

Normality-like Properties, Paraconvexity and Selections

By

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Dedication

*To my mother, Véronique Nkegue,
and my father, Michel Loufouma Makala.*

Preface

The study described in this thesis was carried out in the School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, from June 2009 to December 2012, under the supervision of Professor Valentin Gutev.

This study represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Narcisse Roland Loufouma Makala



December 2012

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I, *Narcisse Roland Loufouma Makala*, declare that

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Declaration 2 - Publications

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and include research presented in this thesis.

▷ Publication 1.

V. Gutev and N.R. Loufouma Makala, *Selections, Extensions and Collectionwise Normality*, J. Math. Anal. Appl. **368** (2010) 573 – 577.

(Sections 3 and 4 of the paper are parts of this PhD thesis, whereas sections 1 and 2 were parts of my MSc dissertation [33]).

▷ Publication 2.

N.R. Loufouma Makala, *Selections for Paraconvex-valued Mappings on Non-paracompact Domains*, Topology Appl. **159** (2012) 153 – 157.

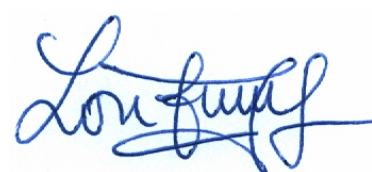
▷ Publication 3.

N.R. Loufouma Makala, *Selections, Paraconvexity and PF-normality*, (in preparation).

Professor V. Gutev



NR. Loufouma Makala



Abstract

In 1956, E. Michael proved his famous convex-valued selection theorems for l.s.c. mappings defined on spaces with higher separation axioms (paracompact, collectionwise normal, normal and countably paracompact, normal, and perfectly normal), [39]. In 1959, he generalized the convex-valued selection theorem for mappings defined on paracompact spaces by replacing “convexity” with “ α -paraconvexity”, for some fixed constant $0 \leq \alpha < 1$ (see, [42]). In 1993, P.V. Semenov generalized this result by replacing α with some continuous function $f : (0, \infty) \rightarrow [0, 1)$ (functional paraconvexity) satisfying a certain property called (PS), [63]. In this thesis, we demonstrate that the classical Michael selection theorem for l.s.c. mappings with a collectionwise normal domain can be reduced only to compact-valued mappings modulo Dowker’s extension theorem for such spaces. The idea used to achieve this reduction is also applied to get a simple direct proof of that selection theorem of Michael’s. Some other possible applications are demonstrated as well. We also demonstrate that the α -paraconvex-valued and the functionally-paraconvex valued selection theorems remain true for $\mathcal{C}'_\alpha(Y)$ -valued mappings defined on τ -collectionwise normal spaces, where τ is an infinite cardinal number. Finally, we prove that these theorems remain true for $\mathcal{C}_\alpha(Y)$ -valued mappings defined on τ -PF-normal spaces; and we provide a general approach to such selection theorems.

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CHAPTER 1

Introduction

1. Historical background: from classical extensions to selections

In his historical review on selections [20], V. Gutev wrote:

In 1856-1857, Dirichlet lectured on potential theory in Göttingen, and notes of those lectures were published by F. Grube in 1876. The subject treated in these notes is mathematical, but the influence of physics is apparent. In this set of lecture notes, Dirichlet's problem and Dirichlet's method of treating it (which came to be called Dirichlet's principle) both appear. The history of Dirichlet's principle is remarkable. Green, Dirichlet, Thomson, and others of their time regarded it as a completely sound method and used it freely. Then Riemann in his complex function theory showed it to be extraordinarily instrumental in leading to major results. However, all of these men were aware that the fundamental existence question was not settled, even before Weierstrass to announce his critique in 1870 which discredited the method for several decades. The demand for rigour in mathematics increased. In 1904 and 1905, Fréchet published a series of short papers on "abstract sets" or "abstract classes" that laid the groundwork of his thesis. In his first paper, the analogy between Weierstrass' theorem and Dirichlet's principle was given as the motivation to develop a general theory of continuous real functions on arbitrary sets that encompasses both theorems. In the second chapter of the first part of the thesis, Fréchet introduced an abstract notion

of distance, which he called “voisinage”. It is remarkable that at one point in the thesis, Fréchet replaced the third “V-class” axiom by a triangle inequality. This is what we nowadays call a metric space (following Hausdorff, 1914). In 1907, Lebesgue (in his paper on Dirichlet’s problem) was interested also in preserving only continuity by a possible extension, and proved an extension theorem for bounded functions on the plane. In 1915, Tietze published his famous extension theorem generalizing Lebesgue’s result. His motivation went beyond Lebesgue’s interpretation of Dirichlet’s problem, he was solving the Carathéodory-Fejér problem about plain Jordan curves. For his approach to solve this problem, Tietze used his extension result for a special case only. Probably, it was also his sense for generalizations which forced him to prove the result for all Euclidean spaces and even more, for metric spaces. In 1923, Tietze defined normal spaces, and in 1925 appeared in print Urysohn’s famous generalization of Tietze’s extension theorem for normal spaces. In many books, Urysohn’s result is described as Tietze’s theorem, but Tietze made no attempt to prove it in his papers, and it does not follow from his methods. This is what we nowadays call the *Tietze-Urysohn extension theorem*. In the mid 1950’s Ernest Michael wrote a series of fundamental papers relating familiar extension theorems to selections, thus laying down the foundation of the *theory of continuous selections*.

2. Michael’s selection theory

Nowadays, selections became an indispensable tool for many mathematicians working in vastly different areas. However, the key importance of Michael’s selection theory is not only in providing a comprehensive solution to diverse selection problems, but also in

the immediate inclusion of the obtained results into the general context of development of topology, [20].

In their paper [56], D. Repovš and P.V. Semenov wrote: “For a large number of those working in topology, functional analysis, multivalued analysis, approximation theory, convex geometry, mathematical economics, control theory, and several other areas, the year 1956 has always been strongly connected with the publication by Ernest Michael of two fundamental papers on continuous selections which appeared in the *Annals of Mathematics* [39, 40]. With sufficient precision that year marked the beginning of the theory of continuous selections of multivalued mappings.

Michael stated in his paper [39] that one of the most interesting and important problem in topology is the extension problem. The following general extension problem is considered: let X and Y be two topological spaces, $A \subset X$ closed, and $f : A \rightarrow Y$ be continuous. Under what conditions on X and Y does f have a continuous extension over X (or at least over some open $U \subset A$)? Sometimes, there are additional requirements on f , which frequently (as in the theory of fibre bundles) take the following form: For every $x \in X$, $f(x)$ must be an element of a pre-assigned subset of Y . This new problem, which we call the *selection problem*, is clearly more general than the extension problem, and presents a challenge even when A is the null set or a 1-point set (where the extension problem is trivial).

One of the versions of the axiom of choice states that for an arbitrary nonempty set X , there exists a mapping f which associates to each nonempty subset Y of X an element of Y , i.e. $f(Y) \in Y$ (see, for instance, [55, Theorem 0.10, (3)]), i.e. the axiom of choice guarantees the existence of a selection for any nonempty family of subsets of X . Note that there is no guarantee that the resulting selection is continuous. In our study, we are concerned about selections (for set-valued mappings $\varphi : X \rightarrow 2^Y$) which are continuous. Such continuous selections do not always exist; and they do exist only for a certain type of topological spaces having some “nice” topological structure (for instance, paracompact spaces, collectionwise normal spaces, \dots) for the domain X of

φ , and a “nice” algebraic structure (for instance, Banach spaces, Fréchet spaces, \dots) for the range Y of φ .

Selections are closely related to extensions, and most of the classical Michael's selection theorems are analogues and in most respects generalizations of ordinary extension theorems. So, in a sequence of Michael's papers [39, 40] such familiar extension theorems as Urysohn's characterization of normality, Kuratowski's extension theorem for finite-dimensional spaces, and the homotopy extension theorem were transformed (and thus essentially generalized) into selection theorems. Moreover, these Michael-type selection theorems establish that the existence of continuous selections for lower semi-continuous mappings $\varphi : X \rightarrow \mathcal{F}(Y)$ is actually equivalent to some higher separation axioms (like paracompactness, collectionwise normality, normality, \dots) of X .

There are four classical selection theorems: convex-valued, zero-dimensional, compact-valued, and finite-dimensional theorems. In our study, we will focus on the first type of theorems, i.e. the convex-valued selection theorems which is undoubtedly one of the most famous Michael's contributions in the theory of continuous selections for set-valued mappings. The paracompact version of the theorem states that a topological space X is paracompact if and only if for every Banach space Y , every lower semi-continuous mapping from X to the set of all nonempty closed and convex subsets of Y has a continuous selection [39], see also [44]. As Michael stated in [39], this result seems interesting from several points of views. In the first place, it fits naturally into the general scheme of extension and selection theorems (\dots), and thus highlights the relation of paracompactness to other separation properties. In the second place, it is the first characterization of paracompactness which deals with *continuous functions* rather than open coverings. Since functions are usually easier to handle than coverings, this feature can be quite helpful in certain proofs (\dots) Finally, and most important, the theorem yields the simple fact that every paracompact space, and hence every metric space, has the property in the theorem.

Michael's convex-valued selection theorem has many applications not only in general topology, but also in functional analysis, multivalued analysis, mathematical economic, convex geometry, and several other areas. Michael [37] later proved the theorem when the Banach space Y is substituted with a Fréchet space (i.e. a completely metrizable locally convex topological vector space). Three year later (in 1959), E. Michael generalized the convex-valued selection theorem for mappings defined on paracompact spaces by replacing "convexity" with " α -paraconvexity", for some fixed constant α such that $0 \leq \alpha < 1$; and proved in [42] (Theorem 2.1) that if X is paracompact, Y is a Banach space, and $\mathcal{F}_\alpha(Y)$ is the family of all nonempty closed and α -paraconvex subsets of Y , then every lower semi-continuous mapping $\varphi : X \rightarrow \mathcal{F}_\alpha(Y)$ has a continuous selection. This result is referred to as the paraconvex-valued selection theorem (for mappings defined on paracompact spaces). In that theorem, α is a fixed constant; and the theorem is true if $0 \leq \alpha < 1$, but fails if $\alpha = 1$. The case $\alpha = 0$ is in fact the convex-valued selection theorem, as α -paraconvexity coincides with convexity for $\alpha = 0$. Regarding this, another question is naturally raised: does this theorem remain true if for each $x \in X$, there is an $\alpha(x) < 1$, possibly different for different x , for which $\varphi(x)$ is $\alpha(x)$ -paraconvex? In 1993, P.V. Semenov [63] suggested a first attempt in answering the above question and generalized Michael's paraconvex-valued selection theorem by replacing the fixed constant α with a continuous function $f : (0, +\infty) \rightarrow [0, 1)$ satisfying a certain property called (*PS*) and introduced the concept of "functional paraconvexity"; and in [58], D. Repovš and P.V. Semenov considered a function $\alpha_P : (0, \infty) \rightarrow [0, 2)$, called the function of nonconvexity associated to each nonempty subset $P \subset Y$. Also, they obtained several applications of selections for paraconvex-valued mappings, see [53, 54, 59, 60, 61, 64].

3. Purpose of this study

All the theorems mentioned above, i.e. Michael's α -paraconvex-valued selection theorem and Semenov's functionally paraconvex-valued selection theorem, were obtained

only for mappings defined on paracompact spaces. The non-paracompact versions (i.e., for mappings defined on τ -collectionwise normal and τ -PF-normal spaces, for instance) of these theorems were not previously examined. This study aims to investigate the existence of continuous selections for paraconvex-valued mappings defined on non-paracompact spaces; in particular, we study the cases where the domain is a τ -collectionwise normal space and a τ -PF normal space. Also, we investigate the equivalence between the classical Michael's selection theorem for $\mathcal{C}'(Y)$ -valued mappings and for $\mathcal{C}(Y)$ -valued mappings, when the domain of the mappings is τ -collectionwise normal.

4. Organisation of the thesis and contributions

This thesis is organized as follows:

In Chapter II, we begin with some background in General Topology which will be very important and will be used in the sequel.

Chapter III is dedicated to the study of Michael's selection theory; in particular, we give some properties and results related to the convex-valued selection theory for set-valued mappings.

In Chapter IV, we prove that the classical Michael selection theorem for l.s.c. mappings with a collectionwise normal domain can be reduced only to compact-valued mappings modulo Dowker's extension theorem for such spaces. The idea used to achieve this reduction is also applied to get more results on selections and collectionwise normality by suggesting an easy and direct proof of the convex-valued selection theorem for mappings defined on collectionwise normal spaces. Also, some results on controlled selections and countable paracompactness are given.

In Chapter V, we prove the non-paracompact versions of these theorems; in particular, we demonstrate that both Michael's paraconvex-valued selection theorem and Semenov's functionally paraconvex-valued selection theorem remain valid for mappings

defined on τ -collectionwise normal spaces, where τ is an infinite cardinal number, provided that the mappings are nonempty compact valued or have images the whole of Y . We also prove the τ -PF-normal version of the theorem for nonempty compact-valued mappings and provide a general approach to such selection theorems.

CHAPTER 2

Preliminaries and Backgrounds

In this chapter, we review some concepts/notions in General Topology which will be very important in the study of the paraconvex-valued selection theory. Most of the information may be found in [13, 14, 32, 33, 62, 69].

1. Topological Spaces

DEFINITION 2.1.1. A *topological space* is a pair (X, τ) consisting of a set X and a family τ of subsets of X satisfying the following conditions:

- (O_1) \emptyset and X belong to τ ,
- (O_2) The union of any family of members of τ is again in τ ,
- (O_3) If $U, V \in \tau$, then $U \cap V \in \tau$.

The family τ is called *topology* for X and its members are called *open sets*.

EXAMPLE 2.1.2. Let X be any set, then the power set $\mathcal{P}(X)$ is a topology for X called the *discrete topology*, and the pair $(X, \mathcal{P}(X))$ is called a *discrete topological space*.

EXAMPLE 2.1.3. Let X be any set, then $\tau = \{\emptyset, X\}$ is a topology on X called the *indiscrete topology*, and the corresponding pair is called an *indiscrete topological space*.

EXAMPLE 2.1.4. Let (X, τ) be a topological space, and let $Y \subset X$. Consider $\tau_Y = \{O \cap Y : O \in \tau\}$. Then τ_Y is a topology on Y called the *subspace topology*, and (Y, τ_Y) is called a *subspace*.

In the sequel, we shall often say that X is a topological space to express that X is a set with a given topology.

If X is a topological space and $x \in X$, a *neighborhood* of x is a set U which contains an open set V_x containing x . A set U is open if and only if for every $x \in U$, there is a neighborhood V_x of x such that $V_x \subset U$.

DEFINITION 2.1.5. A subset F of a topological space X is called *closed* if $X \setminus F$ is an open set.

This leads to the following properties of closed sets:

PROPOSITION 2.1.6. *For any topological space X , the following hold:*

(C₁) \emptyset and X are closed,

(C₂) *The intersection of any nonempty family of closed sets in X is again closed,*

(C₃) *If $F, G \subset X$ are closed, then $F \cup G$ is also closed.*

To any subset $A \subset X$ of a topological space X , one can associated a closed set \bar{A} , called the *closure* of A , which is defined by $x \in \bar{A}$ if and only if $V \cap A \neq \emptyset$ for every neighborhood V of x .

PROPOSITION 2.1.7. *Let X be a space and $A \subset X$. Then,*

$$\bar{A} = \bigcap \{C \subset X : A \subset C \text{ and } C \text{ is closed}\}.$$

DEFINITION 2.1.8. A subset $A \subset X$ of a topological space X is said to be *dense* in X if $\bar{A} = X$. Equivalently, A is dense in X if and only if every nonempty open set in X has a nonempty intersection with A . A topological space containing a countable dense subset is called *separable*.

For any set $A \subset X$, there is a largest open set contained in A . This set is called the *interior* of A and is denoted by $\text{Int}(A)$ or $\overset{\circ}{A}$.

$$\text{Int}(A) = \bigcup \{U \subset X : U \subset A \text{ and } U \text{ is open}\}.$$

A point $x \in \text{Int}(A)$ if and only if there is a neighborhood U of x such that $U \subset A$.

DEFINITION 2.1.9. The *weight* $w(X)$ of a topological space (X, \mathcal{T}) is the least cardinal of a base for the topology of X , i.e.

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\},$$

where $|A|$ denote the cardinality of the set A . Here, a family $\mathcal{B} \subset \mathcal{T}$ is called a *base* for X if for every $x \in X$ and $U \in \mathcal{T}$, with $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subset U$. Note that $\mathcal{B} \subset \mathcal{T}$ is a base for the topology on X if and only if for every $U \in \mathcal{T}$, there is an $\mathcal{U} \subset \mathcal{B}$, with $U = \bigcup \mathcal{U}$.

DEFINITION 2.1.10. A set F is called an F_σ -set if it can be written as a countable union of closed sets; i.e. $F = \bigcup\{A_n : n \in \mathbb{N}\}$, where each A_n , $n \in \mathbb{N}$, is closed. The complement G of an F_σ -set is called a G_δ -set, and can be written as a countable intersection of open sets; i.e., $G = \bigcap\{O_n : n \in \mathbb{N}\}$, where each O_n , $n \in \mathbb{N}$, is open. Obviously, every closed (resp. open) set is an F_σ -set (resp. a G_δ -set).

DEFINITION 2.1.11 (Continuous maps). Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is *continuous* if $f^{-1}(V)$ is open in X for every open subset $V \subset Y$. Equivalently, f is continuous if $f^{-1}(F)$ is closed in X for every closed subset $F \subset Y$.

The following result for continuous maps will be necessary in the sequel (see proofs of part (a) of Theorems 5.1.3, 5.2.6, 5.4.19, and 5.5.21 in Chapter 5).

PROPOSITION 2.1.12. *Let X and Y be spaces, $\{U_n : n \in \mathbb{N}\}$ be an increasing sequence of open subsets of X such that $X = \bigcup\{U_n : n \in \mathbb{N}\}$. Also, for each $n \in \mathbb{N}$, let $A_n = \overline{U_n}$, and $f_n : A_n \rightarrow Y$ be continuous such that $f_{n+1} \upharpoonright A_n = f_n$. Then, the map $f : X \rightarrow Y$ defined by $f(x) = f_n(x)$, $x \in A_n$, is also continuous.*

PROOF. Since $f \upharpoonright A_n$ is continuous, $f \upharpoonright U_n$ is also continuous. So, for any open set $O \subset Y$, $(f \upharpoonright U_n)^{-1}(O)$ is open in U_n , hence it is also open in X and

$$f^{-1}(O) = \bigcup\{(f \upharpoonright U_n)^{-1}(O) : n \in \mathbb{N}\},$$

which is open in X as a union of open sets. So, f is continuous. \square

1.1. Metric Spaces.

DEFINITION 2.1.13. Let X be any set. A function $d : X \times X \rightarrow \mathbb{R}$ is called a *metric* if for every $x, y, z \in X$, the following conditions are satisfied:

- (a) $d(x, y) = 0 \Leftrightarrow x = y$,
- (b) $d(x, y) = d(y, x)$,
- (c) $d(x, y) \leq d(x, z) + d(z, y)$.

It follows from the definition that $d(x, y)$ is always non-negative; it is called the *distance* between the points x and y , and the pair (X, d) is called a *metric space*.

In a metric space (X, d) , we define the *open ball* and the *closed ball* with center x and radius $r > 0$ respectively by

$$B_r^d(x) = \{y \in X : d(x, y) < r\} \text{ and } D_r^d(x) = \{y \in X : d(x, y) \leq r\}.$$

For a nonempty subset $A \subset X$, we use $B_r^d(A) = \{y \in X : d(y, A) < r\}$. Note that for a metric space (X, d) , the distance between a point $x \in X$ and a subset $A \subset X$ is given by $d(x, A) = \inf\{d(x, y) : y \in A\}$; and that $\bar{A} = \{x \in X : d(x, A) = 0\}$.

The family $\mathcal{B} = \{B_r^d(x) : x \in X \text{ and } r > 0\}$ is not necessarily a topology on X , but it is a base for a topology on X which is called the *topology generated by the metric d* . A topological space X is called *metrizable* if its topology is generated by a metric d on X . Let us recall that the *diameter* of a subset A of a metric space (X, d) is given by

$$\text{diam}_d(A) = \sup\{d(x, y) : x, y \in A\}.$$

The subset $A \subset X$ is *bounded* if $\text{diam}_d(A) < \infty$.

A subset A of a topological space X is called a *retract* of X if there is a continuous map $r : X \rightarrow A$ such that the restriction of r to A is the identity map $\text{id}_A : A \rightarrow A$; i.e. $r(a) = a$, for all $a \in A$. The map r is called a *retraction* of X onto A . A metrizable space Y is called an *absolute retract* for the metrizable spaces if whenever Y is a closed subset of a metrizable space X , there exists a retraction $f : X \rightarrow Y$; or equivalently,

if whenever Y is embedded as a closed subset of a metrizable space X , there exists a retraction $f : X \rightarrow Y$.

1.2. Completeness in Metric Spaces. Let (X, d) be a metric space. We say that a sequence $\{x_n\} \subset X$ converges to $x \in X$, and write $x_n \rightarrow x$, if for every $\varepsilon > 0$, there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \text{ for every } n \geq N(\varepsilon)$$

DEFINITION 2.1.14. A sequence $\{x_n\} \subset (X, d)$ is said to be a *Cauchy sequence* if for every $\varepsilon > 0$, there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$d(x_m, x_n) < \varepsilon \text{ for every } m, n \geq N(\varepsilon).$$

Note that in a metric space (X, d) , every Cauchy sequence is bounded, and that every convergent sequence is a Cauchy sequence.

DEFINITION 2.1.15. A metric space (X, d) is *complete* if every Cauchy sequence of X converges in X .

2. Normed Spaces

DEFINITION 2.2.16. Let S be a real vector space. A function $\|\cdot\| : S \rightarrow \mathbb{R}$ is called a *norm* on S if the following conditions are satisfied, for every $x, y \in S$ and $\lambda \in \mathbb{R}$:

- (N_1) $\|x\| = 0$ iff $x = 0$;
- (N_2) $\|\lambda x\| = |\lambda| \|x\|$;
- (N_3) $\|x + y\| \leq \|x\| + \|y\|$.

For every $x \in S$, the non-negative real number $\|x\|$ is called the norm of x .

A vector space equipped with a norm $\|\cdot\|$ is called a *normed space*.

Note that every normed space may be regarded as a metric space; i.e., if $\|\cdot\|$ is a norm on a vector space S , then a metric d on S is defined by letting, for every $x, y \in S$

$$d(x, y) = \|x - y\|.$$

The metric arising from a norm and the corresponding topology are referred to as the metric and the topology *induced (or generated)* by the norm $\|\cdot\|$. Every normed space is a topological vector space with respect to the topology generated by its norm.

2.1. Banach Spaces.

DEFINITION 2.2.17. A normed space $(X, \|\cdot\|)$ is called a *Banach space* if it is a complete metric space with respect to the metric generated by its norm $\|\cdot\|$.

EXAMPLE 2.2.18. The set \mathbb{R} of real numbers, equipped with the metric $d(x, y) = |x - y|$, is a Banach space.

EXAMPLE 2.2.19. For a space X , the function (vector) space

$$C^*(X) = \{f \mid f : X \rightarrow \mathbb{R} \text{ is continuous and bounded}\}$$

equipped with the norm $\|f\| = \sup\{|f(x)| : x \in X\}$, is a Banach space. Here, the norm $\|\cdot\|$ on $C^*(X)$ is called the *sup-norm*. In particular, for a compact space X ,

$$C(X) = \{f \mid f : X \rightarrow \mathbb{R} \text{ is continuous}\} = C^*(X)$$

is a Banach space with respect to the sup-norm on it.

Note that if Y is a real vector space, A and B are two subsets of Y , $u \in Y$ and $k \in \mathbb{R}$, then we have:

- ▷ $u \pm B = \{u \pm v : v \in B\}$,
- ▷ $A + B = \{u + v : u \in A, \text{ and } v \in B\}$,
- ▷ $kA = \{ku : u \in A\}$.

DEFINITION 2.2.20 (Convex set). A subset A of a vector space X is called *convex* if given any two points $u, v \in A$, the segment $[u, v] \subset A$; that is

$$tu + (1 - t)v \in A \text{ for every } u, v \in A \text{ and } 0 \leq t \leq 1.$$

REMARK 2.2.21. A subset A of a vector space X is convex if and only if $tA + (1 - t)A \subset A$, for every $0 \leq t \leq 1$.

PROPOSITION 2.2.22. *If \mathcal{F} is a nonempty family of convex sets in a vector space Y , then its intersection $\bigcap \mathcal{F}$ is also a convex set.*

Note that the union of two convex sets is not always a convex set. However, one has:

PROPOSITION 2.2.23. *If \mathcal{F} is a family of convex sets in a vector space X which is totally ordered by the inclusion, then its union $\bigcup \mathcal{F}$ is also a convex set.*

3. Separation Axioms

3.1. Normality. Recall that a space X is called T_1 if every singleton is closed in X .

DEFINITION 2.3.24 (Normal Space). A space X is *normal* if it is a T_1 -space and given any two disjoint closed subsets A and B of X , there exist disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$. Normal spaces are sometimes called T_4 -spaces.

The following property of normal spaces is well-known (See, for instance, [13, 14, 69]).

LEMMA 2.3.25 (Urysohn's Lemma, see [14, 69] or [33, Lemma 1.4.39]). *A T_1 -space X is normal if and only if whenever A and B are disjoint closed subsets in X , there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

The subsets A and B in Lemma 2.3.25 are said to be *completely separated*, and we say that f separates A and B . Note that if A and B are completely separated, then A and B are disjoint ; and so are their closures.

COROLLARY 2.3.26. *Let X be a normal space, $F \subset X$ be closed and $U \subset X$ be open such that $F \subset U$. Then, there is an open F_σ -set V such that $F \subset V \subset \overline{V} \subset U$.*

PROOF. If $F \subset X$ is closed and $U \subset X$ is open such that $F \subset U$, then $X \setminus U$ and F are two disjoint closed subsets of X . By Lemma 2.3.25, there exists a continuous $f : X \rightarrow [0, 1]$ such that $F \subset f^{-1}(0)$ and $X \setminus U \subset f^{-1}(1)$. Let $V = f^{-1}([0, \frac{1}{2}))$. Then, V is open in X and it is an F_σ -set because $V = \bigcup \{f^{-1}([0, \frac{1}{2} - \frac{1}{4^n}]) : n \in \mathbb{N}\}$. Moreover, we have $F \subset V \subset \overline{V} \subset U$. \square

The following theorem was proved by S. Lefschetz. Its proof can be found in [13, 33, 51, 69]. In this theorem and what follows, a family $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ of subsets of a topological space X is said to be a *cover* of X if $X = \bigcup \{U_\alpha : \alpha \in \mathcal{A}\}$. An *open cover* of X is a cover all of whose members are open sets. \mathcal{U} is said to be *point-finite* if every point in X is contained in a finite number of members of \mathcal{U} ; that is, for every $x \in X$, the set $\{\alpha \in \mathcal{A} : x \in U_\alpha\}$ is finite.

THEOREM 2.3.27 (Lefschetz Theorem). *If $\{U_s : s \in \mathcal{A}\}$ is a point-finite open cover of a normal space X , then there exists an open cover $\{V_s : s \in \mathcal{A}\}$ of X such that $\overline{V_s} \subset U_s$, for every $s \in \mathcal{A}$.*

DEFINITION 2.3.28. Two subsets A and B of a topological space X are said to be *separated* if $A \cap \overline{B} = \emptyset$ and $B \cap \overline{A} = \emptyset$.

Two disjoint sets are separated if and only if neither of them contains accumulation points of the other. In particular, any two disjoint closed sets are separated. Also, any two disjoint open sets are separated. If A and B are separated and $A_1 \subset A$ and $B_1 \subset B$, then A_1 and B_1 are also obviously separated, [13]. The following result is due to Urysohn.

PROPOSITION 2.3.29 ([13, Problem 2.7.2. (a)]). *For every pair A, B of separated F_σ -subsets of a normal space X , there exist open sets $U \subset X$ and $V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

PROOF. Let $A = \bigcup\{A_n : n \in \mathbb{N}\}$ and $B = \bigcup\{B_n : n \in \mathbb{N}\}$, where A_n and B_n are closed for each $n \in \mathbb{N}$. Since $A_n \subset A$ and $A \cap \overline{B} = \emptyset$, we have $A_n \cap \overline{B} = \emptyset$, i.e. $A_n \subset X \setminus \overline{B}$. By Corollary 2.3.26, there are open sets $G_n \subset X$ such that $A_n \subset G_n \subset \overline{G_n} \subset X \setminus \overline{B}$, for each $n \in \mathbb{N}$. Hence, the family $\{G_n : n \in \mathbb{N}\}$ is an open cover of A such that $\overline{G_n} \cap \overline{B} = \emptyset$. Using the same argument, one can construct an open cover $\{H_n : n \in \mathbb{N}\}$ of B such that $\overline{H_n} \cap \overline{A} = \emptyset$. Define

$$U_n = G_n \setminus \bigcup\{\overline{H_k} : k \leq n\} \text{ and } V_n = H_n \setminus \bigcup\{\overline{G_k} : k \leq n\},$$

and let $U = \bigcup\{U_n : n \in \mathbb{N}\}$ and $V = \bigcup\{V_n : n \in \mathbb{N}\}$. Then, U and V are disjoint and $A \subset U$ and $B \subset V$. \square

Proposition 2.3.29 allows us to show that normality is hereditary with respect to F_σ -sets.

PROPOSITION 2.3.30 ([13, Problem 2.7.2 (b)]). *If X is a normal space and $A \subset X$ is an F_σ -set, then A is also normal.*

PROOF. Let $A \subset X$ be an F_σ -set of a normal space X , A_1 and A_2 be two disjoint closed subsets of A . Then, A_1 and A_2 are also F_σ -sets. By Proposition 2.3.29, A_1 and A_2 are separated in X . \square

Let us recall the following separation axioms: A topological space X is called

- ▷ T_2 or *Hausdorff* if for every $x, y \in X$ such that $x \neq y$, there exist disjoint open sets $U, V \subset X$ containing x and y respectively.
- ▷ T_3 or *regular* if it is T_1 and for every $A \subset X$ closed and $x \notin A$, there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $A \subset V$.
- ▷ $T_{3\frac{1}{2}}$ or *completely regular* if it is T_1 and for every $A \subset X$ closed and $x \notin A$, there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$.

Note that we have the following implications: $T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$,

and by Lemma 2.3.25, we have $T_4 \Rightarrow T_{3\frac{1}{2}}$.

The next characterization of normality is due to Tietze and Urysohn (see, [69]).

THEOREM 2.3.31 (Tietze-Urysohn's Extension Theorem). *A T_1 -space X is normal if and only if whenever A is a closed subset of X and $f : A \rightarrow \mathbb{R}$ is continuous, there exists a continuous $g : X \rightarrow \mathbb{R}$ such that $g(x) = f(x)$ for every $x \in A$.*

3.2. Paracompact spaces. A family \mathcal{V} of subsets of a space X is called a *refinement* of a cover \mathcal{U} of X if \mathcal{V} is also a cover of X and for every $V \in \mathcal{V}$, there is some $U \in \mathcal{U}$ such that $V \subset U$. It is called an *open refinement* if all its members are open sets. The family \mathcal{U} is said to be *locally finite* if every point of X has a neighborhood V_x which intersects only finitely many members of the family \mathcal{U} ; that is the set $\{\alpha \in \mathcal{A} : U_\alpha \in \mathcal{U} \text{ and } V_x \cap U_\alpha \neq \emptyset\}$ is finite. It is said to be *σ -locally finite* if $\mathcal{U} = \bigcup\{\mathcal{U}_i : i \in \mathbb{N}\}$, where \mathcal{U}_i is locally finite for each $i \in \mathbb{N}$. Let us also recall that a *finite cover* of a space X is a cover of X which has a finite number of members. A subfamily \mathcal{V} of a cover \mathcal{U} of a space X is called *subcover* if it is also a cover of X .

DEFINITION 2.3.32 (Paracompact spaces). A Hausdorff space X is *paracompact* if every open cover of X has a locally finite open refinement.

PROPOSITION 2.3.33 (see, [13, Theorem 5.1.5]). *Every paracompact space is normal.*

DEFINITION 2.3.34 (Compact spaces). A Hausdorff space X is *compact* if every open cover of X has a finite subcover.

REMARK 2.3.35. Every compact space is paracompact.

The following result is well-known and is due to H. A. Stone.

THEOREM 2.3.36 ([66, Corollary 1]). *Every metric space is paracompact.*

3.3. Partition of Unity.

DEFINITION 2.3.37. Let X be a topological space, and let $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ be an open cover of X . A family $\mathcal{P} = \{p_\alpha : \alpha \in \mathcal{A}\}$ of continuous function $p_\alpha : X \rightarrow [0, 1]$ is called a *partition of unity* on X subordinated to the cover $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ if

- (1) $p_\alpha^{-1}((0, 1]) \subset U_\alpha$ for every $\alpha \in \mathcal{A}$,
- (2) $\sum_{\alpha \in \mathcal{A}} p_\alpha(x) = 1$ for every $x \in X$.

Note that (1) is equivalent to $X \setminus U_\alpha \subset p_\alpha^{-1}(0)$ for every $\alpha \in \mathcal{A}$, i.e. $p_\alpha(x) = 0$ for every $x \in X \setminus U_\alpha$. Also, $\{p_\alpha^{-1}((0, 1]) : \alpha \in \mathcal{A}\}$ is a cover of X , hence it is a refinement of \mathcal{U} . Thus, it makes sense to talk about properties of \mathcal{P} related to covers.

DEFINITION 2.3.38. A partition of unity $\mathcal{P} = \{p_\alpha : \alpha \in \mathcal{A}\}$ is locally finite if $\{p_\alpha^{-1}((0, 1]) : \alpha \in \mathcal{A}\}$ is a locally finite cover of X .

THEOREM 2.3.39 ([51], see also [33]). *If X is a normal space and $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ is a locally finite open cover of X , then there exists a partition of unity subordinated to this cover \mathcal{U} .*

The following theorems give other characterizations of paracompactness.

THEOREM 2.3.40 ([36], see also [33]). *For a T_1 -space X , the following are equivalent:*

- (a) X is paracompact.
- (b) Every open cover of X has a locally finite partition of unity subordinated to it.
- (c) Every open cover of X has a partition of unity subordinated to it.

THEOREM 2.3.41 ([13, 36]). *For a regular space X , the following are equivalent:*

- (a) X is paracompact.
- (b) Every open cover of X has a σ -locally finite open refinement.
- (c) Every open cover of X has a locally finite refinement.

(d) *Every open cover of X has a locally finite closed refinement.*

The next result show that paracompactness is hereditary with respect to F_σ -subsets, hence to closed subsets.

PROPOSITION 2.3.42 ([36, 69]). *Every F_σ -subset of a paracompact space is also paracompact.*

PROOF. Let A be an F_σ -subset of a paracompact space X , i.e. $A = \bigcup\{A_n : n \in \mathbb{N}\}$, where each A_n is closed in X . Let $\{U_s : s \in S\}$ be an open cover of A . For each U_s , there is an open set $V_s \subset X$ such that $U_s = A \cap V_s$. For each $n \in \mathbb{N}$, the family $\{V_s : s \in S\} \cup \{X \setminus A_n\}$ is an open cover of X . Since X is paracompact, it has a locally finite open refinement \mathscr{W}_n . Let

$$\mathscr{V}_n = \{A \cap W : W \in \mathscr{W}_n\}.$$

Then, each \mathscr{V}_n is a locally finite family of open subsets of A and $\bigcup\{\mathscr{V}_n : n \in \mathbb{N}\}$ is a refinement of $\{U_s : s \in S\}$. Thus, $\{U_s : s \in S\}$ has a σ -locally finite open refinement. By Theorem 2.3.41, A is paracompact. \square

3.4. Countably paracompact spaces.

DEFINITION 2.3.43 ([10]). A T_2 -space X is called *countably paracompact* if every countable open cover of X has a locally finite open refinement.

Clearly, every compact, paracompact, or countably compact space is also countably paracompact. Note that unlike paracompact spaces, a countably paracompact space is not necessarily normal, and there are normal spaces which are not countably paracompact. The latter are known as *Dowker spaces*.

The following theorem gives other characterizations of countably paracompact spaces and normal countably paracompact spaces.

THEOREM 2.3.44 ([13, Theorem 5.2.1]). *For a T_2 -space X , the following are equivalent:*

- (a) X is countably paracompact.
- (b) For every countable open cover $\{U_n : n \in \mathbb{N}\}$ of X , there exists a locally finite open cover $\{V_n : n \in \mathbb{N}\}$ of X such that $V_n \subset U_n$, for every $n \in \mathbb{N}$.
- (c) For every increasing sequence $W_1 \subset W_2 \subset \dots$ of open subsets of X satisfying $\bigcup\{W_n : n \in \mathbb{N}\} = X$, there exists a sequence F_1, F_2, \dots of closed subsets of X such that $F_n \subset W_n$, $n \in \mathbb{N}$, and $\bigcup\{\text{Int}(F_n) : n \in \mathbb{N}\} = X$.
- (d) For every decreasing sequence $F_1 \supset F_2 \supset \dots$ of closed subsets of X satisfying $\bigcap\{F_n : n \in \mathbb{N}\} = \emptyset$, there exists a sequence W_1, W_2, \dots of open subsets of X such that $F_n \subset W_n$, $n \in \mathbb{N}$, and $\bigcap\{\overline{W}_n : n \in \mathbb{N}\} = \emptyset$.

PROOF. (a) \Rightarrow (b). Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open cover of a countably paracompact space X . By definition, \mathcal{U} has a locally finite open refinement \mathcal{V} . For each $V \in \mathcal{V}$, choose an $n_V \in \mathbb{N}$ such that $V \subset U_{n_V}$ and let $V_n = \bigcup\{V : n_V = n\}$. The family $\{V_n : n \in \mathbb{N}\}$ is as required.

(b) \Rightarrow (c). Since $\{W_n : n \in \mathbb{N}\}$ is a countable open cover of X , by (b), there is a locally finite open cover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ of X such that $V_n \subset W_n$, for every $n \in \mathbb{N}$. We let $F_n = X \setminus \bigcup\{V_k : k > n\}$. Then, each F_n is closed and $F_n \subset \bigcup\{V_k : k \leq n\}$. Since $\bigcup\{V_k : k \leq n\} \subset \bigcup\{W_k : k \leq n\} = W_n$, we have $F_n \subset W_n$ for each $n \in \mathbb{N}$. Since \mathcal{V} is locally finite, every $x \in X$ has a neighborhood intersecting only finitely many members of \mathcal{V} . This implies that x is in some $\text{Int}(F_n)$, i.e. $\bigcup\{\text{Int}(F_n) : n \in \mathbb{N}\} = X$.

(c) \Leftrightarrow (d). This follows from De Morgan's laws.

(c) \Rightarrow (a). Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open cover of X . The family $\{W_n : n \in \mathbb{N}\}$, with $W_n = \bigcup\{U_k : k \leq n\}$, forms an increasing sequence of open subsets of X such that $\bigcup\{W_n : n \in \mathbb{N}\} = X$. By (c), there exists a sequence F_1, F_2, \dots of closed subsets of X such that $F_n \subset W_n$, $n \in \mathbb{N}$, and $\bigcup\{\text{Int}(F_n) : n \in \mathbb{N}\} = X$. The family $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$, with $V_n = U_n \setminus \bigcup\{F_k : k < n\}$, is a locally finite open refinement of \mathcal{U} . Indeed, since $\bigcup\{F_k : k < n\} \subset \bigcup\{W_k : k < n\} \subset \bigcup\{U_k : k < n\}$, we have by taking the complement that $U_n \setminus \bigcup\{U_k : k < n\} \subset U_n \setminus \bigcup\{F_k : k < n\} = V_n$,

which implies that \mathcal{V} is a cover of X . Moreover, it is clear that each V_n is open and $V_n \subset U_n$ for every $n \in \mathbb{N}$. To show that \mathcal{V} is locally finite, note that every $x \in X$ has a neighborhood $\text{Int}(F_k)$ which does not intersect any of the sets V_n if $n > k$. \square

COROLLARY 2.3.45 ([13]). *A normal space X is countably paracompact if and only if for every decreasing sequence $F_1 \supset F_2 \supset \dots$ of closed subsets of X satisfying $\bigcap \{F_n : n \in \mathbb{N}\} = \emptyset$, there exists a sequence W_1, W_2, \dots of open subsets of X such that $F_n \subset W_n$, for each $n \in \mathbb{N}$, and $\bigcap \{W_n : n \in \mathbb{N}\} = \emptyset$.*

THEOREM 2.3.46 ([13]). *For a T_1 -space X , the following are equivalent:*

- (a) X is countably paracompact and normal.
- (b) For every countable open cover $\{U_n : n \in \mathbb{N}\}$ of X , there exists a locally finite open cover $\{V_n : n \in \mathbb{N}\}$ of X such that $\bar{V}_n \subset U_n$, for every $n \in \mathbb{N}$.
- (c) For every countable open cover $\{U_n : n \in \mathbb{N}\}$ of X , there exists a closed cover $\{F_n : n \in \mathbb{N}\}$ of X such that $F_n \subset U_n$, for every $n \in \mathbb{N}$.

PROOF. (a) \Rightarrow (b). If X is countably paracompact and normal and $\{U_n : n \in \mathbb{N}\}$ is a countable open cover of X , then by (b) of Theorem 2.3.44, there exists a locally finite open cover $\{W_n : n \in \mathbb{N}\}$ of X such that $W_n \subset U_n$, for every $n \in \mathbb{N}$. By Theorem 2.3.27, since $\{W_n : n \in \mathbb{N}\}$ is point finite (being locally finite), there is an open (locally finite) cover $\{V_n : n \in \mathbb{N}\}$ of X such that $\bar{V}_n \subset W_n \subset U_n$, $n \in \mathbb{N}$.

(b) \Rightarrow (c). This is obvious (take $F_n = \bar{V}_n$, for each $n \in \mathbb{N}$).

(c) \Rightarrow (a). Suppose that (c) holds. Let us show that X is normal. If A_1 and A_2 are two disjoint closed subsets of X , then $U_1 = X \setminus A_1$ and $U_2 = X \setminus A_2$ are open subsets that cover X . By (c), there are closed subsets F_1 and F_2 of X such that $F_1 \subset U_1 = X \setminus A_1$ and $F_2 \subset U_2 = X \setminus A_2$, with $F_1 \cup F_2 = X$. This implies that $A_1 \subset X \setminus F_1 = W_1$ and $A_2 \subset X \setminus F_2 = W_2$ and W_1 and W_2 are open subsets of X such that $W_1 \cap W_2 = \emptyset$. Indeed, suppose to the contrary that $W_1 \cap W_2 \neq \emptyset$. Then, there is some $x \in W_1 \cap W_2$, i.e. $x \notin F_1$ and $x \notin F_2$, or $x \notin F_1 \cup F_2 = X$, which is absurd. So, $W_1 \cap W_2 = \emptyset$. Hence,

X is normal. Finally, applying De Morgan's law to (c), one has that for every sequence $\{F_n : n \in \mathbb{N}\}$ of closed subsets of X satisfying $\bigcap\{F_n : n \in \mathbb{N}\} = \emptyset$, there exists a sequence W_1, W_2, \dots of open subsets of X such that $F_n \subset W_n$, for each $n \in \mathbb{N}$, and $\bigcap\{W_n : n \in \mathbb{N}\} = \emptyset$. Hence, by Corollary 2.3.45, X is countably paracompact. \square

The following insertion theorem due to Dowker [10] gives a characterization of countably paracompact and normal spaces in terms of semi-continuous functions. Let us first recall that a function $\xi : X \rightarrow \mathbb{R}$ is *lower (upper) semi-continuous* if the set

$$\{x \in X : \xi(x) > r\} \text{ (respectively, } \{x \in X : \xi(x) < r\})$$

is open in X for every $r \in \mathbb{R}$.

THEOREM 2.3.47 ([10]). *The following are equivalent for a T_1 -space X :*

- (a) X is countably paracompact and normal.
- (b) If $g : X \rightarrow \mathbb{R}$ is lower semi-continuous and $h : X \rightarrow \mathbb{R}$ is upper semi-continuous such that $h(x) < g(x)$ for every $x \in X$, then there exists a continuous $f : X \rightarrow \mathbb{R}$ such that $h(x) < f(x) < g(x)$, for every $x \in X$.

PROOF. (a) \Rightarrow (b). Let X be countably paracompact and normal, $g : X \rightarrow \mathbb{R}$ be lower semi-continuous and $h : X \rightarrow \mathbb{R}$ be upper semi-continuous such that $h(x) < g(x)$, $x \in X$. If $r \in \mathbb{Q}$, we let $O_r = \{x \in X : h(x) < r < g(x)\}$. Since h is upper semi-continuous, the set $H_r = \{x \in X : h(x) < r\}$ is open, and since g is lower semi-continuous, the set $G_r = \{x \in X : g(x) > r\}$ is also open. So, $O_r = H_r \cap G_r$ is also open. Since for every $x \in X$, $h(x) < g(x)$, there is some rational number $r(x)$ such that $h(x) < r(x) < g(x)$, i.e. $x \in O_{r(x)}$. Hence, $\{O_r : r \in \mathbb{Q}\}$ is a countable open cover of X . Since X is countably paracompact, there is a locally finite open cover $\{U_r : r \in \mathbb{Q}\}$ of X such that $U_r \subset O_r$, $r \in \mathbb{Q}$. Since X is normal and every locally finite cover is a fortiori point-finite, by Theorem 2.3.27, there is an open (locally finite) cover $\{V_r : r \in \mathbb{Q}\}$ of X such that $\overline{V_r} \subset U_r$, i.e. $\overline{V_r} \cap (X \setminus U_r) = \emptyset$. Define a continuous function $f_r : X \rightarrow [-\infty, +\infty]$, with $-\infty \leq f_r(x) \leq r$, such that $f_r(x) = -\infty$ if

$x \in X \setminus U_r$ and $f_r(x) = r$ if $x \in \overline{V}_r$. Let $f(x) = \sup\{f_r(x) : r \in \mathbb{Q}\}$. Since $\{U_r : r \in \mathbb{Q}\}$ is locally finite, every $x \in X$ has a neighborhood W_x intersecting only finitely many U_r . Hence, in W_x , $f_r(x) = -\infty$ for a finite number of values of r ; and $f(x) = \sup_{r \in I} f_r(x)$, where $I \subset \mathbb{Q}$ is finite, which means that $f(x)$ is the least upper bound of a finite number of continuous functions f_r , hence f is continuous. In U_r , $f_r(x) \leq r < g(x)$ and in $X \setminus U_r$, $f_r(x) = -\infty < g(x)$. So, $f_r(x) < g(x)$, $x \in X$. This implies that $f(x) < g(x)$, $x \in X$. On the other hand, for every $x \in X$, there is some $r_0 \in \mathbb{Q}$ such that $x \in V_{r_0}$, and $f_{r_0}(x) = r_0$. Thus, $f(x) \geq f_{r_0}(x) = r_0 > h(x)$. Hence $f(x) > h(x)$, $x \in X$. Therefore, $h(x) < f(x) < g(x)$, for every $x \in X$.

(b) \Rightarrow (a). Let X be a space satisfying condition (b), and let A and B be two disjoint closed subsets of X . Let

$$h(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{if } x \in B \\ 2 & \text{if } x \notin B. \end{cases}$$

Then, h is upper semi-continuous, g is lower semi-continuous and $h(x) < g(x)$, $x \in X$. By (b), there exists a continuous $f : X \rightarrow \mathbb{R}$ such that $h(x) < f(x) < g(x)$, $x \in X$. Let $U = \{x \in X : f(x) > 1\}$ and $V = \{x \in X : f(x) < 1\}$. Then, U and V are open (because f is continuous), disjoint and $A \subset U$ and $B \subset V$. This implies that X is normal. To show that X is countably paracompact, let $\{F_n : n \in \mathbb{N}\}$ be a decreasing sequence of closed subsets of X satisfying $\bigcap\{F_n : n \in \mathbb{N}\} = \emptyset$. Define a function g by $g(x) = \frac{1}{n+1}$, $x \in F_n \setminus F_{n+1}$, $n \in \mathbb{N}$, where $F_0 = X$. Let $h(x) = 0$, for every $x \in X$. Then, h is upper semi-continuous, g is lower semi-continuous and $h(x) < g(x)$, $x \in X$. Hence, by (b), there is a continuous $f : X \rightarrow \mathbb{R}$ such that $0 < f(x) < g(x)$, $x \in X$. Let $W_n = \{x \in X : f(x) < \frac{1}{n+1}\}$. Then, $\{W_n : n \in \mathbb{N}\}$ is a family of open subsets of X such that $F_n \subset W_n$ for each $n \in \mathbb{N}$. Since $f(x) > 0$ for all $x \in X$, we also have $\bigcap\{W_n : n \in \mathbb{N}\} = \emptyset$. By Corollary 2.3.45, X is countably paracompact. \square

REMARK 2.3.48. If one replaces the strict inequality “ $<$ ” by “ \leq ” in condition (b) of Theorem 2.3.47, then one gets a characterization of normal spaces, rather than

countably paracompact and normal spaces. This was proved independently by M. Katětov [28] and H. Tong [67] and is referred to as Katětov-Tong insertion theorem.

4. Normal covers

Let X be a topological space. The *star* of a set $A \subset X$ with respect to a cover \mathcal{U} of X is the set $\text{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$. Note that the star of a singleton with respect to a cover is just the union of all members of the cover containing that point. A cover \mathcal{U} of X is said to be a *star-refinement* of another cover \mathcal{V} of X if for each $U \in \mathcal{U}$, there is some $V \in \mathcal{V}$ such that $\text{St}(U, \mathcal{U}) \subset V$.

DEFINITION 2.4.49. A *normal sequence* in a space X is a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X such that \mathcal{U}_{n+1} star-refines \mathcal{U}_n , $n \in \mathbb{N}$. An open cover \mathcal{U} of X is called a *normal cover* if $\mathcal{U} = \mathcal{U}_1$ for some normal sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X .

Let us recall that a subset A of a topological space X is called a *zero-set* if $A = f^{-1}(0)$, for some continuous function $f : X \rightarrow [0, 1]$. The complement of a zero-set in X is called a *cozero-set*. If $B \subset X$ is a cozero-set, then $B = f^{-1}((0, 1])$. Obviously, every zero-set of X is closed in X ; and every cozero-set of X is open in X . The converse is not true. However, if X is a metric space, then any closed (resp. open) set is a zero-set (resp. cozero-set). Note that the union and the intersection of two (hence of finitely many) zero-set (resp. cozero-set) are also a zero-sets (resp. cozero-sets). A countable intersection of zero-sets is also a zero-set. Indeed, if $A_n = f_n^{-1}(0)$ for some continuous functions $f_n : X \rightarrow [0, 1]$, $n = 1, 2, \dots$, then the function $f : X \rightarrow [0, 1]$ defined by $f(x) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} f_n(x)$, $x \in X$, is continuous and $\bigcap\{A_n : n \in \mathbb{N}\} = f^{-1}(0)$. Cozero-sets have a dual property, i.e., a countable union of cozero-sets is also a cozero-set. The inverse image of a zero-set (resp. cozero-set) under a continuous map is also a zero-set (resp. cozero-set). Finally, note that since $\{x \in X : f(x) = 0\} = \bigcap_{n \in \mathbb{N}} \{x \in X : |f(x)| < \frac{1}{n}\}$, one concludes that every zero-set is a G_δ -set (see, for instance, [17]).

PROPOSITION 2.4.50 ([13, Corollary 1.5.12]). *A subset A of a normal space X is a zero-set if and only if it is a closed G_δ -subset of X .*

PROOF. Let $A = f^{-1}(0)$, for some continuous $f : X \rightarrow [0, 1]$. Note that the singleton $\{0\} \subset [0, 1]$ is a closed and G_δ -set. So, since f is continuous, $A = f^{-1}(0)$ is also a closed G_δ -subset of X . Conversely, suppose that A is a closed G_δ -subset of a normal space X . Then, $X \setminus A$ is an F_σ -subset of X ; i.e. $X \setminus A = \bigcup\{A_n : n \in \mathbb{N}\}$, where A_n is closed for each $n \in \mathbb{N}$. By Lemma 2.3.25, for each $n \in \mathbb{N}$, there is a continuous function $f_n : X \rightarrow [0, 1]$ such that $f_n(x) = 0$ for all $x \in A$ and $f_n(x) = 1$ for all $x \in A_n$. Define a function $f : X \rightarrow [0, 1]$ by $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x)$ for $x \in X$. Then, f is a continuous function such that $f(x) = 0$ if $x \in A$. If $x \notin A$, then $x \in A_n$, for some $n \in \mathbb{N}$, and $f(x) \geq \frac{1}{2^n} f_n(x) = \frac{1}{2^n} > 0$, because $f_n(x) = 1$ if $x \in A_n$. Hence, $A = f^{-1}(0)$. \square

Taking complements, one gets the following.

PROPOSITION 2.4.51 ([13, Corollary 1.5.13]). *A subset A of a normal space X is a cozero-set if and only if it is an open F_σ -subset of X .*

Zero-sets are sometime referred to as *functionally closed sets* and cozero-sets are called *functionally open sets*. A cover of X consisting only of cozero-sets is called a *cozero-set cover* of X .

The following characterizations of normal covers were proved by Morita.

THEOREM 2.4.52 ([48, Theorem 1.1]). *An open cover \mathcal{U} of a space X is normal if and only if there exists a metric space T , a continuous map $f : X \rightarrow T$ and an open cover \mathcal{V} of T such that $f^{-1}(\mathcal{V})$ refines \mathcal{U} .*

THEOREM 2.4.53 ([48, Theorem 1.2] and [46, Theorem 1.2]). *Let \mathcal{U} be an open cover of a space X . Then, the following are equivalent:*

- (a) \mathcal{U} is normal.

- (b) \mathcal{U} admits a locally finite cozero-set refinement.
- (c) \mathcal{U} admits a σ -locally finite cozero-set refinement.
- (d) \mathcal{U} has a partition of unity subordinated to it.

PROOF. (a) \Rightarrow (b). If \mathcal{U} is normal, then by Theorem 2.4.52, there exists a metric space T , a continuous map $f : X \rightarrow T$ and an open cover \mathcal{V} of T such that $f^{-1}(\mathcal{V})$ refines \mathcal{U} . Since T is paracompact (by Theorem 2.3.36), \mathcal{V} has a locally finite open refinement \mathcal{W} ; and each (open) member of \mathcal{W} is a cozero-set because T is a metric space. Hence, $f^{-1}(\mathcal{W})$ is also a locally finite cozero-set cover of X , and refines \mathcal{U} .

(b) \Rightarrow (c). This is obvious.

(c) \Rightarrow (d). Suppose that there is a cozero-set cover $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that \mathcal{V} refines \mathcal{U} and each $\mathcal{V}_n = \{G_n^\alpha : \alpha \in \mathcal{A}_n\}$ is locally finite, where each $G_n^\alpha = \{x \in X : g_n^\alpha(x) > 0\}$, for some continuous function $g_n^\alpha : X \rightarrow [0, 1]$. Since \mathcal{V}_n is locally finite, the function $f_n : X \rightarrow [0, +\infty)$ defined by $f_n(x) = \sum_{\alpha \in \mathcal{A}_n} g_n^\alpha(x)$, $x \in X$, is continuous. The function f defined by $f(x) = \sum_{n \in \mathbb{N}} \frac{f_n(x)}{2^n(1+f_n(x))}$, $x \in X$, is also continuous such that $f(x) > 0$ for every $x \in X$. Finally, define $h_n^\alpha : X \rightarrow [0, 1]$ by

$$h_n^\alpha(x) = \frac{g_n^\alpha(x)}{2^n f(x)(1+f_n(x))}, \quad x \in X.$$

Then, $\sum_{n \in \mathbb{N}} \sum_{\alpha \in \mathcal{A}_n} h_n^\alpha(x) = 1$, $x \in X$ and $G_n^\alpha = \{x \in X : h_n^\alpha(x) > 0\}$. So, $\{h_n^\alpha : \alpha \in \mathcal{A}_n, n \in \mathbb{N}\}$ is a partition of unity subordinated to \mathcal{U} .

(d) \Rightarrow (a). Let $\{p_\alpha : \alpha \in \mathcal{A}\}$ be a partition of unity subordinated to \mathcal{U} . Let M be the metric space which consists of the points $\{x_\alpha : \alpha \in \mathcal{A}\}$, with $x_\alpha \in [0, 1]$, such that $\sum x_\alpha = 1$ and the metric is defined by $d(x, y) = \sum |x_\alpha - y_\alpha|$. Define a map $p : X \rightarrow M$ by $p(x) = \{p_\alpha(x) : \alpha \in \mathcal{A}\}$, $x \in X$. Let us show that p is continuous. Let $x_0 \in X$ and $\varepsilon > 0$. Since $\sum p_\alpha(x_0) = 1$, then for some finite subset \mathcal{B} of \mathcal{A} and a neighborhood U of x_0 , we have

$$\sum_{\alpha \notin \mathcal{B}} p_\alpha(x_0) < \frac{1}{4}\varepsilon, \quad \sum_{\alpha \in \mathcal{B}} |p_\alpha(x) - p_\alpha(x_0)| < \frac{1}{4}\varepsilon, \quad \text{for } x \in U.$$

Then for every $x \in U$,

$$\begin{aligned}
\sum_{\alpha \notin \mathcal{B}} p_\alpha(x) &= 1 - \sum_{\alpha \in \mathcal{B}} p_\alpha(x) \\
&= \sum_{\alpha \in \mathcal{B}} (p_\alpha(x_0) - p_\alpha(x)) + \sum_{\alpha \notin \mathcal{B}} p_\alpha(x_0) \\
&\leq \sum_{\alpha \in \mathcal{B}} |p_\alpha(x) - p_\alpha(x_0)| + \sum_{\alpha \notin \mathcal{B}} p_\alpha(x_0) < \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = \frac{1}{2}\varepsilon.
\end{aligned}$$

Hence,

$$\begin{aligned}
d(p(x), p(x_0)) &= \sum |p_\alpha(x) - p_\alpha(x_0)| \leq \sum_{\alpha \notin \mathcal{B}} |p_\alpha(x) - p_\alpha(x_0)| + \sum_{\alpha \in \mathcal{B}} |p_\alpha(x) - p_\alpha(x_0)| \\
&< \sum_{\alpha \notin \mathcal{B}} |p_\alpha(x) - p_\alpha(x_0)| + \frac{1}{4}\varepsilon \\
&\leq \sum_{\alpha \notin \mathcal{B}} |p_\alpha(x) + p_\alpha(x_0)| + \frac{1}{4}\varepsilon \\
&\leq \sum_{\alpha \notin \mathcal{B}} p_\alpha(x) + \sum_{\alpha \notin \mathcal{B}} p_\alpha(x_0) + \frac{1}{4}\varepsilon \\
&< \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon.
\end{aligned}$$

That is, p is continuous. If we let $V_\alpha = \{\{x_\lambda\} : x_\alpha > 0\}$, then V_α are open sets of M and $p^{-1}(V_\alpha) = \{x : p_\alpha(x) > 0\}$. Hence, by Theorem 2.4.52, \mathcal{U} is a normal cover. \square

COROLLARY 2.4.54 ([47, Theorem 1.2]). *If $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a σ -locally finite cozero-set cover of a topological space X , then \mathcal{U} is a normal cover.*

CHAPTER 3

Michael's Selection Theory

In this chapter, we review some concepts and results about the convex-valued selection theory. Most of these results were already extensively studied in [33]; nevertheless, we present them here again for the completeness and self-containedness of this thesis.

1. Set-valued Mappings

Given a set Y , define the subset 2^Y of the power set $\mathcal{P}(Y)$ of Y by

$$2^Y = \{S \in \mathcal{P}(Y) : S \neq \emptyset\}.$$

A map $\Phi : X \rightarrow 2^Y$ is usually called a *set-valued mapping* or *multifunction*. For a topological space Y , if for every $x \in X$, $\Phi(x)$ is closed (resp. compact, convex¹, \dots), then the mapping Φ is said to be closed-valued (resp. compact-, convex-valued, \dots). As a rule, we will use Greek letters such as $\varphi, \Phi, \theta, \dots$ for set-valued mappings, and lower case letters such as f, g, h, \dots for single-valued mappings.

1.1. Continuity of Set-valued Mappings. Recall that for topological spaces X and Y , a map (single-valued mapping) $f : X \rightarrow Y$ is continuous if the preimage of an open set is open; that is

$$f^{-1}(U) \subset X \text{ is open whenever } U \subset Y \text{ is open.}$$

By analogy, we have two continuity properties of a set-valued mapping $\Phi : X \rightarrow 2^Y$ depending on the preimage. Consider the following subsets of X , for $U \subset Y$:

$$\triangleright \text{Big preimage: } \Phi^{-1}(U) = \{x \in X : \Phi(x) \cap U \neq \emptyset\},$$

¹In this case, Y is a topological vector space.

▷ **Small preimage:** $\Phi^\#(U) = \{x \in X : \Phi(x) \subset U\}$.

REMARK 3.1.1. Note that for every $U \subset Y$, we always have:

- $\Phi^\#(U) \subset \Phi^{-1}(U)$, and

$$\begin{aligned} \bullet \Phi^\#(U) &= \{x \in X : \Phi(x) \subset U\} \\ &= \{x \in X : \Phi(x) \cap (Y \setminus U) = \emptyset\} \\ &= X \setminus \{x \in X : \Phi(x) \cap (Y \setminus U) \neq \emptyset\} \\ &= X \setminus \Phi^{-1}(Y \setminus U). \end{aligned}$$

Lower semi-continuity: A set-valued mapping $\Phi : X \rightarrow 2^Y$ is said to be *lower semi-continuous*, or l.s.c., if $\Phi^{-1}(U) \subset X$ is open for every open subset $U \subset Y$.

Upper semi-continuity: A set-valued mapping $\Phi : X \rightarrow 2^Y$ is said to be *upper semi-continuous*, or u.s.c., if $\Phi^\#(U) \subset X$ is open for every open subset $U \subset Y$.

This is equivalent to saying that Φ is u.s.c. if $\Phi^{-1}(F)$ is closed, whenever $F \subset Y$ is closed. It follows from the second part of the previous Remark 3.1.1.

Usco set-valued mappings: A set-valued mapping $\Phi : X \rightarrow 2^Y$ is said to be *usco* if it is a compact valued u.s.c. mapping, i.e. Φ is u.s.c. and $\Phi(x)$ is compact, for every $x \in X$.

REMARK 3.1.2. There are l.s.c. mappings which are not u.s.c. and vice versa. For instance, the mapping $\varphi_1 : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ defined by

$$\varphi_1(x) = \begin{cases} [-1, +1] & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

is l.s.c. but not u.s.c.; whereas the mapping $\varphi_2 : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ defined by

$$\varphi_2(x) = \begin{cases} \{0\} & \text{if } x \neq 0 \\ [-1, +1] & \text{if } x = 0 \end{cases}$$

is u.s.c. but not l.s.c., see [1].

REMARK 3.1.3. The lower semi-continuity plays a very important role in Michael's selection theory. Indeed, l.s.c. mappings are exactly those which admit local selections. In other words, the notion of lower semi-continuity is by definition very close to the notion of a selection [57]. So, in the sequel, we will be mostly interested in it.

1.2. Basic Properties of l.s.c. Mappings. In this section, we give some properties of l.s.c. mappings and other alternative ways to characterize them. These properties play a very crucial role in the proofs of both convex- and paraconvex-valued selection theorems. They are due to Michael [39] (see also [33, 55] for their proofs). Because of their importance, we present here their proofs as they will be used in this thesis (see Chapters 3 and 4). These proofs may be also found in [33].

PROPOSITION 3.1.4 ([39, Proposition 2.1]). *If $\Phi : X \rightarrow 2^Y$, then the following are equivalent:*

- (a) Φ is l.s.c.,
- (b) *If $x \in X$, $y \in \Phi(x)$, and V is a neighborhood of y in Y , then there exists a neighborhood U of x in X such that $\Phi(x') \cap V \neq \emptyset$, for every $x' \in U$.*

PROOF. (a) \Rightarrow (b): Suppose $\Phi : X \rightarrow 2^Y$ is l.s.c. Let $x \in X$, $y \in \Phi(x)$ and $V \subset Y$ be an open set containing y . Then, $y \in \Phi(x) \cap V$, i.e. $\Phi(x) \cap V \neq \emptyset$. This implies $x \in \Phi^{-1}(V)$. So, take $U = \Phi^{-1}(V)$ which is open since Φ is l.s.c.

(b) \Rightarrow (a): Suppose that $\Phi : X \rightarrow 2^Y$ satisfies (b). We must show that Φ is l.s.c. So, take an open set $V \subset Y$. By (b), for each $x \in \Phi^{-1}(V)$ there exists an open set $U_x \subset X$ such that $x \in U_x \subset \Phi^{-1}(V)$. Thus, $\Phi^{-1}(V) = \bigcup \{U_x : x \in \Phi^{-1}(V)\}$ is open. \square

Recall that we use \overline{A} to denote the closure of subset A of X .

PROPOSITION 3.1.5. ([39, Proposition 2.3]). *Let $\Phi : X \rightarrow 2^Y$ be l.s.c. Define $\psi : X \rightarrow 2^Y$ by*

$$\psi(x) = \overline{\Phi(x)} \text{ for every } x \in X.$$

Then ψ is also l.s.c.

PROOF. Let $U \subset Y$ be open and $x \in X$. Then, $\overline{\Phi(x)} \cap U \neq \emptyset$ if and only if $\Phi(x) \cap U \neq \emptyset$. That is, $\psi^{-1}(U) = \Phi^{-1}(U)$, hence by lower semi-continuity of Φ , $\psi^{-1}(U)$ is open, i.e. ψ is also l.s.c. \square

For every subset B of a topological vector space Y , there exists a unique minimal convex subset of Y containing B , which is called *convex hull* of B , and denoted by $\text{conv}(B)$. It is indeed the intersection of all convex subsets containing B ; that is

$$\text{conv}(B) = \bigcap \{C \subset Y : B \subset C \text{ and } C \text{ is convex}\}.$$

The convex hull of B can also be written as follows:

$$\text{conv}(B) = \left\{ \sum_{i=1}^n \alpha_i b_i : n \in \mathbb{N}, b_i \in B, \alpha_i \geq 0, \text{ and } \sum_{i=1}^n \alpha_i = 1 \right\}.$$

DEFINITION 3.1.6. A topological vector space Y is said to be *locally convex* if the origin of Y has a base of neighborhoods consisting of convex sets.

PROPOSITION 3.1.7. ([39, 55]). *If Y is a locally convex topological vector space, and $\Phi : X \rightarrow 2^Y$ is l.s.c., then the multifunction $\psi : X \rightarrow 2^Y$, defined by*

$$\psi(x) = \text{conv}(\Phi(x)), \quad x \in X$$

is also l.s.c.

PROOF. Let $U \subset Y$ be open, and let $x \in \psi^{-1}(U)$. Then $\psi(x) \cap U \neq \emptyset$, i.e. $\text{conv}(\Phi(x)) \cap U \neq \emptyset$. Consider $y \in \text{conv}(\Phi(x)) \cap U$, then $y \in U$ and $y \in \text{conv}(\Phi(x))$. Let $y = \sum_{i=1}^n \lambda_i y_i$, where $n \in \mathbb{N}$, $\lambda_i \geq 0$, $y_i \in \Phi(x)$, and $\sum_{i=1}^n \lambda_i = 1$. Let W be a convex open neighborhood of the origin $0 \in Y$, contained in the neighborhood $U - y$ of the origin 0 of the locally convex topological vector space Y . Since W is an open neighborhood of the origin $0 \in Y$, the sets $y_i + W$ are open neighborhoods of y_i , for all $i = 1, \dots, n$. Since $y_i \in \Phi(x)$, and Φ is l.s.c., the sets $\Phi^{-1}(y_i + W)$ are nonempty open

neighborhoods of x , for all $i = 1, \dots, n$. Now, set $O = \bigcap_{i=1}^n \Phi^{-1}(y_i + W)$. Then, O is open in X and $x \in O \subset \psi^{-1}(U)$. It is clear that $x \in O$. Let us show that $O \subset \psi^{-1}(U)$.

$$\begin{aligned} x' \in O &\Rightarrow x' \in \Phi^{-1}(y_i + W), \text{ for all } i = 1, \dots, n, \\ &\Rightarrow \Phi(x') \cap (y_i + W) \neq \emptyset, \text{ for all } i = 1, \dots, n. \end{aligned}$$

Then, there exists some $y'_i \in \Phi(x') \cap (y_i + W)$, for each $i = 1, \dots, n$; that is, $y'_i \in \Phi(x')$ and $y'_i \in y_i + W$, or $y'_i - y_i \in W$ for all $i = 1, \dots, n$. Therefore, by convexity of W , we obtain for $y' = \sum_{i=1}^n \lambda_i y'_i \in \psi(x')$ that

$$y' - y = \sum_{i=1}^n \lambda_i (y'_i - y_i) \in W \subset U - y,$$

i.e. $y' \in U$. Hence, $y' \in \psi(x') \cap U$, so $\psi(x') \cap U \neq \emptyset$, and $x' \in \psi^{-1}(U)$. Thus, $O \subset \psi^{-1}(U)$ and, by Proposition 3.1.4, ψ is l.s.c.. \square

Before giving the next property, we need some new terminology.

DEFINITION 3.1.8. For a single-valued mapping $f : X \rightarrow Y$, the *graph* of f is the subset of $X \times Y$ defined by

$$\text{Graph}(f) = \{(x, f(x)) : x \in X\}.$$

Similarly, for a set-valued mapping $\Phi : X \rightarrow 2^Y$, the graph of Φ is the subset of $X \times Y$ defined by

$$\text{Graph}(\Phi) = \{(x, y) \in X \times Y : y \in \Phi(x)\}.$$

A mapping $\Phi : X \rightarrow 2^Y$ has an open graph, or is an *open-graph mapping*, if $\text{Graph}(\Phi)$ is open in $X \times Y$.

PROPOSITION 3.1.9 (see, [3]). *If $\Phi : X \rightarrow 2^Y$ has an open graph, then Φ is l.s.c.*

The converse of Proposition 3.1.9 is not true. For instance, for $[0, 1]$ equipped with its usual topology, the mapping $\varphi : [0, 1] \rightarrow 2^{[0,1]}$ defined by $\varphi(x) = [0, 1]$ if $x = 0$ and

$\varphi(x) = [0, 1] \setminus \{x\}$ if $0 < x \leq 1$, is open-valued and l.s.c., and yet it has no open graph (see [49, Example 1.6]). On the other hand, one has:

PROPOSITION 3.1.10 (see, [40]). *If (Y, d) is a metric space, $\Phi : X \rightarrow 2^Y$ is l.s.c. and $\varepsilon > 0$, then $\psi : X \rightarrow 2^Y$ defined by*

$$\psi(x) = B_\varepsilon^d(\Phi(x)), \text{ for every } x \in X,$$

has an open graph.

PROOF. Let $(x, y) \in \text{Graph}(\psi)$. We need to find an open set O such that $(x, y) \in O \subset \text{Graph}(\psi)$. So, $(x, y) \in \text{Graph}(\psi)$ implies that $y \in \psi(x) = B_\varepsilon^d(\Phi(x))$, which means that $\Phi(x) \cap B_\varepsilon^d(y) \neq \emptyset$ i.e. $d(y, \Phi(x)) < \varepsilon$. Let $r = d(y, \Phi(x)) < \varepsilon$, and set $\delta = \frac{\varepsilon - r}{2} > 0$. Then, $B_{r+\delta}^d(y) \cap \Phi(x) \neq \emptyset$, because $d(y, \Phi(x)) = r < r + \delta$. By lower semi-continuity of Φ , the set $U = \Phi^{-1}(B_{r+\delta}^d(y))$ is open in X , and contains x . Take $V = B_\delta^d(y) \ni y$, and set $O = U \times V = \Phi^{-1}(B_{r+\delta}^d(y)) \times B_\delta^d(y)$. Then, $(x, y) \in O \subset \text{Graph}(\psi)$. Clearly, $(x, y) \in O$. Let us show that $O \subset \text{Graph}(\psi)$.

$$\begin{aligned} (s, t) \in O &\Rightarrow s \in \Phi^{-1}(B_{r+\delta}^d(y)) \text{ and } t \in B_\delta^d(y) \\ &\Rightarrow \Phi(s) \cap B_{r+\delta}^d(y) \neq \emptyset \text{ and } t \in B_\delta^d(y) \\ &\Rightarrow d(y, \Phi(s)) < r + \delta \text{ and } d(t, y) < \delta. \end{aligned}$$

Hence, $d(t, \Phi(s)) \leq d(t, y) + d(y, \Phi(s)) < r + 2\delta = \varepsilon$, which implies that $t \in B_\varepsilon^d(\Phi(s)) = \psi(s)$, i.e. $(s, t) \in \text{Graph}(\psi)$. The proof is completed. \square

Note that the intersection of two l.s.c. mappings is not necessarily l.s.c. But one has:

PROPOSITION 3.1.11 ([39]). *If $\Phi : X \rightarrow 2^Y$ is l.s.c., $\psi : X \rightarrow 2^Y$ has an open graph and $\theta(x) = \Phi(x) \cap \psi(x) \neq \emptyset$, for every $x \in X$. Then, $\theta : X \rightarrow 2^Y$ is l.s.c.*

PROOF. Let $U \subset Y$ be open, and $x \in \theta^{-1}(U)$. Then, $\theta(x) \cap U \neq \emptyset$, i.e.

$$(\Phi(x) \cap \psi(x)) \cap U \neq \emptyset.$$

Let $y \in (\Phi(x) \cap \psi(x)) \cap U$. Then, $y \in \psi(x)$ which means that $(x, y) \in \text{Graph}(\psi)$. Since $\text{Graph}(\psi)$ is open, there exist open sets $V_x \subset X$ and $V_y \subset Y$ such that $(x, y) \in V_x \times V_y \subset \text{Graph}(\psi)$. Set $W = \Phi^{-1}(U \cap V_y) \cap V_x$. Then, W is a neighborhood of x contained in $\theta^{-1}(U)$ and, by Proposition 3.1.4, θ is l.s.c. \square

PROPOSITION 3.1.12 ([39]). *Let $\Phi : X \rightarrow 2^Y$ be l.s.c., and let V be an open subset of Y . If $\Phi(x) \cap V \neq \emptyset$ for every $x \in X$, then the multifunction $\psi : X \rightarrow 2^Y$ defined by*

$$\psi(x) = \Phi(x) \cap V, \text{ for every } x \in X,$$

is also l.s.c.

PROOF. This follows from the previous proposition, since the set-valued mapping $\psi : X \rightarrow 2^Y$, defined by $\psi(x) = V$, $x \in X$, has an open graph. Indeed, $\text{Graph}(\psi) = \{(x, y) \in X \times Y : y \in \psi(x) = V\} = X \times V$, which is open in $X \times Y$. \square

From Propositions 3.1.10 and 3.1.11, one has the following:

COROLLARY 3.1.13 ([39]). *If (Y, d) is a metric space, $\Phi : X \rightarrow 2^Y$ is l.s.c., and $f : X \rightarrow Y$ is a continuous map such that $\Phi(x) \cap B_\varepsilon^d(f(x)) \neq \emptyset$, for every $x \in X$. Then the multifunction $\psi : X \rightarrow 2^Y$ defined by*

$$\psi(x) = \Phi(x) \cap B_\varepsilon^d(f(x)), \quad x \in X,$$

is also l.s.c.

2. Extension and Selection Theorems

DEFINITION 3.2.14. Let X and Y be spaces, $A \subset X$ and $g : A \rightarrow Y$. A map $f : X \rightarrow Y$ is an *extension* of g if $f \upharpoonright A = g$, i.e. $f(x) = g(x)$, $\forall x \in A$.

One of the best known extension theorems is the Tietze-Urysohn's characterization of normality, a partial case of it was already mentioned in the previous section (see, Theorem 2.3.31).

THEOREM 3.2.15 (Tietze-Urysohn). *For a T_1 -space X , the following are equivalent:*

- (a) X is normal,
- (b) if $A \subset X$ is a closed subset, then every continuous $g : A \rightarrow \mathbb{R}$ can be continuously extended to the whole of X ,
- (c) if $A \subset X$ is a closed subset, then every continuous $g : A \rightarrow Y$ to a separable Banach space Y can be continuously extended to the whole of X .

Theorem 3.2.15 was naturally generalized by Dowker for the case of arbitrary Banach spaces. In order to state it, let us recall that a T_1 -space X is called *collectionwise normal* if for every discrete collection \mathcal{F} of closed subsets of X , there exists a discrete collection $\{U_F : F \in \mathcal{F}\}$ of open subsets of X such that $F \subset U_F$, for every $F \in \mathcal{F}$.

Here, a family $\mathcal{D} = \{D_\alpha : \alpha \in \mathcal{A}\}$ of subsets of X is called *discrete* if every $x \in X$ has a neighborhood V_x such that

$$|\{\alpha \in \mathcal{A} : D_\alpha \cap V_x \neq \emptyset\}| \leq 1.$$

In what follows, τ will denote an infinite cardinal number, that is $\tau \geq \omega$, where ω is the first infinite cardinal number. Also, recall that $w(X)$ denotes the topological weight of a space X .

DEFINITION 3.2.16. A topological space X is called τ -*collectionwise normal* if it is a T_1 -space and for every discrete collection \mathcal{F} of closed subsets of X , with $|\mathcal{F}| \leq \tau$, there exists a discrete collection $\{U_F : F \in \mathcal{F}\}$ of open subsets of X such that $F \subset U_F$ for every $F \in \mathcal{F}$.

In fact, the “discrete family $\{U_F : F \in \mathcal{F}\}$ ” in the above definition may be replaced by a weaker condition of “pairwise disjoint family $\{U_F : F \in \mathcal{F}\}$ ”.

PROPOSITION 3.2.17 (see [13]). *A topological space X is τ -collectionwise normal if it is a T_1 -space and for every discrete collection \mathcal{F} of closed subsets of X , with $|\mathcal{F}| \leq \tau$,*

there exists a pairwise disjoint family $\{U_F : F \in \mathcal{F}\}$ of open subsets of X such that $F \subset U_F$ for every $F \in \mathcal{F}$.

Every τ -collectionwise normal space is normal. In fact, if τ is allowed to be finite, then X is normal if and only if it is 2-collectionwise normal. This is equivalent to saying that X is normal if and only if X is n -collectionwise normal, for every $n \geq 2$. Also, X is normal if and only if X is ω -collectionwise normal.

A space X is collectionwise normal if and only if X is τ -collectionwise normal, for every τ . However, there are τ -collectionwise normal spaces which are not collectionwise normal. In particular, there are normal spaces which are not collectionwise normal, see Bing's example in [14, Example 5.1.23]. In fact, for every infinite cardinal τ , there is a τ -collectionwise normal space which is not τ^+ -collectionwise normal (see [52]). Here, τ^+ is the next cardinal (i.e., the immediate successor of τ). Clearly, collectionwise normality lies between normality and paracompactness.

LEMMA 3.2.18 ([51]). *A T_1 -space X is τ -collectionwise normal if and only if for every closed subset A of X and every open (in X) cover \mathcal{U} of A , with $|\mathcal{U}| \leq \tau$, which is point finite in A , there exists an open locally finite (in X) cover of A which is a refinement of \mathcal{U} .*

The following proposition is a consequence of Proposition 2.3.29 and shows that collectionwise normality is hereditary with respect to F_σ -sets (and hence, it is hereditary with respect to closed subsets).

PROPOSITION 3.2.19 ([68], see also [13, Problem 5.5.1 (b)]). *If F is an F_σ -subset of a collectionwise normal space, then F is also collectionwise normal.*

THEOREM 3.2.20 (**Dowker's extension theorem**, [9]). *For a T_1 -space X , the following are equivalent:*

- (a) X is τ -collectionwise normal,

- (b) if A is a closed subset of X and Y is a Banach space with $w(Y) \leq \tau$, then every continuous $g : A \rightarrow Y$ can be continuously extended to the whole of X .

As mentioned above, every normal space is ω -collectionwise normal. Hence, the Tietze-Urysohn's theorem is an immediate consequence of Theorem 3.2.20.

By Theorem 3.2.20, we also get the following consequence which was actually proved by Dowker:

COROLLARY 3.2.21. ([9]) For a T_1 -space X , the following are equivalent:

- (a) X is collectionwise normal,
 (b) if $A \subset X$ is a closed subset, then every continuous $g : A \rightarrow Y$ to a Banach space Y can be continuously extended to the whole of X .

2.1. Selections.

DEFINITION 3.2.22. If X and Y are sets and $\Phi : X \rightarrow 2^Y$ is a set-valued mapping, then a *selection* for Φ is a map $f : X \rightarrow Y$ such that

$$f(x) \in \Phi(x), \text{ for every } x \in X.$$

If moreover X and Y are spaces and f is continuous, then f is called a *continuous selection* for Φ .

EXAMPLE 3.2.23 ([39, Example 1.3]). Let $\Phi : X \rightarrow 2^Y$, $A \subset X$ and $g : A \rightarrow Y$ be a selection for $\Phi \upharpoonright A$. Define $\Phi_g : X \rightarrow 2^Y$ by

$$\Phi_g(x) = \begin{cases} \{g(x)\} & \text{if } x \in A \\ \Phi(x) & \text{if } x \in X \setminus A. \end{cases}$$

Then, $f : X \rightarrow Y$ is a selection for Φ_g if and only if f is an extension of g .

From this example, it is clear that any extension problem is a special case of a selection problem for the so defined Φ_g . If X and Y are topological spaces, $A \subset X$ is closed,

and g is continuous, then the “continuous” extension problem is a special case of the “continuous” selection problem for l.s.c. mappings. Namely, we have the following example:

EXAMPLE 3.2.24 ([39, Example 1.3*]). If $\Phi : X \rightarrow 2^Y$ is l.s.c., $A \subset X$ is closed, $g : A \rightarrow Y$, and Φ_g is defined as in Example 3.2.23, then g is continuous if and only if Φ_g is l.s.c.. Moreover, f is a continuous extension of g if and only if f is a continuous selection for Φ_g .

In the sequel, we shall consider the following families of subsets of a space Y . We will denote by $\mathcal{F}(Y)$ the subfamily of 2^Y consisting of all closed members of 2^Y , and we let

$$\begin{aligned}\mathcal{C}(Y) &= \{S \in \mathcal{F}(Y) : S \text{ is compact}\}, \text{ and} \\ \mathcal{C}'(Y) &= \mathcal{C}(Y) \cup \{Y\}.\end{aligned}$$

EXAMPLE 3.2.25. Let X and Y be spaces, $A \subset X$, and $g : A \rightarrow Y$. Define a mapping Ψ_g by

$$\Psi_g(x) = \begin{cases} \{g(x)\} & \text{if } x \in A \\ Y & \text{if } x \in X \setminus A. \end{cases}$$

Then, $\Psi_g : X \rightarrow \mathcal{C}'(Y)$.

3. Selections for Convex-valued Mappings on Paracompact Domains

The following result is known as the convex-valued selection theorem for mappings defined on paracompact spaces and was proved by E. Michael [39] (for mappings defined on collectionwise normal spaces, see Theorem 4.1.1 in Chapter 4).

THEOREM 3.3.26 ([39, Theorem 3.2]). *For a T_1 -space X , the following are equivalent:*

- (a) X is paracompact.
- (b) If Y is a Banach space and $\varphi : X \rightarrow \mathcal{F}(Y)$ is an l.s.c. convex-valued mapping, then φ has a continuous selection.

Note that in most of the convex-valued selection theorems, the resulting selection $f : X \rightarrow Y$ of a given set-valued mapping φ is practically always constructed as a uniform limit of some sequence of approximate selections $f_n : X \rightarrow Y$. A typical difficult situation arises with the limit point $\lim_{n \rightarrow \infty} f_n$. Such a limit point can easily end up in the boundary of the set $\varphi(x)$, rather than in the set $\varphi(x)$, if one does not pay attention to a more careful construction of the uniform Cauchy sequence of approximate selections. The simplest and the most direct way to avoid this possibility is to consider only closed-valued mappings into a complete range or alternatively, to deal only with complete-valued mappings, [57]. Hence, the closedness assumption in Theorem 3.3.26 cannot be dropped. Indeed, as [39, Example 6.3] shows, there exists an l.s.c. set-valued mapping from the closed unit interval $[0, 1]$ (which is paracompact because it is compact) to the set of all nonempty, open, convex subsets of a Banach space Y for which there exists no selection. Also, [39, Example 6.2] shows that Theorem 3.3.26 becomes false if “Banach space” is replaced by “normed space”. As for lower semi-continuity and convexity, the reason for such a restriction is justified by the fact that the range of the set-valued mapping has an algebraic structure, hence it is possible to do convex combinations of points. The lower semi-continuous mappings transform open covers of the range into open covers of the domain, and if the domain is a nice topological space (for instance, paracompact), one can use partitions of unity to arrange coordinates for such convex combinations. This is roughly how one can construct approximate selections. If the range has not only an algebraic structure, but also a complete metric one (i.e., is a Banach space, for instance), then one can manage a Cauchy sequence of approximate selections which will finally result in a continuous selection for the set-valued mapping under interest.

Note that the above theorem (i.e. Theorem 3.3.26) is also true if X is countably paracompact and normal, provided that the Banach space Y is separable; that is, we have the following theorem:

THEOREM 3.3.27 ([39, Theorem 3.1]). *For a T_1 -space X , the following are equivalent:*

- (a) *X is countably paracompact and normal.*
- (b) *If Y is a separable Banach space and $\varphi : X \rightarrow \mathcal{F}(Y)$ is an l.s.c. convex-valued mapping, then φ has a continuous selection.*
- (c) *Every l.s.c. convex-valued mapping $\varphi : X \rightarrow \mathcal{F}(\mathbb{R})$ has a continuous selection.*

CHAPTER 4

Selections, Extension and Collectionwise Normality

1. Selections for Convex-valued mappings on Collectionwise normal Domains

The Dowker extension theorem states that a T_1 -space X is collectionwise normal if and only if for every closed subset $A \subset X$, every continuous map from A to a Banach space Y can be continuously extended to the whole of X (see, Theorem 3.2.20). Generalizing this result, Michael [39] stated the following theorem.

THEOREM 4.1.1 (see [39]). *For a T_1 -space X , the following are equivalent:*

- (a) X is collectionwise normal.
- (b) If Y is a Banach space and $\varphi : X \rightarrow \mathcal{C}'(Y)$ is an l.s.c. convex-valued mapping, then φ has a continuous selection.

The arguments in [39] for (a) \Rightarrow (b) of Theorem 4.1.1 were incomplete and, in fact, working only for the case of $\mathcal{C}'(Y)$ -valued mappings. This fact was pointed out by S. Nedev and the first complete proof of this implication was given by Choban and Valov [7] using a different technique.

In [24], V. Gutev and the author proved the following theorem which demonstrate that the original Michael arguments in [39] have been actually adequate to the proof of Theorem 4.1.1.

THEOREM 4.1.2 ([24, Theorem 1.2]). *For a Banach space Y , the following are equivalent:*

- (a) If X is collectionwise normal and $\varphi : X \rightarrow \mathcal{C}(Y)$ is an l.s.c. convex-valued mapping, then φ has a continuous selection.
- (b) If X is collectionwise normal and $\varphi : X \rightarrow \mathcal{C}'(Y)$ is an l.s.c. convex-valued mapping, then φ has a continuous selection.

Let us mention that Theorem 4.1.2 was also included in [33] with a slight difference in condition (b): the range Y of the mapping φ is a closed convex subset of a Banach space, rather than the entire Banach space. Both proofs of Theorem 4.1.2 in [24] and [33] are identical. For the sake of completeness, we include its proof with (b) as it appears in [24]. Let us also emphasize that the proof of Theorem 4.1.2 is based only on Dowker's extension theorem [9] and its main ingredient is the fact that if $\varphi : X \rightarrow \mathcal{C}'(Y)$ and $g : X \rightarrow Y$, then $\varphi(x)$ is compact for every $x \in X$ for which $g(x) \notin \varphi(x)$. This is further applied in the Section 2 to get with ease a direct proof of a natural generalization of Theorem 4.1.1. Section 3 deals with controlled selections for set-valued mappings defined on countably paracompact or collectionwise normal spaces which are naturally interrelated to Theorem 4.1.2.

The key element in the proof of this theorem is the following construction of approximate selections (see Claim 1). Recall that for $\varepsilon > 0$, a map $g : X \rightarrow Y$ to a metric space (Y, d) is an ε -selection for $\varphi : X \rightarrow 2^Y$, if $d(g(x), \varphi(x)) < \varepsilon$, for every $x \in X$.

CLAIM 1 ([24]). Let $\psi : X \rightarrow 2^Y$ be an l.s.c. convex-valued mapping and $g : X \rightarrow Y$ be a continuous map such that $\psi(x)$ is compact whenever $x \in X$ and $g(x) \notin \psi(x)$. Then, for every $\varepsilon > 0$, ψ has a continuous ε -selection.

PROOF. Let $\varepsilon > 0$ and $A = \{x \in X : d(g(x), \psi(x)) \geq \varepsilon\}$. Then, $A \subset X$ is closed because ψ is l.s.c. and g is continuous. Since $\psi \upharpoonright A : A \rightarrow \mathcal{C}(Y)$ and A is itself collectionwise normal (because A is closed in X), by (a) of Theorem 4.1.2, $\psi \upharpoonright A$ has a continuous selection $h_0 : A \rightarrow E$. Since X is collectionwise normal, by Theorem 3.2.20, there exists a continuous map $h : X \rightarrow E$ such that $h \upharpoonright A = h_0$. Consider the set $U = \{x \in X : d(h(x), \psi(x)) < \varepsilon\}$ which contains A and is open because ψ is l.s.c.

and h is continuous. Finally, since X is normal, by Urysohn's Lemma (Lemma 2.3.25), take a continuous function $\alpha : X \rightarrow [0, 1]$ such that $A \subset \alpha^{-1}(0)$ and $X \setminus U \subset \alpha^{-1}(1)$, and then define a continuous map $f : X \rightarrow Y$ by

$$f(x) = \alpha(x) \cdot g(x) + (1 - \alpha(x)) \cdot h(x), \quad x \in X.$$

This f is as required. Indeed, take a point $x \in X$. If $x \in A$, then $\alpha(x) = 0$ and, therefore, $f(x) = h(x) = h_0(x) \in \psi(x)$. If $x \in X \setminus U$, then $\alpha(x) = 1$ and we now have that $f(x) = g(x)$, so $d(f(x), \psi(x)) = d(g(x), \psi(x)) < \varepsilon$ because $x \notin A$. Suppose finally that $x \in U \setminus A$. In this case, $d(h(x), \psi(x)) < \varepsilon$ and $d(g(x), \psi(x)) < \varepsilon$. Since $\psi(x)$ is convex and $f(x) = \alpha(x) \cdot g(x) + (1 - \alpha(x)) \cdot h(x)$, this implies that $d(f(x), \psi(x)) < \varepsilon$. The proof is completed. \square

Having already established Claim 1, we proceed to the proof of $(a) \Rightarrow (b)$ which is based on standard arguments for constructing continuous selections, see [39].

PROOF OF THEOREM 4.1.2. It only suffices to prove $(a) \Rightarrow (b)$. To this end, suppose that (a) of Theorem 4.1.2 holds, Y is a Banach space, and X is a collectionwise normal space. Here, we use d to denote the metric on Y generated by the norm of Y . Let $\varphi : X \rightarrow \mathcal{C}'(Y)$ be an l.s.c. convex-valued mapping. If $g : X \rightarrow Y$ is any continuous map, say a constant one, then $\varphi(x)$ is compact for every $x \in X$ for which $g(x) \notin \varphi(x)$. Hence, by Claim 1, φ has a continuous 2^{-1} -selection $f_0 : X \rightarrow Y$. Define $\varphi_1 : X \rightarrow \mathcal{F}(Y)$ by

$$\varphi_1(x) = \overline{\varphi(x) \cap B_{2^{-1}}^d(f_0(x))}, \quad x \in X.$$

According to Proposition 3.1.5 and Corollary 3.1.13, φ_1 is l.s.c., and clearly it is convex-valued. Finally, observe that if $f_0(x) \notin \varphi_1(x)$ for some $x \in X$, then $f_0(x) \notin \varphi(x)$ and, therefore, $\varphi(x)$ is compact because it is $\mathcal{C}'(Y)$ -valued. Since $\varphi_1(x)$ is a closed subset of $\varphi(x)$, it is also compact. Hence, by Claim 1, φ_1 has a continuous 2^{-2} -selection f_1 . In particular, f_1 is a continuous 2^{-2} -selection for φ such that

$$d(f_0(x), f_1(x)) < 2^{-1} + 2^{-2} < 2^0, \quad \text{for every } x \in X.$$

Thus, by induction, we get a sequence $\{f_n : n < \omega\}$ of continuous maps such that, for every $n < \omega$ and $x \in X$,

$$(1) \quad d(f_n(x), \varphi(x)) < 2^{-(n+1)},$$

$$(2) \quad d(f_n(x), f_{n+1}(x)) < 2^{-n}.$$

By (2), $\{f_n : n < \omega\}$ is a Cauchy sequence, so it must converge to some continuous $f : X \rightarrow Y$. By (1), $f(x) \in \varphi(x)$ for every $x \in X$. Hence, (b) holds, and the proof of Theorem 4.1.2 is completed. \square

REMARK 4.1.3. Although conditions (a) and (b) in Theorem 4.1.2 are equivalent and (b) gives a characterization of collectionwise normality in terms of continuous selections, condition (a) on the other hand does not characterize collectionwise normality. In fact, there are spaces satisfying (a) which are not collectionwise normal (see Theorems 5.4.15 and 5.4.16 in Chapter 5). Such spaces are called PF-normal and are discussed in Section 4 of Chapter 5.

2. More on Selections and Collectionwise Normality

The proof of Theorem 4.1.2 suggests an easy direct proof of the following theorem (Theorem 4.2.5) which is a natural generalization of Theorem 4.2 in [51]. We first need the following well-known result.

PROPOSITION 4.2.4. *If Y is a space, with $w(Y) \leq \tau$, and \mathcal{V} is a locally finite family of subsets of Y , then $|\mathcal{V}| \leq \tau$.*

PROOF. Let \mathcal{B} be a base for the topology of Y such that $|\mathcal{B}| \leq \tau$. Since \mathcal{V} is locally finite, every $x \in X$ has a neighborhood O_x such that the family $\{V \in \mathcal{V} : V \cap O_x \neq \emptyset\}$ is finite. Since $x \in O_x$ and O_x is open, there is a $B_x \in \mathcal{B}$ such that $x \in B_x \subset O_x$. In particular, the family $\{V \in \mathcal{V} : V \cap B_x \neq \emptyset\}$ is also finite. Let $\mathcal{B}_0 = \{B_x : x \in X\}$. Then, every $B \in \mathcal{B}_0$ meet only finitely many elements of \mathcal{V} , say \mathcal{V}_B . Now, we have

that $\mathcal{V} = \bigcup\{\mathcal{V}_B : B \in \mathcal{B}_0\}$. Since $|\mathcal{B}_0| \leq \tau$, and each \mathcal{V}_B is finite, for every $B \in \mathcal{B}_0$, this implies that $|\mathcal{V}| \leq \tau$. \square

THEOREM 4.2.5 (see, [51]). *Let X be a τ -collectionwise normal space, Y be a Banach space with $w(Y) \leq \tau$, and let $\varphi : X \rightarrow \mathcal{C}'(Y)$ be an l.s.c. convex-valued mapping. Then, φ has a continuous selection.*

PROOF. It only suffices to prove the statement of Claim 1 for this particular case. So, suppose that $\psi : X \rightarrow 2^Y$ is l.s.c. and convex-valued, and $g : X \rightarrow Y$ is a continuous map such that $\psi(x)$ is compact whenever $g(x) \notin \psi(x)$. Also, let $\varepsilon > 0$ and let \mathcal{V} be a locally finite open cover of Y such that $\text{diam}_d(V) < \varepsilon$ for every $V \in \mathcal{V}$. Since g is continuous, $\mathcal{U}_1 = \{g^{-1}(V) \cap \psi^{-1}(V) : V \in \mathcal{V}\}$ is a locally finite family of open subsets of X which refines $\{\psi^{-1}(V) : V \in \mathcal{V}\}$. Then, $A = X \setminus \bigcup \mathcal{U}_1$ is a closed subset of X , while $\psi \upharpoonright A$ is compact-valued. Indeed, if $g(x) \in \psi(x)$, then $x \in \psi^{-1}(V)$ whenever $V \in \mathcal{V}$ and $g(x) \in V$. That is, $x \in A$ implies $g(x) \notin \psi(x)$, so, in this case, $\psi(x)$ must be compact. Thus, $\{\psi^{-1}(V) : V \in \mathcal{V}\}$ is an open (in X) and point-finite (in A) cover of A such that $|\mathcal{V}| \leq \tau$ because \mathcal{V} is locally finite (in X) and $w(Y) \leq \tau$ (see Proposition 4.2.4). Since X is τ -collectionwise normal, by Lemma 3.2.18, $\{\psi^{-1}(V) : V \in \mathcal{V}\}$ has an open and locally finite (in X) refinement \mathcal{U}_2 which covers A . Then, $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ is an open and locally finite cover of X which refines $\{\psi^{-1}(V) : V \in \mathcal{V}\}$. For every $U \in \mathcal{U}$, take a fixed $V_U \in \mathcal{V}$ such that $U \subset \psi^{-1}(V_U)$ and a point $e(U) \in V_U$ provided $V_U \neq \emptyset$. Next, take a partition of unity $\{\xi_U : U \in \mathcal{U}\}$ on X which is index subordinated to the cover \mathcal{U} , see [36] or Theorem 2.3.39. Finally, define a continuous map $f : X \rightarrow Y$ by $f(x) = \{\xi_U(x) \cdot e(U) : U \in \mathcal{U}\}$, $x \in X$. This f is an ε -selection for ψ . \square

As far as the role of the family $\mathcal{C}'(Y)$ is concerned, the arguments in the proof of Theorems 4.1.2 and 4.2.5 were based only on the property that if $\varphi : X \rightarrow \mathcal{C}'(Y)$ and $g : X \rightarrow Y$ is an ε -selection for φ for some $\varepsilon > 0$, then the set-valued mapping $\psi(x) = \overline{\varphi(x) \cap B_\varepsilon^d(g(x))}$, $x \in X$, is such that $\psi(x)$ is compact whenever $g(x) \notin \psi(x)$. That is, this resulting ψ is always as in Claim 1, and the inductive construction can

be carried on.

Motivated by this, we shall say that a mapping $\psi : X \rightarrow \mathcal{F}(Y)$ has a *selection $\mathcal{C}(Y)$ -deficiency* (see, [24]) if there exists a continuous $g : X \rightarrow Y$ such that $\psi(x) \in \mathcal{C}(Y)$ for every $x \in X$ for which $g(x) \notin \psi(x)$. Clearly, every $\varphi : X \rightarrow \mathcal{C}'(Y)$ has this property, for instance take $g : X \rightarrow Y$ to be any constant map. However, there are natural examples of mappings $\varphi : X \rightarrow \mathcal{F}(Y)$ which have a selection $\mathcal{C}(Y)$ -deficiency and are not $\mathcal{C}'(Y)$ -valued, see next section. Related to this, we don't know if such mappings may have a continuous selection in the case of collectionwise normal spaces.

▷ QUESTION 1 ([24]). Let X be a collectionwise normal space, Y be a Banach space, and let $\varphi : X \rightarrow \mathcal{F}(Y)$ be an l.s.c. convex-valued mapping which has a selection $\mathcal{C}(Y)$ -deficiency. Then, is it true that φ has a continuous selection?

Another aspect of improving Michael's convex-valued selection theorem for mappings defined on collectionwise normal spaces is related to the range of the set-valued mapping. In this regard, Theorem 4.2.5 remains valid without any change in the arguments if the Banach space Y is replaced by a closed convex subset E of Y . On the other hand, if E is a completely metrizable absolute retract for metrizable spaces, then for every collectionwise normal space X and a closed $A \subset X$, every continuous map $g : A \rightarrow E$ can be continuously extended to the whole of X , see, Theorem 3.2.20. In particular, this is true for every convex G_δ -subset E of a Banach space Y . Namely, E is an absolute retract for the metrizable spaces being convex (by Dugundji's extension theorem [11]), and is also completely metrizable being a G_δ -subset of a complete metric space. Motivated by this and the relationship between extensions and selections demonstrated in the proof of Theorem 4.1.2, we have also the following question.

▷ QUESTION 2 ([24]). Let Y be a Banach space, $E \subset Y$ be a convex G_δ -subset of Y , X be a collectionwise normal space, and let $\varphi : X \rightarrow \mathcal{C}'(E)$ be an l.s.c. convex-valued mapping. Then, is it true that φ has a continuous selection?

Question 2 is similar to Michael's G_δ -problem [45, Problem 396] whether for a paracompact space X and G_δ -subset E of a Banach space, every l.s.c. convex-valued $\varphi : X \rightarrow \mathcal{F}(E)$ has a continuous selection. In general, the answer to this latter problem is in the negative due to a counterexample by Filippov [15, 16]. However, Michael's G_δ -problem was resolved in the affirmative in a number of partial cases. The solution in some of these cases remains valid for Question 2 as well. For instance, if E is a countable intersection of open convex sets, then the closed convex-hull $\overline{\text{conv}(K)}$ of every compact subset K of E will still be a subset of E , see [45]. In this case, by a result of Choban and Valov [7], there will exist an l.s.c. convex-valued mapping $\psi : X \rightarrow \mathcal{C}(E)$ such that $\psi(x) \subset \varphi(x)$, $x \in X$. Hence, φ will have a continuous selection because, by Theorem 4.1.1, so does ψ . If the covering dimension¹ of X is finite (i.e., $\dim(X) < \infty$), then the answer to Question 2 is also "yes", this follows directly from a selection theorem in [18]. The answer to Question 4.2.5 is also "yes" if X is strongly countable-dimensional (i.e., a countable union of closed finite-dimensional subsets). In this case, there exists a metrizable (strongly) countable-dimensional space Z , a continuous map $g : X \rightarrow Z$ and an l.s.c. mapping $\psi : Z \rightarrow \mathcal{C}(E)$ such that $\psi(g(x)) \subset \varphi(x)$ for every $x \in X$, see, for instance, the proof of [51, Theorem 5.3]. Next, define a mapping $\Phi : Z \rightarrow \mathcal{F}(E)$ by $\Phi(z) = \overline{\text{conv}(\psi(z))}^E$, $z \in Z$, where the closure is in E . According to Propositions 3.1.5 and 3.1.7, Φ remains l.s.c., and, by [19, Corollary 1.2], it admits a continuous selection $h : Z \rightarrow E$. Then, $f = h \circ g$ is a continuous selection for φ because $\Phi(g(x)) \subset \varphi(x)$ for all $x \in X$.

3. Controlled Selections and Countable Paracompactness

If (Y, d) is a metric space, $\varphi : X \rightarrow 2^Y$ and $\eta : X \rightarrow (0, +\infty)$, then we shall say that $g : X \rightarrow Y$ is an η -selection for φ if $d(g(x), \varphi(x)) < \eta(x)$ for every $x \in X$.

¹The covering dimension $\dim(X)$ of a topological space X is the least integer n such that every finite functionally open cover of X has a finite functionally open refinement \mathcal{V} of order at most n , i.e. every point $x \in X$ is contained in at most $n + 1$ members of \mathcal{V} and no point of X lies in more than $n + 1$ members of \mathcal{V} .

In [39], E. Michael proved that a T_1 -space X is countably paracompact and normal if and only if for every separable Banach space Y , every l.s.c. convex-valued mapping $\varphi : X \rightarrow \mathcal{F}(Y)$ has a continuous selection (see Theorem 3.3.27).

In this section, we first prove the following characterization of countably paracompact normal spaces.

THEOREM 4.3.6 ([24, Theorem 4.1]). *For a T_1 -space X , the following are equivalent:*

- (a) *X is countably paracompact and normal.*
- (b) *If Y is a separable Banach space, $\varphi : X \rightarrow \mathcal{F}(Y)$ is an l.s.c. convex-valued mapping, $\eta : X \rightarrow (0, \infty)$ is lower semi-continuous and $g : X \rightarrow Y$ is a continuous η -selection for φ , then φ has a continuous selection $f : X \rightarrow Y$ such that $d(f(x), g(x)) < \eta(x)$ for all $x \in X$.*
- (c) *If $\varphi : X \rightarrow \mathcal{C}(\mathbb{R})$ is an l.s.c. convex-valued mapping, $\varepsilon > 0$ and $g : X \rightarrow \mathbb{R}$ is a continuous ε -selection for φ , then φ has a continuous selection $f : X \rightarrow \mathbb{R}$ such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$.*

PROOF. (a) \Rightarrow (b). Let X be a countably paracompact and normal, and let Y , φ , η , and g be as in (b). Since φ is l.s.c. and g is continuous, $\xi(x) = d(g(x), \varphi(x))$, $x \in X$, is an upper semi-continuous function such that $\xi(x) < \eta(x)$ for all $x \in X$ because g is an η -selection for φ . Since X is countably paracompact and normal, by Theorem 2.3.47 (see, also, [8, 10, 29]), there exists a continuous function $\alpha : X \rightarrow \mathbb{R}$ such that $\xi(x) < \alpha(x) < \eta(x)$ for every $x \in X$. Then, define an l.s.c. mapping $\psi : X \rightarrow \mathcal{F}(Y)$ by $\psi(x) = \overline{\varphi(x) \cap B_{\alpha(x)}^d(g(x))}$, $x \in X$. Since ψ is convex-valued, by Theorem 3.3.27, ψ has a continuous selection $f : X \rightarrow Y$. In particular, $d(f(x), g(x)) < \eta(x)$ for all $x \in X$.

Since (b) \Rightarrow (c) is obvious, we complete the proof showing that (c) \Rightarrow (a). So, suppose that (c) holds. If A and B are disjoint closed subsets of X , then $\varphi(x) = \{0\}$ if $x \in A$, $\varphi(x) = \{1\}$ if $x \in B$, and $\varphi(x) = [0, 1]$ otherwise, is an l.s.c. convex-valued mapping

$\varphi : X \rightarrow \mathcal{C}(\mathbb{R})$. If $g(x) = \frac{1}{2}$, $x \in X$, then g is a continuous 1-selection for φ , and, by (c), φ has a continuous selection $f : X \rightarrow \mathbb{R}$. According to the definition of φ , we get that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$, hence X is normal. To show that X is countably paracompact, let $\{F_n : n < \omega\}$ be a decreasing sequence of closed subsets of X such that $F_0 = X$ and $\bigcap\{F_n : n < \omega\} = \emptyset$. Next, for every $x \in X$, let $n(x) = \max\{n < \omega : x \in F_n\}$. Then, define a convex-valued mapping $\varphi : X \rightarrow \mathcal{C}(\mathbb{R})$ by $\varphi(x) = [0, 2^{-n(x)}]$, $x \in X$. Observe that φ is l.s.c. because $z \in X \setminus F_{n(x)+1}$ implies that $n(z) \leq n(x)$ and, therefore, $\varphi(x) = [0, 2^{-n(x)}] \subset [0, 2^{-n(z)}] = \varphi(z)$. Finally, observe that $g(x) = 1$, $x \in X$, is a continuous 1-selection for φ . Hence, by (c), φ has a continuous selection $f : X \rightarrow \mathbb{R}$ such that $|g(x) - f(x)| = 1 - f(x) < 1$ for every $x \in X$, or, in other words, $f(x) > 0$ for all $x \in X$. Finally, define $W_n = f^{-1}((-\infty, 2^{-n+1}))$, $n < \omega$. Thus, we get a sequence $\{W_n : n < \omega\}$ of open subsets of X such that $F_n \subset W_n$ for every $n < \omega$. Indeed, $x \in F_n$ implies $n \leq n(x)$, so $f(x) \in \varphi(x) = [0, 2^{-n(x)}] \subset [0, 2^{-n}] \subset [0, 2^{-n+1}]$. Since $\overline{W_n} \subset f^{-1}((-\infty, 2^{-n+1}))$, $n < \omega$, and $f(x) > 0$ for every $x \in X$, we have that $\bigcap\{\overline{W_n} : n < \omega\} = \emptyset$. That is, X is countably paracompact by Theorem 2.3.44. \square

For collectionwise normal spaces, we have a very similar result (see Proposition 4.3.8 below) which, in particular, illustrates the difference with countably paracompact one (see (c) of Theorem 4.3.6). Recall that if (Y, d) is a metric space, a mapping $\psi : X \rightarrow 2^Y$ is *d-l.s.c.* (resp. *d-u.s.c.*) if, given $\varepsilon > 0$, every $x \in X$ admits a neighborhood V such that $\psi(x) \subset B_\varepsilon^d(\psi(z))$ (resp. $\psi(z) \subset B_\varepsilon^d(\psi(x))$), for every $z \in V$. A mapping ψ is said to be *d-continuous* if it is both *d-l.s.c.* and *d-u.s.c.*; ψ is *continuous* if it is both l.s.c. and u.s.c.; and ψ is *d-proximal continuous* if it is both l.s.c. and *d-u.s.c.* (see, [23]). Every *d-continuous* or *continuous* mapping is *d-proximal continuous*, but the converse is not true (see, for instance [23, Proposition 2.5]). The *d-proximal continuity* depends on the metric on the range Y which leads us to its following topological version. If Y is a metrizable space, we shall say that $\psi : X \rightarrow 2^Y$ is *proximal continuous* provided there exists a compatible metric d on Y such that ψ is *d-proximal continuous*.

The following lemma also plays an essential role² in the proof of Proposition 4.3.8 and Theorem 4.3.9 below. We first need some new terminology. Following Nedev [51], for a normal space X and a metric space (Y, d) , a mapping $\varphi : X \rightarrow \mathcal{F}(Y)$ has the *Selection Factorization Property* if for every closed subset F of X and every locally finite collection \mathcal{U} of open subsets of Y such that $\varphi^{-1}(\mathcal{U}) = \{\varphi^{-1}(U) : U \in \mathcal{U}\}$ covers F , there exists a locally finite open (in F) cover of F which refines $\varphi^{-1}(\mathcal{U})$. According to [51, Proposition 4.3], if X is normal, Y is a closed convex subset of a Banach space and $\varphi : X \rightarrow \mathcal{F}(Y)$ is convex-valued and has the Selection Factorization Property, then φ has a continuous selection. Recall that a set-valued mapping $\psi : X \rightarrow 2^Y$ is a *multi-selection* (or, a *set-valued selection*) for another set-valued mapping $\varphi : X \rightarrow 2^Y$ if $\psi(x) \subset \varphi(x)$, for every $x \in X$.

LEMMA 4.3.7 ([25, Lemma 4.2]). *Let X be a τ -collectionwise normal space, Y be a Banach space with $w(Y) \leq \tau$, $\psi : X \rightarrow \mathcal{F}(Y)$ be a proximal continuous convex-valued mapping, and $\varphi : X \rightarrow \mathcal{F}(Y)$ be an l.s.c. convex-valued multi-selection for ψ such that $\varphi(x)$ is compact whenever $\varphi(x) \neq \psi(x)$, $x \in X$. Then, φ has a continuous selection.*

PROOF. It suffices to show, by [51, Proposition 4.3], that φ has the Selection Factorization Property. Towards this end, take a closed set $F \subset X$, and a locally finite family \mathcal{U} of non-empty open sets in Y such that $F \subseteq \bigcup \varphi^{-1}(\mathcal{U})$. Since ψ is proximal continuous, by [23, Theorem 3.1], there exists a locally finite open cover $\{V_U : U \in \mathcal{U}\}$ of F such that $V_U \subset \psi^{-1}(U)$, for every $U \in \mathcal{U}$. Set $\mathcal{W}_1 = \{W_U : U \in \mathcal{U}\}$, where $W_U = V_U \cap \varphi^{-1}(U)$, $U \in \mathcal{U}$, and let $W_1 = \bigcup \mathcal{W}_1$. Then, $\varphi(x)$ is compact for every $x \in F \setminus W_1$. Indeed, take a point $x \in F$ and $U \in \mathcal{U}$ such that $x \in V_U$ and $\varphi(x) = \psi(x)$. Since $\varphi(x) \cap U = \psi(x) \cap U$, we get that $x \in W_U \subseteq W_1$. Hence, $x \in F \setminus W_1$ implies $\varphi(x) \neq \psi(x)$ and, by hypothesis, $\varphi(x)$ is compact. As a result, $\varphi^{-1}(\mathcal{U})$ is point-finite at every point of $F \setminus W_1$. On the other hand, by Proposition 4.2.4, $|\mathcal{U}| \leq \tau$. Therefore,

²It also plays a crucial role in the proof of Theorem 5.2.6 and Example 5.5.25.

there exists a locally finite open (in F) cover \mathscr{W}_2 of $F \setminus W_1$ which refines $\varphi^{-1}(\mathscr{U})$ because X is τ -collectionwise normal. Then, $\mathscr{W} = \mathscr{W}_1 \cup \mathscr{W}_2$ is as required. \square

PROPOSITION 4.3.8 ([24, Proposition 4.2]). *Let Y be a Banach space, X be a collectionwise normal space, $\varphi : X \rightarrow \mathcal{C}'(Y)$ be an l.s.c. convex-valued mapping, and let $g : X \rightarrow Y$ be a continuous ε -selection for φ for some $\varepsilon > 0$. Then, φ has a continuous selection $f : X \rightarrow Y$ such that $d(f(x), g(x)) \leq \varepsilon$ for all $x \in X$.*

PROOF. Define a mapping $\psi : X \rightarrow \mathcal{F}(Y)$ by $\psi(x) = \overline{B_\varepsilon^d(g(x))}$, $x \in X$. Then, ψ is convex-valued and d -proximal continuous. Define another mapping $\theta : X \rightarrow \mathcal{F}(Y)$ by $\theta(x) = \overline{\varphi(x) \cap B_\varepsilon^d(g(x))}$, $x \in X$. According to Proposition 3.1.5 and Corollary 3.1.13, θ is l.s.c. and clearly it is also convex-valued. Finally, observe that $\theta(x) \subset \psi(x)$ for every $x \in X$, while $\theta(x) \neq \psi(x)$ implies that $\theta(x)$ is compact. Then, by Lemma 4.3.7, θ has a continuous selection $f : X \rightarrow Y$. This f is as required. \square

Following the idea in the proof of Proposition 4.3.8, one can extend Theorem 4.3.6 to the case of countably paracompact and τ -collectionwise normal spaces.

THEOREM 4.3.9 ([24, Theorem 4.3]). *For a T_1 -space X and an infinite cardinal number τ , the following are equivalent:*

- (a) X is countably paracompact and τ -collectionwise normal.
- (b) If Y is a Banach space with $w(Y) \leq \tau$, $\varphi : X \rightarrow \mathcal{C}'(Y)$ is l.s.c. and convex-valued, $\eta : X \rightarrow (0, +\infty)$ is lower semi-continuous, and $g : X \rightarrow Y$ is a continuous η -selection for φ , then φ has a continuous selection $f : X \rightarrow Y$ such that $d(f(x), g(x)) < \eta(x)$ for all $x \in X$.

PROOF. (a) \Rightarrow (b). As in (a) of the proof of Theorem 4.3.6, there exists a continuous function $\alpha : X \rightarrow (0, +\infty)$ such that $d(g(x), \varphi(x)) < \alpha(x) < \eta(x)$ for every $x \in X$. Next, as in the proof of Proposition 4.3.8, define a (d -proximal) continuous $\psi(x) = \overline{B_{\alpha(x)}^d(g(x))}$, $x \in X$, and another l.s.c. $\theta : X \rightarrow \mathcal{F}(Y)$ by $\theta(x) = \overline{\varphi(x) \cap B_{\alpha(x)}^d(g(x))}$,

$x \in X$. As in the proof of Lemma 4.3.7, this θ has the Selection Factorization Property. Hence, by [51, Proposition 4.3], θ has a continuous selection $f : X \rightarrow Y$. According to the definition of θ , we get that $d(f(x), g(x)) < \eta(x)$ for all $x \in X$.

(b) \Rightarrow (a). This implication is based on standard arguments. In fact, X will be countably paracompact and normal by Theorem 4.3.6. To show that X is also τ -collectionwise normal, let \mathcal{D} be a discrete family of closed subsets of X , with $|\mathcal{D}| \leq \tau$, and let $\ell_1(\mathcal{D})$ be the Banach space of all functions $y : \mathcal{D} \rightarrow \mathbb{R}$, with $\sum\{|y(D)| : D \in \mathcal{D}\} < \infty$, equipped with the norm $\|y\| = \sum\{|y(D)| : D \in \mathcal{D}\}$. Also, let $v(D) = 0$, $D \in \mathcal{D}$, be the origin of $\ell_1(\mathcal{D})$. For every $D \in \mathcal{D}$, consider the function $e_D : \mathcal{D} \rightarrow \mathbb{R}$ defined by $e_D(D) = 1$ and $e_D(T) = 0$ for $T \in \mathcal{D} \setminus \{D\}$. Then, $e_D \in \ell_1(\mathcal{D})$, $D \in \mathcal{D}$, and $\|e_D - v\| = 1$ for every $D \in \mathcal{D}$. Finally, define an l.s.c. mapping $\varphi : X \rightarrow \mathcal{C}'(\ell_1(\mathcal{D}))$ by $\varphi(x) = \{e_D\}$ if $x \in D$ for some $D \in \mathcal{D}$ and $\varphi(x) = \ell_1(\mathcal{D})$ otherwise. Then, φ is convex-valued and $g(x) = v$, $x \in X$, is a continuous 2-selection for φ . Since $w(\ell_1(\mathcal{D})) \leq \tau$, by (b), φ has a continuous selection $f : X \rightarrow \ell_1(\mathcal{D})$. Then, $U_D = f^{-1}(B_1(e_D))$, $D \in \mathcal{D}$, is a pairwise disjoint family open subsets of X such that $D \subset U_D$, $D \in \mathcal{D}$. Since X is normal, by Proposition 3.2.17, X is also τ -collectionwise normal. \square

Exactly the same arguments as those in the proof of Theorem 4.3.9 show that if X is τ -collectionwise normal, Y is a Banach space with $w(Y) \leq \tau$, $\varphi : X \rightarrow \mathcal{C}'(Y)$ is an l.s.c. convex-valued mapping, $\eta : X \rightarrow (0, +\infty)$ is continuous, and $g : X \rightarrow Y$ is a continuous η -selection for φ , then φ has a continuous selection $f : X \rightarrow Y$ such that $d(f(x), g(x)) \leq \eta(x)$ for all $x \in X$. Motivated by this and Proposition 4.3.8, we have the following natural question.

\triangleright QUESTION 3 ([24]). Let X be a collectionwise normal space, Y be a Banach space, $\varphi : X \rightarrow \mathcal{C}'(Y)$ be an l.s.c. convex-valued mapping, and let $g : X \rightarrow Y$ be a continuous η -selection for φ for some lower semi-continuous function $\eta : X \rightarrow (0, \infty)$. Then, does φ have a continuous selection $f : X \rightarrow Y$ with $d(f(x), g(x)) \leq \eta(x)$ for all $x \in X$?

Let us point out that the answer to Question 3 is “yes” if that is the answer to Question 1. Indeed, if $\eta : X \rightarrow (0, +\infty)$ is lower semi-continuous and $g : X \rightarrow Y$ is continuous, then the mapping $\psi(x) = B_{\eta(x)}^d(g(x))$, $x \in X$, will have an open graph. If g is also an η -selection for $\varphi : X \rightarrow \mathcal{C}'(Y)$, then $\theta(x) = \overline{\varphi(x) \cap \psi(x)}$, $x \in X$, will have a selection $\mathcal{C}(Y)$ -deficiency. Finally, if $f : X \rightarrow Y$ is a continuous selection for θ , then $d(f(x), g(x)) \leq \eta(x)$ for all $x \in X$.

Selections for Paraconvex-valued Mappings

1. Selections, Paraconvexity and Paracompactness

Let Y be a normed space and d be the metric on Y generated by the norm of Y .

DEFINITION 5.1.1 (Paraconvex-set, Michael [42]). A subset P of a normed space Y is called α -*paraconvex*, where $0 \leq \alpha \leq 1$, if whenever $r > 0$ and $d(p, P) < r$ for some $p \in Y$, then

$$d(q, P) \leq \alpha r \text{ for all } q \in \text{conv}(B_r^d(p) \cap P).$$

The set P is called *paraconvex* if it is α -paraconvex for some $\alpha < 1$. A closed set is 0-paraconvex if and only if it is convex.

EXAMPLE 5.1.2 (Examples of paraconvex sets).

- (a) Every subset of an inner product space or a two-dimensional space is 1-paraconvex (V.L. Klee [30]).
- (b) The letters **V**, **X**, **Y**, and **Z**, and a circular arc subtending an angle less than π are paraconvex, whereas the letter **U** and a circular arc subtending an angle greater or equal to π are not paraconvex (E. Michael [42]).
- (c) The set $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ or } y \geq 0\}$ is an α -paraconvex subset of \mathbb{R}^2 , with $\alpha = 1/\sqrt{2}$ (G.E. Ivanov [26, Remark 2]).
- (d) Let P be the graph of a Lipschitz function with a constant $k \geq 0$ and with a convex domain. Then, P is an α -paraconvex subset of the plane, where $\alpha = \sin(\arctan k)$ (P.V. Semenov, [63]).

- (e) Let G be the graph of a monotonic continuous function with a convex domain. Then, G is an β -paraconvex subset of the plane, where $\beta = \sqrt{2}/2$ (P.V. Semenov, [63]).

In the sequel, we will use $\mathcal{F}_\alpha(Y)$ (resp. $\mathcal{C}_\alpha(Y)$) to denote the set of all α -paraconvex members of $\mathcal{F}(Y)$ (resp. $\mathcal{C}(Y)$); and $\mathcal{C}'_\alpha(Y) = \mathcal{C}_\alpha(Y) \cup \{Y\}$.

In [42], E. Michael generalized Theorem 3.3.26 by replacing “convexity” with “ α -paraconvexity” for a fixed $\alpha < 1$, and proved the following theorem.

THEOREM 5.1.3 ([42, Theorem 2.1]). *Let X be a paracompact space, Y be a Banach space, and let $\varphi : X \rightarrow \mathcal{F}_\alpha(Y)$ be an l.s.c. mapping, where $\alpha < 1$. Then, the following hold:*

- (a) φ has a continuous selection.
- (b) If $r > 0$ and $g : X \rightarrow Y$ is continuous such that $d(g(x), \varphi(x)) < r$ for all $x \in X$, then there exists $\delta > 0$ and a continuous selection f for φ such that $d(g(x), f(x)) < \delta r$, $x \in X$.

It should be remarked that Theorem 5.1.3 is not true for $\alpha = 1$. Indeed, by Example 5.1.2 (a), for an inner-product space H (in particular, a Hilbert space H), $\mathcal{F}_1(H) = \mathcal{F}(H)$, and not every l.s.c. mapping $\varphi : X \rightarrow \mathcal{F}_1(H)$, from a paracompact space X has a continuous selection.

Notice that in the proof of (a) of Theorem 5.1.3 in [42], to an increasing open cover $\{U_n : n \in \mathbb{N}\}$ of the paracompact X , it was found a refinement $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ such that

- (i) A_n is closed for each $n \in \mathbb{N}$,
- (ii) \mathcal{A} is an increasing family, and
- (iii) \mathcal{A} is locally finite.

This closed cover is used to construct a sequence of continuous selections $f_n : A_n \rightarrow Y$ for $\varphi \upharpoonright A_n$ such that $f_{n+1} \upharpoonright A_n = f_n$, for every $n \in \mathbb{N}$. The resulting continuous selection $f : X \rightarrow Y$ is then defined by $f \upharpoonright A_n = f_n$, $n \in \mathbb{N}$. However, an (infinite) increasing family of nonempty sets cannot be locally finite, since each point will have a neighborhood intercepting infinitely many elements of the family; i.e. conditions (ii) and (iii) cannot coexist. From the proof, it is clear that (i) and (ii) are substantial. Dropping (iii) may lead to the failure of the original arguments that the resulting selection $f : X \rightarrow Y$ is continuous. Here is an example.

EXAMPLE 5.1.4. There exists a metrizable space X , an increasing closed cover $\{A_n : n \in \mathbb{N}\}$ of X and a map $f : X \rightarrow \mathbb{R}$ such that

- (a) $f \upharpoonright A_n$ is continuous, for each $n \in \mathbb{N}$,
- (b) f is not continuous.

PROOF. Take $X = \mathbb{Q}$, and $f : X \rightarrow \mathbb{N} \subset \mathbb{R}$ a bijection. Define $A_n = \{x \in X : f(x) \leq n\}$. Thus, we get an increasing closed cover $\{A_n : n \in \mathbb{N}\}$ of X . Since each set A_n is finite, $f \upharpoonright A_n$ is continuous. However, f is not continuous because \mathbb{N} is discrete while \mathbb{Q} is not. \square

Fortunately, it is easy to correct that part of the proof of Theorem 5.1.3 by slightly changing the original argument. The part (b) of the theorem remains unchanged.

PROOF OF (a) OF THEOREM 5.1.3. Let X , Y and φ be as in that theorem. Following Michael's proof, let $\beta \geq 2$ be such that $\varphi(x) \cap B_\beta^d(0) \neq \emptyset$ for some $x \in X$, where 0 is the origin of Y . For each $n \in \mathbb{N}$, let

$$V_n = \{x \in X : \varphi(x) \cap B_{\beta^n}^d(0) \neq \emptyset\}.$$

Since φ is l.s.c., each V_n is open in X . In fact, $\{V_n : n \in \mathbb{N}\}$ is an increasing open cover of X . Since X is countably paracompact and normal (being paracompact), by Theorem 2.3.46, there exists an open cover $\{U_n : n \in \mathbb{N}\}$ of X such that $\bar{U}_n \subset V_n$, for

each $n \in \mathbb{N}$. The family $\{A_n : n \in \mathbb{N}\}$, with $A_n = \bigcup\{\bar{U}_k : k \leq n\}$, is an increasing closed cover of X such that $A_n \subset V_n$, for each $n \in \mathbb{N}$. Using the original Michael's argument, we get a sequence of continuous selections $f_n : A_n \rightarrow Y$ for $\varphi \upharpoonright A_n$ such that $f_{n+1} \upharpoonright A_n = f_n$, $n \in \mathbb{N}$. Then, f is continuous because each $f \upharpoonright U_n = f_n \upharpoonright U_n$, $n \in \mathbb{N}$, is continuous and $\{U_n : n \in \mathbb{N}\}$ is an open cover of X (see Proposition 2.1.12). The rest of the proof goes along the same lines as that of Theorem 5.1.3 in [42]. \square

COROLLARY 5.1.5 (E. Michael [42]). *If X is a paracompact space, $A \subset X$ is closed, and P is a closed paraconvex subset of a Banach space, then every continuous $g : A \rightarrow P$ can be extended to a continuous $f : X \rightarrow P$.*

Corollary 5.1.5 shows that *every closed paraconvex subset of a Banach space is an absolute retract.*

2. Selections, Paraconvexity and Collectionwise Normality

In this section, we are going to prove a collectionwise normal version of Theorem 5.1.3 (see, Theorem 5.2.6), thus generalizing Theorem 4.2.5 in terms of paraconvex sets. We will also show how our arguments can be used to generalize further some of these results.

THEOREM 5.2.6. *Let X be a τ -collectionwise normal space, Y be a Banach space with $w(Y) \leq \tau$, and $\varphi : X \rightarrow \mathcal{C}'_\alpha(Y)$ be an l.s.c. mapping, for some $\alpha < 1$. Then, the following hold:*

- (a) φ has a continuous selection.
- (b) If $r > 0$ and $g : X \rightarrow Y$ is continuous such that $d(g(x), \varphi(x)) < r$ for all $x \in X$, then there exists $\delta > 0$ and a continuous selection f for φ such that $d(g(x), f(x)) < \delta r$, $x \in X$.

To prepare for the proof of Theorem 5.2.6, we need the following proposition.

PROPOSITION 5.2.7. *Let X be a τ -collectionwise normal space, Y be a completely metrizable space with $w(Y) \leq \tau$, $\{V_n : n \in \mathbb{N}\}$ an increasing open cover of Y , and $\varphi : X \rightarrow \mathcal{C}'(Y)$ an l.s.c. mapping. Then, there exists an increasing closed cover $\{A_n : n \in \mathbb{N}\}$ of X such that $A_n \subset \varphi^{-1}(V_n)$, for every $n \in \mathbb{N}$.*

PROOF. Since $\{V_n : n \in \mathbb{N}\}$ is an increasing open cover of Y and Y is normal and countably paracompact (being metrizable), there exists an increasing closed cover $\{F_n : n \in \mathbb{N}\}$ of Y such that $F_n \subset V_n$, for every $n \in \mathbb{N}$. We then have

$$(3) \quad \varphi^{-1}(F_n) \subset \varphi^{-1}(V_n), \text{ for every } n \in \mathbb{N}.$$

By a result of Choban and Valov [7] (see also Nedev [51]), there exists a u.s.c. multi-selection $\psi : X \rightarrow \mathcal{C}(Y)$ for φ . Since ψ is u.s.c., each $\psi^{-1}(F_n)$, $n \in \mathbb{N}$, is closed in X . Since $\psi(x) \subset \varphi(x)$, $x \in X$, we have

$$\begin{aligned} \psi^{-1}(F_n) &= \{x \in X : \psi(x) \cap F_n \neq \emptyset\} \\ &\subset \{x \in X : \varphi(x) \cap F_n \neq \emptyset\} = \varphi^{-1}(F_n). \end{aligned}$$

The last inclusion and (3) imply that the family $\{A_n : n \in \mathbb{N}\}$, with $A_n = \psi^{-1}(F_n)$, is an increasing closed cover of X such that $A_n \subset \varphi^{-1}(V_n)$, for every $n \in \mathbb{N}$. \square

Let us now prove Theorem 5.2.6.

PROOF OF THEOREM 5.2.6. Let X , Y , α , and φ be as in that theorem. We first prove (b), and then (a).

(b). Since $\alpha < 1$, there exists $\gamma \in \mathbb{R}$ such that $\alpha < \gamma < 1$. Then, $\sum_{i=0}^{\infty} \gamma^i < \infty$ (i.e. the series $\sum_{i=0}^{\infty} \gamma^i$ converges). So, take δ such that $\sum_{i=0}^{\infty} \gamma^i < \delta$. To show that this δ works, by induction, we shall define a sequence of continuous maps $f_n : X \rightarrow Y$, $n < \omega$, with

$f_0 = g$, such that for all n and all $x \in X$,

$$(4) \quad d(f_n(x), \varphi(x)) < \gamma^n r,$$

$$(5) \quad d(f_n(x), f_{n+1}(x)) \leq \gamma^n r.$$

This will be sufficient because by (5), $\{f_n : n < \omega\}$ is a Cauchy sequence, so it must converge to some continuous map $f : X \rightarrow Y$, because Y is complete. By (4), $f(x) \in \varphi(x)$, for every $x \in X$, and by (5)

$$\begin{aligned} d(g(x), f_{n+1}(x)) &= d(f_0(x), f_{n+1}(x)) \\ &\leq d(f_0(x), f_1(x)) + d(f_1(x), f_2(x)) + \cdots + d(f_n(x), f_{n+1}(x)) \\ &\leq r + \gamma r + \gamma^2 r + \gamma^3 r + \cdots + \gamma^n r \\ &= r \sum_{i=0}^n \gamma^i. \end{aligned}$$

Therefore, $d(f(x), g(x)) \leq r \sum_{i=0}^{\infty} \gamma^i < \delta r$.

Let $f_0 = g$, which satisfies (4). Suppose that f_n has been constructed for some $n \geq 0$, and let us construct f_{n+1} . Define a mapping $\psi_{n+1} : X \rightarrow \mathcal{F}(Y)$ by $\psi_{n+1}(x) = \overline{B_{\gamma^n r}^d(f_n(x))}$, $x \in X$. Then, ψ_{n+1} is d -proximal continuous (being d -continuous) and convex-valued. Define another mapping $\varphi_{n+1} : X \rightarrow \mathcal{F}(Y)$ by

$$\varphi_{n+1}(x) = \overline{\text{conv}(\varphi(x) \cap B_{\gamma^n r}^d(f_n(x)))}, \quad x \in X.$$

By the inductive assumption, $\varphi_{n+1}(x)$ is never empty for every $x \in X$, because f_n satisfies (4) above. Furthermore, by Propositions 3.1.5, 3.1.7 and Corollary 3.1.13, φ_{n+1} is l.s.c. and it is clearly convex-valued. Finally, φ_{n+1} is a multi-selection of ψ_{n+1} and $\varphi_{n+1}(x) \neq \psi_{n+1}(x)$ implies that $\varphi_{n+1}(x)$ is compact. Then, by Lemma 4.3.7, φ_{n+1} has a continuous selection $f_{n+1} : X \rightarrow Y$ such that

$$d(f_n(x), f_{n+1}(x)) \leq \gamma^n r,$$

which is (5). Since $\varphi(x)$ is α -paraconvex for every $x \in X$, and $\alpha < \gamma$, we have

$$d(f_{n+1}(x), \varphi(x)) \leq \alpha \cdot \gamma^n r < \gamma^{n+1} r, \text{ for all } x \in X,$$

that is, f_{n+1} satisfies also (4).

(a). Take $\beta > 1$ such that $\varphi(x) \cap B_\beta^d(0) \neq \emptyset$ for some $x \in X$, where 0 is the origin of Y . Let

$$V_n = B_{\beta^n}^d(0), \text{ for each } n \in \mathbb{N}.$$

Then, each V_n is open in Y , and the family $\{V_n : n \in \mathbb{N}\}$ is an increasing open cover of Y . By Proposition 5.2.7, there exists an increasing closed cover $\{B_n : n \in \mathbb{N}\}$ of X such that $B_n \subset \varphi^{-1}(V_n)$, for every $n \in \mathbb{N}$. Since X is normal, there are open sets $U_n \subset X$ such that $B_n \subset U_n \subset \overline{U_n} \subset \varphi^{-1}(V_n)$ and $U_n \subset U_{n+1}$, for each $n \in \mathbb{N}$. Letting $A_n = \overline{U_n}$ and following the construction in the proof of (a) in [42, Theorem 2.1], we get (using (b) above) a sequence of continuous selections $f_n : A_n \rightarrow Y$ for $\varphi \upharpoonright A_n$ such that $f_{n+1} \upharpoonright A_n = f_n$, $n \in \mathbb{N}$. Then, the mapping $f : X \rightarrow Y$ defined by $f \upharpoonright A_n = f_n$, $n \in \mathbb{N}$, is a selection for φ which is continuous because each $f \upharpoonright U_n = f_n \upharpoonright U_n$, $n \in \mathbb{N}$, is continuous and $\{U_n : n \in \mathbb{N}\}$ is an open cover of X (see Proposition 2.1.12). The proof is completed. \square

2.1. Some Possible Generalizations. By Corollary 5.1.5, if X is paracompact, $A \subset X$ is closed and E is a closed paraconvex subset of a Banach space Y , then every continuous $g : A \rightarrow E$ can be extended to a continuous $f : X \rightarrow E$. According to Theorem 3.2.20, this implies that the same remains valid for X being only collectionwise normal. As a rule, the theorems for the existence of continuous selections for l.s.c. mappings originated as a natural generalization of extension theorems, see Michael [39, 40]. In view of the above, this brings the question for a more natural setting of Theorem 5.2.6. Namely, given $0 \leq \alpha < 1$ and a closed α -paraconvex set E of a Banach space Y , let $\mathcal{C}_\alpha(E) = \{S \in \mathcal{C}_\alpha(Y) : S \subset E\}$ and $\mathcal{C}'_\alpha(E) = \mathcal{C}_\alpha(E) \cup \{E\}$. Since E is α -paraconvex, each member of $\mathcal{C}'_\alpha(E)$ is also α -paraconvex, so it is in a good accordance

with the families $\mathcal{F}_\alpha(Y)$ and $\mathcal{C}'_\alpha(Y)$. The following question was posed to the author by V. Gutev.

▷ QUESTION 4. Let X be a τ -collectionwise normal space, Y be a Banach space, $0 \leq \alpha < 1$, and E be a nonempty α -paraconvex closed subset of Y , with $w(E) \leq \tau$. Then, is it true that every l.s.c. $\varphi : X \rightarrow \mathcal{C}'_\alpha(E)$ has a continuous selection?

To resolve Question 4, one can try to follow the proof of (b) of Theorem 5.2.6. A particular difficulty to do this is that even to take $f_0 : X \rightarrow E$ and construct f_1 in a similar way, some values of f_1 may already go out of the set E .

For an infinite cardinal number τ , a T_1 -space X is called τ -paracompact if every open cover \mathcal{U} of X , with $|\mathcal{U}| \leq \tau$, has a locally finite open refinement. In contrast to paracompactness, there are τ -paracompact spaces which are not normal. Of course, a space is paracompact if and only if it is τ -paracompact for every τ .

It is well known that if X is τ -paracompact and normal, Y is a Banach space, with $w(Y) \leq \tau$, then every l.s.c. convex-valued mapping $\varphi : X \rightarrow \mathcal{F}(Y)$ has a continuous selection (see, [51]). Using exactly the same proof as for the case of paracompact spaces and the above fact, one gets the following theorem.

THEOREM 5.2.8. *Let X be a τ -paracompact and normal space, Y be a Banach space, with $w(Y) \leq \tau$, and let $\varphi : X \rightarrow \mathcal{F}_\alpha(Y)$ be an l.s.c. mapping, where $\alpha < 1$. Then, the following hold:*

- (a) φ has a continuous selection.
- (b) If $r > 0$ and $g : X \rightarrow Y$ is continuous such that $d(g(x), \varphi(x)) < r$ for all $x \in X$, then there exists $\delta > 0$ and a continuous selection f for φ such that $d(g(x), f(x)) < \delta r$, $x \in X$.

In the special case of $\tau = \omega$, the above theorem implies the following consequence.

COROLLARY 5.2.9. *Let X be a countably paracompact and normal space, Y be a separable Banach space, and let $\varphi : X \rightarrow \mathcal{F}_\alpha(Y)$ be an l.s.c. mapping, where $\alpha < 1$. Then, the following hold:*

- (a) φ has a continuous selection.
- (b) If $r > 0$ and $g : X \rightarrow Y$ is continuous such that $d(g(x), \varphi(x)) < r$ for all $x \in X$, then there exists $\delta > 0$ and a continuous selection f for φ such that $d(g(x), f(x)) < \delta r$, $x \in X$.

Note that Theorem 5.1.3 remains true for arbitrary domain X , provided that the continuity of $\varphi : X \rightarrow \mathcal{F}_\alpha(Y)$ is strengthened to d -continuity; that is, the following holds.

THEOREM 5.2.10. *Let X be a topological space, Y be a Banach space, and $\varphi : X \rightarrow \mathcal{F}_\alpha(Y)$ be a d -continuous mapping, for some $0 \leq \alpha < 1$. Then, φ has a continuous selection.*

It is unclear whether the above theorem holds when one further relaxes the continuity of the mapping φ to d -proximal continuity. The following question was posed to the author by V. Gutev.

▷ QUESTION 5. Let X be a topological space, Y be a Banach space, and $\varphi : X \rightarrow \mathcal{F}_\alpha(Y)$ be a d -proximal continuous mapping, for some $0 \leq \alpha < 1$. Then, is it true that φ has a continuous selection?

Recently, the question was resolved in the affirmative by Takamitsu Yamauchi [70].

3. Selections, Functional Paraconvexity and Collectionwise Normality

This section is devoted to study of the functionally paraconvex-valued selection theorem for $\mathcal{C}^1(Y)$ -valued mappings defined on collectionwise normal spaces. Note that in Theorem 5.1.3, α is a fixed constant. Regarding this, the following question is naturally

raised: does Theorem 5.1.3 remain true if to each $x \in X$, there corresponds an $\alpha(x) < 1$ (possibly different for different x) for which $\varphi(x)$ is $\alpha(x)$ -paraconvex? A first attempt in answering the above question was proposed by P. Semenov [63], who generalized Theorem 5.1.3 by replacing the constant α by a function $h : (0, +\infty) \rightarrow [0, 1)$ satisfying a certain property (PS), see Theorem 5.3.11 below.

For a function $h : (0, \infty) \rightarrow [0, 1)$, a closed nonempty subset P of a Banach space Y is called *h-paraconvex* (see, [63]) if for any open ball B of radius r that intercepts the set P and for any point $q \in \overline{\text{conv}(P \cap B)}$, we have $d(q, P) \leq h(r) \cdot r$, where d is distance generated by the norm of Y . A nonempty closed subset of Y is called *functionally paraconvex* if it is *h-paraconvex* for some function $h : (0, \infty) \rightarrow [0, 1)$.

For an arbitrary function $H : (0, \infty) \rightarrow [0, 1)$, define a functional sequence $\{H_n : n < \omega\}$ by the equalities

$$H_0(t) = 1 \text{ and } H_{n+1}(t) = H(H_n(t) \cdot t)H_n(t) \text{ for every } n < \omega.$$

A function $H : (0, \infty) \rightarrow [0, 1)$ *strictly dominates* another function $h : (0, \infty) \rightarrow [0, 1)$ if $h(t) < H(t)$, for all $t > 0$. A function $h : (0, \infty) \rightarrow [0, 1)$ has *property (PS)* if there exists a function $H : (0, \infty) \rightarrow [0, 1)$ that strictly dominates h such that the series $\sum_{n=0}^{\infty} H_n(t)$ converges for all $t > 0$ (see, [63]).

THEOREM 5.3.11 ([63]). *Suppose that a function $h : (0, +\infty) \rightarrow [0, 1)$ has property (PS), X is a paracompact space, and Y is a Banach space. Then, every l.s.c. mapping $\varphi : X \rightarrow \mathcal{F}(Y)$ whose values are *h-paraconvex* has a continuous selection.*

The principal purpose of this section is to prove the following result which is the collectionwise normal version of Theorem 5.3.11 and is a generalization of Theorem 5.2.6 in terms of functionally paraconvex sets. The theorem was stated in [35] without a proof.

THEOREM 5.3.12. *Suppose that a function $h : (0, \infty) \rightarrow [0, 1)$ has property (PS), X is a τ -collectionwise normal space, and Y is a Banach space with $w(Y) \leq \tau$. Then, every l.s.c. mapping $\varphi : X \rightarrow \mathcal{C}'(Y)$ whose values are h -paraconvex has a continuous selection.*

Note that if the function h is equal to a constant $\alpha < 1$, then h -paraconvexity is equivalent to α -paraconvexity; and Theorem 5.3.11 obviously implies Theorem 5.1.3, while Theorem 5.2.6 is a consequence of Theorem 5.3.12. Moreover, as it was stated in [35], the above theorem is not true for $\alpha = 1$, for the same reason given in [30, 35]. It should be remarked that the proof of Theorem 5.3.12 follows the ideas in [35, 42, 63].

3.1. Preliminary result. We need the following result for the proof of Theorem 5.3.12. It is an analogue of the Lemma in [63].

PROPOSITION 5.3.13. *Suppose that a function $h : (0, \infty) \rightarrow [0, 1)$ has property (PS) and that $H : (0, \infty) \rightarrow [0, 1)$ is the corresponding function that strictly dominates h . Suppose also that X is a collectionwise normal space, Y is a Banach space with metric d , and $\varphi : X \rightarrow \mathcal{C}'(Y)$ is an l.s.c. h -paraconvex-valued mapping. If for some $r > 0$, there exists a continuous single-valued map $g : X \rightarrow Y$ such that $d(g(x), \varphi(x)) < r$ for all $x \in X$, then φ has a continuous selection $f : X \rightarrow Y$ such that*

$$d(g(x), f(x)) < \tilde{H}(r) \cdot r, \text{ where } \tilde{H}(r) = 1 + \sum_{n=0}^{\infty} H_n(r).$$

PROOF. We shall construct f as a limit of a Cauchy sequence $\{f_n : n < \omega\}$ of continuous maps $f_n : X \rightarrow Y$, $n < \omega$, such that for all n and all $x \in X$,

$$(6) \quad d(f_n(x), \varphi(x)) < H_n(r) \cdot r,$$

$$(7) \quad d(f_n(x), f_{n+1}(x)) \leq H_n(r) \cdot r.$$

This will be sufficient because by (7), $\{f_n : n < \omega\}$ is Cauchy sequence, so it must converge to some continuous map $f : X \rightarrow Y$. By (6), $f(x) \in \varphi(x)$, for every $x \in X$, and by (7) $d(f(x), g(x)) < \tilde{H}(r) \cdot r$.

To this end, let $f_0 = g$ which satisfies (6) (with $n = 0$). Suppose that the functions f_k have been constructed for $0 \leq k \leq n$. Let us construct f_{n+1} . Define a mapping $\psi_{n+1} : X \rightarrow \mathcal{F}(Y)$ by $\psi_{n+1}(x) = \overline{B_{H_n(r) \cdot r}^d(f_n(x))}$, $x \in X$. Then, ψ_{n+1} is d -proximal continuous. Define another mapping $\varphi_{n+1} : X \rightarrow \mathcal{F}(Y)$ by

$$\varphi_{n+1}(x) = \overline{\text{conv}(\varphi(x) \cap B_{H_n(r) \cdot r}^d(f_n(x)))}, \text{ for every } x \in X.$$

Then, φ_{n+1} is an l.s.c. nonempty convex-valued mapping such that $\varphi_{n+1}(x) \subset \psi_{n+1}(x)$, $x \in X$, and $\varphi_{n+1}(x) \neq \psi_{n+1}(x)$ implies that $\varphi_{n+1}(x)$ is compact. By Lemma 4.3.7, φ_{n+1} has a continuous selection $f_{n+1} : X \rightarrow Y$ such that

$$d(f_n(x), f_{n+1}(x)) \leq H_n(r) \cdot r, \text{ which is (7).}$$

Since $\varphi(x)$ is h -paraconvex for each $x \in X$, we have

$$\begin{aligned} d(f_{n+1}(x), \varphi(x)) &\leq h(H_n(r) \cdot r) \cdot (H_n(r) \cdot r) \\ &< H(H_n(r) \cdot r) \cdot H_n(r) \cdot r = H_{n+1}(r) \cdot r, \text{ which is (6).} \end{aligned}$$

Moreover, we also have

$$\begin{aligned} d(g(x), f_{n+1}(x)) &= d(f_0(x), f_{n+1}(x)) \\ &\leq d(f_0(x), f_1(x)) + d(f_1(x), f_2(x)) + \cdots + d(f_n(x), f_{n+1}(x)) \\ &\leq H_0(r) \cdot r + H_1(r) \cdot r + \cdots + H_n(r) \cdot r \\ &= r \sum_{k=0}^n H_k(r). \end{aligned}$$

Taking the limit as n tends to infinity, we get

$$d(f(x), g(x)) \leq r \sum_{k=0}^{\infty} H_k(r) < \tilde{H}(r) \cdot r.$$

The proof is completed. □

3.2. Proof of Theorem 5.3.12. We define a sequence of functions $\tilde{H}_n : [0, \infty) \rightarrow [1, \infty)$ by

$$\tilde{H}_0(t) = 1, \text{ and } \tilde{H}_{n+1}(t) = \tilde{H}(\tilde{H}_n(t) \cdot t)\tilde{H}_n(t) \text{ for every } n < \omega,$$

where the function \tilde{H} was constructed in Proposition 5.3.13. It is clear that for every $t > 0$, $\tilde{H}_{n+1}(t) = \tilde{H}(\tilde{H}_n(t) \cdot t)\tilde{H}_n(t) = \left(1 + \sum_{n=0}^{\infty} H_n(\tilde{H}_n(t) \cdot t)\right)\tilde{H}_n(t) > \tilde{H}_n(t)$. Take $R > 0$ such that $\varphi(x) \cap B_R^d(0) \neq \emptyset$, where 0 is the origin of Y . Let

$$V_n = B_{\tilde{H}_n(R)R}^d(0), \text{ for each } n \in \mathbb{N}.$$

Then, each V_n is open in Y , and the family $\{V_n : n \in \mathbb{N}\}$ is an increasing open cover of Y . By Proposition 5.2.7, there exists an increasing closed cover $\{B_n : n \in \mathbb{N}\}$ of X such that $B_n \subset \varphi^{-1}(V_n)$, for every $n \in \mathbb{N}$. Since X is normal, there exists an open set $U_n \subset X$ such that $B_n \subset U_n \subset \overline{U_n} \subset \varphi^{-1}(V_n)$ and $U_n \subset U_{n+1}$, for each $n \in \mathbb{N}$. We let $A_n = \overline{U_n}$ and the rest of the proof is very similar to the proof Theorem 5.3.11 (see, the main theorem in [63]).

REMARK 5.3.14. In [58, 61], D. Repovš and P.V. Semenov considered a function $\alpha_P : (0, \infty) \rightarrow [0, 2)$ (called the *function of nonconvexity*) associated to each nonempty subset $P \subset Y$ to prove a nonconvex-valued selection theorem for mappings with paracompact domain (see, [61, Theorem 1.3]). Also, they obtained several applications of selections for paraconvex-valued mappings, see [53, 54, 59, 60, 61, 64] and the monograph [55].

4. Selections, Paraconvexity and PF-normality

In [27], T. Kandô proved the following theorem:

THEOREM 5.4.15 ([27, Theorem IV]). *A T_1 -space X is PF-normal if and only if for every Banach space Y , every l.s.c. convex-valued mapping $\varphi : X \rightarrow \mathcal{C}(Y)$ has a continuous selection.*

S. Nedev in [51] proved a more general version of Theorem 5.4.15.

THEOREM 5.4.16 ([51, Theorem 4.1]). *For an infinite cardinal number τ , a T_1 -space X is τ -PF-normal if and only if for every Banach space Y with $w(Y) \leq \tau$, every l.s.c. convex-valued mapping $\varphi : X \rightarrow \mathcal{C}(Y)$ has a continuous selection.*

S. Nedev also proved the following multi-selection theorem.

THEOREM 5.4.17 ([51, Theorem 4.3]). *For an infinite cardinal number τ , a T_1 -space X is τ -PF-normal if and only if for every metric space Y , with $w(Y) \leq \tau$, every l.s.c. mapping $\varphi : X \rightarrow \mathcal{C}(Y)$ has an usco multi-selection.*

In this section, we prove a selection theorem for l.s.c. $\mathcal{C}_\alpha(Y)$ -valued mappings defined on τ -PF-normal spaces (Theorem 5.4.19) which is a generalization of Theorems 5.4.15 and 5.4.16 in terms of paracompact-valued mappings defined on τ -PF-normal spaces, see below for the definition of these spaces. Let us explicitly remark that the proofs of all the Theorems 5.1.3, 5.2.6, 5.3.11 and 5.3.12 utilize the fact that τ -paracompactness and τ -collectionwise normality are hereditary with respect to closed subsets. The challenge in this particular case is that τ -PF-normality is not hereditary with respect to closed subsets. The rest of the arguments are similar to the case of τ -collectionwise normal spaces in [35]. Every τ -collectionwise normal space as well as every τ -paracompact normal space is τ -PF-normal. Based on this fact and the arguments dealing with τ -PF-normal spaces, we also generalize all these theorems from a common point of view, see Theorem 5.5.21.

A T_1 -space X is τ -PF-normal if every point-finite open cover \mathcal{U} of X , with $|\mathcal{U}| \leq \tau$, is normal. A space X is PF-normal if it is τ -PF-normal for every τ ; or if it is normal and every point-finite open cover of X has a locally finite open refinement. Note that in the realm of normal spaces, τ -PF-normal spaces coincide with τ -pointwise- \aleph_0 -paracompact spaces in Nedev's terminology [51]; and PF-normal spaces with point-finitely paracompact spaces in the sense of Kandô [27]. Every collectionwise normal

space is PF-normal (see [43, Theorem 2]) and every PF-normal space is obviously normal. However, none of these implications is invertible (see [43, Examples 1 and 2]). PF-normal spaces were investigated in [25, 27, 43, 65]. In contrast to collectionwise normality and paracompactness, PF-normality is not hereditary with respect to closed subsets (see, [25, Page 506, §4]). However, T. Yamauchi proved in [71] that PF-normality is hereditary with respect to open F_σ -subsets.

PROPOSITION 5.4.18 ([71, Proposition 4.5]). *Every open F_σ -subset of a τ -PF-normal space is also τ -PF-normal.*

PROOF. Let O be an open F_σ -subset of a τ -PF-normal space X . Using the normality of X , take a sequence $\{V_n : n \in \mathbb{N}\}$ of cozero-sets of X such that $O = \bigcup\{\overline{V_n} : n \in \mathbb{N}\}$ and $\overline{V_n} \subset V_{n+1}$ for each $n \in \mathbb{N}$. Let \mathcal{U} be a point-finite open cover of O with $|\mathcal{U}| \leq \tau$. For each $n \in \mathbb{N}$, let $\mathcal{W}_n = \{U \cap V_{n+1} : U \in \mathcal{U}\} \cup \{X \setminus \overline{V_n}\}$. Then, \mathcal{W}_n is a point-finite open cover of X such that $|\mathcal{W}_n| \leq \tau$, and hence it is normal. By Theorem 2.4.53, there is a locally finite cozero-set cover \mathcal{W}'_n of X which refines \mathcal{W}_n . Let $\mathcal{V}_n = \{W \cap V_n : W \in \mathcal{W}'_n\}$. Then, \mathcal{V}_n is a locally finite collection of cozero-sets in X which refines \mathcal{U} . Thus, $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is normal by Corollary 2.4.54. \square

THEOREM 5.4.19 ([34, Theorem 2.1]). *For a Banach space Y , with $w(Y) \leq \tau$, and $0 \leq \alpha < 1$, the following hold:*

- (a) *Whenever X is a τ -PF-normal space, every l.s.c. $\varphi : X \rightarrow \mathcal{C}_\alpha(Y)$ has a continuous selection.*
- (b) *There exists $\delta = \delta(\alpha) > 0$ depending only on α such that if X is a τ -PF-normal space, $\varphi : X \rightarrow \mathcal{C}_\alpha(Y)$ is l.s.c., and $g : X \rightarrow Y$ is a continuous r -selection for φ for some $r > 0$, then φ has a continuous selection $f : X \rightarrow Y$ with $d(g(x), f(x)) < \delta r$, for all $x \in X$.*

To prepare for the proof of Theorem 5.4.19, we need the following proposition which is a τ -PF-normal version of Proposition 5.2.7.

PROPOSITION 5.4.20. *Let X be a τ -PF-normal space, Y a completely metrizable space with $w(Y) \leq \tau$, $\{V_n : n \in \mathbb{N}\}$ an increasing open cover of Y , and $\varphi : X \rightarrow \mathcal{C}(Y)$ an l.s.c. set-valued mapping. Then, there exists an increasing closed cover $\{A_n : n \in \mathbb{N}\}$ of X such that $A_n \subset \varphi^{-1}(V_n)$, for every $n \in \mathbb{N}$.*

PROOF. The proof repeats that one of Proposition 5.2.7, but now using Theorem 5.4.17 instead of the result of Choban and Valov [7]. \square

PROOF OF THEOREM 5.4.19. Let X , Y , α , and φ be as in the theorem. As in [35, 42], we shall first prove (b), then (a).

(b). Since $\alpha < 1$, there exists $\gamma \in \mathbb{R}$ such that $\alpha < \gamma < 1$. Then, $\sum_{i=0}^{\infty} \gamma^i < \infty$ (i.e. the series $\sum_{i=0}^{\infty} \gamma^i$ converges). So, take δ such that $\sum_{i=0}^{\infty} \gamma^i < \delta$. This δ works. Indeed, let $r > 0$ and $g : X \rightarrow Y$ be a continuous r -selection for φ . We shall define by induction a sequence of continuous maps $f_n : X \rightarrow Y$, $n < \omega$, with $f_0 = g$, satisfying the conditions:

$$(8) \quad d(f_n(x), \varphi(x)) < \gamma^n r,$$

$$(9) \quad d(f_n(x), f_{n+1}(x)) \leq \gamma^n r.$$

This will be sufficient because by (9), $\{f_n : n < \omega\}$ is a Cauchy sequence, so it must converge to some continuous map $f : X \rightarrow Y$. By (8), $f(x) \in \varphi(x)$, for every $x \in X$ and, by (9), $d(g(x), f(x)) < \delta r$, $x \in X$.

Let $f_0 = g$, which satisfies (8). Suppose that f_n has been constructed for some $n \geq 0$, and let us construct f_{n+1} . Define a mapping $\varphi_{n+1} : X \rightarrow \mathcal{F}(Y)$ by

$$\varphi_{n+1}(x) = \overline{\text{conv}(\varphi(x) \cap B_{\gamma^n r}^d(f_n(x)))}, \quad x \in X.$$

Then, by the inductive assumption, $\varphi_{n+1}(x)$ is not empty for every $x \in X$. By Propositions 3.1.5, 3.1.7 and Corollary 3.1.13, φ_{n+1} is l.s.c. and it is clearly convex-valued.

Moreover, φ_{n+1} is compact-valued because $\varphi_{n+1}(x)$ is a closed subset of the compact set $\overline{\text{conv}(\varphi(x))}$, for every $x \in X$. Then, by Theorem 5.4.16, φ_{n+1} admits a continuous selection $f_{n+1} : X \rightarrow Y$. By the definition of φ_{n+1} , we have

$$d(f_n(x), f_{n+1}(x)) \leq \gamma^n r,$$

and by α -paraconvexity of φ , we also have

$$d(f_{n+1}(x), \varphi(x)) \leq \alpha \gamma^n r < \gamma^{n+1} r, \text{ for all } x \in X.$$

The proof of (b) is completed.

(a). Take $\lambda \geq 2$ such that $\varphi(x) \cap B_\lambda^d(0) \neq \emptyset$ for some $x \in X$, where 0 is the origin of Y , and let $\beta = \max\{\delta, \lambda\}$, where δ is as in (b) applied to α . Let

$$V_n = B_{\beta^n}^d(0), \text{ for each } n \in \mathbb{N}.$$

Then, the family $\{V_n : n \in \mathbb{N}\}$ is an increasing open cover of Y . By Proposition 5.4.20, there is an increasing closed cover $\{A_n : n \in \mathbb{N}\}$ of X such that $A_n \subset \varphi^{-1}(V_n)$, $n \in \mathbb{N}$. Since X is normal, there is an increasing family $\{G_n : n \in \mathbb{N}\}$ of open sets such that $A_n \subset G_n \subset \overline{G_n} \subset \varphi^{-1}(V_n)$, $n \in \mathbb{N}$. We are going to construct by induction partial selections $f_n : \overline{G_n} \rightarrow V_n$ for $\varphi \upharpoonright \overline{G_n}$ such that $f_{n+1} \upharpoonright \overline{G_n} = f_n$, $n \in \mathbb{N}$. Let us construct $f_1 : \overline{G_1} \rightarrow V_1$. Since $\overline{G_1} \subset \varphi^{-1}(V_1)$, by normality of X , there is an open F_σ -set U_1 such that $\overline{G_1} \subset U_1 \subset \varphi^{-1}(V_1)$. Note that by Proposition 5.4.18, U_1 is also τ -PF-normal. Let $\psi_1 = \varphi \upharpoonright U_1$ which is l.s.c., $g(x) = 0$ for all x , and take r to be β . Then, by (b), ψ_1 has a continuous selection $g_1 : U_1 \rightarrow Y$ with $d(0, g_1(x)) < \delta\beta \leq \beta^2$, $x \in U_1$. So, one can take $f_1 = g_1 \upharpoonright \overline{G_1}$. Suppose that $f_n : \overline{G_n} \rightarrow V_n$ has been constructed for some $n \in \mathbb{N}$. Let us construct $f_{n+1} : \overline{G_{n+1}} \rightarrow V_{n+1}$. One has $\overline{G_{n+1}} \subset \varphi^{-1}(V_{n+1})$, and by normality of X , there is an open F_σ -set U_{n+1} such that $\overline{G_{n+1}} \subset U_{n+1} \subset \varphi^{-1}(V_{n+1})$. Define a mapping $\psi_{n+1} : U_{n+1} \rightarrow \mathcal{C}_\alpha(Y)$ by

$$\psi_{n+1}(x) = \begin{cases} \{f_n(x)\} & \text{if } x \in \overline{G_n} \\ \varphi(x) & \text{if } x \in U_{n+1} \setminus \overline{G_n}. \end{cases}$$

Then, by Example 3.2.24, ψ_{n+1} is l.s.c.; and since $f_n(\overline{G}_n) \subset V_n = B_{\beta^{n+1}}(0)$, it follows from (b), with $g(x) = 0$ for all x , and substituting r with β^{n+1} , that ψ_{n+1} has a continuous selection $g_{n+1} : U_{n+1} \rightarrow Y$ with $d(0, g_{n+1}(x)) < \delta\beta^{n+1} \leq \beta^{n+2}$, $x \in U_{n+1}$. Taking $f_{n+1} = g_{n+1} \upharpoonright \overline{G}_{n+1}$, by the definition of ψ_{n+1} , we have also that $f_{n+1} \upharpoonright \overline{G}_n = f_n$. Hence, for each $n \in \mathbb{N}$, one has a continuous selection $f_n : \overline{G}_n \rightarrow Y$ for $\varphi \upharpoonright \overline{G}_n$ such that $f_{n+1} \upharpoonright \overline{G}_n = f_n$. The map $f : X \rightarrow Y$ defined by $f(x) = f_n(x)$, $x \in \overline{G}_n$, is a continuous selection for φ because each $f \upharpoonright G_n = f_n \upharpoonright G_n$ is continuous; and $\{G_n : n \in \mathbb{N}\}$ is an open cover of X (see Proposition 2.1.12). The proof is completed. \square

5. General approach of paraconvex-valued selection theorems

The method developed to prove the paraconvex-valued theorem for mappings defined on τ -collectionwise normal spaces (Theorem 5.2.6) and that one for τ -PF-normal spaces (Theorem 5.4.19) can be used to design a common approach for proving such selection theorems (see Remark 5.5.23 below). Namely, both proofs were dependent on the existence of usco (u.s.c. and compact valued) multi-selections. They were also dependent on an approximate selection property which is present in the proof of the τ -paracompact case as well. In order to express this in terms of multi-selections, we will say that a mapping $\varphi : X \rightarrow 2^Y$ has the *Dense Multi-selection Property*, or *DMP* for short, where (Y, d) is a metric space, if the following hold:

- (i) φ has an l.s.c. multi-selection $\psi : X \rightarrow \mathcal{C}(Y)$.
- (ii) For every $\varepsilon > 0$, a cozero-set $U \subset X$, and a continuous ε -selection $g : U \rightarrow Y$ for $\varphi \upharpoonright U$, there exists an l.s.c. $\psi : U \rightarrow \mathcal{C}(Y)$ such that

$$\psi(x) \subset \overline{\varphi(x) \cap B_\varepsilon^d(g(x))}, \quad x \in U.$$

We may consider open balls $B_\varepsilon^d(y)$, when $\varepsilon = \infty$. Thus, $B_\infty^d(y) = Y$, and we have $\overline{\varphi(x) \cap B_\infty^d(g(x))} = \overline{\varphi(x) \cap Y} = \overline{\varphi(x)}$, $x \in X$. So, the DMP property of $\varphi : X \rightarrow 2^Y$ can now be simply expressed that for every $0 < \varepsilon \leq \infty$, a cozero-set $U \subset X$, and a

continuous ε -selection $g : U \rightarrow Y$ for $\varphi \upharpoonright U$, there exists an l.s.c. $\psi : U \rightarrow \mathcal{C}(Y)$ such that $\psi(x) \subset \overline{\varphi(x) \cap B_\varepsilon^d(g(x))}$, $x \in U$.

Note that in the realm of normal spaces, cozero-sets coincide with open F_σ -sets (see Proposition 2.4.51). So, if X is normal, $U \subset X$ is an open F_σ -set, and $\varphi : X \rightarrow 2^Y$ has the DMP, then $\varphi \upharpoonright U$ will automatically have the DMP.

CLAIM 2. Let X be a space, (Y, d) be a metric space, $\varphi : X \rightarrow 2^Y$ have the DMP, $A \subset X$ be closed and $U \subset X$ be a neighbourhood of A . If $g : U \rightarrow Y$ is a continuous selection for $\varphi \upharpoonright U$, then the mapping $\varphi_g : X \rightarrow 2^Y$ defined by

$$\varphi_g(x) = \begin{cases} \{g(x)\} & \text{if } x \in A \\ \varphi(x) & \text{if } x \in X \setminus A, \end{cases}$$

also has the DMP.

PROOF. Let $V \subset X$ be a cozero-set and $f : V \rightarrow Y$ be a continuous ε -selection for $\varphi_g \upharpoonright V$. Then, f is also a continuous ε -selection for $\varphi \upharpoonright V$. Since φ has the DMP, there exists an l.s.c. $\psi : V \rightarrow \mathcal{C}(Y)$ such that $\psi(x) \subset \overline{\varphi(x) \cap B_\varepsilon^d(f(x))}$, $x \in V$. Define a mapping $\psi_g : V \rightarrow \mathcal{C}(Y)$ by

$$\psi_g(x) = \begin{cases} \{g(x)\} & \text{if } x \in A \cap V \\ \{g(x)\} \cup \psi(x) & \text{if } x \in (U \cap V) \setminus A \\ \psi(x) & \text{if } V \setminus U. \end{cases}$$

Then, ψ_g is l.s.c. and $\psi_g(x) \subset \overline{\varphi_g(x) \cap B_\varepsilon^d(f(x))}$, $x \in V$. Hence, φ_g has the DMP.

To show that ψ_g is l.s.c., we need to show that $\psi_g^{-1}(O)$ is open in V for every open $O \subset Y$. We claim that

$$\psi_g^{-1}(O) = g^{-1}(O) \cup [(V \setminus A) \cap \psi^{-1}(O)],$$

which is clearly open. Indeed, note that since g is a continuous selection for ψ_g over $U \cap V$, we have $g^{-1}(O) \subset \psi_g^{-1}(O)$. Note also that ψ being a multi-selection for ψ_g

over $V \setminus A$, we also have $(V \setminus A) \cap \psi^{-1}(O) \subset \psi_g^{-1}(O)$. The two inclusions imply that $g^{-1}(O) \cup [(V \setminus A) \cap \psi^{-1}(O)] \subset \psi_g^{-1}(O)$. The other inclusion is straightforward. \square

We are going to show that if Y is a Banach space with $w(Y) \leq \tau$, then every l.s.c. mapping $\varphi : X \rightarrow \mathcal{F}_\alpha(Y)$ which has the DMP has also a continuous selection.

THEOREM 5.5.21 ([34, Theorem 3.1]). *For a Banach space Y , with $w(Y) \leq \tau$, and $0 \leq \alpha < 1$, the following hold:*

- (a) *Whenever X is a τ -PF-normal space, every mapping $\varphi : X \rightarrow \mathcal{F}_\alpha(Y)$ having the DMP has a continuous selection.*
- (b) *There exists $\delta = \delta(\alpha) > 0$ depending only on α such that if X is a τ -PF-normal space, $\varphi : X \rightarrow \mathcal{F}_\alpha(Y)$ has the DMP, and $g : X \rightarrow Y$ is a continuous r -selection for φ for some $r > 0$, then φ has a continuous selection $f : X \rightarrow Y$ with $d(g(x), f(x)) < \delta r$, for all $x \in X$.*

In order to prove Theorem 5.5.21, we need the following proposition which generalizes both Propositions 5.2.7 and 5.4.20.

PROPOSITION 5.5.22. *Let X be a τ -PF-normal space, Y be a completely metrizable space with $w(Y) \leq \tau$, $\{V_n : n \in \mathbb{N}\}$ be an increasing open cover of Y , and $\varphi : X \rightarrow \mathcal{F}(Y)$ have the DMP. Then, there exists an increasing closed cover $\{A_n : n \in \mathbb{N}\}$ of X such that $A_n \subset \varphi^{-1}(V_n)$, for every $n \in \mathbb{N}$.*

PROOF. Since $\{V_n : n \in \mathbb{N}\}$ is an increasing open cover of Y and Y is normal and countably paracompact (being metrizable), there exists an increasing closed cover $\{F_n : n \in \mathbb{N}\}$ of Y such that $F_n \subset V_n$, for every $n \in \mathbb{N}$. We then have

$$\varphi^{-1}(F_n) \subset \varphi^{-1}(V_n), \text{ for every } n \in \mathbb{N}.$$

Since the mapping φ has the DMP, then by (i) of the definition of DMP, there exists an l.s.c. $\psi : X \rightarrow \mathcal{C}(Y)$ such that $\psi(x) \subset \varphi(x)$, $x \in X$. Since X is τ -PF-normal, by

Theorem 5.4.17, ψ has a u.s.c. multi-selection $\phi : X \rightarrow \mathcal{C}(Y)$. We then have

$$\phi^{-1}(F_n) \subset \psi^{-1}(F_n) \subset \varphi^{-1}(F_n) \subset \varphi^{-1}(V_n), \text{ for every } n \in \mathbb{N}.$$

The family $\{A_n : n \in \mathbb{N}\}$, with $A_n = \phi^{-1}(F_n)$, is an increasing closed cover of X such that $A_n \subset \varphi^{-1}(V_n)$, for every $n \in \mathbb{N}$. \square

We are now ready to prove Theorem 5.5.21.

PROOF OF THEOREM 5.5.21. Let X, Y, α , and φ be as in that theorem. We are going to first prove (b), and then (a).

(b) Let $\gamma > 0$ and $\delta > 0$ be as in the proof of (b) of Theorem 5.4.19, and let us show that this δ works. Indeed, let $r > 0$ and $g : X \rightarrow Y$ be as in (b). We shall define a sequence of continuous maps $f_n : X \rightarrow Y$, $n < \omega$, with $f_0 = g$, satisfying the conditions (8) and (9) in the proof of part (b) of Theorem 5.4.19.

Let $f_0 = g$, which satisfies (4). Suppose that f_n has been constructed for some $n \geq 0$, and let us construct f_{n+1} . Define a mapping $\Phi_{n+1} : X \rightarrow \mathcal{F}(Y)$ by

$$\Phi_{n+1}(x) = \overline{\varphi(x) \cap B_{\gamma^n r}^d(f_n(x))}, \quad x \in X.$$

Since φ has the DMP, the mapping Φ_{n+1} has an l.s.c. multi-selection $\psi_{n+1} : X \rightarrow \mathcal{C}(Y)$. Since X is τ -PF-normal, by Theorem 5.4.16, there exists a continuous map $f_{n+1} : X \rightarrow Y$ such that

$$f_{n+1}(x) \in \overline{\text{conv}(\psi_{n+1}(x))}, \quad x \in X.$$

Since $\psi_{n+1}(x) \subset \Phi_{n+1}(x) \subset \overline{\text{conv}(\varphi(x) \cap B_{\gamma^n r}^d(f_n(x)))}$, we also have that

$$f_{n+1}(x) \in \overline{\text{conv}(\psi_{n+1}(x))} \subset \overline{\text{conv}(\varphi(x) \cap B_{\gamma^n r}^d(f_n(x)))}, \quad x \in X.$$

By α -paraconvexity of $\varphi(x)$, we get that

$$d(f_{n+1}(x), \varphi(x)) \leq \alpha \gamma^n r < \gamma^{n+1} r, \text{ for all } x \in X,$$

which is (4). Clearly, we also have

$$d(f_n(x), f_{n+1}(x)) \leq \gamma^n r, \text{ for all } x \in X,$$

which is (5).

(a). We follow the proof of part (a) of Theorem 5.4.19. We take λ and β as in Theorem 5.4.19, and let $V_n = B_{\beta^n}(0)$, $n \in \mathbb{N}$. Since $\{V_n : n \in \mathbb{N}\}$ is an increasing open cover of Y , by Proposition 5.5.22, there is an increasing closed cover $\{A_n : n \in \mathbb{N}\}$ of X such that $A_n \subset \varphi^{-1}(V_n)$, for every $n \in \mathbb{N}$. Since X is normal, there is an increasing family $\{G_n : n \in \mathbb{N}\}$ of open sets such that $A_n \subset G_n \subset \overline{G_n} \subset \varphi^{-1}(V_n)$, $n \in \mathbb{N}$. Since $\overline{G_n} \subset \varphi^{-1}(V_n)$ for every $n \in \mathbb{N}$, there exists an open F_σ -set $U_n \subset X$ such that $\overline{G_n} \subset U_n \subset \varphi^{-1}(V_n)$. We are going to construct by induction partial selections $g_n : U_n \rightarrow V_n$ for $\varphi_n = \varphi \upharpoonright U_n$ such that $g_{n+1} \upharpoonright \overline{G_n} = g_n \upharpoonright \overline{G_n}$, $n \in \mathbb{N}$. To this end, let $g(x) = 0$ for all x , and take r to be β . Since $\varphi_1 = \varphi \upharpoonright U_1$ has the DMP, it follows from (b) that it has a continuous selection $g_1 : U_1 \rightarrow V_1$. Suppose that $g_n : U_n \rightarrow V_n$ is a continuous selection for $\varphi_n : U_n \rightarrow \mathcal{F}_\alpha(Y)$. Next, define a mapping $\psi_{n+1} : U_{n+1} \rightarrow \mathcal{F}_\alpha(Y)$ by

$$\psi_{n+1}(x) = \begin{cases} \{g_n(x)\} & \text{if } x \in \overline{G_n} \\ \varphi(x) & \text{if } x \in U_{n+1} \setminus \overline{G_n}. \end{cases}$$

By Claim 2, ψ_{n+1} has the DMP, and it follows from (b), with $g(x) = 0$ for all x , and substituting r with β^{n+1} , that ψ_{n+1} has a continuous selection $g_{n+1} : U_{n+1} \rightarrow V_{n+1}$. In particular, g_{n+1} is a continuous selection for φ_{n+1} and $g_{n+1} \upharpoonright \overline{G_n} = g_n \upharpoonright \overline{G_n}$. This completes the construction of the partial selections g_n , $n \in \mathbb{N}$. Finally, define $f_n = g_n \upharpoonright \overline{G_n}$, $n \in \mathbb{N}$. Then, f_n is a continuous selection for $\varphi \upharpoonright \overline{G_n}$ such that $f_{n+1} \upharpoonright \overline{G_n} = f_n$, $n \in \mathbb{N}$. This allows us to define a map $f : X \rightarrow Y$ by $f \upharpoonright \overline{G_n} = f_n$, which is continuous (by Proposition 2.1.12), and is a selection for φ . The proof is completed. \square

REMARK 5.5.23. As the following examples show, Theorem 5.5.21 incorporates several results of continuous selections for l.s.c. paraconvex-valued mappings with different

types of domains, where τ is an infinite cardinal number and (Y, d) is a complete metric space with $w(Y) \leq \tau$.

EXAMPLE 5.5.24. If X is τ -paracompact and normal and $\varphi : X \rightarrow \mathcal{F}(Y)$ is l.s.c., then φ has the DMP.

PROOF. Let $\varphi : X \rightarrow \mathcal{F}(Y)$ be l.s.c., $U \subset X$ be a cozero-set, i.e. an open F_σ -subset of X (by Proposition 2.4.51) and $g : U \rightarrow Y$ be a continuous ε -selection for $\varphi \upharpoonright U$. Note that by Proposition 2.3.42, U is τ -paracompact; and by Proposition 2.3.30, U is also normal. Define a mapping $\Phi : U \rightarrow \mathcal{F}(Y)$ by $\Phi(x) = \overline{\varphi(x) \cap B_\varepsilon^d(g(x))}$, $x \in U$. Then, $\Phi(x)$ is never empty because g is a continuous ε -selection for $\varphi \upharpoonright U$. Moreover, by Proposition 3.1.5 and Corollary 3.1.13, Φ is l.s.c. By [38, Theorem 1.1]¹, there exists an l.s.c. $\psi : U \rightarrow \mathcal{C}(Y)$ such that $\psi(x) \subset \Phi(x) = \overline{\varphi(x) \cap B_\varepsilon^d(g(x))}$, $x \in U$; i.e. φ has the DMP. \square

EXAMPLE 5.5.25. If X is τ -collectionwise normal and $\varphi : X \rightarrow \mathcal{C}'(Y)$ is l.s.c., then φ has the DMP.

PROOF. Let $\varphi : X \rightarrow \mathcal{C}'(Y)$ be l.s.c., $U \subset X$ be a cozero-set and $g : U \rightarrow Y$ be a continuous ε -selection for $\varphi \upharpoonright U$. Then, by Proposition 3.2.19, U is also τ -collectionwise normal because it is an (open) F_σ -subset of X . Define a mapping $\Omega : U \rightarrow \mathcal{F}(Y)$ by $\Omega(x) = \overline{B_\varepsilon^d(g(x))}$, $x \in U$. Then, Ω is proximal continuous. Define another mapping $\Phi : U \rightarrow \mathcal{F}(Y)$ by $\Phi(x) = \overline{\varphi(x) \cap B_\varepsilon^d(g(x))}$, $x \in U$. Then, as in Example 5.5.24, Φ is l.s.c. and nonempty-valued. Moreover, Φ is a multi-selection for Ω , and $\Phi(x) \neq \Omega(x)$ implies that $\Phi(x)$ is compact. Note that Φ has the Selection Factorization Property (see the proof of Lemma 4.3.7); and by [51, Proposition 4.1]², there exists an l.s.c.

¹It states that if X is τ -paracompact and normal, Y is completely metrizable with $w(Y) \leq \tau$, and $\Phi : X \rightarrow \mathcal{F}(Y)$ is l.s.c., then there are an l.s.c. $\psi : X \rightarrow \mathcal{C}(Y)$ and a u.s.c. $\theta : X \rightarrow \mathcal{F}(Y)$ such that $\psi(x) \subset \theta(x) \subset \Phi(x)$, $x \in X$.

²It states that if X is a normal space, Y is a metric space with $w(Y) \leq \tau$, and $\Phi : X \rightarrow \mathcal{F}(Y)$ has the Selection Factorization Property, then there are an l.s.c. $\psi : X \rightarrow \mathcal{C}(Y)$ and a u.s.c. $\theta : X \rightarrow \mathcal{F}(Y)$ such that $\psi(x) \subset \theta(x) \subset \Phi(x)$, $x \in X$.

$\psi : U \rightarrow \mathcal{C}(Y)$ such that

$$\psi(x) \subset \Phi(x) = \overline{\varphi(x) \cap B_\varepsilon^d(g(x))}, \quad x \in U.$$

Hence, φ has the DMP. □

EXAMPLE 5.5.26. If X is τ -PF-normal, and $\varphi : X \rightarrow \mathcal{C}(Y)$ is l.s.c., then φ has the DMP.

PROOF. Let $\varphi : X \rightarrow \mathcal{C}(Y)$ be l.s.c., $U \subset X$ be a cozero-set, and $g : U \rightarrow Y$ be a continuous ε -selection for $\varphi \upharpoonright U$. Then, U is also τ -PF-normal because it is a cozero-subset of X . Then, as in Examples 5.5.24 and 5.5.25, the mapping $\Phi : U \rightarrow \mathcal{F}(Y)$ defined by $\Phi(x) = \overline{\varphi(x) \cap B_\varepsilon^d(g(x))}$, $x \in U$, is l.s.c. and nonempty-valued. Moreover, $\Phi(x) \in \mathcal{C}(Y)$ because $\Phi(x)$ is a closed subset of the compact set $\varphi(x)$, $x \in U$. Thus, by [51, Theorem 4.3]³, there exists an l.s.c. $\psi : U \rightarrow \mathcal{C}(Y)$ such that

$$\psi(x) \subset \Phi(x) = \overline{\varphi(x) \cap B_\varepsilon^d(g(x))}, \quad x \in U.$$

Hence, φ has the DMP. □

Note that if $\tau = \omega$ and Y is a separable Banach space, then one gets also the following case:

EXAMPLE 5.5.27. If X is countably paracompact and normal, Y is a separable Banach space, and $\varphi : X \rightarrow \mathcal{F}(Y)$ is l.s.c., then φ has the DMP.

PROOF. Similar to the proof of Example 5.5.24, but using [6, Theorem 11.2] instead of [38, Theorem 1.1]. □

³It states that if X is τ -PF-normal, Y is metric space with $w(Y) \leq \tau$, and $\Phi : X \rightarrow \mathcal{C}(Y)$ is l.s.c., then there are an l.s.c. $\psi : X \rightarrow \mathcal{C}(Y)$ and a u.s.c. $\theta : X \rightarrow \mathcal{F}(Y)$ such that $\psi(x) \subset \theta(x) \subset \Phi(x)$, $x \in X$. Note that in [51], τ -pointwise \aleph_0 -paracompactness corresponds to τ -PF-normality

The author would like to know whether Theorem 5.3.12 is true if X is τ -PF-normal and the mapping φ is compact-valued, or if [27, Theorem IV, (D)] can be generalized to functionally paraconvex-valued mappings; that is, we have the following question:

▷ QUESTION 6. Suppose that $h : (0, \infty) \rightarrow [0, 1)$ has property (PS). Let X be τ -PF-normal, Y be a Banach space with $w(\tau) \leq \tau$, and let $\varphi : X \rightarrow \mathcal{C}(Y)$ be an l.s.c. h -paraconvex valued mapping. Then, is it true that φ has a continuous selection?

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