Mathematical and Numerical Analysis of The Discrete Fragmentation-Coagulation Equation with Growth, Decay and Sedimentation

By

Luke Oluwaseye JOEL

(211559858)

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Westville Campus, South Africa.

Supervisors

Prof. J. Banasiak

Dr. S. Shindin

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DECLARATION

The work described by this thesis was carried out at the University of Kwazulu-Natal, School of Mathematics, Statistics and Computer Science, University of Kwazulu-Natal, Westville Campus, under the supervision of Prof. J. Banasiak and Dr. S. Shindin.

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Dedication

This dissertation is dedicated to God Almighty who guided me into this PhD study and also saw me through it. Glory, Honour and Power be to His Holy name.

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Abstract

Fragmentation-coagulation equations arise naturally in many branches of engineering and science, the applications stretching from astrophysics, blood clotting, colloidal chemistry and polymer science to molecular beam epitaxy. In realistic application, the fragmentation and coagulation are often coupled with growth, decay and/or sedimentation processes. The resulting models are used to describe the evolution of a population in which individuals can grow, coalesce, split or divide, and die. For example, in the phytoplankton dynamics, in addition to forming or breaking of clusters, individuals within them are born or die and so the latter processes must be adequately represented in the models. In the continuous case, the birth or death processes are incorporated into the model by adding an appropriate first order transport term, analogously to the age and size structured McKendrick models. In the discrete case, these vital processes are modelled by adding weighted differences operators.

In this study, we focus on the discrete fragmentation-coagulation models with growth, decay or/and sedimentation. The problem is treated as an infinite-dimensional differential equation, which consists of a linear part (fragmentation, growth, decay and sedimentation term) and a nonlinear part (coagulation term), posed in a suitable Banach space, X. We use the theory of semigroups of linear operators, perturbation of positive semigroups and semilinear operators for the mathematical analysis of these models. The linear part of the models is shown to generate a semigroup which is analytic, compact and irreducible and thus has the asynchronous exponential growth property.

These results are used to demonstrate the existence of global classical solutions to the semilinear fragmentation-coagulation equations with growth, decay and sedimentation for a class of unbounded coagulation kernels. Theoretical conclusions are supported by numerical simulations.

Contents

List of Figures

Outcome of Research Work - Publication

Details of publication that form part and/or include research presented in this thesis.

- Joel L.O., Banasiak J. and Shindin S. Analysis of The Discrete Growth-Fragmentation Equation. Proceedings of the Southern Africa Mathematical Sciences Association (SAMSA2016), Annual Conference, 21 - 24 November 2016, University of Pretoria, South Africa.
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Chapter 1

General Introduction

1.1 Introduction

Coagulation and fragmentation models that describe the processes of objects forming larger clusters or, conversely, splitting into smaller fragments, have received a lot of attention over several decades due to their importance in chemical engineering and other fields of science and technology, see e.g. [34, 74]. One of the most efficient approaches to modelling dynamics of such processes is through the kinetic (rate) equation which describes the evolution of the distribution of interacting clusters with respect to their size/mass. The first model of this kind, consisting of an infinite system of ordinary differential equations, was derived by Smoluchowski, [67, 68], to describe pure coagulation in the discrete case; that is, if the ratio of the mass of the basic building block (monomer) to the mass of a typical cluster is a positive integer, and thus the size of a cluster is a finite multiple of the masses of the monomers. The original Smoluchowski equation reads

$$\frac{df_i(t)}{dt} = \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j} f_{i-j}(t) f_j(t) - \sum_{j=1}^{\infty} k_{i,j} f_i(t) f_j(t), \quad i \geq 1, \tag{1.1.1}$$

where $k_{i,j}$ is the coagulation rate (the rate at which a cluster of mass i and a cluster of mass j join each other to form a cluster of mass $i+j$) and $f_i(t)$ is the number density of clusters of mass i, $i \geq 1$. The number density of clusters is a non-negative function. The first term on the right-hand side describes the emergence of i-clusters by coagulation of smaller clusters, while the second term gives the disappearance of i-clusters due to the coalescence with other clusters. The factor of $\frac{1}{2}$ is to ensure that double counting is avoided.

In many applications, however, it turned out to be advantageous to allow clusters to be composed of particles of any size $x > 0$. This leads to the continuous integro-differential equation that was derived by Müller in the pure coagulation case, [53],

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\int_0^x k(x-y,y)u(t,x-y)u(t,y)\,dy - u(t,x)\int_0^\infty k(x,y)u(t,y)\,dy, \tag{1.1.2}$$

and extended to a continuous fragmentation–coagulation equation in [49]. The discrete version, which involves also the binary fragmentation and the initial condition, is given by

$$\frac{df_i(t)}{dt} = \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j} f_{i-j}(t) f_j(t) - \sum_{j=1}^{\infty} k_{i,j} f_i(t) f_j(t) - \frac{1}{2}\sum_{j=1}^{i-1} \psi_{j,i-j} f_i(t) + \sum_{j=i+1}^{\infty} \psi_{i,j-i} f_j(t), \tag{1.1.3a}$$

$$f_i(0) = f_i^0, \quad i \geq 1. \tag{1.1.3b}$$

In (1.1.3a), the coagulation rate, $k_{i,j}$ and the fragmentation rate, $\psi_{i,j}$ are assumed to be non-negative and satisfy:

$$k_{i,j} = k_{j,i}, \qquad \psi_{i,j} = \psi_{j,i}, \qquad i, j \geq 1. \tag{1.1.4}$$

The fragmentation here is a binary. The first term on the right-hand side represents the formation of i-clusters by binary coalescence of smaller ones and the second term represents the death of the i-clusters by coagulation with other clusters. The third term on the right-hand side represents the fragmentation of i clusters into two smaller ones and the fourth term represents the creation of i-clusters from the breakage of larger ones. The factor of $\frac{1}{2}$ is to ensure that double counting is avoided. Setting $\psi_{i,j} = 0$, gives the pure Smoluchowski coagulation equation introduced earlier. Similarly, setting $k_{i,j} = 0$ gives the pure binary fragmentation equation, see [5, 16, 25, 47, 48, 72, 75, 73] and references therein. The total mass of the system, given by

$$M(t) = \sum_{i=1}^{\infty} i f_i(t), \quad t \geq 0, \tag{1.1.5}$$

is expected to be conserved.

Ball and Carr [4] considered the existence, uniqueness and mass conservation of solutions to problem (1.1.3). Their study is concentrated on the case when the coagulation rates satisfy

$$k_{i,j} \leq k(i+j), \quad i, j \geq 1, \tag{1.1.6}$$

where $k > 0$ is a constant, and with no further assumption on the fragmentation rates. Ball and Carr showed the existence of a mild solution to the equation by taking the limit of solutions to approximating finite-dimensional systems. Laurençot [44] presented another proof of the existence for problem (1.1.3), under the same assumption (1.1.6), via the study of propagation of moments for approximating solutions. Other existence results have been obtained in the literature for various classes of fragmentation rates [28, 69] and when the coagulation rates satisfy the weaker condition

$$k_{i,j} \leq kij, \quad i, j \geq 1. \tag{1.1.7}$$

1.2 Fragmentation-Coagulation Model with Multiple Fragmentation

The discrete fragmentation-coagulation equation with multiple fragmentation was studied in [7, 44, 46, 65]. Multiple fragmentation means that the fragmenting clusters can break into more than two pieces. The coagulation rate is still denoted by $k_{i,j}$, while the fragmentation rate is given by a_i, and $b_{i,j}$ is the average number of i-mers produced after the breakup of a j-mer, with $j \geq i$. The fragmentation-coagulation model with multiple fragmentation is given by

$$\frac{df_i(t)}{dt} = \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j} f_{i-j}(t) f_j(t) - \sum_{j=1}^{\infty} k_{i,j} f_i(t) f_j(t) - a_i f_i(t) + \sum_{j=i+1}^{\infty} a_j b_{i,j} f_j(t), \quad (1.2.1a)$$

$$f_i(0) = f_i^0, \quad i \geq 1. \quad (1.2.1b)$$

Note that for $i = 1$, the first summation on the right-hand side of (1.2.1a) is zero. Also, as clusters can fragment into two or more smaller pieces but not into bigger clusters, it is good to note the following:

$$\begin{aligned} &a_1 = 0, \quad a_i \geq 0, \quad i \geq 2, \\ &b_{i,j} = 0, \quad i \geq j. \end{aligned} \quad (1.2.2)$$

Further, for this system to conserve mass, the following assumption has to be imposed

$$\sum_{i=1}^{j-1} i b_{i,j} = j, \quad j \geq 2. \quad (1.2.3)$$

Indeed, in this case, formal summation yields

$$M(t) = \sum_{i=1}^{\infty} i f_i(t) = \sum_{i=1}^{\infty} i f_i(0), \quad t \geq 0, \quad (1.2.4)$$

since

$$\begin{aligned} \sum_{i=1}^{\infty} i\left(-a_i f_i + \sum_{j=i+1}^{\infty} a_j b_{i,j} f_j\right) &= -\sum_{i=1}^{\infty} i a_i f_i + \sum_{j=2}^{\infty} j a_j f_j \sum_{i=1}^{j-1} i b_{i,j} \\ &= -\sum_{i=1}^{\infty} i a_i f_i + \sum_{j=1}^{\infty} j a_j f_j = 0, \qquad \text{since} \quad a_1 = 0 \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{\infty} i\left(\frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j} f_{i-j} f_j - \sum_{j=1}^{\infty} k_{i,j} f_i f_j\right) &= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty} i k_{i-j,j} f_{i-j} f_j - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i k_{i,j} f_i f_j \\ &= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} (l+j) k_{l,j} f_l f_j - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i k_{i,j} f_i f_j \\ &= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} l k_{l,j} f_l f_j + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} j k_{l,j} f_l f_j \\ &\quad - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i k_{i,j} f_i f_j = 0. \end{aligned}$$

Remark 1.2.1. Equation (1.1.3) is a particular case of (1.2.1). Given the fragmentation rate $\psi_{i,j}$, having the symmetry property as described in (1.1.4), we derive equation (1.2.1) from system (1.1.3) by setting

$$a_j = \frac{1}{2}\sum_{i=1}^{j-1}\psi_{i,j-i} \quad \text{and} \quad b_{i,j} = \frac{\psi_{i,j-i}}{a_j}.$$

Note that with these settings, the total number of clusters obtained after the breaking of a j-mer equals

$$\sum_{i=1}^{j-1} b_{i,j} = \sum_{i=1}^{j-1}\frac{\psi_{i,j-i}}{a_j} = 2, \quad j \geq 2,$$

i.e. the fragmentation process described by (1.1.3) is indeed binary.

In [65, Chapter 3], the author proved the existence and uniqueness of a strong solution to model (1.2.1) via the theory of semigroups with the additional assumptions that the coagulation kernel $k_{i,j}$ is symmetric and is uniformly bounded:

$$k_{i,j} = k_{j,i}, \quad k_{i,j} \leq k, \quad i,j \geq 1. \tag{1.2.5}$$

1.3 Fragmentation-Coagulation with Growth, Decay and Sedimentation

In the last few decades it has been observed that also living organisms form clusters or split into subgroups depending on circumstances, see e.g. [42, 57, 56, 30] for modelling concerning larger animals, or [43, 1] for phytoplankton models. It turns out that also the process of cell division may be modelled within the same framework, see e.g. [19, 64, 60, 18]. What was not always fully recognized in some papers mentioned above was that the living matter has its own vital dynamics; that is, in addition to forming or breaking clusters, individuals within them are born or die and so the latter processes must be adequately represented in the models. In the continuous case, the birth and death processes are incorporated into the model by adding an appropriate first order transport term, analogous to the age or size structured McKendrick model, see [1, 13, 8, 18, 60]. On the other hand, in the discrete case the vital processes are modelled by adding the classical birth-and-death terms to the Smoluchowski equation. Note that e.g. the pure birth terms (or pure death terms) can be obtained by the Euler discretization of the first order differential operator of the continuous case while the full birth-and- death problem can be thought of as the discretization of the diffusion operator. This leads to the generalization of the model in (1.2.1):

$$\begin{aligned}\frac{df_i(t)}{dt} &= g_{i-1}f_{i-1}(t) - g_i f_i(t) + d_{i+1}f_{i+1}(t) - d_i f_i(t) - s_i f_i(t) - a_i f_i(t) \\ &\quad + \sum_{j=i+1}^{\infty} a_j b_{i,j} f_j(t) + \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j} f_{i-j}(t) f_j(t) - \sum_{j=1}^{\infty} k_{i,j} f_i(t) f_j(t),\end{aligned} \tag{1.3.1a}$$

with the initial conditions

$$f_i(0) = f_i^0, \quad i \geq 1. \tag{1.3.1b}$$

The coagulation and fragmentation processes in (1.3.1a) are generally mass conservative, but the growth, decay and sedimentation parts are not. Hence, there is a mass leakage or increase in the system.

Most of the literature [1, 13, 8, 58, 62] deals with the continuous version of this model, hence, the review to be given here will be for the continuous form of systems (1.3.1a). Ackleh and Fitzpatrick [1], in their earlier work on the aggregation and growth processes in an algal population, considered the continuous form of (1.3.1a) without fragmentation, decay and sedimentation term using the theory of semigroups of operators in $L_2([x_0, x_1])$. In [55, Chapter 5], the author considered phytoplankton dynamics using the continuous form of (1.3.1a), without the growth, decay and sedimentation term, with the assumption that only some part of the phytoplankton aggregates, which gives a different coagulation model. The space used for the analysis was

$$X_1 = L_1((x_0, \infty), xdx) = \{\psi : \|\psi\| = \int_{x_0}^{\infty} x|\psi|dx < \infty\}.$$

In [13], the authors carried out the analysis of the continuous model in the space

$$X_{0,1} = L_1([x_0, x_1), (1+x)dx) = L_1([x_0, x_1), dx) \cap L_1([x_0, x_1), xdx),$$

where $0 \leq x_0 < x_1 \leq \infty$, x_0 and x_1 being the minimum and maximum size of the particles. The existence and uniqueness of a strongly differentiable solution was proved in this space using the theory of semigroups of operators.

1.4 Long Time Behaviour

One of the most important problems in the analysis of dynamical systems is to determine their long time behaviour. The first systematic study of the long time dynamics of the binary cell division model was carried out using semigroup theory in [31]; the semigroup approach was significantly extended to more general models in [52]. Recently a number of results have been obtained by the General Relative Entropy (or related) methods that lead to convergence of solutions in spaces weighted by the eigenvector of the adjoint problem, see e.g. [32, 33, 11, 45, 50, 60, 61].

All of the above results concern models with a continuous size distribution. Recently it has been observed, however, that a large class of discrete fragmentation equations has much better properties than their continuous counterparts, especially when considered in spaces where sufficiently high moments of solutions are finite. In particular, the fragmentation operator in such spaces generates a compact analytic semigroup. In this thesis, we explore these ideas for the growth-decay-sedimentation-fragmentation equation, growth-decay-fragmentation equation and the decay-fragmentation equation and we show, in particular, that under some assumptions on the coefficients of the problem the solution semigroup is analytic, compact and irreducible and thus has the Asynchronous Exponential Growth (AEG) property, see [3].

1.5 Outline of Thesis

In this study, we shall focus on the fragmentation-coagulation equation with growth, decay and sedimentation. We investigate the existence of a unique strong, non-negative solution to various forms of the fragmentation-coagualtion equation with growth, decay and sedimentation. We employ the theory of semigroups of linear and semilinear operators, similar to [13, 7, 14, 8] to show under which conditions these solutions exist. We also show under which conditions the semigroups are analytic and compact, and then, we investigate the long-time behaviour of the solutions.

In Chapter 2, we give some definitions and results of the semigroup of linear and semilinear operators used in proving the existence and uniqueness of solutions to the models. In Chapter 3, we study the fragmentation equation with growth, decay and sedimentation, and prove the analyticity, compactness of the solution semigroup and the long time behaviour of the model. We also give some examples of such models with some numerical illustration of their properties. In Chapter 4, we study the fragmentation-coagulation equation with growth, decay and sedimentation. The nonlinear coagulation term is added to the linear terms studied in Chapter 3. The results of the linear parts are used to prove the local and global solution to the general model. Chapter 5 is the conclusion of the thesis.

The work in Chapters 3 and 4 has been submitted to journals for publication. These papers consist of excerpts of this thesis.

Chapter 2

Theoretical Background

2.1 Introduction

This chapter deals with the theoretical tools that are needed for the analysis of the models presented in these studies. The method relies on the techniques of semigroups of linear and semilinear operators. This method can be used to solve a large class of evolution equations which appear in many disciplines such as biology, physics, chemistry, and engineering. Hence, we introduce some basic definitions and results from the theory of semigroups of operators which will be used through all the analyses done in this study.

2.2 Semigroups and their Generators

Let X be a Banach space over $\mathbb{C}$ with norm $\|.\|$ and let $\mathcal{B}(X)$ be the set of all linear bounded (continuous) operators from X to itself.

Definition 2.2.1. *A one-parameter family $(S(t))_{t\geq 0} \subset \mathcal{B}(X)$ of bounded linear operators on X is said to be a semigroup on X, if it satisfies*

(i) $S(0) = I$,

(ii) $S(s)S(t) = S(s+t)$ *for all* $s, t \geq 0$.

Definition 2.2.2. *A semigroup $(S(t))_{t\geq 0}$ is said to be uniformly continuous with respect to the operator norm $\|.\|$ associated with X, if*

$$\lim_{t\to 0^+} \|S(t) - I\| = 0,$$

while it is said to be a strongly continuous or C_0-semigroup, if

$$\lim_{t\to 0^+} \|S(t)x - x\| = 0,$$

for all $x \in X$.

In view of Definition 2.2.1, we expect $(S(t))_{t\geq 0}$ to mimic the behaviour of the classical exponential function. The following result shows that this is true.

Theorem 2.2.1. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup. Then there exist $M \geq 1$, $w \in \mathbb{R}$ such that*

$$\|S(t)\| \leq Me^{wt}, \quad t \geq 0. \tag{2.2.1}$$

The class of C_0-semigroups, that satisfy (2.2.1) *is denoted by $C_0(M, w)$.*

Proof. For the proof see [59, 37]. □

For our solution, in this study, to make physical sense, we need to incorporate positivity into the Banach space setting, and we do this through the notion of a positive cone.

Definition 2.2.3. *A subset E of a Banach space X is said to be a cone if*

(i) $f \in E \Rightarrow tf \in E \quad \forall t \geq 0$, *and*

(ii) $f \in E$ *and* $-f \in E \Rightarrow f = 0_X$,

where 0_X is the zero vector in X.

The cone E is referred to as a positive cone if it is a convex subset of X and is closed with respect to the norm $\|.\|$ on X. In the sequel, we denote a positive cone of X by X_+. The cone X_+ defines a partial order in X. We say that $x \geq y$ iff $x - y \in X_+$. It can be verified that the relation '$\geq$', defined in this way, satisfies all axioms of the partial order.

Definition 2.2.4. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup. We say that $(S(t))_{t\geq 0}$ is*

(i) a C_0-semigroup of contractions *if $(S(t))_{t\geq 0} \in C_0(1, 0)$;*

(ii) a positive semigroup *if $S(t)x \geq 0$ for all x in the positive cone X_+ of X;*

(iii) a sub-stochastic semigroup *if it is a semigroup of contractions and is positive;*

(iv) a stochastic semigroup *if it is positive and $\|S(t)x\| = \|x\|$, for all $x \in X_+$.*

Definition 2.2.5. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup. The linear operator A, with its domain $D(A)$, defined by the identity*

$$Ax = \lim_{h\to 0^+} \frac{S(h)x - x}{h}, \quad D(A) = \left\{x \in X : \lim_{h\to 0^+} \frac{S(h)x - x}{h} \quad \text{exists in } X\right\},$$

is called the infinitesimal generator of the C_0-semigroup $(S(t))_{t\geq 0}$.

Directly from Definition 2.2.5 it follows that to each C_0-semigroup can be assigned a unique infinitesimal generator $(A, D(A))$. The generator is linear, but generally unbounded in X, map from $D(A)$ to X. The converse is not true. Not every linear map $(A, D(A))$ is a generator of a C_0-semigroup. To be an infinitesimal generator of a C_0-semigroup, operator A must satisfy some extra conditions.

Theorem 2.2.2. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup, and let A be its generator. Then*

(i) the domain of A is dense in X;

(ii) the operator A is closed.

Proof. See [38] for the proof. □

2.3 Generation Theorems

There is an intimate connection between the semigroup theory and Abstract Cauchy Problems (ACPs)

$$\frac{du}{dt} = Au, \quad u(0) = f_0,$$

posed in X. In particular, if A is an infinitesimal generator of a C_0-semigroup $(S(t))_{t\geq 0}$, then the solution of the corresponding ACP is given explicitly by $u(t) = S(t)f_0$. Hence, a natural question arises: under what conditions does an operator A generate a C_0-semigroup? Theorems that yield this are known as generation theorems. To formulate the results, we need the following

Definition 2.3.1. *The resolvent set $\rho(A)$ of the operator $(A, D(A))$ is defined by*

$$\rho(A) = \{\lambda \in \mathbb{C} : R(\lambda; A) \in \mathcal{B}(X)\},$$

where $R(\lambda; A) = (\lambda I - A)^{-1}$ is called the resolvent operator associated with operator A.

Theorem 2.3.1 (Hille-Yosida)**.** *A closed linear operator $(A, D(A))$ in X with dense domain $D(A)$, generates a C_0-semigroup of contraction $(S(t))_{t\geq 0}$ if and only if $(0, \infty) \subseteq \rho(A)$ and the following resolvent inequality holds*

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

Proof. See [59] for the proof. □

Theorem 2.3.2 (Feller-Miyadera-Phillips)**.** *A closed linear operator $(A, D(A))$ in X with dense domain $D(A)$, generates a C_0-semigroup $(S(t))_{t\geq 0} \subset C_0(M, w)$ if and only if $(w, \infty) \subseteq \rho(A)$ and the following resolvent inequality holds*

$$\|R(\lambda; A)^n\| \leq \frac{M}{(\lambda - w)^n}, \quad \lambda > w, \quad n \geq 1.$$

Proof. For the proof, see [37, 10]. □

2.4 Perturbation Theorems

Many practical problems take the form:

$$\frac{du}{dt} = Au + Bu, \quad u(0) = f_0.$$

In these settings, we view operator B as a perturbation of A. If A generates a C_0-semigroup and the perturbation B is not too 'bad', it is natural to expect that a suitable realization of the sum $A+B$ still generates a C_0-semigroup. Special literature contains a large number of perturbation results. Below, we mention two that are relevant for our studies.

Theorem 2.4.1 (The Bounded Perturbation Theorem)**.** *Let $(A, D(A))$ be the infinitesimal generator of a $C_0(M,w)$-semigroup $(S_A(t))_{t\geq 0}$ on X. If $B \in \mathcal{B}(X)$, then $(A+B, D(A))$ generates a $C_0(M, w+M\|B\|)$-semigroup $(S_{A+B}(t))_{t\geq 0}$ on X. Moreover, the semigroup $(S_{A+B}(t))_{t\geq 0}$ can be obtained via the Dyson-Philips expansion*

$$S_{A+B}(t)x = \sum_{n=0}^{\infty} V_n(t)x, \;\; V_0(t) = S_A(t), \;\; V_{n+1}(t)x = \int_0^t S_A(t-s)BV_n(s)xds, \;\; n \geq 0. \qquad (2.4.1)$$

Proof. See [59, 37]. □

Theorem 2.4.2 (Kato-Voigt Perturbation Theorem)**.** *Let $(A, D(A))$ and $(B, D(B))$ be two operators in $X = L_1(\Omega, \mu)$, where (Ω, μ) is a measure space. Assume further that*

(i) $(A, D(A))$ generates a sub-stochastic semigroup $(S_A(t))_{t\geq 0}$;

(ii) $D(B) \supseteq D(A)$ and $Bx \geq 0$ for all $x \in D(B)_+ = D(B) \cap X_+$;

(iii) for all $x \in D(A)_+ = D(A) \cap X_+$

$$\int_\Omega (Ax + Bx)d\mu \leq 0.$$

Then there exists a smallest sub-stochastic semigroup $(S_G(t))_{t\geq 0}$ on X whose infinitesimal generator is an extension of $(A+B, D(A))$. The generator G satisfies, for $\lambda > 0$

$$R(\lambda, G)x = \sum_{k=0}^{\infty} R(\lambda, A)[BR(\lambda, A)]^k x. \qquad (2.4.2)$$

Proof. See [10, Corollary 5.17]. □

We state a corollary of the Kato-Voigt Perturbation Theorem which will be useful for our analysis in this thesis. Sometimes, the operator may generate a C_0-semigroup which is not sub-stochastic. The corollary is obtained by adding or substracting a diagonal operator.

Corollary 2.4.3. *Let $(A, D(A))$ and $(B, D(B))$ be two operators in $X = L_1(\Omega, \mu)$, where (Ω, μ) is a measure space. Assume further that*

(i) $(A, D(A))$ generates a positive $C_0(1, w)$-semigroup $(S_A(t))_{t\geq 0}$;

(ii) $D(B) \supseteq D(A)$ and $Bx \geq 0$ for all $x \in D(B)_+ = D(B) \cap X_+$;

(iii) for all $x \in D(A)_+ = D(A) \cap X_+$

$$\int_\Omega (Ax + Bx)d\mu \leq w \int_\Omega x d\mu.$$

Then there exists a smallest positive $C_0(1,w)$-semigroup $(S_G(t))_{t\geq 0}$ on X whose infinitesimal generator is an extension of $(A+B, D(A))$. The generator G satisfies, for $\lambda > w$

$$R(\lambda, G)x = \sum_{k=0}^{\infty} R(\lambda, A)[BR(\lambda, A)]^k x. \tag{2.4.3}$$

Next, we give a definition for an operator B to be a Miyadera Perturbation of operator A and then we state a Miyadera perturbation theorem.

Definition 2.4.1. *Let $(A, D(A))$ generate a C_0-semigroup $(S_A(t))_{t\geq 0}$ on X and let operator $B \in \mathcal{B}(D(A), X)$, i.e B is $A-$bounded. B is a Miyadera perturbation on A if it satisfies*

$$\int_0^{\alpha} \|BS_A(t)x\| dt \leq q\|x\|, \tag{2.4.4}$$

for some α and q with $0 < \alpha < \infty$, $0 \leq q < 1$ and all $x \in D(A)$.

Theorem 2.4.4 (Miyadera Perturbation Theorem)**.** *If B is a Miyadera perturbation of A, then $(A+B, D(A))$ is the generator of a C_0-semigroup $(S_{A+B}(t))_{t\geq 0}$*

Proof. The proof is in [10, Theorem 4.16]. □

2.5 Characterization of the Generator

In the previous section, we noticed the existence of a smallest semigroup $(S_G(t))_{t\geq 0}$ generated by an extension $(G, D(G))$ of $(A+B, D(A))$. This semigroup satisfies

$$\frac{d}{dt} S_G(t)f = GS_G(t)f, \quad \forall f \in X \quad \text{and} \quad t > 0. \tag{2.5.1}$$

Since the Kato-Voigt Perturbation Theorem does not characterize the domain of the generator explicitly, we present a summary of some techniques [10, Chapter 6], that allow us to provide such a characterization. Full details of this can be found in [10, Chapter 6].
Let $X = L_1(\Omega, d\mu)$, A be the generator of a substochastic semigroup on X and let $B : D(A) \to X$ be a positive linear operator such that

$$\int_{\Omega} (A+B)u d\mu = -c(u), \quad u \in D(A)_+, \tag{2.5.2}$$

where c is a non-negative functional defined on $D(A)$, which can be written as an integral functional as follows

$$c(u) = \int_{\Omega} \varsigma(x) u(x) d\mu', \tag{2.5.3}$$

for some positive measurable function ς and positive measure μ' and without any assumption on the closedness or boundedness of c.

Definition 2.5.1. *A positive semigroup $(S_G(t))_{t\geq 0}$ generated by an extension G of the operator $A+B$ is said to be strictly substochastic if* (2.5.2) *holds with $c \neq 0$.*

Definition 2.5.2. *A positive semigroup $(S_G(t))_{t\geq 0}$ generated by an extension G of the operator $A+B$ is said to be honest if c extends to $D(G)$ and for any $\mathring{f} \in X_+$ the solution $u(t) = S_G(t)\mathring{f}$ of (2.5.1) satisfies*

$$\frac{d}{dt}\int_\Omega u(t)d\mu = \frac{d}{dt}\|u(t)\| = -c(u(t)). \tag{2.5.4}$$

For $c = 0$, the honest semigroups are the same as stochastic semigroups.

Theorem 2.5.1. *The semigroup $(S_G(t))_{t\geq 0}$ is honest if and only if $G = \overline{A+B}$.*

Proof. [10, Theorem 6.13]. □

Corollary 2.5.2. *The semigroup $(S_G(t))_{t\geq 0}$ is honest if and only if for any $u \in D(G)_+$ we have*

$$\int_\Omega Gud\mu \geq -c(u). \tag{2.5.5}$$

The statement also holds true if we replace $D(G)_+$ by $R(\lambda, G)X_+$ for any $\lambda > 0$.

Proof. [10, Corollary 6.14]. □

A number of alternative techniques for characterizing generators of semigroups arising in connection with Theorem 2.4.2 are developed in [10, Section 6.3]. They rely on the extension of the operators that appear in the model, since they are easier to construct.
Let $\mathbf{E} := L_0(\Omega, d\mu)$ denote the set of μ-measurable functions that are defined on Ω and take values in the extended set of real numbers, and let $\mathbf{E}_f$ be the subspace of $\mathbf{E}$ consisting of functions that are finite almost everywhere.

Let A, B be as defined in Corollary 2.4.3 and let $\mathbf{F} \subset \mathbf{E}$ be defined by the condition: $\psi \in \mathbf{F}$ if and only if for any non-negative and non-decreasing sequence $(\psi_n)_{n\in\mathbb{N}}$ satisfying $\sup_{n\in\mathbb{N}} \psi_n = |\psi|$, we have $\sup_{n\in\mathbb{N}}(I - A)^{-1}\psi_n \in X$. Under some natural assumptions on B, we construct another subset of $\mathbf{E}$, say $\mathbf{G}$, defined as the set of all functions $\psi \in X$ such that for any non-negative, non-decreasing sequence $(\psi_n)_{n\in\mathbb{N}}$ of elements of $D(B)$ such that $\sup_n \psi_n = |\psi|$, we have $\sup_n B\psi_n < \infty$ almost everywhere. We can then define mappings $\mathbf{L} : \mathbf{F}_+ \to X_+$ and $\mathbf{B} : \mathbf{G}_+ \to \mathbf{E}_+$ by

$$\mathbf{L}\psi := \sup_{n\in\mathbb{N}} R(1, A)\psi_n, \quad \psi \in \mathbf{F}_+,$$

$$\mathbf{B}\psi := \sup_{n\in\mathbb{N}} B\psi_n, \quad \psi \in \mathbf{G}_+,$$

where $0 \leq \psi_n \leq \psi_{n+1}$ for any $n \in \mathbb{N}$, and $\sup_n \psi_n = \psi$. We extend the mapping $\mathbf{L}$ and $\mathbf{B}$ onto $\mathbf{F}$ and $\mathbf{G}$, respectively, by linearity [10, Theorem 2.64]. By [10, Lemma 6.18], $\mathbf{L}$ is one-to-one therefore, we can define the operator $\mathbf{A}$ with $D(\mathbf{A}) = \mathbf{LF} \subset X$ by

$$\mathbf{A}u = u - \mathbf{L}^{-1}u, \tag{2.5.6}$$

so that $\mathbf{A}$ is an extension of A. This ultimately leads us to the following two important theorems [10, Theorem 6.20 and Theorem 6.22] on the extension techniques:

Theorem 2.5.3. *If $(A, D(A))$ and $(B, D(B))$ are operators in X such that $(A, D(A))$ generates a substochastic semigroup $G_A(t)_{t\geq 0}$ on X, $D(B) \supset D(A)$, $Bu \geq 0$ for $u \in D(B)_+$ and*

$$\int_\Omega (Au + Bu)d\mu \leq 0, \tag{2.5.7}$$

for all $u \in D(A)_+$, then, the extension G of $A + B$, that generates the smallest substochastic semigroup on X described by Theorem 2.4.2 is given by

$$Gu = \mathbf{A}u + \mathbf{B}u, \tag{2.5.8}$$

$$D(G) = \{u \in D(\mathbf{A}) \cap D(\mathbf{B}) : \mathbf{A}u + \mathbf{B}u \in X, \lim_{n\to+\infty} \|(\mathbf{LB})^n u\| = 0\}. \tag{2.5.9}$$

Proof. [10, Theorem 6.20]. □

Theorem 2.5.4. *If for any $g \in \mathbf{F}_+$ such that $-g + \mathbf{BL}g \in X$ and $c(\mathbf{L}g)$ exists,*

$$\int_\Omega \mathbf{L}g d\mu + \int_\Omega (-g + \mathbf{BL}g)d\mu \geq c(\mathbf{L}g), \tag{2.5.10}$$

then $G = \overline{A + B}$.

Proof. [10, Theorem 6.22]. □

2.6 Spectral Theory

In this section, our attention is on the qualitative behaviour of C_0-semigroups introduced earlier. We start by noting that the complement of the resolvent set in Definition 2.3.1 is called the spectrum of A, $\sigma(A) := \mathbb{C}\backslash\rho(A)$. This spectrum is a closed subset of $\mathbb{C}$.
We give the definition of different subsets of the spectrum $\sigma(A)$.

Definition 2.6.1. *Let $A : D(A) \subseteq X \to X$ be a closed operator.*

1. *The point spectrum of A is given by*

$$\sigma_p(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not injective}\}.$$

2. *The approximate point spectrum of A is*

$$\sigma_a(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not injective or } \operatorname{range}(\lambda I - A) \text{ is not closed in } X\}.$$

3. *The residual spectrum of A is*

$$\sigma_r(A) := \{\lambda \in \mathbb{C} : \operatorname{range}(\lambda I - A) \text{ is not dense in } X\}.$$

From the definition, we observe that $\sigma_p(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \sigma_r(A)$.

Definition 2.6.2. *Let $A : D(A) \subset X \to X$ be a closed operator. Then*

$$s(A) := \sup\{Re\lambda : \lambda \in \sigma(A)\}$$

is called the spectral bound of A.

Definition 2.6.3. *For a C_0-semigroup $\mathbb{S} = (S(t))_{t\geq 0}$, the quantity*

$$\omega_0 := \omega_0(\mathbb{S}) := \inf\{\omega \in \mathbb{R} : \text{ there exists } M_\omega \geq 1 \text{ such that } \|S(t)\| \leq M_\omega e^{\omega t} \text{ for all } t \geq 0\}$$

is called its growth bound.

We note that an operator A is said to be compact if the image of the unit ball has a compact closure. Let $\mathcal{K}(X)$ denote the set of all compact operators on X. The essential norm, $\|\cdot\|_{ess}$, of an operator $A \in \mathcal{B}(X)$ is the distance from A to the subspace $\mathcal{K}(X)$ of compact operators, i.e.

$$\|A\|_{ess} = \inf\{\|A - K\| : K \in \mathcal{K}(X)\}.$$

Definition 2.6.4. *The essential growth bound of the semigroup $\mathbb{S} = (S(t))_{t\geq 0}$, with generator A is given by*

$$\omega_{ess} := \omega_{ess}(\mathbb{S}) = \omega_{ess}(A) := \inf_{t>0} \frac{1}{t} \log \|S(t)\|_{ess}.$$

Definition 2.6.5. *A C_0- semigroup $(S(t))_{t\geq 0}$ is called eventually norm continuous if there exists $t_0 \geq 0$ such that the function*

$$t \mapsto S(t)$$

is norm continuous from (t_0, ∞) into $\mathcal{B}(X)$. The semigroup is called immediately norm continuous if t_0 can be chosen to be $t_0 = 0$.

Theorem 2.6.1. *(Spectral Mapping Theorem) Let $(S(t))_{t\geq 0}$ be an eventually norm-continuous semigroup with generator $(A, D(A))$ on X. Then*

$$\sigma(S(t))\backslash\{0\} = e^{t\sigma(A)}, \quad t \geq 0.$$

Proof. The proof is in [38, Theorem V.2.8]. □

The above theorem implies that for eventually norm continiuous C_0-semigroups the following is true [38, Corollary V.2.9]:

$$s(A) = \omega_0. \tag{2.6.1}$$

Theorem 2.6.2. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X with generator $(A, D(A))$. Then*

$$\omega_0 = \max\{\omega_{ess}, s(A)\}.$$

Moreover, for every $w > \omega_{ess}$, the set $\sigma_c := \sigma(A) \cap \{\lambda \in \mathbb{C} : Re\lambda \geq w\}$ is either empty or finite. In the latter case, the corresponding spectral projection has finite rank.

Proof. See [37, Corollary IV.2.11]. □

Theorem 2.6.3. *Let $\mathbb{S} = (S(t))_{t\geq 0}$ be a C_0-semigroup with generator A and let $\lambda_1, ..., \lambda_p \in \sigma(A)$ satisfy $Re\lambda_1, ..., Re\lambda_p > \omega_{ess}(\mathbb{S})$. Then $\lambda_1, ..., \lambda_p$ are isolated eigenvalues of A with finite* *algebraic multiplicity. If $P_1, ..., P_p$ denote the corresponding spectral projections and $k_1, ..., k_p$ the corresponding orders of poles of $R(\lambda, A)$, then*

$$S(t) = S_1(t) + ... + S_p(t) + R_p(t),$$

where

$$S_n(t) = e^{\lambda_n t} \sum_{j=0}^{k_n - 1} \frac{t^j}{j!} (A - \lambda_n)^j P_n, \quad \text{for } n = 1, ..., p.$$

Moreover, for every $w > \sup\{\omega_{ess}(\mathbb{S})\} \cup \{Re\lambda : \lambda \in \sigma(A) \backslash \{\lambda_1, ..., \lambda_p\}\}$ there exists $M > 0$ such that

$$\|R_p(t)\| \leq M e^{wt},$$

for all $t \geq 0$.

Proof. See [37, Theorem V.3.1]. □

2.7 Analyticity and Compactness

Definition 2.7.1. *A closed linear operator $(A, D(A))$ with dense domain $D(A)$ in X is called sectorial if there exists $0 < \delta \leq \frac{\pi}{2}$ such that the sector*

$$\Sigma_{\frac{\pi}{2}+\delta} := \Big\{\lambda \in C : |\arg \lambda| < \frac{\pi}{2} + \delta\Big\} \backslash \{0\}$$

is contained in the resolvent set $\rho(A)$, and if for each $\varepsilon \in (0, \delta)$ there exists $M_\varepsilon \geq 1$ such that

$$\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|} \quad \text{for all} \quad 0 \neq \lambda \in \Sigma_{\frac{\pi}{2}+\delta-\varepsilon}. \tag{2.7.1}$$

Definition 2.7.2. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X with infinitesimal generator A. $(S(t))_{t\geq 0}$ is said to be analytic if*

(i) For some $\delta \in (0, \pi/2]$, $S(t)$ can be extended to $\triangle_\delta$ where

$$\triangle_\delta = \{0\} \cup \Sigma_\delta.$$

(ii) For all $t \in \triangle_\delta - \{0\}$, $(S(t))_{t\geq 0}$ is analytic in t in the uniform operator topology.

The following result characterizes generators of bounded analytic semigroups.

Theorem 2.7.1. *For an operator $(A, D(A))$ on X, the following statements are equivalent.*

(a) A generates a bounded analytic semigroup $(S(z))_{z\in\Sigma_{\frac{\pi}{2}+\delta}\cup\{0\}}$ on X.

(b) There exists $\theta \in (0, \frac{\pi}{2})$ such that the operators $e^{\pm i\theta}A$ generate bounded C_0-semigroup on X.

(c) A generates a bounded C_0-semigroup $(S(t))_{t\geq 0}$ on X such that range$(S(t)) \subset D(A)$ *for all $t > 0$, and*

$$M := \sup_{t>0} \|tAS(t)\| < \infty. \tag{2.7.2}$$

(d) A generates a bounded C_0-semigroup $(S(t))_{t\geq 0}$ on X, and there exists a constant $C > 0$ such that

$$\|R(r + is, A)\| \leq \frac{C}{|s|}, \tag{2.7.3}$$

for all $r > 0$ and $0 \neq s \in \mathbb{R}$.

(e) A is sectorial.

Proof. The proof is in [37, Theorem II.4.6]. □

Below, we study positive perturbations of positive analytic semigroups. In this context, the following result (Arendt and Rhandi, [2]) is useful.

Theorem 2.7.2. *Assume that X is a Banach lattice, $(A, D(A))$ is a resolvent positve operator which generates an analytic semigroup and $(B, D(A))$ is a positive operator. If $(\lambda I-(A+B), D(A))$ has a nonnegative inverse for some λ larger than the spectral bound $s(A)$ of A, then $(A+B, D(A))$ generates a positive analytic semigroup.*

Proof. [9, Theorem 19]. □

Definition 2.7.3. *A C_0-semigroup $(S(t))_{t\geq 0}$ is called compact for $t > t_0$ if for every $t > t_0$, $(S(t))_{t\geq 0}$ is a compact operator. $(S(t))_{t\geq 0}$ is called compact if it is compact for $t > 0$.*

Theorem 2.7.3. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup and let A be its infinitesimal generator. The semigroup $(S(t))_{t\geq 0}$ is compact if and only if it is continuous in the uniform operator topology for $t > 0$ and $R(\lambda, A)$ is compact for $\lambda \in \rho(A)$.*

Proof. [59, Theorem 2.3.3]. □

In connection with Theorem 2.7.3, we note that a resolvent $R(\lambda, A)$ of a linear operator A is compact for one $\lambda \in \rho(A)$ if and only if it is compact for all $\lambda \in \rho(A)$. Also, we note that analytic semigroups are norm-continuous [38, Section II.5]. Hence, the results in Theorem 2.6.1 and (2.6.1) hold for compact semigroups and analytic semigroups [38, Corollary V.2.10].

2.8 Interpolation Spaces

In this section, we quote several results from the theory of interpolation spaces. These spaces play an important role in the analysis of the nonlinear fragmentation-coagulation models, when the coefficients of the coagulation kernel are unbounded. Specifically, in Chapter 4 we employ the following space [20, Chapter 3]

$$(X_0, X_1)_{\alpha,p} = \{u \in X_1 : \|u\|_{\alpha,p} < \infty\}, \quad 1 \leq p \leq \infty \quad 0 < \alpha < 1,$$

which is defined in terms of the given Banach spaces $X_0 \subset X_1$ and the functional

$$\|u\|_{\alpha,p} = \left[\int_0^\infty t^{-1-p\alpha} K^p(t, u)dt\right]^{\frac{1}{p}},$$

$$K(t,u) = \inf_{u_0+u_1=u}\left[\|u_0\|_{X_0} + t\|u_1\|_{X_1}\right].$$

The space $(X_0, X_1)_{\alpha,p}$, endowed with the norm $\|.\|_{\alpha,p}$ is an interpolation Banach space [20, Chapter 3]. That is, it is a Banach space such that

(i) for any $0 < \alpha < 1$, we have
$$X_0 \subset (X_0, X_1)_{\alpha,p} \subset X_1,$$
where each embedding is continuous and

(ii) for any $T \subset \mathcal{B}(X_1, Y_1)$, such that the restriction $T : X_0 \to Y_0 \subset Y_1$ is also continuous, the following is true
$$\|T\|_{(X_0,X_1)_{\alpha,p}\to(Y_0,Y_1)_{\alpha,p}} \leq \|T\|^{\alpha}_{X_0\to Y_0}\|T\|^{1-\alpha}_{X_1\to Y_1} \quad 1 \leq p \leq \infty, \quad 0 < \alpha < 1, \tag{2.8.1}$$

where $\|T\|_{(X_0,X_1)_{\alpha,p}\to(Y_0,Y_1)_{\alpha,p}}$ is the norm of bounded operator $T : (X_0, X_1)_{\alpha,p} \to (Y_0, Y_1)_{\alpha,p}$. The inequalities of type (2.8.1) are known as interpolation inequalities. Note also that in (2.8.1) the norm $\|T\|_{(X_0,X_1)_{\alpha,p}\to(Y_0,Y_1)_{\alpha,p}}$ is bounded by the product of norms multiplied by a constant factor of 1. This expresses the fact that the interpolation functor $(.,.)_{\alpha,p}$ is exact, see [20, Definition 2.4.1] for more details.

In our application, X_0, X_1 are weighted Lebesgue type spaces. In this settings, the complete characterization of $(X_0, X_1)_{\alpha,p}$ is contained in the theorem below.

Theorem 2.8.1 (The Interpolation Theorem of Stein-Weiss)**.** *Assume that* $0 < p \leq \infty$ *and* $0 < \alpha < 1$. *Let* $w(x) = w_0^{1-\alpha}(x)w_1^{\alpha}(x)$.
Then
$$(L^p(U, w_0 d\mu), L^p(U, w_1 d\mu))_{\alpha,p} = L^p(U, w d\mu).$$
Moreover, if
$$T \in \mathcal{B}(L^p(U, w_0 d\mu), L^p(V, \tilde{w}_0 dv)) \cap \mathcal{B}(L^p(U, w_1 d\mu), L^p(V, \tilde{w}_1 dv)),$$
then
$$\begin{aligned}\|T\|_{L^p(U,wd\mu)\to L^p(V,\tilde{w}dv)} \leq &\|T\|^{\alpha}_{L^p(U,w_0 d\mu)\to L^p(V,\tilde{w}_0 dv)} \\ &\times \|T\|^{1-\alpha}_{L^p(U,w_1 d\mu)\to L^p(V,\tilde{w}_1 dv)},\end{aligned}$$
where $\tilde{w}(x) = \tilde{w}_0^{1-\alpha}(x)\tilde{w}_1^{\alpha}(x)$.

Proof. The proof is in [20, Theorem 5.4.1] □

2.9 Semilinear Equations

So far, we have dealt with the linear semigroup theory. In this section, we will give some definitions and theorems relating to the theory of semilinear equations, which are useful in analysing more difficult evolution equations.

Definition 2.9.1. *(Semilinear Abstract Cauchy Problem) Let X be a Banach space and let $(G, D(G))$ be an operator in X with associated semigroup $(S_G(t))_{t\geq 0}$. Furthermore, let F be a nonlinear operator which maps $[0,T] \times D$ into X where $D(G) \cap D$ is not empty. Then, the abstract problem*

$$\frac{du(t)}{dt} = Gu(t) + F(t,u), \quad u(0) = f_0 \in D(G) \cap D, \tag{2.9.1}$$

is called a semilinear ACP.

Definition 2.9.2. *A function $u : [0, t_{\max}) \to X$ is said to be a classical solution to the semilinear ACP (2.9.1) on $[0, t_{\max})$ if u is continuous on $[0, t_{\max})$, continuously differentiable on $(0, t_{\max})$, $u(t) \in D(G)$ for $0 < t < t_{\max}$ and (2.9.1) is satisfied on $[0, t_{\max})$.*

Definition 2.9.3. *A function $u : [0, t_{\max}) \to X$ is said to be a mild solution to the semilinear ACP (2.9.1) on $[0, t_{\max})$ if u is continuous on $[0, t_{\max})$, $u(t) \in D$ for all $t \in [0, t_{\max})$, and*

$$u(t) = S_G(t)f_0 + \int_0^t S_G(t-s)F(s, u(s))ds, \quad 0 \leq t \leq t_{\max}. \tag{2.9.2}$$

Definition 2.9.4. *(Local Lipschitz Condition) An operator F on X is said to satisfy a local Lipschitz condition if, for any given $f_0 \in X$, there exists a closed ball*

$$\overline{B}_r(f_0) = \{\psi \in X : \|\psi - f_0\| \leq r\}$$

and a constant L such that $\|F\psi - F\mu\| \leq L\|\psi - \mu\|$ for all $\psi, \mu \in \overline{B}_r(f_0)$, where L may depend on f_0 and r.

Theorem 2.9.1. *(Local Mild Solution) Let F satisfies Definititon 2.9.4. If $(G, D(G))$ is the infinitesimal generator of a C_0-semigroup $(S_G(t))_{t\geq 0}$ on X, then for every $f_0 \in X$, there is a $t_{\max}$ such that the ACP (2.9.1) has a unique mild solution u on $[0, t_{\max})$. Moreover, if $t_{\max} < \infty$ then*

$$\lim_{t \to t_{\max}} \|u(t)\| = \infty.$$

Proof. The proof is in [59, Theorem 6.1.4]. □

Theorem 2.9.2. *(Local Classical Solution) Let $(G, D(G))$ be the infinitesimal generator of the C_0-semigroup $(S_G(t))_{t\geq 0}$ on X. If F is continuously differentiable on X, then the mild solution of ACP (2.9.1) with $f_0 \in D(G)$ is classical.*

Proof. The proof is in [59, Theorem 6.1.5]. □

Chapter 3

The Fragmentation Equation with Growth, Decay and Sedimentation

3.1 Introduction

Recent research on the fragmentation–coagulation models has been extended to include the diffusion [23, 27, 71] or transport of clusters in space, or their decay and/or growth [8, 12, 13, 58, 62]. For instance, in the application to life sciences, the change in size of the clusters is due to their coalescence and splitting as well as due to the death or birth of the organisms inside [56, 57, 42, 43, 51]. Particularly, in the phytoplankton dynamics, the removal of whole clusters caused by their sedimentation is an important process that is responsible for the rapid clearance of the organic material from the surface of the sea. The removal of clusters of suspended solid particles from a mixture is also important in water treatment, biofuel production, or beer fermentation. In all these applications the size distribution of the clusters is a crucial parameter controlling the efficacy of the process, [1, 43, 51]. Thus, models coupling the fragmentation, coagulation, birth, death and removal processes are relevant in many applications and hence in this chapter we focus on analysing the following system

$$\begin{aligned}\frac{df_i}{dt} &= g_{i-1}f_{i-1} - g_i f_i + d_{i+1}f_{i+1} - d_i f_i \\ &\quad - s_i f_i - a_i f_i + \sum_{j=i+1}^{\infty} a_j b_{i,j} f_j \qquad (3.1.1)\\ f_i(0) &= f_i^{in}, \quad i \geq 1,\end{aligned}$$

where $f = (f_i)_{i=1}^{\infty}$ gives the numbers f_i of clusters of mass i, and, to shorten notation, we adopt the convention that $g_0 = f_0 = 0$. The nonnegative coefficients g_i, d_i and s_i, $i \geq 1$, control the growth, the decay and the sedimentation processes, respectively. The fragmentation rates are given by a_i, while $b_{i,j}$ is the average number of i-mers produced after the breakup of a j-mer, with $j > i$. The difference operators $f \to (g_{i-1}f_{i-1} - g_i f_i)_{i=1}^{\infty}$ and $f \to (d_{i+1}f_{i+1} - d_i f_i)_{i=1}^{\infty}$ describe the rate of change of the number of particles due to, respectively, the birth and death process. The form of these operators can be obtained as in the standard birth-and-death Markov process, see e.g. [21], assuming that only one birth or death event occurs in a cluster of cells in a short period of time so that an i-cluster only may become an $i+1$, or an $i-1$-cluster. Setting $g_i = d_i = s_i = 0, i \geq 1$, we arrive at the classical mass-conserving fragmentation-coagulation equation.

In the sequel, it is required that conditions (1.2.3) and (1.2.2) are satisfied. We shall also use the fact that equation (3.1.1) can be written as the growth-sedimentation-fragmentation model

$$\begin{aligned} \frac{df_1}{dt} &= -g_1 f_1 + \sum_{i=2}^{\infty} \mathsf{a}_i \mathsf{b}_{1,i} f_i, \\ \frac{df_n}{dt} &= g_{n-1} f_{n-1} - (g_n + \mathsf{a}_n + s_n) f_n + \sum_{i=n+1}^{\infty} \mathsf{a}_i \mathsf{b}_{n,i} f_i, \quad n \geq 2, \\ f_n(0) &= f_n^{in}, \quad n \geq 1, \end{aligned} \tag{3.1.2}$$

where $\mathsf{a}_n = a_n + d_n$, $n \geq 2$, (with $\mathsf{a}_1 = 0$, $s_1 = 0$) and

$$\mathsf{b}_{n,i} = \begin{cases} \frac{a_{n+1} b_{n,n+1} + d_{n+1}}{a_{n+1} + d_{n+1}}, & i = n+1, \\ \frac{a_i b_{n,i}}{a_i + d_i}, & i \geq n+2. \end{cases} \tag{3.1.3}$$

We note that the fragmentation part of the modified model (3.1.2) no longer is conservative as

$$\sum_{n=1}^{i-1} n \mathsf{b}_{n,i} = i \left(1 - \frac{d_i}{i(a_i + d_i)} \right) < i, \qquad i \geq 2. \tag{3.1.4}$$

Hence formulation (3.1.2) corresponds to the model with the so-called discrete mass-loss scenario, with mass-loss fraction $\lambda_n = d_n / n(a_n + d_n)$, see [22, 36] and [66] for the mathematical analysis.

The analysis of pure fragmentation equations most often is carried out in the weighted space $X_1 := \ell_1^1$, with the norm

$$\|f\|_{[1]} = \sum_{n=1}^{\infty} n |f_n| \tag{3.1.5}$$

which, for a nonnegative f, gives the total mass of the ensemble. However, it is better to consider (3.1.1) in the spaces with finite higher moments, $X_p := \ell_p^1$, with the norm

$$\|f\|_{[p]} = \sum_{n=1}^{\infty} n^p |f_n|, \qquad p \geq 1. \tag{3.1.6}$$

The reason being that in ℓ_1^1, summation on the fragmentation part of the model vanishes, while in ℓ_p^1, summation the on fragmentation term plays a major role by introducing some dissipativity into the system. In the sequel, for any infinite diagonal matrix $\mathcal{P} = \mathrm{diag}(p_n)_{n \geq 1}$, we define the operator P_p in X_p by $P_p f = \mathcal{P} f = (p_n f_n)_{n \geq 1}$ on $D(P_p) = \{f \in X_p;\ \mathcal{P} f \in X_p\}$.

3.2 Analysis of the Subdiagonal Part

In this section, we shall consider the simplified problem corresponding to the subdiagonal part of (3.1.2),

$$\frac{df}{dt} = \mathcal{K} f = \mathcal{G}^- f + (\mathcal{A} + \mathcal{G}^0 + \mathcal{D}^0 + \mathcal{S}) f, \quad f(0) = f^{in}. \tag{3.2.1}$$

where $[\mathcal{G}^- f]_n = g_{n-1}f_{n-1}$, $[\mathcal{G}^0 f]_n = g_n f_n$, $[\mathcal{A}f]_n = a_n f_n$, $[\mathcal{D}^0 f]_n = d_n f_n$, and $[\mathcal{S}f]_n = s_n f_n$. We denote for brevity $\mathcal{T} = \mathcal{A} + \mathcal{G}^0 + \mathcal{D}^0 + \mathcal{S}$ and consider the operator $(T_p, D(T_p))$ defined, as above, by $T_p f = \mathcal{T} f$ on $D(T_p) = \{f \in X_p;\ \mathcal{T} f \in X_p\}$. Then $G_p^- := \mathcal{G}^-|_{D(T_p)}$ is a well defined positive operator in X_p and we can apply the substochastic semigroup theory, [10], to $\mathcal{K}|_{D(T_p)} = T_p + G_p^-$. Let $K_{p,\max}$ denote the maximal extension of K_p; that is, $K_{p,\max} f = \mathcal{T} f + \mathcal{G}^- f$ on

$$D(K_{p,\max}) = \left\{ f \in X_p;\ \sum_{n=2}^{\infty} n^p |a_n f_n + d_n f_n + s_n f_n + g_n f_n - g_{n-1} f_{n-1}| < \infty \right\}.$$

Theorem 3.2.1. *1. If, for some $p > 1$ and $g_n > 0$,*

$$\liminf_{n\to\infty} \left(a_n + d_n + s_n - g_n \frac{(n+1)^p - n^p}{n^p} \right) > -\infty, \tag{3.2.2}$$

then there is an extension K_p of $T_p + G_p^-$ that generates a quasi-contractive (of type $C_0(1,\omega)$ for some $\omega \in \mathbb{R}$) positive semigroup on X_p and, moreover, $K_p = K_{p,\max}$. The resolvent $R(\lambda, K_p)$ for $\lambda > \omega$ is given by

$$[R(\lambda, K_p) f]_n = \sum_{i=1}^{n} \frac{f_i}{\lambda + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j}, \quad n \geq 1, \tag{3.2.3}$$

where $\theta_1 = g_1$ and $\theta_n = g_n + d_n + s_n + a_n, n \geq 1$.

2. If there is $p' > p > 1$ such that[1]

$$\liminf_{n\to\infty} \frac{n(a_n + d_n + s_n)}{g_n} \geq p', \tag{3.2.4}$$

then (3.2.2) is satisfied. Moreover, $D(K_p) = D(A_p) \cap D(D_p^0) \cap D(S_p) \cap D(G_p)$, where $G_p = (\mathcal{G}^- + \mathcal{G}^0)|_{D(G_p)} = \mathcal{G}|_{D(G_p)}$ with

$$D(G_p) = \left\{ f \in X_p;\ \sum_{n=2}^{\infty} |g_n f_n - g_{n-1} f_{n-1}| < \infty \right\}, \tag{3.2.5}$$

and $(K_p, D(K_p)) = \overline{(T_p + G_p^-, D(T_p))}$.

3. If (3.2.4) is satisfied, then $R(\lambda, K_p)$, $\lambda > \omega$, is compact provided

$$\liminf_{n\to\infty} (a_n + d_n + s_n) = \infty \tag{3.2.6}$$

and either

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_0 + \theta_n} < \infty, \quad \text{or} \quad \lim_{n\to\infty} \frac{n}{\lambda_0 + \theta_n} = 0. \tag{3.2.7}$$

for some fixed $\lambda_0 > \omega$

[1] In (3.2.4) and everywhere below, it is assumed that the numerical coefficients like a_n, d_n, g_n, s_n, e.t.c. take their values in the extended half line $\bar{\mathbb{R}}_+ = [0, \infty) \cup \{+\infty\}$, where for any $a \in (0, \infty)$, we let $\frac{a}{0} = \infty$. In particular, according to this convention for a nonnegative sequence $(g_n)_{n\geq 0}$, with $g_n = 0$, $n \geq n_0$, (3.2.4) holds for any finite $m' \geq 0$.

4. If

$$\liminf_{n\to\infty} \frac{a_n + d_n + s_n}{g_n} > 0, \tag{3.2.8}$$

then $(K_p, D(K_p)) = (G_p^- + T_p, D(T_p)) = (G_p^- + G_p^0 + D_p^0 + A_p + S_p, D(T_p))$ *and* $(S_{K_p}(t))_{t\geq 0}$ *is an analytic semigroup. If additionally* (3.2.6) *is satisfied, then* $(S_{K_p}(t))_{t\geq 0}$ *is compact.*

Proof. ad 1.) We denote $\sigma_n = a_n + d_n + s_n, n \geq 1$. Let $\sigma_n - n^{-p} g_n((n+1)^p - n^p) \geq \alpha > -\infty$ for $n \geq 1$. Then, if $\alpha < 0$, for $f \in D(T_p)_+$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^p [(T_p + G_p^-) f]_n &= -\sum_{n=1}^{\infty} n^p \left((\sigma_n + g_n) f_n - g_{n-1} f_{n-1}\right) \\
&= -\sum_{n=1}^{\infty} n^p \sigma_n f_n - \sum_{n=1}^{\infty} n^p g_n f_n + \sum_{n=1}^{\infty} (n+1)^p g_n f_n \\
&= -\sum_{n=1}^{\infty} n^p \sigma_n f_n - \sum_{n=1}^{\infty} ((n+1)^p - n^p) g_n f_n \\
&= -\sum_{n=1}^{\infty} n^p f_n \left(\sigma_n - g_n \frac{(n+1)^p - n^p}{n^p} - \alpha\right) - \alpha \sum_{n=1}^{\infty} n^p f_n \\
&= -c_1(f) + c_0(f),
\end{aligned} \tag{3.2.9}$$

where c_0 is a bounded functional on X_p and c_1 is nonnegative. If $\alpha \geq 0$, we set $\alpha = 0$ in the formulae above. Thus, as in Corollary 2.4.3, there is an extension $K_p \supset G_p^- + T_p$ generating a smallest quasicontractive (with the growth rate ω not exceeding $\|c_0\|$) positive semigroup. By Theorem 2.5.3, $K_p \subset K_{p,\max}$. We note that $\mathrm{Ker}(\lambda I - K_{p,\max}) = \{0\}$ for $\lambda > s(K_p)$, hence [10, Lemma 3.50 & Proposition 3.52] gives $K_p = K_{p,\max}$.

Let $\lambda > \omega$. We use the formula from (2.4.3)

$$R(\lambda, K_p) f = \sum_{k=0}^{\infty} R(\lambda, T_p)[G_p^- R(\lambda, T_p)]^k f, \quad f \in X_p, \quad \lambda > \omega. \tag{3.2.10}$$

Since $R(\lambda, T_p)$ is represented by the matrix $\mathcal{R}(\lambda) = \mathrm{diag}\left(\frac{1}{\lambda+\theta_n}\right)_{n\geq 1}$, and G_p^- is represented by $\mathcal{G}^-$, we have $\mathcal{R}(\lambda)[\mathcal{G}^- \mathcal{R}(\lambda)]^k = (\gamma_{ij}^{(k)})_{i,j\in\mathbb{N}}$, where

$$\gamma_{ij}^{(k)} = \begin{cases} \frac{1}{\lambda+\theta_i} \prod_{l=j}^{i-1} \frac{g_l}{\lambda+\theta_l}, & i \geq k+1, \ \ j = i-k, \\ 0, & \text{otherwise.} \end{cases}$$

Since the convergence in X_p implies the coordinate-wise convergence, we see that for each n the component $[R(\lambda, K_p) f]_n$ of the series (3.2.10) terminates after n terms and hence the resolvent is given by (3.2.3).

ad 2.) Since c_1 extends by monotonic limit to $D(K_p)_+$, for $f \in D(K_p)_+$, [10, Theorem 6.8] we can write

$$\begin{aligned}
\sum_{n=1}^{\infty} n^p (K_p f)_n &= \lim_{l\to\infty} \left(\sum_{n=1}^{l} n^p f_n \left(\sigma_n - g_n \frac{(n+1)^p - n^p}{n^p}\right) - l^p g_l f_l\right) \\
&= c_0(f) - c_1(f) - \lim_{l\to\infty} l^p g_l f_l
\end{aligned} \tag{3.2.11}$$

and hence the last limit exists. Further, we have

$$c_1(f) = \sum_{n=1}^{\infty} n^p f_n \left(-\alpha + \sigma_n - g_n \frac{(n+1)^p - n^p}{n^p}\right)$$
$$= \sum_{n=1}^{\infty} n^p f_n \sigma_n \left(1 - \frac{g_n}{n\sigma_n}\left(p + O\left(\frac{1}{n}\right)\right)\right),$$

as $-\alpha > 0$ (otherwise we set $\alpha = 0$) If (3.2.4) is satisfied, then (possibly adjusting n_0 from the previous part of the proof) for $n \geq n_0$

$$1 - \frac{g_n}{n\sigma_n}\left(p + O\left(\frac{1}{n}\right)\right) \geq 1 - \frac{p}{p'} + \frac{g_n}{n\sigma_n} O\left(\frac{1}{n}\right) \geq c' > 0$$

on account of $p' > p$ and $g_n/n\sigma_n \leq 1/p'$. Since c_1 extends to $D(K_p)_+$ by monotonic limits, we argue as in [7, Theorem 2.1] that any $f \in D(K_p)$ is summable with the weights $(n^p\sigma_n)_{n\geq 1}$ and hence, by (3.2.4), it is also summable with the weight $(n^{p-1}g_n)_{n\geq 1}$. Therefore, in particular, $D(K_p) \subset D(A_p) \cap D(D_p^0) \cap D(S_p)$ and hence also $D(K_p) \subset D(G_p)$ holds by the definition of $D(K_{p,\max})$. The converse inclusion $D(A_p) \cap D(D_p^0) \cap D(S_p) \cap D(G_p) \subset D(K_p) = D(K_{p,\max})$ is obvious. Further, from (3.2.11) we know that $\lim_{l\to\infty} l^p g_l f_l$ exists, and thus it must be 0. Indeed, otherwise $l^p g_l f_l > c$ for some $c > 0$ and large l contradicting the summability of $(n^{p-1}g_n)_{n\geq 1}$. But then (3.2.11) implies that K_p is honest, hence $(K_p, D(K_p)) = \overline{(T_p + G_p^-, D(T_p))}$ by Corollary 2.5.2.

ad 3.) Though not strictly necessary, the estimates of the norm of the resolvent are instructive and used also further down. To simplify the calculations, instead of $\|\cdot\|_{[p]}$, we employ the norm $\|f\|_* := \sum_{n=1}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)}|f_n|$ that is equivalent to $\|\cdot\|_{[p]}$ by virtue of the Gautschi inequality

$$c_p n^p \leq \frac{\Gamma(n+p)}{\Gamma(n)} = C_p n^p, \quad n \geq 1, \tag{3.2.12}$$

which holds uniformly for all $n \geq 1$, and a fixed $p > 0$, with absolute constants $0 < c_p < C_p$, see e.g. [40].

Let $f \in X_p$ and $\lambda > \omega$. Then, changing the order of summation,

$$\|R(\lambda, K_p)f\|_* \leq \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)} \frac{1}{\lambda + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j}$$
$$= \frac{1}{\lambda} \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)} \left(\frac{\lambda + g_n}{\lambda + \theta_n} - \frac{g_n}{\lambda + \theta_n}\right) \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j}$$
$$\leq \frac{1}{\lambda} \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)} \left(\prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j} - \prod_{j=i}^{n} \frac{g_j}{\lambda + \theta_j}\right). \tag{3.2.13}$$

Now, we have

$$
\begin{aligned}
&\sum_{n=i}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)} \left(\prod_{j=i}^{n-1} \frac{g_j}{\lambda+\theta_j} - \prod_{j=i}^{n} \frac{g_j}{\lambda+\theta_j} \right) \\
&\quad = \lim_{N\to\infty} \sum_{n=i}^{N} \frac{\Gamma(n+p)}{\Gamma(n)} \left(\prod_{j=i}^{n-1} \frac{g_j}{\lambda+\theta_j} - \prod_{j=i}^{n} \frac{g_j}{\lambda+\theta_j} \right) \\
&\quad = \frac{\Gamma(i+p)}{\Gamma(i)} \\
&\qquad + \lim_{N\to\infty} \left(\sum_{n=i+1}^{N} \frac{\Gamma(n+p)}{\Gamma(n)} \prod_{j=i}^{n-1} \frac{g_j}{\lambda+\theta_j} - \sum_{n=i}^{N} \frac{\Gamma(n+p)}{\Gamma(n)} \prod_{j=i}^{n} \frac{g_j}{\lambda+\theta_j} \right) \\
&\quad = \frac{\Gamma(i+p)}{\Gamma(i)} + \lim_{N\to\infty} \left(\sum_{n=i+1}^{N} \frac{\Gamma(n+p-1)}{\Gamma(n-1)} \prod_{j=i}^{n-1} \frac{g_j}{\lambda+\theta_j} \right. \\
&\qquad \left. + p \sum_{n=i+1}^{N} \frac{\Gamma(n+p-1)}{\Gamma(n)} \prod_{j=i}^{n-1} \frac{g_j}{\lambda+\theta_j} - \sum_{n=i}^{N} \frac{\Gamma(n+p)}{\Gamma(n)} \prod_{j=i}^{n} \frac{g_j}{\lambda+\theta_j} \right) \\
&\quad = \frac{\Gamma(i+p)}{\Gamma(i)} \qquad\qquad (3.2.14) \\
&\qquad + \lim_{N\to\infty} \left(p \sum_{n=i+1}^{N} \frac{\Gamma(n+p-1)}{\Gamma(n)} \prod_{j=i}^{n-1} \frac{g_j}{\lambda+\theta_j} - \frac{\Gamma(N+p)}{\Gamma(N)} \prod_{j=i}^{N} \frac{g_j}{\lambda+\theta_j} \right).
\end{aligned}
$$

Using (3.2.4), for sufficiently large j we have

$$
\frac{g_j}{\lambda+\theta_j} = \frac{g_j}{\lambda+g_j+(a_j+d_j+s_j)} \le \frac{jg_j}{\lambda j+(j+p')g_j} \le \frac{j}{p'+j}, \qquad (3.2.15)
$$

hence

$$
\begin{aligned}
0 &\le \limsup_{N\to\infty} \frac{\Gamma(N+p)}{\Gamma(N)} \prod_{j=i}^{N} \frac{g_j}{\lambda+\theta_j} \le \limsup_{N\to\infty} \frac{\Gamma(N+p)}{\Gamma(N)} \frac{\Gamma(N+1)\Gamma(i+p')}{\Gamma(N+p'+1)\Gamma(i)} \\
&= \limsup_{N\to\infty} \frac{N}{N+p'} \frac{\Gamma(N+p)\Gamma(i+p')}{\Gamma(N+p')\Gamma(i)} = 0,
\end{aligned}
$$

on account of the Gautschi Inequality, see (3.2.12). Thus, (3.2.14) can be estimated from above by

$$
\begin{aligned}
&= \frac{\Gamma(i+p)}{\Gamma(i)} + p \sum_{n=i+1}^{\infty} \frac{\Gamma(n+p-1)}{\Gamma(n)} \prod_{j=i}^{n-1} \frac{j}{j+p'} \\
&\qquad = \frac{\Gamma(i+p)}{\Gamma(i)} + p \sum_{n=i+1}^{\infty} \frac{\Gamma(n+p-1)\Gamma(i+p')}{\Gamma(i)\Gamma(n+p')} \\
&\qquad = \frac{\Gamma(i+p)}{\Gamma(i)} + p \sum_{n=i+1}^{\infty} \frac{\Gamma(n+p-1)}{\Gamma(i)\Gamma(n-i)} B(n-i, p'+i), \qquad (3.2.16)
\end{aligned}
$$

where B is the Beta function. The sum above can be computed explicitly. Indeed, using the integral representation for the Beta function [63, Page 945], the Taylor series expansion

$$
(1-t)^{-p-i} = \sum_{n=i+1}^{\infty} \frac{\Gamma(n+p-1)}{\Gamma(n-i)\Gamma(p+i)} t^{n-i-1}, \quad |t| < 1,
$$

that converges pointwise in the interval $(-1,1)$, nonnegativity of the partial sums of the above series in $[0,1)$ and the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\sum_{n=i+1}^{\infty} \frac{\Gamma(n+p-1)}{\Gamma(i)\Gamma(n-i)} \int_0^1 (1-t)^{p'+i-1} t^{n-i-1} dt \\
&\quad = \frac{1}{\Gamma(i)} \int_0^1 \left((1-t)^{p'+i-1} \sum_{n=i+1}^{\infty} \frac{\Gamma(n+p-1)}{\Gamma(n)} \frac{d^i}{dt^i} t^{n-1} \right) dt \\
&\quad = \frac{1}{\Gamma(i)} \int_0^1 \left((1-t)^{p'+i-1} \frac{d^i}{dt^i} \sum_{n=0}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n+1)} t^n \right) dt \\
&\quad = \frac{\Gamma(p)}{\Gamma(i)} \int_0^1 (1-t)^{p'+i-1} \left(\frac{d^i}{dt^i} \frac{1}{(1-t)^p} \right) dt \\
&\quad = \frac{\Gamma(p+i)}{\Gamma(i)} \int_0^1 (1-t)^{p'-p-1} dt = \frac{\Gamma(p+i)}{\Gamma(i)} \frac{1}{p'-p}.
\end{aligned}$$

Substituting the above into (3.2.16) and returning to (3.2.13), we obtain

$$\|R(\lambda, K_p)f\|_* \leq \frac{p'}{p'-p} \frac{1}{\lambda - \omega} \|f\|_*. \tag{3.2.17}$$

To prove the compactness, we fix some $\lambda_0 > \omega$ and consider the projections

$$P_N f = (f_1, f_2, \ldots, f_N, 0, \ldots), \quad N \geq 1. \tag{3.2.18}$$

Since $P_N R(\lambda_0, K_p)$ is an operator with finite-dimensional range, it is compact. We consider

$$\begin{aligned}
\|P_{N-1}R(\lambda_0, K_p)f - R(\lambda_0, K_p)f\|_* &\leq \sum_{n=N}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)} \sum_{i=1}^{n-1} \frac{|f_i|}{\lambda_0+\theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0+\theta_j} \\
= \sum_{i=1}^{N-1} |f_i| S_{N,i} + \sum_{i=N}^{\infty} |f_i| S_{i+1,i},
\end{aligned} \tag{3.2.19}$$

where

$$S_{l,i} = \sum_{n=l}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)} \frac{1}{\lambda_0+\theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0+\theta_j}.$$

Now,

$$\begin{aligned}
S_{i+1,i} &= \sum_{n=i+1}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)} \frac{1}{\lambda_0+\theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0+\theta_j} \\
&\leq \sup_{n \geq i+1} \left\{ \frac{1}{\sigma_n} \right\} \sum_{n=i}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)} \frac{\sigma_n}{\lambda_0+\theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0+\theta_j} \\
&\leq \sup_{n \geq i+1} \left\{ \frac{1}{\sigma_n} \right\} \sum_{n=i}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)} \left(1 - \frac{g_n}{\lambda_0+\theta_n} \right) \prod_{j=i}^{n-1} \frac{g_j}{\lambda_0+\theta_j} \\
&= \sup_{n \geq i+1} \left\{ \frac{1}{\sigma_n} \right\} \sum_{n=i}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n)} \left(\prod_{j=i}^{n-1} \frac{g_j}{\lambda_0+\theta_j} - \prod_{j=i}^{n} \frac{g_j}{\lambda_0+\theta_j} \right) \\
&\leq \sup_{n \geq i+1} \left\{ \frac{1}{\sigma_n} \right\} \frac{p'}{p'-p} \frac{\Gamma(i+p)}{\Gamma(i)},
\end{aligned}$$

where we used the estimates for (3.2.13). Hence,

$$\sum_{i=N}^{\infty} |f_i| S_{i+1,i} \leq \sup_{n\geq N+1}\left\{\frac{1}{\sigma_n}\right\}\frac{p'}{p'-p}\sum_{i=N}^{\infty}|f_i|\frac{\Gamma(i+p)}{\Gamma(i)}$$
$$\leq \sup_{n\geq N+1}\left\{\frac{1}{\sigma_n}\right\}\frac{p'}{p'-p}\|f\|_* \tag{3.2.20}$$

and, by (3.2.6), this term tends to 0 as $N\to\infty$, uniformly on the unit ball of X_p.

Since, by (3.2.15),

$$\prod_{j=i}^{n}\frac{g_j}{\lambda_0+\theta_j} \leq \frac{i}{p'+i}\cdot\ldots\cdot\frac{n}{p'+n} \leq \frac{i}{p+i}\cdot\ldots\cdot\frac{n}{p+n} = \frac{\Gamma(n+1)\Gamma(i+p)}{\Gamma(i)\Gamma(n+p+1)}$$
$$= \frac{\Gamma(n)\Gamma(i+p)}{\Gamma(i)\Gamma(n+p)}\frac{n}{n+p},$$

we have

$$S_{N,i} = \sum_{n=N}^{\infty}\frac{\Gamma(n+p)}{\Gamma(n)}\frac{1}{\lambda_0+\theta_n}\prod_{j=i}^{n-1}\frac{g_j}{\lambda_0+\theta_j} = \sum_{n=N}^{\infty}\frac{1}{\lambda_0+\theta_n}\frac{\Gamma(n+p)}{\Gamma(n)}\prod_{j=i}^{n}\frac{g_j}{\lambda_0+\theta_j}$$
$$\leq \frac{\Gamma(i+p)}{\Gamma(i)}\sum_{n=N}^{\infty}\frac{1}{\lambda_0+\theta_n}\frac{n-1}{n-1+p} \leq \frac{\Gamma(i+p)}{\Gamma(i)}\sum_{n=N}^{\infty}\frac{1}{\lambda_0+\theta_n}.$$

Hence,

$$\sum_{i=1}^{N-1}|f_i|S_{N,i} \leq \left(\sum_{n=N}^{\infty}\frac{1}{\lambda_0+\theta_n}\right)\sum_{i=1}^{\infty}|f_i|\frac{\Gamma(i+p)}{\Gamma(i)}$$

and, using the first option of (3.2.7) and combining the above estimate with (3.2.20), we see that

$$\lim_{N\to\infty} P_{N-1}R(\lambda_0, K_p) = R(\lambda_0, K_p)$$

in the uniform operator norm. Therefore $R(\lambda_0, K_p)$ is compact.

To use the second option of (3.2.7), first we re-write the formula for $S_{N,i}$ as

$$S_{N,i} = \sum_{n=N}^{\infty}\frac{\Gamma(n+p)}{\Gamma(n)}\frac{1}{\lambda_0+\theta_n}\prod_{j=i}^{n-1}\frac{g_j}{\lambda_0+\theta_j} = \sum_{n=N}^{\infty}\frac{1}{\lambda_0+\theta_n}\frac{\Gamma(n+p)}{\Gamma(n)}\prod_{j=i}^{n}\frac{g_j}{\lambda+\theta_j}$$
$$\leq \frac{\Gamma(i+p')}{\Gamma(i)}\sum_{n=N}^{\infty}\frac{1}{\lambda_0+\theta_n}\frac{\Gamma(n+p)}{\Gamma(n+p')}.$$

Then, using again the Gautschi inequality, for large i and $N > i$ we can write

$$\frac{\Gamma(i+p')}{\Gamma(i)}\sum_{n=N}^{\infty}\frac{1}{\lambda_0+\theta_n}\frac{\Gamma(n+p)}{\Gamma(n+p')} \leq Ci^pN^{p'-p}\sum_{n=N}^{\infty}\frac{n^{p'-p}}{\lambda_0+\theta_n},$$

for some constant C, since $p'-p>0$. Now, by assumption, $\frac{n^{p-p'}}{\lambda_0+\theta_n}$ is summable (as $\lambda+\theta_n \geq cn$, for some $c>0$ and large values of n, while $p-p'<0$), hence $\sum\limits_{n=N}^{\infty}\frac{n^{p-p'}}{\lambda_0+\theta_n}$ converges to 0 as $N\to\infty$. Since $N^{p-p'}$ monotonically converges to 0, we can use the Stolz–Cesáro theorem [54, Theorem 1.22]. We have

$$\lim_{N\to\infty} N^{p'-p}\sum_{n=N}^{\infty}\frac{n^{p-p'}}{\lambda_0+\theta_n} = \frac{1}{p'-p}\lim_{N\to\infty}\frac{N+1}{\lambda_0+\theta_N} = 0,$$

by assumption, hence we see that

$$\sum_{i=1}^{N-1} |f_i| S_{N,i} \leq \left(N^{p'-p} \sum_{n=N}^{\infty} \frac{n^{p-p'}}{\lambda_0 + \theta_n} \right) \sum_{i=1}^{\infty} |f_i| i^p$$

and for the fixed λ_0 the result follows as above. The general case $\lambda > \omega$, follows from this and the standard resolvent identity $R(\lambda, K_p) = R(\lambda_0, K_p) + (\lambda_0 - \lambda) R(\lambda_0, K_p) R(\lambda, K_p)$.

ad 4.) By (3.2.8), $g_n \leq C(a_n + d_n + s_n)$ for large n and some $C > 0$, hence (3.2.4) holds and thus also the result of 2. holds. Moreover, (3.2.8) implies

$$D(A_p) \cap D(D_p^0) \cap D(S_p) \subset D(T_p)$$

and hence, by 2.), $D(K_p) \subset D(T_p)$. Since K_p is an extension of $(T_p + G_p^-, D(T_p))$, we see that $K_p = T_p + G_p^-$, but then we also have $K_p = A_p + D_p^0 + S_p + G_p^0 + G_p^-$. Further, since $(T_p, D(T_p))$ is a diagonal operator, it generates an analytic semigroup and hence $(K_p, D(T_p))$ also generates an analytic semigroup by Theorem 2.7.2.

Now, the stronger assumption on g_n allows for a simpler proof of the compactness without the need for (3.2.7). By virtue of the above and [10, Theorem 4.3], $I - G_p^- R(\lambda, T_p)$ is invertible and

$$R(\lambda, T_p + G_p^-) = R(\lambda, T_p)[I - G_p^- R(\lambda, T_p)]^{-1}.$$

In view of the last identity, it suffices to show that $R(\lambda, T_p)$ is compact for some $\lambda > 0$. For each $f \in X_p$ with $\|f\|_{[p]} \leq 1$, we have $\|R(\lambda, T_p) f\|_{[p]} \leq 1/\lambda$ and

$$\sum_{n=n_0}^{\infty} n^p \big| [R(\lambda, T_p) f]_n \big| \leq \sup_{n \geq n_0} \frac{1}{\lambda + \theta_n} \sum_{n=n_0}^{\infty} n^p |f_n| \leq \sup_{n \geq n_0} \frac{1}{\lambda + \sigma_n} \sum_{n=n_0}^{\infty} n^p |f_n|.$$

If (3.2.6) holds, we have

$$\lim_{n_0 \to \infty} \sup_{n \geq n_0} \frac{1}{\lambda + \sigma_n} = \frac{1}{\lambda + \liminf_{n \to \infty} \sigma_n} = 0.$$

Hence the image of the unit ball $B = \{f \in X : \|f\|_{[p]} \leq 1\}$ under $R(\lambda, T_p)$ is bounded and uniformly summable and therefore it is precompact, see [35, IV.13.3]. Hence $R(\lambda, T_p)$ is compact and the compactness of $(S_{K_p}(t))_{t \geq 0}$ follows from Theorem 2.7.3. □

Remark 3.2.2. We note that (3.2.8) also allows us to apply the Miyadera perturbation theorem, see Theorem 2.4.4. Indeed, if (3.2.8) is satisfied, then we can find n_0 such that for $n \geq n_0 + 1$

$$\frac{(n+1)^p g_n}{n^p (g_n + \sigma_n)} \leq q < 1$$

and then $\omega > 0$ such that

$$\max_{1 \leq n \leq n_0} \frac{(n+1)^p g_n}{n^p (g_n + \sigma_n + \omega)} \leq q < 1.$$

Since the generation for $T_p + G_p^-$ is equivalent to that for $T_p - \omega I + G_p^-$, [10, Lemma 4.15], the Miyadera condition for $T_p - \omega I + G_p^-$ and $f \in D(T_p)_+$ reads

$$\begin{aligned}\int_0^\delta \|G_p^- S_{T_p-\omega I}(t) f\|_{[p]} dt &= \sum_{n=1}^{\infty} \frac{(n+1)^p g_n (1 - e^{-(g_n+\sigma_n+\omega)\delta})}{n^p (g_n + \sigma_n + \omega)} n^p f_n \\ &\le \sum_{n=1}^{n_0} \frac{(n+1)^p g_n}{n^p (g_n + \sigma_n + \omega)} n^p f_n + \sum_{n=n_0+1}^{\infty} \frac{(n+1)^p g_n}{n^p (g_n + \sigma_n)} n^p f_n \\ &\le q \|f\|_{[p]}.\end{aligned}$$

At the same time, if $\sigma_n / g_n \to 0$ as $n \to \infty$, then $g_n/(\sigma_n + g_n) \to 1$ and the above estimate is not available.

3.3 Growth-Sedimentation-Fragmentation Equation

We introduce the following notation, see [7],

$$\Delta_n^{(p)} := n^p - b_n^{(p)} := n^p - \sum_{k=1}^{n-1} k^p b_{k,n}, \quad n \ge 2, \quad p \ge 0. \tag{3.3.1}$$

Then, for $n \ge 2$,

$$\Delta_n^{(0)} = 1 - b_1^0 \le 0, \quad \Delta_n^{(1)} = 0, \quad \Delta_n^{(p)} \ge 0, \qquad p > 1. \tag{3.3.2}$$

Further, let us recall the notation $\theta_1 = g_1$ and $\theta_n = a_n + g_n + d_n + s_n$, $n \ge 2$.

Theorem 3.3.1. *1. Let* (3.2.4) *be satisfied. If for some $p' > p > 1$*

$$\liminf_{n\to\infty} \frac{a_n}{\sigma_n} \frac{\Delta_n^{(p)}}{n^p} > \frac{p}{p'} \tag{3.3.3}$$

holds, where as before $\sigma_n = a_n + d_n + s_n$, then

$$\begin{aligned}(Y_p, D(K_p)) &= (K_p + D_p^+ + B_p, D(K_p)) \\ &= \overline{(T_p + G_p^- + D_p^+ + B_p, D(T_p))}\end{aligned} \tag{3.3.4}$$

generates a positive semigroup in X_p. If additionally (3.2.6) *and* (3.2.7) *are satisfied, $R(\lambda, Y_p)$ is compact for sufficiently large λ.*

2. If for some $p_0 > 1$

$$\liminf_{n\to\infty} \frac{a_n}{\theta_n} \frac{\Delta_n^{(p_0)}}{n^{p_0}} > 0 \tag{3.3.5}$$

holds, then for any $p > 1$

$$(Y_p, D(T_p)) = (A_p + S_p + G_p + D_p + B_p, D(T_p)) = (Y_p, D(Y_p)) \tag{3.3.6}$$

generates a positive, analytic semigroup in X_p.

Proof. ad 1.) Repeating the calculations in (3.2.9) for the full operator using $f \in D(K_p)_+ \subset D(A_p)_+ \cap D(D_p^0)_+ \cap D(S_p)_+$ and (3.2.5), we obtain

$$\sum_{n=1}^{\infty} n^p[(K_p + D_p^+ + B_p)f]_n = \sum_{n=1}^{\infty} n^p[(A_p + D_p + S_p + B_p)f]_n + \sum_{n=1}^{\infty} n^p[G_p f]_n$$

Now, using the convention that $g_0 f_0 = 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^p[G_p f]_n &= \lim_{l\to\infty} \sum_{n=1}^{l} n^p(g_{n-1}f_{n-1} - g_n f_n) \\ &= \lim_{l\to\infty} (\sum_{n=1}^{l}((n+1)^p - n^p)g_n f_n - (l+1)^p g_{l+1} f_{l+1}) \\ &= \sum_{n=1}^{\infty}((n+1)^p - n^p)g_n f_n\end{aligned}$$

by the proof of Theorem 3.2.1, part 2. Hence

$$\begin{aligned}\sum_{n=1}^{\infty} n^p[(K_p + D_p^+ + B_p)f]_n &= \sum_{n=1}^{\infty} \sigma_n n^p f_n \left(\left(\left(1+\frac{1}{n}\right)^p - 1\right)\frac{g_n}{\sigma_n}\right. \\ &\quad \left. -\frac{a_n}{\sigma_n}\left(1 - \frac{1}{n^p}\sum_{k=1}^{n-1} k^p b_{k,n}\right) - \frac{d_n}{\sigma_n}\left(1 - \left(1-\frac{1}{n}\right)^p\right) - \frac{s_n}{\sigma_n}\right) \\ &=: -\sum_{n=1}^{\infty} \Lambda_n \sigma_n n^p f_n. \end{aligned} \tag{3.3.7}$$

Thus, if $\Lambda_n \geq 0$ for large n, then there is an extension $(Y_p, D(Y_p))$ of $(K_p + D_p^+ + B_p, D(A_p) \cap D(D_p^0) \cap D(S_p) \cap D(G_p))$ generating a positive semigroup. Since

$$\Lambda_n = \frac{a_n}{\sigma_n}\frac{\Delta_n^{(p)}}{n^p} + \frac{d_n}{\sigma_n}O\left(\frac{1}{n}\right) - \frac{g_n}{n\sigma_n}\left(p + O\left(\frac{1}{n}\right)\right) + \frac{s_n}{\sigma_n},$$

where both $\frac{d_n}{\sigma_n}O\left(\frac{1}{n}\right)$ and $\frac{g_n}{n\sigma_n}O\left(\frac{1}{n}\right)$ converge to zero due to the boundedness of d_n/σ_n and $\frac{g_n}{n\sigma_n} \leq \frac{1}{p'} + \varepsilon_n$, with $\varepsilon_n \to 0^+$ as $n \to \infty$ (see (3.2.4)), we have $\Lambda_n \geq \Lambda > 0$ for some Λ and large n, provided (3.3.3) is satisfied. We observe that, in view of (3.3.1) and the trivial inequality $\frac{a_n}{\sigma_n} \leq 1$, $\Lambda \leq 1$ so that $1 < p < p'$ is a necessary condition for (3.3.3) to hold. Hence, if (3.3.3) holds, $D(Y_p) \subset D(A_p) \cap D(D_p^0) \cap D(S_p)$. Then, since $D(B_p), D(D_p^+) \subset D(A_p) \cap D(D_p^0) \cap D(S_p)$ and Y_p is a restriction of the maximal operator, $D(Y_p) \subset D(G_p)$ and hence the first part of (3.3.4) is proved. To prove the second part, we note that $(K_p, D(K_p)) = \overline{(T_p + G_p^-, D(T_p))}$. Since $K_p + D_p^+ + B_p$ is the generator, it is closed and thus

$$\begin{aligned}\overline{(T_p + G_p^- + D_p^+ + B_p, D(T_p))} &\subset \overline{(K_p + D_p^+ + B_p, D(K_p))} \\ &= (K_p + D_p^+ + B_p, D(K_p)).\end{aligned} \tag{3.3.8}$$

On the other hand, $D(K_p) \subset D(D_p^0) \cap D(A_p) = D(D_p^+) \cap D(B_p)$, hence $D_p^+ + B_p$ is K_p-bounded by [10, Lemma 4.1 & Theorem 2.65]. Let $f \in D(K_p)$. Then $f = \lim_{n\to\infty} f_n$ with $f_n \in D(T_p)$ and $\lim_{n\to\infty} K_p f_n = \lim_{n\to\infty}(T_p + G_p^-)f_n = K_p f$. By K_p-boundedness, $((D_p^+ + B_p)f_n)_{n\in\mathbb{N}}$ converges. By (3.3.8), $T_p + G_p^- + D_p^+ + B_p$ is closable and hence

$$K_p f + D_p^+ f + B_p f = \lim_{n\to\infty}(T_p + G_p^- + D_p^+ + B_p)f_n = \overline{(T_p + G_p^- + D_p^+ + B_p)}f.$$

Thus

$$K_p + D_p^+ + B_p \subset \overline{T_p + G_p^- + D_p^+ + B_p}$$

and (3.3.4) follows.

The compactness of $R(\lambda, Y_p)$ follows from

$$R(\lambda, Y_p) = R(\lambda, K_p)[I - (B_p + D_p^+)R(\lambda, K_p)]^{-1},$$

where the second term on the right-hand side is a bounded operator by $D(B_p + D_p^+) = D(A_p) \cap D(D_p^0) \supset D(K_p)$. Thus the proof of the compactness of $R(\lambda, Y_p)$ follows as in item 4.) of Theorem 3.2.1.

ad 2.) By [7, Theorem 2.1], if (3.3.5) holds for some p_0, then it holds for any $p \geq p_0$. Hence, we can fix a p for which (3.3.5) holds. Then, for $f \in D(T_p) = D(A_p + G_p^0 + D_p^0 + S_p) = D(A_p) \cap D(G_p^0) \cap D(D_p^0) \cap D(S_p)$, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} n^p[(G_p + D_p + S_p + A_p + B_p)f]_n &= \sum_{n=1}^{\infty} \theta_n n^p f_n \left(\left(\left(1 + \frac{1}{n}\right)^p - 1\right) \frac{g_n}{\theta_n} \right. \\ &\left. - \frac{a_n}{\theta_n}\left(1 - \frac{1}{n^p}\sum_{k=1}^{n-1} k^p b_{k,n}\right) - \frac{d_n}{\theta_n}\left(1 - \left(1 - \frac{1}{n}\right)^p\right) - \frac{s_n}{\theta_n}\right) \\ &=: -\sum_{n=1}^{\infty} \theta_n n^p f_n \Theta_n. \end{aligned} \tag{3.3.9}$$

Then we proceed as above. Since

$$\Theta_n = \frac{a_n}{\theta_n}\frac{\Delta_n^{(p)}}{n^p} + \frac{d_n}{\theta_n} O\left(\frac{1}{n}\right) - \frac{g_n}{\theta_n} O\left(\frac{1}{n}\right) + \frac{s_n}{\theta_n},$$

where the terms $\frac{g_n}{\theta_n} O\left(\frac{1}{n}\right)$, $\frac{d_n}{\theta_n} O\left(\frac{1}{n}\right)$ and $\frac{s_n}{\theta_n}$ converge to zero due to the boundedness of $\frac{g_n}{\theta_n}$, $\frac{d_n}{\theta_n}$ and $\frac{s_n}{\theta_n}$, $\Theta_n \geq c > 0$ for large n if and only if (3.3.5) is satisfied. Hence $(Y_p, D(T_p)) := (G_p + D_p + S_p + A_p + B_p, D(A_p + G_p^0 + D_p^0 + S_p))$ generates an analytic semigroup as a positive perturbation of the diagonal operator $(T_p, D(T_p))$. However, by the closedness of $K_p + D_p^+ + B_p$,

$$\begin{aligned}(A_p + G_p + D_p + S_p + B_p, D(A_p + G_p^0 + D_p^0 + S_p)) &= (T_p + G_p^- + D_p^+ + B_p, D(T_p)) \\ &= \overline{(T_p + G_p^- + D_p^+ + B_p, D(T_p))} \\ &= (Y_p, D(Y_p)).\end{aligned}$$

To conclude the proof, we show that if (3.3.5) holds for some $p_0 > 1$, then it also holds for all $p \in (1, p_0]$ and thus the argument of the proof applies for all $p > 1$. We let $\phi_i(p) = \frac{\Delta_i^{(p)}}{i^p}$, $i \geq 2$, $p > 1$. It can be verified that for any $i \geq 2$ and $p > 1$, $0 < \phi_i(p) < 1$. This indicates, in particular, that condition (3.3.5) is equivalent to the existence of constants $\alpha > 0$ and $\beta > 0$ such that

$$\inf_{i \geq 2} \frac{a_i}{\theta_i} = \alpha > 0,$$

and

$$\inf_{i \geq 2} \phi_i(p_0) = \beta > 0. \tag{3.3.10}$$

Straightforward computations yield for $p > 1, i \geq 2$,

$$\phi_i'(p) = \frac{1}{i^p}\sum_{j=1}^{i-1} \ln\Big(\frac{i}{j}\Big) j^p b_{j,i} > 0, \quad \phi_i''(p) = -\frac{1}{i^p}\sum_{j=1}^{i-1} \ln^2\Big(\frac{i}{j}\Big) j^p b_{j,i} < 0,$$

so that each function $\phi_i(p)$ is monotone increasing and strictly concave with respect to p on $(1,\infty)$. The monotonicity ensures that if (3.3.10) is satisfied for some p_0, then it is satisfied for any $p > p_0$. On the other hand, since $\phi_i(1) = 0$, $i \geq 2$, the concavity implies that for $p \in (1, p_0]$

$$\phi_i(p) \geq \frac{\phi(p_0)}{p_0 - 1}(p-1) \geq \beta\frac{p-1}{p_0-1}$$

and hence

$$\phi_i(p) \geq \min\left\{1, \frac{p-1}{p_0-1}\right\}\beta > 0, \quad p > 1.$$

Hence (3.3.5) holds for all $p > 1$ and we conclude that for each $p > 1$, the sum $(Y_p, D(Y_p)) = (T_p + G_p^- + D_p^+ + B_p, D(T_p))$ generates a positive analytic C_0- semigroup $(S_{Y_p}(t))_{t\geq 0}$ in X_p. □

Remark 3.3.2. We note that (3.3.5) implies that both $\frac{a_n}{\theta_n}$ and $\frac{\Delta_n^{(p)}}{n^p}$ must be bounded away from 0 and thus, in particular, (3.2.8) is satisfied so that $(S_{K_p}(t))_{t\geq 0}$ is an analytic and compact semigroup in its own right.

We mention that if the sedimentation is sufficiently strong, the generation result extends to $p = 1$. This is demonstrated in the following corollary.

Corollary 3.3.3. *If the following assumption*

$$\limsup_{i\to\infty}\left(\frac{i\theta_i}{is_i + d_i - g_i}\right) \leq c' < \infty, \tag{3.3.11}$$

holds, then the sum $(Y_1, D(Y_1)) = (T_1 + D_1^+ + B_1 + G_1^-, D(T_1))$ *generates a positive quasi-contractive analytic C_0-semigroup $(S_{Y_1}(t))_{t\geq 0}$ in X_1. If in addition, equation* (3.2.6) *is satisfied, then $(S_{Y_1}(t))_{t\geq 0}$ is compact.*

Proof. The operator T_1 is diagonal and it can be verified that the following is true:

(i) The resolvent is given by $[R(\lambda, T_1)f]_n = \frac{f_n}{\lambda+\theta_n}$, $n \geq 1$, where $\theta_n = g_n + s_n + \mathsf{a}_n$.

(ii) $\|R(\lambda, T_1)\| \leq \frac{1}{\lambda}$, for $\lambda > 0$.

(iii) The operator $(T_1, D(T_1))$ generates a substochastic semigroup $(S_{T_1}(t))_{t\geq 0}$ in ℓ_1^1.

(iv) The semigroup $(S_{T_1}(t))_{t\geq 0}$ is analytic.

Since $(G_1^-, D(G_1^-))$, $(D_1^+, D(D_1^+))$ and $(B_1, D(B_1))$ are positive operators and $D(T_1) \subseteq D(G_1^- +$

$D_1^+ + B_1$), we can apply Corollary 2.4.3. For $f \in D(T_1)_+$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n[T_1 f + G_1^- f + D_1^+ + B_1 f]_n &= \sum_{n=1}^{\infty} n\Bigg[g_{n-1}f_{n-1} - (g_n + s_n + a_n + d_n)f_n + d_{n+1}f_{n+1} \\
&\quad + \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j\Bigg] \\
&= \sum_{n=1}^{\infty} n(g_{n-1}f_{n-1} - g_n f_n) - \sum_{n=1}^{\infty} n s_n f_n + \sum_{n=1}^{\infty} n(d_{n+1}f_{n+1} - d_n f_n) \\
&\quad - \sum_{n=1}^{\infty} n a_n f_n + \sum_{n=1}^{\infty} n \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j \\
&= \sum_{n=1}^{\infty} g_n f_n - \sum_{n=1}^{\infty} n s_n f_n - \sum_{n=1}^{\infty} d_n f_n = \sum_{n=1}^{\infty} (g_n - n s_n - d_n) f_n \\
&= -\Bigg[\sum_{n=1}^{n_0-1} + \sum_{n=n_0}^{\infty}\Bigg] n(s_n - \frac{d_n - g_n}{n}) f_n \\
&= c_0(f) - c_1(f). \qquad (3.3.12)
\end{aligned}$$

Hence, there exists a smallest positive semigroup $(S_{Y_1}(t))_{t\geq 0}$ generated by an extension $(Y_1, D(Y_1))$ of $T_1 + G_1^- + B_1$. Note that $c_1(f)$ extends to $D(Y_1)$.
On the other hand, for $f \in D(Y_1)$, using condition (3.3.11) for $n \geq n_0$, we have

$$\begin{aligned}
\sum_{n=n_0}^{\infty} n\theta_n f_n &= \sum_{n=n_0}^{\infty} (n s_n + d_n - g_n) \frac{n\theta_n}{(n s_n + d_n - g_n)} f_n \\
&\leq c' \sum_{n=n_0}^{\infty} (n s_n + d_n - g_n) f_n \leq c' c_1(f), \qquad (3.3.13)
\end{aligned}$$

i.e. $D(Y_1) \subset D(T_1)$. Hence, $(Y_1, D(Y_1)) = (T_1 + G_1^- + D_1^+ + B_1, D(T_1))$.
By Theorem 2.7.2, $(T_1 + G_1^- + D_1^+ + B_1, D(T_1))$ generates an analytic semigroup, since $(T_1, D(T_1))$ is a diagonal operator that generates an analytic semigroup and $(G_1^-, D(G_1^-))$, $(D_1^+, D(D_1^+))$ and $(B_1, D(B_1))$ are well defined positive operators.
The proof of the last statement of the corollary follows as in the proof of Theorem 3.2.1(ad 4). □

If in equation (3.1.1) $g_i = 0$, then the associated semigroup $(Y_1, D(Y_1))$ is substochastic.

Corollary 3.3.4. *Assume the coefficients of the decay-sedimentation-fragmentation model, that is when $g_i = 0$, satisfy*

$$\limsup_{n\to\infty} \frac{a_n}{s_n} < \infty. \qquad (3.3.14)$$

Then the operator $(Y_1, D(Y_1)) = (A_1 + D_1^0 + S_1 + D_1^+ + B_1, D(A_1 + D_1^0 + S_1))$ generates a substochastic C_0- semigroup $(S_{Y_1}(t))_{t\geq 0}$ in X_1. The semigroup $(S_{Y_1}(t))_{t\geq 0}$ is analytic in X_1 provided for some $C > 0$

$$C d_n \leq s_n, \quad n \geq 1. \qquad (3.3.15)$$

If in addition, equation (3.2.6) is satisfied, then $(S_{Y_1}(t))_{t\geq 0}$ is compact.

3.4 Asynchronous Exponential Growth

Proposition 3.4.1. *If assumptions* (3.2.6) *and* (3.3.5) *hold, then the semigroup* $(S_{Y_p}(t))_{t\geq 0}$, $p > 1$, *is analytic and compact.*

Proof. By (3.3.6), $(S_{Y_p}(t))_{t\geq 0}$ can be viewed as the semigroup generated by the perturbation $Y_p = T_p + (G_p^- + D_p^+ + B_p)$ so, as in the proof of Theorem 3.2.1, $R(\lambda, T_p)X_p \subset D(Y_p)$ implies

$$R(\lambda, Y_p) = R(\lambda, T_p)(I - (G_p^- + D_p^+ + B_p)R(\lambda, T_p))^{-1},$$

where, by the identity

$$(\lambda I - T_p)R(\lambda, Y_p) = (I - (G_p^- + D_p^+ + B_p)R(\lambda, T_p))^{-1},$$

the operator $(I - (G_p^- + D_p^+ + B_p)R(\lambda, T_p))^{-1}$ is bounded. Hence, $R(\lambda, Y_p)$ is compact, provided $R(\lambda, T_p)$ is compact and that was proved in Theorem 3.2.1. Since $(S_{Y_p}(t))_{t\geq 0}$ is analytic, its compactness follows from Theorem 2.7.3. □

Proposition 3.4.2. *The semigroup* $(S_{Y_p}(t))_{t\geq 0}$ *is irreducible if and only if* $g_n > 0$ *for all* $n \geq 1$.

Proof. *N*ecessity. If $g_{n_0} = 0$ for some $n_0 \geq 1$, then the closed n_0-dimensional subspace of X_p, spanned by the canonical vectros $e_n = (\delta_{i,n})_{i\geq 1}$, $1 \leq n \leq n_0$ is invariant under the action of the generator $(Y_p, D(T_p))$ and hence the semigroup $(S_{Y_p}(t))_{t\geq 0}$ is not irreducible.

*S*ufficiency. By [26, Proposition 7.6] it suffices to show that $R(\lambda, Y_p)e_n > 0$ for all $n \geq 1$ and for some fixed $\lambda > s(Y_p)$. To simplify the calculations, we use the representation

$$(Y_p, D(T_p)) = (K_p + D_p^+ + B_p, D(T_p)) =: (K_p + \mathsf{B}_p, D(T_p)),$$

(see (3.3.6), corresponding to (3.1.2) and (3.1.3)), which combined with the resolvent formula from (2.4.3), gives

$$R(\lambda, Y_p)f = \sum_{k=0}^{\infty} R(\lambda, K_p)[\mathsf{B}_p R(\lambda, K_p)]^k f, \quad f \in X_p, \quad \lambda > s(Y_p). \tag{3.4.1}$$

Let $n_0 \geq 1$ be fixed. From (3.4.1) and (3.2.3), we infer

$$\begin{aligned}&\left[R(\lambda, Y_p)e_{n_0}\right]_n \geq \left[R(\lambda, K_p)(I + \mathsf{B}_p R(\lambda, K_p))e_{n_0}\right]_n \\ &= \sum_{i=1}^{n} \frac{1}{\lambda + \theta_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + \theta_j} \left(\delta_{i,n_0} + \sum_{j=i+1}^{\infty} \mathsf{a}_j \mathsf{b}_{i,j} \left(\sum_{s=1}^{j} \frac{\delta_{s,n_0}}{\lambda + \theta_j} \prod_{l=s}^{j-1} \frac{g_l}{\lambda + \theta_l} \right) \right).\end{aligned}$$

In view of our assumptions, the last formula indicates that $\left[R(\lambda, Y_p)e_{n_0}\right]_n > 0$, provided $n \geq n_0$.

Moreover, for $n_0 > 1$,

$$\begin{aligned}&\left[R(\lambda, K_p)\mathsf{B}_p R(\lambda, K_p))e_{n_0}\right]_{n_0-1} \\ &= \sum_{i=1}^{n_0-1} \frac{1}{\lambda + \theta_{n_0-1}} \prod_{j=i}^{n_0-2} \frac{g_j}{\lambda + \theta_j} \left(\sum_{j=i+1}^{\infty} \mathsf{a}_j \mathsf{b}_{i,j} \frac{1}{\lambda + \theta_j} \prod_{l=n_0}^{j-1} \frac{g_l}{\lambda + \theta_l} \right) > 0\end{aligned} \tag{3.4.2}$$

for, if not, then

$$\mathsf{b}_{i,n_0} = 0, \quad 1 \le i \le n_0 - 1,$$

as all other terms and multipliers are positive. This contradicts (3.1.4) that requires

$$\sum_{i=1}^{n_0-1} i\mathsf{b}_{i,n_0} = n_0 - \frac{d_{n_0}}{a_{n_0} + d_{n_0}} > 0, \qquad n_0 > 1.$$

Thus, if $n_0 = 2$, then $\left[R(\lambda, Y_p)e_{n_0}\right]_n > 0$ for $n \ge 1$. If $n_0 > 2$, then we consider the third term in (3.4.1), evaluated at e_{n_0},

$$\begin{aligned} R(\lambda, K_p)[\mathsf{B}_p R(\lambda, K_p)]^2 e_{n_0} &= R(\lambda, K_p)\mathsf{B}_p[R(\lambda, K_p)\mathsf{B}_p R(\lambda, K_p)e_{n_0}] \\ &= R(\lambda, K_p)\mathsf{B}_p\Phi, \end{aligned}$$

where $\Phi = (\phi_n)_{n\ge1}$ is a sequence with $\phi_n > 0$ for $n \ge n_0 - 1$. Since in the proof of (3.4.2) we only used the fact that $R(\lambda, K_p)\mathsf{B}_p$ acted on a sequence with a positive n_0 entry, we see in the same way that

$$[R(\lambda, K_p)[\mathsf{B}_p R(\lambda, K_p)]^2 e_{n_0}]_{n_0-2} > 0.$$

Repeating the argument, for $n_0 > k$ we have

$$\left[R(\lambda, K_p)[B_p R(\lambda, K_p]^k e_{n_0}\right]_{n_0-k} > 0, \quad 1 \le k \le n_0 - 1$$

and hence $R(\lambda, Y_p)e_{n_0} > 0$. Since any $f \in X_{p,+}$, $f \ne 0$, is bounded from below by a finite linear combination of elements from $\{e_n\}_{n\ge1}$, we conclude that $R(\lambda, Y_p)$, and hence $(S_{Y_p}(t))_{t\ge0}$, are irreducible. □

Thus [38, Theorem VI.3.5] yields the following result.

Theorem 3.4.1. *Assume that $g_n > 0$, $n \ge 1$ and* (3.2.6) *and* (3.3.5) *are satisfied. Then there exist a strictly positive $e \in X_p$, a strictly positive $h \in X_p^*$, $M \ge 1$ and $\epsilon > 0$ such that for any $f^{in} \in X_n$ and $t \ge 0$*

$$\|e^{-s(Y_p)t} S_{Y_p}(t) f^{in} - \langle h, f^{in}\rangle e\|_{[p]} \le M e^{-\epsilon t}. \tag{3.4.3}$$

Note that in the absence of growth, $g_i = 0$, the resulting decay-fragmentation equation mentioned in Corollary 3.3.4 also have the property of Asynchronous Exponential Growth. However, the proof is quite different from the one presented above because the substochastic semigroup $(S_{Y_1}(t))_{t\ge0}$ associated with this model is reducible.

Theorem 3.4.2. *Assume $g_n = 0$, $n \ge 1$ and assume that the decay-sedimentation-fragmentation semigroup $(S_{Y_p}(t))_{t\ge0}$ satisfies either $p = 1$ and conditions* (3.3.14), (3.3.15) *and* (3.2.6) *or $p > 1$ and conditions* (3.3.5) *and* (3.2.6). *Let*

$$\lambda_1 := -\min_{n\in\mathbb{N}} \theta_n \tag{3.4.4}$$

be the negative strict minimum of the sequence $(\theta_n)_{n=1}^{\infty}$. *Then there exist constants* $\epsilon > 0$, $M \geq 1$ *and* $e \in X_p, e^* \in X_p^*$ *such that for any* $f \in X_p$

$$\|e^{-\lambda_1 t} S_{Y_p}(t)f - \langle e^*, f\rangle e\| \leq Me^{-\epsilon t}. \tag{3.4.5}$$

Proof. Let us introduce the dual space to $X_p = \ell_1^p$,

$$X_p^* = \left\{ f^* : \|f^*\|_{X_p^*} := \sup_{k\geq 1} \frac{1}{k^p}|f_k| < \infty \right\}$$

with the duality pairing

$$\langle f^*, f\rangle := \sum_{i=1}^{\infty} f_i^* f_i, \quad f \in X_p, f^* \in X_p^*.$$

The following result is analogous to classical theorems on asynchronous exponential growth, see e.g. [38, Theorem VI.3.5] since, however, $(S_{Y_p}(t))_{t\geq 0}$ is not irreducible, it requires a separate proof. We follow the proof of [15, Theorem 4.3] with some modifications. Under the adopted assumptions, $R(\lambda, Y_p)$ is compact, hence $\sigma(Y_p)$ is countable and consists of poles of $R(\lambda, Y_p)$ of finite algebraic multiplicity, [37, Corollary V.3.2]. Thus, in particular, the essential radius of the semigroup and its essential growth rate satisfy, respectively, $r_{ess}(S_{Y_p}(t)) = 0$ and $\omega_{ess}(Y_p) = -\infty$. Hence, any half-plane $\{\Re\lambda > a \ : a > -\infty\}$ contains only a finite number of eigenvalues of Y_p. Since, in addition, $(S_{Y_p}(t))_{t\geq 0}$ is positive, it follows that the peripheral spectrum of Y_p is additively cyclic and, being finite, consists of the single point $s(Y_p)$ that is the dominant eigenvalue of Y_p, see e.g. [6, Theorems 48 & 49].

To prove that $\lambda_1 = s(Y_p)$, first we observe that, by (3.2.6), the infimum of θ_n is attained. Let P_N be the projection operator on X defined by

$$P_N u = (u_1, u_2, ..., u_N, 0, ...) \quad \text{for fixed} \quad N \in \mathbb{N}. \tag{3.4.6}$$

The key observation is that the space $P_N X_p$ is invariant under $Y_p = A_p + D_p^0 + S_p + D_p^+ + B_p$ (and thus under S_{Y_p}) for each N – this follows from the upper triangular structure of $A_p + D_p^0 + S_p + D_p^+ + B_p$. Denoting $Y_{p_N} = Y_p|_{P_N X_p}$, we have $\sigma(Y_{p_N}) = \{-\theta_1, \ldots, -\theta_N\}$ and $s(Y_p) \geq \lambda_1$ as, due to the invariance, any eigenvalue of Y_{p_N} is an eigenvalue of Y_p. On the other hand, assume $s := s(Y_p) > \lambda_1$. Then s is an isolated eigenvalue of Y_p and hence there is a decomposition $X_p = X_{p_1} \oplus X_{p_2}$, where X_{p_1} is the spectral subspace corresponding to s, whose dimension is at least 1, while X_{p_2} is a closed complementary subspace invariant under Y_p on which $sI - Y_p$ is invertible. In particular, $(sI - Y_p)X_p \subset X_{p_2}$. On the other hand, since $s \notin \sigma(Y_{p_N})$ for any N

$$(sI - Y_p)X_p \supset (sI - Y_p) \bigcup_{N=2}^{\infty} P_N X_p = \bigcup_{N=2}^{\infty} P_N X_p$$

and the latter is dense in X_p. This contradiction shows $s(Y_p) = \lambda_1$.

By [37, Corollary V.3.2] we can write

$$S_{Y_p}(t) = S_{Y_p}^1(t) + R(t), \tag{3.4.7}$$

where there is $\epsilon > 0$ and $M \geq 1$ such that

$$\|R(t)\| \leq M_\epsilon e^{(\lambda_1-\epsilon)t}, \quad t \geq 0. \tag{3.4.8}$$

The operator $S^1_{Y_p}$ has finite rank and hence is given by

$$S^1_{Y_p}(t) = \left(e^{\lambda_1 t}\sum_{j=0}^{k-1}\frac{t^j}{j!}\right)\Pi, \tag{3.4.9}$$

where k is the order of the pole $\lambda_1 = s(Y_p)$ and Π is the corresponding spectral projection onto the subspace of X_p whose dimension equals the algebraic multiplicity m of λ_1. We shall prove that $k = m = 1$.

The semigroup $(S^1_{Y_p}(t))_{t\geq 0}$ is finite-dimensional and it is generated by an operator which has $\lambda_1 = s(Y_p)$ as the only eigenvalue. Since, by the previous part of the proof, $\|e^{-\lambda_1 t}S_{Y_p}(t)\| \leq 1$, also $\|e^{-\lambda_1 t}S^1_{Y_p}(t)\| \leq 1$ and thus $k = 1$, for otherwise $\|e^{-\lambda_1 t}S^1_{Y_p}(t)\|$ would grow polynomially in t. Using the inequality $m_g + k - 1 \leq m \leq m_g k$, where m_g is the geometric multiplicity, see e.g. [6, p. 19], for $k = 1$, we find $m_g = m$ which shows that to prove $m = 1$, it is sufficient to show $m_g = 1$.

We prove that $m_g = 1$ by examining the adjoint operator Y_p^*. Arguing as in [15, Theorem 4.3], we find that the operator Y_p^* defined by

$$\begin{aligned}(Y_p^*f^*)_1 &:= -\theta_1 f_1^*,\\ (Y_p^*f^*)_j &:= -\theta_j f_j^* + d_j f_{j-1}^* + a_j\sum_{i=1}^{j-1} b_{i,j}f_i^*, \quad j = 2,3,\dots,\end{aligned} \tag{3.4.10}$$

with the domain

$$D(Y_p^*) := \left\{f^* \in X^* : \sup_{j\geq 2}\frac{1}{j^p}\left|-\theta_j f_j^* + d_j f_{j-1}^* + a_j\sum_{i=1}^{j-1} b_{i,j}f_i^*\right| < \infty\right\}, \tag{3.4.11}$$

is the adjoint of Y_p.

Suppose that $e^* = (e_1^*, e_2^*, \dots)$ is an eigenvector of Y_p^* corresponding to the eigenvalue $\lambda_1 = s(Y_p)$ and assume $\lambda_1 = -\theta_{N_0}$. Then we see that $e_i^* = 0$ for $1 \leq i \leq N_0 - 1$, the N_0-th equation is satisfied irrespectively of $e_{N_0}^*$, so $e_{N_0}^*$ can be chosen in an arbitrary way and then e_N^* for $N > N_0$ can be recursively evaluated in a unique way (for a given $e_{N_0}^*$) as

$$e_N^* = \frac{1}{\theta_N - \theta_{N_0}}\left(d_N e_{N-1}^* + a_N\sum_{j=N_0}^{N-1} b_{j,N}e_j^*\right). \tag{3.4.12}$$

Therefore, the geometric multiplicity of $\lambda_1 = s(Y_p)$ is at most 1. Hence, $\lambda_1 = s(Y_p)$ is a simple dominating eigenvalue of Y_p. To complete the proof, let us find the explicit form of the spectral projection Π. For this we find the eigenvector $e = (e_1, e_2, \dots)$ of Y_p belonging to $\lambda_1 = -\theta_{N_0}$. We observe that e satisfies

$$
\begin{aligned}
-\theta_{N_0} e_1 &= -\theta_1 e_1 + d_2 e_2 + \sum_{j=2}^{\infty} a_j b_{i,j} e_j,\\
\vdots &= \vdots,\\
-\theta_{N_0} e_{N_0-1} &= -\theta_{N_0-1} e_{N_0-1} + d_{N_0} e_{N_0} + \sum_{j=N_0}^{\infty} a_j b_{i,j} e_j,\\
0 &= d_{N_0+1} e_{N_0+1} + \sum_{j=N_0+1}^{\infty} a_j b_{i,j} e_j,\\
-\theta_{N_0} e_{N_0+1} &= -\theta_{N_0+1} e_{N_0+1} + d_{N_0+2} e_{N_0+2} + \sum_{j=N_0+2}^{\infty} a_j b_{i,j} e_j,\\
\vdots &= \vdots.
\end{aligned}
$$

We see that the equations for $N \geq N_0 + 1$ decouple from the system and are solved by $e_N = 0, N \geq N_0 + 1$. Then the N_0-th equation is trivially satisfied and the remaining $N_0 - 1$ equations form an upper triangular system with N_0 unknowns that can be solved recursively setting $e_{N_0} = 1$,

$$
e_N = \frac{1}{\theta_N - \theta_{N_0}} \left(d_{N+1} e_{N+1} + \sum_{j=N+1}^{N_0} a_j b_{i,j} e_j \right), \quad 1 \leq N \leq N_0 - 1. \tag{3.4.13}
$$

Choosing e^* with $e^*_{N_0} = 1$, we obtain $\langle e^*, e \rangle = 1$, hence, by using the standard linear algebra formula on $P_N X_p$ and passing to the limit, we obtain the spectral projection

$$
\Pi f = \left(\sum_{k=1}^{\infty} e^*_k f_k \right) e = \langle e^*, f \rangle e
$$

and, in view of (3.4.9),

$$
S^1_{Y_p}(t) f = e^{\lambda_1 t} \Pi f = e^{\lambda_1 t} \langle e^*, f \rangle e. \tag{3.4.14}
$$

The last formula combined with (3.4.8) yields (3.4.5). □

3.5 An Alternative View of the Model

Previously, in Theorem 3.2.1, we have seen a regularizing role played by the diagonal operator induced by $\mathcal{T}$ even in the case not involving the full fragmentation operator. In many applications, however, (3.1.1) models a combination of two independent processes – the birth-and-death process and the fragmentation process and it is important to investigate when they exist irrespective of each other. In other words, we consider (3.1.1) as

$$
\frac{d}{dt} f = Gf + Df + Sf + (A + B) f, \qquad f(0) = f^{in}. \tag{3.5.1}
$$

The pure birth-and-death problem

$$
\frac{d}{dt} f = \mathcal{V} f = \mathcal{G} f + \mathcal{D} f + \mathcal{S} f, \qquad f(0) = f^{in}, \tag{3.5.2}
$$

has been extensively analysed in the space X_0, see e.g. [10, Chapter 7]. Its behaviour in X_p, $p > 1$, creates, however, unexpected challenges. First, we observe

Example 3.5.1. If there is C such that

$$g_n \leq Cn, \quad n \geq 1, \tag{3.5.3}$$

then there is a realization of the growth expression $\mathcal{G}$ that generates a C_0-semigroup in X_p. Indeed, this again follows from the Kato–Voigt theorem. We consider G as the perturbation of G^0 by G^-; that is, we introduce $G_p^0 = G^0|_{D(G_p^0)}$, with

$$D(G_p^0) = \{f \in X_p;\ G^0 f \in X_p\}.$$

Then, as in (3.2.9), for $f \in D(G_p^0)$,

$$\sum_{n=1}^{\infty} n^p [(G_p^0 + G_p^-)f]_n = \sum_{n=0}^{\infty} n^p f_n \left(g_n \frac{(n+1)^p - n^p}{n^p} \right) \leq C' \|f\|_{[p]}, \tag{3.5.4}$$

for some constant C'. Hence, there is an extension of $G_p^0 + G_p^-$ generating a C_0-semigroup in X_p. On the other hand, if for some $c, C > 0$

$$cn^q \leq g_n \leq Cn^q, \quad n \geq 1, q > 1, \tag{3.5.5}$$

then there is no realization of G with resolvent bounded in X_p with $q \leq p+1$. Indeed, the resolvent of the generator, if it exists, must be given by (3.2.3),

$$[R_\lambda f]_n = \sum_{i=1}^{n} \frac{f_i}{\lambda + g_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + g_j}, \quad n \geq 1. \tag{3.5.6}$$

Let us fix λ. Then

$$\prod_{j=i}^{n-1} \frac{g_j}{\lambda + g_j} \geq g_\lambda := \prod_{j=1}^{\infty} \frac{g_j}{\lambda + g_j},$$

where $g_\lambda \neq 0$, and, for $f \in X_{p,+}$,

$$\begin{aligned} \|R_\lambda f\|_{[p]} &= \sum_{n=1}^{\infty} n^p \sum_{i=1}^{n} \frac{f_i}{\lambda + g_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + g_j} = \sum_{i=1}^{\infty} f_i \sum_{n=i}^{\infty} \frac{n^p}{\lambda + g_n} \prod_{j=i}^{n-1} \frac{g_j}{\lambda + g_j} \\ &\geq g_\lambda \sum_{i=1}^{\infty} f_i \sum_{n=i}^{\infty} \frac{n^p}{(\lambda + g_n)} \geq g_\lambda C^{-1} \sum_{i=1}^{\infty} f_i \sum_{n=i}^{\infty} \frac{1}{n^{q-p}}. \end{aligned}$$

Hence R_λ is not bounded if (3.5.5) is satisfied and hence, in particular, there is no realization of G generating a C_0-semigroup in X_p. We note that for $q = 2$ and $p = 1$ we have a discrete version of the nonexistence result obtained in [17, Remark 2].

Let us return to the full birth-and-death model (3.5.1). As before, we introduce $V_p^0 + V_p^1 := G_p^0 + D_p^0 + S_p + G_p^- + D_p^+$ on

$$D(V_p^0) = \{f \in X_p;\ (G^0 + D^0 + S)f \in X_p\}.$$

We have

Theorem 3.5.1. *1. If*

$$\limsup_{n\to\infty} \Gamma_n \leq C \tag{3.5.7}$$

for some constant $C \in \mathbb{R}$, where

$$\Gamma_n = g_n\left(\left(1+\frac{1}{n}\right)^p - 1\right) - d_n\left(1 - \left(1-\frac{1}{n}\right)^p\right) - s_n,$$

then there is an extension V_p of $V_p^0 + V_p^1$ that generates a quasicontractive semigroup $(S_{V_p}(t))_{t\geq 0}$ on X_p.

2. Condition (3.5.7) *is satisfied if either*

a) (3.5.3) *is satisfied, or*

b) $\limsup_{n\to\infty} \frac{g_n}{d_n} \leq 1$, $g_n + d_n = O(n^2)$, $s_n = O(n^2)$.

3. If any of the conditions of point 2. is satisfied, then $V_p = \overline{V_p^0 + V_p^1}$.

Proof. ad 1.) of the theorem follows in a standard way as an application of the Kato–Voigt theorem. For $f \in D(V_p^0)_+$ we have

$$\sum_{n=1}^{\infty} n^p[(G_p^0 + D_p^0 + S_p + G_p^- + D_p^+)f]_n$$
$$= \sum_{n=1}^{\infty} n^p f_n\left(\left(\left(1+\frac{1}{n}\right)^p - 1\right)g_n - d_n\left(1-\left(1-\frac{1}{n}\right)^p\right) - s_n\right) = \sum_{n=1}^{\infty} n^p f_n \Gamma_n.$$

ad 2.) we observe that

$$\left(1+\frac{1}{n}\right)^p - 1 = \frac{p}{n} + \frac{p(p-1)}{2n^2} + O\left(\frac{1}{n^3}\right),$$
$$1-\left(1-\frac{1}{n}\right)^p = \frac{p}{n} - \frac{p(p-1)}{2n^2} + O\left(\frac{1}{n^3}\right).$$

Thus, if 2a) is satisfied, then the positive part of Γ_n is bounded. If 2b) is satisfied, then

$$\Gamma_n = g_n\left(\frac{p}{n} + \frac{p(p-1)}{2n^2} + O\left(\frac{1}{n^3}\right)\right) - d_n\left(\frac{p}{n} - \frac{p(p-1)}{2n^2} + O\left(\frac{1}{n^3}\right)\right) - s_n$$
$$\leq d_n\left(\frac{m(m-1)}{n^2} + O\left(\frac{1}{n^3}\right)\right)$$

for sufficiently large n and hence Γ_n is bounded from above.

ad 3.) To prove the last statement, we use the approach of [10, Theorem 7.11], based on the technique which allows us to characterise the domain of the generator explicitly, see Theorem 2.5.4. Let $f \in D(V_p)_+$. Then

$$\sum_{n=1}^{\infty} n^p(-(g_n + d_n + s_n)f_n + g_{n-1}f_{n-1} + d_{n+1}f_{n+1})$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} k^p(-(g_k + d_k + s_k)f_k + g_{k-1}f_{k-1} + d_{k+1}f_{k+1})$$
$$= \sum_{k=1}^{\infty} k^p f_k \Gamma_k + \lim_{n\to\infty}(-g_n f_n + d_{n+1}f_{n+1})n^p, \tag{3.5.8}$$

where the limit exists. For honesty, it suffices to prove that for any $f \in D(V_p)_+$

$$\lim_{n\to\infty}(-g_n f_n + d_{n+1}f_{n+1})n^p \geq 0.$$

Assume, to the contrary, that for some $0 \leq f \in D(V_p)_+$, the limit is negative so that there exists $b > 0$ such that

$$(-g_n f_n + d_{n+1}f_{n+1})n^p \leq -b, \tag{3.5.9}$$

for all $n \geq n_0$ with large enough n_0. Thus, for $n \geq n_0$ we have

$$f_n \geq \frac{b}{n^p g_n} + \frac{d_{n+1}}{g_n} f_{n+1}$$

and, by induction,

$$\begin{aligned} f_{n+1} \geq& \frac{b}{(n+1)^p g_{n+1}} + \frac{d_{n+2}}{g_{n+1}} f_{n+2} \\ f_{n+2} \geq& \frac{b}{(n+2)^p g_{n+2}} + \frac{d_{n+3}}{g_{n+2}} f_{n+3} \\ f_{n+3} \geq& \frac{b}{(n+3)^p g_{n+3}} + \frac{d_{n+4}}{g_{n+3}} f_{n+4} \\ \vdots& \end{aligned}$$

So,

$$f_n \geq \frac{b}{n^p g_n} + \frac{b}{(n+1)^p g_n}\frac{d_{n+1}}{g_{n+1}} + \frac{b}{(n+2)^p g_n}\frac{d_{n+1}d_{n+2}}{g_{n+1}g_{n+2}} + \frac{b}{(n+3)^p g_n}\frac{d_{n+1}d_{n+2}d_{n+3}}{g_{n+1}g_{n+2}g_{n+3}} + \ldots$$

Hence, for arbitrary k

$$f_n \geq \frac{b}{g_n}\left(\sum_{i=0}^{k}\frac{1}{(n+i)^p}\prod_{j=1}^{i}\frac{d_{n+j}}{g_{n+j}}\right).$$

Because k is arbitrary, we obtain

$$f_n \geq \frac{b}{g_n}\left(\sum_{i=0}^{\infty}\frac{1}{(n+i)^p}\prod_{j=1}^{i}\frac{d_{n+j}}{g_{n+j}}\right), \quad n \geq n_0.$$

Thus, if

$$\sum_{n=1}^{\infty}\frac{n^p}{g_n}\left(\sum_{i=0}^{\infty}\frac{1}{(n+i)^p}\prod_{j=1}^{i}\frac{d_{n+j}}{g_{n+j}}\right) = +\infty \tag{3.5.10}$$

(where we put $\prod_{j=1}^{0}\cdot = 1$) is satisfied, then $\sum_{n=0}^{\infty} n^p f_n = +\infty$ which contradicts $f \in D(V_p)_+$.

Now, if (3.5.3) is satisfied, we have

$$\sum_{n=1}^{\infty}\frac{n^p}{g_n}\left(\sum_{i=0}^{\infty}\frac{1}{(n+i)^p}\prod_{j=1}^{i}\frac{d_{n+j}}{g_{n+j}}\right) \geq \sum_{n=1}^{\infty}\frac{1}{g_n} = +\infty.$$

Similarly, if assumption 2.b) is satisfied, we have

$$\sum_{n=1}^{\infty}\frac{n^p}{g_n}\left(\sum_{i=0}^{\infty}\frac{1}{(n+i)^p}\prod_{j=1}^{i}\frac{d_{n+j}}{g_{n+j}}\right) \geq \sum_{n=1}^{\infty}\frac{n^p}{g_n}\left(\sum_{i=n}^{\infty}\frac{1}{i^p}\right) \geq C\sum_{n=1}^{\infty}\frac{1}{n} = +\infty,$$

where we used the integral estimate for the inner sum. Hence, the series is divergent. □

By [7, Theorem 2.1], under standard assumptions on the fragmentation coefficients $F_p = \overline{A_p + B_p}$ generates a quasi-contractive semigroup $(S_{F_p}(t))_{t\geq 0}$.

Theorem 3.5.2. *Assume the conditions of Theorem 3.3.1, item 1. and of Theorem 3.5.1, item 3. are satisfied. Then* $Y_p = \overline{V_p + F_p}$ *and*

$$S_{Y_p}(t)f = \lim_{n\to\infty}\left(S_{V_p}\left(\frac{t}{n}\right)S_{F_p}\left(\frac{t}{n}\right)\right)^n f, \quad f \in X_p, \tag{3.5.11}$$

uniformly on bounded time intervals.

Proof. First, we observe that $D(V_p)\cap D(F_p) \supset D(G_p^0)\cap D(D_p^0)\cap D(S_p)\cap D(A_p)$ and the latter is dense in X_p. Next, we see that

$$\begin{aligned}
&[\lambda I - (V_p + F_p)]D(V_p)\cap D(A_p)\\
&\qquad \supset [\lambda I - (V_p + F_p)]D(G_p^0)\cap D(D_p^0)\cap D(S_p)\cap D(A_p)\\
&\qquad = [\lambda I - (T_p + G_p^- + D_p^+ + B_p)]D(T_p).
\end{aligned}$$

Since $\overline{(T_p + G_p^- + D_p^+ + B_p, D(T_p))}$ is the generator of a semigroup, $[\lambda I-(T_p+G_p^-+D_p^++B_p)]D(T_p)$ is dense in X_p for sufficiently large λ. Indeed, if $f \in X_p$, then $f = (\lambda I - \overline{T_p + G_p^- + D_p^+ + B_p})u$ for some $u \in D(\overline{T_p + G_p^- + D_p^+ + B_p})$ and $u = \lim_{n\to\infty} u_n$ with $u_n \in D(T_p)$ and $\lim_{n\to\infty}(T_p + G_p^- + D_p^+ + B_p)u_n = \overline{T_p + G_p^- + D_p^+ + B_p}u$. But then $f = \lim_{n\to\infty}(\lambda u_n - (T_p + G_p^- + D_p^+ + B_p)u_n)$; that is, $f \in \overline{[\lambda I - (T_p + G_p^- + D_p^+ + B_p)]D(T_p)}$.

Since both $(S_{V_p}(t))_{t\geq 0}$ and $(S_{F_p}(t))_{t\geq 0}$ are quasi-contractive, [59, Corollary 3.5.5] implies that $\overline{V_p + F_p}$ is the generator of a quasi-contractive semigroup. Now

$$\begin{aligned}
\lambda I - Y_p &= \lambda I - \overline{T_p + G_p^- + D_p^+ + B_p} = \lambda I - \overline{(D_p^0 + D_p^+ + G_p^- + G_p^0 + S_p) + (A_p + B_p)}\\
&\subset \lambda I - \overline{V_p + F_p}
\end{aligned}$$

and, since both Y_p and $\overline{V_p + F_p}$ are generators, we must have $Y_p = \overline{V_p + F_p}$. Then (3.5.11) follows from [59, Corollary 3.5.5]. $\square$

3.6 Examples

We present, in this section, some examples to illustrate our theoretical investigation.

Example 3.6.1. We begin by considering the following growth-fragmentation problem

$$\begin{aligned}
\frac{df_1}{dt} &= -g_1 f_1 + \sum_{i=2}^{\infty} a_i b_{1,i} f_i,\\
\frac{df_n}{dt} &= g_{n-1}f_{n-1} - (a_n + g_n)f_n, \quad n \geq 2\\
f_n(0) &= f_n^{in}, \quad n \geq 1,
\end{aligned} \tag{3.6.1}$$

where

$$b_{n,i} = \begin{cases} i & \text{for} \quad n = 1,\\ 0 & \text{otherwise;}\end{cases}$$

that is, any particle breaks down into monomers. Since $d_n = 0$ for all n, we take any unbounded $(a_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ satisfying

$$0 < \gamma a_n \le g_n \le g a_n, \quad n \ge 2, \tag{3.6.2}$$

for some $0 < \gamma \le g$. We note that in this settings

$$\Delta_n^{(p)} = n^p - \sum_{k=1}^{n-1} k^p b_{k,n} = n^p - n$$

and hence (3.3.5) is satisfied for any $p > 1$. Hence, the semigroup $(S_{Y_p}(t))_{t\ge 0}$ that solves (3.6.1) is analytic and compact in X_p for any $p > 1$ and Theorem 3.4.1 holds. Moreover, we observe that

$$\begin{aligned}\lambda f_1 &= -g_1 f_1 + \sum_{i=2}^{\infty} a_i b_{1,i} f_i,\\ \lambda f_n &= g_{n-1} f_{n-1} - (a_n + g_n) f_n, \quad n \ge 2\end{aligned} \tag{3.6.3}$$

can be explicitly solved. Indeed, let $\lambda \ge 0$ and, starting from the second equation, we get

$$f_{n,\lambda} = \frac{g_1 f_{1,\lambda}}{\lambda + g_n + a_n} \prod_{j=2}^{n-1} \frac{g_j}{\lambda + g_j + a_j}, \quad n \ge 2$$

and

$$\sum_{n=2}^{\infty} a_n b_{1,n} f_{n,\lambda} = g_1 f_{1,\lambda} \sum_{n=2}^{\infty} \frac{a_n n}{\lambda + a_n + g_n} \prod_{j=2}^{n-1} \frac{g_j}{\lambda + g_j + a_j}.$$

Now, by (3.6.2), $\frac{g_j}{\lambda + g_j + a_j} \le c = \frac{g}{1+g} < 1$ and thus

$$g_1 \sum_{n=2}^{\infty} \frac{a_n n}{\lambda + a_n + g_n} \prod_{j=2}^{n-1} \frac{g_j}{\lambda + g_j + a_j} \le g_1 \sum_{n=2}^{\infty} n c^{n-2} < \infty. \tag{3.6.4}$$

Hence, after dividing by $g_1 f_{1,\lambda} \ne 0$, the first equation in (3.6.3) takes the form

$$\psi(\lambda) := \frac{\lambda + g_1}{g_1} = \sum_{n=2}^{\infty} \frac{a_n n}{\lambda + a_n + g_n} \prod_{j=2}^{n-1} \frac{g_j}{\lambda + g_j + a_j} =: \phi(\lambda)$$

and the eigenvalue problem reduces to the algebraic equation $\psi(\lambda) = \phi(\lambda)$. By (3.6.4), the series defining ϕ is uniformly convergent on $[0, \infty)$, hence ϕ is continuous there and

$$\phi(0) = \sum_{n=2}^{\infty} \frac{a_n n}{g_n + a_n} \prod_{j=2}^{n} \frac{g_j}{g_j + a_j}.$$

Using (3.6.2), we have, for $q = \frac{\gamma}{1+\gamma}$,

$$\phi(0) \ge \frac{1}{g+1} \sum_{n=2}^{\infty} n q^{n-1} = \frac{1}{g} \frac{d}{dq} \sum_{n=2}^{\infty} q^n = \frac{1}{g} \frac{d}{dq} \frac{q^2}{1-q} = \frac{1}{g} \frac{2q - q^2}{(1-q)^2} = \frac{1}{g} \left(\frac{1}{(1-q)^2} - 1 \right);$$

that is,

$$\phi(0) \ge \frac{(\gamma+1)^2 - 1}{g} > 1,$$

provided

$$g + 1 < (\gamma + 1)^2 \le (g+1)^2, \tag{3.6.5}$$

where the second inequality follows from $\gamma \le g$, implied by (3.6.2). We see that, in particular, if $\gamma = g$; that is, $g_n = ga_n$, (3.6.5) is satisfied. Also, $\lim_{\lambda\to\infty}\phi(\lambda) = 0$. On the other hand, $\psi(0) = 1$ and $\lim_{\lambda\to\infty}\psi(\lambda) = +\infty$. Since ϕ is decreasing and ψ is increasing, there is exactly one $\lambda_0 > 0$ for which (3.6.3) has a solution (with arbitrary f_1 that can be set to 1). Moreover, we see that

$$\sum_{n=1}^{\infty} a_n n^p f_{n,\lambda} = g_1 \sum_{n=2}^{\infty} \frac{a_n n^p}{\lambda + a_n + g_n} \prod_{j=2}^{n-1} \frac{g_j}{\lambda + g_j + a_j} \le g_1 \sum_{n=2}^{\infty} n^p c^{n-2} < \infty,$$

and thus $f_{\lambda_0} = (f_{n,\lambda_0})_{n\in\mathbb{N}}$ is the Perron eigenvector of the generator Y_p.

Example 3.6.2. The dominant eigenvalue λ_0 can be explicitly found in certain cases. Let us consider problem (3.1.1) with $g_n = gn, d_n = 0, s_n = 0$ for all $n \in \mathbb{N}$ and some $g > 0$ and with other coefficients satisfying the assumptions of Theorem 3.4.1. Let $f_\lambda = (f_{n,\lambda})_{n\in\mathbb{N}} \in D(Y_p)$ satisfy

$$\begin{aligned} \lambda f_{1,\lambda} &= -g f_{1,\lambda} + \sum_{i=2}^{\infty} a_i b_{1,i} f_{i,\lambda}, \\ \lambda f_{n,\lambda} &= g(n-1) f_{n-1,\lambda} - (a_n + gn) f_{n,\lambda} + \sum_{i=n+1}^{\infty} a_i b_{n,i} f_{i,\lambda}, \quad n \ge 2. \end{aligned} \tag{3.6.6}$$

Multiplying the n-th equation by n and summing them, we obtain

$$\lambda \sum_{n=1}^{\infty} n f_{n,\lambda} = g \sum_{n=1}^{\infty} n f_{n,\lambda}.$$

The above is satisfied if either $\lambda = g$ or $\sum\limits_{n=1}^{\infty} n f_{n,\lambda} = 0$. Since we know that the Perron eigenvector must be positive, we obtain that $\lambda_0 = g$ is the Perron eigenvalue. As a byproduct, we see that any eigenvector f_λ belonging to an eigenvalue $\lambda \ne g$ must satisfy $\sum\limits_{n=1}^{\infty} n f_{n,\lambda} = 0$.

To conclude, let us consider the transposed matrix

$$\mathcal{Y}^T = \begin{pmatrix} -g_1 & g_1 & 0 & 0 & 0 & \dots \\ a_2 b_{1,2} & -(g_2 + a_2) & g_2 & 0 & 0 & \dots \\ a_3 b_{1,3} & a_3 b_{2,3} & -g_3 + a_3 & g_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n b_{1,n} & a_n b_{2,n} & \dots & -(g_n + a_n) & g_n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Let Y_p^* be the adjoint to Y_p acting in $X_p^* = \{(v_n)_{n\in\mathbb{N}} \,; \sup\limits_{n\in\mathbb{N}} n^{-p}|v_n| < \infty\}$ and let $f^* \in D(Y_p^*)$. Then, by definition

$$\langle Y_p^* f^*, f\rangle = \langle f^*, Y_p f\rangle, \quad f \in D(Y_p).$$

Taking $f = (\delta_{n,N})_{n\in\mathbb{N}}$, we see that

$$[Y_p^* f^*]_N = \langle f^*, Y_p f\rangle = \sum_{n=1}^{N-1} f_n^* a_N b_{n,N} - (g_N + a_N) f_N^* + g_N f_{N+1}^* = [\mathcal{Y}^T f^*]_N,$$

hence Y_p^* is a restriction of $\mathcal{Y}^T$ to $D(Y_p^*) \subset D(Y_{p,\max}^*) = \{f \in X_p^* \;:\; \mathcal{Y}^T f \in X_p^*\}$. On the other hand, let $f^* \in D(Y_{p,\max}^*)$, $f \in D(Y_p)$. Then, since $D(Y_p)$ is a weighted ℓ^1 space, $\bigcup_{N=1}^{\infty} P_N D(Y_p)$, where P_N is the projection defined in (3.2.18), is a core for Y_p. Using the fact that $Y_p P_N D(Y_p)$ is finite-dimensional, for each N

$$\langle \mathcal{Y}^T f^*, P_N f\rangle = \langle f^*, Y_p P_N f\rangle = \langle Y_p^* f^*, P_N f\rangle$$

and hence, passing to the limit with $N \to \infty$, $f^* \in D(Y_p^*)$. Thus $Y_p^* = \mathcal{Y}^T$ with $D(Y_p^*) = D(Y_{p,\max}^*)$.

Using the assumption that $g_n = gn$, we see that $h = (1, 2, \ldots, n, \ldots) \in D(Y_p^*)$ for any $m \geq 1$ and

$$Y_p^* h = gh.$$

Thus, by Theorem 3.4.1,

$$Y_p(t) f^{in} = e^{gt} \left(\sum_{n=1}^{\infty} n f_n^{in} \right) e + O(e^{g' t})$$

for some $g' < g$, where e is the Perron eigenvector with unit mass; that is $e = f_{\lambda_0} / \sum_{n=1}^{\infty} n f_{n,\lambda_0}$.

To illustrate the formulas derived in the last two examples, we let $p = 2$, $g = 1$, $a_n = 2n$, $f_n^{in} = \delta_{n,10} 10$ and integrate (3.6.1) numerically in the time interval $t \in [0, 20]$. As evident from Figure 3.1 and 3.2, the solution $f(t)$ very quickly settles to its asymptotic limit $\langle h, f^{in}\rangle e$ (see Figure 3.1 (right)), while in complete agreement with Theorem 3.4.1, the deviation $\|e^{-gt} S_{Y_p}(t) f^{in} - \langle h, f^{in}\rangle e\|_{[p]}$ decreases exponentially as t increases (see Figure 3.2 (left)).

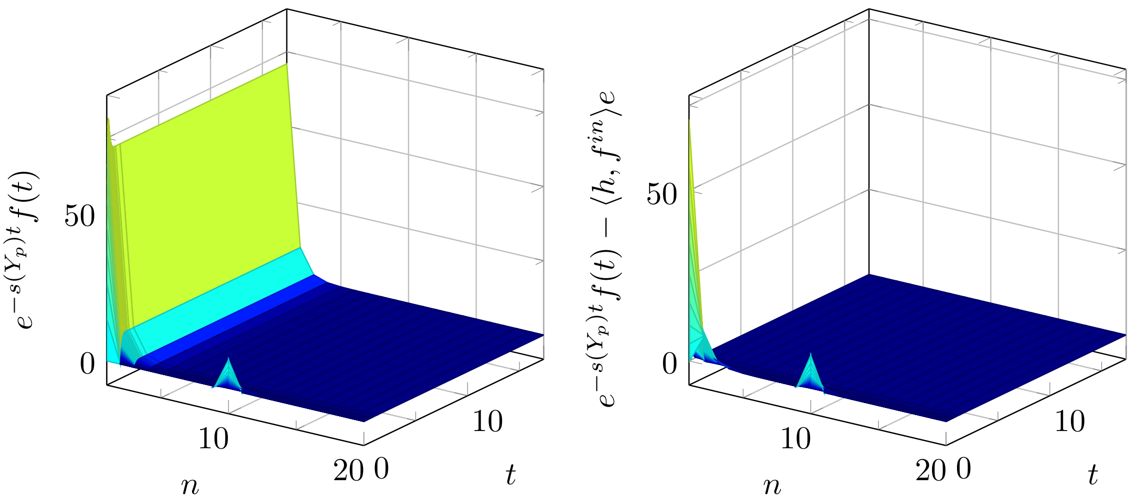

Figure 3.1: The long time behavior of (3.6.1). The semigroup solution $f(t) = S_{Y_p}(t) f^{in}$ of (3.6.1) (left); the asymptotic error $e^{-s(Y_p)t} f(t) - \langle h, f^{in}\rangle e$ (right).

A crucial role in the analysis is played by (3.3.5). It ensures that most of the mass of the daughter particles is concentrated in smaller particles, [7]. A large class of fragmentation kernels, that can be considered to be a discrete equivalent of the homogeneous kernels in continuous fragmentation, satisfying (3.3.5), is presented in the next example.

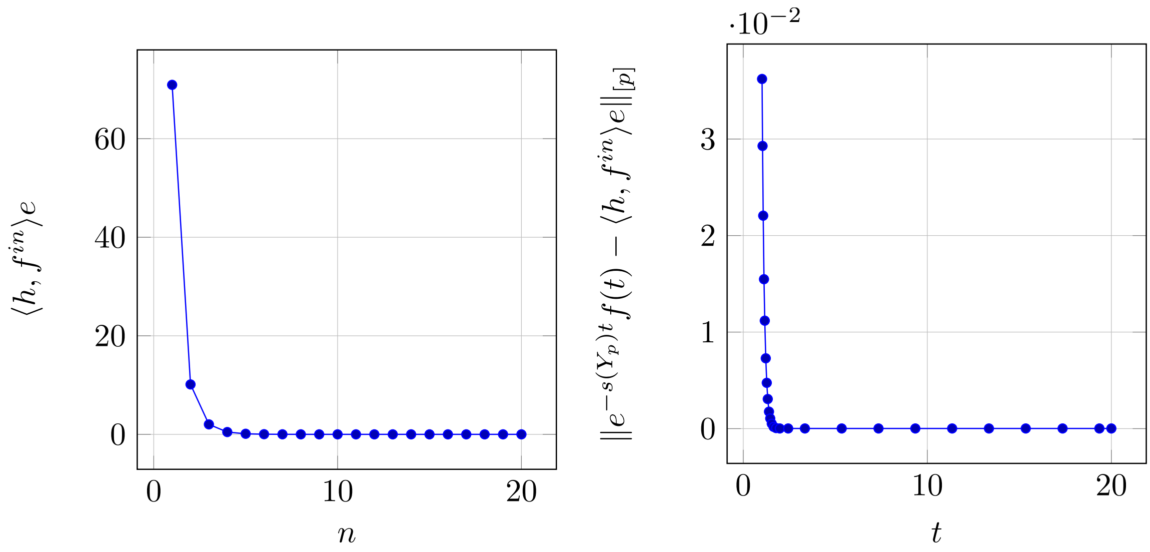

Figure 3.2: The long time behavior of (3.6.1). The asymptotic mass distribution $\langle h, f^{in}\rangle e$ (left) and the evolution of the asymptotic error $\|e^{-s(Y_p)t}f(t) - \langle h, f^{in}\rangle e\|_{[p]}$, for $t \geq 1$ (right).

Example 3.6.3. Assume that $b_{k,n}$ can be written as [7],

$$b_{k,n} = \zeta(n)h\left(\frac{k}{n}\right), \quad 1 \leq k \leq n-1, \quad n \in \mathbb{N}, \tag{3.6.7}$$

where h is a Riemann integrable function on $[0,1]$ and $\zeta(n)$ is an appropriate sequence that ensures that (1.2.3) is satisfied. By (1.2.3), we have

$$1 = \zeta(n)(n-1)\sum_{k=1}^{n-1}\frac{k}{n}h\left(\frac{k}{n}\right)\frac{1}{n-1}.$$

Since

$$\frac{k-1}{n-1} \leq \frac{k}{n} \leq \frac{k}{n-1},$$

for $1 \leq k \leq n$, we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n}h\left(\frac{k}{n}\right)\frac{1}{n-1} = \int_0^1 zh(z)dz$$

and thus

$$\lim_{n\to\infty}(n-1)\zeta(n) = \frac{1}{\int_0^1 zh(z)dz}.$$

Therefore

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^p b_{k,n} &= \lim_{n\to\infty}\zeta(n)(n-1)\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^p h\left(\frac{k}{n}\right)\frac{1}{n-1}\\ &= \frac{\int_0^1 z^p h(z)dz}{\int_0^1 zh(z)dz} < 1.\end{aligned}$$

Thus

$$\liminf_{n\to\infty}\frac{\Delta_n^{(p)}}{n^p} = \lim_{n\to\infty}\frac{n^p - \sum\limits_{k=1}^{n-1}k^p b_{k,n}}{n^p} > 0$$

and hence (3.3.5) is satisfied.

We note that (3.6.7) is satisfied by the binary uniform fragmentation

$$b_{n,i} = \frac{2}{i-1}, \quad n = 1, \ldots, i-1.$$

Another example is offered by the binary fragmentation written in terms of a symmetric infinite matrix $(\psi_{i,j})_{i,j\geq 1}$ in equation (1.1.3a), when there is no coagulation occurring, $k_{i,j} = 0$. Typical cases in the polymer degradation are

$$\psi_{i,j} = (i+j)^\beta,$$
$$\psi_{i,j} = (ij)^\beta.$$

The first case gives $a_n = \frac{1}{2}n^\beta(n-1)$ and $b_{n,i} = \frac{2}{i-1}$ and hence it is a uniform binary fragmentation (see the long time behavior of $S_{Y_p}(t)f^{in}$, with $p = 2$, $\beta = \frac{1}{10}$, $g_n = d_n = n^{1+\beta}$ and $f_n^{in} = \delta_{10,n}10$, in Figure 3.3 and 3.4). In the second case, we have

$$b_{n,i} = \frac{n^\beta(i-n)^\beta}{a_i} = \frac{i^{2\beta}}{a_i}\left(\frac{n}{i}\right)^\beta\left(1-\frac{n}{i}\right)^\beta$$

and (3.6.7) is satisfied with

$$\zeta(n) = \frac{n^{2\beta}}{a_n} \quad \text{and} \quad h(z) = z^\beta(1-z)^\beta.$$

(the typical qualitative behavior of $S_{Y_p}(t)f^{in}$, with $p = 2$, $\beta = \frac{1}{10}$, $d_n = g_n = n^{1+\beta}$ and $f_n^{in} = \delta_{10,n}10$, is shown in Figure 3.5 and 3.6).

Example 3.6.4. On the other hand, the fragmentation process given by

$$\begin{aligned} &b_{1,2} = 2, \quad \text{and} \quad b_{1,i} = b_{i-1,i} = 1, \\ &b_{n,i} = 0, \quad i \geq 2, \quad 2 \leq n \leq i-2, \end{aligned} \tag{3.6.8}$$

does not satisfy (3.3.5) and, in fact, the corresponding semigroup is neither analytic, nor compact, see [7], in any X_p.

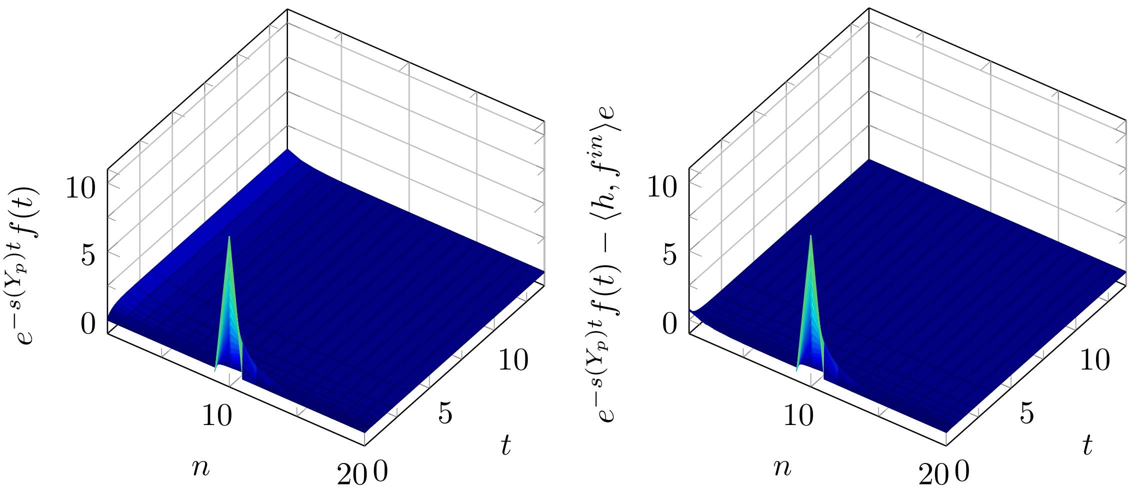

Figure 3.3: The long time behavior of (3.1.1), $\psi_{i,j} = (i+j)^\beta$. The semigroup solution $f(t) = S_{Y_p}(t)f^{in}$ of (3.1.1) (left); the asymptotic error $e^{-s(Y_p)t}f(t) - \langle h, f^{in}\rangle e$ (right).

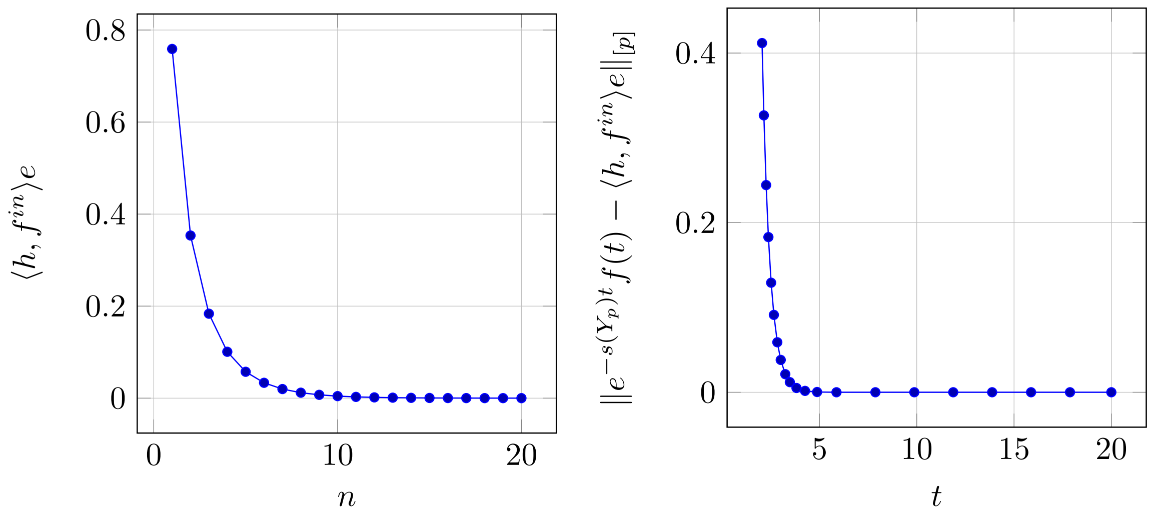

Figure 3.4: The long time behavior of (3.1.1), $\psi_{i,j} = (i+j)^\beta$. The asymptotic mass distribution $\langle h, f^{in}\rangle e$ (left) and the evolution of the asymptotic error $\|e^{-s(Y_p)t}f(t) - \langle h, f^{in}\rangle e\|_{[p]}$, for $t \geq 1$ (right).

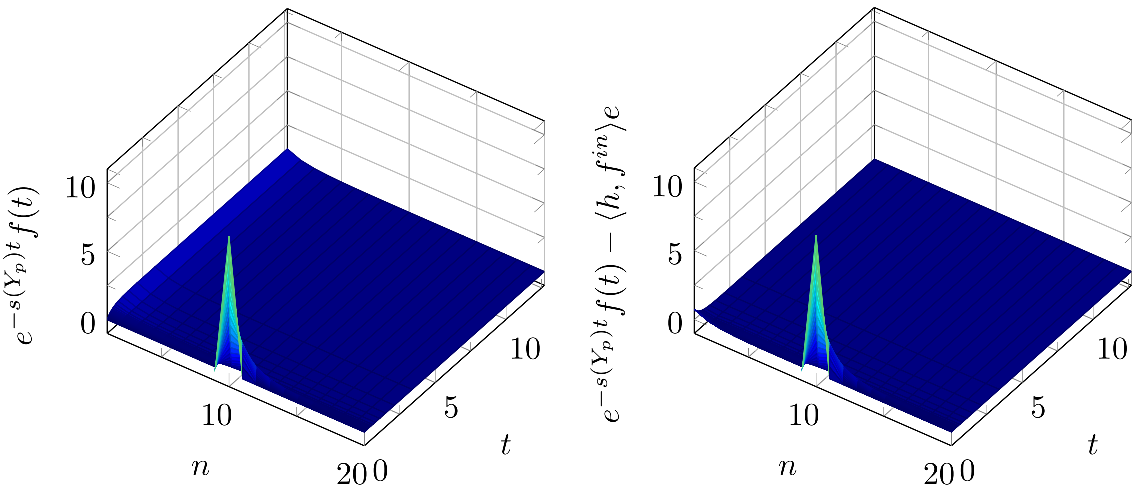

Figure 3.5: The long time behavior of (3.1.1), $\psi_{i,j} = (ij)^{\beta}$. The semigroup solution $f(t) = S_{Y_p}(t)f^{in}$ of (3.1.1) (left); the asymptotic error $e^{-s(Y_p)t}f(t) - \langle h, f^{in}\rangle e$ (right).

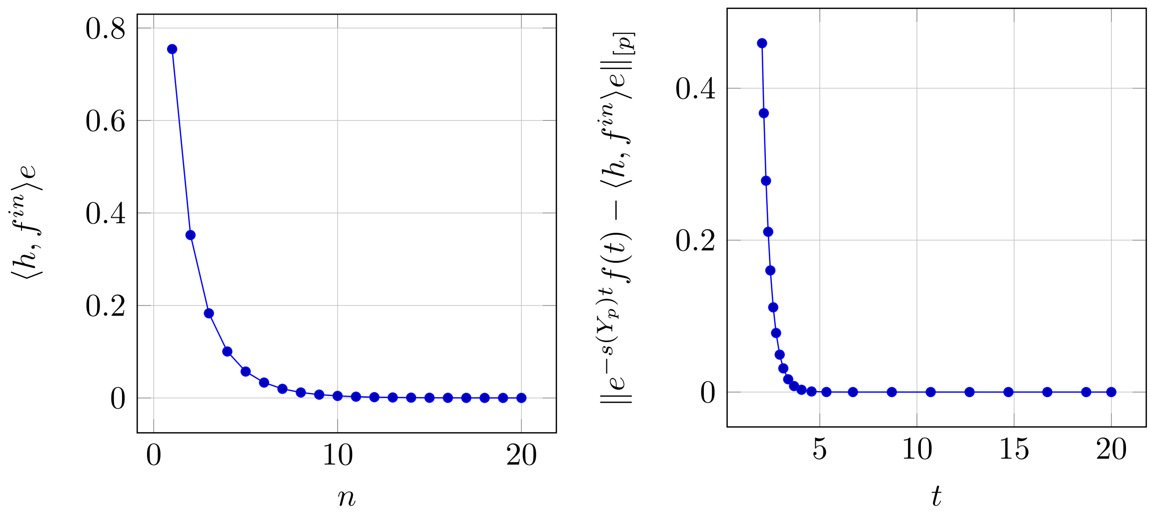

Figure 3.6: The long time behavior of (3.1.1), $\psi_{i,j} = (ij)^{\beta}$. The asymptotic mass distribution $\langle h, f^{in}\rangle e$ (left) and the evolution of the asymptotic error $\|e^{-s(Y_p)t}f(t) - \langle h, f^{in}\rangle e\|_{[p]}$, for $t \geq 1$ (right).

Chapter 4

The Fragmentation-Coagulation Equation with Growth, Decay and Sedimentation

4.1 Introduction

We will now focus on the semilinear equation:

$$\begin{aligned}\frac{df_i}{dt} &= g_{i-1}f_{i-1} - g_if_i + d_{i+1}f_{i+1} - d_if_i - s_if_i - a_if_i \\ &\quad + \sum_{j=i+1}^{\infty} a_jb_{i,j}f_j + \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j}f_{i-j}f_j - \sum_{j=1}^{\infty} k_{i,j}f_if_j, \\ f_i(0) &= \mathring{f}_i, \quad i \geq 1,\end{aligned} \tag{4.1.1}$$

The main aim of this chapter is to prove the existence of global classical solutions to (4.1.1) and provide a working numerical scheme for solving it. Thanks to the results in Chapter 3 showing that the linear part of the problem generates an analytic semigroup, $(S_{Y_p}(t))_{t\geq 0}$, in this chapter we significantly extended well-posedness results existing in the literature, see e.g. [7, 28], by considering more general models, removing many constraints on the coefficients of the problem and proving all results for both mild and classical solutions considered by most earlier works.

4.2 Preliminaries

In view of the second part of Theorem 3.3.1, we define

$$X_{p,1} = \{f \,:\, f \in X_p \cap D(T_p), \|f\|_{p,1} = \|(1+\theta)f\|_p\},$$

where $\theta := (\theta_i)_{i=1}^{\infty}$, (recall $\theta_i = a_i + g_i + d_i + s_i$) and consider the intermediate spaces $X_{p,\alpha} = (X_p, X_{p,1})_{\alpha,1}$, $0 < \alpha < 1$, where $(\cdot,\cdot)_{\alpha,1}$ is the standard real interpolation functor, see Section 2.8.

We observe that both spaces X_p and $X_{p,1}$ are weighted versions of ℓ^1, consequently, Theorem 2.8.1 applies and the norm in $X_{p,\alpha}$ is given by the expression

$$\|f\|_{p,\alpha} = \sum_{i\geq 1} i^p (1+\theta_i)^\alpha |f_i|, \quad 0<\alpha<1. \tag{4.2.1}$$

The interpolation functor $(\cdot,\cdot)_{\alpha,1}$ is known to be exact, see Section 2.8 and equation (2.8.1). In our case we observe that $\|S_{Y_p}(t)\|_{X_p\to X_p} \leq c_{0,p}e^{\omega_p t}$, for all $t\geq 0$ and some fixed $c_{0,p}, \omega_p > 0$, while, due to the analyticity, $\|S_{Y_p}(t)\|_{X_p\to X_{p,1}} \leq \frac{c_{1,p}}{t}e^{\omega_p t}$, $t>0$, see Theorem 2.7.1(c). It follows that $S_{Y_p}(t)\in\mathcal{B}(X_p, X_{p,\alpha})$ and

$$\begin{aligned}\|S_{Y_p}(t)\|_{X_p\to X_{p,\alpha}} &\leq \|S_{Y_p}(t)\|^{1-\alpha}_{X_p\to X_p}\|S_{Y_p}(t)\|^{\alpha}_{X_p\to X_{p,1}}\\ &\leq \frac{c_{0,p}^{1-\alpha}c_{1,p}^{\alpha}}{t^\alpha}e^{\omega_p t} =: \frac{c_{\alpha,p}}{t^\alpha}e^{\omega_p t}, \quad t>0, \quad 0<\alpha<1. \end{aligned}\tag{4.2.2a}$$

In addition, since $S_{Y_p}(t)\in\mathcal{B}(X_{p,1}, X_{p,1})\cap\mathcal{B}(X_p, X_p)$, by Theorem 2.7.1(c), similar arguments of analyticity imply

$$\|S_{Y_p}(t)\|_{X_{p,\alpha}\to X_{p,\alpha}} \leq c_{0,p}e^{\omega_p t}, \quad t\geq 0, \quad 0<\alpha<1 \tag{4.2.2b}$$

and

$$\begin{aligned}\|S_{Y_p}(t)\|_{X_{p,\alpha}\to X_{p,1}} &\leq \|S_{Y_p}(t)\|^{1-\alpha}_{X_p\to X_{p,1}}\|S_{Y_p}(t)\|^{\alpha}_{X_{p,1}\to X_{p,1}}\\ &\leq \frac{c_{0,p}^{\alpha}c_{1,p}^{1-\alpha}}{t^{1-\alpha}}e^{\omega_p t} =: \frac{c'_{\alpha,p}}{t^{1-\alpha}}e^{\omega_p t}, \quad t>0, \quad 0<\alpha<1. \end{aligned}\tag{4.2.2c}$$

In the sequel, we make use of the operator

$$(Y_{\gamma,p,\beta}, D(T_p)) := (Y_p+\gamma T_{p,\beta}, D(T_p)), \quad [T_{p,\beta}f]_i = -(1+\theta_i)^\beta f_i, \quad i\geq 1,$$

where γ is a positive parameter and $0\leq\beta\leq 1$. Using [59, Corollary 3.2.4] for $0\leq\beta<1$ (and obvious addition if $\beta=1$) and an argument analogous to that in the proof of [7, Theorem 5.1], we verify that under the assumptions of Theorem 3.3.1, $(Y_{\gamma,p,\beta}, D(T_p))$ generates a positive analytic C_0-semigroup $(S_{Y_{\gamma,p,\beta}}(t))_{t\geq 0}$ in X_p for all $p>1$ and $\gamma>0$. Furthermore, for any $f\in X_{p,+}$, we have

$$S_{Y_{\gamma,p,\beta}}(t)f = S_{Y_p}(t)f - \int_0^t S_{Y_p}(t-s)\big[\gamma T_{p,\beta}S_{Y_{\gamma,p,\beta}}(s)f\big]ds,$$

where the equality holds in the sense $X_{p,1}$. Since all the operators $(S_{Y_p}(t))_{t\geq 0}$, $(S_{Y_{\gamma,p,\beta}}(t))_{t\geq 0}$, $-T_{p,\beta}$ are positive, we obtain initially

$$\|S_{Y_{\gamma,p,\beta}}(t)\|_{X_p\to X_p} \leq \|S_{Y_p}(t)\|_{X_p\to X_p}, \quad \|S_{Y_{\gamma,p,\beta}}(t)\|_{X_p\to X_{p,1}} \leq \|S_{Y_p}(t)\|_{X_p\to X_{p,1}}$$

and then

$$\|S_{Y_{\gamma,p,\beta}}(t)\|_{X_p\to X_{p,\alpha}} \leq \|S_{Y_p}(t)\|_{X_p\to X_{p,\alpha}}, \quad \|S_{Y_{\gamma,p,\beta}}(t)\|_{X_{p,\alpha}\to X_{p,1}} \leq \|S_{Y_p}(t)\|_{X_{p,\alpha}\to X_{p,1}},$$

uniformly in $\gamma>0$ and $0\leq\beta\leq 1$. So the estimates (4.2.2), with the constants $c_{0,p}$, $c_{1,p}$, $c_{\alpha,p}$ and $c'_{\alpha,p}$, hold for the operator $S_{Y_{\gamma,p,\beta}}(t)$ as well. In fact, $(S_{Y_{\gamma,p,\beta}}(t))_{t\geq 0}$ is substochastic when $\gamma>0$ is sufficiently large, i.e. if (4.2.2) holds with $\omega_p=0$.

4.3 Global well-posedness

In this section, we provide a well-posedness analysis of the complete semilinear model (4.1.1). We assume that the condition (3.3.5) of Theorem 3.3.1 is satisfied. In addition, we impose the following bound on the coefficients of the coagulation kernel

$$k_{i,j} \le \kappa((1+\theta_i)^\alpha + (1+\theta_j)^\alpha), \quad i,j \ge 1,\ 0<\alpha<1. \tag{4.3.1}$$

The analysis proceeds in a number of technical steps. For the readers convenience the proofs of main results are broken into a sequence of short independent statements.

4.3.1 Local analysis

The local analysis of the semilinear equation is presented below. We convert (4.1.1) into a Volterra type integral equation and then employ a variant of the classical Picard-Lindelöf iterations to obtain local mild solutions. Then, with some additional work we verify that the mild solutions are classical. The calculations are similar to that of e.g. [59, Section 6.3] or [7] but, as some intermediate estimates are needed for calculations in Section 4.4, we provide an outline of the proofs.

Lemma 4.3.1. *Assume for some $p>1$ conditions* (3.3.5) *and* (4.3.1) *are satisfied. Then for each $f_0 \in X_{p,\alpha,+}$ and some $T>0$, the initial value problem* (4.1.1) *has a unique non-negative mild solution $f \in C([0,T], X_{p,\alpha})$.*

Proof. (a) To begin, we cast the equation (4.1.1) in the form of an Abstract Cauchy Problem (ACP), i.e.

$$\frac{df}{dt} = Y_{\gamma,p,\alpha} f + F_{\gamma,\alpha}(f), \quad f(0) = f_0 \in X_{p,\alpha},$$

where

$$\begin{aligned}[F_{\gamma,\alpha}(f)]_i &:= \gamma(1+\theta_i)^\alpha f_i + [F_1(f)]_i - [F_2(f)]_i \\ &:= \gamma(1+\theta_i)^\alpha f_i + \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j} f_{i-j} f_j - \sum_{j=1}^{\infty} k_{i,j} f_i f_j, \quad i \ge 1,\end{aligned} \tag{4.3.2}$$

and $\gamma = (1+\omega_p+2\kappa)(1+c_{0,p}\|f_0\|_{p,\alpha})$. A positive function $\gamma(1+\theta_i)^\alpha f_i$ is added to the nonlinear term in order to guarantee positivity. As noted above, $(S_{Y_{\gamma,p,\alpha}}(t))_{t\ge 0}$ is substochastic in X_p and for all $t \in [0,T]$ and some fixed $T>0$, classical solutions of (4.1.1) satisfy

$$f(t) = S_{Y_{\gamma,p,\alpha}}(t) f_0 + \int_0^t S_{Y_{\gamma,p,\alpha}}(t-\tau) F_{\gamma,\alpha}(f(\tau))d\tau. \tag{4.3.3}$$

We demonstrate that the integral equation (4.3.3) is locally solvable.

(b) The map $F_{\gamma,\alpha} : X_{p,\alpha} \to X_p$ is bounded and locally Lipschitz continuous provided (4.3.1) holds. The argument here is the same as in the proof of [7, Theorem 5.1], leading to

$$\begin{aligned}\|F_{\gamma,\alpha}(f)\|_p &= \sum_{i=1}^{\infty} i^p \Big|\gamma(1+\theta_i)^\alpha f_i + \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j} f_{i-j} f_j - \sum_{j=1}^{\infty} k_{i,j} f_i f_j\Big| \\ &\le \gamma\|f\|_{p,\alpha} + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty} i^p k_{i-j,j}|f_{i-j}||f_j| + \sum_{i=1}^{\infty} i^p \sum_{j=1}^{\infty} k_{i,j}|f_i||f_j| \\ &= \gamma\|f\|_{p,\alpha} + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(i+j)^p k_{i,j}|f_i||f_j| + \sum_{i=1}^{\infty} i^p \sum_{j=1}^{\infty} k_{i,j}|f_i||f_j|.\end{aligned}$$

Using $(i+j)^p \le 2^p(i^p + j^p)$, we get

$$\begin{aligned}\|F_{\gamma,\alpha}(f)\|_p &\le \gamma\|f\|_{p,\alpha} + \kappa 2^{p-1}\sum_{j=1}^{\infty}|f_j|\sum_{i=1}^{\infty} i^p((1+\theta_i)^\alpha + (1+\theta_j)^\alpha)|f_i| \\ &+ \kappa 2^{p-1}\sum_{j=1}^{\infty} j^p|f_j|\sum_{i=1}^{\infty}((1+\theta_i)^\alpha + (1+\theta_j)^\alpha)|f_i| \\ &+ \kappa\sum_{i=1}^{\infty} i^p|f_i|\sum_{j=1}^{\infty}((1+\theta_i)^\alpha + (1+\theta_j)^\alpha)|f_j| \\ &= \gamma\|f\|_{p,\alpha} + \kappa 2^{p-1}\sum_{j=1}^{\infty}|f_j|\sum_{i=1}^{\infty} i^p(1+\theta_i)^\alpha|f_i| + \kappa 2^{p-1}\sum_{j=1}^{\infty}(1+\theta_j)^\alpha|f_j|\sum_{i=1}^{\infty} i^p|f_i| \\ &+ \kappa 2^{p-1}\sum_{j=1}^{\infty} j^p|f_j|\sum_{i=1}^{\infty}(1+\theta_i)^\alpha|f_i| + \kappa 2^{p-1}\sum_{j=1}^{\infty} j^p(1+\theta_j)^\alpha|f_j|\sum_{i=1}^{\infty}|f_i| \\ &+ \kappa\sum_{i=1}^{\infty} i^p(1+\theta_i)^\alpha|f_i|\sum_{j=1}^{\infty}|f_j| + \kappa\sum_{i=1}^{\infty} i^p|f_i|\sum_{j=1}^{\infty}(1+\theta_j)^\alpha|f_j| \\ &\le \big(\gamma + 2^{p+2}\kappa\|f\|_{p,\alpha}\big)\|f\|_{p,\alpha} \qquad (4.3.4)\end{aligned}$$

and

$$\begin{aligned}\|F_{\gamma,\alpha}(f) - F_{\gamma,\alpha}(g)\|_p &= \sum_{i=1}^{\infty} i^p \Big|\Big(\gamma(1+\theta_i)^\alpha f_i + \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j} f_{i-j} f_j - \sum_{j=1}^{\infty} k_{i,j} f_i f_j\Big) \\ &- \Big(\gamma(1+\theta_i)^\alpha g_i + \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j} g_{i-j} g_j - \sum_{j=1}^{\infty} k_{i,j} g_i g_j\Big)\Big| \\ &\le \gamma\|f-g\|_{p,\alpha} + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty} i^p k_{i-j,j}|f_{i-j}f_j - g_{i-j}g_j| \\ &+ \sum_{i=1}^{\infty} i^p \sum_{j=1}^{\infty} k_{i,j}|f_i f_j - g_i g_j| \\ &\le \gamma\|f-g\|_{p,\alpha} + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(i+j)^p k_{i,j}|f_i||f_j - g_j| \\ &+ \frac{1}{2}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(i+j)^p k_{i,j}|g_j||f_i - g_i| \\ &+ \sum_{i=1}^{\infty} i^p\sum_{j=1}^{\infty} k_{i,j}|f_i||f_j - g_j| + \sum_{i=1}^{\infty} i^p|f_i - g_i|\sum_{j=1}^{\infty} k_{i,j}|g_j|.\end{aligned}$$

Again, using $(i+j)^p \le 2^p(i^p + j^p)$ and (4.3.1), we get

$$
\begin{aligned}
\|F_{\gamma,\alpha}(f) - F_{\gamma,\alpha}(g)\|_p \le{}& \gamma\|f-g\|_{p,\alpha} + \kappa 2^{p-1}\sum_{j=1}^{\infty}|f_j - g_j|\sum_{i=1}^{\infty} i^p((1+\theta_i)^\alpha + (1+\theta_j)^\alpha)|f_i| \\
&+ \kappa 2^{p-1}\sum_{j=1}^{\infty} j^p|f_j - g_j|\sum_{i=1}^{\infty}((1+\theta_i)^\alpha + (1+\theta_j)^\alpha)|f_i| \\
&+ \kappa 2^{p-1}\sum_{j=1}^{\infty}|g_j|\sum_{i=1}^{\infty} i^p((1+\theta_i)^\alpha + (1+\theta_j)^\alpha)|f_i - g_i| \\
&+ \kappa 2^{p-1}\sum_{j=1}^{\infty} j^p|g_j|\sum_{i=1}^{\infty}((1+\theta_i)^\alpha + (1+\theta_j)^\alpha)|f_i - g_i| \\
&+ \kappa\sum_{i=1}^{\infty} i^p|f_i|\sum_{j=1}^{\infty}((1+\theta_i)^\alpha + (1+\theta_j)^\alpha)|f_j - g_j| \\
&+ \kappa\sum_{i=1}^{\infty} i^p|f_i - g_i|\sum_{j=1}^{\infty}((1+\theta_i)^\alpha + (1+\theta_j)^\alpha)|g_j| \\
&\le \gamma\|f-g\|_{p,\alpha} + \kappa 2^{p-1}\sum_{j=1}^{\infty}|f_j - g_j|\sum_{i=1}^{\infty} i^p(1+\theta_i)^\alpha|f_i| \\
&+ \kappa 2^{p-1}\sum_{j=1}^{\infty}(1+\theta_j)^\alpha|f_j - g_j|\sum_{i=1}^{\infty} i^p|f_i| \\
&+ \kappa 2^{p-1}\sum_{j=1}^{\infty} j^p|f_j - g_j|\sum_{i=1}^{\infty}(1+\theta_i)^\alpha|f_i| \\
&+ \kappa 2^{p-1}\sum_{j=1}^{\infty} j^p(1+\theta_j)^\alpha|f_j - g_j|\sum_{i=1}^{\infty}|f_i| \\
&+ \kappa 2^{p-1}\sum_{j=1}^{\infty}|g_j|\sum_{i=1}^{\infty} i^p(1+\theta_i)^\alpha|f_i - g_i| \\
&+ \kappa 2^{p-1}\sum_{j=1}^{\infty}(1+\theta_j)^\alpha|g_j|\sum_{i=1}^{\infty} i^p|f_i - g_i| \\
&+ \kappa 2^{p-1}\sum_{j=1}^{\infty} j^p|g_j|\sum_{i=1}^{\infty}(1+\theta_i)^\alpha|f_i - g_i| \\
&+ \kappa 2^{p-1}\sum_{j=1}^{\infty} j^p(1+\theta_j)^\alpha|g_j|\sum_{i=1}^{\infty}|f_i - g_i| \\
&+ \kappa\sum_{i=1}^{\infty} i^p(1+\theta_i)^\alpha|f_i|\sum_{j=1}^{\infty}|f_j - g_j| + \kappa\sum_{i=1}^{\infty} i^p|f_i|\sum_{j=1}^{\infty}(1+\theta_j)^\alpha|f_j - g_j| \\
&+ \kappa\sum_{i=1}^{\infty} i^p(1+\theta_i)^\alpha|f_i - g_i|\sum_{j=1}^{\infty}|g_j| + \kappa\sum_{i=1}^{\infty} i^p|f_i - g_i|\sum_{j=1}^{\infty}(1+\theta_j)^\alpha|g_j| \\
&\le \gamma\|f-g\|_{p,\alpha} + \kappa 2^{p+2}\|f-g\|_{p,\alpha}\|f\|_{p,\alpha} + \kappa 2^{p+2}\|f-g\|_{p,\alpha}\|g\|_{p,\alpha} \\
&= \big(\gamma + 2^{p+2}\kappa(\|f\|_{p,\alpha} + \|g\|_{p,\alpha})\big)\|f-g\|_{p,\alpha}. \qquad (4.3.5)
\end{aligned}
$$

We use estimates (4.3.4) and (4.3.5) to show in (c) that the nonlinear map

$$M(f) = S_{Y_{\gamma,p,\alpha}}(t)f_0 + \int_0^t S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(f(\tau))d\tau,$$

is a contraction in the closed ball $B_r(f^0) = \{f : \|f - f^0\|_{C([0,T],X_{p,\alpha})} \le r\}$ with $0 < r < 1$ and

$$0 < T \le \Big(\frac{r(1-\alpha)}{2(1+\omega_p + 2^{p+4}\kappa)c_{\alpha,p}(1+c_{0,p}\|f_0\|_{p,\alpha})^2}\Big)^{\frac{1}{1-\alpha}}. \tag{4.3.6}$$

(c) Let $f^0(t) = S_{Y_{\gamma,p,\alpha}}(t)f_0$, $t \in [0,T]$. First, we show that $B_r(f^0)$ is invariant under the action of M. Indeed, for any $f \in B_r(f^0)$

$$\begin{aligned}
\|f^0 - M(f)\|_{C([0,T],X_{p,\alpha})} &= \Big\| f^0 - S_{Y_{\gamma,p,\alpha}}(t)f_0 - \int_0^t S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(f(\tau))d\tau \Big\|_{p,\alpha} \\
&\le \big\|f^0 - S_{Y_{\gamma,p,\alpha}}(t)f_0\big\|_{p,\alpha} + \Big\|\int_0^t S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(f(\tau))d\tau\Big\|_{p,\alpha} \\
&\le \max_{0\le t\le T}\int_0^t \|S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(f(\tau))\|_{p,\alpha}d\tau \\
&= \frac{c_{\alpha,p}T^{1-\alpha}}{1-\alpha}\big[\gamma + 2^{p+2}\kappa\|f\|_{C([0,T],X_{p,\alpha})}\big]\|f\|_{C([0,T],X_{p,\alpha})} \\
&\le \frac{c_{\alpha,p}T^{1-\alpha}}{1-\alpha}(1+\omega_p + 2^{p+3}\kappa)(1+c_{0,p}\|f_0\|_{p,\alpha})^2 \le r,
\end{aligned}$$

where we used the inequality

$$\begin{aligned}
\|f\|_{C([0,T],X_{p,\alpha})} &= \|f - f^0 + f^0\|_{C([0,T],X_{p,\alpha})} \\
&\le \|f - f^0\|_{C([0,T],X_{p,\alpha})} + \|f^0\|_{C([0,T],X_{p,\alpha})} \\
&\le r + \|f^0\|_{C([0,T],X_{p,\alpha})} \le 1 + c_{0,p}\|f_0\|_{p,\alpha},
\end{aligned}$$

combined with (4.2.2a), (4.2.2b), (4.3.4) and our definition of γ. Furthermore, with the aid of (4.3.5), (4.3.6), in the same manner as above we have for $f, g \in B_r(f^0)$,

$$\begin{aligned}
\|M(f) - M(g)\|_{C([0,T],X_{p,\alpha})} &= \|S_{Y_{\gamma,p,\alpha}}(t)f_0 + \int_0^t S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(f(\tau))d\tau \\
&\quad - S_{Y_{\gamma,p,\alpha}}(t)g_0 - \int_0^t S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(g(\tau))d\tau\|_{p,\alpha} \\
&\le \|S_{Y_{\gamma,p,\alpha}}(t)f_0 - S_{Y_{\gamma,p,\alpha}}(t)g_0\|_{p,\alpha} \\
&\quad + \|\int_0^t S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(f(\tau))d\tau - \int_0^t S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(g(\tau))d\tau\|_{p,\alpha} \\
&\le \|S_{Y_{\gamma,p,\alpha}}(t)\|_{p,\alpha}\|f_0 - g_0\|_{p,\alpha} \\
&\quad + \int_0^t \|S_{Y_{\gamma,p,\alpha}}(t-\tau)\|_{C([0,T],X_{p,\alpha})}\|F_{\gamma,\alpha}(f(\tau)) - F_{\gamma,\alpha}(g(\tau))\|_{p,\alpha}d\tau \\
&\le \frac{c_{\alpha,p}T^{1-\alpha}}{1-\alpha}(1+\omega_p + 2^{p+4}\kappa)(1+c_{0,p}\|f_0\|_{p,\alpha})^2\|f-g\|_{C([0,T],X_{p,\alpha})} \\
&< \tfrac{1}{2}\|f-g\|_{C([0,T],X_{p,\alpha})}.
\end{aligned}$$

Hence, $M : B_r(f^0) \to B_r(f^0)$ is a contraction and the classical Banach fixed point theorem yields a unique, mild solution of (4.1.1) in $B_r(f^0) \subset C([0,T], X_{p,\alpha})$.

(d) To complete the proof we note that the maps F_1 and F_2, defined in (4.3.2), are non-negative in $X^+_{p,\alpha}$. Assuming that $f \in B_r(f_0)^+$, we have

$$\begin{aligned}[F_2(f)]_i &= \sum_{j=1}^{\infty} k_{i,j} f_i f_j \leq 2\kappa \|f\|_{p,\alpha}(1+\theta_i)^\alpha f_i \\ &\leq 2\kappa(r + \|f_0\|_{C([0,T],X_{p,\alpha})})(1+\theta_i)^\alpha f_i \leq \gamma(1+\theta_i)^\alpha f_i\end{aligned}$$

and then $[F_{\gamma,\alpha}(f)]_i \geq 0$, $i \geq 1$. The last inequality indicates that $B_r(f_0)^+$ is invariant under the action of the map M and hence the local mild solution f is non-negative. □

To proceed further, we make use of the following modification of the Gronwall inequality, sometimes called the singular Gronwall inequality, see e.g. [24, Lemma 8.1.1]. Since we need some specific aspects of it, we shall provide an elementary proof.

Lemma 4.3.2. *Let $u \in L_{\infty,loc}((0,T]) \cap L_1((0,T))$, $0 < T < \infty$, be a nonnegative function satisfying*

$$u(t) \leq \frac{c}{t^\gamma} + c_1 \int_0^t u(\tau)(t-\tau)^{-\alpha} d\tau, \qquad t \in (0,T], \tag{4.3.7}$$

where $\gamma < 1$, $0 < \alpha < 1$ and $c, c_1 > 0$. Then there is a constant $C(\gamma,\alpha,T)$, independent of c, such that

$$u(t) \leq \frac{cC(\gamma,\alpha,T)}{t^\gamma}, \qquad t \in (0,T]. \tag{4.3.8}$$

Proof. First we observe that, for any $\beta < 1, \delta < 1$ and $a < b < \infty$, we have

$$\begin{aligned}\int_a^b (b-t)^{-\beta}(t-a)^{-\delta} dt &= (b-a)^{-\beta-\delta+1} \int_0^1 (1-v)^{-\beta} v^{-\delta} dv \\ &= B(1-\beta, 1-\delta)(b-a)^{-\beta-\delta+1},\end{aligned} \tag{4.3.9}$$

where B is the beta function. Since u satisfies (4.3.7), it follows from (4.3.9) that

$$\begin{aligned}\int_0^t \frac{u(\tau)}{(t-\tau)^\alpha} d\tau &\leq c\int_0^t \frac{1}{\tau^\gamma(t-\tau)^\alpha} d\tau + c_1 \int_0^t \frac{1}{(t-\tau)^\alpha}\left(\int_0^\tau \frac{u(s)}{(\tau-s)^\alpha} ds\right) d\tau \\ &= c(\theta_\gamma * \theta_\alpha)(t) + c_{1,\alpha} \int_0^t u(s)(t-s)^{1-2\alpha} ds,\end{aligned} \tag{4.3.10}$$

where $*$ denotes the Laplace convolution, $\theta_\kappa(t) = t^{-\kappa}$ and $c_{1,\alpha} = c_1 B(1-\alpha, 1-\alpha)$. Inserting (4.3.10) into (4.3.7), we obtain

$$u(t) \leq c\theta_\gamma(t) + cc_1(\theta_\gamma * \theta_\alpha)(t) + c_{2,\alpha} \int_0^t u(\tau)(t-\tau)^{1-2\alpha} d\tau, \qquad t \in (0,T], \tag{4.3.11}$$

with $c_{2,\alpha} = c_1\, c_{1,\alpha}$. Note that the convolution $\theta_\gamma * \theta_\kappa$ exists for any choice of $\gamma < 1$ and $\kappa < 1$, since

$$(\theta_\gamma * \theta_\kappa)(t) = B(1-\gamma, 1-\kappa)\, t^{1-\gamma-\kappa} = B(1-\gamma, 1-\kappa)\, \theta_{\gamma+\kappa-1}(t). \tag{4.3.12}$$

Furthermore,

$$(\theta_\gamma * \theta_\kappa)(t) \leq \frac{\bar{C}(\gamma,\kappa,T)}{t^\gamma}, \qquad t \in (0,T], \tag{4.3.13}$$

where $\bar{C}(\gamma,\kappa,T)$ is a positive constant.

If $1-2\alpha \geq 0$, then we can infer from (4.3.11) and (4.3.13) that

$$u(t) \leq \frac{c(1+\bar{C}(\gamma,\alpha,T))}{t^{\gamma}} + c_{2,\alpha} t^{1-2\alpha} \int_0^t u(\tau) d\tau,$$

and then apply the standard arguments used to establish Gronwall-type inequalities to obtain the desired result. Otherwise, we repeat the above process inductively, using (4.3.12) and (4.3.9), until we arrive at

$$u(t) \leq c\Theta(t) + c^{(k)}_{2^k,\alpha}(u * \theta_{2^k\alpha - 2^k + 1})(t), \tag{4.3.14}$$

where $k \in \mathbb{N}$ is such that $2^k(1-\alpha) - 1 \geq 0$,

$$\Theta(t) = \theta_{\gamma}(t) + \sum_{r=1}^{2^k-1} c^{(k)}_{r,\alpha}(\theta_{\gamma} * \theta_{r\alpha - r + 1})(t),$$

and each $c^{(k)}_{r,\alpha}$, $r = 1, 2, \ldots k$, is a positive constant, independent of c. Hence,

$$u(t) \leq c\Theta(t) + c^{(k)}_{2^k,\alpha} t^{2^k(1-\alpha)-1} \int_0^t u(\tau) d\tau. \tag{4.3.15}$$

Since by $\alpha < 1$, we have $r\alpha - r + 1 < 1$ for any $r \geq 1$, from (4.3.13) we infer that there is a constant $C_1(\gamma,\alpha,T) > 0$ such that

$$\Theta(t) \leq \frac{C_1(\gamma,\alpha,T)}{t^{\gamma}}$$

and Θ is integrable on $[0,T]$. Hence, a classical Gronwall inequality leads to

$$u(t) \leq \frac{cC(\gamma,\alpha,T)}{t^{\gamma}},$$

for some constant $C(\gamma,\alpha,T)$ and this gives (4.3.8). □

To simplify the notation, in the calculations below, we employ the symbol c to denote a positive constant whose particular value is irrelevant.

Lemma 4.3.3. *Under assumptions of Lemma 4.3.1, the mild solution $f(t)$ is Hölder continuous with exponent $1-\alpha$, i.e.*

$$\|f(t+h) - f(t)\|_{p,\alpha} \leq \frac{ch^{1-\alpha}}{t^{1-\alpha}}, \tag{4.3.16}$$

for all $t \in (0,T]$ and some $c > 0$.

Proof. By virtue of (4.3.3), we have

$$\begin{aligned}
\|f(t+h) - f(t)\|_{p,\alpha} \leq\ & \|(S_{Y_{\gamma,p,\alpha}}(t+h) - S_{Y_{\gamma,p,\alpha}}(t))f_0\|_{p,\alpha} \\
& + \left\| \int_t^{t+h} S_{Y_{\gamma,p,\alpha}}(t+h-\tau) F_{\gamma,\alpha}(f(\tau)) d\tau \right\|_{p,\alpha} \\
& + \left\| \int_0^t S_{Y_{\gamma,p,\alpha}}(\tau) \Big(F_{\gamma,\alpha}(f(t+h-\tau)) - F_{\gamma,\alpha}(f(t-\tau)) \Big) d\tau \right\|_{p,\alpha} \\
& =: J_1 + J_2 + J_3.
\end{aligned}$$

First we infer, by (4.2.2a) and (4.2.2c),

$$J_1 \le \int_0^h \Big\| S_{Y_{\gamma,p,\alpha}}(\tau)\big[Y_{\gamma,p,\alpha} S_{Y_{\gamma,p,\alpha}}(t) f_0\big]\Big\|_{p,\alpha} d\tau \le c \|S_{Y_{\gamma,p,\alpha}}(t) f_0\|_{p,1} \int_0^h \frac{d\tau}{\tau^\alpha}$$
$$\le \frac{ch^{1-\alpha}}{t^{1-\alpha}} \|f_0\|_{p,\alpha} \le \frac{ch^{1-\alpha}}{t^{1-\alpha}}.$$

For J_2 and J_3, in the same manner as in part (c) of Lemma 4.3.1, we obtain

$$J_2 \le ch^{1-\alpha} \|f\|^2_{C([0,T],X_{p,\alpha})}, \quad J_3 \le c \int_0^t \|f(\tau+h) - f(\tau)\|_{p,\alpha} \frac{d\tau}{(t-\tau)^\alpha}.$$

Combining the estimates, we get

$$\|f(t+h) - f(t)\|_{p,\alpha} \le \frac{ch^{1-\alpha}}{t^{1-\alpha}} + c\int_0^t \|f(\tau+h) - f(\tau)\|_{p,\alpha} \frac{d\tau}{(t-\tau)^\alpha}.$$

Hence, the bound (4.3.16), with a constant $c > 0$ that depends on α, T and the initial data f_0 only, follows directly from Lemma 4.3.2 with $\gamma = 1 - \alpha$. □

Lemmas 4.3.1 and 4.3.3, combined together, yield

Theorem 4.3.4. *Assume that conditions* (3.3.5) *and* (4.3.1) *are satisfied. Then, for each* $f_0 \in X_{p,\alpha}$ *there is* $T = T(f_0) > 0$ *such that the initial value problem* (4.1.1) *has a unique non-negative classical solution* $f \in C([0,T], X_{p,\alpha}) \cap C^1((0,T), X_p) \cap C((0,T), X_{p,1})$.

Proof. First we prove the differentiability of f in X_p for $t > 0$. From (4.3.3),

$$\frac{f(t+h) - f(t)}{h}$$
$$= \frac{S_{Y_{\gamma,p,\alpha}}(h) - I}{h} S_{Y_{\gamma,p,\alpha}}(t) f_0 + \frac{1}{h}\int_t^{t+h} S_{Y_{\gamma,p,\alpha}}(t+h-\tau) F_{\gamma,\alpha}(f(\tau)) d\tau$$
$$+ \frac{1}{h}\int_0^t \Big(S_{Y_{\gamma,p,\alpha}}(t+h-\tau) - S_{Y_{\gamma,p,\alpha}}(t-\tau)\Big) F_{\gamma,\alpha}(f(\tau)) d\tau := I_1 + I_2 + I_3.$$

We observe that, by the analyticity, $S_{Y_{\gamma,p,\alpha}}(t) f_0 \in D(T_p)$ for $t > 0$, so that

$$\lim_{h\to 0} I_1 = \lim_{h\to 0} \frac{S_{Y_{\gamma,p,\alpha}}(h) - I}{h} S_{Y_{\gamma,p,\alpha}}(t) f_0$$
$$= Y_{\gamma,p,\alpha} S_{Y_{\gamma,p,\alpha}}(t) f_0$$

in X_p. By (4.2.2c) we have

$$\big\|\lim_{h\to 0} I_1\big\| = \|Y_{\gamma,p,\alpha} S_{Y_{\gamma,p,\alpha}}(t) f_0\|_p \le \|S_{Y_{\gamma,p,\alpha}}(t) f_0\|_{p,1} \le \frac{c}{t^{1-\alpha}} \|f_0\|_{p,\alpha}. \tag{4.3.17}$$

The strong continuity of $(S_{Y_{\gamma,p,\alpha}}(t))_{t\ge 0}$, the continuity of f (see Lemma 4.3.1) and estimates (4.3.4), (4.3.5) combined together, show that in X_p

$$\big\|\lim_{h\to 0} I_2\big\|_{X_p} = \Big\| \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} S_{\gamma,p,\alpha}(t+h-\tau) F_{\gamma,\alpha}(f(\tau)) d\tau \Big\|_p$$
$$= \|F_{\gamma,\alpha}(f(t))\|_p \le c.$$

To find $\lim_{h\to 0} I_3$, we first show that

$$\Big\|\frac{1}{h}\int_0^t \Big(S_{Y_{\gamma,p,\alpha}}(t+h-\tau) - S_{Y_{\gamma,p,\alpha}}(t-\tau)\Big)F_{\gamma,\alpha}(f(\tau))d\tau\Big\|_p < \infty.$$

By (4.3.16), we have

$$\begin{aligned}\|\lim_{h\to 0} I_3\| &= \Big\|\lim_{h\to 0}\frac{1}{h}\int_0^t \Big(S_{Y_{\gamma,p,\alpha}}(t+h-\tau) - S_{Y_{\gamma,p,\alpha}}(t-\tau)\Big)F_{\gamma,\alpha}(f(\tau))d\tau\Big\|_p \qquad (4.3.18)\\ &= \Big\|\int_0^t Y_{\gamma,p,\alpha}S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(f(\tau))d\tau\Big\|_p\\ &\le \int_0^t \|Y_{\gamma,p,\alpha}S_{Y_{\gamma,p,\alpha}}(t-\tau)(F_{\gamma,\alpha}(f(\tau)) - F_{\gamma,\alpha}(f(t)))\|_p d\tau\\ &\quad + \Big\|\int_0^t Y_{\gamma,p,\alpha}S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(f(t))d\tau\Big\|_p\\ &\le c\int_0^t \|S_{Y_{\gamma,p,\alpha}}(t-\tau)(F_{\gamma,\alpha}(f(\tau)) - F_{\gamma,\alpha}(f(t)))\|_{p,1}d\tau\\ &\quad + \|(S_{Y_{\gamma,p,\alpha}}(t) - I)F_{\gamma,\alpha}(f(t))\|_p\\ &\le c\int_0^t \frac{1}{t-\tau}\|f(\tau)-f(t)\|_{p,\alpha}d\tau + c \le ct^{\alpha-1}\int_0^t (t-\tau)^{-\alpha}d\tau + c \le c, \qquad (4.3.19)\end{aligned}$$

where $c > 0$ depends on $t > 0$, α, T and κ, the constants that appear in (4.2.2), and the initial data f_0. Thus,

$$\lim_{h\to 0} I_3 = \int_0^t Y_{\gamma,p,\alpha}S_{Y_{\gamma,p,\alpha}}(t-\tau)F_{\gamma,\alpha}(f(\tau))d\tau.$$

Combining all our calculations,

$$\Big\|\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}\Big\| = \|\lim_{h\to 0} I_1\| + \|\lim_{h\to 0} I_2\| + \|\lim_{h\to 0} I_3\| \le 3c$$

we conclude that $\frac{df}{dt} \in X_p$ for any $t > 0$ and the continuity of each of the above limits shows that $f \in C^1((0,T), X_p)$ is a classical solution. The same calculations (specifically the variation of costant formula (4.3.3), the estimate (4.3.19) and the inclusion $S_{Y_{\gamma,p,\alpha}}(t)X_p \subset X_{p,1}$, $t > 0$) demonstrate also that $f \in C((0,T), X_{p,1})$. □

Remark 4.3.5. The calculations presented above, in particular (4.3.17) and the last but one inequality in (4.3.19), show that $\|Y_p f(t)\|_p \le \frac{c}{t^{1-\alpha}}$, $t > 0$, provided $f_0 \in X_{p,\alpha}$. Since the graph norm $\|\cdot\|_p + \|Y_p\cdot\|_p$ and $\|\cdot\|_{p,1}$ are equivalent in $X_p \cap D(T_p)$ (as $D(T_p) = D(Y_p)$ and both operators are closed), it follows that

$$\|f(t)\|_{p,1} \le \frac{c}{t^{1-\alpha}}, \quad t > 0, \qquad (4.3.20)$$

for $f_0 \in X_{p,\alpha}$ and hence $\|f\|_{L^1([0,T],X_{p,1})} < \infty$. The last fact is crucial for the numerical analysis presented in Section 4.4.

4.3.2 Global non-negative solutions

Below we show that classical solutions of (4.1.1) emanating from non-negative initial data are globally defined. Our analysis requires the following observation.

Lemma 4.3.6. *Assume that $f_0 \in X_{p,\alpha,+}$ and for some ω_1*

$$\frac{g_i - d_i}{i} - s_i \le \omega_1, \quad i \ge 1 \tag{4.3.21}$$

Then, under the assumptions of Theorem 4.3.4, the local solution satisfies

$$\|f\|_1 \le e^{\omega_1 t}\|f_0\|_1, \qquad t \in (0, T(f_0)). \tag{4.3.22}$$

Proof. Since $f \in X_{p,\alpha,+}$, we know that every term of (4.1.1) is separately well-defined for $t \in (0, T(f_0))$ (as the solution takes values in $D(X_{p,1})$) and is differentiable in $X_{p,1}$, and hence in X_1. Thus

$$\frac{d}{dt}\|f(t)\|_1 \le \sum_{i=1}^{\infty}\Big(-s_i + \frac{g_i - d_i}{i}\Big) i f_i \le \omega_1 \|f(t)\|_1$$

and (4.3.22) follows from the standard Gronwall inequality. □

Two remarks are in place here. First, in the case of pure fragmentation-coagulation models ($s_i = g_i = d_i = 0$, $i \ge 1$) or in the absence of growth ($g_i = 0$, $i \ge 1$), we have $\omega_1 \le 0$. Second, even in the absence of sedimentation, the bound (4.3.22) still holds provided there is a reasonable balance between the growth and the death processes.

Theorem 4.3.7. *Under the assumptions of Theorem 4.3.4 and Lemma 4.3.6, any solution of* (4.1.1) *with $f_0 \in X_{p,\alpha,+}$, $p > 1$, is global in time.*

Proof. (a) To begin, we observe that for any $f \in X_{p,\alpha,+}$ we have

$$\sum_{i=1}^{\infty} i^p [Y_p f]_i = -\sum_{i=1}^{\infty} i^p \theta_i f_i \Bigg[\frac{a_i}{\theta_i}\frac{\triangle_i^{(p)}}{i^p} + \Big(1 - \Big(1 - \frac{1}{i}\Big)^p\Big)\frac{d_i}{\theta_i} - \Big(\Big(1 + \frac{1}{i}\Big)^p - 1\Big)\frac{g_i}{\theta_i} - \frac{s_i}{\theta_i}\Bigg] \le -c_p\|f\|_{p,1} + \beta_p\|f\|_p,$$

where, by (3.3.5), c_p and β_p are positive constants that only depend on the coefficients of (4.1.1) and p (in fact one can take c_p to be any positive constant smaller than $\liminf_{i\to\infty} \frac{a_i}{\theta_i}\frac{\triangle_i^{(p)}}{i^p}$). By (4.3.1), the nonlinearity F admits the bound

$$\begin{aligned}\sum_{i=1}^{\infty} i^p F(f)_i &= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}((i+j)^p - i^p - j^p)k_{i,j} f_i f_j \\ &\le \frac{2^p - 1}{2}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(i^{p-1}j + ij^{p-1})k_{i,j} f_i f_j = c_2(\|f\|_1\|f\|_{p-1,\alpha} + \|f\|_{p-1}\|f\|_{1,\alpha}),\end{aligned}$$

with an absolute constant $c_2 > 0$, where we used the estimate [7, Equation (5.21)]

$$(i+j)^p - i^p - j^p \le (2^p - 1)(i^{p-1}j + j^{p-1}i)$$

for the weight. By (4.3.23) and the non-negativity of the local classical solution $f(t)$, for $t \in (0, T)$ we obtain

$$\frac{d}{dt}\|f\|_p \le -c_p\|f\|_{p,1} + \beta_p\|f\|_p + c_2(\|f\|_1\|f\|_{p-1,\alpha} + \|f\|_{p-1}\|f\|_{1,\alpha}). \tag{4.3.23a}$$

On the other hand, again by (4.3.1), we have

$$\|F(f)\|_p \le (1+2^p)\sum_{j=1}^{\infty}|f_j|\sum_{i=1}^{\infty} i^p k_{i,j}|f_i| \le 2^{p+2}\kappa\|f\|_p\|f\|_{p,\alpha},$$

while the variation of constants formula and the analyticity of the semigroup $(S_{Y_p}(t))_{t\ge 0}$ (see estimates (4.2.2)) imply

$$\|f(t)\|_{p,\alpha} \le c_{0,p}e^{\omega_p t}\|f_0\|_{p,\alpha} + 2^{p+2}\kappa c_{\alpha,p}\int_0^t \frac{e^{\omega_p(t-\tau)}}{(t-\tau)^\alpha}\|f(\tau)\|_p\|f(\tau)\|_{p,\alpha}d\tau. \tag{4.3.23b}$$

We use estimates (4.3.23) to demonstrate that the non-negative local classical solutions cannot blow up in a finite time. For technical reasons, we separately consider two cases, $1 < p \le 2$ and $2 < p < \infty$.

(b) Let $1 < p \le 2$. Then (4.3.23a) implies

$$\frac{d}{dt}\|f\|_p \le -c_p\|f\|_{p,1} + \beta_p\|f\|_p + 2c_2\|f\|_1\|f\|_{1,\alpha}.$$

To bound the product term, we use an approach similar to that of [28] and employ Hölder's inequality with the exponent $q = \frac{1}{\alpha} > 1$ to obtain

$$\|f\|_{1,\alpha} \le \|f\|_{p,1}^{\alpha}\|f\|_{\frac{1-p\alpha}{1-\alpha}}^{1-\alpha} \le \|f\|_{p,1}^{\alpha}\|f\|_1^{1-\alpha}$$

and then, using Young's inequality,

$$2c_2\|f\|_1\|f\|_{1,\alpha} \le c_p\|f\|_{p,1} + (2c_2)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{c_p}\right)^{\frac{\alpha}{1-\alpha}}\|f\|_1^{\frac{2-\alpha}{1-\alpha}}.$$

Hence

$$\frac{d}{dt}\|f(t)\|_p \le \beta_p\|f\|_p + (2c_2)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{c_p}\right)^{\frac{\alpha}{1-\alpha}}\|f\|_1^{\frac{2-\alpha}{1-\alpha}},$$

so that the Gronwall inequality, combined with (4.3.22), gives us the bound

$$\|f(t)\|_p \le \left[1 + (2c_2)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{c_p}\right)^{\frac{\alpha}{1-\alpha}}\right]e^{\omega_p' t}\|f_0\|_p =: \beta_{\alpha,p}e^{\omega_p' t}\|f_0\|_p,$$

where $\omega_p' \le \max\left\{\beta_p, \frac{2-\alpha}{1-\alpha}\omega_1\right\}$. We combine this with (4.3.23b) to obtain

$$e^{-\omega_p t}\|f(t)\|_{p,\alpha} \le c_{0,p}\|f_0\|_{p,\alpha} + 2^{p+2}\kappa c_{\alpha,p}\beta_{\alpha,p}e^{\omega_p' t}\int_0^t \frac{e^{-\omega_p\tau}\|f(\tau)\|_{p,\alpha}}{(t-\tau)^\alpha}d\tau.$$

Proceeding as in the proof of Lemma 4.3.2, we conclude that

$$\|f(t)\|_{p,\alpha} \le C_{p,\alpha}(\|f_0\|_{p,\alpha})e^{\Omega_{p,\alpha}t}, \tag{4.3.24}$$

where $C_{p,\alpha}(\|f_0\|_{p,\alpha}) > 0$ depends on the coefficients of the model (4.1.1), parameter $1 < p \le 2$ and the norm $\|f_0\|_{p,\alpha}$ of the initial data, while the exponent $\Omega_{p,\alpha} > 0$ is completely controlled by the parameter $1 < p \le 2$ and the coefficients of (4.1.1) only. Hence, the case $1 < p \le 2$ is settled.

(c) When $2 \le p < \infty$, we use Hölder's inequality with the exponent $q = p' := \frac{p}{p-1} > 1$. Since $0 < \alpha < 1$, we have $\frac{p}{q}(q\alpha - 1) \le \alpha$, consequently

$$\|f\|_{p-1,\alpha} \le \|f\|_{p,1}^{\frac{1}{q}}\left(\sum_{i=1}^{\infty}(1+\theta_i)^{\frac{p}{q}(q\alpha-1)}f_i\right)^{\frac{1}{p}} \le \|f\|_{p,1}^{\frac{p-1}{p}}\|f\|_{1,\alpha}^{\frac{1}{p}}$$

and, by Young's inequality,

$$c_2\|f\|_1\|f\|_{p-1,\alpha} \leq \frac{c_p}{2}\|f\|_{p,1} + \Big(1-\frac{1}{p}\Big)^{1-p}\Big(\frac{2c_2}{c_p}\Big)^p\|f\|_{1,\alpha}^{p+1}.$$

A similar procedure yields also

$$c_2\|f\|_{p-1}\|f\|_{1,\alpha} \leq \frac{c_p}{2}\|f\|_{p,1} + \Big(1-\frac{1}{p}\Big)^{1-p}\Big(\frac{2c_2}{c_p}\Big)^p\|f\|_{1,\alpha}^{p+1}.$$

Hence, using (4.3.23a), we obtain

$$\frac{d}{dt}\|f\|_p \leq \beta_p\|f\|_p + \gamma_p\|f\|_{1,\alpha}^{p+1},$$

where $\gamma_p > 0$ only depends on $p > 1$ and the parameters of the model (4.1.1). From part (b) and the continuity of the embedding $X_{1,\alpha} \subset X_{2,\alpha}$, we have

$$\|f(t)\|_{1,\alpha} \leq \|f(t)\|_{2,\alpha} \leq C_{2,\alpha}(\|f_0\|_{2,\alpha})e^{\Omega_{2,\alpha}t},$$

that is, $\|f(t)\|_{1,\alpha}$ grows at most exponentially. Hence, the classical Gronwall inequality yields

$$\|f(t)\|_p \leq \beta_{\alpha,p}e^{\omega_p' t}\|f_0\|_p$$

also for $2 < p < \infty$, where constants $\beta_{\alpha,p}, \omega_p' > 0$ depend on p and the parameters of (4.1.1) only. As in part (b) of the proof, the last estimate, together with the inequality (4.3.23b), yields the exponential bound (4.3.24) for $2 < p < \infty$. We conclude that for any $p > 1$, the norm $\|f(t)\|_{p,\alpha}$ of the local solution f emanating from a non-negative initial datum cannot blowup in a finite time. Hence, any such solution is defined globally. □

Remark 4.3.8. In the strong sedimentation case, (3.3.11), the analysis of Theorems 4.3.4 and 4.3.7 extends to the case of $p = 1$, since then we also have the analytic fragmentation semigroup in X_1 and the estimates can be repeated almost verbatim. In fact, the analysis of Theorem 4.3.7 becomes much simpler as the X_1 norm of the solution does not blow up in finite time by Lemma 4.3.6 provided (4.3.21) is satisfied and thus (4.3.23b) is immediately applicable with $p = 1$.

Remark 4.3.9. We note that Theorem 4.3.7 significantly extends global solvability results obtained earlier in the context of the pure fragmentation-coagulation model (see [7]), where the existence of global solutions is established under much more restrictive assumptions that $\theta_i = a_i \leq ci^s$, $i \geq 1$, for some constants $c, s > 0$ and the exponent α of (4.3.1) satisfies $0 < \alpha s \leq 1$.

4.4 Numerical Simulations

4.4.1 The Truncated Problem

In numerical simulations, we approximate the original infinite-dimensional system (4.1.1) by the following finite-dimensional counterpart:

$$\begin{aligned}\frac{du_i}{dt} &= g_{i-1}u_i - \theta_i u_i + d_{i+1}u_{i+1} + \sum_{j=i+1}^{N} a_j b_{i,j} u_j \\ &\quad + \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j}u_{i-j}u_j - \sum_{j=1}^{N} k_{i,j}u_i u_j + \frac{\delta_{N,i}}{N}\sum_{j=1}^{N}\sum_{n=N+1-j}^{N} jk_{n,j}u_n u_j, \\ u_i(0) &= u_{0,i}, \quad 1 \le i \le N.\end{aligned} \tag{4.4.1}$$

The quadratic penalty term ensures that the discrete coagulation process is conservative – this property is important when dealing with pure fragmentation-coagulation models.

Let $P_N : X_p \to \mathbb{R}^N$ and $I_N : \mathbb{R}^N \to X_p$ denote the projector from X_p onto $\mathbb{R}^N$ and the embedding from $\mathbb{R}^N$ into X_p, respectively. Below, we shall show that if $u^{(N)}$ is the solution of the truncated problem (4.4.1) with the initial condition $u_0^{(N)}$, then the sequence $I_N u^{(N)}$ approaches f as the truncation index N increases.

Theorem 4.4.1. *Assume* (3.3.5), (4.3.1) *and* (4.3.21) *hold. The truncated problem in* (4.4.1) *is locally solvable, i.e. for each* $p > 1$ *there exists some* $T > 0$ *such that for each* N

$$u^{(N)} \in C([0,T], X_{p,\alpha}) \cap C^1((0,T), X_p) \cap C((0,T), X_{p,1}), \tag{4.4.2}$$

and the respective norms of $u^{(N)}$ *are bounded independently of* N*. If, in addition, the initial datum* $u_0^{(N)}$ *is non-negative,* (4.4.2) *holds for any fixed* $T > 0$*. Finally, if for some* $q > p-1$, $q \ge 0$ *we have* $f_0 \in X^+_{q+1,\alpha}$ *and* $\lim_{N\to\infty} \|I_N u_0^{(N)} - f_0\|_{p,\alpha} = 0$*, then* $I_N u^{(N)} \to f$ *in* $C([0,T], X_{p,\alpha})$ *as* $N \to \infty$*.*

Proof. (a) System (4.4.1) is an ODE with a smooth vector field, hence it is locally solvable for any $N > 0$. Let

$$\begin{aligned}&[Y_N f]_i = g_{i-1}f_i - \theta_i f_i + d_{i+1}f_{i+1}, \quad 1 \le i \le N, \quad [Y_N f]_i = 0, \quad i > N, \\ &[G_N f]_i = \delta_{N+1,i}g_{i-1}f_{i-1}, \quad i \ge 1,\end{aligned}$$

where for each $i \in \mathbb{N}$, $(\delta_{ij})_{j=1}^{\infty}$ is the Kronecker delta concentrated at i. We see that the linear part of the truncated equation (4.4.1) acts on the elements of the finite-dimensional subspace $I_N(\mathbb{R}^N) \subset D(T_p)$ according to the formula

$$Y_N f = Y_p f - G_N f.$$

Since the operator G_N is non-negative and bounded, direct application of the variation of constants formula implies that the semigroup $(S_N(t))_{t\geq 0}$ generated by $(Y_N, D(T_p))$ satisfies

$$\|S_N(t)\|_{X_p\to X_p} \leq \|S_{Y_p}(t)\|_{X_p\to X_p}, \quad \|S_N(t)\|_{X_p\to X_{p,\alpha}} \leq \|S_{Y_p}(t)\|_{X_p\to X_{p,\alpha}}$$
$$\|S_N(t)\|_{X_p\to X_{p,1}} \leq \|S_{Y_p}(t)\|_{X_p\to X_{p,1}}, \tag{4.4.3}$$

so that all estimates involving $(S_N(t))_{t\geq 0}$ are uniform in $N > 0$. Hence, the analysis of Theorems 4.3.4 applies, i.e. for some $T > 0$ (that, in general, depends on $p > 1$, the initial condition and the coefficients of the problem) inclusion (4.4.2) holds and the respective norms are bounded independently of N.

Assuming that the initial datum $u_0^{(N)}$ is non-negative, we proceed as in Theorem 4.3.7 to show that the inclusion (4.4.2) holds for any fixed $T > 0$ uniformly in N. Hence, the first two claims of Theorem 4.4.1 are settled.

(b) To prove the last claim, we derive the equation governing the evolution of the numerical error $e^{(N)}(t) := P_N f(t) - u^{(N)}(t) \in \mathbb{R}^N, t \geq 0$. We have

$$\begin{aligned}
\frac{de_i^{(N)}}{dt} &= g_{i-1}e_{i-1}^{(N)} - \theta_i e_i^{(N)} + d_{i+1}e_{i+1}^{(N)} + \sum_{j=i+1}^{\infty} a_j b_{i,j} e_j^{(N)} + \delta_{N,i} d_{i+1} f_{i+1} \\
&+ \frac{1}{2}\sum_{j=1}^{i-1} k_{i-j,j}\big(e_{i-j}^{(N)} f_j + u_{i-j}^{(N)} e_j^{(N)}\big) - \sum_{j=1}^{N} k_{i,j}\big(e_i^{(N)} f_j + u_i^{(N)} e_j^{(N)}\big) \\
&+ \frac{\delta_{N,i}}{N}\sum_{j=1}^{N}\sum_{n=N+1-j}^{N} j k_{j,n}\big(e_j^{(N)} f_n + e_n^{(N)} u_j^{(N)}\big) \\
&- \frac{\delta_{N,i}}{N}\sum_{j=1}^{N}\sum_{n=N+1-j}^{N} j k_{j,n} f_j f_n - \sum_{j=N+1}^{\infty} k_{i,j} f_i f_j, \\
e_i^{(N)}(0) &= e_{0,i}^{(N)}, \quad 1 \leq i \leq N,
\end{aligned}$$

or, in a compact form,

$$\frac{de^{(N)}}{dt} = Y_N e^{(N)} + H_N(t)e^{(N)} + \Big(E_N^0 f - E_N^1 f - E_N^2 f\Big), \quad e^{(N)}(0) = e_0^{(N)},$$

where, for a given f and $u^{(N)}$, $H_N(t)e^{(N)}$ is linear in $e^{(N)}$ and

$$[E_N^0 f]_i = \delta_{N,i} d_{i+1} f_{i+1}, \quad [E_N^1 f]_i = \frac{\delta_{N,i}}{N}\sum_{j=1}^{N}\sum_{n=N+1-j}^{N} j k_{j,n} f_j f_n,$$
$$[E_N^2 f]_i = \sum_{j=N+1}^{\infty} k_{i,j} f_i f_j, \quad 1 \leq i \leq N.$$

In what follows we will use two inequalities based on the properties of the function $[0,a] \ni x \mapsto \phi(x) := x^r(a-x)^r$, $a > 2, r > 0$. Clearly, ϕ is symmetric, nonnegative with $\phi(0) = \phi(a) = 0$ and has a single maximum at $x = a/2$. Thus, for $x \in [1, a-1]$ we have $\phi(x) \geq (a-1)^r$. In particular, for $q \geq 0$ and $a = N+1$ we have

$$N^q \leq j^q(N+1-j)^q, \quad 1 \leq j \leq N, \tag{4.4.4}$$

where the inequality for $q = 0$ is trivial, and for $p \geq 1$, using (4.4.4) and $j \leq N$

$$j^{p-1}(N+1-j)^p = \frac{j^p(N+1-j)^p}{j} \geq N^{p-1}, \quad 1 \leq j \leq N. \tag{4.4.5}$$

Then, by (4.3.1), (4.3.4), the fact that f is globally defined by Theorem 4.3.7, and by (4.4.5), $H_N e^{(N)}$ satisfies

$$\begin{aligned}
\|H_N e^{(N)}\|_p &\leq \frac{1}{2}\sum_{i=1}^{N} i^p \sum_{j=1}^{i-1} k_{i-j,j}\big(|e_{i-j}^{(N)}||f_j| + |u_{i-j}^{(N)}||e_j^{(N)}|\big) \\
&+ \sum_{i=1}^{N} i^p \sum_{j=1}^{N} k_{i,j}\big(|e_i^{(N)}||f_j| + |u_i^{(N)}||e_j^{(N)}|\big) \\
&+ N^{p-1}\sum_{j=1}^{N}\sum_{n=N+1-j}^{N} jk_{j,n}\big(|e_j^{(N)}||f_n| + |u_j^{(N)}||e_n^{(N)}|\big) \\
&\leq (1+2^{p+2})\kappa\|e^{(N)}\|_{p,\alpha}(\|f\|_{p,\alpha} + \|u^{(N)}\|_{p,\alpha}) \leq \bar{c}\|e^{(N)}\|_{p,\alpha},
\end{aligned}$$

where (4.4.5) was used to get

$$\begin{aligned}
&N^{p-1}\sum_{j=1}^{N}\sum_{n=N+1-j}^{N} jk_{j,n}\big(|e_j^{(N)}||f_n| + |u_j^{(N)}||e_n^{(N)}|\big) \\
&= \sum_{j=1}^{N} jN^{p-1}\sum_{n=N+1-j}^{N} k_{j,n}\big(|e_j^{(N)}||f_n| + |u_j^{(N)}||e_n^{(N)}|\big) \\
&\leq \sum_{j=1}^{N} j^p \sum_{n=N+1-j}^{N} (N+1-j)^p k_{j,n}\big(|e_j^{(N)}||f_n| + |u_j^{(N)}||e_n^{(N)}|\big) \\
&\leq \sum_{j=1}^{N} j^p \sum_{n=N+1-j}^{N} n^p k_{j,n}\big(|e_j^{(N)}||f_n| + |u_j^{(N)}||e_n^{(N)}|\big).
\end{aligned}$$

Similarly, using (4.4.4) to estimate $E_N^1 f$, we have

$$\begin{aligned}
&\|E_N^0 f\|_p \leq (N+1)\theta_{N+1}|f_{N+1}|, \\
&\|E_N^1 f\|_p \leq \bar{c}N^{p-q-1}\|f\|_{q,\alpha}\|f\|_{q+1,\alpha} \leq \bar{c}N^{p-q-1}, \\
&\|E_N^2 f\|_p \leq \bar{c}\|(I-P_N)f\|_{p,\alpha},
\end{aligned} \tag{4.4.6}$$

where all generic constants $\bar{c} > 0$ are uniform in $N > 0$. The last four bounds, combined with the variation of constants formula,

$$e^{(N)}(t) = S_N(t)e_0^{(N)} + \int_0^t S_N(t-\tau)(H(\tau)e^{(N)}(\tau) + E_N^0 f(\tau) - E_N^1(\tau) - E_N^2(\tau))d\tau,$$

(4.4.3) and (4.2.2), yield

$$\begin{aligned}
\|e^{(N)}(t)\|_{p,\alpha} &\leq \bar{c}\|e_0^{(N)}\|_{p,\alpha} + \bar{c}\|(I-P_N)f\|_{C([0,T],X_{p,\alpha})} + \bar{c}N^{p-q-1} \\
&+ \bar{c}\int_0^t \frac{\|e^{(N)}(\tau)\|_{p,\alpha}}{(t-\tau)^\alpha}d\tau + \bar{c}\int_0^t \frac{\|E_N^0 f(\tau)\|_p}{(t-\tau)^\alpha}d\tau, \quad t \in [0,T],
\end{aligned}$$

with a constant $\bar{c} > 0$ that does not depend on the truncation parameter $N > 0$. Further, by (4.4.6) and (4.3.20), we have

$$\int_0^t \frac{\|E_N^0 f(\tau)\|_p}{(t-\tau)^\alpha} d\tau \le \int_0^t \frac{(N+1)\theta_{N+1}|f_{N+1}(\tau)|}{(t-\tau)^\alpha} d\tau \le N^{1-p} \int_0^t \frac{\|f(\tau)\|_{p,1}}{(t-\tau)^\alpha} d\tau$$
$$\le \bar{c} N^{1-p} \int_0^t \tau^{\alpha-1}(t-\tau)^{-\alpha} d\tau = \bar{c} B(\alpha, 1-\alpha) N^{1-p} = \bar{c} N^{1-p},$$

where, as before, $\bar{c} > 0$ is independent of $N > 0$. Thus, using (4.3.8) with $\gamma = 0$ and

$$c = \bar{c}(\|e_0^{(N)}\|_{p,\alpha} + \|(I - P_N)f\|_{C([0,T],X_{p,\alpha})} + N^{p-q-1} + N^{1-p})$$

in a fixed finite time interval $[0, T]$, we conclude that

$$\|e^{(N)}(t)\|_{p,\alpha} \le \bar{C}\Big[\|e_0^{(N)}\|_{p,\alpha} + \|(I - P_N)f\|_{C([0,T],X_{p,\alpha})} + N^{p-q-1} + N^{1-p}\Big],$$

with $\bar{C} > 0$ independent of $N > 0$. Note that $\lim_{N\to\infty} \|e_0^{(N)}\|_{p,\alpha} = 0$, by our assumptions, and the convergence of $\|(I - P_N)f(t)\|_{X_{p,\alpha}}$ to zero is indeed uniform on $[0, T]$ by Dini's theorem [54, Theorem 8.4]. Hence,

$$\lim_{N\to\infty} \|I_N u^{(N)} - f\|_{C([0,T],X_{p,\alpha})} = 0$$

and the last claim of the theorem is settled. □

4.4.2 Simulations

Below, we provide several numerical illustrations to the theory developed above. In our simulations, we make use of the following two fragmentation kernels:

$$b_{i,j} = \frac{2}{j-1}, \tag{4.4.7a}$$

$$b_{i,j} = \frac{i^\sigma (j-i)^\sigma}{\alpha_j}, \quad \alpha_j = \frac{1}{j}\sum_{i=1}^{j-1} i^{1+\sigma}(j-i)^\sigma, \quad \sigma > -1. \tag{4.4.7b}$$

The coagulation process is driven by one of the unbounded kernels (see e.g. [14, 41, 29] for the references and particular applications)

$$k_{i,j} = k_1(i^{1/3} + j^{1/3})^{\frac{7}{3}}, \tag{4.4.8a}$$

$$k_{i,j} = k_2(i + k_3)(j + k_3), \tag{4.4.8b}$$

where k_1, k_2 and k_3 are positive constants. The transport, the sedimentation and the fragmentation rates are chosen to be

$$g_i = g i^\alpha, \quad d_i = d i^\beta, \quad s_i = s i^\gamma, \quad a_i = a i^\delta,$$

for all $i \ge 1$, except for $d_1 = a_1 = 0$.

In view of Theorem 3.3.1, in the calculations below it is assumed that either

$$\max\{\alpha, \beta, \gamma\} \le \delta, \quad p > 1, \tag{4.4.9a}$$

or

$$\max\{\beta, \delta\} \leq \gamma, \quad p = 1. \tag{4.4.9b}$$

The conditions ensure that the associated semigroups $(S_{Y_p}(t))_{t \geq 0}$, equipped with either of the fragmentation kernels (4.4.7a) or (4.4.7b), are analytic in X_p, $p \geq 1$.

4.4.2.1 The pure fragmentation-coagulation scenario

Example 1. To begin, we consider (4.1.1) with $g = d = s = 0$, fragmentation kernel (4.4.7a) and coagulation kernel (4.4.8a). Here, the coagulation coefficients satisfy $k_{i,j} = \mathcal{O}(i^{\frac{7}{9}} + j^{\frac{7}{9}})$ hence Threorem 4.3.7 applies, provided $\delta > \frac{7}{9}$. In our simulations, we let: $N = 200$, $a = 1$, $\delta = 1$ and $k_1 = 5 \cdot 10^{-3}$. Since N is fixed, we shorten the notation setting $u^{(N)} = u$. As the initial conditions, we take

$$u_n(0) = 10, \quad 5 \leq n \leq 20 \quad \text{and} \quad u_n(0) = 0 \quad \text{otherwise}$$

and integrate (4.4.1) in time interval $[0, 1]$ using `ode15s` built-in `Matlab` ODE solver. The results of simulations are shown in Figure 4.1, 4.2 and 4.3.

At the initial stage (the left diagram in Figure 4.1), the coagulation process does generate large clusters with $n > 20$. However, due to the fragmentation, the densities associated with very large particles steadily go to zero and the solution settles near a steady state distribution. The right diagram in Figure 4.1 further illustrates this, where the evolution of mass $nu_n(t)$ concentrated at the clusters of size $1 \leq n \leq 80$ is plotted. As predicted by Theorem 4.3.7, the strong fragmentation processes acting in the model prevents uncontrollable mass absorption by the clusters of extremely large sizes. One can see that after a short transition stage, the mass distribution (concentrated initially in the aggregates of size $5 \leq n \leq 20$) quickly settles near a fixed state, in which the bulk mass of the ensemble accumulates in clusters of moderate size.

The behaviour of the total number of particles $\|u\|_0$ and the total mass of the system $\|u\|_1$ are shown in Figure 4.2 while the higher order moments $\|u\|_2$, $\|u\|_3$ are shown in Figure 4.3. The right diagram of Figure 4.2 shows, in particular, that the process is conservative (the total mass of the ensemble does not change), while Figure 4.2(left), and Figure 4.3 indicate that the solution settles near a steady state.

Example 2. In our second example, we employ the fragmentation kernel (4.4.7b) with $\sigma = 10^{-1}$ and the coagulation kernel (4.4.8b) with $k_2 = 5 \cdot 10^{-3}$ and $k_3 = 1$. Note that $k_{i,j} = \mathcal{O}(i^2 + j^2)$ and, in view of (4.3.1), we let $\delta = 2.5$. The remaining set of parameters is identical to those used in Example 1.

In the settings described above, the growth rate of the quantities $k_{i,j}$ is superlinear. Hence, the pure coagulation models lead to a formation of a massive particle outside the system (the so called gelation phenomenon, see [70] and references therein). In addition, the moment conditions, proposed in [7] in context of the discrete pure fragmentation-coagulation models, are also not

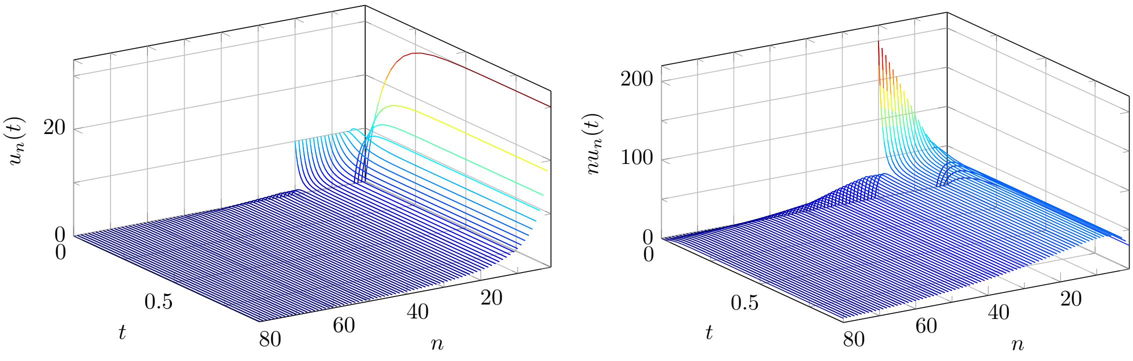

Figure 4.1: Evolution of the pure fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8a) and the fragmentation kernel (4.4.7a): number of clusters $u_n(t)$ (left); distribution of cluster masses $nu_n(t)$ (right).

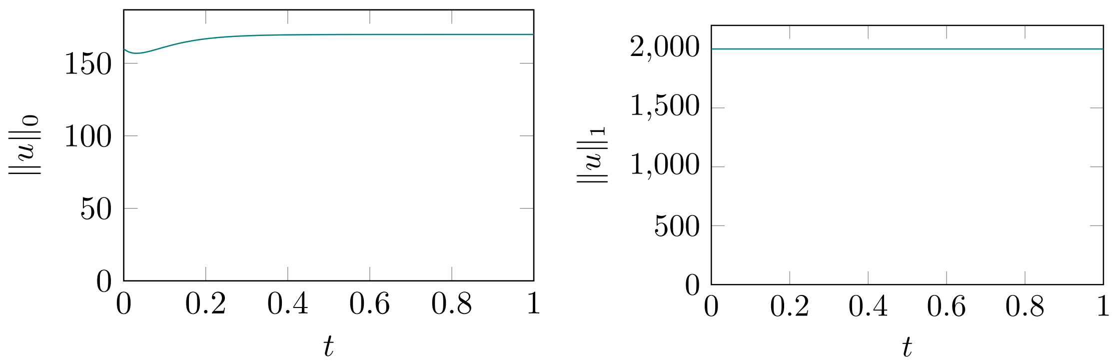

Figure 4.2: Evolution of the pure fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8a) and the fragmentation kernel (4.4.7a): the total number of particles (left); the total mass (right).

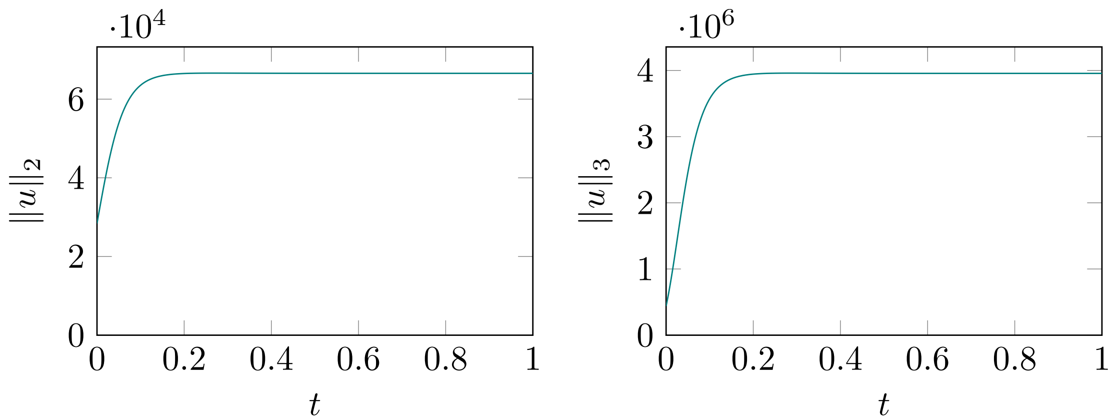

Figure 4.3: Evolution of the pure fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8a) and the fragmentation kernel (4.4.7a): higher order moments.

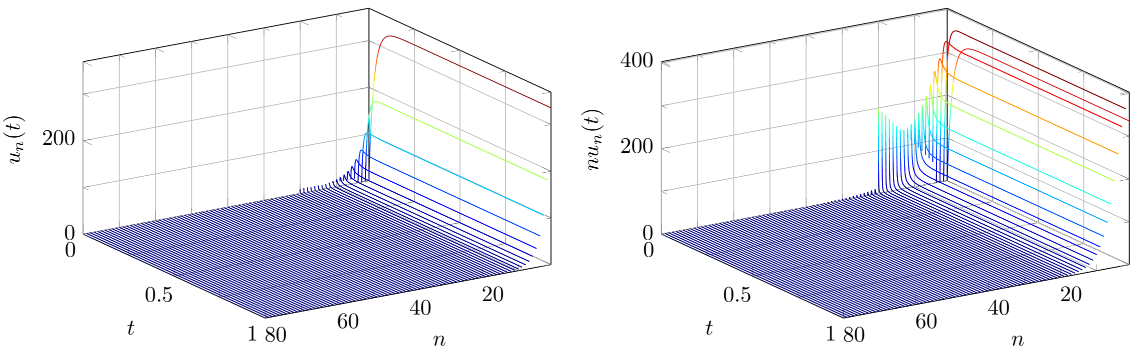

Figure 4.4: Evolution of the pure fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8b) and the fragmentation kernel (4.4.7b): number of clusters $u_n(t)$ (left); distribution of cluster masses $nu_n(t)$ (right).

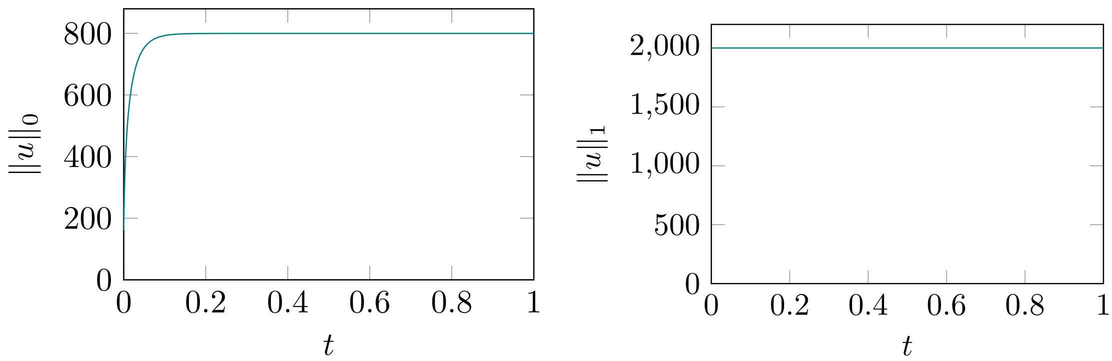

Figure 4.5: Evolution of the pure fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8b) and the fragmentation kernel (4.4.7b): the total number of particles (left); the total mass (right).

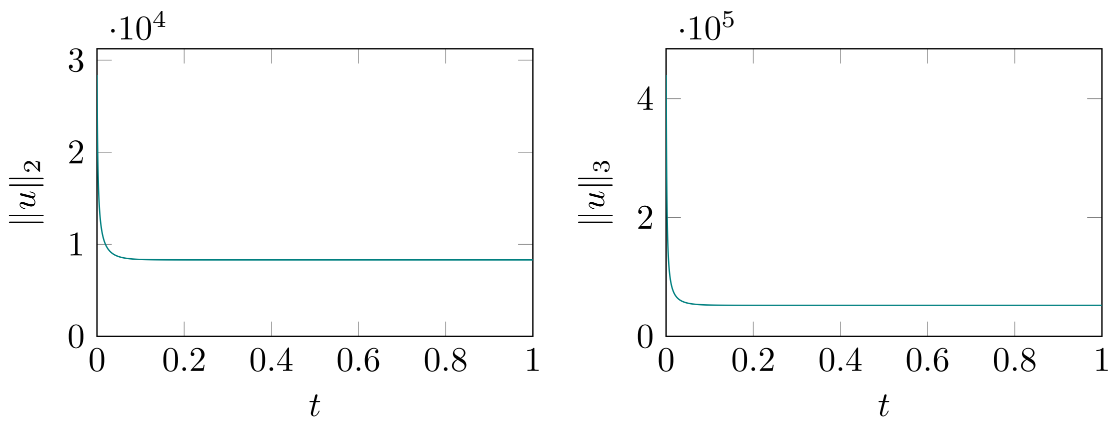

Figure 4.6: Evolution of the pure fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8b) and the fragmentation kernel (4.4.7b): the higher order moments.

satisfied. Nevertheless, the example falls in the scope of Theorem 4.3.7 and, as predicted by the theory, the numerical solution demonstrates qualitative features similar to those observed in Example 1, see Figure 4.4, 4.5 and 4.6. The total mass is preserved (i.e. no shattering and/or gelation occur) and after a short transition stage the numerical trajectory settles near a stationary particles/mass distribution.

4.4.2.2 The growth-decay-sedimentation-fragmentation-coagulation scenario

Example 3. We consider the complete model (4.1.1), with $g = d = s = a = 1$, $\beta = \gamma = 0$ and $\alpha = \delta = 1$. The fragmentation and the coagulation processes are controlled respectively by the kernels (4.4.7a) and (4.4.8a), with $k_1 = 5 \cdot 10^{-3}$. The truncation index N, the time interval $[0, T]$ and the initial condition u_0 are chosen to be the same as in Examples 1 and 2.

As demonstrated by Figures 4.7, 4.8 and 4.9 in the presence of the transport processes the qualitative dynamics of the model (4.1.1) changes (compare Figures 4.7, 4.8 and 4.9 with the figures in examples 1 and 2. The death and the sedimentation processes dominate and yield a slow decay in each of the moments $\|u\|_p$, $p = 0, 1, 2, 3$ as time increases.

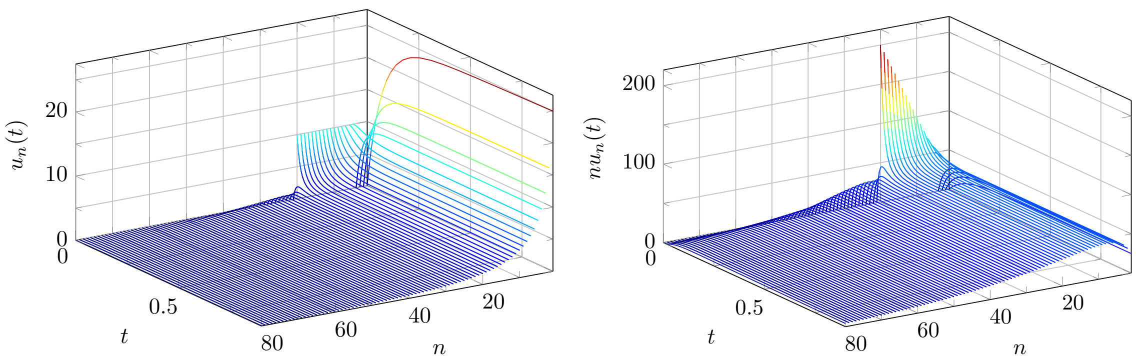

Figure 4.7: Evolution of the growth-decay-fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8a) and the fragmentation kernel (4.4.7a): number of clusters $u_n(t)$ (left); distribution of cluster masses $nu_n(t)$ (right).

Example 4. To provide a further illustration of the effect of transport processes on the dynamics of (4.1.1), we repeat the computations but with the fragmentation and the coagulation kernels from Example 2. To ensure global solvability of the model, we let $g = d = s = a = 1$, $\beta = \gamma = 0$ and $\alpha = \delta = 2.5$.

With these settings, the birth and the fragmentation terms dominate and we expect the total mass of the ensemble to grow. As shown in Figures 4.10, 4.11 and 4.12, this is indeed the case for t close to zero. However, as time goes on, the contributions of the growth and the decay/sedimentation processes compensate each other and the numerical solution settles near an equilibrium state.

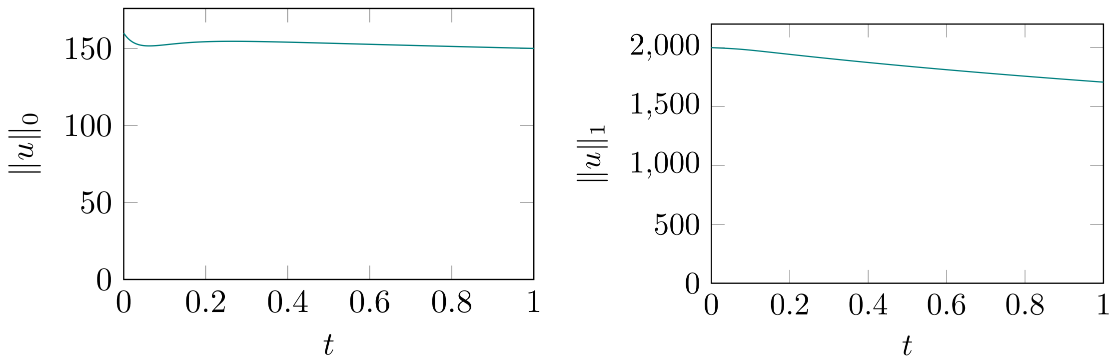

Figure 4.8: Evolution of the growth-decay-fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8a) and the fragmentation kernel (4.4.7a): the total number of particles (left); the total mass (right).

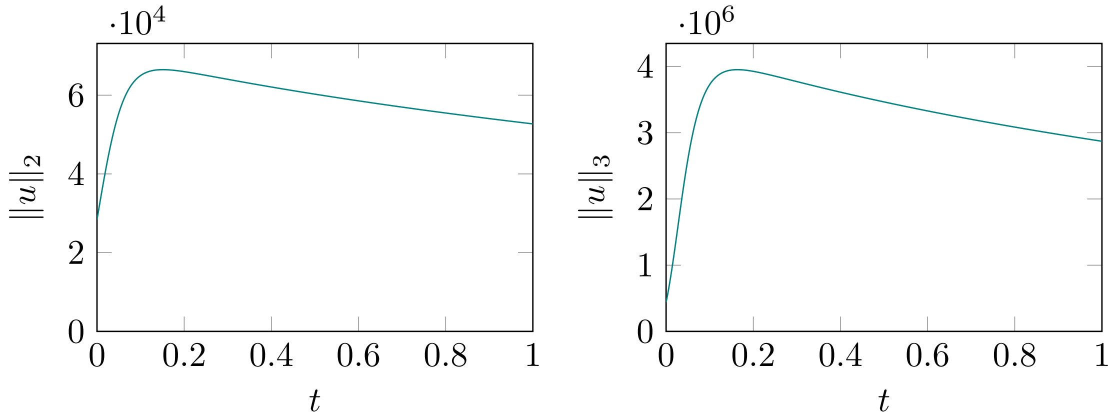

Figure 4.9: Evolution of the growth-decay-fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8a) and the fragmentation kernel (4.4.7a): higher order moments.

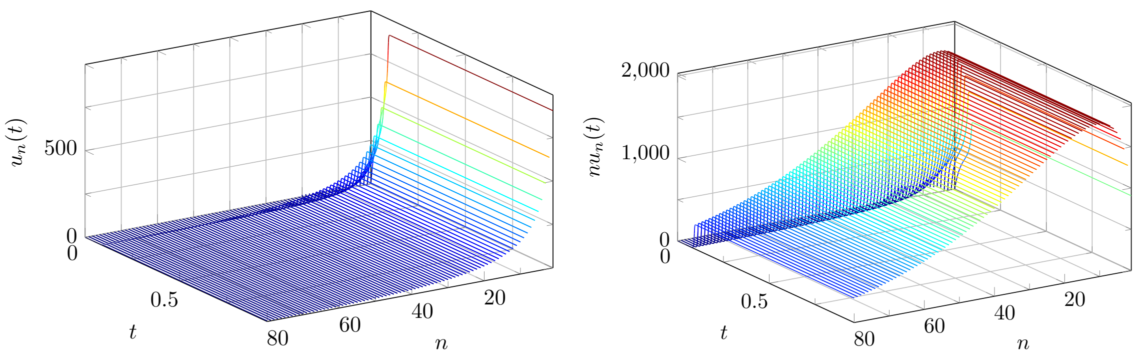

Figure 4.10: Evolution of the growth-decay-fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8b) and the fragmentation kernel (4.4.7b): number of clusters $u_n(t)$ (left); distribution of cluster masses $nu_n(t)$ (right).

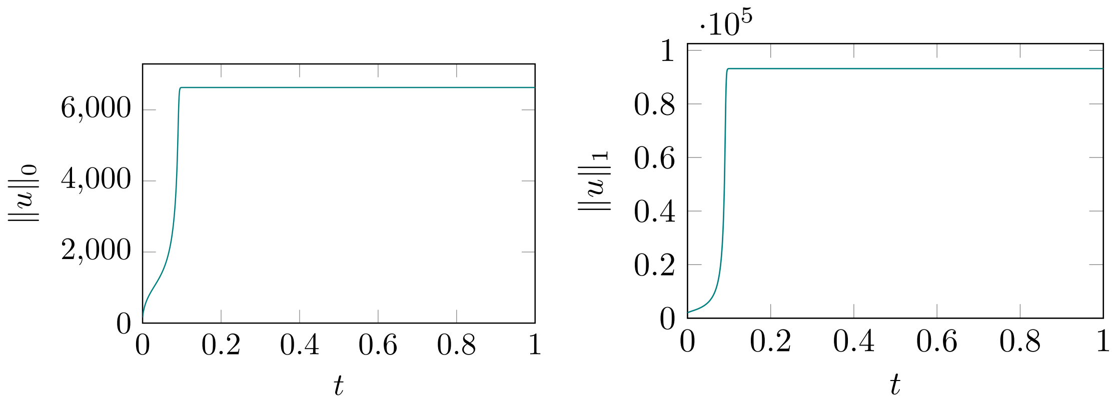

Figure 4.11: Evolution of the growth-decay-fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8b) and the fragmentation kernel (4.4.7b): the total number of particles (left); the total mass (right).

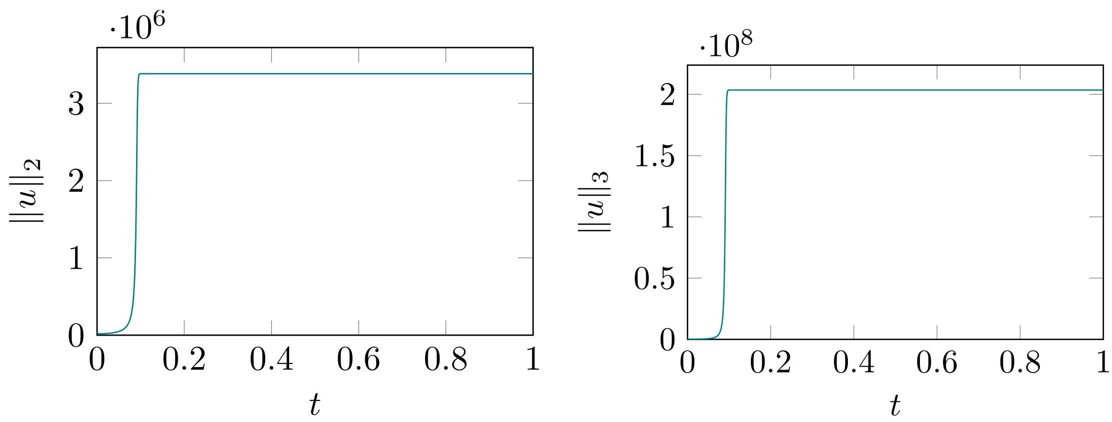

Figure 4.12: Evolution of the growth-decay-fragmentation-coagulation model (4.1.1) with the coagulation kernel (4.4.8b) and the fragmentation kernel (4.4.7b): higher order moments.

The example demonstrates a certain degree of flexibility of model (4.1.1). A proper interplay between the fragmentation and the transport components of the equation allows for simulation of a wide range of realistic scenarios arising within coupled transport-fragmentation-coagulation systems.

4.4.2.3 The Strong Sedimentation Case

Our last two examples demonstrate behaviour of (4.1.1) with sufficently strong sedimentation. In this settings, the model is globally well posed in X_1, provided (3.3.11) and (4.3.1) are satisfied.

Example 5. We let $g = 1$, $d = s = a = 1$, $\alpha = \beta = \delta = 1$ and $\gamma = 2$. The fragmentation and the coagulation kernels and all other parameters are the same as in Example 1.
The results of simulations are shown in Figures 4.13 and 4.13. The strong sedimentation (see

condition (3.3.11)) describing the death of clusters, prevents uncontrolled mass absorption by the clusters of large sizes. The right diagram in Figure 4.13 demonstrates the bulk mass of the system remains concentrated in clusters of moderate size. As time goes on, both processes lead to a steady decay in the total mass of the system.

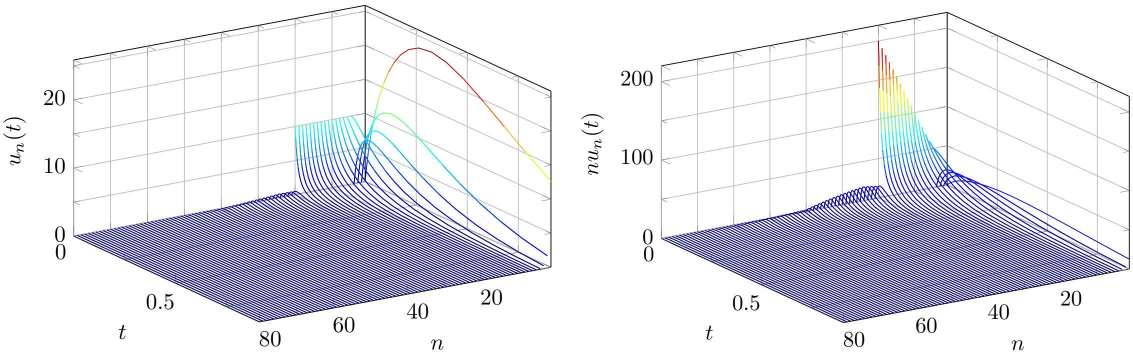

Figure 4.13: Evolution of the growth-decay-sedimentation-fragmentation-coagulation model (4.1.1) in X_1 with the coagulation kernel (4.4.8a) and the fragmentation kernel (4.4.7a): number of clusters $u_n(t)$ (left); distribution of cluster masses $nu_n(t)$ (right).

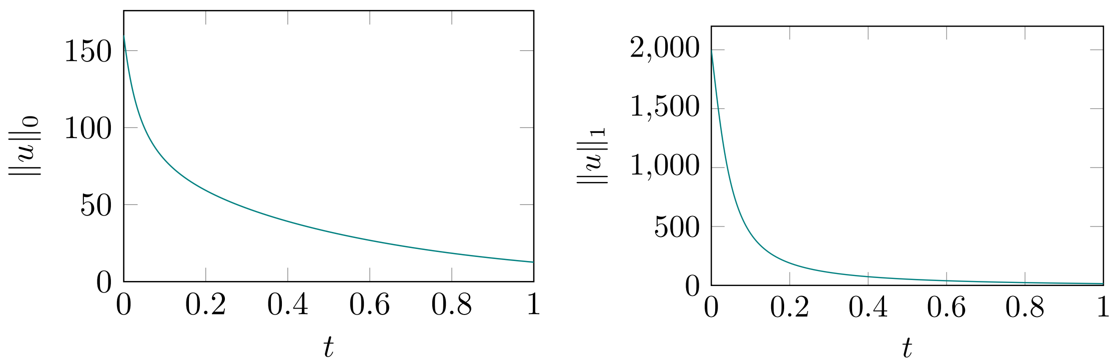

Figure 4.14: Evolution of the growth-decay-sedimentation-fragmentation-coagulation model (4.1.1) in X_1 with the coagulation kernel (4.4.8a) and the fragmentation kernel (4.4.7a): the total number of particles (left) and the total mass (right).

Example 6. In our last example, we make use of the fragmentation and the coagulation kernels from Examples 2 and 4. Further, we set $g = 1$, $d = s = a = 1$, $\alpha = \beta = \delta = 1$ and $\gamma = 2$.
As mention earlier, the growth rate of the quantities $k_{i,j}$ is superlinear and one expects gelation in context of pure coagulation models. Nevertheless, in complete agreement with the theory, the simulations show (see the evolution of the clusters masses in the right diagram of Figure 4.15) that in the presence of a sufficiently strong decay-sedimentation process the latter scenario is impossible, and the solution remains bounded in X_1 settings (see the right diagram in Figure 4.16). It is worth

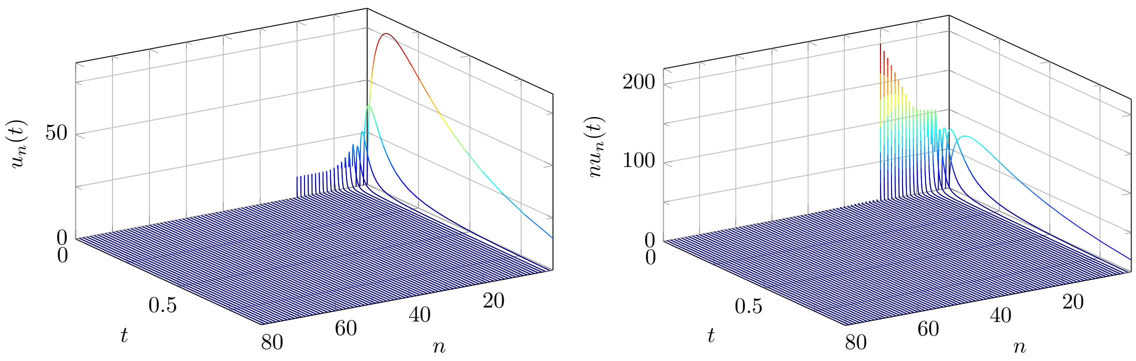

Figure 4.15: Evolution of the growth-decay-sedimentation-fragmentation-coagulation model (4.1.1) in X_1 with the coagulation kernel (4.4.8b) and the fragmentation kernel (4.4.7b): number of clusters $u_n(t)$ (left); distribution of cluster masses $nu_n(t)$ (right).

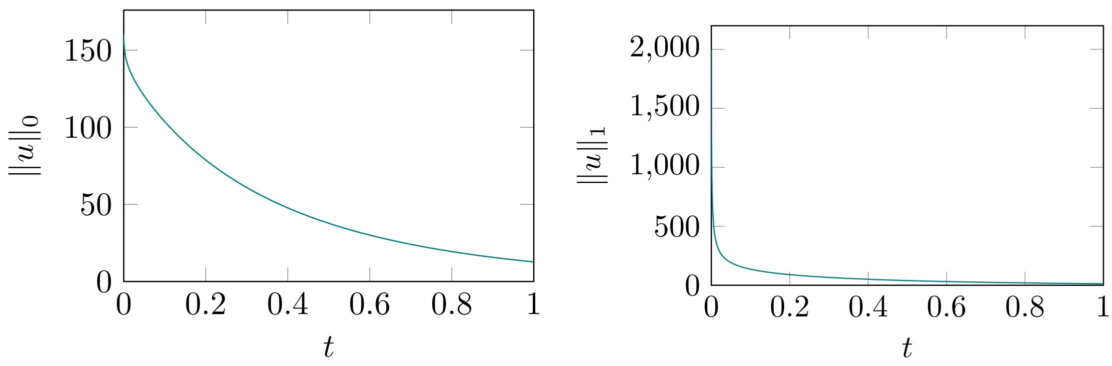

Figure 4.16: Evolution of the growth-decay-sedimentation-fragmentation-coagulation model (4.1.1) in X_1 with the coagulation kernel (4.4.8b) and the fragmentation kernel (4.4.7b): the total number of particles (left) and the total mass (right).

mentioning that in this example the mechanism preventing gelation is connected with the strong sedimentation, in contrast to Examples 2 and 4, where the central role is played by the strong fragmentation.

Chapter 5

Conclusion

We studied the discrete fragmentation-coagulation equation with growth, decay and sedimentation. Most papers on this type of equation deals with the continuous case. However, the discrete case has some important properties, as we have demonstrated in our study.

We considered the discrete coagulation–fragmentation models with growth, decay and sedimentation. We proved that for a large class of growth-decay-sedimentation-fragmentation problems the solution semigroup is analytic and compact and thus has the Asynchronous Exponential Growth property. In Corollary 3.3.3, we demonstrated that for a strong sedimentation term, s_i, $i \geq 1$, the generation results extends to the case when $p = 1$. We noted that a calculation procedure similar to the ones for general model in X_p could be repeated for the fragmentation equation with growth and decay only; that is, when the sedimentation term is zero, $s_i = 0$. An alternative view of the fragmentation equation with growth and decay only was given in Section 3.5. The chapter was concluded with some examples of this equation with some numerically illustrations.

In Chapter 4, we looked at the fragmentation-coagulation equation with growth, decay and sedimentation. That is, a nonlinear coagulation term is added to the linear part of Chapter 3. Using the theory of interpolation spaces and under some condition on the coagulation term, we prove the existence of the local and global classical solution to the semilinear equation. The analysis presented in Section 4.3 shows that, irrespective of the coagulation rates, the model is always globally well posed, provided the fragmentation (in the case of $p > 1$), or the sedimentation (for $p = 1$) dominate. This is in contrast to pure coagulation models, see e.g. [70] but confirms earlier results obtained in a more restricted setting in the discrete, [7, 28], and continuous, [39], cases. Theoretical conclusions are completely supported by the numerical simulations presented in Section 4.4.

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