AN EXPLORATION OF THE LEARNERS' COMMOGNITION WHEN ENGAGING WITH GEOMETRY IN THE GRADE ELEVEN CLASSROOM

By

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ABSTRACT

This study investigates the learner's commognition when engaging with geometry in a grade 11 class. The theory of commognition, propounded by Anna Sfard, underpinned this study with particular focus on the four characteristics of mathematical discourse; namely, specific purpose word use, visual mediators, substantiated narratives, and routines.

The phenomenological qualitative research design approach was utilized to investigate 3 grade 11 learners' commognition. The study was conducted at a high school in the province of KwaZulu-Natal of the republic of South Africa. Data was collected by means of a written mathematics activity that required participants to use straightedge and compass to construct a circle, followed by proving one Euclidean proof. The participants then participated in a semi focused group interview based on the written task activity. Data was analysed by means of two analytical tools; the Discourse Profile of the Tangent adapted from Mpofu, and Pournara and others, was used to examine the interview transcripts. The Realization Tree Assessment, initiated by Weingarden with others, and adapted by Haghjoo and his colleagues was used to investigate the written task activity. The examination of the results revealed that the participants word use was problematic and affected their level of objectification. Therefore, it was concluded that the participants discourse was ritualistic.

These results suggest that attending to the degree of objectification in the initial stages of transforming ritual to explorations is cardinal before embarking on the process of deritualisation.

DECLARATION

I declare that this research report is my own work, except as suggested in the acknowledgements, the text and references. All phrases, sentences and paragraphs taken directly from other works have been cited and the reference recorded in full in the reference list. The research report is submitted to the Faculty of Humanities in fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics Education at the university of KwaZulu-Natal. The research report has not been submitted before for any degree or examination purpose at any other higher education institution.

Signature: Date: 2024/06/20

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ABBREVIATIONS AND ACRONYMS IN THE STUDY

CAPS	Curriculum and Assessment Policy Statement
FET	Further Education and training
DBE	Department of Basic Education
ECZ	Examination Council of Zambia
NSC	National Senior Certificate
ZPD	Zone of Proximal development
ADP	Arithmetic Discourse Profile
DPH	Discourse Profile of the Hyperbola

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CHAPTER 1

INTRODUCTION

1.1 Background and overview of the study

Mathematics is one of the important subjects taught in schools (primary and secondary phases) and is a compulsory subject in most of the school curricula, the Curriculum and Assessment Policy Statement in South Africa (CAPS, DBE, 2011), the Examination Council of Zambia (ECZ,2016). In the 21st century, data and statistics underpin every job and as such, the need for analytical skills is more critical than ever before.

Mathematics has been defined in different ways. However, according to the Curriculum and Assessment Policy Statement (CAPS, DBE, 2011 p. 8) of South Africa, mathematics is defined as follows:

"Mathematics is a language that uses symbols and notations to describe numerical, geometric, and graphical relationships. It is a human activity that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomena and between mathematical objects. It helps to develop mental processes that enhance logical and critical thinking, accuracy, and problem-solving, which will contribute to decision-making. Mathematical problem-solving enables us to

understand the world (physical, social and economic) around us, and most of all to teach us to think creatively"

From this definition, it can be deduced that the core activity in mathematics learning is about problems and solutions. And to emphasise this critical activity in mathematics, Halmos (1980) says, *"mathematicians' main reason for existence is to solve problems"* (p. 519).

Problem-solving is a critical component of mathematics. A number of countries in their school curricular, emphasise problem-solving as a required life skill in society, as noted in the South African mathematics school curriculum (CAPS,2011) and the Zambian mathematics school curriculum (ECZ,2016).

1.2 Why mathematics is important

Mathematics is among the most critical subjects in the school learning curriculum worldwide. From South Africa, examining the CAPS document, one can assert that mathematics, in terms of rank, is second to the critical discipline of language and is given unquestionable importance regarding time allocation. At FET and senior phase levels, mathematics is allocated 4.5 hours together with languages, but the home language has 5 hours in the senior phase. Mathematics is essential as it is needed in most careers. Human proficiency in many spheres of daily life is needed to enhance society. In this respect, mathematics is essential in developing such needed human proficiency. According to van de Walle (2013) and his colleagues, mathematics develops conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

1.3 Achievement in mathematics

Mathematics is essential and is a prerequisite in many careers. However, many students continue to perform poorly in the subject in many countries, such as in Tanzania (Kyaruzi, 2023; Mazana et al., 2020) in the Philippines (Capate & Lapinid, 2015), and in South Africa (Graham et al., 2021).

In South Africa, students have consistently performed poorly in mathematics compared to other school subjects at the matric level (grade 12), as shown in Table (1). This perspective reflects the student's performance in mathematics covering a period ranging from 2019 to 2022 is shown in the table. Table 1 shows the comparison of mathematics achievement

against other school subjects at matric level over a period of four years (2019, 2020, 2021, 2022).

Table 1.1: Comparison of performance in gateway subjects (2019 to 2022).

Source: National Senior Certificate (NSC) Examination Report 2022 (DBE, 2022, p. 8).

Subject Description	2019	2020	2021	2022
Accounting	78,4%	75,5%	74,7%	75,4%
Agricultural Sciences	74,6%	72,7%	75,4%	75,8%
Business Studies	71,0%	77,9%	80,5%	76,7%
Economics	69,3%	68,8%	67,9%	71,5%
Geography	80,5%	75,3%	74,3%	81,3%
History	90,0%	92,1%	89,5%	88,2%
Life Sciences	72,3%	71,0%	71,5%	71,5%
Mathematical Literacy	80,6%	80,8%	74,5%	85,7%
Mathematics	54,6%	53,8%	57,6%	55,0%
Physical Sciences	75,5%	65,8%	69,0%	74,6%

Regarding some country's average in terms of mathematics achievement, the International Mathematics and Science Study (TIMSS, 2019) report placed the top two achieving countries in rank as Singapore at (625) and Hong Kong at (605). The bottom five countries that were ranked achievers are Morocco at (383), Kuwait at (383), South Africa at (374), Pakistan at (328) and the Philippines at (297). The information from the TIMMS (2019) report highlights the general status of the teaching and learning of mathematics in many countries, including South Africa.

1.4 Students' attitude towards mathematics

Since many students find mathematics difficult (Angateeah, 2017; Sholihah & Afriansyah, 2018), many students tend to develop a poor attitude towards the subject. Numerous studies have shown a close relationship between attitude and achievement in mathematics

(Steinkamp, 1982; Kloosterman, 1991; Minato & Yanase, 1984). Attitude has been singled out as a predominant predictor concerning achievement in mathematics (Callaman & Itaas, 2020). According to the TIMSS report (2019), there is a positive correlation between achievement and confidence levels. This report stated that 41% of 8th-grade students dislike mathematics, while 20% of 4 graders do not like the subject. The report stated that 44% of 8th graders do not feel confident, while in the 4th grade, 23% did not feel confident. However, regarding valuing mathematics, it was reported that 84% of 8th graders value the subject. A similar study at Stanford University revealed that a positive attitude towards mathematics enhances the brain's memory and better mathematical performance (Chen et al., 2018).

1.5 Geometry as one of the branches of mathematics

Geometry is one of the critical branches of the mathematics syllabus globally, and it is taught in all school grades (i.e., from kindergarten to high school). Geometric shapes are all around us in the environment, and people encounter these shapes constantly in daily life. In geometry, students learn about shapes and space (Gokbulut & Ubuz, 2013). According to Bishop (1985), the main focus of geometry is space and its mathematical interpretation. The geometric content and goals of the school curriculum comprise shapes and their properties (in two and three dimensions), transformations (translations, reflections, rotations, enlargements/reductions), location (coordinate geometry), visualisation (recognition of shapes from different viewpoints), (van de Walle et al., 2013).

1.6 Why is geometry important?

As a branch of mathematics, Geometry plays a significant role in advancing human life. Geometric skills are a prerequisite in most careers. Careers that require these skills include, among others, engineering, architectural design, and construction work, (Abudullah & Zakaria, 2013a; Georg, 2012; Gronmo et al., 2016; Alex & Mammen, 2018). For this reason,

Clement and Sarama (2011a) assert that geometry is more relevant than other areas of mathematics. Given its importance in society, in most mathematics syllabi, geometry continues to receive more attention and emphasis than other areas of mathematics, especially at senior levels of schooling. For example, in the CAPS document (South Africa), geometry in the FET phase comprises about 60% of the questions in the matric (grade 12) mathematics paper and two final examinations.

1.6.1 Spatial reasoning

Geometry has the ability to enhance spatial reasoning. According to Sarama & Clements (2009), understanding shapes and the relationships between shapes is critical to mathematical study, just like numbers are. The development of spatial reasoning is connected to the comprehending of three interwoven properties, viz; (1) an awareness of space itself and in terms of distance and dimensions, (2) the representation of spatial information in terms of graphics such as diagrams and maps, and (3) the reasoning involved in the interpretation and manipulation of spatial information for problem-solving and decision making (Carroll, 1993; National et al., 2006). Spatial reasoning is an important skill needed in the 21st century. Science, Technology, Engineering, and Mathematics, also referred to as STEM subjects, are highly valued school subjects. Developing high-order thinking and student performance in STEM subjects has been linked to spatial reasoning (Lean & Clements, 1981; Lowrie & Diezmann, 2007).

However, despite the importance of geometry and its practical relevance in people's daily lives, teaching and learning this area of mathematics is often inadequate and contributes to low learning outcomes (Risma et al., 2013; Susilwati et al., 2017). Much concern regarding the difficulty of teaching and learning geometry has been raised by scholars such as Harel and Sowder (2007), Clements and Battista (1992), Clements (2003), Battista (1999), and Yi,

Flores and Wang (2020). These concerns include students' inabilities regarding abstract thinking, failure to analyse properties of geometric shapes (Musdi et al., 2020), students' geometric thinking being at lower levels than expected (Md Yunus et al., 2019; Tan et al., 2015), and the limited knowledge of basic geometry of some foundation phase teachers (Luneta, 2014).

1.7 Some of the theories of teaching and learning mathematics

The need to improve the teaching and learning of geometry has given rise to various theories and models that seek to develop insight into how students construct their meaning of geometry concepts. Some of the theories and models are van Hiele's theory, Fischbein's notion of figural concepts, Duval's assumption of figural apprehension, Wessels and van Niekerk's Spatial Operational Capacity model, and Sfard's commognition theory (Sharma, 2019). The van Hiele (1985) model of geometric thinking has been widely used among these theories and models. The van Hiele (1985) model is divided into 5 levels of geometric thinking. The first 4 are basic levels these are level 1: visualisation, the object of thought are shapes and what they "look like"; level 2: analysis, the object of thought are classes of shapes rather than individual shapes; level 3: informal deduction, the object of thought are the properties of shapes, level 4: deduction, the object of thought are relationships between properties of geometric objects (Bergstrom & Zhang, 2016; Swoboda & Vighi, 2016). The last level is advanced, level 5: rigour, the object of thought is deductive axiomatic systems for geometry (van de Walle et al., 2013). van Hiele proposes that these learning phases assist students to navigate from one thought level to the next (Sharma, 2019).

1.8 The research problem

> *In this changing world, those who understand and can-do mathematics will have significantly enhanced opportunities and options for shaping their futures. Mathematical*

competence opens doors to productive futures. A lack of mathematical competence keeps those doors closed…. All students have the opportunity and the support necessary to learn significant mathematics with depth and understanding. (*NCTM 2000, p.* 50).

Mathematics is one of the most important subjects that students learn in schools. Mathematics is taught in primary school and university. Mathematics (geometry) develops thinking skills needed in higher-order thinking (Sarama, 2011b). Thinking skills such as spatial reasoning and problem-solving are vital in engineering or architecture (Abdullah & Zakaria, 2013a; Gorg, 2012; Gronmo et al., 2013). Science, Technology, Engineering, and Mathematics (STEM) subjects significantly benefit from spatial reasoning (Lean & Clements, 1988; Lowrie & Diezmann, 2007).

One condition that prevents enhancing spatial sense and reasoning is attributed to low level of thinking skills. Studies have found that most students are at level one and level two of geometric thinking of van Hiele levels (Alex & Mammen, 2011a; Alebe & Schafer, 2008; Md et al., 2019; Masiru et al., 2019).

Geometry is one of the main components of mathematics in the mathematics school curriculum. In South Africa, geometry accounts for about 60% of the marks out of 150 on paper two at the matric level. This large examinable percentage is a testament to its importance in the school curriculum. Several studies have been undertaken, and the knowledge gained from research has been applied to illuminate the content of school mathematics as well as ways of teaching mathematics (de Walle et al., 2013). However, despite all these meaningful strides in this area of teaching mathematics and geometry in particular, students still find mathematics difficult (Angateeah, 2017; Sholihah & Afriansyah, 2017), and the majority of students consistently perform poorly in the subject (Risma, Putri,

& Hartono, Y. (2013); Susilawati, Suryadi, & Dahlan (2017); Kyaruzi, 2023; Mazana, Montero, & Casmir, (2020).

One way to improve underachievement in mathematics, particularly in geometry, is to conceptualise teaching and learning as communication (thinking). Human developments are driven by effective communication. Learners in today's classroom come from diverse backgrounds and different experiences. Because of these unique experiences, communication failure is bound to exist in the classroom whenever they engage in particular discourses. This calls for the shift to conceptualising teaching and learning as communication and its significance in learning. The following are what the teacher who is the expert discursant needs to prioritise while in the teaching and learning situation:

- Teaching 1. Focus on the discourse -; Are there signs of communication gaps in the classroom?
- Teaching 2. Focus on the nature of the gaps -; What are the sources of these communication gaps?
- Teaching 3. Focus on closing or reducing the gaps -; How can I turn the existence of these gaps into opportunities to learn?
- Teaching 4. Focus on opportunities-; What pedagogical moves do opportunities to learn to invite? (Sfard,2021)

1.9 Research objectives

Isam and Samsadin (2020) postulate that the research's significance is finding answers to questions by applying scientific procedures. This implies that research work cannot be carried out without objectives, which is equivalent to an aimless study. Each scientific inquiry holds

particular importance and is underpinned by its objectives. This study was underpinned by three pre-determined objectives. The three objectives are:

1. *To explore the types of discourses learners engage with when learning geometry.*

Geometry, as one of the learning areas of mathematics, is important since geometric shapes are all around us (Van Walle, 2013). Exploring the types of discourse, learners engage with through the commognitive components would provide insight into the spatial world the learner lives in.

2. *To explore how these discourses influence the way learners understand geometric concepts.*

Exploring how these discourses influence the way learners understand concepts would provide a window into their commognition.

3. *To explore how these discourses may be influenced to improve the learning and understanding of geometric concepts.*

Exploring how these discourses may be influenced would enable the educator to be aware of significant teaching and learning activities that would foster the structuring of the learner's spatial world.

1.10 Research questions

Research questions are essential to all scientific inquiry in that they guide the study. Alvesson and Sandberg (2013) argue that posing research questions is fundamental to constructing our knowledge regarding a particular subject. Research questions are a researcher's construct and define what the study is about. This research project had three main questions that were informed by the researcher's own curiosity and underpinned by Sfard's (2008) commognitive theory together with the study approach.

The questions that underpinned this study were:

1. What discourses do learners engage in when learning geometry?
2. How do these discourses influence the way they understand geometric concepts?
3. How can these discourses be influenced to improve the understanding of geometric concepts?

1.11 Significance of the study

The importance of mathematics in society is well documented as it is a discipline of several different aspects (Steen, 1978; Wan, 1989). Sfard (2015) states that mathematics is a gatekeeper to many coveted jobs and, thus, a key to social mobility. However, many learners perceive mathematics concepts as complex and perform poorly in this valued subject (Adler, 2011; Seng, 2010; Sfard & Cobb, 2014). Table 1 illustrates students' mathematics performance at the matric (grade 12) level compared to nine other subjects over four years. For example, in 2019, performance in mathematics was 54.6%; in 2020, it was 53.8%; in 2021, reflected 57.6%; and in 2022, it showed 55.0%. On the other hand, over the same period, the performance in mathematical literacy in 2019 was 80.6%, 2020 was 80.8%, 2021 was 74.5% and so on. Compared to the other nine subjects, the performance in mathematics was the lowest for the four-year period under review. The trend of poor performance in mathematics, as illustrated in Table 1, was my primary motivation to undertake this study, as I desire to improve my performance in this core subject and, which would ultimately lead to a better attitude of the learners towards mathematics. I purposely focused on grade eleven since the significant section of Euclidean geometry is covered in this grade.

One of the goals of education research is to seek ways to improve practice, keeping in line with the NCTM teaching principle that "effective *teaching requires continually seeking improvement"* (NCTM, 2000, p. 19), and this study will, therefore, improve my own practice.

This study focused on the student's mathematical thinking and reasoning as it was about the learners' discourse when engaging with geometry. The study highlighted the role of rituals in the teaching and learning of mathematics. The findings of this study resonate with Larvie and Sfard (2019) when they say, *"Many of the routines taught in school are learned as rituals"* (p. 22). My study underscores the call to regard rituals as an integral part of the initial stages of the learning process and offers suggestions to teachers about appropriate pedagogical moves in the learning environment that may induce, as well as spur, the de-ritualization process.

The study will improve my practice as I have learned about the differences in learning caused by the difference between object-level and metalevel learning and how word use permeates these two levels of learning. Armed with this new knowledge, I am in a position to be fully aware of the vital role and contingency of metalevel learning, the nature of the desired change in learning and how this desired change takes place. The new insight I have gained in this regard is valuable and necessary to improve my practice. Furthermore, educators teaching other subjects would benefit from the findings of this study and the suggested implications to practice since they apply to diverse academic disciplines. Another significant factor of the study is that it will guide me when I undertake future investigations. The study will also contribute to the growing volume of literature on the process of de-ritualization in the teaching and learning of mathematics. McMillan (2010) postulates that exploratory studies add to the literature by building rich descriptions of complex situations and directing future research.

1.12 Definition of terms

Here are some of the keywords used in this study. These words are defined to convey meaning and clarity to the reader.

Acquisitionist: Refers to people who regard the metaphor of learning as acquiring knowledge.

Commognitive conflict: Refers to a situation when incommensurable discourses come together.

Commognition: Refers to thinking as an act of thinking.

Discursant: Refers to a person who is a participant in a discourse.

Incommensurable: Refers to when two discourses seem vary with respect to word use or with respect to routines of enacting as well as endorsing narratives.

Learner: Refers to a person who is learning a subject or skill.

Learning: Refers to the perceived change in discourse.

Mathematical discourse: Refers to all the communication of mathematical concepts between members of the mathematics community.

Matric: Refers to the final year of secondary school.

Participationist: Refers to people that regard a metaphor of learning as concerning in advancing the improvement of human participation in activities.

1.13 Structure of the thesis

This study report comprises of seven chapters.

In chapter 1, I offer the introduction. This chapter opens with a discussion on the background of the study. Next is the presentation of the research problem, objectives/aims and the research questions guiding this study. The chapter ends with a discussion of the significance of the study and the definition of terms used in the study.

In chapter 2, I offer a review relevant literature concerning the discourses students engage in when learning mathematics, the challenges of learning and teaching geometry, and instructional strategies to mitigate some of these challenges.

In chapter 3, I present the theoretical framework that underpinned the study, Sfard’s theory of commognition (Sfard, 2008). This theory underscores a sociocultural and discursive perspective. I present the main tenets of the theory and its constituents and then I proceed and discusses the concepts relevant to my study.

In chapter 4, I give details of the methodology of the study. I first describe my research design which is the interpretive qualitative paradigm. I then in general terms discuss the context where my study took place, namely at a secondary school in Kwa Zulu-Natal, in South Africa. I then proceed and provide a detailed elaboration of the method, procedure and essence of employing them in the research is presented. The following areas are described: research instruments, population and sampling procedures, analysis method, informed consent, and quality assurance.

In chapter 5 has three sections. The first section deals with the analysis of the data. The second section is the presentation of data. The third section is the presentation of the findings.

In chapter 6 begins with an explanation of the interpretation of the findings and is then followed by a discussion of the findings.

In chapter 7 summarises some of the study's findings and proposes some recommendations to improve practice arising from some of these findings. The chapter also has some recommendations to inform future research and as well as some few sentences explaining the study’s limitations. Also included is a paragraph on the project’s contribution to mathematical knowledge.

Chapter 2

Literature Review

2.1 Introduction

Geometry is one of the topics that students learn in mathematics, and Euclidean geometry is a critical component of geometry. In Euclidean geometry, learners experience and work on proving some theorems of circles and other shapes. When engaged in geometry problem-solving activities, learners are required in most cases to provide reasons for their statements. This study's topic is learner's commognition when engaging with geometry. In other words, learner's communication (thinking).

The specific area of focus for this study is to interpret the learner's commognition when engaging with geometry and how their commognition may be influenced to enhance the understanding of geometry concepts.

This study was conducted in the grade 11 classroom and was underpinned by the qualitative methodological approach. The rationale for adopting a qualitative approach is to interpret the learner's commognition, and the learners had to engage with authentic geometry tasks in the natural classroom environment. Denzin and Lincoln (2000) adeptly say this about the role of the qualitative researcher: "This means that qualitative researchers study things in their natural settings, attempting to make sense of or interpret *phenomena in terms of meanings people bring to them"* (p. 3). Similarly, other commognitive researchers such as Sfard (2007), Nachlieli and Tabach (2019), Zayyadi et al., (2019), Viirman and Nardi (2019), Mpofu and Mudaly (2019), used the same approach and also conducted their studies in the classroom.

The review was organised under the following three main sections, and these are:

1. Identification of the type of mathematical discourse.

2. Factors that may contribute to the identified type of discourse
3. How the discourse may influence the learners' understanding of geometric concepts.

The sources that I included are those not older than 6 years. The rationale for focusing on sources over 6 years old was to draw lessons from recent studies. The lessons of interest were those about trends in methodological resources and trends in how the various elements of the commognitive framework were being employed.

A growing number of research studies have used the theory of commognition. However, from the literature I reviewed, the published scholarship trend appeared to lean more towards interpreting the learners' mathematical discourse in either ritualistic or exploration. Based on my review, few published scholarships have focused on transforming rituals to spur the development of explorative participation in mathematical discourse.

Sfard (2008) describes learning as change as discourse. Sfard further points out that the primary goal of learning mathematics is to explore mathematical objects. Considering the trends that I observed in the published scholarships, I wrote this review to interpret the learner's mathematical discourse and further review how their discourse may be influenced to spur the development of the discourse of mathematics from that of ritualistic participation to explorative participation.

One of the particular forms of human activity is communication. Communication plays a vital role, considering that it is through communication that human beings can cope with and navigate the environment. In this regard, Sfard (2008) points out, "*The development of a discourse is shaped by its interaction with other discourses".* (p. 293).

This study was conducted in South Africa, where geometry is one of the critical areas of the mathematics school curriculum. This area of mathematics is introduced in the early grades and is taught right up to the university level because it is needed in many careers.

2.2 The context of the teaching and learning of geometry in South Africa

Geometry is one of the core topics in the school mathematics curriculum. However, with the dawn of the democratic government in 1994, many reforms have been aimed at transforming and providing equity in the education sector. This has resulted in the implementation of several curriculum reforms, such as the Outcomes-Based Education curriculum, (OBE,1997), the National Curriculum Statement (NCS,2002), and The Curriculum and Assessment Polocy Statement (2013). Curriculum reforms have affected several areas of education, including that of instruction in mathematics and geometry. The enacted changes have affected both the teachers and learners. For the teachers, the impact has been more on their content and pedagogical knowledge (Luthuli, 1996; Jones & Tzekaki, 2016). The changes have particularly impacted the teaching of the section of Euclidean geometry. In the process of these curriculum reforms, this geometry section has been within and out of the school syllabus at different points (Tachie, 2020). The ramifications of this move have been that a particular cohort of graduate teachers did not learn Euclidean geometry, and yet, with the reintroduction of the section, these teachers are expected to teach it (Tachie, 2020).

In South Africa, the Curriculum and Assessment Policy Statement (DBECAPS,2011), the document that informs teaching and learning, specifies the following as one of the specific skills set of learning mathematics: To develop essential mathematics skill, the learner should "*Communicate appropriately by using descriptions with words, graphs, symbols, tables and diagrams"* (p. 8).

The participants in this study were grade 11 learners. The grade 11 Euclidean geometry content is focused on the following;

1. To investigate and prove theorems of the geometry of circles assuming results from earlier grades, together with one other result concerning tangents and radii of circles;

2. To solve circle geometry problems, provide reasons for statements when required.

The geometry and measurements content weighting in grade 11 is 50 ± 3, out of 150 marks in paper two. This translates to about 33.3% in percentage terms.

The van Hiele theory (1957) describes how learners learn geometry, and the theory postulates that grade 11 learners (in the context of South Africa) are at level 3.

Level 3. *Deduction (Formal deduction).* At this level, students can provide deductive geometric proofs. They can differentiate between necessary and sufficient conditions. They identify which properties are implied by others. They understand the role of definitions, theorems, axioms and proofs.

Looking at the grade 11 geometry content (South Africa) and the alignment with level 3 demonstrates that the van Hiele theory of geometric understanding was considered when designing the mathematics content of South African education.

2.3 Identification of the type of the learners' discourse

Nisa, and Lukito (2021) conducted a study investigating students' mathematical discourse in solving absolute value equations. They used a qualitative approach, and their aim was to describe and analyse the students' mathematical discourse when solving the absolute value. Ten grade 10 students were randomly selected for this study. The learners' discourses were analysed using the commognitive framework theory lens. The analysis was done at two levels. The level one analysis was on the discursive features of mathematical discourse, viz; word use, visual mediators, narratives, and routines. The level two analysis was concerned with classifying their discourses based on the mathematical features of ritualised and explorative discourses. The findings from the study revealed the following at level one analysis: the participants used a combination of mathematical and colloquial words, and the participants were able to use visual mediators. However, their narratives were not endorsed

and did not fit the mathematical theorem. The participants solved the absolute value equation in one way only. The participants solved the equation at level two analysis and produced a valid narrative. Their performance was not influenced by other authorities. Although most participants only made realisation trees with single branches, they could use mathematical definitions to solve the equations. These results suggest that the learners' discourse was ritualised even though some aspects of explorative discourse were also evident.

In another study, Roberts and Le Roux (2019) qualitatively investigated grade 8 and grade 9 learners thinking about linear equations from a commognitive perspective. They focused on the learners thinking as they talked about their solutions to linear equations and non-verbal communication during the interview. The study participants were eleven grade 8 and four grade 9 learners. They were selected from two high schools based on the marks from their written assessment. Data was collected and then analysed using the commognitive framework and its characteristic features of mathematical discourse, namely; word use, visual mediators, narratives, and routines. The findings of this study revealed the following: (1) although some learners used keywords in a phrase-driven way to produce phrases that are mathematically endorsable, most of the learners used colloquial words and talked about dis-objectified entities, (2) learners were able to produce endorsed narratives about positive and negative integers and, (3) learners used spatial arrangement and physical appearance as sources of narratives and performed routines for and with others. These findings indicated a ritualistic rather than explorative nature of discourse.

Similarly, Mpofu and Mudaly (2019) investigated learners' perspectives on the concept of the asymptotes of a hyperbola and exponential functions. In their study, they implemented some exploratory, descriptive and interpretive features into a qualitative design. The study focused on identifying the participant's discourses regarding their viewpoint with respect to the asymptote of the hyperbola and exponential functions. A sample of 112 grade 11 participants

from four secondary schools were recruited. They collected data from the participant's responses to the written mathematics test on hyperbola and exponential function and from focus group interviews. Data was then analysed using the commognitive framework and the Realisation Tree of a function, adapted from the Realisation Tree Assessment tool of Weingarten, Heyd-Metzuyaninm and Nachliel (2019). The study's findings demonstrated that although learners could generally recognise the asymptotes of functions from equations or formulas during teaching and learning, they failed to recognise the asymptotes when presented in an unfamiliar format. Other findings were that learners' use of mathematical words and symbols was not connected with the realisation trees of the mathematical objects, and learners' use of different visual mediators was unsatisfactory as they had different interpretations of each visual mediator. Based on these findings, Mudaly and his colleague concluded that the learner's routines were ritualised.

In another related study, Mpofu and Pournara (2018) investigated learner participation in the functions discourse. Their study focused on the learner's discourse on algebraic, graphic, and numeric representations of the hyperbola. The study participants comprised of five grade 10 learners sampled from one secondary school. The instruments employed for the study were task-problem activity and task-based interviews. Data was obtained by focusing on the consistency in how participants spoke about the various representations of the function. The commognitive framework theory lens and the Arithmetic Discourse Profile (ADP) adapted from Ben-Yahuda (1985) were used for this analysis. The study's findings revealed that, while the participants' graphs were generally correct, their word use was colloquial, and their representation of mediators and narratives was visual. These findings suggest the ritualised nature of their routines.

Also, Thoma, and Nardi (2018) investigated the presence of unresolved commognitive conflict in first-year university students. They aimed to explore the transition from school

mathematics to university mathematics by analysing end-of-year examination transcripts. Their focus was mainly on the manifestations of unresolved commognition conflict on task problems of the module on Sets, Numbers, Proofs, and Probability. The examination of the data revealed the presence of unresolved conflict with visual mediators on the following concepts: (1) the rules of school algebra and Set Theory discourses, (2) the rules of the Probability and Set Theory discourses, and (3) the rules of Probability Theory discourses. Other findings from the study include the identification of consistent work concerning the appropriate numerical context of examination tasks. They came to conclude that these findings were the cause of students' errors in the final examination.

Another researcher, Ioannou (2017), conducted a study investigating the discursive shift in the learning of Group Theory. The aim was to investigate the learning experiences of undergraduate mathematical students in a Group Theory course. The study was aimed at determining how the participants applied object-level and meta-level rules of mathematical concepts regarding group, subgroups, sets and their elements and the resultant commognitive conflicts. Data was collected from multiple sources that involved; (1) Lecturer notes and field notes, (2) Audio-recording from interviews with lecturers and with students, and (3) Student coursework, markers' comments, and student examination scripts. The commognitive framework theory lens was used to analyse the data regarding object rules concerning the elements of sets or subsets and meta-level rules concerning the proof of algebraic structure confirming a subgroup. The study's results revealed manifestations of commognitive conflict arising from applying object-level rules of the concept of group on sets without defining binary operation. Other revelations were on commognitive conflict on applying meta-level rules arising from the problematic notation of an empty set, non-empty set, or a subset. These findings suggested that in spite of some students having a structural knowledge of the required proof, the students did not possess the necessary procedural understanding for

implementing the proof. Therefore, the researcher concluded that the student's Group Theory discourse was ritualistic.

Lastly, Tasara (2017) studied the commognitive analysis of an educator's discourse on the derivative. The study aimed to describe the participant's mathematical discourse concerning the four commognitive constructs. These are word use, visual mediators, endorsed narratives and routines. The participant in the study was a teacher at a post-16 education college. Data was collected through an interview and observation of some of the lessons. The interview as well as the lesson were video recorded. The examination of the data focused on two commognitive constructs, viz; word use and endorsed narratives. The study's findings revealed (1) Contradictory narratives concerning the definition of the gradient and (2) inconsistent word use regarding the gradient at the object level inconsistent with literate mathematical discourse.

In the sources reviewed above, the participants of the studies range from learners in the secondary school phase to teachers, pre-service teachers and other undergraduates in tertiary learning. All the studies were conducted in classroom environments across different countries and provided a heterogeneous data corpus. These sources all claim the collective manifestations of rituals in the participant's mathematical discourse.

Azizah, Wulan, and Rahayu (2023) conducted an investigation on learners' commognitive processes when solving problems related to the polyhedron. Their study approach was qualitative in nature. The sample for the study were learners of class VIII (8). The focus of this study was to describe the commognitive components in the participant's discourse when solving problems of the polyhedron. Data in this investigation came from the interview transcripts and the participant's responses to a test task. The analysis of the data indicated that participants were more focused on the final results of their performance and ignored the

constituents of the discourse of mathematics namely, the use of words, the use of visual mediators, routines, as well as narratives, which are essential components of the problem-solving process. Based on these findings, Azizah and his colleagues made the following conclusions: (1) narratives play a critical role in understanding mathematical concepts, and (2) the students' success in solving problems is affected by the relationship between narratives and understanding of mathematical concepts.

Zayyadi, Nusantara, Subanji, Sulandra and Hidayanto (2019) investigated students' errors when solving problems by focusing on high-order thinking skills. They used a descriptive approach to show students' errors when solving questions that target problems with high-order thinking skills in trigonometry. The investigation was conducted online with 3 participants from a secondary school. Data was collected from a written test and a semi-structured interview. The commognitive framework was used to analyse the data, with the focus on the four attributes of the discourse of mathematics discourse, viz: the use of words, the use of visual mediators, the construction of narratives and the employment of routines. The results revealed that the participants experienced problems in (1) word use, where errors were made in writing mathematical symbols; (2) visual mediators, here errors in drawings or not using visual mediators; (3) narratives by not writing the formula but writing the solution directly and; (4) routines by inaccurate arithmetic operations and substituting values incorrectly in the formula or not knowing the strategy to use. The results suggested that students made procedural errors in building systematic hierarchical steps to answer a problem.

Fujita and Shinno (2022) conducted a study that characterised the development of the onset of how and when a way of proving in a mathematics classroom at primary school. They aimed to characterise a way of proving which can be produced and to explore the factors that lead to changes in the way of proving. A proving task was designed and given to grade 5

participants who formed the sample. They employed the commognitive framework theory to examine the data. The unit of analysis was the construction and substantiation associated with the discursive features of mathematical discourse, namely, word use, visual mediators, narratives, and routines during the proving process. The study's results showed that the interplay between construction and substantiation developed progressively.

Lavie et al, (2019) argue that although explorative participation in learning mathematics is the desired goal, rituals are unavoidable in the early stages of the process. However, there are several factors that enable rituals to last and not readily turn into explorations. These factors may be attributed to the socio-cultural context of mathematical instruction and its dominant discourses. Robert and Graven (2019) argue that attention should be given to how and why context matters concerning teaching practices and bi/multilingual learners. They highlighted the inherent constraints that educators and learners face in some contexts. When teaching in the bi/lingual context, educators and learners work in the second language, English. This view is supported by Adler (2019), who points to context and conditions that limit learning by restricting learning opportunities to ritualistic style.

Thoma and Nardi (2018) investigated discursive shifts from school to university mathematics and lecturer assessment practices. The study aimed to examine first-year mathematics students' participation in the university mathematics discourse and characterise the lecturers' assessment discourse on the choice of examination tasks and the wording of the tasks. The study focused on two aspects, viz: (1) students' errors regarding variables when engaging in a Number Theory task and (2) lecturers' ways of assisting the students to avoid the errors. Data was collected from the module on Number Theory in the examination paper and from interviews with the lecturers. The commognitive framework was employed, and the analytical lens was provided. The outcome of the study found the following: (1) Some the participants used symbols without providing information about the domain of the variable

and therefore leaving parts of the task unanswered, and (2) some the participants were able to work out but leaving the part of defining. These revelations show the differences between the two discourses: the secondary school discourse that does not focus on the routines of defining, and the lecturers' discourse that constrains the mathematical discourse at the university level.

In the next section, I reviewed sources that focused on some of these factors that perpetuate rituals in the context of mathematical instruction.

2.4 Factors that may contribute to the identified type of discourse

2.4.1 Language

The language used in the mathematics classroom plays a vital role in teaching and learning mathematical concepts and procedures (Le Fevre et al., 2010; Powell et al., 2017; Powell & Nelson, 2017).

This view is supported by several studies (Forsyth & Powell, 2017; Powell, Driver, Robert, & Fall, 2017; Purpura et al.; 2017; Purpura et al., 2019; Purpura & Reid, 2016) that pointed to the significance of mathematics vocabulary in students' mathematics performance.

Robertson and Graven (2019) conducted a study that explored mathematics talk in a second language from a sociolinguistic perspective. They aimed to identify literacy learning opportunities experienced by students when learning mathematics in a language that is not their mother tongue. The study determined how mathematics teachers' classroom talk facilitates or constrains mathematical meaning-making. The study participants were two grade 4 teachers in two different schools. The bilingual model of instruction was practised at these two schools. Data collected was from a range of classroom observations spanning a period of 4 weeks and from video-recorded interviews with the participants. The transcription of interviews was verbatim. Results from the analysis of observed classroom talk based on

Cummins' cognitive and contextual demands framework (1994) revealed that while the teachers’ questions were more open and probing, the students were unable to move from quadrant A (context-embedded, cognitively undemanding) to quadrant B (context embedded but more cognitively demanding). These results suggested that the students' language limitation hindered moving across the four quadrants of Cummins' cognitive and contextual demands framework (1994).

In a study related to Robertson and Graven, Riordian and Flanagan (2020) examined the use of languages by two bilingual undergraduate learners when learning mathematics. They sought to investigate whether language use by bilingual undergraduate mathematics learners impacts metal-level mathematical processes when learning function concepts. The commognitive theoretical framework lens was used to analyse the data. The study revealed that learners relied on their first language when engaging in meta-level discourse. Another finding was the impact of the choice of language used in the discourse when they worked in pairs on function problems. They concluded that learners’ language preferences for communicating mathematics learning are highly situated and impact the discursive processes of bilingual learners.

Azrou (2020) also investigated how learners in a multilingual context experience instruction in mathematics. In this context three different languages of expression were distinguished: Arabic, French, and a local dialect. The participants were teachers, and the study focused on their awareness of the differences between the three languages when teaching some essential concepts in mathematics and the resulting difficulties experienced by learners. Data was collected through interviews. The examination of data revealed the existence of a relationship between weak language proficiency and expanded teaching of procedural mathematics. Azrou (2020) concluded that these findings are related to the poor level of learners' mastery of the three languages.

In contrast, Neugebauer and Prediger (2023) investigated whether students with different language backgrounds can profit from a language-responsive instructional approach for percentages concepts. Their focus was on the effects of the instructional approach on students with low academic language proficiency in linguistically diverse classrooms. 655 participants in the seventh grade were investigated. They utilised two groups, the intervention and the control group. The findings revealed that the intervention group developed significantly more conceptual understanding of percentages than the control group.

Another similar study is that of Sharma (2022), who investigated the interactive role of prosody in multilingual children's utterances during a geometry lesson. The participants were 9-11 years old primary school learners. 15 learners and their teachers were investigated. Of the 15 learners, 9 were multilingual. Data collected included field notes, lesson video recordings, and learner-written work samples. The findings of the results showed that multilingual children use some prosody characteristics of their multiple languages as interactional tools when working on geometry problems. Sharma (2022) concluded that the learner's use of prosody features positively influenced the learning of mathematics.

Also, Maluleke (2019) conducted a qualitative study that investigated how teachers can use multiple languages as a teaching method in the classroom as a way of improving learners' performance in mathematics. This study revealed several findings, including (1) that code-switching helps teachers to evaluate whether learners understand the mathematics content being taught, (2) code-switching, as a teaching strategy, increases learners' participation in class when used successfully, can facilitate teaching and learning of mathematics, and (3) code-switching helped teachers and learners to share knowledge in an active way rather than when the subject matter is presented in English to passive learners. Maluleke (2019) concluded that using multiple languages as an instruction method can successfully facilitate learning in a mathematics classroom.

A study by Lin and Peng (2019) focused on terminology. Lin and his colleagues conducted a study to investigate whether there is a relationship between mathematics vocabulary and performance in this subject among fourth graders. They aimed to explore how these various forms of mathematics vocabulary found in the syllabus are related to separate mathematics outcomes (i.e., mathematics performance and cognitive skills). The study participants included 237 (125 boys and 112 girls) fourth graders in an elementary school. The instruments in this investigation were two tests to measure calculations, two tests of word problems, and an oral question test, which was videotaped. Mathematics vocabulary, general vocabulary, cognitive skills (IQ, working memory, and processing speed) and mathematics problems (calculations and word problems) informed the data. The findings revealed that even after controlling for the other variables, mathematics vocabulary still made a contribution to mathematics performance. However, the effects of mathematics vocabulary on the different mathematics outcomes varied from concept to concept. They concluded that mathematics vocabulary contributed significantly to word problems but not calculations. Other findings were that vocabulary related to measurement and geometry was more useful to word problems than vocabulary related to numerical operations.

The sources reviewed above on the role of context in mathematical instruction about language and geometry terminology all point to how ritualistic performance by learners may perpetuate and hinder the development of explorations. Erath et al. (2021) emphasise the vital role of language in developing mathematical concepts. Bi/multilingual learners faced with this dilemma of having a weak language base in the second language may not be able to grasp the concepts quickly, and therefore, their only recourse is to imitate the expert performer. This standpoint concerning bi/multilingual learners is supported by Cummins (1979), when he points out that these learners with a weak grasp of language will underperform in school. Other studies (Neville-Barton & Sneddon, 2005; Clarkson, 1992;

Dawe, 1983) have also found a relationship between language proficiency and mathematics achievement.

Cummins (1979) also argues that bi/multilingual learners possess some advantages over mono-lingual learners. In his argument, Cummins says learners who master two or more languages will outperform and surpass those with have one language. Although mathematics is regarded as a language with its own words and symbols, it is a language created for communication in mathematics and is, therefore, not a natural language. In other words, it's nobody's first language, but as a language, it could be advantageous to use second language approaches in classroom teaching (Phummer, 2022). The second language can be profitably used as a resource in the teaching and learning mathematics. This has been demonstrated in the studies reviewed above (Neugebauer & Prediger, 2022; Sharma, 2022; Maluleka, 2019).

2.4.2 Patterns of thinking

Ordiz and Mecate (2022) investigated clusters of prevalent patterns of geometric thinking levels among mathematics students. They aimed to determine the number of patterns that occur and to describe their level of geometric thinking. The participants in the study were 153 university education students. The van Hiele (1982) achievement test (VHGT) was administered to the participants and formed the data source. The participant's performance was measured at the following levels: visualisation, analysis, informal deduction, deduction, and rigour. The findings showed that 13.1% of the participants were at the third level of the van Hiele levels, while 43.1% could not be placed. Other findings revealed the common patterns that helped to describe the consequences of geometric thinking ability at level 0. These patterns informed the grouping into clusters of core-remedial, topic-corrective, and close-corrective groups. They concluded that these clusters could enable universities to systematically address the student's individual gaps in geometry.

Nasution, Yulia, Anggraini, Putri and Sari (2021) studied the correlation between mathematical creativity, thinking ability and disposition in geometry among junior high school students. Their focus was to analyse the correlation between creative thinking and creative disposition concerning the area and volume of flat geometrical object-related problems. Two groups, the experimental group and the control group were used for this study. A creative thinking test and a questionnaire were administered to the participants after a learning period. Data was analysed using Bivariate Correlation through SPSS. The results found the correlation coefficient at the middle level of the quantum. These findings indicate a significant correlation between students' creative thinking ability and creative thinking disposition.

Ngirishi and Bansilal (2019) conducted a study that explored how secondary school learners understand some geometric concepts. Their aim was to explore the understanding of basic geometry concepts among grade 10 and grade 10 learners in terms of van Hiele's levels of geometric thinking. A total of 147 participants drawn from three high schools were recruited for this study. The study's findings revealed that the participants experienced difficulties with problems involving definitions of geometric terms and had challenges concerning the interrelations of shapes and class inclusion properties. These findings indicate that the participants generally operated at the visual and analysis levels of the van Hiele (1995) geometric thinking model.

Amidu and Nyarko (2019) investigated the van Hiele's thinking levels of geometry of junior secondary school students. They aimed to measure van Hiele's levels of geometric thinking attained by junior high school students before writing the school certificate examination. The study used a quantitative approach, and the participants were 105 randomly selected students from four schools. The VHGT served as the instrument to collect the data. The results revealed the following: 21% were at level 0, 62% were at level 1, 16% were at level 2, and

1% were at level 3. No participants were found to be at level 4. The researchers also compared the achievement levels of participants from private and public schools. An independent t-test yielded no statistically significant difference between public and private school participants' geometric thinking.

Asemani and his colleagues (2017) carried out a study to determine the van Hiele's levels of geometric thinking obtained by learners and the end of secondary school learning. The employed a qualitative research design and randomly selected 200 participants from across three secondary schools. The results of their study revealed that about 0.5% of the participants functioned at level 4 of geometric thinking while the majority of the paricipants were between levels 1 and 2. They concluded few learners function at level 4 at the end of their secondary school learning.

2.4.3 Teacher pedagogical and content knowledge

Mbhiza (2021) investigated some teachers' teaching a section from the topic of algebraic functions at one of the schools in the rural areas. This qualitative multiple case study explored grade 10 teachers' pedagogical approaches to teaching algebraic functions. Five participants were recruited for this study, and it was conducted at five different rural secondary schools. The data was gathered through semi-structured classroom observation and video-stimulated (VSRI) interviews. Data analysis focused on capturing how teachers used examples to facilitate learners' understanding of the essential concepts in the different types of functions. The results of worked examples' generalisations revealed that most teachers dominated this process and did not create opportunities to probe learners' cognitive skills or provide a chance to embark on mean-making on their own. Regarding the learners' approach to generalisation, only one participant out of five exhibited more explorative discourse during the course of their instructions. The researchers concluded that the teachers did not create enough

opportunities for explorative discourse but relied heavily on drill as a method of instruction in the mathematics classroom.

Pavlovicova and Bockova (2021) explored the geometric thinking of future teachers for primary education. They focused on geometric thinking and solutions to geometric tasks among prospective primary school teachers, and 59 master's participants drawn from the student population under the teacher training programme were recruited for the study. The van Hiele geometry test (VHGT) was administered. Two geometric multi-item tasks were selected from this test, and the participants' solutions were then evaluated qualitatively and quantitatively. The statistical software C.H.I.C tool was used to analyse the data. The analysis results revealed that 40% of the participants were below the required level of geometric thinking.

Zayyadi, Nusantara and Hidayanto (2020) investigated prospective teachers' content and pedagogical knowledge. Their aim was to describe the content and pedagogical skills of prospective teachers in learning mathematics from a commognitive perspective. The participants were drawn from 7th-semester mathematics education university students. This study used the qualitative approach to gather data. The participants were video recorded while doing peer teaching. The video recordings were then analysed to observe the participants' ability to content and pedagogical knowledge. The commognitive lens used to analyse the data revealed significant fundamental differences in components of content and pedagogical knowledge of prospective teachers.

Salifu, Yakubu and Ibrahim (2018) studied the van Hiele thinking levels of geometry of some student teachers. Three hundred and fifty-one participants (133 female, and 218 male) were randomly selected from the population of student teachers at a college of education. The instrument for this study was the van Hiele geometry test (VHGT). The analysis of the data

revealed the following: 56.5% were found to be at level of 1, 48.9% were at level of 2, 36.2% were at level of 3, 21.7% were at level of 4, and 15.7% were at level of 5.

2.5 How the discourse may be influenced

Learning mathematics is a change in discourse (Sfard, 2008). The change in the learner's discourse can be facilitated through several pedagogical moves and strategies that may be employed by the teacher to enhance the understanding of mathematical concepts (viz., geometric concepts in this study). Among the pedagogical moves that may influence the mathematics discourse of learner's is teaching problem-solving methods to enhance the understanding of geometric concepts.

Researchers such as (Mudaly & Zulu, 2023; Yuan, 2016; Sangcap, 2010) state that problem-solving skills are essential in teaching and learning mathematics. Qomariyal et al. (2023) assert that the ability to solve mathematical problems is an important skill the student should possess. Problems are found in every sphere of humanity. The experience to engage in solve problems does not end in mathematics as such, the same extends to other subjects and everyday life (Valdez & Bungihan, 2019; Harisman et al., 2017). One way to measure the student learning outcomes in mathematics is through proficiency in problem-solving.

Multiple studies have indicated that mathematical objects are abstract in nature, and as a result, many students face challenges in problem-solving in mathematics (Gafoor et al., 2015; Mierzwa, 2014; 4). As a result of the abstract nature of mathematics and the numerous challenges inherent in mathematics, the problem-solving abilities of many students are poor due to some underlying factors (Sangcap, 2010). Some of these factors include the challenges in unravelling unfamiliar text, fathoming the meaning when new vocabulary is used in mathematics, comprehending the way the words are framed in the problem statement, and determining the best strategy to employ in solving the given problem (Mierzwa, 2014; 4).

However, as Mudaly and Zulu (2023) point out concerning the importance of problem-solving in mathematics, possessing this ability can enhance cognitive abilities (Pehkonen, 1997). Enhancing cognitive abilities results in the development of high-order thinking skills. One of the characteristics inherent in problem-solving involves high-order thinking, which may be challenging to students as they make necessary predictions to obtain evidence and answers (Ahdhianto et al., 2020). Given the importance of problem-solving in mathematics and its relationship with high-order thinking, the teacher as the expert discursant has an essential role in providing a conducive learning and teaching environment that can enhance and improve the ability of students regarding problem-solving (Naufal et al., 2021). There are many problem-solving strategies. The teacher needs to teach multiple problem-solving strategies, given that mathematical problem-solving is essential in enhancing much-needed 21st-century skills and offers learners cross-curricular experiences with real-world meaning (Szabo et al., 2020). One practical teaching approach that has been used to facilitate student problem-solving abilities is to provide open-ended questions about the student's thinking ability and a conducive, non-threatening teaching and learning environment for students to explain their thinking processes (Hwang et al., 2014; Kosyvas, 2016). Balta and Asikainen (2019) reaffirm the vital role of teaching problem-solving methods and point out what students need to utilise their qualitative understanding of the problem to arrive at a quantitative solution.

Another instructional strategy to facilitate learners' understanding of geometrical concepts is to teach using visualisation. As mentioned earlier, mathematical concepts are abstract, and students often face challenges in understanding mathematical concepts. Nafasov et al. (2023) emphasise the need for teachers to use the array teaching strategy to meet the diverse learning needs of students, which can help them benefit from mathematics. Arcavi (2003) contends that visualisation is critical in the teaching and learning of mathematics and facilitates various

mathematical thinking processes. As a tool, visualisation facilitates problem-solving skills and aids in the understanding of mathematical concepts (Tiwari et al., 2021). Teaching using visualisation has several benefits. These benefits include facilitating student's cognitive abilities by way of construction of mental images (Brunner, 1996; Giaquinto, 2007; Schafer, 2021; Skemp, 1989), enhancing spatial reasoning skills (Tiwari et al., 2021; Yilmaz & Argun, 2018), giving pictorial representation to solve word problems (Bruter, 2013; Hanna & Villiers, 2012), assisting students to construct connections between concepts, patterns, and relationships to gain a deeper understanding of mathematics (Rathour et al., 2022), and communicating information as well as aiding reasoning in (Arcavi, 2003).

Teachers can utilise various visualisation tools to deepen their understanding of mathematical concepts. Teachers can use hand-drawn diagrams as a form of external visualisation (Kozhevnikov, 1999), gestures and body movement (Presmeg, 1992; Radford, 2003), or they can utilise digital media and hand-held graphing technologies as well as geometry software (Rivera & Becker, 2004), to construct visual representation. Visualisation as a teaching method is valuable for geometry and can be applied to other domains of mathematics (Arcavi, 2003; Presmeg, 2006).

However, the use of visualisation can sometimes cause problems for learners. Tversky (2013) takes a swipe at the concept of visualisation by cautioning teachers to ensure that the visual representations used are as accurate as possible. In this vein, Tversky points out that images can confuse students if inaccurate and may make them unable to comprehend the meaning embedded in a given image.

Another teaching strategy that may be used to influence learner's mathematic discourse is differentiating instruction. Today's classrooms are diverse regarding gender, language, culture, socioeconomic status, interests, and learning styles. Given this prevailing diversity in

the present-day classroom, all students can't participate in teaching and learning the same way. Therefore, morally and professionally, such an assumption by a teacher would be wrong (Kado et al., 2022). Magableh and Abdullah (2020) point out that differentiated instruction is a pedagogical theory premised on mitigating learners' academic and learning profile needs and is tailored by adapting instruction based on their diversity. By using differentiation of instruction, equity in teaching and learning that appears to be lacking when other teaching methods are used may be achieved by overcoming the challenges due to the unequal dispensation of knowledge (Ariss, 2017). Because differentiated instruction is based on the individual needs of learners and is focused on different levels of learning, all learners in diverse classrooms are therefore supported to develop knowledge and skills at their own pace (Rock et al., 2008; Schleicher, 2016). Differentiated instruction as a teaching and learning strategy has gained comprehensive support from learning theories such as Vygotsky’s (1978) Social Learning that incorporates the concepts of Zone of Proximal Development (ZPD) and Gardner’s (1996) Theory of Multiple Intelligences (Magableh & Abdullah, 2020; Kado et al., 2022)

Differentiating instruction entails four aspects, namely: (1) the content knowledge as prescribed in the curriculum- in this regard, Tomlison and Moon (2013) suggest that knowledge information should, if possible, be delivered using multiple modalities such as visual representation, text, or video. (2) The instruction process- in this regard, Aliakbari and Haghighi (2014) call for tired instruction and assessments coupled with flexible grouping. (3) The end product- in this regard, different assessments based on learner’s ability levels or affording the learners some form of choice in terms of working individually or in groups on their product are proposed (Kado et al., 2022). (4) The teaching and learning environment- in this regard, consideration should be given to learners’ readiness to learn, interest, and their individual learning profiles (Tomlison & Imbeau, 2010).

Therefore, differentiating instruction that caters to diverse learners' individual learning needs is an effective pedagogical strategy that may influence learners' mathematical discourse.

Another pedagogical method that may influence learner's geometric discourse is cooperative group work. Sarikaya and Egmir (2023) describe cooperative learning as a teaching method whereby learners are assigned to work in specific groups to achieve a defined goal. All group members are assigned a different role to play and thus collectively work towards a particular set goal.

Numerous studies have demonstrated that cooperative learning as a method of instruction positively affects both academic achievement and attitude towards mathematics among learners (Karali, 2017; Susanto et al., 2018; Hossain & Rezal, 2018; Zakaria et al., 2016).

Unlike other methods of learning supported by a traditional type of learning and are usually teacher-centred, such as the lecture method, cooperative learning as a teaching method is learner-centred. In cooperative learning, learners are not passive but are active members of a group and actively participate in the learning process (Sarikaya & Egmir, 2023). In cooperative learning, students, as members, bring different abilities, experiences and individual skills to the group. As they cooperate on a task, learners share and transfer their individual experiences and knowledge to one another and collectively strive to find solutions to problems with a common goal.

Cooperative learning as a method of instruction has several academic and social benefits. On the academic side, one such benefit is that in cooperative learning, learners have opportunities to work together and communicate with other members. Working, communicating and sharing knowledge with other group members is more beneficial than working alone on a task (Rienties et al., 2020). When learners work together, they share their individual experiences, learn and transfer skills from one another while at the same time

having opportunities to look at a problem from different perspectives. On the social side, the learning environment provided by working in cooperative groups is non-threatening. During cooperative learning, learners have abundant opportunities to cooperate with one another while speaking, and at the same time exchange information with other members freely (Turgut & G, Turget, 2018). Other academic and social benefits of cooperative learning include: (1) interdependency among members, (2) each member has a role of sharing his/her individual experiences and knowledge, (3) each member has equal opportunities to learn, and (4) member learn and develop cooperative skills which are essential and desired in society (George & Adu, 2018).

Whereas several studies such as those of Susanto, Bharata and Dahlan (2018), Hossain and Rezal (2018) and Zakaria et al., (2016), demonstrated the effectiveness of cooperative instruction, its implementation in the classroom may at times pose a challenge to teachers. Some challenges include group dynamics due to the diversity of learners, uneven workloads given to different group members, and classroom management challenges on the part of the teacher. Therefore, given these challenges, it is imperative for the teacher to consider these impediments well before implementing cooperative learning instruction in the classroom.

Another pedagogical move that may influence a learner’s mathematical discourse is the method of infusing van Hiele's (1986) model in the teaching and learning of geometry. Several studies have demonstrated the van Hiele model's effectiveness in teaching and learning geometry (Usiskin, 1982).

As mentioned earlier, many students experience challenges in understanding geometric concepts, and thus, their mathematical discourse is impacted (Md Yunus et al., 2019). Further studies have found that many students operate at level one and level two of five levels of geometric thinking, which is the major contributing factor to the challenges that relate to poor

understanding of geometric concepts (Usiskin, 1982; Abdullah & Zakaria, 2013; Solaima et al., 2017; Fatih & Ilham, 2018; Mdyunus et al., 2019; Nasiru et al., 2019). According to van Hiele's model, levels one and two are among the first categories and cover visualisation and analysis. These two levels do not enable the learner to participate fully in mathematical discourse.

Given the affordance of the van Hiele (1986) model as a pedagogical tool to develop geometric thinking in learners, teachers may utilise this tool and base the instruction sequence on the model's five levels. In this way, students can have a hierarchical sequence of learning on which to constructively build their knowledge (Ismail, 2020). Planning and sequencing instructions from visualisation to deductive reasoning and rigour are crucial in developing the learners' logical and geometric thinking (Battista et al., 1982).

In this regard, Nasiru, Abdullah, and Norulhuda (2019) suggest several instructional strategies, including that of using technology as one of the methods. Technology as an instructional tool is better positioned to be effective in many areas of mathematics learning as it is learner-centred. Studies by Nasiru, Abdullah, and Norulhuda have demonstrated that infusing technology with van Hiele's theory positively enhances geometric thinking. They suggest that technologies such as GeoGebra, Game-based, Touch screen, smartboards and Geometer sketchpads impact geometric thinking positively. Another suggested instructional strategy is to engage learners with challenging tasks and later participate in small and whole class discussions. Such a participatory instructional approach promotes thinking in geometry and allows learners to demonstrate their learning through discussions (Alex, 2016). In group discussions as well as in whole class discussions, learners have a good chance to share their learning while sharpening their communication skills (Fujita et al., 2017; Pasani, 2019).

2.5.1 Cooperative learning

Gago, Geronomo, Huanuco and Sanchez (2021) investigated cooperative learning and notions of geometry in 5-year-old children. Their aim was to assess the incidences of cooperative learning in geometry among the participants of the study. A qualitative descriptive approach was adopted in this study. The participants were 78 children from three preschools. An observation form with a checklist provided data for the study. The checklist had five levels for analysis: identification of basic geometric shapes, different geometric shapes, recognition of geometric shapes in the environment, exploration and production of shapes using tangram, and using shapes to reproduce objects from imagination. The results were then measured against the following indicators: (1) expected accomplishment, (2) in process, and (3) at the beginning of the process. The results showed that more than half of the participants were in the intermediate learning process, and the majority had an excellent social and personal development capacity. The results show the benefit of communication among the students and the impact of the relationship between cooperative learning and the notions of geometry in 5-year-old children.

Elmagustilla and Masrukan (2021) investigated mathematical representation ability on geometry material from the interest in learning in jigsaw cooperative learning performance. They aimed to explore the impact of cooperative jigsaw material geometry on mathematical representation ability. Participants for these mixed methods qualitative-quantitative study were learners of class VII at a junior high school. Participants were then divided into the experimental and control groups. The experimental group received jigsaw cooperative learning instruction with performance assessment, while the control group were given discovery learning instruction. The instruments used to collect data were a test activity, a questionnaire, and an interview. The data analysis focused on comparing mathematical representation abilities between the two groups. The study revealed that the mathematical

representation abilities of participants who received cooperative learning jigsaw instruction had higher performance capabilities after the intervention than participants who received discovery learning instruction.

Tristanti and Hidayati (2020) investigated the implementation of cooperative learning type-assisted individualisation for teaching 3D geometry. They aimed to investigate the effectiveness of cooperative learning type TAI strategy for teaching 3D geometry. They adopted an experimental pre-test and post-test control approach design. The study recruited 70 participants, randomly divided into experimental and control groups. The experimental group received instruction using the TAI strategy, while the traditional strategy was administered to the control group. Data was collected from multiple sources, including (1) responses to a written task, (2) observation of the student's cooperative skills, (3) responses from a questionnaire, and (4) a test task for learning outcomes. The data analysis revealed the following results: (1) 88.8% on learning completeness when cooperative learning type TAI was implemented. (2) Student activities and their cooperative skills were considered practical. (3) Regarding the effectiveness of the learning process, the participants' responses were positive. The results suggest that the cooperative learning type TAI effectively taught 3D geometry.

Gates (2019) investigated a study that compared cooperative learning and collaboration in high school geometry, including geometry in construction. This study had two aims: (1) to explore how the use of cooperative strategies in the learning of geometry impacts students' understanding and views of the attainment of 21st-century skills of teamwork and collaboration, and (2) to determine the differences in the implementation of cooperative strategies in geometry construction and other geometry classes. This study project adopted a mixed methods approach. Participants included five geometry teachers. Data sources came from observation in some geometry classes, pre-test and post-test strategies, and a teacher

survey. The data analysis showed the following results: (1) there were no differences between the two groups' use of cooperative strategies and views of attaining 21st-century skills. (2) Implementing cooperative strategies was more effective in geometry construction than in other geometry classes.

Samura (2018) conducted a study that compared STAD-type and TPS-type cooperative learning in middle school geometry learning. The study aimed to describe and compare the effectiveness of these types of cooperative learning. A quasi-experiment pre-test and post-test approach was used in this study. The participants were 60 randomly selected junior high school students. The participants were tested before and after the intervention and provided quantitative data. Qualitative data for the study was derived from observations and one interview. Data analysis revealed the following results: (1) on the competency description, the standard pre-test score mean was 75, and the post-test mean was more than 75. (2) Regarding mathematical communication skills, the first test had a mean score of 75, and the second test mean was more than 75. (3) MAVOVA result data revealed that SDAT had a higher significance level than TPS. These results suggest that the SDAT type impacts competency standard achievement, mathematical communication skills and mathematical thinking skills more than TPS-type cooperative learning. The SDAT type was concluded to be more effective than the TPS type.

2.5.2 Problem-solving

Halim, Nurhidayati and Zayyadi (2020) investigated mathematics prospective teacher problem solving abilities on logarithms. They used a qualitative approach and aimed at describing the problem-solving abilities using the commognitive analysis of the characteristic features of the discourse of mathematics, that is, the use of words, the use of visual mediators, the construction of narratives, and the employment of routines. The instruments for the study

were mathematical problems and semi-structured interviews. The study results revealed that the participants showed significant differences in all the features. There were differences in word use to inform the comprehension, differences in the use of objects to aid mediation, differences in narratives to aid systematic solving of problems, and differences in routines, leading to poor understanding of basic logarithm concepts. They concluded that prospective mathematics teachers make mistakes when working on mathematics problem-solving due to poor skills.

Hassan and Rahman (2017) investigated how problem-solving skills, metacognition awareness, and mathematical achievement can be used as a mediation model. The study explored the relationships between problem-solving skills, metacognition awareness, and mathematical achievement. They focused on how metacognition awareness can be used as a mediation tool. The participants were 333 learners drawn from ten secondary schools. Two sets of data were collected for this investigation. The first data sources were those from students' responses to the questionnaire, and the second set was school management's information on students' mathematics achievement. Researchers then employed the Structural Equation Modelling (SEM) technique to analyse the data. The results revealed a strong correlation between all the variables (problem-solving, metacognition awareness, and mathematical achievement). These findings suggest that developing metacognition awareness can improve problem-solving skills and enhance achievement in mathematics. In my study, the participants worked individually on the task and participated in the focus group interview based on the task. This activity enabled to directly experience and apply problem-solving strategies. At the focus group interview, the participants were expected to observe alternative solutions to the problem and different perspectives. This served as an opportunity to enhance their understanding of the concepts. The activity resonates with Suryanto and his colleagues

(2021) who call for the need to train students on problem-solving and to socially convey their ideas as one of the goals of learning.

Negara el at., (2022) investigated the differences in student reasoning abilities between those using Geogebra-assited social cognitive learning and Geogebra-assited problem based learning. In their study, they employed the quantitative and a quasi-experimental nonequivalent pre and post-test control group approach. The experimental group utilized the social cognitive and the control the problem based software. The findings from the study revealed significant improvement in mathematical reasoning abilities in favour of the experimental group.

Geometric reasoning entails alternating between the visual and the conceptual (Duval, 2017). The Geogebra software has built-in capabilities to offer both visual diagrams and the related properties at the same time. In this regard, the software can facilitate the aspect of argumentation necessary in geometric tasks. Student working on geometric tasks often have to justify their statements. The criteria for a geometric argument is that the argument should sufficiently be general and valid and must hold on all cases for a particular domain (Ramirez & Hidalgo, 2022).

2.5.3 Differentiation

Magableh and Abdullah (2020) investigated the influence of differentiated instruction on the overall achievement of English language students. This study recruited 60 participants drawn from two randomly selected schools. The participants were then assigned to two different groups: the experiment and control groups. The experimental group received differentiated instructions, while the control group did not. Data was obtained from the pre-test and post-test results. These results were compared. The results obtained after the t-test analysis revealed significant differences in achievement between the experimental and control groups.

The results favoured the experimental group than the control group. They concluded that differentiated instruction is effective on overall student achievement.

Another study by Awofola and Lawani (2020) examined the effectiveness of differentiated instruction on secondary school students. A sample of 220 participants was chosen for the study. The participants were then divided into two groups: the experimental group and the control group. The experimental group received differentiated instructions, and the control group received conventional instructions. Data was obtained from three instruments, namely, the Mathematics Achievement Test (KR-20 = 0.89), Felder-Soloman Index of Learning Styles (Cronbach α = 0.92), and McKenzie Multiple Intelligences Inventory (Cronbach α = 0.90). The results from the tests revealed that participants in the experimental group outperformed those from the control group. Another finding was that gender had no effect on achievement in mathematics. They concluded that administering differentiated instruction as a strategy has great potential towards achievement and offer suggestion for its adoption in the mathematics classroom.

Al-Shehri (2020) also conducted a study exploring the effectiveness of differentiated instruction on achievement and the development of critical thinking skills among 6th-grade science students. In this study, 50 participants were selected and assigned to the experiment and control groups. Differentiated instructions were administered to the experimental group, and the control group received conventional instruction. Two sets of tests were administered. Each set had a pre-test and a post-test. One set of tests was on academic achievement, while the other set was on critical thinking. Data obtained from these tests was then examined. The results revealed that the experimental group improved academic achievement and critical thinking more than the control group. She concluded that differentiated instruction is adequate for academic achievement and developing critical thinking skills.

Kotob and Abadi (2019) conducted a study investigating the role of differentiated instruction on student achievement. In their study, they aimed to explore teaching strategies that are based on the principles of differentiation in mixed-ability classrooms. A sample of 20 participants of mixed-ability students was selected for the study. The participants were grouped into the experimental and control groups. The intervention group received instructions based on the principles of differentiation, while the control group received conventional methods of instruction. The results obtained from the scores in the test before the intervention and test after the intervention revealed improved achievement with respect to the intervention group compared to the control group. Another finding was that low-ability students demonstrated remarkable improvement in achievement in the intervention group between the pre-test and post-test results. They concluded that differentiated instruction positively affects low-ability students' academic achievement. The participants in my study had different geometric abilities. However, the cognitive demands of the mathematical tasks were at two levels (Low and High) as suggested by Polat and Dede (2023). They were required to think conceptually and then make connection to the geometric relationships. Because of their different abilities, they would profit from differentiated instruction.

2.5.4 Visualisation

Lowrie, Logan and Hegarty (2019) investigated the influence of spatial visualisation training on students; spatial reasoning and mathematics performance. The study aimed to determine the effectiveness of a spatial visualisation intervention program on increasing student spatial reasoning and mathematics performance. This quantitative study recruited 327 participants (age range 10-12) from 10 primary schools. The participants were divided into two groups: experimental and control groups. The intervention program was carried out by classroom teachers undertaking a Professional Development (PD) workshop regarding the Spatial Reasoning program. Two instruments were used in this study: The Spatial Reasoning

Instrument (RST) to measure spatial reasoning and a mathematics test to measure achievement. The experimental group received the intervention program, while the control group received standard instruction. Visualisation data were divided into three constructs: spatial visualisation, spatial orientation, and mental rotation. The test had three categories, viz: geometry based, word problem, and non-geometry graphics. The findings from the study revealed the following results: (i) on spatial visualisation, the intervention group scored a mean average of 5.49 in the post-test, and the control group had 5.18 (ii) on the mathematics test, the intervention group scored a mean average of 2.6 in the post-test while the control group scored 2.5. These results showed that the spatial reasoning enrichment program enhanced spatial reasoning and mathematics performance.

Target and Turgut (2018) conducted a study on the effects of visualisation on mathematics achievement in a thesis study. The purpose of the study was to review literature on the effects of visualisation in mathematics instruction. This meta-analysis study was aimed at employing two statistical methods. The Meta-analysis method was used to analyse the results of individual analyses and combine the findings, while the MetaWin examined the normal distribution of effect sizes. The study's findings showed the following: (I) Regarding the effect of visualisation on mathematics achievement, the effect size value from 42 studies was calculated at 0.811. This value demonstrates that visualisation in the learning of mathematics positively influences achievement. (ii) regarding effect sizes in terms of educational levels between middle school, high school, and university, the effect size was calculated at a value of 0.056. This is regarded as medium level, indicating that visualisation has a similar positive effect at determining the effect of visualisation on mathematical achievement. There were 34 theses, 9 of which were doctoral dissertations, and 26 were master's theses.

The cornerstone of success in geometry is premised on the creation of correct geometric conception and visualisation is fundamental for the creation of correct mental imagery vital in

the development of geometric thinking. In my study, participants worked on a geometry problem and participated in the focus group interview based on the activity. Duval (1995) suggests three types of cognitive processes that a crucial to the development of geometric thinking, namely; visualisation, construction and argumentation processes. On the other hand, geometric problems involve creating representations of abstract objects. Brunner and Haste (2010) subdivide the representations into enactive, iconic and symbolic representation. In my study, the participants were required to use pencil and paper to construct the mathematical object, in other words this involved iconic and symbolic representation of the object. Iconic relates to imagery of mental thoughts and symbolic focuses on describing associated terminology.

2.5.5 van Hiele's theory

Machisi and Feza (2021) investigated grade 11 students' geometric proof competencies using van Hiele theory-based instruction. The aim of the study was to test the effects of van Hiele's theory-based instruction on students' proof abilities. A quasi-experiment approach was used to study 186 participants from four secondary schools. A geometry proof test provided the data for the study, while the analysis tool was the nonparametric analysis of covariance. The results from the analysis showed a statistically significant difference in performance between the experimental and control groups. The results favoured the experimental group that received instruction based on van Hiele's theory.

Jogymol and Mammen (2018) studied students' understanding of geometry terminology through the lens of the van Hiele theory. The participants of the study were first-year university mathematics students. Data for the study came from responses to multiple-choice items on a questionnaire. The items covered 30 geometry terms, half presented verbally and the other half visually. The participants were asked to link the verbal items to their

corresponding visual images for this activity. The van Hiele descriptions served as the lens for the analysis. The following results were obtained: the participants scored 64% on basic geometry terminology, 68% on visually presented items and 59% on verbally presented items. These findings suggest that a teaching approach that combines verbal and visual representations impacts conceptual understanding in geometry.

Astuti, Suryadi and Turmudi (2018) conducted a study that analysed the geometry skills of junior high school students on the congruence concept based on the van Hiele geometric thinking level. They aimed to describe the geometric skills of learners. The study employed a qualitative descriptive approach. Six participants were recruited for this investigation. The VHGT informed the data collection instrument. The study's findings showed that participants at levels 1, 2, and 3 of van Hiele's thinking model had all the skills, such as visual, verbal, drawing, and logic. However, there were differences in their application of the characteristics for each level.

Mbusi and Luneta (2021) investigated the reasoning of university education students faulty reasoning in relation to translations when designing phase-based instructions using the van Hiele levels of geometric thinking. Their findings showed that students had several misconceptions with geometric translations. The errors the students made were divided into two groups. The first group had errors pertaining to incorrect use of properties of translation. While the second group comprised of errors related to basic mathematics operations and wrong diagrammatic representations. They concluded that specific instructional approaches that target van Hiele levels of geometric thinking should be implemented to address the students' reasoning skills for effective learning.

My study was underpinned by the commognition theoretical framework to explain the participants' geometrical thinking through their discourse while working on a geometrical

problem activity. On the other hand, van Hiele theory delineate the levels of geometrical thinking during one's geometry development. According to the van Hiele theory, there are five fixed levels of geometrical thinking, and one cannot proceed to the next level without mastery of the previous level. The development of geometrical thinking increases with participation in geometrical discourse. Having knowledge of one's level would serve as a useful indicator of one's geometrical thinking. Mahlaba and Mudaly (2022) emphasise that commognition and van Hiele theories serve a similar purpose and work in tandem with one another. Therefore, taking into consideration the similarities between these theories, the van Hiele theory is consequential in explaining the participant's geometrical discourse.0

2.5.6 Technology

Frank-Baccaglini (2020) investigated how dynamic interactive mediators can foster explorative participation in mathematical discourse. The study focused on students' mathematical discourse emerging from interactions with the GeoGebra environment. Two cohorts of students were recruited as participants in the study. One cohort comprised standard-to-high achieving students who did not use GeoGebra, and the other cohort involved demotivated, low-achieving students who used GeoGebra. The discourse developed by the two cohorts of participants to an unfamiliar interview question was then analysed. The findings from the study revealed that participants who used GeoGebra had more significant development in characteristics of their discourse towards the explorative end of the spectrum than the cohort that did not use GeoGebra.

Hassan, Abdullah and Ismail (2020) studied the consequences of integrating interventions based on van Hiele model. They concluded that combining technology-based intervention with van Hiele phases in teaching geometry concepts strengthened the students' geometric thinking skills. The van Hiele model has different phases of students' geometric thinking. In

this study, they reviewed literature results from the study indicated that technology-based intervention had more significant size effects than manipulation of various existing research studies between (199 and 2019) on levels of geometric thinking while at the same time reviewing the approaches employed in these studies. They focused on two categories of interventions: those that used technology and those that used manipulatives.

Mudaly and Fletcher (2019) investigated the effectiveness of GeoGebra when teaching linear functions using the iPad. This study's qualitative approach focused on understanding the learners' opinions and reactions to using the GeoGebra app to explore the concepts of linear function. Convenient sampling techniques were to select 27 grade 9 students from a secondary school as participants. These participants had no prior knowledge of the concept of equations of a straight line. The data sources were from the task sheet that participants completed and from interview transcripts of five participants who volunteered to be interviewed. In assessing the task, the researchers focused on participants' understanding of the properties of linear functions, while the interview questions were aimed at eliciting the participants' opinions concerning their perceptions and challenges of working with the app. The data analysis revealed that (1) 67% of the participants could match questions to their corresponding graphs. (2) 74% managed to identify how the shape of the graph changes with corresponding changes in the value of the gradient. (3) 81% could identify the y-intercept given the equation. (4) 93% showed knowledge of the shape of the graph as the y-intercept changed. Concerning the interview response, the participants indicated that they all enjoyed working with the app and said the app facilitated their understanding of concepts of linear functions. They further revealed that they struggled to work with the app because they were unfamiliar. These findings show the app's effectiveness in discovering properties of straight-line equations.

Yunis, Ayub and Hock (2019) investigated the geometric thinking of Malaysian elementary school students. They aimed to determine van Hiele's levels of geometric thinking among elementary school learners. A true experimental approach was used in this study of 95 year-five participants. Three different instruments were utilised to gather the data. These included (i) van Hiele's phases learning module, (ii) van Hiele's theory integrated with the Google SketchUp software (VH_GSU) module, and (iii) the conventional instruction (NVH-CI) module without the use of van Hiele's theory of learning or the use of any teaching and learning software. WU's Geometry test (WGT) and sample responses identified by Fuys, Geddes, and Tischler (1988) informed data collection and analysis. The results revealed that while the participants at the start of the experiment were operating at the lower levels of van Hiele's levels of geometric thinking, the WGT results showed that most of the participants attained higher levels of geometric thinking after the intervention in all three groups.

Ozcakir, Konca and Arikan (2019) investigated children's mathematical understanding through geometric shapes. They focused on the effects of digital learning tools on learning about geometric shapes in early childhood education. The aim was to determine the learning progress as the participants engaged in digital learning activities. Six participants from a kindergarten were recruited for the study. The participants used a tablet to work on various task activities and had task-based interview sessions. The study results showed that the participants significantly improved cognitive abilities and understanding of geometry concepts.

Lu, Zhang and Stephen (2019) conducted a study on visualising the commognitive processes in computer-supported one-to-one tutoring. Their descriptive study focused on understanding the interaction during computer-supported teaching and learning. The study participants included a senior mathematics high school online student and a mathematics tutor. Data were collected by reviewing the commognitive processes during the online tutoring sessions

between the student and the tutor and from the interview with the tutor conducted after the course. Data was analysed at two levels; from the teacher-lead commognitive process and the student-lead commognitive process. The study's findings revealed that in sessions 1 and 2, the tutor mainly taught the meaning of sets and guided the students about set presentations. The student did not ask many questions. During this session there was a significant rise in teacher-led commognition over student-led. In sessions 3 and 4, there was an increase in the frequency of questions from the students, resulting in the rise of student-led commognition than teacher-led. The results suggest that student can better construct their knowledge and achieve better learning outcomes when they dominate commognition.

Singh (2018) investigated the impact of using GeoGebra software on students' achievement in geometry. This quasi-experimental research design study was conducted on 44 secondary school students. The participants were then equally placed into two groups, the experimental and control groups. The participants in the experiment group used GeoGebra software for the learning activities, while the control group used traditional methods to interact with the activities. The data were collected after 3 weeks of application. The Mann-Whitney u test was used to analyse the data. The findings from the study showed a significant difference in achievement between the two groups. The experimental group using the GeoGebra software performed better than the control group using traditional methods.

Sibiya and Mudaly (2018) investigated the effects of Geoboard on learners' understanding of geometry theorems. They used a qualitative approach to study participants from two secondary schools. The participants were grouped into four focus groups of 50 each and were taught geometry theorems for a period of two weeks using the Geoboard. The thematic analysis strategy informed data analysis. The study's results significantly improved learners' understanding of geometric theorems, especially about geometric terminology and reasoning.

They concluded that using Geoboard enhanced learners’ understanding of geometric theorems.

Stumbles (2018) studied the factors contributing to students' understanding of geometry in a bilingual classroom. The study had two aims: (1) to investigate the effects of using Dynamic Geometric software (DGS) on student understanding of geometry as well as student motivation for learning geometry in a bilingual context, and (2) to investigate how a teaching approach based on van Hiele teaching phases can assist teachers in unpacking of subject-content and language-content in a bilingual context. Data was collected using a mixed-method research approach. The study showed that Dynamic Geometric software increased motivation for learning geometry rather than traditional, paper-based methods. Another study finding revealed that the teaching approach of utilising van Hiele teaching phases of geometry thinking when instruction is delivered in the second language of the student contributed successfully to their geometric understanding.

The studies reviewed above all suggest the effectiveness of using technology in enhancing mathematical concepts. In geometry, perception is fundamental in symbolic and conceptual processes. Perception underpins mathematical practice (Marghetis et al., 2016). Technology has the ability to demonstrate multiple representation of geometry concepts. Zakelj and Klancar (2022) point out that when information is easier to understand and memorise when it is conveyed through images. In my study, the participants were required to recall from previous experience how to construct the mathematical object and implement the correct procedures as well as relationships between the objects. The participants in this study were to profit from utilizing technology to improve their competences in solving geometry problems.

2.5.7 Other teaching strategies

Siregar, Rosli and Maat (2019) investigated the effects of a discovery learning module on geometry in improving students' mathematical reasoning skills, communication, and self-confidence. Their aim was to study the effectiveness of the module. They used a quasi-experimental approach with a non-equivalent control group with pre-test and post-test designs. A total of 128 participants, based on gender, were placed into the experimental and control groups. The instruments for the study included aspects of mathematical reasoning skills, communication, and self-confidence. Data was analysed using the two-way MANOVA method. The results from the examination of data demonstrated that participants from the intervention group achieved increased levels of reasoning of mathematical reasoning skills, and their communication and self-confidence improved compared to the participants from the control group. These findings suggest that the discovery learning module provided more opportunities to engage with the content, improving reasoning skills, mathematical communication, and self-confidence.

Ngin (2018) examined teachers' use of multiple representations in teaching percentages to improve communication and connection among primary school learners. Goldin (2000) defines multiple representations as using external modes of representation such as symbols, graphs, or diagrams. The study aimed to explore the effect of multiple representations in the teaching and learning of mathematics. The study participants were the teacher and seven learners from her class. The data analysis focused on the transitions among multiple representations in teaching percentages. The lens of the commognitive framework was used to analyse the congruence among the four-characteristic feature of the discourse of mathematics. The findings of the study revealed the following: (1) the participants used mathematical words and everyday words to connect different representations, and (2) the participants used flowcharts to make the representations visible. The results suggest that the

use of multiple representations using the mediation flowchart was effective in increasing the teacher's awareness and increased communication in the classroom.

Ramachandran, Gartland, Huang and Scassellati (2018) conducted an investigation by integrating the think-aloud teaching strategy and a tutoring robot to enhance learning. Their study explored the effects of a robot platform and the think-aloud strategy on learning outcomes in a one-to-one tutoring interaction. The sample for this study was drawn from a population of primary school students. A total of 52 participants were recruited for the investigation. The effects of using the support of the social robot in delivering problem-solving activities were evaluated immediately after each lesson. The findings revealed that using social robots during one-to-one tutoring and the thinking-aloud strategy enhanced immediate learning gains. Ramachandran et al. (2018) concluded that using social robots to support a think-aloud teaching strategy can enhance metacognitive thinking skills in young learners.

Fachrudin, Ekawati, Kohar, Widadah, Kusumawati and Setianingsh (2019) investigated how ancient Chinese history can support students' geometric reasoning and mathematical literacy in learning the Pythagoras. The aim of their study was to explore and identify the learning opportunities provided by integrating ancient China’s geometric diagrams in learning Pythagoras's theorem. They focused the study on the learning process, students' visual reasoning, and mathematical literacy. The study's results revealed that using figures and diagrams containing written information supported performing algebraic calculations and improved visual reasoning among students. They concluded that embedding ancient history in the learning of geometry has the potential that may increase opportunities to students to improve their performance in mathematics.

Viirman (2021) conducted a study that investigated university lecturing as a means of modelling mathematical discourse. Data was collected from first-semester lectures given by seven mathematics lecturers on the concept of functions. Data was analysed using the commognitive lens, focusing on meta-level aspects of mathematical discourse, namely introducing new mathematical objects and what counts as a valid mathematical endorsement of a narrative. The study's findings revealed several metarules about modelling mathematical reasoning, behaviour, and rules concerning constructing and endorsing narratives. Virman (2021) concluded that the modelling aspects of the mathematics lecturing format do not deepen understanding of the function concepts.

Hwang, Purba, Liu and Zhang (2018) investigated the effects of measuring authentic context on geometry learning achievement. The focus of the study was to determine the effects of using a Ubiquitous geometry (UG) system on student geometry estimation ability, geometry learning achievement, spatial ability, and learning geometry concepts and explore the correlations. The study participants were 82 fifth-grade students who were divided into three groups: the experimental group, the ruler measurement group, and the control group. The experimental and ruler groups performed measurement exercises on authentic objects in the environment, while the control group worked on paper-based exercises. The experimental group used a UG system, the ruler group used rulers, and the control group worked without tool support. The study's findings revealed that the participants in the experimental group performed significantly better in spatial ability, estimation ability, and geometry learning achievement than the other groups. Another study found a close relationship between estimation ability, spatial ability, and geometry learning achievement.

Ombay and Roble (2020) investigated using repetition as a teaching strategy with simple and complex variations on students' mathematical thinking. The study aimed to determine the influence of repetition with a complex variation on students' commognition using a pre-test –

post-test approach. The participants were 66 grade 10 students at a high school. Two groups were formed: the experimental and control groups. In this study, the students were not aware that they were participants. This was done deliberately to lessen what is known as the Hawthorne effects. The experimental group received instructions that applied repetition with complex variations, while the control group had instruction with repetition with simple variations. The collected data analysed participants’ commognition abilities on explaining, interpreting and applying aspects. The study results showed the experimental group's commognition mean score was 11.4 on the pre-test and 88.1 on the post-test, while the group that did not receive intervention had a mean score was at 9.5 on the first test and at 48.4 on the second test. Regarding the effect of the treatment, the ANCOVA results showed a probability value of 0.000009, which is much less than the significant level of 0.05. This demonstrates that the substantial difference between the two group was caused by was in the intervention group. Based on these two results, the researchers concluded that repetition with complex variation as a teaching and learning strategy effectively enhances mathematical commognition.

Ahdhianto, Marsigit, Haryanto and Nurfauzi (2020) investigated improving fifth-grade students' mathematical and critical thinking skills using problem-based learning. This quasi-experimental non-equivalent control group study aimed to determine the effects of problem-based learning in improving students' problem-solving and critical thinking skills. Their participants were 78 fifth graders at a public elementary school. The two classes involved in this study were then divided, employing simple sampling techniques, into the experimental group and the other as the control group. The experimental group received instruction using the problem-based learning model, while the control group received instruction using conventional learning methods. During the course of the intervention, the participants were given two tests as part of the pre-test and post-test. One test focused on problem-solving

skills, and the other on critical thinking skills. Data was analysed using the MAVOVA statistical tool. The results from the analysis revealed the following: (1) the pre-test score showed no significant differences between the two groups regarding problem-solving skills and critical thinking skills. (2) The post-test score showed significant differences in favour of the control group regarding problem-solving skills and critical thinking. These results suggest that problem-based learning has an impact on enhancing students' problem-solving and critical-thinking skills.

Johansson and Kilhamn (2022) conducted a study investigating teachers' discourse in introductory algebra instruction in grade 6. The participants of this case study were three grade 6 teachers from three different schools. The introductory lessons on algebra were video recorded. Detailed transcripts from the recordings were then made and focused on the teachers' utterances and gestures. Examination of data was conducted at two levels. At level one, was on the four-features of the discourse of mathematics, viz: the use of words, the use of visual mediators, the construction of narratives, and the employment of routines. At level two, the examination was on the differences and similarities regarding objectification among the participants. The results showed the following: (1) At level one, discursive changes occurred in T2 through consistent demonstration of the processes of reification and alienation. Discursive changes in T1 and T3 discourses were minimal. Their word use was inconsistent and showed few signs of reification and alienation. (2) At level two, aspects of objectified discourse were evident in the discourse of T2 on the use of formal algebraic symbols as visual mediators, generalisations of routines and changes from processual to objectual, T1 and T3 were inconsistent in word use, as well as in the use of informal symbols such as visual mediators and routines. These results suggest that some progress in objectification occurred in a few participants.

Hidayat and Rosnawati (2020) explored students' critical thinking skills in a geometry lesson. Their study aimed to describe three aspects of students' thinking skills: analysing, evaluating and concluding when presented with questions accompanied by pictures and without pictures. This survey research design had 17 participants selected from class VII at the high school level. Data was collected by exploring participant's answers from a geometry test. Results of the analysis of each indicator of critical thinking revealed that there were more correct answers from questions that were accompanied by pictures than from those that did not have pictures. These results suggest that students' understanding is enhanced when geometry material is accompanied by pictures or images.

Bayuningsih, Usodo and Subanti (2018) investigated the degree of critical thinking levels among students in geometry based on self-regulated learning. Their qualitative descriptive study aimed to determine students' level of critical thinking when solving geometry problems in self-regulated learning. For this study, they recruited 32 randomly selected participants from class VII. The participants were allowed to choose either a high, a medium or a low learning task to work on in a self-regulated learning environment. The tasks were then analysed to determine the level of critical thinking, and an interview was conducted to determine the participants' critical thinking ability. Analysis of the results showed the following: (1) On a high category, the degree of critical thinking was found to be at level 2. (2) In the medium category, the degree of critical thinking was at level 1. (3) In the low category, the degree of critical thinking was at level 0. These results suggest that low-engagement tasks do not facilitate the development of critical thinking since these tasks have few opportunities to engage students fully in the principles of problem-solving.

Pratiwi, Nusantara, Susiswo and Muksar (2020) carried out a study to reveal sources of students' commognitive conflict in solving problems. Their qualitative study aimed at determining the source of the commognitive conflict experienced by first-year students in

solving mathematical problems regarding improper fractions. The study participants were 38 first-year university students who had taken the number concept course. Two data-gathering instruments were used in the study. One comprised of the tasks given to participants on improper fractions. These tasks were explicitly designed to bring up commognitive conflict. The other instrument was semi-structured interviews. The interview questions focused on distinguished four features of mathematical discourse, which are; how words are used, how visual mediators are used, how narratives are constructed and how routines are employed. The analysis of data revealed that; (1) The redrawing of visual mediators in detail caused a shift in textual meaning, resulting in commognition conflict. (2) The participants changed strategies for solving the problem, resulting in narratives that could not be endorsed. The new strategy and the old strategy, therefore, caused commognition conflict. (3) Word use on new mediators was consistent and properly used, no commognition conflict arose from this characteristic. (4) The new strategy employed in solving the problem had a clear series of sequential steps characteristic of routines. Therefore, no commognition conflict arose on this aspect. These results suggest that the sources of commognition conflict were from visual mediators and the construction of narratives.

Ziyyadi (2019) conducted a study to describe the student's ability to solve mathematical problems among junior high school students. A qualitative descriptive design was used in this study and was framed by commognitive theory propounded by Sfard (2008). The project comprised of 10 participants chosen from a population of 35 high-ability students. The participants were sampled based on their individual performance in a written task. Out of 10 participants, five had correct answers but did not use a problem-solving strategy; three had wrong answers, though they had used problem-solving, while two used problem-solving, and their answers were correct. These participants were put into two groups, a problem-solving group and one without using problem-solving. The participants were then given a

mathematics algebra problem task to answer. While working on the task, they were required to verbalise their thoughts loudly. As they worked on the task, all observable behaviour and utterances were video recorded. The study showed that 3 participants from the problem-solving group gave incorrect answers, and 2 had correct answers. Two participants from a problem-solving group gave correct answers, while 3 had incorrect answers. Analysis of features of mathematical discourse from both groups revealed that their word use, visual mediators and routines were appropriate, but some participants' narratives were not fully endorsed. These results suggest that students focus more on finding the solution and do not look back to analyse their steps. This shows that they do not use the IDEAL stage of problem-solving.

Nachlieli and Tabach (2022) carried out a study to examine the usefulness of conceptualisation routines in terms of tasks, procedures, and pairs in learning. A sample of 18 prospective elementary teachers at a college of education enrolled in a course on solving arithmetic questions participated in this study. These participants had also learned arithmetic and sequencing in their high school years. For their course, the participants worked on tasks involving three types of questions, focusing on learning specific procedures. Data was collected from the lessons' video recording and the participants' written tasks. Data from each learning episode was then scrutinised in terms of tasks, procedures, and pairs while focusing on the routines employed by participants to solve questions.

CHAPTER 3

Theoretical framework

3.1 Introduction

In this chapter, I present Sfard's (2008) theory of commognition that underpin this study. The theory of commognition is rooted in the sociocultural perspective of human development with the focus on the individual as a participant in collective cultural activities. I used this theory to explore the learner's commognition when engaging with geometry in the grade 11 classroom. I discuss the theory in relation to the learner's mathematical discourse concerning the teaching and learning of geometry

3.2 Background

Mathematics teaching and learning have constantly challenged both the teacher and the learner. The birth of commognition was initiated due to the observed deficiencies in the traditional views of human development. Central to this human development is the process of learning. However, conventional scholars in learning conceptualised learning in different ways. Anna Sfard (2008), positioning herself behind reputable scholars such as Vygotsky (1978) and Wittgenstein (1953), refers to thinking as a process of personal communication. Combining the words communication and cognition, she coins a new word commognition. To arrive at a better understanding of human development and address the quandaries regarding this development, commognition is used to examine the mathematical discourse in classrooms.

3.3 The theory of commognition

The theoretical framework underpinning this study is Sfard's (2008) theory of commognition. According to Sfard, the phenomenon we call thinking and the phenomenon we regard as communication emanates from the same process. *"Thinking is an individualised version of (interpersonal) communicating"* (Sfard, 2008, p. 81).

Meanwhile, in order to communicate, participants in this collective human activity, perform an action in response to a previous action of another participant.

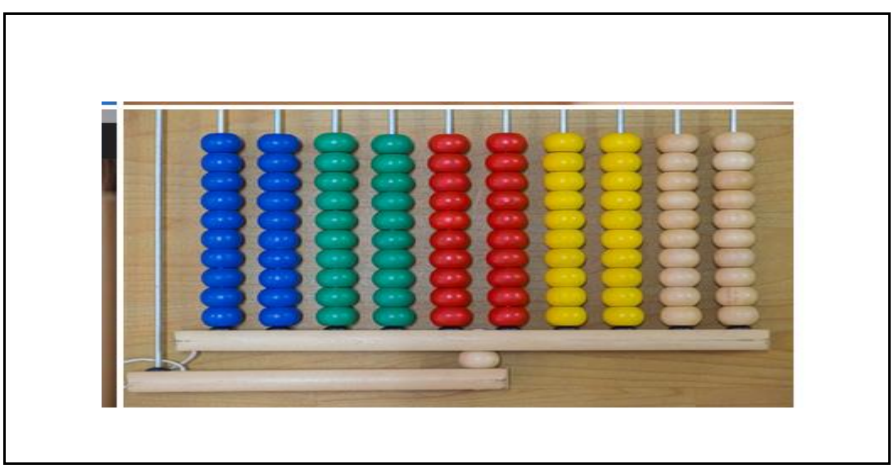
Therefore, in recognition of the two cognitive processes and to symbolise their unity. According to Presmeg (2016) Sfard determined to amalgamate the nouns "communication" and "cognition" into the term ''commognition'' (p. 434).

Communication, now referred to as thinking, is, like most human activities, repetitive patterned ways of doing and is quite often governed by rules. Participants in communicational acts, like players in a game such as rugby or soccer, must follow a prescribed set of rules to be participants. The act of communication has many features. These features include, among others, the agency of the participants. It means participation in communication is a matter of individual choice. As members of a community, we communicate through a unique language and are implicitly bound by some pact and a particular set of rules that hold the community together (Whorf, 1940).

Another feature of communication is about the objects of communication. According to Sfard (2008), there are objects that we use in communicational actions. The object of the communication act is what informs the re-action the actor has to perform. The other feature is that in order to participate in communication, participants have recourse to communicational mediators. This implies that any material may be used or probably adapted and is used as a

means of communication. These mediators may be visual (representations), auditory, symbolic, or in the form of text, artefacts, gestures, and the list goes on.

Figure 3.1: Abacus as an example of the communicational mediator. Source: https://www.123rf. com/photo_56625177_colorful-abacus-toy-for-kids-on-wooden-floor.html



Human beings who form part of the collective engage in many different types of activities, and that being the case, an individual is at liberty to decide whether to take part or not in any of these different activities. The decision to participate in any given activity is a matter of contingency. For example, some learners may participate in a chess game in the school environment. However, the learner who chooses to participate in the game of chess must first familiarise himself or herself with the names of the various chess pieces, know how these chess pieces are placed on the chess board, and still how these chess pieces are moved around on the chess board from one square to another, their names and their individual roles in the game of chess. On the other hand, some learners might regard chess as a complex game, and they may decide to exclude themselves from this game. Instead, these learners may opt to participate in other games they consider not too complex. There are many different types of games, and these games are often played under different rules and perhaps with different

tools. In the same manner, there are many different types of communication that differ from one another regarding the rules that govern participation, the objects of the particular communication, and the types of communication mediators used. Therefore, like in chess, individual participants in any type of communication have contingency and may decide to be excluded from one particular communicational activity while opting to participate in other activities. Sfard (2008) says this about the different types of communication, "*The different types of communication, and thus of commognition, that draw some individuals together while excluding some others will be called discourses".* (p. 91).

3.3.1 Tenet of commognition

The theory of commognition is based on the participationist viewpoint of "*patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual"* (Sfard, 2008, p. 78). Inspired by Wittgenstein (1953) and later by Vygotsky (1978), participationists assert that higher cognitive processes, in this case thinking starts in social interaction with other members of the collective. That learning is not developed naturally, like other processes in the human body, but happens because of association with other collective members (Sfard, 2008). The human agency plays a significant role in spurring the individual's quest to individualise (internalise) any human activity. Sfard (2008) refers to this process of individualisation as turning discourses from others into discourses of one's - self.

Vygotsky (1978), contends that learning occurs when there is communication between a learner and a more skill-full adult or peer. The precise role of the adult or peer is to steer the learner into a process that may lead the learner into more complex problem-solving situations in a collaborative environment. This learning occurs in what Vygotsky refers to as the Zone of Proximal Development (ZPD).

For the acquisitionists, they regard the metaphor of learning as acquiring knowledge, and in this learning act, the resulting product is an increase in individual possession (knowledge, skills, mental schemas).

However, to the particitionist, what is fundamental is the tendency for individualisation, that is, the turning of patterned collective doings into activities for an individual.

The unit of analysis for the participationist is, therefore, both collective and individual, while for the acquisitionist, the focus is on the individual. For the particpationist, developmental transformations arising from learning result from two complementary processes: individualisation of the collective and collectivisation of the individual. This is what is known as learning.

The distinctions as a result of the acquisitionist and the participationist viewpoints on the version of human well-being, and summarised by Sfard (1998) are illustrated in Table 2.1 below.

Table 3.1: Acquisition and Participation Metaphors (Adapted from Sfard, 1998, p. 7)

The Metaphorical Mappings

Acquisition metaphor		Participation metaphor
Individual enrichment	Goal of learning	Community building
Acquisition of something	Learning	Becoming a participant
Recipient (consumer), re-constructor	Student	Peripheral participant. Apprentice
Provider, facilitator, mediator	Teacher	Expert participant, preserver of practice/discourse

Property, possession, commodity (individual, public)	Knowledge, concept	Aspect of practice/discourse/activity
Having, possessing	Knowing	Belonging, participation, communicating

3.3.2 Properties of commognition

There are two closely related aspects of commognition (thinking). Those that are related to commognitive objects and those that are related to the performer of the discursive action. These related aspects mentioned above are bounded with necessary conditions by what Sfard (2008) refers to as recursivity of human commognition. Recursion is a,

"Feature of language thanks to which every legitimate linguistic construct may give rise to a new, more complex one, provided we replace some of its simple elements with more complex linguistic constructs. Recursivity allows for turning one commognitive act into an object of another and thus accords the property of reflexivity to our discourse". (Sfard, 2008, p. 301).

These conditions are now presented:

(1) Reasoning;

Reasoning is the act of the examination of utterances in order to arrive at an appropriate reaction to new communicative action. Sfard (2008) defines reasoning as; "the art of systematic derivation of utterances from other utterances" (2008, p. 110). This ability to examine relations between utterances is part of recursion.

(2) Abstraction;

Abstracting refers to considering different groups of processes and presenting them as a single unit in the form of a name or symbol. For example, the word *multiplication* implies several processes of addition.

$$2x2x2x2x2 = 2^5$$

Repeated addition = multiplication.

(3) Objectifying;

Objectification is the main goal of learning mathematics. For any mathematical concept, objectification is central and is, therefore, a prerequisite for achieving the goal of learning mathematics. There are two crucial fundamental elements of objectification. These are word use and the use of mathematical symbols. Words and symbols play an important role in mathematics. Therefore, objectification is pivotal to how languages or words are used, more especially regarding instruction in the mathematics classroom. Words are vital, and the convey meaning in a language. Language is a medium of communication made possible by a collection of random symbols and governed by a particular grammar which informs this set of symbols (Sfard, 2008).

There are also two processes that deserve attention in objectification. These are the concepts of reification and alienation. According Sfard's (2008) theory, reification is to talk about a mathematical object instead of talking about the process.

In other words, a reified utterance calls for introducing nouns or written symbols. For example, "*The learners are unable to comprehend this algorithm in simple terms*". The reified version of this utterance would be "*They have a learning disability*".

Alienation is applying discursive forms to introduce phenomena in an impersonal way without referring to human participation in the activity (Sfard, 2008).

For example, *"I add five plus seven and I get twelve".* This utterance is alienated and represents the presence of the participation of a human being. The alienated version of this utterance would be "*add five to seven equals twelve*". In this utterance human participation in the discourse is entirely eliminated.

There is another aspect of word use that concerns objectification. This aspect is about reification. When the focus is on mathematical processes, the discourse can be characterised as process-oriented, while if the focus is on mathematical objects, then the discourse is regarded as object-oriented.

These two aspects are now shown in table (3.2).

Table 3.2: Examples of how different ways of using natural language concerning the symbolic expression 5 +7 = 12 results in different degree of objectification of the mathematical discourse. Sources*:* Researcher's own initiative derived from the review of literature.

5+7=12	Process-Oriented	Object-Oriented
Personified	If we add five and seven, we get twelve	We have five plus seven which is equal to twelve
Alienated	Five plus seven becomes twelve	Five plus seven is equal to twelve

Table 3.3: A Sample of some mathematical nouns. Source: Researcher's own inititive derived form the review of literature.

A sample of some mathematical nouns
Arabic numerals, Roman numerals, Fractions, Variables, Expressions, Figures, Infinity, Pi and Imaginary numbers like i

(4) Subjectifying;

Sfard (2008) refers to subjectifying (identifying) as an exceptional conditioning of objectification that happens when attention is diverted from the actions and their objects to the performers themselves.

For example, subjectification may be examined in the following utterances;

1. Mrs. Zulu teaches physics.
2. Mrs. Zulu is a physician.

In analysing the first utterance, one would notice that the focus is on Mrs Zulu's action (teaching). When examining the second utterance, the focus shifts to Mrs. Zulu as a physician by profession. The second utterance is a reified version of the first and shows attempts of reflection and some elements of permanency concerning the subjects' properties.

(5) Consciousness;

Sfard (2008) refers to consciousness as an aptitude to act now and then make judgement regarding one's on thinking. To put it in other words, it is the ability to monitor meta utterances and assist concerning the necessary and appropriate course of action. For example,

knowing about the existence of object B means there is interpersonal communication (commognition) about object *B*.

3.3.3 Why do discourses change?

Discourse is a patterned form of collective human activity. As a patterned activity, discourses are processes and are therefore liable to change over time and space. Discourses are means of communication, and people are often interested in increasing the effectiveness of their communication and practical activities. Sfard (2008) regards change in discourse as a product of the historical change in human ways of doing, characteristic of unvarying rhetorical thinking, is responsible for intricate development in discourse is informed by changes in all spheres of human communication.

There are two causes of discourse change; one is intentional change, and the other is not intentional change.

As independent agents, members of a particular discourse often intentionally modify and individualise their discourses' rules for individual expression. However, individual expression is needed so that it feeds into communalism to avoid a mismatch in certain aspects of the discourse. Communication breakdown is avoided when individualised expressions are creatively embedded in the broader discourse. The new embedded expressions then increase communication effectiveness and may improve practical activities.

On the other hand, an intentional change in discourse does not happen without the conscious effort of an individual. This type of change happens over time and space and results from the broader developmental change in the community. Human activity of doing, through innovation, changes from generation to generation. The need for change is propelled by the need to make practical activities effective. This change, therefore, causes a change in discourse.

3.3.4 How do discourses change?

In human forms of commognition, recursivity as a property of commognition, plays a central part in the developmental process discourses. New layers of discursive discourse about discourse are often added in succession. With objectification, new discourses begin to build on previous discourses. This building process results in an increase in the complexity of the resultant discourse.

There are two types of discursive expansion of discourses. One type is called endogenous (from within) expansion, and the other is exogenous (from outside) expansion. In this context endogenous expansion is a result of progressive growth in the size and intensity of discursive routines as well as narratives that are endorsed in one’s discourse, whereas exogenous expansion is the discursive multiplication that results in new discourses (Sfard, 2008). Endogenous expansion is often caused by constant usage of the discourse's general use over time and space. Due to its constant use, the complexity increases, and the need to compress the resultant long chain of the new discourse comes in. An example of this type of expansion is illustrated in Table (3.4).

Table 3.4: An Example of changes resulting from discursive expansion and successive compression. Source: Researcher's own initiative.

The following is an example of a description of calculating the future value in an investment. *Add one to the interest rate, then raise this sum to the power of the number of payments made; from this value, subtract one and then multiply by the value of the monthly payment* An equivalent discursive procedure can be presented in the compressed form as $Fv = \frac{X[(1+i)^n - 1]}{i}$

Where x= the monthly payment

i = the interest rate

n = the number of payments made

Fv= the future value

The other, exogenous expansion is caused by the need to grow a new discourse. An illustration of exogenous expansion is given in the following example;

A learner in grade 4 familiar with whole numbers may be able to understand that,

$$5 < 7 \text{ and } 2 > 1$$

or

$$1 multplyby 5 < 1 multiplyby 7$$

and

$$1 multiplyby 2 > 1 multiplyby 1$$

However, when the learner is now introduced to negative numbers, the learner may not be able to understand that,

$$-1 multiplyby 5 > -1 multiplyby 7$$

and

$$-1 multiplyby 2 < -1 multiplyby 1$$

In the example given above, there is a need to grow a new discourse to enable the grade 4 learner to understand. This new discourse that is introduced when the learner is exposed to a negative number will often require the intervention of an expert for it to be added to the old discourse that the learner is already familiar with.

3.3.5 Mathematics as a form of discourse

There are many discourses, such as the discourse of biology or chemistry, and the list goes on. Mathematics is also a discourse focusing on mathematical objects such as geometrical shapes, numbers and sets, just to mention a few. However, it must be argued that mathematical discourse is very distinct (different) from these other discourses found in the classroom. What makes mathematical discourse distinguishable from the other discourses is a result of two aspects. The first difference is in the aspect the unique nature of the objects of mathematical discourse. The objects employed in mathematical discourse are not tangible, but abstract constructs (Sfard, 2008). Unlike in other discourses, for example, the discourse of biology or the discourse of chemistry whose objects are perceptible, mathematical objects are not perceptible. The second aspect is that the object of talk is about mathematical objects and are discursive entities.

Mathematics as a discourse differs from other discourses by its characteristic use of key specific purpose words (mathematical keywords and keywords from ordinary language that have mathematical meaning), its perceptually attainable use of visual mediators to aid communication, and its use of routines. With these discursive mechanisms at their disposal, participants in mathematical discourse can communicate and construct narratives about mathematical objects.

However, there are two different ways in which mathematical communication is carried out. One type of mathematical communication is regarded as colloquial mathematical discourse. This type of mathematical discourse is constituted by everyday language. The other type of communication is called school mathematical discourse or literate mathematical discourse and is constituted by specialised symbols and the rigorous use of key-specific purpose words. This type of communication is practised in the mathematics classroom. School mathematical

discourse is mathematicians' focus and differs from colloquial in some features. The illustration in Figure 3.2 shows a sample of these two types. Sample 1 is colloquial mathematical discourse, while sample 2 is mathematical discourse practised in the mathematics classroom.

Fig 3.2: Samples of mathematical discourse. Source: Researcher's own illustration.

<table>
<tr><td colspan="2">

Sample 1: A conversation with a 6-year-old boy

Themba: Sibusiso, how much do you have?
Sibusiso: 10 rands.
Themba: What about Pholile, how much does she have?
Sibusiso: 14 rands.
Themba: Then, who has more money?
Sibusiso: Ohm…. I'm not sure.
Themba: Think about it.
Sibusiso: Pholile has more.
Themba: How do you know?
Sibusiso: ten, eleven, twelve, thirteen, fourteen
{after each number word, he bends a finger}

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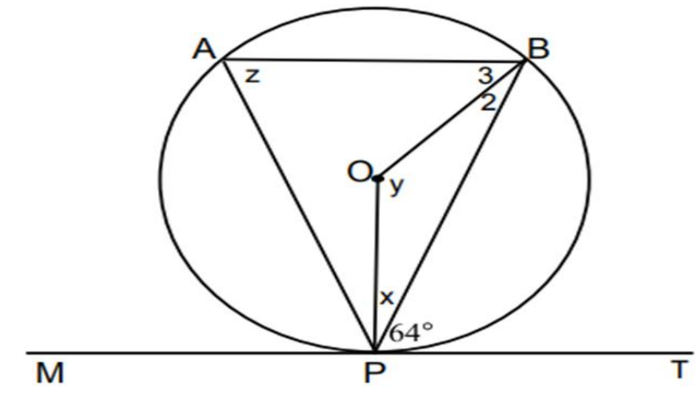
Sample 2: A school problem



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Solution

$x = 90^0 - 64^0$
$x = 26^0$
OP is the radius ∴ is perpendicular to MT
$x = OBP$, *base <s of Isosceles* Δ
$\therefore y = 180^0 - 26^0 - 26^0$
$y = 128$,*sum of* Δs

</td></tr>
</table>

	$z = \frac{128}{2}$, *Angle at centre theorem* $\therefore z = 64^0$

There are four critical properties that determine whether a particular discourse can be regarded as mathematical. These are word use, visual mediators, narratives, and routines.

(1). *Word use.* A unique property of mathematical discourse is the rigorous use of key-specific purpose words. Using keywords for specific purposes in mathematics reflects how the user interprets the environment. There are specialised words in mathematics, such as mathematical shapes, words that signify quantities, number words, words about mathematical relationships, etc. These words include tangent, perpendicular, bisector, triangle, polygon, radius, quadrilateral, five, eight, sum, and subtract. And lastly, words from an ordinary language have mathematical meanings, like the point where the acute angle meets the line.

(2). *Visual mediators.* Visual mediators are visual objects employed to aid communication. These are symbolic artefacts such as algebraic notations, diagrams, graphs, scientific notations, tables of functions, and so on. These visual mediators are used in mathematical discourse to mediate mathematical objects to make communication effective.

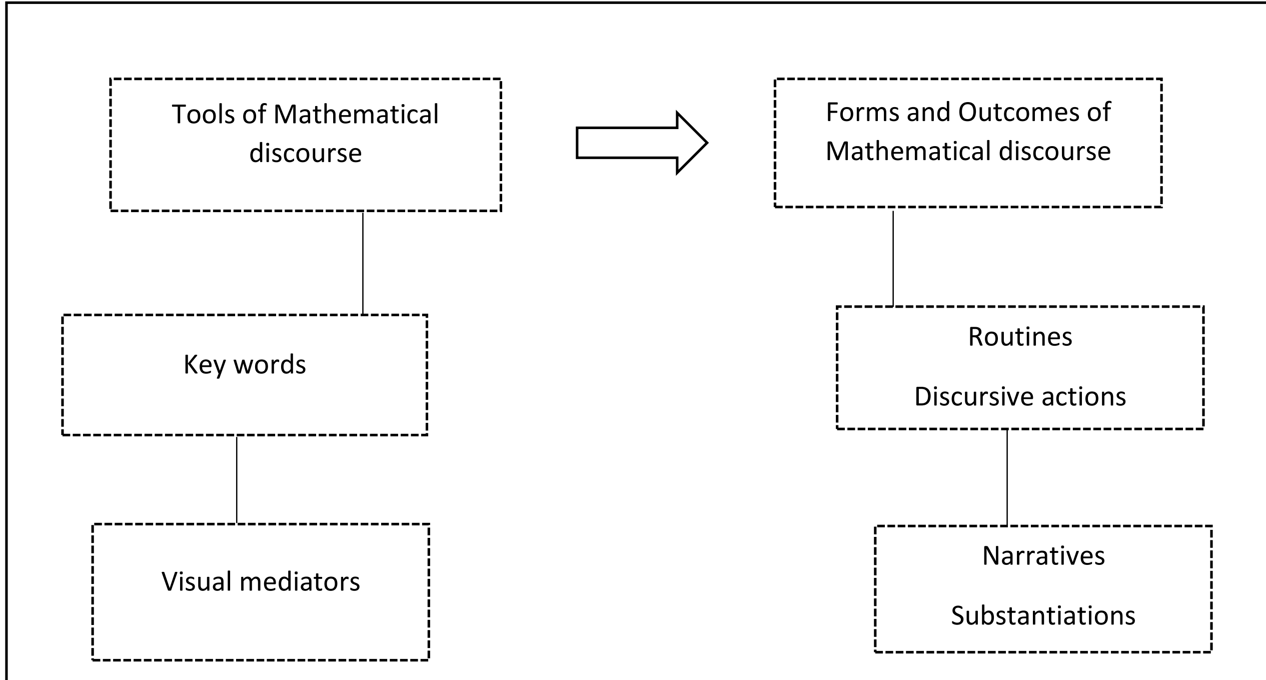
(3). *Narratives.* Narratives are stories about mathematical objects, relationships between mathematical objects, processes of mathematical objects, or processes with mathematical objects. Narratives face endorsement or rejection by way of substantiation procedures. Endorsed narratives are considered authentic if the narrative meets the acceptable requirements recognised by the community of mathematicians. Definitions, proofs, theorems and axioms are some examples of endorsed narratives. However, the terms and criteria for endorsement may vary depending on a particular discourse. For example, in discourse

practised by academicians, the criteria for endorsement are much more stringent than the ones used in school mathematical discourse.

(4). *Routines.* Routines are repetitive sequence of aspects of a particular discourse (Sfard, 2008). Mathematical discourse is laden with repetitive patterns, such as the process of generating substantiated narratives, the enactment of procedures when solving algebraic equations, or the process of counting even /odd numbers. According to Sfard (2008), repetitive patterns can also be noticed in most collective human activities

Figure 3.3: The Four Distinguishing Aspects of Mathematical Discourses.

Source: Researcher's own illustration.



3.3.6 Visual Realization of Mathematical Signifiers.

Mathematical communication relies on the use of objects that can be perceptually accessed. Sfard (2008) call these perceptually accessible objects signifiers and realisations. *"Realization of the signifier S is a perceptually accessible thing S' so that every endorsed narrative about*

S can be translated according to well-defined rules into an endorsed narrative about S' ". (p. 154).

Here are examples of signifier-realization pairs;

(1). $54 + 26 = 80$

(2) quadratic equation = Table of values

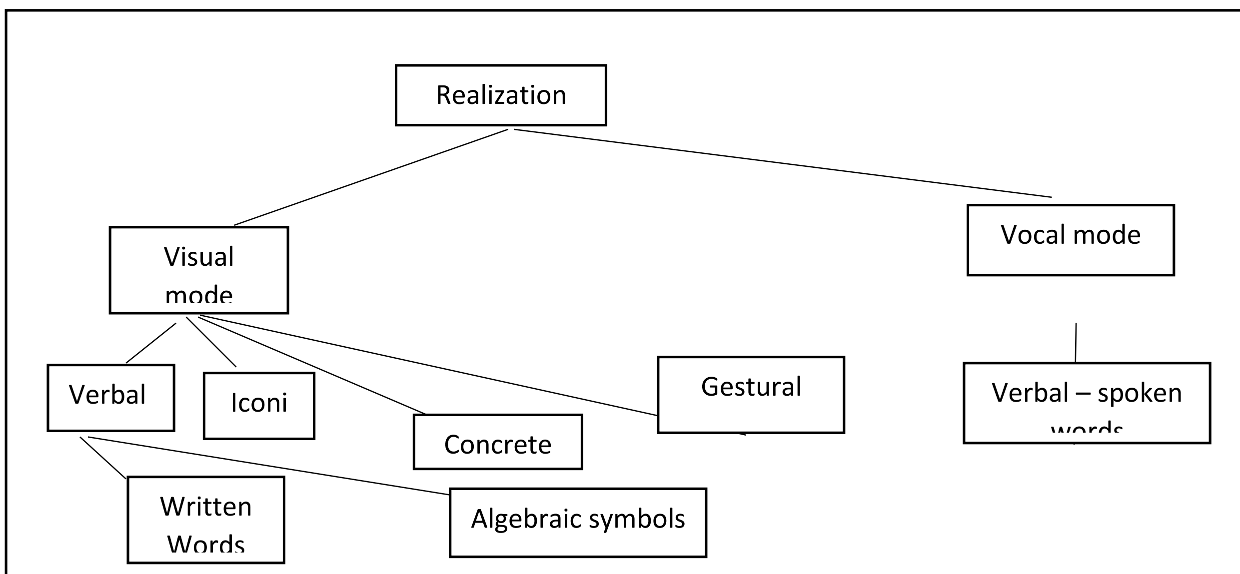
(3) $80 = 26 + 54$

A signifier in one relation can become a realisation in the same relation as in the first and third examples.

According to Sfard (2008), presenting signifier-realization pairs in visual media is essential to communicating mathematical objects. Presenting mathematical objects in signifier-realisation pairs makes it possible for both the doer and the observer to navigate to endorsed narratives more quickly.

However, several different modalities can also mediate signifiers and realisations in mathematical discourse. Figure 3.4 shows some different modalities of signifier realisation in mathematical discourse.

Figure 3.4: Different types (modalities) of signifiers' realisation in mathematical discourse (Adapted from Sfard, 2008, p. 155)



3.3.7 Learning mathematics

In learning mathematics, two types of learning are distinguished, viz: meta-level and object-level learning.

3.3.7.1 Meta-level learning

Meta rules are narratives that govern what may be considered as acceptable discursive patterns. For example, the rules by which narratives may be rejected or endorsed, like geometrical theorems, proofs, or definitions.

Concerning narratives, meta-level learning also entails changes even in the rules that govern a particular discourse (Sfard, 2012). An example would be a participant who is about to be introduced to the discourse of fractions when this particular participant already knows the discourse of natural numbers and might require some level of meta-level intervention to work on fractions.

Meta-level learning involves changes in;

(1) Endorsed narratives,

(2) Changes in rules that govern the discourse,

(3) Expansion of endorsed narratives with recourse of pre-existing rule

3.3.7.2 Object-level learning

Object-level learning is concerned with mathematical objects. Learning about mathematical objects means learning about the properties of these objects and their processes.

An example of object-level learning;

Table 3.5: The Commutative property on whole numbers. Source: Researcher's own initiative.

The Commutative property
$12 + 13 = 25 \wedge 13 + 13 = 25$
$10 \times 30 = 300 \wedge 30 \times 10 = 300$

Table 3.6: The Associative property on whole numbers. Source: Researcher's own initiative.

The Associative property
$(10 + 20) + 5 = 30 + 5 = 35 \wedge 10 + (20 + 5) = 10 + 25 = 35$
$(2 \times 12) \times 4 = 24 \times 4 = 96 \wedge 2 \times (12 \times 4) = 2 \times 48 = 96$

Therefore, the product of object-level learning is an increase about the number of routines and endorsed narratives. As one learns more about these mathematical objects and processes, the degree in the complexity of the discourse also increases (Sfard, 2008).

When learners are exposed to more complex discourse, they strive in this aspect as a result of contingency, to individualise (internalise) their discourse and move away from participating in the discourse for others to participating in discourse for themselves. Through their participation in the discourse, and the fact that they are new entrants to the discourse, commognitive conflict is unavoidable. Commognitive conflict may be regarded as originating from condition when conversational partners use seemingly un related discourses in the course of their communication (Sfard, 2008). However, commognitive conflict is not a hindrance, but is regarded as a necessary part of learning. For example, just like a child being initiated into walking, stumbling along the way may not be avoided, as it is a necessary and vital part of the learning curve on the way to learning how to walk.

If the situation arising out of commognitive conflict is left unresolved, the situation may result in negative implications on the learning process. One way of resolving commognitive

conflict is by bringing together all the participants in the discourse and coming to a consensus (Sfard, 2008; Ben-Zvi & Sfard, 2007). There should be consensus on (1) defining the role of the teacher and the learners, in other words, the teacher agreeing to lead and the learners ageing to be led, (2) a determination about the leading discourse and (3) a determination about the nature of the expected change. These agreements together constitute what is referred to as a learning-teaching agreement.

Sfard (2008) provides a caution about commognitive conflict and the notion cognitive conflict, viewed by acquisitionists as unnecessary in learning. The fundamental point of departure between the acquisitionists and commognitivists is that they have different standpoints about the locus of the conflict. Some of these differences are given in the following table.

Table 3.7 below summarises these differences concerning the conflict.

Table 3.7: Comparison of concepts: cognitive conflict verses commognitive conflict (Adapted from Sfard, 2008, p. 258)

	Cognitive conflict	Commognitive conflict
The conflict is between:	The interlocutor and the world	Incommensurable discourse
Role in Learning	Is an optional way to remove mis-conceptions	Practically indispensable for meta-level learning
How is it resolved	By learner's rational effort	By learner's acceptance and rationalisation (individualisation) of the discursive ways of the expert interlocutor

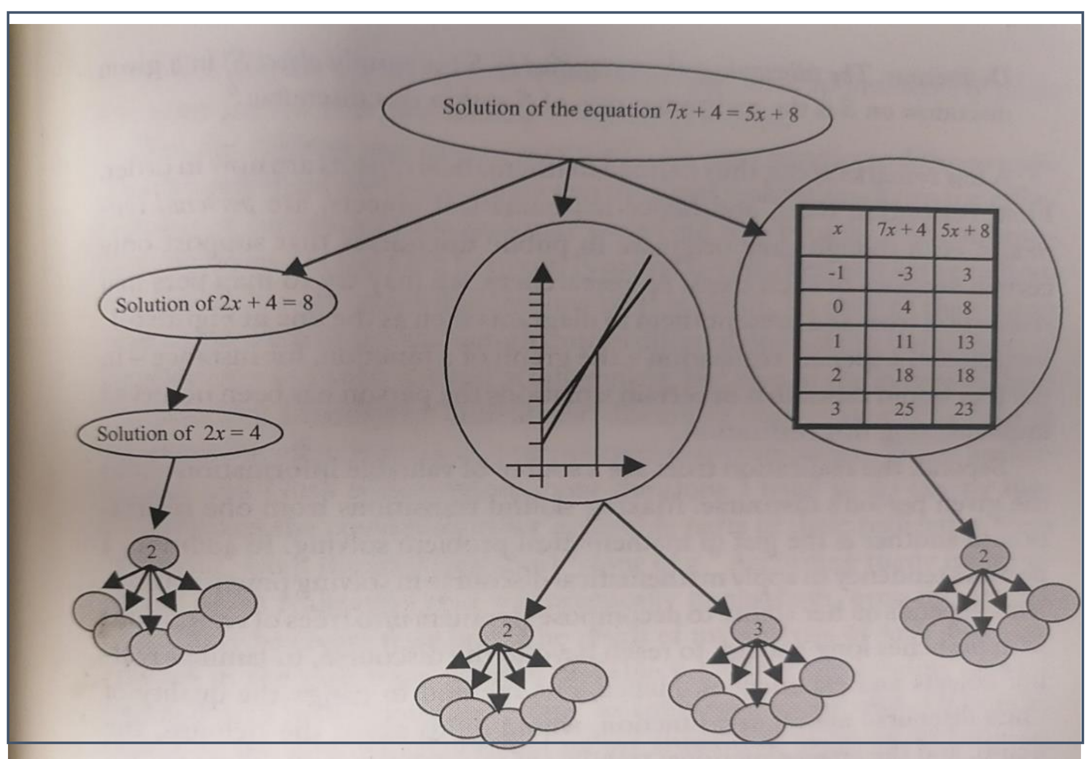
3.3.8 Mathematical objects

Mathematics uses symbols and representations in the form of shapes, diagrams, graphs, tables and many more. Mathematical objects are a product of human construction and are intangible discursive objects (Sfard, 2008).

3.3.8.1 Discursive object

In the theory of commognition, mathematical objects may also be referred to as realisations. The term realisation is preferred for two reasons; the first reason is to enable mathematical objects, which are often intangible, to be perceptually accessible (realisations are tangible). Second, a given signifier may be realised in multiple ways. Mathematical signifiers often have multiple numbers of realisations of solutions of a particular mathematical object. The gist of mathematical learning is recourse to multiple solutions to a given task. Sfard (2008) goes on to define a realisation as "*The (discursive) object signified by S (or simply object S) in a given discourse on S in the realisation tree of S within this discourse"* (p. 166.). This implies that any given realisation in one situation, has the potential be turned into a signifier and further realised in the same or similar situation. Signifier and realisation form a pair, of which several pairs can be formed. Figure 3.5 illustrates a realisation tree and the signify-realisation pairs.

Figure 3.5: An example of a realisation tree with a number of branches of the signifier "solution of the equation" (Adapted from Sfard, 2008, p. 165).



x	$7x+4$	$5x+8$
-1	-3	3
0	4	8
1	11	13
2	18	18
3	25	23

In the example in Figure (3.5), the realisation to the solution of the equation

$7x + 4 = 5x + 8$."

is realised through the use of three discourses.

The first realisation is by way of algebraic discourse to simplify the equation as follows;

$7x + 4 = 5x + 8$ (Signifier)

$2x + 4 = 8$ (Realization)

$2x = 4$ (Signifier)

$x = 2$ (Realization)

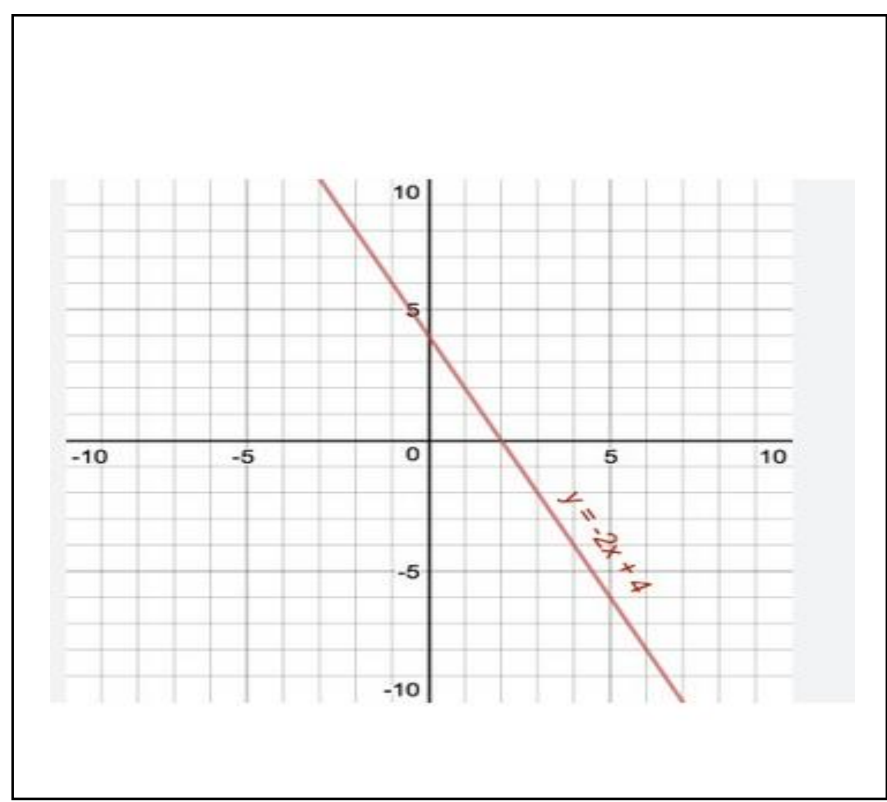
In this example, there are two pairs of signifier-realization pairs.

The second realisation is by way of the discourse of graphs;

The graph of $2x = 4$

$y = -2x + 4$

Figure 3.6: Realisation of y = 2x + 4. Source: Researcher's own initiative.



The third realisation is by way of the discourse of ordered sets of pairs

Table 3.8: Realisation of 7x + 4 = 5x + 8. Source: Researcher's own initiative.

x	$7x + 4$	$5x + 8$
-1	-3	3
0	4	8
0	11	13
2	18	18
3	25	23

According to Sfard (2008), realisations have the following inherent properties which have to be considered;

- *Personal constructs* - Participants in mathematical discourse are individuals in this collective human activity. However, their participation is varied. Just like the variations found in their participation, the same variation is found in individual thinking processes, and realisations being perceptually accessible may show these individual variations. The individualisation of the discourse accounts for these variations. Therefore, considering the above, one learner's realisation tree may be different from another learner's realisation tree in terms of structure and the information shown
- S*ource of information* - A realisation tree holds valuable information about the given person′s discourse. A realisation tree provides a window to discern an individual's commognition. Examination of the realisation tree provides information about the richness or otherwise of the individual's discourse. The implementation of the *how* and *when* of the routines and how the procedures feed into each other, as well as the nature of endorsed narratives, can be observed when examining the realisation tree of the individual.
- *Interlocutors* - Distinct interlocutors may realise the same signifier in multiple ways. A single signifier often would have multiple representations to arrive at the same solution. This implies that these multiple representations of the solution of the given signifier may be located in different interlocutors. An individual who has recourse to multiple representations of signifier-realization pairs demonstrates the effectiveness of his or her own interpersonal communication. (Sfard, 2008, p. 166).

3.3.8.2 Individualisation of mathematical objects

Individualisation of the discourse is one of the products of learning. Individuals have agency, and by participating in human forms of "doing" while reflecting on their participation, the need to individualise their actions arises. Since mathematics is a discourse, and participation

in this type of discourse requires one to manipulate mathematical objects, the need to individualise these mathematical objects is imperative.

However, participation in mathematical discourse involves using features like keywords and symbols. Using keywords and symbols may contribute to some of the challenges new entrants to mathematical discourse face. Newcomers to mathematical discourse may not have acted in similar situations before and, therefore, have no precedents to fall back on. Consequently, in their new-found situation, the newcomer may have no other recourse but to imitate expert performers or too familiar colloquial discourses.

Individualisation of mathematical objects is a process. To aid this process, the newcomer, with the help of an expert performer, has to be gradually introduced to mathematical objects. The individual needing individualising mathematical objects must constantly participate in the new discourse. The aim of learning according to Sfard (2008), is to able to repeating a preliminary set of series of actions when faced with a new situation. By constantly participating and simultaneously reflecting on the new discourse, the newcomer, who now may have an array of precedents to fall on, may indulge in individualisation and turn the discourse from that of others to that of himself.

3.3.9 Four-stage model of the development of word use

Word use is essential in developing any discourse, particularly mathematical discourse. Sfard (2008) proposes four stages in the development of word use. These are:

Passive use: This is the first stage in the development of word use. Participants cannot use words independently but respond to others using the word.

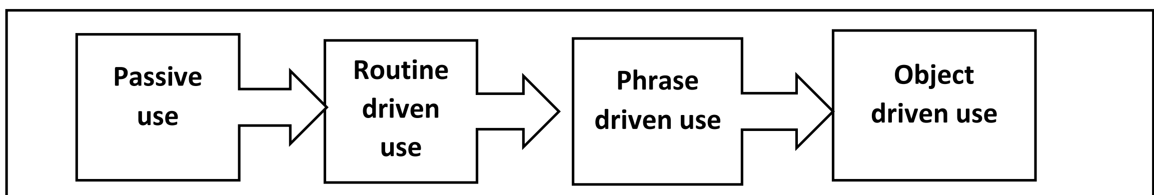
Routine-driven use: Participants at this stage learn to use the word in their utterances in the context of mathematical routines. They do this without imitating others, but only in a few routines.

Phrase-driven use: As their confidence in using the word grows, participants may begin to use it independently in various routines but in repeated phrases.

Object-driven use: As their confidence expands through constant participation, participants learn to use the nouns more independently across different contexts. The only restriction they might face concerning word use is probably related to grammar. Their word use at this stage can be said to be objectified.

Participants initiated into mathematical discourse begin by using mathematical terminology passively, though they cannot use the words (keywords) independently.

Fig 3.7: Four-stage model of the development of word use (Adapted from Sfard, 2008, p. 182)



3.3.10 Routines

Repetitiveness underpins human collective activities of doing. Repetition plays an integral part in communicative activities.

Having to repeat the same action over and over again is the starting point of making communication effective. When an individual reacts to a given situation, it is on the basis that the individual might recall having been in a familiar situation before that he can recall how to carry out an action resembling the previous one.

Rules of Discourse: Like any other discourse, mathematical discourse is guided by rules. The rules are distinguished between object-level rules and meta-discursive rules.

(a). Object-level rules are narratives about the behaviour of properties of mathematical objects of the discourse. For example, *the angles of equilateral triangles are equal to = 60⁰.* Similarly, *the angle sum of a triangle equals 180⁰.* These are object-level rules.

(b). Meta discursive rules concern the patterned activity regarding enacting and substantiating object-level rules. Meta-discursive rules are, therefore, focused on the actions of the discursant and not on mathematical objects. For example, *to multiply the sum of two and six by five, one might multiply 5 x 2 and 5 x 6 and then add the products.*

Variability. Meta-discursive rules undergo variability from time to time. These rules gradually evolve over time and space. Over time, discursive patterns are transformed through the modification of discourse. Participants of mathematical discourse can reflect on the discourse in their community and, where necessary, may modify the discourse to make communication more effective. The step-by-step transformation of metarules in a student's mathematical discourse is among the aims of school mathematics (Sfard, 2008).

Tacitness. Metarules have two subsets, and these are endorsed and enacted. This implies that constituted metarules save the purpose of describing the discourse as is, whereas endorsed metarules conveys the way the discourse is perceived with respect to the participants (Sfard, 2008). Discursants, as individuals, bring their varied subjective experiences and multiple perspectives to the discourse. Because of varied individual experiences, their interpretation of phenomena is also varied. As discursants participate in the discourse, they continuously reflect on the emerging patterns in their own actions and develop individual ways of enunciating metarules. An example of such enunciating can be observed when discursants define a given definition. Two participants may define a particular concept using different words.

Value-Ladedness. Metarules are practised by the broader community of participants in mathematical discourse. Because most members practise this, metarules become valued and develop normative characteristics.

Flexibility. When discursants participate in mathematical discourse, they often reflect on the metarules of the discourse. It is through constant reflection that discursants develop the ability to realise different possibilities of manipulating metarules to have the same outcome. For example,

$$f(x) = x^2 + x - 30$$

In this function of x, there are many possible ways in which the solution can be realised.

Contingency. Contingency applies to mathematical metarules. The procedure involved when discursants associate endorsed and enacted rules due to contingency inherent in mathematical metarules. Mathematical communication becomes effective by associating the endorsed rules with their counterpart enacted rules.

3.3.11 Learning as routinization

Patterns are noticeable in all forms of human activity, and learning is one of them.

According to Lavie et al. (2019), people learn by associating present and similar past actions. For example, a learner who can use BODMAS would work out the problem and find the solution as shown;

$$2 \times 4 - 5 = 3$$

On another day, the learner is then required to work out another problem;

2 x 6 divided by 2 + 4

The learner might recall BODMAS and work it out like this

2 x 6 divided by 2 + 4 = 10

In this example, the learner can associate the latter with the former. The ability to align present actions to similar but previously done actions in a patterned way is called the routinisation of actions.

Participants in collective human forms of doing develop routines by individualising those actions done by more experienced performers (Lavie et al., (2019). This individualisation enables new entrants to the discourse to navigate from being peripheral participants to becoming fully-fledged members who can now fully participate. Individualisation develops by constantly imitating other people's actions until those actions become owned by the individual. When the individual owns those actions, she becomes an independent performer capable of implementing the routines with an agency.

The table 3.9 below is another example of routinisation

Table 3.9: An example of routinisation. Source: Researcher's own initiative.

	Name of Speaker	What is said by speaker	What is done by speaker
1.	Simphiwe:	It looks like a zebra, what about the other animals?	Looks at the magazine, scratches her shoulder, moves her lips as if trying to speak.
2.	Mikaela:	Can I move this animal to….?	Moves the animal to the other group, looks at her grandfather.
3.	Simphiwe:	Why…do you want to do that?	Looks at the magazine again, scratches her shoulder once more, tries to speak in a low voice.
4.	Mikaela:	Because…	Moves the animal back to the previous group and starts to counting the legs, one, two, three, four.
5.	Simphiwe:	What are you counting?!	Looks at Mikaela, and then at grandfather.
6.	Mikaela:	The legs, four legs	Looks at the magazine, and then at Simphiwe and lastly at her

			grandfather and smiles.
7.	Simphiwe:	Yes, four legs all the animal with four legs are in this group. Now there are five animals in the group. How many legs are there altogether in this group? Twenty because you have to multiply five by four. Another to do this is first draw a circle around all the legs to count all the legs in the group. Then start counting, One, two, three, four, five, six and so on. I don't know which method is best for you.	

In the example of routinisation, the participants' discursive actions are governed by metarules. This set of patterned metarules is further broken down into two subsets:

(1) The *how* of a routine: This subset contains metarules concerning the way to approach and implement the strategy in a given task.

(2) The *when* of a routine: This is a subset of metarules that govern when the situation is appropriate to call for implementing those actions needed to perform the performance.

The *how* of the routine is concerned with the course of action in the task situation, whereas the *when* of the routine is concerned with the application conditions and closing conditions of the routine. Individual participants may implement the *how* and the *when* of the routines in different ways. Sfard (2008) points out that implementing these subsets of routines is a personal construct.

In the context of learning mathematics, Lavie el at (2019) say that routine can be defined as "*a routine performed in a given task situation but a given person in the task situation, as seen by the performer together with the procedure she executed to perform the task*". (p. 161).

They point out that a similar activity in the former constitutes what they referred to as *precedent-search-space* (PSS). They assert that in a task situation, a participant is given to act on specific requirements that need to be fulfilled. These requirements are as follows:

(1) To act in a task situation, the participant should have recourse to precedents.

(2) Identifying the appropriate procedure requires a certain degree of pre-selection of the precedent-search space (PSS).

(3) That Precedent-Search-Space can be found with the aid of precedent identifiers.

For example, learners are given a task situation where they are to work in groups to solve simultaneous equations by using three methods, viz; elimination, substitution, and graphical methods. On another given day, the learner is required to work on her own to solve simultaneous equations. The learner might recall acting in a similar situation. The precedent identifier might be the group she participated in while solving the equations. When the precedent identifier is in place, the learner will identify and enact the three methods of solving simultaneous equations, given that PSS is in place.

Participants in any form of discourse, such as mathematical discourse strive to individualise it based on their commognition. The implications for individualising the discourse are that the interpretation of any aspect varies from individual to individual. Therefore, identifying an appropriate routine in a given task situation is an interpretive activity for participants and the observer of the discourse.

3.3.12 Kinds of routines

Two kinds of routines may be distinguished. The first kind is referred to as practical routine, and the second kind is a discursive routine. These two kinds of routines differ by the nature of the desired outcome of the performance.

According to Lavie et al, (2019), a practical routine mainly focuses on the object of the discourse. The end result of a practical routine in a task situation is a change in the object. For example, opening a previously closed door, riding a bicycle from one place to another, or preparing lunch are practical routines that change the object. On the other hand, a discursive routine has its main focus on relaying communication. Discursive routines are patterning those participants in a task situation as they implement communicative actions. The required communication may be interpersonal or with other participants of the discourse.

Table 3.10: Classification of routines according to Lavie et al., (2019, p. 166)

Routine	Process-Oriented	Product-Oriented
Practical	Ritual	Deed
Discursive	Ritual	Exploration

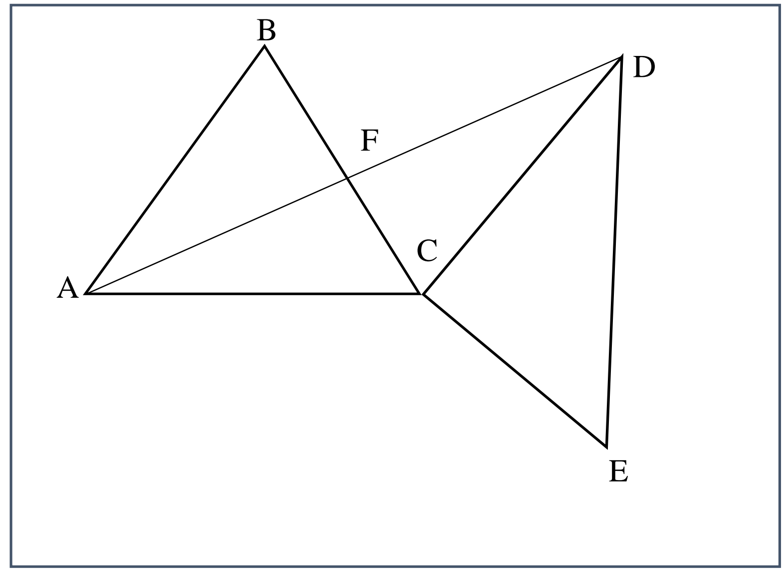
3.3.13 Forms of routines

There are three forms of mathematical routines. Given that one of the objectives of mathematical discourse is the construction of narratives, the three forms of mathematical routines are, therefore, focused on narratives. These forms of routines are explorations, deeds and rituals

3.3.13.1 Explorations

In commognition, explorations are distinct routines that end up in the proffering of endorsed narratives (Sfard, 2008). Endorsed narratives are those narratives that are endorsed by members of the mathematical community. Examples of endorsed narratives include, among others, axioms, theorems, and definitions. Therefore, routines such as solving equations in a task situation, defining, or proving are some examples of explorations. The following table (2.11) illustrates an example of explorations.

Table 3.11: An example of explorations. Source: Researcher’s own initiative.

<table>
<tr><td colspan="2">Δs ABC & CDE are equilateral triangles of the same size. If AC = 10 and <BCD = 80, find the area Δ ABF.

</td></tr>
<tr><td>Since <ACB = 60, <ACD = 60 + 80 = 140
Since Δ ACD is isosceles, <CAD = <ADC = $\frac{180-140}{2}$ = 20
It then follows that <BAF = <BAC - <DAC = 60 – 20 = 40
By the exterior angle theorem, <BFA = 20
By applying sine law on Δ BAF, we have
$\frac{BF}{sin40} = \frac{10}{sin80} \rightarrow$ BF = $\frac{sin40}{sin80}(10)$
Area of Δ =$\frac{1}{2}(AB)(BF)$
$= \frac{1}{2}(10)\left(\frac{sin40}{sin80}\right)(10) = 28.263$</td><td>Signifier
Realization
Signifier

Realization

Signifier
Realization</td></tr>
</table>

Explorations can be divided into three subsets. Each of these subsets comprise of elements that have the features of the endorsed narrative. The features are construction, substantiation, and recall. Construction as a process involves recycling previously constructed narratives to generate new endorsed narratives. Substantiation pertains to determining whether to endorse or reject the new narratives. Recalls related to engaging in efforts to retrieve from memory past endorsed narratives.

(1) Construction of narratives

Narratives in mathematical discourse are often constructed with the aid of realisation procedures. It must be remembered that, unlike in colloquial mathematical discourse where the object of talk about concrete objects, the object of talk in literate mathematical discourse is about relations and properties of intangible objects. Therefore, given this difference between colloquial and literate mathematical discourse, it is imperative that narratives in literate mathematical discourse are constructed with the aid of perceptual realisations procedures. Sfard's (2008) theory of commognition suggests that it is advantageous to construct narratives based on multistep mediated realisation procedures than single step realisations. According to Sfard (2008), the expectation that is valued in school discourse is when participants split a given mathematics task situation into two: the first one being the act of recognition and the second being the act of naming, each act having its own particular narratives. In this regard, Sfard suggests that the discourse practiced in school should strive towards displacing the direct recognition approach with the discursively implemented one.

The construction of narratives is based on meta-discursive rules among which are; deduction, induction, and abduction.

Deduction refers to *"when a new narrative is obtained from previously endorsed natives with the help of well-defined inferring operations"* (Shard, 2008, p. 229). For example;

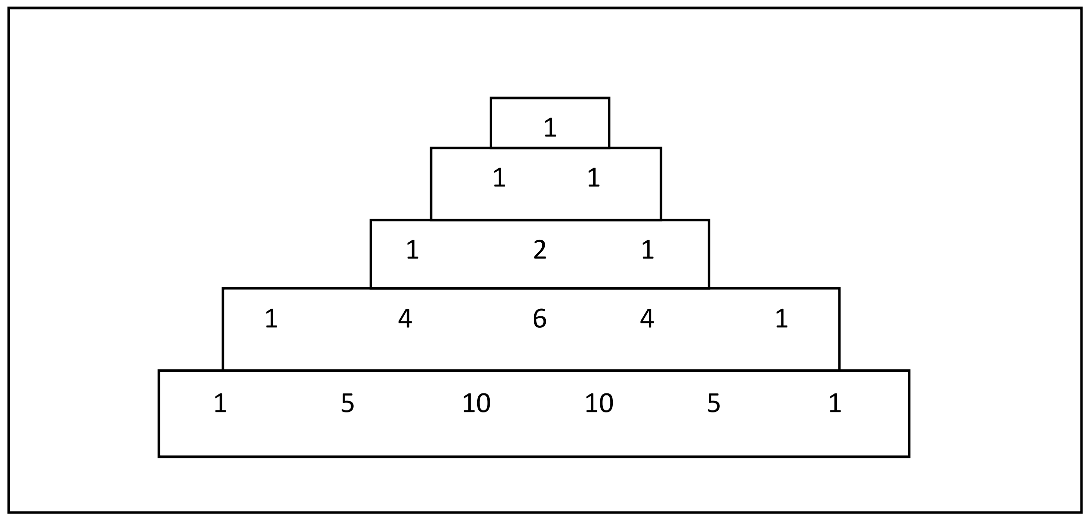
$$If\ T \rightarrow Y\ is\ an\ already\ endorsed\ narrative, Y can\ be\ endorsed\ as\ well.$$

The endorsement may be like this;

$The\ baseanglesofanisosceles \triangle are\ equal. If the\ base\ angleso\ f\ a \triangle areequal$
$\triangle isi\ sosceles.$

Induction refers to *"A process in which a new narrative on any object of a given type is obtained from a finite number of already endorsed narratives on specific instances of this type".* (Sfard, 2008, p. 229). Pascal's triangle provides an example of the induction process:

Figure 3.8: Pascal's triangle. Source: Researcher's own illustration.



$$The \sum of\ the\ numbers \in each\ row\ ofPascal'striangle\ is\ equal 2^n,$$
$$where\ n\ representst\ he\ row\ of\ numbers\ starting\ at\ n = 0.$$

Abduction stands for a series of steps with the potential to endorse a new constructed narrative that the necessary conditions have been achieved (Sfrad, 2008).

An example:

$$In\ ferring\ TT \rightarrow Y.$$
$$The \sum of\ the\ exterior\ angles\ of\ apolygon\ is \frac{180^0(n-2)}{n}.$$
$$Whenn = 4, the\ polygon\ is\ a\ quadrilateral.$$

During the construction of narratives, two interrelated routine processes are required to aid substantiation. These are the *when* of the routine and the *how* of the routine of

the substantiation. The implementation of *when* of the routine is implemented when there is sufficient evidence that points to where the substantiation should originate from. For the *how* of the routine, the implementation requires the participant to have recourse to what kind of discursive action is needed to count as satisfactory substantiation. It should be realised that in order to implementation of the *when* and *how* of the routine of substantiation, the participant in the discourse need not look else-where. This is because the whole process (enactment of the *when* and *how*, as well as substantiation) is informed by the evidence arising from "inside" the discourse itself and not from "outside".

As mentioned earlier, not all narratives, even when constructed with the help of mediated realization procedures are endorsable. Participants in mathematical discourse may be able to construct narratives, however, significant narratives are only those that can be endorsed by the mathematics community. Some examples of endorsed narratives include, theorem, definitions, and axioms.

(2) Recalling

Summoning memory to remember former narratives is part and parcel of explorations. Since exploratory participants often have at their disposal multiple representations of the same outcome to a task situation, recalling some former narratives is important. To the explorative participant, the act of remembering becomes significantly enhanced when such recall is mediated by special routines. The ability to recall some former narratives and build new narratives with less effort of memory is regarded as an aspect of discursive fluency.

(3) Characteristics of exploration.

(1.1). Flexibility: Is the ability to perform a task situation in multiple ways. An increase in the aspect of flexibility enables the learner to have at his disposal various alternative ways of implementing the performance and arriving at the same outcome. For example, a participant who performs an alternative procedure to yield the same come as seen in the problem below.

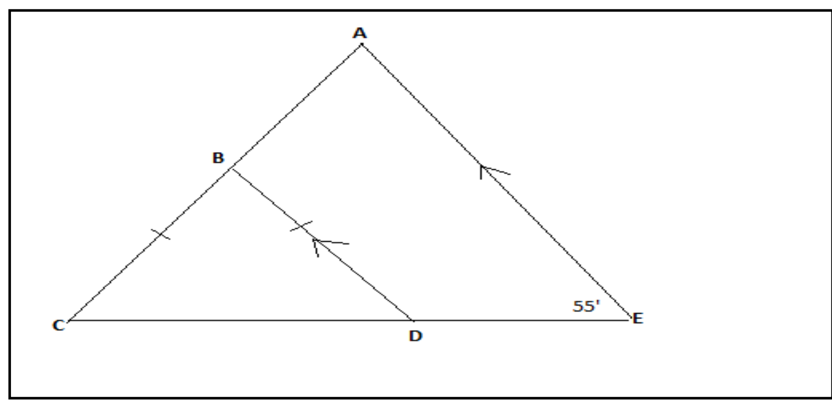
$5(2+5)$ or $5(2+5)$

$10+25=35$ $5(7)=35$

(1.2). Bondedness: Refers to a routine in which the output of the previous step, in terms of procedures that serves as the input in the next procedure.

For example,

Figure 3.9: An Example of Bondedness. Source: Researcher's own initiative.



.

G$iven\ that\ \Delta CBD\ is\ isosceles \wedge angle\ AED\ is\ 55°. Calculate\ the\ angle\ A.$

In this example, the output of finding the value of angle O in the isosceles triangle will serve as the input for finding the value of angle A.

(4) Applicability: Refers to the potential of a routine in a task situation to act as a precedent in a range of similar future task situations.

(5) Performer's agentivity: Refers to the performer moving away from depending on others to implement a routine to be an independent performer. An independent performer can implement the *how* and *when* of the routine and evaluate the task on his or her own.

(6) Objectification: Refers to the ability to derive the necessary procedures in a task situation from the properties of the mathematical objects and increase in the degree in the use of nouns. There are two processes involved in objectification. One is reification, which is talking about objects instead of processes. The other one is alienation, which is talking about mathematical objects without the mention of human action. For example, "*two times four is eight*".

(7) Substantiability: A narrative is substantiated when a series of steps convince mathematicians to accept and endorse the narrative. For example, "*the base angles of an isosceles triangle are equal*", or *"the diagonals of a square bisect each other".*

Table 3.12: An example of substantiated narratives and evidence of substantiation arising from the discourse. Source: Researcher's own initiative.

	Statement	Reason
A C 108° x y B D	$x = 108°$ $y = 180 - 108°$ $y = 72°$	Corr ∠'s ; AB//CD Adj ∠'s on a str line

3.3.13.2 Deeds

Deeds like explorations are product-oriented; however, they differ in their focus. Sfard (2008) defines deeds as "*a set of rules for a patterned sequence of actions that unlike explorations, produce or change objects, not just narratives"* (p. 237). Since deeds are focused on the changes of objects, they are regarded as belonging to the category of

discursive object level, whereas explorations are in the category of discursive meta-level. Other than what has been described above, there is a thin line between deeds and explorations, and distinguishing one from another is not easy. Because of their similarity, participants in mathematical discourse may alternate between deeds and exploratory modes (Sfard, 2008).

3.3.13.3 Rituals

Rituals are routines that focus on performance. Unlike other attributes of mathematical discourse whose principal role may be in growing endorsed narrative and change in object, rituals are more concerned with making and maintaining a social bond with other performers (Sfard, 2008). For this reason, ritual routine performers seem to differ from explorative and deeds performers in several unrelated features. According to Sfard (2008), some of the features of ritualistic performers include the following;

(1) To the ritualistic performer, the primary concern of participating in the discourse is creating social bonds with others. A ritualistic performer, therefore, strives to gain approval of her performance from colleagues or authority.

(2) Ritualistic performers do not recognise applicability conditions on their own. More often than not, a ritualistic performer relies on the prevailing environment, such as additional mediational means or the availability of prompts from other people.

(3) Ritualistic performers often follow a rigid line of procedural action without regard to variations. They do not recognise the need to explore multiple realisations of procedures that would lead to the same outcome.

(4) Because ritualistic performers follow a rigid line of action and have no recourse to alternative action, rituals that do not yield the desired outcomes are often repeated using the same procedures.

(5) Ritualistic performers retain satisfaction from their performance and are not often concerned with constructing endorsed narratives.

Table 3.13: A comparison of: deeds, explorations and rituals (Adapted from Sfard, 2008, p. 243).

	Deed	**Ritual**	**Exploration**
Closing condition/goal.	A change in environment.	Relationships with others (Improving one's position with respect to others).	Description of the world (production of endorsed narratives about the world).
By whom the routine is performed. For who the routine is performed.	No special requirement. No special requirement.	With (scaffolding by others). Others (authoritative discourse).	No need for scaffolding- can be performed individually. Others and oneself (internally persuasive discourse).
Applicability (changing the *when*, keeping the *how* constant).		Restricted-the procedure is highly situated.	Broad- the procedure is applicable in a wide range of situations.
Flexibility (changing the *how*, keeping the *when* constant).		Almost no degree of freedom in the course of action.	The procedure is a whole class of equivalence of different courses of action.
Correctability.		Cannot be logically corrected- has to be reiterated in its entirety.	Parts can be logically replaced with an equivalent subroutine.
Acceptability condition.	Result-change in environment- must count as adequate; no need for human mediation of the acceptance- it depends on the environment.	The activity has to be shown to adhere strictly to the rules defining the routine procedure- the acceptance depends on other people.	The narrative produced through the performance must be substantiated in such a way that the acceptance is independent of other people.
Words and mediators' use.	Possibly not active use of keywords.	Phrase-driven use of keywords- as descriptors of extra discursive mediators.	Objectified use of keywords-as signifying objects in their own right.

3.3.14 Thoughtful imitation and the ritualised beginnings

When one takes certain aspects into consideration, such as the outcome of routines, rituals on their own are perhaps overlooked when pitted with the other two forms of routines. However, all routines, whether practical or process-oriented, are by nature interwoven and feed into one another. Having said this, rituals, like explorations, play a significant role in mathematics learning. Regarding the significance of rituals in learning, for Lavie (2019) and her colleagues although rituals may be highly situated, the play an important part in the starting phases of experiencing new mathematical objects.

In the mathematics classroom, explorations are desired over rituals. However, participation in collective human doings, including discourse, starts from the initiation stage. At the initiation stage, new participants in the discourse are regarded as novices and have no experience. Their only way of participating in the new discourse is by imitating experts already familiar with the rules of the discourse. Novices may acquire the necessary experience through participation after being exposed to a particular discourse.

According to Lavie et al., (2019), a new entrant in a discourse, imitated others not for any reason other than being new to a given task situation. This type of participation is ritualised. Because of being new to the discourse, the participant does not yet have recourse to what is known as *Precedent-search-space* (PSS). According to the theory of cognition, when a participant is newly introduced to a task situation, the participant can act in the new situation by recalling previous experiences from similar past situations. Participants act in a new task by relating to similar past situations (Lavie et al., 2019). The formerly performed task situation constitutes the environment that contains sets of *precedent identifiers* that one relies

on when faced with a new but similar task situation. The environment containing these sets of precedents is referred to as *Precedent-search-space* (PSS) (Lavie et al., 2019).

It is through imitating others that a participant in the new discourse can gain valuable experience. The gain in experience is spurred by constant reflection on the actions. When the participant has gained enough experience, his or her PSS widens, and he or she can utilise precedent identifiers on his or her own without the help of others. As the participant's confidence and the independence to act on his or her own grows, the human quest to individualise this newly learned discourse comes to the foil. The participant does not want to be seen as imitating the expert anymore. The need to individualise the discourse is the human desire, that is, wanting to be an individual actor even when participating in collective human activities.

Considering all that has been explained, one may now recognise the importance of rituals in an initial encounter with a new discourse, the role of thoughtful imitation, and ritualised beginnings.

Two routines that play an essential role in learning mathematics are distinguished. These perceptual forms/kinds of routines are ritual and exploration. Ritual and exploration differ in the mechanisms these routines employ while attending to the participant's past experiences.

3.3.15 Transforming ritual into exploration

Explorations, deeds, and rituals are the discursive routines that play an essential role in the teaching and learning of mathematics. These routines spur one another, but they differ in the types of tasks they accomplish. In this respect, explorations are the most desired routines in mathematical discourse among these three routines. To grow new routines, especially those that include new metarules and mathematical objects, performers must go through a series of ritualised performance (Sfard, 2008). However, attainment of explorations is not easy.

This stems from what has been said: newcomers to a discourse often do not have sufficient sets of precedent-search-space (PSS) from which precedents can be sourced. In order to act successfully in a task situation, the learner needs to have recourse to several precedent identifiers within the given space. These precedent identifiers aid the implementation of the *who* of the routine and the *when* of the routine. The lack of these identifiers implies that the learner cannot yet perform independently but has to rely on others and thus act ritualistically.

By constantly participating in the new discourse, the proficiency in the new discourse will eventually grow and the former ways of imitating others, that were often employed used in the process of learning may lead to the individualisation of the new discourse. When the discourse has been individualised, and the level of confidence has grown, the performer, who was once regarded as a newcomer to the discourse, is now in a position to turn into an independent participant and able to make decisions independently. Turning rituals into explorations is the desired outcome of learning. However, turning rituals into explorations is not an automatic process. According to the theory of commognition, transforming any routine, in this case, ritual to exploration, is a gradual process. Routines, initially performed as rituals may ultimately convert to new mathematical explorations (Sfard,2008).

As has been mentioned earlier, rituals play an essential role in the teaching and learning of mathematics. Their place in this process is inevitable. When one considers that both rituals and explorations are observable, it is possible to transform rituals into explorations because they are observable. What may aid this transformation is to deliberately focus on those observable aspects that are visible to the observer and not on the inner experiences of the learner, which are often covert.

According to the theory of commognition, transforming rituals into exploration should, ideally, happen simultaneously with the initiation stage of the learner into a new discourse. Three reasons are advanced for transforming ritual into exploration simultaneously with each

other. The first reason is that at this initiation stage, the learner imitates others and is therefore prepared to follow a rule enacted by the expert to "please others". Secondly, if the new learner is given space to become a skilful performer, the learner may be less motivated to accept an alternative procedure, given that she may not appreciate the advantages of another procedure. And thirdly, Sfard (2008) points out that rituals are generally highly situated and rigid. Turning "old" rituals into explorations would not be easy. Fourthly, a tenable method to grow a new discourse is through attending to the old discourse with which the learner is already familiar. This is possible because new discourse emerges from old, recycled discourses.

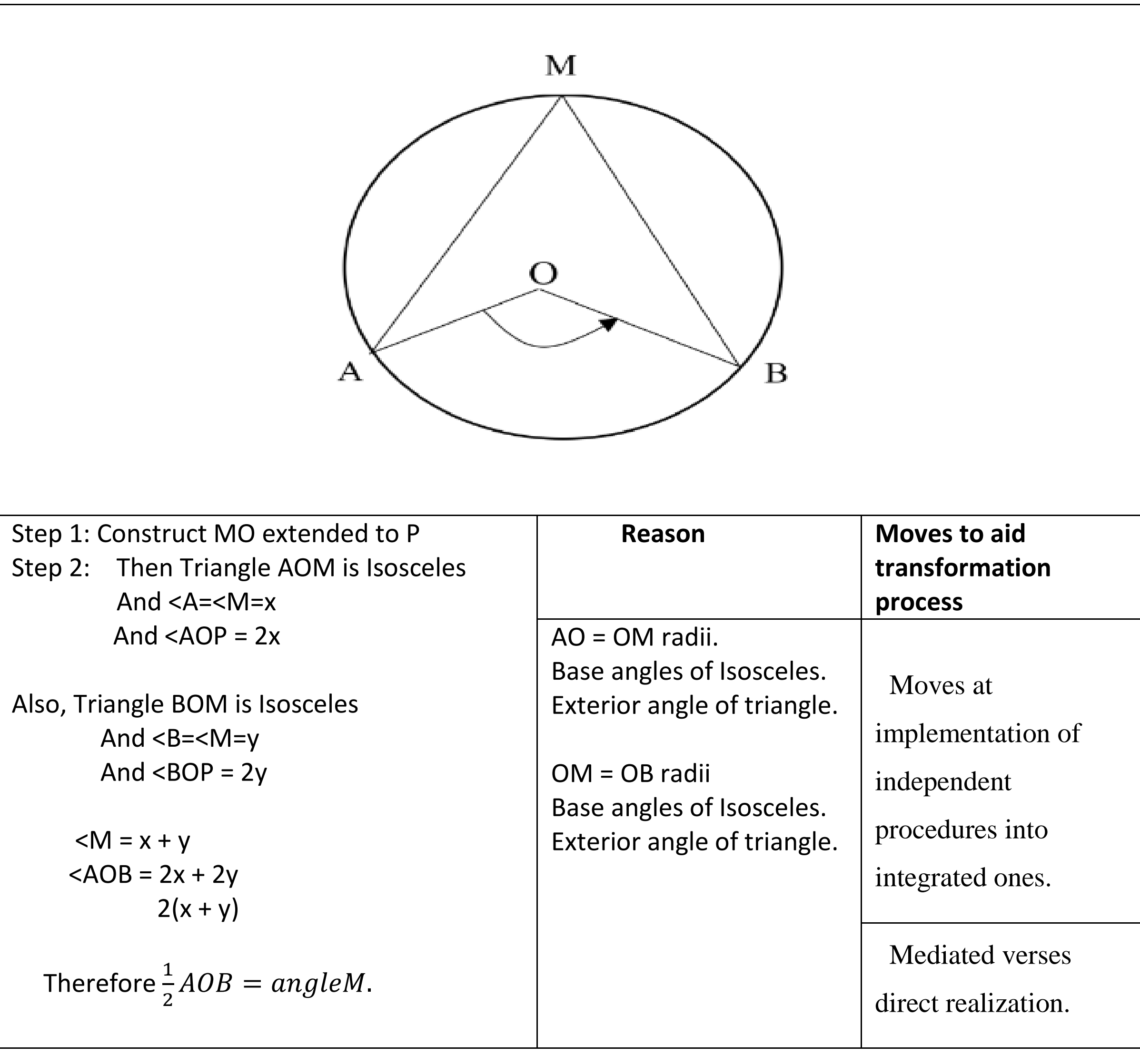
To start transforming rituals into explorations, Sfard (2008) suggests that possible trajectories that must be targeted should be those characteristics that help distinguish rituals from explorations. These characteristics are summarised in the table below;

Table 3.14: Characteristics to help distinguish ritualistic from exploratory aspects of routines (Adapted from Kuchle, 2021, p. 24)

Characteristic	Ritualistic aspects of a routine	Exploratory aspects of a routine
Flexibility	The performer knows and follows only one procedure	Performers can vary procedures used to perform routine
Bondedness	Routine has not become 'bonded	"Routine is bonded, that is, the relation between steps is transparent to the person
Applicability	The routine is evoked only in situations remarkably similar to the situation in which it was first learned	The routine can be applied in various different contexts.
Performer's Agentivity	Performer is dependent upon and performs for others, actual or imagined.	The performer feels free to make choices about the performance of the routine.
Objectification of the Discourse	The performer relies on concrete examples/objects to perform routine. The performer's discourse is not yet objectified.	The performer has abstracted concrete objects into mathematical objects and uses objectified discourse.
Substantiability	Performer cannot substantiate the validity of their routine and rely on external authority.	Performer can assess the validity of their own performance and do not rely on external authority.

As mentioned earlier in this report, the ritualistic performer is more concerned with the process of his or her performance and creating a bond with others, while the explorative performer is concerned about the outcome of his or her performance. Some of the moves to aid in the transformation process are given below with the help of the task situation in figure (2.8) proof of the angle at the centre theorem, while table (2.13) contains a sample of some characteristics of exploration when a performer working on a task situation.

Figure 3.10: An example of how to prove the theorem of the angle at the centre is twice that at the circumference. Source: Researcher's own illustration.



Step 1: Construct MO extended to P Step 2: Then Triangle AOM is Isosceles And <A=<M=x And <AOP = 2x Also, Triangle BOM is Isosceles And <B=<M=y And <BOP = 2y <M = x + y <AOB = 2x + 2y 2(x + y) Therefore $\frac{1}{2}AOB = angleM$.	**Reason** AO = OM radii. Base angles of Isosceles. Exterior angle of triangle. OM = OB radii Base angles of Isosceles. Exterior angle of triangle.	**Moves to aid transformation process** Moves at implementation of independent procedures into integrated ones. Mediated verses direct realization.

		Ability to answer the "why" with "because" with independent reasons.
		Transforming loose collection of discourse into integrated discourse.

Table 3.15: Explorative Participation (Adapted from Emre-Akdojan, & Gurbuz, 2023, p. 33)

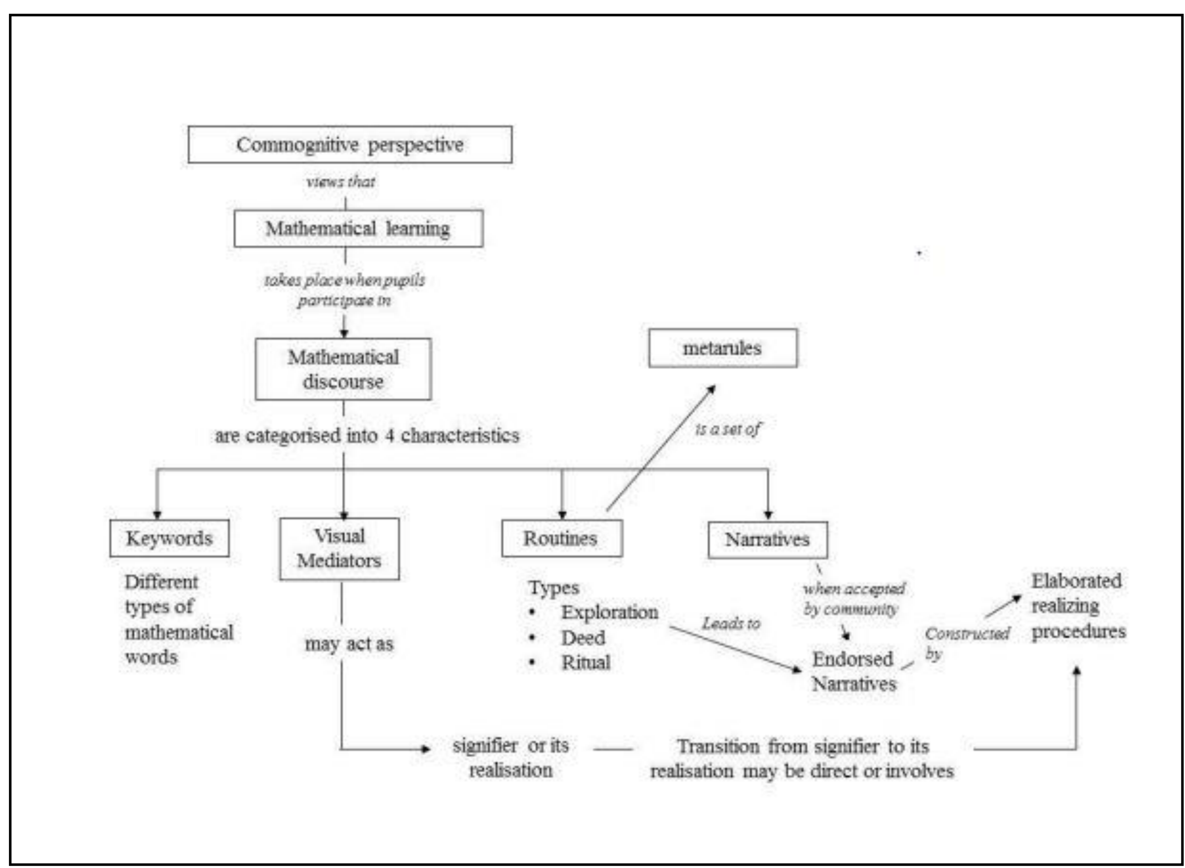
	Explorative participation	
Initiation	What is the question the teacher poses (raises)?	What is it I want to get
Procedure	How is the procedure of the routine determined?	Students are expected to choose from alternative procedures. They are expected to make independent decisions
Closure	What type of answer does the teacher expect	Indicating the new narrative produced

3.3.16 Summary of terms used in commognition

There are several terms commonly used in commognition. These terms are sometimes unique to commognition.

In the figure 3.11 is the illustration of a summary of terms used in commognition.

Figure 3.11: Summary of commognitive terms used in this study (Adapted with permission from Chia, 2017)



3.3 Conclusion

This chapter was divided into two sections. The first section was on the literature review, while the second was on the study's theoretical framework.

This study focused on the learners' commognition while engaging with geometry in the grade 11 classroom. The chapter was divided into two sections. In the first section, a review of the studies on the learner’s commognition in mathematics in general, particularly in geometry, was discussed.

Firstly, most of the sources reviewed used the qualitative approach in their studies, and the primary method of collecting data was from analyses of interview transcripts and written task

documents. The main finding from the sources reviewed was that the nature of participants' mathematical discourses was ritualistic (Thoma & Nardi, 2018; Mpofu & Mudaly, 2019; Robert & Le Roux, 2019); Nisa et al., 2021).

Secondly, a review of some of the challenges experienced by learners that might contribute to ritualistic performance in mathematics learning was done. The findings of these sources revealed several challenges as causes of ritual participation in geometric discourse by learners. The sources highlighted some challenges that face the teaching and learning of mathematics. According to these findings, some contributing factors to ritual performance included the context of teaching and learning mathematics in a second language (Robertson & Graven, 2019; Riordian et al., 2022), the abstract nature of mathematics and its unique vocabulary (Lin & Peng, 2019), and issues related to the expert discursant in terms of pedagogical content knowledge (Pavlovicova & Bockova, 2021; Mbhiza, 2021; Zayyadi et al., 2020).

Thirdly, a review of how to spur the de-ritualization process in the learners was described. The findings from the sources revealed many practical pedagogical methods. These instructional methods include; imparting problem-solving skills (Ziyyadi, 2019; Fratiwe et al., 2020; Bayuningsih et al., 2018; Ahdhianto et al., 2020), differentiation of instruction (Tomlinson et al., 2006), teaching using technological tools to spur visualisation (Turget & Turgut, 2018; Zhang & Stephen, 2019; Mudaly & Fletcher, 2019), cooperative teaching (Gates, 2019; Gogo et al., 2021), and teaching using the van Hiele model (Yunis et al., 2019; Machisi & Feza, 2021).

From the sources reviewed, an evaluation of the current state of scholarship is that most of the studies are focused on describing the nature of learners' discourses, and most of these studies have not gone further to suggest suitable pedagogical strategies to employ that might

effectively spur de-ritualization. Since rituals are essential in the initial stages of receiving instructions, it is necessary to engage in further research in order to acquire knowledge on how to infuse suitable pedagogical strategies in teaching mathematics and other school subjects.

This section of the chapter discussed the theory that underpinned this study. Anna Sfard's (2008) theory of commognition framed the study. The lens of commognition was used to examine the participants' mathematical discourse.

The section opened with a brief background of the theory of commognition. The need to develop the theory was informed by conventional deficiencies in conceptualising the learning process. Lamenting the lack of operationality about conceptualisation of the process of learning in the context of human development, Sfard (2008) viewpoint is that in the absence of clear sets of definitions our thoughts are bound by metaphors to import concepts from what is known to the unknown world. While endorsing the works of researchers such as Vygotsky (1978) and Wittgenstein (1953), she proposes conceptualising thinking as individual communication and learning as a change in discourse. Sfard (2008) merges the words communication and cognition to develop a single term and refer to it as commognition. By doing this, Sfard stresses the interrelationship between individual thinking and interpersonal communication to be the same.

Armed with the operationalisation of the concept of the learning process, Sfard (2008) concludes that mathematics is a form of communication characterised by specialised word use, visual mediators, narratives, and routines. Sfard (2008) stresses that as mathematics is a discourse, unique objects are used. These mathematical objects are abstract and, therefore, inaccessible to our senses. However, using visible realisations, mathematical objects can become visible to our senses. $x^2 + 5x - 10$ is an example of the realisation of a quadratic

function. The theory of commognition stresses that participants in mathematical use routines such as rituals, deeds, or explorations to mathematise mathematical objects. According to Sfard (2008), explorations are the desired outcome of learning in mathematics.

According to commognition theory, there are four distinguishable features of mathematical discourse. These specific attributes form the lens used to describe mathematical discourse. The four features are word use, visual mediators, routines, and narratives. Routines are broken down into three types, viz; rituals, deeds, and explorations. Whereas ritual participation is encountered during the initial stages of learning, explorative participation in mathematical discourse is desired in the mathematics classroom (Sfard, 2008). On learning, commognition theory proposes that learning takes place on two levels. The first level of learning is object-level learning, and the second type is metalevel learning. Both of these levels play a critical role in learning.

In the next chapter, I elaborate on the methodology employed to carry out the research study.

Chapter 4

Methodology

4.1 Introduction

A qualitative researcher seeks to understand phenomena from the participants' perspective. In this scientific inquiry, I intended to examine learners' commognition when engaging with geometry in a grade 11 class.

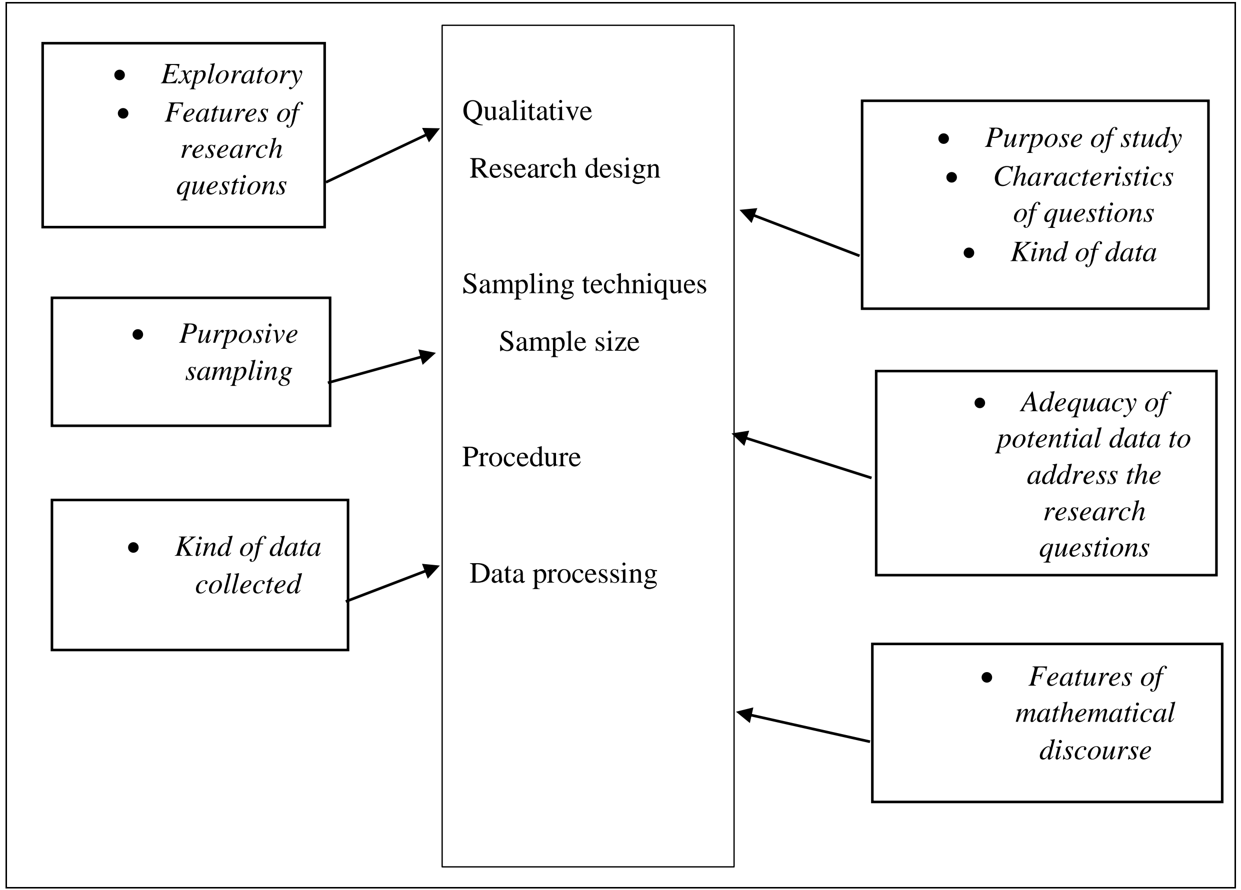
This section of the study covers the overall scheme. Yin (2014) standpoint regarding the research design is that the scheme embraces all parts of the research and connects them to specific units for examination so as to reach the study conclusions. In this respect, the chapter opens with an elaboration of the philosophical paradigms that underpinned the study. Further, there is a section that discusses the particular research design in detail and the rationale behind its employment. The methods and rationale of selecting a particular group of participants, its procedures, and the sample size of interest are also described. Further on, research instruments, systematically focused data collection procedures and data analysis processes that enable answering the research questions are included. Also shared in the chapter are the researcher's background, beliefs, biases, and assumptions. This is imperative because the researcher is regarded as an instrument in qualitative studies. Then, the chapter discusses some critical aspects of scientific inquiry about quality assurance regarding a qualitative research approach.

4.2 Research design

Research design is about the researcher’s plan of managing the various units of the research process. Kumar (2012) asserts that a research design is a researcher’s construct of various procedures and strategies to inform the processes of obtaining answers to research questions.

In the figure below (Figure 4.1), I illustrate the conceptualisation of the critical design decisions that underpinned the study.

Figure 4.1: Key design decisions. Source: Researcher's initiative



The choice of methodology depends on the nature of the research question(s), and is also informed by certain assumptions or characteristics. This study sought to explore a phenomenon from the participants' experiences in a natural setting. As McMillan (2012) pointed out, one characteristic of the qualitative approach is that the inquiry is conducted in its natural setting. Unlike a quantitative approach, this design is best suited to answer the specific questions posed by this study.

This study was informed by Sfard's (2008) theory of commognition. In this theory, thinking is regarded as the individualised form of the activity of communicating, and this personal

communication does not necessarily have to be heard or seen by an observer and does not necessarily have to be in the form of words or text (Sfard, 2008; Harre & Gillett, 1995). Changeaux, cited in Sfard (2008), further explains that the phenomenon we call thinking depends on how an individual manipulates internal representations. Therefore, considering what Chaneaux (1995) says, it was imperative that I adopt a qualitative exploratory design as the most suitable method to study the phenomenon under investigation. In a qualitative approach, individuals are considered to be participants of the collective and because of this, they present multiple as well as subjective representations and perspectives of reality based on their individual and unique experiences. Creswell (2012) articulates that one of the common goals of qualitative study is to try to understand perplexing situations. And as stated elsewhere in this report, McMillan (2012) asserts that one of the characteristics of qualitative studies, among others, is that the study is conducted in a natural setting. The intent is to allow behaviour to be studied as it occurs naturally. McMillan (2012) further states that to better understand the phenomena under study, the inquirer has to collect data from credible sources who experience the phenomena. Qualitative inquiry is an established undertaking and is made up of an interpretive to a given phenomenon for the exclusive purpose of enlightening the world (Denzin & Lincoln, 2011). It is this interpretive approach that makes generalisation of the results in most instances un attainable.

Hancock and Algozzine (2017) add that qualitative research aims to understand the situation under investigation primarily from the participants' perspective, not that of the researcher.

4.2.1 Philosophical assumptions

The characteristics of qualitative research.

(1) *Natural setting-* Qualitative researchers often collect data in the field at the site where participants experience the issue or problem under study.

(2) *Researcher as a key instrument*- The qualitative researchers collect data by examining documents, observing behaviour, and interviewing participants.

(3) *Multiple methods*- Qualitative researchers typically gather multiple forms of data, such as interviews, observations, and documents, rather than rely on a single data source.

(4) *Complex reasoning through inductive and deductive logic*- Qualitative researchers build their patterns, categories, and themes from the "bottom-up" by organising the data inductively into increasingly more abstract information units.

(5) *Participants' multiple perspectives and meanings*- In the entire qualitative research process, the researchers focus on learning the meaning that the participants hold about the problem or issue, not the meaning that the researchers bring to the research or write from the literature.

(6) *Context-dependent*- The research is situated within the context or setting of the participants or site.

(7) *Emergent design*- The research process for qualitative researchers is emergent

(8) *Reflexivity*- Researchers "position themselves" in a qualitative study.

(9) *Holistic account*- Qualitative researchers try to develop a complex picture of the problem or issue under study.

(Creswell, J & Poth, C 2016, p. 38).

For this study, I adopted two approaches, viz; the phenomenological approach and the case study approach. The aim was to gain a deeper understanding of the phenomena from the participant's point of view. From the literature review, I gathered that no matter which approach is adopted, each approach has its own strengths and limitations. Therefore, my rationale for using the two approaches, that is, the phenomenological and case study, was to complement the strengths and reduce the limitations inherent in adopting just one approach.

For example, when we consider the case study approach, Yin (2014) says, one of the major strengths of the case study is in data collection, as the method allows the researcher to use different sources of evidence.

Furthermore, Giorgi (2009) and Mosatakas (1994) define a phenomenological design as an inquiry in which the researcher has an opportunity to reflect on the phenomenon as described by several participants who have all experienced the phenomenon. In addition, Giorgi (2009) and Mosatakas (1994) assert that phenomenological study aims to describe the ordinary meaning of several individuals' lived experiences of a particular concept or phenomenon.

For phenomenologists, the focus is on describing what all participants have in common as they experience a phenomenon. For Manen (1990) the primary objective of the phenomenological approach to shift from focusing on individual experiences to a more expansive understanding and applicable to the whole universe.

The assumption here is that there is an essence or central meaning of an experience shared by individuals that can be investigated and explained through research. In phenomenological studies, different people's experiences are analysed to describe the variant structures of a phenomenon (Hancock & Algozzine, 2014).

The philosophical assumptions of phenomenology rests on the following;

(1). the study of lived experiences of persons.

(2). the view that these experiences are conscious ones. (Van Manen, 2014).

(3). and the development of descriptions of the essences of these experiences, not explanations or analyses (Moustakas, 1994).

On the other hand, Stake (1995) and Yin (2009 & 2012) describe a case study as an inquiry in which the researcher employs multiple data collection procedures to develop an in-depth

analysis of a well-defined case. (Yin, 2014) further describes case study research as an approach that involves the study of a case (or cases) within a real-life, contemporary context or setting. This case or cases may be a concrete entity, such as an individual, a small group, an organisation, or a partnership. At a less concrete level, it may be a community, a relationship, a decision process, or a specific project (Yin, 2014). Through case studies, researchers hope to understand the situations and meanings of those involved (Hancock & Algozzine, 2014). Table 4.1 below presents the strengths and limitations inherent in a qualitative approach articulated by Christiansen and his colleagues (2021).

Table 4.1: Strengths and limitations of qualitative research. Conceptualised by Mwita, 2022, pp. 618-625.

Strengths	Limitations
• Fewer limitations: *Qualitative studies are less stringent than quantitative ones. Out of the box answers to questions, opinions, and beliefs are included in data collection and data analysis.*	• Sample sizes: *A big enough group of participants ensure results are accurate. A small sample is not enough to show a reliable picture. Data collected may be insufficient.*
• More versatile: *Researchers, can adjust questions, adapt to circumstances that change or change the environment to optimize results.*	• Bias: *For internal qualitative studies, for example, participants may give a popular answer that colleagues agree with rather than a true opinion. This can negatively influence the outcome of the study.*
• Greater speculation: *Researchers can speculate more on what answers to drill down into and how to approach them. They can use instinct and subjective experience to identify and extract good data.*	• Self-selection bias: *researchers who call on volunteers to answer questions worry that participants who respond are not reflective of the greater group.*
• More targeted: *The process can*	• Quality questions: *It’s hard to know*

target any area of concern. Researchers can concentrate on specific target/s to collect valuable information. This requires fewer resources than quantitative studies.	*whether researcher questions are of sufficient quality or not because they are all subjective.*

4.2.2 Justification for this design

Based on the above-stated assumptions as articulated by Moustakas (1994) and van Manen (2023), and taking into consideration the nature of the specific questions posed in the study and from what Hesse-Biber and Leavy (2006) point out, I made the determination that the phenomenological and case study are the most congruent and appropriate approaches to adopt. The determinant here rests on the nature of the research questions posed in the study. I wanted to collect rich information from the participants in their natural context. The type of research questions that I posed (viz. "What" and "How") were aimed at exploring the phenomenon, as well as capturing participants' unique experiences while they were actively engaged in the collective activities and also further examining how they make sense of their individual meaning to the phenomena.

Another consideration that I looked at when choosing the phenomenological and case study approaches was the methods these two approaches used to collect data. I collected data in the form of written responses from the mathematics task activity that the participants wrote and from verbatim transcriptions of focus group interviews highlighting what was said and done by the participants. Another determining factor was being the class teacher of the participants, whom I had known for an extended period of time, whereby a close relationship was developed. Based on the nature of this close relationship of trust, I was better positioned to collect rich data in a social and natural context. As I mentioned earlier in this report, the data sources for the study were the audio-video recording of the focus group interview with

the participants and the pencil and paper mathematics task activity (document); this required the participants to freely talk and engage in the activity. The kind of data from transcripts were verbatim accounts (what was said and what was done) by individual participants while in direct interaction with others, while from the pencil and paper mathematics task, the data comprised the participant's responses (text).

The specific philosophical paradigm that I linked to in this research study was the social constructive paradigm. Creswell (2012) posits that social constructivists assume that individuals seek an understanding of the world in which they live and work. Creswell (2012) adds that because individuals have unique experiences, they develop subjective meanings of their experiences, which are directed towards particular objects or things. Furthermore, he says that though they may experience the same phenomenon, their meaning is subjective because of the influence of their different backgrounds and experiences. Constructivists also believe that knowledge is something individuals construct for themselves using previously derived knowledge and experiences, Brower, 1996; Derry,1992). The ontological, epistemological and axiological stances of constructivism that informed my study are illustrated in Table 4.2.

4.2.3 Social constructivism Philosophical paradigm and Assumptions

Table 4.2: Ontological, Epistemological, and Axiological Stance (Adapted from Creswell, 2012, pp. 36-37)

Ontological, epistemological and axiological stance	
Ontological stance	Participants and researcher develop multiple realities through social interaction.
Epistemological stance	Co-creation of reality between participants and researcher.

Axiological stance	Belief and values are socially constructed.

The ontological stance of the social constructivist paradigm states that the participants and the researcher are developers of multiple realities through social interaction. This stance resonates with phenomenological design because the researcher goes to the participant's natural settings to collect their stories based on their experiences. The view is that though the researcher and the participants may sometimes have passed through similar realities, their interpretation of reality might be different given their experiences and diverse backgrounds.

About the epistemological stance, social constructivists hold that when it comes to reality, reality is co-created as a result of the interaction between the participants and the researcher. Therefore, this aspect of the co-creation of reality is consistent with the phenomenological design, as the researcher is regarded as an instrument himself/herself (Sfard, 2009; Lincoln & Guba, 1985). The researcher plays a double role of being a researcher and simultaneously being a participant. As a result of this dual role, the researcher and the participants are co-creators of a new reality.

The axiological stance of constructivists is also consistent with the phenomenological design. According to this stance, beliefs and values are socially constructed through active participation in collective activities. For example, when considering phenomenological design, the researcher goes to the participant's environment and gains an understanding of the context, beliefs and values and, most importantly, develops a cordial relationship of trust with the participant. The researcher also strives to build an equal and healthy balance of power with the participants, and this is done well before the beginning of data collection.

Considering what I have stated above, I decided that the social constructivist paradigm, the phenomenological, and the case study approaches were consistent with the chosen design for this study.

4.2.4 Qualitative research and Commognitive research

When one considers qualitative and commognitive research, many parallels can be drawn between these approaches, making them particularly compatible. For example, in qualitative inquiry, Stake (1995) points to three aims of qualitative research, viz; (1) understanding as the purpose of inquiry, (2) personal role of the researcher, and (3) knowledge construction (Stake; 1995, p. 37). Therefore, these qualitative research aims are compatible with commognitive research when Sfard (2008) talks about communication as the primary object of focus in commognitive research together with the researcher is an instrument in the data collection process while observing the principle of utmost verbal fidelity. Another parallel that comes to the fore between the two approaches is their emphasis on natural settings. Qualitative inquiry emphasises natural settings and commognitive researchers mostly conduct their inquiry in the natural setting of a school classroom (Gcsamba, 2014; Wang & Kinzel, 2014; Mudaly & Mpofu, 2019; Zayaddi, 2019; Weingarden et al., 2019; Roberts & Le Roux, 2019).

As this study sought to gain insight into the social phenomenon of exploring grade 11 learners' commognition when engaging with geometry, I gathered that qualitative research and commognitive research were compatible. I looked at the methods of data collection and analysis and noticed that they were primarily qualitative. I also examined my dual role as the researcher and participant, which put me in a position to gain an in-depth understanding of the phenomena under investigation. The methodology I embraced for this project does not allow the findings to be separated from the context. However, I must stress that, as it is true

of qualitative research, the characteristic of this type of research is not to generalise the findings to a larger population.

Based on what I have elaborated in this section, I concluded that Commognitive approach and qualitative paradigm are compatible. The table (4.3) below compares some commonly agreed-upon characteristics of qualitative research with some goals and characteristics of a commognitive approach to research

Table 4.3 Comparison of Qualitative and Commognitive Research Characteristics. Source: Researcher's own initiative derived from the review of literature.

Qualitative Characteristics	Commognitive goals and Characteristics
Inductive Inquiry	The commognitive researcher searches for patterns in observed instances to "extrapolate beyond the present set of data into a range of future situations (Sfard 2008, p. 200)
Understanding Social Phenomena	The unit of analysis in commognitive research is discourse- a special type of communication (collective activity) that follows patterned actions and reactions that evolve across time within a community. Participation in discourse is viewed as a communal/social activity.
Atheoretical	Commognititve research is less about generating theory and more about identifying and describing discursive patterns.
Holistic Inquiry	Commognitive research assumes holistic
Context-Specific	

	interpretation and avoids exclusivity in its claims.
Observer-Participant	Commognitive researchers are interested in discourse defined by the context of the community in which it occurs (e.g., classroom, school, and academia).
Narrative Description	The commognitive researcher is a participant and observer.
	The results of commognitive research are non-statistical. They are instead narrative descriptions of the circumstances and evolution of discourse

4.3 Researchers' background, beliefs and biases

I'm a qualified mathematics and physical sciences educator and have been teaching for the past 25 years. I am passionate about teaching and learning, and as a result, I'm a lifelong learner. I'm an academic staff member at the institution where I conducted the study.

I believe in the social constructivism paradigm. I concur with the paradigm as it regards learning as a social activity. Social constructivism represents knowledge as a human product that is socially and culturally constructed (Ernest, 1999; Gredler, 1997; Prawat & Floden, 1994), cited in (Kim, 2001). For Schreiber and Valle (2013), the participant's role in social constructivism is to be an active member of the collective in creating their own knowledge. However, Vygotsky (1978) points out that what the individual learns depends on what the learning leader (or teacher) provides.

In the next section elaborate on the social constructive paradigm that has shaped my views on teaching and learning.

4.3.1 The origins of constructivism

The origins of constructivism can be traced to traditional scholars such as Piaget (1967) and Bruner (1966, 1996), Vygotsky (1978). These traditional scholars propose that in the teaching and learning situation, learners should be given adequate opportunities to talk with each other and be allowed to interpret and construct what they may not know by asking questions (Erdem, 2001).

4.3.2 What is social constructivism?

Constructivism is an amalgamation of several theories that have a similar regard to the approach to how people learn. According to Merriam and Caffarella (1999), The "*constructivist stance maintains that learning is a process of constructing meaning; it is how people make sense of their experience"* p. 260.

4.3.3 The social constructivism view of learning

Social constructivism regards learning as a process of constructing one's own knowledge. As a stakeholder in the learning situation, the learner derives new understanding using prior knowledge. The learner should be regarded as an active participant in the learning situation and should negotiate his or her understanding in a given new learning situation. If the new knowledge to be gained/acquired falls short of the learner's current understanding, the learner should be provided with recourse to assimilate and accommodate this new knowledge.

4.3.4 The social constructivism view of teaching

To the social constructivist, the learner is an essential partner in the learning situation. The constructivist view of teaching puts the learner at the centre of all teaching and learning

activities. In this regard, Hoover (1996) argues that teaching is not one-way traffic that transmits knowledge from the more knowledgeable (expert) to the less knowledgeable (novice). In other words, teaching is not a monologue or one-direction act. Instead, it is about delving into the learner's prior knowledge while at the same time providing appropriate learning opportunities to challenge the learner's current knowledge and understanding (Clements, 2010; Hoover, 1996).

4.3.5 The social constructivism view of the learner

Social constructivism views the learner as an individual who comes to the teaching and learning situation with prior knowledge shaped by his or her individual experiences. While participating in meaning-making, the learner interprets the experience individually. Huber and Moallem (2000) point out that in constructivism, the learner is viewed simultaneously as a constructor and an examiner of knowledge while collaborating with others. While regarded as an individual, constructivism calls for the learner not to be passive but a dynamic participant in the teaching and learning situation.

4.3.6 The social constructivism view of the teacher

In constructivism, the teacher plays a vital role as a significant stakeholder in the teaching situation. In this regard, constructivism views the teacher as the facilitator in the teaching and learning situation. This view requires the facilitator to meaningfully contribute to learning, creating a safe and cooperative environment where scaffolding instruction and cognitive apprenticeship may occur (Brown, 1994 & Rogoff, 1998).

Social constructivism has specific elements and principles shaping how theory may be applied in learning. Some of these principles are;

(1) Knowledge is constructed.

(2) People learn to learn as they learn.

(3) Learning is an active process.

(4) Learning is a social activity.

(5) Learning is contextual.

(6) Knowledge is personal.

(7) Learning exists in the mind.

(8) Motivation is the key to learning.

(Kim, 2001, pp. 1-10).

4.3.7 Ontological, epistemological, and axiological stance of social constructivism

Concerning the ontological and epistemological stance, the social constructivism paradigm has favourable views to this study.

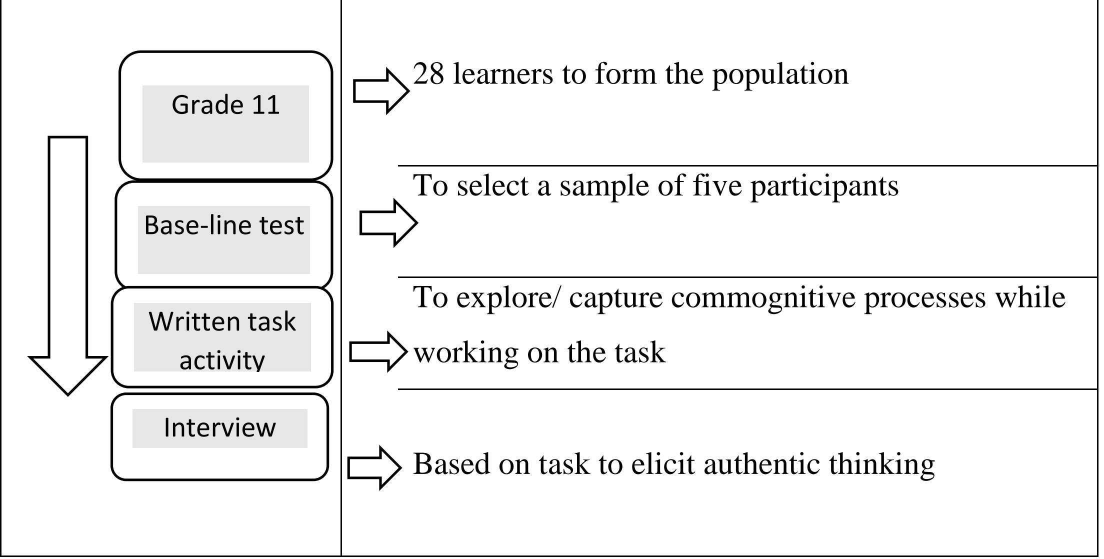
The social constructivism stances are given in (Section 4.2.3, Table 4.2).

Being a mathematics teacher for over 25 years, my interest in this study stems from my long desire to improve the delivery of mathematics teaching and learning in general and geometry. Geometry is an essential area of the mathematics curriculum (Van de Walle, Karp & Bay-Williams, 2013). This significance of geometry has been recognised by many scholars and researchers, such as (Clements & Sarama, 2011a; Abdullah & Zakara, 2013b; Atebe & Schafer, 2008; Ngirishi & Bansil, 2019; Vojkuvkova, 2012). Van de Walle et al. (2013) point out that a rich understanding of geometry is necessary because of its importance and has important implications for other curriculum areas, like algebra, proportional reasoning and measurements.

4.4 Population, sampling technique and participants

In table 4.4 I illustrate how I conceptualised the process from the population to the interview.

Table 4.4: Population, Sample, Written task, and Interview. Source: Researcher's own initiative.



4.4.1 Population

A population in research studies is the entire group you want to draw conclusions about (Creswell, 2012). A population, therefore, does not necessarily always refer to people. A population can contain elements of anything you want to study, such as objects, events, organisations, countries, species, and organisms (Fuso et al., 2017).

This research project was conducted at a secondary school in KwaZulu-Natal. The school has a population of about 500 learners (grades 1 to 12). These are boys and girls drawn from the surrounding communities. The majority of the learners at this school are from what is regarded as slow-socioeconomic status (SES) households. The subjects in the study were grade 11 learners who are offered the core mathematics syllabus. The average age of the participants was 16 years old.

Mathematics in the FET phase covers ten main content areas. Each content area is designed in such a way as to contribute towards the acquisition of specific skills. Some of these skills associated with geometry include the:

(1) Development of appropriate use of mathematics language.
(2) Use of spatial skills and properties of shapes and objects to identify, pose and solve problems creatively and critically (DBE, 2013, pp. 8-9).

Euclidean geometry, or circle geometry, is dealt with in grade 11. The written task activity, one of the data sources for this study, was designed based on the content of Euclidean geometry covered in the grade 11 syllabus. It was convenient that I selected participants from this population (grade 11) as I perceived them to be the most appropriate source of the data I required to answer the research questions.

4.4.2 Sampling technique

Regarding sampling, Mudaly and Singh (2016) assert that there is a close relationship between the methods of sampling and the population. They point out that sampling is informed by the population on which the study is based. Siding with this stance, I adopted the purposive sampling technique for this study. Patton (2002) articulates that purposeful sampling is widely used in qualitative research to identify and select information-rich cases for effective use when limited resources are available. Upon critically reflecting on the suggestions offered by both Mudaly (2016) and Paton (2002) about what informs a suitable sampling method /s, and considering that I had limited resources for the study, I based my rationale for adopting a purposeful strategy on the aims and objectives of the study. (Refer to section 1.8).

I purposefully sampled the participants. The reason for purposive sampling was that I needed specific kinds of participants who may have held rich information on the phenomena at hand. (Mason, 2002; Robinson, 2014; Trost, 1986).

4.4.3 Participants

Sample size in research refers to the number of participants included in the sample. This group, now called a sample, is selected from the general population and considered representative enough of the real population. According to Baker and Edwards (2012), qualitative researchers do not generally study many people and delve more deeply into those selected individuals, settings, subcultures, and scenes, hoping to generate a subjective, informed understanding of how and why people perceive, reflect, role-take, interpret, and interact. In addition, Baker and Edwards (2012) point out that in qualitative research, the determination of sample size is contextual; therefore, they offer the following suggestions as determining factors;

(1) The research approach chosen.

(2) Recommendations made by qualitative researchers.

(3) Homogeneity of participants' background.

(4) Accessibility of participants.

(5) Attainment of saturation.

(6) Availability of time and resources.

(7) Adequacy of the potential data to address the research questions

(Barker & Edward, 2012, pp. 4-10).

In this study, I extensively evaluated existing literature on commognition studies by researchers such as (Nisa Z et al, 2021; Thoma & Nardi, 2018; Shinno & Fujita, 2022; Nachlieli & Tabach, 2019; Nachlieli & Heyd-Metzuyanim, 2022; Mpofu & Mudaly, 2019).

I then proceeded to analyse and reflect on the considerations suggested by Barker and Edwards (2012). Based on the information I gleaned from these authors and the evaluation of existing literature on commognitive studies, I initially selected a sample size of 5 participants. However, due to unforeseen circumstances, two participants dropped out at the very last minute, and therefore, only 3 participants were available to write the mathematics task and later participate in the focus group interview.

4.5 Procedure

A scientific procedure means a procedure through which a given task related to the research and reaching the research aim is successively implemented.

4.5.1 Ethical consideration

Hornby (2000) defines ethics as a system of moral principles, or rules of behaviour, connected with beliefs about right or wrong. McMillan (2012) adds that within the context of gathering data from subjects or using data in which subjects are identified, ethics are concerned with what is right or wrong, good or bad, or proper or improper. According to Yin (2014), studying “a contemporary phenomenon in its real-world context" obligates the researcher to practice essential ethical practices. Yin goes on to say that the researchers, or researchers, are responsible for conducting their studies with special care and sensitivity, going beyond the research design and other technical considerations.

McMillan (2010) also postulates that researchers should generally be open and honest with participants in all study aspects. He adds that informed consent is achieved by providing subjects with an explanation of the whole research, an opportunity to terminate their participation at any time (with no penalty), and full disclosure of any risks associated with the study.

I obtained informed consent from the participants and from their parents/guardians as well as permission from all the relevant stakeholders, in this case, the school at which the study was conducted and from the Institutional Ethics Board of the University of KwaZulu-Natal. The consent form I forwarded to the parents was written in two languages, English and the local language of IsiZulu. I used these two official languages because they are the most common languages spoken by the majority of the population of KwaZulu-Natal. Sending consent in two languages was done to cater to parents/guardians who might experience challenges understanding the nature and scope of the research project and probably some of the words/terms used in this study. I described the purpose of the study to the participants and outlined what the research participants' involvement in the study would entail. I also explained to the participants that by volunteering, they should not expect to receive favours from me or any staff member for their participation. In this report, I avoided identifying the participants by name; instead, I used pseudonyms. Furthermore, I only provided a general description of the school and area where the study was conducted. Additionally, before recording the interviews, I sought permission to videotape the participants (as stated on the consent form). I also undertook appropriate steps to blur their faces. Interview transcripts were all labelled with pseudonyms, and I did this in order to protect the identities and privacy of the participants.

The safety of the participants, as outlined by the researcher's professional responsibility, was another aspect that I adhered to. This implied that the responsibility of ensuring that the study was conducted in a safe and clean environment lay with me as the researcher. Furthermore, I ensured and maintained the highest research standards by being honest and accurate when gathering and interpreting the data. This standard enables the researcher to report the findings that would be confirmed with a high degree of veracity. This guided me to seek and consult peers and experts who reviewed the report. The idea was to allow peer and expert review

before reporting the findings. Another step I took was to protect the participants' privacy; it meant securing all data in a locked cupboard so that their privacy and identities were kept confidential. In addition, I forwarded all electronic data to the supervisor, who locked this data in electronic folders and ensured that the code was not to be shared with anybody outside of the study.

4.5.2 Base-line test

Before administering the base line test, I ensured that we had completed the Euclidean section of the grade 11 syllabus with the whole cohort of grade 11 mathematics learners during regular teaching lessons. I also ensured that I reviewed some basic concepts, like how to use a compass on construction tasks, as this was covered in earlier grades. I used the van Hiele test as my base-line test instrument.

4.5.2.1 van Hiele test

A base line test was one of the preliminary steps I needed to start this investigation. I embarked on the task of identifying the most suitable instrument for the pre-test by reviewing related literature (Machisi & Feza, 2021; Salifu et al., 2018; Asuti et al., 2018; Amidu & Nyarko, 2019) about the van Hiele model and its affordance as a tool to measure and identify the thinking level of learners. Based on this review, I decided that the van Hiele test that was used in the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project (Usiskin, 1982) and had been used by many researchers (Wang, 2013; Zakaria, 2013; Baffoe, 2010), was therefore the appropriate tool to be used as a pre-test instrument. The van Hiele Geometry Test (VHGT) contains twenty-five multiple-choice items distributed into the five van Hiele levels. These items identify the learner’s geometric thinking at the five levels. I used this van Hiele Geometry Test to provide information on the

learner's level of thinking for the sample selection. The table below provides these van levels of geometric thinking.

Table 4.5: van Hiele levels of geometric thinking (Adapted from Karakus & Peker, 2015, p.339)

Levels	***Characteristics***
Level 1 (Visualisation)	*Students recognise figures by their appearance. They make decisions based on intuition not reasoning.*
Level 2 (Analysis)	*Students recognise figures by their properties. They can analyse and name properties between these properties.*
Level 3 (Informal deduction)	*Students can distinguish between necessary and sufficient conditions for a concept. They can form meaningful definitions and give informal arguments to justify their reasoning.*
Level 4 (Deduction)	*Students can construct theorems within an axiomatic system. They know the meaning of necessary and sufficient conditions of a theorem.*
Level 5 (Rigor)	*Students understand relationships between various systems of geometry. They can compare, analyse and create proofs under different geometric systems*

In this model, there are five levels of geometric thinking. These are; (1) visualisation, (2) analysis, (3) informal deduction, (4) deduction, and (5) rigor). van Hiele and other researchers (Abdulla & Zakara, 2013; Al-ebous, 2016) have stated that in order to develop geometric thinking, a student must progress through these levels sequentially. Pierre and Dana (1985) developed their theory on three learning assumptions in developing their model. These assumptions are: (1) that learning is a discontinuous process characterised by qualitatively different levels of thinking; (2) that the levels are sequential, invariant, and

hierarchical. The process depends upon instruction, not age, and (3) concepts implicitly understood at one level become explicitly understood at the next level (van Walle et al., 2013). Each level is equipped with its own set of unique terminology, symbols, concepts and reasoning strategies.

Many researchers have widely acknowledged the practicability of the van Hele model when describing the sequential levels of development of geometric thinking. Some of these researchers who have used the model include (Wang, 2014; Chang & Bhagat, 2015; Baffoe & Mereku, 2010).

4.5.3 Selection of participants

According to the school mathematics syllabus used in South Africa (DBE, 2013), the main concepts of Euclidean geometry, particularly those concerning the theorems of the circle, are introduced in grade 10 and reinforced using the spiral approach in grade 11. The grade 11s were my target, and it is from this population that I drew a sample. As stated on page 103, Mudaly (2016) claims that sampling matters arise directly from defining the population on which the research will focus. This study was focused on obtaining information from grade 11 learners regarding their commognition while engaging with geometry. I used purposive sampling procedures to select participants to obtain rich information. I started the sampling process by letting the whole grade 11 class, including those who had not consented to participate, write the base-line test of geometric thinking. The reason for giving the VHGT as a base-line test to the whole class was two-fold. The first reason was to gain information about the whole class concerning their level of geometric thinking. This information would be helpful when conducting remedial teaching to the class. The second reason was to target those consented and choose the sample from among them. I had four goals in giving the pre-test: the first goal was to encourage geometric thinking in the participants, the second was to

provide me with a representative sample of participants, the third was to ensure that the participants acquired sufficient experience with what was being studied (McMillan, 2012), and the fourth and final goal, was to gather valuable information on the thinking levels of all the learners in the class. Regarding the participants, the information from the pre-test was vital to me as the investigator. I needed this information to have sufficient knowledge to assess their level of geometric thinking before implementing the data collection process. The aim was to recruit participants who were at least on par with their peers concerning their level of geometric thinking and could provide information on the phenomenon. The other objective was that commognitive researchers in their data analysis, as articulated by Sfard (2008), often focus their attention on every detail such as what was said, and what was done by the participants. Sfard (2008) refers to this as the principle of completeness. I then used the results of the VGHT base-line test to select five participants for the study rather than a large number (only 3 participants were available to write the mathematics task and participate in the focus group interviews). As stated in the report, McMillan (2012) postulates that since the aim of qualitative research is to understand phenomena in depth, it is vital to select subjects that will provide the richest source of information. Therefore, reflecting on what McMillan says, I decided that the five participants I selected were sufficient and would provide enough data to give me a greater understanding and description of the phenomenon. To ensure transparency and fairness in the selection process, I recruited a representative sample of the general population. I decided that this sample of five participants of mixed gender and of diverse cultures should comprise two participants of high cognitive ability, one of average cognitive ability, and two of low cognitive ability.

Table 4.6: Selecting participants from representative group. Source: Resaercher's own initiative derived from the review of the literature.

SELECTING INTERVIEWEE FROM REPRESENTATIVE GROUP	
GEOMETRIC THINKING ASSESSMENT	
High level geometric ability	Two interviewees
Average level geometric ability	One interviewee
Low level geometric ability	Two interviewees

Considering the seven suggestions offered by Baker and Edwards (2012), as given at the beginning of this chapter, the review of literature on commognitive studies that I conducted, and Sfard (2008) suggestions of focusing attention on every detail of what was said and done, I concluded that a sample of five participants would be adequate for my study. (However, due to unforeseen circumstances, two participants dropped out, so I remained with three participants in the study).

4.5.4 Data collection tools

Qualitative researchers use several tools to gather data from participants. I used a written geometry task activity and focus group interviews for this study.

4.5.4.1 Written task activity and interview

The choice of which instruments to use is informed by the research design. McMillan (2012) posits that qualitative researchers approach a situation assuming nothing is trivial or unimportant. This view is also shared by Sfard (2008) concerning commognitive researchers. Keeping the views of McMillan and Sfard in mind, I decided that the kind of data needed, the strategies of gathering and analysing the data, the ways of merging all this to answer the research questions, required that I select the most appropriate and effective instruments. The

appropriate instrument in this regard needed to have certain features that would facilitate the following: (1) provide opportunities for participants to participate and experience solving a mathematical problem, (2) provide opportunities to engage in mathematical discourse, and (3) provide opportunities to demonstrate the use of the commognitive components, (word use, mediator use, narratives and routines).

As the researcher, I decided to use a task based on Euclidean geometry as this section has great potential to engage the participants in geometric construction problems. This geometric activity also has the potential to provide adequate opportunities for participants to form their own conjectures and engage in argumentation processes. In this regard, Kim and his colleagues (2017) point out that suitable task questions should enable participants to engage in discourse and create data with the potential to answer research questions. I collected data from these two sources: (1) the responses to the pencil and paper written task (document) and (2) the focus group interview.

The goals of the written task activity and focus group interview were:

(1) To examine participants' abilities when verifying their claims and derive mathematical proofs;
(2) To elicit authentic participants' thinking and capture related commognitive processes;
(3) To explore participants' geometric discourse;

Using the compass was explicitly to realise the graphic representation of circles and diagrams.

Marriott (2001) postulates that students can more potentially comprehend mathematical proofs if they engage in argumentation processes which lead to forming conjectures rather than merely reading and following pre-prepared proofs.

It is also commonly thought that geometrical constructions allow students to form conjectures and consider why their construction worked.

Geometrical construction provides valuable experience in prof and proving geometrical statements.

The selection of the task activity was informed by epistemological, pedagogical purpose, pedagogical, and discursive considerations. These considerations are elaborated below.

Epistemological consideration- Different epistemological approaches to mathematical learning will have different implications on the task. The task was considered to be a mediator for mathematical discourse.

Students acquire knowledge by learning specific mathematics language/content. Because mathematics is a discourse, learning mathematics is participating in this discourse.

Pedagogical purpose- This can be considered a function of how the researcher perceives mathematical knowledge epistemologically.

Tool representational considerations- The way a chosen tool represents mathematical knowledge. Symbolic representation, mathematical concept? Is the tool capable of representing the targeted mathematical knowledge parallel to the corresponding symbolic representation?

Pedagogical consideration- Can the task be supported by a suitable pedagogical environment?

Discursive consideration- Sfard (2008) sees mathematics discourse as a type of communication with unique features: keywords, visual mediators, routines, and narratives. The task should be a means to facilitate the development of these features through task design. Table 4.7 below illustrates the selection of mathematical task considerations.

Table 4.7: Selection of mathematics task considerations. Source: Researcher's own initiative derived from the review of the literature.

	Consideration	Explanation (Brief)
1.	Curriculum alignment	The task should be in line with curriculum requirements and grade level
2.	Epistemological and mathematical	The task should be aligned to the epistemological approach
3.	Pedagogical	Task should be supported by a suitable pedagogical environment and resources.
4.	Discursive	Task should be means to facilitate the development of the features of mathematical discourse.
5.	van Hiele level of geometric thinking	The task should be at the appropriate level of geometric thinking

The second step that I took was to examine the task in detail and decide whether the task was adequately aligned to the research questions. I drew up a table to conceptualise this process. See Table 4.8.

Table 4.8: Alignment of research instruments with research questions. Source: Researcher's own initiative.

Research Questions	**Research instrument**
1. What discourses do learners engage in when learning geometry? 2. How do these discourses influence the way they	Written task activity and interview

understand geometric concepts? 3. How can these discourses be influenced to improve understanding of geometric concepts?	

Participants worked individually on the task. After participating in the written task activity, the participants proceeded to do the focus group interview.

The interview questions were of two kinds: semi-structured questions and open-ended questions. The primary purpose of these questions was to get answers from the participants that would enable me to have information and engage in attempts to analyse the research questions. In designing the questions, I was guided by the suggestions of Zazkis and Hazzan (1999) regarding the features of the mathematics questions and the design considerations. My questions had some of the following properties of interviewing questions;

(1) Performance questions - *"How would you"*

(2) Unexpected questions - "Why *do you think"*

(3) Twist questions - *"How could you"*

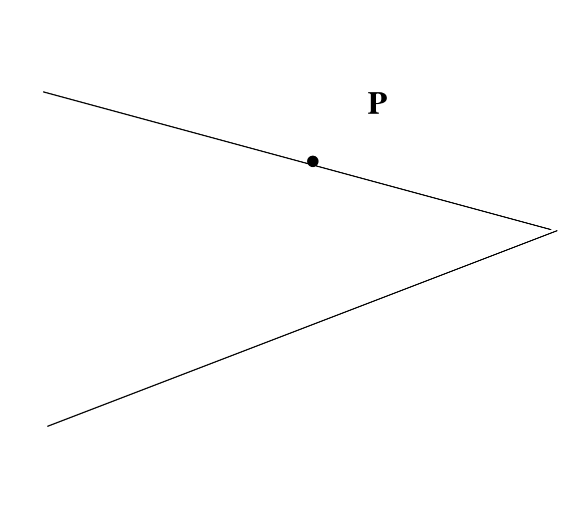
(4) Construct task questions - *"Define"*

My choice for these questions was influenced by the theoretical lens of commognition. I designed them so that the questions corresponded to the theoretical framework, as I needed the participants to engage in mathematical discourse and respond by serving as identifiers of different features of mathematical discourse. The interview was conducted by answering these semi-structured and open-ended questions based on the task (Goldin, 2000). The open-ended questions served as a mode of prompting participants' responses and extending the discourse. Using such questions aimed to articulate the participants' thought processes as they

worked on the pencil and paper construction task. The participants were asked to explain how they obtained their solutions. The participants talked freely, and I only interjected to probe when I needed more clarification. In some instances, I provided prompts to deliberately encourage the participants to talk more or respond in a different way. At most, the participants were required to explain all the steps they followed along with their reasoning. The purpose of this focus group interview was to create an environment for easy interaction with one another, to enable the participants to explain their mathematical processes as they engaged in the written task from a commognitive point of view and to listen to different points of views. The interview was video recorded, the goal being to have a visual record of their participation in this research project. See Appendix D for the interview protocol.

A sample of the task activity is given in Table 4.9.

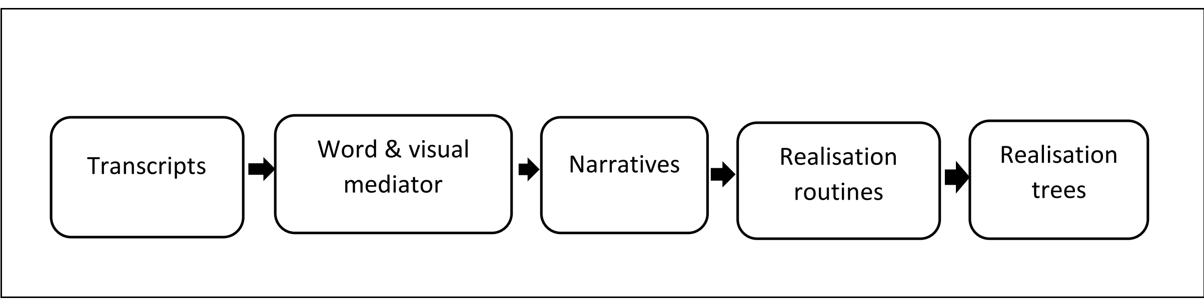
Table 4.9: Written task activity. Source: Researcher's own elaboration

<table>
<tr><td>1. You are given two intersecting straight lines and a point P marked on one of them as shown in the diagram.
Show how to construct, using straightedge and compass, a circle that is tangent to both lines and has the point P as the point of tangency to one of the lines.</td><td></td></tr>
<tr><td colspan="2">2. Prove that two lines from the same point which is outside a circle, such that both lines are tangent to the circle, then their lengths are the same.</td></tr>
</table>

I chose the task questions to give the participants ample opportunities to practice and experience mathematical discourse while demonstrating word use, use of visual mediators, routines and constructing narratives.

Figure 4.2 illustrates how I conceptualised my steps as I progressed from the transcripts to the realisation trees.

Figure 4.2: Steps in progression from transcripts to realisation trees (Adapted from Roberts & le Roux, 2019, p. 7)



4.6 Data processing

In scientific enquiry, data processing is collecting and translating a data set into valuable, usable information. Jansen and Vilminko-Heikkinon (2016) point out that data is a set of characters that may have no meaning unless seen in the context of usage. Therefore, the objective of undertaking this process is data reduction in readiness for analysis. In this section of the report, I outline the step-by-step actions I undertook to reduce sets of data for the purposes of gaining a better position to answering the research questions.

4.6.1 Pre-coding stage

The qualitative researcher is regarded as an instrument in that he/she is an outsider and, simultaneously, a participant. According to Sfard (2008), scientific researchers who attempt and adopt a commognitive perspective commit themselves to take up the dual position of being both as an observer and a participant of the discourse under study, and this in their analyses entails to periodically navigate between these two positions. Because of this dual

role, Creswell (2014) points to the need to identify personal values, assumptions and possible biases at the onset. In this regard, (Locke et al., 1987) add that an investigator's contribution to the research setting can be useful and positive rather than detrimental.

"With *these concerns in mind, inquirers explicitly identify reflexively their biases, values, and personal background, such as gender, history, culture and socioeconomic status (SES) that shape their interpretations formed during a study"* (John W Creswell 2014, p. 165)

I took measures to be aware of and reflect on the factors influencing data processing and analysis. I understood to bracket these factors, including, among others, the researcher's background, beliefs and biases, interests, and the philosophical paradigm. The measures to bracket these influences were documenting personal reflections and impressions in a memo. I also documented my thoughts about the process at hand. The following are some of the reflections and impressions I wrote in my memo;

(1) What the investigator was observing in the field.

(2) The experiences during the data collection process.

(3) The investigator's interaction with the participants

(4) The investigator's thoughts about the codes and their meaning.

(5) The relationships among codes, categories, and themes.

4.6.2 Kinds of data

I collected raw data from two sources: a pencil and paper geometric written task and, later, a focus group interview. The data from the written task was in text format, while the focus group interview data was in video/audio recording. The recording provided evidence of the participant's interactions in mathematical discourse. The data from the interviews was transcribed verbatim into text with the aid of a computer program called Otter.ai, and later, I

worked through the transcript manually to capture what the participants said and what was done while speaking.

Table 4.10: Sample of some of the data collected

Turn	***Speaker***	***What is said***	***What is done***
8.	S2:	It's a line outside a circle…..outside the circle.	While looking at S1.
9.	S2:	Any line that is a tangency to a circle doesn't go through the circle………through.	Raises the hand to indicate through.
10.	S2:	Cuts the circle.	Looks at S1.
11.	S3:	Doesn't cut the circle.	
12.	S2:	Touches the circle.	
13.	S1:	It touches the circle….	Bends head to one side while looking at the interviewer

4.6.3 The coding processes

A code is a researcher–generated construct that symbolises or "translates" data analysis (Vogt et al., 2014, p.13) and thus attributes interpreted meaning to each individual datum for later purposes of pattern detection. Saldana (2014) suggests various methods of coding data in the first and second cycles of the coding process. These methods include, among others, attribute coding, descriptive coding, In Vivo coding, process coding and theming data.

However, before selecting the appropriate method for this study, I reviewed the literature on related commognition research studies and looked at other researchers' unique methods. These are some of the studies that I reviewed (Nisa Z et al., 2021; Thoma & Nardi, 2017; Shinno & Fujita, 2021; Nachlieli & Tabach, 2018; Nachieli & Heyd-Metzuyanim, 2022; Mudaly & Mpofu, 2019), and Patton (2014) says this about qualitative research "*Because each qualitative study is unique, the analytical approach used will be unique".* (p. 522).

After studying related literature, I reflected upon the nature of my research questions and how I had framed them, ie. using "*what"* and "*how*". This consideration was fundamental in influencing my specific coding choices and the types of answers needed. Trede and Higgs (2009) suggest that research question framing should harmonise with the ontological, epistemological stance informing the study. These stances are presented in Table (4.2) in this report. I summarised these stances for the reader. The ontological stance is that participants and researchers develop multiple realities through social interaction, and the epistemological stance is that there is a co-creation of reality between participants and the researcher.

The study had three research questions:

1. What discourses do learners engage in when learning geometry?
2. How do these discourses influence the way they understand geometric concepts?
3. How can these discourses be influenced to improve the understanding of geometric concepts?

The first question using "*What"* is an ontological question as it addresses the nature of participants' multiple realities. Saldana (2014) points out that these types suggest exploring personal, interpretive meanings found within data. To better reveal these ontologies, the best approach was theming the data. The following two questions using *"How"* focus on understanding the phenomenon and, therefore, are epistemological. These questions suggest

exploring the participant’s action processes and perceptions within the data (Saldana, 2014). The appropriate methods for questions 2 and 3 are descriptive coding, In Vivo coding and theming data. Vivo coding refers to a word or short phrase from the language found in the qualitative data record, i.e., “*the terms used by participants themselves*” (Strauss, 1987, p. 33).

After deciding on the most appropriate method of coding data, my next task was to decide whether to code data electronically or manually. Using the commognitive lens and what I had discerned from the review of related literature, I decided to first electronically code and then manually code the rich and in-depth data regarding what was said and what was done by the participants.

4.6.3.1 First cycle coding

Having decided on the methods to code data, I then reflected on the method suggested by Miles and Huberman (1994) and used a predetermined list from the cognitive conceptual framework. I perused the transcripts, searching for both significant and minor illuminative quotations and coded them using the categories that had been identified.

4.6.3.2 After the first cycle of coding

Saldana (2014) offers some suggestions on how to conduct the after-first-cycle coding. These suggestions are what I used to categorise the codes and group them under four themes: (a) the use of words, (b) the use of visual mediators, (c) the construction of narratives, and (d) the employment of routines. These four themes were derived from the discursive characteristics of mathematics expounded by Sfard (2008). Following Saldana's (2014) suggestions, I then grouped them based on (a) the relationship between codes, (b) the code frequencies, and (c) the underlying meaning across codes. I exemplify the process that I took in table 4.1

Table 4.11: Coding process in inductive analysis (Adapted from Creswell, 2002, p. 266)

Initial read though text data	Identify specific segments of information	Label the segments of information to create categories	Reduce overlaps redundancy among the categories	Create a model incorporating most important categories
Many pages of text	Many segments of text	30 – 40 categories	15 – 20 categories	3 – 8 categories

4.7 Quality assurance

In this section, I discuss quality assurance. Quality assurance is an essential aspect of scientific inquiry as it deals with the validity and reliability of the study's findings (Brink, 1993). Quality assurance is even more essential in qualitative studies as the quality assurance measures inform the difference between what is considered a good and or an alarming study. The reason is that, unlike quantitative studies, qualitative studies are regarded as subjective in nature as the findings often rely on the researcher's interpretation. (Brink, 1993). In this section, I discuss the measures I used to enhance the truthfulness and accuracy of the study. The strategies to enhance rigour are presented in Table (4.12) below.

Table 4.12: Strategies to determine rigour. Source: Researcher's own initiative derived from the review of literature.

Approach to rigour	Strategies
1. Credibility	
2. Transferability	Thick descriptions
3. Dependability	Audit trail Reflexivity
4. Confirmability	Audit trail Reflexivity

4.7.1 Credibility

Credibility in qualitative research is about establishing to what extent the findings agree with reality, Merriam (2019). In a qualitative study, researchers go back to their data “o*ver and over again to see if the constructs, categories, explanations, and interpretations make sense* (Paton, 1980, p. 339). Lincoln and Guba (1985) point out that ensuring credibility is crucial to establishing the accuracy of the findings. Therefore, in qualitative research, credibility is attained by the research answering the following questions:

(1) Truth value?

(2) Do I have complete information?

(3) Is the information true for the participants?

The rationale for answering these questions is that, unlike a quantitative study, the interpretation of the findings is the inquirer’s intuition in a qualitative study. The role of the researcher in qualitative inquiry, much like that of a commognitive researcher, alternates between being an “insider” and “outsider” to the discourse under study (Sfard, 2008; p. 278). Because of this double role, the researcher is regarded as an instrument himself/ herself, and therefore, the information gathered must be credible (Rodger & Cowless, 1993).

In this study, the participants were grade 11 learners in the investigator’s mathematics class. As reported earlier, I have taught in the school for several years and was familiar with the context, environment and culture. Given my familiarity with the environment and adequate understanding of the context, I was known to the participants and had built a relationship of trust long before the first data collection dialogue. This meant that the participants felt free to give out information.

To further ensure credibility, I used a purposive sampling technique, enabling me to select participants equitably for the study. The participants first had to write a van Hiele geometric

test to assess their geometric thinking level and prepare them for the task. Then, from the results of this test, I could select participants who would provide rich information. I further took measures to ensure that the sample was representative of the population. Another strategy I used was the triangulation method, which uses multiple sources to gather data. I triangulated data from multiple sources by looking at both supporting and unsupportive evidence that I could gather. The benefits of using triangulation, according to Guba (1981), Brewer, and Hunter (1989) is that each method of collecting data has its own individual strengths and limitations, and using multiple sources of data simultaneously is one way of compensating for the individual limitations of the methods.

A further strategy to ensure credibility was subjecting the study to constant scrutiny. This research project was scrutinised from different stages in its course by peers and the supervisor. The advice from both the supervisor and the peers was acted upon. I also took time to extensively examine previous research findings of related literature. This was an essential aspect of ensuring credibility. The rationale for using this strategy was to assess the degree to which the results from this study agreed with those of past studies (Silverman, 2013).

Another strategy of credibility that I employed related to interviewing techniques. This included looking for cues and gestures to gauge the participants' understanding of the questions. Probes were used to uncover information or to clarify some concepts. Some of the questions that appeared to be unclear were reframed, and in this way, I was able to ascertain whether the participants understood the question and were giving consistent information.

4.7.2 Transferability

In qualitative research, transferability refers to whether or not particular findings can be transferred to another similar context or situation while still preserving the meanings and inferences from the completed study (Leininger, 1994).

The research project was conducted at a secondary school in KwaZulu-Natal. The school has a population of about 500 learners that range from grade 1 to grade 12. The school caters to boys and girls from the surrounding communities. Most of the learners at this school are from households with low economic status. The learners who participated in this study project were grade 11 in the core mathematics class. The average age of the participants was 16 years old.

The qualitative phenomenological and case study approaches were used in this project. The rationale for choosing these two approaches was to complement each other while exploring the participant's lived experiences.

4.7.3 Dependability and Confirmability

In scientific inquiry, dependability refers to data stability (Graneheim &Lundman, 2004; Tobin & Begley, 2004; Shah & Corley, 2006; Rolfe, 2006). On the other hand, confirmability refers to the neutrality and accuracy of data (Tobin & Begley, 2004).

To ensure the trustworthiness of the research project, I employed the strategy Lincoln and Guba (1982) suggested of recording comprehensive reflective notes on the study's progress. The aim of an audit was to record every stage of the inquiry and outline how I arrived at decisions made throughout the research project. Koch (1994) suggests that although the readers of the report are not obliged to influence a researcher’s interpretation, they should be able to understand how it has been determined.

The following are some of the subheadings I wrote notes on:

(1) Perceptions of the researcher's prior belief of how the research process would unfold.

(2) A day-to-day report in chronological format describes what was accomplished.

(3) All methodological decisions and the rationale for the action taken.

(4) Own thoughts and feelings.

(5) Summary of where researchers change position concerning knowledge of the phenomena and a working hypothesis about it.

(Lincoln & Guba, 1982).

Another strategy I used was that of maintaining a reflexive diary. The rationale for using this strategy was to record my day-to-day reflections. Qualitative inquiry is a subjective process as the researcher is also an instrument, and therefore, the credibility of the study is underpinned by the procedures implemented and the self-awareness of the researcher that is maintained throughout the research process (Stoecker, 1991; Rodgers & Cowles, 1993; Mantzoukas, 2005). The audit trail is illustrated in table 3.12 below.

Table 4.13: Audit trail. Source: Researcher's own initiative.

1.	A log of evolving perceptions
2.	A log of day –to – day procedures
3.	A log of methodological decision points
4.	A log of day-to-day personal introspection
5.	A log of developing insight

4.8 Conclusion

A research design primarily consists of ways of gathering data, examination informs interpretation and presenting these sets of data in scientific inquiries (Creswell & Plano, 2007).

The specific qualitative approach (interpretive approach) was adopted for the study. The rationale for adopting this approach was informed by the following; the purpose of the study, the characteristics of the research questions, viz. “What” and “How” are the kinds of data and the data sources. The study was underpinned by the qualitative philosophical assumptions expounded by Creswell and Poth (2018). Some of these assumptions include;

(1) Natural setting by collecting data in the field at the site where participants experience the phenomena.
(2) Multiple methods of collecting data.
(3) Participants have multiple perspectives and meanings.

I further elaborated on the two qualitative approaches in great detail, informing the study and the rationale behind the combination of the two approaches. The approaches that I combined are phenomenological and case studies, and the rationale was the need to investigate the phenomenon within its real-life context and derive multiple sources of evidence (Yin, 1992). The two approaches complement each other regarding the strengths and limitations inherent in adopting one approach.

I described my personal background, beliefs, and biases that were relevant to the study.

According to Marrow (2005), qualitative trustworthiness is carefully associated with the paradigm that informs a given discipline linked with the investigation.

The intent was to enable the reader to better understand the investigator's actions and the study's findings. The richness of my study would come from a full description of the context, its limitations, issues unique to it, etc.

I described the population and community being studied by presenting the context/setting in detail. I have given details of the participants I was focusing on and covered the reasons and rationale for why this particular population was determined to be the most appropriate data source. The sampling technique (purposive sampling) used for this study was informed by several factors. These factors included the need for a representative sample of participants best positioned to provide information to answer the research questions. Furthermore, the number of participants (4) (only 3 were available on the day of the interview) was determined by the factors articulated by Barker and Edward (2012), on page 129 of this report, as well as the review of the literature of commognitive researchers ((Nisa Z et al, 2021; Thoma & Nardi, 2017; Shinno & Fujita, 2021; Nachlieli & Tabach, 2018; Nachieli & Heyd-Metzuyanim, 2021; Mudaly & Mpofu, 2019).

I elaborated on the detailed step-by-step process of collecting data from the participants. I also explained where the data was collected and the participants' names. The data collected were verbatim transcriptions from the focus group interview and responses from the mathematics task activity (text). I elaborated on my actions, such as getting informed consent from the parents/guardian and participants and getting demographic information. Before getting consent from the participants and parents, I explained the nature of the research project and that their participation was voluntary. I also explained that their names would remain anonymous and not be stated anywhere in the report.

I further elaborated on the qualitative data collection strategies used in the study. These strategies are focus group interviews and document analysis (written mathematics task activity). These instruments provided data in the form of audio/video and text.

I described the data process that was adopted. First and foremost, considering this was a qualitative study, describing how information was analysed in detail was essential. I described the data analysis process and how the research questions were addressed. I then went on to present the coding process of data and how I coded them into four themes (word use, visual mediator use, narratives and routines) that are characteristic of commognitive studies (Nachlieli & Tabach, 2018; Nachlieli & Heyd-Metzuyanim, 2021; Mpofu & Mudaly, 2019). I then elaborated on the coding strategies used in this study. Saldana (2013) suggested that these strategies involved the pre-coding stage and the after-first cycle coding.

Towards the last segment of this chapter, I elaborated on the strategies I employed to demonstrate that the study was conducted rigorously. This was important because of the subjective nature of the qualitative approach. The commognitive researcher alternates between being an observer and a participant, and in analysis, it is crucial to demonstrate rigour, Sfard (2008). The strategies covered in the study are credibility, transferability, dependability and confirmability.

(1) Credibility. Regarding credibility, I demonstrated the accuracy of the data findings by clearly reflecting on the participants' experiences and showing the direct connection between the findings and the data collected. The context and the triangulation of data sources were also described in detail.

(2) Transferability. Regarding transferability, I clearly described the context and detailed the research assumptions that inform the study. The essence was to demonstrate that the findings may be transferable to a similar context.

(3) Dependability and confirmability. I presented the adopted step-by-step data collection and analysis process in this regard. I described the paradigm that informs the study and the researcher's role in the study, bias, and background.

Chapter 5

DATA ANALYSIS, PRESENTATION, AND FINDINGS

5.1 Introduction

My study was aimed at interpreting the discourses of grade 11 learners' commognition when engaging with geometry. The analytical lens employed was Sfard's (2008) theory of commognition. In this chapter 4 of the report, I present and elaborate on the data analysis, present some findings to the reader, and discuss some of these findings. The chapter is divided into three main sections. In the first section, I provided details of how the data analysis was done, the next section dealt with data presentation, and lastly, in the third section, I presented the findings from the analysis.

5.2 Data analysis

Given the nature of my study, I began the data analysis by ensuring that I adhered and followed the guidance that Sfard (2013) suggests to commognitive researchers. These suggested principles to the analysts are;

(1) The principle of completeness.

(2) The principle of interdiscursivity.

(3) The principle of alternating perspectives.

Adhering to these principles meant that I focused my analysis on what happened by considering the entire context within which the discourse emerged. Thus, my data included what was said by the participants and what was done by the participants. While performing the analysis, I was also aware that because the study was conducted in a natural environment, the participants, as members of the collective, also participated in other discourses found in their environment. Therefore, their participation in mathematical discourse was affected by

other multiple discourses. Furthermore, about the principle of alternating perspective, I was reminded to navigate as objectively as possible as I alternated between the roles of a researcher and myself being an instrument and also being mindful of the possibility that my mathematical discourse (as a teacher) may have been different from that of the participants (as learners being new to the discourse). This meant that I considered every participant's utterance in its discursive context.

The essence of employing these principles in my study was to ensure that I analysed the information provided regarding what actually happened and then transformed the information into a coherent account of what I found.

Data analysis, according to Merriam-Webster (2014), is a detailed examination of anything complex to understand its nature or determine its essential features. In a research study, the purpose of data analysis is to extract useful information from data and transform it into a coherent account of what was found to answer the research questions. There are different forms of analysis, which are informed by the type of approach employed. For example, when one considers qualitative studies, the form of the analyses is determined by the specific qualitative approach taken and the form of data collected. Patton (1999) illustrates that any qualitative analysis is a creative process that depends to a more significant extent on the insight and conceptual capabilities of the analyst.

The design for this study was qualitative in nature and employed multiple approaches: the phenomenological and case study approaches. Informed by this type of design, I made the decision to use the inductive method of data analysis because the inductive method gives step-by-step details on how to perform the analysis.

5.2.1 Inductive analysis

Inductive analysis in qualitative enquiry provides a model that guides the researcher to derive concepts from raw data readings. In this model, inquirers are called upon to analyse raw data at greater lengths while seeking for concepts and themes to assist with interpretation (Thomas, 2006).

The inductive approach serves three purposes. These are (1) to enable the researcher to change data that is in raw form into some form of a brief summary, (2) to connect the research study objectives with the derived findings, and (3) to structure the processes evident in the data to the theory.

Thomas (2006) also proposes nine steps that may guide researchers in inductive analysis. These steps are to;

(1) Peruse through the data and come up with main ideas for examination.

(2) Identify areas based on similarity relationships discovered within frames of examinations.

(3) Identify salient domains, assign them a code, and put others aside.

(4) Re-read data, refining salient domains and keep a record of where relationships are found in the data.

(5) Decide if your domains are supported by the data and search data for examples that do not fit with or run counter to the relationships in your domains.

(6) Complete an analysis with domains.

(7) Search for themes across domains.

(8) Create a master outline expressing relationships within and among domains.

(9) Select data excerpts to support the elements of your outline.

(Thomas, 2006; p. 241).

5.2.2 Coding process in inductive analysis

In addition to the above steps in inductive analysis, I also utilised Creswell's (2003) suggestions that he offers analysts to employ regarding the coding process in inductive analysis. According to Creswell (2003), the coding process has five stages that have to be followed. These stages that inform the coding process were conceptualised in (Section 4.6.3.2 the table 4.11).

5.2.3 Instruments

Qualitative inquiry has some characteristics that set it apart from other scientific enquiries. One of these characteristics of qualitative inquiry is that the inquiry is a subjective process based on the researcher's context and intuition. This characteristic resonates with some of the characteristics of commognitive research. Because of this subjective characteristic, Sfard (2008) points out that the data analysis process is the researchers' construct. Hatch (2002) also adds that because qualitative research is subjective, the interpretation relies on the researcher's skills and experience to make sense of the data. Hatch (2002) and Sfard (2008) imply that the specific qualitative approach together with the form and type of available data inform the chosen analysis. For this study, I used two data sources, and the goal was to triangulate the data to adequately answer the research questions. The sources used were the written mathematics task and a focus group interview. These two sources yielded two different kinds of data: text, while the other was a verbatim account of what the participants said and their actions. To make sense of these different data sets, I employed two analytical tools and these are the Adapted Profile of Tangent (Adapted from Mpofu & Pournara, 2018) and the Realisation Tree Assessment of the tangent (Adapted from Knox & Kontorovich, 2022). The opportunities created by using these two analytical tools allowed me to get as close as possible to the data, greatly facilitating the examination. These two analytical tools are described in detail in the following sections of this chapter.

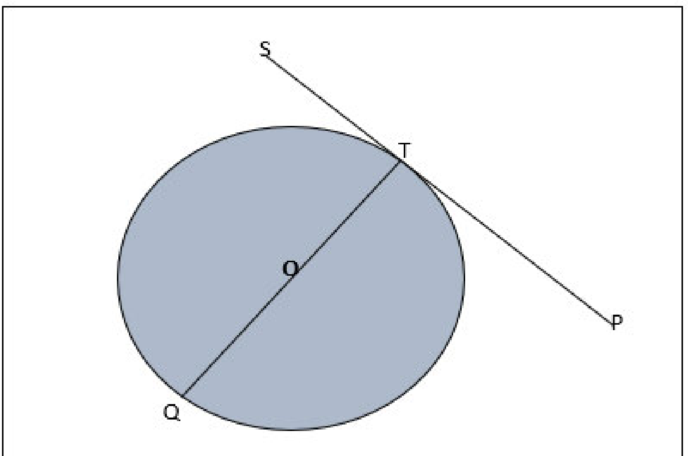
Before embarking on the data analysis process, I aligned the research questions with the data collection and analysis methods. From the onset, the idea was to determine the affordance of these tools in extracting useful information from data for answering the research questions. I used Table 5.1 below to conceptualise this process.

Table 5.1: Alignment of Research Questions with Data Collection and Data Analysis and Methods. Source: Researcher's own initiative derived from the review of literature.

Research Question	*Data Collection*	*Data Analysis*
1. What are the discourses do? Learners engage in When Learning geometry?	Written task activity Video/audio of Focus Group Interview	Realization Tree Assessment Adapted profile of the tangent (DPT)
2. How do these discourses influence the way they understand geometric Concepts?	Written task activity Video/audio of Focus Group interview	Realization Tree Assessment Adapted profile of the tangent (DPT)
3. How can these discourses be influenced to improve the understanding of geometric concepts?	Written task activity Video/audio of Focus Group interview	Realization Tree Assessment Adapted profile of the tangent (DPT)

5.2.3.1 School mathematics discourse on the tangent at a point

Figure 5.1. The point of tangency. Source: Researcher’s own illustration.



Note: *The point of tangency is where line segment QT intersects line segment SP at poit T.*

The concept of the tangent line appears in different domains of mathematics, such as; in geometry-tangent to a circle, algebra-tangent to a parabola, calculus-tangent to a function at a point, and so on.

In this section, I describe school mathematics discourse on a tangent at a point (geometry) about 11th-grade learners in schools. The core of describing mathematical discourse is twofold; the first is to provide the reader with the context and insight as to the acceptable school mathematical discourse on the concept of the tangent, and the second is to provide guidance to me when performing the analysis.

5.2.3.2 The point of tangency

The point of tangent is defined as "a straight line that touches a curve at a single point” Merriam-Webster, (2020).

The tangent has three main properties;

(1) A tangent intersects a curve at only one point.

(2) A tangent is a line that never enters the circle's interior.

(3) A tangent intersects the circle's radius at the point of tangency at a right angle.

Besides these properties, a tangent to the circle is also associated with some mathematical theorems. The relevant theorems to this study that are associated with the tangent theorems and found in the grade 11 syllabus are;

(1) The tangent at any point of a circle is perpendicular to the radius through the point of contact.

(2) If two lines are drawn to a circle from one external point, they have equal length.

Table 5.2: Grade 11 Curriculum Statement on Euclidean Geometry.

CAPS FET Mathematics (DBE, 2011, p. 34)

GRADE 11: TERM 3			
No. of weeks	**Topic**	**Curriculum Statement**	**Clarification**
1	**Measurement**	1. Revise the Grade 10 work.	
3	**Euclidean Geometry**	Accept results established in earlier grades as axioms and also that a tangent to a circle is perpendicular to the radius, drawn to the point of contact. Then investigate and prove the theorems of the geometry of circles: • The line drawn from the centre of a circle perpendicular to a chord bisects the chord; • The perpendicular bisector of a chord passes through the centre of the circle; • The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre); • Angles subtended by a chord of the circle, on the same side of the chord, are equal; • The opposite angles of a cyclic quadrilateral are supplementary; • Two tangents drawn to a circle from the same point outside the circle are equal in length; • The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment. Use the above theorems and their converses, where they exist, to solve riders.	**Comments:** Proofs of theorems can be asked in examinations, but their converses (wherever they hold) cannot be asked. **Example:** 1. *AB* and *CD* are two chords of a circle with centre O. *M* is on *AB* and *N* is on *CD* such that *OM* ⊥ *AB* and *ON* ⊥ *CD*. Also, *AB* = 50mm, *OM* = 40mm and *ON* = 20mm. Determine the radius of the circle and the length of *CD*. (C) 2. O is the centre of the circle below and $\hat{O}_1 = 2x$. 2.1. Determine $\hat{O}_2$ and $\hat{M}$ in terms of x. (R) 2.2. Determine $\hat{K}_1$ and $\hat{K}_2$ in terms of x. (R) 2.3. Determine $\hat{K}_1 + \hat{M}$. What do you notice? (R) 2.4. Write down your observation regarding the measures of $\hat{K}_2$ and $\hat{M}$. (R)

5.2.4 Features of mathematical discourse

Commognition theory regards thinking and communication as one. Sfard (2008) states that "thinking is an individualised version of interpersonal communicating" (p. 81). To put it another way, thinking is a form of communication with oneself. Commognition as a theory has, therefore, been used by researchers in analysing mathematical discourse (Gcsamba, 2014; Mudaly, 2015; Roberts & Le Roux, 2019; Mudaly & Mpofu, 2019). According to this theory, mathematical discourse has four features: word use, visual mediators, narratives and routines. These four features are used to describe someone's mathematical discourse.

(1) Word use refers to key specific words used in the discourse

(2) Visual mediators are visual objects that operate as part of the discourse process

(3) Narrative is any sentence (story), written or oral, to describe an object, a relationship between objects, or a process subject to endorsement.

(4) Routine is a pattern or regularity of participant's actions while using keywords and visual mediators, and their use in narratives.

The four commognitive features are the pre-determined themes that informed my data codes.

5.2.5 Analysis of interview data

Data from the focus group interview was collected from a video documentation of the focus group interview. The transcription of the data from the interview was informed by the principles suggested by Sfard (2013). These principles are;

(1) Requirement of the principle of verbal fidelity: The requirement is for the researcher to document as precisely as possible what the participant said

(2) Requirement of the principle of multimodality: The requirement is for the researcher to document as accurately as possible all the actions done by the participant, including non-verbal while in the process of making an utterance.

With the data from the video recordings, I first cleaned and transcribed the data verbatim using the computer software Otter.ai. Since the software did not capture everything correctly, I had to review the transcript (Otter.ai) while at the same time listening to the original video recording and embarked on manually making corrections to some of the words that the software had wrongly translated or spelt. The purpose of doing this exercise was to ensure that I did not miss out on anything that was said or done by the participants. After I had made the corrections to the transcript, my next step was to review the literature to gain insight into how other commognitive researchers did in their studies ((Ben-Yehuda et al.,2005; Gcsamba, 2014; Mudaly & Mpofu, 2019). The essence of reviewing the literature on other commognitive researchers was determining how best to analyse interview data. This aspect led me to the third step of analysing the interview. Hatch (2002) offers some suggestions which I had to follow. Hatch's suggestion identifies 9 steps that may be followed to aid the analysis of interview data. These steps are as follows;

(1) Identify typologies to be analysed.

(2) Read the data, marking entries related to your typologies.

(3) Read entries by typologies, recording the main ideas in entries on a summary sheet.

(4) Look for patterns, relationships, and themes with typologies.

(5) Read data and code entries according to patterns identified, and keep a record of what entries go with elements of your patterns.

(6) Decide if your patterns are supported by the data, and search the data for non-examples of patterns.

(7) Look for relationships among the patterns identified.

(8) Write your patterns as one-sentence generalisations.

(9) Select data excerpts that support your generalisation

(Hatch, 2002; p. 153).

5.2.6 The analytical instrument for analysis of focus group interview

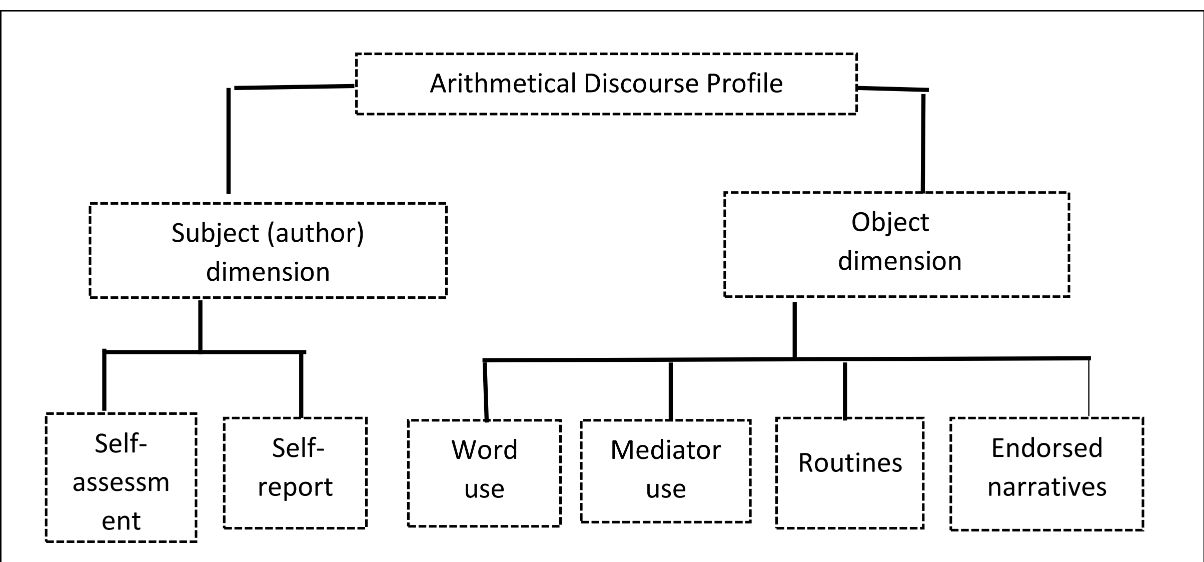
I begin by elaborating on the background of the analytical tool I used to analyse. I further provide a description of this analytical tool and how it fits with the constituents of mathematical discourse that characterised the participants' talk about the geometric object. These four features of mathematical discourse are word use, visual mediator use, narratives, and routines.

Ben-Yahuda, Lavy, Linchevski, and Sfard2005) developed a tool for assessing students' arithmetic discourse by utilising transcriptions of interviews, and they named this tool the Arithmetic Discourse Profile (ADP). The ADP is aligned with Sfard's theory of commognition. With this analytical tool, Ben-Yehuda (2005) and colleagues were able to focus on assessing what was said by the participants (verbal), as well as focusing on their actions, that is, what was done while speaking (non-verbal). In other words, they focus on the context of what happened.

In order to analyse the interviewee's discursive actions and the linguistic aspects of communication, Ben-Yehuda (2005) and colleagues divided the ADP into two main dimensions. These are the subject dimension and the object dimension. The subject dimension is focused on the interviewee identities articulated through remarks about themselves. The object dimension is about the object of mathematical discourse and actions with and about the object, in this case, arithmetic discourse (numerical calculation). These

main dimensions were then divided into sub-dimensions in alignment with the commognitive features of mathematical discourse. A summary of the ADP is provided in Figure 5.2 below.

Figure 5.2. The structure of the Arithmetical Discourse Profile (ADP). Source: Researcher's own initiative derived from the review of literature.



Although Ben-Yahuda, Lavy, Linchevski, and Sfard (2005) used the ADP to examine arithmetical discourse and considered the affordance of the ADP as an analytical tool in terms of commognition, other Commognitive researchers have adapted this tool to assess students' mathematical discourse in line with their study focus (Gcsamba, 2014; Mpofu & Pournara, 2018). For example, Gcsamba's (2014) study focused on linear function discourse and adapted the ADP to function discourse profile (FDP) to analyse the participants' discourse, as the study was not focused on arithmetic. In the case of Mpofu and Pournara (2018), though their study was on functions, they focused particularly instead on the components of the asymptotes of the hyperbola function discourse about learners' use of routines and their narratives. Because of their focus, they adapted the tool to the Hyperbola Discourse Profile (HDP). I illustrate the adapted HDP in the table 5.3 on the next page.

Table 5.3: The Adapted Discourse Profile of the Hyperbola (DPH) Mpofu and Pournara, 2018, p. 6).

Four key characteristics of mathematical discourse	**Classification**	**Description**	**Example**
Word use	Colloquial Literate	Combination of literate and colloquial. Mathematical	An asymptote is an imaginary line that a graph can't pass An asymptote is a line or curve that approaches a given curve arbitrarily closely (Weisstein, 2016)
Visual mediators	Iconic Symbolic	Viewed Drawn Viewed Formulated/generated	Identify asymptote from graphs Sketching graph; table of values Identifying asymptote from equation Generating equation from graph or table
Narratives	Substantiation Memorization	Justifications and reasons Formula/rule Visual (based on what learner can see)	justifications for actions, e.g., it is an asymptote because.... $y = \frac{a}{x-p} + q$ $y = q$
Routines Kinds of routines Properties of routines	Ritualized Exploratory Applicability Corrigibility Flexibility	Correct procedure but wrong or/no justification Verification of narratives, working with unfamiliar tasks Solving equations Drawn Use of a table of values Use of key features Using visual trigger Correction Use of multiple routines Translating	Sketching graph showing that y-axis is an asymptote but talking as if there is no vertical asymptote Choosing values in table to show function values moving towards a limit Solving equation to find intercepts, asymptote Sketching graph from equation or table Identifying key features from table of values Sketching curve using intercepts, asymptotes and equation Asymptote signifies vertical or horizontal translation Self-evaluating and correcting Using key features and/or table of values to sketch graphs Being able to transform an equation to standard form

In this study, my general focus area was Euclidean geometry while the specific focus was on the concept of the tangent. I decided to adapt the HDP while focusing on aspects of the

tangent concepts associated with geometrical constructions, axioms, theorems and proofs. In this study, the adapted HDP will now be indicated to as the adapted Tangent Discourse Profile.

Table 5.4: The Adapted Tangent Discourse Profile (DPT) (Adapted from Mpofu & Pournara, 2018, p. 6)

Four key characteristics of mathematical discourse	**Classification**	**Description**	**Example**
Word use	Colloquial Literate	Combination of literate and colloquial. Mathematical	A tangent is a line that cuts a circle. A line that touches the circle at a single point is known as a tangent to a circle.
Visual mediators	Iconic Symbolic	Viewed Drawn Viewed	Identify tangent from a circle. Sketching a circle and tangent line. Identifying the equation of tangent. Generating equation of a tangent line given a point of intersection with the circle.
Narratives	Substantiation Memorisation	Justifications and reasons Formula/rule Visual (based on what learner can see)	justifications for actions, e.g., it is a tangent because....
Routines Kinds of routines Properties of routines	Ritualised Exploratory Applicability Corrigibility	Correct procedure but wrong or/no justification Verification of narratives, working with unfamiliar tasks Solving equations Drawn Use of a diagram Use of key features Using visual trigger Correction Use of multiple routines	Sketching the tangent line to a circle but talking as if the line does not touch at a single point. Choosing other line segments to show the point of tangency with a circle. Solving equation to find tangent line, point of intersection with a circle. Sketching from equation. Identifying key features from the diagram. Sketching tangent line and circle using point of intersection and radius of a circle. Point of intersection with a circle, radius and equation of a line. A tangent touches a circle at a single point. Self-evaluating and correcting. Using key features to sketch a

	Flexibility	Translating	tangent line. Being able to use different modes to describe tangents.

5.2.7 Objectification

Sfard (2008) defines objectification as the "*process in which a noun begins to be used as if it signified an extra discursive, self-sustained entity (object), independent of human agency"* p. 300. Commognition is underpinned by objectification, which involves alienation and reification. In reification, participants, in their utterances, substitute talking about actions with talking about mathematical objects. In alienation, participants present phenomena without reference to human agency. It is through objectification that mathematical discourse is set apart from other discourses.

This study was an exploration of learners' commognition while engaging with geometry. In order to explore their commognition, the focus of the study was on participants' interaction with objects of the discourse of mathematics. In other words, the participant's direct actions on objects and their communicational actions about objects. These communicational actions have distinct characteristics, including the;

(1) Are rule-driven.

(2) Are a made by the participants through of independent decisions.

(3) Are carried out informed by appropriate perceptual mediators.

(4) Are generally about a specific object.

According to Sfard (2008) mathematical objects (realisation trees) are discursive constructs, and developed in the interest of communication. As a particular form of speech, discussants are able to say more in fewer words. In this study, some mathematical objects include circles, perpendicular, congruent, tangent, radius, algebraic symbols, iconic symbols, etc.

Considering the above and to answer the posed questions of the study, my focus of the

project was to delve into the verbatim transcriptions of the participant's stories and endeavour to map discursive objects. Mathematical discourse has four features with subsets from which discursive actions could be analysed. These features are;

1. words or word use,
2. Visual mediators (symbolic, iconic, concrete),
3. Narratives;
    - Endorsed,
    - Derivation,
    - Recall.
4. Routines (meta-rules and "how" and "when" routines)
    - Applicability (amending the how, but keeping the when unchanged),
    - Flexibility (amending the how, but keeping the when unchanged)
    - Corrigibility (ability to correct procedure)

These four main features and their subdivisions were used to aid the coding and inform the analysis process.

5.2.8 Coding the transcript for analysis

In this section, I present how the coding for the analysis using the Adapted Tangent Discourse Profile (ATDP) was done. The coding was informed by four pre-determined themes. These themes are the four attributes of mathematical discourse: word use, visual mediators, narratives, and routines.

Use of words:

Table 5.5: A Sample of Word use.

Turn	Speaker	What was said	What was done
10	S2	Cuts the circle	Looks at S1
11	S3	Doesn't cut the circle	
13	S1	It touches the circle	
80	S3	I bisected the line	

Mathematical discourse entails the disciplined use of purpose keywords. Word use was, therefore, divided into two codes: mathematical and colloquial. In order to be coded mathematically, the purpose keywords are those that signified quantities, shapes, and so on. Examples under this code from the transcript include "*I bisected the line*" (in [80]).

The words coded colloquial included words used in everyday life or used spontaneously. Examples under this code from the transcript include "*Doesn't cut the circle*" (in [11]).

Visual mediators use:

Table 5.6: A sample of Visual Mediator use.

Turn	Speaker	What was said	What was done
41	S1	Draw a circle along the line down the spot here	Points the at the line and moves the finger around
69	S2	Between these two lines	Points the finger at the two lines
178	S1	Angle B and Q that is hundred and eighty degrees, so ninety degrees plus ninety degrees plus three hundred and sixty degrees minus hundred and eighty degrees. Because you have to get hundred and eighty from angle O and from angle T	Pointing with the fingers while saying this.

In mathematical discourse, visual mediators refer to visible objects operated upon in communication. Visual mediators were divided into two categories: iconic, and symbolic. To be coded iconic (pictorial means), the participant would say as if deriving “in the head”. Examples under this code from the transcript include *“You draw touching P and the other line”* (in [67]).

To be coded "*concrete"* (tangible objects), the participant's derivation would be through tangible mediation. Examples from the transcript include “Between these two lines” and what is done, that is, pointing the finger at the two lines (in [69]).

To be coded "*symbolic," the participant derivation would be done by* scanning numerical symbols mentally. An example from the transcript includes “*Angle B and Q that is hundred and eighty degrees, so ninety plus ninety degrees plus hundred and sixty degrees minus hundred and eighty degrees. Because you have to get hundred and eighty from angle O and from angle T”* (in [178]).

Construction of narratives:

Table 5.7: A Sample of Narratives.

Turn	Name of Speaker	What was said by speaker	What was done by speaker
17	S1	I would say tangent is the term used when two lines.... or any line that doesn’t go through the circle	Raise the hand, then moves hand forward, while looking at S2
107	S2	The radius is a line from the centre of the circle to the circumference	Turns the head and looks at S1, while smiling

Narratives are any arrangement of expressions used to describe objects, relations between objects, or processes with objects. The narratives were divided into two principal codes and separate subsets, one of the main codes for endorsed narratives and another main code for narratives that are not endorsed. Those that were coded endorsed include substantiated narratives. Examples under this code from the transcript include *“The radius is a line from*

the centre of the circle to the circumference" (in [107]). Not endorsed were those that could not be substantiated. Examples under this code from the transcript include "*I would say a tangent is the term used when two lines…or any line that doesn't go through the circle"* (in [17]). The subset codes under derivation had two codes (2 where the derivation was from memory or the object's properties). Example of a code from memory *"BT is equal to QT…."* (in [143]). Example of a code from the object's properties *"Angle B and Q that is hundred and eighty degrees, so plus ninety degrees plus three hundred and sixty degrees minus hundred and eighty. Because you have to get hundred and eighty from angle O and from angle T"* (in [178]).

Enacting routines:

Routines are repetitive patterns noticed when using words or mediators to create substantiated narratives. Routines were divided into two groups. One code for the process followed and another code for how the process was enacted. For the coded process, I looked at whether the steps were feeding into each other. The previous step's output becomes the input for the next step. Example of this code from the transcript *"I got the distance to R, drew a line intersecting P. Then I joined together to make the centre of the circle"* (in [73]).

To check the code for how the routine was enacted, I looked at whether the implementation was done in one step or whether it was mediated. An example of one-step code from the transcript is *"I bisected the line"* (in [80]). An example of mediated code is *"I would say tangent is the term used when two lines…or any line that doesn't go through the circle".* What was done: Raises the hand, indicating to go through, while looking at participant S2 (in [17]).

The second group of routines was about the properties of routines. This group was divided into three codes; namely applicability, corrigibility, and flexibility. To code *applicability*, the

investigator looked at the routine course of action (the when and *how* of the routine) by considering the circumstances of the given action.

To code *corrigibility*, the investigator looked at the participant's self-correction. Examples under this code from the transcript include *"Cuts…I don't know. But it touches the circle"* (in [18]).

Table 5.8: A Sample of Corrigibility.

Turn	Speaker	What was said	What was done
47	S1	Draw a circle along the line down the spot here	Points the finger at the line and moves the finger around
178	S1	Angle B and Q that is hundred and eighty degrees, so ninety degrees plus ninety degrees plus three hundred and sixty degrees minus hundred and eighty degrees. Because you have to get hundred and eighty from angle O and from angle T	Counting with the fingers

To be coded *flexibility*, the participant had to demonstrate individual variation to reach the same outcome. I considered the participant's use of multiple modes of mediators in their communication. Examples of this code from the transcript include "Draw a circle along the line down the spot here", and what was done, point the finger at the line and move the finger around (in [47]).

5.2.9 Instrument for mathematical assessment of the written task activity

5.2.10 Realisation Tree Assessment

The mathematical task was analysed with the help of the Realisation Tree Assessment (RTA). Sfard (2008) has initiated a r*ealisation tree* (p. 165) as an examination tool for visualising one's mathematical objects.

The term *realisation* used in commognition theory is the same as the common term used in ordinary language. However, the term realisation is endorsed to stress that mathematical objects are a product of human discourses, and nothing independent of human language can definitely be described (Weingarden, Heyd-Metzuyanim, & Nachlieli, (2019). Therefore, a realisation tree is a graph connecting many signifier-realisation relations (Sfard, 2012). A realisation tree is defined as a;

> *"Hierarchically organised set of all the realisations of the given signifier, together with the realisations of these realisations, as well as the realisations of these latter realisations, and so forth"* (Sfard, 2008, p. 301).

In other words, a mathematical object can be regarded as the sum of signifiers and realisations a participant enacted while participating in a mathematical discourse. Building on this definition, Weingarden and colleagues determined that realisation trees provide worthwhile information to ascertain a learner's discourse and could potentially be employed as an assessment tool to explore the degree of a students' participation in the discourse in terms of explorative (Weingarden et al., 2019). Based on this, the Realisation Tree Assessment (RTA) tool (Weingarten & Heyd-Metzuyanim, 2017, 2018; Weingarten et al., 2017) was designed to highlight the mathematical objects that emerge as participants engage with a task together with its various realisations. They contend that the Realisation Tree Assessment (RTA) tool offers a graphical representation of the mathematical ideas, students get exposed to and engage in through a specific task. Several researchers have since adopted and adapted the Realisation Tree Assessment Tree (RTA) tool and have used the tool to assess participants' tasks. Examples include the discourse in the form of text concerning the derivative at a point (Haghjoo et al., 2023), students collaborating with one another while working in a small group (Knox & Kontorovich (2022), an examination of the methods used in three calculus textbooks regarding the derivative at a point and as a function (Park, 2016),

similarities and differences of lessons based on two different tasks (Weingarden & Heyd-Metzuyanim, (2019), the examination of learners' viewpoints with regard to on the asymptotes of a hyperbola and exponential function (Mudaly & Mpofu, 2019).

In this study, my focus was on exploring the learners' discourse when engaging with geometry; based on the affordance of the Realisation Tree Assessment (RTA) tool as well as the review of literature, I, therefore decided to use the Realisation Tree Assessment tool as the most appropriate instrument to analyse the participants pencil and paper mathematics construction task. Sfard (2008) contends that realisation trees may be used in not less than three ways;

(1) To assess the standard of one's discourse about a mathematical object, for example, by alluding to the branches' richness and depth.

(2) By analysing the developments in one's tree, for example, by referring to the introduction of new signifiers that may lead to changes in the hierarchical structures.

(3) By analysing communication effectiveness, considering that different interlocutors may realise the same signifier differently.

(Sfard, 2008, p. 167).

I decided that by adopting this tool to analyse the mathematics construction task, I would gain possible insight and rich information about the participants' mathematical object, "tangency at a point of a circle", in their discourse.

5.2.11 The *a priori* realisation tree

Haghjoo et al., (2023) developed the Realisation Assessment Tree of the derivative at a point. They used this tool to analyse the written discourse about the derivative at a point in the

function. Their unit of examination included written texts, examples, various activities, and some exercises from textbooks. I reviewed Weingarten et al. (2019) Realisation Tree Assessment (RTA) and studied their explanations regarding the task assessment. Then, I reviewed Haghjoo et al. (2023) Realisation Tree Assessment of the derivative. From the literature review, I decided that though Haghjoo and colleagues investigated the derivative at a point, some concepts employed in their tool could be adapted and be made to suit and align with the concepts of the mathematics task for this study, which was about the tangent at a point. I, therefore, adapted the Realisation Tree Assessment of the derivative developed by Haghjoo and colleagues to the Realisation Tree Assessment of the tangent at a point. From now on in this report, this will be referred to as the adapted Realisation Tree Assessment (RTA) of the tangent at a point on a circle.

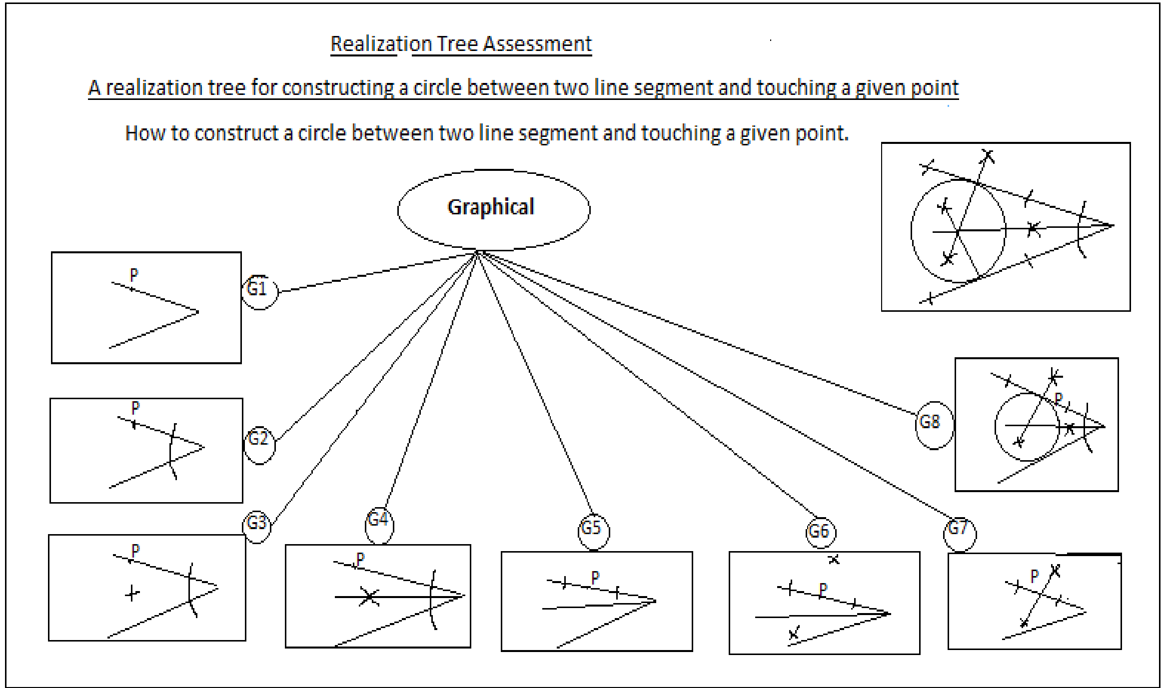
Weingarden, Heyd-Metzuyanim, and Nachlieli, (2019) suggest that the first step in using and operating the RTA tool is to build the realisation tree per the task. I began the process of constructing the RTA by first consulting some mathematics textbooks that are used in grade 11. I then determined that the tasks in these textbooks are those that the students are familiar with. Then, I develop the theoretical realisation tree a priori, following some possible realisations of the tangent at a point that they (participants) might have been exposed to while working on specific geometrical activities found in the textbooks. In this study, participants were required to utilise two fundamental realisations of a tangent to answer the questions using graphical and algebraic equations. However, it must be emphasised here that there are many possible solutions that may express the tangent at a point, and it is not possible to include all such solutions in one RTA. What was considered in this particular RTA was to include only solutions commonly used in grade 11 textbooks, that is, solutions and expressions that participants may have encountered before while working on activities in the classroom environment.

In constructing the RTA, Weingarten and colleagues (2019) point out that the starting point is the root node. The root node is the signifier that suits the object that has to be samed. The realisation tree is a set of all realisations of signifiers. Each node is a realisation of the node above or below it. In the case of the mathematics construction task for this study, the tree has the root in the signifier "the tangent at a point". Knox and Kontorovich (2022) contend that the research aims and questions should inform decisions about the signifier at the tree's root. I developed two realisation tree assessments to assess the questions posed in the mathematics construction task. For the first question of the task, I named the RTA (1). For the second question of the task, I named the RTA (2). For example, in this study, in the root of the realisation tree, RTA (1) was labelled Graphical and RTA (2) Algebraic.

From this root (Graphical), eight branches with realisations were developed. The branches have roots containing visual realisations based on visual mediators used when constructing a circle between two lines from the same point and tangents on both lines. The leaves for this branch are numbered G1, G2, G3 up to G8 according to the order in which they are realised. This realisation is for question number one of the written task activity (The realisation of constructing a circle at a point and between two lines that are tangent to the circle).

RTA (2) has two main branches; the first branch has roots numbered A1.1, A 1.2 up to A1.8, again according to the order of written (text and keywords) realisations of reasoning involved when determining that the lines formed are equal. The second branch has leaves numbered A2.1, A2.2 up to A2.8. This is for question two of the task (Realisation of how to prove that lines from the same point and tangents to a circle are equal in length). Sfard (2008) reminds us that the *"realisation tree is a personal construct"* p. 166. The meaning Sfard is conveying is that the realisation tree can be different for each person. The two RTAs are presented in Figures 5.3, and 5.4.

Figure 5.3: Realisation Tree Assessment RTA number1. A realisation tree for the tangent at a point on the circle. Source: Researcher's own initiative derived from the review of the literature.



5.2.12 How the analysis was conducted

The first question of the task was about constructing a circle between two-line segments and touching a given point. I was able to locate 8 different roots for this mathematical object.

Root 1 and 2: Graphical realisation (G1, G2): These two realisations focus on constructing an arc to cut the two-line segments.

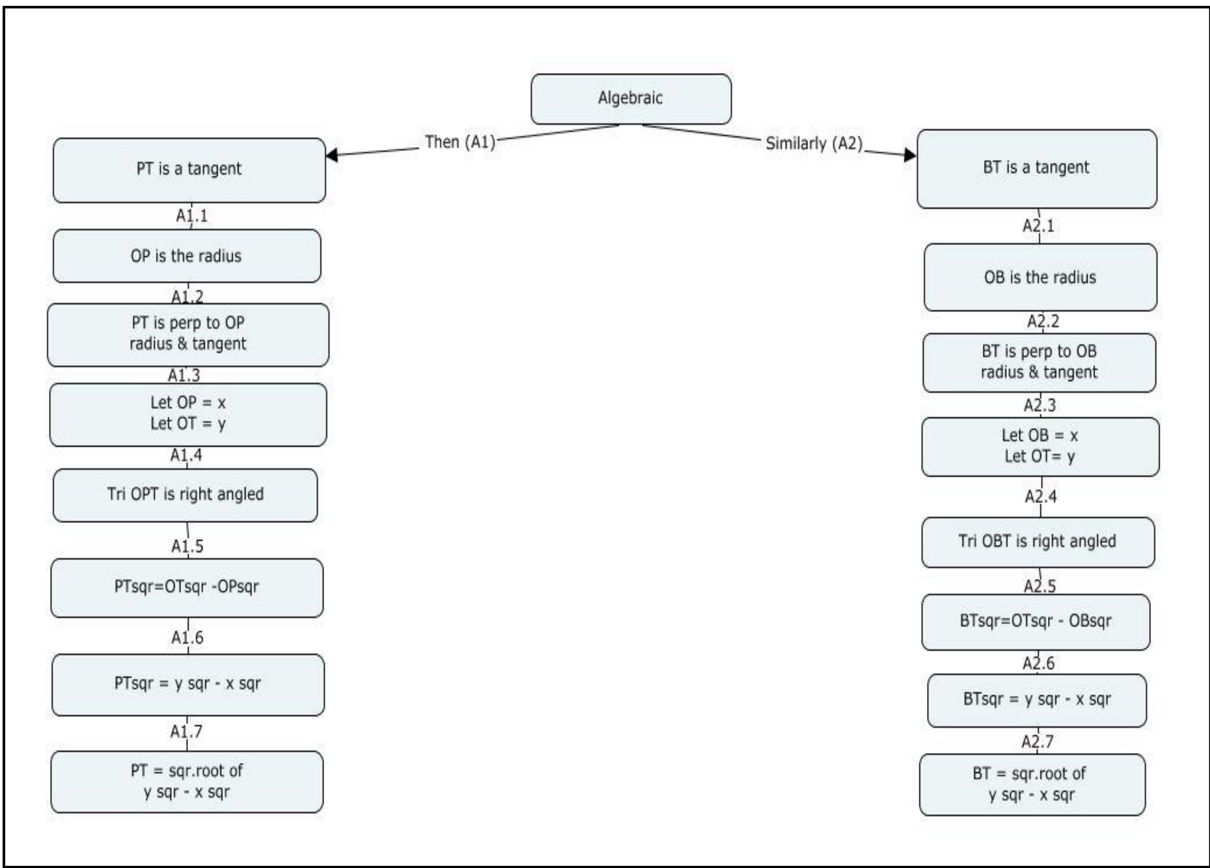
Root 3 and 4: Graphical realisation (G3, G4): The focus is on drawing a bisector from where the two-line segments meet in the subsequent two realisations. To obtain this realisation, the compass is placed on the arc that cuts the first line segment and an arc is drawn. Similarly, the compass is then placed on the second line segment to cut the arc. The bisector is drawn

from where the two-line segment meets to where the arcs meet. The line bisects the two-line segments.

Root 5, 6 and 7: Graphical realisation (G5, 6, 7): In the following three realisations, to draw a perpendicular line through the point P, the compass is placed at P, and the line segment is cut on either side of P (near P). While the compass is at one of the places where the line segment is cut, an arc is drawn above and below P. Similarly, the same process is repeated while the compass is placed on the line segment on the other side of P. A line is drawn, joining the two arcs and passing through P.

Root 8: Graphical realisation (G8): In this realisation, draw a circle and P as a point of tangency. The point where the two bisectors meet is the centre of the circle. The compass is placed where the two bisectors meet; the distance from this point to P is the radius, and a circle is drawn touching the two-line segments. The point P is the point of tangency.

Figure 5.4: Realisation Tree Assessment RTA number 2. A realisation tree for two lines drawn those tangents and drawn from the same point outside a circle are equal in length. Source: Researcher's own initiative derived from the review of the literature.



The second question of the task is about proving that lines drawn from the same point outside a circle are tangents and equal in length. I identified 8 different leaves for this mathematical object.

Root 1: Algebraic realisation (A1, A2): In this realisation, the focus was on naming the points as P, T, O, and B, respectively. Then OT and OP are joined. Line segment OT is considered in both triangle POT and triangle BOT. Keyword tangent

Root 2: Algebraic realisation (A1.1, A2.1): These two realisations consider OP as a perpendicular bisector of line segment PT. The angle formed OPT is considered to be 90^{0}.

Similarly, OB should be considered the perpendicular bisector of line segment BT. The angle formed by OBT is considered to be 90^0. Keywords: perpendicular, bisect, radius.

Root 3: Algebraic realisation (A1.2, A2.2): In these two realisations, consider the derivation of the variable for the equation. For example, let OB = x.

Root 4: Algebraic realisation (A1.3, A2.3): In this realisation, line segment PO and line segment BO are the radius of the circle. Consider that line segment OP and line segment OB are equal.

Root 5: Algebraic realisation (A1.4, A2.4): Consider triangle OPT a right-angled triangle in this realisation. Similarly, consider triangle OBT a right-angled triangle.

Root 6 and 7: Algebraic realisation (A1.5, A1.6, A2.5, A2.6): The focus is on line segment PT and line segment PB. Triangle OPT is a right-angled triangle. The length of line segment PT can be calculated using the Pythagoras rule. Similarly, triangle OBT is a right-angled triangle. The length of line segment BT is calculated using the same Pythagoras rule.

Root 8: Algebraic realisation (A1.7, A2.7): In these two realisations, the focus is on the two-line segments PT and BT. Consider line segment OB and line segment OP radii. Similarly, line segment OT in triangle OBT and line segment OT in triangle OPT are common.

In this realisation, triangle OPT and triangle OBT are presented as congruent. Consider SAS or RHS.

5.2.13 How the assessment was done

The main focus of the analysis was to qualitatively describe the characteristics of each participant's realisations and gain helpful information for answering the research questions. As pointed out by Sfard (2008), commognition describes mathematical discourse as involving continuous transitions between signifiers. She stresses that Words and symbols that are used

as nouns in sentences and their realisation as accessible objects may work in line with the production of narratives and its signifiers.

Therefore, the analysis was informed by the main elements characterising mathematical discourse while utilising the indicators of the commognition used in solving mathematical problems.

Since the task activity given to the participants was in a problem-solving format, I reviewed the problem-solving tasks of other commognitive researchers (Zayyadi et.al.2019) to gain insight into how they implemented the analysis in their study. These researchers identified the cognitive components and assigned each to indicators found in the problem-solving process. The table below exemplifies the commognitive components and their indicators.

Table 5.9: Indicators of the Commognitive Components Used in Solving Mathematical Problems (Adapted from Zayyadi et al., 2019, p. 93)

Commognitive Component	**Indicator**
Word use	Write and recites words including algebraic, numerical, and geometric terms, equations, and other terms used by students in solving mathematical problems
Visual mediator	Uses objects such as graphics, images, diagrams, and others in solving mathematical problems
Narrative	Describes mathematical facts such as axioms, definitions, and theorems that are used by students in solving mathematical problems
Routines	Explains the steps followed to solve the problems given

From my experience as an educator, I needed to have an assessment scheme to guide the task assessment. I developed a rubric Table 5.9. I used the rubric and Zayyadi et al., (2019) commognitive components and indicators to perform the analysis.

The analytical scheme for the RTA was designed to aid in the RTA assessment. I adapted the RTA (1) and RTA (2) schemes. The analysis was organised into two groups. The first group dealt with the mathematical objects in focus. Guiding questions were then asked. For example, the following question was used: "What can be seen about the objects and their properties or unrealised symbols and one's action with them?" This question allowed me to look for communication indicators like terms or visual mediators used or signifiers done. The second group considered the task. The guiding question was, "Is the task familiar to the participants, and is this to perform a procedure or produce an endorsed narrative?" This type of question directed me to the commognition communication indicators like the restatement of the task and words or visual mediators prescribing action in the given task situation. For a more detailed understanding of the scheme, see table 5.10.

Table 5.10: Analytical scheme for the RTA. Source: Researcher's own initiative.

Aspects of the discourse	***Questions guiding the analysis***	***Communication indicators***
1. Objects in focus	What can be seen about *objects and their properties or unrealized symbols* and one's actions with these symbols? How many realizations of the same signifier are used? How are the *transitions between realisations* made?	(a) Terms or visual mediators used as precedents identifiers. (b) Signifiers done: in relation explicitly to the outcome of the routine, or only in relation to memorised steps of procedure. (c) Addition realisations done. (d) Presence of transitions between realisations.
2. Task	Is the task familiar (are the students able to identify a precedent)? Is this to perform a *procedure* or to produce an *endorsed narrative*	(e) The construction is unrealised symbol.

	(f) The construction is realisation of mathematical object. (a) Restatement of the task. (b) Words or visual mediators prescribing action in the given task situation (in relation to endorsed narrative, or simply about what needs to be done in the procedure). (c) Sub-tasks feeding into the procedure

The participant's tasks were then assessed by looking at the richness and depth of the realisations with the aid of the scheme. Each task was classified at three levels based on criteria provided in the scheme. The table below shows how the classification was done and the designated levels.

Table 5.11: Scoring Rubric (Adapted from Mpofu & Mudaly, 2019, p. 739)

Classification	**Description**	**Examples**
Level 1	Marginal or no objectification of the mathematical object	No or little substantiation of statements and connections between the basic representations.
Level 2	Partial objectification of the mathematical object	Some substantiation of statements and connections between the basic representations
Level 3	Objectified communication on representations of a tangent	Able to substantiate statements and connections between the basic representation

5.2.14 Coding

I used the three tools, that is, the commognitive components and indicators used in problem-solving tasks (Zayyadi et al., (2019), the adapted analytical scheme and RTA (1) and RTA (2) to aid in the coding. However, the commognitive components and indicators used in problem-solving tasks were to verify the codes. I now describe how the coding was done.

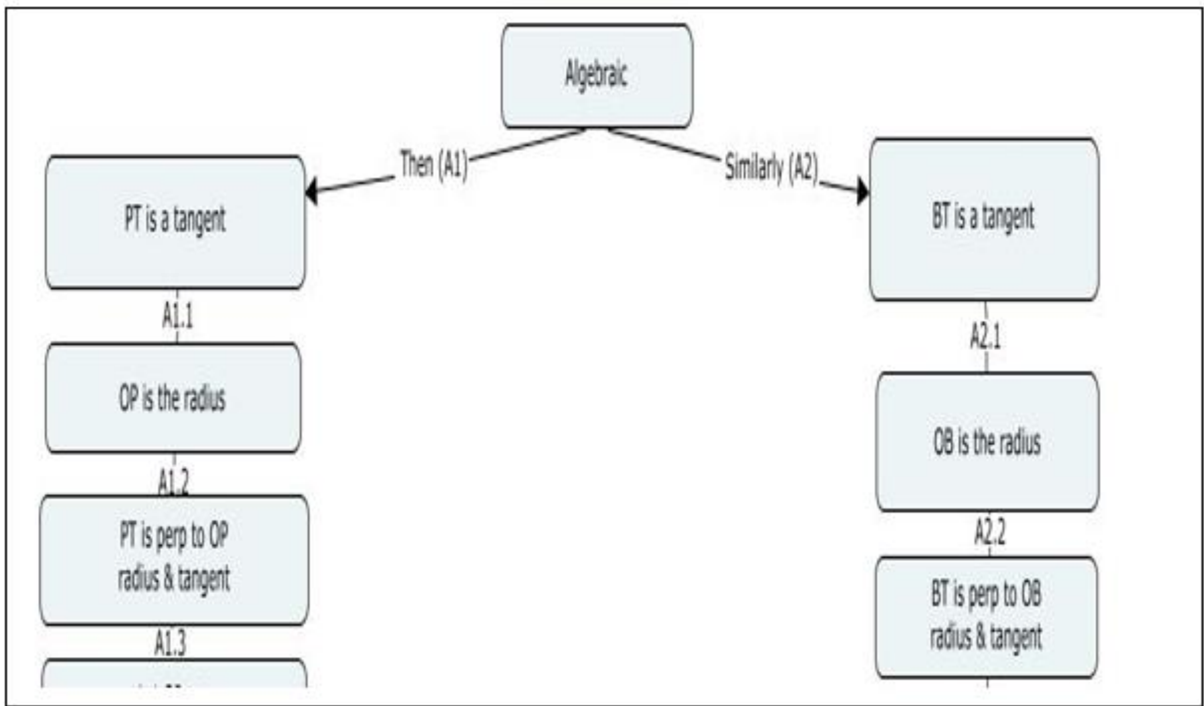
5.2.15 How the coding was done.

Data was analysed by attending to the components of mathematical discourse and was aimed at identifying the characteristics of participants' discourse and its development while focusing on the transitions between signifiers and their realisations for producing narratives. The coding was conducted according to the closeness of the participant's realisation with one of the descriptions given in RTA (1) and RTA (2). The closeness was established according to the inferences of the researcher, who knows the endorsed narratives associated with this section of Euclidean geometry.

Word use: The participants' realisation tree was analysed for mathematical terms: written text or symbols. These were coded as *word use*. Under this category, two types of codes were used. A code referred to as mathematical was assigned to words that are mathematical and have unique meanings acceptable by the mathematical community; other than that, the code not mathematical would be used. An example of the mathematical code from the participant's realisation is the word "Tangent".

An example of this code is Figure 5.5.

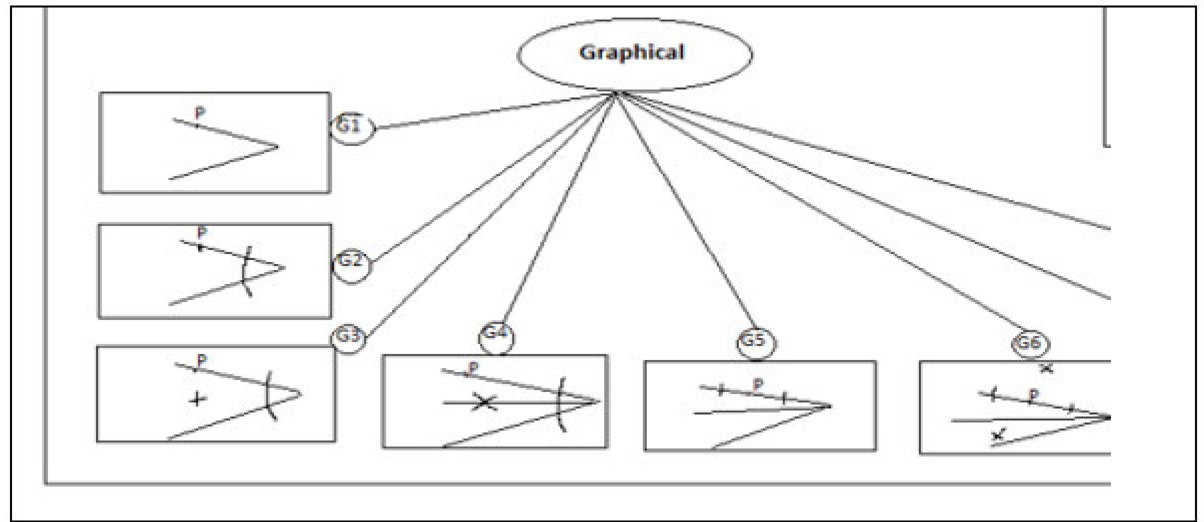
Figure 5.5: Realisation tree of word use. Source: Researcher's own initiative.



Mediator use: The participant's realisation tree was analysed for numerals, algebraic symbols or graphical presentations. To be coded under this code, the participants may provide images to identify the object and coordinate their communication. An example of this code is the diagram of a circle or line segment.

Example of this code is illustrated in Figure 5.6.

Figure 5.6: Realisation tree of visual mediator. Source: Researccher's own initiative.



Routines: The codes for routines were divided into types: those that represented types of the routine, those that represented the properties of the routine, and those that represented the procedures. To be coded under this code, the participants' signifier -realisation pairs must contain a combination of metarules that describe discursive actions and procedures. An example of this code from the participant's realisations is the substitution of the known into the formula/theorem.

Types of routines: The *How* of the routine and the *When* of the routine. An example of this code is Figure 5.7

Figure 5.7: Realization tree of routines. Source: Researcher's own initiative.

Endorsed narratives:

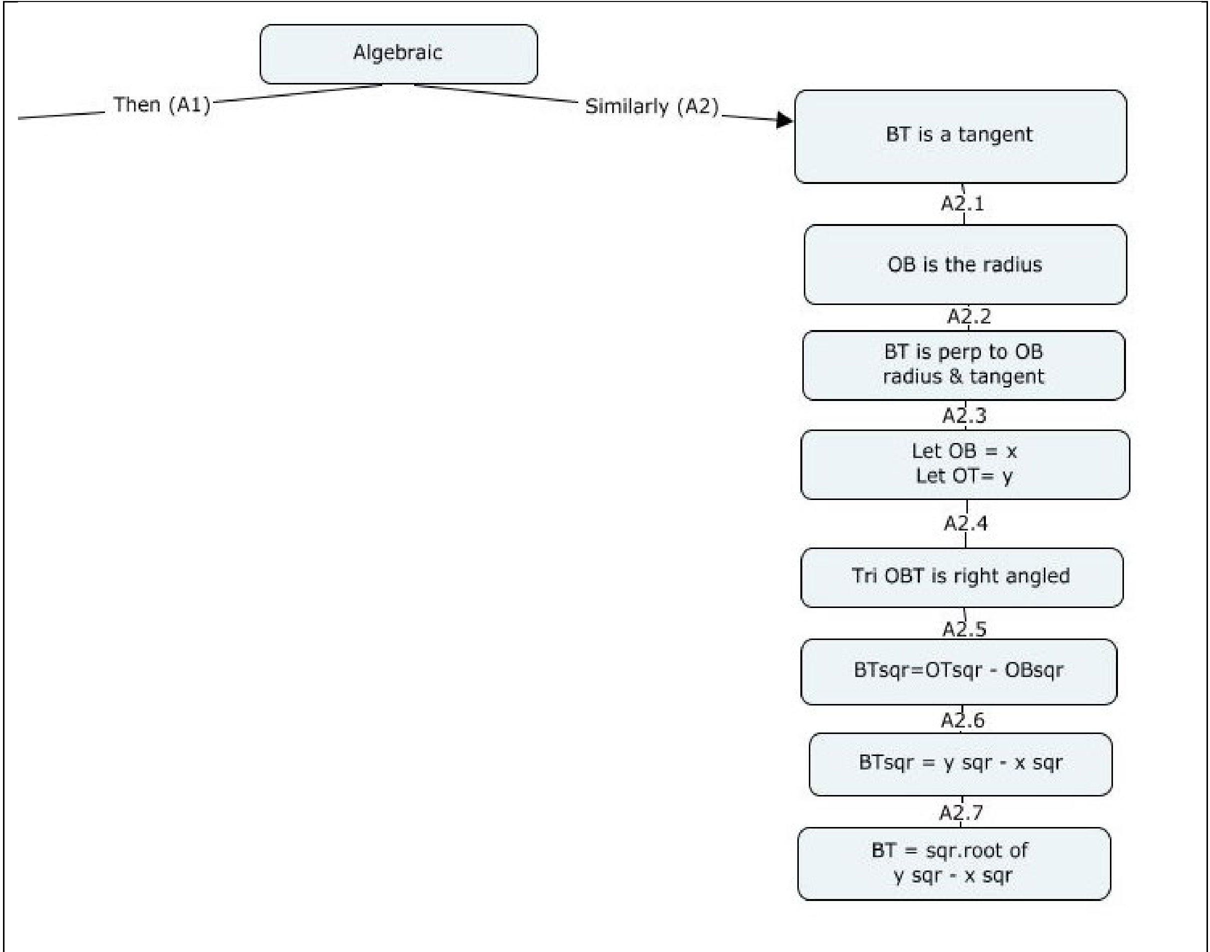
Substantiation. Two codes were used for substantiation, one for endorsed narratives and another for not endorsed narratives. Endorsed include theorems, conjectures and so on.

Narratives:

To be coded under this code, the participant's realisation-signifier pairs must contain texts/written work, which describe objects and may be labelled as true or false. An example

of this code from the participant's realisation tree is the formula or theorem. An example of this code is Figure 5.8.

Figure 5.8: Realisation tree of Narratives. Source: Researcher's own initiative.



5.3 Presentation

In this section, I present the data collected through the two instruments employed in the study. These instruments are the focus group interview transcript and the mathematics task activity. I have used a tabular format as the best approach to summarise, organise and present data to the reader. This method of presentation is informed by the commognitive method of analysis guided by Sfard's (2013) two principles, the principle of verbal fidelity and the principle of completeness. These principles require that the researcher pays equal attention to all the information within the context, such as what was said, what was done (action,

gestures), and so on. A sample of participant's word use is in Table 5.12.

Table 5.12: Sample of the participants' word use

Turn	*Speaker*	*What was said*	*What was done*
[8]	S2	It's a line outside a circle...outside the circle	Looks at S1, nods head, turns and looks at S3.
[9]	S2	Any line that is a tangent to a circle doesn't go through the circle...through.	Raises the hand to indicate through
[17]	S1	I would say tangent is the term used when two lines...or any line that doesn't go through the circle.	Raises the hand indicating to go through, while looking at S2.
[18]	S2	Cuts... I don't know. But it touches the circle.	Looks at S1.
[19]	S1	It touches the circle...just outside.	
[32]	S1	I constructed a circle.	Points with finger, then move finger around as if drawing a circle.
[47]	S1	Draw a circle along the down the spot here.	Points the finger at the line and moves finger around the point.
[59]	S1	We can make P a chord	Looks at interviewer while bending the head.
[80]	S3	I bisected the line.	
[83]	S3	Cutting a line	
[85]	S3	To cut into half. To cut into two halves.	
[95]	S2	The touch each other.	
[96]	S1	The point where the lines meet, that's the point of intersection	
[107]	S2	The radius is a line from the Centre to the circumference.	Turn the head and looks at S1, then smiles.

Table 5.12 has information about the participants' word use and forms of expression. The data contains a sample of representative utterances of the participants during the focus interview. This sample data relates to the utterances of the participant's use of purpose keywords of geometrical discourse and includes number words, operation words, specific words, and words that announce results of geometric of participant's use of purpose keywords of geometrical discourse and includes number words, operation words, specific words, and words that announce the operations.

Table 5.13 combines visual mediators, routines and narratives in the sample.

Table 5.13: Sample of Visual Mediator, Routines, and Narratives

Turn	*Speaker*	*What was said*	*What was done*
[32]	S1	I constructed a circle	Points with finger, then move finger around as if drawing a circle.
[47]	S1	Draw a circle along the line down the spot here.	Points the finger at the line and moves finger around the point.
[55]	S2	It's the only way to draw a circle	Moves the finger around, as if drawing a circle.
[56]	S1	I would a circle around the entire line, leaving P in the middle, Midpoint, then it makes more sense to draw a circle passing the …. P	Moves the finger around, as if drawing a circle.
[60]	S2	It has to be between the two lines...So you draw it here...here.	Moves the finger around a point, while looking at S1.
[62]	S2	The circle will touch these two lines. Has to touch both lines.	Points finger at the lines.
[69]	S2	Between these two lines.	Points finger at the lines.
[78]		I got the distance to R, drew a line intersecting P. Then I joined together to make the centre of the circle.	Point with finger at P.
[87]	S3	With a compass and pencil	Moves the finger as if making an arc.
[96]	S1	The point where the lines meet, that's the point of intersection	Bends the head.
[105]	S2	We drew the radius, we labelled the centre O, and we drew a line from O to P. That was the radius.	
[109]	S2	Well, where the line OP meets, we labelled, we labelled that something, and we tried to show that it was a right angle, to prove that P is a tangent.	Raises head and bends to one side, then looks at interviewer.
[114]	S1	Because tangents make a right angle. The radius and the tangent make a right angle.	Raise hand to demonstrate the right angle.
[117]	S1	That a tangent is perpendicular to the radius	Raise hand to demonstrate perpendicular.
[121]	S1	Yes…meaning ninety degrees.	Clasps the fingers
[122]	S1	he lines meet at ninety degrees	
[136]	S2	You see, because there is an angle. Right angle and OB is equal to OQ…. that is the radius, that is how you prove	Looks sideways at SI, then turns and looks at S3.
[137]	S1	I said that B and Q are ninety degrees, added together is hundred and eighty degrees. So, to measure..., say angle o and angle T, we need to know, angle B and angle Q are hundred and eighty degrees and opposite sides of a quadrilateral are supplementary…because both are hundred and eighty degrees and to get three hundred and sixty degrees we have to add.	Raises hand to demonstrate a quadrilateral.
[157]	S2	Then, BT and QT are equal because there are tangents coming from the same point.	
[178]	S1	Angle B and Q that is hundred and eighty degrees. So, ninety degrees plus ninety degrees plus three hundred and sixty degrees minus hundred and eighty degrees. Because you have to get hundred and eighty from angle O and from angle T.	Count with fingers.
[182]	S1	Because the radius meets the tangent, and all tangents are perpendicular to the radius.	

Table 5.13 shows participants' utterances regarding the use of mediators, routines and narratives. In geometrical communication, participants use several mediational modes. The representative sample also contains routines of a set of meta-rules that govern both the *when* and the *how* of repetitive discursive action. Included are narratives that are endorsed or not.

Table 5.14: Results RTA of the mathematics written task activity

Aspect of discourse	Participant	Commognitive indicators	
		Word use	Mediators, narrative, and routines
1.Object in focus	S1	Level 2	Level 2
	S2	Level 2	Level 1
	S3	Level 2	Level 2
2.Task	S1	Level 2	Level 2
	S2	Level 2	Level 1
	S3	Level 2	Level 3

Table 5.14 is a sample of information derived from the analysis of the written task according to the commognitive features of mathematical discourse.

Key: Level 1 = low. Level 2 = medium. Level 3 = high.

Figure 5.9: Sample of solution to written task activity.

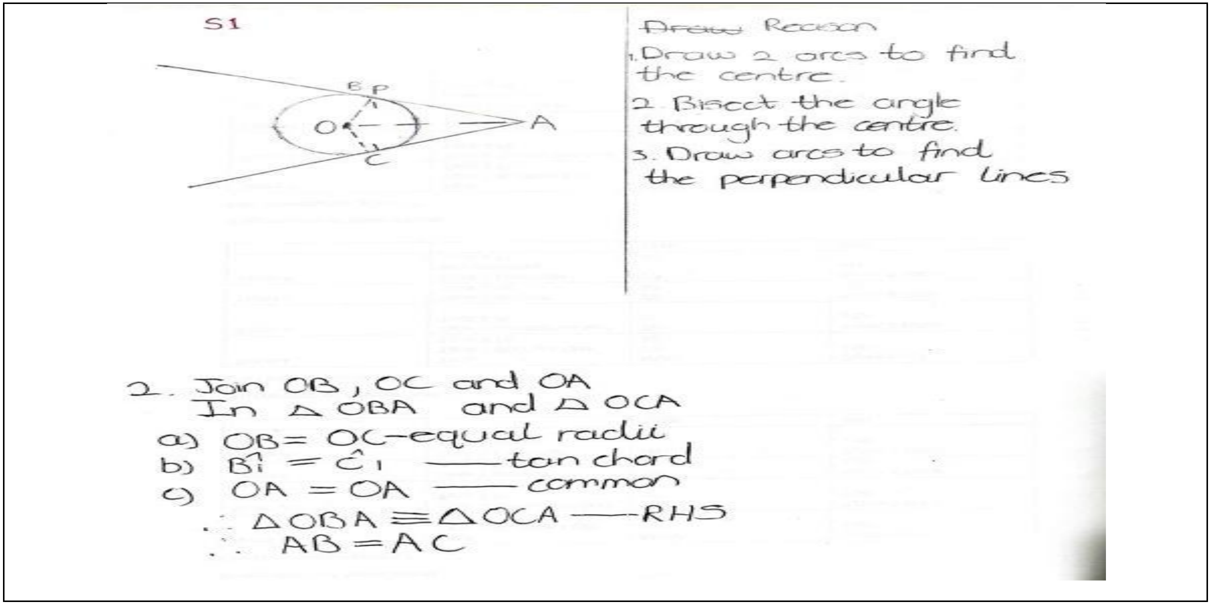
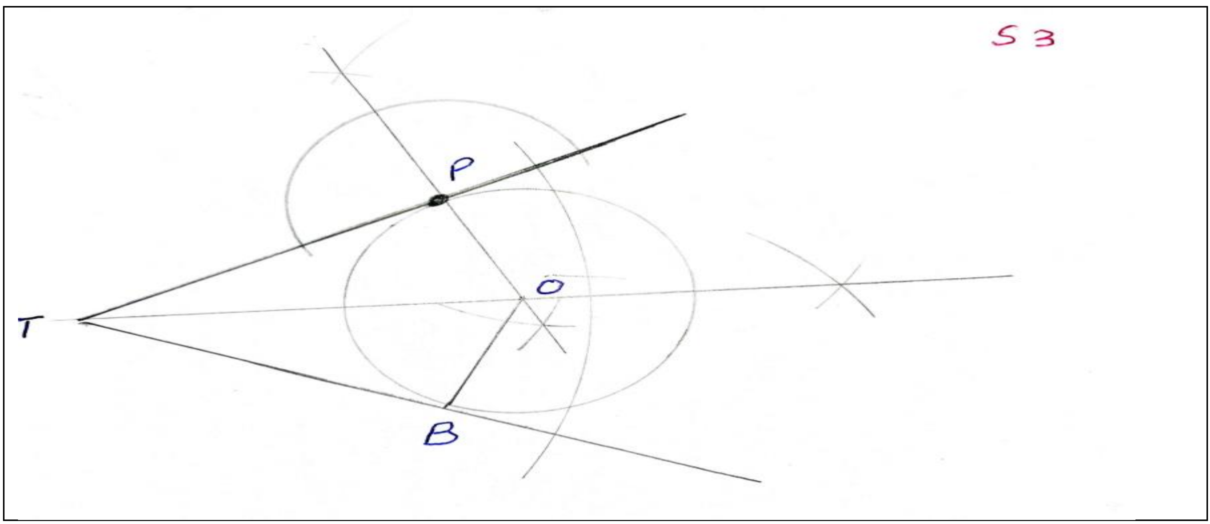


Figure 5.9 is a sample of the solution to the written task problem, highlighting the steps and procedures necessary to arrive at the solution.

Figure 5.10: Sample of pencil and paper construction



In figure 5.10 is a sample of data derived from the pencil and paper construction of the circle between two- line segments.

5.4 Findings

5.4.1 Overview

The current study was focused on exploring the learners' commognition when engaging with geometry in a grade 11 classroom. The study had three research questions. Two sets of data were generated in order to answer these research questions. The first data set was derived from the learner's responses while working on the pencil and paper task (document), and the second set was from the focus group interview. Given that I had two sets of data, I had to divide the presentations of findings into two sections. In the first section, I present findings derived from the analysis of the written task activity, and in the second section, I present findings derived from the analysis of the focus group interview.

5.4.2 Participants characteristics

There were initially five participants in this study, of which two were female and two were male. All the participants wrote the task activity. However, only three were available on the day of the focus group interview. The data from the three participants who wrote the written task and were available for the interview were the ones considered for this study. The average age of the participants was 16 years. The individual demographics of the three participants with respect to their gender, age, ethnicity or race are shown in the table below.

Table 5.15: Participants' demographics

Participant's ID	Gender	Age	Ethnicity
S1	Female	16	Asian
S2	Female	16	Asian
S3	Male	16	Asian

5.4.3 The role of the researcher

Since qualitative study is a subjective process, and the researcher alternates between a participant and an instrument, Sfard (2013) offers a set of principles to guide the analyst while presenting the findings to mitigate this subjectivity. These principles, among others, include the;

(1) Principle of ontological fidelity.

(2) Principle of epistemological fidelity.

To preserve these fidelities, I have decided to present the findings from the focus group interview by using reported speech (the participant said that), that is, what was said or done by the participant. The essence is to present what they did. In other words, the narratives are

about the participant's stories and not the world. I have also endeavoured to present my own stories in the first person.

5.4.4 Findings from the focus group interview

In this section, I present the findings by applying Sfard’s (2008) commognitive frame to each participant’s mathematical discourse in the interview. I structured these findings according to the features of mathematical discourse.

The aim of this study was to interpret the grade 11 learner discourses while engaging with geometry in the classroom. The analysis of the transcript of the interview data informed by the four features of mathematical discourse and was determined by the following examination questions:

(1) *What features of mathematical discourse to learners engage with in their discourses?*

(2) *How do these features influence the learner's discourse?*

(1). *Word use.*

One distinctive characteristic feature of mathematical discourse is using purpose keywords. The keywords in mathematical discourse, and in this case, geometrical discourse, are those words that signify quantities, number words, operation words, geometric shapes, and words that signify results of geometric operations. Under word use my interpretation of the findings was directed at three characteristics of word use. These are literate, colloquial and objectification.

- *Words for operations.* Participant's discourses showed some expressions of numerical operations. When prompted by the interviewer about “*how to prove that its congruent”*, subject of S1 responds by saying “...B and Q are ninety degrees, added together is hundred and eighty degrees…” (in [137])

Table 5.16. Examples of number words, operations words, words representing shapes, and words representing results derived from the transcript

S1	S2	S3
[137] both are hundred and eighty degrees and to get three hundred and sixty degrees we have to add [137] opposite sides of a quadrilateral.	[116] ninety degrees	[80] I bisected the line

- *Literate* are those words used in literate discourse, the discourse practised in schools, and are mediated by symbolic artefacts.

  In the analysis, I found that the participants used literate words. For example, when asked by the interviewer on "*their understanding of tangency*", participant S2 responded by saying it's a line that *"touches the circle"* (in [12]). The word "touches" in this utterance, referring to a line segment and a circle, is classified as literate as used in literate discourse by mathematicians. Participant S1 also seems to concur while saying, "It *touches the circle"* (in [13]). Furthermore, the utterance of participant S3 saying *"drew the line intersecting P..."* (in [78]) is also classified as literate. The term "intersection" by participant S3 is in keeping with what the mathematical community would refer to as a point where line segments "meet". Further examples of the use of words that I classified as literate can be found in utterances ([49,105,117,132]).
- *Colloquial* words are used in everyday life and are regarded as not part of the mathematical community. These words are often mediated visually by primary objects.

  In the analysis, when the participants were asked about "*their understanding of the term tangency*", participant S1 responded that it's a line that "*cuts the circle"* (in [10]), while participant S3 seemed not to agree with this statement but added,

"*doesn't cut the circle"* (in [11]). The word "cuts" is interpreted as colloquial and is used in everyday discourse. When asked, "How do you decide on the length of the radius?", the utterance of participant S1 ", We *draw a line from the middle of the circle to P"* (in [71]). The word "middle" used by participant S1 to describe the point of a circle is coded as colloquial. Further examples of the use of keywords classified as colloquial can be found in utterances of ([74,83,8595]).

- *Phrase-driven*

In the examination of the participant's utterances, there was evidence of the use of active voice on nouns, material process verbs, and pronouns and adverbs by using specific purpose keywords in phrase driveways. In response to the question *"how do you know that the triangles are congruent"*, participant S1 uttered the following sentences;

"I said that B and Q are ninety degrees, added together is hundred and eighty degrees, so to measure…, say angle O and angle T, we need to know, angle B and angle Q are hundred and eighty degrees and opposite sides of a quadrilateral are supplementary…….because both are hundred and eighty degrees and to get three hundred and sixty degrees we have to add" (in [137]).

I classified this as a phrase-driven use of words.

- *Objectification*

When asked, "How do you know where to place the compass?" participant S3 responded, *"I got the distance to R, drew a line intersecting P. Then I joined to make the centre of the circle"* (in [78]). I classified this word use as dis-objectified discourse.

In responding to the question, "Is there another way to show that the two triangles are congruent?" participant S1 responded by saying, "Angle *B and Q that is hundred and*

eighty degrees, so ninety degrees plus ninety degrees plus three hundred and sixty degrees minus hundred and eighty degrees. Because you have to get hundred and eighty from angle O and from angle T" and what was done, counting with the fingers (in [178]). I classified this word use as dis-objectified.

(2) *Visual mediator.* Visual mediators are those visual realisations of the object of discourse (Sfard, 2008). In literate discourse, participants use visible objects as part of the communication process. This is done, for example, by scanning the mediator with one's eyes, drawing, or using written symbols. In geometric discourse these objects are communicated by using different mediational modes. Examples of mediational modes in geometric discourse include symbolic, iconic, and concrete. I now present the findings regarding occurrences of these modes in the participant's utterances.

- *Iconic.* When the interviewer asked *"how to explain tangency"*, participant S1 responded by saying, "I would say tangent is the term used when two lines...or any line that doesn't go through the circle" what was done, raised and moved the hand forward (in [17]). When asked*, "How would you construct a circle that would pass through the point P?"* participant S1's *response was ", I draw a circle along the line down the spot here" What was done, he pointed* the finger at the line and moves the finger around (in [47]). Further, when prompted about "*what is a tangent?"* participant S1 responded, "that a tangent is perpendicular to the radius", and what was done raises two fingers and makes a cross (in117]). These two gestures, that is, raising the fingers and making a cross with hands as a way to communicate, are classified as iconic mediation.
- *Symbolic.* When asked *"if it is congruent"*, participant S2 response is *"you see, because there is an angle...right angle and OB is equal to OQ...that is the radius, that*

is how to prove" (in136]). The reference to the line segments and OQ is classified as symbolic mediation.

- *Concrete.* When asked "*If you have to explain this to somebody, how would you explain" participant* S1 responds by saying "*I would say tangent is the term used when two lines or any line that doesn't go through the circle"* (in17]). In this response, participant SI is using the word "through", implying that a circle is a concrete object. The use of *through* in the utterance I classified as concrete. Further, when asked "W*here the circle is supposed to be drawn?*", participant S1 responds by saying that "*It has to be between these two lines... so it has to be here"* and what was done, points at the lines with a finger (in 60]). Participant S1's use of the noun "it" in reference to the circle is classified as concrete. Later when asked about the tangent, participants S1 and S2 responded, "It *starts off at B, yes it starts at B"* (in [202]), while referring to a line segment.

(3) *Narrative.* Endorsed narratives can be substantiated by actions through which participants determine to endorse previously constructed narratives.

- *Substantiation*

  When responding to the question posed by the interviewer, *"What do you mean when you say that point is a right angle?".* Participant S1 responds by saying*, "Because tangents make a right angle. The radius and the tangent make a right angle*" (in115]). Further, in a follow-up question on the meaning of a right angle, participant S2 responds by saying "*ninety degrees*" (in [116]). I classified this as substantiation. When asked, "How do you know it's a tangent?" participant S1 responds and says, "because the radius meets P at the *right angle"* (in [124]). Participant S1 in the previous narrative says, "*The lines meet at ninety degrees"* (in [122]). I classified these actions as substantiation of the previously endorsed narrative.

When further probed about the tangent, participant S1 responded and said *"because the radius meets the tangent, and all tangents are perpendicular to the radii"* (in [182]). I classified these actions as substantiation of previously endorsed narratives.

- *Memorization.* Refers to recall from memory.

  When asked "*Why did you choose to start your construction in this way?"* Participant S1 says, *"I would draw a circle around the entire line, leaving P in the middle.... midpoint, then it makes more sense to draw a circle passing the.... P"* (in [56]). In this utterance, participant S1 pauses before completing the sentence. I classified this utterance as recalling from memory. Participant S2, in response to the question, "Is there another way to show that the two triangles are congruent?" says, "…I don't know", and what was done, scratches their head while saying this (in [173]). I classified this recalling from memory. Later, still, in responding to the same question from the interviewer, participant S1 says, "I'm thinking there is another way to show that all angles are equal to three hundred and sixty degrees", and what was looked up to the ceiling while saying this (in [174]). I classified this as trying to recall from memory. Further on, while responding to the interviewer's prompt about the tangent being at point B, participant S1 responds, *"BT is…I think BT is a tangent"* (in [200]). The pause in Participant S1's statement is classified as trying to recall from memory.

(4). *Routine.* The word routines stand for a collection of meta-rules that determine both *when* and *how* repetitive discursive action is performed. The *how* of the routine is a set of discursive procedures to be implemented to bring the action to a conclusion, while the *when* of the routine is a set of prompts that signify when to perform the discursive action.

The *when* of the routine is further divided into two sub-divisions of conditions. The first division concerns the *applicability conditions* and the second division concerns the *closure conditions.* The elements of *Applicability conditions are* meta-rules that determine under

what circumstances to implement the routine procedure. Signals for application may be verbal or from the surroundings and maybe self-evoked or given by others. The elements of *closure conditions* are the meta-rules that make up the closure conditions signal to the performer the circumstances that regard the completion of the performance.

When asked about where to place the compass, participant S3 responded, *"I got the distance to R, drew a line intersecting P. Then I joined together to make the centre of the circle"* (in [78]). Further, in responding to the same question, participant S2 said, *"We drew the radius, we labelled the centre O, and we drew a line from O to P. That was the radius"* (in [105]). I classified these utterances as routines.

- *Corrigibility.*

  In responding to the question "*what is to bisect a line*", participant S3 says *"cutting a line"* (in [83]). But when further prompted by the interviewer, participant S3 says *"to cut into half"* and repeats the same utterance *"to cut into half"* (in [85]). I classified this response as corrigibility.

  Later, when asked about the meaning of the term perpendicular, the participant says, "Yes, meaning ninety degrees" (in [121]), but then follows up and says, "The *lines meet at ninety degrees"* (in [122]). I classified this as an attempt at corrigibility.

(4). *Other related findings.* The findings in this section are also related to commognition.

- Mathematics as a social activity.

Table 5.17: Mathematics as a social acivity.

Turn	Speaker	What was said	What was done
107	S2	The radius is what we call a line from the centre of the circle to the circumference.	Turns her head and looks at S1
184	S1	It is important that we know that the angle is ninety degrees in our proof.	S1 and S2 look at each other and the both smile

The utterances illustrated in table (5.17) demonstrates evidence of the participants participating in mathematical conversation. I classified this as a social activity.

- Agentivity.

Table 5.18: Sample of the performer's agentitvity.

No:		What was said	What was done
59	S1:	We can make P a chord.	Looks at the interviewer.
72	S1:	How do you know where to put the compass?	While saying this looks at S2.
101	S1:	We would draw the center line.	
103	S1:	Draw the circle…. Yeah	Turn her head and looks at S2.
107	S2:	The radius is the line from the centre to the circumference.	Turns her head and looks at S1.
108	S2:	Then we prove that…isn't this the radius?	Turns her head and looks at S1.

The utterances illustrated in table 5.18 demonstrates evidence of the participants making choices with regard to how to proceed towards the solution of the problem. I classified this as performer's agentivity.

- Material entities.

Table 5.19: Some examples of material entities.

Turn	Speaker	What was said
11	S3	Doesn't cut the circle
13	S2	It touches the circle
17	S1	Any line that doesn't go through the circle
137	S1	Added together

The utterances illustrated in table 5.19 shows evidence of the participants making references on mathematical objects as if refering to real things. I classified this as examples of material entities.

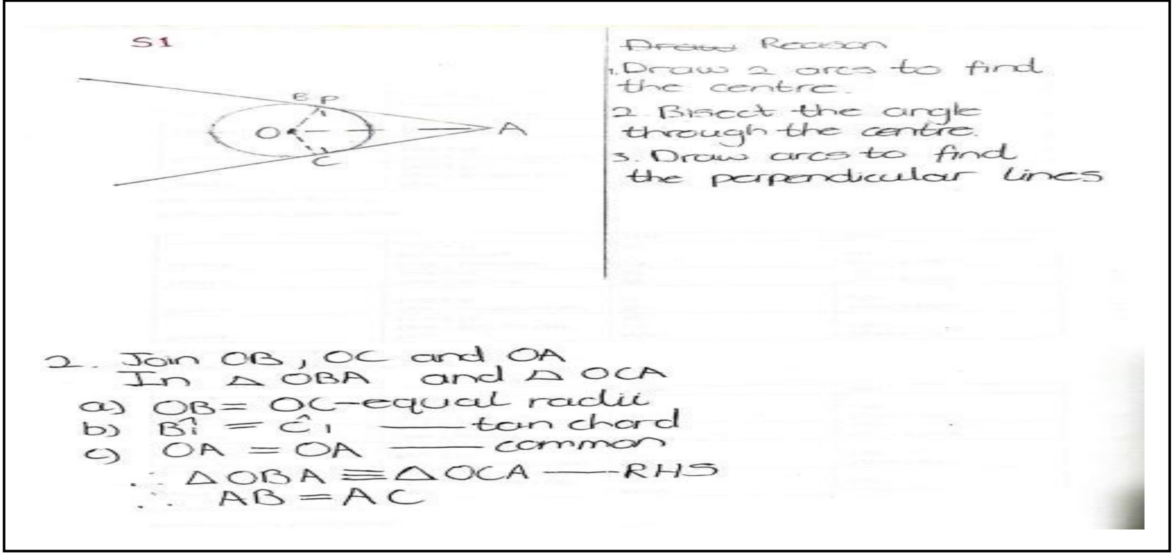
5.4.5 Findings from the Written task activity

In this section, I present findings of each participant's mathematical discourse by analyzing the written responses to the task questions using Sfard's (2008) commognitive approach to thinking.

5.4.5.1 Word use

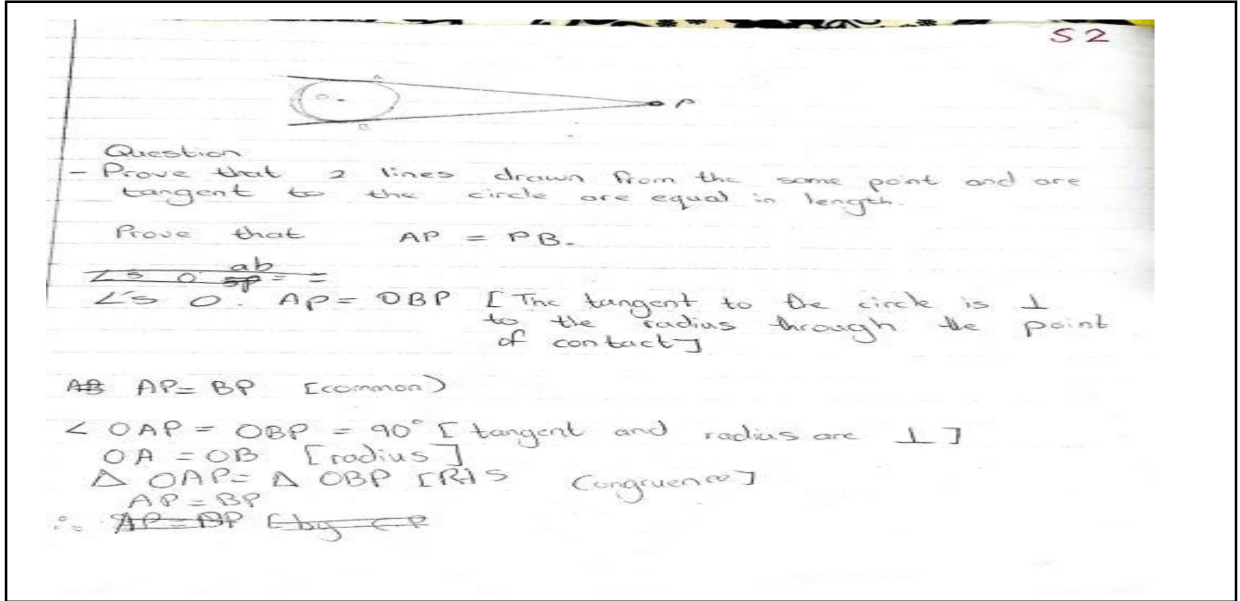
About word use, analysis of the participant's written activity revealed the following, depicted in figure 5.11.

Figure 5.11: Participant S1's solution to the written task activity.



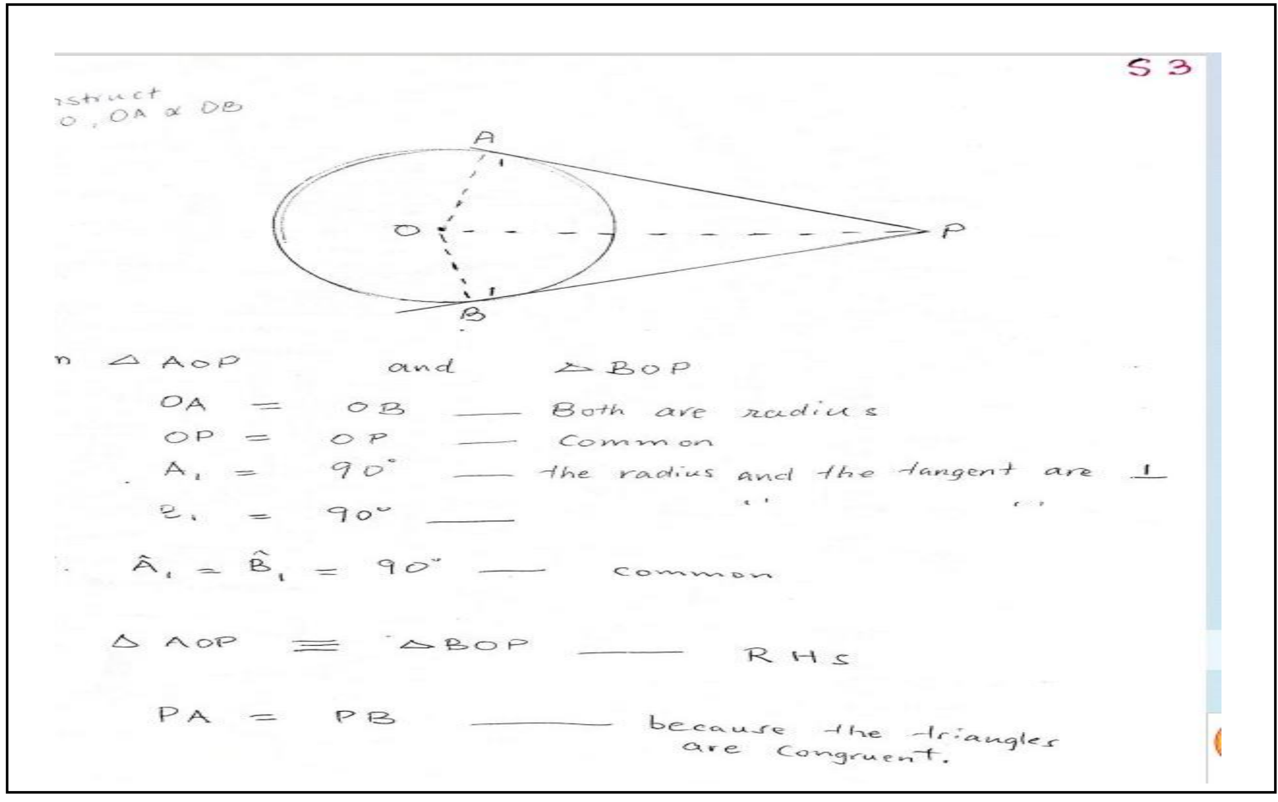
Analysis of Figure 5.11 shows evidence of word use by participant S1. The diagram was labelled (A, B, C) using text and (algebraic, numerical and geometric terms) as precedent identifiers. These words have specific meanings in mathematical discourse and are acceptable to the mathematicians' community. I coded this as a literate use of words.

Figure 5.12: Participant S2's solution to the written task activity.



Analysis of Figure 5.12 shows evidence of word use. Participant S2's diagram included labels in the form of text and used (algebraic, numerical, and geometric terms) as precedent identifiers. These words have specific meanings in mathematical discourse. I classified this as a literate use of words.

Figure 5.13: Participant S3's solution to the task activity.

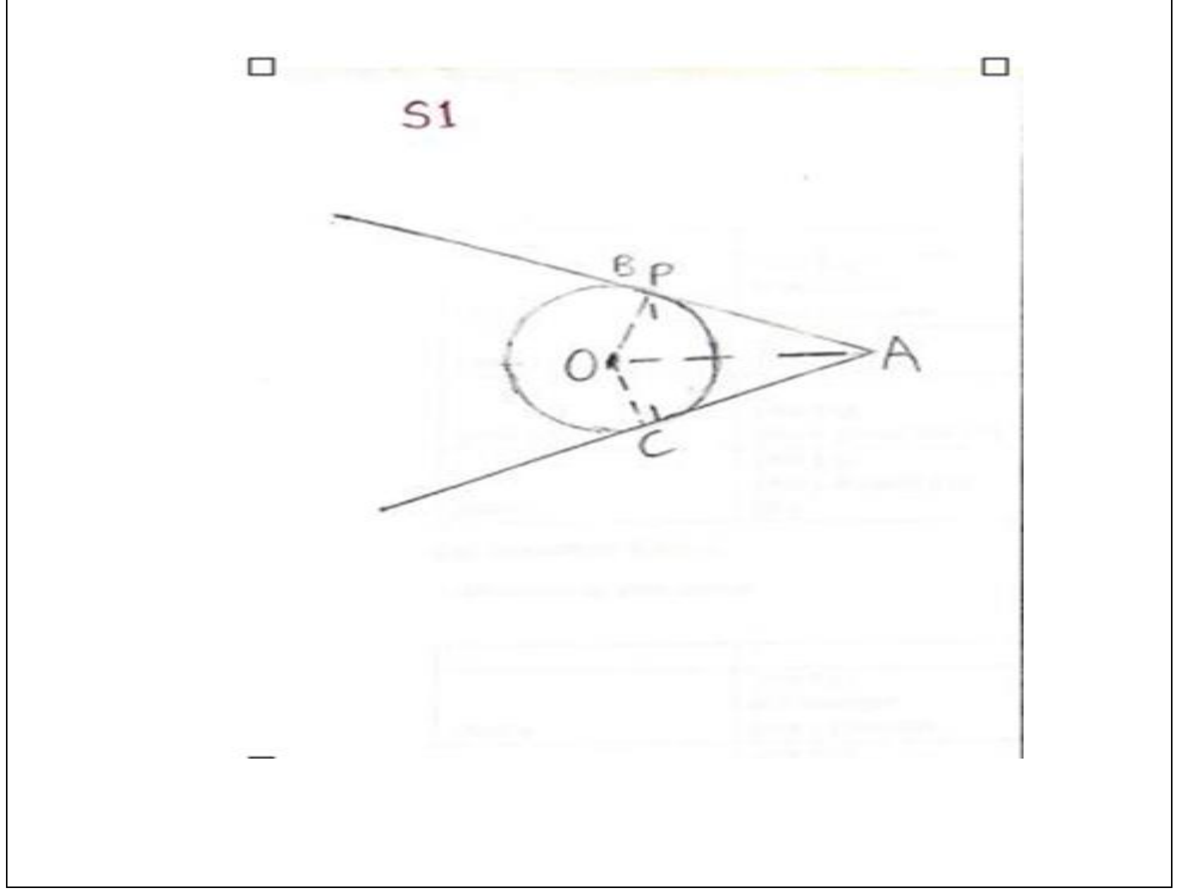


The words and text used by Participant S3 are classified as mathematical. The first signifier-relationship pair shows evidence of the use of specific words. These words (algebraic, numerical, and geometric terms) are used by participant S3 as precedent identifiers. I classified this as a literate use of words.

5.4.5.2 Mediator use

Objects such as graphs, images, and diagrams are mediators.

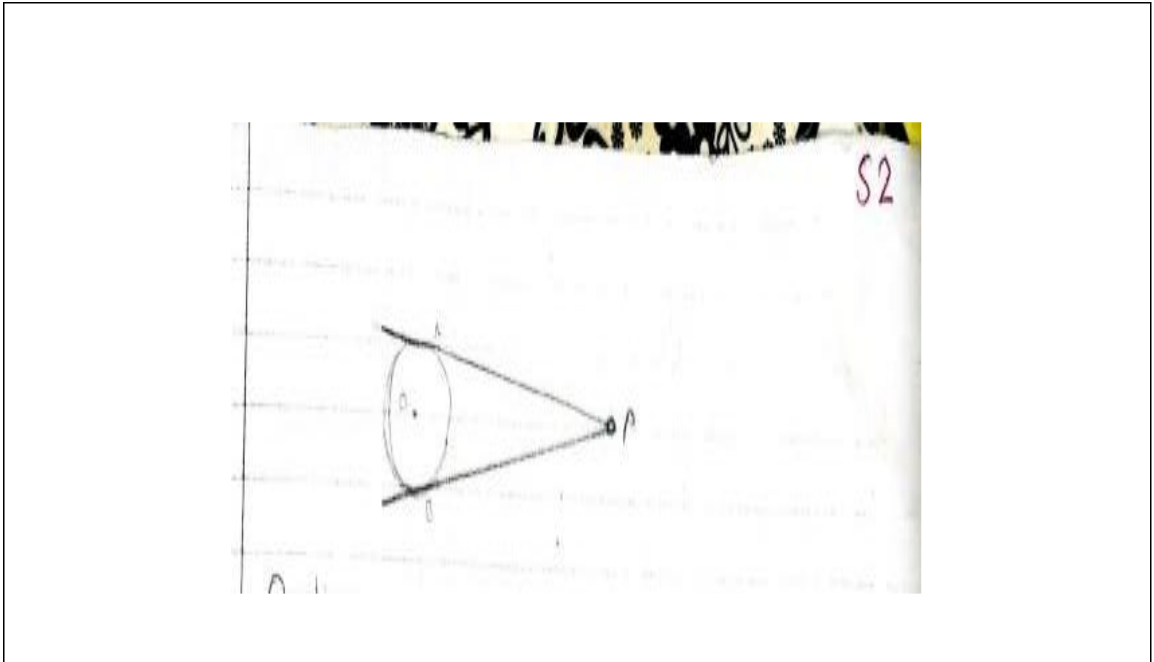
Figure 5.14: Participant S1: Mediator use.



The above figure shows that Participant S1 draws a line between segment BP and CA. Participant S1 bisects angle A and marks a point along the bisector. Participant S1 then draws a circle centre O between the two-line segments and the radii OP and OC. The diagram is accurate, and the labeling is congruent with what is expected regarding the solution to the

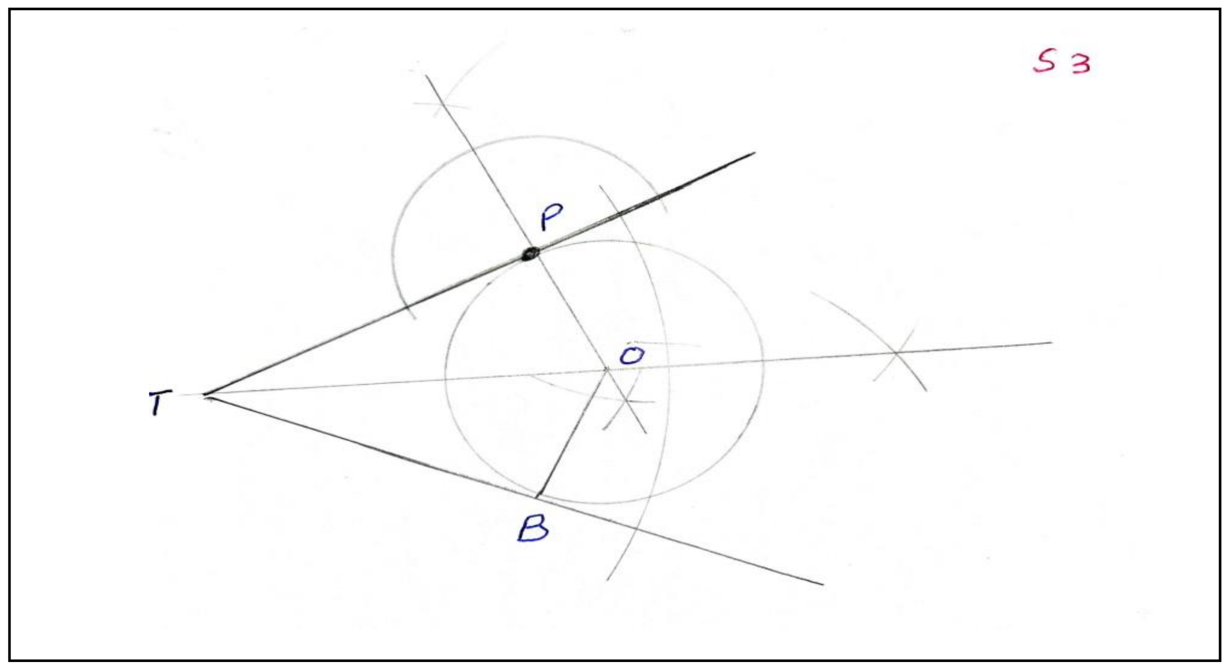
problem. This evidence of the use of graphic and symbolic representation shows mediator use. I classified this as mediator use.

Figure 5.15: Participant S2: Mediator use



The examination of participant S2's task shows a drawing of the circle centre O between two-line segments CP and BP. While the sketch seems to be inaccurate, the labeling on the sketch matches what the solution to the task would look like, and this in itself indicates that participant S2 comprehended the problem. This evidence of iconic (graphic) and symbolic representation shows mediator use. I classified this as *one-step mediator use.*

Figure 5.16: Participant S3: Mediator use.

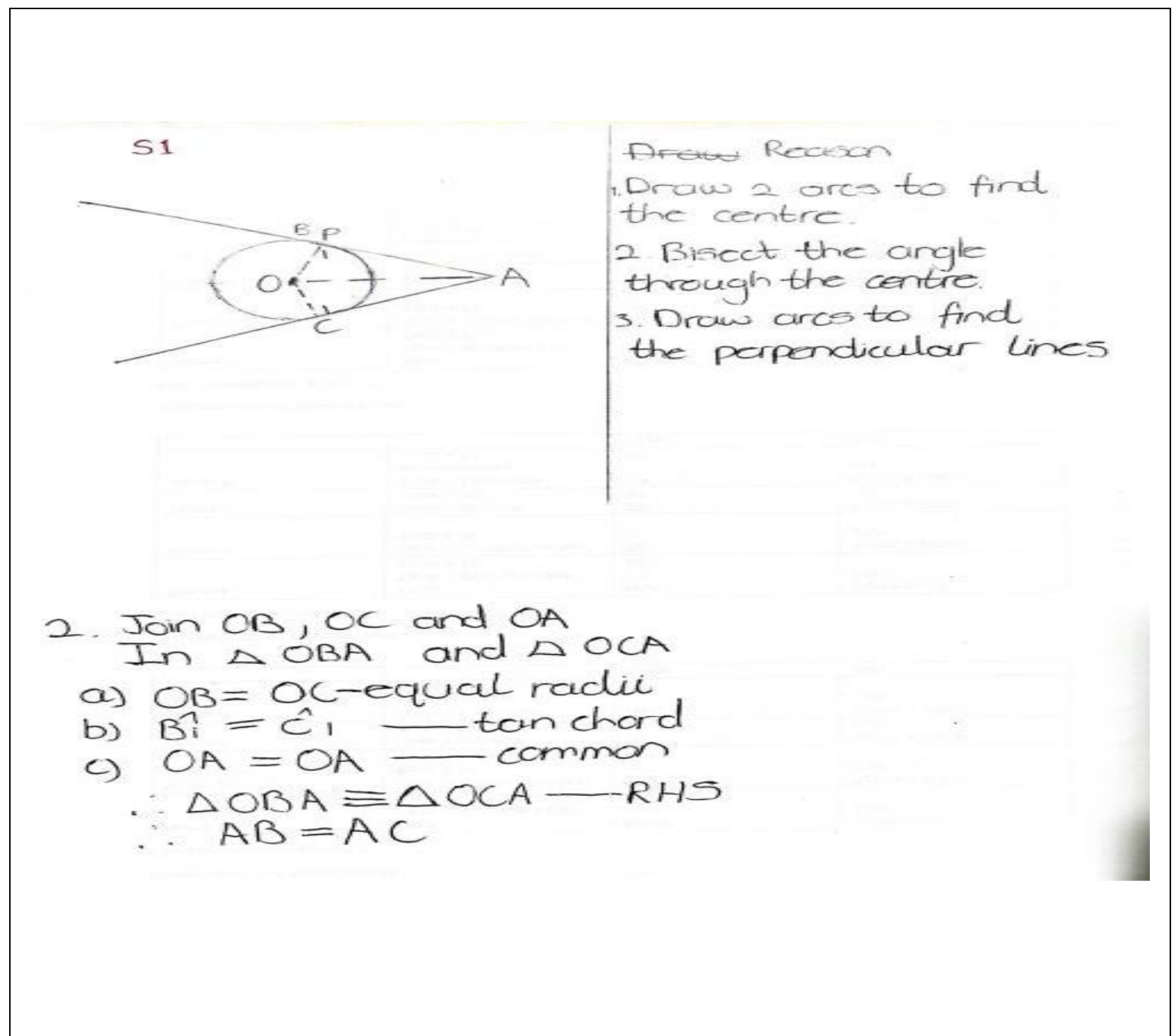


Examining Participant S3's response to the task question shows the construction of two-line segments, TB and TP. Participant S3 constructs the bisector of angle T. Participant S3 constructs the perpendicular bisector BO and PO. Participant S3 then constructs the circle centre O. This evidence shows mediator use. I categorised this as mediator use.

5.4.5.3 Narratives and Routines

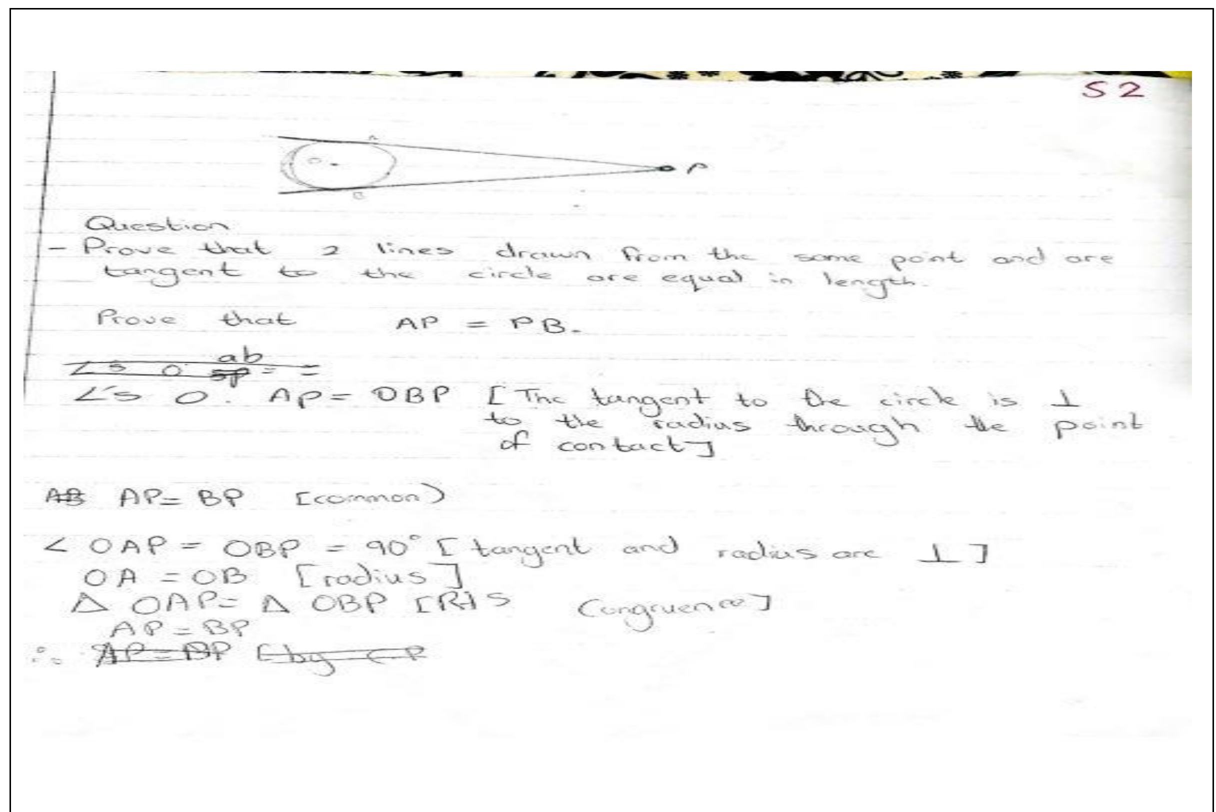
These explain the steps followed (the output of each procedure step serves as the input for the next) and describe mathematical facts such as axioms, definitions, and theorems

Figure 5.17: Participant SI: Repetitive procedures (Routines).



Examining participant SI's responses to the task question is evidence of repetitive procedures. For example, in question number 2, participant SI creates five signifier-realization pairs. The sequence of steps shows evidence of a signifier-realization pair feeding into the next pair coupled with justifications. The last signifier-realization pair leads to endorsed narratives. I classified this as examples of endorsed narratives derived from the properties of the mathematical object within the discourse.

Figure 5.18: Participant S2: Repetitive procedures (Routines).


S2

Question
- Prove that 2 lines drawn from the same point and are tangent to the circle are equal in length.

Prove that AP = PB.

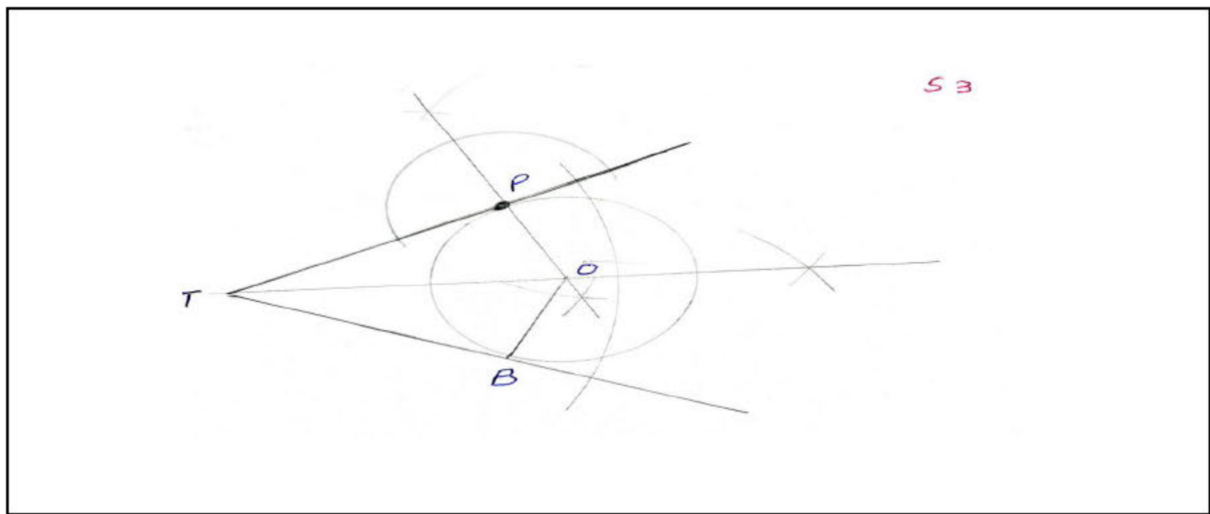
∠'s O. Ap = OBP [The tangent to the circle is ⊥ to the radius through the point of contact]

AP = BP [common)

< OAP = OBP = 90° [tangent and radius are ⊥]
OA = OB [radius]
Δ OAP = Δ OBP [RHS congruence]
AP = BP

The examination of participant S2’s response to the task question shows evidence of repetitive procedure. The work has more than four signifier-realization pairs. There is evidence of the previous pair feeding into the next pair. Each pair is justified, and there is evidence of recycling old narratives to create new endorsed narratives. I classified this as examples of endorsed narratives derived from the properties of the mathematical object.

Figure 5.19: Participant S3: Signifier-realization pairs.



Analysis of participant S3's response to the construction question shows evidence of an elaborate realisation procedure. Each signifier realisation is achieved with the help of construction lines. The diagram has more than five signifier-realization pairs. The pair of steps feed into the next pair. The construction lines are the narratives, and the last signifier-realization pair leads to endorsed narratives acceptable to the mathematician community. I coded this as an endorsed narratives derived from the properties of the mathematical object within the discourse.

I have elaborated on the findings revealed from the data derived from the interview and the data derived from the written task activities. I have now summarised the main findings and presented them in table format, Table 5.20, given below.

Table 5.20: Summary of some of the main findings (Interview and written Task activity)

Category	Line number	Example of evidence	Brief meaning
Word use	12 13 78	"Touches the circle". " It touches the circle…." " I got the distance to R, drew a line intersecting P. Then I joined together to make the centre of the circle".	Literate
	10	"Cuts the circle".	Colloquial

	11 71	“Doesn’t cut the circle”. “We draw a line from the middle of the circle to P…… You measure first”.	
	13 17 32 48 49 56 74	“It touches the circle….” “I would say tangent is the term used when two lines………or any line that doesn’t go through the circle”. ” I constructed a circle”. “I drew a circle…….” ” I would draw a circle passing through point P. leaving P on the circumference of the circle”. “I would draw a circle around the entire line, leaving P in the middle……midpoint, then it makes more sense to draw a circle passing the ……P”. “I would measure with a pencil… make sure it doesn’t go beyond this line… doesn’t touch the line”.	Dis-objectified
Visual mediators			Single media
	What was said	*What was done*	Multi modal
Routines			Single step

<table>
<tr><td></td><td colspan="2"></td><td>Multi step</td></tr>
<tr><td rowspan="3">Narratives (Endorsed)</td><td>114

122
124</td><td>“Because tangents make a right angle. The radius and the tangent make a right angle”.
“The lines meet at ninety degrees”.
” Because the radius meets P at right angle”.</td><td rowspan="2">Source from inside/with-in discourse</td></tr>
<tr><td colspan="2">2. Join OB, OC and OA
In △OBA and △OCA
a) OB = OC – equal radii
b) B̂1 = Ĉ1 —— tan chord
c) OA = OA —— common
∴ △OBA ≡ △OCA —— RHS
∴ AB = AC</td></tr>
<tr><td>56

171

180
200</td><td>“I would draw a circle around the entire line, leaving P in the middle……midpoint, then it makes more sense to draw a circle passing the ……P”.
“BT will be equal to QT. Proving both lines are tangents”.
“Because we proved that”.
“BT is …… I think BT is a tangent”.</td><td>Source from outside discourse</td></tr>
</table>

5.5 Conclusion

This chapter had three main sections: presentation, data analysis and the findings. Data was collected through sources of data. One source was through the focus group interview, and the other was through the mathematics written task activity. Data from the interview in the form of transcripts was transcribed and then analysed using the Adapted Tangent Discourse Profile

instrument (ADPT). The profile adapted Mpofu and Pournara's (2018) instrument. Data from the mathematics written task was analysed using the Realization Tree Assessment (RTA). The Realization Tree Assessment was an adaptation of Knox and Kontovich (2022) instrument. The analysis of data was informed by predetermined themes emanating from the attributes of mathematical discourse. These attributes include word use, visual mediators, routines, and narratives. The qualitative data was then presented to the reader in tabular and figurative format. Presentation of data in these formats, enabled me to present the information as accurately as possible while adhering to the two principles of verbal and epistemological fidelity (what was said, and what was done) that guide commognitive analysis of data. This method was precisely informed by the type of methods used in the analysis of data.

The last section of the chapter dealt with the findings after examining the data. For the reader's sake, I combined the data from the two separate instruments. Some of the main findings from the examination of data were summarised and presented in Table 5.16.

In the next chapter, I deliberate upon the findings and interpret them, based on my experience and according to the context of the commognition theory.

Chapter 6

Interpretation and Discussion of the findings

6.1 Introduction

This chapter comprises two sections: the interpretation and discussion of some of the findings. In the first section, the qualitative interpretation of these findings is presented, while the next section that follows deals with the discussion of some of the findings. The commognitive perspective underpins the deliberations of these two sections.

6.2 Interpretation

Creswell (2009), points out that the researcher has several roles in scientific enquiry. One of these roles of the qualitative researcher is to make meaning of the data and influence the interpretation. Sfard (2000) offers some guidance to commognitive researchers regarding their role in interpreting the findings. In her guidance, she urges commognitive researchers to focus on the following things;

1. What is said and what is done,
2. What the participant is looking at, listening to,
3. The image the participant perceives (imagination),
4. Attend to the participant is procedure while scanning the image.

(Sfard, 2000, p. 304)

The discussion that follows is on the researcher's interpretation of some of the findings of the study on learner's commognition while engaging with geometry. The researcher must answer the research question using the collected data to complete this study. My interpretation of some of the findings was underpinned by three cardinal sources used when describing mathematical discourse. These pillars are;

1. The degree of objectification in the participants' use of words.

2. The sources of their mathematical narratives and subsequent endorsement.

3. The goal/s of the activity.

For the sake of positioning the reader in this section, I organised interpretation around the critical questions on which the study in order to answer the research questions that hinged the study.

6.2.1 Research Question One

What discourses do learners engage in when learning geometry in a grade 11 classroom?

6.2.1.1 The degree of objectification in the participants' use of words

Word use plays a significant role in making communication effective in human communication. Mathematical discourse revolves around talking about intangible mathematical objects. Some of the findings from the analysis of the utterances of all participants in the study revealed that their mathematical discourse concerning word use was disobjectified and, at times, colloquial; therefore, the intended meaning embedded in the utterances was more likely to be distorted. For example, in their discourse, the participant's word use was not objectified in instances where they referred to numeral numbers as stand-alone entities, such as (in [116, 178]). In other instances, the participants referred to mathematical objects with words such as those referring to a physical object out there in the world (in [8, 13]. Other instances of dis-objectification included utterances such as (in [56]) and casual words (in [155]). In these utterances, ample evidence suggests the presence of human agency. The participant's use of the words "*we*" and "*I"* indicates human participation. The interpretation is that the process of alienation, which is an integral part of objectification, has not been given attention. The interpretation drawn from the findings

concerning word use is that the participants' degree of objectification is low. The participants regard mathematical objects as objects that exist in the world and as self-sustained entities. Other findings from the examination of the utterances showed that the participants' use of specific purpose keywords was phrase-driven. Mathematics is a language used in the classroom and participants draw on mathematical terminology to communicate their mathematical thinking. Plannas (2018) asserts that language has a vital role in mean making. Examination of the utterance of participant S3 (in [78]) is phrase-driven. The interpretation is that although specific purpose keywords were used in some instances on entities, most of the utterances were not object-driven.

In examining the utterances (in [11], [13], [17] and [137]) all the participants S1, S2, and S3 refer to mathematical objects not as abstract objects by nature but as tangible material objects found in the world. The evidence shown above further suggests that the participants have not yet attained the stage of abstraction but still regard mathematical objects as existing in the world and concrete by nature. Since mathematical objects are abstract by nature, the stage of abstraction is one of the properties of objectification. These are characteristics of ritual participation. Abstraction in mathematics draws the learner to focus on the properties of the perceived mathematical object as well as the actions on object. Gray and Tall (2007) make the assertion that the goal is to reduce the cognitive strain by only concentrating on some aspects while ignoring others in the situation. Abstraction is vital in constructing mathematical knowledge. This claim is supported by the results of Ozmen his colleague (2020) who found abstraction to have a positive effect on students' knowledge construction.

6.2.1.2 The sources of participants' mathematical narratives and subsequent endorsement

Sfard (2008) points out that narratives germinate from word use. Narratives can materialise as spoken or written text and is generated for the sake of describing the object or the relationship between objects. Mpofu and Pournara (2018) divide narratives into two categories: (1) Memorisation narratives and (2) Substantiation narratives. They state that memorisation narratives are characterised by formulae and rules, while substantiation narratives take the form of justification and reasoning for particular actions. Mpofu and Pournara's (2018) illustration of narratives underpinned my interpretation of the sources and subsequent endorsement of the participant's narratives.

Regarding the origin of mathematical narratives and their endorsement. Some of the findings from the analysis of two of the participants' (S2 and S3) task-procedure pairs in their realisation trees showed slight evidence of vertical and horizontal equivalence, see Figure (5.1). There was evidence of vertical bonding in that the outcome of the previous step was used as the input for the next step. However, the horizontal equivalence branches did not go far enough to constitute horizontal bonding and were limited, see figure (5.1) for participant S2 and S3 task responses. These findings suggest that there was a lack of alignment between the vertical and horizontal equivalences, and the recognition by participants S2 and S3 that although these equivalences, vertical and horizontal, were used separately, they would lead to the same outcomes. Vertical and horizontal bonding that would result in a single procedure was missing. The significance of this interpretation is that the mechanism of developing a new discourse as a result of several routines bonding into a single one was absent, and this was partly because the focus of these two participants suggests being more on the processes of their performance rather than on the end product. Hoeizontal bonding is related to the learner’s prior knowledge while vertical bonding concerns the conversion of symbols to a

more abstract symbolisation. These two types of thinking processes involve deductive and inductive reasoning. Mathematical reasoning is crucial for successful implementation of tasks (Batlolona, 2018). Adams and colleagues (2021) investigated the impact of inductive and deductive approach and found that both types of thinking play a significant role in problem-solving situations. Other such as Shodikin, (2017), and Singa and colleagues (2023) also found the same results and the effectivenesss in enhancing critical thinking skills as well as learning motivation. As for the remaining participant S1, see Figure (4.17), the analysis of the realisation tree showed evidence that the implementation of the task was done through one-step mediation instead of multiple mediations. This participant's realisation tree (figure 4.17) showed that the solution to the task was implemented in one step. This suggests that Participant S1 had not internalised the discourse's individualisation process but was simply imitating some other discussants. This aspect of individualising the discourse is a crucial element of the commognitive framework. There was no evidence in her implementation to show that one particular procedure was feeding into another, ending up with the use of multiple routines to arrive at the solution to the task. It suggests that Participant S1 could only recall what the end product looked like from the past but did not yet possess the necessary procedural steps to arrive at the end product. Participant S1 did not engage in a number of the needed discourses identified as requirements for the successful implementation of the task. Therefore, the stand-alone routine, mediated with one step, was process-oriented and not product-oriented.

An examination of some of the utterances suggests that the narratives' endorsement sources were the appearance of the mathematical entities and not their properties. For example, participant S2 said she did not measure, but just guessed (in [75]), while participant S1 justified her statement from the appearance of the mathematical object (in [134]) when probed about the sources of their endorsement of the narratives. In saying they look

congruent, participant S1 was implying that the triangle is a concrete physical object whose properties can be scanned with our eyes and not an abstract mathematical object with properties that can be derived from the object's relationships. The same can be said about Participant S2's statement when she says that she never measured the distance. This suggests that participant S2 scanned a concrete physical object herself and estimated the distance by looking at the appearance of the concrete object. The interpretation of such an utterance is that the participant's narratives were memorised, so they could not justify their actions. This interpretation resonates with the findings of Robert and Le Roux (2019), who found that the participants derived their narratives from the appearance of mathematical entities and could not justify their actions on the entities when constructing endorsed narratives.

6.2.1.3 The goal/s of the activity

About the goal of the activity, the utterances under *What was said* and the actions of the participants depicted under *What was done* were simultaneously analysed. Under *What was done,* the three participants frequently looked at the interviewer or at one another immediately after constructing an endorsed narrative as if to seek approval from someone else regarding their performance. Some of these instances where the participant looked at the interviewer after making an utterance are illustrated in the following (in [59]), looks at the interviewer (in [90]), turns the head and looks at participant S1 (in [107]), turns the head and looks at participant S2 (in [124]). The significance of looking at the interviewer or at one another suggests seeking approval of their actions from an external authority (the interviewer) and social approval from their peers. This suggests that the participants regarded the interviewer as the authority and the expert discursant whose approval was required to signify their successful task implementation. Their performance when they looked at one another or smiled at each other suggests that their participation in the discourse was geared more towards others and not for themselves. It means that the participants had not yet

individualised their participation in the discourse. This evidence suggests that the participants have low self-reliance. Among the learner's abilities, self-reliance improves mathematics learning. Nur Aisyah (2020) and colleagues point out that learners who demonstrate a high level of self-reliance tend to have confidence and push themselves to attain higher goals without seeking direction from others. Nur Aisyah and colleagues in their study found that learning anxiety and self-reliance significantly influenced mathematics learning performance. Further in my study, under *What was said*, the data showed many instances where the participants utterances began with the word, *We*. Instead of applying the processes of alienation in their utterances, the use of the word " we " signifies that the participants were acting in solidarity with others while executing the task.

The use of the word *we*, in their utterances and actions of solidarity, (in [107] and [184]), suggests that the goal of their performance was geared towards others and not for themselves. Upon scrutiny of what was done, the participants in this example, they looked at each other and smiled after making their utterances.

From the evidence presented in the above discussion of some of the findings, I laid claim that the participants did not give the required significance to the end products of their actions but rather gave more significance to the process of their actions in their performance. I therefore concluded that the participants' discourse was more ritualistic participation than exploratory participation. Further evidence to justify this claim can be found in the noticeable absence of explorations in their utterances, as illustrated in the following sentences; (in [74,80]), and in single word utterances even when prompted by the interviewer, such as (in [52, 58, 92]). Some casual utterances begin with the word *because* such as those in the utterances (in [114,124]). Sometimes, they simply repeated what was said before, such as in those found in sentences (in [19, 21,65,67]). The use of the word *because* in their speech suggests they were simply repeating what they had said previously. It may be argued that the participants did not

realise that the prompts' purpose was invitations to give alternate responses to the ones given before while engaging in explorations. Their inability to seek alternative ways of implementing the task was evident when participants S1, S2, and S3 were asked if they could have worked out the task in another way. In responding to this particular prompt, they all uttered in unison, (in [41]). This demonstrates a rigid approach to the task. This evidence suggests that the participants were more focused on the process of their performance in implementing the task rather than the end product of the procedures. Their participation in mathematical discourse had some elements characteristic of ritual participation.

This interpretation resonates with some of the findings of other commognitive researchers such as Nisa, Lukito, & Masriyah (2020), Roberts & le Roux (2019), Mpofu & Mudaly (2019), Mpofu & Pournara (2019), Thoma & Nardi (2017), Tasra (2017), who found in their various studies the participants' discourses to be ritualistic.

6.2.2 Research question two:

How do these discourses influence the way they understand of geometric concepts?

To position the reader, let me draw the reader to the definition of a concept. A concept in commognition is *"a symbol together with its uses"* (Sfard, 2008, p. 111). "x and 2 are examples of symbols in mathematics. Mathematical objects are abstract in nature and, therefore, not tangible. As a result, symbols are used to make mathematical objects perceptual.

Word use;

Since mathematical communication involves building new utterances upon former utterances (Sfard, 2008), telling stories about mathematical objects requires using words and, more importantly, mathematical keywords. Mathematical keywords have specific purposes and, in most cases, have meanings that may differ in colloquial discourse. For example, using

"integrate" or "differentiate" in mathematical discourse differs from its colloquial usage. As I laid claim above (under question one), the participant's discourse was ritualistic. One of the characteristics of ritual performers, among others, is their inclination to alternate between the colloquial use of words and the mathematical use of words in the same discourse when telling stories about mathematical objects. Another significant characteristic of ritual performers is that of a relatively high degree of dis-objectified word use, as noted in these utterances, (in [8,13,56]).

The participants frequent reliance on the use of colloquial words such as in those utterances (in [13,56,71]), and the high degree of dis-objectified word use becomes a contributing factor in the effectiveness of communication. The act of communication involves two sets of acting, firstly the action of an individual and secondly this is followed by the re-action of another (action-re-action pair). The action and re-action pairs lead to new communicative actions. Mathematical language has its own terminology and colloquial words do not fit as its usage everyday language is different. Meaning is often lost when words are imported from one language to another. Therefore, when communication is ineffective due to word use or disobjectification, meaning is lost, and different interlocutors may react differently to the same action. The breakdown in the action re-action process due to two or more incommensurable discourses may influence the participant's comprehension of mathematical concepts. Objectification with its related components of reification, and alienation are key to the development of communication and practical effectiveness. Through the use of symbols and words, objectification can be observed in learners written narratives. Kontorovich (2018) found that learners with a low degree of objectification struggle to coordinate words and symbols into coherent narratives. Vygotsky (1978) underscores the importance of word use when he points out that the learning system is supported by language. The interpretation drawn is that word use influences the participants understanding of geometric concepts. This

interpretation resonates with Zulu and Mudaly (2023), whose study unveiled the problem-solving strategies of pre-service mathematics teachers and established that word use was a factor in self-discursive activities as it contributed to successful visualisation. They concluded that word use influences the participants' understanding of geometric concepts.

Concerning the sources of their mathematical narratives and endorsements. How the ritualistic and explorative performers construct endorsed narratives is different. For the explorative performer, the sources of his or her mathematical narratives and endorsement of these narratives are derived from within the discourse, that is, from the relationships and properties of the mathematical objects. But for the ritual performer, this may not be the case. For example, the analysis of participant S2's task responses and the task-procedures pairs showed that in her implementation of the *how* of the routine, the decision to enact the routine was made arbitrary rather than systematic (figure 5.18). This is the case since the solution to the construction task was implemented due to direct mediation and did not contain as many sub-procedures as required. This evidence points to the inability of the participant to perform and enact a range of necessary procedures to address the task solution. The absence of sub-procedures in the routine resulted in a lack of endorsed narratives that may be derived from within the discourse and not from outside the discourse. The implementation of the *when* of the routine with regard to the opening and closing conditions was also lacking, as seen from the evidence so far. During the interview, when asked if there was another way of arriving at the same outcome, all the participants uttered in one voice that they only learnt one method of the solution to the problem (in [41]). This evidence suggests that they implemented their task solutions in a rigid way and did not exercise flexibility in their routines, a characteristic of ritual performers. Another set of evidence that suggests the lack of flexibility in their discourse is in the use of utterances that begin with the word *because* and *one-word* utterances or repeating the same procedure when prompted for an alternative response. There

were many instances when the participants replied to the prompts with one-word statements and statements starting with because. Examples can be found (in [52,58,92, 116]).

In addition to the evidence that suggested the lack of, or absence of flexibility in the participant's performance, further scrutiny of participants' (S1, S2, S3) solutions to the construction task pointed at the rigid implementation of the procedures. The interpretation of their performance as rigid is the case with participant S3, whose construction was precise and accurate, mathematically correct and acceptable by the community of mathematicians. This demonstrated that participant S3 had procedural knowledge to implement the task but his conceptual knowledge was lacking. Conceptual knowledge and procedural knowledge work hand in hand. Baroody (2013) asserts that conceptual knowledge includes awareness of the principles and relationships that underpin a particular domain. Enhanced conceptual understanding is vital as it enables learners to summon multiple strategies in problem-solving. Gilmore and colleagues (2017) found that conceptual understanding plays a vital role in the relationship between procedural skills and mathematics achievement. However, the evidence from his speech during the discussion all points to the role of rituals and how rituals influence the comprehension of geometric concepts. This resonates with Mpofu and Pournara (2018) when they assert that ritualised routine materialises when the students can implement the necessary procedures, but when probed to explain, they cannot provide an acceptable justification for the actions and the answer obtained. The evidence presented so far in the above discussion is corroborated by the findings from the study of Thoma and Nardi (2017), who found the existence of unresolved conflict with visual mediators on the rules of school algebra and the rules of probability. They concluded that the participant's discourse was ritualistic. Another study that came up with the same conclusion is that of Ioannou (2017), who found that participants had commognitive conflict on applying metarules. Further, the study of Tasara (2017) concluded that the participant's discourse was ritualistic. In his study,

Tasara found the existence of contradictory narratives in the participant's discourse about the definition of the gradient. With evidence from the findings, I presented in this section, I claim that the participants' discourse was ritualistic. My interpretation is that their discourse is not fully fledged as it is influenced by the presence of rituals.

Visual mediator use:

The interpretation that visual mediators use influences their understanding of geometric concepts, as corroborated by Desi et al. (2022), who studied students' statistical reasoning processes that inform decision-making. They found that visual mediators used in presenting data were instrumental in successfully analysing and interpreting data. Tabach and Nachlieli (2011) contend that visual mediators positively influence a person's ideas about what is being discussed and the type of discursive action to undertake.

Routines and narratives:

Participant S3's realisation routines of the solution to the problem of constructing two tangents to the circle and the subsequent discursive narratives about the mathematical objects and gestures were examined to interpret the discourses and how these discourses influence the understanding of geometric concepts. Realisation trees, which are visual mediators, are constructed based on the individual's understanding of the problem and his own intuitions. Sfard (2008) points out that realisation trees contain valuable information about the personal understanding of the problem.

In examining Figure 5.19, participant S3 could perform a series of actions on the mathematical object to realise the solution to the problem. Participant S3's first procedure was to bisect angle P mediated by the signifier arcs along the bisector. His utterance (in [36]) supports this assertion regarding his starting procedure. This signifier-realisation pair feeds into another set of actions that resulted in the realisation of the perpendicular bisectors of the

line segments TP and TB. The signifier realisation of this action feeds into the realisation of the point where the two perpendicular bisectors intersect and are used to realise the centre of the circle. The evidence presented in figure 5.19 led to my claim that Participant S3 demonstrated how he performed multiple routines and participated in multiple discourses about these routines to arrive at the solution to the problem. In contrast, examining another participant's solution to the same problem did not result in the same claim. When Participant S2's responses, as illustrated in figure 5.18, were examined, the realisation tree was short in length and, therefore, did not contain much valuable information to constitute a fair attempt at the solution to the problem. During the interview, she alluded to the fact that she understood the problem (in [6] and [26]), but her routines did not constitute a series of step-by-step actions on the mathematical object. The visual mediators in her task do not depict the use of construction lines to realise the bisector angle of any line segments in the diagram or the realisation of the perpendicular bisectors to realise the centre of the circle. My claim is that the realisation routines did not contain multiple actions and therefore multiple discourses on the mathematical object and solution to the problem, was realised with one step mediation instead of multiple mediation. The claim that she did not participate in multiple routines and discourse is supported by the utterance she made when she admitted not have measured, but just guessed where to place the compass, (in [75]). The realisation tree does not demonstrate the use of some signifier realisation pairs feeding into one another to realise the solution to the problem that would constitute an acceptable solution. I, therefore, claim that Participant S2's discourse influenced her understanding of geometric concepts.

This claim corroborates the findings of Fernadez-Leon et al. (2019), whose study involved identifying the routines of undergraduate students when defining. They found the existence of commognitive conflict between the routines used when describing, and when defining. Tasara (2017) also found contradictory narratives in the participant's discourse when defining

the gradient. During the semi-structured focus interview, all the participants claimed that they were familiar with the question posed in the task and therefore, the search space for precedents would have been limited to those from among past classroom situations with the likelihood to associate the situation with the discourses, viz. (in [5]), and (in [6]). The written task activity required the participants to implement a series of interrelated procedural steps as well as conceptual procedures. Each of these requirements called for the participants to engage in a number of discourses. Some of these discourses were; (1) the discourse of bisecting the angle, (2) the discourse of constructing a perpendicular to the diameter in a circle, (3) the discourse of constructing a tangent to a circle, (4) the discourse of implementing the theorems in geometry for the execution of the task and, (5) the discourse of implementing the sequence of the necessary and appropriate procedures.

Participant *S1:* Whe the sketch in figure 5.17 is analysed, it shows no evidence of construction but evidence of drawing with the help of a pencil and a ruler. There are attempts at indulging in the discourse of bisecting the angle, constructing the perpendicular to the diameter of a circle, and constructing the tangent to a circle. However, there is evidence of indulgence in the discourse of theorems in geometry and in the discourse of implementing the sequence of the appropriate procedures.

Participant *S2:* The analysis of the sketch in figure 5.18 shows no evidence of construction using a pencil and compass; however, what is evident is a sketch of what the constructed diagram would look like done with the help of a pencil and probably a ruler. There is no evidence of indulging in the discourse of bisecting the angle, constructing the perpendicular to the diameter of a circle or constructing the tangent to a circle. However, there is evidence of indulgence in the discourse of theorems in geometry and in implementing the sequence of the appropriate procedures. In order to execute a mathematical task successfully, the learner need to summon and simultaneously move between forms of knowledge that underpin

mathematical tasks. These forms of knowledge are conceptual and procedural knowledge. For one to gain adequate understanding of mathematics, the pre-requisite is to have a solid conceptual and procedural knowledge. The evidence presented suggests that the participants had difficult in acquiring either conceptual or procedural knowledge to execute the task. Zulnaidi and Zamri (2017) found that procedural knowledge significantly mediates between conceptual knowledge and achievement in mathematics.

Participant *S3:* The examination of the sketch in figure 5.19 shows evidence of construction with the use of a compass and ruler and therefore indulging in the discourse of bisecting the angle, the discourse of constructing the perpendicular to the diameter of a circle, the discourse of constructing the tangent to a circle, the discourse of geometry theorems for the execution of the task, and the discourse of the sequence of the appropriate procedures.

The interpretation derived from this analysis is that the execution of the task required the coalescence of the mentioned discourses into a single one. However, though participant S3 could be regarded as having recourse to the set of discourses necessary for the successful implementation of the task and made attempts to indulge in these discourses, both participants, S1 and S2, did not seem to show evidence of recourse to all the elements of the set of discourses required therefore their execution of the task was inadequately done. What is evident in the sketches of participants S1 and S2 is that they made some attempts to recall from memory the necessary discourses required to successfully execute the task; however, not all elements of the discourses could be adequately remembered. Sfard (2007) underscores the role of recall routines in generating discursive fluency in mathematical activities. For participants S1 and S2, their only recourse was to imitate what they had previously seen. This interpretation resonates with Ioannou (2017), who, in his study, found that students may sometimes have a structural understanding of the required proof but may not possess the necessary procedural understanding for implementing the proof. The study of Ioannou

(2017), therefore, underscores the influence of the discourses in the execution of mathematical tasks.

Communication gaps:

Another aspect that may be considered concerning how these discourses influence their understanding of geometry concepts is to look at the existence or non-existence of communication gaps between the participants and the educators. If there are indeed communication gaps, that would simply mean communication (thinking) is impacted. Examination of the participants' utterances revealed that instances of where the interviewer and the participants were talking about different things.

At the start of the semi-structured focus interview, participants S1, S2 and S3 all claimed that they were familiar with the question posed to execute the task. Their responses to the interviewer's question (in [46]) will help examine whether the participants S1, S2 and S3 responded to the same question.

Participant *S1*: She would draw a circle around the line, with the point P in the middle (in [56]).

Participant *S2*: Talks about drawing with the compass touching the two lines (in [65]).

Participant *S3*: Also talks about drawing with the compass touching the points, and he points at P (in [78]).

What can be inferred from the examination of these utterances is that the participants did not fully comprehend what was being asked. This may be attributed to their mathematical register. For one to participate fully in problem-solving activities, knowledge previously deployed to the mathematical register must be accessed. Learners need to have adequate experience to present mathematical ideas in a variety of ways including language and non-

language. According to Wilkinson (2018), learners who have not developed adequate command of their mathematical register do not explain their mathematical knowledge in a coherent and accurate way. While Participant S1 was talking about where she would execute the task and not about the process of constructing a circle, participants S2 and S3 talked about the instruments they used to execute the task and did not mention the process of executing the task. The interpretation drawn from this exchange is that participants S1, S2, and S3 did not fully understand that what was being asked was the process of constructing a circle that would pass through point P. This evidence suggests the existence of a discursive gap between the interviewer and the three participants. The failure in communication, in this case, would lead to an inadequate attempt at the solution of the task. Further evidence of the existence of communication gaps can be found in the examination of exchanges between the interviewer and the participants regarding their understanding of the concept of tangency. One of the participants pointed out that a tangent line does not cut the circle, but runs near to a circle. To emphasise her point, she demonstrated with her hands. This was in response to another participant who had said the same about the tangent. Examination of this exchange suggested that the participants did not understand the line of questioning. The participants did not seem to realise that the question required them to provide the definition of tangency. They did not mention the tangent as being a point. Instead, the participants talked about the lines and how these line segments related to the circle. This evidence suggests that all the participants were talking about different things at that moment. Sfard (2008) asserts that communication is not regarded as effective unless at any particular instance, all the discussants are of one accord, that is, concerning what they are talking about and with regard to whether they are using the same words. Therefore, the communication gaps influence their understanding of geometric concepts. This interpretation is corroborated by Lomibao et al. (2016), who studied the influence of mathematical communication on students' mathematics achievement and

anxiety. They found that students exposed to mathematical communication performed significantly better had higher conceptual understanding, and had reduced anxiety than their peers who were not exposed to mathematical communication.

6.3 Discussion

In this section, I present a discussion informed by the interpretation of of the findings from the previous section.

The discussion in this section was divided into two parts. The first is a discussion based on the first question that guided this study. The second is a discussion of this study's second and third questions.

6.3.1 QUESTION 1. *What discourses do the learners engage in when learning geometry?*

Concerning the first question, the reader is reminded that according to Sfard's (2008) theory of commognition, mathematical discourse has four key features. These four features are words/word use, visual mediators, routines, and narratives (endorsed). These features of mathematical discourse are closely intertwined and intrinsically associated with one another, the implication being that because of this close relationship, these features are often investigated together.

Word/word use; Word use was analysed in terms of the level of objectification when talking about the tangent notion. The interpretation of the findings was that the participants' degree of objectification was low.

Visual mediators: mediators were analysed in terms of tools of communicators. The interpretation of the findings was that though some of the participants used visual mediators adequately enough as communication tools, other participants' use of these tools was inadequate and did not possess the desired effects.

Routines: About routines, the focus was on participants' actions while talking, writing, or pointing in order to bring to bear the "when" of the routine and the "how" of the routine when operating on the tangent notion. The interpretation was that the participants' use of the *"when"* of the routine and the *'how"* of the routine was highly situated and rigid.

Narratives: Concerning endorsed narratives, the focus was on the construction and recycling of narratives in the form of definitions, axioms, or theorems. The interpretation was that some participants could construct new narratives while others constructed with the help of scaffolding.

The four features of mathematical discourse are interwoven into one consolidated discourse. This means that all the features are to be taken together as inadequate participation in one aspect affects proper participation in the other aspects of the features. Given their low participation levels in some of these features of mathematical discourse, the participants' discourse was, therefore, found to be characteristically ritualised. My claim that the participants' discourse was more ritualistic than explorative corroborates the findings of other commognitive researchers such as (Nisa et al., 2020; Mpofu & Mudaly, 2019: Mpofu & Pournara, 2018; Thoma & Nardi, 2017; Ioannou, 2017). These scholars concluded in their studies that the learners' discourse was ritualised.

6.3.2 QUESTION 2. *How do these discourses influence the way they understand geometric concepts?*

Let me begin by first stating that the commognitive perspective of a concept is that it is a word or all realizations and signifiers used in the discourse (Sfard, 2008).

As earlier mentioned in the report (p. 82), mathematical objects are abstract in nature and are therefore not tangible. Given the nature of mathematical objects, mathematicians use symbols to represent mathematical objects to enable the objects to at least become visible to our

senses (Sfard, 2008). In this regard, while talking, participants of mathematical discourse often have to engage in abstraction, the process of formulating intangible concepts.

One of the characteristics of ritual participation is to imitate the expert discursant (Sfard, 2008). The reason for engaging in imitation is that the participant has not yet attained the stage where the discourse may be individualised. In order to successfully implement a task, there are multiple routines and, therefore, discourses to attend to. The ritual participant comes across multiple discourses that must be subsumed into a single discourse. All the features of mathematical discourse play an essential role in bonding the discourses. An inability to fully use one of these features of mathematical discourse harms making communication effective. For example, in word use, ritualistic participants often transition between colloquial and mathematical word use. This interpretation is corroborated by Robert and Le Roux's (2019) findings, who found that the participants used both colloquial and mathematical words in their narratives. Another study that supports this interpretation that the different discourses that are characteristic of the features of mathematical discourse each influence a particular discourse is that of Zulu and Mudaly (2023). Their study found that language use (word use) was a factor in self-discursive activity during problem-solving and was vital for successful visualisation. The participants' lack of bonding multiple discourses into one resulted in their inability to comprehend the mathematical concepts and, therefore, failure to individualise the discourse. Conceptual comprehension is fundamental to successfully learning mathematics (Klipatrick et al., 2001). The low level of conceptual comprehension, characteristic of ritual performers, affects routines' performance and implementation. Examination of some of the utterances of the participant revealed the influence of ritualistic performance on participants' discourse.

Analysis of Participant S1's utterance [41] and Participant S2's utterance [42] is evidence of a lack of flexibility; they cannot vary procedures used to perform routine, bondedness, and

applicability of the routine. This characteristic of ritual performers can be attributed to low conceptual comprehension that affects their mathematical discourse. This revelation is corroborated by the study of Roberts and Le Roux (2019), who concluded that ritualistic performance was the course of errors while solving algebraic equations.

Agentivity:

Another aspect that influences the ritualistic performer is agentivity. In examining the utterances of the participants such as (in [101, 103, 107], and [108]) it is evident that the performance of participants S1 and S2 is process-oriented rather than product-oriented. Their performance is geared toward social approval, and these characteristic influences conceptual comprehension.

6.3.3 QUESTION 3: *How can these discourses be influenced to improve the understanding of geometry?*

There are several principles that inform teaching and learning. One of the principles is the (NTCM, 2000, p 16) Teaching Principle,

> *"Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well"*

This teaching principle underscores the importance of the teacher concerning teaching and learning mathematics. The teacher has the leading voice regarding the affairs of the classroom environment. The teacher's discursive acts in the classroom, particularly in designing learning activities that enable learners to participate in classroom discussions, can facilitate learning (Michael & O'Connor, 2015; Stein et al., 2015; Sfard et al., 1998).

However, Sfard (2008) reminds us of the who are the main stakeholders in teaching and learning: the teacher and the learner. In this regard, she suggests several vital conditions that must be implemented to facilitate teaching and learning. She goes on to say that since one of

the characteristics of the ritualistic participant is to participate in discourse with others, it is vital that these conditions are present from the onset. Once these conditions are in place, it may be possible for the ritualistic performer to move from participating in discourse for others to participating in discourse for themselves. These necessary conditions, as articulated by Sfard (2008) include, among others;

(1) Agreement on the leading discourse. The implication is that from the onset, participants should have a sense of the power relations at play and the expert discourse in the classroom.
(2) Agreement on the interlocutors' rules. Every discourse has a unique set of rules that govern a particular discourse. The participants in the discourse must, therefore, be familiar with the rules of the discourse at hand.
(3) Agreement on the necessary course of the discourse. A change in discourse implies that new narratives are recycled from old ones, and agreement is therefore vital on what direction the new discourse should take.
(4) Realization that commognitive conflict is a necessary part of learning. Commognitive conflict may occur at both the object-level or the meta-level. As learning results in a change in discourse, this change does not happen simultaneously in participants. Just like in other human collective activities, change is not uniform from one individual to another. Because of the different paces at which change happens, one individual may still be acting using old rules while another has moved on and is acting on new rules, and this may result in a communication breakdown. When this happens, it is necessary to resolve this, allow the different interlocutors to be governed by the new set of rules, and make communication effective again.

Once these conditions are put in place and attempts are made at every opportunity to resolve any commognitive conflict that may arise, it becomes possible for the teacher to influence the

participants' discourse. A ritual performer has the following characteristics: (a) knows and follows only one procedure, (b) routine has not become bonded, (c) routine is evoked only in a situation similar to the previous one, (d) performs for others, (e) relies on concrete objects, and (f) can not substantiate the validity of their routines.

One of the goals of learning mathematics is to enable participants in mathematical discourse to become explorers (Sfrad, 2008). However, she also stresses that the cause of action for explorations is eventually by way of rituals.

The above statement implies that rituals are part and parcel of the learning process and should not be overlooked. Sfard (2008) further suggests that one milestone on the path towards de-ritualization is considering perceptual aspects of commognition such as routines. In this regard, the educator should bear in mind the following;

(1) New mathematical objects as pre-requisite for new explorations - New mathematical objects should be gradually introduced by building on previous knowledge. The spiral approach method is best suited in this regard. The spiral approach is premised on the constructivist epistemological belief of multiple realities and that learning is a process characterised by the learner constructing his or her own knowledge based on what is known (Martin, 2000). This approach also utilises Vygotsky's (1978) concepts of scaffolding, the *Zone of Proximal Development,* and discovery learning by Bruner (1968). According to Vygotsky (1978) while referring to the ZPD, points out that in the course of learning, there a number of tasks that a learner may not be able to fully accomplish on his/her own. Such tasks may require the assistance of a mentor or more experienced peer.

(2) Thoughtful imitation as the ritualised beginnings - The implication here is that thoughtful imitation or mimicking is part of established forms of human development. However, Vygotsky (1987) cautions by saying that "*The child can only imitate only*

what lies within the zone of his intellectual potential" (p. 209). A novice participant in the discourse may not have recourse to some rules that govern a particular discourse. For a participant who finds himself or herself in this position, it is also highly likely that in enacting a routine, the novice performer is confronted with challenges emanating from applying the when and how of the routine. Another challenge to the novice performer may be poised by the dual nature of mathematical objects (Heyd-Metzuuanim, 2015). Therefore, mimicking becomes the only vehicle to participate in the new discourse. The driving force comes from the human desire to individualise the discourse on becoming a full participant. (Bakhtin,1999, p. 130) says this about mimicking,

> "*The unique speech experience of each individual is shaped and developed continuous and constant interaction with others' individual utterance. This experience can be characterized to some degree as the process of assimilation – more or less creative for others' words (and not the words of a language) …. These words of others carry with them their own expression, their own evaluative tone, which we assimilate, rework and re-accentuate"*

(3) Continuity of discourse - Sfard (2008) lays emphasis on the continuity of discourse as the foundation stone on which discourse development is premised. Active participation in the discourse may result in the expansion of the discourse. This expansion may be endogenous from the outside or exogenous from the inside. Lavie and Sfard (2019) further point out that "in the process of learning, constant changes occur not only in learners' procedures – in the way they do things – but also in their interpretations of task situations" (Lavie & Sfard, 2019 p. 458). Like in any other collective human activity, development tends to be slow without continuity. But in

situations of constant active participation, development is increased. Through continuity, one becomes an expert in a particular given discourse.

The process of de-ritualization involves attending to the characteristics of explorative participation outlined by Lavie and Sfard (2019). These characteristics were presented in earlier chapters of the study. However, I present them again for the reader who might just want to re-read this section. These characteristics of de-ritualization are;

development in the domains of flexibility, bondedness, applicability, performer agentivity, objectification, and substantiability.

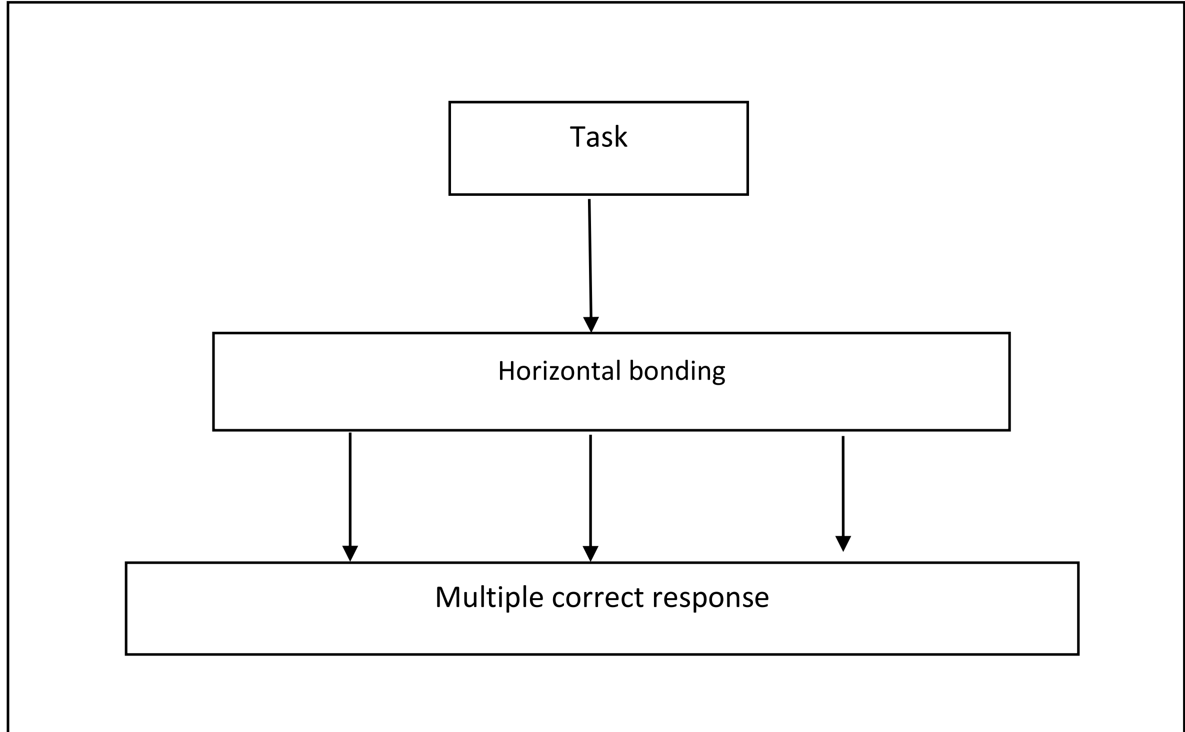
The choice of tasks and teachers moves.

In order to spur learners' thinking and facilitate the process of de-ritualization, appropriate learning tasks must be utilised in the classroom. Anthony and Walshaw (2009) assert that successful teachers are those that recognise the critical place of appropriate tasks as well as examples to develop sense making in mathematics.

The above statement by Anthony and Walshaw has implications in the role of tasks in teaching and learning. Worthwhile tasks add value to learning and generate multiple opportunities and a variety of opportunities to learn new mathematics. For example, a student can practice multiplication and addition when learning about perimeter and area. When students can engage with tasks, they can gain insight and develop mathematical ideas (Hodge et al.; 2007).

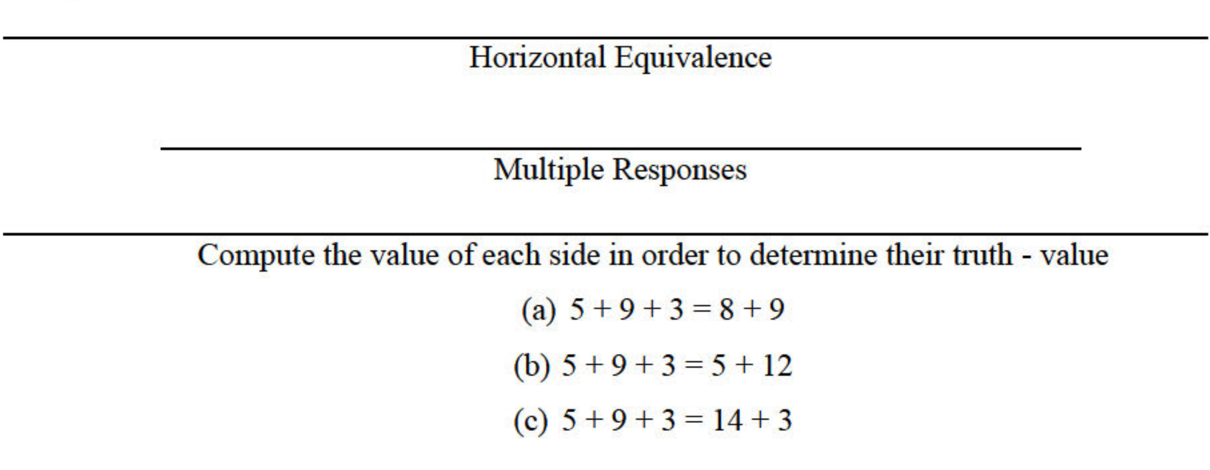
Anthony and Walshaw (2009) provide attributes of tasks that have the potential to propel students into constructive struggle in the mathematics classroom. These include, among others, problem-solving tasks, open-ended tasks, discovery learning tasks and investigative tasks.

Figure 6.1: Multiple responses in task situation. Source: Researcher's own initiative.



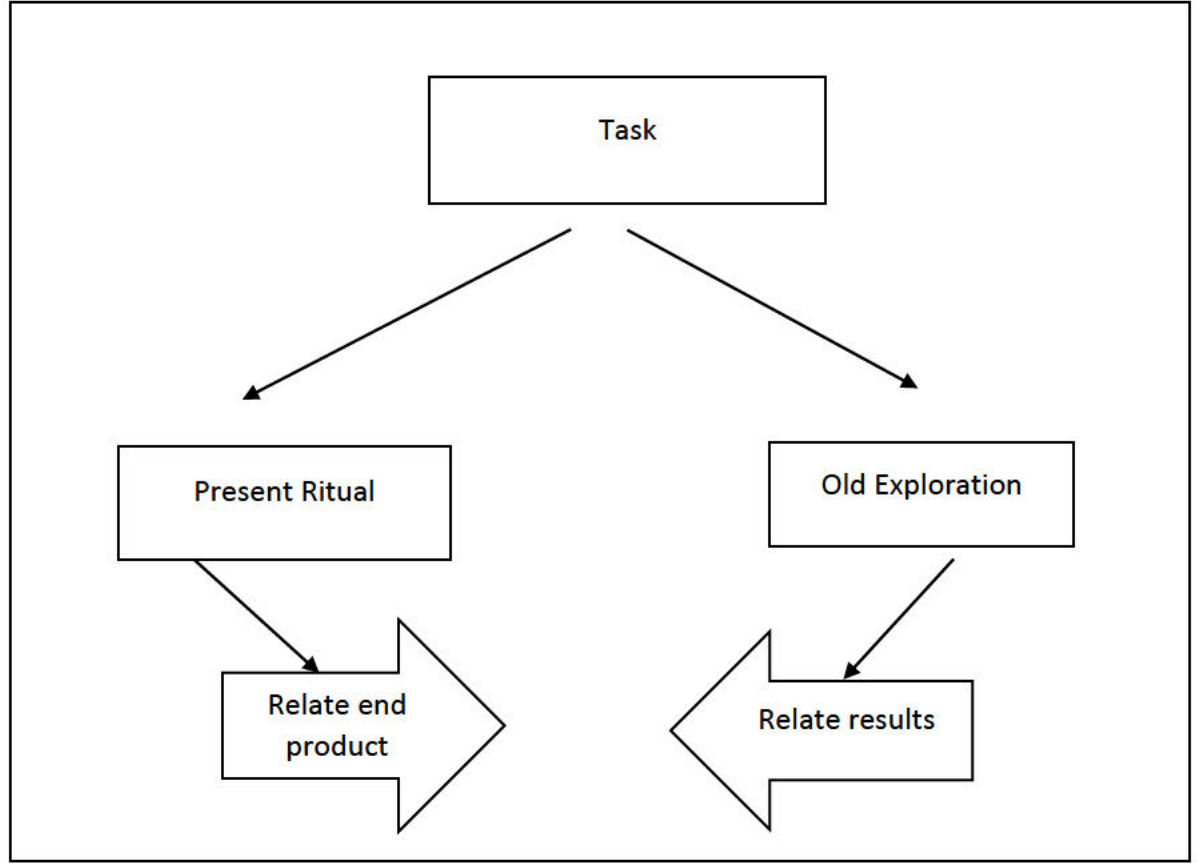
Lavie and Sfard (2019) of insight to the educator regarding particular moves that may spur de-ritualization in the classroom. They suggest that to attain de-ritualization, the educator should provide the learner with support in suitable task situations with opportunities to practice horizontal equivalence (left–proper equivalence). They contend that horizontal bonding is realised when a learner has opportunities to practice horizontal equivalence. Horizontal bonding is achieved when learners gain insight and realise that in a given task situation, there are opportunities for multiple responses. Insight about multiple responses in a task situation enhances reflexivity, a feature of explorations. Table 6 .1 illustrates horizontal equivalence.

Table 6.1: Horizontal equivalence (Adapted from Kieran & Martinez-Hernadez, 2022, p. 1218).



Horizontal Equivalence
Multiple Responses
Compute the value of each side in order to determine their truth - value
(a) 5 + 9 + 3 = 8 + 9
(b) 5 + 9 + 3 = 5 + 12
(c) 5 + 9 + 3 = 14 + 3

Figure 6.2: Ritualised verses Explorative Opportunities. Source: Researcher's own initiative derived from the review of the literature.



Another suggestion from Sfard and Lavie (2019) regarding de-ritualization concerning the

teacher's moves are the recognition of providing opportunities that focus on the characteristics of the ritual, the performer and the explorative performer. A characteristic of a ritualistic performer is to focus on the end product, while the exploratory performer focuses on the results. They point out that task situations should have opportunities to relate end products to results.

Task situations that have opportunities to relate the end products of one situation to the results of another but more complex situations have the potential to influence the ritualistic performer. Sfard and Lavie (2019) contend that the procedures that the learner employs ritualistically to produce end products should be compared to the results of those situations that have already been practiced as explorations. They point out that by providing these opportunities in task situations, the ritualistic performer can be introduced to and practice vertical equivalence. In vertical equivalence, the routine of horizontal feeds into the routine of vertical equivalence and creates new endorsed narratives. According to Lavie and Sfard (2019), a clear sign of ritualisation is when routines do not contain evidence of horizontal or vertical equivalence. Table 5.8 illustrates vertical equivalence.

Table 6.2: Vertical equivalence (Adapted from Kieran & Martinez-Hernadez, 2022, p. 1218).

Vertical Equivalence
Rewrite in different ways the equalities so as to show their truth - value Numeric $765 + 237 = 967 + 35$ $700 + 65 + 200 + 37 = 700 + 265 + 2 + 35$ $700 + 65 + 200 + 2 + 35 = 700 + 65 + 200 + 2 + 35$ Algebraic $(x - 3)(4x - 3) =$

$$4x^2 - 3x - 12x + 9 = x^2 - 3x - 3x + 9 + 3x^2 - 9x$$
$$4x^2 - 15x + 9 = 4x^2 - 15x + 9$$

Guided and Recreated Opportunities to Learn

Nachliel and Tabach (2021) concurred with Lavie and Sfard's (2019) mechanisms of spurring de-ritualization in task situations. They elected to call their strategy a hybrid between rituals and exploration. They premised their strategy on Sfard's (2008) assertion that within a diverse classroom, some learners may perform at the level of rituals in one task situation and at the level of explorations in another. Also, some learners move between rituals and explorations within the same given task situation. Lessons may flow back and forth between ritualisation and deritualisation (Warshauer, 2015).

In Nachiel and Tabach's (2019) strategy, two tasks are used: one for guided opportunities to learn (OTLs) and the other for created opportunities to learn (OTLs). The teachers' focus is to influence de-ritualization by targeting specific features of rituals and explorations. Regarding rituals, the target is on the learner’s agentivity, while regarding explorations, the target is on substantiation.

Agentivity is chosen as a target as it is a crucial feature of the ritualistic performer. A ritualistic performer participates in the discourse for others (social). Nachliel and Tabach (2022) and Sfard (2020) point out that ritualistic performers do not exert enough effort to individualise the discourse. The performer is content with just participating to socialise with others. On the other hand, the exploratory performer is focused on the results and participates in the discourse for himself or herself. Therefore, focusing on developing agentivity may move the learner to become independent in decision-making (Heyd-Metzuyanin et al., 2022; Lavie et al., 2019).

Substantiation is chosen as a target as it is a crucial routine of mathematical discourse. The routine of substantiating and endorsing narratives is critical to mathematical discourse. The process of substantiating often involves all the other properties of mathematical discourse. The focal point of the four attributes of mathematical is in routines (Mpofu and Pournara, 2018).

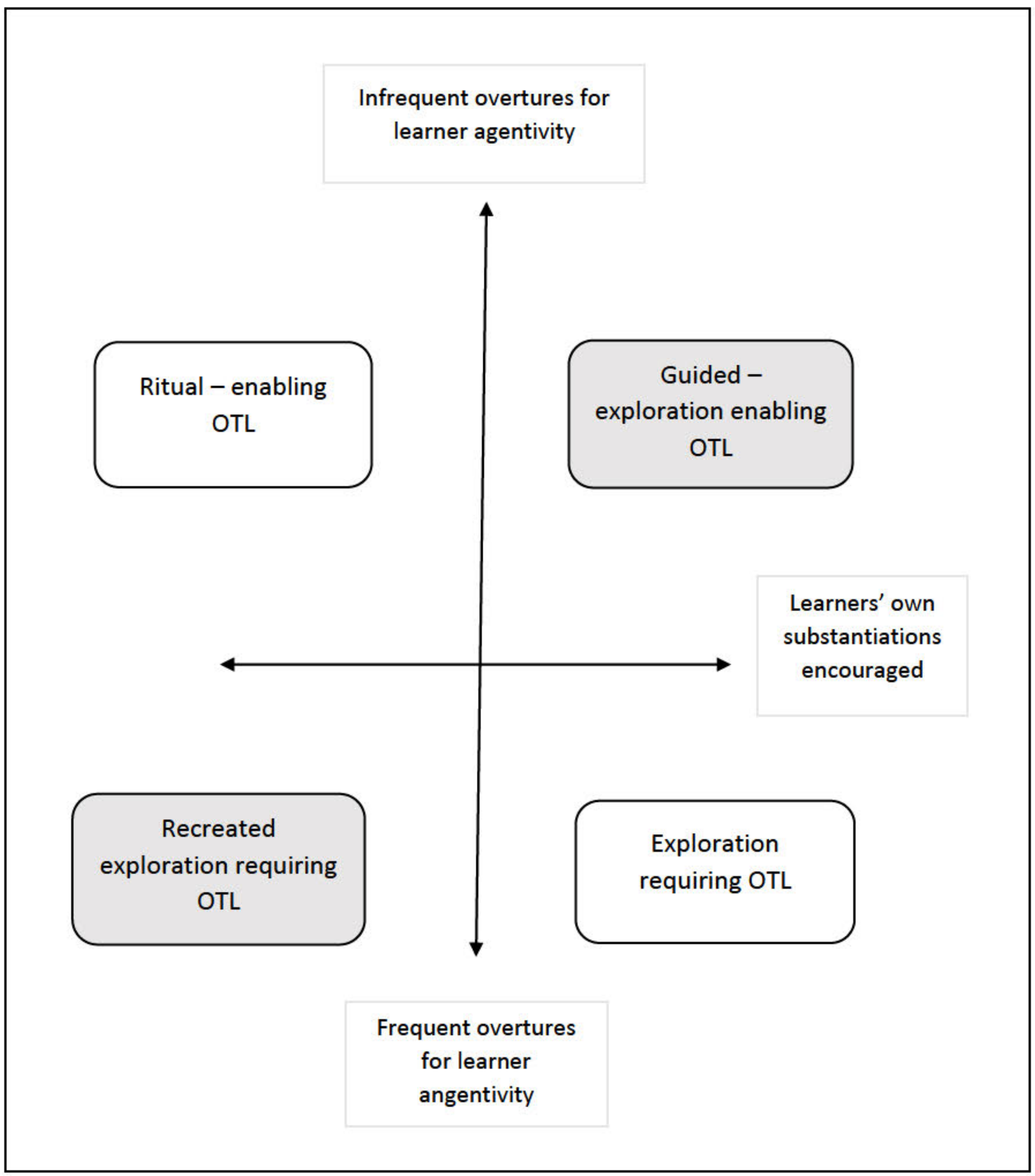
In Nachiel and Tabach's (2019) strategy, the guided OTLs are characterised by inviting learners to real opportunities to engage in substantiations but with less independence. Recreated OTLs are characterised by inviting learners to opportunities of real independence but with reduced opportunities to develop their own substantiations.

(1) Learner agentivity becomes central to quadrant 2.

(2) Learner substantiation becomes central to quadrant 1.

The figure 6.3 is given conceptualises the whole of the OTLs.

Fig 6.3: Dimensions in the plane intersecting the Opportunities To Learn space with ritual enabling Opportunities To Learn and exploration requiring Opportunities To Learn at opposite extremes (Adapted from Christiansen et, al., 2023, p. 56)

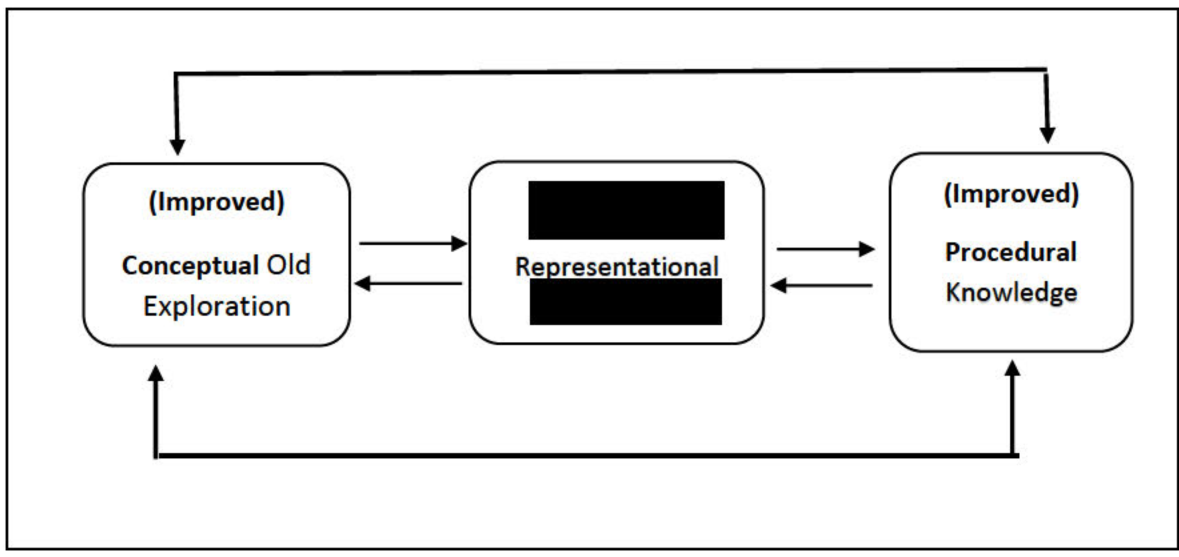


6.3.4 Implications of these characteristics

Understanding geometrical shapes and relations requires learners to utilize the two forms of knowledge, that is, conceptual knowledge and procedural knowledge to solve geometry problems. Rittle-Johnson and Siegler (2001) outline that procedural knowledge concerns with a number of related actions enacted during problem-solving while conceptual knowledge relates to relational understanding of the fundamental principles underpinning a particular domain of knowledge units.

They conducted a study that examined the development of conceptual and procedural knowledge of decimal fractions in learners. They used the iterate model to assess the iterate development of these two types of knowledge. This experimental study had two parts; the first examined learner's individual difference in prior knowledge concerning decimal fractions. The second experiment involved interviews, and learners were provided with manipulatives to support forming of the correct representation. The results from these two experiments were then analysed. The findings revealed that using correct problem representation in the form of manipulatives positively impacted the acquisition of conceptual knowledge. Another finding was that the gains in conceptual knowledge resulted in improvement of procedural knowledge.

Figure 6.4: Iterative model for the development of conceptual and procedural knowledge (Adapted from Rittle-Johnson & Siegler, 2001, p. 347)



The relationships between conceptual and procedural knowledge play an important role in the learning and teaching activity. Kilpatrick (2001) and colleagues have this to say about these relationships. When students are unable to understand or associate concepts that inform procedures, they are generally prone to procedures that culminate in mistakes (Kilpatrick 2001).

6.3.5 How the discourses influence the learning of geometry?

As described earlier in this discussion, a ritualistic performer does not participate fully in mathematical discourse. This stance on its own is enough to imply that the ritualistic performer is content in taking on the role of an outsider or observer in the discourse of mathematics and argues that the performer does not take recourse to extend and explore the discourse any further.

Sfard's (2008) in her theory, gives five properties of commognition that influence thinking. These properties of commognition are:

(1) Reasoning - as the art of systematic derivation of utterances from other utterances.

(2) Abstracting - the ability to create concepts that do not refer to tangible, concrete objects.

(3) Objectifying - the process in which a noun begins to be used as if it signified an extra discursive, self-sustained object, independent of human agency.

(4) Subjectifying - the activity of objectifying that occurs when the discursive focus shifts from actions and their objects to the performers of the actions.

(5) Consciousness - an awareness of one's own self and the surroundings.

Sfard (2008, pp. 110-114).

By considering the characteristics of the ritualistic performer, that is, by not being able to be a full participant in mathematical discourse, the degree of commognition development in the ritualistic performer is low due to the absence or slow pace of successive addition of ever new discursive layers.

6.3.6 How the discourses can be influenced to enhance learning of geometry?

The use of visuals in geometry enhances correct representation of geometry shapes. Learners' geometrical discourse can be influenced by instruction that attends to the role visualization. Arcavi (2003) defines visualization as *the ability, process and product of creation, interpretation, use of reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools* (p. 217). Mudaly and Reddy (2015) investigated the role of visualisation in solving Euclidean geometry problems. They pinned their focus on whether visualisation was necessary for proof and why. They also investigated whether learners could use diagrams to solve problems when given guidance on using diagrams. During the interviews with participants, learners revealed that diagrams helped them to make sense of the question. Therefore, they valued the use of diagrams when solving geometric problems. These findings imply that diagrams had a positive impact and allowed the learners to change

or cultivate new knowledge. Based on their findings, Mudaly and Reddy (2015) therefore concluded that visual representation supports and communicates mathematical ideas, that diagrams contain conceptual information and facilitate a better understanding of the mathematical problem.

Another important aspect of visualisation worth considering is in its role regarding multiple intelligences. According to Gardner (2011), humans' cognitive ability is pluralistic by nature, and these strong different types of intelligence can be developed. Gardner says that for learning instruction to be regarded as practical, learners should be given opportunities to learn in ways that allow them to benefit from their particular intelligence strengths. In this regard, visualisation can potentially address multiple intelligences in diverse classrooms. Visuals accompanied by text, colour or numbers have the potential to address such intelligences as verbal-linguistic, spatial, and logical-mathematical intelligences, just to name a few.

6.3.7 Some instructional moves that may enhance objectification of the discourse

There are several instructional moves that can enhance objectification. In this section, 1 shall only discuss two of such moves.

6.3.7.1 Definitions

In Euclidean geometry, learners must know the definitions of the various geometric shapes. In proving problems, they must state the definition and substantiate their discursive actions. To enhance the understanding of the relation of definitions to geometric shapes, learners should be given opportunities to engage in metalevel learning during instruction.

Gucler (2015) conducted a study that focused on making implicit metalevel rules of the discourse on function explicit. He used a strategy that enabled university students to reflect on their previous discourse during classroom discussions. Gucler's experimental method required students to reflect on particular metarules on function concepts. The experiment

focused on metarules of endorsed narratives concerning definitions and their discursive actions. The experiment had four stages. The initial stage involved students completing a survey in which they were required to describe in their own words what a function is. The second and third stages were classroom activities where they defined a function using one word, and later on in the fourth stage, they defined a function using as many words as possible. During the discussion stage of the experiment, the students were given definitions taken from calculus textbooks. The student reflected on their formulated definitions and compared them to the definitions from the calculus textbooks. The finding revealed changes in the definitions provided during the interviews from the ones provided during the survey. Improvements concerning objectification were noticeably present in their new discourse. These findings suggest the potential affordance of learning instructions that target the metarules of a particular discourse. The teaching approach used by Gucler (2015) can influence the learners' discourse to move from ritualised discourse towards exploratory discourse.

6.3.7.2 Questioning techniques

The teacher's discursive moves and actions that may influence the learners' discourse from ritual to explorative include paying attention to the questioning techniques. Questions that can facilitate and enable the learner to expand his or her discourse are regarded as vital components in the development of explorations. Questioning based on the learners' geometric developmental level is effective in teaching and learning. In this regard, Krathwohl (2002) revised Bloom's Taxonomy, offering six different cognitive levels of questioning. These include knowledge, comprehension, application, analysis, synthesis, and evaluation. The first three are low-order questions, and the last three are high-order types. Examples of questions include; *"How can you use this theorem to get to the next step?"* This type of question calls for the learner to use the given principle and is at the comprehension level. The application

level could be *“What is the next statement?”*. Here, the learner has to summon the deductive principle and apply it to the given situation. Asking high-order questions is particularly helpful in extending the learner's thinking capabilities. Studies such as those of Gall (1970) and Winne (1979) have shown the positive effect of high-order questions on learner achievement.

Other moves to incorporate with questioning may include probes, re-phrasing of the question and adjusted response time given to learners. Probes are particularly helpful in eliciting higher cognitive level skills in the learners’ thinking processes. Examples of probing questions include words like *“What?”, “How?”* and *“Why?”*. Such oral questions are helpful in that they give learners opportunities to clarify and extend their thinking and perhaps modify the new utterance to include words or terms that might have been absent in their previous utterance. According to Sfard (2008), such moves by the learner would be regarded as attempts at flexibility and corrigibility, which are exploration features.

6.4 Conclusion

This chapter was about the interpretation and discussion of the findings. The findings were those stated in the previous chapter.

In the first segment of the chapter, I discussed some of the findings of the study, and I provided evidence to support my claim that the participants’ discourse was closer to the ritual than the exploration form. I cited similar studies that corroborate my interpretation of the participants’ discourse. I discussed how indulging in rituals may influence participants' comprehension of mathematical concepts and cited similar studies to corroborate my assertion.

The interpretation was based on the researcher’s personal construct and, therefore, subjective in nature, as noted by (Sfard, 2008). The findings were interpreted around the three

mathematical discourse questions commonly used in interpreting mathematical discourse. These questions are:

1. What is the degree of objectification in the participant's use of words?
2. What are the sources of participant's mathematical narratives and their subsequent endorsement?
3. What are the goals of participation in the mathematical activity?

The interpretation was organised concerning the research questions that guided the study.

Claim 1. Based on the cardinal questions used in interpreting mathematical discourse, the claim was made that the degree of objectification was high; that sources of the mathematical narratives and endorsement were from outside the discourse and not from inside the discourse, and that the focus of learners' participation was on the process and not on the end product of their performance. Therefore, their discourse was interpreted as being ritualistic.

Claim 2. Participants' participation was for others and not for themselves; their participation was yet to be individualised, and their only recourse for participation was to imitate others, and these factors negatively influenced their discourse.

Claim 3. That their ritual participation can become the basis of the process of deritualisation.

Sfard (2008) points out that the most appropriate and viable method to grow a new discourse is through measured transformation of the previous discourse in which the child was immersed.

The third claim was therefore used to underpin the discussion on influencing the participants' discourse and facilitating their understanding of mathematical concepts. Through the review of the literature, some instructional moves were identified as potential candidates to spur the de-ritualization process. Among the pedagogical moves suggested as

effective by other researchers were employing higher-order questioning techniques (Gall, 1970; Winne, 1979) and teaching definitions (Gucler, 2015), just to mention a few.

The next chapter (7) is the conclusion and shall serve as the last chapter of the thesis.

CHAPTER 7

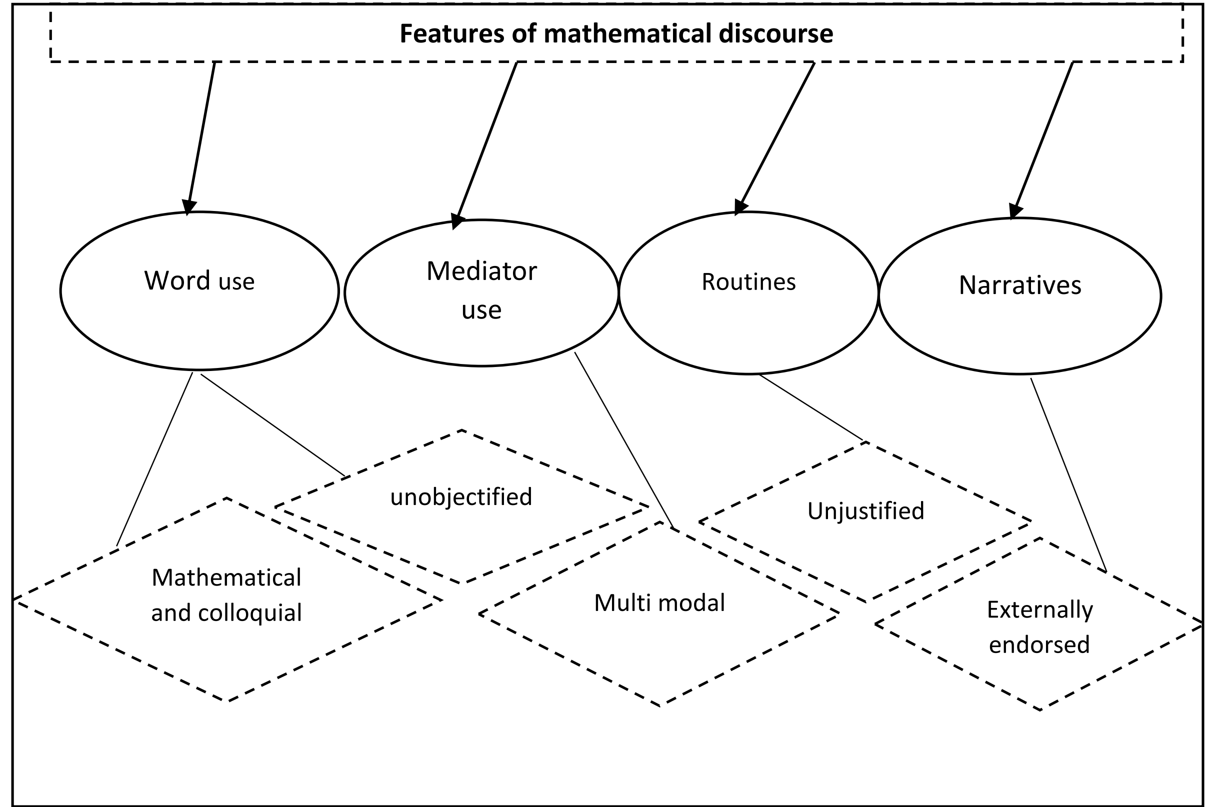
CONCLUSION

7.1 Introduction

This chapter presents the conclusion of this study, which examined the discourses of the learners when participating in geometric discourse in the classroom. The chapter opens with an outline and summary of the findings. These are presented under the research questions that guided the study. The findings unearthed pertinent questions that point to future research on the role of rituals and the resultant communication gaps between participants of mathematical discourse. The chapter offers some practice recommendations. In addition, a discussion on the summary of the key findings and limitations of the study is presented. Lastly, the chapter pens off with some few remarks to conclude the report.

7.2 Summary of some of the main findings

Figure 7.1: Researcher's summary of the findings. Source: Researcher's own initiative



Word use: Mathematical discourse is characterised by specific purpose word use that the participants employ when discussing mathematical objects. These keywords have specific meanings. The study found that the participants moved between mathematical use of words and colloquial use, in their discourse.

Visual mediators: Participants in mathematical discourse use visual mediators as perceptual tools to aid their communication when discussing mathematical objects. There are different modes of visual mediators. The visual mediators could be in symbols, diagrams, gestures or scanned with their eyes. The study found that the participants used different modes of visual mediators in their discourse.

Routines: Participants in mathematical discourse perform repetitive acts that regulate their actions when interacting with mathematical objects. These routines are necessary for the successful implementation of the task. The study found that the participants did not adequately implement the *when* and *how* of the routine in their performance. They also failed to justify their actions.

Narratives: Participants in mathematical discourse construct narratives when they talk about mathematical objects. These narratives may or may not be endorsed by the community of mathematicians. Such narratives are constructed principally from the properties of the mathematical objects that participants talk about. The study found that the participants' narratives were not based on the properties of mathematical objects.

The findings that I have illustrated above here are summarised in Figure 7.1

This study was guided by the following questions:

1. What discourses do learners engage in when learning geometry?
2. How do these discourses influence the way they understand geometric concepts?

3. How can these discourses be influenced to improve understanding of these geometric concepts?

The following were the objectives that were covered by the main questions that the study sought. The stated objectives served as milestones and as the means to measure whether the outcomes' endpoints had been achieved (Dhir & Gupta, 2021). The objectives were:

1. To explore the types of discourse that learners engage in when learning geometry.
2. To explore how these discourses may influence how they understand geometric concepts.
3. To explore how these discourses may be influenced to improve learners' understanding of geometric concepts.

7.2.1 What discourses do learners engage in when learning geometry?

The first question that guided this study was to ascertain the types of discourses learners engage in when learning geometry. These types of discourses were measured against the following three cardinal questions of examining mathematical discourse,

1. The degree of objectification in the participants’ use of words?

2. What are the sources of their mathematical narratives and subsequent endorsement?

3. The goal/s of the activity?

Mathematical objects are human socially constructed objects. These objects are intangible and not concrete objects. They are symbolic objects and do not exist in the world. Examination of the participants’ utterance of the study revealed low a degree of objectification in the participants’ mathematical discourse ([136], [155]). Other revelations were that the derivation of their mathematical narratives and their subsequent endorsement were from outside the discourse and not from within ([90], [124]). They did not recycle old

narratives to form new narratives. Their endorsements were not from recall but were mainly from scaffolding by the expert discursant ([106], [107]). The third revelation was that the goal/s of the activity were socially oriented and not product-oriented ([10], [107]). The claim was made that their discourse was ritualistic. This claim is corroborated by the findings of Mpofu and Mudaly (2019) and Nisa et al. (2020), who found their participant's discourse ritualistic.

7.2.2 How do these discourses influence the way they understand geometric concepts?

The second question that guided the study was to ascertain how these discourses influence learners' understanding of geometric concepts. Further examination of the participants' discourse revealed their understanding depended on the discourse type. Ritual performers are prone to imitating others as they participate in mathematical discourse. They do not yet individualise their participation and often perform for others. The reason is that they are more focused on the processes and procedures in their performance rather than on the end product. These characteristics result in a lack of understanding of geometric concepts. The claim was made that ritualistic performance influences the understanding of geometric concepts. This claim is corroborated by the findings of Ioannou (2017), who found that the participants did not possess the necessary procedural knowledge to implement proofs due to their ritualistic discourse. Thoma and Nardi (2017) also found that the participant's cause of errors at the final examinations was ritualisation.

7.2.3 How can these discourses be influenced to improve the understanding of geometry?

The third question was how to enhance these discourses to improve the understanding of geometry. The study revealed many teaching and learning strategies that can be effective to spur the de-ritualization process. The study affirmed that rituals are inevitable in learning

mathematics and should be used as a stepping stone to de-ritualization. One of the strategies deemed to be effective in activating de-ritualization is teaching through differentiation (Magableh & Abdullah, 2019; Rock et al., 2008; Schleicher, 2019).

7.3 Recommendations

This study was born out of the need to improve the teaching and learning of mathematics in general and in geometry, in particular. See Table 1 for a comparison of mathematics performance against other subjects at grade 12 level. The study explored the types of discourses learners engage with when learning geometry. Given the findings of this study, some recommendations are proposed for consideration.

7.3.1 Recommendation pertaining to the Curriculum Development at Department of Basic Education.

This study was focused on identifying the type of discourses that learners engage in when participating in geometric discourse. Among the findings was that the participants failed to communicate effectively about the sources of their narratives. This finding highlights the role of narratives in understanding mathematical concepts. This study urges the Department of Basic Education to elevate communication as central to our approach to teaching and learning mathematics. Therefore, this study recommends that students from grade 8 onward be required to state their reasoning for the various steps they enact while working out solutions in all mathematics topics, not just geometry.

7.3.2 Recommendation to Pre-service and In-service teacher education training institutions.

This study and other researchers highlighted the role of rituals as legitimate in learning and transitioning to explorations. This study recommends that teacher training institutions design

a teachers' guide booklet and include suitable tasks for engaging in significant learning as well as the goals of mathematical activity.

7.3.3 Recommendation pertaining to teaching and learning.

This study revealed that participation in geometric discourse was ritualistic. These findings are corroborated by other researchers. To transition from rituals to explorations, this study recommends to teachers of geometry that in their teaching and learning of geometry, they conceptualise communication in the classroom as follows:

1. Shift the focus to what needs to change in the learner's discourse. This implies an awareness of the existence of communication gaps in the classroom.
2. Shift the focus to how the learners and the teacher navigate towards this change. This implies investigating the sources of the communication gaps.
3. Direct attention to whether the envisaged change has occurred. This implies focusing on what the students did (not on the errors) as opportunities for new learning and further applying suitable teaching strategies incorporating object and meta-level learning to extend the learning to close or reduce the communicational gaps.

7.3.4 Recommendation for future research.

This study explored the discourses the learners engage in when learning geometry. The study found that the learners engage in ritualistic discourses.

The study ascertained that as ritualistic participants, their focus in learning geometry is on processes and not on the end product. The study established that such focus on processes, not the end product, negatively influenced their understanding of geometric concepts.

As rituals are inevitable in mathematics learning, the study suggested that before introducing pedagogical methods on how the ritualistic discourses may be influenced to improve the learner's understanding of geometric concepts, discursive gaps that perpetuate ritual participation should be reduced, if not closed. In light of this, further research is recommended to focus on identifying the discursive gaps in mathematics classrooms and further explore the causes of these discursive gaps.

7.4 Limitations of the study

This study was an interpretive qualitative study that employed phenomenological and case study approaches. And like any other scientific inquiry, the methods employed have potentialities and limitations.

The study had some limitations. Initially, the sample size for the study was 5. Only 4 participants arrived and could write the mathematical task (document), while on another day, only 3 participants were available for the focus interview. Since the participants were learners at the school, their availability was problematic because, during the pandemic (Cv-19), learner attendance at school was poor since some parents chose to keep their children at home. The participants were drawn from one of the institutions in KwaZulu Natal. Although English is the medium of instruction, for some participants, English is their second language. During the focus interview session, getting all the participants fully involved was difficult. Some participants could not articulate fully their feelings and thoughts related to the experience being investigated. Another limitation is that the researcher (and interviewer) was also the participants’ classroom educator; this is an advantage and, at the same time, a limitation in that some participants may have been afraid to express themselves freely.

Researchers could obtain different results using different contexts, sample sizes, and other instruments besides pencil and paper tasks and focus group interviews. Also, this study used

one approach (qualitative); different researchers may use other approaches or a combination of approaches to mitigate the limitations of using just one approach. However, the study's goal was not to generalise the results to other contexts.

7.5 Summary of the study

7.5.1 Background of the study

Mathematics is one of the important subjects taught in school (primary and secondary levels) and is a compulsory subject in most of the school curricula (CAPS, 2013; ECZ,2016).

Steen (1997) says, *"As information becomes ever more quantitative and as society relies increasingly on computers and the data they produce, an innumerate citizen today is as vulnerable as the illiterate peasant of Gutenberg's time* (p. xv).

Teaching and learning mathematics in South Africa has been made compulsory from grade 1 to grade 9. In the FET phase, mathematics is split into core mathematics and mathematical literacy. Learners must take up one of the two streams of mathematics once they enter the FET phase.

Knowledge of mathematics is needed in most careers. Mathematical skills are needed in the job market and in many spheres of daily life. One such skill that can be gained from learning mathematics is problem-solving. The skills gained from problem-solving include approaching problems from a mathematical perspective to find multiple solutions to real-life situations. In this regard, mathematics is essential in developing this much-needed human proficiency. Mathematics develops conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (van de Walle, 2013).

However, many students find mathematics difficult (Angateeah, 2017; Sholihah & Afriansyah, 2018). They, therefore, perform poorly in mathematics (Kyaruzi, 2023; Mazana

et al., 2020; Graham et al., 2021). Students perform poorly in mathematics compared to other school subjects. Table 1, presented in chapter one, compares performance in gateway subjects between 2019 and 2021. Table 1 confirms that learners consistently performed poorly during this period.

Geometry is one of the main components of the mathematics syllabus as it is recognised as an essential branch. In South Africa's grade 12 final examination, geometry (Analytical and Euclidean) accounts for about 60% of 150 marks in paper one. (DBE, 2011).

In geometry, students learn about shapes, their properties and their interpretation (Bishop (1983). The geometric content that is taught in schools comprises of Shapes and properties (in two and three dimensions), Transformations (translations, reflections, rotations, enlargements/reductions), Locations (coordinate geometry) and Visualisation (recognition of shapes from different viewpoints), (van de Walle et al., 2013).

Geometry has the ability to enhance spatial reasoning. According to Sarama & Clements (2009), understanding shapes and the relationships between shapes is critical to mathematical study, just like numbers. Spatial reasoning is an essential skill for the 21st century. Spatial reasoning is valued in Science, Technology, Engineering, and Mathematics (STEM) subjects. Spatial reasoning has a role in developing high-order thinking and is linked to student performance in STEM subjects (Lean & Clements, 1981; Lowrie & Diezmann, 2007).

Despite the importance attached to geometry and its practical relevance, students find geometry concepts challenging to learn, while teachers often find geometry difficult to teach (Wu & Ma, 2006; Clements, 2003). Students find geometry difficult, including inabilities concerning abstract thinking, a failure to analyse properties of geometric shapes (Wiska et al., 2020), and geometric thinking being at lower levels than expected (Chew, 2009).

To answer these concerns, the need to improve the teaching and learning of geometry is paramount. This study was born out of the need to improve the teaching and learning of geometry.

7.5.2 Theoretical framework

The theoretical framework underpinning this study is Sfard's (2008) theory of commognition.

> According to Sfard, the phenomenon we call thinking, and the phenomenon we regard as communication emanate from the same process. For Sfard, "*Thinking is an individualised version of (interpersonal) communicating"* (Sfard, 2008, p. 81)

Sfard (2008) adds that in order to communicate, participants in this collective human activity, perform an action in response to a previous action of another participant.

Therefore, in recognition of the two cognitive processes and to symbolise their unity, Sfard (2008) determined to amalgamate the nouns "communication" and "cognition" into the term ''commognition''. Thus, the theory is called commognition.

7.5.2.1 Mathematics as a form of discourse

Discourses are sets of tools to construct narratives about specific categories of objects. In the case of school, there is the discourse of Biology, Physics, and so on. Mathematics also focuses on abstract mathematical objects, such as geometrical shapes, numbers, and sets. However, mathematical discourse is distinct (different) from these other discourses. Mathematics as a type of communication has four unique features. These are:
(1) Specific purpose keywords with their uses- For example, triangle, rectangle, differentiation and, integration. (2) Set of visual mediators of tangible entitles which participants of mathematical discourse utilise to communicate what they are talking about. Examples are drawings, graphs, symbols, labels and so on. (3) Set of routines in the form of

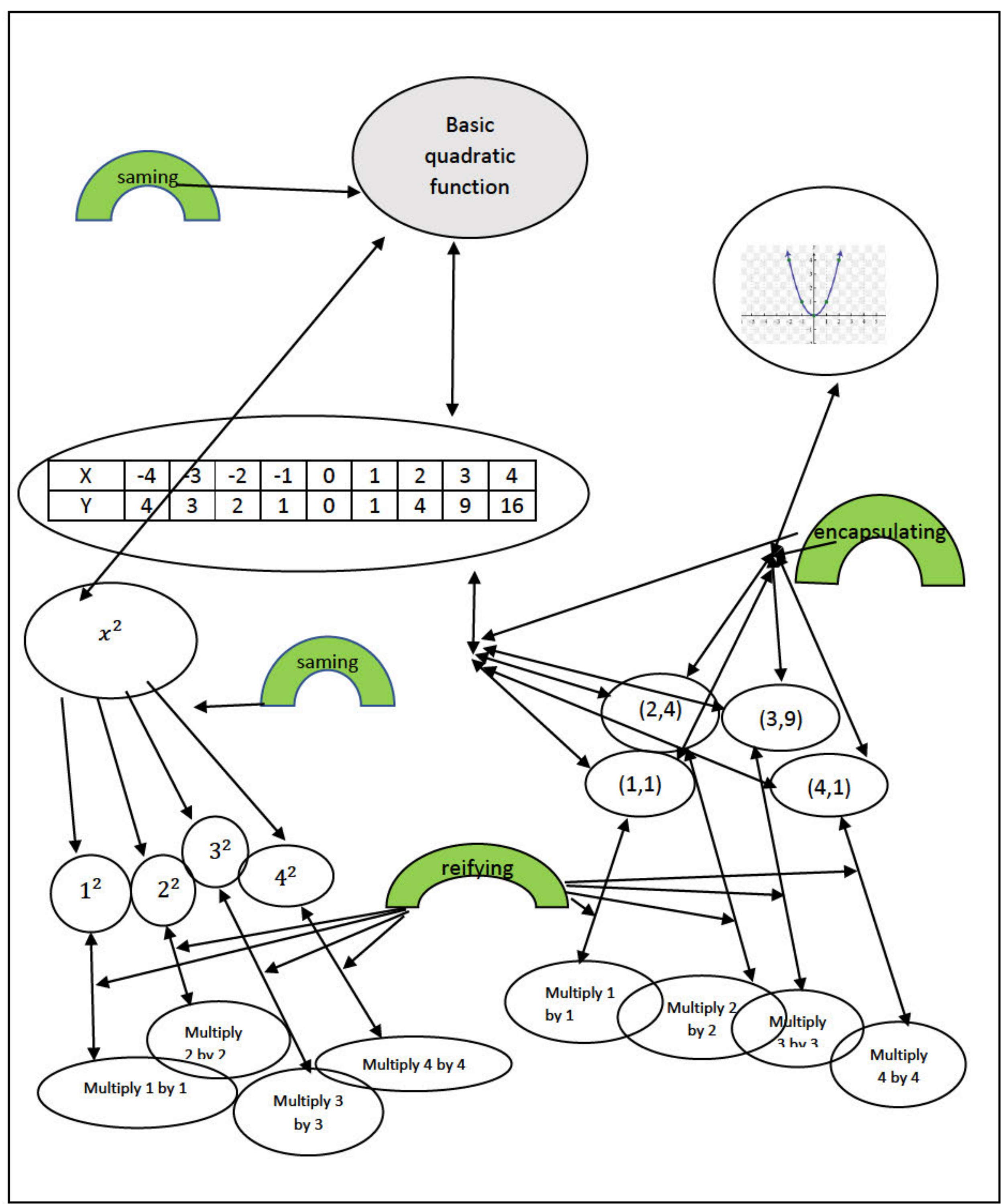
repetitive patterns of actions that participants of mathematical discourse perform in the act of communication. (4) Set of narratives participants construct with the use of routines. Some of these narratives on mathematical objects may be endorsed if they adhere to the defining rules of the discourse, while others may not be endorsed by the mathematics community. Some examples of endorsed narratives are theorems, axioms and conjectures. With these discursive mechanisms at their disposal, participants in mathematical discourse can communicate about mathematical objects.

7.5.2.2 Mathematical objects

Mathematical objects are abstract objects. Unlike other objects of discourses learned in schools, such as biology or geography, mathematical objects are not physical in nature but abstract and do not exist prior to and independently of the discourse. Mathematical objects (numbers, functions, symbols, diagrams, etc.) are created to enable discourse participants to talk about these entities.

Since mathematics is thinking, the goal of learning is communicating about mathematical objects, which is carried out through objectification. Objectification is the process in which mathematical objects (discursive objects) are discursively constructed. Objectification involves two sets of processes. The first set has elements known as alienation and reification. Alienation is the act of removing the presence of human beings and replacing phenomena with an independent status. Reification is the act of replacing a verb with a noun in the phrase. Through reification, participants talk about objects instead of talking about processes. The second set has elements referred to as saming and encapsulation. Saming gives one name to various things that were previously considered to be different. Encapsulation is the act of using a noun or pronoun to a set of discursive objects previously referred to in plural but which can now be referred to in singular.

Figure 7.2: Mathematical objects called basic quadratic function as a tree of signifiers interrelated by operations of saming, encapsulating and reifying (Adapted from Sfard, 2023, p. 577)



7.5.2.3 Routines: rituals and explorations

All human activities, of which teaching and learning are one, are achieved through collectively established routines (Sfard, 2007). A routine has three components: initiation (the *when* of the routine), procedure (the *how* of the routine), and closure (the *when* of the routine). The commognitive framework distinguishes two types of discursive routines. These two types are rituals and explorations. These two types differ in the way they are evaluated. Explorations are routines that measure whether a new endorsed narrative has been constructed (Lavie et al., 2018). The explorative participant focuses on the end product.

On the other hand, rituals are geared towards social acceptance by others (Heyd-Metzuyanim & Graven, 2016; Sfard, 2008). The ritual is mainly concerned with the process and not the product. As a result, a ritualistic performer is not independent and tends to imitate others in a rigid way.

The characteristics to help distinguish ritualistic from exploratory aspects of routines were exemplified in table 3.14. These are:

(1) Flexibility: Were the ritualistic follows only one procedure while the exploratory may vary.
(2) Bondedness: Not boded for the ritualistic but bonded for the exploratory.
(3) Applicability: Ritualistis can only evoke in similar situations, but exploratory can in different contexts.
(4) Performer’s agentivity: Ritualistic is dependent upon others, but exploratory is independent.
(5) Objectification of the discourse: Ritualistic relies on concrete objects, but exploratory can abstract.

(6) Substantiation: Ritualistic may not substantiate while the exploratory can assess the validity on their own.

7.5.2.4 Learning Mathematics as Change in Discourse

Learning mathematics is a change in discourse and involves individualisation of the discourse. The change in discourse happens through expanding the discourse of mathematical objects. Thus, learning takes place at two levels: at the object level and at the meta-level. Object-level learning involves the expansion of the discourse of mathematical objects by way of procedures and endorsing new narratives about mathematical objects. Metal-level learning involves expanding the discourse of meta-rules through different words (Sfard 2012). The routine involving metarules has two subsets, and these are (1) routine procedure, that is, the *how* of the routine that determines how the procedure is going to be performed, and (2) the *when* of the routine, that is, opening condition regarding the circumstances in which the routine is summoned, and the closing condition regarding the circumstances in which the routine is deemed as complete.

As meta-level learning involves changes in mathematical word use and how tasks are performed, this often results in the advent of incommensurable narratives with some of those previously endorsed. This leads to a situation referred to as commognitive conflict, when participants use incompatible narratives in the process of moving between some different discourses. Incommensurability is vital to the growth of mathematical thinking. However, mathematical discourse participants often cannot independently accommodate incommensurable narratives. They may require the intervention of the expert discursant. Meta-level learning begins with ritual performance (Nachlieli & Tabach, 2012).

7.5.3 Literature review

A literature review was undertaken in Chapter 2. The review was divided into three sections to give insight into the questions that guided this study.

The first section reviewed the literature on the phenomena under study. This section dealt with literature identifying the discourses participants engage in in mathematical discourse. Several studies on the forms of discourses participants engage in were reviewed. These studies included, among others, those of (Nia et al., 2020; Roberts & le Roux, 2019; Mpofu & Mudaly, 2019; Mpofu & Poutnara, 2018; Thoma & Nardi, 2017).

The second segment of the review dealt with what could have been the contributing factors to the phenomena revealed in the studies mentioned in the first section. Several studies that pointed to some contributing factors to the identified phenomena were reviewed. These studies included those of (Robertson & Graven, 2019; Riordian & Flanagan, 2018; Mbhiza, 2020; Zayyadi et al., 2020).

The first section was the review of the literature on how to mitigate the findings highlighted in the first section review of the literature. The findings were: (1) The participants used colloquial and mathematical words in their discourse. (2) The degree of objectification was low. (3) Their mediator use was multi-modal. (4) Their narratives were not justified, and the sources of endorsement were from external discourses. The literature review suggested several effective instructional strategies to mitigate the first section's findings. These studies included those of (Mudaly & Zulu, 2023; Tiwari et al., 2021; Magableh & Abdullah, 2019; Karali, 2017; Turgut & G, Turgut, 2018).

7.5.4 Methodology

The qualitative research approach was adopted for this study. This particular approach was underpinned by Creswell's (2014) assumptions that inform qualitative studies. These

assumptions were elaborated on in Chapter 4 of this study. The aim of using the qualitative approach, among others, was that qualitative studies are characterised by natural settings to study behaviour by collecting data from informed sources who had experienced the phenomena (McMillan, 2012).

The study population was grade 11 learners at a secondary school in KwaZulu- Natal, South Africa. Purposive sampling techniques were used. The goal was to select participants with rich information about the phenomena under investigation. A pre-test was first administered, and from the results obtained, 5 participants were selected (only three eventually participated). The instrument for the pre-test was the van Hiele test used for the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project (Usiskin, 1982). Population and sampling procedures were addressed in 4.4.1 of Chapter 4.

Two instruments were used to collect data. These instruments were the focus group interview and the mathematical written task. The interview was video recorded and then transcribed. This was outlined in 4.5.2 of chapter 4.

Before collecting data from the participant's, informed consent was obtained from their parents/guardians, and permission was obtained from the university and the secondary school authorities. This was discussed in 4.5.1 of chapter 4.

Issues about quality assurance were strictly addressed and adhered to. This included credibility, transferability, dependability and confirmability. These were described in 4.7 of chapter 4.

7.5.5 Findings and discussion

This study found that the participant's word use was a mixture of colloquial and mathematical words, that the degree of objectification was low, that the sources of their narratives were not from within the discourse but from outside the discourse and that they relied on prompts to recall their endorsement. These findings led to the claim that the participants' discourse was ritualistic.

The review of literature accompanying the study established that the participant's discourse may be transformed into an explorative discourse through appropriate teacher pedagogical moves. Several effective teaching strategies are suggested by some scholars, such as (Turgut & G Turgut, 2018), which were reviewed and discussed. These teaching strategies include differentiation of instruction, teaching through problem-solving, use of technology, and teaching considering the van Hiele model, just to mention a few.

7.6 Findings in relation to previous research

This study found that the participants' discourse was ritualistic, mainly due to inconsistency in word use. The participants' word use was problematic because they moved between colloquial and mathematical word use in the same discourse, which impacted the other attributes of mathematical discourse. Robert and Le Roux (2019) also found the same. In this study, whereas some participants could construct mathematical objects, they failed to realise that deriving substantiable narratives was supposed to come from within, that is, from the constructed objects, and not from outside the discourse. The properties of the mathematical objects constructed were to be the source of their narratives. In this study, there was an instance of a participant, saying that she did not measure the distance between two points but just guessed to come up with the middle point. In another instance, when asked to examine

the two triangles, the participant responded, that the triangles are right angled, while another added on and said they looked congruent.

The study corroborates the findings of other researchers, such as those Wang et al. (2014) and Mpofu and Pournara (2018), that revealed that participants relied on visual recognition to aid substantiation. In Wang's study, participants justified the properties of quadrilateral by "looking" at the mathematical object. and Mpofu and Pournara (2018) revealed that participants relied on visual recognition. In the study of Mpofu and Pournara, the participants could draw the asymptote but did not mention it in their narratives. These findings reveal the challenges the learners experience in mathematics, linking object-level and meta-level learning together as they strive to be explorative participants in mathematical discourse.

7.7 Reflection on new knowledge

According to Harel (2006), the endpoint of receiving mathematics instruction is to enable students to find ways of thinking that move in synch with those of renowned mathematicians.

This study explored the learner's commognition when engaging with geometry. My primary motivation for conducting this study was to improve my own practice as a mathematics educator. In this section, I share my reflection on the new knowledge I gained from the study.

Learning mathematics is mainly aimed at changing mathematical discourse, attained through explorations. However, as mentioned earlier in this report (p. 81), rituals are a necessary part of this process, and for this reason, they play an essential part in the initial stages of teaching and learning. Therefore, in this regard, the notion of rituals cannot be downplayed. However, even though the necessity of ritual in the initial stages of learning is fundamental, rituals stymie the learning of mathematics (Sfard, 2008). For this reason, the mathematics educator is implored to strive and make a concerted effort to facilitate the transition of moving the learner from ritual participation to explorative participation in mathematical discourse.

However, this transition from rituals to explorations has challenges imposed by communicational gaps in any form of communication. For this reason, it is incumbent on the mathematics educator to mitigate these communication gaps by reducing and closing them while shaping a new discourse in the classroom. Mathematics is a language with its own terminology and register. Locating mathematics as a language, underscores the notion of being a collective human activity. Thus, learning mathematics, just like any language is a social activity and effective communication is fundamental.

The evidence from the study demonstrates that engaging in dialogue is one method that may highlight communication gaps in the discourse. When the discursant uttterances are analysed, one may notice evidence that suggests that the participant and the interviewer participated in different discourses on several occasions. In one instance, the prompts did not seem to move the participant from her initial discourse as she simply replied, and very casually for that matter, by starting her utterance with "that" and using the exact words used before to emphasise the statement. The participant did not realise that the intervention from the interviewer was meant to facilitate the scaffolding of information stored in the participant's mathematical register. However, it was only when prompted further, at the third attempt, that she seemed to move from her initial discourse and attempted to enter the new discourse that was being suggested all along. She changed her response and answered in a different way. This analysis supports my conclusion that since learning mathematics is a social activity, it is through dialogue that the communication gaps between the participants in the discourse, is made explicit.

Further evidence from the analysis suggested that the participant and the interviewer participated in different discourses especially in the initial stages of the interactions.

Within a given task situation, learners often engage in multiple discourses that they have to navigate to arrive at the solution to the task problem. Most mathematical tasks involve different kinds of concepts, and to successfully workout the solution, learners required to simultaneously navigate between structural versions of concepts and operational versions (Sfard, 1991). This interplay requires one to engage in a number of multiple as well as related discourses. These multiple discourses often build upon one another and are interwoven in nature. Therefore, any breakdown in communication in one of them raises the potential to impede the transfer of concepts from one discourse to another. When this process is stymied, introducing commensurable and incommensurable discourses, which are the necessary ingredients for the new discourses, becomes problematic. The merging of commensurable and incommensurable discourses is fundamental to explorations as the process learners experience inductive and deductive thinking.

This can be done by giving the learner a range of differentiated task activities specially tailored at different levels and paces of learning to enable learners to explore and gradually experience both low-level and challenging tasks. While the learners are busy working, the educator may challenge them to explain their routines, listen attentively to their word use and examine the visual mediator use. At this stage, scaffolding may be necessary by working out some examples with them. When the learners have mastered the low-level tasks, the educator may now gradually infuse in commensurable and in-commensurable discourses. At this time, little or no scaffolding may be done to allow learners to learn from their mistakes. While the learners are engaging in these seemingly different discourses, the focus in the activities should be on exploring similarities and differences between the discourses. The educator should still be focused on word use and visual mediator use. At the end of these task activities, learners can now be engaged in a whole discussion on the task activities. The

purpose of the whole class discussion is for learners to listen to their peers, expose communication gaps, and evaluate their learning.

The figure (6.2) below illustrates my conceptualising learning regarding some new knowledge.

1. *Teaching focus 1*

Here, the educator is mainly concerned with paying specific attention to word use and visual mediators in the students' narratives and routines, as there are potential sources of communicational gaps.

2. *Teaching focus 2*

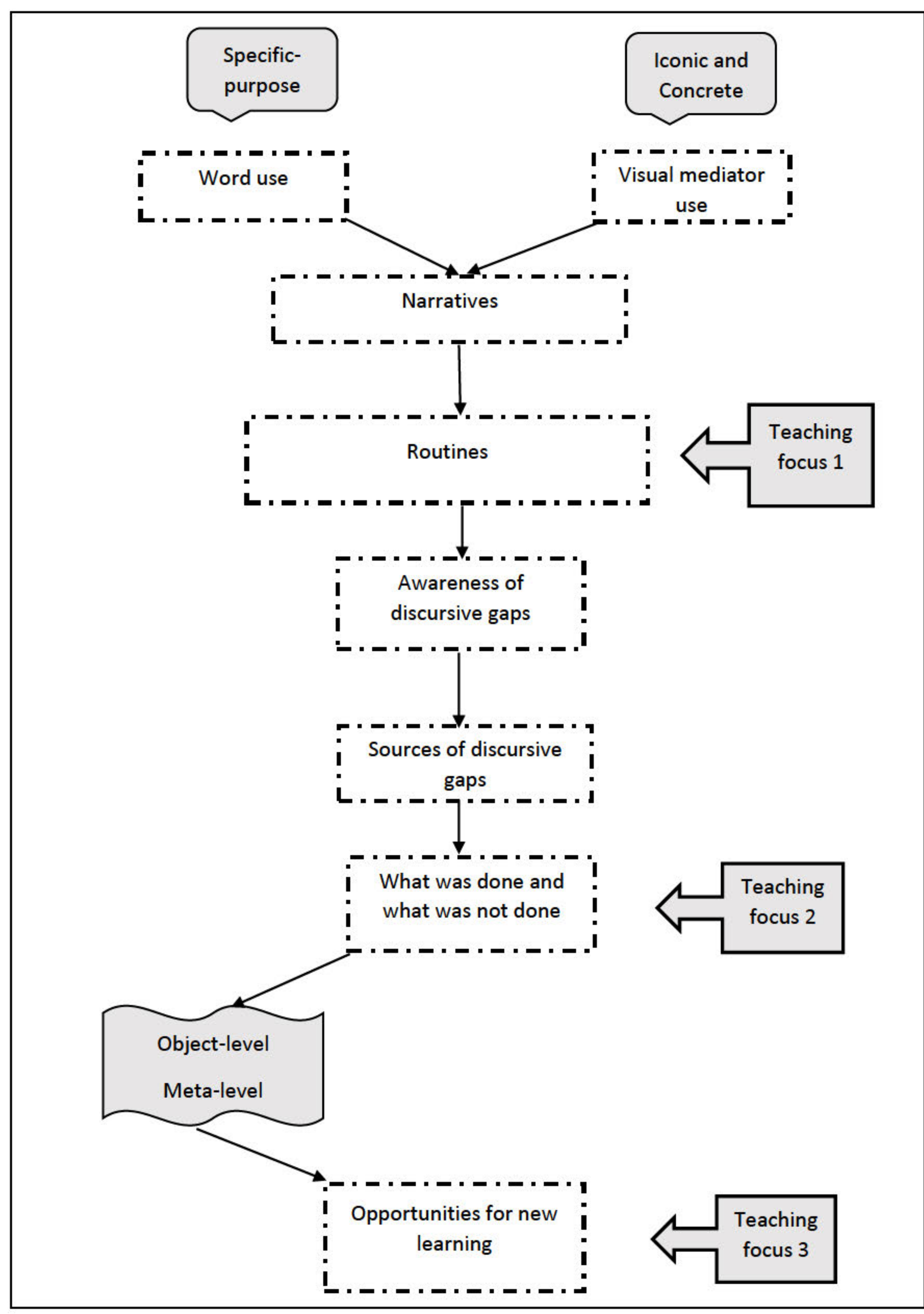
Once the sources of communicational gaps have been identified. The educator may now embark on a critical analysis of what was done and what was not done by the students before the educator moves to formulate appropriate strategies to mitigate the closure or reduction of the gaps.

3. Teaching focus 3

At this stage of teaching and learning, the focus shifts to linking object-level and meta-level learning. Here, the educator gradually initiates a new discourse that contains commensurable and incommensurable discourses. Here, the students' focus as they work on the tasks is to look for similarities and differences between the old and new discourses.

Just as rituals are highly situated (Sfard, 2008), so is practice. With this new knowledge gained, and considering Harel (2006) suggestions, I intend to utilise my new knowledge of conceptualising learning as communication and explore alternate ways of improving my practice as a mathematics educator.

Figure 7.3: Conceptualisation of teaching and learning. Source: Researcher's own initiative



7.8 Concluding remarks

This study established that the participant's discourse was ritualistic. The participants demonstrated that their discourse was not coherent as they exhibited inadequate use of some of the four commognitive features of mathematical discourse. The study established that the interlink between these features of mathematical discourse was essential and that inadequate use of any feature had significant consequences on the other features. The study also established that word use is central to these features. Participants' use of crucial purpose words permeated all the other three features of mathematical discourse. The implication is that emphasis in teaching and learning should be placed on word use to enable all participants to be of one accord concerning what is being discussed. Effective communication is premised on suitable word use. Proper word use can support visual mediators, routines and narratives in enhancing the understanding of mathematical concepts.

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Appendices

Appendix A: Ethical Clearance Letter

03 December 2019

Mr Alexander Mulenga (218086151)
School Of Education
Edgewood Campus

Dear Mr Mulenga,

Protocol reference number: HSSREC/00000421/2019
Project title: An exploration of learners commognition when engaging with geometry in a grade eleven class.

Approval Notification – Expedited Application

This letter serves to notify you that your application received on 29 August 2019 in connection with the above, was reviewed by the Humanities and Social Sciences Research Ethics Committee (HSSREC) and the protocol has been granted **FULL APPROVAL**

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number. PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

This approval is valid for one year from 03 December 2019
To ensure uninterrupted approval of this study beyond the approval expiry date, a progress report must be submitted to the Research Office on the appropriate form 2 - 3 months before the expiry date. A close-out report to be submitted when study is finished.

Yours sincerely,

Professor Urmilla Bob
University Dean of Research

/dd

Humanities & Social Sciences Research Ethics Committee
Dr Rosemary Sibanda (Chair)
UKZN Research Ethics Office Westville Campus, Govan Mbeki Building
Postal Address: Private Bag X54001, Durban 4000
Website: http://research.ukzn.ac.za/Research-Ethics/

Founding Campuses: Edgewood · Howard College · Medical School · Pietermaritzburg · Westville

Appendix B: Written task activity

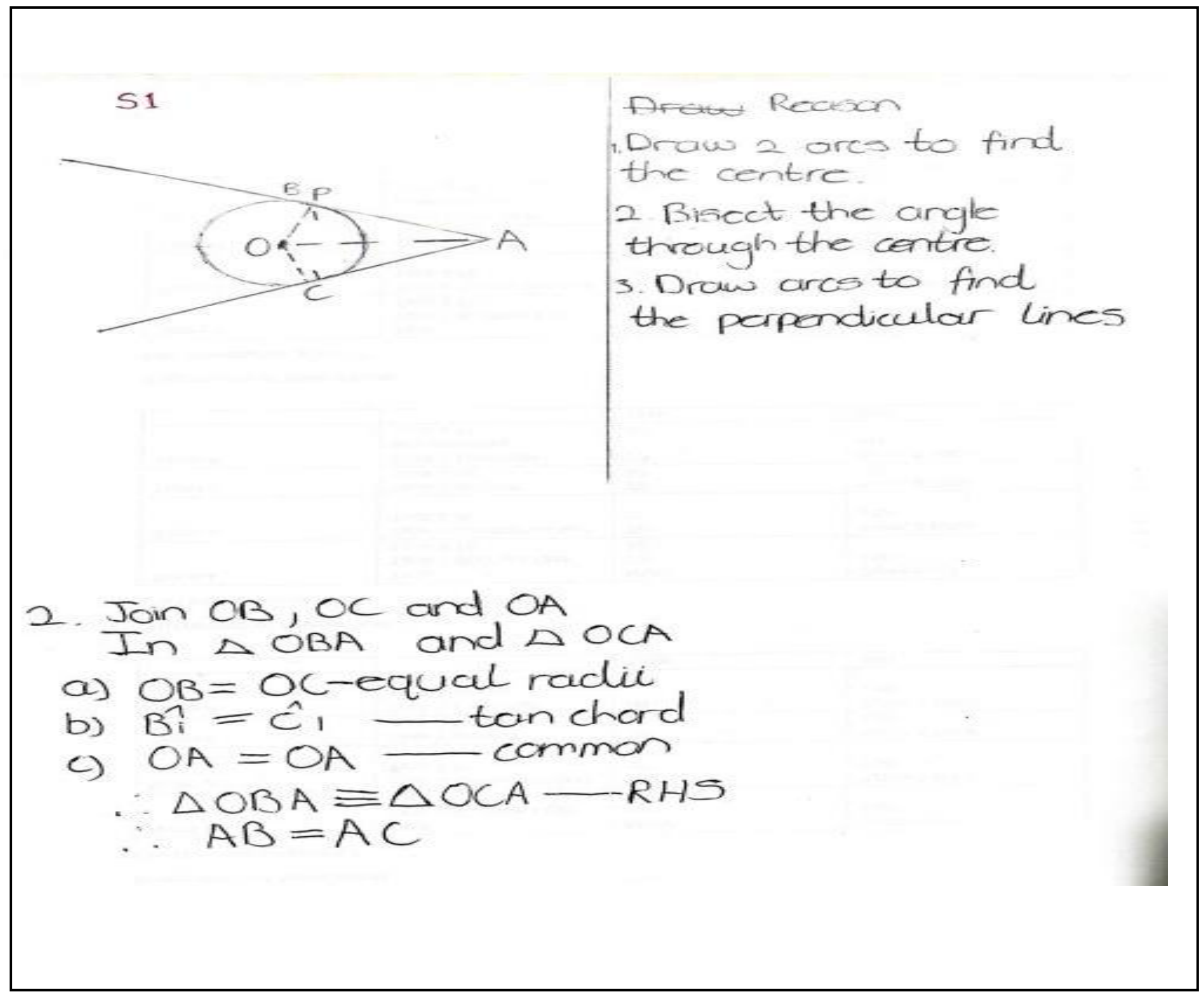
Task Activity

P

1. You are given two intersecting straight lines, and a point P marked on one of them as shown in the diagram. Show how to construct, using straightedges and compass, a circle that is tangent to both lines and has the point P as the point of tangency to one of the lines.

2. Prove that two lines from the same point which is outside a circle, such that both lines are tangent to the circle, then their lengths are the same.

Appendix C: Sample of participant's written work



Appendix D: Sample of Interview Protocol

You are given two intersecting straight lines and a point P marked on one of them, as in the figure below. Show how to construct, using straightedge and compass, a circle that is tangent to both lines and that has the point P as its point of tangency to one of the lines.

A. The concept of a tangent.

What is your understanding of the concept of a tangent to a circle' how can you explain to your colleague the concept of a tangent to a circle?

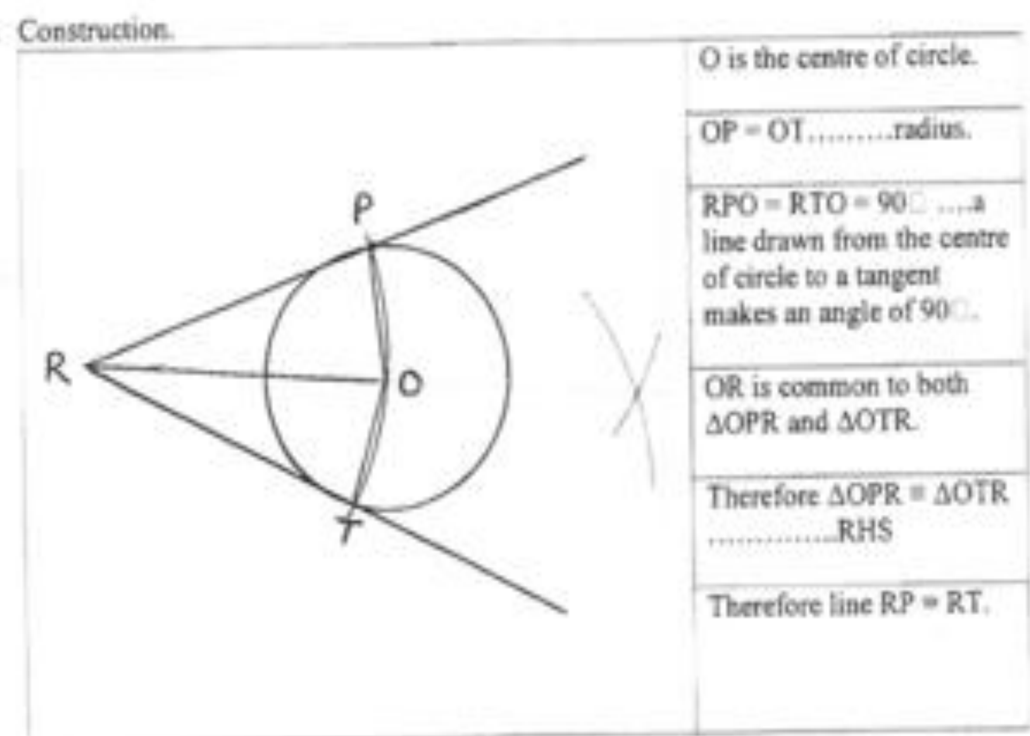
They might explain it

(1) Using terms like radius, perpendicular (This shall help me identify the words they are using).
(2) They might draw it (This shall help me identify the visual mediator).
(3) How they are explaining/answering shall help me identify the routines.
(4) All the above shall help me identify their narratives.

Follow up questions shall be informed by

- The words they use.
- Or if they could not all together give an understanding of the concept of a tangent to a circle.

B. Construction.



Let us talk about the construction process.

- How would you construct a circle that would pass through the point P?
  How they answer this question shall help me identify (words, routine, visual mediator).
- Why have you chosen to start the construction in this way?
  How they answer this question shall help me identify (words, routine, narrative).
- Can you show me how you are going to start?
  How they answer shall help me identify (visual mediator).
- Why do you draw this line?
  How they answer shall help me identify (narrative).
- What it is the name of this line?
  How they answer shall help me identify (words).
- What does this line (OP) you joined to the centre of the circle represent/what distinguishes it from the lines?
  How they answer shall help me identify (narratives. routines, words).
- What is the value of this angle (OPR)/ how do you know?
  How they answer shall help me identify (words, visual mediator, routines).
- What other forms/ways can help you to identify that this angle is a right angle?
  How they answer shall help me identify (routines, visual mediators, words, narratives).
- How, why do you know that this point P is a tangent?
  How they answer shall help me identify (words, routines, visual mediators, narratives).
- Is this point (T) on the other side of the circle also a tangent/why and how do you know?
  How they answer shall help me identify (words, routines, narratives).
- Are these two triangles congruent/why and how do you know?
  How they answer shall help me identify (words, visual mediators, routines, narratives).
- Is there another way to show that they are congruent?
  How the answer shall help me identify (words, visual mediators, routines, narratives).
- What can you conclude about line AP and AT?
  How they answer shall help me identify (words, routines, narratives).

The end.

Appendix E: Sample of the Transcriptions

Turn	Speaker	What is said	What is done
1.	Interviewer	Can you introduce yourselves	
2.	S1:	S1	Nods the head and looks at S2.
3	S2:	S2	
4	S3:	S3	
5.	Interv:	All right we are about to start the investigation, and ……ah….. Are you familiar with the question?	
6.	S1:	Yes.	
7.	Interv:	Ok, what is your understanding of tangency? When we talk about tangency, what is understanding?	
8.	S2:	It's a line outside a circle…..outside the circle.	While looking at S1, nods the head, then looks at S3
9.	S2:	Any line that is a tangency to a circle doesn't go through the circle………through.	Raises the hand to indicate through.
10.	S2:	Cuts the circle.	Looks at S1.
11.	S3:	Doesn't cut the circle.	
22.	Interv:	When you say this question, what were your first thoughts? What did you think about?	
23.	S2:	I didn't understand it.	
24.	S3:	I could draw a circle.	Leans forward.
25.	S1:	I was …….I was afraid that I would fail.	Looks at S2 and at S3, then smiles
26.	Interv:	You were afraid that you would fail?	
27.	S2:	Kind of…… I didn't understand it.	
28.	S1:	When I looked closer, I did understand.	

29.	Interv:	When you began to answer the question, why did you start from that point?	
30.	S1:	Which point?	Looks at the task closely.
31.	Interv:	The point where you began answering the question from? What did you do first?	
32.	S1:	I constructed a circle.	Points with the finger, then moves the finger as if drawing a circle.
33.	S2:	I don't remember.	Shakes the head while saying this.
34.	Interv:	You don't remember?	
35.	S2:	No.	Nods the head.
36.	S3:	First time, I first drew a circle.	
37.	Interv:	You drew a circle?	
38.	S2:	Yeah.	Nods the head.
53.	Interv:	Why did you choose to start your construction in this way?	
54.	S1:	It seemed the most logical..... And.....	Looks sideways.
55.	S2:	It's the only way to draw a circle.	Moves the finger as if drawing a circle.
56.	S1:	I would draw a circle around the entire line, leaving P in the middle,…..midpoint, then it makes more sense to draw a circle passing the …..P.	Moves the finger as if drawing a circle.
57.	Interv:	But remember, the circle has to be between those two lines?	
58.	S2:	Yes……	
59.	S1:	We can make P a chord?	Looks at interviewer, then bends the head.
60.	S2:	It has to be between the two lines……	Points at the lines with a finger, while looking at S1.
61.	Interv:	Should the circle be in between these two line?	
62.	S2:	The circle will touch these two lines. Has to touch both lines.	Points finger at a line

63.	S1:	Oh…yeah. ...... yeah.	Nods the head.
64.	Interv:	Then, how do you draw the circle?	
65.	S2:	You draw with the compass touching the two lines.	
66..	Interv:	Yes, you draw with a compass.	
67.	S2:	You draw touching P and the other line.	
68.	S1:	Here...	Points with the finger.
69.	S2:	Between these two lines.	Points with the finger at the two lines.
70.	Interv:	How do you decide on the length of your radius?	
71.	S1:	We draw a line from the middle of the circle to P…… You measure first.	
72.	S2:	How do you know where to put the compass?	While saying this Looks at S1.
73.	Interv:	Yes, how do you know where to place the compass?	
74.	S1:	I would measure with a pencil,… make sure it doesn't go beyond this line,.. doesn't touch the line.	
75.	S2:	I never measured….I just guessed.	Looks at the interviewer.
94.	Interv:	What do you mean by point of intersection?	
95.	S2:	The touch each other.	
96.	S1:	The point where the lines meet, that's the point of intersection.	
97.	S1:	Yeah….	Nods the head.
98.	Interv:	Would that help you in identifying where to place the compass?	
99.	S2:	Yeah….	
100.	Interv:	Then what would you do?	
101.	S1:	We would draw the centre line….	
102.	Interv:	And then?	

103.	S1:	Draw the circle….. Yeah….	Turns the head and looks at S2.
104.	Interv:	And there after?	
105.	S2:	We drew the radius, we labelled the centre O, and we drew a line from O to P. That was the radius.	
106.	Interv:	What do you mean when you say radius?	
107.	S2:	The radius is a line from the centre to the circumference.	Turn the head and looks at S1, then smiles.
108.	S2:	Then we prove that…… isn't the line here the radius?	Turns the head and looks at S1, while pointing with the finger.
109.	S2:	Well, where the line OP meet, we labelled, we label that something, and we tried to show that it was a right angle, to prove that P is a tangent.	Looks at the interv.
120.	Interv:	You are using the word perpendicular?	
121.	S1:	Yes,… meaning ninety degrees.	Demonstrates with the finger.
122.	S1:	The lines meet at ninety degrees.	
123.	Interv:	How do you know that point P is a tangent?	
124.	S1:	Because the radius meets P at right angel.	Turns the head and looks at S2.
125.	Interv:	The line would meet at a right angle?	
126.	S1:	Yes.	
127.	Interv:	Look at the diagram, what can you say about those two triangles?	
128.	S2:	There are both right angled.	
129.	Interv:	At what point are they right angled?	
130.	S2:	At Q and B.	
131.	Interv:	And then?	
132.	S1:	The look congruent.	Looks at interv.
133.	Interv:	The look congruent?	
134.	S2:	Yes.	

135.	Interv:	How do you know? You said the look congruent? Can you prove that?	
136.	S2:	You see, because there is an angle….right angle and OB is equal to OQ,…. That is the radius that is how you prove.	Looks sideways at s1, then at S3.
137.	S1:	I said that B and Q are ninety degrees, added together is hundred and eighty degrees, so to measure…, say angle O and angle T, we need to know, angle B and angle Q are hundred and eighty degrees and opposite sides of a quadrilateral are supplementary……..because both are hundred and eighty degrees and to get three hundred and sixty degrees we have to add.	Demonstrates with fingers.
138.	Interv:	You said those two triangles are congruent. Is the line BT equal to the line QT?	
139.	S2:	Yes, because both tangents come from the same point.	
140.	Interv:	You say because both tangents come from the same point?	
141.	S2:	Yes, that means there are equal.	
142.	Interv:	Then, how can you prove that BT is equal to QT? If you look at the diagram, how can you prove?	
143.	S1:	BT is equal to QT……..	
144.	S2:	To prove that BT is a tangent and QT is a tangent. That is how you prove.	Looks at interv.
145.	Interv:	What properties did you use to prove that the triangles are congruent?	
146.	S1, S2, S3:	Ham …………………………….	They look at one another
147.	Interv:	Are the triangles congruent? How?	
148.	S2:	There are equal.	
149.	S1:	OB is equal to OQ, because both are radii.	
150.	Interv:	OB is equal to?	
151.	S1:	OQ.	
152.	Interv:	Why do you say that?	

153.	S1:	…………I said OB is equal to OQ.	
154.	Interv:	Are you saying that side is equal to the other side? Then you have one reason.	
155.	S2:	Yeah, then we say angle B and angle Q are equal. We now have two reasons.	
156.	Interv:	Yes. Go on.	
157.	S2:	Then, BT and QT are equal because there are tangents coming from the same point.	
158.	Interv:	But that we want to prove. We want to prove that BT equal to QT. So we can't use this statement before we prove it.	
159.	Interv:	That should come in our conclusion.	
160.	S2:	Oh…..ahm…. and…….	
161.	S3:	OT?	Looks at S1 while saying this.
162.	Interv:	What about OT?	
163.	S2:	OT is a common line, so it is equal.	
164.	Interv:	It is common? When you say this what do you mean?	
165.	S3:	There are in both triangles. Lines are in both triangles.	Raises the finger.

Appendix F: Informed consent letter to School Principal

Informed Consent Letter to School Principal

Dear Principal

Mr. M Y Gouse

Name of school: Verulam Islamic School

Re: Permission to conduct a research study in your school

I am writing to request your permission to conduct a research study in your school. This research study is entitled:

An exploration of learner's commognition when engaging with geometry in a grade 11 class.

My name is Alex Mulenga and I am currently studying towards a PhD Degree at the University of KwaZulu-Natal (UKZN). As part of the requirements of this degree, I am required to complete a research thesis. This study focuses on learner's commognition.

I require about 6 grade 11 participants of mixed gender to participate in this research. I would be very grateful if you would consent to these learners participating in this study. They will be selected from your school.

If you agree to this, they will be invited to respond to questions in a focus group discussion and in depth individual interview.

All discussions, interviews and dialogues with participants will be audio recorded using a Dictaphone, and thereafter transcribed verbatim to produce transcriptions. This research information (data) is required for the analysis of data and completion of the actual write up of the thesis. Collecting research information for this study will take approximately 3 months. All focus group discussions, in depth individual interviews and group and individual activities will take place at Verulam Islamic School. Times and dates will be discussed and arranged with you and the participants at a later stage. I will try to ensure that this takes place during their free periods of time, in an attempt to avoid any disruptions during lessons. Participants will also be encouraged to eat their lunch during discussions,

interviews and activities, as well as make use of the school toilet should the need arise. I will not deprive them of these opportunities, especially since I intend to use some of their free time in order to collect sufficient data for my study.

Data generation activities will also take place Verulam Islamic School with your consent. If I am unable to collect my data during school hours, I will make arrangements with your consent and that of my participants' to perhaps do this after school hours, on days when school closes early or during weekends.

Please note:

* Times and dates of this data generation process will be at your sole discretion. I have merely presented you with an outline of what I intend to do, however you are free to make any changes and suggestions, if necessary.
* Participation is completely voluntary and participants have the right to withdraw from this study at any time. They will not be penalised if they choose to do so.
* Confidentiality and anonymity will be maintained at all times. The identity of your school and all participants will not be revealed at any time, as pseudonyms (different names) will be used to protect everyone's right to privacy.
* Any information provided by the participants will not be used against them, or against the school, and will be used for purposes of this research only.
* Participation in this study <u>will not</u> result in any cost to your school or the participants.
*Neither the participants nor your school will receive financial remuneration. However costs incurred by participants as a result of their involvement in this project will be covered.
* This study does not intend to harm the participants in any way.
* Both parents/guardians as well as participants will be handed letters of consent which they will have to carefully read and sign, before I begin data collection.

I may be contacted at:
Email address; [redacted]
Tel: [redacted]

My supervisor's contact details are:
Prof Vimolan Mudaly
Email address; mudalyv@ukzn.ac.za

Cell: [redacted]

Tel: 031 2607326

You may also contact the Research Office through:

Mariette Snyman

HSSREC Research Office,

Tel: 031 260 8350 E-mail: snymanm@ukzn.ac.za

If you would like any further information or if you are unclear about anything, please feel free to contact me at any time. Your co-operation and consent will be greatly appreciated.

If you grant permission to conduct this research at your school, please complete the form below and return to me.

Warm regards

Alex Mulenga

DECLARATION

I ...MAHOMED YASEEN GAUSE................ (full name/s of school principal) of ...VERULAM ISLAMIC........ (name of school) hereby confirm that I understand the contents of this document and the nature of this research project, and I consent to the learners participating in this research project. I also grant permission for my school to be used as the research site.

Additional consent

I understand that interviews will be audio-recorded and I grant permission for this. YES/NO

I understand that the learners and the school are free to withdraw from the research project at any time. YES/NO

SIGNATURE OF SCHOOL PRINCIPAL **DATE**

.... 21/11/19..........

VERULAM ISLAMIC SCHOOL
P.O. BOX 944, VERULAM 4340
TELE/FAX: 032 533 1258
email: m[illegible]@mweb.co.za

Appendix G: Letter of introduction

LETTER OF INTRODUCTION AND INFORMED CONSENT FOR PARTICIPATION IN ACADEMIC RESEARCH

Title of study: *An exploration of learner's commognition when engaging with geometry in a grade eleven class.*

Principal investigator:

Alexander Mulenga
University of KwaZulu Natal

Supervisor:

Prof Vimolan Mudaly
University of KwaZulu Natal
mudalyv@ukzn.ac.za

Purpose of study

You are being asked to take part in a research study. Before you decide to participate in this study, it is important that you understand why the research is being done and what it will involve. Please read the following information carefully. Please ask the researcher if there is anything that is not clear or if you need more information.

Terminology:

1. Commognition is a combination of the words, communication and cognition. In commognition, thinking is a form of communication. Commognition stresses that interpersonal communication and individual thinking are two sides of the same phenomenon.

The purpose of the study is to explore the development of mathematical thinking when engaging with geometry. The study will employ the commognitive analysis of mathematical discourse (stories).

The results of the study may be published in an academic journal. You will be provided with a summary of our findings on request. No participants names will be used in the final publication.

Study procedure

1. You will write a baseline test on geometry. The baseline line test will provide information that will enable the researcher to select a sample of 5 participants.
2. The 5 participants will be given a task on geometry to answer. While answering the task, the participants will be video- taped. The amount of time on the task will be approximately 40 minutes.
3. The participants will then be interviewed based on their written work. The interview process will also be video/audio-taped. The interviews will take place in focus groups of two or three. Each session will be approximately one hour. The number of sessions will be informed by the responses. However, the duration of these sessions will not be more than three months.

Risks

I do not foresee any risks associated with your participation in the study, beyond those experienced in everyday life.

You may decline answer any or all questions of the study and you may terminate your involvement at any time if you choose.

Benefits

Your participation in the study will contribute to the body of knowledge on the theory of commognition.

Confidentiality

Your responses to the study will be anonymous. Please do not write any identifying information on your scripts. Every effort will be made by the researcher to preserve your confidentiality including the following:

- Assigning code names/numbers for participants that will be used on all research notes and documents.
- Keeping notes, interview transcripts, and any other identifying participant information in a locked file cabinet in the possession of the researcher.

Participant data will be kept confidential except in cases where the researcher is legally obligated to report specific incidents.

Contact information

If you have questions at any time about this study, you may contact the researcher whose contact information is provided on the first page. If you have questions regarding your rights as a research participant, or if problems arise which you do not feel you can discuss with the Primary Investigator, please contact the supervisor or the research office through:

Mariette Snyman

HSSREC Research Office,

Tel: 031 260 8350. E-mail: snymanm@ukzn.ac.za

Voluntary participation

Your participation in this study is voluntary. It is up to you to decide to whether or not to take part in this study. If you decide to take part in this study, you will be asked to sign a consent form. After you sign the consent form, you are still free to withdraw at any time and without giving reasons. Withdrawing from this study will not affect the relationship you have, if any with the researcher. If you withdraw from the study before data collection is completed, your data will be destroyed.

CONSENT FORM

Title of project: An exploration of learner's commognition when engaging with geometry in a grade 11 class

Name of researcher: Alexander Mulenga

I ……………………………………………………………agree to participate in this research project.

The research has been explained to me and I understand what my participation will involve.

(Tick in the box).

I agree that my participation will remain anonymous.	Yes ☐	No ☐
I agree that the researcher may use anonymous quotes in the research report.	Yes ☐	No ☐
I agree that the interview may be video/audio recorded.	Yes ☐	No ☐
I agree that the information I provide may be used anonymously by other researchers following this study.	Yes ☐	No ☐

Name of participant: ______________________ **Signature:** ___________ **Date:** ________

Researcher: _____________________________ **Signature:** __________ **Date:** _______

Appendix H: Letter of introduction in Isi Zulu

IFOMU YEMVUME YOMZALI /UMQAPHI WENGANE

Isihloko: *Ukuhlolwa komfundi indlela acabanga ngayo emqondweni (phecelezi commognition) uma ebhekene negeometry kwibanga leshumi nanye eklasini.*

Ngingumfundi owenza ucwaningo eSchool of Education at University of KwaZulu Natal.

Njengenxenye yesifundo sami ngenza ucwaningo endimeni yezibalo. Ngenza uphenyo mayelana nendlela umfundi acabanga ngayo emqondweni uma ebhekene nezibalo.

Uma wena nengane yakho nivuma ukuba ingxenye yalolu cwaningo, umfundi uzobuzwa imibuzo okuzomele ayiphendule emayelana nomsebenzi wegeometry aphinde futhi abe sengxoxweni elalelwayo eqoshiwe. Konke lokhu kuzokwenzeka ngesikhathi sesikole kodwa ngaphandle kwesikhathi sokufunda emaklasini.

Akukho zinkinga nobungozi obungabhekana nengane yakho uma izibandakanya nalocwaningo ngaphandle kwalokhu okujwayelekileyo empilweni. Ulwazi oluzotholakala kulolucwaningo luzobe luyimfisho. Akukho lwazi kanye nemiphumela mayelana nengane yakho okuzophuma kuye emphakathini. Lonke ulwazi olutholakala kulocwaningo luzogcinwa ngokuphepha njengemthetho-mgomo yenyuvesi. Abantu abazoba nalolwazi ilabo abayinxenye yalocwaningo kanye nabahloli.

Imiphumela yalolu cwaningo izobandakanywa embhalweni onobuhlakani wokuthola iziqu enyuvesi kanti futhi kungenzeka kukhulunywe ngawo ezingqungqutheleni iphinde futhi ishicilelwe ezincwadini.

Ingane yakho ayiphoqelekile ukuba ibe yingxenye yalolu cwaningo uma ingathandi. Ingahoxa noma yinini ngaphandle kwesizathu. Uma unemibuzo noma kukhona okungakucacelanga, ungambuza umcwaningi akucacisele.

Okokugcina, ngiyabonga ngesikhathi sakho sokufunda lokhu.

Imininingwane yokuxhumana yomcwaningi: Alex Mulenga

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Imininingwane yokuxhumana yomphathi: Prof Vimolan Mudaly

mudalyv@ukzn.ac.za

0829770577

IFOMU YEMVUME YOMZALI /UMQAPHI WENGANE

Isihloko somsebenzi: *Ukuhlolwa komfundi indlela acabanga ngayo emqondweni uma ebhekene negeometry (phecelezi commognition) kwibanga leshumi nanye eklasini.*

Ingane yakho ayiphoqiwe ukuba izibandakanye kulolucwaningo, uma evuma ukuba yingxenye yalolu cwaningo, kodwa akhethe ukuhoxa esephakathi nalo ucwaningo unalo ilungelo lokuhoxa noma inini.

Ngicela uphendule lemibuzo elandelayo (beka umaka ebhokisini okhetha impendulo yalo).

Ngiyifunde ngayiqonda yonke imibandela mayelana nemibandeko yaleliphepha.	Yebo	Cha
Ngiyaqonda ukuthi ucwaningo lumayelana nani nokuthi imiphumela izosentshenziselwani.	Yebo	Cha
Ngiyayiqonda yonke inqubo mgomo ebandakanya ingane yami, ubungozi kanye nenzuzo ehlangene nalolu cwaningo.	Yebo	Cha
Ngiyazi ukuthi ukuzibandakanya kwengane yami okokuzikhethela futhi ingahoxa noma inini ngaphandle kwesizathu.	Yebo	Cha
Ngiyazi ukuthi imiphumela yengane yami izoba yimfihlo.	Yebo	Cha
Ngiyavuma ukuba ingane yami izibandakanye kulolu cwaningo.	Yebo	Cha

________________________________ ________________

Ukusayina komzali/ komqaphi Usuku

________________________________ ________________

Ukusayina komcwaningi Usuku

Appendix I: Parent/Guardian information sheet

PARENT/GUARDIAN INFORMATION SHEET

Title: *An exploration of learner's commognition when engaging with geometry in a grade eleven class*

I am a research student in the School of Education at University of KwaZulu Natal.

As part of my course, I am carrying out research in the area of mathematics. The thesis is investigating the learner's commognition when engaging with geometry.

If you and your child agree to take part in this study (s) he will be asked to answer questions in a task on geometry and take part in an audio recorded interview. Both will take place during school hours but not during class time.

I foresee no risks for your child's participation in the study, beyond those experienced in everyday life. The information gathered will be treated with privacy and anonymity. No information regarding your child will be revealed in the research. Information will be stored safely as per university regulations, with access only available to the research team and examiners and it will be destroyed.

The anonymised results from the study will be included in a thesis and may be discussed at conferences or published in a book or a journal.

Your child doesn't have to take part in this study if you / they don't want to and you/they can withdraw from the study at any time, without saying why. If you have any questions or if anything is unclear, just ask the researcher to explain it for you.

Finally, thank you for taking time to read this.

Researcher contact details: Alexander Mulenga

m

Supervisor contact details: Prof Vimolan Mudaly

mudalyv@ukzn.ac.za

Appendix J: Parent/Guardian consent form

PARENT/GUARDIAN CONSENT FORM

Title of Project: *An exploration of the learner's commognition when engaging with geometry in a grade eleven class.*

Your child is under no obligation to participate in this study, if they agree to participate, but at a later stage feel the need to withdraw, they are free to do so. It will not affect them in any way.

Please answer all of the following (tick the appropriate box)

I have read and understand the subject information sheet. Yes ☐ No ☐

I understand what the project is about, and what the results will be used for Yes ☐ No ☐

I am fully aware of all the procedures involving my child and any of the risks and benefits associated with the study. Yes ☐ No ☐

I know that my child's participation is voluntary and that I can withdraw from the project at any stage without giving any reason. Yes ☐ No ☐

I am aware that my child's results will be kept anonymous. Yes ☐ No ☐

I agree for my child to participate in the above study. Yes ☐ No ☐

________________________________ ________________

Signature of Parent/Guardian Date

________________________________ ________________

Signature of researcher Date

Appendix K: Editor's certificate

To whom it may concern

This is to confirm that the doctoral thesis submitted by

ALEXANDER MULENGA

has been language edited

TOPIC : An Exploration of Learners' Commognition when Engaging with Geometry in a Grade 11 Class

M. Govender

Date: 14/02/2024

Appendix L: Turnitin Report

AN EXPLORATTION OF THE LEARNERS' COMMOGNITION WHEN ENGAGING WITH GEOMETRY IN THE GRADE ELEVEN CLASSROOM

ORIGINALITY REPORT

17%	14%	11%	0%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

#	Source	Type	%
1	researchspace.ukzn.ac.za	Internet Source	2%
2	hdl.handle.net	Internet Source	1%
3	repository.stkipjb.ac.id	Internet Source	1%
4	uir.unisa.ac.za	Internet Source	<1%
5	files.eric.ed.gov	Internet Source	<1%
6	wiredspace.wits.ac.za	Internet Source	<1%
7	www.pmena.org	Internet Source	<1%
8	www.saarmste.org	Internet Source	<1%