \frac{\sigma^2}{2})\Delta t \sum_{i=1}^{n} i$$

$$(r - \frac{\sigma^2}{2}) \left[(T_0 - t) + \frac{n+1}{2n}(T - T_0) \right]$$

and the variance

$$\sigma^2 (T_0 - t) + \frac{1}{n^2} \sigma^2 \Delta t \sum_{i=1}^{n} i^2 = \sigma^2 [(T_0 - t) + \frac{(n+1)(2n+1)}{6n^2}(T - T_0)]$$

let $\tau = T - t$, where τ is the time to expiry. Suppose we write

$$\sigma_G^2 \tau = \sigma^2 \left[\tau - [1 - \frac{(n+1)(2n+1)}{6n^2}](T - T_0) \right]$$

$$(\mu_G - \frac{\sigma_G^2}{2})\tau = (r - \frac{\sigma^2}{2})\left[\tau - \frac{(n-1)}{2n}(T - T_0)\right]$$

then the transaction density function of A_{Gn} at time T, given the asset price $S(t)$ at an earlier time $t < T_0$ can be expressed as

$$\psi(A_{Gn}; S(t)) = \frac{1}{A_{Gn}\sqrt{2\pi\sigma_I^2\tau}} \exp\left(-\frac{(\ln A_{Gn} - [\ln(S(t)) + (\mu - \frac{\sigma_I^2}{2})\tau])^2}{2\sigma_G^2\tau}\right).$$

By the risk neutral discounted expectation approach, the price of the European fixed strike call option with discrete geometric averaging is given by

$$\begin{aligned} C_G(S(t), \tau) &= e^{-r\tau}\int_K^\infty (A_{Gn} - K)\psi(A_{Gn}; S(t))\mathrm{d}A_{Gn} \\ &= e^{-r\tau}[S(t)e^{\mu\tau}N(d_1) - KN(d_2)], \end{aligned}$$

$\tau > T - T_0$, where

$$\begin{aligned} d_1 &= \frac{\ln(S(t)/K) + (\mu_G + \frac{\sigma_G^2}{2})\tau}{\sigma_G\sqrt{\tau}} \\ d_2 &= d_1 - \sigma_G\sqrt{\tau} \end{aligned} \tag{4.26}$$

the value for put is

$$p(t) = e^{-r\tau}[KN(-d_2) - S(t)e^{\mu\tau}N(-d_1)].$$

By considering the two extreme cases where $n = 1$ and $n \to \infty$. When we have $n = 1$, $\sigma_G^2\tau$ and $(\mu_G - \frac{\sigma_G^2\tau}{2})\tau$ reduce to $\sigma^2\tau$ and $(r - \frac{\sigma^2}{2})\tau$, respectively, so that the call price reduces to that of a European vanilla call option. We observe that σ_G^2 is a decreasing function of n, which is consistent with the intuition that more frequent we take the averaging, the lower the volatility. When $n \to \infty$, $\sigma_G^2\tau$ and $(\mu_G - \frac{\sigma_G^2}{2})\tau$ tend to $\sigma^2[\tau - \frac{2}{3}(T - T_0)]$ and $(r - \frac{\sigma^2}{2})[\tau - \frac{T-T_0}{2}]$, respectively; and correspondingly, the discrete geometric averaging becomes continuous geometric averaging. In particular, the price of a European fixed strike call with continuous geometric averaging at $t = T_0$ is found to be

$$C_G(S(T_0, T - T_0) = S(T_0)e^{-\frac{1}{2}(r+\frac{\sigma^2}{6})(T-T_0)}N(\tilde{d}_1) - Ke^{-r(T-T_0)}N(\tilde{d}_2)$$

where

$$\tilde{d}_1 = \frac{\ln(S(T_0)/K) + \frac{1}{2}(r + \frac{\sigma^2}{6})(T - T_0)}{\sigma\sqrt{\frac{T-T_0}{3}}}$$

$$\tilde{d}_2 = \tilde{d}_1 - \sigma\sqrt{\frac{T - T_0}{3}} \tag{4.27}$$

the value for put is

$$P_G = e^{-r\tau}[KN(-\tilde{d}_2) - \tilde{S}(t)e^{\tilde{\mu}\tau}N(-\tilde{d}_1)].$$

Next, suppose the current time t is within the averaging period, that is, $t \geq T_0$ where $t = t_k + \xi\Delta t$ for some integer k, $0 \leq k \leq n-1$ and $0 \leq \xi \leq 1$. Now, $S(t_1), S(t_2), S(t_3), \cdots, S(t_k), S(t)$ are known quantities, and the price ratios $\frac{S(t_{k+1})}{S(t)}$, $\frac{S(t_{k+2})}{S(t_{k+1})}, \cdots, \frac{S(t_n)}{S(t_{n-1})}$ are independent lognormal random quantities. We may write

$$A_{Gn} = [S(t_1)\cdots S(t_k)]^{\frac{1}{n}} S(t)^{\frac{(n-k)}{n}} \left\{ \frac{S(t_n)}{S(t_{n-1})} \left[\frac{S(t_{n-1})}{S(t_{n-2})}\right]^2 \cdots \left[\frac{S(t_{k+1})}{S(t)}\right]^{n-k} \right\}^{\frac{1}{n}}$$

so that

$$\ln\frac{A_{Gn}}{\tilde{S}(t)} = \frac{1}{n}[\ln R_n + 2\ln R_{n-1} + \cdots + (n-k-1)\ln R_{k+2} + (n-k)\ln R_t]$$

where

$$\tilde{S}(t) = [S(t_1)\cdots S(t_k)]^{\frac{1}{n}} S(t)^{\frac{(n-k)}{n}} = A_{G_k}^{\frac{k}{n}} S(t)^{\frac{(n-k)}{n}}$$

and

$$R_t = \frac{S(t_{k+1})}{S(t)}.$$

Let the variance and mean of $\ln\left(\frac{A_{Gn}}{\tilde{S}(t)}\right)$ be denoted by $\tilde{\sigma}_G^2\tau$ and $\left(\tilde{\mu}_G - \frac{\tilde{\sigma}^2}{2}\right)\tau$, respectively. They are found to be

$$\tilde{\sigma}_G^2\tau = \sigma^2\Delta t\left[\frac{(n-k)^2}{n^2}(1-\xi) + \frac{(n-k-1)(n-k)(2n-2k-1)}{6n^2}\right]$$

and

$$(\tilde{\mu}_G - \frac{\tilde{\sigma}_G^2}{2})\tau = (r - \frac{\sigma^2}{2})\Delta t \left[\frac{n-k}{n}(1-\xi) + \frac{(n-k-1)(n-k)}{2n} \right].$$

Then the value of a call option for a discrete average

$$C_G(S(t), \tau) = e^{-r\tau}[\tilde{S}(t)e^{\tilde{\mu}_G \tau} N(\tilde{d}_1) - KN(\tilde{d}_2)], t \geq T_0,$$

where

$$\tilde{d}_1 = \frac{\ln(\tilde{S}(t)/K) + (\tilde{\mu}_G + \frac{\tilde{\sigma}_G^2}{2})\tau}{\sigma_G \sqrt{\tau}}$$

$$\tilde{d}_2 = \tilde{d}_1 - \tilde{\sigma}_G \sqrt{\tau} \tag{4.28}$$

and the value for put option is

$$P_G = e^{-r\tau}[-\tilde{S}(t)e^{\tilde{\mu}_G \tau} N(\tilde{d}_1) - Ke^{-r\tau} N(\tilde{d}_2)].$$

By taking the limits $n \to \infty$, the discrete geometric averaging becomes a continuous geometric averaging. We can observe that the limiting values of $\tilde{\sigma}_G^2$, $\tilde{\mu}_G^2 - \frac{\tilde{\sigma}_G^2}{2}$ and $\tilde{S}(t)$ become $\lim_{n\to\infty} \tilde{\sigma}_G^2 = (\frac{T-t}{T-T_0})^2 \frac{\sigma^2}{3}$, $\lim_{n\to\infty} \tilde{\mu}_G - \frac{\tilde{\sigma}_G^2}{2} = (r - \frac{\sigma^2}{2})(\frac{T-t}{2(T-T_0)})$ also $\lim_{n\to\infty} \tilde{S}(t) = S(t)^{\frac{T-t}{T-T_0}} A_G(t)$ where $A_G(t) = e^{\frac{1}{T-T_0}\int_{T_0}^{t} \ln S(\tau) \mathrm{d}\tau}$. By subtituting the above limits to the equation of the discrete geometric average call option, we can obtain the price of the continuous fixed strike of geometric average call option. Then the value of a continuous call option takes the form

$$\begin{aligned} C_G(S(t), \tau) &= e^{-r(T-t)}[S(t)\exp((r - \frac{1}{2}\sigma^2)(T-t)^2/2(T-T_0) \\ &\quad + \sigma^2(T-t)^3/6(T-T_0)^2)N(d_1) - KN(d_2)] \end{aligned}$$

and the value of a put is

$$\begin{aligned} P_G(S(t), \tau) &= e^{-r(T-t)}[KN(-d_2) - S(t)\exp((r - \frac{1}{2}\sigma^2)(T-t)^2/2(T-T_0) \\ &\quad + \sigma^2(T-t)^3/6(T-T_0)^2)N(-d_1)] \end{aligned}$$

where

$$d_1 = \frac{(T-T_0)\ln(\frac{\tilde{S}(t)}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)^2/2 + \sigma^2(T-t)^3/3(T-T_0)}{\sigma\sqrt{(T-t)^3/3}}$$

$$d_2 = \frac{(T-T_0)\ln(\frac{\tilde{S}(t)}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)^2/2}{\sigma\sqrt{(T-t)^3/3}}. \tag{4.29}$$

The closed form of a geometric average can also be derived from a closed form of a vanilla Black-Scholes option see [7].

2.5 Floating strike options with continuous geometric averaging

Consider a European average value call option with continuous geometric averaging whose terminal payoff function at expiration time T is given by
$\max\left(S_T - \exp\frac{I(T)}{T}, 0\right)$, where $I(t)$ is defined by

$$I(t) = \int_0^t \ln S(\tau)\mathrm{d}\tau, 0 \le t \le T,$$

and K is the strike price. Then the governing equation (1.5) for the price of the above European call, and using the following transformation of variable with final condition

$$C(S,I,T) = \left(S - \exp\left(\frac{I}{\int_0^T p(\tau)\mathrm{d}\tau}\right)\right)^+$$

$$y = I - t\ln S$$

$$V(y,t) = \frac{c(S,I,t)}{S}.$$

Using the above equations, and following the same procedure as in the previous section, where we derive the PDE of a fixed strike Asian option we obtain the governing equation of a floating strike Asian option to be

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 t^2 \frac{\partial^2 c}{\partial y^2} - (r - q + \frac{\sigma^2}{2})t\frac{\partial c}{\partial y} - rc = 0 \tag{5.30}$$

$0 \leq t \leq T$, $-\infty < y < \infty$

$$V(y,T) = \left(1 - \exp\left(\frac{y}{\int_0^T \rho(\tau)\mathrm{d}\tau}\right)\right)^+ .$$

Asian option with the early exercise feature Here we will see how Lixin Wu, Yue Kuen Kwok, Hong Yu derive the PDE of a floating strike of an early exercise of an American Asian option. Consider American Asian option whose payoff depends on continuous Geometric averaging. The terminal payoff function of the value option with continuous geometric average of the asset price is

$$C(S_T, I_T, T) = \max(S_T - I_T), 0)$$

where t is the current time and T is the expiration time, S_T is the asset price at time T, I_T continuous geometrical averaging of the asset price,

$$I_T = \exp\left(\frac{1}{t}\int_0^t \ln S_u \mathrm{d}u\right), \quad 0 < t < T.$$

S_t is assumed to follow the risk-neutral lognomal process

$$\mathrm{d}S_t = (r-q)S_t\mathrm{d}t + \sigma S_t \mathrm{d}Z(t)$$

where r is a constant riskless rate, q is a dividend yield, σ is a constant volatility and $Z(t)$ is a standard Weiner process. By the above equations we obtain

$$\ln S_T = \ln S_t + (r - q - \frac{\sigma^2}{2})(T-t) + \sigma[Z(T) - Z(t)]$$

$$\ln I_T = \frac{t}{T}\ln I_t + \frac{1}{T}\left[(T-t)\ln S_t + (r-q-\frac{\sigma^2}{2})(\frac{T-t}{2})^2\right] + \frac{\sigma}{T}\int_t^T [Z(T) - Z(t)]\mathrm{d}u$$

where

$$Z(T) - Z(t) = \phi(0, \sqrt(T-t))$$

$$\int_t^T [Z(u) - Z(t)]\mathrm{d}u = \phi\left(0, \frac{1}{\sqrt{3}}(T-t)^{\frac{3}{2}}\right)$$

and $\phi(\mu, \sigma)$ denotes the normal distribution with mean μ and standard deviation σ. I_T is also a lognormally distribution. If we apply the no-arbitrage argument and following the riskless hedging approach, the governing equation of the European counterpart of the above Asian option call is

$$\frac{\partial c}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 I}{\partial S^2} + \left(\frac{I}{t} \ln \frac{S}{I}\right) \frac{\partial c}{\partial I} + (r-q) S \frac{\partial c}{\partial S} - rc = 0, \ 0 < t < T \tag{5.31}$$

with terminal condition

$$c(S, I, T) = \max(S - I, 0).$$

If one let $S^*(I, t)$ denote the optimal exercise asset price above which is optimal to exercise the American Asian option. By Jamshidian, the governing equation of the above American Asian option is obtained by modifying the above governing equation as

$$\begin{aligned} &\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 I}{\partial S^2} + \left(\frac{I}{t} \ln \frac{S}{I}\right) \frac{\partial C}{\partial I} + (r-q) S \frac{\partial C}{\partial S} - rC \\ &= \begin{cases} 0 & \text{if } S \leq S^*(I, t) \\ -qS - \frac{\mathrm{d}I}{\mathrm{d}t} + rI, & \text{if } S > S^*(I, t). \end{cases} \end{aligned} \tag{5.32}$$

We propose the following choice for the asset of similarity variables

$$y = t \ln \frac{I}{S},$$

$$V(y, t) = \frac{C(S, I, t)}{S}$$

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} t^2 \frac{\partial^2 V}{\partial y^2} - (r - q + \frac{\sigma^2}{2}) t \frac{\partial V}{\partial y} - qV \\ &= \begin{cases} 0, & \text{if } y \geq y^*(t), \\ -q + re^{\frac{y}{t}} + \frac{y}{t^2} e^{\frac{y}{t}}, & \text{if } y < y^*(t) \end{cases} \end{aligned} \tag{5.33}$$

and

$$V(y, T) = \max(1 - e^{y/T}, 0) \tag{5.34}$$

in the stopping region, the American Asian option value is given by

$$V(y,t) = 1 - e^{y/t},\ \ y < y^*(t).$$

Now we can derive the pricing formula for American option.

The integral representation of the early exercise The solution for the American call option value obtained from the above pricing model can be formally represented as an integral involving the Green function of the governing equation. Let $I(y,t,Y,T)$ be the Green function which satisfies the reduced equation (6.70). Then the Green function is

$$I(y,t,Y,T) = N\left(\frac{Y - y + \mu\int_t^T u\mathrm{d}u}{\sigma\sqrt{\int_t^T u^2\mathrm{d}u}}\right)$$

where $\mu = r - q + \frac{1}{2}\sigma^2$ and $N(x)$ is the standard normal density function. The solution to (6.70) and (5.34)

$$\begin{aligned} V(y,t) &= e^{-q(T-t)}\int_{-\infty}^{\infty}\max(1-e^{Y/T},0)I(y,t;Y,T)\mathrm{d}Y \\ &\quad + \int_t^T e^{-q(u-t)}\int_{-\infty}^{y^*(u)}\left(q - re^{\frac{Y}{u}} - \frac{Y}{u^2}e^{\frac{Y}{u}}\right)I(y,t;Y,u)\mathrm{d}Y\mathrm{d}u \quad (5.35)\end{aligned}$$

$y^*(u)$ is the critical value of y at time u such that $y \leq y^*(u)$. In (5.34) if we multiply the first term by S, it gives the option value of the European counter parts of (5.31), the present American Asian call option.

If we let the early exercise premium $C_I^e(S,I,t) = C(S,I,t) - c(S,I,t)$. Let the second integral in the above $V_e(y,t)$ such that $e(S,I,t) = SV_e(y,t)$. If we integrate this, the integral representation of the early exercise premium is found to be

$$\begin{aligned} e(S,I,t) &= S\int_t^T\Bigg(qe^{-q(u-t)}N(\tilde{d}_1) - (\frac{I}{S})^{\frac{t}{u}}e^{-q(u-t)}e^{\frac{\frac{\sigma^2}{3}(u^3-t^3)}{2u^2}} \\ &\quad -\frac{\mu(u^2-t^2)}{2u}[(r+\tilde{d}_3)N(\tilde{d}_2) - \frac{\sigma}{u^2}n(\tilde{d}_2)]\Bigg)\mathrm{d}u\end{aligned}$$

where

$$\tilde{d}_1 = \frac{u \ln \frac{I}{S^*(I,u)} - t \ln \frac{I}{S} + \frac{\mu}{2}(u^2 - t^2)}{\tilde{\sigma}}$$

$$\tilde{d}_2 = \tilde{d}_1 - \frac{\tilde{\sigma}}{u}$$

$$\tilde{d}_3 = \frac{t \ln \frac{S}{I} - \frac{\mu}{2}(u^2 - t^2) + \frac{\tilde{\sigma}^2}{u}}{u^2}. \tag{5.36}$$

This early exercise premium resembles that of American vanilla option

2.6 Numerical methods

First we look at a numerical method which is used for a two dimensional partial differential equation (PDE) for continuous arithmetic averaging, called the fractional step method (FSM). Later we take a look at another numerical method for continuous geometric averaging, called binomial method.

2.6.1 The fractional step method

Hajime Fujiwara in [9] discovered that when applying the finite difference methods for the degenerated PDE gives an inaccurate solution due to numerical diffusion and spurious oscillations (Zvan, Forsth and Vetzal 1998). To get the solution, which they say it is an accurate solution, they combined three methods which were originally developed in the computational fluid dynamics: the fractional steps methods, the Interpolated Differential Operator method (IDO), the Cubic Interpolated Propagation method (CIP).

$$\begin{aligned}
&\mathrm{d}S_t = \mu(t, S_t)\mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t \\
&\mu(\cdot,\cdot) : R^+ \times R \to R, \\
&\sigma(\cdot,\cdot) : R^+ \times R \to R
\end{aligned}$$

where W_t is a standard Brownian motion. Let

$$I(t) = \int_0^t S_u \mathrm{d}u. \tag{6.37}$$

The value of the derivative securities with the maturity T is given by the following PDE, this PDE is a forward equation in time by replacing t with $\tau = T - t$.

$$\frac{\partial V}{\partial \tau} - S\frac{\partial V}{\partial I} - \mu(\tau, S)\frac{\partial V}{\partial S} = \frac{1}{2}\sigma(\tau, S)^2\frac{\partial^2 V}{\partial S^2} - rV \tag{6.38}$$

where $V(\tau = 0, I, S) = f(I, S)$, $f(I, S)$ represent the payoff at maturity. The above PDE is splits it into three equations that is: advection equation, diffusion equation, discount equation, into a small interval. Different numerical method which is appropriate for each equation was applied, since the equation can be solved separately. By dividing the time interval $[0, T]$ into time steps, i.e $(0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_N = T)$. For each time step $\Delta t_n = t_{n+1} - t_n = [t_n, t_{n+1}], n = 0, 1, 2, \cdots, N-1$. The CIP method is used to evaluate the advection equation. It is used with the time evolution of the derivative and is affective to avoid numerical diffusion and spurious oscillation. The IDO method is used for diffusion equation, it is an extension of the CIP method. Although the diffusion method can be solved by the standard finite difference method accurately, the IDO is used because it can provide the time evolution of the derivatives that the CIP method needs. And finally the discount method can be solved analytically. The fractional steps method split the above PDE into three equations

An advection equation

$$\frac{\partial V}{\partial \tau} + u\frac{\partial V}{\partial I} + v\frac{\partial V}{\partial \tau} = 0 \tag{6.39}$$

A diffusion equation

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} \tag{6.40}$$

A discount equation

$$\frac{\partial V}{\partial \tau} = -rV \tag{6.41}$$

$\tau \in [t_n, t_{n+1}]$ where $u = -S$ and $v = -\mu(\tau, S)$. Suppose one want to have the value $V_{t_{n+1}}$ given V_{t_n}, $V_{I,t_n} = \frac{\partial V_{t_n}}{\partial I}$ and $V_{S,t_n} = \frac{\partial V_{t_n}}{\partial S}$. First, having the solutions V^{cip}, V_I^{cip} and V_S^{cip} of the advection equation by the CIP method with V_{t_n}, V_{I,t_n} and V_{S,t_n} as the initial value. Next by using the solutions of the advection equation as the initial value, one have the solutions V^{ido}, V_I^{ido} and V_S^{ido} of the diffusion equation by IDO method. The discount equation can be finally solved with the previous solutions as the initial value. Then the value V_{t_n+1}, V_{I,t_n} and V_{S,t_n} can be found. The same procedure is repeated to every time step starting from the time $\tau = 0$, and the solution of the PDE is obtained at $\tau = T$.

The diffusion equation The IDO method is used to solve the diffusion equation. If given V on the grid points, $S_j,\ j = 1, 2, \cdots, N_S$. Since it is an extension of the CIP method it gives the time evolution of the value and the derivative. The interpolating function is given as $S_{j+\eta} \in [S_{j-1}, S_{j+1}]$. It can be solved by finite difference method.

$$V(S_j + \eta) = G_j(\eta) = b_1\eta^5 + b_2\eta^4 + b_3\eta^3 + b_4\eta^2 + b_5\eta + b_6 \tag{6.42}$$

the coefficients $\{b_i\}_{i=1}^6$ are calculated from V and V_S on the grid points S_j, S_{j+1} and S_{j-1}. Subtituting the above formula into the diffusion equation, one obtain

$$\frac{\partial V(S_j)}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V(S_j)}{\partial S^2} = \frac{1}{2}\sigma^2 \frac{\partial^2 G(0)}{\partial \eta^2} = \sigma^2 b_4. \tag{6.43}$$

The time evolution of the derivative for the diffusion equation can be derived. Differentiating the diffusion equation (6.40) with respect to S, the following is obtained

$$\frac{\partial V_S(S_j)}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 V_S(S_j)}{\partial S^2} = \frac{1}{2}\sigma^2 \frac{\partial^3 G(0)}{\partial \eta^3} = 3\sigma^2 b_3. \tag{6.44}$$

By applying the finite difference method in the above two equation in the time evolution, and using the coefficients b_3 and b_4 explicitly, the solutions are

$$\begin{aligned} V_{t_{n+1}}(S_j) &= V_{t_n}(S_j) + \sigma^2 \Delta t b_4 \\ &= V_{t_n}(S_j) + \tfrac{\sigma^2 \Delta t}{\Delta S^2}[V_{t_n}(S_{j+1}) - 2V_{t_n}(S_j) + V_{t_n}(S_{j-1})] \\ &\quad -\tfrac{\sigma^2 \Delta t}{4\Delta S}[V_{S,t_n}(S_{j+1}) - V_{S,t_n}(S_{j-1})] \end{aligned}$$

and

$$\begin{aligned} V_{S,t_{n+1}}(S_j) &= V_{S,t_n}(S_j) + 3\sigma^2 \Delta t b_3 \\ &= V_{S,t_n}(S_j) + \tfrac{15\sigma^2 \Delta t}{4\Delta S^3}[V_{t_n}(S_{j+1}) - V_{t_n}(S_{j-1})] \\ &\quad -\tfrac{3\sigma^2 \Delta t}{4\Delta S^2}[V_{S,t_n}(S_{j+1}) + 8V_{S,t_n}(S_j) + V_{S,t_n}(S_{j-1})]. \end{aligned}$$

It can be seen that IDO method for the diffusion equation is a sort of high order finite difference method.

The discount equation It can be solved analytically, given V_{t_n}, V_{I,t_n} and V_{S,t_n} and one can obtain the following solution.

$$\begin{aligned} V_{t_{n+1}} &= V_{t_n} e^{(-r\Delta t_n)} \\ V_{I,t_{n+1}} &= \frac{\partial V_{t_{n+1}}}{\partial I} = V_{I,t_n} e^{-r\Delta t_n} \\ V_{S,t_{n+1}} &= \frac{\partial V_{t_{n+1}}}{\partial S} = V_{S,t_n} e^{-r\Delta t_n}. \end{aligned}$$

The advection equation There is a numerical solution of this method by the CIP method. The solution of the advection equation satisfies the following relation

$$V_{t_{n+1}}(I, S) = V_{t_n}(I - u\Delta t_n, S - v\Delta t_n)\Delta t = t_{n+1} - t_n.$$

Suppose that one have the value V_{t_n} on the spatial grid $\{(I_i, S_j)\}_{i=1,2,\cdots,N_I}^{j=1,2,\cdots,N_S}$. In the CIP method the following polynomial is used as the interpolating function (see Takewaki and Yabe 1987) for $(I_i + \S_i, S_j + \eta) \in [I_i, S_j] \times [I_{i+1}, x_{j+1}]$

$$\begin{aligned} F_{i,j}(\xi, \eta) &= a_1\xi^3 + a_2\xi^2 + a_3\xi + a_4 + a_5\eta^3 + a_6\eta^2 + a_7\eta + a_8\xi^2\eta \\ &\quad + a_9\xi\eta + a_{10}\xi\eta^2 + a_{11}\xi^3\eta + a_7 12\xi\eta^3. \end{aligned} \tag{6.45}$$

The coefficients $\{a_i\}_{i=1}^{12}$ are determined from V, V_I and V_S on the four points, (I_i, S_j), (I_{i+1}, S_j) and (I_{i+1}, S_{j+1}). With the above interpolating function, one can express the solution as

$$V_{t_{n+1}}(I_i, S_j) = V_{t_n}(I_i - u\Delta t, S_j - v\Delta t) = F_{k,l}(-u\Delta t, -v\Delta t) \tag{6.46}$$

where $k = i(u < 0)$ or $k = i - 1(u > 0)$ and $l = j(v < 0)$ or $l = j - 1(v > 0)$. The time evolution of the derivatives (V_I and V_S) has to be calculated, because they are used to determine the coefficients of the interpolating function at time step. If we differentiate the advection equation with respect to I and S, we then obtain the following PDEs for V_I and V_S

$$\frac{\partial V_I}{\partial \tau} + u\frac{\partial V_I}{\partial I} + v\frac{\partial V_I}{\partial S} = 0 \tag{6.47}$$

and

$$\frac{\partial V_S}{\partial \tau} + u\frac{\partial V_S}{\partial I} + v\frac{\partial V_S}{\partial S} = -\frac{\mathrm{d}u}{\mathrm{d}S}V_I. \tag{6.48}$$

By applying the fractional steps method for the equation and split it into two equations

$$\frac{\partial V_S}{\partial \tau} + u\frac{\partial V_S}{\partial I} + v\frac{\partial V_S}{\partial S} = 0 \tag{6.49}$$

$$\frac{\partial V_S}{\partial \tau} = -\frac{\mathrm{d}u}{\mathrm{d}S}V_I \tag{6.50}$$

since (6.46)and (6.47) are advection equation, and their solution can be represented respectively as follows

$$V_{I,t_{n+1}}(I_i, S_j) = V_{I,t_n}(I_i - u\Delta t, S_j - v\Delta t) = \frac{\partial F_{k,l}(-u\Delta t, -v\Delta t)}{\partial \xi}$$

and

$$V_{S,t_{n+1}}(I_i, S_j) = V_{S,t_n}(I_i - u\Delta t, S_j - v\Delta t) = \frac{\partial F_{k,l}(-u\Delta t, -v\Delta t)}{\partial \eta}$$

where

$$\frac{\partial F_{k,l}(\xi, \eta)}{\partial \xi} = 3a_1\xi^2 + 2a_2\xi + a_3 + 2a_8\xi\eta + a_9\eta + a_{10}\eta^2 + 3a_{11}\xi^2\eta + a_{12}\eta^3$$

and

$$\frac{\partial F_{i,j}(\xi, \eta)}{\partial \eta} = 3a_5\eta^2 + 2a_6\eta + a_7 + a_8\xi^2 + a_9\xi + 2a_{10}\xi\eta + a_{11}\xi^3 + 3a_{12}\xi\eta^2$$

and the solution is given by the finite difference method

$$V_{S,t_{n+1}} = V_{S,t_n} - \Delta t_n \frac{\partial u}{\partial S} V_{I,t_n}.$$

Summarizing the result, for advection equation. Given V, V_I, V_S on the grid points at time t_n, at time t_{n+1} these values are obtained

$$V_{t_{n+1}}(I_i, S_j) = F_{k,l}(-u\Delta t, -v\Delta t)$$

$$V_{I,t_{n+1}}(I_i, S_j) = \frac{\partial F_{k,l}(-u\Delta t, -v\Delta t)}{\partial \eta}$$

$$V_{S,t_{n+1}}(I_i, S_j) = \frac{\partial F_{k,l}(-u\Delta t, -v\Delta t)}{\partial \eta} - \Delta t_n \frac{\mathrm{d}u}{\mathrm{d}S} \frac{\partial F_{k,l}(-u\Delta t, -v\Delta t)}{\partial \xi}.$$

Note that time evolution of the derivatives is deduced from the original advection equation.

In the next examples (Tables) are fixed strike put and floating strike put options with various maturities and strikes. These option do not have an analytical solution, Hajime Fujiwara in [9] compared the fractional step method with Monte Carlo simulation (MC), one dimensional of Rogers and Shi (RS), modified binomial method of and Zvan, Forsth and Vetzal 1998 (ZFV). The fractional step method (FSM) for a fixed strike call option is consistent with those of Monte Carlo and comparable with those of one dimensional.

TABLE 2.2

σ	T	K	FSM	$MC(std)$	RS	ZFV
		95	6.119	6.114(0.0010)	6.114	6.113
	0.25	100	1.850	1.841(0.0007)	1.841	1.793
		105	0.151	0.147(0.0002)	0.162	0.162
		95	7.221	7.220(0.0014)	7.216	7.244
0.1	0.50	100	3.073	3.069(0.0011)	3.064	3.052
		105	0.716	0.712(0.0005)	0.712	0.712
		95	9.289	9.289(0.0017)	9.286	9.289
	1.00	100	5.257	5.254(0.0013)	5.254	5.254
		105	2.297	2.294(0.0012)	2.295	2.294

		95	6.488	6.488(0.0016)	6.461	6.488
	0.25	100	2.2924	2.2924(0.0013)	2.293	2.928
		105	0.948	0.944(0.0007)	0.944	0.958
		95	7.899	7.898(0.0023)	7.898	7.890
0.2	0.50	100	4.510	4.502(0.0018)	4.502	4.511
		105	2.208	2.204(0.0012)	2.206	2.229
		95	10.299	10.297(0.0031)	10.294	10.309
	1.00	100	7.046	7.042(0.0026)	7.041	7.042
		105	4.512	4.506(0.0024)	4.508	4.519
		95	8.126	8.093(0.0028)	8.097	8.123

In Table 2.2, a European fixed strike call option, $N_t = 90$, for $T = 0.25$ and $T = 0.5$, $N_t = 120$ for $T = 1.0$, where std is the standard deviation of Monte Carlo simulation.

American options Note that American Asian options can not be priced by Monte Carlo simulation.

TABLE 2.3

σ	T	FSM	ZFV
	0.25	1.217	1.359
0.1	0.50	1.573	1.601
	1.00	1.927	1.952
	0.25	2.831	2.867
0.2	0.50	3.833	3.806
	1.00	4.953	4.932
	0.25	6.076	6.111
0.4	0.50	8.436	8.361
	1.00	11.367	11.352

In Table 2.3, an American Floating strike put option, $N_t = 90$, for $T = 0.25$ and $T = 0.5$, $N_t = 120$ for $T = 1.0$, $\frac{\mathrm{d}I}{\mathrm{d}t} = 0.05$.

2.6.2 Binomial models

In this section we look at the numerical method for continuous geometric averaging, by Min Dai. Let t be the expiration date, $[0, T]$ be the life of the option and r, q and σ be interest rate, continuous dividend rate and volatility respectively. If N is the number of discrete time points, one has time points $n\Delta t, n = 0, 1...N$, with $\Delta t = T/N$. Let $V^n(S_n, I_n)$ be the option price at time point $n\Delta t$ with underlying asset price S_n and geometric average value $e^{\frac{I_n}{n+1}}$, where $I_N = \sum_{i=1}^{n} \ln S_i$. Assume that S_N will be either $S_n u$ for upwards movement with probability p or $S_n d$ for downwards movement with probability $1-p$ at time $(n+1)\Delta t$. Suppose

$$\mu = e^{\sigma\sqrt{\Delta t}}, d = \frac{1}{u}, p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

where I_n is either

$$I_{n+1}^u = I_n + \ln(S_n u) = I_n + \ln S_n + \sigma\sqrt{\Delta t} \tag{6.51}$$

or

$$I_{n+1}^d = I_n + \ln(S_n d) = I_n + \ln S_n - \sigma\sqrt{\Delta t}.$$

By the Cox, Ross and Rubinstein arbitrage arguments, the following is obtained

$$V^n(S_n, I_n) = \frac{1}{\rho}[\rho V^{n+1}(S_n u, I_{n+1}^u) + (1-\rho)V^{n+1}(S_n d, I_{n+1}^d)] \tag{6.52}$$

where $\rho = e^{r\sqrt{\Delta t}}$.

Floating strike option

Consider the case of floating strike case, and where the payoff is given by

$$V^N(S_N, I_N) = (S_N - e^{\frac{I_N}{N+1}})^+.$$

Suppose that

$$V^n(S_n, I_n) = S_n V^n(y_n) \tag{6.53}$$

where

$$y_n = I_n - (n+1)\ln S_n \tag{6.54}$$

(6.53) and (6.54) are equivalent to the probabilistic technique: change of numeraire, using the underlying asset as the numeraire for a martingale measure instead of using the bank account, according to (6.51)

$$\begin{aligned} I^u_{n+1} - (n+2)\ln(S_n u) &= I_n + \ln S_n + \sigma\sqrt{\Delta t} - (n+2)\ln S_n - (n+2)\sigma\sqrt{\Delta t} \\ &= I_n - (n+1)\ln(S_n) - (n+1)\sigma\sqrt{\Delta t} \\ &= y_n - (n+1)\sigma\sqrt{\Delta t}. \end{aligned} \tag{6.55}$$

If we rewrite

$$V^{n+1}(S_n u, I^u_{n+1}) = S_n u V^{n+1}(y_n - (n+1)\sigma\sqrt{\Delta t}).$$

So

$$V^{n+1}(S_n d, I^d_{n+1}) = S_n d V^{n+1}(y_n + (n+1)\sigma\sqrt{\Delta t}). \tag{6.56}$$

Then from (6.52)-(6.56) it follows that

$$\begin{aligned} V^n(y_n) &= \frac{1}{\rho}[puV^{n+1}(y_n + (n+1)\sigma\sqrt{\Delta t}) \\ &\quad +(1-p)dV^{n+1}(y_n - (n+1)\sigma\sqrt{\Delta t})]. \end{aligned} \tag{6.57}$$

At the expiry date, the following equation is obtained

$$V^N(y_N) = \left(1 - \frac{e^{\frac{I_N}{N+1}}}{S_N}\right)^+ = \left(1 - e^{\frac{y_N}{N+1}}\right)^+. \tag{6.58}$$

Write

$$V^n(j) = V^n(j\sigma\sqrt{\Delta t})$$

$I_0 = \ln S_0$ and then $y_0 = 0$. In order to compute $V^0(S_0, \ln S_0) = S_0 V^0(0)$, at time $n\Delta t$, the values of W^n should be given at the nodes in the interval

$$-\sum_{k=0}^{n-1}(k+1)\sigma\sqrt{\Delta t}, \sum_{k=0}^{n-1}(k+1)\sigma\sqrt{\Delta t} = (-\frac{n(n+1)}{2}\sigma\sqrt{\Delta t}, \frac{n(n+1)}{2}\sigma\sqrt{\Delta t}).$$

By (6.57) and (6.58), one can calculate the option price using the backward induction

$$V^n(j) = \frac{1}{\rho}[puV^{n+1}(j-n-1) + (1-p)dV^{n+1}(j+n+1)] \tag{6.59}$$

$$\forall j \in [0, N-1],\ j = -n(n+1)/2, -n(n+1)/2+2, \cdots, n(n+1)/2$$

$$V^N(j) = (1 - e^{\frac{j\sigma\sqrt{\Delta t}}{N+1}})^+ = (1 - u^{\frac{j}{N+1}})^+ \tag{6.60}$$

$$\forall j = -N(N+1)/2, -N(N+1)/2+2, \cdots, N(N+1)/2$$

with $V^0(S_0, \ln S_0) = S_0 V^0(y_0)$.

TABLE 2.4

N	0.1	0.2	0.3
4	1.948	3.248	4.601
12	1.974	3.286	4.650
60	1.984	3.300	4.669
120	1.986	3.302	4.670
240	1.986	3.302	4.670
Analytic	1.986	3.303	4.671

American case

$$
\begin{aligned}
V^n(j) &= \max\{\frac{1}{\rho}[puV^{n+1}(j-n-1)+(1-p)\mathrm{d}V^{n+1}(j+N+1)],(1-u^{\frac{j}{n+1}})^+\} \quad (6.61)\\
&\quad \forall j\in[0,N-1], j=-n(n+1)/2,-n(2N-n+1)/2+2,\cdots,n(n+1)/2\\
V^N(j) &= (1-u^{\frac{j}{N+1}})^+ \ \forall j=-N(N+1)/2,-N(N+1)/2+2,\cdots,N(N+1)/2.
\end{aligned}
\tag{6.62}
$$

TABLE 2.5

N	0.1 correct soln≈ 2.39	0.2 correct soln≈ 4.25	0.3 correct soln≈ 6.14	CPU
60	2.333	4.137	5.967	0.06
120	2.359	4.191	6.046	0.3
240	2.375	4.222	6.092	2.7
480	2.383	4.239	6.117	23

Fixed strike

$$V^N(S_N,I_N)=(e^{\frac{I_N}{N+1}}-K)^+ \tag{6.63}$$

where K is the strike price. In this case, if we write

$$V^n(S_n,I_n)=V^n(y_n), y_n=I_n+(N-n)\ln S_n$$

similarly, we can derive from (6.51), (6.52) and (6.63)

$$V^n(y_n)=\frac{1}{\rho}[pV^{n+1}(y_n+(N-n)\sigma\sqrt{\Delta t})+(1-p)dV^{n+1}(y_n-(N-n)\sigma\sqrt{\Delta t})]$$

and

$$V^N(y) = (e^{\frac{y}{N+1}} - K)^+.$$

Write

$$V^n(j) = V^n(y_o + j\sigma\sqrt{\Delta t}).$$

Here $y_0 = I_0 + N \ln S_0 = (N+1)\ln S_0$, $V^0(S_0, \ln S_0) = V^0(y_0)$, at time Δt, it is enough to give the values of V^n at the nodes in the interval

$$\begin{aligned}
&(y_0 - \sum_{k=0}^{n-1}(N-k)\sigma\sqrt{\Delta t}), y_0 + \sum_{k=0}^{n-1}(N-k)\sigma\sqrt{\Delta t}) \\
&= (y_0 - \frac{n(2N-n+1)}{2}\sigma\sqrt{\Delta t}, y_0 + \frac{n(2N-n+1)}{2}\sigma\sqrt{\Delta t}). \qquad (6.64)
\end{aligned}$$

$$\begin{aligned}
V^n(j) &= \frac{1}{\rho}[pV^{n+1}(j+N-n) + (1-p)V^{n+1}(j-N+n)] \qquad (6.65) \\
&\quad \forall j \in [0, N-1], j = -n(2N-n+1)/2 - n(2N-n+1)/2 + 2, \cdots, \\
&\qquad n(2N-n+1)/2 \\
V^N(j) &= (e^{\frac{y_0 + j\sigma\sqrt{\Delta t}}{N+1}} - K)^+ = (S_0 u^{\frac{j}{N+1}} - K)^+ \qquad (6.66) \\
&\quad \forall\, j = -N(N+1)/2, -N(n+1)/2 + 2, \cdots, N(N+1)/2.
\end{aligned}$$

TABLE 2.6

N	0.1	0.2	0.3
4	6.300	6.691	7.485
12	6.313	6.743	7.538
60	6.319	6.755	7.571
120	6.319	6.758	7.576
240	6.320	6.760	7.578
Analytic	6.320	6.761	7.581

Min Dai in [25], said that the one state variable binomial model for European-style fixed strike option cannot be extended to price its American-style counterparts because the transformation fails.

Theorem 1 Neglecting higher order terms of Δt, the binomial models (6.59) for European style floating strike geometric Asian options (or (6.65) for the fixed strike case) are equivalent to certain explicit difference schemes of (6.70) (or 4.25) respectively.

Proof Given mesh size Δy, $\Delta t > 0$, $N\Delta t = T$. Let $Q = \{(j\Delta y, n\Delta t) : j \in Z, 0 \leq n \leq N\}$ and let V_j^n be the value of the numerical approximation at $(n\Delta t, j\Delta y)$, then the floating strike Asian option of a European style is reduced to

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} t^2 \frac{\partial^2 V}{\partial y^2} - (r - q - \frac{\sigma^2}{2}) t \frac{\partial V}{\partial y} - qV = 0$$

for $0 \leq t \leq T, -\infty < y < \infty$, and $V(y,T) = (1 - e^{\frac{y}{T}})$.

Making use of the approximations

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_{(j\Delta y,(n+1)\Delta t)} = \frac{V_{j+n+1}^{n+1} + V_{j-n-1}^{n+1} - 2V_j^{n+1}}{(n+1)^2 \Delta y^2}$$

$$\left.\frac{\partial V}{\partial y}\right|_{(j\Delta y,(n+1)\Delta t)} = \frac{V_{j+n+1}^{n+1} - V_{j-n-1}^{n+1}}{2(n+1)\Delta y}$$

for space and taking the explicit difference for time at the nodes $(j\Delta y, (n+1)\Delta t)$, we have

$$\begin{aligned} &\frac{V_j^{n+1} - V_j^n}{\Delta t} + \frac{\sigma^2}{2}(n+1)^2 \Delta t^2 \frac{V_{j+n+1}^{n+1} + V_{j-n-1}^{n+1} - 2V_j^{n+1}}{(n+1)^2 \Delta y^2} \\ &-(r - q - \frac{\sigma^2}{2})(n+1)\Delta t \frac{V_{j+n+1}^{n+1} - V_{j-n-1}^{n+1}}{2(n+1)\Delta y} - qV_j^n = 0. \end{aligned} \tag{6.67}$$

If we take

$$\Delta y - \sigma \Delta t^{\frac{3}{2}}$$

we obtain

$$V_j^n = \frac{1}{1+q\Delta t}[a - V_{j-n-1}^{n+1} + (q-a)V_{j+n+1}^{n+1}] \tag{6.68}$$

where

$$a = \frac{a}{2} - \frac{\sqrt{\Delta t}}{2\sigma}(q - r - \frac{\sigma^2}{2})$$

and the final condition is given by

$$V_j^n = (1 - e^{\frac{j\Delta y}{T}})^+ = (1 - e^{\frac{j\sigma\sqrt{\Delta y}}{N}})^+. \tag{6.69}$$

We can easily check that

$$\frac{1}{\rho}pu = \frac{a}{1+q\Delta t} + \mathcal{O}(\Delta t^{\frac{3}{2}})$$

$$\frac{1}{\rho}(1-p)d = \frac{1-a}{1+q\Delta t} + \mathcal{O}(\Delta t^{\frac{3}{2}}).$$

Then the binomial model (6.59) is equivalent to the explicit difference scheme (6.68) and (6.69) on neglecting the higher order terms of Δt.

Theorem 2 Neglecting higher order terms of Δt, the binomial models (6.61) for American style floating strike geometric Asian options are equivalent to certain explicit difference schemes of

$$\begin{aligned} \min\{-\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}t^2\frac{\partial^2 V}{\partial y^2} - (-r + q - \frac{\sigma^2}{2})t\frac{\partial V}{\partial y} + qV, \\ V - (1 - e^{y/t})^+\} = 0 \end{aligned} \tag{6.70}$$

$0 \le t \le T, -\infty < y < \infty.$

Proof The proof is similar to that of theorem 1. Note that a one-dimensional binomial model cannot be constructed in the case of American style fixed strike.

Chapter 3

Martingale approach on Asian options

Since it had been said that there are no analytical solution to the values of European call or put written on the arithmetic average when the underlying follows a lognormal process. Almost all over-the-counter path-dependent contracts are based on arithmetic rather than geometric averages. Different techniques are used to price the values of an arithmetic averages. We will now use martingale approach to derive these averages. We will also show another method where an Arithmetic average can be reasonably approximated by geometric with an appropriate adjusted mean and volume. The advantage of the martingale approach is that a lower and upper bound can be obtained. Analytical approximation methods for geometric averages are derived and will be used to derive the arithmetic averages. We will look at how to derive the upper and lower bounds according to Rogers and Shi and Thomson, also look at the variance reduction techniques and the discritization.

3.1 Floating strike Asian option

Asbjorn T Hansen and Peter Lochte Jorgensen in [2] claim that their approach is an exact and an easily implementable analytical formula for American-style Asian options. The exact value of a floating geometric averaging was used to approximate the analytical formulas for floating arithmetic averaging. In the case where averaging is geometric, the distribution of $x_G(t)$ can be determined and the valuation expression from Theorem 1 can thus be further manipulated.

Let us consider a financial market in which all activity occurs on filtered probability space $(\Omega, \mathcal{F}_t, (\mathcal{F}), \mathbb{P})$ supporting Brownian motion on the finite time interval $[0, T]$. $\mathbb{P}$ is the standard risk-neutral probability measure in which the price S, of the basic risky asset (the stock) of the economy evolves according to the stochastic differential equation

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W^{\mathbb{P}}(t)$$

r is the constant and positive risk-free rate of interest, σ is the constant volatility of stock returns, and $W^{\mathbb{P}}$, a standard Brownian motion under $\mathbb{P}$. The solution of this stochastic differential, is given by

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W^{\mathbb{P}}(t)}, 0 \le t \le T.$$

The risk-free asset of the economy the-money market-account has dynamics

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \quad B(0) = 1$$

$$B(t) = e^{rt}.$$

Here the standard assumptions about continuous-time perfect markets is maintained. Assume that the assets trade continuously and that there are no frictions, e.g no transaction costs or taxes of any kind. The contracts are initiated at time zero and the payoffs upon exercise at time t are given as

$$\text{Payoff} = [\rho(S_t - A_t)]^+$$

where $\rho = 1$ for a floating strike call and $\rho = -1$ for a floating strike put. From [22] where the general treatment of the the fair valuation of American-type contingent claims, let $V(t)$ denote the option value at time t, then we have

$$V(t) = \text{ess-sup}_{r\in\mathcal{F}_{t,\tau}}\mathbb{E}_t^{\mathbb{P}}[e^{-r(\tau-t)}[(\rho S(t) - A(t))]^+]$$

where $\Im_{t,T}$ denotes the class of $(\Im_t)$ stopping times taking values in $[t, T]$ and $\mathbb{E}_t^{\mathbb{P}}$ is the $\mathbb{P}$-expectation conditional on $(\Im_t)$. If one define the $\mathbb{P}$-martingale

$$\xi(t) = e^{-rt}\frac{S_t}{S_0} = e^{-\frac{1}{2}\sigma^2+\sigma W^{\mathbb{P}}(t)}$$

and the new equivalent measure $\mathbb{P}'$, by

$$\mathrm{d}\mathbb{P}' = \xi(t)\mathrm{d}\mathbb{P}.$$

By Girsanov's theorem the process

$$W^{\mathbb{P}'}(t) = W^{\mathbb{P}}(t) - \sigma t$$

is the standard Brownian motion under $\mathbb{P}'$. The stock price evolves under $\mathbb{P}'$ according to the stochastic differential equation

$$\mathrm{d}S_t = (r + \sigma^2)S_t\mathrm{d}t + \sigma S_t\mathrm{d}W^{\mathbb{P}'}(t).$$

When the measure $\mathbb{P}'$ is discounted by (measured in units of) the stock price, all asset price will be $\mathbb{P}'$-martingales. i.e $e^{-rt}f(t) = E_t^{\mathbb{P}}[e^{-rT}f(T)]$ we obtain

$$\begin{aligned} f(t) &= E_t^{\mathbb{P}}\{\tfrac{\xi(t)}{\xi(T)}f(T)\} \\ &= E_t^{\mathbb{P}}\{S(t)\tfrac{f(T)}{S(T)}\}. \end{aligned}$$

Therefore $\frac{f(\cdot)}{S(\cdot)}$ is a $\mathbb{P}$-martingale. Applying the rules of conditional expectation, invoking the optional sampling theorem, and defining $x(t) = \frac{A_t}{S_t}$ one can write

$$\begin{aligned}
V(t) &= \text{ess-sup}_{r\in\Im_{t,\tau}} \mathbb{E}_t^{\mathbb{P}}[e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+] \\
&= \text{ess-sup}_{r\in\Im_{t,\tau}} \mathbb{E}_t^{\mathbb{P}'}[\tfrac{\xi(t)}{\xi(T)} e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+] \\
&= \text{ess-sup}_{r\in\Im_{t,\tau}} \mathbb{E}_t^{\mathbb{P}'}[\tfrac{S(\tau)}{e^{rt}} e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+ \mathbb{E}_\tau^{\mathbb{P}'}(\tfrac{e^{rT}}{S(\tau)})] \\
&= \text{ess-sup}_{r\in\Im_{t,\tau}} \mathbb{E}_t^{\mathbb{P}'}[\tfrac{S(t)}{e^{rt}} e^{-r(\tau-t)}[\rho(S(t) - A(\tau))]^+ (\tfrac{e^{\tau t}}{S(\tau)})] \\
&= \text{ess-sup}_{r\in\Im_{t,\tau}} \mathbb{E}_t^{\mathbb{P}'}[S(t)[(\rho(1 - \tfrac{A(\tau)}{S(\tau)})]^+] \\
&= \text{ess-sup}_{r\in\Im_{t,\tau}} S(t)\mathbb{E}_t^{\mathbb{P}'}[S(t)[\rho(1 - x(\tau))]^+].
\end{aligned}$$

Now if $x(\cdot)$ is a Markov process on the filtration generated by $x(\cdot)$. From (Oksendal 1992) it follows that the optimal stopping time for this problem contained in the set

$$\Im_{T,t}^x = [\tau(w,u) \in \Im_{t,T} | \tau \equiv f(x_u, u), \text{f-measurable}].$$

Using Ito's lemma we obtain an SDE equation describing the evolution in $x(.)$:

$$\begin{aligned}
\mathrm{d}x(t) &= \mathrm{d}(\tfrac{A(t)}{S(t)}) \\
&= \tfrac{1}{S(t)}\mathrm{d}A(t) - \tfrac{A(t)}{S^2(t)}\mathrm{d}S(t) + \tfrac{A(t)}{S^3(t)}(\mathrm{d}S(t))^2 \\
&= x(t)\tfrac{\mathrm{d}A(t)}{A(t)} - x(t)((r+\sigma^2)\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{P}'}(t)) + x(t)\sigma^2\mathrm{d}t \\
&= x(t)\tfrac{\mathrm{d}A(t)}{A(t)} - rx(t)\mathrm{d}t - \sigma x(t)\mathrm{d}W^{\mathbb{P}'}(t).
\end{aligned}$$

The exact form of the term $\frac{\mathrm{d}A(t)}{A(t)}$ depends on the type of averaging. If the average is arithmetic then we have

$$\frac{\mathrm{d}A(t)}{A(t)} = \frac{1}{t}(x^{-1}(t) - 1)\mathrm{d}t$$

and for geometric

$$\frac{\mathrm{d}A(t)}{A(t)} = -\frac{1}{t}\ln x(t)\mathrm{d}t.$$

Regardless of the type of averaging, the average process is a process of bounded variation. Define arithmetic and geometric averaging by using the subscripts A and G on $x(\cdot)$

$$\mu_A(x_A(t),t) = \frac{1}{t}(x_A^{-1}(t) - 1) - r$$

$$\mu_G(x_G(t),t) = -[\frac{1}{t}\ln x_G(t) + r] \tag{1.1}$$

and thus (1.1) can be written as, for arithmetic form

$$\frac{\mathrm{d}x_A}{x_A(t)} = \mu_A(x_A(t),t) - \sigma x(t)\mathrm{d}W^{\mathbb{P}'}(t)$$

and for geometric form

$$\frac{\mathrm{d}x_G}{x_G(t)} = \mu_G(x_G(t),t) - \sigma x(t)\mathrm{d}W^{\mathbb{P}'}(t).$$

We conclude that $x_A(\cdot)$ and $x_G(.)$ are Markov processes on the filtration generated by $x(\cdot)$. Therefore (1.1) becomes

$$V(t) = \text{ess-sup}_{r\in\Im^x_{t,\tau}} S(t)\mathbb{E}_t^{\mathbb{P}'}[S(t)[\rho(1 - x(\tau))]^+].$$

This allow us to characterise the optimal stopping rule relating to Problem (1.1) as follows

$$\tau_t^* = \inf[s \in [t,T]x(s) = x^*(s)].$$

If we denote the stock price denominated value of the American-style floating strike Asian option by $V(x(t),t)$ then one can state the following theorem.

Theorem 1 The value of a stock price of a floating strike Asian option at time t, is given by

$$\begin{aligned}
\tilde{V}(x_i(t),t) &= \bar{v}(x_i(t),t) + \bar{e}(x_i(t),t) \\
\text{where} & \\
\bar{v}(x_i(t),t) &= \mathbb{E}_t^{\mathbb{P}'}[(\rho(1 - x_i(T)))^+]
\end{aligned}$$

and

$$\bar{e}(x_i(t),t) = \mathbb{E}_t^{\mathbb{P}'}\left[\int_t^T \rho\mu_i(x_i(u),u)x_i(u)1_{\Im}(x_i(u))\mathrm{d}u\right]$$

where $\rho = 1$ for a call option, $\rho = -1$ for a put option and $i \in [A, G]$.

One can easily establish that using similar economic argument, but working under the risk neutral measure $\mathbb{P}$ an alternative characterization of the early exercise premium is

$$E_t^{\mathbb{P}'}\left[\int_t^T e^{-r(u-t)}\rho(\frac{\mathrm{d}A(u)}{\mathrm{d}u} - rA(u))1_{\Im}(A(u),S(u),u)\mathrm{d}u\right].$$

Proof By considering separately price changes on continuation and stopping regions. By Ito's lemma

$$\begin{aligned}
\mathrm{d}\bar{V} &= \tfrac{\partial\bar{V}}{\partial x}\mathrm{d}x + \tfrac{1}{2}\tfrac{\partial^2\bar{V}}{\partial x^2}(\mathrm{d}x)^2 + \tfrac{\partial\bar{V}}{\partial t}\mathrm{d}t \\
&= \tfrac{\partial\bar{V}}{\partial x}(\mu(x(t),t)x(t)\mathrm{d}t - \sigma x(t)\mathrm{d}W^{\mathbb{P}'}(t)) + \tfrac{1}{2}\sigma^2x^2(t)\tfrac{\partial^2\bar{V}}{\partial x^2}\mathrm{d}t + \tfrac{\partial\bar{V}}{\partial t}\mathrm{d}t \\
&= [\mu(x(t),t)x(t)\tfrac{\partial\bar{V}}{\partial x} + \tfrac{1}{2}\sigma^2x^2(t)\tfrac{\partial^2\bar{V}}{\partial x^2} + \tfrac{\partial\bar{V}}{\partial t}]\mathrm{d}t - \sigma x(t)\tfrac{\partial\bar{V}}{\partial x}\mathrm{d}W^{\mathbb{P}'}(t) \\
&= -\sigma x(t)\tfrac{\partial\bar{V}}{\partial x}\mathrm{d}W^{\mathbb{P}'}(t)
\end{aligned}$$

This follows that $\bar{V}$ is a $\mathbb{P}'-$ Martingale, where the functional argument of $\bar{V}$ and subscript on μ and x have been suppressed for ease of notation. On the stopping region $\Im$, the option value (denominated by the stock price) is simply the intrisic value of the option

$$\bar{V}(x(t),t) = \rho(1 - x(t))$$

so that on $\Im$ one must have

$$\begin{aligned}
\mathrm{d}\bar{V}(x(t),t) &= -\rho\mathrm{d}x(t) \\
&= -\rho(\mu(x(t),t)x(t)\mathrm{d}t - \sigma x(t)\mathrm{d}W^{\mathbb{P}'}(t)).
\end{aligned}$$

Hence,

$$\mathrm{d}\bar{V}(x(t),t) = -\rho 1_{\Im}(x(t))\mu(x(t),t)x(t)\mathrm{d}t - \mathrm{d}M^{\mathbb{P}'}(t)$$

where $M^{\mathbb{P}'}(t)$ is a $\mathbb{P}$-Martingale. Integrating the above equation and taking the expectation gives the desired result.

Analytical valuation of a continuous floating strike options Firstly, consider the case of geometric averaging, then derive formulas for American-style floating strike Asian options. And in the case of Arithmetic averaging the approximation formulas with characteristics in common with the geometric case. Consider the following lemma:

Lemma 1 For $u > t$

$$\ln x_G(u)|\Im_t \sim N(\alpha_G(t,u), \beta_G^2(t,u)) \tag{1.2}$$

where

$$\alpha_G(t,u) = \frac{t}{u}\ln x_G(t) - \frac{u^2 - t^2}{2u}(r + \frac{1}{2}\sigma^2) \tag{1.3}$$

and

$$\beta_G^2(t,u) = \frac{\sigma^2}{3u^2}(u^3 - t^3). \tag{1.4}$$

For the proof of this lemma see [2].

3.1.1 Floating strike Asian option of a geometric averaging

Theorem 2 (Valuation of the European-type floating strike Asian option) Let $C_G(t)$ be a call option and $P_G(t)$ be a put option of the European-type floating strike Asian option for geometric averaging on the interval $[0, T]$. The value of α and β are given in (1.3) and (1.4) respectively. By theorem 1 and lemma 1, the following result result is obtained, the value for call option is

$$\begin{aligned} C_G(t) &= S(t)\left[N(-\frac{\alpha_G(t,T)}{\beta_G(t,T)}) - e^{\alpha_G(t,T)+\frac{1}{2}\beta_G^2(t,T)}N(-\frac{\alpha_G(t,T)}{\beta_G(t,T)} - B_G(t,T))\right] \\ &= e^{-r(T-t)}[S(t)\exp((r - \frac{1}{2}\sigma^2)(T-t)^2/2(T-T_0) \\ &\quad +\sigma^2(T-t)^3/6(T-T_0)^2)N(d_1) - KN(d_2)] \end{aligned}$$

and the value for put option is

$$\begin{aligned}
P_G(t) &= S(t)\left[e^{\alpha_G(t,T)+\frac{1}{2}\beta_G^2(t,T)}N(\frac{\alpha_G(t,T)}{\beta_G(t,T)}+\beta_G(t,T))-N(\frac{\alpha_G(t,T)}{\beta_G(t,T)})\right] \\
&= e^{-r(T-t)}[KN(-d_2)-S(t)\exp((r-\frac{1}{2}\sigma^2)(T-t)^2/2(T-T_0) \\
&\quad +\sigma^2(T-t)^3/6(T-T_0)^2)N(-d_1)]
\end{aligned}$$

where

$$d_1=\frac{(T-T_0)\ln(\frac{\tilde{S}(t)}{K})+(r-\frac{1}{2}\sigma^2)(T-t)^2/2+\sigma^2(T-t)^3/3(T-T_0)}{\sigma\sqrt{(T-t)^3/3}}$$

$$d_2=\frac{(T-T_0)\ln(\frac{\tilde{S}(t)}{K})+(r-\frac{1}{2}\sigma^2)(T-t)^2/2}{\sigma\sqrt{(T-t)^3/3}}. \tag{1.5}$$

Theorem 3 (The early exercise premiums) Let $C_G^e(t)$ be an early exercise call option price and $P_G^e(t)$ be an early exercise put option price geometric for averaging of an American-style floating strike Asian option on the interval $[0,T]$. Then the following is obtained

$$\frac{C_G^e(t)}{S(t)}=\int_t^T e^{\alpha_G(t,u)+\frac{1}{2}\beta_G^2(t,u)}\times[\frac{\beta_G(t,u)}{u}n(\gamma_G(t,u))-N(-(\gamma_G(t,u))(\frac{\alpha_G(t,u)+\beta_G^2(t,u)}{u}+r)]\mathrm{d}u \tag{1.6}$$

and

$$\frac{P_G^e(t)}{S(t)}=\int_t^T e^{\alpha_G(t,u)+\frac{1}{2}\beta_G^2(t,u)}\times[\frac{\beta_G(t,u)}{u}n(\gamma_G(t,u))+N(\gamma_G(t,u))(\frac{\alpha_G(t,u)+\beta_G^2(t,u)}{u}+r)]\mathrm{d}u$$

where α and β are given in (1.3) and (1.4) respectively

$$\gamma_G(t,u)=\frac{(\alpha_G(t,u)+\beta_G^2(t,u))-\ln x_G^*(u)}{B_G(t,u)}$$

and x_G^* depends on the value of ρ.

Proof For put option $\rho=-1$ then we have

$$\frac{P_G^e(t)}{S(t)}=-\mathbb{E}_t^{\mathbb{Q}'}[\mu_G(x_G(u),u)x_G(u)1_{(x_G(u)\geq x_G^*(u))}\mathrm{d}u]$$

If we substitute $\mu_G(x_G(u),u) = -[\frac{1}{u}\ln x_G(u) + r]$ and interchange the order of integration we obtain

$$\begin{aligned}\frac{P^e_G(t)}{S(t)} &= \int_t^T \mathbb{E}_t^{\mathbb{Q}'}[(\tfrac{1}{u}\ln x_G(u) + r) \times x_G(u) 1_{(x_G(u) \geq x^*_G(u))}]\mathrm{d}u \\ &= \int_t^T (\tfrac{1}{u}\mathbb{E}_t^{\mathbb{Q}'}(\ln x_G(u) x_G(u) 1_{(x_G(u)) \geq x^*_G(u))} \\ &\quad + r\mathbb{E}_t^{Q'}(x_G(u)1_{(x_G(u)\geq x^*_G(u)))})\}\mathrm{d}u \\ &= \int_t^T \tfrac{1}{u} e^{\alpha_G(t,u)+\frac{1}{2}\beta^2_G(t,u)}[\beta_G(t,u)n(\gamma_G(t,u)) + (\alpha_G(t,u) + \beta^2_G(t,u))N((\gamma_G(t,u))] \\ &\quad + re^{\alpha_G(t,u)+\frac{1}{2}\beta^2_G(t,u)}N(\gamma_G(t,u)))\mathrm{d}u\end{aligned}$$

then the value of the European put option is

$$P_G = \int_t^T e^{\alpha_G(t,u)+\frac{1}{2}\beta^2_G(t,u)} \times [\frac{\beta_G(t,u)}{u} n(\gamma_G(t,u)) + N(\gamma_G(t,u))(\frac{\alpha_G(t,u)+\beta^2_G(t,u)}{u} + r)]\mathrm{d}u$$

The value of the American floating geometric call option can also be obtain by making a choice of $\rho = 1$

Arithmetic option Here we will see how the exact formula was used to derive Arithmetic analytical of a European option as well American option. The exact analytical evaluation of the expectation term of the arithmetic valuation formula equation (1.2) is not possible because the distribution of $x_A(.)$ is unknown. However, using the same ideas from the literature of the European style Asian options, we can utilize the general formula as a basis for a derivation of approximation formula, for average based American-style floating strike option. Since we are dealing with American-style option we need more than just one approximation distribution of the variables at the expiration date. but a whole family of approximating distributions since as clarified, by lemma 1, all the conditional distributions of $x_A(u)|\Im$ for $0 \leq t \leq u \leq T$ must be characterised. And approximating t at any time the remainder of the stochastic process $(x_A(u))_{u\geq t}$ by a geometric Brownian motion $(\hat{x}_A(u))_{u\geq t}$ augmented with appropriate time-varying coefficients. By approximating the log of the average divided by the log of the contemporaneous stock price to a normal distribution instead of approximating the log of the average and the log of the contemporaneous stock price to a bivariate normal distribution. Then the log of the fraction between these two variable is also normally distributed, the reverse is in general not true. To be more specific for $u > t$, consider $\ln \hat{x}_A(t,u)|\Im_t$, which is normal with mean $\alpha_A(t,u)$ and variance parameters $\beta_A^2(t,u)$. By Wilkinson approximation

$$\alpha_A(t,u) = 2\ln \mathbb{E}_t^{\mathbb{P}'}[(x_A(u))] - \frac{1}{2}\ln \mathbb{E}_t^{\mathbb{P}'}[(x_A^2(u))]$$

and

$$\beta_A^2(t,u) = \ln \mathbb{E}^{\mathbb{P}'}[(x_A^2(u))] - 2\ln \mathbb{E}_t^{\mathbb{P}'}[(x_A(u))].$$

The above equation under $\mathbb{P}'$ of $x_A(u)$ can be calculated analytically. This lead to

$$\mathbb{E}_t^{\mathbb{P}'}[(x_A(u))] = \frac{t}{u}x_A(t)e^{-r(u-t)} + \frac{1}{ru}(1 - e^{-r(u-t))})$$

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{P}'}[(x_A^2(u))] \quad &= \quad \left(\frac{t}{u}\right)^2 x_A^2(t)e^{(-2r-\sigma^2)(u-t)} \\
&+ \frac{2(r-\sigma^2) - (4r - 2\sigma^2)e^{-r(u-t)} + 2re^{-(2r-\sigma^2)(u-t)}}{u^2 r(2r-\sigma^2)(r-\sigma^2)} \\
&+ x_A(t)\frac{2te^{-r(u-t)}}{u^2(r-\sigma^2)}(1 - e^{-(r-\sigma^2)(u-t)}).
\end{aligned}$$

Now the American-style based on arithmetic can be derived, we can now state the following theorem.

3.1.2 Floating strike Asian option of an Arithmetic averaging

Theorem 4 (Valuation of the European-type strike Asian option of an Arithmetic average) Let $\tilde{C}_A(t)$ be an approximated value of a call option and

$\tilde{P}_A(t)$ be an approximated value of put option on the interval $[0,T]$ of continuous Arithmetic averaging. Also let $\tilde{C}^e_A(t)$ be an approximated early exercise value of call option and $\tilde{P}^e_A(t)$ be the approximated early exercise price of a put option of the American-style option. Then the value for an Arithmetic call option is

$$\tilde{C}_A(t) = S(t)[N(-\frac{\alpha_A(t,T)}{\beta_A(t,T)}) - e^{\alpha_A(t,T)+\frac{1}{2}\beta^2_A(t,T)}N(-\frac{\alpha_A(t,T)}{\beta_A(t,T)}) - \beta_A(t,T)] \qquad (1.7)$$

and the value for put option is

$$\tilde{P}_A(t) = S(t)[e^{\alpha_A(t,T)+\frac{1}{2}\beta^2_A(t,T)}N(\frac{\alpha_A(t,T)}{\beta_A(t,T)}) + \beta_A(t,T)) - N(\frac{\alpha_A(t,T)}{\beta_A(t,T)})] \qquad (1.8)$$

where α_G and $\beta_G(t,u)$ are defined in (1.3) and (1.4) and N is the standard normal density function and the early exercise values are

$$\tilde{C}^e_A(t) = S(t)\int_t^T [\frac{1}{u}N(\beta_A(t,u) - \gamma_A(t,u)) - (r+\frac{1}{u})e^{\alpha_A(t,u)+\frac{1}{2}\beta^2_A(t,u)}N(-\gamma_A(t,u))]\mathrm{d}u \qquad (1.9)$$

and

$$\tilde{P}^e_A(t) = S(t)\int_t^T [(r+\frac{1}{u})e^{\alpha_A(t,u)+\frac{1}{2}\beta^2_A(t,u)}N(\gamma_A(t,u)) - \frac{1}{u}N(\gamma_A(t,u) - \beta_A(t,u))]\mathrm{d}u \qquad (1.10)$$

where

$$\gamma_A(t,u) = \frac{(\alpha_A(t,u) + \beta^2_A(t,u)) - \ln \tilde{x}^*_A(u)}{B_A(t,u)}.$$

Proof From theorem 1

$$\frac{C^e_A(t)}{S(t)} = \mathbb{E}^{\mathbb{P}'}_t[\int_t^T \mu_A(\hat{x}_A(u),u)\hat{x}_A(u)1_{\hat{x}_A(u)\leq\hat{x}^*_A(u)}\mathrm{d}u].$$

Substituting $\mu_A(\hat{x}_A(u),u) = -[\frac{1}{t}\ln x_A(u) + r]$ and interchanging the order of integration one get

$$\begin{aligned}\tfrac{C^e_A(t)}{S(t)} &= \textstyle\int_t^T \mathbb{E}^{\mathbb{P}'}_t[[(\frac{1}{u}\hat{x}^{-1}_A(t,u) - 1) - r] \times \hat{x}_A(t,u)1_{\hat{x}_A(u)\leq\hat{x}^*_A(u))}]\mathrm{d}u \\ &= \textstyle\int_t^T (\frac{1}{u}\mathbb{E}^{\mathbb{P}'}_t(1_{\hat{x}_A(u))\leq\hat{x}^*_A(u)} - (r+\frac{1}{u})\mathbb{E}^{\mathbb{P}'}_t(\hat{x}_A(t,u)1_{(\hat{x}_A(u)\leq\hat{x}^*_A(u))})\mathrm{d}u.\end{aligned}$$

Then the value of the European put option is

$$\int_t^T (\frac{1}{u}N(\beta_A(t,u)-\gamma_A(t,u))-(r+\frac{1}{u})e^{\alpha_A(t,u)+\frac{1}{2}\beta_A^2(t,u)}\times N(-\gamma_A(t,u)))\mathrm{d}u.$$

Numerical Result To calculate any of the early exercise price in principle, $x^*(u)$ for $u \in [t,T]$ must be known, with the boundary condition

$$\bar{V}(x^*(t),t)=\rho(1-x^*(t))$$

at the maturity date $x^*(T)=1$. To obtain the intermediate value of x^* a discretization scheme is used. The time interval $[0,T]$ is divided into n interval of equal length Δt defining the time points $0=t_0<t_1<\cdots<t_n=T$ when $\Delta t=t_{i+1}-t_i=\frac{T}{n}$. The methodology is to work towards $t=0$. To work out the value of $x^*(t_i)$ is contingent on $x^*(t_{i+1})\cdots x^*(t_n)$ having been found already. When $x^*(0)$ has been determined at $t=0$ the early exercise price can be determined by one final numerical evaluation of the integral in question.

Table 3.1, shows the result from [2] a floating strike Asian call of an American and European style option, with $\sigma=010$, T=1 and $S_0=100$, based on an approximation. In the case of Arithmetic average the values were compared with the values of a finite difference method. The finite difference pricing result for the geometric average based options were found to be exactly to the result obtained from the implementation when measured to three decimal places.

TABLE 3.1

			Geo Av (ExactForm)		Ari Av (ApprForm)		(FinDiffSoln)	
r	T	σ	American	European	American	European	American	European
	$\frac{1}{12}$	0.20	1.955	1.406	1.953	1.392	1.949	1.392
		0.30	2.909	2.088	2.905	2.056	2.895	2.056
		0.40	3.864	2.776	3.857	2.721	3.838	2.720
0.03	$\frac{4}{12}$	0.20	4.007	2.967	3.998	2.907	3.980	2.907
		0.30	5.912	4.358	5.897	4.232	5.854	4.228
		0.40	7.818	5.774	7.799	5.560	7.718	5.548
	$\frac{7}{12}$	0.20	5.382	4.056	5.366	3.950	5.334	3.949
		0.30	7.898	5.917	7.874	5.697	7.796	5.688
		0.40	10.412	7.820	10.388	7.452	10.238	7.425
	$\frac{1}{12}$	0.20	1.988	1.449	1.985	1.434	1.981	1.435
		0.30	2.941	2.130	2.936	2.098	2.926	2.097
		0.40	3.895	2.817	3.888	2.762	3.869	2.761
0.05	$\frac{4}{12}$	0.20	4.138	3.143	4.127	3.079	4.110	3.079
		0.30	6.039	4.528	6.021	4.396	5.980	4.393
		0.40	7.942	5.941	7.918	5.719	7.839	5.709
	$\frac{7}{12}$	0.20	5.615	4.369	5.593	4.252	5.564	4.253
		0.30	8.119	6.217	8.088	5.982	8.014	5.975
		0.40	10.626	8.111	10.593	7.724	10.448	7.701

3.2 Fixed strike Asian option

3.2.1 Fixed strike Asian option of a Geometric Averaging

Let $S(t)$ be the price of the traded security at time t, let $A_A(t_0, t)$ and $A_G(t_0, t)$ be an average value of an Arithmetic and Geometric respectively at time t. Where $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ with $t_k < t < t_{k+1}$, and $\Delta t = t_{i+1} - t_i$ Let $V(t; t_0, t_n)$ be the value of the replicating portfolio at time $t, t_k \leq t < t_{k+1}$ where $k = 0, 1, \cdots, n-1$. Then

$$V(t; t_0, t_n) = e^{-r(t_n - t)} \tilde{\mathbb{E}}_t\{A(t_0, t_n)\} \tag{2.11}$$

where the expectation is taken with respect to the martingale measure that prices all the securities, see Harrison and Pliska in [10]. For arithmetic and geometric mean, equation (2.11) reduces to

$$V_A(t; t_0, t_n) = e^{-r(t_n - t)} [\frac{k}{n} A_A(t_0, t_k) + \frac{n-k}{n} \tilde{\mathbb{E}}_t\{A_A(t_k, t_n)\}] \tag{2.12}$$

$$V_G(t; t_0, t_n) = e^{-r(t_n - t)} [A_G(t_0, t_k)]^{\frac{k}{n}} \tilde{\mathbb{E}}_t[\{A_G(t_k, t_n)\}^{\frac{(n-k)}{n}}]. \tag{2.13}$$

Assume that prices follows a Geometric Weiner process with μ and σ representing the instantaneous expected return and volatility respectively. Prices are reset points t_i and t_{i+1} are related as

$$S(t_{i+1}) = t_i e^{\alpha \Delta t + \sqrt{\Delta t} \tilde{Z}_i} \tag{2.14}$$

where $\alpha = \mu - \frac{1}{2}\sigma^2$, and $\tilde{Z}_i$ is a standard normal random variable, under the equivalent martingale measure $\alpha = r - \frac{1}{2}\sigma^2$. According to [10], given this process the expectation in (2.12) and (2.13) can be computed as

$$\tilde{\mathbb{E}}_t\{A_A(t_k, t_n)\} = v(k) S(t) \tag{2.15}$$

$$\tilde{\mathbb{E}}_t[\{A_G(t_k,t_n)\}^{\frac{(n-k)}{n}}]e^{a_k+b_k^2/2}(S_t)^{\frac{(n-k)}{n}} \tag{2.16}$$

where

$$\begin{aligned}
v(k) &= \frac{e^{r(t_{k+1}-t)}\left[e^{(n-k)rT/n}-1\right]}{(n-k)(e^{rT/n-1})},\\
a_k &= \alpha(n-k)\left[\frac{t_{k+1}-t}{n}+T\frac{n-k-1}{2n^2}\right],\\
b_k^2 &= \sigma^2\left[(\frac{n-k}{n})^2(t_{k+1}-t)+\frac{T}{3n^2}(n-k-1)(n-k-\frac{1}{2})(n-k)\right].
\end{aligned}$$

Substituting (2.15)and (2.16) into (2.12)and (2.13) we obtain

$$V_A(t;t_0,t_n) = e^{-r(t_n-t)}[\frac{k}{n}A_A(t_0,t_k)+\frac{n-k}{n}v(k)S(t)]$$

$$V_G(t;t_0,t_n) = e^{-r(t_n-t)}[A_G(t_0,t_k)]^{\frac{k}{n}}e^{a_k+b_k^2/2}(S_t)^{\frac{(n-k)}{n}}.$$

At time t, the value of the European averaging is obtained by computing the appropriate expectations of the terminal payoffs under the equivalent martingale measure and discounted by the riskless rate. Since the distribution of the underlying geometric averaging is lognormal, the price of European geometric averaging option on the geometric mean can readily be establish. Then the value of averaging option on the geometric mean is obtained as

$$\begin{aligned}
C(t) &= V_G(t;t_0,t_n)N(d_1)-Ke^{-r(t_n-t)}N(d_2)\\
&\text{where}\\
d_1 &= \tfrac{\ln(V_G(t;t_0,t_n)/K)+r(t_n-t)+b_k^2/2}{b_k}\\
d_2 &= d_1-b_k.
\end{aligned} \tag{2.17}$$

When the number of reset points n equals unity, $C(t)$ reduces to the Black-Scholes equation. Further as the number of reset points increases, the price of the option decline and converge asymptotically to

$$\begin{aligned}
C(t) &= S(t_0)e^{-(r+\sigma^2/6)T/2}N(d_1) - Ke^{-rt}N(d_2) \\
\text{where} & \\
d_1 &= \tfrac{\ln(S(t_0)/K)+(r+\sigma^2/6)T/2}{\sigma\sqrt{T/3}} \\
d_2 &= d_1 - \sqrt{T/3}
\end{aligned} \tag{2.18}$$

This equation was first developed by Kemna and Vorst (1990) as we have seen in chapter 2. It provides the value of the averaging option on the geometric average with continuum of reset points. It can be interpreted as the Black-Scholes formula for a call but where the volatility parameter is $\nu \equiv \sigma/\sqrt{3}$ and where the dividend yield parameter is $\delta \equiv (r+\nu^2/2)/2$. Sine $\nu < \sigma$ and $\delta > 0$, a call on a continuously reset geometric average is worth less than a call (a call on stock price) on an average set once at expiration.

An analytical approximation for European options on the Arithmetic mean Under the assumption that the price process is proportional (lognormal), the distribution of the arithmetic average is the convolution of the finite number of lognormal distributions. Since no analytical solution for convoluted distribution exist, the approach that we will show in this section is based on approximating the true distribution by some standard distribution. Analytical approximation are useful because explicit expressions can be developed for comparative static measures such as delta, gamma, theta and vega values.

3.2.2 Fixed strike Asian option of an Arithmetic averaging

The value for Arithmetic averaging is derive by using the approximation approach based on approximating the true distribution by some standard distribution, the

Edgeworth series expansion. Let $f_t(x)$ and $f_a(x)$ represent the density functions of the true and approximating distribution, and let κ_m^t and κ_m^a denote their mth order cumulants respectively. Edgeworth series expansion is given by

$$f_t(x) = f_a(x) + \sum_{i=1}^{\infty}(-1)^l \frac{\theta_l}{l!}\frac{\partial f_a(x)}{\partial x^l}$$

where θ_l is the coefficient of $w^l/l!$ in the expansions of

$$\prod_{j=1}^{\infty}\left[1 + \frac{1}{1!}(\frac{(k_j^t - k_j^a)w^j}{j!}) + \frac{1}{2!}(\frac{(k_j^t - k_j^a)w^j}{j!})^2 + \frac{1}{3!}(\frac{(k_j^t - k_j^a)w^j}{j!})^3 + \cdots\right]$$

where the first four coefficients are given by

$\theta_1 = \kappa_1^t - \kappa_1^a,$

$\theta_2 = \kappa_2^t - \kappa_2^a + \theta_1^2,$

$\theta_3 = \kappa_3^t - \kappa_3^a + 3\theta_1(\kappa_2^t - \kappa_2^a) + \theta_1^3,$

$\theta_4 = \kappa_4^t - \kappa_4^a + 4\theta_1(\kappa_3^t - \kappa_3^a) + 3(\kappa_2^t - \kappa_2^a)^2 + 6\theta_1^2(\kappa_2^t - \kappa_2^a) + \theta_1^4.$

By truncating the expansion after a finite number of steps, an approximation to the true distribution is obtained.

The Edgeworth expansion needs the moments of the true and approximating distributions. The average $A(t_0, t_n)$ can be written as

$$A(t_0, t_n) = [Y_1 + Y_1Y_2 + Y_1Y_2Y_3 + Y_1Y_2Y_3Y_4 + \cdots + Y_1Y_2Y_3\cdots Y_n]S(t_0)/n \quad (2.19)$$

where

$Y_i = S(t_i)/S(t_{i-1})$ is the price relative between t_{i-1} and t_i The price relatives are independent and identically distributed lognormal random variable with

$\tilde{\mathbb{E}}_{t_0}\{Y_i\} = e^{\frac{rT}{n}}$

and

$\tilde{\mathbb{V}ar}_{t_0}\{Y_i\} = e^{\frac{2rT}{n}}(e^{\frac{\sigma^2 T}{n}} - 1).$

Hence the pth moments of the average is given by

$\tilde{\mathbb{E}}_{t_0}\{A(t_0, t_n)^p\} = [\frac{S(t_0)}{n}]^p \tilde{\mathbb{E}}_{t_0}\{\sum_{k_1,k_2,\cdots,k_n\geq 0}(\frac{p!}{k_1!k_2!\cdots k_n!})Y_1^{a_1}Y_2^{a_2}Y_3^{a_3}\cdots Y_n^{a_n}\}$

where $a_j = k_j + k_{j+1} + \cdots + k_n$, and $k_1 + k_2 + \cdots + k_n = p$. Computing this expectation gives

$$\begin{aligned}\tilde{\mathbb{E}}_{t_0}\{A(t_0,t_n)^p\} &= \left[\frac{S(t_0)}{n}\right]^p \sum_{k_1,k_2,\cdots,k_n\geq 0}\left(\frac{p!}{k_1!k_2!\cdots k_n!}\right) \\ &\quad \times \exp\left(\sum_{j=1}^{n}[a_j(r-\sigma^2/2)(T/n)+a_j^2\sigma^2T/(2n)]\right). \quad (2.20)\end{aligned}$$

Since all the moments of the lognormal distribution can be obtained, the value of an arithmetic options can be approximated to any desired level of accuracy. A two term Edgeworth expansion yields

$f_t(x) = f_a(x) - \theta_1 \frac{\partial f_a(x)}{\partial x} + \frac{\theta_2}{2}\frac{\partial^2 f_a(x)}{\partial x^2} + \varepsilon$

$\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 = k_3^t - k_3^a$, $\sigma_4 = k_4^t - k_4^a$, $\sigma_5 = k_5^t - k_5^a$, $\sigma_6 = k_6^t - k_6^a + 10\theta_3^2$.

The expectation of the terminal average is obtained as in (2.16) and the variance is obtained from (2.20) as

$$\begin{aligned}\tilde{\mathbb{V}ar}_{t_0}\{A(t_0,t_n)\} &= (\frac{S(t_0)}{n})^2(h_1(2r+\sigma^2) - h_1(2r) + 2f_0(r+\sigma^2)h_2(2r+\sigma^2) \\ &\quad -2f_1(r+\sigma^2)h_2(r) - 2f_0(r)h_2(2r) + 2f_1(r)h_2(r)) \qquad (2.21)\end{aligned}$$

where

$h_1(x) = \frac{e^{x\Delta t}(e^{nx\Delta t}-1)}{e^{x\Delta t}-1}$,

$h_2(x) = \frac{e^{2x\Delta t}(e^{(n-1)x\Delta t}-1)}{e^{x\Delta t}-1}$,

$f_0(x) = \frac{1}{e^{x\Delta t}-1}$,

$f_1(x) = \frac{e^{x\Delta t}}{e^{x\Delta t}-1}$.

The analytical approximation model for averaging can now be established. Given the logarithmic variance of the average $\xi^2 T$ as

$$\xi^2 T = \ln\left(1 + \frac{\tilde{\mathbb{V}ar}_{t_0}(A(t_0,t_n))}{[\mathbb{E}_{t_0}(A(t_0,t_n))]^2}\right) \qquad (2.22)$$

Then the approximated value for an arithmetic was found.

Proposition The value of an arithmetic averaging option can be approximated as

$$C_A(t_0) \approx V_A(t_0; t_0, t_n)N(d_1) - Ke^{-r(t_n - t_0)}N(d_2) \tag{2.23}$$

where

$$d_1 = \frac{\ln(V_A(t_0; t_0, t_n)/K) + (r + \xi^2/2)T}{\xi\sqrt{T}}$$
$$d_2 = d_1 - \xi\sqrt{T}.$$

3.3 Pricing Asian Option by Lower and Upper Bounds

In this section we show how to derive the formulas for lower and upper bounds, to price the value of an Arithmetic averaging in both fixed and floating strike Asian options according to (Rogers and Shi) and Thompson. We will firstly look at Rogers and Shi's lower and upper bounds.

Rogers and Shi 's Lower and Upper bounds

The Lower bound Rogers and Shi (1995) exploits the inequality:

$$\mathbb{E}[Y^+] = \mathbb{E}[\mathbb{E}(Y^+|Z)] \geq \mathbb{E}[\mathbb{E}(Y + |Z)^+] \tag{3.24}$$

which holds for any random variables Y and Z.
For a fixed strike Asian option, we choose $Y = \int_0^1 S_t \mathrm{d}t - K$ and $Z = \int_0^1 B_t \mathrm{d}t$. Since the inner expectation is

$$\int_0^1 \mathbb{E}(Se^{\alpha t + \sigma B_t}| \int_0^1 B_s \mathrm{d}s)\mathrm{d}t - K \tag{3.25}$$

and conditional

$$\int_0^1 B_t \mathrm{d}t = z \tag{3.26}$$

B_t is normal with mean $3t(1-t/2)z$ and variance $t - 3t^2(1-t/2)^2$ then the lower bound is

$$V_{fixed} \geq V_{low} \equiv e^{-r} \int_{-\infty}^{\infty} \sqrt{3}\phi(\sqrt{3}z)[\int_0^1 S_0 e^{3\sigma t(1-t/2)z+\alpha t+1/2\sigma^2(t-3t^2(1-t/2)^2)}\mathrm{d}t - K]^+ \mathrm{d}z \tag{3.27}$$

and for floating strike option we choose $Y = \int_0^1 S_t \mathrm{d}t - S_1$, $Z = \int_0^1 B_t \mathrm{d}t - B_1$ and we get

$$V_{floating} \geq V_{low} \equiv e^{-r} \int_{-\infty}^{\infty} \sqrt{3}\phi(\sqrt{3}z)[\int_0^1 S_0 e^{\frac{-3\sigma tz}{2}+\alpha t+1/2\sigma^2(t-3t^4/4)}\mathrm{d}t - Se^{\frac{-3\sigma z}{2}+\alpha+\sigma^2/8}]^+ \mathrm{d}z. \tag{3.28}$$

Both of these equations are slightly tricky to evaluate, since the outer integration has a non-smooth integrand.

The Upper bound Rogers and Shi obtained an upper bound by considering the error on the lower bound. For their accuracy if $\sigma = 0.3$, $\rho = 0.09$, $S = 100$, $T = 1$ and $k = 100$ the lower bound for both (fixed and floating option) is 8.8275, then the upper bound is 9.039, see Table 3.2.

Thompson 's Lower and Upper bounds Thompson derived the lower and upper bound which gives the same results as that of Rogers and Shi. This was proven in [8]. It is much easier to compute the lower bound, since it involves only a one-dimensional integral.

The Lower bound Let $\mathcal{A} = \{w : \int_0^1 S_t \mathrm{d}t > K\}$ and note that

$$\mathbb{E}[(\int_0^1 Se^{\alpha t+\sigma B_t} - K)^+] = \int_0^1 \mathbb{E}[(Se^{\alpha t+\sigma B_t} - K)I(\mathcal{A})]\mathrm{d}t. \tag{3.29}$$

By replacing $\mathcal{A}$ by some other event $\mathcal{A}'$, they no longer have equality in (3.29), the right hand side is now a lower bound. Use $\mathcal{A}' = \{\int_0^1 B_t \mathrm{d}t > \gamma\}$. The optimal value of γ is determined by letting $N_t = \alpha t + \sigma B_t + \ln S$ and for any random variable X with density $f_X(x)$

$$\frac{\partial}{\partial \gamma} \int_0^1 \mathbb{E}(e^{N_t} - K; X > \gamma)\mathrm{d}t = \int_0^1 \mathbb{E}(e^{N_t} - K | X = \gamma)(-f_X(\gamma))\mathrm{d}t. \tag{3.30}$$

Thus the optimal value of γ, γ^* satisfies

$$\int_0^1 \mathbb{E}(e^{N_t} - K | X = \gamma^*)\mathrm{d}t = K \tag{3.31}$$

with the choice of $X = \int_0^1 B_t \mathrm{d}t$, it was concluded that

$$\int_0^1 S e^{3\gamma^* t(1-t/2)+\alpha t+1/2\sigma^2(t-3t^2(1-t/2)^2)}\mathrm{d}t = K \tag{3.32}$$

γ^* is uniquely determined. Now the bound is

$$V_{fixed} \geq e^{-r} \int_0^1 \mathbb{E}[(Se^{\alpha t+\sigma B_t} - K)I(\int_0^1 B_s \mathrm{d}s > \gamma^*)]\mathrm{d}t. \tag{3.33}$$

Now we can calculate the expectation. Fix $t \in (0, 1)$ and let $\Phi_1 = \alpha t + \sigma B_t + \ln S$ and $\Phi_2 = \int_0^1 B_s \mathrm{d}s - \gamma^*$. Let $\mu_i = \mathbb{E}(\Phi_i)$, $\sigma_i^2 = \mathbb{V}ar(\Phi_i)$ and $c = \mathbb{C}ov(\Phi_1, \Phi_2)$, then using

$$\mathbb{E}[(e^{\Phi_1} - K)I(\Phi_2 > 0)] = e^{\mu_1+1/2\sigma_1^2} N(\frac{\mu_2 + c}{\sigma_2}) - KN(\frac{\mu_2}{\sigma_2}) \tag{3.34}$$

and substituting $\mu_1 = \alpha t + \ln S$, $\mu_2 = -\gamma^*$, $\sigma_1^2 = \sigma^2 t$, $\sigma_2^2 = 1/3$, $c = \sigma t(1 - t/2)$, the lower bound is

$$V_{fixed} \geq V_{low} \equiv e^{-r}[\int_0^1 S e^{\alpha t+\frac{1}{2}\sigma^2 t} N(\frac{-\gamma^* + \sigma t(1 - t/2)}{1\sqrt{3}})dt - KN(\frac{-\gamma^*}{1\sqrt{3}})] \tag{3.35}$$

for floating strike option, let $\mathcal{A} = \{w : \int_0^1 S_t \mathrm{d}t > S_1\}$ and use an approximation to $\mathcal{A}$ of the form $\mathcal{A}' = \{\int_0^1 B_t \mathrm{d}t - B_1 > \gamma\}$, with the choice γ^*, the optimal value of γ, satisfies

$$\mathbb{E}[(\int_0^1 Se^{\alpha t+\sigma B_t} - Se^{\alpha+\sigma B_1})\mathrm{d}t | \int_0^1 B_s \mathrm{d}s - B_1 = \gamma^*] = 0. \tag{3.36}$$

which gives

$$e^{3\gamma^*\sigma/2-\alpha-\sigma^2/8}\int_0^1 e^{-3\gamma^*\sigma t^2/2+\alpha t+1/2\sigma^2(t-3t^4/4)}\mathrm{d}t = 1 \tag{3.37}$$

which has a unique solution.

$$V_{floating} \geq \mathbb{E}[(\int_0^1 Se^{\alpha t+\sigma B_t} - Se^{\alpha+\sigma B_1})I(\int_0^1 B_s\mathrm{d}s - B_1 > \gamma^*)]\mathrm{d}t \tag{3.38}$$

reduces to

$$V_{floating} \geq V_{low} \equiv e^{-r}[\int_0^1 Se^{\alpha t+\frac{1}{2}\sigma^2 t}N(\frac{-\gamma^* - \sigma t^2/2}{1\sqrt{3}})dt - Se^{\alpha+\frac{1}{2}\sigma^2 t}N(\frac{-\gamma^* - 1/2}{1\sqrt{3}})] \tag{3.39}$$

where

$$\gamma^* \approx \frac{1}{\sigma}[\alpha/2 + \ln(2 - e^{-\alpha/4-\sigma^2/8})\int_0^1 e^{\alpha t-3\alpha t^2/4+1/2\sigma^2(t-3t^4/4)}\mathrm{d}t)]. \tag{3.40}$$

The Upper bound Let X be a random variable, and let $f_t(w)$ be a random function with $\int_o^1 f_t(w)\mathrm{d}t = 1$ for all w. Then

$$\begin{aligned}\mathbb{E}[(\int_0^1 Se^{\alpha t+\sigma B_t}\mathrm{d}t - X)^+] &= \mathbb{E}[(\textstyle\int_0^1(Se^{\alpha t+\sigma B_t} - Xf_t)\mathrm{d}t)^+] \\ &\leq \mathbb{E}[\textstyle\int_0^1(Se^{\alpha t+\sigma B_t} - Xf_t)^+\mathrm{d}t] \\ &= \textstyle\int_0^1 \mathbb{E}[(Se^{\alpha t+\sigma B_t} - Xf_t)^+]\mathrm{d}t.\end{aligned} \tag{3.41}$$

Use $f_t = \mu_t + \sigma(B_t - \int_0^1 B_s\mathrm{d}s)$, for both fixed and floating strike cases, where μ_t is a deterministic function satisfying $\int_0^1 \mu_t\mathrm{d}t = 1$, and derive an expression for μ_t, which is approximately optimal in each case. In the case of a fixed strike option, take $X = K$ in (3.41), and consider the choice $f_t = \mu_t$. To choose μ_t, we will minimize the right hand side of (3.41) over the set of deterministic functions f_t such that $\int_0^1 f_t\mathrm{d}t = 1$. Let

$$L(\lambda, \{f_t\}) = \mathbb{E}[\int_0^1 (Se^{\alpha t+\sigma B_t} - Xf_t)^+\mathrm{d}t] - \lambda(\int_0^1 \mathrm{d}t - 1) \tag{3.42}$$

be the Lagrangian, and consider stationary with respect to $\{f_t\}$ for the unconstrained problem. This gives the condition

$$\int_0^1 (-K\mathbb{P}(Se^{\alpha t+\sigma B_t} \geq Kf_t) - \lambda)\epsilon_t \mathrm{d}t = 0 \tag{3.43}$$

where ϵ_t is some small deterministic perturbation. $\mathbb{P}(Se^{\alpha t+\sigma B_t} \geq Kf_t)$ must be independent of t. Also $\ln(Kf_t/S) - \alpha t = \gamma\sigma\sqrt{t}$ for some constant γ for the choice of $\int_0^1 f_t \mathrm{d}t = 1$. Thus the optimal choice for f_t is

$$f_t = (S/K)e^{\alpha t+\sigma\gamma\sqrt{t}}. \tag{3.44}$$

Since $\int_0^1 f_t \mathrm{d}t$ is monotone increasing in γ, the correct value for γ is easy to estimate numerically. If instead $f_t = \mu_t + \sigma(B_t - \int_0^1 B_s \mathrm{d}s)$, the condition for stationary with respect to small deterministic perturbation is

$$\mathbb{P}[(Se^{\alpha t+\sigma B_t} \geq K(\mu_t + \sigma(B_t - \int_0^1 B_s \mathrm{d}s))] = \lambda, \forall t \tag{3.45}$$

This cannot be easily arranged to give the dependence of μ_t on λ Instead they use the aproximation $e^{\sigma B_t} \approx 1 + \sigma B_t$, for small values of σ. This leads to a condition $\mathbb{P}(Se^{\alpha t}(1+\sigma B_t) \geq Kf_t = \lambda$, $\forall t$. By letting $N(t) = (Se^{\alpha t}\sigma - K\sigma)B_t + K\sigma\int_0^1 B_s \mathrm{d}s$, one can conclude that $\mathbb{P}(N_t \geq K\mu_t)$ must be independent of t. Using the fact about the joint distribution of $(B_t, \int_0^1 B_s \mathrm{d}s)$, they deduce that

$$\mu_t = \frac{1}{K}(Se^{\alpha t} + \gamma\sqrt{v_t}) \tag{3.46}$$

where

$$v_t = \mathbb{V}ar(N_t) = c_t^2 + 2(K\sigma)c_t t(1-t/2) + (K\sigma)^2/3 \tag{3.47}$$

$$c_t = Se^{\alpha t}\sigma - K\sigma. \tag{3.48}$$

Imposing $\int_0^1 \mu_t \mathrm{d}t = 1$ gives

$$\gamma = (K - S(e^{\alpha}-1)/\alpha)/\int_0^1 \sqrt{v_t}\mathrm{d}t. \tag{3.49}$$

$\int_0^1 \sqrt{v}_t \mathrm{d}t$ is estimated numerically. Since γ is known and hence the function μ_t for the upper bound:

$$V_{fixed} \le e^{-r} \int_0^1 \mathbb{E}[(Se^{\alpha t+\sigma B_t} - K(\mu_t + \sigma(B_t - \int_0^1 B_s \mathrm{d}s)))^+]\mathrm{d}t, \tag{3.50}$$

when $B_t = x$, this becomes

$$V_{fixed} \le e^{-r} \int_0^1 \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} n(\frac{x}{\sqrt{t}}) \mathbb{E}[a(t,x) + b(t,x)N)^+]\mathrm{d}x\mathrm{d}t \tag{3.51}$$

where N has $N(0,1)$ distribution, and the functions a and b are given by

$$a(t,x) = Se^{\sigma x+\alpha t} - K(\mu_t + \sigma x) + K\sigma(1-t/2)x \tag{3.52}$$

$$b(t,x) = K\sigma\sqrt{1/3 - t(1-t/2)^2}. \tag{3.53}$$

By calculating $\mathbb{E}[(a+bN)^+]$, we obtain $aN(a/b) + bn(a/b)$. In the form of (3.51) the integrand behave badly near $(0,0)$ so we perfom the change of variables $v = \sqrt{t}$, $w = x/\sqrt{t}$, giving

$$V_{fixed} \le V_{upper} \equiv e^{-r}[\int_0^1 \int_{-\infty}^{\infty} 2vn(w)[a(t,x)N(\frac{a(t,x)}{b(t,x)}) + b(t,x)dn(\frac{a(t,x)}{b(t,x)})]\mathrm{d}w\mathrm{d}v. \tag{3.54}$$

This expression, combined with (3.46), (3.47), (3.49) and (3.52), constitutes the upperbound in the case of a fixed strike.

For the case of a floating strike option, we take $X = S_1$, setting $Z = \int_0^1 B_t \mathrm{d}t$, the condition for stationarity with respect to small deterministic perturbations analogous to (3.45) is

$$\mathbb{E}[-Se^{\alpha+\sigma B_1}; Se^{\alpha t+\sigma B_t} \ge (\mu_t + \sigma(B_t - Z))Se^{\alpha+\sigma B_1}] = \lambda, \forall t \tag{3.55}$$

This is approximated by

$$\mathbb{P}[Se^{\alpha+\sigma B_t} \ge \mu_t + \sigma(B_t - Z))Se^{\alpha+\sigma B_1} = \lambda'. \tag{3.56}$$

Further using approximation $e^{\sigma B_t+\alpha t} \approx 1 + \sigma(B_t - B_1)$, by

$$\mathbb{P}[e^{\alpha(t-1)}(1 + \sigma(B_t - B_1) \ge \mu_t + \sigma(B_t - Z))] = \lambda'', \forall t \tag{3.57}$$

for some γ

$$\begin{aligned} u_t &= e^{\alpha(t-1)} + \gamma\sqrt{v}_t \quad (3.58)\\ v_t &= \mathbb{V}ar[e^{\alpha(t-1)}(1+\sigma(B_t - B_1) - \sigma(B_t - Z)]. \quad (3.59)\end{aligned}$$

Since $\int_0^1 \sqrt{v}_t \mathrm{d}t$, one must have

$$\gamma = (1-(1-e^{\alpha})/\alpha)/\int_0^1 \sqrt{v}_t \mathrm{d}t. \quad (3.60)$$

Therefore the bound is given by

$$\begin{aligned} V_{floating} &\le e^{-r}\int_0^1 \mathbb{E}[(Se^{\sigma B_t+\alpha t} - (\mu_t + \sigma(B_t - Z))Se^{\sigma B_1+\alpha t})^+]\mathrm{d}t \\ &= e^{-r}\int_0^1 \mathbb{E}[(e^{N_1(t)} - N_2(t)e^{N_3(t)})^+]\mathrm{d}t \end{aligned} \quad (3.61)$$

where

$N_1(t) = \sigma B_t + \alpha t + \ln S$, $N_2(t) = \mu_t + \sigma(B_t - \int_0^1 B_s \mathrm{d}s)$

and

$N_3(t) = \sigma B_1 + \alpha + \ln S$.

If $N_2 = x$ the remaining expectation can be perform analytically. Let

$\mu_i(t) = \mathbb{E}(N_i(t))$, $\sigma_{ij}(t) = \mathbb{C}ov(N_i(t), N_j(t))$,

$N_2(t) = x : \tilde{\mu}_i(t,x) = \mu_i + (x-\mu_2)\sigma_{i2}/\sigma_{22}$

and

$\tilde{\sigma}_{ij} = \sigma_{11} - \sigma_{i2}\sigma_{j2}/\sigma_{22}$

$v^2 = \mathbb{V}ar(N_1(t) - N_3(t)|N_2(t) = x) = \tilde{\sigma}_{11} - 2\tilde{\sigma}_{13} + \tilde{\sigma}_{33}$

$$\begin{aligned} V_{floating} \;\le\; & V_{low} \equiv e^{-r}\int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma}_{22}} n(\frac{x-\mu_2}{\sqrt{\sigma}_{22}}) \\ & \times\Bigg[e^{\tilde{\mu}_1+\frac{1}{2}\tilde{\sigma}_{11}} N(\frac{\tilde{\mu}_1 - \tilde{\mu}_3 - \ln(x) + \tilde{\sigma}_{11} - \tilde{\sigma}_{13}}{v}) \\ & - xe^{\tilde{\mu}_3+\frac{1}{2}\tilde{\sigma}_{33}} N(\frac{\tilde{\mu}_1 - \tilde{\mu}_3 - \ln(x) + \tilde{\sigma}_{13} - \tilde{\sigma}_{33}}{v})\Bigg]\mathrm{d}x\mathrm{d}t. \end{aligned} \quad (3.62)$$

Table 3.2 is an example of a fixed strike option from [8], which shows the upper bound of Rogers and Shi (RS) (1995), Thomson's lower and upper bounds that has been derived above and Monte Carlo result of Levy and Turnbull (1992). Assume $\rho = 0.09$, an initial stock price of $S = 100$, and an expiry time of 12 months. The approximation time taken (on an HP 9000/730) for Thomson's lower and upper bound is paranthesized.

TABLE 3.2

σ	K	Thomson(LB)	Monte Carlo	Thomson(UB)	R-S(UB)
	95	8.8088	8.81	8.8089	8.821
		(0.0019)	(0.00)	(0.013)	
0.05	100	4.3082	4.31	4.3084	4.318
		(0.0011)	(0.00)	(0.019)	
	105	0.9583	0.95	0.9585	0.968
		(0.0011)	(0.00)	(0.019)	
	95	8.9118	8.91	8.9130	8.95
		(0.0018)	(0.00)	(0.019)	
0.10	100	4.9150	4.91	4.9155	5.10
		(0.017)	(0.00)	(0.020)	
	105	2.0699	2.06	2.0704	2.34
		(0.0018)	(0.00)	(0.021)	
	95	14.9827	14.96	14.9929	15.194
		(0.0019)	(0.01)	(0.024)	
0.30	100	8.8275	8.81	8.8333	9.039
		(0.0019)	(0.01)	(0.024)	
	105	4.6949	4.68	4.7027	4.906
		(0.0018)	(0.01)	(0.028)	

Table 3.2 for Fixed strike Asian Option of $S = 100$, $\rho = 0.09$, at $T = 1$. Estimates of standard errors (from Curran(1992)) are given in brackets.

3.4 PDE's for Arithmetic averaging

Rogers and Shi (1995) formulated a one-dimensional construct that can model both fixed and floating strike options. Suppose that the price at time t, S_t of some risky asset is given by

$$S_t = S_0 \exp((r - \frac{1}{2}\sigma^2)t + \sigma W_t)$$

where W_t is a standard one-dimensional Brownian motion and r is the riskless interest rate. The problem of computing the value of an Asian (call) option with maturity T and strike price K, written on this risky asset, is mathematically equivalent to calculating

$$\mathbb{E}(\int_0^T S_u \mu(\mathrm{d}u) - K)^+.$$

For a fixed strike Asian option, the measure μ is given $\mu(\mathrm{d}u) = \frac{I}{T}(u)\mathrm{d}u$. If we take $\mu(\ du) = \delta_t(\mathrm{d}u)$, then we have the classic European call option, and if we take $\mu(\ du) = \frac{I}{T}(u)\mathrm{d}u - \delta_T(\mathrm{d}u)$ together with $K = 0$, where δ is a delta function, then we have the floating strike Asian option whose price at time zero

$$\mathbb{E}\left(\frac{1}{T}\int_0^T S_u \mathrm{d}u - S_T\right)^+.$$

A PDE of an Asian option Assume that the probability measure μ has density ρ_t in $(0, T)$ and the maturity of the option T is fixed. Define

$$V(t,x) = \mathbb{E}\left[(\int_t^T S_u \mu(\mathrm{d}u) - x)^+ \mid S_t = 1\right]$$

where S is given by the standard Brownian equation, by Martingale

$$\begin{aligned} V_t &\equiv \mathbb{E}[(\textstyle\int_0^T S_u \mu(\mathrm{d}u) - K)^+ \mid \Im_t] \\ &= \mathbb{E}[(\textstyle\int_t^T S_u \mu(\mathrm{d}u) - (K - \int_t^T S_u \mu(\mathrm{d}u)))^+ \mid \Im_t] \\ &= S_t \mathbb{E}[(\textstyle\int_t^T \frac{S_u}{S_t}\mu(\mathrm{d}u) - \frac{K - \int_0^T S_u \mu(\mathrm{d}u)}{S_t})^+ \mid \Im_t] \\ &= S_t V(t, x_t). \end{aligned} \tag{4.63}$$

V is jointly continuous, decreasing in t and decreasing convex in x, from its definition. By Ito's formula

$$\mathrm{d}x_t = -\rho_t \mathrm{d}t + (-\alpha \mathrm{d}t - \sigma \mathrm{d}W_t + \sigma^2 \mathrm{d}t)x_t$$

$$\begin{aligned}
\mathrm{d}V &= V\mathrm{d}S + S(\tfrac{\partial V}{\partial x}\mathrm{d}t + \tfrac{\partial V}{\partial x}\mathrm{d}x + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial x^2}\mathrm{d}[x]^2 + \mathrm{d}S\mathrm{d}V \\
&= rVS\mathrm{d}t + S(\tfrac{\partial V}{\partial t} + \tfrac{\partial V}{\partial x}(-\rho_t - rx + \sigma^2 x) + \tfrac{1}{2}\sigma^2 x^2 \tfrac{\partial^2 V}{\partial x^2})\mathrm{d}t - \sigma S\tfrac{\partial V}{\partial x}\sigma x \mathrm{d}t \\
&= S[rV + \tfrac{\partial V}{\partial t} - (\rho_t + rx)\tfrac{\partial V}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 \tfrac{\partial^2 V}{\partial x^2}]\mathrm{d}t
\end{aligned}$$

which implies that

$$\frac{\partial V}{\partial t} - (\rho_t + rx)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rV = 0. \tag{4.64}$$

We write

$$V(t,x) \equiv e^{-r(T-t)} f(t,x) \tag{4.65}$$

then f solves

$$\frac{\partial V}{\partial t} + \mathcal{G}f = 0 \tag{4.66}$$

where the operator $\mathcal{G}$ is

$$\mathcal{G} \equiv -(\rho_t + rx)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}.$$

The boundary conditions depend on the problem, in the case of the fixed strike Asian option

$$f(T,x) = x^-$$

and in the case of the floating strike Asian option we have

$$f(T,x) = (1+x)^-.$$

Now PDE (4.64) can be solved numerically. Let us denote the solution to this PDE with the fixed strike boundary condition by ϕ and the solution with floating strike boundary condition by ψ. In the case where μ is uniform on $[0,T]$, the price of Asian option with maturity T, fixed strike price K, and initial price S_0 is

$$e^{-rT}\mathbb{E}(\int_0^T \frac{1}{T}(S_u - K)\mathrm{d}u)^+ = S_0 V(0, KS_0^{-1}) \equiv e^{-rT} S_0 \phi V(0, KS_0^{-1})$$

and price of Asian option with maturity T and floating strike is

$$e^{-rT}\mathbb{E}(\int_0^T \frac{1}{T}S_u \mathrm{d}u - S_T)^+ \equiv e^{-rT}S_0\psi V(0,0).$$

For $x \leq 0$

$$\phi(t,x) = r^{-1}(e^{r(T-t)} - 1) - x.$$

Also for large negative x, $\psi(t,x)$ is very close to

$$\mathbb{E}\left(\int_0^T \frac{1}{T}S_u \mathrm{d}u - S_T - x \mid S_t = 1)\right)^+ = \frac{e^{r(T-t)} - 1}{rT} - e^{r(T-t)} - x$$

which can be used to set boundary values for numerical methods. One can derive other formula from this. For an example the price for an Asian option with strike K and maturity T, whose average is computed over the interval $[T-t,T]$, $0 < t < T$ is

$$e^{-rT}\int_0^\infty \mathbb{P}(S_{T-t}\epsilon \mathrm{d}x)x\phi(T-t,K/x) \tag{4.67}$$

where ϕ is computed using the measure μ which is uniform on $[T-t,T]$, if function $\phi(T-t,\cdot)$ is known.

3.4.1 Pricing PDEs for fixed and floating strike options

It was discovered that some numerical PDE techniques commonly used in finance for standard options are inaccurate in the case of Asian options and illustrate modifications which alleviate the problem. We will now show the two different method to price the partial differential equation of a fixed and floating strike Asian option for Arithmetic averaging, a variable reduction method and a discretization method.

Variable reduction methods

Variable reduction methods is applicable for average fixed strike option if the underlying is a lognormal process. It transforms the valuation problem of a fixed strike option into an evaluation of a conditional expectation that is determined by one-dimensional Markov process. This method has reduced the dimensionality of pricing the fixed strike option by one, which makes the pricing more efficient in terms of computing time. This method disagrees with the result shown in Ingersoll (1987) and Wilmott, Dewynne and Howison (1993) that the variable reduction only works in average floating strike option. Consider a fixed strike of a call option, $C(t)$, with a maturity date T and a strike price K which can be evaluated by

$$C_t = e^{-r(T-t)}\mathbb{E}_t^*[\frac{1}{T}A_T - K)+], \tag{4.68}$$

wher $\mathbb{E}^*$ denotes the conditional expectation under the risk neutral probability distribution, or equivalently, the equivalent martingale measure $\mathbb{Q}$. Note that

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t \tag{4.69}$$

$$\mathrm{d}A_t = S_t\mathrm{d}t \tag{4.70}$$

S and A together form a two dimensional Markov process under $\mathbb{Q}$. Thus the value of a fixed strike call option at $t(< T)$ must be a function of S_t, A_t and t, i.e $C = C(S_t, A_t, t)$. Moreover, since $e^{-rt}C$ must be a martingale under $\mathbb{Q}$, the drift of $e^{-rt}C$ under $\mathbb{Q}$ must be zero. This leads to the following second-order PDE equation for C, with two-dimensional space variable and one time variable,

$$\frac{\partial C}{\partial t} + (r-q)\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + S\frac{\partial C}{\partial A} - rC = 0. \tag{4.71}$$

Now we introduce the variable reduction method which transform (4.71) into a PDE with only one state variable and one time variable. To motivate this transformation, let us write the valuation equation (4.68) as follows

$$C_t = e^{-r(T-t)}\mathbb{E}[\frac{1}{T}A_t - K + \frac{1}{T}\int_t^T S_u \mathrm{d}u]^+ \tag{4.72}$$

$$= \frac{S_t}{T}e^{-r(T-t)}\mathbb{E}[x_t + \int_t^T \frac{S_u}{S_t}\mathrm{d}u]^+ \tag{4.73}$$

where x is determined by

$$x_t = \frac{1}{S}(A_t - TK) \tag{4.74}$$

Since $\frac{S_u}{S_t}(u > t)$ is independent of S up to t, the conditional expectation in the above equation must be a function of x_t. Thus, C can be written as a function of x_t and t multiplied by S_t, i.e

$$C(S_t, A_t, t) = S_t(V(x_t, t))$$

where

$$\begin{aligned} \frac{\partial C}{\partial S} &= V - x\frac{\partial}{\partial x} \\ \frac{\partial^2 C}{\partial S^2} &= \frac{1}{S}x^2\frac{\partial^2 V}{\partial x^2} \\ \frac{\partial C}{\partial A} &= \frac{\partial V}{\partial x} \\ \frac{\partial C}{\partial t} &= S\frac{\partial V}{\partial t} \end{aligned} \tag{4.75}$$

for some function V of x and t only:

$$V(x,t) = e^{-r(T-t)}\mathbb{E}[x + \int_t^T \frac{S_u}{S_t}\mathrm{d}u]^+$$

The PDE for V is

$$\frac{\partial V}{\partial t} + (1 - \alpha x)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} - qV = 0. \tag{4.76}$$

The boundary condition is

$$V(x,T) = \frac{1}{T}\max[x, 0] \tag{4.77}$$

Proposition 1 The value of a fixed strike call option is determined by $S_t V(x_t, t)$, where f satisfies the PDE (4.76) and the boundary condition (4.77), and where $(A_t - TK)/S$. Note that the stochastic process x is a diffusion process by itself, i.e.

$$\mathrm{d}x_t = (1 - \alpha x_t + \sigma^2 x_t)\mathrm{d}t - \sigma x_t \mathrm{d}W_t$$

Moreover, if we introduce a pseudo probability measure $\mathbb{Q}'$ in such a way that

$$\mathrm{d}x_t = (1 - \alpha x_t)\mathrm{d}t - \sigma x_t \mathrm{d}W'_t$$

where W' is a standard Brownian motion under $\mathbb{Q}'$, then (4.76) is equivalent to the statement that under $\mathbb{Q}'$, the discounted value V is a martingale, while the discount rate is the implicit payout rate q i.e.

$$V(x_t, t) = \mathbb{E}'_t[\frac{e^{-q(T-t)}}{T} \max[x_T, 0]] \tag{4.78}$$

where the expectation is taken under $\mathbb{Q}'$. Equation (4.78) is also called the Feyman-Kac representation of the PDE (4.76). The explicit formula for V when $x_t \geq 0$ or $(t < T)$

$$V(x, t) = \frac{1}{T} e^{-r(T-t)} x + \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r-q)}. \tag{4.79}$$

For $x_t < 0$ numerical techniques are needed, such as finite difference method or Monte Carlo simulations method to evaluate V from the PDE (4.76) or from conditional expectation that define V in (4.78). If we apply the finite difference method, then we need to use the boundary condition (4.79), $\lim_{x \to -\infty} V(x, t) = 0$ for small values of x and (4.77) for large values of x. The variable reduction method can also be applied to a fixed strike options where the averaging is taken at a discrete set of time points i.e., discrete averaging. Assume that averaging takes place at points $0 = t_1 < t_2 < \cdots < t_n = T$. Define

$$A_{t_k} \equiv \sum_{i=1}^{k} S_{t_i}$$

$$x_{t_k} \equiv \frac{A_{t_k} - nK}{S_{t_k}}, k = 1, 2, \cdots, n.$$

By applying the variable reduction technique, one can show that

$$C_{t_k} = e^{-r(t_n - t_k)} \frac{S_{t_k}}{n} E^*_{t_k}[x_{t_k} + \sum_{i=k+1}^{n} \frac{S_{t_i}}{S_{t_k}}]^+$$

$$\equiv S_{t_k} V(x_{t_i}, t_k)$$

where

$$V(x_{t_k}, t_k) = \frac{e^{-r(t_n - t_k)}}{n} E^*_{t_k}[x_{t_k} + \sum_{i=k+1}^{n} \frac{S_{t_i}}{S_{t_k}}]^+$$

$$V(x_{t_k}, t_k) = \frac{1}{n} x^+_{t_n}.$$

It follows then, that

$$V(x_{t_k}, t_k) = \frac{e^{-r(t_{k+1} - t_k)}}{n} E^*_{t_k}[\frac{S_{t_{k+1}}}{S_{t_k}} f(x_{t_{k+1}}, t_{k+1})] \tag{4.80}$$

$$x_{t_{k+1}} = \frac{S_{t_k}}{S_{t_{k+1}}} x_{t_k} + 1$$

$$\frac{S_{t_{k+1}}}{S_{t_k}} = e^{(r - q - \frac{1}{2}\sigma^2)\Delta t_k + \sigma\sqrt{\Delta}\epsilon}$$

$$\frac{S_{t_k}}{S_{t_{k+1}}} = e^{-(r - q - \frac{1}{2}\sigma^2)\Delta t_k + \sigma\sqrt{\Delta}\epsilon}$$

where ϵ is a random variable distributed as $N(0,1)$, and $\Delta t_k = t_{k+1} - t_k$. Thus (4.80) can be solved recursively by numerical integrations. We can fix a set of grid points for x, and evaluate V over these points recursively. For those points that are not on the grids, a second order interpolation can be used to find the value of V on these points. Alternatively, V can also be determined, by

$$V(x_{t_k}, t_k) = \frac{e^{-q(t_n - t_k)}}{n} E^*_t[\max[x_{t_n}, 0]].$$

This formula is useful if we would like to value V by Monte-Carlo simulation. In this case we will be simulating the process x under the probability measure Q'.

Discretization

Since usual methods generally produce spurious oscillations, flux limiting techniques which are total variation diminishing and hence free of spurious oscillation, (for non-conservative PDEs such as those typically encounted in finance, for fully explicit and implicit schemes) was adapted in the field of computational fluid dynamics in order to rapidly obtain accurate solutions to PDEs. Also Van Leer flux limiter was modified so that the second-order total variation diminishing property is preserved for non-uniform grid spacing. The two-dimensional PDE for floating strike option can be reduced to a one-dimensional PDE. In the previous section it appears that Rogers and Shi (1995) formulated a one-dimensional that can model both fixed and floating strike options. But this can not be applied to American options which can be solved only in 2-dimensional PDE. It is not easy to solve it numerically, since the diffusion term is very small for values of interest on the finite-difference grid. Discretization method solve both two-dimension and one-dimension in cases of a little or no diffusion in second-order derivative term, in a space dimension. To eliminate the oscillation caused by centrally weighted scheme (Rouch, 1972) one must use first-order upstream weighting for the convective time. By Ingersoll, the value of an Asian option is given by the following PDE in two-dimension in terms of the running sum (I), the solution to the above equation is represented as

$$V(S_t, A_t, t) = e^{-r(T-t)} E[g(S_T, I_T, T)]$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0$$

and an alternative formula given in terms of the average (A),

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{1}{T}(S - A)\frac{\partial V}{\partial A} - rV = 0$$

the solution to the above equation is represented as

$$V(S_t, A_t, t) = e^{-r(T-t)} E[g(S_T, I_T, T)]$$

As we have seen in chapter 2 different terminal boundary conditions may be used to price different types of securities.

The only analytical solution which is known is for a fixed strike case when $K = 0$. An early exercise constraint $V(S(\tau), \tau) \geq \max(K - S(\tau), 0)$ may be applied to value the American-style Asian options. Note that the above equations have no diffusion term in the I direction and similarly in the A direction. A PDE for Asian type options was reduced to a one-dimensional PDE in Rogers and Shi (1995). To value a fixed strike options with early exercise opportunities, we must solve the two-dimensional PDE.

One dimension model

The result obtained in the discritization in European vanilla option, in [32], will now examine treating convection using the Van Leer flux limiter of one- and two-dimensional PDE models of Asian options. Since (Rogers and Shi (1995)) model can be used to price both fixed and floating strike Asian option, but cannot handle the early exercise feature, we will also consider a two-dimensional PDE models for an early exercise feature. Consider Rogers and Shi's one dimensional equation

$$\frac{\partial V}{\partial t} - (\rho_t + rS)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{4.81}$$

after converting (4.81) equation to a forward equation, by substituting t with $\tau = T - t$, and discretizing this equation, using the finite volume approach. The resulting discretization with temporal weighting for the value at cell i at time step $n+1$ written in general form is

$$\begin{aligned}\frac{V_i^{n+1} - V_i^n}{\Delta\tau} &= \theta F_{i-\frac{1}{2}}^{n+1} - \theta F_{i+\frac{1}{2}}^{n+1} \theta f_i^{n+1} + (1-\theta) F_{i-\frac{1}{2}}^n \\ &\quad - (1-\theta) F_{i+\frac{1}{2}}^n + (1-\theta) F_i^n\end{aligned}$$

where

$\theta =$ temporal weighting $(0 \leq \theta \leq 1)$

$F_{i-\frac{1}{2}} =$ flux entering cell i at interface $i - \frac{1}{2}$

$F_{i+\frac{1}{2}} =$ flux entering cell i at interface $i + \frac{1}{2}$

$f_i =$ source /sink term

We let $\theta = 1$ for fully-implicit method, for $\theta = \frac{1}{2}$ we have the Crank-Nicolson method and we let $\theta = 0$ for a fully-explicity method. The flux functions are

$$F_{i-\frac{1}{2}}^{n+1} = \frac{1}{\Delta S_i}[(-\frac{1}{2}\sigma^2 S_i^2)\frac{(V_i^{n+1} - V_{i-1}^{n+1})}{\Delta S_{i-\frac{1}{2}}} + (-rS_i)V_{i-\frac{1}{2}}^{n+1} \tag{4.82}$$

$$F_{i+\frac{1}{2}}^{n+1} = \frac{1}{\Delta S_i}\left[(-\frac{1}{2}\sigma^2 S_i^2)\frac{(V_{i+1}^{n+1} - V_i^{n+1})}{\Delta S_{i+\frac{1}{2}}} + (\rho^{n+1} + rS_i)V_{i+\frac{1}{2}}^{n+1}\right] \tag{4.83}$$

and the source term

$$f_i^{n+1} = 0. \tag{4.84}$$

Note that S takes on negative values and thus $(\rho^{n+1} + rS_i)$ may take on both negative and positive values. The discretization of $V_{i+\frac{1}{2}}^{n+1}$ must take this into account. If we use a point-distributed finite volume scheme, then

$$\begin{aligned} \Delta S_i &= \frac{S_{i+1} - S_{i-1}}{2} \\ \Delta S_{i+\frac{1}{2}} &= S_{i+1} - S_i. \end{aligned} \tag{4.85}$$

Note that the flux functions (4.82) and (4.83) allow for non-uniform grid spacing. Thus we can construct grids which will make the numerical computations more efficient by having a fine grid spacing near and at the exercice price and coarse grid away from the exercise price. We must first examine handling the convective term $V_{i+\frac{1}{2}}^{n+1}$ in (4.83) using the following central weighting scheme

$$V_{i+\frac{1}{2}}^{n+1} = \frac{V_{i+1}^{n+1} - V_i^{n+1}}{2}$$

which has second-order accuracy for uniform grids. We must also ensure that solutions produced using central weighting are free of spurious oscillations, we must satisfy the Peclet condition

$$\frac{1}{\Delta S_{i-\frac{1}{2}}} > \frac{r}{\sigma^2 S_i} \tag{4.86}$$

and the additional condition

$$\frac{1}{(1-\theta)\Delta\tau} > \frac{\sigma^2 S_i^2}{2}\left(\frac{1}{\Delta S_{i-\frac{1}{2}}\Delta S_i} + \frac{1}{\Delta S_{i+\frac{1}{2}}\Delta S_i}\right) + r \tag{4.87}$$

for all cells i. For the discretization of a European vanilla option and precise definition of spurious oscillations, see [32].

Two-Dimensional Models

Since we cannot use the one-dimensional models to price American-style fixed strike options. Only two-dimensional can be used, so the following equation was chosen to be the suitable one

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{1}{T}(S-A)\frac{\partial V}{\partial A} - rV = 0. \tag{4.88}$$

Note that at $t = 0$ a singularity exists in the above equation because of the $\frac{1}{t}(S - A)\frac{\partial V}{\partial A}$ term. To avoid this, let $S = A$, thus equation becomes the Black-Scholes equation at $t = 0$. After converting equation (4.88) to a forward PDE, the finite volume discretization is

$$\begin{aligned}\frac{V_i^{n+1} - V_i^n}{\Delta\tau} \quad = \quad & \theta F^{n+1}_{i-\frac{1}{2},j} - \theta F^{n+1}_{i+\frac{1}{2},j} \\ & +\theta F^{n+1}_{i,j-\frac{1}{2}} - \theta F^{n+1}_{i,j+\frac{1}{2}} + \theta f^{n+1}_{i,j} \\ & +(1-\theta)F^n_{i-\frac{1}{2},j} - (1-\theta)F^n_{i+\frac{1}{2},j} + (1-\theta)F^n_{i,j-\frac{1}{2}} \\ & +(1-\theta)F^n_{i,j+\frac{1}{2}} + (1-\theta)f^n_{i,j}\end{aligned}$$

where

$$F^{n+1}_{i,j+\frac{1}{2}} = \frac{1}{\Delta A_{i,j}} \left[\frac{1}{t}(A_{i,j} - S_{i,j}) V^{n+1}_{i,j+\frac{1}{2}} \right]$$

and

$$F^{n+1}_{i+\frac{1}{2},j} = \frac{1}{\Delta S_{i,j}} [-\frac{1}{2}\sigma^2 S^2_{i,j} \frac{V^{n+1}_{i+1,j} - V^{n+1}_{i,j}}{\Delta S_{i+\frac{1}{2},j}} + (-rS_i) V^{n+1}_{i+\frac{1}{2},j}]$$

$$f^{n+1}_{i,j} = (-r) V^{n+1}_{i,j}. \tag{4.89}$$

The discritization of $V^{n+1}_{i,j+\frac{1}{2}}$ using the Van Leer limiter must take into account the fact that $\frac{1}{t}(A_{i,j} - S_{i,j})$ will take on negative and non-negative values. Also if at j_{max} there exist $A_{i,j_{max}}$ which are less than $S_{i,j_{max}}$ then the appropriate boundary must be imposed at these point. The Van Leer can also be applied to other financial PDE models that have the problem of convection dominance. Since the method is non-linear it can be easily extended to solve non-linear option models.

TABLE 3.3

				European		American
σ	$T-t$	K	LB	$1-D$	$2-D$	$2-D$
	0.25	95	6.118	6.114	6.133	6.646
		100	1.851	1.841	1.793	1.903
		105	0.148	0.162	0.162	0.161
0.10	0.50	95	7.220	7.216	7.244	7.687
		100	3.104	3.064	3.052	3.180
		105	0.714	0.718	0.726	0.733
	1.00	95	9.285	9.286	9.316	9.662
		100	5.255	5.254	5.261	5.398
		105	2.294	2.295	2.314	2.340
	0.25	95	6.476	6.461	6.501	7.521
		100	2.932	2.923	2.928	3.224
		105	0.947	0.958	0.971	1.009
0.20	0.50	95	7.891	7.890	7.921	8.908
		100	4.505	4.502	4.511	4.901
		105	2.211	2.206	2.229	2.337
	1.00	95	10.295	10.294	10.309	11.295
		100	7.042	7.041	7.042	7.548
		105	4.509	4.508	4.519	4.742

Table 3.3 for American and European fixed strike call with r= 0.10 and $S_0 = 100$.

Lower bound (LB) and one-dimensional PDE (1-D) by (Rogers and Shi (1995)), two-dimensional PDE (2-D) values obtained by using the Van Leer limiter with $\theta = \frac{1}{2}$ and non spatial $A \times S$ grid of 41×45. Δt^* was set to one day, two days and three days for maturities of three, six and twelve months respectively. Mean execution times are for runs performed on a DEC Alpha. Readings were taken from [32].

Chapter 4

Asian option under Stochastic volatility

It is difficult to price Asian options when volatility is stochastic. Essentially the price now depends on three variables. The underlying asset price, the averaging variable and the stochastic volatility, which become more challenging. We will consider a one-factor and a two-factor stochastic volatility model. Fouque and Tullie proposed to use approximations of European options prices obtained from singular perturbation expansions for the important sampling techniques. They discovered that the first order correction term added to zeroth order option price approximation dramatically reduce the variance. Arithmetic Asian option can be priced by the Monte Carlo simulations. The two-factor stochastic volatility model also does not have closed form solution. The Asian option is called fresh when the current time t is exactly at the contract starting date 0, and is called season when it is between the contract starting date 0 and maturity date T.

4.1 Asian option for a one factor stochastic volatility

Firstly, we look at how to derive Asian price in a one-factor stochastic volatility.

4.1.1 Arithmetic Asian option of a one-factor stochastic volatility

To perform an asymptotic analysis, introduce a small parameter ϵ such that the rate of mean reversion defined by $\alpha = \frac{1}{\epsilon}$ becomes large. To capture the volatility clustering behaviour, define $\nu^2 = \frac{\beta}{2\alpha}$ to be fixed $\mathcal{O}(1)$ constant. Since rate of mean reversion of the volatility process depends on ϵ, define ϵ-independence for S_t and Y_t by S_t^ϵ the stock price and Y_t^ϵ the volatility factor respectively. Assume that the market is pricing the derivatives under a risk-neutral probability measure P^*. Using the Girsanov's theorem, the above model under P^* is

$$\mathrm{d}S_t^\epsilon = rS_t^\epsilon \mathrm{d}t + f(Y_t^\epsilon)S_t^\epsilon \mathrm{d}W_t^*, \tag{1.1}$$

$$\mathrm{d}Y_t^\epsilon = [\frac{1}{\epsilon}(m - Y_t^\epsilon - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(Y_t^\epsilon)]\mathrm{d}t + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}(\rho \mathrm{d}W_t^* + \sqrt{1-\rho^2}\mathrm{d}Z_t^*) \tag{1.2}$$

where W_t^* and Z_t^* are correlated Brownian motions. The combined market price of risk is defined as

$$\Lambda(y) = \rho\frac{\mu - r}{f(y)} + \gamma(y)\sqrt{1-\rho^2}$$

which describe the relationship between the physical measure under the stock price as observed, and the risk-neutral measure under which the market prices derivative securities are computed.

Assume that the risk-free interest rate r is constant, and that the market price of volatility risk $\gamma(y)$ is bounded, and depends only on the volatility level y. At the leading order $\frac{1}{\epsilon}$ in (1.2), that is omitting the Λ-term in the drift , Y_t is an OU process which is fast mean-reverting with normal invariant distribution $N(m, \nu^2)$. The volatility factor Y_t^ϵ fluctuates randomly around its mean level m and the long run magnitude v of volatility fluctuations remains fixed for values of ϵ. Furthermore, due to the presence of the other Brownian motion Z^* there exists a y-dependent family of equivalent risk-neutral measures. However we assume that the market chooses one measure through market price of volatility risk y.

Three-dimensional pricing PDE with stochastic volatility If we let

$$I_t^\epsilon = \int_0^t S_t^\epsilon \mathrm{d}S. \tag{1.3}$$

Assume that the stochastic volatility models obeys (1.1) and (1.2) in addition to the differential form of (1.3)

$$\mathrm{d}I_t^\epsilon = S_t^\epsilon \mathrm{d}t$$

the process $(S_t^\epsilon, Y_t\epsilon, I_t^\epsilon)$ is a Markov process under the risk-neutral probability measure P^*. The price of a call of a floating strike Asian option with stochastic volatility at $0 \leq t \leq T$ is given by

$$P^\epsilon(t, s, y, I) = E^*[e^{-r(T-t)}(S_T^\epsilon - \frac{I_T^\epsilon}{T})^+ \mid S_t^\epsilon = s, Y_t^\epsilon = y, I_t^\epsilon = I].$$

The following three-dimensional PDE is obtained from the Feynman-Kac formula

$$\frac{\partial P^\epsilon}{\partial t} + \frac{1}{2}f(y)^2 s^2 + \frac{\partial^2 P^\epsilon}{\partial s^2} + r(s\frac{\partial P^\epsilon}{\partial s} - P^\epsilon) + \frac{\rho\nu\sqrt{2}}{\sqrt{\epsilon}}f(y)\frac{\partial^2 P^\epsilon}{\partial s \partial y}$$
$$+\frac{\nu^2}{\epsilon}\frac{\partial^2 P^\epsilon}{\partial y^2} + (\frac{1}{\epsilon}(m-y) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(y))\frac{\partial P^\epsilon}{\partial y} + s\frac{\partial P^\epsilon}{\partial I} = 0$$

with terminal condition $P^\epsilon(t, s, y, I) = (s - \frac{I}{T})^+$. The Asian option is obtain by solving the above PDE equation, at time t, given S_t^ϵ and driving volatility Y_t^ϵ. This equation needs to be reduced to a two-dimensional case, since numerical scheme for a three-dimensional PDE needs a significant computation effort. The Vecer'technique of dimension reduction was used. From the Feynman-Kac formula, the following two-dimensional PDE is obtained

$$\begin{aligned}\frac{\partial P^\epsilon}{\partial t} + \frac{1}{2}(\psi - q(t))^2 f(y)^2 s^2 + \frac{\partial^2 P^\epsilon}{\partial s^2} + \frac{\rho\nu\sqrt{2}}{\sqrt{\epsilon}}(q(t) - \psi) f(y) \frac{\partial^2 P^\epsilon}{\partial s \partial y} \\ + \frac{\nu^2}{\epsilon}\frac{\partial^2 P^\epsilon}{\partial y^2} + (\frac{1}{\epsilon}(m - y) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}(\Lambda(y) - \rho f(y)))\frac{\partial P\epsilon}{\partial y} = 0 \qquad (1.4)\end{aligned}$$

with terminal condition $P^\epsilon(T, \psi, y; T, K_1, K_2) = h(\psi - K_1)$ for the derivation of the above PDE see [17].

Asymptotics We apply the asymptotic analysis to the fresh case, since the seasoned Asian option price can be deduced from the fresh Asian option prices. We look for the solution of a two-dimensional PDE of the form

$$u^\epsilon(t, \psi, y) = u_0(t, \psi, y) + \sqrt{\epsilon} u_1(t, \psi, y) + \cdots . \qquad (1.5)$$

which solves a two-dimensional PDE (1.4). Differential operators on the left hand side of a two-dimensional PDE can be decompose to

$$\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1(t) + \mathcal{L}_2(t)$$

where the operations are defined as

$$\mathcal{L}_0 = (m - y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2} \qquad (1.6)$$

$$\mathcal{L}_1(t) = \rho\sqrt{2}\nu(q(t) - \psi) f(y) \frac{\partial^2}{\partial y \partial \psi} - \sqrt{2}\nu(\Lambda(y) - \rho f(y))\frac{\partial}{\partial y} \qquad (1.7)$$

$$\mathcal{L}_2(t) = \frac{\partial}{\partial t} + \frac{1}{2}(q(t) - \psi)^2 f(y)^2 \frac{\partial^2}{\partial \psi^2}. \qquad (1.8)$$

If we substitute (1.5)-(1.8) into a PDE (1.4), the expansion follows

$$\frac{1}{\epsilon}\mathcal{L}_0 u_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_1(t)u_0 + \mathcal{L}_0 u_1) + (\mathcal{L}_0 u_2 + \mathcal{L}_1(t)u_1 + \mathcal{L}_2(t)u_2)$$
$$+\sqrt{\varepsilon}(\mathcal{L}_2(t)u_1 + \mathcal{L}_1(t)u_1 + \mathcal{L}_0 u_3) + \cdots = 0 \tag{1.9}$$

with the terminal condition

$$u_0(T,\psi,y) + \sqrt{\epsilon}u_1(T,\psi,y) + \cdots = (\psi - K_1)^+. \tag{1.10}$$

To obtain the expression for u_0 and u_1 by successively equating the four leading order terms in (1.9) to zero. Let $\langle\cdot\rangle$ denote the averaging with respect to the invariant distribution $N(m, v^2)$ of the OU process Y_t namely

$$\langle g\rangle = \frac{1}{v\sqrt{2\pi}}\int g(y)e^{\frac{-(m-y)^2}{2v^2}}\mathrm{d}y. \tag{1.11}$$

It necessary to solve the Poisson equation associated with $\mathcal{L}_0$

$$\mathcal{L}_0 x + g = 0 \tag{1.12}$$

which requires the solvability condition

$$\langle g\rangle = 0. \tag{1.13}$$

Equating the terms of order $\frac{1}{\epsilon}$, one obtain

$$\mathcal{L}_0 u_0 = 0 \tag{1.14}$$

choose u_0 to be independent of y, to avoid solutions that exhibit unreasonable growth at infinity. Equating the terms of order $\frac{1}{\sqrt{\epsilon}}$, we obtain

$$\mathcal{L}_0 u_1 + \mathcal{L}_1(t)u_0 = 0 \tag{1.15}$$

since $\mathcal{L}_1(t)$ contains only terms with derivatives in y, it reduces to $\mathcal{L}_0 u_1 = 0$. Also u_1 is chosen to be independent of y. The order one term gives

$$\mathcal{L}_0 u_2 + \mathcal{L}_1(t)u_1 + \mathcal{L}_2(t)u_0 = 0. \tag{1.16}$$

The above equation reduces to the Poisson equation in u_2, since $\mathcal{L}_1(t)u_0 = 0$, we obtain

$$\mathcal{L}_0 u_2 + \mathcal{L}_2(t)u_0 = 0. \tag{1.17}$$

Its solvability condition becomes

$$\langle \mathcal{L}_2(t)u_0 \rangle + \langle \mathcal{L}_2(t) \rangle u_0 = 0 \tag{1.18}$$

the averaged differential operator $\langle \mathcal{L}_2(t) \rangle$ denoted by $\mathcal{L}_2(t;\bar{\sigma})u_0$, for which $\bar{\sigma}^2$ is defined as $\langle f^2 \rangle$. Hence the leading term u_0 solves

$$\mathcal{L}_2(t;\bar{\sigma})u_0 = \frac{\partial u_0}{\partial t} + \frac{1}{2}(q(t)-\psi)^2\bar{\sigma}^2\frac{\partial^2 u_0}{\partial \psi^2} = 0$$

with the terminal boundary condition

$$u_0(T,\psi,y) = (\psi - K_1)^+$$

then

$$\frac{\partial u_0}{\partial t} + \frac{1}{2}(q(t)-\psi)^2\bar{\sigma}^2\frac{\partial^2 u_0}{\partial \psi^2} = 0.$$

Observe that the second-order correction u_2 is given by

$$u_2 = -\mathcal{L}_0^{-1}(\mathcal{L}_2(t) - \mathcal{L}_2(t;\bar{\sigma}))u_0. \tag{1.19}$$

The order $\sqrt{\epsilon}$ term gives the Poisson equation in u_3

$$\mathcal{L}_0 u_3 + \mathcal{L}_1(t)u_2 + \mathcal{L}_2(t)u_1 = 0. \tag{1.20}$$

Its solvability condition

$$\begin{aligned} 0 &= \langle \mathcal{L}_2(t)u_1 + \mathcal{L}_1(t)u_2 \rangle \\ &= \mathcal{L}_2(t;\bar{\sigma})u_1 - \langle \mathcal{L}_1(t)\mathcal{L}_0^{-1}(\mathcal{L}_2(t) - \mathcal{L}_2(t;\bar{\sigma}))\rangle u_0. \end{aligned} \tag{1.21}$$

Thus we derive

$$\begin{aligned} \mathcal{L}_2(t;\bar{\sigma})u_1 &= \langle \mathcal{L}_1(t)\mathcal{L}_0^{-1}(\tfrac{1}{2}(q(t)-\psi)^2(f(y)^2-\bar{\sigma}^2))\rangle \tfrac{\partial^2 u_0}{\partial \psi^2} \\ &= \langle \mathcal{L}_1(t)\phi(y)\rangle \tfrac{1}{2}(q(t)-\psi)^2\tfrac{\partial^2 u_0}{\partial \psi^2} \end{aligned} \tag{1.22}$$

with zero terminal condition, and the function ϕ solves the Poisson equation

$$\mathcal{L}_0\phi(y) = f(y)^2 - \bar{\sigma}^2 \tag{1.23}$$

one can compute the differential operator, using (1.7)

$$\langle \mathcal{L}_1(t)\phi(y)\rangle = \rho\sqrt{2}\nu\langle f(y)\phi'(y)\rangle(q(t)-\psi)\frac{\partial}{\psi} - \sqrt{2}\nu(\langle\Lambda(y)\phi'(y)\rangle - \rho\langle f(y)\phi'(y)\rangle)$$

then the PDE (1.4) for $\tilde{u}_1 = \sqrt{\epsilon}u_1$ is finally derived that is

$$\mathcal{L}_2(t;\bar{\sigma})\tilde{u}_1 = \bar{V}_2(q(t)-\psi)^2\frac{\partial^2 u_0}{\psi^2} + \bar{V}_3(q(t)-\psi)^3\frac{\partial^3 u_0}{\psi^3} \tag{1.24}$$

with zero terminal condition, and where

$$\bar{V}_2 = \frac{\nu\sqrt{\epsilon}}{\sqrt{2}}(-\rho\langle f(y)\phi'(y)\rangle - \langle\Lambda(y)\phi'(y)\rangle) \tag{1.25}$$

and

$$\bar{V}_3 = \frac{\rho\nu\sqrt{\epsilon}}{\sqrt{2}}\langle f(y)\phi'(y)\rangle. \tag{1.26}$$

We can see that

$$\bar{V}_2 = V_2 - 3V_3$$

$$\bar{V}_3 = V_3.$$

By LMMR

$$\bar{V}_2 = \bar{\sigma}((\bar{\sigma}-b) - a(r - \frac{3}{2}\bar{\sigma}^2))$$

$$\bar{V}_3 = -a\bar{\sigma}^3.$$

The accuracy of the corrected Black-Scholes price

$$|P^\epsilon(t,s,y) - (P_0(t,s) + \tilde{P}_1(t,s))| = \mathcal{O}(\varepsilon). \tag{1.27}$$

4.1.2 Geometric Asian option of a one-factor stochastic volatility

Assume S_t follows a geometric Brownian motion whose volatility depends on a mean-reverting process Y_t

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t \\ \sigma_t &= f(Y_t) \end{aligned}$$

$$\mathrm{d}Y_t = \alpha(m - Y_t) + \beta(\rho \mathrm{d}W_t + \sqrt{1-\rho^2}\mathrm{d}Z_t)$$

$$I_t = \exp\left(\frac{1}{t}\int_0^t \ln S_\tau \mathrm{d}\tau\right)$$

where we denote

S_t, $f(Y_t)$, α, m, ρ, W_t and Z_t are defined as in chapter 1 and Y_t is a Gaussian process where I_t is

$$Y_t \sim N(Y_0 e^{-\alpha t} + m(1 - e^{-\alpha t}), \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t}))$$

which derives a unique invariant distribution $N(m, \nu^2)$, where $\nu^2 = \frac{\beta^2}{(2\alpha)}$ for Y_t when α or t is big. We write

$$Y_t - Y_\infty \approx N(m, v^2) \tag{1.28}$$

in distribution. Let $V(t, S, I, Y)$ be the price function for an Asian option whose payoff function is $H(S_T, I_T)$. By Ito's lemma the approach of Fouque et al (2000), the PDE of the governing equation is

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}f(y)^2 s^2 + \frac{\partial^2 V}{\partial s^2} + r(s\frac{\partial V}{\partial s} - V) + \frac{\rho\nu\sqrt{2}}{\sqrt{\epsilon}}f(y)s\frac{\partial^2 V}{\partial s \partial y} + \frac{\nu^2}{\epsilon}\frac{\partial^2 V}{\partial y^2} \\ &+[\frac{1}{\epsilon}(m - y) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(t,y)]\frac{\partial V}{\partial y} + \frac{I}{t}\ln\frac{s}{I}\frac{\partial V}{\partial I} = 0 \end{aligned} \tag{1.29}$$

$V(T, s, I, Y) = H(S, I)$ where $\epsilon = \frac{1}{\alpha}$ and $\nu = \frac{\beta\sqrt{\epsilon}}{\sqrt{2}}$, is regarded as a fixed value whereas β is large of order $\frac{1}{\sqrt{\epsilon}}$. This is because ν^2 is the variance for the invariant distribution in (1.28) for large t.

$$\Lambda(t,y) = \rho\frac{(\mu - r)}{f(y)} + \gamma(t,y)\sqrt{1-\rho^2}$$

with $\gamma(t,y)$ being an arbitrary function calibrated from market prices of the equity. Applying a transformation to independent of S and I as

$$s = t\ln\frac{I}{S} \tag{1.30}$$

and

$$z = \ln S \tag{1.31}$$

then equation (1.29), becomes

$$\begin{aligned}(\mathcal{L}_0 + \sqrt{\epsilon}\mathcal{L}_1 + \epsilon\mathcal{L}_2)V &= 0 \\ V(T,S,I,y) &= H(S,I),\end{aligned} \tag{1.32}$$

where the operations are defined as

$$\mathcal{L}_0 V = (m-y)\frac{\partial V}{\partial y} + \nu^2\frac{\partial^2 V}{\partial y^2}$$

$$\mathcal{L}_1 V = \nu\sqrt{2}[\rho f(y)(\frac{\partial}{\partial z} - t\frac{\partial}{\partial s})\frac{\partial V}{\partial y} - \Lambda(y)\frac{\partial V}{\partial y}]$$

$$\mathcal{L}_2 V = \frac{\partial V}{\partial t} + \frac{1}{2}f(y)^2 + (\frac{\partial}{\partial z} - t\frac{\partial}{\partial s})^2 V + (r - \frac{f(y)^2}{2})(\frac{\partial}{\partial z} - t\frac{\partial}{\partial s})V - rV, \tag{1.33}$$

$f(y)$ is a positive constant function.

Pricing Geometric Asian options in asymptotic case Here we are trying to solve the above equation in asymptotic expansions under the assumption of fast mean reverting rate i.e $0 < \epsilon << 1$.

One can achieve this by considering Asian option prices of the form

$$V = V_0 + \sqrt{\epsilon}V_1 + \epsilon V_2 + \epsilon\sqrt{\epsilon}V_3 + \cdots \tag{1.34}$$

where $V_0, V_1, V_2, \cdots$ are the functions of (t, s, I, Y) that are solved one by one until a certain accuracy is attained.

Subtituting (1.34) into (1.32) and collecting $\mathcal{O}(1)$ yields

$$\mathcal{L}_0 V_0 = 0.$$

Collect the term of $\mathcal{O}(\sqrt{\epsilon})$ to yield

$$\mathcal{L}_0 V_1 + \mathcal{L}_1 V_0 = 0 \Rightarrow \mathcal{L}_0 V_1 = 0$$

$\mathcal{L}_1$ involves y differentials and V_1 is independent of y

continuing to obtain $\mathcal{O}(\epsilon)$ term, we have

$$\mathcal{L}_0 V_2 + \mathcal{L}_1 V_1 + \mathcal{L}_2 V_0 = 0 \Rightarrow \mathcal{L}_0 V_2 + \mathcal{L}_2 V_0 = 0. \tag{1.35}$$

Given the function of V_0, the above equation can be viewed as a first order linear ODE whose unique solution exists within at most polynomially growing to infinity if

$$E_y(\mathcal{L}_2 V_0) = 0, y \sim N(m, \nu^2).$$

This fact is known as Fredholm solvability for Poisson equations. Since V_0 is a function independent of y, the expectation only affect the operator $\mathcal{L}_2$ through the function $f(y)$. Specifically,

$$\begin{aligned} E_y(\mathcal{L}_2 V_0) &= E_y(\mathcal{L}_2) V_0 = 0 \\ V_0(T, S, I) &= H(S, I). \end{aligned} \tag{1.36}$$

A more explicit expression for the above can be obtain through denoting

$$E_y(f(y)^2) = \bar{\sigma}^2, y \sim N(m, \nu^2) \tag{1.37}$$

and a standard Black-Scholes operator for Asian option

$$E_y(\mathcal{L}_2) = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2 + (\frac{\partial}{\partial z} - t\frac{\partial}{\partial s})^2 + (r - \frac{1}{2}\bar{\sigma}^2)(\frac{\partial}{\partial z} - t\frac{\partial}{\partial s}) - r.$$

To summarize the above we state the following theorem

Theorem 1 The zeroth order approximations for the values of any geometric Asian contingent claim in a fast mean-reverting stochastic volatility economy are the Black-Scholes prices of the claims with constant (long term) volatility, $\bar{\sigma}$, defined in (1.37), consider $\mathcal{O}(\epsilon^{\frac{3}{2}})$

$$\mathcal{L}_0 V_3 + \mathcal{L}_1 V_2 + \mathcal{L}_2 V_1 = 0$$

which leads to the result

$$E_y(\mathcal{L}_0 V_3) = 0 \Rightarrow E_y(\mathcal{L}_1 V_1) + E_y(\mathcal{L}_2) V_1 = 0$$

since V_1 is independent of y and by applying Fredholm solvability, to solve V_1 we express V_2 in terms of V_0, using (1.35)

$$E_y(\mathcal{L}_2) V_1 = E_y(\mathcal{L}_1 \mathcal{L}_0^{-1} \mathcal{L}_2 V_0) \tag{1.38}$$

$$V_1(T, S, I) = 0 \tag{1.39}$$

The above equation (1.38) becomes the Black-Scholes equation, by Fouque et al (2000)

$$\begin{aligned} E_y(\mathcal{L}_1 \mathcal{L}_0^{-1} \mathcal{L}_2 V_0) \quad = \quad & P_1[(t\frac{\partial}{\partial s} - \frac{\partial}{\partial z}) + (t\frac{\partial}{\partial s} - \frac{\partial}{\partial z})^2] V_0 \\ & + P_2[(t\frac{\partial}{\partial s} - \frac{\partial}{\partial z})^2 + (t\frac{\partial}{\partial s} - \frac{\partial}{\partial z})^3] V_0 \end{aligned} \tag{1.40}$$

where the constant parameters

$$\begin{aligned} P_1 &= \tfrac{\rho(\mu-r)}{\sqrt{2}\nu} E_y[\hat{F}(y)(f(y)^2 - \bar{\sigma}^2)] + \tfrac{\sqrt{1-\rho^2}}{\sqrt{2}\nu} E_y[\Gamma(y)(f(y)^2 - \bar{\sigma}^2)] \\ P_2 &= \tfrac{\rho}{\sqrt{2}\nu} E_y[F(y)(f(y)^2 - \bar{\sigma}^2)] \end{aligned} \tag{1.41}$$

with

$$F(y) = \int f(y) \mathrm{d}y, \hat{F}(y) = \int \frac{1}{f(y)} \mathrm{d}y, \Gamma(y) = \int \gamma(y) \mathrm{d}y. \tag{1.42}$$

Theorem 2 Suppose that a function $F(t,s,z)$ satisfies

$$E_y(\mathcal{L}_2)F(t,s,z) = \sum_{i=1}^{n} f_i(t) I_i(t,s,z) F(T,s,z) = 0$$

where $I_i(t,s,z), i = 1,2,\cdots,n$ are homogeneous solutions to the PDE

$$E_y(\mathcal{L}_2)I(t,s,z) = 0$$

then $F(t,s,z)$ has the form

$$F(t,s,z) = -\sum_{i=1}^{n}[\int_t^T f_i(\tau)\mathrm{d}\tau] I_i(t,s,z).$$

For the proof of the above theorem see [12]. Now we can demonstrate the application of the theorems in pricing geometric averages with stochastic volatility

A floating strike Asian call option

A floating strike Asian call option has a payoff $C(T,S_T,I_T) = \max(S_T - I_T, 0)$ by (1.38).

The zero order approximation to the option price equal to the Black-Scholes pricing formula for geometric floating strike call with constant volatility $\bar{\sigma}$

$$C(t,S,I) \;=\; e^z U(t,s) = e^z[N(d_1) - \exp(\frac{s}{T} + R(t,T))N(d_2)] \tag{1.43}$$

where

$$d_1 \;=\; \frac{-s + (r + \bar{\sigma}^2/2)(T^2 - t^2)/2}{\sqrt{\bar{\sigma}^2(T^3 - t^3)/3}} \tag{1.44}$$

$$d_2 \;=\; d_1 - \sqrt{\bar{\sigma}^2(T^3 - t^3)/3T^2} \tag{1.45}$$

$$R(t;T) = (r - \bar{\sigma}^2/2)(T-t)^2/2T + \bar{\sigma}^2(T-t)^3/6T^2 - r(T-t).$$

The first order correction term can be derived via (1.43), as the form (1.43) guarantees

$$\frac{\partial^{m+n}C_0}{\partial s^m \partial z^n} = \frac{\partial^n}{\partial z^n}(\frac{\partial^m C_0}{\partial s^m}) = \frac{\partial^m C_0}{\partial s^m} \tag{1.46}$$

$\forall m, n = 0, 1, 2, \cdots$ then the $E_y(\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_2V_0)$ in (1.40) is reduced to

$$E_y(\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_2C_0) = t(P_2 - P_1)\frac{\partial C_0}{\partial s} + t^2(P_1 - 2P_2)\frac{\partial^2 C_0}{\partial s^2} + t^3 P_2\frac{\partial^3 C_0}{\partial s^3} \tag{1.47}$$

s-differential operators of C_0 are homogeneous solution to the PDE

$$E_y(\mathcal{L}_2)I = 0. \tag{1.48}$$

Applying theorem 2 we obtain

$$C_1(t, s, I) = (P_1 - P_2)(T^2 - t^2)/2\frac{\partial C_0}{\partial s} + (2P_2 - P_1)(T^3 - t^3)/3\frac{\partial^2 C_0}{\partial s^2} - P_2(T^4 - t^4)/4\frac{\partial^3 C_0}{\partial s^3}. \tag{1.49}$$

If we combine the above equation with with the zeroth order solution, the first order approximation to the floating strike geometric Asian call is

$$C = C_0(t, s, I_t) + \sqrt{\epsilon}C_1(t, s, I_t) + \cdots . \tag{1.50}$$

A fixed strike Asian call option

A fixed strike Asian call option has a payoff $C(T, S_T, I_T) = \max(I_T - K, 0)$.
The zero order approximation

$$C(t, S, I) = \exp(\frac{s}{T} + z + R(t, T))N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{1.51}$$

where

$$d_1 = \frac{T(z - \ln K) + s + (r - \bar{\sigma}^2/2)(T - t)^2/2 + \bar{\sigma}^2(T - t)^3/3T}{\sqrt{\bar{\sigma}^2(T - t)^3/3}}$$

$$d_2 = d_1 - \sqrt{\bar{\sigma}^2(T - t)^3/3T^2}$$

$$R(t; T) = (r - \bar{\sigma}^2/2)(T - t)^2/2T + \bar{\sigma}^2(T - t)^3/6T^2 - r(T - t).$$

By (1.51)

$$\frac{\partial^{m+n} C_0}{\partial s^m \partial z^n} = \frac{\partial^n}{\partial z^n}(\frac{\partial^m C_0}{\partial s^m}) = T^n \frac{\partial^{m+n} C_0}{\partial s^{m+n}} \tag{1.52}$$

$\forall m, n = 0, 1, 2, \cdots$, which is then substituted into (1.40), then the expression is reduced

$$E_y(\mathcal{L}_1\mathcal{L}_0^{-1}\mathcal{L}_2C_0) = -P_1(T-t)\frac{\partial C_0}{\partial s} + (P_1+P_2)(T-t)^2\frac{\partial^2 C_0}{\partial s^2} - P_2(T-t)^3\frac{\partial^3 C_0}{\partial s^3} \quad (1.53)$$

$$C_1 = \frac{1}{2}(T-t)^2P_1\frac{\partial C_0}{\partial s} - \frac{1}{3}(T-t)^3(P_1+P_2)\frac{\partial^2 C_0}{\partial s^2} + (T-t)^4/4P_2\frac{\partial^3 C_0}{\partial s^3} \quad (1.54)$$

the first order approximation to the fixed strike geometric Asian call is

$$C = C_0(t, s, I) + \sqrt{\epsilon}C_1(t, s, I) + \cdots . \quad (1.55)$$

4.2 Asian option of a Multiscale stochastic volatility

Now we look at how to derive Asian price in a multi-factor stochastic volatility, note that we will only consider a two-factor stochastic volatility.

4.2.1 Arithmetic Asian option under Multi stochastic volatility

A four-dimensional price PDE is reduced to a three-dimensional PDE. Consider a discrete-sampled scenario with multiscale stochastic volatility model using time-dependent. A trading strategy function, given by

$$q_t = e^{-rt}\int_t^T e^{rs}\mathrm{d}\lambda(s)$$

is a finite-variation process q_t is the number of units held at time t of the underlying stock. The quantity $(X_t - q_tS_t)e^{-rt}$ is the number of units held in bonds. The price of the bond is e^{rt}. Assume this portfolio is self-financing so that the variation of the wealth process can be expressed in differentiation form as

$$\begin{aligned} dX_t &= q_t \mathrm{d}S_t + (X_t - q_t S_t)e^{-rt}\mathrm{d}(e^{rt}) \\ &= q_t \mathrm{d}S_t + r(X_t - q_t S_t)\mathrm{d}t. \end{aligned}$$

The payoff of the Asian contract can be replicated by X_T, which is

$$X_T = \int_0^T S_t \mathrm{d}\lambda(t) - K_2, \tag{2.56}$$

if the initial wealth is chosen to be

$$X_0 = q_0 S_0 - e^{-rT} K_2.$$

The general payoff function of arithmetic average Asian option

$$h\left(\int_0^T S_t \mathrm{d}\lambda(t) - K_1 S_T - K_2\right) = h(X_T - K_1 S_t)$$

where $\lambda(t) = t/T$. When $K_1 = 0$, we have a fixed strike Asian option and when $K_2 = 0$ we have the floating strike Asian option. And the price of an arithmetic average Asian option with multiscale stochastic volatility is given by

$$P^{\epsilon,\delta}(0, s, y, z; T, K_1, K_2) = e^{-rT} E^*[h(X_T - K_1 S_T | S_0 = s, Y_0 = y, Z_0 = z] \tag{2.57}$$

under the risk-neutral measure P^*. By change of numeraire

$$\psi_t = X_t / S_t,$$

and by Ito's formula, the dynamics of this numeraire process is given

$$\mathrm{d}\psi_t = (q_t - \psi_t) f(Y_t) \mathrm{d}\tilde{W}_t^*, \tag{2.58}$$

where the shifted Brownian motion $\tilde{W}_t^*$ is defined by

$$\tilde{W}_t^* = W_t^* - \int_0^t f(Y_s, Z_s) \mathrm{d}s.$$

By Girsanov theorem, under the probability measure $\tilde{P}^*$ defined by

$$\begin{aligned} \mathrm{d}\tilde{P}^*/\mathrm{d}P^* &= e^{-rT} S_T/S_0 \\ &= e^{\int_0^T f(Y_t,Z_t)\mathrm{d}W_t^* - \frac{1}{2}\int_0^T f(Y_t,Z_t)^2 \mathrm{d}t}. \end{aligned}$$

The process $\tilde{W}_t^*$ by the above equation, becomes a standard Brownian motion. Hence the driving volatility processes can be expressed as

$$\begin{aligned} \mathrm{d}Y_t^\epsilon &= [\alpha(m_f - Y_t^\epsilon) - \nu_f\sqrt{2\alpha}\Lambda(Y_t, Z_t)]\mathrm{d}t + \nu_f\sqrt{2\alpha}(\rho \mathrm{d}W_t^{(0)} + \sqrt{1-\rho_1^2}\mathrm{d}W_t^{(1)}) \\ \mathrm{d}Z_t &= [\delta(m_s - Z_t) - \nu_s\sqrt{2\delta}\Gamma(Y_t, Z_t)]\mathrm{d}t + \nu_s\sqrt{2\delta}(\rho_2 \mathrm{d}W_t^{(0)} + \rho_{12}\mathrm{d}W_t^{(1)} \\ &\quad + \sqrt{1-\rho_2^2-\rho_{12}^2}\mathrm{d}W_t^{(2)}). \end{aligned}$$

where $(W_t^{(0)}, W_t^{(1)}, W_t^{(2)})$ are independent standard Brownian motions, and the function Λ and Γ are given by

$$\Lambda(t,s,y) = \frac{\rho_1(u-r)}{f(y,z)} + \gamma(t,s,y)\sqrt{1-\rho_1^2}$$

$$\Gamma(t,s,y) = \frac{\rho_2(u-r)}{f(y,z)} + \gamma(t,s,y)\rho_{12} + \epsilon(s,y,z)\sqrt{1-\rho_2^2-\rho_{12}}. \tag{2.59}$$

Assume that the payoff function h satisfies the homogeneous property i.e

$$h(sy) = sh(y) \quad \text{for } s > 0. \tag{2.60}$$

When $t = 0$ the Asian option price (2.57) becomes

$$\begin{aligned} &xE^*[e^{-rT}\tfrac{S_T}{S_0}h(\tfrac{X_T}{S_T} - K_1)|S_0 = s, Y_0 = y, Z_0 = z] \\ &= x\tilde{E}^*[h(\psi_T - K_1)|\psi_0 = \psi, Y_0 = y, Z_0 = z]. \end{aligned}$$

By (2.56) we have

$$\psi = \frac{s}{x} = q(0) - e^{-rT}\frac{K_2}{x}.$$

The quantity of interest $u^{\epsilon,\delta}$ is defined by

$$u^{\epsilon,\delta}(0,\psi,y,z;T,K_1,K_2) = \tilde{E}^*[h(\psi_T - K_1|\psi_0 = \psi, Y_0 = y, Z_0 = z]$$

such that the Asian option price (2.57) is

$$P^{\epsilon,\delta}(0,s,y,z;T,K_1,K_2) = su^{\epsilon,\delta}(0,\psi,y,z;T,K_1,K_2).$$

Note that from (2.58) and (2.59) the joint process (ψ, Y_t, Z_t) is Markovian. If for $t \le T$

$$u^{\epsilon,\delta}(t,\psi,y,z;T,K_1,K_2) = \tilde{E}^*[h(\psi_T - K_1|\psi_t = \psi_t, Y_t = y, Z_t = z].$$

By Feynman-Kac formula, $u^{\epsilon,\delta}$ solves

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2 + \sqrt{\delta}\hat{\mathcal{M}}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3)u^{\epsilon,\delta} = 0 \tag{2.61}$$

with terminal condition $u^{\epsilon,\delta}(T,\psi,y,z) = h(\psi - K_1)$, where

$$\hat{\mathcal{L}}_2(f(y,z)) = \frac{\partial}{\partial t} + \frac{1}{2}(\psi - q(t))^2 f(y)^2 + \frac{\partial^2}{\partial \psi^2}$$

$$\hat{\mathcal{M}}_1 = -(\sqrt{2}\nu\Gamma(y,z) - \rho_2\sqrt{2}\nu f(y,z))\frac{\partial}{\partial z} + \rho_2\sqrt{2}\nu f(y,z)(q(t)-\psi)\frac{\partial^2}{\partial s \partial z}.$$

Now PDE (2.61) has one less spatial dimension.

Asymptotics Expand the solution $u^{\epsilon,\delta}$ of (2.61) in powers of $\sqrt{\delta}$

$$u^{\epsilon,\delta}(t,\psi,y,z) = u_0^{\epsilon}(t,\psi,y,z) + \sqrt{\delta}u_1^{\epsilon}(t,\psi,y,z) + \delta u_2(t,\psi,y,z) \tag{2.62}$$

and substitute into (2.61) to get

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2)u_0^{\epsilon} + \sqrt{\delta}((\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2)u_1^{\epsilon} + \hat{\mathcal{M}}_1 u_0^{\epsilon} + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3 u_0^{\epsilon}) + \cdots = 0. \tag{2.63}$$

The leading order term $u_0^{\epsilon,\delta}$ solves the problem

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2)u_0^{\epsilon} = 0$$

with the terminal condition $u_0^\epsilon = h(\psi - K_1)$. When the single perturbation was performed, the following approximation was obtained

$$u_0^\epsilon \approx u_0(t,\psi,z) + \tilde{u}_{1,0}(t,\psi,z) \tag{2.64}$$

where the leading order term $u_0(t,\psi,z)$ solves

$$\begin{aligned} \langle \hat{\mathcal{L}}_2 \rangle u_0 &= \frac{\partial u_0}{\partial t} + \frac{1}{2}(\psi - q(t))^2 \langle f(y,z)^2 \rangle \frac{\partial^2 u_0}{\partial \psi^2} = 0 && (2.65) \\ u_0(T,\psi,z) &= h(\psi - K_1) && (2.66) \end{aligned}$$

and the correction $\tilde{u}_{1,0}(t,\psi,z) \equiv \sqrt{\epsilon} u_{1,0}(t,\psi,z)$ solves

$$\begin{aligned} \langle \hat{\mathcal{L}}_2 \rangle \tilde{u}_{1,0} &= (\bar{V}_2(z)(q_t - \psi)^2 \frac{\partial^2}{\partial \psi^2} + \bar{V}_3(z)(q_t - \psi)^3 \frac{\partial^3}{\partial \psi^3}) u_0 && (2.67) \\ \tilde{u}_{1,0}(T,\psi,z) &= 0 && (2.68) \end{aligned}$$

The z-independent functions are given by

$$\bar{V}_2(z) = \frac{\nu\sqrt{\epsilon}}{\sqrt{2}}(-\rho_1 \langle f(y,z) \frac{\partial \phi(y,z)}{\partial z} \rangle - \langle \Lambda(y,z) \frac{\partial \phi(y,z)}{\partial z} \rangle)$$

$$\bar{V}_3(z) = \frac{\rho_1 \nu \sqrt{\epsilon}}{\sqrt{2}} \langle f(y,z) \frac{\partial \phi(y,z)}{\partial z} \rangle$$

The term $u_1^\epsilon(t,\psi,y,z)$ in (2.62) solves

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2) u_1^\epsilon = -(\hat{\mathcal{M}}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3) u_0^\epsilon \tag{2.69}$$

with the zero terminal condition, we are looking for the solution whose expansion is given by

$$u_1^\epsilon(t,\psi,y,z) = u_{0,1}(t,\psi,y,z) + \sqrt{\epsilon} u_{1,1}(t,\psi,y,z) + \epsilon u_{2,1}(t,\psi,y,z) + \cdots$$

substituting the above into PDE (2.69), using the expansion (2.64) for u_0^ϵ, it follows that $u_{0,1}$, $u_{1,1}$ and $u_{2,1}$ solve the following PDE

$$\begin{aligned} &\mathcal{L}_0 u_{0,1} = 0 \\ &\mathcal{L}_1 u_{0,1} + \mathcal{L}_0 u_{1,1} = -\mathcal{M}_3 u_0 = 0 \\ &\hat{\mathcal{L}}_2 u_{0,1} + \mathcal{L}_1 u_{1,1} + \mathcal{L}_0 u_{2,1} = -\hat{\mathcal{M}}_1 u_0. \end{aligned}$$

In conclusion, $u_{0,1}$ and $u_{1,1}$ are independent of variable y, and $u_{0,1}$ solves

$$\langle\hat{\mathcal{L}}_2\rangle u_{0,1} = -\hat{\mathcal{M}}_1 u_0 \tag{2.70}$$

where the homogenized partial differential operator

$$\begin{aligned}\langle\hat{\mathcal{M}}_1\rangle &= (-\sqrt{2}\nu\langle\Gamma(y,z)\rangle - \rho_2\sqrt{2}\nu\langle f(y,z)\rangle) \\ &\quad +\rho_2\sqrt{2}\nu\langle f(y,z)\rangle(q_t-\psi)\frac{\partial}{\partial\psi})\bar{\sigma}'\frac{\partial}{\partial\sigma}\end{aligned}$$

and $\bar{\sigma}'$ denotes the derivative with respect to z. Define $\tilde{u}_{0,1} = \frac{\sqrt{\delta}}{2}u_{0,1}$, so that $\tilde{u}_{0,1}(t,\psi,z)$ solves

$$\langle\hat{\mathcal{L}}_2\rangle\tilde{u}_{0,1} = (\frac{1}{\bar{\sigma}}(\bar{V}_0^\delta\frac{\partial}{\partial\sigma} + \bar{V}_1^\delta(q_t-\psi)\frac{\partial^2}{\partial\psi\partial\sigma})u_0 \tag{2.71}$$

$$\tilde{u}_{0,1}(T,\psi,z) = 0 \tag{2.72}$$

where

$$\begin{aligned}\bar{V}_0^\delta &= \tfrac{\sqrt{\delta}}{\sqrt{2}}(\sqrt{2}\nu\langle\Gamma(y,z)\rangle - \rho_2\sqrt{2}\nu\langle f(y,z)\rangle)\bar{\sigma}' \\ \bar{V}_1^\delta &= -\tfrac{\sqrt{\delta}}{\sqrt{2}}\rho_2\sqrt{2}\nu\langle f(y,z)\rangle)\bar{\sigma}'.\end{aligned} \tag{2.73}$$

If we summarize this we obtained

$$u^{\epsilon,\delta}(t,\psi,y,z) = u_0(t,\psi,z) + \tilde{u}_{1,0}(t,\psi,z) + \tilde{u}_{0,1}(t,\psi,z) + \mathcal{O}(\epsilon+\delta+\sqrt{\epsilon\delta})$$

where u_0 solves (2.65). $\tilde{u}_{1,0}$ is of order $\mathcal{O}(\sqrt{\epsilon})$ which solves (2.67) and $\tilde{u}_{1,0}$ is of order $\mathcal{O}(\sqrt{\delta})$ which solves (2.71). The price approximation for a fresh Asian is given by

$$P^{\epsilon,\delta}(0,s,y,z) = su_0(0,\psi,z) + s\tilde{u}_{1,0}(0,\psi,z) + s\tilde{u}_{0,1}(0,\psi,z) + \mathcal{O}(\epsilon+\delta+\sqrt{\epsilon\delta}).$$

The leading term $u_0(t,\psi,z)$ solves

$$\langle\mathcal{L}_2\rangle u_0 \equiv \frac{\partial u_0}{\partial t} + \frac{1}{2}(\psi-q(t))^2\frac{\partial^2 u_0}{\partial\psi^2} = 0 \tag{2.74}$$

with the terminal condition $u_0^\epsilon(t,\psi,z) = h(\psi-K_1)$ and the sum of $\tilde{u}_{1,0}$ and $\tilde{u}_{0,1}$ solves the source problem

$$\begin{aligned}\langle\hat{\mathcal{L}}_2\rangle(\tilde{u}_{1,0}+\tilde{u}_{0,1}) &= \bar{V}_2^\varepsilon(z)(q(t)-\psi)^2\frac{\partial^2 u_0}{\partial\psi^2}) + \bar{V}_3^\epsilon(z)(q(t)-\psi)^3\frac{\partial^3 u_0}{\partial\psi^3}) \\ &\quad +\frac{1}{\bar{\sigma}}(\bar{V}_0^\delta\frac{\partial u_0}{\partial\sigma} + \bar{V}_1^\delta(q(t)-\psi) + \frac{\partial^2 u_0}{\partial\psi\partial\bar{\sigma}}\end{aligned}$$

Then we obtain the linear relation

$\bar{V}_2^\epsilon = V_2^\epsilon - V_3^\epsilon$

$\bar{V}_3^\epsilon = V_3^\epsilon$

$\bar{V}_0^\delta = V_0^\delta + V_1^\delta$

$\bar{V}_1^\delta = V_1^\delta$

The correction for the homogenized Asian option need to be solved numerically. Since there are no closed form solution for two-factor geometric average option Jean-Pierre Fouque and Chuan-Hsiang Han propose to evaluate geometric average Asian option by Monte Carlo simulations using the variance reduction technique see [16].

Seasons Asian Option prices and Asian Put-Call Parity The argument for continuous sampled season for Asian option prices under multiscale stochastic volatility model is the same as that of a one-factor stochastic volatility . Denote $\mathcal{F}_t$ the σ-algebra denoted by three-dimensional $(S_u, Y_u, Z_u, 0 \leq u \leq t)$. The price of Asian call at time t is given by

$$\begin{aligned}
&E^*[e^{-r(T-t)}(\frac{1}{T}\int_0^T S_u \mathrm{d}u - K_1 S_t - K_2)^+ | F_t] \\
&= E^*[e^{-r(T-t)}(\frac{1}{T}\int_0^T S_u \mathrm{d}u - K_1 S_t - K_2)^+ | S_t = s, Y_t = y, Z_t = z, G_t = t] \\
&= \frac{\tau}{T} E^*[e^{-r\tau}(\frac{1}{\tau}\int_0^\tau S_t \mathrm{d}t - \hat{K}_1 S_\tau - \hat{K}_2)^+ | S_0 = s, Y_0 = y, Z_0 = z] \\
&= \frac{\tau}{T} P_{call}^{\epsilon,\delta}(0, s, y, z; \tau, \hat{K}_1, \hat{K}_2) \qquad (2.75)
\end{aligned}$$

and the Asian put option is

$$\begin{aligned}
&\frac{\tau}{T} E^*[e^{-r\tau}(\frac{1}{\tau}\int_0^\tau S_t \mathrm{d}t - \hat{K}_1 S_\tau - \hat{K}_2)^- | S_0 = s, Y_0 = y, Z_0 = z] \\
&= \frac{\tau}{T} P_{put}^{\epsilon,\delta}(0, s, y, z; \tau, \hat{K}_1, \hat{K}_2) \qquad (2.76)
\end{aligned}$$

where we denote $\tau = T - t$ the time to maturity, and the updated strikes $\hat{K}_1 = \frac{T}{\tau}K_1$ and $\hat{K}_2 = \frac{T}{\tau}K_2 + \frac{1}{\tau}I$. For a put-call Asian option parity we have

$$\begin{aligned}
&\frac{\tau}{T}P^{\epsilon}_{call}(0, s, y, z; \tau, \hat{K}_1, \hat{K}_2) + \frac{\tau}{T}P^{\epsilon}_{put}(0, s, y, z; \tau, \hat{K}_1, \hat{K}_2) \\
&\quad = \frac{\tau}{T}E^*[e^{-r\tau}(\frac{1}{\tau}\int_0^{\tau} S_t \mathrm{d}t - \hat{K}_1 S_\tau - \hat{K}_2)^+ | S_0 = s, Y_0, y, Z_0 = z] \\
&\quad = \frac{s}{T}\frac{1 - e^{-r\tau}}{r} + \frac{\tau}{T}\hat{K}_1 s - \frac{\tau}{T}e^{-r\tau}\hat{K}_2. \qquad (2.77)
\end{aligned}$$

4.2.2 Geometric Asian option under Multi stochastic volatility

Denote the price of a fixed strike of a Geometric Asian option by $P^{\varepsilon,\delta}$ and apply a Feynman-Kac formula to the price of a geometric average Asian call option

$$P(t, v, L)\mathbb{E}^*\{e^{-r(T-t)}(\exp(\frac{L_T}{T}) - S_T - K)^+ | V_t = v, L_t = L\} \qquad (2.78)$$

where

$$\mathrm{d}L_t = \ln S_t \mathrm{d}t \qquad (2.79)$$

then $P^{\epsilon,\delta}(t, s, y, z, L)$ solves a four-dimensional PDE

$$\begin{aligned}
&\mathcal{L}_L^{\epsilon,\delta} P^{\epsilon,\delta} &&= 0 \\
&P^{\epsilon,\delta}(T, s, y, z, L) &&= (\exp(\tfrac{L}{T}) - K)^+,
\end{aligned} \qquad (2.80)$$

where the partial differential operator $\mathcal{L}_L^{\epsilon,\delta}$ is denoted by

$$\mathcal{L}_L^{\epsilon,\delta} = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_L + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3$$

and

$$\mathcal{L}_L(f(y, z)) = \mathcal{L}_2(f(y, z)) + \ln s \frac{\partial}{\partial L}, \qquad (2.81)$$

see [19] for details on derivation of a geometric average option with floating strikes. By the change of variables

$$\hat{s} = L - t\ln s \tag{2.82}$$

and

$$\hat{z} = \ln s \tag{2.83}$$

a modified PDE, is obtained

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\hat{\mathcal{L}}_1 + \mathcal{L}_2 + \sqrt{\delta}\hat{\mathcal{M}}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3\right) P^{\epsilon,\delta} = 0 \tag{2.84}$$

$$P^{\epsilon,\delta}(T,\hat{s},y,z,\hat{z}) = \left(\exp(\frac{(\hat{s}+T\hat{z})}{T}) - K\right)^+$$

where

$$\hat{\mathcal{L}}_1 = \nu_f\sqrt{2}[\rho_1 f(y,z)(\frac{\partial}{\partial\hat{z}} - t\frac{\partial}{\partial\hat{s}})\frac{\partial}{\partial y} - \Lambda(y,z)\frac{\partial}{\partial y}]$$

$$\mathcal{L}_2(f(y,z)) = \frac{\partial}{\partial t} + \frac{1}{2}f(y,z)^2(\frac{\partial}{\partial\hat{z}} - t\frac{\partial}{\partial\hat{s}})^2 + (r - \frac{1}{2}f(y,z)^2)(\frac{\partial}{\partial\hat{z}} - t\frac{\partial}{\partial\hat{s}}) - r.$$

$$\hat{\mathcal{M}}_1 = \nu_s\sqrt{2}[\rho_2 f(y,z)(\frac{\partial}{\partial\hat{z}} - t\frac{\partial}{\partial\hat{s}})\frac{\partial}{\partial z} - \Gamma(y,z)\frac{\partial}{\partial z}].$$

Consider an asymptotic expansion in powers of $\sqrt{\delta}$

$$P^{\epsilon,\delta}(t,\hat{s},y,z,\hat{z}) = P_0^\epsilon(t,\hat{s},y,z,\hat{z}) + \sqrt{\delta}P_1^\epsilon(t,\hat{s},y,z,\hat{z}) + \delta P_2(t,\hat{s},y,z,\hat{z}) \tag{2.85}$$

and substitute this into the modified PDE (2.84) we then obtain

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\hat{\mathcal{L}}_1 + \mathcal{L}_2)P_0^\epsilon + \sqrt{\delta}((\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\hat{\mathcal{L}}_1 + \mathcal{L}_2)P_1^\epsilon + \hat{\mathcal{M}}_1 P_0^\epsilon + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3 P_0^\epsilon) + \cdots = 0. \tag{2.86}$$

P_0^ϵ solves the singular perturbation problem with an additional z-dependent variable

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\hat{\mathcal{L}}_1 + \mathcal{L}_2)P_0^\epsilon = 0 \tag{2.87}$$

with terminal condition

$$P^{\epsilon,\delta}(T,\hat{s},y,z,\hat{z}) = \left(\exp(\frac{(\hat{s}+T\hat{z})}{T}) - K\right)^+.$$

If we perform the singular perturbation, see [12] for details, the following approximation is obtained

$$P_0^\epsilon \approx P_0(t,\hat{s},z,\hat{z}) + \tilde{P}_{1,0}(t,\hat{s},z,\hat{z}) \tag{2.88}$$

where the leading order term $P_0(t,\hat{s},z,\hat{z})$ solves

$$\langle \mathcal{L}_2 \rangle P_0 \ = 0 \tag{2.89}$$

$$P_0(t,s,z,\hat{z}) = \left(\exp(\frac{(\hat{s}+T\hat{z})}{T}) - K \right)^+ \tag{2.90}$$

and $\tilde{P}_{1,0}(t,\hat{s},z,\hat{z}) \equiv \sqrt{\epsilon} P_{1,0}(t,\hat{s},z,\hat{z})$ solves

$$\begin{aligned} \langle \mathcal{L}_2 \rangle \tilde{P}_{1,0}(t,s,z,L) &= -V_2((T-t)^2 \frac{\partial^2}{\partial \hat{s}^2} - (T-t)\frac{\partial}{\partial \hat{s}})P_0 \\ &\quad -V_3((T-t)^3 \frac{\partial^3}{\partial \hat{s}^3} - (T-t)^2\frac{\partial^2}{\partial \hat{s}^2})P_0 \\ \tilde{P}_{1,0}(t,\hat{s},z,\hat{z}) &= 0. \end{aligned} \tag{2.91}$$

There exists explicit solutions in terms of (s,z,L) for these two PDEs

$$\begin{aligned} P_0(t,s,L;\bar{\sigma}) &= \exp\left(\frac{L - t\ln s}{T} + \ln s + R(t,T,z))N(d_1(s,z,L) \right) \\ &\quad -Ke^{-r(T-t)}N(d_2(s,z,L)) \end{aligned} \tag{2.92}$$

is the price of a fixed strike geometric average Asian option under the effective volatility $\bar{\sigma}(z)$.

$$\begin{aligned} \tilde{P}_{1,0}(t,s,z,L) &= -(T-t)^2/2V_2 \frac{\partial P_0}{\partial s} + (T-t)^3/3(V_2 - V_3)\frac{\partial^2 P_0}{\partial s^2} \\ &\quad +(T-t)^4/4V_3 \frac{\partial^3 P_0}{\partial s^3}. \end{aligned} \tag{2.93}$$

If we consider the expansion $P_1^\epsilon(t,\hat{s},y,z,\hat{z})$, which solves

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\hat{\mathcal{L}}_1 + \mathcal{L}_2)P_1^\epsilon = -(\hat{\mathcal{M}}_1 + \frac{\mathcal{M}_3}{\sqrt{\epsilon}})P_0^\epsilon \tag{2.94}$$

with zero terminal condition. We look at the expansion of the following form

$$P_1^\epsilon(t,\hat{s},y,z,\hat{z}) = P_{0,1}(t,\hat{s},y,z,\hat{z}) + \sqrt{\epsilon}P_{1,1}(t,\hat{s},y,z,\hat{z}) + \epsilon P_{2,1}(t,\hat{s},y,z,\hat{z}) + \cdots$$

substituting the above expansion into (2.94) and using the expansion (2.88), it follows that $P_{0,1}$, $P_{1,1}$ and $P_{2,1}$ solve the following PDEs

$$
\begin{aligned}
&\mathcal{L}_0 P_{0,1} = 0\\
&\hat{\mathcal{L}}_1 P_{0,1} + \mathcal{L}_0 P_{1,1} = -\mathcal{M}_3 P_0 = 0\\
&\mathcal{L}_2 P_{0,1} + \hat{\mathcal{L}}_1 P_{1,1} + \mathcal{L}_0 P_{2,1} = -\hat{\mathcal{M}}_1 P_0.
\end{aligned}
$$

In conclusion, $P_{0,1}$ and $P_{1,1}$ are independent of the variable y, and $P_{0,1}$ solves

$$\langle \mathcal{L}_2 \rangle P_{0,1} = -\langle \hat{\mathcal{M}}_1 \rangle P_0 \tag{2.95}$$

where the homogenized partial differential operator $\langle \hat{\mathcal{M}}_1 \rangle$ is written as

$$\langle \hat{\mathcal{M}}_1 \rangle = \nu_s \sqrt{2} (\rho_2 \langle f(y,z) \rangle (\frac{\partial}{\partial \hat{z}} - t \frac{\partial}{\partial \hat{s}}) \frac{\partial}{\partial z} - \langle Lambda(y,z) \rangle \frac{\partial}{\partial z}$$

using the homogeneous property of the solution P_0

$$\frac{\partial^n P_0}{\partial \hat{z}^n} = T^n \frac{\partial^n P_0}{\partial \hat{s}^n} \tag{2.96}$$

we simplify

$$\langle \hat{\mathcal{M}}_1 \rangle P_0 = (T-t) \nu_s \sqrt{2} \rho_2 \langle f(y,z) \rangle \bar{\sigma}'(z) \frac{\partial^2 P_0}{\partial \hat{s} \partial \sigma} - \nu_s \sqrt{2} \langle \Gamma(y,z) \rangle \bar{\sigma}'(z) \frac{\partial P_0}{\partial \sigma} \tag{2.97}$$

where the Vega of P_0 in terms of $\hat{s}, y, z$ and $\hat{z}$ is

$$\frac{\partial P_0}{\partial \sigma} = (T-t)^3/3 \frac{\partial^2 P_0}{\partial \hat{s}^2} - (T-t)^2/2\sigma \frac{\partial P_0}{\partial \sigma}. \tag{2.98}$$

Then we obtain the following explicit solution

$$P_{0,1} = \frac{T^2 - t^2}{2} \nu_s \sqrt{2} \rho_2 \langle f(y,z) \rangle \bar{\sigma}'(z) \frac{\partial^2 P_0}{\partial \hat{s} \partial \sigma} + (T-t) \nu_s \sqrt{2} \langle \Gamma \rangle \bar{\sigma}'(z) \frac{\partial P_0}{\partial \sigma}$$

or in terms of (t, s, z, L) with the definition $\tilde{P}_{0,1} = \sqrt{\delta} P_{0,1}$

$$\tilde{P}_{0,1} = (T+t) s V_1 \frac{\partial^2 P_0}{\partial s \partial \sigma} - (T-t) \sqrt{2} V_0 \frac{\partial P_0}{\partial \sigma}.$$

Remark To obtain an accuracy result of the approximation

$$P_G^{\epsilon,\delta}(t,s,y,z,L) \approx \tilde{P}_G((t,s,y,z,L) = P_0 + \tilde{P}_{1,0} + \tilde{P}_{0,1} \tag{2.99}$$

which is

$$\begin{aligned} P_G^{\epsilon,\delta}(t,s,y,z,L) &= P_0 + (T+t)sV_1\frac{\partial^2 P_0}{\partial s\partial\sigma} - (T-t)\sqrt{2}V_0\frac{\partial P_0}{\partial\sigma} - (T-t)^2/2V_2\frac{\partial P_0}{\partial s} \\ &\quad +(T-t)^3/3(V_2-V_3)\frac{\partial^2 P_0}{\partial s^2} + (T-t)^4/4V_3\frac{\partial^3 P_0}{\partial s^3}. \end{aligned} \tag{2.100}$$

For details see [19], and the main result is presented here. For any given point $t < T, s \in \Re^+$, and $(y,z,L) \in \Re^3$, the accuracy of the approximation for fixed strike Asian call option is given by

$$|P_G^{\epsilon,\delta}(t,s,y,z,L) - \tilde{P}_G^{\epsilon,\delta}((t,s,z,L)| \leq C\max\{\epsilon,\delta,\sqrt{\epsilon\delta}\} \tag{2.101}$$

for all $0 < \delta < \bar{\delta}$ and $0 < \epsilon < \bar{\epsilon}$.

Appendix A

Appendix

Given V, V_I and V_S at four grid points $(I_i, S_j), (I_{i+1}, S_j), (I_i, S_{j+1})$ and (I_{i+1}, S_{j+1}), we have twelve equations to determine the coefficients $\{a_i\}_{i=1}^{12}$ of the interpolating function (6.45) with $\Delta I = I_{i+1} - I_i$ and $\Delta S = S_j$ at (I_i, S_j)

$$F_{i,j}(0,0) = V(I_i, S_j)$$

$$\frac{\partial F_{i,j}(0,0)}{\partial \xi} = V_I(I_i, S_j)$$

$$\frac{\partial F_{i,j}(0,0)}{\partial \eta} = V_S(I_i, S_j)$$

at (I_{i+1}, S_j)

$$F_{i,j}(\Delta I, 0) = V(I_{i+1}, S_j)$$

$$\frac{\partial F_{i,j}(\Delta I, 0)}{\partial \xi} = V_I(I_{i+1}, S_j)$$

$$\frac{\partial F_{i,j}(\Delta I, 0)}{\partial \eta} = V_S(I_{i+1}, S_j)$$

at (I_i, S_{j+1})

$$F_{i,j}(0, \Delta S) = V(I_i, S_{j+1})$$

$$\frac{\partial F_{i,j}(0, \Delta S)}{\partial \xi} = V_I(I_i, S_{j+1})$$

$$\frac{\partial F_{i,j}(0, \Delta S)}{\partial \eta} = V_S(I_i, S_{j+1})$$

at (I_{i+1}, S_{j+1})

$$F_{i,j}(\Delta I, \Delta S) = V(I_{i+1}, S_{j+1})$$

$$\frac{\partial F_{i,j}(\Delta I, \Delta S)}{\partial \xi} = V_I(I_{i+1}, S_{j+1})$$

$$\frac{\partial F_{i,j}(\Delta I, \Delta S)}{\partial \eta} = V_S(I_{i+1}, S_{j+1})$$

From these equations, the coefficients are

$$
\begin{aligned}
a_1 &= \frac{V_I(I_{i+1},S_j)+V_I(I_i,S_j)}{\Delta I^2} - 2\frac{V(I_{i+1},S_j)-V_I(I_i,S_j)}{\Delta I^3} \\
a_2 &= -\frac{V_I(I_{i+1},S_j)+2V_I(I_i,S_j)}{\Delta I} + 3\frac{V(I_{i+1},S_j)-V_I(I_i,S_j)}{\Delta I^2} \\
a_3 &= V_I(I_i,S_j) \\
a_4 &= V(I_i,S_j) \\
a_5 &= \frac{V_S(I_i,S_{j+1})+V_S(I_i,S_j)}{\Delta S^2} - 2\frac{V(I_i,S_{j+1})-V_I(I_i,S_j)}{\Delta S^3} \\
a_6 &= -\frac{V_S(I_i,S_{j+1})+2V_S(I_i,S_j)}{\Delta S} + 3\frac{V(I_i,S_{j+1})-V_I(I_i,S_j)}{\Delta S^2} \\
a_7 &= V_S(I_i),S_j) \\
a_8 &= -\frac{V_I(I_{i+1},S_{j+1})-V_I(I_{i+1},S_j)+2V_I(I_i,S_{j+1})-2V_I(I_i,S_j)}{\Delta I \Delta S} \\
&\quad +3\frac{V(I_{i+1},S_{j+1})-V(I_i,S_{j+1})-V(I_{i+1},S_j)+V(I_i,S_j)}{\Delta I^2 \Delta S} \\
a_9 &= \frac{V_I(I_i,S_{j+1})-V_I(I_i,S_j)}{\Delta S} + \frac{V_S(I_{i+1},S_j)-V_S(I_i,S_j)}{\Delta I} \\
&\quad -\frac{V(I_{i+1},S_{j+1})-V(I_{i+1},S_j)-V(I_i,S_{j+1})+V(I_i,S_j)}{\Delta I \Delta S} \\
a_{10} &= -\frac{V_S(I_{i+1},S_{j+1})-V_S(I_i,S_{j+1})+2V_S(I_{i+1},S_j)-2V_S(I_i,S_j)}{\Delta I \Delta S} \\
&\quad +3\frac{V(I_{i+1},S_{j+1})-V(I_{i+1},S_j)+2V(I_i,S_{j+1})+V(I_i,S_j)}{\Delta I \Delta S^2} \\
a_{11} &= \frac{V_I(I_{i+1},S_{j+1})-V_I(I_{i+1},S_j)+V_I(I_i,S_{j+1})-V_I(I_i,S_j)}{\Delta I^2 \Delta S} \\
&\quad -2\frac{V(I_{i+1},S_{j+1})-V(I_{i+1},S_j)-V(I_i,S_{j+1})+V(I_i,S_j)}{\Delta I^3 \Delta S} \\
a_{12} &= \frac{V_S(I_{i+1},S_{j+1})-V_S(I_i,S_{j+1})+V_S(I_{i+1},S_j)-V_S(I_i,S_j)}{\Delta I \Delta S^2} \\
&\quad -2\frac{V(I_{i+1},S_{j+1})-V(I_{i+1},S_j)-V(I_i,S_{j+1})+V(I_i,S_j)}{\Delta I \Delta S^3}
\end{aligned}
$$

Coefficients of the IDO method Given V, V_I and V_S at three grid points S_{j-1}, S_j, and S_{j+1}, we have then have six equations to determine the coefficients $b_{i\,i=1}^{\,6}$ of the interpolating function (6.45) with $\Delta S = S_{j+1} - S_j = S_j - S_{j-1}$

at S_{j-1}

$$G_j(-\Delta S) = V(S_{j-1})$$

$$\frac{\partial G_j(-\Delta S)}{\partial \eta} = V_S(S_{j-1})$$

at S_j

$$G_j(0) = V(S_j)$$

$$\frac{\partial G_j(0)}{\partial \eta} = V_S(S_j)$$

at S_{j+1}

$$G_j(\Delta S) = V(S_{j+1})$$

$$\frac{\partial G_j(\Delta S)}{\partial \eta} = V_S(S_{j+1})$$

$$b_1 = -\frac{3}{4}\frac{V(S_{j+1}) - V(S_j)}{\Delta S^5} + \frac{1}{4}\frac{V_S(S_{j+1}) + 4V_S(S_j) + V_S(S_{j-1})}{\Delta S^4}$$

$$b_2 = -\frac{1}{2}\frac{V(S_{j+1}) - 2V(S_j) + V(S_{j-1})}{\Delta S^4} + \frac{1}{4}\frac{V_S(S_{j+1}) - V_S(S_{j-1})}{\Delta S^3}$$

$$b_3 = -\frac{5}{4}\frac{V(S_{j+1}) - V(S_{j-1})}{\Delta S^3} - \frac{1}{4}\frac{V_S(S_{j+1}) + 8V_S(S_j) + V_S(S_{j-1})}{\Delta S^2}$$

$$b_4 = \frac{V(S_{j+1}) - 2V(S_j) + V(S_{j-1})}{\Delta S^2} - \frac{1}{4}\frac{V_S(S_{j+1}) - V_S(S_{j-1})}{\Delta S}$$

$$b_5 = V_S(S_j)$$

$$b_6 = V(S_j)$$

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