

**FISCHER-CLIFFORD MATRICES OF THE  
GENERALIZED SYMMETRIC GROUP**

**AND**

**SOME ASSOCIATED GROUPS**

by

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Dedication  
to  
My mother Tifwilemo, my wife Beatrice,  
and  
my children Memory and Victoria

The Fischer method known as the technique of the **Fischer-Clifford Matrices** makes use of Clifford Theory [11]. Given a group extension  $\overline{G} = H.G$  such that every irreducible character of  $H$  extends to its inertia group, then for each conjugacy class  $[g]$  of  $G$  we construct a matrix  $M(g)$  called a **Fischer-Clifford matrix**. By using these matrices together with the fusion maps and character tables of some subgroups of  $G$  called **inertia factor groups** we can construct the complete character table of  $\overline{G}$ . The Fischer method has been used in many works both on split and non-split extensions (see for example [1], [2], [5], [14], [18], [36], [47], [51], [65]). However in this thesis we apply the Fischer method to construct the character tables of split extensions, these are generalized symmetric groups and some associated groups of order  $m^{n-1}n!$ .

The Fischer-Clifford matrices of a group satisfy several known properties which can be used to construct these matrices. However we will use combinatorics to construct the Fischer-Clifford matrices of the groups discussed here. In [5] Almetady has presented a combinatorial method for constructing the Fischer-Clifford matrices of generalized symmetric groups. As an example, Almetady has applied the combinatorial method to construct the Fischer-Clifford matrices of the group  $B(3, 4)$ . However the manual construction of these matrices for large values of  $m$  and  $n$  using this method is not an easy task. The partial aim of this work was to use the combinatorial method to develop a computer programme for computing the Fischer-Clifford matrices of  $B(m, n)$ . Here we give a series of computer programmes written for GAP, that give various parameters for the Fischer-Clifford matrices of  $B(m, n)$ . Many of these programmes have been used in the development of the main Programme 5.2.4 that combinatorially computes matrices which are row equivalent to the Fischer-Clifford matrices of  $B(m, n)$ . Thus the matrices computed by Programme 5.2.4 require to have their rows collected together into blocks that correspond to particular inertia factor groups. The Programme 5.2.2 has been developed to compute some combinatorial objects called  **$m$ -compositions of  $n$**  which describe the inertia factor groups to which the rows of the matrices computed by Programme 5.2.4 correspond. A particular  $m$ -composition of  $n$  distinguishes the rows of these matrices that correspond to the same inertia factor group. Such rows of a matrix computed by Programme 5.2.4 are then manually collected together in a block corresponding to a particular inertia factor group, to give a Fischer-Clifford matrix of  $B(m, n)$ . As examples, an application of the Programme 5.2.4 to construct the Fischer-Clifford matrices of the

# Abstract

With the classification of finite simple groups having been completed in 1981, recent work in group theory has involved the study of the structures of simple groups. The character tables of maximal subgroups of simple groups give substantive information about these groups. Most of the maximal subgroups of simple groups are of extension type. Some of the maximal subgroups of simple groups contain constituents of the generalized symmetric groups. Here we shall be interested in discussing such groups which we may call groups associated with the generalized symmetric groups.

There are several well developed methods for calculating the character tables of group extensions. However Fischer [17] has given an effective method for calculating the character tables of some group extensions including the generalized symmetric group  $B(m, n)$ . Actually work on the characters of wreath products with permutation groups dates back to Specht's work [61], through the works of Osima [49] and Kerber [33]. And more recently other people have worked on characters of wreath products with symmetric groups, these amongst others include Darafshesh and Iranmanesh [14], List and Mahmoud [36], Puttaswamiah [52], Read [55, 56], Saeed-Ul-Islam [59] and Stembridge [64].

It is well known that the character table of the generalized symmetric group  $B(m, n)$ , where  $m$  and  $n$  are positive integers, can be constructed in GAP [22] with  $B(m, n)$  considered as the wreath product of the cyclic group  $\mathbb{Z}_m$  of order  $m$  with the symmetric group  $S_n$ . For example Pfeiffer [50] has given programmes which compute the character tables of wreath products with symmetric groups in GAP. However it may be necessary to obtain the partial character table of a group in hand rather than its complete character table. Further due to limited workspace in GAP, the wreath product method can only be used to compute character tables of  $B(m, n)$  for small values of  $m$  and  $n$ . It is for these reasons amongst others that Fischer's method is sometimes used to construct the character tables of such groups.





groups  $B(2, 6)$  and  $B(3, 5)$  of orders 46080 and 29160 is done here. We have also used Programme 5.2.4 to construct the Fischer-Clifford matrices of the groups  $B(2, 12)$  and  $B(4, 5)$  of orders  $2^{22} \times 3^5 \times 5^2 \times 7 \times 11$  and  $2^{13} \times 3 \times 5$  respectively. Due to lack of space here we have given the Fischer-Clifford matrices of  $B(2, 12)$  and  $B(4, 5)$  on the compact disk submitted with this thesis. However note that these matrices are the equivalent form of the Fischer-Clifford matrices of  $B(2, 12)$  and  $B(4, 5)$ .

In [35] R.J. List has presented a method for constructing the Fischer-Clifford matrices of group extensions of an irreducible constituent of the elementary abelian group  $2^n$  by a symmetric group. The other aim of our work is to adapt the combinatorial method in [5] to the construction of the Fischer-Clifford matrices of some group extensions associated with  $B(m, n)$ , using a similar method as the one used in [35]. Examples are given on the application of this adaptation to some groups associated with the groups  $B(2, 6)$ ,  $B(3, 3)$  and  $B(3, 5)$ .

In this thesis we have constructed the character tables of the groups  $B(2, 6)$  and  $B(3, 5)$  and some group extensions associated with these two groups and  $B(3, 3)$ . We have also constructed the character tables of the groups  $B(2, 12)$  and  $B(4, 5)$  in our work, these character tables are given on the compact disk submitted with this thesis. The correctness of all the character tables constructed in this thesis has been tested in GAP.

The main working programmes (Programme 2.2.3, Programme 3.1.9, Programme 3.1.10, Programme 5.2.1, Programme 5.2.4 and Programme 5.2.2) are given on the compact disk submitted with this thesis. It is anticipated that with further improvements, a number of the programmes given here will be incorporated into GAP. Indeed with further research work the programmes given here should lead to an alternative programme for computing the character table of  $B(m, n)$ .

# Preface

The work presented in this thesis has been carried out on part-time under the supervision and direction of Professor Jamshid Moori, School of Mathematical Sciences, University of KwaZulu-Natal, Pietermaritzburg, from February 1999 to April 2005.

The thesis represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:

Handwritten signature of Kenneth Zimba in black ink, featuring a stylized 'K' and 'Z'.

.....  
Kenneth Zimba (student)

Handwritten signature of J. Moori in black ink, with a large 'J' and 'M'.

.....  
Professor Jamshid Moori (supervisor)

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May I thank my supervisor, Professor Jamshid Moori, for his advice, support, encouragement and guidance during my part-time studies. This has undoubtedly led to the completion of this thesis. His love for mathematics is hugely amazing. Certainly, I will always be indebted to him for his dedication and constant inspiration which he has given me.

I am grateful for the facilities provided to me by the School of Mathematical Sciences, University of KwaZulu-Natal, Pietermaritzburg. I also wish to express my sincere gratitude to my colleagues in the School of Computational and Mathematical Sciences at the University of Limpopo (formerly the University of the North) for allowing me to take study leave and for supporting me in many ways during my part-time studies.

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My special thanks go to my wife Beatrice, my children Memory and Victoria for managing to spend considerable time without my presence, they made so much sacrifice to get this work completed. Special thanks are also due to my parents, my father Wyson Zimba and my mother Tifwilemo Zimba, who could have chosen not to have me at-all and there-after whose love, constant encouragement and support resulted in these blessings.

I am grateful to Professor Herbert Pahlings at RWTH, Aachen (Germany) for verifying the consistency, accuracy and correctness of some of the character tables presented here and also for helping to re-write the programmes in GAP syntax. Lastly

but not the least I express my sincere gratitude to my study-mates Bernado Rodrigues, Debusho Leggesse and Winter Sinkala, for being ready to assist me. Their presence during my part-time studies at the University of KwaZulu-Natal meant that I was constantly well versed with the goings on there.

# Notation

Throughout this thesis all groups discussed are finite. We will use the following notation from the ATLAS [12] unless stated otherwise in the text.

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$\mathbb{Z}_m$	residues (mod $m$ ) or cyclic group of order $m$
$F$	a field
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\Omega$	a set containing $n$ elements
$[\lambda] = [1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}]$	a partition of $n$
$\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$	$n$ -tuple of powers in $[\lambda]$
$\bar{\lambda}_0 = (\lambda_s \mid \lambda_s \neq 0, s = 1, 2, \dots, n)$	all the non zero entries of $\bar{\lambda}$
$(k_1, k_2, \dots, k_m)$	an $m$ -composition of $n$
$G, H, M, N, N_1, N_2, S, Q$	groups
$1_G$	identity element of $G$
$H \leq G$	$H$ is a subgroup of $G$
$N \trianglelefteq G$	$N$ is a normal subgroup of $G$
$G'$	the commutator subgroup of $G$
$\bar{G} = N.G$	a group extension of $N$ by $G$
$N:G$	a split extension or semi-direct product
$N \cdot G$	a non-split extension of $N$ by $G$
$o(g)$	order of an element $g$

$d^g$	action of $g$ on $d$
$[g]$	a conjugacy class with representative $g$
$d^G$	an orbit of the action of $G$ on $d$
$C_G(g)$	centralizer of $g$ in $G$
$Hg$	the right coset of $H$
$N_G(H)$	normalizer of $H$ in $G$
$Irr(G)$	the set of irreducible characters of $G$
$I_G$	identity character of $G$
$\chi_H$ or $\chi \downarrow H$	the restriction of a character $\chi$ to $H$
$\theta^G$ or $\theta \uparrow G$	the induction of a character $\theta$ to $G$
$\chi(G/H)$	permutation character of $G$ on $H$
$S_n$	the symmetric group on $\Omega$
$\mathbb{Z}_m^n$	the direct product of $n$ copies of $\mathbb{Z}_m$
$m^n$	for prime $m$ , an elementary abelian group of order $m^n$
$B(m, n)$	the generalized symmetric group $(m^n:S_n)$
$B_A(m, n)$	a group associated with $B(m, n)$ , where $A = S$ or $Q$
$(a_{is})$	$m$ -set of partition or cycle type of a conjugacy class of $B(m, n)$
$GL(n, F)$	general linear group of non-singular $n \times n$ matrices over $F$
$M_m(X)$	set of all $m \times m$ matrices over $X$

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# Chapter 1

## Preliminaries

### 1.1 Introduction

Since the completion of the classification of finite simple groups in 1981, recent work in group theory involves the study of the structures of simple groups. The structures and character tables of maximal subgroups of simple groups give substantive information about these groups. Most of the maximal subgroups of simple groups are of extension type (see [12]). However, here we are particularly interested in special group extensions which are wreath products of cyclic groups with the symmetric group  $S_n$ , specifically called generalized symmetric groups  $B(m, n)$ , for positive integers  $m$  and  $n$ . Actually work on the characters of wreath products with permutation groups dates back to Specht's work [61], through the works of Osima [49] and Kerber [33]. And more recently other people have worked on characters of wreath products with symmetric groups, these amongst others include Almetady [5], Darafshesh and Iranmanesh [14], List and Mahmoud [36], Puttaswamiah [52], Read [55, 56], Saeed-Ul-Islam [59] and Stembridge [64].

There are several well developed methods for constructing the character tables of group extensions. However Fischer [17] has given an effective method for constructing the character tables of some group extensions including the generalized symmetric group  $B(m, n)$ . This method known as the technique of the Fischer-Clifford Matrices makes use of Clifford Theory [11]. Given a group extension  $\overline{G} = H.G$  such that every irreducible character of  $H$  can be extended to its inertia group, for each conjugacy class of  $G$  we construct a matrix called a Fischer-Clifford matrix of  $\overline{G}$ . By using the

Fischer-Clifford matrices of  $\overline{G}$  together with the fusion maps and the character tables of subgroups of  $G$  called inertia factor groups, we are able to construct the complete character table of  $\overline{G}$ . The technique of the Fischer-Clifford Matrices has been used in many works both on split and non-split extensions (see for example [1], [2], [5], [14], [18], [36], [47], [51], [65]). Here we use the technique of the Fischer-Clifford Matrices to construct the character tables of examples of generalized symmetric groups and some associated groups of order  $m^{n-1}n!$ .

In [5] Almetady has presented a combinatorial method for constructing the Fischer-Clifford matrices of generalized symmetric groups. As a few examples for small values of  $m$  and  $n$  show the manual construction of these matrices using the combinatorial method is not an easy task. The partial aim of our work was to use the combinatorial method to develop a computer programme for computing the Fischer-Clifford matrices of  $B(m, n)$ . Here we give a series of computer programmes, written for GAP, that give various parameters of Fischer-Clifford matrices of  $B(m, n)$ . Many of these programmes have been used in the development of the main Programme 5.2.4 that combinatorially computes matrices which are row equivalent to the Fischer-Clifford matrices of  $B(m, n)$ . Thus the matrices computed by Programme 5.2.4 require to have their rows collected together into blocks that correspond to particular inertia factor groups. The Programme 5.2.2 has been developed to determine some combinatorial objects called  $m$ -compositions of  $n$  which index the inertia factor groups corresponding to the rows of the matrices computed by Programme 5.2.4. A particular  $m$ -composition of  $n$  distinguishes the rows of these matrices that correspond to the same inertia factor group. Such rows of a matrix computed by Programme 5.2.4 are then manually collected together in a block corresponding to a particular inertia factor group, to give a Fischer-Clifford matrix of  $B(m, n)$ . As examples, an application of Programme 5.2.4 to construct the Fischer-Clifford matrices of the groups  $B(2, 6)$  and  $B(3, 5)$  of orders 46080 and 29160 is done here. We have also used Programme 5.2.4 to construct the Fischer-Clifford matrices of the groups  $B(2, 12)$  and  $B(4, 5)$  of orders  $2^{22} \times 3^5 \times 5^2 \times 7 \times 11$  and  $2^{13} \times 3 \times 5$  respectively. Due to lack of space here we have given the Fischer-Clifford matrices of  $B(2, 12)$  and  $B(4, 5)$  on the compact disk submitted with this thesis. However note that these matrices are the equivalent form of the Fischer-Clifford matrices of  $B(2, 12)$  and  $B(4, 5)$ .

In [35] List has presented a method for constructing the Fischer-Clifford matri-

ces of group extensions of a faithful irreducible constituent of the elementary abelian group  $2^n$  by symmetric groups. This was done by choosing entries from appropriate rows and columns of the Fischer-Clifford matrices of  $B(2, n)$ . The other aim of our work is to adapt the combinatorial method in [5] to construct the Fischer-Clifford matrices of some group extensions associated with  $B(m, n)$  using a similar method as the one used in [35]. The Fischer-Clifford matrices so constructed are used in constructing the character tables of the respective groups  $B(m, n)$  and some group extensions associated with them. In this thesis we have constructed the character tables of only the groups  $B(2, 6)$ ,  $B(3, 5)$  and some group extensions associated with these groups and  $B(3, 3)$ . We have also constructed the character tables of the groups  $B(2, 12)$  and  $B(4, 5)$ . However due to lack of space here the character tables of the groups  $B(2, 12)$  and  $B(4, 5)$  are given on the compact disk submitted with this thesis.

The following is the arrangement of the chapters of this thesis. This preliminary chapter presents general aspects on the work being discussed here. We discuss the theory of group extensions in Section 1.2 followed by representations and characters of finite groups in Section 1.3. We describe Clifford Theory and define Fischer-Clifford matrices in Sections 1.4 and 1.5 respectively. In Chapter 2 we discuss some Combinatorics required for the combinatorial method used for constructing the Fischer-Clifford matrices of generalized symmetric groups (see [5], [46]). In Section 2.2, we define special partitions of a positive integer  $n$ , called  $m$ -sets of partitions of  $n$ , which index the conjugacy classes of  $B(m, n)$ . In Section 2.3 we define some combinatorial objects called magic matrices, these will be used in constructing the Fischer-Clifford matrices of generalized symmetric groups later. We devote Chapter 3 to the description of the groups studied in this thesis. We discuss the generalized symmetric group and its conjugacy classes in Section 3.1. Here, we also give two programmes developed to compute orders of centralizers and orders of elements of conjugacy classes respectively, of  $B(m, n)$ . This is equivalent to coset analysis (see Subsection 1.2.1) for calculating the conjugacy classes of group extensions  $\overline{G} = H.G$ , where  $H$  is abelian. We give the orbits of the actions of the symmetric group  $S_n$  on the group  $N = \mathbb{Z}_m^n$  and on the set  $Irr(N)$  of irreducible characters of  $N$  in Section 3.2. We discuss the groups  $B_S(m, n)$  and  $B_Q(m, n)$  associated with the generalized symmetric group  $B(m, n)$  in Section 3.3. We also give the orbits of the actions of  $S_n$  on the subgroup  $S = \mathbb{Z}_m^{n-1}$ , on the quotient  $Q = \overline{\mathbb{Z}_m^n}$  of  $N$  and on the irreducible characters of  $S$  and  $Q$  in Subsections 3.3.1 and 3.3.2 respectively. This information is useful in describing the inertia factor

groups required for constructing the character table of each respective group.

In Chapter 4, we apply Fischer-Clifford theory to the generalized symmetric group  $B(m, n)$  and discuss the results involved in the combinatorial method for constructing the Fischer-Clifford matrices of  $B(m, n)$ . These results including Theorem 4.3.2 and Proposition 4.3.4 (due to Almestady [5]) and Theorem 4.3.5 (due to List and Mahmoud [36]) have been used to develop Programme 5.2.4 for computing matrices which are row equivalent to Fischer-Clifford matrices of  $B(m, n)$ . We present Programme 5.2.4 and its application to construct the Fischer-Clifford matrices of the groups  $B(2, 6)$  and  $B(3, 5)$  in Chapter 5. We will use the Fischer-Clifford matrices of the groups  $B(2, 6)$  and  $B(3, 5)$  to construct the character tables of these groups in Chapter 6. The combinatorial methods adapted to construct the Fischer-Clifford matrices of the groups  $B_S(p, n)$ , where  $p$  is prime, and  $B_Q(m, n)$  associated with  $B(m, n)$  are described in Chapters 7 and 8 respectively. As examples we will construct the character tables of the groups  $B_S(2, 6)$ ,  $B_S(3, 3)$ ,  $B_S(3, 5)$ ,  $B_Q(2, 6)$  and  $B_Q(3, 5)$ . The correctness of all the character tables constructed in this thesis has been tested in GAP, these tables are given in Appendices A, B, C, D, E, F and G. The results given here are numbered *c.s.n* meaning a result number  $n$  in Section  $s$  of Chapter  $c$ .

For further information on the material given in this chapter, readers may consult the references [1], [3], [4], [5], [6], [7], [8], [9], [12], [13], [17], [18], [23], [24], [25], [26], [28],[29], [30], [31], [33], [47], [53], [57], [60], [62], [66] and any other relevant sources.

## 1.2 Group Extensions

Since all the groups studied here are group extensions we discuss some aspects of the theory of group extensions in this section.

**Definition 1.2.1** *Let  $N$  and  $G$  be groups. Then an **extension** of  $N$  by  $G$  is a group  $\overline{G}$  such that*

$$(i) \quad N \triangleleft \overline{G}.$$

$$(ii) \quad \overline{G}/N \cong G.$$

Alternatively a group  $\overline{G}$  is a group extension of  $N$  by  $G$  if in the following sequence of groups and maps, we have  $\ker(\sigma) = \text{Im}(\delta)$ , where  $\delta$  is a homomorphism from  $N$

to  $\overline{G}$  and  $\sigma$  is a homomorphism from  $\overline{G}$  to  $G$ :

$$\{1\} \rightarrow N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \rightarrow \{1\}.$$

Let  $\overline{G} = N.G$  denote an extension of  $N$  by  $G$ .

**Definition 1.2.2** Let  $\overline{G} = N.G$  and  $\{1\} \rightarrow N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \rightarrow \{1\}$  be the corresponding sequence. Let  $g \in G$  and  $\overline{g} \in \overline{G}$  such that  $\sigma(\overline{g}) = g$ . Then  $\overline{g}$  is called a **lifting** of  $g$  in  $\overline{G}$ .

**Proposition 1.2.3** ([57],[65]) Let  $\overline{G}$  be an extension of  $N$  by  $G$  where  $N$  is abelian. Then there is a homomorphism  $\theta : G \rightarrow \text{Aut}(N)$  such that  $\theta_g(n) = \overline{g}n(\overline{g})^{-1}$ ,  $n \in N$  and  $\theta$  is independent of the choice of liftings  $\{\overline{g} \mid g \in G\}$ .

**Proof.** Let  $a \in \overline{G}$  and  $\gamma_a$  denote conjugation by  $a$ . Since  $N$  is a normal subgroup of  $\overline{G}$ ,  $\gamma_a \downarrow_N \in \text{Aut}(N)$  and the function  $\mu : \overline{G} \rightarrow \text{Aut}(N)$  defined by  $\mu(a) = \gamma_a \downarrow_N$  is a homomorphism. If  $a \in N$ , then since  $N$  is abelian we have  $\mu(a) = I_N$ . Thus there is a homomorphism  $\mu^* : \overline{G}/N \rightarrow \text{Aut}(N)$  which is given by  $\mu^*(Na) = \mu(a)$ . However  $G \cong \overline{G}/N$  and for any lifting  $\{\overline{g} \mid g \in G\}$ , the function  $\phi : G \rightarrow \overline{G}/N$  defined by  $\phi(g) = N\overline{g}$  is an isomorphism. If  $\{\overline{g}_1 \mid g \in G\}$  is another choice of liftings, then  $\overline{g} \overline{g}_1^{-1} \in N$  for every  $g \in G$  and thus  $N\overline{g} = N\overline{g}_1$ . Therefore the isomorphism  $\phi$  is independent of the choice of liftings. Let  $\theta : G \rightarrow \text{Aut}(N)$  be the composition  $\mu^* \circ \phi$ . For  $g \in G$  and  $\overline{g}$  a lifting of  $g$ , then  $\theta(g) = \mu^*(\phi(g)) = \mu^*(N\overline{g}) = \mu(\overline{g}) \in \text{Aut}(N)$  and thus for  $n \in N$ , we have  $\theta_g(n) = \mu(\overline{g})(n) = \overline{g}n(\overline{g})^{-1}$ . Hence the result.  $\square$

**Definition 1.2.4** A group extension  $\overline{G}$  of  $N$  by  $G$  is called a **split extension** if there is a homomorphism  $\varphi$  of  $G$  into  $\overline{G}$  such that  $\sigma\varphi = I_G$ , where  $\sigma$  is the homomorphism from  $\overline{G}$  to  $G$  with  $N = \text{Ker}(\sigma)$ . Let  $\overline{G} = N:G$  denote a split extension of  $N$  by  $G$ .

**Definition 1.2.5** A group  $\overline{G}$  is a **semi-direct product** of  $N$  by  $G$  if

- (i)  $N$  and  $G$  are subgroups of  $\overline{G}$
- (ii)  $\overline{G} = NG$ .
- (ii)  $N \triangleleft \overline{G}$ .
- (iii)  $N \cap G = \{1\}$ .

**Theorem 1.2.6** *Every split extension of  $N$  by  $G$  is equivalent to a semi-direct product of  $N$  by  $G$ .*

**Proof.** See Remark 2.2.5 in [47] or [60]  $\square$

Let  $\overline{G}$  be a split extension of  $N$  by  $G$ , then every  $\overline{g} \in \overline{G}$  can be written uniquely as  $\overline{g} = ng$ , where  $n \in N$  and  $g \in G$ . The composition of elements of  $\overline{G}$  is given by  $(n_1g_1).(n_2g_2) = n_1n_2^{g_1}g_1g_2$ .

**Corollary 1.2.7** *If  $\overline{G}$  is a semi-direct product of  $N$  by  $G$  then there is a homomorphism  $\tau : G \rightarrow \text{Aut}(N)$  defined by  $\tau(g) = \tau_g$  for each  $g \in G$ , where  $\tau_g$  is an automorphism of  $N$  given by  $\tau_g(n) = n^g = gng^{-1}$ .*

**Proof.** See Proposition 1.2.3 or [27]  $\square$

**Theorem 1.2.8** *Let  $N$  and  $G$  be groups,  $\theta \in \text{Hom}(G, \text{Aut}(N))$ ,  $\overline{G} = N \times G$  as a set with multiplication defined by  $(n_1, g_1)(n_2, g_2) = (n_1\theta_{g_1}(n_2), g_1g_2)$ . Let  $\delta, \sigma$  and  $\lambda$  be functions given by  $\delta(n) = (n, 1_G)$ ,  $\sigma(n, g) = g$  and  $\lambda(g) = (1_N, g)$ . Then*

- (i)  $\{1\} \rightarrow N \xrightarrow{\delta} \overline{G} \xrightarrow{\sigma} G \rightarrow \{1\}$  is an extension of  $N$  by  $G$
- (ii)  $\delta$  is an isomorphism of  $N$  with a subgroup  $N_1$  of  $\overline{G}$
- (iii)  $\lambda$  is an isomorphism of  $G$  with a subgroup  $G_1$  of  $\overline{G}$
- (iv)  $\overline{G}$  is a semidirect product of  $N_1$  by  $G_1$  that realizes a homomorphism  $\psi$  satisfying  $[\psi(\lambda(g))](\delta(n)) = \delta(\theta_g(n))$ , for all  $n \in N, g \in G$
- (v)  $\sigma\lambda = I_G$ .

**Proof.** See Theorem 9.2.1 in [60].  $\square$

**Definition 1.2.9** *A group  $N$  is called an elementary abelian- $p$  group if it is abelian and every non-trivial element has the same prime order  $p$ .*

Let  $p$  be a prime number. Any elementary abelian  $p$ -group can be considered as a vector space over the field of order  $p$ , and is therefore isomorphic to the direct sum of  $n$  copies of the cyclic group of order  $p$ . Conversely, any such direct sum is obviously an elementary abelian  $p$ -group. In many of our examples here we shall deal with elementary abelian  $p$ -groups  $p^n$  for some prime  $p$  and  $n \in \mathbb{N}$ .

### 1.2.1 Conjugacy Classes of Group Extensions

The conjugacy classes of a finite group give a lot of important information about the structure of the group. Since the number of irreducible ordinary characters of a finite group is equal to the number of conjugacy classes of the group, having information about conjugacy classes of a finite group is the best point to start constructing the character table of the group. Several works have been done on the properties of conjugacy classes as well as comparisons between results on conjugacy classes and characters of finite groups. For example we have following result.

**Proposition 1.2.10** *Let  $\overline{G} = N.G$ ,  $\overline{g}$  a lifting of  $g \in G$ ,  $C$  be the centralizer of  $N\overline{g}$  in  $G$  and  $\overline{C}$  be the complete preimage in  $\overline{G}$  of  $C$ . Then*

- (i) *the union of the cosets  $N\overline{x}$  which are conjugate in  $G$  to  $N\overline{g}$ , is the union of the conjugacy classes  $L_1, L_2, \dots, L_r$  of  $\overline{G}$ ,*
- (ii)  *$\overline{C}$  acts on the coset  $N\overline{g}$  by conjugation,*
- (iii)  *$\overline{C}$  has  $r$  orbits in its action on  $N\overline{g}$  and the orbit representatives  $\overline{g}_1, \overline{g}_2, \dots, \overline{g}_r$  are representatives of the conjugacy classes  $L_1, L_2, \dots, L_r$  of  $\overline{G}$ ,*
- (iv) *the centralizer  $C_{\overline{G}}(\overline{g}_i)$  for  $1 \leq i \leq r$  is the stabilizer of  $\overline{g}_i$  in  $\overline{C}$  in its action on  $N\overline{g}$ .*

**Proof.** See [8].  $\square$

The use of computational methods to determine the conjugacy classes of a group is another aspect to be cited. For example in [8] and [9], Butler gives various algorithms which can be used for computing conjugacy classes in finite groups and in permutation groups respectively. The technique of **coset analysis**, used for the determination of the conjugacy classes of elements of both split and non-split extensions  $\overline{G} = N.G$  where  $N$  is an abelian normal subgroup of  $\overline{G}$ , was developed by Moori in [37]. For each conjugacy class  $[g]$  in  $G$  with representative  $g$ , we analyse the coset  $N\overline{g}$ , where  $\overline{g}$  is a lifting of  $g$  in  $\overline{G}$ . For each such a class, we define

$$C_{\overline{g}} = \{x \in \overline{G} \mid x(N\overline{g}) = (N\overline{g})x\},$$

the stabilizer of  $N\overline{g}$  in  $\overline{G}$  under the action on  $N\overline{g}$  by conjugation of  $\overline{G}$ . Since  $N$  is normal in  $\overline{G}$ , it is clear that  $N$  is a normal subgroup of  $C_{\overline{g}}$ .

**Lemma 1.2.11** [65]  $C_{\bar{G}/N} = C_{\bar{G}/N}(N\bar{g})$ .

**Proof.** Let  $k \in \bar{G}$ . Then

$$\begin{aligned}
Nk \in C_{\bar{G}/N}(N\bar{g}) &\Leftrightarrow Nk(N\bar{g})(Nk)^{-1} = N\bar{g} \\
&\Leftrightarrow NkN\bar{g}Nk^{-1} = N\bar{g} \\
&\Leftrightarrow NkN\bar{g}k^{-1} = N\bar{g} \\
&\Leftrightarrow NkNn\bar{g}k^{-1} = N\bar{g} \quad \forall n \in N \\
&\Leftrightarrow Nkn\bar{g}k^{-1} = N\bar{g}, \quad \forall n \in N \\
&\Leftrightarrow kn\bar{g}k^{-1} \in N\bar{g}, \quad \forall n \in N \\
&\Leftrightarrow k \in C_{\bar{g}} \\
&\Leftrightarrow Nk \in C_{\bar{G}/N}. \quad \square
\end{aligned}$$

Since  $N \trianglelefteq C_{\bar{g}}$  and by Lemma 1.2.11, it follows that  $C_{\bar{g}} = N.C_{\bar{G}/N}(N\bar{g})$ . For  $\bar{g}$  a lifting of  $g \in G$  in  $\bar{G}$ , we can identify  $C_{\bar{G}/N}(N\bar{g})$  with  $C_G(g)$  and write  $C_{\bar{g}} = N.C_G(g)$ . For each conjugacy class  $[g]$  of  $G$ , the conjugacy classes of  $\bar{G}$  where  $N$  is abelian will be determined by the action by conjugation of  $C_{\bar{g}}$  on the elements of  $N\bar{g}$ . To act  $C_{\bar{g}}$  on the elements of  $N\bar{g}$ , we first act  $N$  and then act  $\{\bar{h} \mid h \in C_G(g)\}$ , where  $\bar{h}$  is a lifting of  $h$  in  $\bar{G}$ . We outline this action as follows:

**STEP 1: The action of  $N$  on  $N\bar{g}$ :** Let  $C_N(\bar{g})$  be the stabilizer of  $\bar{g}$  in  $N$ . Then for any  $n \in N$  we have

$$\begin{aligned}
x \in C_N(n\bar{g}) &\Leftrightarrow x(n\bar{g})x^{-1} = n\bar{g} \\
&\Leftrightarrow xnx^{-1}x\bar{g}x^{-1} = n\bar{g} \\
&\Leftrightarrow n(x\bar{g}x^{-1}) = n\bar{g}, \quad \text{since } N \text{ is abelian} \\
&\Leftrightarrow x\bar{g}x^{-1} = \bar{g} \\
&\Leftrightarrow x \in C_N(\bar{g}).
\end{aligned}$$

Thus  $C_N(\bar{g})$  fixes every element of  $N\bar{g}$ . Now let  $|C_N(\bar{g})| = k$ . Then under the action of  $N$ ,  $N\bar{g}$  splits into  $k$  orbits  $Q_1, Q_2, \dots, Q_k$ , where

$$|Q_i| = [N : C_N(\bar{g})] = \frac{|N|}{k},$$

for  $i \in \{1, 2, \dots, k\}$ .

**STEP 2: The action of  $\{\bar{h} \mid h \in C_G(g)\}$  on  $N\bar{g}$ :** Since the elements of  $N\bar{g}$  are now in the orbits  $Q_1, Q_2, \dots, Q_k$  from Step 1 above, we need only act  $\{\bar{h} \mid h \in C_G(g)\}$  on these  $k$  orbits. Suppose that under this action  $f_j$  of these orbits  $Q_1, Q_2, \dots, Q_k$  fuse together to form one orbit  $\Delta_j$ , then we have

$$|\Delta_j| = f_j \times \frac{|N|}{k} .$$

Thus for  $x = d_j\bar{g} \in \Delta_j$ , we obtain that

$$\begin{aligned} |[x]_{\bar{G}}| &= |\Delta_j| \times |[g]_G| \\ &= f_j \times \frac{|N|}{k} \times \frac{|G|}{|C_G(g)|} \\ &= f_j \times \frac{|\bar{G}|}{k|C_G(g)|} \end{aligned}$$

and thus we obtain that

$$|C_{\bar{G}}(x)| = \frac{|\bar{G}|}{|[x]_{\bar{G}}|} = |\bar{G}| \times \frac{k|C_G(g)|}{f_j|\bar{G}|} = \frac{k|C_G(g)|}{f_j} .$$

Thus to calculate the conjugacy classes of  $\bar{G} = N.G$ , we find the values of  $k$  and the  $f_j$ 's for each class representative  $g \in G$ .

However for the special case of a split extension  $\bar{G} = N:G$ , we identify  $C_{\bar{g}}$  with  $C_g = \{x \in \bar{G} \mid x(Ng) = (Ng)x\}$ , where the lifting of  $g$  in  $\bar{G}$  is  $g$  itself since  $G \leq \bar{G}$ .

**Corollary 1.2.12** [47] *If  $\bar{G} = N:G$ , then  $C_g = N:C_G(g)$ .*

**Proof.** We have that  $N$  is a normal subgroup of  $C_g$ . Now we show that  $C_G(g) \leq C_g$  and that  $N \cap C_G(g) = \{1\}$ . Let  $x \in C_G(g)$ . Then we obtain  $(Ng)^x = x(Ng)x^{-1} = xNgx^{-1} = Nxgx^{-1} = Ng$ . Thus  $x \in C_g$  and hence  $C_G(g) \leq C_g$ . Since  $N \cap C_G(g) \leq N \cap G = \{1_G\}$ , then we have that  $N \cap C_G(g) = \{1_G\}$ . Hence the result.  $\square$

Thus in the case of a split extension  $\bar{G} = N:G$ , we analyse the coset  $Ng$  instead of  $N\bar{g}$ . Under the action of  $N$  on  $Ng$ , we always assume that  $g \in Q_1$ . Also instead of acting  $\{\bar{h} \mid h \in C_G(g)\}$  on the  $k$  orbits  $Q_1, Q_2, \dots, Q_k$  we just act  $C_G(g)$  on these orbits.

The technique of coset analysis for computing conjugacy classes group extensions  $\bar{G} = N.G$  has since been used in several works (see [1], [2], [37], [41], [47], [65]). The

orders of the elements of conjugacy classes of  $\overline{G} = N:G$  may be determined in different ways. However the following results are useful in the determination of the orders of the elements of a group  $\overline{G} = N:G$ .

**Theorem 1.2.13** [47] *Let  $\overline{G} = N:G$  and  $dg \in \overline{G}$  where  $d \in N$  and  $g \in G$  such that  $o(g) = m$  and  $o(dg) = k$ . Then  $m$  divides  $k$ .*

**Proof.** We have that

$$1_{\overline{G}} = (dg)^k = dd^g d^{g^2} d^{g^3} \dots d^{g^{k-1}} g^k .$$

Since  $G$  acts on  $N$  and  $d \in N$ , we have  $d, d^g, d^{g^2}, \dots, d^{g^{k-1}} \in N$ . Hence  $dd^g d^{g^2} \dots d^{g^{k-1}} \in N$ . Thus we must have that  $dd^g d^{g^2} \dots d^{g^{k-1}} = 1_N$  and  $g^k = 1_G$ . Hence  $m$  divides  $k$ .  $\square$

**Theorem 1.2.14** [47] *Let  $\overline{G} = N:G$  such that  $N$  is an elementary abelian  $p$ -group, where  $p$  is prime. Let  $dg \in \overline{G}$  where  $d \in N$  and  $g \in G$  such that  $o(g) = m$  and  $o(dg) = k$ . Then either  $k = m$  or  $k = pm$ .*

**Proof.** See Theorem 2.3.10 in [47].  $\square$

**Remark 1.2.15** [47] *Let  $\overline{G} = N:G$  where  $N$  is an elementary abelian  $p$ -group. Let  $dg \in \overline{G}$  where  $d \in N$  and  $g \in G$  such that  $o(g) = m$  and  $o(dg) = k$ . Then we have that*

$$(dg)^m = dd^g d^{g^2} d^{g^3} \dots d^{g^{m-1}} g^m .$$

Since  $g^m = 1_G$ , we obtain that  $(dg)^m = w$ , where  $w \in N$ . Now applying Theorem 1.2.14 we have if  $w = 1_N$  then  $k = m$  and if  $w \neq 1_N$  then  $k = pm$ .

In [47] Mpono has developed computer programmes in CAYLEY which are used for computing the conjugacy classes and the orders of the conjugacy class representatives of the group extension  $\overline{G} = N:G$  where  $N$  is an elementary abelian  $p$ -group for prime  $p$  on which a linear group  $G$  acts. These programmes can similarly be applied to the group extension  $\overline{G} = N:G$  where  $G$  is a permutation group, by considering a subgroup of a linear group which is isomorphic to  $G$ .

### 1.3 Representations and Characters of Finite Groups

In this section we discuss some results on group representations and characters which are useful for the technique of the Fischer-Clifford matrices to be described in Section 1.5. We will not deal much with modular characters, therefore we devote our discussion mostly to ordinary representations and characters, that is representations and characters of a finite group  $G$  over the complex field  $\mathbb{C}$ . In most of the following material the proofs are omitted but reference is made to [28] for an extensive treatment of character theory. For further reading on representations and characters of finite groups, readers are encouraged to consult [3], [6], [7], [13], [21], [25], [28], [29], [30], [31], [39], [47] and other relevant sources.

**Definition 1.3.1** *Let  $G$  be a group,  $F$  a field and  $GL(n, F)$  the general linear group (or the multiplicative group of all nonsingular  $n \times n$  matrices over  $F$  for some integer  $n$ ). Then a **representation of  $G$  over  $F$**  is a homomorphism  $\rho : G \rightarrow GL(n, F)$  from  $G$  to  $GL(n, F)$ . The degree of the representation  $\rho$  is the integer  $n$ . Define a function  $\chi : G \rightarrow F$  by  $\chi(g) = \text{trace}(\rho(g))$ . Then  $\chi$  is called the **character** of  $G$  afforded by the representation  $\rho$ . The degree of the character  $\chi$  is the degree of the representation  $\rho$ .*

**Definition 1.3.2** *Let  $\rho_1$  and  $\rho_2$  be representations of  $G$  over  $F$ . We say  $\rho_1$  and  $\rho_2$  are **equivalent** if there exists  $P \in GL(n, F)$  such that  $\rho_1(g) = P\rho_2(g)P^{-1}$  for all  $g \in G$ . We say a representation  $\rho$  of  $G$  is **reducible** if it is equivalent to a representation  $\rho'$  given by*

$$\rho'(g) = \begin{pmatrix} \phi_1(g) & 0 \\ \gamma(g) & \phi_2(g) \end{pmatrix}$$

*for all  $g \in G$ , where  $\phi_1, \phi_2$  are representations of  $G$  and  $\gamma(g)$  is any matrix. A representation which is not reducible is said to be **irreducible**.*

**Theorem 1.3.3 (Schur's Lemma)** *Let  $\rho_1 : G \rightarrow GL(n, F)$  and  $\rho_2 : G \rightarrow GL(m, F)$  be two irreducible representations of a group  $G$  over a field  $F$ . Let  $P$  be a matrix such that  $P\rho_1(g) = \rho_2(g)P$  for all  $g \in G$ . Then either  $P$  is the zero matrix or  $P$  is nonsingular so that  $\rho_1$  and  $\rho_2$  are equivalent.*

**Proof.** See Theorem 1.8 in [39].  $\square$

It is clear that equivalent representations afford the same character, since similar matrices have the same trace. The character afforded by an irreducible representation is called an **irreducible character**. Let  $Irr(G)$  denote the set of irreducible characters of a group  $G$ .

**Corollary 1.3.4** [39] *If  $\rho : G \rightarrow GL(n, F)$  is an irreducible representation of a group  $G$  over an algebraically closed field  $F$ , then the only matrices which commute with all matrices  $\rho(g)$ ,  $g \in G$  are scalar matrices  $aI_n$ , where  $a \in F$  and  $I_n$  is the  $n \times n$  identity matrix.*

**Proof.** Let  $P$  be an  $n \times n$  matrix such that  $P\rho(g) = \rho(g)P$  for all  $g \in G$ . Then for any  $a \in F$  we have that

$$(aI_n - P) \cdot \rho(g) = \rho(g) \cdot (aI_n - P), \quad \forall g \in G. \quad (1)$$

Let  $m(x) = \det(xI_n - P)$  be the characteristic polynomial of  $P$ . Since  $m(x)$  is a polynomial over  $F$  and  $F$  is algebraically closed, then there exists  $a_1 \in F$  such that  $m(a_1) = 0_F$ . Hence  $\det(a_1I_n - P) = 0_F$  and thus  $a_1I_n - P$  is singular. Then from relation (1) above and Schur's Lemma, we obtain that  $a_1I_n - P = 0$  and hence  $a_1I_n = P$ .  $\square$

**Definition 1.3.5** *Let  $G$  be a group,  $F$  a field and  $\phi : G \rightarrow F$  be a function which is constant on the conjugacy classes of  $G$ . Then  $\phi$  is called a **class function** of  $G$ .*

It is clear that characters are class functions. It is also known (see for example [3]) that the set  $Irr(G)$  of irreducible characters of  $G$  forms a basis for the space of class functions on  $G$ . From now on, we will consider ordinary representations and characters. We note the following from character theory (see [39]).

- (i) Two representations of  $G$  are equivalent if and only if they have the same character.
- (ii) The number of irreducible characters of  $G$  is equal to the number of conjugacy classes of  $G$ .
- (iii) Any character of  $G$  can be written as a sum of irreducible characters.

**Lemma 1.3.6** *Let  $G$  be a group,  $\rho$  be a representation of  $G$  which affords the character  $\chi$ . Let  $g \in G$  be an element of  $G$  of order  $n$ . Then the following conditions hold*

(i)  $\rho(g)$  is similar to a diagonal matrix  $\text{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$

(ii)  $\varepsilon_i^n = 1$

(iii)  $\chi(g) = \sum_i \varepsilon_i$

(iv)  $|\chi(g)| \leq \chi(1_G) = \text{degree of } \chi$

(v)  $\chi(g^{-1}) = \overline{\chi(g)}$ , the complex conjugate of  $\chi(g)$ .

**Proof.** See Lemma 2.15 in [28].  $\square$

**Definition 1.3.7** *Let  $G$  be a group and  $\text{Irr}(G) = \{\chi_1, \chi_2, \dots, \chi_r\}$ . Let  $\chi$  be a class function of  $G$  such that  $\chi = \sum_{i=1}^r n_i \chi_i$ . Then those characters  $\chi_i$  for which  $n_i > 0$  are called the irreducible constituents of  $\chi$ .*

**Definition 1.3.8** *Let  $\chi$  and  $\psi$  be class functions of a group  $G$ . Then the inner product of  $\chi$  and  $\psi$  is defined by*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

The following theorems are called the first (or row) and second (or column) orthogonality relations respectively.

**Theorem 1.3.9** [28]/**(First Orthogonality Relation)** *Let  $G$  be a group and  $\text{Irr}(G) = \{\chi_1, \chi_2, \dots, \chi_r\}$ . Then*

$$\sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \delta_{ij} |G|.$$

**Proof.** See [28]  $\square$

**Theorem 1.3.10** [28]/**(Second Orthogonality Relation)** *Let  $G$  be a group and  $\text{Irr}(G) = \{\chi_1, \chi_2, \dots, \chi_r\}$  and  $\{g_1, g_2, \dots, g_r\}$  be a set of representatives of the conjugacy classes of  $G$ . Then*

$$\sum_{\chi \in \text{Irr}(G)} \chi(g_i) \overline{\chi(g_j)} = \delta_{ij} |C_G(g_i)|.$$

**Proof.** See Theorem 3.19 in [47] or [28]  $\square$

We now discuss the relationships between the characters of a group  $G$  and those of its subgroups.

### 1.3.1 Restriction of Characters

**Definition 1.3.11** *Let  $H$  be a subgroup of a finite group  $G$ . If  $\rho$  is a representation of  $G$ , then the **restriction of  $\rho$  to  $H$**  is a representation of  $H$ , which is denoted by  $\rho_H$  or  $\rho \downarrow H$ . If  $\chi$  is a character of  $G$  afforded by  $\rho$ , then the restriction of  $\chi$  to  $H$  is a character of  $H$  afforded by the representation  $\rho_H$ , and is denoted by  $\chi_H$  or  $\chi \downarrow H$ .*

The characters  $\chi_H$  and  $\chi$  take on the same values on the elements of  $H$ . If  $\chi_H$  is irreducible, then  $\chi$  is irreducible in  $G$ . The converse is not true in general. But under certain conditions the irreducibility of  $\chi$  implies that  $\chi_H$  is irreducible, as in the following result.

**Theorem 1.3.12** *Suppose that  $H$  is a subgroup of  $G$  of index 2 and  $\chi \in \text{Irr}(G)$ . Let  $\lambda$  be a linear character of  $G$  taking value 1 on  $H$  and -1 otherwise. Then the following conditions are equivalent*

- (i)  $\chi_H$  is irreducible,
- (ii)  $\chi(g) \neq 0$  for some  $g \in G - H$ ,
- (iii)  $\chi \neq \chi\lambda$ .

**Proof.** See Proposition 20.9 in [30].  $\square$

In-fact Karpilovsky in [31] has also proved a theorem of Gallagher (1966) that if  $H \leq G$  and  $\chi \in \text{Irr}(G)$  such that  $\chi(g) \neq 0 \forall g \in G - H$ , then  $\chi_H$  is irreducible.

### 1.3.2 Induced Characters

Let  $H$  be a subgroup of a group  $G$  with transversal the set  $\{x_1, x_2, \dots, x_r\}$  in  $G$ . Let  $\phi$  be a representation of  $H$  of degree  $n$ . Then we define  $\phi^*$  on  $G$  as follows:

$$\phi^*(g) = \begin{pmatrix} \phi(x_1gx_1^{-1}), \phi(x_1gx_2^{-1}), \dots, \phi(x_1gx_r^{-1}) \\ \phi(x_2gx_1^{-1}), \phi(x_2gx_2^{-1}), \dots, \phi(x_2gx_r^{-1}) \\ \vdots \\ \phi(x_rgx_1^{-1}), \phi(x_rgx_2^{-1}), \dots, \phi(x_rgx_r^{-1}) \end{pmatrix}$$

where  $\phi(x_i g x_j^{-1})$  are  $n \times n$  sub-matrices of  $\phi^*(g)$  such that  $\phi(x_i g x_j^{-1}) = 0 \quad \forall x_i g x_j^{-1} \notin H$ . Then  $\phi^*$  is a representation of  $G$  of degree  $n[G : H]$ .

**Definition 1.3.13** Let  $G, H, \phi$  and  $\phi^*$  be as above. Then the representation  $\phi^*$  is called the **representation of  $G$  induced from the representation  $\phi$  of  $H$** . The induced representation of  $\phi$  is denoted by  $\phi^* = \phi^G$  or  $\phi^* = \phi \uparrow G$ .

**Definition 1.3.14** Let  $G$  be a group and  $H \leq G$ . Let  $\theta$  be a class function of  $H$ . Then we define  $\theta^G$  as follows:

$$\theta^G(g) = \frac{1}{|H|} \sum_{x \in G} \theta^\circ(x g x^{-1}) \quad ,$$

where

$$\theta^\circ(h) = \begin{cases} \theta(h) & \text{if } h \in H \\ 0 & \text{otherwise} \end{cases} .$$

Then  $\theta^G$  is a class function of  $G$ , called the **induced class function of  $G$** , of degree  $[G : H] \deg(\theta)$ .

Let  $\phi$  be a representation of  $H$  that affords a character  $\theta$  of  $H$ . Then  $\theta^G$  is a character of  $G$  afforded by the induced representation  $\phi^G$  of  $G$ . The character  $\theta^G$  is called the **induced character of  $G$** . We note that the induction and restriction processes do not necessarily preserve irreducibility of characters.

**Theorem 1.3.15** [28](**Frobenius Reciprocity Theorem**) Let  $G$  be a group,  $H \leq G$  and suppose that  $\theta$  is a class function of  $H$  and  $\chi$  is a class function of  $G$ . Then

$$\langle \theta, \chi_H \rangle = \langle \theta^G, \chi \rangle$$

**Proof.** We obtain that

$$\langle \theta^G, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \theta^G(g) \overline{\chi(g)} = \frac{1}{|G|} \frac{1}{|H|} \sum_{g \in G} \sum_{x \in G} \theta^\circ(x g x^{-1}) \overline{\chi(g)} .$$

Putting  $y = x g x^{-1}$  and since  $\chi$  is a class function, then we obtain that  $\chi(y) = \chi(g)$ . Hence we have

$$\begin{aligned} \langle \theta^G, \chi \rangle &= \frac{1}{|G|} \frac{1}{|H|} \sum_{g \in G} \sum_{x \in G} \theta^\circ(x g x^{-1}) \overline{\chi(g)} = \frac{1}{|G|} \frac{1}{|H|} \sum_{y \in G} \sum_{x \in G} \theta^\circ(y) \overline{\chi(y)} \\ &= \frac{1}{|H|} \sum_{y \in H} \theta(y) \overline{\chi(y)} = \langle \theta, \chi_H \rangle. \quad \square \end{aligned}$$

**Theorem 1.3.16** *Let  $G$  be a group and  $H \leq G$ . Let  $\theta$  be a character of  $H$ ,  $g \in G$  and  $\{x_1, x_2, \dots, x_m\}$  be a set of representatives of the conjugacy classes of elements of  $H$  which fuse into  $[g]$  in  $G$ . Then we obtain that*

$$\theta^G(g) = |C_G(g)| \sum_{i=1}^m \frac{\theta(x_i)}{|C_H(x_i)|} ,$$

where we have that  $\theta^G(g) = 0$  whenever  $H \cap [g] = \emptyset$ .

**Proof.** We have that

$$\theta^G(g) = \frac{1}{|H|} \sum_{x \in G} \theta^\circ(xgx^{-1}).$$

If  $H \cap [g] = \emptyset$ , then  $xgx^{-1} \notin H$  and thus  $\theta^\circ(xgx^{-1}) = 0 \ \forall x \in G$  and hence  $\theta^G(g) = 0$ . Now if  $H \cap [g] \neq \emptyset$ , then let  $h \in H \cap [g]$ . Then as  $x$  runs over  $G$ , then  $xgx^{-1} = h$  for exactly  $|C_G(g)|$  values of  $x$ . Hence we obtain that

$$\theta^G(g) = \frac{1}{|H|} \sum_{x \in G} \theta(xgx^{-1}) = \frac{|C_G(g)|}{|H|} \sum_{h \in H \cap [g]} \theta(h) = |C_G(g)| \sum_{i=1}^m \frac{\theta(x_i)}{|C_H(x_i)|}. \quad \square$$

### 1.3.3 Lifted Characters

Let  $G$  be a group and  $\chi$  be a character of  $G$  afforded by a representation  $\rho$ . Then

$$\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1_G)\},$$

can be shown easily to be a normal subgroup of  $G$  (see [65]). Further every normal subgroup of  $G$  is an intersection of some of the  $\ker(\chi_i)$ , where  $\chi_i \in \text{Irr}(G)$ . If  $N$  is a normal subgroup of  $G$  and  $\rho$  is representation of  $G$  such that  $N \subseteq \ker(\rho)$ , then there exists a unique representation  $\hat{\rho}$  of  $G/N$  defined by  $\hat{\rho}(Ng) = \rho(g)$ . If  $\rho$  affords a character  $\chi$  of  $G$ , then  $\hat{\rho}$  affords a character  $\hat{\chi}$  of  $G/N$ .

**Theorem 1.3.17** *There is a one to one correspondence between the set of characters of  $G/N$  and the set of characters of  $G$  which contain  $N$  in their Kernel. Furthermore*

$$\text{Irr}(G/N) = \{\chi \in \text{Irr}(G) \mid N \subseteq \ker(\chi)\}.$$

**Proof.** See Theorem 17.3 in [30].  $\square$

Following from above we have the following definition.

**Definition 1.3.18** Let  $G$  be a group,  $N$  a normal subgroup of  $G$  and  $\hat{\chi}$  be a character of  $G/N$ . Then the character  $\chi$  of  $G$  defined by

$$\chi(g) = \hat{\chi}(Ng), \forall g \in G$$

is called a **lifting** of  $\hat{\chi}$  to  $G$ .

Thus by lifting the characters of  $G/N$  we can obtain some of the characters of  $G$ . It is clear from the definition that the degree of the lifted character  $\chi$  is equal to the degree of the character  $\hat{\chi}$ . Further  $\chi$  is irreducible if and only if  $\hat{\chi}$  is irreducible.

### 1.3.4 Permutation Characters

Let  $\Omega$  be a set. A group  $G$  is said to act on  $\Omega$  if there is a homomorphism  $\phi : G \rightarrow S_\Omega$ , where  $S_\Omega$  is the symmetric group on  $\Omega$ . If  $\phi$  is a monomorphism, we say that  $G$  acts faithfully on  $\Omega$ . In this case  $G$  can be identified with a subgroup of  $S_\Omega$  and  $G$  becomes a **permutation group** on  $\Omega$ .

**Definition 1.3.19** A group  $G$  is said to be **transitive** on  $\Omega$  if it has only one orbit on  $\Omega$ .

If  $G$  acts on  $\Omega$ , we define a representation  $\pi : G \rightarrow GL(n, \mathbb{C})$ , where  $n = |\Omega|$ . Let  $\Omega = \{x_1, x_2, \dots, x_n\}$ . For each  $g \in G$  we define  $\pi_g = (a_{ij})$  by

$$a_{ij} = \begin{cases} 1 & \text{if } x_i^g = x_j \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\pi_g$  is a permutation matrix of the action of  $g$ . The representation  $\pi$  defined above is called the **permutation representation** of  $G$  obtained from the action of  $G$  on  $\Omega$ . The character afforded by the permutation representation  $\pi$  denoted by  $\chi(G|\Omega)$ , is called the **permutation character** of  $G$  associated with the action of  $G$  on  $\Omega$ . Further for  $g \in G$  we have

$$\chi(G|\Omega)(g) = |\{x \in \Omega \mid x^g = x\}| = \text{the number of points of } \Omega \text{ fixed by } g.$$

Suppose that  $G$  acts transitively on  $\Omega$  and  $G_x$  is the stabilizer of  $x \in \Omega$ . Then the action of  $G$  on  $\Omega$  is the same as the action of  $G$  on the cosets of  $H = G_x$  in  $G$ . Hence  $\forall g \in G$ ,  $\chi(G|\Omega)(g)$  also gives the number of cosets of  $H = G_x$  in  $G$  which are fixed by  $g \in G$  and in this case we denote this number by  $\chi(G|H)(g)$ . It follows that  $\chi(G|H) = \chi(G|\Omega)$ .

**Theorem 1.3.20** *Let  $G$  be a group acting transitively on a set  $\Omega$ . Let  $\alpha \in \Omega$ ,  $H = G_\alpha$  and  $\chi(G|H)$  be the permutation character of this action. Then*

$$\chi(G|H) = (I_H)^G.$$

**Proof.** We have that

$$(I_H)^G(g) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} I_H(xgx^{-1}) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} 1.$$

Now if  $xgx^{-1} \in H$ , then  $xg \in Hx$ . Thus  $Hxg = Hx$  and hence  $Hx$  is fixed by  $g \in G$ . However the summation is taken over all  $x \in G$  such that  $xgx^{-1} \in H$ . Hence the summation is taken over all  $x \in G$  for which the coset  $Hx$  is fixed by  $g \in G$ . But  $\forall y \in Hx$ ,  $Hx = Hy$  and thus we obtain that

$$\sum_{x \in G, xgx^{-1} \in H} 1 = |H| |\{Hx \mid Hxg = Hx\}|$$

and hence we obtain that

$$(I_H)^G(g) = \frac{1}{|H|} |H| |\{Hx \mid Hxg = Hx\}| = |\{Hx \mid Hxg = Hx\}| = \chi(G|H)(g). \quad \square$$

Let  $G$  be a group,  $H \leq G$  and  $\chi = \chi(G|H)$ . The following are some properties of permutation characters (see Theorem 2.5.6 in [65]).

- (i)  $\deg(\chi)$  divides  $|G|$ .
- (ii)  $\langle \chi, \psi \rangle \leq \deg(\psi)$  for all  $\psi \in \text{Irr}(G)$ .
- (iii)  $\langle \chi, I_G \rangle = 1$ .
- (iv)  $\chi(g) \in \mathbb{N} \cup \{0\}$  for all  $g \in G$ .
- (v)  $\chi(g) \leq \chi(g^m)$  for all  $g \in G$  and  $m \in \mathbb{N} \cup \{0\}$ .
- (vi)  $\chi(g) = 0$  if  $o(g)$  does not divide  $|G|/\deg(\chi)$ .
- (vii)  $\chi(g) \frac{|g|}{\deg(\chi)}$  is an integer for all  $g \in G$ .

## 1.4 Clifford Theory

In this section we discuss the technique of Fischer-Clifford matrices. Later we will use this technique to construct the character tables of generalized symmetric groups and some associated groups. For the theory on Fischer-Clifford matrices, we follow the works of Ali [1], Almestady [5], Mpono [47] and Whitely [65]. However for the actual construction of the Fischer-Clifford matrices of  $B(m, n)$  and associated groups we will follow the works of Almestady [5] and List [35].

**Definition 1.4.1** *Let  $G$  be a group,  $H \leq G$  and  $\theta$  be a character of  $H$ . Then for  $g \in G$ , we define  $\theta^g : gHg^{-1} \rightarrow \mathbb{C}$  by  $\theta^g(t) = \theta(gtg^{-1})$  for all  $t \in gHg^{-1}$ . Then  $\theta^g$  is said to be a  $G$ -conjugate of  $\theta$ . If  $H$  is a normal subgroup of  $G$  and  $\theta^g = \theta$  for all  $g \in G$ , then  $\theta$  is said to be  $G$ -invariant.*

It is clear that  $\theta^g$  is a character of  $gHg^{-1}$ .

**Theorem 1.4.2 [28](Clifford's Theorem)** *Let  $G$  be a group,  $H$  a normal subgroup of  $G$  and  $\chi \in \text{Irr}(G)$ . Let  $\theta$  be an irreducible constituent of  $\chi_H$  and  $\theta_1, \theta_2, \dots, \theta_n$  be distinct conjugates of  $\theta$  in  $G$  such that  $\theta_1 = \theta$ . Then*

$$\chi_H = e \sum_{i=1}^n \theta_i, \quad \text{where } e = \langle \chi_H, \theta \rangle.$$

**Proof.** For  $h \in H$  we have

$$\theta^G(h) = \frac{1}{|H|} \sum_{x \in G} \theta^\circ(xhx^{-1}) = \frac{1}{|H|} \sum_{x \in G} \theta^x(h) \quad .$$

Thus we obtain that

$$(\theta^G)_H = \frac{1}{|H|} \sum_{x \in G} \theta^x \quad .$$

Let  $\phi \in \text{Irr}(H)$  such that  $\phi \notin \{\theta_i \mid 1 \leq i \leq n\}$ . Then we obtain that

$$\left\langle \sum_{x \in G} \theta^x, \phi \right\rangle = 0$$

and hence  $\langle (\theta^G)_H, \phi \rangle = 0$ . However by the Frobenius reciprocity theorem, we obtain that  $\langle \chi_H, \theta \rangle = \langle \chi, \theta^G \rangle$ . Hence  $\chi$  is an irreducible constituent of  $\theta^G$ . Since  $\langle (\theta^G)_H, \phi \rangle =$

0, then  $\langle \chi_H, \phi \rangle = 0$ . Thus  $\phi$  is not an irreducible constituent of  $\chi_H$ . Hence all the irreducible constituents of  $\chi_H$  are among the  $\theta_i$  and thus we obtain that

$$\chi_H = \sum_{i=1}^n \langle \chi_H, \theta_i \rangle \theta_i = \sum_{i=1}^n \langle \chi_H, \theta \rangle \theta_i = \langle \chi_H, \theta \rangle \sum_{i=1}^n \theta_i = e \sum_{i=1}^n \theta_i \quad ,$$

where  $e = \langle \chi_H, \theta \rangle$ .  $\square$

**Definition 1.4.3** *Let  $\phi$  be a representation of  $G$  and  $\alpha$  an automorphism of  $G$ . Then  $\phi^\alpha$  is a representation of  $G$  given by*

$$\phi^\alpha(x) = \phi(x^\alpha) \quad \text{and} \quad \phi^\alpha(xy) = \phi^\alpha(x)\phi^\alpha(y)$$

for  $x, y \in G$ . If the representation  $\phi$  affords a character  $\chi$  of  $G$ , then the representation  $\phi^\alpha$  affords a character  $\chi^\alpha$  of  $G$  which is given by  $\chi^\alpha(x) = \chi(x^\alpha)$  for  $x \in G$ . Then the representation  $\phi^\alpha$  and the character  $\chi^\alpha$  are called the **algebraic conjugates** of  $\phi$  and  $\chi$  respectively induced by the automorphism  $\alpha$ .

Let  $X = (\chi_i(x_j))$  be the character table of  $G$ , where  $\chi_i \in Irr(G)$ ,  $1 \leq i \leq n$  and  $x_j$ ,  $1 \leq j \leq n$  are representatives of the conjugacy classes of elements of  $G$ . Then the automorphism  $\alpha$  of  $G$  induces a permutation on the conjugacy classes of  $G$  and therefore also on the columns of  $X$ . For each  $\chi_i \in Irr(G)$ , we deduce that  $\chi_i^\alpha \in Irr(G)$ . Hence  $\alpha$  induces a permutation on the irreducible characters  $\chi_i$  of  $G$  and therefore also on the rows of  $X$ . Moreover since  $\chi_i^\alpha(x_j) = \chi_i(x_j^\alpha)$ , then the matrices obtained from  $X$  by these two operations are identical. We have the following result known as Brauer's Theorem.

**Theorem 1.4.4** [23](**Brauer's Theorem**) *Let  $G$  be a group and  $K$  be a group of automorphisms of  $G$ . Then the number of orbits of  $K$  as a group of permutations on the irreducible characters of  $G$  is the same as the number of orbits of  $K$  as a group of permutations on the conjugacy classes of  $G$ .*

**Proof.** Let  $X$  be the character table of  $G$ . Then as a matrix,  $X$  is square and nonsingular. Let  $\alpha$  be an automorphism of  $G$  such that  $\alpha \in K$ . Then  $\alpha$  induces a permutation on the conjugacy classes of  $G$  and thus induces a permutation on the columns of  $X$ . Hence  $K$  acts on the conjugacy classes of  $G$ . Since  $\alpha \in K$ , then to each character  $\chi$  of  $G$ , we obtain a character  $\chi^\alpha$  of  $G$  such that  $\chi^\alpha \in Irr(G)$  whenever

$\chi \in \text{Irr}(G)$ . For  $y \in G$ , we obtain that  $\chi^\alpha(y) = \chi(y^\alpha)$ . Thus  $\alpha$  induces a permutation on the rows of  $X$ . Hence  $K$  acts on the irreducible characters of  $G$ . Let  $X^\alpha$  denote the image of  $X$  under  $\alpha$ . Then we obtain that

$$P(\alpha)X = X^\alpha = XQ(\alpha) \quad ,$$

where  $P(\alpha), Q(\alpha)$  are appropriate permutation matrices which are uniquely determined by  $\alpha \in K$ . Suppose that  $\alpha, \beta \in K$ . Then we obtain that  $X^{\alpha\beta} = (X^\alpha)^\beta$ . Also we have that

$$P(\alpha\beta)X = X^{\alpha\beta} = (X^\alpha)^\beta = (P(\alpha)X)^\beta = P(\beta)P(\alpha)X$$

and hence  $P(\alpha\beta) = P(\beta)P(\alpha)$ . We also have that  $X^{\alpha\beta} = XQ(\alpha\beta)$  and  $(X^\alpha)^\beta = (XQ(\alpha))^\beta = XQ(\alpha)Q(\beta)$ . Since  $X^{\alpha\beta} = (X^\alpha)^\beta$ , we obtain that  $XQ(\alpha\beta) = XQ(\alpha)Q(\beta)$ . The non-singularity of  $X$  implies that  $Q(\alpha\beta) = Q(\alpha)Q(\beta)$ . Define mappings  $\pi_1$  and  $\pi_2$  on  $K$  by  $\pi_1(\alpha) = (P(\alpha))^t$  and  $\pi_2(\alpha) = Q(\alpha)$ , where  $t$  denotes the transpose operation on matrices. Then  $\pi_1$  and  $\pi_2$  are permutation representations of  $K$ . Let  $\theta_1$  and  $\theta_2$  be the permutation characters afforded by  $\pi_1$  and  $\pi_2$  respectively. Since  $X^{-1}P(\alpha)X = Q(\alpha)$ ,  $P(\alpha)$  and  $Q(\alpha)$  are similar and thus have the same trace. Since  $\text{trace}(P(\alpha))^t = \text{trace}(P(\alpha))$ , we have that  $\text{trace}(P(\alpha))^t = \text{trace}(Q(\alpha))$ . Hence  $\theta_1 = \theta_2$  and  $\pi_1$  and  $\pi_2$  are equivalent. Let  $d_1, d_2$  be the number of orbits of  $K$  on the irreducible characters and on the conjugacy classes of  $G$  respectively. Thus we observe that  $d_1$  is the number of orbits of  $\pi_1(K)$  in its action as a group of permutations. Also  $d_2$  is the number of orbits of  $\pi_2(K)$  in its action as a group of permutations. Since  $\theta_1$  is the permutation character of  $K$  acting on the irreducible characters of  $G$ , we obtain that  $\langle \theta_1, I_K \rangle = d_1$ . Also for  $\theta_2$ , we obtain that  $\langle \theta_2, I_K \rangle = d_2$ . However  $\theta_1 = \theta_2$  and thus  $\langle \theta_1, I_K \rangle = \langle \theta_2, I_K \rangle$  and hence  $d_1 = d_2$ .  $\square$

**Definition 1.4.5** Let  $\theta$  be a character of a subgroup  $H$  of a group  $G$ . Let

$$I_G(\theta) = \{g \in N_G(H) \mid \theta^g = \theta\}.$$

Then we call  $I_G(\theta)$  the inertia group of  $\theta$  in  $G$ . If  $H$  is normal in  $G$ , then

$$I_G(\theta) = \{g \in G \mid \theta^g = \theta\}.$$

We observe that  $N_G(H)$  acts on the characters of  $H$  by  $g : \theta \mapsto \theta^g$  for all  $g \in N_G(H)$ . Then the inertia group of  $\theta$  is the stabilizer of  $\theta$  in  $N_G(H)$ . Hence  $I_G(\theta) \leq N_G(H) \leq G$  and it is clear that  $H$  is a normal subgroup of  $I_G(\theta)$ .

**Theorem 1.4.6** [28] *Let  $G$  be a group,  $H$  a normal subgroup of  $G$ ,  $\theta \in \text{Irr}(H)$  and  $T = I_G(\theta)$ . Let  $A = \{\psi \in \text{Irr}(T) \mid \langle \psi_H, \theta \rangle \neq 0\}$  and  $B = \{\chi \in \text{Irr}(G) \mid \langle \chi_H, \theta \rangle \neq 0\}$ . Then*

- (a) *If  $\psi \in A$ , then  $\psi^G \in \text{Irr}(G)$ .*
- (b) *If  $\psi^G = \chi$  and  $\psi \in A$ , then  $\langle \psi_H, \theta \rangle = \langle \chi_H, \theta \rangle$ .*
- (c) *If  $\psi^G = \chi$  and  $\psi \in A$ , then  $\psi$  is the unique irreducible constituent of  $\chi_T$  which sits in  $A$ .*
- (d) *The map  $\psi \mapsto \psi^G$  is a bijection of  $A$  to  $B$ .*

**Proof.**

- (a) Let  $\psi \in A$  and  $\chi$  be an irreducible constituent of  $\psi^G$ . Then  $\psi$  is an irreducible constituent of  $\chi_T$ . Since  $\theta$  is an irreducible constituent of  $\psi_H$ ,  $\theta$  is an irreducible constituent of  $\chi_H$  and thus  $\chi \in B$ . Now suppose that  $\theta_1, \theta_2, \dots, \theta_n$  are the distinct conjugates of  $\theta$  in  $G$ , where  $\theta_1 = \theta$ . Then we obtain that  $[G : T] = n$  and from Clifford's theorem, we deduce that  $\chi_H = e \sum_{i=1}^n \theta_i$  for some  $e \in \mathbb{N}$ , where  $e = \langle \chi_H, \theta \rangle$ . Since  $\theta$  is invariant in  $T$ ,  $\theta$  is self-conjugate in  $T$ . Hence by Clifford's theorem (applied to  $T$ ,  $H$  and  $\psi$ ) we get that  $\psi_H = k\theta$  for some  $k \in \mathbb{N}$  where  $k = \langle \psi_H, \theta \rangle$ . Since  $\psi$  is an irreducible constituent of  $\chi_T$ , then we obtain that  $k \leq e$ . Hence we have

$$en\theta(1_H) = \chi(1_G) \leq \psi^G(1_G) = n\psi(1_T) = kn\theta(1_H) \leq en\theta(1_H)$$

and thus equality holds throughout. In particular, from this equality we obtain that  $\psi^G(1_G) = \chi(1_G)$  and hence we obtain that  $\psi^G = \chi$ . Therefore  $\psi^G \in \text{Irr}(G)$ .

- (b) We have that  $\langle \chi_H, \theta \rangle = e$  and  $\langle \psi_H, \theta \rangle = k$  and from the equality in part(a), we obtain that  $k = e$  and thus  $\langle \chi_H, \theta \rangle = \langle \psi_H, \theta \rangle$ .
- (c) Let  $\phi \in A, \phi \neq \psi$  and  $\phi$  be an irreducible constituent of  $\chi_T$ . Then we obtain that

$$\langle \chi_H, \theta \rangle \geq \langle (\phi + \psi)_H, \theta \rangle = \langle \phi_H, \theta \rangle + \langle \psi_H, \theta \rangle > \langle \psi_H, \theta \rangle$$

which is a contradiction by part(b). Hence the result.

- (d) The map  $\psi \mapsto \psi^G$  is well-defined by part(a). Also we obtain that for any  $\psi \in A$ ,  $\psi^G \in B$  by part(b). By the uniqueness assertion given by part(c), the map  $\psi \mapsto \psi^G$  is one-to-one. Then suffices to show that the map is onto  $B$ . Let  $\chi \in B$ . Then  $\theta$  is an irreducible constituent of  $\chi_H$  and hence there exists an irreducible constituent  $\psi$  of  $\chi_T$  such that  $\langle \psi_H, \theta \rangle \neq 0$ . Thus  $\psi \in A$  and we have that  $\chi$  is an irreducible constituent of  $\psi^G$ . Hence we obtain that  $\chi = \psi^G$  since  $\psi^G \in Irr(G)$  by part(a).  $\square$

**Definition 1.4.7** Let  $G$  be a group,  $H$  a normal subgroup of  $G$ ,  $\theta \in Irr(H)$  and  $T = I_G(\theta)$ . Since  $H$  is normal in  $T$ , we obtain the factor group  $T/H$  called the inertia factor of  $T$ .

Let  $\bar{G} = N:G$ . Then for all  $\theta \in Irr(N)$ , define

$$\bar{H} = \{x \in \bar{G} \mid \theta^x = \theta\} = I_{\bar{G}}(\theta)$$

$$H = \{y \in G \mid \theta^y = \theta\} = I_G(\theta).$$

Then it can be shown that  $\bar{H} = N:H$ .

**Definition 1.4.8** Let  $G$  be a group,  $H$  a subgroup of  $G$ ,  $\theta \in Irr(H)$  and  $\chi \in Irr(G)$  such that  $\chi_H = \theta$ . Then  $\theta$  is said to be **extendible** to an irreducible character of  $G$ .

If  $\theta$  is extendible to an irreducible character of  $G$ , we will say that  $\theta$  is extendible to  $G$ .

**Definition 1.4.9** Let  $G$  be a group and  $F$  be a field. Then the map  $\rho : G \rightarrow GL(n, F)$  such that

(i)  $\rho(1_G) = I$ , where  $I$  is the identity  $n \times n$  matrix,

(ii) for all  $x, y \in G$ , there exists a map  $\alpha : G \times G \rightarrow F^*$  such that

$$\rho(x)\rho(y) = \alpha(x, y)\rho(xy) \quad \text{where } \alpha(x, y) \in F^* .$$

Then  $\rho$  is called a **projective representation** of  $G$  over  $F$  of degree  $n$ . The map  $\alpha$  is called the **factor set** associated with  $\rho$ .

Now if  $\alpha(x, y) = 1_F$  for all  $x, y \in G$ , we obtain that  $\rho(xy) = \rho(x)\rho(y)$ , that is  $\rho$  is an ordinary representation of  $G$ . Thus projective representations are generalizations of ordinary representations.

**Theorem 1.4.10** [32] *Let  $N$  be a normal subgroup of  $G$ ,  $F = \mathbb{C}$ ,  $\chi \in \text{Irr}(N)$ , where  $\chi$  is  $G$ -invariant and let  $\Gamma$  be a matrix representation of  $N$  which affords  $\chi$ . Then*

- (i) *there exists a projective representation  $\rho$  of  $G$  such that  $\Gamma(n) = \rho(n)$  and  $(\rho(g))^{o(g)} = I$ , for all  $n \in N, g \in G$  where  $I$  is the identity matrix.*
- (ii) *If  $G = N.H$  for some  $H \leq G$  and if  $\rho_H$  is an ordinary representation of  $H$ , then  $\chi$  can be extended to  $G$ .*

**Proof.**

- (i) Let  $g \in G$ . Since  $\chi$  is  $G$ -invariant, then the representations  $\Gamma$  and  $\Gamma^g$  of  $N$  are equivalent. Hence there is an invertible matrix  $\theta(g)$  such that  $(\theta(g))^{-1}\Gamma(n)\theta(g) = \Gamma^g(n)$ , where  $g \in G, n \in N$ . We may assume that  $\theta(n) = \Gamma(n)$  for all  $n \in N$ . We have that  $\theta : G \rightarrow GL(k, F)$ , where  $k = \text{deg}(\Gamma)$ , and that  $\theta_N = \Gamma$ . Now let  $g_1, g_2 \in G$ , then we obtain that

$$\begin{aligned} (\theta(g_1g_2))^{-1}\Gamma(n)\theta(g_1g_2) &= \Gamma^{g_1g_2}(n) = (\Gamma^{g_1})^{g_2}(n) = (\theta(g_2))^{-1}\Gamma^{g_1}(n)\theta(g_2) \\ &= (\theta(g_2))^{-1}(\theta(g_1))^{-1}\Gamma(n)\theta(g_1)\theta(g_2). \end{aligned}$$

So that

$$\theta(g_1)\theta(g_2)(\theta(g_1g_2))^{-1}\Gamma(n) = \Gamma(n)\theta(g_1)\theta(g_2)(\theta(g_1g_2))^{-1} .$$

Thus for all  $n \in N$ ,  $\theta(g_1)\theta(g_2)(\theta(g_1g_2))^{-1}$  commutes with  $\Gamma(n)$  and thus by the Corollary 1.3.4, we can define a function  $\alpha : G \times G \rightarrow F^*$  such that  $\theta(g_1)\theta(g_2) = \alpha(g_1, g_2)\theta(g_1g_2)$ . Since  $\Gamma$  is a representation of  $N$ , we obtain that  $\theta(1_N) = \Gamma(1_N) = I$ . Hence  $\theta$  is a projective representation of  $G$  with associated factor set  $\alpha$ . Let  $o(g) = m$  and  $g \in N$ . Then we obtain that  $(\theta(g))^m = I$ . However if  $g \in G - N$ , then since  $\theta(g^m) = \theta(1_G) = I$ , there exists  $\lambda(g) \in F^*$  such that  $(\theta(g))^m = \lambda(g)I$ . Now let  $\mu(g) \in F^*$  such that  $(\mu(g))^m = (\lambda(g))^{-1}$  and let  $\mu(n) = 1_F$  for all  $n \in N$ . Then the projective representation  $\rho$  of  $G$  given by  $\rho(g) = \mu(g)\theta(g)$  is such that  $\rho(n) = \mu(n)\theta(n) = \theta(n) = \Gamma(n)$  for all  $n \in N$ . Also we have that

$$(\rho(g))^m = (\mu(g)\theta(g))^m = (\mu(g))^m(\theta(g))^m = (\lambda(g))^{-1}\lambda(g)I = I .$$

Hence property (i) is established.

- (ii) Let  $T$  be a transversal for  $N \cap H$  in  $H$  containing  $1_H$ . Then every  $g \in G$  has a unique expression of the form  $g = tn$ , where  $t \in T, n \in N$ . Now let  $g_1 \in G, g_1 \neq g$  be given by  $g_1 = t_1n_1$ , where  $t_1 \in T, n_1 \in N$ . Since  $t, t_1 \in T$ , then  $t, t_1 \in H$  and hence  $tt_1 \in H$ . Now let  $tt_1 = t_2n_2$ , where  $t_2 \in T$  and  $n_2 \in N \cap H$ . Define  $\psi$  on  $G$  by  $\psi(g) = \rho(t)\rho(n)$ . Since  $n_2t_1^{-1}nt_1n_1 \in N$ , we obtain that

$$\psi(gg_1) = \psi(tnt_1n_1) = \psi(tt_1t_1^{-1}nt_1n_1) = \psi(t_2n_2t_1^{-1}nt_1n_1) = \rho(t_2)\rho(n_2t_1^{-1}nt_1n_1) .$$

Also we have

$$\begin{aligned} \psi(g)\psi(g_1) &= \rho(t)\rho(n)\rho(t_1)\rho(n_1) = \rho(t)\rho(t_1)(\rho(t_1))^{-1}\rho(n)\rho(t_1)\rho(n_1) \\ &= \rho(t)\rho(t_1)[(\rho(t_1))^{-1}\rho(n)\rho(t_1)]\rho(n_1). \end{aligned}$$

However from the proof of part(i) above we have that  $(\rho(g))^{-1}\Gamma(n)\rho(g) = \Gamma^g(n)$  and  $\rho(n) = \Gamma(n)$  for all  $n \in N, g \in G$ . Since  $t_1^{-1}nt_1 \in N$ , then we obtain that

$$\rho(t_1^{-1}nt_1) = \Gamma(t_1^{-1}nt_1) = \Gamma^{t_1}(n) = (\rho(t_1))^{-1}\Gamma(n)\rho(t_1) = (\rho(t_1))^{-1}\rho(n)\rho(t_1) .$$

Since by the assumption  $\rho$  is an ordinary representation on  $H$  and  $tt_1 \in H$  we have  $\rho(tt_1) = \rho(t)\rho(t_1)$ . We deduce that

$$\begin{aligned} \psi(g)\psi(g_1) &= \rho(t)\rho(n)\rho(t_1)\rho(n_1) \\ &= \rho(t)\rho(t_1)(\rho(t_1))^{-1}\rho(n)\rho(t_1)\rho(n_1) \\ &= \rho(t)\rho(t_1)[(\rho(t_1))^{-1}\rho(n)\rho(t_1)]\rho(n_1) \\ &= \rho(t)\rho(t_1)\rho(t_1^{-1}nt_1)\rho(n_1) = \rho(tt_1)\rho(t_1^{-1}nt_1)\rho(n_1) \\ &= \rho(t_2n_2)\rho(t_1^{-1}nt_1)\rho(n_1) = \rho(t_2)\rho(n_2t_1^{-1}nt_1n_1). \end{aligned}$$

Hence we obtain that  $\psi(gg_1) = \psi(g)\psi(g_1)$ . Therefore  $\psi$  is an ordinary representation of  $G$ . However  $\forall n \in N$ , we obtain that  $\psi(n) = \rho(n) = \Gamma(n)$  and thus the character afforded by the representation  $\psi$  of  $G$ , extends  $\chi$  to  $G$ .  $\square$

**Theorem 1.4.11** ([13])(Mackey's Theorem) *Let  $N$  be a normal subgroup of  $\overline{G}$  and  $\theta$  be a  $\overline{G}$ -invariant irreducible character of  $N$ . If  $N$  is abelian and  $\overline{G}$  splits over  $N$ , then  $\theta$  can be extended to  $\overline{G}$ .*

**Proof.** Let  $\overline{G} = N:G$ . Since  $\overline{G}$  is a semidirect product of  $N$  by  $G$ , then any  $x \in \overline{G}$  can be expressed uniquely as  $x = ng$ , where  $n \in N, g \in G$ . Define  $\chi$  on  $\overline{G}$  by  $\chi(ng) = \theta(n)$ . Since  $N$  is abelian,  $\theta$  has degree 1 and thus is linear. The invariance of  $\theta$  in  $\overline{G}$  implies that  $\theta(n) = \theta(xnx^{-1})$  for all  $x \in \overline{G}$ . Now let  $x_1 = n_1g_1, x_2 = n_2g_2$  be elements of  $\overline{G}$ . Then we obtain that

$$\begin{aligned}\chi(x_1x_2) &= \chi(n_1g_1n_2g_2) = \chi(n_1n_2^{g_1}g_1g_2) = \theta(n_1n_2^{g_1}) \\ &= \theta(n_1)\theta(n_2^{g_1}) = \theta(n_1)\theta(n_2) = \chi(x_1)\chi(x_2).\end{aligned}$$

Therefore  $\chi$  is a linear character of  $\overline{G}$  such that  $\chi_N = \theta$ .  $\square$

**Theorem 1.4.12** [32] *Let  $\overline{G} = N:G$  where  $N$  is a normal subgroup of  $\overline{G}$ , and  $G \leq \overline{G}$  such that  $N \cap G \subseteq N'$ . If  $\theta$  is an irreducible  $\overline{G}$ -invariant character of  $N$  such that  $(\deg(\theta), |G|) = 1$ , then  $\theta$  can be extended to  $\overline{G}$ .*

**Proof.** See Corollary 7.1.2 in [32].  $\square$

**Theorem 1.4.13** *Suppose  $\overline{G}$  is a splitting extension of a normal subgroup  $N$ , then any linear character  $\theta \in \text{Irr}(N)$  can be extended to its inertia group  $I_{\overline{G}}(\theta)$ .*

**Proof.** Let  $\overline{G} = N:G$  and  $\theta \in \text{Irr}(N)$  be linear. Let  $\overline{H} = I_{\overline{G}}(\theta)$ , then we obtain that  $\overline{H} = N:H$ , where  $H = I_G(\theta)$ . Since  $\overline{H}$  is a split extension, we obtain that  $N \cap H = \{1\} \leq N'$ . Also we have that  $(\deg(\psi), |H|) = (1, |H|) = 1$  and clearly  $\theta$  is  $\overline{H}$ -invariant. Thus the conditions of Theorem 1.4.12 are satisfied and hence  $\theta$  can be extended to  $\overline{H}$ .  $\square$

**Theorem 1.4.14** ([28],[65])(**Gallagher's Theorem**) *Let  $N$  a normal subgroup of  $\overline{G}$ ,  $\theta \in \text{Irr}(N)$  and  $\overline{H} = I_{\overline{G}}(\theta)$ . If  $\theta$  can be extended to  $\psi \in \text{Irr}(\overline{H})$  then as  $\beta$  ranges over all the irreducible characters of  $\overline{H}$  which contain  $N$  in their kernels,  $\beta\psi$  ranges over all the irreducible characters of  $\overline{H}$  which contain  $\theta$  in their restriction to  $N$ .*

**Proof.** Since  $\overline{H} = I_{\overline{G}}(\theta)$ , then  $\theta$  is self-conjugate in  $\overline{H}$  and thus by Clifford's theorem we obtain that  $(\theta^{\overline{H}})_N = f\theta$  for some positive integer  $f$ . Comparing degrees we have  $(\theta^{\overline{H}})_N = [\overline{H} : N]\theta$  and so  $\langle \theta^{\overline{H}}, \theta^{\overline{H}} \rangle = \langle \theta, (\theta^{\overline{H}})_N \rangle = [\overline{H} : N]$ . Now we claim that  $\theta^{\overline{H}} = \sum_{\beta} \beta(1_{\overline{G}})\beta\psi$ , where  $\beta$  ranges over all the irreducible characters of  $\overline{H}$  that contain  $N$  in their kernels. Both  $\theta^{\overline{H}}$  and  $\sum_{\beta} \beta(1_{\overline{G}})\beta\psi$  are zero off  $N$  since for  $g \notin N, xgx^{-1} \notin N$  for all  $x \in \overline{G}$  and thus  $\theta^{\overline{H}}(g) = 0$ . Also for  $g \notin N$ , by the orthogonality

of the columns of the character table of  $\overline{H}/N$  we have that  $\sum_{\beta} \beta(1_{\overline{G}})(\beta\psi)(g) = [\sum_{\beta} \beta(1_{\overline{G}})\beta(g)]\psi(g) = 0$ . We also have that  $(\theta^{\overline{H}})_N = [\overline{H} : N]\theta = (\sum_{\beta} \beta(1_{\overline{G}})\beta\psi)_N$ , since for  $g \in N$ ,  $\sum_{\beta} \beta(1_{\overline{G}})\beta(g)\psi(g) = \sum_{\beta} (\beta(1_{\overline{G}}))^2\psi(g) = [\overline{H} : N]\psi(g) = [\overline{H} : N]\theta(g)$ . Hence we obtain that  $\theta^{\overline{H}} = \sum_{\beta} \beta(1_{\overline{G}})\beta\psi$ . So we have

$$[\overline{H} : N] = \langle \theta^{\overline{H}}, \theta^{\overline{H}} \rangle = \langle \sum_{\beta} \beta(1_{\overline{G}})\beta\psi, \sum_{\tau} \tau(1_{\overline{G}})\tau\psi \rangle = \sum_{\beta, \tau} \beta(1_{\overline{G}})\tau(1_{\overline{G}})\langle \beta\psi, \tau\psi \rangle .$$

The diagonal terms contribute at least  $\sum (\beta(1_{\overline{G}}))^2 = [\overline{H} : N]$ , so the  $\beta\psi$  are irreducible and distinct, and are all the irreducible constituents of  $\theta^{\overline{H}}$  and so are all the irreducible characters of  $\overline{H}$  that contain  $\theta$  in their restriction to  $N$ , since for  $\phi \in Irr(\overline{H})$  such that  $\langle \phi_N, \theta \rangle \neq 0$ , we obtain that  $\langle \phi_N, \theta \rangle = \langle \phi, \theta^{\overline{H}} \rangle$  which implies that  $\phi$  is an irreducible constituent of  $\theta^{\overline{H}}$  and hence is of the form  $\beta\psi$ .  $\square$

## 1.5 Fischer-Clifford Matrices

Let  $\overline{G} = N.G$  such that every irreducible character of  $N$  is extendible to its inertia group. We have that  $\overline{G}$  permutes  $Irr(N)$  by  $x : \chi \mapsto \chi^x$ , where  $x \in \overline{G}$  and  $\chi \in Irr(N)$ . Now let  $\chi_1, \chi_2, \dots, \chi_t$  be representatives of the orbits of  $\overline{G}$  on  $Irr(N)$ ,  $\overline{H}_i = I_{\overline{G}}(\chi_i)$ ,  $1 \leq i \leq t$ ,  $\psi_i \in Irr(\overline{H}_i)$  be an extension of  $\chi_i$  to  $\overline{H}_i$  and  $\psi \in Irr(\overline{H}_i)$  such that  $N \subseteq ker(\psi)$ . Then by Theorem 1.4.14 and Theorem 1.4.6 all irreducible characters of  $\overline{G}$  will be of the form  $(\psi_i\psi)^{\overline{G}}$ ,  $1 \leq i \leq t$ . Thus

$$Irr(\overline{G}) = \bigcup_{i=1}^t \{(\psi_i\psi)^{\overline{G}} \mid \psi \in Irr(\overline{H}_i), N \subseteq ker(\psi)\} .$$

Therefore the irreducible characters of  $\overline{G}$  fall into blocks, each block corresponding to an inertia group  $\overline{H}_i$ .

### 1.5.1 Defining Fischer-Clifford Matrices

Let  $\overline{G} = N.G$  with the property that every irreducible character of  $N$  is extendible to its inertia group. Let  $\overline{g} \in \overline{G}$  be a lifting of  $g \in G$  under the natural homomorphism  $\overline{G} \rightarrow G$  and  $[g]$  be a conjugacy class of  $G$  with representative  $g$ . Let  $X(g) = \{x_1, x_2, \dots, x_{c(g)}\}$  be a set of representatives of the conjugacy classes of  $\overline{G}$  from the coset  $N\overline{g}$  whose images under the natural homomorphism  $\overline{G} \rightarrow G$  are in  $[g]$  and we take  $x_1 = \overline{g}$ . Let  $\{\chi_1, \chi_2, \dots, \chi_t\}$  be a set of representatives of the orbits of  $\overline{G}$  on

$Irr(N)$  such that for  $1 \leq i \leq t$ , we have  $\overline{H}_i = I_{\overline{G}}(\chi_i)$  with  $H_i = \overline{H}_i/N \leq G$  and that  $\psi_i \in Irr(\overline{H}_i)$  is an extension of  $\chi_i$  to  $\overline{H}_i$ . Then without loss of generality suppose that  $\chi_1 = I_N$  is the identity character of  $N$ . Then  $\overline{H}_1 = \overline{G}$  and  $H_1 = G$ . Now choose  $y_1, y_2, \dots, y_r$  to be the representatives of the conjugacy classes of  $H_i$  which fuse into  $[g]$  in  $G$ . Since  $y_k \in H_i$  for  $1 \leq k \leq r$ , then we define  $y_{\ell_k} \in \overline{H}_i$  such that  $y_{\ell_k}$  ranges over all the representatives of the conjugacy classes of  $\overline{H}_i$  which map to  $y_k$  under the homomorphism  $\overline{H}_i \rightarrow H_i$  whose kernel is  $N$ . Let  $\psi \in Irr(\overline{H}_i)$  such that  $N \subseteq \ker(\psi)$ . Then  $\psi$  is a lifting of  $\hat{\psi} \in Irr(H_i)$  such that  $\psi(y_{\ell_k}) = \hat{\psi}(y_k)$  for any lifting  $y_{\ell_k} \in \overline{H}_i$  of  $y_k \in H_i$ . We obtain that

$$\begin{aligned} (\psi_i \psi)^{\overline{G}}(x_j) &= \sum_{1 \leq k \leq r} \sum_{\ell}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|} \psi_i \psi(y_{\ell_k}) \\ &= \sum_{1 \leq k \leq r} \sum_{\ell}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|} \psi_i(y_{\ell_k}) \psi(y_{\ell_k}) \\ &= \sum_{1 \leq k \leq r} \left( \sum_{\ell}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|} \psi_i(y_{\ell_k}) \right) \hat{\psi}(y_k) \end{aligned}$$

where  $\sum_{\ell}'$  is the summation over all  $\ell$  for which  $y_{\ell_k} \sim x_j$  in  $\overline{G}$ . Now we define a matrix  $M_i(g)$  by  $M_i(g) = (a_{uv})$ , where  $1 \leq u \leq r$  and  $1 \leq v \leq c(g)$ , and

$$a_{uv} = \sum_{\ell}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|} \psi_i(y_{\ell_k}) \quad .$$

Then we obtain that

$$(\psi_i \psi)^{\overline{G}}(x_j) = \sum_{1 \leq k \leq r} a_{uv} \hat{\psi}(y_k) \quad .$$

By doing this for all  $1 \leq i \leq t$  such that  $H_i$  contains an element in  $[g]$  we obtain the matrix  $M(g)$  given by

$$M(g) = \begin{bmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g) \end{bmatrix} ,$$

where  $M_i(g)$  is the sub-matrix corresponding to the inertia group  $\overline{H}_i$  and its inertia factor  $H_i$ . If  $H_i \cap [g] = \emptyset$ , then  $M_i(g)$  will not exist and  $M(g)$  does not contain  $M_i(g)$ . The size of the matrix  $M(g)$  is  $p \times c(g)$  where  $p$  is the number of conjugacy classes of the inertia factors  $H_i$ 's for  $1 \leq i \leq t$  which fuse into  $[g]$  in  $G$  and  $c(g)$  is the number of

conjugacy classes of  $\overline{G}$  which correspond to the coset  $N\overline{g}$ . The matrix  $M(g)$  is called the **Fischer-Clifford matrix** of  $\overline{G}$  corresponding to the coset  $N\overline{g}$ . Let

$$R(g) = \{(i, y_k) \mid 1 \leq i \leq t, H_i \cap [g] \neq \emptyset, 1 \leq k \leq r\}.$$

We note that  $y_k$  runs over representatives of the conjugacy classes of  $H_i$  which fuse into  $[g]$  in  $G$ . Following the notation used in [18], [47] and [65] we write  $M(g) = (a_j^{(i, y_k)})$ , where

$$a_j^{(i, y_k)} = \sum_{\ell} \frac{|C_{\overline{G}}(x_j)|}{|C_{H_i}(y_{\ell_k})|} \psi_i(y_{\ell_k}),$$

with columns indexed by  $X(g)$  and rows indexed by  $R(g)$ . Then the partial character table of  $\overline{G}$  on the classes  $\{x_1, x_2, \dots, x_{c(g)}\}$  is given by

$$\begin{bmatrix} C_1(g)M_1(g) \\ C_2(g)M_2(g) \\ \vdots \\ C_t(g)M_t(g) \end{bmatrix}$$

where the Fischer-Clifford matrix  $M(g)$  is divided into blocks  $M_i(g)$  corresponding to an inertia group  $\overline{H}_i$  and  $C_i(g)$  is the partial character table of  $H_i$  consisting of the columns corresponding to the classes that fuse into  $[g]$  in  $G$ . We also see that the number of irreducible characters of  $\overline{G}$  is the sum of the numbers of irreducible characters of the inertia factors  $H_i$ 's.

There is yet another method for constructing the Fischer-Clifford matrices of an extension of an elementary abelian  $p$ -group (see [18] and [35]). This method will be used later to construct the Fischer-Clifford matrices of groups associated with  $B(p, n)$ , where  $p$  is prime, therefore we describe it below. Suppose that  $N$  is an elementary abelian  $p$ -group, that is  $N = p^n$  is a vector space over  $\mathbb{Z}_p$ . Let  $\overline{g} \in \overline{G}$  and  $d \in N$ . Then the map  $\phi_{\overline{g}} : N \rightarrow N$  such that  $\phi_{\overline{g}}(d) = d(d^{-1})^{\overline{g}}$  defines an endomorphism of  $N$ . It is clear that  $Im(\phi_{\overline{g}})$  and  $ker(\phi_{\overline{g}})$  are subspaces (subgroups) of  $N$ . Let  $C_{\overline{g}} = \{x \in \overline{G} \mid x(N\overline{g}) = (N\overline{g})x\} = N.C_{\overline{G}}(N\overline{g})$  as defined earlier.

**Lemma 1.5.1** *The subspaces  $Im(\phi_{\overline{g}})$  and  $ker(\phi_{\overline{g}})$  are  $C_{\overline{g}}$ -invariant.*

**Proof.** We show that  $Im(\phi_{\overline{g}}) \triangleleft C_{\overline{g}}$ , that is for  $u \in C_{\overline{g}}$  and  $d(d^{-1})^{\overline{g}} = [d, \overline{g}] \in Im(\phi_{\overline{g}})$  we obtain that  $[d, \overline{g}]^u \in Im(\phi_{\overline{g}})$ . Thus

$$\begin{aligned}
[d, \bar{g}]^u &= u[d, \bar{g}]u^{-1} \\
&= d^u \bar{g}^u (d^{-1})^u (\bar{g}^{-1})^u \\
&= d^u \bar{g}^u (d^u)^{-1} (\bar{g}^u)^{-1} \\
&= [d^u, \bar{g}^u] \in \text{Im}(\phi_{\bar{g}}),
\end{aligned}$$

that is  $\text{Im}(\phi_g)$  is  $C_{\bar{g}}$ -invariant. Now since  $N$  and  $N/\text{Ker}(\phi_{\bar{g}})$  ( $\cong \text{Im}(\phi_{\bar{g}})$ ) are  $C_{\bar{g}}$ -invariant, we conclude that  $\text{Ker}(\phi_{\bar{g}})$  is  $C_{\bar{g}}$ -invariant.  $\square$

We see that  $N$  acts on  $N\bar{g}$  by conjugation and  $K = \text{Im}(\phi_{\bar{g}})$  acts on  $N\bar{g}$  by left multiplication such that the resulting orbits of the two actions are the same. Hence the action of  $C_{\bar{g}}$  on the orbits of  $N$  acting on  $N\bar{g}$  is the same as the action of  $C_{\bar{g}}$  on the module  $N/K$ . Thus the orbits of the action of  $K$  on  $N\bar{g}$  can be identified with the elements of  $N/K$ . Let  $\theta_i \in \text{Irr}(N)$ ,  $\psi_i \in \text{Irr}(\overline{H}_i)$  and  $\psi_i$  be an extension of  $\theta_i$  to  $\overline{H}_i$ . Then  $\psi_i$  is constant on the orbits of  $N$  acting on  $N\bar{g}$ . So we may define a class function  $\mu$  on  $N/K$  by  $\mu(Kd_j\bar{g}) = \psi_i(d_j\bar{g})$ , where  $d_j \in N$ ,  $d_j\bar{g} \in Q_j$  is a representative of the  $j$ -th orbit of  $N$  acting on  $N\bar{g}$  and  $d_1 = 1_N$ . Then  $\mu(K\bar{g}) = \psi_i(\bar{g})$ . Let  $\hat{\mu}$  be an extension of  $\mu$  to the inertia group of  $\mu$  in  $C_{\bar{g}}$ . Then induction of  $\hat{\mu}$  to  $\overline{G}$  evaluated on the elements of  $N\bar{g}$  is equivalent to the induction of  $\hat{\mu}$  to  $C_{\bar{g}}/K$  evaluated on the elements of  $N/K$ . However for  $\overline{G}$  a non-split extension, it may happen that  $\mu$  is not a character of  $N/K$ . In that case  $\xi\mu$  will be a character of  $N/K$ , where  $\xi$  is an appropriate  $p$ -th root of unity.

**Proposition 1.5.2** *Let  $N$  be an elementary abelian  $p$ -group. Let  $\bar{g} \in \overline{G}$  and  $d \in N$ . Define an endomorphism  $\phi_{\bar{g}}$  of  $N$  by  $\phi_{\bar{g}}(d) = d(d^{-1})^{\bar{g}}$ , with image  $K = \text{Im}(\phi_{\bar{g}})$ .*

- (i) *If  $\overline{G}$  is a split extension, then the Fischer-Clifford matrix at a non-identity coset of  $N$  in  $\overline{G}$  is the matrix of orbit sums of  $C_{\bar{g}}$  acting on the rows of the character table of  $N/K$  with duplicate columns discarded.*
- (ii) *If  $\overline{G}$  is a non-split extension, then the Fischer-Clifford matrix is the matrix of orbit sums of  $C_{\bar{g}}$  acting on the rows of the character table of  $N/K$  with duplicate columns discarded and with each row multiplied by an appropriate  $p$ -th root of unity.*

**Proof.** See Lemma 5.3 in [18] or page 119 in [35].  $\square$

### 1.5.2 Properties of Fischer-Clifford Matrices

The properties of Fischer-Clifford matrices have been discussed in [1], [35], [36], [47], [51], [65], we also discuss them in this subsection. Let  $K$  be a group and  $A \leq \text{Aut}(K)$ . Then by Theorem 1.4.4 the group  $A$  acts on the conjugacy classes of elements of  $K$  and on the irreducible characters of  $K$  resulting in the same number of orbits.

**Lemma 1.5.3** *Suppose we have the following matrix describing the above actions:*

$$\begin{array}{c} 1 = l_1 \quad l_2 \quad \cdots \quad l_j \quad \cdots \quad l_t \\ \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_i \\ \vdots \\ s_t \end{array} \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2t} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{it} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tj} & \cdots & a_{tt} \end{pmatrix} \end{array}$$

where  $a_{1j} = 1$  for  $j \in \{1, 2, \dots, t\}$ ,  $l_j$ 's are lengths of orbits of  $A$  on the conjugacy classes of  $K$ ,  $s_i$ 's are lengths of orbits of  $A$  on  $\text{Irr}(K)$  and  $a_{ij}$  is the sum of  $s_i$  irreducible characters of  $K$  on the element  $x_j$ , where  $x_j$  is an element of the orbit of length  $l_j$ . Then the following relation holds for  $i, i' \in \{1, 2, \dots, t\}$ :

$$\sum_{j=1}^t a_{ij} \overline{a_{i'j}} l_j = |K| s_i \delta_{ii'}.$$

**Proof.** See Lemma 4.2.2 in [65].  $\square$

Let  $x_j \in X(g)$  and define  $m_j = [C_{\overline{G}} : C_{\overline{G}}(x_j)]$ . The Fischer-Clifford matrix  $M(g)$  is partitioned row-wise into blocks, where each block corresponds to an inertia group. The columns of  $M(g)$  are indexed by  $X(g)$  and to each  $x_j \in X(g)$ , corresponds  $|C_{\overline{G}}(x_j)|$  of a conjugacy class of  $\overline{G}$ . The rows of  $M(g)$  are indexed by  $R(g)$  and on the left of each row we write  $|C_{H_i}(y_k)|$ , where  $y_k$  fuses into  $[g]$  in  $G$ . The following result gives the orthogonality relation for  $M(g)$ .

**Proposition 1.5.4** [65] (*Column orthogonality*) *Let  $\overline{G} = N.G$ , then*

$$\sum_{(i, y_k) \in R(g)} |C_{H_i}(y_k)| a_j^{(i, y_k)} \overline{a_{j'}^{(i, y_k)}} = \delta_{jj'} |C_{\overline{G}}(x_j)|.$$

**Proof.** See Proposition 4.2.3 in [65].  $\square$

**Theorem 1.5.5**  $a_j^{(1,g)} = 1$  for all  $j = \{1, 2, \dots, c(g)\}$

**Proof.** For  $y_{\ell_k} \sim x_j$  in  $\overline{G}$ , we have  $|C_{\overline{G}}(x_j)| = |C_{\overline{H}_1}(y_{\ell_k})|$ . Thus we obtain that

$$a_j^{(1,g)} = \sum_{\ell} \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_1}(y_{\ell_k})|} \psi_1(y_{\ell_k}) = \sum_{\ell} 1 = 1 \quad . \quad \square$$

**Proposition 1.5.6** ([35], [65], [47]) *The matrix  $M(1_G)$  is the matrix with rows equal to the orbit sums of the action of  $\overline{G}$  on  $\text{Irr}(N)$  with duplicate columns discarded. For this matrix we have  $a_j^{(i,1_G)} = [G : H_i]$ , and an orthogonality relation for rows:*

$$\sum_{j=1}^t \frac{1}{|C_{\overline{G}}(x_j)|} a_j^{(i,1_G)} a_j^{(i',1_G)} = \frac{1}{|C_{H_i}(1_G)|} \delta_{ii'} = \frac{1}{|H_i|} \delta_{ii'} \quad .$$

**Proof.** The  $(i, 1_G), j^{\text{th}}$  entry of  $M(1_G)$  is given by

$$a_j^{(i,1_G)} = \sum_{\ell} \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{\ell_k})|} \psi_i(y_{\ell_k}) \quad ,$$

where we sum over representatives of conjugacy classes of  $\overline{H}_i$  which fuse into  $[x_j]$  in  $\overline{G}$ . Therefore  $a_j^{(i,1_G)} = \psi_i^{\overline{G}}(x_j)$ . By Theorem 1.4.6 we have  $\psi_i^{\overline{G}} \in \text{Irr}(\overline{G})$  and we obtain that  $\langle (\psi_i^{\overline{G}})_N, \theta_i \rangle = \langle (\psi_i)_N, \theta_i \rangle = 1$ . Therefore by Clifford's theorem  $(\psi_i^{\overline{G}})_N = \sum_{\alpha} \theta_{\alpha}$ , where the summation is taken over all  $\theta_{\alpha} \in \text{Irr}(N)$  such that  $\theta_{\alpha}$  is conjugate to  $\theta_i$ . So for  $x_j \in N$  we obtain that  $a_j^{(i,1_G)} = \sum_{\alpha} \theta_{\alpha}(x_j)$ . The orthogonality relation follows by Lemma 1.5.3.  $\square$

Following from Lemma 1.5.3, Proposition 1.5.4 and the results proved by Fischer in [18], the Fischer-Clifford matrix  $M(g)$  satisfies the following properties:

- (a)  $|X(g)| = |R(g)|$
- (b)  $\sum_{j=1}^{c(g)} m_j a_j^{(i,y_k)} \overline{a_j^{(i',y'_k)}} = \delta_{(i,y_k),(i',y'_k)} \frac{|C_G(g)|}{|C_{H_i}(y_k)|} |N|$
- (c)  $\sum_{(i,y_k) \in R(g)} a_j^{(i,y_k)} \overline{a_{j'}^{(i,y_k)}} |C_{H_i}(y_k)| = \delta_{jj'} |C_{\overline{G}}(x_j)|$
- (d)  $M(g)$  is square and nonsingular.

For  $N$  is elementary abelian,  $M(g)$  also satisfies the following

$$(e) \ a_1^{(i, y_k)} = \frac{|C_G(g)|}{|C_{H_i}(y_k)|}$$

$$(f) \ |a_1^{(i, y_k)}| \geq |a_j^{(i, y_k)}|, 1 < j \leq c(g).$$

Now let  $\bar{G} = N:G$  be a split extension and  $N$  be an elementary abelian 2-group. By Proposition 1.5.2 the Fischer-Clifford matrix  $M(g)$  is given by

$$M(g) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2j} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ d_{i1} & d_{i2} & d_{i3} & \cdots & d_{ij} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ d_{t1} & d_{t2} & d_{t3} & \cdots & d_{tj} & \cdots & \end{pmatrix}.$$

where  $d_{ij}$ 's are the orbit sums of  $C_{\bar{g}}$  acting on the rows of the character table of  $N/K$ .

**Proposition 1.5.7** ([47])  $d_{i1} \in \mathbb{N}$  for all  $i \in \{2, 3, \dots, t\}$ .

**Proof.** By Proposition 1.5.2, we obtain that

$$d_{i1} = \sum_{\chi \in \Delta_i} \chi(1_{N/K})$$

where  $\Delta_i$ 's are the orbits of  $C_g$  acting on  $\text{Irr}(N/K)$ . Since  $\chi(1_{N/K}) = \text{deg}(\chi)$ , we have  $d_{i1} \in \mathbb{N} \ \forall i \in \{2, 3, \dots, t\}$ .  $\square$

For  $j \geq 2$ , we obtain that

$$d_{ij} = \sum_{\chi \in \Delta_i} \chi(\bar{x}_j)$$

where  $\bar{x}_j \in N/K$  is a representative of the  $j$ -th orbit under the action of  $C_g$  on the elements of  $N/K$ . Since  $\chi(\bar{x}_j) \in \{-1, 1\}$ , we have  $d_{ij} \in \mathbb{Z}$ .

## Chapter 2

# Combinatorics

As is well known, the ordinary irreducible representations of the symmetric group  $S_n$  may be described combinatorially, that is in terms of partitions of a positive integer  $n$  (see [13], [29] and [66]). Since the generalized symmetric group is an extension of the symmetric group, we note that the ordinary irreducible representations of the generalized symmetric group will somehow also be described in terms of partitions of  $n$ . In this chapter we give the definition of a special partition of a positive integer  $n$  called an  **$m$ -set of partition**  $[\lambda]$  of  $n$  (see [66]), elsewhere called an  $r$ -tuple of partitions of  $n$  (see for example section 4 in [50]). We also define some combinatorial objects called **magic matrices**, which will be used in Chapter 4 to construct the Fischer-Clifford matrices of the generalized symmetric groups. As indicated in Chapter 1, the Fischer-Clifford matrices of the generalized symmetric group are used to construct its character table. For details on combinatorics, the reader may consult Stanley ([62], [63]) and other relevant references.

Let  $m \in \mathbb{N}$ , the set of positive integers. Let  $\mathbb{Z}_m = \{0, 1, \dots, m-1\}$  be the set of residues modulo  $m$ , also considered as the cyclic group of order  $m$ .

### 2.1 Compositions

Usually, a composition of a positive integer  $n$  is defined as an ordered collection of positive integers whose sum is  $n$ . However for our purposes we give the following modified definition.

**Definition 2.1.1** Let  $m, n \in \mathbb{N}$ .

(i) An  **$m$ -composition** of  $n$  is an  $m$ -tuple  $(a_1, a_2, \dots, a_m)$  such that  $\sum_{j=1}^m a_j = n$ , where  $a_j \in \mathbb{N} \cup \{0\}$ .

(ii) Let  $(a_1, a_2, \dots, a_m)$  be an  $m$ -composition of  $n$ . The expression

$$\binom{n}{a_1 \ a_2 \ \dots \ a_m} = \frac{n!}{a_1! a_2! \cdots a_m!}$$

is called a **multi-nomial coefficient**.

Let  $A(n, m)$  denote the set of  $m$ -compositions of  $n$ .

**Proposition 2.1.2** ([62]) The number  $N(n, m)$  of  $m$ -compositions of  $n$  is given by

$$N(n, m) = \binom{n+m-1}{m-1}.$$

**Proof.** We consider successive values of  $m$  for a fixed value of  $n$  as follows.

- $m = 1$ : In this case we have only one 1-composition of  $n$  namely  $(n)$ .
- $m = 2$ : In this case we produce  $n + 1 = \binom{n+1}{1}$  distinct 2-compositions of  $n$  namely  $(n, 0), (n-1, 1), (n-2, 2), (n-3, 3), \dots, (1, n-1), (0, n)$ .
- $m = 3$ : Using the 2-compositions of  $n$  listed above, by breaking the first entry of each 2-composition of  $n$  we have:
  - $(n, 0)$  produces  $n + 1$  distinct 3-compositions of  $n$  namely  $(n, 0, 0), (n-1, 1, 0), (n-2, 2, 0), \dots, (1, n-1, 0), (0, n, 0)$ .
  - $(n-1, 1)$  produces  $n$  distinct 3-compositions of  $n$  namely  $(n-1, 0, 1), (n-2, 1, 1), (n-3, 2, 1), \dots, (1, n-2, 1), (0, n-1, 1)$ .
  - $(n-2, 2)$  produces  $n-1$  distinct 3-compositions of  $n$  namely  $(n-2, 0, 2), (n-3, 1, 2), (n-4, 2, 2), \dots, (1, n-3, 2), (0, n-2, 2)$ .
  - $\vdots$
  - $(0, n)$  produces only one 1-composition of  $n$  namely  $(0, 0, n)$ .

Thus for  $m = 3$  we obtain

$$\begin{aligned} (n+1) + n + (n-1) + (n-2) + \dots + 2 + 1 &= \frac{1}{2}(n+1)(n+2) \\ &= \frac{1}{2!}(n+1)(n+2) \\ &= \binom{n+2}{2} \end{aligned}$$

3-compositions of  $n$ .

- $m = 4$ : We work as for  $m = 3$  above, that is breaking the first entry of each 3-composition of  $n$  to get the 4-compositions of  $n$ . We obtain sets with the following respective numbers of 4-compositions of  $n$  grouped in sets for each 3-composition of  $n$ :  $\{n+1, n, n-1, \dots, 1\}$ ,  $\{n, n-1, \dots, 1\}$ ,  $\{n-1, n-2, \dots, 1\}$ ,  $\dots$  and  $\{1\}$  giving a total of

$$\begin{aligned} 1(n+1) + 2n + 3(n-1) + \dots + (n-1)3 + (n)2 + (n+1)1 \\ &= \frac{1}{2}[(n+1)(n+2) + n(n+1) + (n-1)n + \dots + 2(3) + 1(2)] \\ &= \frac{1}{2} \times \frac{1}{3}(n+1)(n+2)(n+3) \\ &= \frac{1}{3!}(n+1)(n+2)(n+3) \\ &= \binom{n+3}{3} \end{aligned}$$

4-compositions of  $n$ .

- On generalizing for  $m = k$  there are

$$\frac{1}{(k-1)!}(n+1)(n+2)(n+3) \dots (n+k-1) = \binom{n+k-1}{k-1}$$

$k$ -compositions of  $n$ .  $\square$

We have developed the following programmes in GAP [22] for computing  $m$ -compositions of  $n$  and multi-nomial coefficients respectively.

**Programme 2.1.3 (*m*-compositions of *n*)**

```

mcompsn:=function(m,n)
local i, mcn;
for i in [1..n] do
mcn:=Reversed(Filtered(Tuples([0..i],m),j->Sum(j)=i));
od;
return mcn;
end;

```

**Programme 2.1.4 ( *Multi-nomial coefficients* )**

```

mcompsn:=function(m,n)
local mcn;
return Reversed(Filtered(Tuples([0..n],m),j->Sum(j)=n));
end;

multifact:=function(p)
local i,res;
res:=Factorial(0);
for i in p do
res:=Factorial(i)*res;
od;
return res;
end;

multinomialcoefs:=function(m,n)
local i,mnc,res;
mnc=[];
for i in mcompsn(m,n) do
res:=Factorial(n)/multifact(i);
Add(mnc,res);
od;
return mnc;
end;

```

**Examples 2.1.5** We give examples of sets of  $m$ -compositions of  $n$  and corresponding multi-nomial coefficients below.

(a)  $A(6, 2) = \{(6, 0), (5, 1), (4, 2), (3, 3), (2, 4), (1, 5), (0, 6)\}$ ,  
 where  $N(6, 2) = 7$ , with multi-nomial coefficients  
 $[1, 6, 15, 20, 15, 6, 1]$ .

(b)  $A(5, 3) = \{(5, 0, 0), (4, 1, 0), (4, 0, 1), (3, 2, 0), (3, 1, 1), (3, 0, 2), (2, 3, 0),$   
 $(2, 2, 1), (2, 1, 2), (2, 0, 3), (1, 4, 0), (1, 3, 1), (1, 2, 2), (1, 1, 3), (1, 0, 4),$   
 $(0, 5, 0), (0, 4, 1), (0, 3, 2), (0, 2, 3), (0, 1, 4), (0, 0, 5)\}$ ,  
 where  $N(5, 3) = 21$ , with multi-nomial coefficients  
 $[1, 5, 5, 10, 20, 10, 10, 30, 30, 10, 5, 20, 30, 20, 5, 1, 5, 10, 10, 5, 1]$ .

(c)  $A(3, 4) = \{(3, 0, 0, 0), (2, 1, 0, 0), (2, 0, 1, 0), (2, 0, 0, 1), (1, 2, 0, 0), (1, 1, 1, 0),$   
 $(1, 1, 0, 1), (1, 0, 2, 0), (1, 0, 1, 1), (1, 0, 0, 2), (0, 3, 0, 0), (0, 2, 1, 0), (0, 2, 0, 1),$   
 $(0, 1, 2, 0), (0, 1, 1, 1), (0, 1, 0, 2), (0, 0, 3, 0), (0, 0, 2, 1), (0, 0, 1, 2), (0, 0, 0, 3)\}$  ,  
 where  $N(3, 4) = 20$ , with multi-nomial coefficients  
 $[1, 3, 3, 3, 3, 6, 6, 3, 6, 3, 1, 3, 3, 3, 6, 3, 1, 3, 3, 1]$ .

## 2.2 Partitions

There are several ways to define a partition of a positive integer  $n$ . For our purposes we will take the following definition.

**Definition 2.2.1** Let  $n \in \mathbb{N}$ . Then  $[\lambda] = [1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}]$  is called a **partition** of  $n$  if  $\sum_{s=1}^n s\lambda_s = n$ , where  $\lambda_s \in \mathbb{N} \cup \{0\}$ .

**Definition 2.2.2** Let  $[\lambda] = [1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}]$  as above and  $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$  be the  $n$ -tuple of powers in  $[\lambda]$ .

(1) We call  $\beta = ((a_{11}, a_{12}, \dots, a_{1n}), (a_{21}, a_{22}, \dots, a_{2n}), \dots, (a_{m1}, a_{m2}, \dots, a_{mn}))$  such that  $\sum_{i=1}^m a_{is} = \lambda_s$ , an  **$m$ -set of partition**  $[\lambda]$ . However for computation purposes  $\beta$  is written in the form

$$\beta = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{is}).$$

- (2) We say  $\beta$  is an **even**  $m$ -set of partition  $[\lambda]$  if  $\sum_{i=1}^m \sum_{s=1}^n (i-1)sa_{is} \equiv 0(\text{mod } m)$  (or  $\sum_{i=1}^m (i-1)a_i \equiv 0(\text{mod } m)$ , where  $a_i = \sum_{s=1}^n sa_{is}$ ).
- (3) We say  $\beta$  is a **type-zero**  $m$ -set of partition  $[\lambda]$  if  $\sum_{i=1}^m \sum_{s=1}^n (i-1)a_{is} \equiv 0(\text{mod } m)$ .
- (4) Let  $\mu = ((b_{11}, \dots, b_{1n}), (b_{21}, \dots, b_{2n}), \dots, (b_{m1}, \dots, b_{mn}))$  be an  $m$ -set of partition  $[\lambda]$ . Then  $\beta$  and  $\mu$  are said to be
- (i) **equivalent** if  $\mu$  can be obtained by cyclically re-arranging the rows of  $\beta$ .  
It is clear that the number of  $m$ -sets of partition  $[\lambda]$  equivalent to  $\beta$  is  $m$ .
  - (ii) **odd equivalent** if  $\mu$  can be obtained by cyclically re-arranging the entries at all  $s$  where  $m \nmid s$  in the rows of  $\beta$ .
- (5)  $\beta$  is said to be **self equivalent (self odd equivalent)** if cyclically re-arranging the rows (or entries at all  $s$  where  $m \nmid s$  in the rows) of  $\beta$  does not yield any  $m$ -set of partition  $[\lambda]$  different from  $\beta$ .
- (6) The  $m$ -sets of partition  $[\lambda]$ , may be arranged in the following order: For each fixed  $m$ -composition of  $\lambda_s$  vary the  $m$ -compositions of  $\lambda_{s-1}$ . Then vary the entries in this collection for each fixed  $m$ -composition of  $\lambda_{s+1}$ . The above is repeated for each  $\lambda_s$ , giving all the  $m$ -sets of partition  $[\lambda]$ . For example we have the following 3-sets of partition  $[2^1 3^1]$  arranged in the above ordering.

$$\begin{aligned} & \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \end{aligned}$$

- (7) When  $\lambda_s = 0$  and the lengths  $s$  are clear we will write the  $m$ -sets of partition  $[\lambda]$  omitting the zero columns to which such  $s$  refers.
- (8) Let  $\bar{\lambda}_0 = (\lambda_s \mid \lambda_s \neq 0, s = 1, 2, \dots, n)$  be a tuple of all the non zero entries of  $\bar{\lambda}$ . Then  $\bar{\lambda}_0$  is a  $q$ -tuple of non zero powers in a partition  $[\lambda]$  of  $n$ ,  $q \leq n$ .

The combinatorial basis for the construction of the character table of the group  $B(m, n)$  will be a mechanism to produce all the  $m$ -sets for all partitions of a positive

integer  $n$ . To my knowledge while there are functions in GAP to produce partitions of  $n$  there is none to produce  $m$ -sets for all partitions of  $n$ . We have developed the following programme for constructing  $m$ -sets for all partitions of  $n$  in GAP.

**Programme 2.2.3** ( *$m$ -sets of partition  $[\lambda]$* )

```

mspntype:=function(m,n)
local i,mcn,res,a1,b1,mspn1,L,p,q;
mcn:=[];
for i in [0..n] do
res:=Reversed(Filtered(Tuples([0..i],m),j->Sum(j)=i));
Add(mcn,res);
od;
L:=List([1..n], i->n+1-i);;
p:=Partitions(n);; q:=List([1..Number(p)],i->List(L,j->Number(p[i],k->k=j)));;
a1:=List(q, i->Cartesian(List(i,j->mcn[j+1])));;
b1:=List(a1, i->List(i, j->TransposedMat(j)));;
mspn1:=List(b1, i->List(i, j->List(j, k->List([1..Number(k)], l->k[Number(k)+1-l]))));;
return mspn1;
end;

```

## 2.3 Magic Matrices

In this section, we define some combinatorial objects called magic matrices which will be used in constructing the Fischer-Clifford matrices of generalized symmetric groups in Chapter 4.

**Definition 2.3.1** Let  $\bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ ,  $\beta = (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_m)$  and  $\delta = (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_m)$ , where  $\bar{a}_i = (a_{i1}, a_{i2}, \dots, a_{in})$ ,  $a_i = \sum_{s=1}^n s a_{is}$ ,  $\bar{k}_j = (k_{j1}, k_{j2}, \dots, k_{jn})$ ,  $k_j = \sum_{s=1}^n s k_{js}$  such that  $(a_{1s}, a_{2s}, \dots, a_{ms})$ ,  $(k_{1s}, k_{2s}, \dots, k_{ms}) \in A(\lambda_s, m)$ ,  $i, j = 1, 2, \dots, m$ ,  $s = 1, 2, \dots, n$ . Let  $M(\beta, \lambda_s) = \{(r_{ij}^s) \in M_m(\mathbb{N} \cup \{0\}) \mid r_i^s = (r_{i1}^s, r_{i2}^s, \dots, r_{im}^s) \in A(a_{is}, m), i = 1, 2, \dots, m\}$  and  $M(\delta, \lambda_s) = \{(r_{ij}^s) \in M_m(\mathbb{N} \cup \{0\}) \mid r_j^s = (r_{1j}^s, r_{2j}^s, \dots, r_{mj}^s) \in A(k_{js}, m), j = 1, 2, \dots, m\}$ . Define  $M(\beta, \delta, \lambda_s) = M(\beta, \lambda_s) \cap M(\delta, \lambda_s) = \{(r_{ij}^s) \in M_m(\mathbb{N} \cup \{0\}) \mid r_i^s = (r_{i1}^s, \dots, r_{im}^s) \in A(a_{is}, m), r_j^s = (r_{1j}^s, r_{2j}^s, \dots, r_{mj}^s) \in A(k_{js}, m), (i, j =$

$1, 2, \dots, m\}$ . Then the  $m \times mn$  matrix

$$M_{\beta, \delta, n} = [M_{\beta, \delta, \lambda_1} | M_{\beta, \delta, \lambda_2} | \dots | M_{\beta, \delta, \lambda_n}],$$

where  $M_{\beta, \delta, \lambda_s} \in M(\beta, \delta, \lambda_s)$ , is called a **magic matrix**. The matrices  $M_{\beta, \delta, \lambda_s}$  may be called **partial magic matrices**.

For computation purposes a magic matrix will be presented as

$$\left[ \begin{array}{c|c|c|c} r_{11}^1 r_{12}^1 \dots r_{1m}^1 & r_{11}^2 r_{12}^2 \dots r_{1m}^2 & \dots & r_{11}^q r_{12}^q \dots r_{1m}^q \\ r_{21}^1 r_{22}^1 \dots r_{2m}^1 & r_{21}^2 r_{22}^2 \dots r_{2m}^2 & \dots & r_{21}^q r_{22}^q \dots r_{2m}^q \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ r_{m1}^1 r_{m2}^1 \dots r_{mm}^1 & r_{m1}^2 r_{m2}^2 \dots r_{mm}^2 & \dots & r_{m1}^q r_{m2}^q \dots r_{mm}^q \end{array} \right], \quad (2.1)$$

where  $k_{js} = \sum_{i=1}^m r_{ij}^s, 0 \leq r_{ij}^s \leq [a_{is} - (\sum_{l=1}^{j-1} r_{il}^s)]$ . The set of magic matrices will be denoted by  $M(\beta, \delta, n)$ . We may obtain the matrices  $M_{\beta, \delta, \lambda_s}$  directly from  $M(\beta, \lambda_s)$  (or  $M(\delta, \lambda_s)$ ) for each  $s$  using the following method.

**Method 2.3.2** Let  $n, \beta, \delta, M(\beta, \lambda_s)$  and  $M(\delta, \lambda_s)$  as defined above. The partial magic matrices  $M_{\beta, \delta, \lambda_s}$  are obtained by choosing from  $M(\beta, \lambda_s)$  ( $M(\delta, \lambda_s)$ ) those matrices whose column (row) entries add up to  $k_{1s}, k_{2s}, k_{3s}, \dots, k_{ms}$  ( $a_{1s}, a_{2s}, a_{3s}, \dots, a_{ms}$ ).

We remark that in using method 2.3.2, it is helpful to work with which ever  $M(\beta, \lambda_s)$  or  $M(\delta, \lambda_s)$  that involves relatively small values of the  $a_{is}$  or  $k_{js}$ . However for most of the results here we require only the partial magic matrices  $M_{\beta, \delta, \lambda_s}$ . Using Definition 2.3.1 we have constructed the following programme which is used in computing the partial magic matrices  $M_{\beta, \delta, \lambda_s}$  for each  $\lambda_s = 1, \dots, n$ .

**Programme 2.3.3** (*Magic matrices for  $\lambda_s = 1, \dots, n$* )

```
Mmat:=function(m,n)
local L,mcn1,mcn2,ma,mk,g;
L:=List([1..n]);;
mcn1:=List(L,i → Filtered(Tuples([0..i],m),j → Sum(j)=i));;
mcn2:=List(L,i → List([0..Number(mcn1[i])-1],j → mcn1[i][Number(mcn1[i])-j]));;
g:=List(mcn2, i → List(i, i → List([1..m],j → Filtered(Tuples([0..i][j],m),
k → Sum(k)=i[j]))));;
```





## Chapter 3

# The Group $B(m, n)$ and Some Associated Groups

The generalized symmetric group  $B(m, n)$  occurs as a constituent of other important groups such as simple groups, for example the group  $B(2, 3)$  is isomorphic to the group extension  $2^3:SP(2, 2)$  (see Lemma 10.2.57 in [53]). In order to understand the structure of such composite groups we may need the character table of  $B(m, n)$ . Since this thesis is devoted to the construction of the character tables of generalized symmetric groups and associated groups, we give a resume of facts about these groups in this chapter. We discuss the general theory of the generalized symmetric group and its conjugacy classes in Section 3.1. Here we also give two programmes used to compute the orders of centralizers and the orders of elements of conjugacy classes respectively, of the generalized symmetric group. This is equivalent to coset analysis (see Subsection 1.2.1) for calculating the conjugacy classes of group extensions  $N.G$ , where  $N$  is abelian. We give the orbits of the actions of the symmetric group  $S_n$  on the group  $N = \mathbb{Z}_m^n$  and on the set  $Irr(N)$  of irreducible characters of  $N$  in Section 3.2. In Section 3.3 we discuss the groups  $B_S(m, n)$  and  $B_Q(m, n)$  associated with the group  $B(m, n)$ . We also give the orbits of the actions of  $S_n$  on the subgroup  $S = \mathbb{Z}_m^{n-1}$  and on the set  $Irr(S)$  of the irreducible characters of  $S$ , on the quotient  $Q = \overline{\mathbb{Z}_m^n}$  of  $N$  and on the set  $Irr(Q)$  of the irreducible characters of  $Q$  in Subsections 3.3.1 and 3.3.2 respectively. The orbits of the action of  $S_n$  on the irreducible characters in each case give information about the inertia factor groups required for the construction of the character table of each respective group.

### 3.1 The Generalized Symmetric Group $B(m, n)$

The representation theory of the symmetric group has been studied extensively. For the necessary details, readers may consult James and Kerber [29], Robinson [54], Sagan [58] and other relevant references.

**Definition 3.1.1** Let  $\mathbb{Z}_m$  be the cyclic group of order  $m$  and  $N$  be the direct product of  $n$  copies of  $\mathbb{Z}_m$ . Let  $S_n$  be the symmetric group of degree  $n$ . The wreath product of  $\mathbb{Z}_m$  with  $S_n$  is a split extension of  $N$  by  $S_n$  called the **generalized symmetric group**,  $B(m, n)$ . When  $m = 2$ , the group  $B(m, n)$  is the Weyl group of type  $B_n$ , called the **hyper-octahedral group** (see [5], [30], [33], [34], [36], [49], [59], [66]).

The group  $B(m, n)$  has a presentation given by

$$\begin{aligned} B(m, n) = \langle & s_i (i = 1, 2, \dots, n-1), w_j (j = 1, 2, \dots, n) \mid \\ & s_i^2 = (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1, |j - i| \geq 2, w_j^m = 1, \\ & w_i w_j = w_j w_i (i \neq j), s_i w_i = w_{i+1} s_i, s_i w_j = w_j s_i, j \neq i, i + 1 \rangle, \end{aligned}$$

that is we identify  $s_i$  with the transposition  $(i \ i + 1)$  and  $w_j$  with the mapping  $j \rightarrow \xi j$ ,  $i \rightarrow i$ ,  $i \neq j$ . The elements of  $B(m, n)$  permute the symbols of  $\Omega = \{1, 2, \dots, n\}$  as well as multiplying any number of the symbols of  $\Omega$  by some powers of a primitive  $m^{\text{th}}$  root  $\xi$  of unity. Let  $g \in S_n$ . Then the elements of  $B(m, n)$  may be written  $\bar{g} = dg$ , where  $d$  is a product of  $w_j$ 's belonging to  $N$ . The operation composition on  $B(m, n)$  is defined by  $(dg) \cdot (d'g') = d(d')^g g g'$ , where  $(w_j)^g = w_{g(j)}$  is the action of  $g$  on  $w_j$ . An element  $g \in S_n$ , that is

$$g = \begin{pmatrix} 1 & 2 & \dots & n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}, \quad b_j \in \{1, 2, \dots, n\},$$

can be written uniquely (up to reordering) as a product of disjoint  $t_i$ -cycles as  $g = g_1 g_2 \dots g_t$ , where  $g_i = (b_{i1} \ b_{i2} \ \dots \ b_{it_i})$ . It is clear that each  $t_i$ -cycle  $g_i$  can be written as a product of a  $(t_i - 1)$ -cycle and a transposition as  $(b_{i1} \ b_{i2} \ \dots \ b_{it_i}) = (b_{i2} \ \dots \ b_{it_i})(b_{i1} \ b_{it_i})$ . Let  $\bar{g} \in B(m, n)$ . Then

$$\bar{g} = \begin{pmatrix} 1 & 2 & \dots & n \\ \xi^{k_1} b_1 & \xi^{k_2} b_2 & \dots & \xi^{k_n} b_n \end{pmatrix},$$

where  $k_j \in \{1, 2, \dots, m\}$  and the  $k_j$ 's are called **associated powers** of  $\bar{g}$ . Further we say that  $\bar{g}$  can be expressed uniquely as a product of disjoint cycles  $\bar{g}_i$ ,  $i = 1, 2, \dots, t$ , where the  $g_i$ 's are disjoint cycles of  $g$ . Thus  $\bar{g} = \bar{g}_1 \bar{g}_2 \cdots \bar{g}_t$  such that

$$\bar{g}_i = \begin{pmatrix} b_{i1} \\ \xi^{k_{i1}} b_{i1} \end{pmatrix} \begin{pmatrix} b_{i2} \\ \xi^{k_{i2}} b_{i2} \end{pmatrix} \cdots \begin{pmatrix} b_{it_i} \\ \xi^{k_{it_i}} b_{it_i} \end{pmatrix} \begin{pmatrix} b_{i1} & b_{i2} & \cdots & b_{it_i} \\ b_{i2} & b_{i3} & \cdots & b_{i1} \end{pmatrix} = c g_i$$

where  $i = 1, 2, \dots, t$ ,  $b_{ij} \in \{1, 2, \dots, n\}$ ,  $k_{ij} \in \{1, 2, \dots, m\}$ , the length  $t_i$  of cycle  $g_i$  is the length of  $\bar{g}_i$  (see [59]), and the expression

$$c = \begin{pmatrix} b_{i1} \\ \xi^{k_{i1}} b_{i1} \end{pmatrix} \begin{pmatrix} b_{i2} \\ \xi^{k_{i2}} b_{i2} \end{pmatrix} \cdots \begin{pmatrix} b_{it_i} \\ \xi^{k_{it_i}} b_{it_i} \end{pmatrix}$$

is called the **cycle-product** of  $\bar{g}_i$  (see [33]), and the  $k_{ij}$ 's are powers associated with  $\bar{g}_i$ . The map  $\zeta : B(m, n) \rightarrow S_n$  defined by

$$\zeta(\bar{g}) = \begin{pmatrix} 1 & 2 & \cdots & n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} \quad (3.1)$$

is a natural homomorphism of  $B(m, n)$  with  $\text{Ker}(\zeta) = N$ .

**Definition 3.1.2** ([59], [66]) Let  $\bar{g} = \bar{g}_1 \bar{g}_2 \cdots \bar{g}_t \in B(m, n)$ . Define  $P(\bar{g})$  by  $P(\bar{g}) = \sum_{i=1}^t P(\bar{g}_i)$  where  $P(\bar{g}_i) = \sum_{j=1}^{t_i} k_{ij}$ . Let  $a_{rs}(\bar{g})$  be the number of cycles  $\bar{g}_i$  of  $\bar{g}$  of length  $s$  such that  $P(\bar{g}_i) \equiv (r-1) \pmod{m}$ ,  $1 \leq r \leq m$ ,  $1 \leq s \leq n$ . The  $m \times n$  matrix  $(a_{rs}(\bar{g}))$  is called the **type** of  $\bar{g}$ , which is denoted by  $Ty(\bar{g})$ .

It is clear from above that  $Ty(\bar{g})$  is equal to a unique  $m$ -set of partition  $[\lambda]$  for some  $\bar{g} \in B(m, n)$  (see Definition 2.2.2). We remark that alternatively  $Ty(\bar{g})$  is expressed in terms of lengths of cycles of  $\bar{g}$  (see [5]) as

$$(1^{a_{11}} 2^{a_{12}} \cdots n^{a_{1n}}, 1^{a_{21}} 2^{a_{22}} \cdots n^{a_{2n}}, \dots, 1^{a_{m1}} 2^{a_{m2}} \cdots n^{a_{mn}}).$$

**Proposition 3.1.3** Two elements  $\bar{g}$  and  $\bar{g}'$  of  $B(m, n)$  are conjugate if and only if  $Ty(\bar{g}) = Ty(\bar{g}')$ .

**Proof.** See result 3.7 in [33] or result 3.2.4 in [66].  $\square$

Since an  $m$ -set of partition  $[\lambda]$  is equal to  $Ty(\bar{g})$  for some conjugacy class of  $B(m, n)$  containing  $\bar{g}$ , we will say an  $m$ -set of partition  $[\lambda]$  indexes a conjugacy class of  $B(m, n)$ . It follows that the number of conjugacy classes of  $B(m, n)$  is equal to the number of  $m$ -sets for all partitions of  $n$ . Therefore by using Programme 2.2.3, we can easily determine the number of conjugacy classes of  $B(m, n)$ . In the following we give an alternative formula for calculating the number of conjugacy classes of  $B(m, n)$  in terms of the conjugacy classes of  $S_n$ .

**Theorem 3.1.4** *The number of conjugacy classes of  $B(m, n)$  is given by*

$$\sum_{[g]} \prod_{s=1}^n N(\lambda_s, m),$$

where the summation is taken over the classes  $[g]$  of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .

**Proof.** To determine the number of conjugacy classes of  $B(m, n)$  we calculate the total number of  $m$ -sets for all partitions of  $n$ . Let  $[g]$  be a conjugacy class of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . Now for each  $m$ -composition of  $\lambda_s$  in  $[\lambda]$ ,  $s$  fixed, there is a unique  $m$ -set of partition  $[\lambda]$ . However by Proposition 2.1.2, the number of  $m$ -compositions of  $\lambda_s$  is  $N(\lambda_s, m)$ . Thus the class  $[g]$  of  $S_n$ , of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  produces

$$\prod_{s=1}^n N(\lambda_s, m)$$

$m$ -sets of partition  $[\lambda]$ . By an earlier remark, this is the number of conjugacy classes of  $B(m, n)$  arising from a conjugacy class  $[g]$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . On summing over all the conjugacy classes  $[g]$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of  $S_n$  we obtain that the total number of conjugacy classes of  $B(m, n)$  is equal to  $\sum_{[g]} \prod_{s=1}^n N(\lambda_s, m)$ .  $\square$

Note that similar results to Theorem 3.1.4 are given elsewhere, for example see result 3.8 in [33] and Lemma 4.2.9 in [29].

**Lemma 3.1.5** *Let  $\bar{g}_i$  be a cycle of an element  $\bar{g}$  of  $B(m, n)$  of length  $s$ . Let  $r \in \mathbb{Z}_m$ . Then the number of arrangements of powers  $\xi^{k_{ij}}$  in  $\bar{g}_i$  such that  $\sum_{i=1}^s k_{ij} \equiv r \pmod{m}$  is given by  $m^{s-1}$ .*

**Proof.** The proof is by induction on the length  $s$  of  $\bar{g}_i$ . Let  $s = 1$ . Then  $\bar{g}_i = \begin{pmatrix} b_{ij} \\ \xi^{k_{ij} b_{ij}} \end{pmatrix}$ . Clearly there is only one arrangement of a power of  $\xi$ , namely  $\xi^{k_{ij}}$  such that  $k_{ij} \equiv r \pmod{m}$ . In this case the result says the number of arrangements of a power of  $\xi$  such that  $k_{ij} \equiv r \pmod{m}$  is  $m^{1-1} = m^0 = 1$ , which is true. Further let  $s = 2$ , that is we consider a 2-cycle. Then

$$\bar{g}_i = \begin{pmatrix} b_{i1} & b_{i2} \\ \xi^{k_{i1} b_{i2}} & \xi^{k_{i2} b_{i1}} \end{pmatrix},$$

with  $k_{i1} + k_{i2} - r = mv$  for some  $v \in \mathbb{N}$  and  $0 \leq r \leq m - 1$ . Since  $k_{ij} < m$  for each  $j = 1, 2$ , we must have  $k_{i1} + k_{i2} - r = m$ . Now

$$\begin{aligned} k_{i1} + k_{i2} - r = m &\Rightarrow k_{i1} + k_{i2} = m + r \\ &\Rightarrow (k_{i1}, k_{i2}) \in A(m + r, 2), \end{aligned}$$

where  $|A(m + r, 2)| = m + r + 1$ . But for the powers of  $\xi$ , the pairs  $(k_{i1}, k_{i2})$  are such that  $([(m + r) - u], u) = ([r - u], u + m)$ , where  $u = 0, 1, 2, \dots, r$ , since  $m \equiv 0$ . So we subtract  $(r + 1)$  from  $m + r + 1$  for the repeated 2-compositions of  $m + r$ , giving  $m$  arrangements of powers of  $\xi$  in a 2-cycle such that  $\sum_{i=1}^2 k_{ij} \equiv r \pmod{m}$ . Now the result says the number of arrangements of powers of  $\xi$  in a 2-cycle such that  $\sum_{i=1}^2 k_{ij} \equiv r \pmod{m}$  is  $m^{2-1} = m$ , which is true.

We also assume the result is true for  $s = l$ , that is the number of arrangements of powers of  $\xi$  in an  $l$ -cycle such that  $\sum_{i=1}^l k_{ij} \equiv r \pmod{m}$  is  $m^{l-1}$ . We prove that for  $s = l + 1$ , the number of arrangements of powers of  $\xi$  in an  $l + 1$ -cycle such that  $\sum_{i=1}^{l+1} k_{ij} \equiv r \pmod{m}$  is  $m^l$ . But recall that an  $(l + 1)$ -cycle can be written as a product of an  $l$ -cycle and a 2-cycle. The result  $m^l$ , follows from the product of  $m^{l-1}$  for the  $l$ -cycle and  $m$  for the 2-cycle.  $\square$

**Example 3.1.6** Let  $m=4, n=3$ , then for various  $s$  and  $r$ , the following table shows collections of  $s$ -tuples each whose entries have a sum congruent to  $r \pmod{m}$ .

s	r	s-tuples	$m^{s-1}$
1	0	(0)	1
	1	(1)	1
	2	(2)	1
	3	(3)	1
2	0	(0,0), (2,2), (3,1),(1,3)	4
	1	(1,0), (0,1), (3,2),(2,3)	4
	2	(1,1), (2,0), (0,2),(3,3)	4
	3	(2,1), (1,2), (3,0),(0,3)	4
3	0	(0,0,0), (0,2,2), (2,0,2), (2,2,0), (1,1,2), (1,2,1), (2,1,1), (0,3,1), (0,1,3), (1,0,3), (1,3,0), (3,0,1), (3,1,0), (3,2,3), (3,3,2), (2,3,3),	16
	1	(1,0,0), (0,1,0), (0,0,1), (2,2,1), (2,1,2), (1,2,2), (3,1,1), (1,3,1), (1,1,3), (3,2,0), (3,0,2), (2,3,0), (2,0,3), (0,3,2), (0,2,3), (3,3,3)	16

**Proposition 3.1.7** *Let  $\bar{g} = dg$  be a class representative of  $B(m, n)$  with  $Ty(\bar{g}) = ((a_{11}, a_{12}, \dots, a_{1n}), (a_{21}, a_{22}, \dots, a_{2n}), \dots, (a_{m1}, a_{m2}, \dots, a_{mn}))$ . Then the*

(i) *order of the centralizer of  $\bar{g}$  in  $B(m, n)$  is given by*

$$\prod_{s=1}^n \prod_{i=1}^m (sm)^{a_{is}} (a_{is})!,$$

(ii) *order  $o(\bar{g})$  of  $\bar{g}$ , is the least common multiple of the products  $s \times r$  for all positive  $a_{is}$ , where  $r$  is the order of the cycle-product  $c$  of a cycle of  $\bar{g}$  of length  $s$ .*

**Proof.**

(i) We first determine the size of the conjugacy class of type  $Ty(\bar{g})$ . We determine the number of ways of arranging the symbols of  $\Omega$  together with powers of  $\xi$  which produce elements, with repetitions, of  $B(m, n)$  of type  $Ty(\bar{g})$ . We

will also determine the number of arrangements of symbols of  $\Omega$  together with powers of  $\xi$  that yield the same element of type  $Ty(\bar{g})$ , this number is used to factor out repetitions. Now we consider an element  $\bar{g} \in B(m, n)$ , with a fixed arrangement of powers of  $\xi$ , of type  $Ty(\bar{g})$ . This is equivalent to considering  $g \in S_n$  such that  $g = \zeta(\bar{g})$  where  $\zeta$  is the natural homomorphism of  $B(m, n)$ , defined in (3.1). We know from  $S_n$  that there are  $n!$  ways of arranging symbols of  $\Omega$  producing elements of type  $Ty(\bar{g})$  (or equivalently of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ , where  $\lambda_s = \sum_{i=1}^m a_{is}$ ). Now for fixed  $s$ , we consider the cycles of  $\bar{g}$  of count  $a_{is}$  such that  $\sum_{i=1}^m a_{is} = \lambda_s$  instead of the cycles of count  $\lambda_s$ . Thus for all  $s$ , the number of arrangements out of the  $n!$  arrangements above which yield the same element of type  $Ty(\bar{g})$  is  $\prod_{s=1}^n (\prod_{i=1}^m s^{a_{is}})(a_{is}!)$ . Hence the number of arrangements out of the  $n!$  arrangements which yield elements of type  $Ty(\bar{g})$  is

$$\frac{n!}{\prod_{s=1}^n (\prod_{i=1}^m s^{a_{is}})(a_{is}!)}. \quad (3.2)$$

Also from Lemma 3.1.5, there are  $m^{s-1}$  ways of arranging powers of  $\xi$  in each  $a_{is}$  cycle, that do not change type  $Ty(\bar{g})$ . Now for all the  $a_{is}$  cycles we have a total of  $(m^{s-1})^{a_{is}}$  ways of arranging powers of  $\xi$  that do not change type  $Ty(\bar{g})$ . Doing this over  $i$  and  $s$  we obtain a total of

$$\prod_{s=1}^n \prod_{i=1}^m m^{(s-1)a_{is}} \quad (3.3)$$

ways of arranging the powers of  $\xi$  in  $\bar{g}$  that do not change type  $Ty(\bar{g})$ . However there is only one way of arranging powers of  $\xi$  in  $\bar{g}$  which yields the same element of type  $Ty(\bar{g})$ . Now to obtain the size of the conjugacy class of type  $Ty(\bar{g})$ , we multiply the two values in (3.2) and (3.3) above, that is

$$\frac{\prod_{s=1}^n \prod_{i=1}^m m^{(s-1)a_{is}} n!}{\prod_{s=1}^n (\prod_{i=1}^m s^{a_{is}})(a_{is}!)} = \frac{m^n n!}{\prod_{s=1}^n \prod_{i=1}^m (sm)^{a_{is}} (a_{is}!)}.$$

It follows that  $|C_{B(m,n)}(\bar{g})| = \prod_{s=1}^n \prod_{i=1}^m (sm)^{a_{is}} (a_{is}!)$ , as required.

(ii) See result 3.12 in [33], and for prime  $m$ , see Theorem 2.7 in [41].  $\square$

**Remark 3.1.8** *The formula for the orders of centralizers in  $B(m, n)$  may be expressed as*

$$|C_{B(m,n)}(\bar{g}_j)| = m^{\sum_{s=1}^n \lambda_s} \prod_{s=1}^n (s^{\lambda_s} \prod_{i=1}^m a_{is}!).$$

We give examples on the construction of the orders of centralizers and orders of class representatives of  $B(2, 6)$  corresponding to the conjugacy class of  $S_6$  of type  $(1^4 2^1)$  using Proposition 3.1.7. Since for this class of  $S_6$  we have  $\bar{\lambda}_0 = (4, 1)$ , there are  $N(2, 4) \times N(2, 1) = 10$  conjugacy classes of  $B(2, 6)$  of types  $(a_{is})$ , given by the following 2-sets of partition  $[1^4 2^1]$ .

$$\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix}.$$

However we can obtain all the 2-sets for all partitions  $[\lambda]$  of 6 by supplying to GAP the value  $n = 6$  and  $m = 2$  and running the Programme 2.2.3. Let the cycle  $g = (1\ 2)$  be the class representative of type  $(1^4 2^1)$  in  $S_6$ . Now by Definition 3.1.2 of  $Ty(\bar{g})$  we can identify the conjugacy class representatives of  $B(2, 6)$  corresponding to  $g$ , respectively as shown in Table 3.1.

Now, taking the ten  $\bar{g}$ 's in the order of the 2-sets of partition  $[1^4 2^1]$  given above and using the formula in Remark 3.1.8, we obtain the orders of the centralizers of  $B(2, 6)$ . For example if  $\bar{g}$  is of type  $\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}$ , then  $\lambda_1 = 4$ ,  $\lambda_2 = 1$  for lengths  $s = 1$  and  $s = 2$  respectively. Also we have  $a_{11} = 4$ ,  $a_{12} = 1$ ,  $a_{21} = 0$ ,  $a_{22} = 0$  for which we obtain

$$\begin{aligned} |C_{B(2,6)}(\bar{g})| &= 2^{\lambda_1 + \lambda_2} [1^{\lambda_1} (a_{11}!) (a_{21}!)] [2^{\lambda_2} (a_{12}!) (a_{22}!)] \\ &= 2^{4+1} [1^4 (4!) (0!)] [2^1 (1!) (0!)] \\ &= 2^5 \times 24 \times 2 = 32 \times 48 = 1536. \end{aligned}$$

Similarly for  $\bar{g}$  of type  $\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$ , we have  $\lambda_1 = 4$ ,  $\lambda_2 = 1$  for lengths  $s = 1$  and  $s = 2$  respectively. Also we have  $a_{11} = 3$ ,  $a_{12} = 1$ ,  $a_{21} = 1$ ,  $a_{22} = 0$  for which we obtain

$$|C_{B(2,6)}(\bar{g})| = 2^{4+1} [1^4 (3!) (1!)] [2^1 (1!) (0!)] = 2^5 \times 6 \times 2 = 32 \times 12 = 384.$$

The rest of the orders of the centralizers in  $B(2, 6)$  corresponding to the conjugacy class of  $S_6$  of type  $(1^4 2^1)$  are obtained in the same way and given in column 3 of

Table 3.1:  $(1^4 2^1)$ -Conjugacy classes of  $B(2, 6)$

$(1^4 2^1)$ -Class representative $\bar{g}$	Order of $\bar{g}$	$ C_{B(2,6)}(\bar{g}) $
$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = g$	$Lcm[(1 \times 1), (2 \times 1)] = 2$	1536
$\begin{pmatrix} 3 & \\ -3 & \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = e_3 g$	$Lcm[(1 \times 1), (1 \times 2), (2 \times 1)] = 2$	384
$\begin{pmatrix} 3 & & \\ -3 & -4 & \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = e_3 e_4 g$	$Lcm[(1 \times 1), (1 \times 2), (2 \times 1)] = 2$	256
$\begin{pmatrix} 3 & & & \\ -3 & -4 & -5 & \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = e_3 e_4 e_5 g$	$Lcm[(1 \times 1), (1 \times 2), (2 \times 1)] = 2$	384
$\begin{pmatrix} 3 & & & & \\ -3 & -4 & -5 & -6 & \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = e_3 e_4 e_5 e_6 g$	$Lcm[(1 \times 2), (2 \times 1)] = 2$	1536
$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = e_1 g$	$Lcm[(1 \times 1), (2 \times 2)] = 4$	1536
$\begin{pmatrix} 3 & & \\ -3 & -2 & 1 \end{pmatrix} = e_1 e_3 g$	$Lcm[(1 \times 1), (1 \times 2), (2 \times 2)] = 4$	384
$\begin{pmatrix} 3 & & & \\ -3 & -4 & -2 & 1 \end{pmatrix} = e_1 e_3 e_4 g$	$Lcm[(1 \times 1), (1 \times 2), (2 \times 2)] = 4$	256
$\begin{pmatrix} 3 & & & & \\ -3 & -4 & -5 & -2 & 1 \end{pmatrix} = e_1 e_3 e_4 e_5 g$	$Lcm[(1 \times 1), (1 \times 2), (2 \times 2)] = 4$	384
$\begin{pmatrix} 3 & & & & & \\ -3 & -4 & -5 & -6 & -2 & 1 \end{pmatrix} = e_1 e_3 e_4 e_5 e_6 g$	$Lcm[(1 \times 2), (2 \times 2)] = 4$	1536

Table 3.1. Further we have used Proposition 3.1.7 (ii) to calculate the orders of the representatives of the conjugacy classes of  $B(2, 6)$  corresponding to the class of  $S_6$  of type  $(1^4 2^1)$ , these are also shown in column 2 of Table 3.1.

Using Proposition 3.1.7 and Programme 2.2.3, we have developed the following programmes for computing the orders of centralizers and orders of class representatives of  $B(m, n)$  respectively. Both the orders of centralizers and orders of class representatives are arranged in order of the arrangement of the  $m$ -sets of partitions of  $n$  as given by Programme 2.2.3.

**Programme 3.1.9 (Orders of Centralizers in  $B(m, n)$ )**

```

cordBmn:=function(m,n)
local l1,mcn,L,p,q,a,b,mspn1,a1,u,v,t,q1,z,h,co;
l1:=List([0,1..n]);;
mcn:=List(l1,i->Reversed(Filtered(Tuples([0..i],m),j->Sum(j)=i)));;
p:=Partitions(n);;
L:=List([1..n], i->n+1-i);;
q:=List([1..Number(p)],i->List(L,j->Number(p[i],k->k=j)));;
a:=List(q, i->Cartesian(List(i,j->mcn[j+1])));;
b:=List(a, i->List(i, j->TransposedMat(j)));;
mspn1:=List(b, i->List(i, j->List(j, k->List([1..Number(k)],
l->k[Number(k)+1-l]))));;
a1:=List(mspn1,i->List(i,j->TransposedMat(j)));;
u:=List(a1,l->List(l,i->Product(List(i,j->Product(List(j,

```

```

k- > Factorial(k))))));;
v:=List(q, i- > Sum(i));; t:=List(v, i- > m^i);;
q1:=List([1..Number(p)], i- > List([1..n], j- > Number(p[i], k- > k=j)));;
z:=List([1..Number(p)], i- > Product(List([1..n], j- > j^(q1[i][j]))));;
h:=List([1..Number(p)], i- > z[i]*t[i]);;
return List([1..Number(p)], i- > u[i]*h[i]);;
end;

```

**Programme 3.1.10 (Orders of class representatives in  $B(m, n)$ )**

```

ord1:=function(m,a)
local i,res;
for i in [1..m] do
if a[i]=0 then a[i]:=1; else
a[i]:=m/Gcd(i-1,m);
fi;
od;
return a;
end;

ord2:=function(m,n)
local mcn,u,res;
mcn:=List([1..n], i- > Reversed(Filtered(Tuples([0..i],m), j- > Sum(j)=i)));;
for u in mcn do
res:=List(u, k- > ord1(m,k));
od;
return res;
end;

```

```

elodsBmn:=function(m,n)
local mcnc,i,L,p,q,r,ctn,ctn1,lths,mults,elods;
mcnc:= List([1..n], i- > List(ord2(m,i), j- > Lcm(j)));;
L:=List([1..n], i- > n+1-i);;
p:=Partitions(n);;
q:=List([1..Number(p)], i- > List(L, j- > Number(p[i], k- > k=j)));;

```

```

r:=List(q, i- >Filtered(i, j- > j > 0));;
ctn:=List(r, i- >Cartesian(List(i, j- > mcnc[j])));;
ctn1:=List(ctn, i- >List(i, j- >List([1..Number(j)], k- > j[Number(j)+1-k])));;
lths:=List(p, i- >Union(List(i, j- > [j])));;
mults:=List([1..Number(ctn1)], i- > (List([1..Number(ctn1[i]), j- >
List([1..Number(ctn1[i][j]), k- > ctn1[i][j][k]*lths[i][k]))));;
return List(mults, i- >List(i, j- > Lcm(j)));
end;
    
```

We obtain all the orders of centralizers in  $B(2, 6)$  by supplying  $n := 6$  and  $m := 2$  to GAP and running the Programme 3.1.9. Further we obtain all the orders of class representatives of  $B(2, 6)$  including those corresponding to the conjugacy class of  $S_6$  of type  $(1^4 2^1)$  as given in Table 3.1 by supplying  $n := 6$  and  $m := 2$  to GAP and running the Programme 3.1.10. Consequently we obtain that  $B(2, 6)$  has a total of 65 conjugacy classes as listed in Tables 6.1 and 6.2.

We remark that apart from giving the conjugacy classes of  $B(m, n)$ , Programmes 3.1.9 and 3.1.10 also give parameters for the Fischer-Clifford matrices of  $B(m, n)$ . This is equivalent to coset analysis (see Subsection 1.2.1) for calculating the conjugacy classes of group extensions  $N.G$ , where  $N$  is abelian. Further analogous programmes for computing the orders of centralizers and orders of class representatives of  $B(m, n)$  in GAP are given in [50]. However Programmes 3.1.9 and 3.1.10 compute these values in order of the arrangements of the columns of the combinatorially computed matrices which are row equivalent to the Fischer-Clifford matrices of  $B(m, n)$  to be discussed later.

### 3.2 The Actions of $S_n$ on $N$ and $Irr(N)$

In this section, we give results which describe the orbits of the action of  $S_n$  on the group  $N$  and its irreducible characters. Let  $N = \langle e_1, e_2, \dots, e_n \rangle$ , regarded as a  $\mathbb{Z}_m$ -module. Then every element  $d$  of  $N$  can be written additively as  $d = \sum_{s=1}^n l_s e_s$ . The group  $S_n$  acts on  $N$  by permuting the generators  $e_s$  of  $N$ , that is  $(e_s)^g = e_{g(s)}$  where  $s = 1, 2, \dots, n$ . Explicitly if  $\phi : S_n \rightarrow \text{Aut}(N)$  is a homomorphism defining the

action of  $S_n$  on  $N$ , then

$$\phi(g)(d) = \sum_{s=1}^n l_s e_{g(s)}, \quad l_s \in \mathbb{Z}_m.$$

**Theorem 3.2.1** *The symmetric group  $S_n$  acts on  $N$  producing  $\binom{n+m-1}{m-1}$  orbits of lengths  $\binom{n}{k_1 \ k_2 \ \dots \ k_m}$  with orbit representatives respectively given by  $d = (l_1, l_2, \dots, l_n)$ , where  $k_i$  is the number of entries  $l_s$  equal to  $i - 1$  in  $d$ ,  $i = 1, 2, \dots, m$  and  $(k_1, k_2, \dots, k_m) \in A(n, m)$ .*

**Proof.** Let  $g \in S_n$  and  $d = \sum_{s=1}^n l_s e_s \in N$ , where  $l_s \in \mathbb{Z}_m$ . Then  $d^g = (\sum_{s=1}^n l_s e_s)^g = \sum_{s=1}^n l_s e_{g(s)} = \sum_{s=1}^n l_{(g^{-1}(s))} e_s$ , effectively permuting the coefficients  $l_s$  as the  $e_s$ 's can be re-arranged. Now considering  $d = (l_1, l_2, \dots, l_n)$ , an orbit  $d^{S_n}$  of the action of  $S_n$  on  $N$  containing  $d$ , will contain all elements of  $N$  which arise from  $d$  by permutating the  $l_s$ 's in  $d$ . Let  $k_i$  be the number of entries  $l_s$  equal to  $i - 1$  in  $d$ , ( $i = 1, 2, \dots, m$ ). Then permuting  $k_i$  equal  $l_s$ 's produces  $k_i!$  identical elements of  $N$ . Thus the orbit containing  $d$  has length

$$\frac{n!}{k_1! \ k_2! \ \dots \ k_m!} = \binom{n}{k_1 \ k_2 \ \dots \ k_m}.$$

We claim that  $(k_1, k_2, \dots, k_m) \in A(n, m)$ . But  $k_i$  is clearly the number of positions in  $d$  with  $i - 1$  as an entry. Thus  $\sum_{i=1}^m k_i = n$ , the total number of positions in  $d$  as claimed. We have seen that for each orbit  $d^{S_n}$  of the action of  $S_n$  on  $N$ , there is an  $m$ -composition  $(k_1, k_2, \dots, k_m)$  of  $n$  such that  $d^{S_n}$  has length  $\binom{n}{k_1 \ k_2 \ \dots \ k_m}$ . Now by Proposition 2.1.2,  $N(n, m) = \binom{n+m-1}{m-1}$ , which is the required number of orbits.  $\square$

However for  $m = 2$ , we have the following special case of Theorem 3.2.1

**Corollary 3.2.2** *The group  $S_n$  acts on  $N = 2^n$  giving  $n + 1$  orbits of lengths  $\binom{n}{n-k \ k}$ ,  $k = 0, 1, \dots, n$ , with respective orbit representatives given by  $d = (l_1, l_2, \dots, l_n)$ , where  $n - k$  is the number of  $l_s$ 's equal to 0 and  $k$  is the number of  $l_s$ 's equal to 1 in  $d$ , and  $(n - k, k) \in A(n, 2)$ .*

**Proof.** Follows from Theorem 3.2.1  $\square$

We use Programmes 2.1.3 and 2.1.4 to compute the  $m$ -compositions of  $n$  and the corresponding multi-nomial coefficients respectively. These are then used in writing down the lengths of the orbits of the action of  $S_n$  on  $N = \mathbb{Z}_m^n$  as given by Theorem 3.2.1.

**Example 3.2.3** The group  $S_6$  acts on  $2^6$  giving  $\binom{6+2-1}{2-1} = 7$  orbits of lengths

$$\binom{6}{6\ 0} = 1, \quad \binom{6}{5\ 1} = 6, \quad \binom{6}{4\ 2} = 15, \quad \binom{6}{3\ 3} = 20, \quad \binom{6}{2\ 4} = 15, \quad \binom{6}{1\ 5} = 6, \quad \binom{6}{0\ 6} = 1,$$

with the following respective orbit representatives:  $(0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 1), (0, 0, 0, 0, 1, 1), (0, 0, 0, 1, 1, 1), (0, 0, 1, 1, 1, 1), (0, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1)$ .

We note from Theorem 3.2.1 that an orbit of length  $\binom{n}{k_1\ k_2\ \dots\ k_m}$  contains elements of  $N$  with  $k_i$  entries equal to  $i - 1$ ,  $i = 1, 2, \dots, m$ . In view of this we obtain the following definition.

**Definition 3.2.4** An element  $d \in N$  is said to define the  $m$ -composition  $(k_1, k_2, \dots, k_m)$  of  $n$  if  $d$  contains  $k_i$  entries equal to  $i - 1$ , ( $i = 1, 2, \dots, m$ ). Since every  $d \in d^{S_n}$  defines the same  $m$ -composition of  $n$ , we shall equivalently say that the orbit  $d^{S_n}$  defines  $(k_1, k_2, \dots, k_m)$ . Let  $A(n, m)_{d^{S_n}}$  denote the  $m$ -composition of  $n$  defined by  $d^{S_n}$ .

We remark that it is known (see [5]) that the inertia factor groups of  $B(m, n)$  are given by the young subgroups  $\bar{I} = S_{k_1} \times S_{k_2} \times \dots \times S_{k_m}$ , where  $(k_1, k_2, \dots, k_m) \in A(n, m)$  and  $S_{k_i}$  is a symmetric group of degree  $k_i$ . In [47], Mpono has developed a computer programme in CAYLEY [10] which computes the elements of the orbits of the action of  $S_n$  on  $N$ , but this programme does not explicitly give the lengths of the orbits nor any information about the inertia factor groups.

We now consider the action of  $S_n$  on the irreducible characters of  $N$ . Let  $\xi$  be a primitive  $m^{\text{th}}$  root of unity and  $\mathbb{Z}_m = \langle e \rangle$ . Then  $\text{Irr}(\mathbb{Z}_m) = \{\theta_i \mid \theta_i(l.e) = \xi^{l(i-1)}, i = 1, 2, \dots, m, l = 0, 1, 2, \dots, m-1\}$  is the set of irreducible characters of  $\mathbb{Z}_m$ . We choose not to distinguish between the  $\theta_i$  taking the same value  $\xi^{l(i-1)}$  on elements of different copies of  $\mathbb{Z}_m$ . Therefore  $\xi^{(\sum_{s=1}^{k_i} l_s)(i-1)}$  is the value of  $\theta_i^{\otimes k_i} = \theta_i \times \theta_i \times \theta_i \times \dots \times \theta_i$  ( $k_i$  times) on the element  $\sum_{s=1}^{k_i} l_s e_s$  of  $k_i$  copies of  $\mathbb{Z}_m$ .

**Definition 3.2.5** Let  $\theta_i^{\otimes k_i}$  as above. Then  $\theta = \theta_1^{\otimes k_1} \theta_2^{\otimes k_2} \dots \theta_m^{\otimes k_m}$  is an irreducible character of  $N$ . For each  $(k_1, k_2, \dots, k_m) \in A(n, m)$  we define  $\Theta(k_1, k_2, \dots, k_m)$  to be the set of all irreducible characters obtained from  $\theta$  by permuting  $\theta_i^{\otimes u_i}$  and  $\theta_{i'}^{\otimes u_{i'}}$  for  $i \neq i' \in \{1, 2, \dots, m\}$ ,  $u_i \in \{1, 2, \dots, k_i\}$  and  $u_{i'} \in \{1, 2, \dots, k_{i'}\}$ . Thus an irreducible character of  $N$  may be written as  $\theta = \prod_i \theta_i^{\otimes u_i}$ , where some of the  $\theta_i$ 's may be equal for different  $i$ 's. We say that  $\theta$  is defined by  $(k_1, k_2, \dots, k_m)$  if  $\theta \in \Theta(k_1, k_2, \dots, k_m)$ .

It is clear that  $\xi^{\sum_{i=1}^m (\sum_{s=1}^{k_i} l_s)(i-1)}$  is the character value of  $\theta$  on an element  $d = \sum_{s=1}^n l_s e_s$ , of  $N$ . However since  $k_i$  is the number of entries  $l_s$  equal to  $i - 1$  in  $d$ , ( $i = 1, 2, \dots, m$ ), we also have

$$\theta(d) = \xi^{\sum_{i=1}^m (\sum_{s=1}^{k_i} l_s)(i-1)} = \xi^{\sum_{i=1}^m \sum_{j=1}^m (i-1)(j-1)r_{ij}},$$

with  $\sum_{i=1}^m \sum_{s=1}^{k_i} l_s$  rewritten as  $\sum_{j=1}^m k_j(j-1)$  where  $k_j = \sum_{i=1}^m r_{ij}$ . We also observe that

$$\text{Irr}(N) = \bigcup_{(k_1, k_2, \dots, k_m) \in A(n, m)} \Theta(k_1, k_2, \dots, k_m). \quad (3.4)$$

**Examples 3.2.6** (a) Let  $m = 4$  and  $n = 3$ . Then we have

$$\text{Irr}(\mathbb{Z}_4^3) = \bigcup_{(k_1, k_2, k_3, k_4) \in A(3, 4)} \Theta(k_1, k_2, k_3, k_4),$$

where for instance

$$\Theta(3, 0, 0, 0) = \{\theta_1^{\otimes 3}\}, \quad \Theta(0, 2, 1, 0) = \{\theta_2^{\otimes 2} \theta_3^{\otimes 1}, \theta_2^{\otimes 1} \theta_3^{\otimes 1} \theta_2^{\otimes 1}, \theta_3^{\otimes 1} \theta_2^{\otimes 2}\},$$

and so on.

(b) To illustrate the use of the formula  $\theta(d) = \xi^{\sum_{i=1}^m (\sum_{s=1}^{k_i} l_s)(i-1)}$  in calculating the value of  $\theta = \theta_2^{\otimes 2} \theta_3^{\otimes 1}$  on elements  $2e_2 + 3e_3$  and  $e_1 + 2e_2 + e_3$  of  $N = \mathbb{Z}_4^3$ , we have

$$(i) \theta_2^{\otimes 2} \theta_3^{\otimes 1}(0e_1 + 2e_2 + 3e_3) = \xi^{(0+2)(2-1)} \times \xi^{3(3-1)} = \xi^{(2+6)} = \xi^8 = 1.$$

$$(ii) \theta_2^{\otimes 2} \theta_3^{\otimes 1}(e_1 + 2e_2 + e_3) = \xi^{(1+2)(2-1)} \times \xi^{1(3-1)} = \xi^{(3+2)} = \xi^5 = \xi.$$

**Theorem 3.2.7** The symmetric group  $S_n$  acts on the irreducible characters of  $N$  producing  $\binom{n+m-1}{m-1}$  orbits  $\Theta(k_1, k_2, \dots, k_m)$ , of lengths  $\binom{n}{k_1 k_2 \dots k_m}$  with orbit representatives respectively given by  $\theta = \theta_1^{\otimes k_1} \theta_2^{\otimes k_2} \dots \theta_m^{\otimes k_m}$ , where  $(k_1, k_2, \dots, k_m) \in A(n, m)$ .

**Proof.** Let  $\theta \in \Theta(k_1, k_2, \dots, k_m)$  such that  $\theta(d) = \xi^{\sum_{i=1}^m (\sum_{s=1}^{k_i} l_s)(i-1)}$ , where  $d = \sum_{s=1}^n l_s e_s \in N$ . Let  $g \in S_n$ . We show that  $\theta^g \in \Theta(k_1, k_2, \dots, k_m)$ , that is  $\theta^g$  is defined by the same  $m$ -composition  $(k_1, k_2, \dots, k_m)$  of  $n$  as  $\theta$ . We have

$$\theta^g \left( \sum_{s=1}^n l_s e_s \right) = \theta \left( \sum_{s=1}^n l_s e_{g(s)} \right) = \theta \left( \sum_{s=1}^n l_{(g^{-1}(s))} e_s \right) = \xi^{\sum_{i=1}^m (\sum_{s=1}^{k_i} l_{(g^{-1}(s))})(i-1)}.$$

The above value involves the same  $k_i$ 's in the same order as in  $\theta(d)$  since only the  $l_s$ 's change to  $l_{(g^{-1}(s))}$ . Therefore  $\theta^g \in \Theta(k_1, k_2, \dots, k_m)$ . We conclude that  $\Theta(k_1, k_2, \dots, k_m)$ , where  $(k_1, k_2, \dots, k_m) \in A(n, m)$ , is an orbit of the action of  $S_n$  on  $Irr(N)$  with representative  $\theta = \theta_1^{\otimes k_1} \theta_2^{\otimes k_2} \dots \theta_m^{\otimes k_m}$ . Thus the number  $N(n, m)$  of  $m$ -compositions of  $n$  is equal to the number of orbits of the action of  $S_n$  on  $Irr(N)$ . By Proposition 2.1.2,  $N(n, m) = \binom{n+m-1}{m-1}$ , which is the number of orbits of the action of  $S_n$  on  $Irr(N)$ .

Now, since  $\sum_{i=1}^m k_i = n$ , each  $\theta$  contains a total of  $n$  characters  $\theta_i$  with each  $\theta_i$  repeated  $k_i$  times. From Definition 3.2.5 an orbit of the action of  $S_n$  on  $Irr(N)$  is obtained by permuting  $\theta_i^{\otimes u_i}$ 's of  $\theta$  for different  $i$ 's but permuting  $\theta_i^{\otimes u_i}$ 's with the same  $i$  produces identical characters of  $N$ . Therefore the length  $|\Theta(k_1, k_2, \dots, k_m)|$  of each orbit of the action of  $S_n$  on  $Irr(N)$  is equal to  $\binom{n}{k_1 \ k_2 \ \dots \ k_m}$ .  $\square$

However for  $m = 2$ , we have a special case of Theorem 3.2.7

**Corollary 3.2.8** *The group  $S_n$  acts on the irreducible characters of  $N = 2^n$  giving  $n+1$  orbits of lengths  $\binom{n}{n-k \ k}$ , with irreducible character representatives for each orbit respectively given by  $\theta_1^{\otimes(n-k)} \theta_2^{\otimes k}$ , where  $k = 0, 1, 2, \dots, n$ .*

**Proof.** The result follows from Theorem 3.2.7.  $\square$

For an example, the action of  $S_3$  on  $Irr(\mathbb{Z}_4^3)$  produces  $N(3, 4) = 20$  orbits, namely  $\Theta(k_1, k_2, k_3, k_4)$  where  $(k_1, k_2, k_3, k_4) \in A(3, 4)$ .

We remark that according to Theorem 1.4.4 (Brauer's Theorem) the actions of  $S_n$  on  $N$  and  $Irr(N)$  produce the same number of orbits (not necessarily of the same lengths), which is the case here (compare Theorems 3.2.1 and 3.2.7 above).

**Remark 3.2.9** *Since  $|Irr(N)| = m^n$ , we have shown by construction that*

$$\sum_{(k_1, k_2, \dots, k_m) \in A(n, m)} \binom{n}{k_1 \ k_2 \ \dots \ k_m} = m^n.$$

### 3.3 The Groups $B_S(m, n)$ and $B_Q(m, n)$

In this section, we discuss two groups of order  $m^{n-1}n!$ , that is a subgroup  $B_S(m, n)$  of index  $m$  in  $B(m, n)$  and a quotient  $B_Q(m, n)$  of  $B(m, n)$  by a normal subgroup.

**Definition 3.3.1** Let  $N = \mathbb{Z}_m^n$  as above. We define the following sets.

- (i)  $M = \{l \sum_{s=1}^n e_s \mid l \in \mathbb{Z}_m\}$ .
- (ii)  $S = \{\sum_{s=1}^n l_s e_s \in N \mid \sum_{s=1}^n l_s \equiv 0 \pmod{m}\}$ .
- (iii)  $Q = N/M$ , the set of left cosets of  $N$  by  $M$ .

**Proposition 3.3.2** Let  $S$ ,  $M$  and  $Q$  as above. Then

- (i)  $M = \langle \sum_{s=1}^n e_s \rangle$  is a normal  $S_n$ -invariant subgroup of  $N$ .
- (ii)  $S = \langle e_1 + (m-1)e_{s+1} \mid s = 1, 2, \dots, n-1 \rangle$  is a subgroup of  $N$ . We write  $S = \mathbb{Z}_m^{n-1}$ .
- (iii)  $Q = \langle e_s + M \mid s = 1, 2, \dots, n-1 \rangle$  is a factor group of  $N$  by  $M$ . We write  $Q = \overline{\mathbb{Z}_m^n}$ .

Further the groups  $S$  and  $Q$  are  $S_n$ -invariant.

**Proof.**

- (i) By definition of  $M$ , we have  $v = \sum_{s=1}^n e_s$  generates the subspace  $M$ . Now  $lv + l'v = (l+l')v = l''v \in M$  for  $l, l', l'' \in \mathbb{Z}_m$  implies that  $M$  is a sub-group of  $N$ . Since  $N$  is abelian, its sub-groups are normal in  $N$  and so is  $M$ . Also since  $v = \sum_{s=1}^n e_s$  is a linear combination of all the generators of  $N$ , and  $S_n$  permutes the generators of  $N$  it follows that  $S_n$  fixes  $v$ . Therefore  $S_n$  fixes all scalar multiples of  $v$ . Hence  $M$  is  $S_n$ -invariant.
- (ii) Let  $a = \sum_{s=1}^n l_s e_s, b = \sum_{s=1}^n l'_s e_s \in S$ . Then  $\sum_{s=1}^n l_s = mk$  and  $\sum_{s=1}^n l'_s = mk'$  for some  $k, k' \in \mathbb{N}$ . We have

$$\sum_{s=1}^n (l_s - l'_s) = \sum_{s=1}^n l_s - \sum_{s=1}^n l'_s = mk - mk' = m(k - k') \equiv 0 \pmod{m}.$$

This shows that  $a - b \in S$ . Hence  $S$  is a subgroup of  $N$ . Now we show that the set  $\{e_1 + (m-1)e_{s+1} \mid s = 1, 2, \dots, n-1\}$  generates  $S$ . Let  $d \in S$ . We aim to find  $l'_s \in \mathbb{Z}_m$  such that

$$d = \sum_{s=1}^{n-1} l'_s [e_1 + (m-1)e_{s+1}].$$

Now we assume the above is the case and find  $l'_s$  in terms of  $l_s$ . Thus

$$\begin{aligned}
 d &= \sum_{s=1}^n l_s e_s = \sum_{s=1}^{n-1} l'_s [e_1 + (m-1)e_{s+1}] \\
 &\Rightarrow \sum_{s=1}^n l_s e_s = \sum_{s=1}^{n-1} l'_s e_1 + \sum_{s=1}^{n-1} l'_s (m-1)e_{s+1} \\
 &\Rightarrow l_1 = \sum_{s=1}^{n-1} l'_s, \quad l_s = (m-1)l'_{s-1}, \quad s = 2, 3, \dots, n \\
 &\Rightarrow l'_{s-1} = (m-1)l_s, \quad s = 2, 3, \dots, n.
 \end{aligned}$$

Hence  $S = \langle e_1 + (m-1)e_{s+1} \mid s = 1, 2, \dots, n-1 \rangle$  as required. Further it is clear that any element of the set  $\{e_1 + (m-1)e_{s+1} \mid s = 1, 2, \dots, n-1\}$  cannot be written as a linear combination of the other elements. Thus this is one of the smallest subsets of  $N$ , with  $n-1$  elements, which generates  $S$ .

(iii) Since  $M$  is normal in  $N$ ,  $Q$  is the factor group of  $N$  by  $M$ . We have  $N = \langle e_1, e_2, \dots, e_n \rangle$  implies that  $Q = \langle e_1 + M, e_2 + M, \dots, e_n + M \rangle$ . We show that  $e_n + M = l(\sum_{s=1}^{n-1} e_s) + M$  for some  $l \in \mathbb{Z}_m$ . Now

$$\begin{aligned}
 \left( \sum_{s=1}^n e_s \right) + M = M &\Leftrightarrow \left( \left( \sum_{s=1}^{n-1} e_s \right) + e_n \right) + M = M \\
 &\Leftrightarrow \left( \sum_{s=1}^{n-1} e_s + M \right) + (e_n + M) = M \\
 &\Leftrightarrow e_n + M = - \sum_{s=1}^{n-1} e_s + M.
 \end{aligned}$$

Thus  $Q = \langle e_1 + M, e_2 + M, \dots, e_{n-1} + M \rangle$ .

Let  $d \in S$  and  $g \in S_n$ . Then  $d^g$  contains the same number say  $k_j$  of  $l_s$ 's equal to  $j-1$  as  $d$ . Therefore the sum of the  $l_s$ 's in  $d^g$  is a multiple of  $m$ . Thus  $d^g \in S$  and hence  $S$  is  $S_n$ -invariant. Also for  $d + M \in Q$  we have  $(d + M)^{S_n} = d^{S_n} + M \in Q$  since  $M$  is  $S_n$ -invariant. Hence  $Q$  is  $S_n$ -invariant.  $\square$

**Remark 3.3.3** Let  $d = (l_1, l_2, \dots, l_n) \in N$ .

(i) Let  $k_j$  be the number of  $l_s$ 's equal to  $j - 1$  in  $d$ . Then

$$S = \bigcup_{d \in N} \left\{ d^{S_n} : |d^{S_n}| = \binom{n}{k_1 \ k_2 \ \dots \ k_m}, \sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m} \right\},$$

that is the group  $S$  is necessarily a collection of the orbits of the action of  $S_n$  on  $N$ , of length  $\binom{n}{k_1 \ k_2 \ \dots \ k_m}$  such that  $\sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}$ .

(ii) We have another generating set for  $S$  given by

$$S = \langle e_s + (m-1)e_{s+1} \mid s = 1, 2, \dots, n-1 \rangle.$$

Let  $\binom{n}{i}$  be a binomial coefficient,  $0 \leq i \leq n$ . Then by using elementary mathematical techniques we can easily prove that  $\sum_{i \equiv 0 \pmod{2}} \binom{n}{i} = 2^{n-1}$ , which also follows from the following theorem.

**Theorem 3.3.4** Let  $r \in \mathbb{Z}_m$  and  $(k_1, k_2, \dots, k_m) \in A(n, m)$ . Then

$$\sum_{\sum_{j=1}^m (j-1)k_j \equiv r \pmod{m}} \binom{n}{k_1 \ k_2 \ \dots \ k_m} = m^{n-1}.$$

**Proof.** By Lemma 3.1.5, the number of ways of arranging elements  $k_{ij}$  (with repetitions and in any order) in  $n$  positions such that  $\sum_{i=1}^n k_{ij} \equiv r \pmod{m}$  is  $m^{n-1}$ . Let  $k_j$  be the number of  $k_{ij}$ 's equal to  $j - 1$ ,  $j = 1, 2, \dots, m$ . Then  $\sum_{j=1}^m k_j = n$ . Now the number of ways of choosing  $k_j$  positions, all  $j = 1, 2, \dots, m$ , out of  $n$  is

$$\binom{n}{k_1 \ k_2 \ \dots \ k_m},$$

and varying the  $k_j$ 's such that  $\sum_{j=1}^m k_j = n$  gives the set  $A(n, m)$ . However we require those  $k_{ij}$ 's such that  $\sum_{i=1}^n k_{ij} \equiv r \pmod{m}$ , that is  $\sum_{j=1}^m (j-1)k_j \equiv r \pmod{m}$ . The result follows on summing over  $A(n, m)$  such that  $\sum_{j=1}^m (j-1)k_j \equiv r \pmod{m}$ .  $\square$

**Theorem 3.3.5** Let  $S = \mathbb{Z}_m^{n-1}$  as defined above. Then  $S.S_n$  defines a split extension of  $S$  by  $S_n$ .

**Proof.** Since  $S$  is an  $S_n$ -invariant subgroup of  $N$ ,  $S.S_n$  is a subgroup of  $B(m, n)$ . We also have  $S \cap S_n \subset N \cap S_n = 1_N$  and hence  $S.S_n$  is a semi-direct product.  $\square$

Let  $B_S(m, n)$  denote the split extension of  $S$  by  $S_n$  given in Theorem 3.3.5. The group  $B_S(m, n)$  has a presentation given by

$$\begin{aligned} B_S(m, n) = \langle & s_i (i = 1, 2, \dots, n-1), w_1 w_{j+1}^{m-1} (j = 1, 2, \dots, n-1) \mid \\ & s_i^2 = (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1, |j-i| \geq 2, w_j^m = 1, \\ & w_i w_j = w_j w_i (i \neq j), s_i w_i = w_{i+1} s_i, s_i w_j = w_j s_i, j \neq i, i+1 \rangle, \end{aligned}$$

where the elements of  $S$  are written multiplicatively.

**Theorem 3.3.6** *Let  $Q = \overline{\mathbb{Z}_m^n}$  as defined above. Then  $Q.S_n$  defines a split extension of  $Q$  by  $S_n$ .*

**Proof.** Since  $Q$  is  $S_n$ -invariant, we define a semi-direct product of  $Q$  by  $S_n$  that realizes this action. That is the set of ordered pairs  $(q, g)$  with  $q \in Q$  and  $g \in S_n$  with multiplication given by  $(q_1, g_1)(q_2, g_2) = (q_1 q_2^{g_1}, g_1 g_2)$ .  $\square$

Let  $B_Q(m, n)$  denote the split extension of  $Q$  by  $S_n$  given in Theorem 3.3.6. The group  $B_Q(m, n)$  has a presentation given by

$$\begin{aligned} B_Q(m, n) = \langle & s_i (i = 1, 2, \dots, n-1), w_j + M (j = 1, 2, \dots, n-1) \mid \\ & s_i^2 = (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1, |j-i| \geq 2, w_j^m = 1, \\ & w_i w_j = w_j w_i (i \neq j), s_i w_i = w_{i+1} s_i, s_i w_j = w_j s_i, j \neq i, i+1 \rangle, \end{aligned}$$

where the elements of  $Q$  are written multiplicatively.

Let  $\alpha = (1 \ 2 \ \dots \ m) \in S_m$  and  $(k_1, k_2, \dots, k_m) \in A(n, m)$ . Define an action of  $\alpha$  on  $(k_1, k_2, \dots, k_m)$  by  $(k_1, k_2, \dots, k_m)^\alpha = (k_{\alpha(1)}, k_{\alpha(2)}, \dots, k_{\alpha(m)})$ . Let  $[(k_1, k_2, \dots, k_m)]_\alpha$  denote the set of all the  $m$ -compositions of  $n$  obtained by acting  $\alpha^l$  on  $(k_1, k_2, \dots, k_m)$  for all  $l \in \mathbb{Z}_m$ . If  $A(n, m)_{dS_n} = (k_1, k_2, \dots, k_m)$  then we will write  $[A(n, m)_{dS_n}]$  in place of  $[(k_1, k_2, \dots, k_m)]_\alpha$ . We note that  $|[(k_1, k_2, \dots, k_m)]_\alpha| = o(\alpha) = m$ . Let  $(k_1, k_2, \dots, k_m)$  be the representative of the set  $[(k_1, k_2, \dots, k_m)]_\alpha$ . In the following two sections we discuss the actions of  $S_n$  on  $S$ ,  $Q$ ,  $Irr(S)$  and  $Irr(Q)$  respectively.

### 3.3.1 The Actions of $S_n$ on $S$ and $Irr(S)$

Since  $S_n$  acts on  $S$  and  $Irr(S)$  as it acts on  $N$  and  $Irr(N)$  respectively, the corresponding orbits of the actions of  $S_n$  on  $S$  and  $Irr(S)$  can easily be determined. We discuss the orbits of these actions below.

**Theorem 3.3.7** *The group  $S_n$  acts on  $S$  giving  $\left\lceil \frac{1}{m} \binom{n+m-1}{m-1} \right\rceil$  orbits of lengths  $\binom{n}{k_1 k_2 \dots k_m}$ , where  $(k_1, k_2, \dots, k_m) \in A(n, m)$  such that  $\sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}$ .*

**Proof.** Let  $d = \sum_{s=1}^n l_s e_s \in S$ . Let  $d^{S_n}$  be the orbit of the action of  $S_n$  on  $N$  containing  $d$ . Since  $g$  acts on  $d$  by permuting the entries  $l_s$ , every element of  $d^{S_n}$  contains the same number  $k_j$  of  $l_s$ 's equal to  $j-1$  as  $d$ , where  $j = 1, 2, \dots, m$ . Thus  $d^{S_n} \subset S$ . And by Remark 3.3.3 the group  $S$  is a collection of the orbits of the action of  $S_n$  on  $N$  containing  $\binom{n}{k_1 k_2 \dots k_m}$  elements such that  $\sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}$ . Clearly these orbits are retained in the action of  $S_n$  on  $S$ . Since  $\sum_{j=1}^m (j-1)k_j$  may be congruent  $\pmod{m}$  to  $m$  possible values, we divide the number of orbits of the action of  $S_n$  on  $N$  by  $m$  but taking the upper part  $\left\lceil \frac{1}{m} \binom{n+m-1}{m-1} \right\rceil$  as required.  $\square$

**Example 3.3.8** *The group  $S_6$  acts on  $S = 2^5$  giving four orbits of lengths 1, 15, 15, 1.*

We define an action of  $\alpha$  on an irreducible character  $\theta = \prod_i \theta_i^{\otimes u_i}$  of  $N$  by  $\theta^\alpha = \prod_i \theta_{\alpha(i)}^{\otimes u_i}$ . Let  $[\theta]_\alpha = \{\theta^{\alpha^l}, l \in \mathbb{Z}_m\}$ . It is clear that  $\alpha$  acts on  $(k_1, k_2, \dots, k_m)$ , hence  $\alpha$  also acts on the orbits  $\Theta(k_1, k_2, \dots, k_m)$  of the irreducible characters of  $N$  as

$$(\Theta(k_1, k_2, \dots, k_m))^\alpha = \Theta(k_{\alpha(1)}, k_{\alpha(2)}, \dots, k_{\alpha(m)})$$

giving the orbit

$$[\Theta(k_1, k_2, \dots, k_m)]_\alpha = \{\Theta(k_{\alpha^l(1)}, k_{\alpha^l(2)}, \dots, k_{\alpha^l(m)}) \mid l \in \mathbb{Z}_m\},$$

with representative orbit  $\Theta(k_1, k_2, \dots, k_m)$ . We have

$$Irr(N) = \bigcup_{(k_1, k_2, \dots, k_m) \neq (k_{\alpha^l(1)}, k_{\alpha^l(2)}, \dots, k_{\alpha^l(m)})} [\Theta(k_1, k_2, \dots, k_m)]_\alpha,$$

where the union is taken over representatives  $(k_1, k_2, \dots, k_m)$ , of the orbits  $[(k_1, k_2, \dots, k_m)]_\alpha$ .

**Lemma 3.3.9** *Let  $\theta \in \Theta(k_1, k_2, \dots, k_m)$ . Then  $\theta^{\alpha^l}, l \neq 0$  does not belong to  $\Theta(k_1, k_2, \dots, k_m)$ .*

**Proof.** Since  $\theta \in \Theta(k_1, k_2, \dots, k_m)$  implies that  $\theta$  contains  $k_i$  copies of  $\theta_i$  while  $\theta^{\alpha^l}$  contains  $k_i$  copies of  $\theta_{\alpha^l(i)}$ , and  $\theta_{\alpha^l(i)} \neq \theta_i$  for  $l \neq 0$ , it follows that  $\theta^{\alpha^l}$  does not belong to  $\Theta(k_1, k_2, \dots, k_m)$ .  $\square$

**Lemma 3.3.10** *Let  $\theta^{\alpha^l}$  as above. Then  $\theta \downarrow S = \theta^{\alpha^l} \downarrow S, l \in \mathbb{Z}_m$ .*

**Proof.** We show that for  $d \in S$  we have  $\theta(d) = \theta^{\alpha^l}(d)$ , that is

$$\begin{aligned}
 & \xi^{\sum_i (\sum_{s=1}^{u_i} l_s)(i-1)} = \xi^{\sum_i (\sum_{s=1}^{u_i} l_s)(\alpha^l(i)-1)} \\
 \Leftrightarrow & \sum_i \sum_{s=1}^{u_i} l_s(i-1) = \left[ \sum_i \sum_{s=1}^{u_i} l_s(\alpha^l(i)-1) \right] \pmod{m} \\
 \Leftrightarrow & \sum_i \sum_{s=1}^{u_i} l_s(i-1) - \left[ \sum_i \sum_{s=1}^{u_i} l_s(\alpha^l(i)-1) \right] \equiv 0 \pmod{m} \\
 \Leftrightarrow & \sum_i \sum_{s=1}^{u_i} l_s[(i-1) - (\alpha^l(i)-1)] \equiv 0 \pmod{m} \\
 \Leftrightarrow & \sum_i \sum_{s=1}^{u_i} l_s(i - (i+l)) \equiv 0 \pmod{m} \\
 \Leftrightarrow & \sum_i \sum_{s=1}^{u_i} l_s(-l) \equiv 0 \pmod{m} \\
 \Leftrightarrow & (-l) \sum_i \sum_{s=1}^{u_i} l_s \equiv 0 \pmod{m} \\
 \Leftrightarrow & (-l) \sum_{s=1}^n l_s \equiv 0 \pmod{m}.
 \end{aligned}$$

This is so for elements  $d$  of  $S$ . Hence  $\theta \downarrow S = \theta^{\alpha^l} \downarrow S, l \in \mathbb{Z}_m$ .  $\square$

We remark that Lemma 3.3.10 shows that the irreducible characters in the orbit  $[\theta]_{\alpha}$  are equal on  $S$ .

**Example 3.3.11** We consider the group  $N = 3^3$  with  $d = 2e_1 + e_2 \in S = 3^2$ . With the 3-cycle  $\alpha = (1\ 2\ 3)$ , we obtain that  $\theta_1^{\otimes 1} \theta_2^{\otimes 1} \theta_3^{\otimes 1}(d) = \theta_2^{\otimes 1} \theta_3^{\otimes 1} \theta_1^{\otimes 1}(d) = \theta_3^{\otimes 1} \theta_1^{\otimes 1} \theta_2^{\otimes 1}(d) = \xi$ .

**Proposition 3.3.12 (Irreducibles of  $S$ )**

$$\text{Irr}(S) = \bigcup_{(k_1, k_2, \dots, k_m) \neq (k_{\alpha^l(1)}, k_{\alpha^l(2)}, \dots, k_{\alpha^l(m)})} \Theta(k_1, k_2, \dots, k_m),$$

where the union is taken over representatives  $(k_1, k_2, \dots, k_m)$ , of the orbits  $[(k_1, k_2, \dots, k_m)]_\alpha$ .

**Proof.** This follows from Lemmas 3.3.9 and 3.3.10.  $\square$

We now describe the action of  $S_n$  on  $\text{Irr}(S)$  in the following.

**Theorem 3.3.13** Let  $(k_1, k_2, \dots, k_m)$  be a representative of  $[(k_1, k_2, \dots, k_m)]_\alpha$ . The symmetric group  $S_n$  acts on  $\text{Irr}(S)$  producing

- (i)  $\frac{1}{m} \left\{ \binom{n+m-1}{m-1} - 1 \right\}$  orbits of lengths  $\binom{n}{k_1\ k_2\ \dots\ k_m}$ , with  $k_i \neq k_j$  for some  $i \neq j$  and one orbit of length  $\frac{1}{m} \binom{n}{k\ k\ \dots\ k}$  if  $m \mid n$  and  $k = \frac{n}{m}$ ,
- (ii)  $\frac{1}{m} \binom{n+m-1}{m-1}$  orbits of lengths  $\binom{n}{k_1\ k_2\ \dots\ k_m}$ , with  $k_i \neq k_j$  for some  $i \neq j$ , if  $m \nmid n$ ,

Further an orbit representative of the action of  $S_n$  on  $\text{Irr}(S)$  is given by  $\theta_1^{\otimes k_1} \theta_2^{\otimes k_2} \dots \theta_m^{\otimes k_m}$  for each representative  $(k_1, k_2, \dots, k_m)$  of  $[(k_1, k_2, \dots, k_m)]_\alpha$ .

**Proof.**

- (i) Let  $m \mid n$  where  $k = \frac{n}{m}$ , for some positive integer  $k$ . This implies that there is one  $m$ -composition of  $n$  given by  $(k, k, \dots, k)$ . Thus we obtain  $\binom{n+m-1}{m-1} - 1$   $m$ -compositions of  $n$  different from  $(k, k, \dots, k)$ . By Lemmas 3.3.9 and 3.3.10 and Proposition 3.3.12, some of the orbits of the action of  $S_n$  on  $\text{Irr}(N)$  contain characters which are equal on  $S$ . This happens in cases where the lengths of the orbits of the action of  $S_n$  on  $\text{Irr}(N)$  employs  $(k_1, k_2, \dots, k_m) \in [(k_1, k_2, \dots, k_m)]_\alpha$ . In such cases the length of the orbit of the action of  $S_n$  on  $\text{Irr}(S)$  is given by  $\binom{n}{k_1\ k_2\ \dots\ k_m}$ . Since  $o(\alpha) = m$  the above happens  $m$  times and so we obtain  $\frac{1}{m} \left\{ \binom{n+m-1}{m-1} - 1 \right\}$  orbits of length  $\binom{n}{k_1\ k_2\ \dots\ k_m}$ . However since there is only one

orbit corresponding to  $(k, k, \dots, k)$ , the length of the orbit of the action of  $S_n$  on  $Irr(N)$  must be divided by  $m$  to give the length of the orbit of the action of  $S_n$  on  $Irr(S)$  as  $\frac{1}{m} \binom{n}{k \ k \ \dots \ k}$ .

- (ii) Clearly if  $m \nmid n$ , all the  $m$ -compositions of  $n$  are different from  $(k, k, \dots, k)$  where  $k = \frac{n}{m}$ . Applying part (i) above with the case where the  $m$ -compositions of  $n$  are different from  $(k, k, \dots, k)$ , the result follows.

Further the orbit representative of the action of  $S_n$  on  $Irr(S)$  above follows from the fact that the orbits of the action of  $S_n$  on  $Irr(S)$  are also orbits of the action of  $S_n$  on  $Irr(N)$ .  $\square$

**Remark 3.3.14** *Following from the action of  $S_n$  on  $Irr(S)$  as described in Theorem 3.3.13, the required inertia factor groups of  $B_S(m, n)$  are chosen from those young subgroups  $S_{k_1} \times S_{k_2} \times \dots \times S_{k_m}$  of  $B(m, n)$  corresponding to representative  $m$ -compositions of  $n$  for each orbit  $[(k_1, k_2, \dots, k_m)]_\alpha$ . However in the case where  $m \mid n$  or the  $m$ -composition of  $n$  given by  $(k, k, \dots, k)$  exists, the corresponding Inertia factor group is given by the wreath product  $S_{\frac{n}{m}} \text{Wr} C_m$ , namely the split extension  $(S_{\frac{n}{m}})^m : C_m$ .*

**Corollary 3.3.15** *The group  $S_n$  acts on  $Irr(2^{n-1})$ , giving  $\lceil \frac{n+1}{2} \rceil$  orbits of lengths*

- (i)  $\left\{ \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \frac{1}{2} \binom{n}{\frac{n}{2}} \right\}$ , for even  $n$ . In this case the inertia factor groups of  $B_S(2, n)$  are given by young subgroups  $S_{n-k} \times S_k$ ,  $k = 0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$  and the split extension  $(S_{\frac{n}{2}})^2 : C_2$ ,
- (ii)  $\left\{ \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{\frac{n-1}{2}} \right\}$ , for odd  $n$ . In this case the inertia factor groups of  $B_S(2, n)$  are given by young subgroups  $S_{n-k} \times S_k$ ,  $k = 0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$ .

**Proof.** The proof follows from Theorem 3.3.13 for  $m = 2$ . The inertia factor groups follow from the lengths given by the binomial coefficients  $\binom{n}{k} = \binom{n}{n-k}$  of the orbits giving the corresponding inertia factor groups  $S_{n-k} \times S_k$ .  $\square$

**Examples 3.3.16** *Using Theorem 3.3.13, Remark 3.3.14 and Corollary 3.3.15 above we have that*

- (i) for  $n = 5$  and  $m = 2$ ,  $S = 2^4$ , the action of  $S_5$  on  $Irr(2^4)$  yields three orbits of lengths 1, 5, and 10, and the inertia factor groups of  $B_S(2, 5)$  are :  $S_5$ ,  $S_4 \times S_1$ , and  $S_3 \times S_2$  of indices 1, 5 and 10 in  $S_5$  respectively,

- (ii) for  $n = 6$  and  $m = 2$ ,  $S = 2^5$ , the action of  $S_6$  on  $\text{Irr}(2^5)$  yields four orbits of lengths 1, 6, 15 and 10, and the inertia factor groups of  $B_S(2, 6)$  are :  $S_6$ ,  $S_5 \times S_1$ ,  $S_4 \times S_2$  and  $(S_3)^2:C_2$  of indices 1, 6, 15 and 10 in  $S_6$  respectively,
- (iii) for  $n = 7$  and  $m = 2$ ,  $S = 2^6$ , the action of  $S_7$  on  $\text{Irr}(2^6)$  yields four orbits of lengths 1, 7, 21 and 35, and the inertia factor groups of  $B_S(2, 7)$  are :  $S_7$ ,  $S_6 \times S_1$ ,  $S_5 \times S_2$  and  $S_4 \times S_3$  of indices 1, 7, 21 and 35 in  $S_7$  respectively,
- (iv) for  $n = 8$  and  $m = 2$ ,  $S = 2^7$ , the action of  $S_8$  on  $\text{Irr}(2^7)$  yields five orbits of lengths 1, 8, 28, 56 and 35, and the inertia factor groups of  $B_S(2, 8)$  are :  $S_8$ ,  $S_7 \times S_1$ ,  $S_6 \times S_2$ ,  $S_5 \times S_3$  and  $(S_4)^2:C_2$  of indices 1, 8, 28, 56, and 35 in  $S_8$  respectively,
- (v) for  $n = 12$  and  $m = 2$ ,  $S = 2^{11}$ , the action of  $S_{12}$  on  $\text{Irr}(2^{11})$  yields seven orbits of lengths 1, 12, 66, 220, 495, 792, and 462, and the inertia factor groups of  $B_S(2, 12)$  are :  $S_{12}$ ,  $S_{11} \times S_1$ ,  $S_{10} \times S_2$ ,  $S_9 \times S_3$ ,  $S_8 \times S_4$ ,  $S_7 \times S_5$  and  $(S_6)^2:C_2$  of indices 1, 12, 66, 220, 495, 792, and 462 in  $S_{12}$  respectively,
- (vi) for  $n = 3$  and  $m = 3$ ,  $S = 3^2$ , the action of  $S_3$  on  $\text{Irr}(3^2)$  yields four orbits of lengths 1, 3, 3 and 2, and the inertia factor groups of  $B_S(3, 3)$  are :  $S_3$ ,  $S_2 \times S_1$ ,  $S_2 \times S_1$  and  $(S_1)^2:C_3$  of indices 1, 3, 3, and 2 in  $S_3$  respectively,
- (vii) for  $n = 5$  and  $m = 3$ ,  $S = 3^4$ , the action of  $S_5$  on  $\text{Irr}(3^4)$  yields seven orbits of lengths 1, 5, 5, 10, 20, 10 and 30, and the inertia factor groups of  $B_S(3, 5)$  are:  $S_5$ ,  $S_4 \times S_1$ ,  $S_4 \times S_1$ ,  $S_3 \times S_2$ ,  $S_3 \times S_1 \times S_1$ ,  $S_3 \times S_2$  and  $S_2 \times S_2 \times S_1$  of indices 1, 5, 5, 10, 20, 10 and 30 in  $S_5$  respectively.

### 3.3.2 The Actions of $S_n$ on $Q$ and $\text{Irr}(Q)$

Since  $S_n$  acts on  $Q$  by acting on the transversal elements it follows that  $S_n$  acts on  $Q$  and  $\text{Irr}(Q)$  as it acts on  $N$  and  $\text{Irr}(N)$  respectively, and the corresponding orbits of the actions of  $S_n$  on  $Q$  and  $\text{Irr}(Q)$  can easily be determined. We begin by discussing the action of  $S_n$  on  $Q$  in the following.

**Lemma 3.3.17** *Let  $d, d' \in N$ . Let  $A(n, m)_{dS_n} = (k_1, k_2, \dots, k_m)$ . Then  $d + M = d' + M$  if and only if  $d' = d + l(1, 1, \dots, 1)$ , where  $l(1, 1, \dots, 1) \in M$  for some  $l \in \mathbb{Z}_m$ . Further we have that  $A(n, m)_{(d')S_n} \in [A(n, m)_{dS_n}]$ .*

**Proof.** Since  $d+M = d'+M \Leftrightarrow d-d' \in M$ , equivalently we have that  $d' = d+d''$ , for some  $d'' = l \sum_{s=1}^n e_s = l(1, 1, \dots, 1)$ ,  $l \in \mathbb{Z}_m$ . We note that the addition of  $l \sum_{s=1}^n e_s$  to  $d$  as found above, moves the  $k_j$  in  $A(n, m)_{dS_n}$  cyclically such that  $d'$  will have  $k_j$  entries equal to  $j$  while  $d$  has  $k_j$  entries equal to  $j-1$ . Thus as  $d''$  is added to  $d$  to give  $d'$ ,  $A(n, m)_{dS_n}$  correspondingly goes through the following  $m$ -compositions of  $n$ :  $(k_1, k_2, \dots, k_m), (k_m, k_1, \dots, k_{m-1}), (k_{m-1}, k_m, k_1, \dots, k_{m-2}), \dots, (k_2, k_3, \dots, k_m, k_1)$ , all of which belong to  $[A(n, m)_{dS_n}]$ . Thus  $A(n, m)_{(d')S_n} \in [A(n, m)_{dS_n}]$  as required.  $\square$

**Theorem 3.3.18** *The symmetric group  $S_n$  acts on  $Q$  producing*

- (i)  $\frac{1}{m} \left\{ \binom{n+m-1}{m-1} - 1 \right\}$  orbits of lengths  $\binom{n}{k_1 k_2 \dots k_m}$  with  $k_i \neq k_j$  for some  $i \neq j$  and one orbit of length  $\frac{1}{m} \binom{n}{k k \dots k}$ , if  $m \mid n$ ,
- (ii)  $\frac{1}{m} \binom{n+m-1}{m-1}$  orbits of lengths  $\binom{n}{k_1 k_2 \dots k_m}$  with  $k_i \neq k_j$  for some  $i \neq j$ , if  $m \nmid n$ ,

where  $(k_1, k_2, \dots, k_m) \in [A(n, m)_{dS_n}]$ ,  $d \in N$ . Further the orbit representatives are given by  $d+M$  such that  $d = (l_1, l_2, \dots, l_n)$ , where  $k_j$  is the number of entries  $l_s$  equal to  $j-1$  in  $d$ .

**Proof.** Let  $d' \in d^{S_n}$  and  $g \in S_n$  such that  $d' = d^g$ . First, we note that since

$$d' + M = d^g + M = (d + M)^g \in (d + M)^{S_n},$$

each orbit  $d^{S_n}$  of the action of  $S_n$  on  $N$  gives an orbit  $(d + M)^{S_n}$  of the action of  $S_n$  on  $Q$ , and all the orbits of the action of  $S_n$  on  $Q$  are produced this way. Next we determine the precise number of orbits of the action of  $S_n$  on  $Q$ .

- (i) Let  $m \mid n$  where  $k = \frac{n}{m}$  for some positive integer  $k$ . Then there is one  $m$ -composition of  $n$  defined by  $d_k^{S_n}$  such that  $A(n, m)_{d_k} = (k, k, \dots, k)$ . Hence there are  $\binom{n+m-1}{m-1} - 1$   $m$ -compositions of  $n$  such that  $A(n, m)_d \neq (k, k, \dots, k)$ . Now consider the case where  $d_k^{S_n}$  defines  $(k, k, \dots, k)$ . Suppose that there is another orbit  $d_s^{S_n}$  such that  $d_s^{S_n} \neq d_k^{S_n}$  with  $(d + M)^{S_n} = (d' + M)^{S_n}$  for some  $d \in d_k^{S_n}$  and  $d' \in d_s^{S_n}$ . Then

$$d = (d')^g + l(1, 1, \dots, 1)$$

for some  $g \in S_n$  and some  $l \in \mathbb{Z}_m$ . Now by Lemma 3.3.17,  $d$  and  $(d')^g$  respectively define  $A(n, m)_{dS_n}$  and  $A(n, m)_{(d')S_n}$  both belonging to  $[A(n, m)_{dS_n}]$ . But

$[A(n, m)_{d^{S_n}}] = \{(k, k, \dots, k)\}$ , that is  $A(n, m)_{d^{S_n}} = A(n, m)_{(d')^{S_n}}$  which contradicts the assumption  $d_s^{S_n} \neq d_k^{S_n}$ . Therefore we must have  $(d + M)^{S_n} \neq (d' + M)^{S_n}$ . We conclude that the only orbit producing the orbit  $(d_k + M)^{S_n}$  is  $d_k^{S_n}$ . Now let  $(d_k + M)^{g_1}, (d_k + M)^{g_2} \in (d_k + M)^{S_n}$  such that  $(d_k + M)^{g_1} = (d_k + M)^{g_2}$ , for some  $g_1, g_2 \in S_n$ . Then

$$d_k^{g_1} = d_k^{g_2} + l(1, 1, \dots, 1),$$

when  $l \in \mathbb{Z}_m$ . Thus for each chosen element  $d_k^{g_2}$  of  $d_k^{S_n}$ , there are  $m$  elements in  $(d_k + M)^{S_n}$  such that  $d_k^{g_1} + M = d_k^{g_2} + M$  for some  $g_1 \in S_n$ . Hence the length of  $(d_k + M)^{S_n}$  is given by  $\frac{1}{m} \binom{n}{k \dots k}$ , where  $\binom{n}{k \dots k} = |d_k^{S_n}|$ .

Now consider the rest of the  $A(n, m)_{d^{S_n}} \neq (k, k, \dots, k)$ . Suppose that  $d_1^{S_n} \neq d_2^{S_n}$  with  $(d + M)^{S_n} = (d' + M)^{S_n}$  for some  $d \in d_1^{S_n}$  and  $d' \in d_2^{S_n}$ . Then

$$d = (d')^g + l(1, 1, \dots, 1)$$

for some  $g \in S_n$ . Again by Lemma 3.3.17,  $d$  and  $(d')^g$  respectively define  $A(n, m)_{d^{S_n}}$  and  $A(n, m)_{(d')^{S_n}}$  both belonging to  $[A(n, m)_{d^{S_n}}]$ . However  $[A(n, m)_{d^{S_n}}]$  has  $m$  elements. Thus for each element  $d \in d_1^{S_n}$  there are  $m$  orbits  $d_2^{S_n}$  defining the elements of  $[A(n, m)_{d^{S_n}}]$  such that  $(d + M)^{S_n} = (d' + M)^{S_n}$  for some  $d' \in d_2^{S_n}$ . Now since there are  $\binom{n+m-1}{m-1} - 1$ ,  $m$  compositions of  $n$  different from  $(k, k, \dots, k)$ , the number of orbits defining  $A(n, m)_{d^{S_n}} \neq (k, k, \dots, k)$  and producing the orbits  $(d + M)^{S_n}$  is equal to

$$\frac{1}{m} \left\{ \binom{n+m-1}{m-1} - 1 \right\}.$$

Now consider the orbit  $(d_1 + M)^{S_n}$  under the action of  $S_n$  on  $Q$ , defining the elements of  $[A(n, m)_{d_1^{S_n}}]$ , that is

$$(d_1 + M)^{S_n} = \{(d_1 + M)^g : g \in S_n\} = \{d_1^g + M : g \in S_n\}.$$

Clearly  $|(d_1 + M)^{S_n}| = |\{d_1^g + M : g \in S_n\}| \leq |d_1^{S_n}|$ . Therefore to show that  $|(d_1 + M)^{S_n}|$  is equal to  $|d_1^{S_n}|$ , we only need to show that  $d_1^{g_1} \neq d_1^{g_2}$  for some  $g_1, g_2 \in S_n$  implies  $d_1^{g_1} + M \neq d_1^{g_2} + M$ . Now assume that  $d_1^{g_1} + M = d_1^{g_2} + M$  for some  $g_1, g_2 \in S_n$ . Then

$$d^{g^1} = d^{g^2} + l(1, 1, \dots, 1).$$

Since  $d_1^{g^1}$  and  $d_2^{g^2}$  are in  $d_1^{S_n}$ , they define the same  $m$ -compositions of  $n$  and hence  $l = 0$ . Thus  $d_1^{g^1} = d_1^{g^2}$ , which contradicts the assumption  $d_1^{g^1} \neq d_1^{g^2}$ . Hence the length of  $(d_1 + M)^{S_n}$  is equal to  $|d_1^{S_n}| = \binom{n}{k_1 \ k_2 \ \dots \ k_m}$ , where  $(k_1, k_2, \dots, k_m) \in [A(n, m)_{d^{S_n}}]$ . This proves case (i).

(ii) Now suppose that  $m \nmid n$ . Then  $A(n, m)_{d^{S_n}}$  cannot be of type  $(k, k, \dots, k)$ ,  $k = \frac{n}{m}$  for all  $d \in N$ . Thus the proof of this case is similar to the proof of the second part of case (i) above. The  $m$ -compositions of  $n$  are collected together as  $[A(n, m)_{d^{S_n}}]$  according to Lemma 3.3.17. Since  $|A(n, m)| = \binom{n+m-1}{m-1}$  and  $|[A(n, m)_{d^{S_n}}]| = m$ , the number of orbits  $(d + M)^{S_n}$  is  $\frac{1}{m} \binom{n+m-1}{m-1}$ . The lengths of the orbits  $(d + M)^{S_n}$  are also found as in the second part of case (i) above to be  $\binom{n}{k_1 \ k_2 \ \dots \ k_m}$ , where  $(k_1, k_2, \dots, k_m) \in [A(n, m)_{d^{S_n}}]$ .

It is clear that the element  $d + M$ , where  $d = (l_1, l_2, \dots, l_n)$  defines an  $m$ -composition  $(k_1, k_2, \dots, k_m)$  of  $n$ , in  $[A(n, m)_{d^{S_n}}]$  such that  $k_j$  is the number of entries  $l_s$  equal to  $j - 1$  in  $d$ , is a representative for an orbit of the action of  $S_n$  on  $Q$ .

□

**Remark 3.3.19** The number of orbits of the action of  $S_n$  on  $Q$  is given by  $\left\lceil \frac{1}{m} \binom{n+m-1}{m-1} \right\rceil$ , since  $\frac{1}{m} \left\{ \binom{n+m-1}{m-1} - 1 \right\} + 1 = \left\lceil \frac{1}{m} \binom{n+m-1}{m-1} \right\rceil$  and the upper part does not affect the number  $\frac{1}{m} \binom{n+m-1}{m-1}$  for both cases  $m \mid n$  and  $m \nmid n$  respectively.

**Corollary 3.3.20** The group  $S_n$  acts on  $Q = 2^{n-1}$ , producing  $\lceil \frac{n+1}{2} \rceil$  orbits of lengths

$$(i) \ \left\{ \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{\frac{n+1}{2}} \right\}, \text{ for odd } n,$$

$$(ii) \ \left\{ \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \frac{1}{2} \binom{n}{\frac{n}{2}} \right\}, \text{ for even } n.$$

**Proof.** The proof follows from Theorem 3.3.18 with  $m = 2$ . □

**Examples 3.3.21** *In the following we apply Theorem 3.3.18.*

- (a) Let  $N = 2^6$ . Now  $S_6$  acts on  $Q = 2^5$ . Since  $2 \mid 6$ , we have  $(3, 3) \in A(6, 2)$ . Therefore we have an orbit of the action of  $S_6$  on  $Q$  of length  $\frac{1}{2} \binom{6}{3 \ 3} = 10$ . The action of  $S_6$  on  $Q = 2^5$  yields 4 orbits of lengths

$$\binom{6}{6 \ 0} = 1, \quad \binom{6}{5 \ 1} = 6, \quad \binom{6}{4 \ 2} = 15, \quad \frac{1}{2} \binom{6}{3 \ 3} = 10$$

with respective orbit representatives given by  $M$ ,  $(0, 0, 0, 0, 0, 1) + M$ ,  $(0, 0, 0, 0, 1, 1) + M$ ,  $(0, 0, 0, 1, 1, 1) + M$ .

- (b) Let  $N = 3^5$ . Now  $S_5$  acts on  $Q = 3^4$ . Since  $3 \nmid 5$ , we do not have a 3-composition of 5 of type  $(k, k, k)$ . Thus all the orbits of the action of  $S_5$  on  $Q = 3^4$  have lengths  $\binom{5}{k_1 \ k_2 \ k_3}$  such that  $(k_1, k_2, k_3) \in [A(n, m)_{dS_n}]$  for some  $d \in N$ . Further the element  $(1, 0, 0, 0, 0)$  is in an orbit of length  $\binom{5}{4 \ 1 \ 0}$ , the element  $(2, 1, 1, 1, 1)$  is in an orbit of length  $\binom{5}{0 \ 4 \ 1}$  and the element  $(0, 2, 2, 2, 2)$  is in an orbit of length  $\binom{5}{1 \ 0 \ 4}$  of the action of  $S_5$  on  $N = 3^5$ . However the orbits containing the elements in  $\{(1, 0, 0, 0, 0), (2, 1, 1, 1, 1), (0, 2, 2, 2, 2)\}$  become one orbit of the action of  $S_5$  on  $Q = 3^4$  of length  $\binom{5}{k_1 \ k_2 \ k_3}$ , where  $(k_1, k_2, k_3)$  is any element of  $\{(4, 1, 0), (0, 4, 1), (1, 0, 4)\}$ . The set  $\{(1, 0, 0, 0, 0), (2, 1, 1, 1, 1), (0, 2, 2, 2, 2)\}$  is obtained by adding  $(1, 1, 1, 1, 1)$  consecutively to  $d = (1, 0, 0, 0, 0)$  as described in the proof of Theorem 3.3.18. The action of  $S_5$  on  $Q = 3^4$  yields 7 orbits of lengths

$$\binom{5}{5 \ 0 \ 0} = 1, \quad \binom{5}{4 \ 1 \ 0} = 5, \quad \binom{5}{4 \ 0 \ 1} = 5, \quad \binom{5}{3 \ 2 \ 0} = 10, \quad \binom{5}{3 \ 1 \ 1} = 20,$$

$$\binom{5}{3 \ 0 \ 2} = 10, \quad \binom{5}{2 \ 2 \ 1} = 30$$

with respective orbit representatives given by  $M$ ,  $(0, 0, 0, 0, 1) + M$ ,  $(0, 0, 0, 0, 2) + M$ ,  $(0, 0, 0, 1, 1) + M$ ,  $(0, 0, 0, 1, 2) + M$ ,  $(0, 0, 0, 2, 2) + M$ ,  $(0, 0, 1, 1, 2) + M$ .

We now discuss the action of  $S_n$  on the set  $\text{Irr}(Q)$  of irreducible characters of  $Q$ . It is clear that  $\text{Irr}(N) = \Delta_M \cup \Delta$ , where  $\Delta_M = \{\theta \in \text{Irr}(N) \mid \ker(\theta) \supset M\}$  and  $\Delta = \{\theta' \in \text{Irr}(N) \mid \theta'(\sum_{i=1}^n e_s) \neq 1\}$ .

**Lemma 3.3.22** *The sets  $\Delta_M$  and  $\Delta$  are invariant under  $S_n$ .*

**Proof.** Let  $g \in S_n$ ,  $\theta \in \Delta_M$  and  $\theta' \in \Delta$ . Then, since  $S_n$  permutes the elements of  $N$  we have

$$\begin{aligned} \theta^g\left(\sum_{s=1}^n e_s\right) &= \theta\left(\sum_{s=1}^n e_s\right)^g \\ &= \theta\left(\sum_{s=1}^n e_{g(s)}\right) = 1, \end{aligned}$$

which implies  $\theta^g \in \Delta_M$ . Similarly we have

$$\begin{aligned} \theta'^g\left(\sum_{s=1}^n e_s\right) &= \theta'\left(\sum_{s=1}^n e_s\right)^g \\ &= \theta'\left(\sum_{s=1}^n e_{g(s)}\right) \neq 1, \quad \square \end{aligned}$$

which implies  $\theta'^g \in \Delta$ .

**Proposition 3.3.23** *Let  $\theta = \theta_1^{\otimes k_1} \theta_2^{\otimes k_2} \dots \theta_m^{\otimes k_m} \in \text{Irr}(N)$ . Then  $\text{Ker}(\theta) \supset M$  if and only if  $\sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}$ .*

**Proof.**

$$\begin{aligned} \text{Ker}(\theta) \supset M &\Leftrightarrow \theta\left(\sum_{s=1}^n e_s\right) = 1 \\ &\Leftrightarrow \xi^{\sum_{j=1}^m (j-1)k_j} = 1 \\ &\Leftrightarrow \sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}. \quad \square \end{aligned}$$

**Proposition 3.3.24** *The set of irreducible characters of  $Q$  is given by*

$$\text{Irr}(Q) = \bigcup_{(k_1, k_2, \dots, k_m)} \{\Theta(k_1, k_2, \dots, k_m) \mid \sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}\}.$$

**Proof.** Since  $\text{Irr}(Q) = \Delta_M$ , this follows from Proposition 3.3.23.  $\square$

**Theorem 3.3.25** *The symmetric group  $S_n$  acts on the irreducible characters of  $Q$  producing  $\left\lceil \frac{1}{m} \binom{n+m-1}{m-1} \right\rceil$  orbits of lengths  $\binom{n}{k_1 \ k_2 \ \dots \ k_m}$ , with orbit representatives respectively given by  $\theta = \theta_1^{\otimes k_1} \theta_2^{\otimes k_2} \dots \theta_m^{\otimes k_m}$ , where  $(k_1, k_2, \dots, k_m) \in A(n, m)$  such that  $\sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}$ .*

**Proof.** We know that  $\text{Irr}(N) = \Delta_M \cup \Delta$  and that  $\text{Irr}(Q) = \Delta_M$ . Now by Lemma 3.3.22  $\theta^g \neq \theta'$  for  $\theta \in \Delta_M$ ,  $\theta' \in \Delta$ , and  $g \in S_n$ . Thus the orbits of the action of  $S_n$  on  $\text{Irr}(Q)$  are equal to those of the action of  $S_n$  on  $\Delta_M$  in the action of  $S_n$  on  $\text{Irr}(N)$ , and are of the same lengths. But by Proposition 3.3.24,  $\text{Irr}(Q) = \bigcup_{(k_1, k_2, \dots, k_m)} \{\Theta(k_1, k_2, \dots, k_m) \mid \sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}\}$ . Considering the action of  $S_n$  within  $\Delta_M$ , and from Lemma 3.3.22, we choose those orbits of lengths  $\binom{n}{k_1 \ k_2 \ \dots \ k_m}$  of the action of  $S_n$  on  $\text{Irr}(N)$  such that  $\sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}$ . The required number of orbits follows from Theorem 1.4.4 (Brauers Theorem) and Theorem 3.3.18.  $\square$

**Remark 3.3.26** Since  $\text{Irr}(Q)$  contains irreducible characters  $N$  with  $M$  in their kernel, for the Inertia factor groups of  $B_Q(m, n)$  we choose those Inertia factor groups  $S_{k_1} \times S_{k_2} \times \dots \times S_{k_m}$  of  $B(m, n)$  corresponding to  $m$ -compositions of  $n$  such that  $\sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}$ .

**Corollary 3.3.27** Let  $Q = 2^{n-1}$ . The group  $S_n$  acts on  $\text{Irr}(Q)$  giving  $\lceil \frac{n+1}{2} \rceil$  orbits of lengths

- (i)  $\{\binom{n}{0}, \binom{n}{2}, \binom{n}{4}, \dots, \binom{n}{n}\}$ , where  $n$  is even, with inertia factor groups of  $B_Q(2, n)$  given by the young subgroups  $S_{n-k} \times S_k$ ,  $k = 0, 2, 4, \dots, n$ ,
- (ii)  $\{\binom{n}{0}, \binom{n}{2}, \binom{n}{4}, \dots, \binom{n}{n-1}\}$ , where  $n$  is odd, with inertia factor groups of  $B_Q(2, n)$  given by the young subgroups  $S_{n-k} \times S_k$ ,  $k = 0, 2, 4, \dots, n-1$ .

**Proof.** Follows from Theorem 3.3.25 with  $k_2 = k$  and  $k_1 = n - k$  where  $k$  is even. Thus the orbit lengths are given by  $\binom{n}{n-k \ k}$  where  $k$  is even. It follows that the required inertia factor groups of  $B_Q(2, n)$  are those young subgroups  $S_{n-k} \times S_k$ , where  $k$  is even, corresponding to the orbits of length  $\binom{n}{n-k \ k}$  in the action of  $S_n$  on  $\text{Irr}(N)$ .  $\square$

**Examples 3.3.28** We apply Theorem 3.3.25 in the following.

- (a) The action of  $S_6$  on  $\text{Irr}(Q)$ ,  $Q = 2^5$ , yields 4 orbits of lengths

$$\binom{6}{6 \ 0} = 1, \binom{6}{4 \ 2} = 15, \binom{6}{2 \ 4} = 15, \binom{6}{0 \ 6} = 1,$$

where  $k_2 \equiv 0 \pmod{2}$ . The respective orbit representatives are given by

$$\theta_1^{\otimes 6} \theta_2^{\otimes 0} = \theta_1^{\otimes 6}, \theta_1^{\otimes 4} \theta_2^{\otimes 2}, \theta_1^{\otimes 2} \theta_2^{\otimes 4}, \theta_1^{\otimes 0} \theta_2^{\otimes 6} = \theta_2^{\otimes 6}.$$

(b) The action of  $S_5$  on  $\text{Irr}(Q)$ ,  $Q = 3^4$ , yields 7 orbits of lengths

$$\binom{5}{5 \ 0 \ 0} = 1, \binom{5}{3 \ 1 \ 1} = 20, \binom{5}{2 \ 3 \ 0} = 10, \binom{5}{2 \ 0 \ 3} = 10, \binom{5}{1 \ 2 \ 2} = 30, \binom{5}{0 \ 4 \ 1} = 5, \binom{5}{0 \ 1 \ 4} = 5.$$

where  $k_2 + 2k_3 \equiv 0 \pmod{3}$ . The respective orbit representatives are given by

$$\begin{aligned} \theta_1^{\otimes 5} \theta_2^{\otimes 0} \theta_3^{\otimes 0} &= \theta_1^{\otimes 5}, \theta_1^{\otimes 3} \theta_2^{\otimes 1} \theta_3^{\otimes 1}, \theta_1^{\otimes 2} \theta_2^{\otimes 3} \theta_3^{\otimes 0} = \theta_1^{\otimes 2} \theta_2^{\otimes 3}, \\ \theta_1^{\otimes 2} \theta_2^{\otimes 0} \theta_3^{\otimes 3}, \theta_1^{\otimes 1} \theta_2^{\otimes 2} \theta_3^{\otimes 2}, \theta_1^{\otimes 0} \theta_2^{\otimes 4} \theta_3^{\otimes 1}, \theta_1^{\otimes 0} \theta_2^{\otimes 1} \theta_3^{\otimes 4}. \end{aligned}$$

**Remark 3.3.29** Using Theorem 3.3.25, Remark 3.3.26 and Corollary 3.3.27 we have found that

- (i) for  $n = 5$  and  $m = 2$ ,  $Q = 2^4$ , the action of  $S_5$  on  $\text{Irr}(Q)$  yields three orbits of lengths 1, 10 and 5, and the inertia factor groups of  $B_Q(2, 5)$  are:  $S_5$ ,  $S_3 \times S_2$ ,  $S_1 \times S_4$  of indices 1, 10, and 5 in  $S_5$  respectively,
- (ii) for  $n = 7$  and  $m = 2$ ,  $Q = 2^6$ , the action of  $S_7$  on  $\text{Irr}(Q)$  yields four orbits of lengths 1, 21, 35 and 7, and the inertia factor groups of  $B_Q(2, 6)$  are:  $S_7$ ,  $S_5 \times S_2$ ,  $S_3 \times S_4$ , and  $S_1 \times S_6$  of indices 1, 21, 35 and 7 in  $S_7$  respectively,
- (iii) for  $n = 8$  and  $m = 2$ ,  $Q = 2^7$ , the action of  $S_8$  on  $\text{Irr}(Q)$  yields five orbits of lengths 1, 28, 70, 28 and 1, and the inertia factor groups of  $B_Q(2, 8)$  are:  $S_8$ ,  $S_6 \times S_2$ ,  $S_4 \times S_4$ ,  $S_2 \times S_6$  and  $S_8$  of indices 1, 28, 70, 28 and 1 in  $S_8$  respectively,
- (iv) for  $n = 10$  and  $m = 2$ ,  $Q = 2^9$ , the action of  $S_{10}$  on  $\text{Irr}(Q)$  yields six orbits of lengths 1, 45, 210, 210, 45 and 1, and the inertia factor groups of  $B_Q(2, 10)$  are:  $S_{10}$ ,  $S_8 \times S_2$ ,  $S_6 \times S_4$ ,  $S_4 \times S_6$ ,  $S_2 \times S_8$  and  $S_{10}$  of indices 1, 45, 210, 210, 45 and 1 in  $S_{10}$  respectively,
- (v) for  $n = 3$  and  $m = 3$ ,  $Q = 3^2$ , the action of  $S_3$  on  $\text{Irr}(3^2)$  yields four orbits of lengths 1, 6, 1 and 1, and the inertia factor groups of  $B_Q(3, 3)$  are:  $S_3$ ,  $S_1 \times S_1 \times S_1$ ,  $S_3$  and  $S_3$  of indices 1, 6, 1, and 1 in  $S_3$  respectively.

## Chapter 4

# Fischer-Clifford Matrices of $B(m, n)$

In this chapter, we discuss the combinatorial method given in [5] for constructing the Fischer-Clifford matrices of the generalized symmetric group  $B(m, n)$ . We apply Fischer-Clifford theory to the group  $B(m, n)$  in Sections 4.1 and 4.2. We discuss the combinatorial method for constructing the Fischer-Clifford matrices of  $B(m, n)$  in Section 4.3. In particular we give the following results which are very useful for this work: Theorem 4.3.2, Proposition 4.3.4 and Theorem 4.3.5. In Section 4.4 we give examples on the application of the combinatorial method to construct the Fischer-Clifford matrices of the groups  $B(2, 6)$  and  $B(3, 5)$ .

### 4.1 Fischer-Clifford Theory Applied to $B(m, n)$

In this section, we apply Fischer-Clifford theory to the generalized symmetric group  $B(m, n)$ . This describes the construction of the irreducible characters of  $B(m, n)$  from those of its subgroups.

Let  $Irr(\mathbb{Z}_m) = \{\theta_1, \theta_2, \dots, \theta_m\}$  be the set of irreducible characters of  $\mathbb{Z}_m$ . The action of  $S_n$  on  $Irr(\mathbb{Z}_m^n)$  was discussed in Chapter 3. Let  $\{\chi_1, \chi_2, \dots, \chi_t\}$  be the set of representatives of the orbits of  $S_n$  acting on  $Irr(N)$ , where  $\chi_l = \theta_1^{\otimes k_1} \theta_2^{\otimes k_2} \dots \theta_m^{\otimes k_m}$ . Following Definition 3.2.5 and Theorem 3.2.7 we have that for each  $m$ -composition  $(k_1, k_2, \dots, k_m)$  of  $n$ , there is an orbit of the action of  $S_n$  on  $Irr(N)$  of length  $\binom{n}{k_1 \ k_2 \ \dots \ k_m}$ . It is clear that for each  $l = 1, 2, \dots, t$ ,  $I_{\chi_l} = B(m, k_1) \times \dots \times B(m, k_m)$

is the inertia group of a character  $\chi_l$ , where  $(k_1, k_2, \dots, k_m) \in A(n, m)$  and  $B(m, k_j)$  is a generalized symmetric group on  $k_j$  symbols. It is also clear that the inertia factor group of  $\chi_l$  is the young subgroup,  $\bar{I}_{\chi_l} = S_{k_1} \times \dots \times S_{k_m}$  of  $S_n$ . Now by Theorem 1.4.11 (Mackey's Theorem (1958)) and since  $B(m, n)$  is a split extension of an abelian group  $N$  with  $S_n$ , every irreducible character of  $N = Z_m^n$  extends to an irreducible character of its inertia group in  $B(m, n)$ . And by Theorem 1.4.6 (Gallagher's theorem) all irreducible characters of  $B(m, n)$  are of form  $(\widehat{\chi}_l \psi) \uparrow B(m, n)$ , where  $\widehat{\chi}_l$  is an extension of an irreducible character  $\chi_l$  of  $N$  to its inertia group  $I_{\chi_l}$ ,  $\psi$  is a character of  $I_{\chi_l}$  such that  $N \subseteq \text{Ker}(\psi)$  and  $\widehat{\chi}_l \psi$  is a tensor product of characters. Thus the irreducible characters of  $B(m, n)$  are given by

$$\text{Irr}(B(m, n)) = \bigcup_{l=1}^t \{(\widehat{\chi}_l \psi) \uparrow B(m, n) \mid \psi \in \text{Irr}(I_{\chi_l}), N \subseteq \text{Ker}(\psi)\}.$$

## 4.2 Defining Fischer-Clifford Matrices of $B(m, n)$

Let  $[g]$  be a conjugacy class of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  and  $X(g) = \{b_1, b_2, \dots, b_{c(g)}\}$  be a set of representatives of the conjugacy classes of  $B(m, n)$  which map to  $[g]$  under the natural homomorphism  $\zeta : B(m, n) \rightarrow S_n$ . Let  $L_1, L_2, \dots, L_r$  be the conjugacy classes of the inertia factor group  $\bar{I}_{\chi_l}$  which fuse to  $[g]$  and  $g_j \in L_j$ . Let  $b_k \in B(m, n)$ . Then by Theorem 1.3.16, we have

$$\begin{aligned} (\widehat{\chi}_l \psi) \uparrow^{B(m, n)} (b_k) &= \sum_{1 \leq j \leq r} \sum_l \frac{|C_{B(m, n)}(b_k)|}{|C_{I_{\chi_l}}(b_{l_j})|} (\widehat{\chi}_l \psi)(b_{l_j}) \\ &= \sum_{1 \leq j \leq r} \left( \sum_l \frac{|C_{B(m, n)}(b_k)|}{|C_{I_{\chi_l}}(b_{l_j})|} \widehat{\chi}_l(b_{l_j}) \right) \psi(g_j), \end{aligned}$$

where for each  $j$ ,  $g_j \in L_j$ , the elements  $b_{l_j}$  run over a set of representatives of the conjugacy classes of  $I_{\chi_l}$  which map to  $L_j$  under the map  $I_{\chi_l} \rightarrow \bar{I}_{\chi_l}$ . Now for each inertia group  $I_{\chi_l}$  we set  $x_{lj}(g) = \sum_l \frac{|C_{B(m, n)}(b_k)|}{|C_{I_{\chi_l}}(b_{l_j})|} (\widehat{\chi}_l(b_{l_j}))$ . The matrix  $(x_{lj}(g))$ ,  $1 \leq l \leq t, 1 \leq j \leq r$ , that is

$$(x_{lj}(g)) = \begin{bmatrix} x_{1j}(g) \\ x_{2j}(g) \\ \vdots \\ x_{tj}(g) \end{bmatrix}$$

is the **Fischer-Clifford matrix** of  $B(m, n)$  at  $[g]$ , which we denote by  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .

Following Almestady in [5], we denote an entry of the Fischer-Clifford matrix  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  by

$$f_{\bar{\lambda}_0}[(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_m), (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_m)],$$

where  $\bar{\lambda}_0$  is a  $q$ -tuple of non-zero powers in  $[\lambda]$ ,  $\bar{a}_i = (a_{i1}, a_{i2}, \dots, a_{in})$  and  $\bar{k}_j = (k_{j1}, k_{j2}, \dots, k_{jn})$ . This entry lies in a column indexed by  $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_m)$  or  $(1^{a_1} 2^{a_2} \dots n^{a_n})$  where  $a_i = \sum_{s=1}^n s a_{is}$ , corresponding to a conjugacy class of  $B(m, n)$  of type

$$(1^{a_{11}} 2^{a_{12}} \dots n^{a_{1n}}, 1^{a_{21}} 2^{a_{22}} \dots n^{a_{2n}}, \dots, 1^{a_{m1}} 2^{a_{m2}} \dots n^{a_{mn}}) \quad (4.1)$$

which under the natural homomorphism  $\zeta : B(m, n) \rightarrow S_n$ , maps to a conjugacy class of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . This entry also lies in a row indexed by  $(\bar{k}_1, \bar{k}_2, \dots, \bar{k}_m)$  or  $(1^{k_1} 2^{k_2} \dots m^{k_m})$ , where  $k_j = \sum_{s=1}^n s k_{js}$  corresponding to the inertia group  $I_{\chi_l}$  of the character  $\chi_l$  of  $N$ . The conjugacy classes of the inertia factor groups, which fuse to a conjugacy class of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ , are of type

$$(1^{k_{11}} 2^{k_{12}} \dots n^{k_{1n}}, 1^{k_{21}} 2^{k_{22}} \dots n^{k_{2n}}, \dots, 1^{k_{m1}} 2^{k_{m2}} \dots n^{k_{mn}}), \quad (4.2)$$

where  $\sum_{j=1}^m k_{js} = \lambda_s$ . The conjugacy classes of the elements of  $I_{\chi_l}$ , which map to a conjugacy class  $[y_k]$  of the inertia factor group  $\bar{I}_{\chi_l}$  under the natural homomorphism  $\zeta : B(m, n) \rightarrow S_n$  and fuse to a conjugacy class  $[b_k]$  of  $B(m, n)$ , are of type

$$((1^{r_{11}^1} 2^{r_{11}^2} \dots n^{r_{11}^n}, \dots, 1^{r_{m1}^1} 2^{r_{m1}^2} \dots n^{r_{m1}^n}); \dots; (1^{r_{1m}^1} 2^{r_{1m}^2} \dots n^{r_{1m}^n}, \dots, 1^{r_{mm}^1} \dots n^{r_{mm}^n})),$$

where  $\sum_{j=1}^m r_{ij}^s = a_{is}$ ,  $\sum_{i=1}^m r_{ij}^s = k_{js}$ .

The character table of  $B(m, n)$  is constructed by multiplying the partial character tables of the inertia factor groups  $\bar{I}_{\chi_l}$  of  $B(m, n)$  by the corresponding rows of the Fischer-Clifford matrices of  $B(m, n)$ , according to the fusions of the conjugacy classes of  $\bar{I}_{\chi_l}$  into  $[g]$  as indicated above.

### 4.3 Fischer-Clifford Matrices of $B(m, n)$ - a Combinatorial Approach

As indicated in the introduction to Chapter 1, we will construct the Fischer-Clifford matrices of  $B(m, n)$  using a combinatorial method rather than the properties of Fischer-Clifford matrices given in Section 1.5.2. For most of the following results we follow closely the works of Almestady [5] and List and Mahmoud [36].

**Theorem 4.3.1** ([5]) *An entry of the Fischer-Clifford matrix  $F(m, 1^n)$  which is in the column indexed by  $(1^{a_1} 2^{a_2} \dots m^{a_m})$  and row indexed by  $(1^{k_1} 2^{k_2} \dots m^{k_m})$  is given by*

$$f_{\lambda_0}[(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_m), (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_m)] = \sum_{M(\beta, \delta)} \left( \prod_{i=1}^m \binom{a_i}{R_{a_i}} \right) \xi^{\sum_{i=1}^m \sum_{j=1}^m (i-1)(j-1)(r_{ij})},$$

where  $\xi$  is a primitive  $m^{\text{th}}$  root of unity,  $M(\beta, \delta)$  is a set of magic matrices with  $i^{\text{th}}$  row  $R_{a_i} = (r_{i1} r_{i2} \dots r_{im})$ .

**Proof.** (See result 4.2.1 in [5]) Let  $g \in S_n$  of type  $(1^n)$ . Let  $\chi_l = \theta_1^{\otimes k_1} \theta_2^{\otimes k_2} \dots \theta_m^{\otimes k_m} \in \text{Irr}(N)$  where  $\sum_{j=1}^m k_j = n$ , be a representative of an orbit of the action of  $S_n$  on  $\text{Irr}(N)$ . Then  $\bar{I}_{\chi_l} = S_{k_1} \times S_{k_2} \times \dots \times S_{k_m}$  is the corresponding inertia factor group of  $\chi_l$  in  $B(m, n)$ . Let  $[b_k]$  be a class of  $B(m, n)$  which maps to a class  $[g]$  of  $S_n$  under the homomorphism  $\zeta : B(m, n) \rightarrow S_n$  from  $B(m, n)$  to  $S_n$ . Then as indicated in section 4.2,  $b_k$  has type  $(1^{a_1}, 1^{a_2}, \dots, 1^{a_m})$ ,  $\sum_{i=1}^m a_i = n$ . If  $L_j$  is a class of  $\bar{I}_{\chi_l}$  fusing to  $[g]$  in  $S_n$ , then  $L_j$  corresponds to the partition  $(1^{k_1}, 1^{k_2}, \dots, 1^{k_m})$ ,  $\sum_{i=1}^m k_i = n$ . And so the classes of the inertia group  $I_{\chi_l} = B(m, k_1) \times B(m, k_2) \times \dots \times B(m, k_m)$  which map to the class  $L_j$  and fuse to the class  $[b_k]$  of  $B(m, n)$  are the classes  $[b_{l_j}]$  of type

$$((1^{r_{11}}, 1^{r_{21}}, \dots, 1^{r_{m1}}); \dots; (1^{r_{1m}}, 1^{r_{2m}}, \dots, 1^{r_{mm}})),$$

where  $\sum_{i=1}^m r_{ij} = k_j$ ,  $\sum_{j=1}^m r_{ij} = a_i$ ,  $i, j = 1, 2, \dots, m$ .

Let  $M \in M(\beta, \delta)$ . Then the entry  $f_{\lambda_0}[(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_m), (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_m)]$  of  $F(m, 1^n)$  is  $x_{lj}(g) = \left( \sum_l \frac{|C_{B(m, n)}(b_k)|}{|C_{I_{\chi_l}}(b_{l_j})|} (\widehat{\chi}_l(b_{l_j})) \right)$ , which is equal to

$$\begin{aligned}
& \sum_{\substack{\sum_{j=1}^m r_{ij} = a_i \\ \sum_{i=1}^m r_{ij} = k_j}} \frac{m^n (a_1)! (a_2)! \cdots (a_m)!}{m^n (r_{11})! (r_{21})! \cdots (r_{mm})!} \left( \prod_{i=1}^m \prod_{j=1}^m (\theta_i ((j-1)x))^{(r_{ij})} \right) \\
&= \sum_{M=(r_{ij}) \in M(\beta, \delta)} \binom{a_1}{r_{11} \ r_{12} \ \cdots \ r_{1m}} \binom{a_2}{r_{21} \ r_{22} \ \cdots \ r_{2m}} \\
&\quad \cdots \binom{a_m}{r_{m1} \ r_{m2} \ \cdots \ r_{mm}} \left( \prod_{i=1}^m \prod_{j=1}^m (\theta_i ((j-1)x))^{(r_{ij})} \right) \\
&= \sum_{M(\beta, \delta)} \left( \prod_{i=1}^m \binom{a_i}{R_{a_i}} \right) \xi^{\sum_{i=1}^m \sum_{j=1}^m (i-1)(j-1)(r_{ij})}. \quad \square
\end{aligned}$$

The following is an important result used in this thesis for the development of a computer programme for computing matrices which are row equivalent to the Fischer-Clifford matrices of  $B(m, n)$ .

**Theorem 4.3.2** ([5]) *The entries of the Fischer-Clifford matrix  $F(m, 1^n)$ , which are in a column indexed by  $(1^{a_1} 2^{a_2} \dots m^{a_m})$  and rows indexed by  $(1^{k_1} 2^{k_2} \dots m^{k_m})$  where  $(k_1, k_2, \dots, k_m) \in A(n, m)$ , are given by the coefficients of  $x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$  in the expansion*

$$\prod_{i=1}^m \left( \sum_{j=1}^m \xi^{(i-1)(j-1)} x_j \right)^{a_i}.$$

**Proof.**

$$\begin{aligned}
\prod_{i=1}^m \left( \sum_{j=1}^m \xi^{(i-1)(j-1)} x_j \right)^{a_i} &= \prod_{i=1}^m \left( \sum_{\sum_{j=1}^m r_{ij}=a_i} \binom{a_i}{r_{i1} \ r_{i2} \ \dots \ r_{im}} \prod_{j=1}^m \xi^{(i-1)(j-1)(r_{ij})} x_j^{(r_{ij})} \right) \\
&= \prod_{i=1}^m \left( \sum_{R_{a_i}} \binom{a_i}{R_{a_i}} \left( \prod_{j=1}^m \xi^{(i-1)(j-1)(r_{ij})} x_j^{(r_{ij})} \right) \right) \\
&= \sum_{R_{a_i}} \left( \prod_{i=1}^m \binom{a_i}{R_{a_i}} \left( \prod_{j=1}^m \xi^{(i-1)(j-1)(r_{ij})} x_j^{(r_{ij})} \right) \right) \\
&= \sum_{R_{a_i}} \left( \prod_{i=1}^m \binom{a_i}{R_{a_i}} \xi^{\sum_{j=1}^m (i-1)(j-1)(r_{ij})} \left( \prod_{j=1}^m x_j^{(r_{ij})} \right) \right) \\
&= \sum_{R_{a_i}} \left( \prod_{i=1}^m \binom{a_i}{R_{a_i}} \right) \xi^{\sum_{i=1}^m \sum_{j=1}^m (i-1)(j-1)(r_{ij})} \left( \prod_{i=1}^m \prod_{j=1}^m x_j^{(r_{ij})} \right) \\
&= \sum_{R_{k_j}} \left( \sum_{R_{a_i}} \left( \prod_{i=1}^m \binom{a_i}{R_{a_i}} \right) \xi^{\sum_{i=1}^m \sum_{j=1}^m (i-1)(j-1)(r_{ij})} \right) \prod_{j=1}^m x_j^{(R_{k_j})} \\
&= f_{\bar{\lambda}_0} [(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_m), (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_m)] \prod_{j=1}^m x_j^{(R_{k_j})}
\end{aligned}$$

by Theorem 4.3.1. Hence  $f_{\bar{\lambda}_0} [(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_m), (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_m)]$  is a coefficient of  $x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$  in the expansion

$$\prod_{i=1}^m \left( \sum_{j=1}^m \xi^{(i-1)(j-1)} x_j \right)^{a_i} . \quad \square$$

We confirm that, as it is also stated in [5], the evaluation of the expression  $\prod_{i=1}^m \left( \sum_{j=1}^m \xi^{(i-1)(j-1)} x_j \right)^{a_i}$  for large values of  $m$  and  $n$  is not an easy task. However we have developed a computer programme (see Programme 5.2.4) to do this. With the necessary changes, the proof of the following theorem is similar to that of Theorem 4.3.1.

**Theorem 4.3.3** ([5]) *An entry of the Fischer-Clifford matrix  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  which is in the column indexed by  $(1^{a_1} 2^{a_2} \dots m^{a_m})$  and row indexed by  $(1^{k_1} 2^{k_2} \dots m^{k_m})$  is given by*

$$f_{\bar{\lambda}_0}[(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_m), (\bar{k}_1, \bar{k}_2, \dots, \bar{k}_m)] \\ = \sum_{M(\beta, \delta, n)} \left( \prod_{s=1}^n \prod_{i=1}^m \binom{a_{is}}{R_{a_i, \lambda_s}} \right) \xi^{\sum_{i=1}^m \sum_{j=1}^m (i-1)(j-1) (\sum_{s=1}^n r_{ij}^s)},$$

where  $\xi$  is a primitive  $m^{\text{th}}$  root of unity,  $R_{a_i, \lambda_s} = (r_{i1}^s r_{i2}^s \dots r_{im}^s)$  is an  $i^{\text{th}}$  row of a partial matrix at  $\lambda_s$ , of a magic matrix in  $M(\beta, \delta, n)$ .

**Proof.** See result 4.3.1 in [5].  $\square$

**Proposition 4.3.4** [36] *Let  $g$  be an element of  $S_n$ , of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . Then  $F(m, s^{\lambda_s})$  is a Fischer-Clifford matrix of a subgroup  $B(m, s\lambda_s)$  of  $B(m, n)$  and*

$$F(m, s^{\lambda_s}) = F(m, 1^{\lambda_s}),$$

for each  $s = 1, 2, \dots, n$ , where  $F(m, 1^0)$  is defined to be the identity matrix (1).

**Proof.** Since  $s^{\lambda_s}$  is a partition of  $s\lambda_s$  it follows that  $F(m, s^{\lambda_s})$  is a Fischer-Clifford matrix of the group  $B(m, s\lambda_s)$ . Further we note that the formula in Theorem 4.3.1 for calculating the entries of  $F(m, 1^{\lambda_s})$  depends on  $\xi$  and  $a_i$ . But for any Fischer-Clifford matrix of  $B(m, k)$ ,  $k \leq n$ ,  $m$  is fixed and therefore so is  $\xi$ . But  $a_i$  is an entry of an  $m$ -composition of  $\lambda_s$ , and since for each  $s$  the  $\lambda_s$  are equal in  $F(m, s^{\lambda_s})$  and  $F(m, 1^{\lambda_s})$ , we use the same  $a_i$ 's  $i = 1, 2, \dots, m$  in calculating the entries of these matrices. The result is clear (see also result 4.3.5 in [5] or result (3.1) in [36]).  $\square$

**Theorem 4.3.5** *Let  $[g]$  be a conjugacy class of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . Then the matrix given by*

$$F^l(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}) = \otimes_{s=1}^n F(m, 1^{\lambda_s}),$$

where  $\otimes$  is the tensor product over  $s$ , is row equivalent to the Fischer-Clifford matrix  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of  $B(m, n)$  at  $[g]$ .

**Proof.** In result (3.3) of [36], it is proved that the Fischer-Clifford matrix of  $B(m, n)$  at  $[g]$  is given by

$$F(m, 1^{\lambda_1}) \otimes F(m, 1^{\lambda_2}) \otimes \dots \otimes F(m, 1^{\lambda_n}).$$

However the above tensor product is row equivalent to  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  as seen from the following. Since  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  and  $(n^{\lambda_n} (n-1)^{\lambda_{n-1}} \dots 2^{\lambda_2} 1^{\lambda_1})$  are the same partition of  $n$ , the Fischer-Clifford matrices  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  and  $F(m, n^{\lambda_n} (n-1)^{\lambda_{n-1}} \dots 2^{\lambda_2} 1^{\lambda_1})$  should be equal. However as we know, the two tensor products  $F(m, 1^{\lambda_1}) \otimes F(m, 2^{\lambda_2}) \otimes \dots \otimes F(m, n^{\lambda_n})$  and  $F(m, n^{\lambda_n}) \otimes F(m, (n-1)^{\lambda_{n-1}}) \otimes \dots \otimes F(m, 2^{\lambda_2}) \otimes F(m, 1^{\lambda_1})$  are not equal. We can obtain one from the other by re-arranging the rows and columns. It follows that both tensor products must be equivalent to the Fischer-Clifford matrix  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . However since the columns of a Fischer-Clifford matrix can be arranged in any order, we do not need to re-arrange the columns of either tensor product.  $\square$

Following from Theorem 4.3.5 we may denote a Fischer-Clifford matrix of  $B(m, n)$  by either  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  or  $F(m, n^{\lambda_n} (n-1)^{\lambda_{n-1}} \dots 2^{\lambda_2} 1^{\lambda_1})$ . In this thesis, we prefer to use the notation  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  which is not as cumbersome as the other notation. However we prefer the tensor product  $F(m, n^{\lambda_n}) \otimes F(m, (n-1)^{\lambda_{n-1}}) \otimes \dots \otimes F(m, 2^{\lambda_2}) \otimes F(m, 1^{\lambda_1})$  where fewer rows need re-arrangement than in the tensor product  $F(m, 1^{\lambda_1}) \otimes F(m, 2^{\lambda_2}) \otimes \dots \otimes F(m, n^{\lambda_n})$ . An example (see Example 4.4.1 (e)) on re-arranging the rows of  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  to obtain  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  is given in subsection 4.4.1.

**Corollary 4.3.6** *Let  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  and  $(1^{\rho_1} 2^{\rho_2} \dots n^{\rho_n})$  be two partitions of  $n$  such that  $\bar{\lambda}_0 = \bar{\rho}_0$ , where  $\bar{\lambda}_0$  denotes a  $q$ -tuple of non zero powers in  $[\lambda]$ . Then  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}) = F(m, 1^{\rho_1} 2^{\rho_2} \dots n^{\rho_n})$ .*

**Proof.** This result follows clearly from Theorem 4.3.5 and Proposition 4.3.4.  $\square$

## 4.4 Examples

We now apply the results in Section 4.3 to construct the Fischer-Clifford matrices of the groups  $B(2, 6)$  and  $B(3, 5)$ . The work of this chapter forms the background for constructing the Fischer-clifford matrices of some groups associated with  $B(m, n)$ , which we will do later. In the next section we show how to construct the Fischer-Clifford matrices of an example of the generalized symmetric groups  $B(2, n)$ .

#### 4.4.1 Constructing the Fischer-Clifford Matrices of $B(2, 6)$

Using the results in Section 4.3, for each conjugacy class of  $S_6$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$ , we construct the Fischer-Clifford matrix  $F(2, 1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$  of  $B(2, 6)$ .

- (a) We use Theorem 4.3.1 to calculate the entries of the Fischer-Clifford matrix  $F(2, 1^6)$ . From  $\lambda_1 = 6$  we obtain the following set of 2-sets of partition  $[1^6]$ .

$$\left\{ \binom{6}{0}, \binom{5}{1}, \binom{4}{2}, \binom{3}{3}, \binom{2}{4}, \binom{1}{5}, \binom{1}{6} \right\}.$$

The number seven of 2-sets of partition  $[1^6]$  found above indicates that the Fischer-Clifford matrix  $F(2, 1^6)$  has order 7. However since the use of Theorem 4.3.1 involves constructing a number of magic matrices for each entry of a Fischer-Clifford matrix we show how to calculate only the entry of  $F(2, 1^6)$  given by  $f_{(6)}[(3, 3), (4, 2)]$ . Using Definition 2.3.1 with  $\beta = (3, 3)$  and  $\delta = (4, 2)$ , we obtain that

$$M(\beta, 6) = \left\{ \begin{array}{l} \binom{3}{3} \binom{0}{0}, \binom{3}{2} \binom{0}{1}, \binom{3}{1} \binom{0}{2}, \binom{3}{0} \binom{0}{3}, \binom{2}{3} \binom{1}{0}, \binom{2}{2} \binom{1}{1}, \\ \binom{2}{1} \binom{1}{2}, \binom{2}{0} \binom{1}{3}, \binom{1}{2} \binom{1}{3}, \binom{1}{2} \binom{2}{1}, \binom{1}{2} \binom{1}{2}, \binom{1}{2} \binom{0}{3}, \\ \binom{0}{3} \binom{3}{0}, \binom{0}{3} \binom{2}{1}, \binom{0}{3} \binom{1}{2}, \binom{0}{3} \binom{0}{3} \end{array} \right\}$$

and

$$M(\delta, 6) = \left\{ \begin{array}{l} \binom{4}{0} \binom{2}{0}, \binom{4}{0} \binom{1}{1}, \binom{4}{0} \binom{0}{2}, \binom{3}{1} \binom{2}{0}, \binom{3}{1} \binom{1}{1}, \binom{3}{1} \binom{0}{2}, \\ \binom{2}{2} \binom{2}{0}, \binom{2}{2} \binom{1}{1}, \binom{2}{2} \binom{0}{2}, \binom{1}{3} \binom{2}{0}, \binom{1}{3} \binom{1}{1}, \binom{1}{3} \binom{0}{2}, \\ \binom{0}{4} \binom{2}{0}, \binom{0}{4} \binom{1}{1}, \binom{0}{4} \binom{0}{2} \end{array} \right\}.$$

Intersecting the sets  $M(\beta, 6)$  and  $M(\delta, 6)$  above, we obtain that

$$M(\beta, \delta, 6) = M(\beta, 6) \cap M(\delta, 6) = \left\{ \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \right\}.$$

We remark that the set  $M(\beta, \delta, 6)$  above is easily computed by supplying  $m = 2$  and  $n = 6$  to GAP and reading the Programme 2.3.3. We obtain the above magic matrices by considering the programme result for  $\lambda_s = 6$ , to be the set of matrices at column 4 (the position of  $\beta$  in the set of 2-sets of partitions of  $[1^6]$ ) and row 3 (the position of  $\delta$  in the set of 2-sets of partitions of  $[1^6]$ ) (see the highlighted text in Example 2.3.4 for Programme 2.3.3). Now by Theorem 4.3.1 together with the above magic matrices we have

$$\begin{aligned} f_{(6)}[(3, 3), (4, 2)] &= \binom{3}{3 \ 0} \binom{3}{1 \ 2} \xi^{(1-1)(1-1)3+(1-1)(2-1)0+(2-1)(1-1)1+(2-1)(2-1)2} \\ &\quad + \binom{3}{2 \ 1} \binom{3}{2 \ 1} \xi^{(1-1)(1-1)2+(1-1)(2-1)1+(2-1)(1-1)2+(2-1)(2-1)1} \\ &\quad + \binom{3}{1 \ 2} \binom{3}{3 \ 0} \xi^{(1-1)(1-1)1+(1-1)(2-1)2+(2-1)(1-1)3+(2-1)(2-1)0} \\ &= 3 - 9 + 3 = -3. \end{aligned}$$

The rest of the entries of  $F(2, 1^6)$  are obtained in the same way. The Fischer-Clifford matrix  $F(2, 1^6)$  is given in Table 5.1.

- (b) For another example we consider the construction of the Fischer-Clifford matrix  $F(2, 1^4 2)$ . Here  $\lambda_1 = 4$  and  $\lambda_2 = 1$  give the following set of 2-sets of partition  $[1^4 2]$ .

$$\left\{ \begin{array}{l} \left( \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix} \right\}.$$

We see that the Fischer-Clifford matrix  $F(2, 1^4 2)$  has order 10. We calculate only the entry given by  $f_{(4,1)}[((3, 0), (1, 1)), ((1, 1), (3, 0))]$ . Using Definition 2.3.1

with  $\beta = ((3, 0), (1, 1))$ ,  $\delta = ((1, 1), (3, 0))$  where  $a_{11} = 3, a_{12} = 0; k_{11} = 1, k_{12} = 1$  and  $a_{21} = 1, a_{22} = 1; k_{21} = 3, k_{22} = 0$ , we obtain that

$$M(\beta, \lambda_1) = \left\{ \begin{array}{l} \left( \begin{array}{cc} 3 & 0 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 3 & 0 \\ 0 & 1 \end{array} \right), \\ \left( \begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 2 & 1 \\ 0 & 1 \end{array} \right), \\ \left( \begin{array}{cc} 1 & 2 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right), \\ \left( \begin{array}{cc} 0 & 3 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 3 \\ 0 & 1 \end{array} \right) \end{array} \right\}; \quad M(\beta, \lambda_2) = \left\{ \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \right\}$$

$$M(\delta, \lambda_1) = \left\{ \begin{array}{l} \left( \begin{array}{cc} 1 & 3 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right), \\ \left( \begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right), \\ \left( \begin{array}{cc} 0 & 3 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 2 \\ 1 & 1 \end{array} \right), \\ \left( \begin{array}{cc} 0 & 1 \\ 1 & 2 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 1 & 3 \end{array} \right) \end{array} \right\}; \quad M(\delta, \lambda_2) = \left\{ \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \right\}.$$

Now intersecting the above sets for each  $\lambda_s, s = 1, 2$  we obtain

$$M(\beta, \delta, \lambda_1) = M(\beta, \lambda_1) \cap M(\delta, \lambda_1) = \left\{ \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 3 \\ 1 & 0 \end{array} \right) \right\};$$

$$M(\beta, \delta, \lambda_2) = M(\beta, \lambda_2) \cap M(\delta, \lambda_2) = \left\{ \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \right\}.$$

By Definition 2.3.1 we obtain that

$$M(\beta, \delta, 6) = \left\{ \left( \begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right), \left( \begin{array}{cc|cc} 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right) \right\}.$$

Now by Theorem 4.3.3 together with the above magic matrices we obtain

$$\begin{aligned} f_{(4,1)[((3,0),(1,1));((1,1),(3,0))]} &= \begin{pmatrix} 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 & 0 \end{pmatrix} \xi^{\mathcal{A}} + \begin{pmatrix} 3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 & 0 \end{pmatrix} \xi^{\mathcal{B}} \\ &= (3)(-1) + 1(1) = -2, \end{aligned}$$

where

$$\begin{aligned} \mathcal{A} &= \{(1-1)(1-1)\}\{(1+0)\} + \{(1-1)(2-1)\}\{(2+0)\} + \{(2-1)(1-1)\}\{(0+1)\} \\ &\quad + \{(2-1)(2-1)\}\{(1+0)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{B} &= \{(1-1)(1-1)\}\{(0+1)\} + \{(1-1)(2-1)\}\{(1+0)\} + \{(2-1)(1-1)\}\{(1+0)\} \\ &\quad + \{(2-1)(2-1)\}\{(0+1)\}. \end{aligned}$$

The rest of the entries of  $F(2, 1^4 2^1)$  are obtained in the same way. The Fischer-Clifford matrix  $F(2, 1^4 2)$  is given in Table 5.1.

However, we can efficiently calculate the entries of the Fischer-Clifford matrices of  $B(2, 6)$  using Theorems 4.3.2 and 4.3.5. We apply these theorems to calculate the entries of the Fischer-Clifford matrices  $F(2, 1^6)$  and  $F(2, 1^4 2^1)$  of  $B(2, 6)$  in the following.

- (c) For the Fischer-Clifford matrix  $F(2, 1^6)$ , we have  $\lambda_1 = 6$ . The 2-sets of partition  $[1^6]$  indexing the columns of  $F(2, 1^6)$  are:  $(6, 0), (5, 1), (4, 2), (3, 3), (2, 4), (1, 5), (0, 6)$ . Now by Theorem 4.3.2 we expand  $(x_1 + x_2)^{a_1}(x_1 - x_2)^{a_2}$ . For example, for  $a_1 = 6, a_2 = 0$ , we obtain the expansion

$$(x_1 + x_2)^6(x_1 - x_2)^0 = x_1^6 + 6x_1^5x_2 + 15x_1^4x_2^2 + 20x_1^3x_2^3 + 15x_1^2x_2^4 + 6x_1x_2^5 + x_2^6,$$

with coefficients 1, 6, 15, 20, 15, 6, 1. Also for  $a_1 = 3, a_2 = 3$  we obtain the expansion

$$(x_1 + x_2)^3(x_1 - x_2)^3 = x_1^6 + 0x_1^5x_2 + 3x_1^4x_2^2 + 0x_1^3x_2^3 + 3x_1^2x_2^4 + 0x_1x_2^5 - x_2^6,$$

with coefficients 1, 0, -3, 0, 3, 0, -1. We do the same for the rest of the 2-sets of partition  $[1^6]$  to obtain other sets of coefficients. The sets of coefficients found above give the respective columns of the Fischer-Clifford matrix  $F(2, 1^6)$  as

$$F(2, 1^6) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 & -2 & -4 & -6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & -20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & -4 & 2 & 0 & -2 & 4 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}.$$

(d) For the Fischer-Clifford matrix  $F(2, 1^4 2^1)$  we have  $\lambda_1 = 4$  and  $\lambda_2 = 1$ . We first consider the Fischer-Clifford matrices  $F(2, s^{\lambda_s})$  of  $B(2, s^{\lambda_s})$  for  $s = 1, 2$ . We obtain

(i) for  $\lambda_1 = 4$ , the following set of 2-sets of partition  $[1^4]$

$$\left\{ \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix} \right\},$$

(ii) for  $\lambda_1 = 1$ , the following set of 2-sets of partition  $[2^1]$

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Using the above 2-sets of partitions  $[1^4]$  and  $[2^1]$  and Theorem 4.3.2 we construct the respective matrices :

$$F(2, 1^4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}, \quad F(2, 2^1) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now by Theorem 4.3.5 the tensor product  $F(2, 2^1) \otimes F(2, 1^4)$  gives the matrix  $F'(2, 1^4 2^1)$  (or  $F'(2, 2^1 1^4)$ ) as

$$F'(2, 1^4 2^1) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 & 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 & 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 & 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 4 & 2 & 0 & -2 & -4 & -4 & -2 & 0 & 2 & 4 \\ 6 & 0 & -2 & 0 & 6 & -6 & 0 & 2 & 0 & -6 \\ 4 & -2 & 0 & 2 & -4 & -4 & 2 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

- (e) Further, from the result of the tensor product  $F(2, 2^1) \otimes F(2, 1^4)$ , we see that the rows (indexed by the 2-sets of partition  $[1^4]$ ) of  $F(2, 1^4)$  are fixed while the rows (indexed by the 2-sets of partition  $[2^1]$ ) of  $F(2, 2^1)$  vary. This results in the rows of  $F'(2, 1^4 2^1)$ , indexed by the 2-sets of partition  $[1^4 2^1]$  arranged as follows.

$$\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix}.$$

Now from the remarks in Section 4.2, about the fusion of the conjugacy classes of the inertia factor groups into  $S_n$ , for each row of  $F'(2, 1^4 2^1)$  we calculate the 2-compositions  $(k_1, k_2)$  of 6 corresponding to the inertia factor groups  $S_{k_1} \times S_{k_2}$  as:  $(6, 0), (5, 1), (4, 2), (3, 3), (2, 4), (4, 2), (3, 3), (2, 4), (1, 5), (0, 6)$ .

The repeated 2-compositions of 6 appearing in non-adjacent positions of the above set indicates that we require to have the pairs of rows 3 and 6, 4 and 7, 5 and 8 of  $F'(2, 1^4 2^1)$  collected into blocks corresponding to inertia factor groups as shown below. Thus

$$F(2, 1^4 2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 & 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 & 6 & 0 & -2 & 0 & 6 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 4 & -2 & 0 & 2 & -4 & 4 & -2 & 0 & 2 & -4 \\ 4 & 2 & 0 & -2 & -4 & -4 & -2 & 0 & 2 & 4 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 6 & 0 & -2 & 0 & 6 & -6 & 0 & 2 & 0 & -6 \\ 4 & -2 & 0 & 2 & -4 & -4 & 2 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix} \begin{matrix} S_6 \\ S_5 \\ S_{42} \\ S_{33} \\ S_{24} \\ S_5 \\ S_6 \end{matrix} \quad (4.3)$$

The Fischer-Clifford matrices of  $B(2, 6)$  are given in Table 5.1.

#### 4.4.2 Constructing the Fischer-Clifford Matrices of $B(3, 5)$

We determine the Fischer-Clifford matrices of  $B(3, 5)$  using the results in section 4.3. For each conjugacy class of  $S_5$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 5^{\lambda_5})$  we construct the Fischer-Clifford matrix  $F(3, 1^{\lambda_1} 2^{\lambda_2} \dots 5^{\lambda_5})$  of  $B(3, 5)$ .

- (a) For the Fischer-Clifford matrix  $F(3, 1^5)$ , we have  $\lambda_1 = 5$  and  $m = 3$ . We obtain following set of twenty one 3-sets of partition  $[1^5]$ .

$$\left\{ \begin{matrix} \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \\ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} \end{matrix} \right\}$$

Thus the Fischer-Clifford matrix  $F(3, 1^5)$  has order 21. As we have done for  $B(2, 6)$ , we calculate only one entry of  $F(3, 1^5)$  given by  $f_{(5)}[(3, 1, 1), (4, 0, 1)]$ . Using Definition 2.3.1, with  $\beta = (3, 1, 1)$  and  $\delta = (4, 0, 1)$ , we obtain that  $M(\delta, 5)$  contains 45 ( $= 15 \times 1 \times 3$ ) magic matrices. Thus



easily (see Method 2.3.2), by choosing from  $M(\delta, 5)$  only those matrices with respective row entries adding up to  $a_1 = 3, a_2 = 1, a_3 = 1$ . Now by Theorem 4.3.1 together with the above magic matrices we obtain

$$\begin{aligned}
 f_{(5)}[(3, 1, 1), (4, 0, 1)] &= \begin{pmatrix} 3 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 & 0 & 1 \end{pmatrix} \xi^A + \begin{pmatrix} 3 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 & 0 & 0 \end{pmatrix} \xi^B \\
 &\quad + \begin{pmatrix} 3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 & 0 & 0 \end{pmatrix} \xi^C \\
 &= \xi^{0+0+4} + \xi^{0+2+0} + 3\xi^{0+0+0} \\
 &= 3 + \xi + \xi^2 = 2,
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{A} &= \{(1-1)(1-1)\}3 + \{(1-1)(2-1)\}0 + \{(1-1)(3-1)\}0 \\
 &\quad + \{(2-1)(1-1)\}1 + \{(2-1)(2-1)\}0 + \{(2-1)(3-1)\}0 \\
 &\quad + \{(3-1)(1-1)\}0 + \{(3-1)(2-1)\}0 + \{(3-1)(3-1)\}1, \\
 \mathcal{B} &= \{(1-1)(1-1)\}3 + \{(1-1)(2-1)\}0 + \{(1-1)(3-1)\}0 \\
 &\quad + \{(2-1)(1-1)\}0 + \{(2-1)(2-1)\}0 + \{(2-1)(3-1)\}1 \\
 &\quad + \{(3-1)(1-1)\}1 + \{(3-1)(2-1)\}0 + \{(3-1)(3-1)\}0, \\
 \mathcal{C} &= \{(1-1)(1-1)\}2 + \{(1-1)(2-1)\}0 + \{(1-1)(3-1)\}1 \\
 &\quad + \{(2-1)(1-1)\}1 + \{(2-1)(2-1)\}0 + \{(2-1)(3-1)\}0 \\
 &\quad + \{(3-1)(1-1)\}1 + \{(3-1)(2-1)\}0 + \{(3-1)(3-1)\}0.
 \end{aligned}$$

The rest of the entries of  $F(3, 1^5)$  are obtained in the same way. The Fischer-Clifford matrix  $F(3, 1^5)$  is given in Table 5.2.

- (b) For the Fischer-Clifford matrix  $F(3, 1^3 2^1)$ , we have  $\lambda_1 = 3, \lambda_2 = 1$  and  $m = 3$ . We obtain the following set of thirty 3-sets of partition  $[1^3 2]$ .

$$\left( \begin{array}{cccccccc} \left( \begin{array}{c} 31 \\ 00 \\ 00 \end{array} \right), & \left( \begin{array}{c} 21 \\ 10 \\ 00 \end{array} \right), & \left( \begin{array}{c} 21 \\ 00 \\ 10 \end{array} \right), & \left( \begin{array}{c} 11 \\ 20 \\ 00 \end{array} \right), & \left( \begin{array}{c} 11 \\ 10 \\ 10 \end{array} \right), & \left( \begin{array}{c} 11 \\ 00 \\ 20 \end{array} \right), & \left( \begin{array}{c} 01 \\ 30 \\ 00 \end{array} \right), & \left( \begin{array}{c} 01 \\ 20 \\ 10 \end{array} \right) \\ \left( \begin{array}{c} 01 \\ 10 \\ 20 \end{array} \right), & \left( \begin{array}{c} 01 \\ 00 \\ 30 \end{array} \right), & \left( \begin{array}{c} 30 \\ 01 \\ 00 \end{array} \right), & \left( \begin{array}{c} 20 \\ 11 \\ 00 \end{array} \right), & \left( \begin{array}{c} 20 \\ 01 \\ 10 \end{array} \right), & \left( \begin{array}{c} 10 \\ 21 \\ 00 \end{array} \right), & \left( \begin{array}{c} 10 \\ 11 \\ 10 \end{array} \right), & \left( \begin{array}{c} 10 \\ 01 \\ 20 \end{array} \right) \\ \left( \begin{array}{c} 00 \\ 31 \\ 00 \end{array} \right), & \left( \begin{array}{c} 00 \\ 21 \\ 10 \end{array} \right), & \left( \begin{array}{c} 00 \\ 11 \\ 20 \end{array} \right), & \left( \begin{array}{c} 00 \\ 01 \\ 30 \end{array} \right), & \left( \begin{array}{c} 30 \\ 00 \\ 01 \end{array} \right), & \left( \begin{array}{c} 20 \\ 10 \\ 01 \end{array} \right), & \left( \begin{array}{c} 20 \\ 00 \\ 11 \end{array} \right), & \left( \begin{array}{c} 10 \\ 20 \\ 01 \end{array} \right) \\ & \left( \begin{array}{c} 10 \\ 10 \\ 11 \end{array} \right), & \left( \begin{array}{c} 10 \\ 00 \\ 21 \end{array} \right), & \left( \begin{array}{c} 00 \\ 30 \\ 01 \end{array} \right), & \left( \begin{array}{c} 00 \\ 20 \\ 11 \end{array} \right), & \left( \begin{array}{c} 00 \\ 10 \\ 21 \end{array} \right), & \left( \begin{array}{c} 00 \\ 00 \\ 31 \end{array} \right) \end{array} \right)$$

Thus the Fischer-Clifford matrix  $F(3, 1^3 2)$  has size 30. We calculate only one entry of  $F(3, 1^3 2)$  given by  $f_{(3,1)}[((2, 1), (1, 0), (0, 0)), ((1, 1), (1, 0), (1, 0))]$ . Using Definition 2.3.1, with  $\beta = ((2, 1)(1, 0), (0, 0))$  and  $\delta = ((1, 1), (1, 0), (1, 0))$  where  $a_{11} = 2, a_{12} = 1, k_{11} = 1, k_{12} = 1; a_{21} = 1, a_{22} = 0, k_{21} = 1, k_{22} = 0$  and  $a_{31} = 0, a_{32} = 0, k_{31} = 1, k_{32} = 0$ , we obtain that

$$M(\beta, \lambda_1) = \left( \begin{array}{cccccc} \left( \begin{array}{ccc} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), & \\ \left( \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), & \\ \left( \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), & \\ & \left( \begin{array}{ccc} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right), & \left( \begin{array}{ccc} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right) \end{array} \right)$$

and

$$M(\beta, \lambda_2) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Now we obtain  $M(\beta, \delta, \lambda_1)$  and  $M(\beta, \delta, \lambda_2)$  directly by choosing matrices from  $M(\beta, \lambda_s)$  with respective column entries adding up to  $k_{11} = 1, k_{21} = 1, k_{31} = 1$  and  $k_{12} = 1, k_{22} = 0, k_{32} = 0$  for each  $\lambda_s$  respectively. We obtain that

$$M(\beta, \delta, \lambda_1) = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

and

$$M(\beta, \delta, \lambda_2) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Therefore we have

$$M(\beta, \delta, 5) = \left\{ \left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right), \left( \begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\}.$$

Now by Theorem 4.3.3 together with the above magic matrices and for  $\beta = ((2, 1), (1, 0), (0, 0))$ ,  $\delta = ((1, 1), (1, 0), (1, 0))$  we obtain

$$\begin{aligned} f_{(3,1)}[\beta, \delta] &= \binom{2}{1 \ 1 \ 0} \binom{1}{0 \ 0 \ 1} \binom{0}{0 \ 0 \ 0} \binom{1}{1 \ 0 \ 0} \binom{0}{0 \ 0 \ 0} \binom{0}{0 \ 0 \ 0} \xi^A \\ &\quad + \binom{2}{1 \ 0 \ 1} \binom{1}{0 \ 1 \ 0} \binom{0}{0 \ 0 \ 0} \binom{1}{1 \ 0 \ 0} \binom{0}{0 \ 0 \ 0} \binom{0}{0 \ 0 \ 0} \xi^B \\ &\quad + \binom{2}{0 \ 1 \ 1} \binom{1}{1 \ 0 \ 0} \binom{0}{0 \ 0 \ 0} \binom{1}{1 \ 0 \ 0} \binom{0}{0 \ 0 \ 0} \binom{0}{0 \ 0 \ 0} \xi^C \\ &= 2\xi^{0+2+0} + 2\xi^{0+1+0} + 2\xi^{0+0+0} = 2\xi^2 + 2\xi + 2 = 0, \end{aligned}$$

where

$$\begin{aligned}
\mathcal{A} &= \{(1-1)(1-1)\}\{(1+1)\} + \{(1-1)(2-1)\}\{(1+0)\} + \{(1-1)(3-1)\}\{(0+0)\} \\
&\quad + \{(2-1)(1-1)\}\{(0+0)\} + \{(2-1)(2-1)\}\{(0+0)\} + \{(2-1)(3-1)\}\{(1+0)\} \\
&\quad + \{(3-1)(1-1)\}\{(0+0)\} + \{(3-1)(2-1)\}\{(0+0)\} + \{(3-1)(3-1)\}\{(0+0)\}, \\
\mathcal{B} &= \{(1-1)(1-1)\}\{(1+1)\} + \{(1-1)(2-1)\}\{(0+0)\} + \{(1-1)(3-1)\}\{(1+0)\} \\
&\quad + \{(2-1)(1-1)\}\{(0+0)\} + \{(2-1)(2-1)\}\{(1+0)\} + \{(2-1)(3-1)\}\{(0+0)\} \\
&\quad + \{(3-1)(1-1)\}\{(0+0)\} + \{(3-1)(2-1)\}\{(0+0)\} + \{(3-1)(3-1)\}\{(0+0)\}, \\
\mathcal{C} &= \{(1-1)(1-1)\}\{(0+1)\} + \{(1-1)(2-1)\}\{(1+0)\} + \{(1-1)(3-1)\}\{(1+0)\} \\
&\quad + \{(2-1)(1-1)\}\{(1+0)\} + \{(2-1)(2-1)\}\{(0+0)\} + \{(2-1)(3-1)\}\{(0+0)\} \\
&\quad + \{(3-1)(1-1)\}\{(0+0)\} + \{(3-1)(2-1)\}\{(0+0)\} + \{(3-1)(3-1)\}\{(0+0)\}.
\end{aligned}$$

The rest of the entries of  $F(3, 1^3 2^1)$  are obtained in the same way. The Fischer-Clifford matrix  $F(3, 1^3 2^1)$  is given in Tables 5.3 to 5.5.

We now alternatively apply Theorems 4.3.2 and 4.3.5 to construct Fischer-Clifford matrices  $F(3, 1^5)$  and  $F(3, 1^3 2^1)$  in the following.

- (c) For the Fischer-Clifford matrix  $F(3, 1^5)$ , we have  $\lambda_1 = 5$  and  $m = 3$ . The 3-sets of partition  $[1^5]$  indexing the columns and rows of  $F(3, 1^5)$  are as given in Example 4.4.2(a) above. Let  $p_1 = x_1 + x_2 + x_3$ ,  $p_2 = x_1 + \xi x_2 + \xi x_3$  and  $p_3 = x_1 + \xi^2 x_2 + \xi^4 x_3$ . Then with for example  $a_1 = 5, a_2 = 0, a_3 = 0$ , we obtain

$$\begin{aligned}
p_1^5 p_2^0 p_3^0 &= x_1^5 + 5x_1^4 x_2 + 5x_1^4 x_3 + 10x_1^3 x_2^2 + 20x_1^3 x_2 x_3 + 10x_1^3 x_3^2 + 10x_1^2 x_2^3 + 30x_1^2 x_2^2 x_3 + 30x_1^2 x_2 x_3^2 \\
&\quad + 10x_1^2 x_3^3 + 5x_1 x_2^4 + 20x_1 x_2^3 x_3 + 30x_1 x_2^2 x_3^2 + 20x_1 x_2 x_3^3 + 5x_1 x_3^4 + x_2^5 + 5x_2^4 x_3 + 10x_2^3 x_3^2 \\
&\quad + 10x_2^2 x_3^3 + 5x_2 x_3^4 + x_3^5,
\end{aligned}$$

giving the coefficients

$$1, 5, 5, 10, 20, 10, 10, 30, 30, 10, 5, 20, 30, 20, 5, 1, 5, 10, 10, 5, 1$$

for column 1 of  $F(3, 1^5)$ . Similarly for  $a_1 = 4, a_2 = 1, a_3 = 0$  we obtain

$$\begin{aligned}
p_1^4 p_2^1 p_3^0 &= x_1^5 + (-3\xi - 4\xi^2)x_1^4 x_2 + (-4\xi - 3\xi^2)x_1^4 x_3 + (-2\xi - 6\xi^2)x_1^3 x_2^2 + 8x_1^3 x_2 x_3 \\
&+ (-6\xi - 2\xi^2)x_1^3 x_3^2 + (2\xi - 4\xi^2)x_1^2 x_2^3 - 6\xi^2 x_1^2 x_2^2 x_3 - 6\xi x_1^2 x_2 x_3^2 + (-4\xi + 2\xi^2)x_1^2 x_3^3 \\
&+ (3\xi - \xi^2)x_1 x_2^4 + 8\xi x_1 x_2^3 x_3 - 6x_1 x_2^2 x_3^2 + 8\xi^2 x_1 x_2 x_3^3 + (-\xi + 3\xi^2)x_1 x_3^4 + \xi x_2^5 \\
&+ (4\xi + \xi^2)x_2^4 x_3 + (6\xi + 4\xi^2)x_2^3 x_3^2 + (4\xi + 6\xi^2)x_2^2 x_3^3 + (\xi + 4\xi^2)x_2 x_3^4 + \xi^2 x_3^5,
\end{aligned}$$

giving the coefficients

$$\begin{aligned}
&1, (-3\xi - 4\xi^2), (-4\xi - 3\xi^2), (-2\xi - 6\xi^2), 8, (-6\xi - 2\xi^2), (2\xi - 4\xi^2), -6\xi^2, -6\xi, (-4\xi + 2\xi^2), \\
&(3\xi - \xi^2), 8\xi, -6, 8\xi^2, (-\xi + 3\xi^2), \xi, (4\xi + \xi^2), (6\xi + 4\xi^2), (4\xi + 6\xi^2), (\xi + 4\xi^2), \xi^2
\end{aligned}$$

for column 2 of  $F(3, 1^5)$ . And for  $a_1 = 3, a_2 = 1, a_3 = 1$  we obtain

$$\begin{aligned}
p_1^3 p_2^1 p_3^1 &= x_1^5 + 2x_1^4 x_2 + 2x_1^4 x_3 + x_1^3 x_2^2 - x_1^3 x_2 x_3 + x_1^3 x_3^2 + x_1^2 x_2^3 - 6x_1^2 x_2^2 x_3 \\
&- 6x_1^2 x_2 x_3^2 + x_1^2 x_3^3 + 2x_1 x_2^4 - x_1 x_2^3 x_3 - 6x_1 x_2^2 x_3^2 - x_1 x_2 x_3^3 + 2x_1 x_3^4 \\
&+ x_2^5 + 2x_2^4 x_3 + x_2^3 x_3^2 + x_2^2 x_3^3 + 2x_2 x_3^4 + x_3^5,
\end{aligned}$$

giving the coefficients

$$1, 2, \mathbf{2}, 1, -1, 1, 1, -6, -6, 1, 2, -1, -6, -1, 2, 1, 2, 1, 1, 2, 1,$$

including the entry **2** found in Example 4.4.2(a) above, for column 5 of  $F(3, 1^5)$  and so on. The sets of coefficients found as above give the columns of the Fischer-Clifford matrix  $F(3, 1^5)$ , given in Table 5.2.

- (d) For the Fischer-Clifford matrix  $F(3, 1^3 2^1)$ ,  $n = 5$ ,  $\lambda_1 = 3$ ,  $\lambda_2 = 1$  and  $m = 3$ . We obtain the respective Fischer-Clifford matrices  $F(3, 1^3)$  and  $F(3, 1^1)$  as has been done for  $F(3, 1^5)$  above. Now by Theorem 4.3.5, the tensor product  $F(3, 2^1) \otimes F(3, 1^3)$  gives the matrix  $F'(3, 1^3 2^1)$ . Lastly we note that rows of the matrix  $F'(3, 1^3 2^1)$  require to be re-arranged to give the Fischer-Clifford matrix  $F(3, 1^3 2^1)$  (see Example 5.3.2 (b) in Chapter 5).

The Fischer-Clifford matrices of  $B(3, 5)$  are given in Tables 5.2 to 5.5.

Before we end this chapter, we restate Propositions 1.5.2 and 1.5.6 for an alternative method for constructing the Fischer-Clifford matrices of  $B(m, n)$ , where  $m$  is prime.

**Proposition 4.4.1** *The Fischer-Clifford matrix  $F(m, 1^n)$  is the matrix with rows equal to the orbit sums of the action of  $B(m, n)$  on the rows of the character table of  $N$  with duplicate columns discarded.*

**Proof.** See Proposition 1.5.6.  $\square$

**Proposition 4.4.2** *Let  $\bar{g} = dg \in B(m, n)$ , where  $m$  is prime. Let  $K$  be the image of the endomorphism  $\phi_{\bar{g}}$  of  $N$  defined by  $\phi_{\bar{g}}(d) = d - d\bar{g}$ . Then the Fischer-Clifford matrix  $F(m, 1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots n^{\lambda_n})$  at a non-identity element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  is the matrix with rows equal to the orbit sums of the action of the stabilizer  $N:C_{S_n}(g)$  on the rows of the character table of the quotient  $N/K$ , with duplicate columns discarded.*

**Proof.** See Proposition 1.5.2.  $\square$

However in Proposition 4.4.2, we take  $\phi_{\bar{g}}$  as  $\phi_g$  since  $N$  is invariant under  $N$  and  $\bar{G}$  is a split-extension with  $\bar{g} = dg$ .

## Chapter 5

# Computing Fischer-Clifford Matrices of $B(m, n)$

### 5.1 Introduction

In Chapter 4, we discussed a combinatorial method for constructing the Fischer-Clifford matrices of generalized symmetric groups. However since the combinatorial method involves protracted calculations for each entry of a Fischer-Clifford matrix, it is not such an efficient method. Indeed as a few examples for small values of  $m$  and  $n$  show, the manual construction of the Fischer-Clifford matrices of  $B(m, n)$  using the combinatorial method is not an easy task. However we can use computers, which have become very much an integral part of research in Group Theory (see [8], [9], [22], [42], [48], [50], [51], [58]), to simplify the task of combinatorially constructing the Fischer-Clifford matrices of  $B(m, n)$ .

The partial aim of our work was to use the combinatorial method to develop a computer programme for computing the Fischer-Clifford matrices of the generalized symmetric groups  $B(m, n)$ . In Chapters 2 and 3 we have given a series of computer programmes, written in GAP, that give various parameters for Fischer-Clifford matrices of  $B(m, n)$ . Many of these programmes have been used in the development of the main Programme 5.2.4 that combinatorially computes matrices which are row equivalent to the Fischer-Clifford matrices of  $B(m, n)$ . Thus the matrices computed by Programme 5.2.4 require to have their rows collected together into blocks that correspond to particular inertia factor groups. We use Programme 5.2.2 to deter-

mine the  $m$ -compositions of  $n$  corresponding to the rows of the matrices computed by Programme 5.2.4. A particular  $m$ -composition of  $n$  distinguishes the rows of these matrices that correspond to the same inertia factor group. All the rows corresponding to a particular  $m$ -composition  $(k_1, k_2, \dots, k_m)$  of  $n$  and therefore to the inertia factor group  $S_{k_1} \times S_{k_2} \times \dots \times S_{k_m}$  form a block of a Fischer-Clifford matrix of  $B(m, n)$ . Such rows of a matrix computed by Programme 5.2.4 are then manually collected together in a block corresponding to a particular inertia factor group, to give a Fischer-Clifford matrix of  $B(m, n)$ . We give the Programmes 5.2.2 and 5.2.4 in Section 5.2. However we also give a special programme (Programme 5.2.1) for computing matrices which are row equivalent to the Fischer-Clifford matrices of the groups  $B(2, n)$  here.

As examples, we apply the Programmes 5.2.2 and 5.2.4 to construct the Fischer-Clifford matrices of the groups  $B(2, 6)$  and  $B(3, 5)$  of orders 46080 and 29160 in Section 5.3. We have also used Programme 5.2.4 to construct the Fischer-Clifford matrices of the groups  $B(2, 12)$  and  $B(4, 5)$  of orders respectively  $2^{22} \times 3^5 \times 5^2 \times 7 \times 11$  and  $2^{13} \times 3 \times 5$ . Due to lack of space here we have not given the Fischer-Clifford matrices of  $B(2, 12)$  and  $B(4, 5)$ . However we have given the matrices which are row equivalent to the Fischer-Clifford matrices of  $B(2, 12)$  and  $B(4, 5)$  on the compact disk submitted with this thesis. Further a summary on the Fischer-Clifford matrices of the groups  $B(2, 12)$  and  $B(4, 5)$  is given in Subsection 5.3.3.

## 5.2 Programmes Written for GAP

We discuss the programmes for  $m = 2$  and a general  $m$  separately in the following subsections. It was relatively easy to develop the programme for  $m = 2$  than for a general  $m$ .

### 5.2.1 Programme for $B(2, n)$

During the earlier stages of the development of Programme 5.2.4, we had developed a special programme for computing the matrices  $F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  which are row equivalent to the Fischer-Clifford matrices  $F(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of the generalized symmetric groups  $B(2, n)$ . We note that the main Programme 5.2.4 generally requires more workspace to compute the matrices  $F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . Thus it is more effi-

cient and therefore advisable to use this special programme when constructing the Fischer-Clifford matrices of  $B(2, n)$ . Using the fore-going Programmes 2.1.3 and 2.2.3, Theorem 4.3.5 and Corollary 4.3.6, we developed Programme 5.2.1 for computing the matrices  $F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . To use the Programme 5.2.1, we first initialize the value of  $c$  in the programme by entering "c:=0;" once. There-after we supply each value of  $n$  to GAP and run the Programme 5.2.1. The programme result is displayed as

$$[\bar{\lambda}_0, "F'(2, i) := ", F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})],$$

where the index  $\bar{\lambda}_0$  appears first and the matrix  $F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  appears after "F'(2, i) :=", which essentially indicates that this is a matrix row equivalent to a Fischer-Clifford matrix of  $B(2, n)$ . Since the Fischer-Clifford matrices are indexed by  $\bar{\lambda}_0$ , the matrices  $F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  which correspond to equal  $\bar{\lambda}_0$  appear once.

Now as indicated in the introduction to this chapter, the matrices produced by Programme 5.2.1 require to have their rows collected together into blocks that correspond to particular inertia factor groups to form Fischer-Clifford matrices of  $B(2, n)$ . The Programme 5.2.2 is used to determine  $m$ -compositions of  $n$  (in this case  $m=2$ ) which correspond to the rows of the matrices  $F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . All the rows corresponding to a particular 2-composition  $(k_1, k_2)$  of  $n$  and therefore to the inertia factor group  $S_{k_1} \times S_{k_2}$  form a block of the Fischer-Clifford matrix of  $B(2, n)$ .

**Programme 5.2.1 (Matrices  $F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of  $B(2, n)$ )**

```
n:=n;; m:=2;;
c:=c+1;;
p:=Partitions(n);;
l:=List([1..n]);;
L:=List([1..Number(p)], i -> List(l, j -> Number(p[i], k -> k=j)));;
r:=List(L, i -> Filtered(i, j -> j < 0));;
s:=Set(r);;
t:=List([1..n+5], i -> Filtered(s, j -> Number(j)=i));;
x:=X(Rationals, "x");;
f:=1+x;; g:=1-x;;
S:=List(l, i -> List([0..i], j -> f^(i-j) * g^j ));;
Q:=List(S, i -> List(i, j -> PolynomialCoefficientsOfPolynomial(j, c)));;
a:=List(Q, i -> TransposedMat(i));;
```

```
f1:=List(t[1], i -> [i,,a[i[1]]]);;
f2:=List(t[2],i -> [i,,KroneckerProduct(a[i[2]],a[i[1]])]);;
f3:=List(t[3],i -> [i,,KroneckerProduct(a[i[3]],KroneckerProduct(a[i[2]],
a[i[1]]))]);;
f4:=List(t[4], i -> [i,,KroneckerProduct(a[i[4]],KroneckerProduct(a[i[3]],
KroneckerProduct(a[i[2]],a[i[1]]))]);;
f5:=List(t[5], i -> [i,,KroneckerProduct(a[i[5]],KroneckerProduct(a[i[4]],
KroneckerProduct(a[i[3]], KroneckerProduct(a[i[2]],a[i[1]])))]);;
```

(may be extended at  $a[i[5]]$  depending on the size of  $n$  ; taken from  $a[i[5++]]$  to  $a[i[1]]$  for correct evaluation. )

```
Display(["partition powers ",t]);
Dis:= "Fischer-Clifford matrices grouped by powers [i] of partitions in t,
Match with t";;
FM:= Difference( List([f1,f2,f3,f4,f5]),[]);;
Display("Type FM to print all Fischer-Clifford Matrices, otherwise Type f1,
f2, f3, f4, f5 , etc to print sets of  $F'(2,n)$  with the same number of
partition powers, otherwise Type fq[j] to print required particular  $F'(2,n)$ ");;
```

### Programme 5.2.2 (Identifying rows of $F'(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ )

```
IdIFGBmn:=function(m,n)
local l,L,p,q,mcn,r,a,b,mspn1,lths,IFGnos;
l:=List([1..n]);;
L:=List(l, i- >n+1-i);;
p:=Partitions(n);;
q:=List([1..Number(p)],i- >List(L,j- >Number(p[i],k- >k=j)));;
mcn:=List(l,i- >Reversed(Filtered(Tuples([0..i],m),j- >Sum(j)=i)));;
r:=List(q, i- >Filtered(i, j- > j > 0));;
r1:=List(r,i- >Reversed(i));;
a:=List(r, j- >Cartesian(List([1..Number(j)], k- >mcn[j[k]])));;
b:=List(a, j- >List(j, k- >List([1..Number(k)], u- >k[Number(k)+1-u]));;
mspn1:=List(b, j- >List(j, k- >TransposedMat(k)));;
```

```

lths:=List(p,i->Union(List(i,j->[j])));;
IFGnos:=List([1..Number(mspn1)],i->["λ0=", Reversed(r[i])
,,,,,,,,, "corresponding mcn",List([1..Number(mspn1[i])],
j->List([1..Number(mspn1[i][j])],k->Sum(List([1..Number(mspn1[i][j][k])],
u->lths[i][u]*mspn1[i][j][k][u]))))));;
return IFGnos;
end;

```

However the aim of our work was to develop a programme for computing Fischer-Clifford matrices of  $B(m, n)$ , for positive integers  $m$  and  $n$ .

### 5.2.2 Programme for $B(m, n)$

To develop a programme for computing Fischer-Clifford matrices of  $B(m, n)$  (or matrices row equivalent to these), we needed to modify Programme 5.2.1. This resulted in the following problems.

- (i) We needed to define a function to generate a sum of  $m$  variables as given in Theorem 4.3.1, that is to replace statements  $f$  and  $g$  of Programme 5.2.1. Similarly it is required that the powers  $i - j$  and  $j$  of the expression at statement  $S$  of Programme 5.2.1 are replaced by the parts  $k_j$  of the  $m$ -compositions of  $n$ .
- (ii) The GAP function **PolynomialCoefficientsOfPolynomial(j,c)** of Programme 5.2.1 does not apply to multi-variable polynomials, and so there was a problem of extracting the coefficients of the multi-variable polynomials produced as indicated in (i) above. The function required to do this, is not available in GAP.
- (iii) The function **KroneckerProduct** does not apply to a set of more than two matrices. This was the reason why we used the statements  $fi$  of Programme 5.2.1. However this means that where some collections of numbers in statement  $t$  have more than five numbers, the programme cannot apply. The Programme 5.2.1 would have to be extended manually by adding new lines  $f6, f7, \dots$
- (iv) The rows of the matrices computed by Programme 5.2.1 are not given in blocks, corresponding to each particular inertia factor group.

All of the above problems except (iv) have been completely solved (see Programme 5.2.4) as follows.

- (i) We wrote the function *pfm* of Programme 5.2.4 to construct the partial Fischer-Clifford matrices  $F(m, 1^{\lambda_s})$  given by Theorem 4.3.1. The statements *v0*, *ind* and *res1* of *pfm* are used to construct the expressions to be expanded by the programme according to Theorem 4.3.2. The function *mcompsn* is used to construct *m*-compositions of *n* which replace the powers  $i - j$  and  $j$  in Programme 5.2.1.
- (ii) The GAP function **ExtRepPolynomialRatFun** collects the coefficients (without zeros) of a multi-variable polynomial but mixed with other information. In order to introduce the zero coefficients, we had to add the expression  $(1/(zimba) * i[1] + j)$  to the polynomials on which the function "ExtRepPolynomialRatFun" applies, this appears in statement *res3*. The function **ExtRepPolynomialRatFun** is used in statement *res4*. The removal of the extra information in the result of the function **ExtRepPolynomialRatFun** and the coefficients of the expression  $(1/(zimba) * i[1] + j)$  is done in statements *nat* and *res6* of *pfm* respectively.
- (iii) Problem (iii) has been solved by writing the Programme 5.2.3 given below.

**Programme 5.2.3 (Multiple-Kronecker product)**

```
setkroneckpdt:=function(pm)
local i,res;
res:=[[1]];
for i in pm do
res:=KroneckerProduct(i ,res);
od;
return res;
end;
```

- (iv) The problem (iv) is not solved yet. However the Programme 5.2.2 was developed to determine the *m*-compositions of *n* which correspond to the rows of the

matrices  $F'(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . All the rows corresponding to a particular  $m$ -composition  $(k_1, k_2, \dots, k_m)$  of  $n$  and therefore to the inertia factor group  $S_{k_1} \times S_{k_2} \times \dots \times S_{k_m}$  form a block of a Fischer-Clifford matrix of  $B(m, n)$ .

Using the fore-going Programmes 2.1.3, 2.2.3 and 5.2.1, Theorems 4.3.2 and 4.3.5 and Corollary 4.3.6, we have developed Programme 5.2.4 for computing the matrices  $F'(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  which are row equivalent to the Fischer-Clifford matrices  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of  $B(m, n)$ . To use the Programme 5.2.4, we run it in GAP. Then we supply  $m$  and  $n$  to GAP by entering the statement  $Fcm(m, n)$ ; with the required values of  $m$  and  $n$ . Just as for the Programme 5.2.1, the matrices  $F'(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  which correspond to equal  $\bar{\lambda}_0$  appear once. Also the Programme 5.2.2 is used to determine  $m$ -compositions of  $n$  which correspond to the rows of the matrices  $F'(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . All the rows corresponding to a particular  $m$ -composition  $(k_1, k_2, \dots, k_m)$  of  $n$  and therefore to the inertia factor group  $S_{k_1} \times S_{k_2} \times \dots \times S_{k_m}$  form a block of a Fischer-Clifford matrix of  $B(m, n)$ .

**Programme 5.2.4 (Matrices  $F'(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of  $B(m, n)$ )**

```

mcompsn:=function(m,n)
  local res,i,mcn;
  res:=[];
  for i in [1..n] do
    mcn:= Reversed(Filtered(Tuples([0..i],m),j->Sum(j)=i));
    Add(res,mcn);
  od;
  return res;
end;

partpowers:=function(m,n)
  local p;
  p:=Partitions(n);
  return Set(List([1..Number(p)],u->List([1..n],
    i->Number(p[u],j->j=i))),k->Filtered(k, l->l > 0));
end;

pfm:=function(m,n)

```

```

local zimba,j1,j2,j3,j4,j5,j6,res1,res2,res3,res4,res5,res6,
v0,v1,ind,po,kap,no,qo,so,oo,tam,nat,zim;
zimba:=n * m * 100;
v0:=List([1..m],i- >E(m)^i);
v1:=VectorSpace(Rationals,v0);
ind:=List([1..m],i- >Indeterminate(Rationals,i));;
po:=[];
for j1 in [1..m] do
res1:=Sum(List([1..m], j- >E(m)^(j-1)*(j1-1) * ind[j]));
Add(po,res1);
od;
kap:=[];
for j2 in mcompsn(m,n) do
res2:=List(j2,j- >Product(List([1..m], k- > po[k]^j[k])););
Add(kap,res2);
od;
no:=[];
for j3 in kap do
res3:=List(j3,j- >(1/(zimba) * j3[1] +j);
Add(no,res3);
od;
qo:=[];
for j4 in no do
res4:=List(j4, ext- > ExtRepPolynomialRatFun (ext));
Add(qo,res4);
od;
so:=[];
for j5 in qo do
res5:=List(j5, j- >Filtered(j, x- >x in v1));
Add(so,res5);
od;
oo:=List(kap, i- >i[1]);;
tam:=List(oo, ext- > ExtRepPolynomialRatFun (ext));;
nat:=List(tam, j- >Filtered(j, x- >x in v1));;

```

```

zim:=[];
for j6 in [1..Number(so)] do
res6:=List(so[j6],j->(-1/(zimba) * nat[j6]) +j);
Add(zim,res6);
od;
return List(zim, i->TransposedMat(i));
end;

skroneckp1:=function(m,n,pi)
local i,res;
res:=[[1]];
for i in pi do
res:=KroneckerProduct(pfm(m,n)[i],res);
od;
return res;
end;

Fcm:=function(m,n)
return List(partpowers(m,n),i->[ i,,,,,, skroneckp1(m,n,i)]);
end;

```

To illustrate how the Programme 5.2.4 works, we consider the construction of the Fischer-Clifford matrices of  $B(2, 6)$ . We run the Programme 5.2.4 and supply  $m = 2$  and  $n = 6$  to GAP by reading the statement "Fcm(2,6);" to obtain the matrices  $F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$ . It can be seen that Programme 5.2.4 is composed of five functions (each function terminating with an end statement). These functions do the following, with respect to  $m = 2$  and  $n = 6$ .

- (i) The function **mcompsn** of Programme 5.2.4 constructs the  $m$ -compositions of  $n$  for all possible  $\lambda_s$ , in this example  $m = 2$ ,  $n = 6$  and  $\lambda_s = 1, 2, 3, 4, 5, 6$ .
- (ii) The function **partpowers** constructs  $\bar{\lambda}_0$  for each conjugacy class of  $S_n$ . It also lists distinct  $\bar{\lambda}_0$  for all the conjugacy classes of  $S_n$ . The result of the function partpowers in this example is:  
 $[[ 1 ], [ 1, 1 ], [ 1, 1, 1 ], [ 2 ], [ 2, 1 ], [ 2, 2 ], [ 3 ], [ 3, 1 ], [ 4, 1 ], [ 6 ]]$ .

- (iii) The function **pfm** constructs the Fischer-Clifford matrices  $F(m, 1^{\lambda_s})$  for all  $\lambda_s$ .
- (iv) The function **skroneckp1** defines the kronecker product of a number of matrices at once.
- (v) The function **Fcm** does the kronecker products as given in Theorem 4.3.2 and according to the  $\bar{\lambda}_0$ 's constructed by the function **partpowers** in (ii) above. For example the Fischer-Clifford matrix  $F(2, 1^4 2^1)$  corresponds to  $\bar{\lambda}_0 = (4, 1)$ . Thus the programme computes  $F'(2, 1^4 2^1)$  (or  $F'(2, 2^1 1^4)$ ) by evaluating the kronecker product  $F(2, 2^1) \otimes (2, 1^4)$ .
- (vi) All the matrices  $F'(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  are printed by entering  $Fcm(m, n)$  with values  $m=2$  and  $n=6$  as  $Fcm(2, 6)$ . For example the matrix  $F'(2, 1^4 2^1)$  appears in position 9 (the position of  $\bar{\lambda}_0 = [4, 1]$  in the result of the function **partpowers**) of the programme result as

```
[[ [4,1] ],,,,,, [ [ 1, 1, 1, 1, 1, 1, 1, 1, 1 ], [ 4, 2, 0,-2, -4, 4, 2, 0, -2, -4 ], [ 6, 0, -2, 0, 6, 6, 0, -2, 0, 6 ], [
4, -2, 0, 2, -4, 4, -2, 0, 2, -4 ], [ 1, -1, 1, -1, 1, 1, -1, 1, -1, 1 ], [ 1, 1, 1, 1, 1, -1, -1, -1, -1, -1 ], [ 4, 2, 0,
-2, -4, -4, -2, 0, 2, 4 ], [ 6, 0, -2, 0, 6, -6, 0, 2, 0, -6 ], [ 4, -2, 0, 2, -4, -4, 2, 0, -2, 4 ], [ 1, -1, 1, -1, 1, -1,
1, -1, -1 ] ]],
```

showing the index  $\bar{\lambda}_0 = [4, 1]$  first and followed by the matrix  $F'(2, 1^4 2^1)$ .

### 5.3 Examples

We use Programmes 5.2.1, 5.2.4 and 5.2.2 to construct the Fischer-Clifford matrices of  $B(2, 6)$  and  $B(3, 5)$  in the following subsection.

#### 5.3.1 Computing the Fischer-Clifford Matrices of $B(2, 6)$

- (a) We supply to GAP the values  $c = 0$ ,  $n = 6$  and run Programme 5.2.1, which we have named **F(2,n)** in GAP. Then we enter **FM**; for which we obtain the output as given below.



- (c) Also on using Programme 5.2.2 which we have named IdIFGBmn in GAP, we obtain the following collections of 2-compositions of 6 (here printed with square brackets) corresponding to the matrices  $F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$  for  $B(2, 6)$  and mcn stands for  $m$ -compositions of  $n$ .

Read("IdIFGBmn");; IdIFGBmn(2,6);

[  $\bar{\lambda}_0 = [ 6 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 5, 1 ], [ 4, 2 ], [ 3, 3 ], [ 2, 4 ], [ 1, 5 ], [ 0, 6 ] ] ],

[  $\bar{\lambda}_0 = [ 4, 1 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 5, 1 ], [ 4, 2 ], [ 3, 3 ], [ 2, 4 ], [ 4, 2 ], [ 3, 3 ], [ 2, 4 ], [ 1, 5 ], [ 0, 6 ] ] ],

[  $\bar{\lambda}_0 = [ 2, 2 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 5, 1 ], [ 4, 2 ], [ 4, 2 ], [ 3, 3 ], [ 2, 4 ], [ 2, 4 ], [ 1, 5 ], [ 0, 6 ] ] ],

[  $\bar{\lambda}_0 = [ 3 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 4, 2 ], [ 2, 4 ], [ 0, 6 ] ] ] ,

[  $\bar{\lambda}_0 = [ 3, 1 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 5, 1 ], [ 4, 2 ], [ 3, 3 ], [ 3, 3 ], [ 2, 4 ], [ 1, 5 ], [ 0, 6 ] ] ],

[  $\bar{\lambda}_0 = [ 1, 1, 1 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 5, 1 ], [ 4, 2 ], [ 3, 3 ], [ 3, 3 ], [ 2, 4 ], [ 1, 5 ], [ 0, 6 ] ] ],

[  $\bar{\lambda}_0 = [ 2 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 3, 3 ], [ 0, 6 ] ] ],

[  $\bar{\lambda}_0 = [ 2, 1 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 5, 1 ], [ 4, 2 ], [ 2, 4 ], [ 1, 5 ], [ 0, 6 ] ] ],

[  $\bar{\lambda}_0 = [ 1, 1 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 4, 2 ], [ 2, 4 ], [ 0, 6 ] ] ],

[  $\bar{\lambda}_0 = [ 1, 1 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 5, 1 ], [ 1, 5 ], [ 0, 6 ] ] ],

[  $\bar{\lambda}_0 = [ 1 ]$ ,,,,,,,,,,,,, corresponding mcn, [ [ 6, 0 ], [ 0, 6 ] ] ] ]

If a 2-composition of 6 appears more than once in non adjacent positions of a collection, then the rows of the matrix  $F'(2, 1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$  corresponding to such a 2-composition of 6 are collected together to give a Fischer-Clifford matrix  $F(2, 1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$  of  $B(2, 6)$ . For this example only the collection

$$[[6, 0], [5, 1], [4, 2], [3, 3], [2, 4], [4, 2], [3, 3], [2, 4], [1, 5], [0, 6]],$$

has 2-compositions of 6 appearing more than once in non adjacent positions and these are  $[4, 2]$ ,  $[3, 3]$  and  $[2, 4]$  each appearing twice. Therefore for the matrix  $F'(2, 1^4 2)$  corresponding to the above collection of 2-compositions of 6, we collect rows 3 and 6, 4 and 7, 5 and 8 corresponding to  $[4, 2]$ ,  $[3, 3]$ ,  $[2, 4]$  each respectively into a block as given in Example 4.4.1(e)(see (4.3)). The Fischer-Clifford matrices of  $B(2, 6)$  are given in Table 5.1.

Table 5.1: Fischer-Clifford matrices of  $B(2, 6)$

$F(2, 1^6) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 & -2 & -4 & -6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & -20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & -4 & 2 & 0 & -2 & 4 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$	$F(2, 2^3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix}$
$F(2, 1^4 2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 & 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 & 6 & 0 & -2 & 0 & 6 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 4 & -2 & 0 & 2 & -4 & 4 & -2 & 0 & 2 & -4 \\ 4 & 2 & 0 & -2 & -4 & -4 & -2 & 0 & 2 & 4 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 6 & 0 & -2 & 0 & 6 & -6 & 0 & 2 & 0 & -6 \\ 4 & -2 & 0 & 2 & -4 & -4 & 2 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$	$F(2, 3^2) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$
$F(2, 1^2 2^2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & -2 & 2 & 0 & -2 & 2 & 0 & -2 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 2 & 2 & 2 & 0 & 0 & 0 & -2 & -2 & -2 \\ 4 & 0 & -4 & 0 & 0 & 0 & -4 & 0 & 4 \\ 2 & -2 & 2 & 0 & 0 & 0 & -2 & 2 & -2 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 2 & 0 & -2 & -2 & 0 & 2 & 2 & 0 & -2 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$	$F(2, 1^2 4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & -2 & 2 & 0 & -2 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 2 & 0 & -2 & -2 & 0 & 2 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$
$F(123) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$	$F(2, 24) = F(2, 15) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$
$F(2, 1^3 3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 & 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 & 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 3 & 1 & -1 & -3 & -3 & -1 & 1 & 3 \\ 3 & -1 & -1 & 3 & -3 & 1 & 1 & -3 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix}$	$F(2, 6) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

### 5.3.2 Computing the Fischer-Clifford Matrices of $B(3, 5)$

- (a) For another example on the application of Programme 5.2.4, we describe the construction of the Fischer-Clifford matrices of the group  $B(3, 5)$ . We run Programme 5.2.4 into GAP and enter the statement "Fcm(3,5);" to obtain the matrices  $F'(3, 1^{\lambda_1} 2^{\lambda_2} \dots 5^{\lambda_5})$  of  $B(3, 5)$ . We give only the Fischer-Clifford matrices  $F'(3, 5)$  and  $F'(3, 23)$  of  $B(3, 5)$  here due to limited space. These matrices respectively appear in the programme result as

$$[[1, \dots, [[1, 1, 1], [1, E(3), E(3)^2], [1, E(3)^2, E(3)]]],$$

and

$$\begin{aligned} & [ [ 1, 1 ], \dots, [[1, 1, 1, 1, 1, 1, 1, 1, 1], [1, E(3), E(3)^2, 1, E(3), E(3)^2, 1, E(3), E(3)^2], \\ & [1, E(3)^2, E(3), 1, E(3)^2, E(3), 1, E(3)^2, E(3)], [1, 1, 1, E(3), E(3), E(3), E(3)^2, E(3)^2, E(3)^2], \\ & [1, E(3), E(3)^2, E(3), E(3)^2, 1, E(3)^2, 1, E(3)], [1, E(3)^2, E(3), E(3), 1, E(3)^2, E(3)^2, E(3), 1], \\ & [1, 1, 1, E(3)^2, E(3)^2, E(3)^2, E(3), E(3), E(3)], [1, E(3), E(3)^2, E(3)^2, 1, E(3), E(3), E(3)^2, 1], \\ & [1, E(3)^2, E(3), E(3)^2, E(3), 1, E(3), 1, E(3)^2]]]. \end{aligned}$$

- (b) From the result of Programme 5.2.2 for  $n = 5$  and  $m = 3$ , we see that only the collection of 3-compositions of 5, given by

$$\begin{aligned} & [ "\bar{\lambda}_0=" , [ 3, 1 ], \dots, "corresponding mcn", [ [ 5, 0, 0 ], [ 4, 1, 0 ], [ 4, 0, 1 ], [ 3, \\ & 2, 0 ], [ 3, 1, 1 ], [ 3, 0, 2 ], [ 2, 3, 0 ], [ 2, 2, 1 ], [ 2, 1, 2 ], [ 2, 0, 3 ], [ 3, 2, 0 ], [ \\ & 2, 3, 0 ], [ 2, 2, 1 ], [ 1, 4, 0 ], [ 1, 3, 1 ], [ 1, 2, 2 ], [ 0, 5, 0 ], [ 0, 4, 1 ], [ 0, 3, 2 \\ & ], [ 0, 2, 3 ], [ 3, 0, 2 ], [ 2, 1, 2 ], [ 2, 0, 3 ], [ 1, 2, 2 ], [ 1, 1, 3 ], [ 1, 0, 4 ], [ 0, \\ & 3, 2 ], [ 0, 2, 3 ], [ 0, 1, 4 ], [ 0, 0, 5 ] ] ], \end{aligned}$$

has 3-compositions of 5 appearing more than once. Thus only the matrix  $F'(3, 1^3 2^1)$  requires to have its rows re-arranged into blocks corresponding to inertia factor groups.

The Fischer-Clifford matrices of  $B(3, 5)$  are given in Tables 5.2 to 5.5. The large Fischer-Clifford matrices of  $B(3, 5)$  have been split by | to fit on an A4 page.

### 5.3.3 A Note on the Fischer-Clifford Matrices of $B(2, 12)$ and $B(4, 5)$

We give a summary on the groups  $B(2, 12)$  and  $B(4, 5)$ , for which we have used Programme 5.2.4 to construct their Fischer-Clifford matrices.

- (a) The group  $B(2, 12)$  has order  $2^{22} \times 3^5 \times 5^2 \times 7 \times 11$  and has 1165 conjugacy classes. We have used the Programme 5.2.4 to construct the Fischer-Clifford matrices of  $B(2, 12)$ . The largest Fischer-Clifford matrix is  $F(2, 1^5 2^2 3^1)$  of order 36.
- (b) The group  $B(4, 5)$  has order  $2^{13} \times 3 \times 5$  and has 252 conjugacy classes. We have used the Programme 5.2.4 to construct the Fischer-Clifford matrices of  $B(4, 5)$ . The largest Fischer-Clifford matrix of  $B(4, 5)$  is  $F(4, 1^3 2^1)$  of order 80.

Due to lack of space here we have given the Fischer-Clifford matrices of  $B(2, 12)$  and  $B(4, 5)$  on the compact disk submitted with this thesis. However note that these matrices are the equivalent form of the Fischer-Clifford matrices of  $B(2, 12)$  and  $B(4, 5)$ .





Table 5.4: Fischer-Clifford matrices of  $B(3, 5)$  (continued)

$F(3, 1^3 2)(\text{contd.}) =$	<table border="0" style="width: 100%; text-align: center;"> <tr><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr> <tr><td>3</td><td><math>-\xi - 2\xi^2</math></td><td><math>-2\xi - \xi^2</math></td><td><math>\xi - \xi^2</math></td><td>0</td><td><math>-\xi + \xi^2</math></td><td><math>3\xi</math></td><td><math>2\xi + \xi^2</math></td><td><math>\xi + 2\xi^2</math></td><td><math>3\xi^2</math></td><td><math>2\xi + \xi^2</math></td><td><math>\xi + 2\xi^2</math></td><td><math>2\xi + \xi^2</math></td><td><math>3\xi</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>3\xi</math></td></tr> <tr><td>3</td><td><math>-2\xi - \xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi + \xi^2</math></td><td>0</td><td><math>\xi - \xi^2</math></td><td><math>3\xi^2</math></td><td><math>\xi + 2\xi^2</math></td><td><math>2\xi + \xi^2</math></td><td><math>3\xi^2</math></td><td><math>2\xi + \xi^2</math></td><td><math>\xi + 2\xi^2</math></td><td><math>2\xi + \xi^2</math></td><td><math>3\xi</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>3\xi</math></td></tr> <tr><td>3</td><td><math>\xi - \xi^2</math></td><td><math>-\xi + \xi^2</math></td><td><math>2\xi + \xi^2</math></td><td>0</td><td><math>\xi + 2\xi^2</math></td><td><math>3\xi^2</math></td><td><math>2\xi + \xi^2</math></td><td><math>\xi + 2\xi^2</math></td><td><math>3\xi^2</math></td><td><math>2\xi + \xi^2</math></td><td><math>\xi + 2\xi^2</math></td><td><math>2\xi + \xi^2</math></td><td><math>3\xi</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>3\xi</math></td></tr> <tr><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td></tr> <tr><td>6</td><td>0</td><td>0</td><td>0</td><td>-3</td><td>0</td><td>6</td><td>0</td><td>0</td><td>6</td><td>0</td><td>0</td><td>0</td><td>6</td><td>0</td><td>0</td><td>0</td><td>6</td></tr> <tr><td>3</td><td><math>-\xi + \xi^2</math></td><td><math>\xi - \xi^2</math></td><td><math>\xi + 2\xi^2</math></td><td>0</td><td><math>2\xi + \xi^2</math></td><td><math>3\xi</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>3\xi</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>3\xi</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>-\xi - 2\xi^2</math></td><td><math>3\xi</math></td></tr> <tr><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td><td><math>\xi^2</math></td></tr> <tr><td>1</td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td>1</td><td><math>\xi</math></td><td>1</td><td><math>\xi</math></td><td><math>\xi</math></td><td>1</td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td>1</td><td><math>\xi</math></td><td><math>\xi</math></td><td><math>\xi</math></td><td>1</td></tr> <tr><td><math>3\xi</math></td><td><math>2\xi + \xi^2</math></td><td><math>\xi - 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\xi^2$	$\xi + 2\xi^2$	0	$-\xi - 2\xi^2$	$3\xi^2$	$-\xi + \xi^2$	$-\xi + \xi^2$	$3\xi^2$	$-\xi + \xi^2$	$-\xi + \xi^2$	$-\xi + \xi^2$	$3\xi^2$	$-\xi + \xi^2$	$-\xi + \xi^2$	$-\xi + \xi^2$	$3\xi^2$	3	$2\xi + \xi^2$	$\xi + 2\xi^2$	$-2\xi - \xi^2$	0	$-\xi - 2\xi^2$	$3\xi$	$-\xi + \xi^2$	$-\xi + \xi^2$	$3\xi$	$-\xi + \xi^2$	$-\xi + \xi^2$	$-\xi + \xi^2$	$3\xi$	$-\xi + \xi^2$	$-\xi + \xi^2$	$-\xi + \xi^2$	$3\xi$	$3\xi$	$\xi - \xi^2$	$2\xi + \xi^2$	$-\xi - 2\xi^2$	0	$\xi + 2\xi^2$	3	$-2\xi - \xi^2$	$-2\xi - \xi^2$	3	$-2\xi - \xi^2$	$-2\xi - \xi^2$	$-2\xi - \xi^2$	3	$-2\xi - \xi^2$	$-2\xi - \xi^2$	$-2\xi - \xi^2$	3	3	$\xi + 2\xi^2$	$2\xi + \xi^2$	$-\xi - 2\xi^2$	0	$-2\xi - \xi^2$	$3\xi^2$	$\xi - \xi^2$	$\xi - \xi^2$	$3\xi^2$	$\xi - \xi^2$	$\xi - \xi^2$	$\xi - \xi^2$	$3\xi^2$	$\xi - \xi^2$	$\xi - \xi^2$	$\xi - \xi^2$	$3\xi^2$	$3\xi^2$	$-\xi + \xi^2$	$\xi + 2\xi^2$	$-2\xi - \xi^2$	0	$2\xi + \xi^2$	3	$-\xi - 2\xi^2$	$-\xi - 2\xi^2$	3	$-\xi - 2\xi^2$	$-\xi - 2\xi^2$	$-\xi - 2\xi^2$	3	$-\xi - 2\xi^2$	$-\xi - 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Table 5.5: Fischer-Clifford matrices of  $B(3, 5)$  (continued)

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## Chapter 6

# The Character Table of $B(m, n)$

It is well known that the character table of the generalized symmetric group  $B(m, n)$  can be computed in GAP, with  $B(m, n)$  considered as the wreath product of the cyclic group  $\mathbb{Z}_m$  of order  $m$  with the symmetric group  $S_n$ . For example we can use the following code in GAP to compute the character table of  $B(m, n)$ .

```
cm:= CharacterTable( "Cyclic", m );;  
wr:= CharacterTableWreathSymmetric( cm, n );;  
Display( wr );
```

Indeed in [50], Pfeiffer has given programmes in GAP which compute the character tables of wreath products with symmetric groups. However it may not be necessary to obtain the full character table of a group in hand, rather we may wish to obtain a partial character table of the group. Further due to limited workspace in GAP, we can only compute character tables of  $B(m, n)$  for small values of  $m$  and  $n$ . It is for these reasons amongst others that the technique of Fischer-Clifford matrices is sometimes used to construct the character tables of generalized symmetric groups. In this chapter we use the Fischer-Clifford matrices constructed in Chapters 4 and 5 to construct the character tables of the groups  $B(2, 6)$  and  $B(3, 5)$ . In Section 5.3.3, we give a summary on the character tables of the groups  $B(2, 12)$  and  $B(4, 5)$ .

As indicated in Section 4.2, the character table of  $B(m, n)$  is constructed by multiplying the partial character tables of the inertia factor groups  $S_{k_1} \times S_{k_2} \times \cdots \times S_{k_m}$  by the corresponding rows of the Fischer-Clifford matrices according to the fusions of the conjugacy classes of  $S_{k_1} \times S_{k_2} \times \cdots \times S_{k_m}$  to a class of  $S_n$ . However before

the multiplication using GAP is done, the rows within the blocks of the Fischer-Clifford matrices should be re-arranged appropriately. Now in order to arrange the rows within the blocks of the Fischer-Clifford matrices appropriately, we determined how the columns of the character tables of the inertia factor groups computed in GAP are arranged. It was found that in cases where there is more than one conjugacy class of the inertia factor group fusing to a class of  $S_n$ , the corresponding rows of a block in the matrix  $F'(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  should be arranged starting with the one at the bottom to the one at the top (see Subsections 6.1.3 and 8.3.3 for examples). In Sections 6.1 and 6.2, we use the Fischer-Clifford matrices given in Table 5.1 and Tables 5.2 to 5.5 to construct the character tables of the groups  $B(2, 6)$  and  $B(3, 5)$ , respectively.

## 6.1 The Group $B(2, 6)$

We construct the character table of  $B(2, 6)$  using the Fischer-Clifford matrices given in Table 5.1. We first give the conjugacy classes of  $B(2, 6)$  in Subsection 6.1.1, describe the inertia factor groups of  $B(2, 6)$  in Subsection 6.1.2 and construct the character table of  $B(2, 6)$  in Subsection 6.1.3.

### 6.1.1 The Conjugacy Classes of $B(2, 6)$

It was indicated in Section 3.1 that Programmes 3.1.9 and 3.1.10 give parameters for Fischer-Clifford matrices of  $B(m, n)$ . This is equivalent to coset analysis for calculating the conjugacy classes of group extensions where  $N$  is abelian (see [37] and [41]). Examples on the calculation of the orders of centralizers and orders of class representatives of  $B(2, 6)$  corresponding to the conjugacy class of  $S_6$  of type  $(1^4 2^1)$  using Proposition 3.1.7 were given in Section 3.1. It was also mentioned in Section 3.1 that using Programmes 3.1.9 and 3.1.10 we obtain a total of 65 conjugacy classes of  $B(2, 6)$ . We give the conjugacy classes of  $B(2, 6)$  in Tables 6.1 and 6.2. Both the second and third columns of these tables show alternative types of the conjugacy classes of  $B(2, 6)$ . The fourth column shows the orders of the class representatives of  $B(2, 6)$ . The fifth and last columns show the orders of centralizers and sizes of conjugacy class of  $B(2, 6)$  respectively.

Table 6.1: Conjugacy classes of  $B(2, 6)$

$[\mathfrak{g}]_{S_6}$	2-sets of partition $[\lambda]$	$(1^{a_{11}} 2^{a_{12}} \dots 6^{a_{16}}, 1^{a_{21}} 2^{a_{22}} \dots 6^{a_{26}})$	$\bar{g}$	$ C_G(\bar{g}) $	$ \bar{g} $
$(1^6) = 1a$	(6, 0)	$(1^6, 1^0)$	1A	46080	1
	(5, 1)	$(1^5, 1^1)$	2A	7680	6
	(4, 2)	$(1^4, 1^2)$	2B	3072	15
	(3, 3)	$(1^3, 1^3)$	2C	2304	20
	(2, 4)	$(1^2, 1^4)$	2D	3072	15
	(1, 5)	$(1^1, 1^5)$	2E	7680	6
	(0, 6)	$(1^0, 1^6)$	2F	46080	1
$(1^4 2) = 2a$	$((4, 1), (0, 0))$	$(1^4 2^1, 1^0 2^0)$	2G	1536	30
	$((3, 1), (1, 0))$	$(1^3 2^1, 1^1 2^0)$	2H	384	120
	$((2, 1), (2, 0))$	$(1^2 2^1, 1^2 2^0)$	2I	256	180
	$((1, 1), (3, 0))$	$(1^1 2^1, 1^3 2^0)$	2J	384	120
	$((0, 1), (4, 0))$	$(1^0 2^1, 1^4 2^0)$	2K	1536	30
	$((4, 0), (0, 1))$	$(1^4 2^0, 1^0 2^1)$	4A	1536	30
	$((3, 0), (1, 1))$	$(1^3 2^0, 1^1 2^1)$	4B	384	120
	$((2, 0), (2, 1))$	$(1^2 2^0, 1^2 2^1)$	4C	256	180
	$((1, 0), (3, 1))$	$(1^1 2^0, 1^3 2^1)$	4D	384	120
	$((0, 0), (4, 1))$	$(1^0 2^0, 1^4 2^1)$	4E	1536	30
$(1^2 2^2) = 2b$	$((2, 2), (0, 0))$	$(1^2 2^2, 1^0 2^0)$	2L	256	180
	$((1, 2), (1, 0))$	$(1^1 2^2, 1^1 2^0)$	2M	128	360
	$((0, 2), (2, 0))$	$(1^0 2^2, 1^2 2^0)$	2N	256	180
	$((2, 1), (0, 1))$	$(1^2 2^1, 1^0 2^1)$	4F	128	360
	$((1, 1), (1, 1))$	$(1^1 2^1, 1^1 2^1)$	4G	64	720
	$((0, 1), (2, 1))$	$(1^0 2^1, 1^2 2^1)$	4H	128	360
	$((2, 0), (0, 2))$	$(1^2 2^0, 1^0 2^2)$	4I	256	180
	$((1, 0), (1, 2))$	$(1^1 2^0, 1^1 2^2)$	4J	128	360
	$((0, 0), (2, 2))$	$(1^0 2^0, 1^2 2^2)$	4K	256	180
$(2^3) = 2c$	(3, 0)	$(2^3, 2^0)$	2O	384	120
	(2, 1)	$(2^2, 2^1)$	4L	128	360
	(1, 2)	$(2^1, 2^2)$	4M	128	360
	(0, 3)	$(2^0, 2^3)$	4N	384	120
$(1^3 3) = 3a$	$((3, 1), (0, 0))$	$(1^3 3^1, 1^0 3^0)$	3A	288	160
	$((2, 1), (1, 0))$	$(1^2 3^1, 1^1 3^0)$	6A	96	480
	$((1, 1), (2, 0))$	$(1^1 3^1, 1^2 3^0)$	6B	96	480
	$((0, 1), (3, 0))$	$(1^0 3^1, 1^3 3^0)$	6C	288	160
	$((3, 0), (0, 1))$	$(1^3 3^0, 1^0 3^1)$	6D	288	160
	$((2, 0), (1, 1))$	$(1^2 3^0, 1^1 3^1)$	6E	96	480
	$((1, 0), (2, 1))$	$(1^1 3^0, 1^2 3^1)$	6F	96	480
	$((0, 0), (3, 1))$	$(1^0 3^0, 1^3 3^1)$	6G	288	160

Table 6.2: Conjugacy classes of  $B(2, 6)$  (continued)

$[[\mathbf{g}]]_{S_6}$	2-sets of partition $[\lambda]$	$(1^{a_{11}} 2^{a_{12}} \dots 6^{a_{16}}, 1^{a_{21}} 2^{a_{22}} \dots 6^{a_{26}})$	$\bar{g}$	$ C_G(\bar{g}) $	$  \bar{g}  $
$(123) = 6a$	$((1, 1, 1), (0, 0, 0))$	$(1^1 2^1 3^1, 1^0 2^0 3^0)$	6H	48	960
	$((0, 1, 1), (1, 0, 0))$	$(1^0 2^1 3^1, 1^1 2^0 3^0)$	6I	48	960
	$((1, 0, 1), (0, 1, 0))$	$(1^1 2^0 3^1, 1^0 2^1 3^0)$	12A	48	960
	$((0, 0, 1), (1, 1, 0))$	$(1^0 2^0 3^1, 1^1 2^1 3^0)$	12B	48	960
	$((1, 1, 0), (0, 0, 1))$	$(1^1 2^1 3^0, 1^0 2^0 3^1)$	6J	48	960
	$((0, 1, 0), (1, 0, 1))$	$(1^0 2^1 3^0, 1^1 2^0 3^1)$	6K	48	960
	$((1, 0, 0), (0, 1, 1))$	$(1^1 2^0 3^0, 1^0 2^1 3^1)$	12C	48	960
	$((0, 0, 0), (1, 1, 1))$	$(1^0 2^0 3^0, 1^1 2^1 3^1)$	12D	48	960
	$(3^2) = 3b$	$(2, 0)$	$(3^2, 3^0)$	3B	72
$(1, 1)$		$(3^1, 3^1)$	6L	36	1280
$(0, 2)$		$(3^0, 3^2)$	6M	72	640
$(1^2 4) = 4a$	$((2, 1), (0, 0))$	$(1^2 4^1, 1^0 4^0)$	4O	64	720
	$((1, 1), (1, 0))$	$(1^1 4^1, 1^1 4^0)$	4P	32	1440
	$((0, 1), (2, 0))$	$(1^0 4^1, 1^2 4^0)$	4Q	64	720
	$((2, 0), (0, 1))$	$(1^2 4^0, 1^0 4^1)$	8A	64	720
	$((1, 0), (1, 1))$	$(1^1 4^0, 1^1 4^1)$	8B	32	1440
	$((0, 0), (2, 1))$	$(1^0 4^0, 1^2 4^1)$	8C	64	720
	$(24) = 4b$	$((1, 1), (0, 0))$	$(2^1 4^1, 2^0 4^0)$	4R	32
$((0, 1), (1, 0))$		$(2^0 4^1, 2^1 4^0)$	4S	32	1440
$((1, 0), (0, 1))$		$(2^1 4^0, 2^0 4^1)$	8D	32	1440
$((0, 0), (1, 1))$		$(2^0 4^0, 2^1 4^1)$	8E	32	1440
$(15) = 5a$	$((1, 1), (0, 0))$	$(1^1 5^1, 1^0 5^0)$	5A	20	2304
	$((0, 1), (1, 0))$	$(1^0 5^1, 1^1 5^0)$	10A	20	2304
	$((1, 0), (0, 1))$	$(1^1 5^0, 1^0 5^1)$	10B	20	2304
	$((0, 0), (1, 1))$	$(1^0 5^0, 1^1 5^1)$	10C	20	2304
$(6) = 6b$	$(1, 0)$	$(6^1, 6^0)$	6N	12	3840
	$(0, 1)$	$(6^0, 6^1)$	12E	12	3840

Table 6.3: Character table of  $S_4 \times S_2$

Class	1a	2a	2b	2c	3a	6a	2d	2e	4a	4b
Type	$(1^4, 1^2)$	$(1^4, 2)$	$(1^2 2, 1^2)$	$(1^2 2, 2)$	$(13, 1^2)$	$(13, 2)$	$(2^2, 1^2)$	$(2^2, 2)$	$(4, 1^2)$	$(4, 2)$
$\psi_1$	1	1	1	1	1	1	1	1	1	1
$\psi_2$	1	1	-1	-1	1	1	1	1	-1	-1
$\psi_3$	1	-1	1	-1	1	-1	1	-1	1	-1
$\psi_4$	1	-1	-1	1	1	-1	1	-1	-1	1
$\psi_5$	2	2	0	0	-1	-1	2	2	0	0
$\psi_6$	2	-2	0	0	-1	1	2	-2	0	0
$\psi_7$	3	-3	-1	1	0	0	-1	1	1	-1
$\psi_8$	3	-3	1	-1	0	0	-1	1	-1	1
$\psi_9$	3	3	-1	-1	0	0	-1	-1	1	1
$\psi_{10}$	3	3	1	1	0	0	-1	-1	-1	-1

Table 6.4: Fusion of inertia factor groups into  $S_6$

$S_6$	$S_5$	$S_4 \times S_2$	$S_3 \times S_3$	$S_2 \times S_4$
$(1^6)$	$(1^5)$	$(1^4, 1^2)$	$(1^3, 1^3)$	$(1^2, 1^4)$
$(1^4 2)$	$(1^3 2)$	$(1^2 2, 1^2), (1^4, 2)$	$(12, 1^3), (1^3, 1^2)$	$(1^2, 1^2 2), (2, 1^4)$
$(1^2 2^2)$	$(12^2)$	$(1^2 2, 2), (2^2, 1^2)$	$(12, 12)$	$(2, 1^2 2), (1^2, 2^2)$
$(2^3)$	-	$(2^2, 2)$	-	$(2, 2^2)$
$(1^3 3)$	$(1^2 3)$	$(13, 1^2)$	$(3, 1^3), (1^3, 3)$	$(1^2, 1^3)$
$(12 3)$	$(23)$	$(13, 2)$	$(3, 12), (12, 3)$	$(2, 1^3)$
$(3^2)$	-	-	$(3, 3)$	-
$(1^2 4)$	$(14)$	$(4, 1^2)$	-	$(1^2, 4)$
$(24)$	-	$(4, 2)$	-	$(2, 4)$
$(15)$	$(5)$	-	-	-
$(6)$	-	-	-	-

### 6.1.2 The Inertia Factor Groups of $B(2, 6)$

From the 2-compositions of 6, we obtain that the inertia factor groups of  $B(2, 6)$  are  $S_6, S_5 \times S_1, S_4 \times S_2, S_3 \times S_3, S_2 \times S_4, S_5$  and  $S_6$ . The character tables of these inertia factor groups are easily computed. Following from the remarks in Section 4.2 we have that the conjugacy classes of the inertia factor groups which fuse to a class of  $S_6$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$  are of type  $(1^{k_{11}} 2^{k_{12}} \dots 6^{k_{16}}, 1^{k_{21}} 2^{k_{22}} \dots 6^{k_{26}})$ , where  $\sum_{j=1}^2 k_{js} = \lambda_s$ . The fusion of the conjugacy classes of the inertia factor groups into  $S_6$  is given in Table 6.4.

### 6.1.3 The Character Table of $B(2, 6)$

The character table of  $B(2, 6)$  is constructed by multiplying the partial character tables of the inertia factor groups by the corresponding rows of the Fischer-Clifford matrices in Table 5.1 according to the fusions given in Table 6.4. For example, from Table 6.4 we see that the conjugacy classes  $(1^4, 2)$  and  $(1^2 2, 1^2)$  of  $S_4 \times S_2$  fuse to the class  $(1^4 2)$  of  $S_6$ . So we multiply the partial character table of  $S_4 \times S_2$  at the elements  $(1^4, 2)$  and  $(1^2 2, 1^2)$  with rows 3 and 4 of the Fischer-Clifford matrix  $F(2, 1^4 2)$  as follows:

$$\begin{aligned}
 & \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \\ 2 & 0 \\ -2 & 0 \\ -3 & -1 \\ -3 & 1 \\ 3 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 6 & 0 & -2 & 0 & 6 & 6 & 0 & -2 & 0 & 6 \end{bmatrix} \\
 & = \begin{bmatrix} 7 & 1 & -1 & 1 & 7 & 5 & -1 & -3 & -1 & 5 \\ -5 & 1 & 3 & 1 & -5 & -7 & -1 & 1 & -1 & -7 \\ 5 & -1 & -3 & -1 & 5 & 7 & 1 & -1 & 1 & 7 \\ -7 & -1 & 1 & -1 & -7 & -5 & 1 & 3 & 1 & -5 \\ 2 & 2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 & -2 & 2 & 2 & 2 & 2 & 2 \\ -9 & -3 & -1 & -3 & -9 & -3 & 3 & 5 & 3 & -3 \\ 3 & -3 & -5 & -3 & 3 & 9 & 3 & 1 & 3 & 9 \\ -3 & 3 & 5 & 3 & -3 & -9 & -3 & -1 & -3 & -9 \\ 9 & 3 & 1 & 3 & 9 & 3 & -3 & -5 & -3 & 3 \end{bmatrix},
 \end{aligned}$$

which appears in block 3 of Table A.1 in the Appendix A under class  $2a$  of  $S_6$ . This is done for each Fischer-Clifford matrix to give the full character table of  $B(2, 6)$  given in Tables A.1 to A.4 in the Appendix A. The orders of centralizers of  $B(2, 6)$  are listed in Tables 6.1 and 6.2 and are not included in the character table of  $B(2, 6)$ . The correctness of the character table of  $B(2, 6)$  has been tested in GAP.

## 6.2 The Group $B(3, 5)$

In this section we construct the character table of another example of the generalized symmetric group  $B(m, n)$ , but for prime  $m \neq 2$  and in particular we consider the group  $B(3, 5)$ , which has a higher order than  $B(3, 4)$  done in [5]. We construct the character table of  $B(3, 5)$  using the Fischer-Clifford matrices given in Tables 5.2 to 5.5. We first give the conjugacy classes of  $B(3, 5)$  in Subsection 6.2.1, describe the inertia factor groups of  $B(3, 5)$  in Subsection 6.2.2 and construct the character table of  $B(3, 5)$  in Subsection 6.2.3.

### 6.2.1 The Conjugacy Classes of $B(3, 5)$

Using Programmes 2.2.3, 3.1.9 and 3.1.10 we calculated the orders of centralizers and orders of class representatives of the group  $B(3, 5)$ . We obtain that the group  $B(3, 5)$  has a total of 108 conjugacy classes. The conjugacy classes of  $B(3, 5)$  including the respective orders of centralizers are given in Tables 6.5, 6.6 and 6.7.

Table 6.5: Conjugacy classes of  $B(3, 5)$

$[g]_{S_5}$	3-sets of partition $[\lambda]$	$(1^{a_{11}}2^{a_{12}} \dots 5^{a_{15}}, 1^{a_{21}}2^{a_{22}} \dots 5^{a_{25}}, 1^{a_{31}}2^{a_{32}} \dots 5^{a_{35}})$	$\bar{g}$	$ C_G(\bar{g}) $	$ \bar{g} $
$(1^5) = 1a$	(5, 0, 0)	$(1^5, 1^0, 1^0)$	1A	29160	1
	(4, 1, 0)	$(1^4, 1^1, 1^0)$	3A	5832	5
	(4, 0, 1)	$(1^4, 1^0, 1^1)$	3B	5832	5
	(3, 2, 0)	$(1^3, 1^2, 1^0)$	3C	2916	10
	(3, 1, 1)	$(1^3, 1^1, 1^1)$	3D	1458	20
	(3, 0, 2)	$(1^3, 1^0, 1^2)$	3E	2916	10
	(2, 3, 0)	$(1^2, 1^3, 1^0)$	3F	2916	10
	(2, 2, 1)	$(1^2, 1^2, 1^1)$	3G	972	30
	(2, 1, 2)	$(1^2, 1^1, 1^2)$	3H	972	30
	(2, 0, 3)	$(1^2, 1^0, 1^3)$	3I	2916	10
	(1, 4, 0)	$(1^1, 1^4, 1^0)$	3J	5832	5
	(1, 3, 1)	$(1^1, 1^3, 1^1)$	3K	1458	20
	(1, 2, 2)	$(1^1, 1^2, 1^2)$	3L	972	30
	(1, 1, 3)	$(1^1, 1^1, 1^3)$	3M	1458	20
	(1, 0, 4)	$(1^1, 1^0, 1^4)$	3N	5832	5
	(0, 5, 0)	$(1^0, 1^5, 1^0)$	3O	29160	1
	(0, 4, 1)	$(1^0, 1^4, 1^1)$	3P	5832	5
	(0, 3, 2)	$(1^0, 1^3, 1^2)$	3Q	2916	10
	(0, 2, 3)	$(1^0, 1^2, 1^3)$	3R	2916	10
	(0, 1, 4)	$(1^0, 1^1, 1^4)$	3S	5832	5
(0, 0, 5)	$(1^0, 1^0, 1^5)$	3T	29160	1	
$(1^3 2) = 2a$	$((3, 1), (0, 0), (0, 0))$	$(1^3 2^1, 1^0 2^0, 1^0 2^0)$	2A	972	30
	$((2, 1), (1, 0), (0, 0))$	$(1^2 2^1, 1^1 2^0, 1^0 2^0)$	6A	324	90
	$((2, 1), (0, 0), (1, 0))$	$(1^2 2^1, 1^0 2^0, 1^1 2^0)$	6B	324	90
	$((1, 1), (2, 0), (0, 0))$	$(1^1 2^1, 1^2 2^0, 1^0 2^0)$	6C	324	90
	$((1, 1), (1, 0), (1, 0))$	$(1^1 2^1, 1^1 2^0, 1^1 2^0)$	6D	162	180
	$((1, 1), (0, 0), (2, 0))$	$(1^1 2^1, 1^0 2^0, 1^2 2^0)$	6E	324	90
	$((0, 1), (3, 0), (0, 0))$	$(1^0 2^1, 1^3 2^0, 1^0 2^0)$	6F	972	30
	$((0, 1), (2, 0), (1, 0))$	$(1^0 2^1, 1^2 2^0, 1^1 2^0)$	6G	324	90
	$((0, 1), (1, 0), (2, 0))$	$(1^0 2^1, 1^1 2^0, 1^2 2^0)$	6H	324	90
	$((0, 1), (0, 0), (3, 0))$	$(1^0 2^1, 1^0 2^0, 1^3 2^0)$	6I	972	30
	$((3, 0), (0, 1), (0, 0))$	$(1^3 2^0, 1^0 2^1, 1^0 2^0)$	6J	972	30
	$((2, 0), (1, 1), (0, 0))$	$(1^2 2^0, 1^1 2^1, 1^0 2^0)$	6K	324	90
	$((2, 0), (0, 1), (1, 0))$	$(1^2 2^0, 1^0 2^1, 1^1 2^0)$	6L	324	90
	$((1, 0), (2, 1), (0, 0))$	$(1^1 2^0, 1^3 2^1, 1^0 2^0)$	6M	324	90
	$((1, 0), (1, 1), (1, 0))$	$(1^1 2^0, 1^1 2^1, 1^1 2^0)$	6N	162	180
	$((1, 0), (0, 1), (2, 0))$	$(1^1 2^0, 1^0 2^1, 1^2 2^0)$	6O	324	90
	$((0, 0), (3, 1), (0, 0))$	$(1^0 2^0, 1^3 2^1, 1^0 2^0)$	6P	972	30
	$((0, 0), (2, 1), (1, 0))$	$(1^0 2^0, 1^2 2^1, 1^1 2^0)$	6Q	324	90
	$((0, 0), (1, 1), (2, 0))$	$(1^0 2^0, 1^1 2^1, 1^2 2^0)$	6R	324	90
	$((0, 0), (0, 1), (3, 0))$	$(1^0 2^0, 1^0 2^1, 1^3 2^0)$	6S	972	30
	$((3, 0), (0, 0), (0, 1))$	$(1^3 2^0, 1^0 2^0, 1^0 2^1)$	6T	972	30
	$((2, 0), (1, 0), (0, 1))$	$(1^2 2^0, 1^1 2^0, 1^0 2^1)$	6U	324	90
	$((2, 0), (0, 0), (1, 1))$	$(1^2 2^0, 1^0 2^0, 1^1 2^1)$	6V	324	90
	$((1, 0), (2, 0), (0, 1))$	$(1^1 2^0, 1^2 2^0, 1^0 2^1)$	6W	324	90
	$((1, 0), (1, 0), (1, 1))$	$(1^1 2^0, 1^0 2^0, 1^1 2^1)$	6X	162	180
	$((1, 0), (0, 0), (2, 1))$	$(1^1 2^0, 1^0 2^0, 1^2 2^1)$	6Y	324	90
	$((0, 0), (3, 0), (0, 1))$	$(1^0 2^0, 1^3 2^0, 1^0 2^1)$	6Z	972	30
	$((0, 0), (2, 0), (1, 1))$	$(1^0 2^0, 1^2 2^0, 1^1 2^1)$	6AB	324	90
	$((0, 0), (1, 0), (2, 1))$	$(1^0 2^0, 1^1 2^0, 1^2 2^1)$	6AC	324	90
	$((0, 0), (0, 0), (3, 1))$	$(1^0 2^0, 1^0 2^0, 1^3 2^1)$	6AD	972	30

Table 6.6: Conjugacy classes of  $B(3, 5)$  (continued)

$[\mathfrak{g}]_{S_5}$	3-sets of partition $[\lambda]$	$(1^{a_{11}}2^{a_{12}} \dots 5^{a_{15}}, 1^{a_{21}}2^{a_{22}} \dots 5^{a_{25}}, 1^{a_{31}}2^{a_{32}} \dots 5^{a_{35}})$	$\bar{g}$	$ C_G(\bar{g}) $	$  \bar{g}  $	
$(1^1 2^2) = 2b$	$((1, 2), (0, 0), (0, 0))$	$(1^1 2^2, 1^{02^0}, 1^{02^0})$	2B	216	135	
	$((0, 2), (1, 0), (0, 0))$	$(1^{02^2}, 1^1 2^0, 1^{02^0})$	6AE	216	135	
	$((0, 2), (0, 0), (1, 0))$	$(1^{02^2}, 1^{02^0}, 1^1 2^0)$	6AF	216	135	
	$((1, 1), (0, 1), (0, 0))$	$(1^1 2^1, 1^{02^1}, 1^{02^0})$	6AG	108	270	
	$((0, 1)(1, 1), (0, 0))$	$(1^{02^1}, 1^1 2^1, 1^{02^0})$	6AH	108	270	
	$((0, 1), (0, 1), (1, 0))$	$(1^{02^1}, 1^{02^1}, 1^1 2^0)$	6AI	108	270	
	$((1, 1), (0, 0), (0, 1))$	$(1^1 2^1, 1^{02^0}, 1^{02^1})$	6AJ	108	270	
	$((0, 1), (1, 0), (0, 1))$	$(1^{02^1}, 1^1 2^0, 1^{02^1})$	6AK	108	270	
	$((0, 1), (0, 0), (1, 1))$	$(1^{02^1}, 1^{02^0}, 1^1 2^1)$	6AL	108	270	
	$((1, 0), (0, 2), (0, 0))$	$(1^1 2^0, 1^{02^2}, 1^{02^0})$	6AM	216	135	
	$((0, 0)(1, 2), (0, 0))$	$(1^{02^0}, 1^1 2^2, 1^{02^0})$	6AN	216	135	
	$((0, 0), (0, 2), (1, 0))$	$(1^{02^0}, 1^{02^2}, 1^1 2^0)$	6AO	216	135	
	$((1, 0), (0, 1), (0, 1))$	$(1^1 2^0, 1^{02^1}, 1^{02^1})$	6AP	108	270	
	$((0, 0)(1, 1), (0, 1))$	$(1^{02^0}, 1^1 2^1, 1^{02^1})$	6AQ	108	270	
	$((0, 0), (0, 1), (1, 1))$	$(1^{02^0}, 1^{02^1}, 1^1 2^1)$	6AR	108	270	
	$((1, 0), (0, 0), (0, 2))$	$(1^1 2^0, 1^{02^0}, 1^{02^2})$	6AS	216	135	
	$((0, 0), (1, 0), (0, 2))$	$(1^{02^0}, 1^1 2^0, 1^{02^2})$	6AT	216	135	
	$((0, 0), (0, 0), (1, 2))$	$(1^{02^0}, 1^{02^0}, 1^1 2^2)$	6AU	216	135	
	$(1^2 3) = 3a$	$((2, 1), (0, 0), (0, 0))$	$(1^2 3^1, 1^{03^0}, 1^{03^0})$	3U	162	180
		$((1, 1), (1, 0), (0, 0))$	$(1^1 3^1, 1^1 3^0, 1^{03^0})$	3V	81	360
		$((1, 1), (0, 0), (1, 0))$	$(1^1 3^1, 1^{03^0}, 1^1 3^0)$	3W	81	360
$((0, 1), (2, 0), (0, 0))$		$(1^{03^1}, 1^2 3^0, 1^{03^0})$	3X	162	180	
$((0, 1), (1, 0), (1, 0))$		$(1^{03^1}, 1^1 3^0, 1^1 3^0)$	3Y	81	360	
$((0, 1), (0, 0), (2, 0))$		$(1^{03^1}, 1^{03^0}, 1^2 3^0)$	3Z	162	180	
$((2, 0), (0, 1), (0, 0))$		$(1^2 3^0, 1^{03^1}, 1^{03^0})$	9A	162	180	
$((1, 0), (1, 1), (0, 0))$		$(1^1 3^0, 1^1 3^1, 1^{03^0})$	9B	81	360	
$((1, 0), (0, 1), (1, 0))$		$(1^1 3^0, 1^{03^1}, 1^1 3^0)$	9C	81	360	
$((0, 0), (2, 1), (0, 0))$		$(1^{03^0}, 1^2 3^1, 1^{03^0})$	9D	162	180	
$((0, 0), (1, 1), (1, 0))$		$(1^{03^0}, 1^1 3^1, 1^1 3^0)$	9E	81	360	
$((0, 0), (0, 1), (2, 0))$		$(1^{03^0}, 1^{03^1}, 1^2 3^0)$	9F	162	180	
$((2, 0), (0, 0), (0, 1))$		$(1^2 3^0, 1^{03^0}, 1^{03^1})$	9G	162	180	
$((1, 0), (1, 0), (0, 1))$		$(1^1 3^0, 1^1 3^0, 1^{03^1})$	9H	81	360	
$((1, 0), (0, 0), (1, 1))$		$(1^1 3^0, 1^{03^0}, 1^1 3^1)$	9I	81	360	
$((0, 0), (2, 0), (0, 1))$		$(1^{03^0}, 1^2 3^0, 1^{03^1})$	9J	162	180	
$((0, 0), (1, 0), (1, 1))$		$(1^{03^0}, 1^1 3^0, 1^1 3^1)$	9K	81	360	
$((0, 0), (0, 0), (2, 1))$		$(1^{03^0}, 1^{03^0}, 1^2 3^1)$	9L	162	180	

Table 6.7: Conjugacy classes of  $B(3, 5)$  (continued)

$[g]_{S_5}$	3-sets of partition $[\lambda]$	$(1^{a_{11}} 2^{a_{12}} \dots 5^{a_{15}}, 1^{a_{21}} 2^{a_{22}} \dots 5^{a_{25}}, 1^{a_{31}} 2^{a_{32}} \dots 5^{a_{35}})$	$\bar{g}$	$ C_G(\bar{g}) $	$  \bar{g}  $
$(23) = 6a$	$((1, 1), (0, 0), (0, 0))$	$(2^1 3^1, 2^0 3^0, 2^0 3^0)$	6AV	54	540
	$((0, 1), (1, 0), (0, 0))$	$(2^0 3^1, 2^1 3^0, 2^0 3^0)$	18A	54	540
	$((0, 1), (0, 0), (1, 0))$	$(2^0 3^1, 2^0 3^0, 2^1 3^0)$	18B	54	540
	$((0, 1), (0, 1), (0, 0))$	$(2^0 3^1, 2^0 3^1, 2^0 3^0)$	18C	54	540
	$((0, 0), (1, 1), (0, 0))$	$(2^0 3^0, 2^1 3^1, 2^0 3^0)$	18D	54	540
	$((0, 0), (0, 1), (1, 0))$	$(2^0 3^0, 2^0 3^1, 2^1 3^0)$	18E	54	540
	$((1, 0), (0, 0), (0, 1))$	$(2^1 3^0, 2^0 3^0, 2^0 3^1)$	18F	54	540
	$((0, 0), (1, 0), (0, 1))$	$(2^0 3^0, 2^1 3^0, 2^0 3^1)$	18G	54	540
	$((0, 0), (0, 0), (1, 1))$	$(2^0 3^0, 2^0 3^0, 2^1 3^1)$	18H	54	540
$(14) = 4a$	$((1, 1), (0, 0), (0, 0))$	$(1^1 4^1, 1^0 4^0, 1^0 4^0)$	4A	36	810
	$((0, 1), (1, 0), (0, 0))$	$(1^0 4^1, 1^1 4^0, 1^0 4^0)$	12A	36	810
	$((0, 1), (0, 0), (1, 0))$	$(1^0 4^1, 1^0 4^0, 1^1 4^0)$	12B	36	810
	$((0, 1), (0, 1), (0, 0))$	$(1^0 4^1, 1^0 4^1, 1^0 4^0)$	12C	36	810
	$((0, 0), (1, 1), (0, 0))$	$(1^0 4^0, 1^1 4^1, 1^0 4^0)$	12D	36	810
	$((0, 0), (0, 1), (1, 0))$	$(1^0 4^0, 1^0 4^1, 1^1 4^0)$	12E	36	810
	$((1, 0), (0, 0), (0, 1))$	$(1^1 4^0, 1^0 4^0, 1^0 4^1)$	12F	36	810
	$((0, 0), (1, 0), (0, 1))$	$(1^0 4^0, 1^1 4^0, 1^0 4^1)$	12G	36	810
	$((0, 0), (0, 0), (1, 1))$	$(1^0 4^0, 1^0 4^0, 1^1 4^1)$	12H	36	810
$(5) = 5a$	$(1, 0, 0)$	$(5^1, 5^0, 5^0)$	5A	15	1944
	$(0, 1, 0)$	$(5^0, 5^1, 5^0)$	15A	15	1944
	$(0, 0, 1)$	$(5^0, 5^0, 5^1)$	15B	15	1944

Table 6.8: Character table of  $S_3 \times S_2$

Class	1a	2a	2b	2c	3a	6a
Type	$(1^3, 1^2)$	$(1^3, 2)$	$(12, 1^2)$	$(12, 2)$	$(3, 1^2)$	$(3, 2)$
$\psi_1$	1	1	1	1	1	1
$\psi_2$	1	1	-1	-1	1	1
$\psi_3$	1	-1	1	-1	1	-1
$\psi_4$	1	-1	-1	1	1	-1
$\psi_5$	2	-2	0	0	-1	1
$\psi_6$	2	2	0	0	-1	-1

Table 6.9: Fusion of inertia factor groups into  $S_5$

$S_5$	$S_4$	$S_3$	$S_3 \times S_2$	$S_2 \times S_3$	$S_2 \times S_2$
$(1^5)$	$(1^4)$	$(1^3)$	$(1^3, 1^2)$	$(1^2, 1^3)$	$(1^2, 1^2)$
$(1^3 2)$	$(1^2 2)$	$(12)$	$(12, 1^2), (1^3, 2)$	$(1^2, 12), (2, 1^3)$	$(1^2, 2), (2, 1^2)$
$(12^2)$	$(2^2)$	-	$(12, 2)$	$(2, 12)$	$(2, 2)$
$(1^2 3)$	$(13)$	$(3)$	$(3, 1^2)$	$(1^2, 3)$	-
$(23)$	-	-	$(3, 2)$	$(2, 3)$	-
$(14)$	$(4)$	-	-	-	-
$(5)$	-	-	-	-	-

### 6.2.2 The Inertia Factor Groups of $B(3, 5)$

From the 3-compositions of 5 we obtain that the distinct inertia factor groups of  $B(3, 5)$  are  $S_5, S_4, S_3 \times S_2, S_3, S_2 \times S_3$  and  $S_2 \times S_2$ . The character tables of these inertia factor groups are easily computed. Following from the remarks in Section 4.2 we have that the conjugacy classes of the inertia factor groups which fuse to a class of  $S_5$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 5^{\lambda_5})$  are of type

$$(1^{k_{11}} 2^{k_{12}} \dots 5^{k_{15}}, 1^{k_{21}} 2^{k_{22}} \dots 5^{k_{25}}, 1^{k_{31}} 2^{k_{32}} \dots 5^{k_{35}}),$$

where  $\sum_{j=1}^3 k_{js} = \lambda_s$ . The fusion of the conjugacy classes of the inertia factor groups into  $S_5$  is given in Table 6.9.

### 6.2.3 The Character Table of $B(3, 5)$

The character table of  $B(3, 5)$  is constructed by multiplying the partial character tables of the inertia factor groups by the corresponding rows of the Fischer-Clifford matrices in Tables 5.2 to 5.5 according to the fusions given in Table 6.9. For example, from Table 6.9 we see that the conjugacy class  $(3, 1^2)$  of  $S_3 \times S_2$  fuses to the class  $(1^2 3)$  of  $S_5$ . So we multiply the partial character table of  $S_3 \times S_2$  at the element  $(3, 1^2)$  with row 4 of the Fischer-Clifford matrix  $F(3, 1^2 3)$  as follows:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & \xi^2 & 1 & \xi \end{bmatrix} =$$

$$\begin{bmatrix} 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & \xi^2 & 1 & \xi \\ 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & \xi^2 & 1 & \xi \\ 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & \xi^2 & 1 & \xi \\ 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & \xi^2 & 1 & \xi \\ -1 & -\xi & -\xi^2 & -\xi^2 & -1 & -\xi & -1 & -\xi & -\xi^2 & -\xi^2 & -1 & -\xi & -1 & -\xi & -\xi^2 & -\xi^2 & -1 & -\xi \\ -1 & -\xi & -\xi^2 & -\xi^2 & -1 & -\xi & -1 & -\xi & -\xi^2 & -\xi^2 & -1 & -\xi & -1 & -\xi & -\xi^2 & -\xi^2 & -1 & -\xi \end{bmatrix},$$

which appears in block 4 of Table B.11 in the Appendix B under the class  $3a$  of  $S_5$ . This is done for each Fischer-Clifford matrix to give the full character table of  $B(3, 5)$  given in Tables B.1 to B.20 in the Appendix B. The orders of centralizers of  $B(3, 5)$  are listed in Tables 6.5 to 6.7 and are not included in the character table of  $B(3, 5)$ . The correctness of the character table of  $B(3, 5)$  has been tested in GAP.

### 6.3 A Note on the Character tables of $B(2, 12)$ and $B(4, 5)$

The group  $B(2, 12)$  has 1165 conjugacy classes. We have used the Programme 5.2.4 to construct the Fischer-Clifford matrices of  $B(2, 12)$ . We have constructed the character table of  $B(2, 12)$  and tested its correctness in GAP. The character table of  $B(2, 12)$  is given on the compact disk submitted with this thesis.

The group  $B(4, 5)$  has 252 conjugacy classes. We have used the Programme 5.2.4 to construct the Fischer-Clifford matrices of  $B(4, 5)$ . We have constructed the character table of  $B(4, 5)$  and tested its correctness in GAP. The character table of  $B(4, 5)$  is given on the compact disk submitted with this thesis.

## Chapter 7

# The Group $B_S(p, n)$

We recall from Section 3.3 that the group  $B_S(p, n)$  is a split extension of  $S = \mathbb{Z}_p^{n-1}$  by the symmetric group  $S_n$ . Such group extensions may occur as constituents of finite simple groups and therefore we need to know their structures. For example the group  $B_S(2, 6)$  is an affine subgroup of the symplectic group  $SP(6, 2)$ . To construct the character table of the group  $2^6:SP(6, 2)$ , which is a maximal subgroup of the Fischer group  $F_{i_{22}}$ , using the method of Fischer-Clifford matrices we require the character table of  $B_S(2, 6)$  (see [12], [47]). In [47], Mpono has used the properties of Fischer-Clifford matrices given in Section 1.5.2 to construct the Fischer-Clifford matrices and therefore the character table of  $B_S(2, 6)$ . In this chapter we will not use the properties of Fischer-Clifford matrices to construct these matrices but rather we use a unified combinatorial method to construct the Fischer-Clifford matrices of the groups  $B_S(2, 6)$ . We also use the Fischer-Clifford matrices of the groups  $B_S(2, 6)$  to construct its character table.

In [35] List has presented a method for constructing the Fischer-Clifford matrices of a group extension of a faithful irreducible- $S_n$  constituent of  $N = 2^n$ , which we may refer to as a group associated with  $B(2, n)$ . This was done by constructing the Fischer-Clifford matrices of the associated group from the Fischer-Clifford matrices of  $B(2, n)$ . In this chapter we use a similar technique as in [35] to adapt the combinatorial method, for constructing the Fischer-Clifford matrices of  $B(p, n)$ , to construct the Fischer-Clifford matrices of  $B_S(p, n)$ , where  $p$  is prime. In Section 7.1, we present some results which describe this adaptation to constructing the Fischer-Clifford matrices of  $B_S(p, n)$ . In Sections 7.2, 7.3 and 7.4, we will construct the Fischer-Clifford matrices

and the character tables of the groups  $B_S(2, 6)$ ,  $B_S(3, 3)$  and  $B_S(3, 5)$  respectively.

## 7.1 Fischer-Clifford Matrices of $B_S(p, n)$

Here we use the method presented by Proposition 1.5.2 to construct the Fischer-Clifford matrices of  $\overline{G} = B_S(p, n)$ . Thus for each conjugacy class  $[g]$  of  $S_n$ , we consider the action of  $C_G(g)$  on the group  $S/K_S$ , where  $K_S = \phi_g(S)$  and  $\phi_g : N \rightarrow N$  is the map defined by  $\phi_g(d) = d + (p-1)d^g$  (see Proposition 1.5.2), but here restricted to  $S$ . It is clear from Lemma 1.5.1, that  $K_S$  is also  $C_G(g)$ -invariant. And from the remarks in [18] and [35], we infer that the actions of  $C_G(g)$  on  $S/K_S$  and  $\text{Irr}(S/K_S)$  will be isomorphic if  $S$  is an elementary abelian  $p$ -group. It is for this reason that in this chapter we take  $m$  to be prime so that  $S$  is an elementary abelian  $p$ -group. The following result generalizes List's result (see [35] page 120).

**Proposition 7.1.1** *Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots n^{\lambda_n})$  containing a cycle  $g_i = (\alpha_1, \alpha_2, \dots, \alpha_s)$  of length  $s$ . Let  $K = \phi_g(N)$  and  $K_S = \phi_g(S)$ , where  $\phi_g : N \rightarrow N$  is defined by  $\phi_g(d) = d + (p-1)d^g$ . Then for  $k \in \mathbb{N} \cup \{0\}$  we have the following.*

(i) *If  $\lambda_s \neq 0$  for some  $s \neq pk$ , then  $K_S = K$ .*

(ii) *If  $\lambda_s = 0$  for all  $s = pk + k'$ ,  $k' = 1, 2, \dots, p-1$ , then for each cycle  $g_i$  of length  $s = pk$  we have  $\left\{ l[e_{\alpha_1} + (p-1)e_{\alpha_{1+k'}}] \mid l, k' = 1, 2, \dots, p-1 \right\} \subseteq K \setminus K_S$ .*

**Proof.** Since  $g_i = (\alpha_1, \alpha_2, \dots, \alpha_s)$  is an  $s$ -cycle of  $g$ , we have

$$\phi_g(e_{\alpha_j}) = e_{\alpha_j} + (p-1)e_{\alpha_{j+1}}, \quad j = 1, 2, \dots, s-1, \quad \phi_g(e_{\alpha_s}) = e_{\alpha_s} + (p-1)e_{\alpha_1}.$$

Thus we have

$$\phi_g(\langle e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_s} \rangle) = \langle \{e_{\alpha_j} + (p-1)e_{\alpha_{j+1}} \mid j = 1, 2, \dots, s-1\} \cup \{e_{\alpha_s} + (p-1)e_{\alpha_1}\} \rangle,$$

and  $C_G(g)$  acts on the  $\lambda_s$   $s$ -cycles as  $S_{\lambda_s}$ . We obtain that

$$K = \phi_g(N) = \langle \{e_{\alpha_j} + (p-1)e_{\alpha_{j+1}} \mid j = 1, 2, \dots, s-1\} \cup \{e_{\alpha_s} + (p-1)e_{\alpha_1}\} \rangle,$$

where  $g_i = (\alpha_1, \alpha_2, \dots, \alpha_s)$  runs over the cycles of  $g$ . Clearly  $K \subseteq S < N$ . Now let  $d$  be an element of  $\langle e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_s} \rangle$ . If  $s = 1$ , then  $\phi_g(d) = 0$ . Let  $1 \leq r < s$ . Then

for  $l \in \{1, 2, \dots, p-1\}$  we have

$$\phi_g(d) = l[e_{\alpha_1} + (p-1)e_{\alpha_{1+r}}] \Leftrightarrow d = l \sum_{j=1}^r e_{\alpha_j} \text{ or } d = (p-1)l \sum_{j=r+1}^s e_{\alpha_j}. \quad (7.1)$$

Now since  $K \subseteq S = \langle e_1 + (p-1)e_{1+j}, j = 1, 2, \dots, n-1 \rangle$  (see Proposition 3.3.2), it follows that every element of  $K$  is generated by the elements  $e_{\alpha_1} + (p-1)e_{\alpha_{1+r}}$ , where  $1 \leq r < s$  and  $g_i = (\alpha_1, \alpha_2, \dots, \alpha_s)$  runs over the cycles of  $g$ .

Now we consider the following.

- (i) Let  $\lambda_s \neq 0$  for some  $s \neq pk$ . Then either  $g$  contains only cycles of length  $s = pk + k'$  or  $g$  contain both cycles of length  $s = pk + k'$  and cycles of length  $s = pk, k' \in \{1, 2, \dots, p-1\}$ .
- (a) Now let  $g$  contain only cycles of length  $s = pk + k', k' \in \{1, 2, \dots, p-1\}$ . Then by statement (7.1), for  $l \in \{1, 2, \dots, p-1\}$  we have

$$\phi_g(d) = l[e_{\alpha_1} + (p-1)e_{\alpha_{1+k'}}] \Leftrightarrow d = l \sum_{j=1}^{k'} e_{\alpha_j} \text{ or } d = (p-1)l \sum_{j=k'+1}^{pk+k'} e_{\alpha_j}.$$

Since  $pk(p-1)l \equiv 0 \pmod{p}$  we have that  $(p-1)l \sum_{j=k'+1}^{pk+k'} e_{\alpha_j} \in S$ . Thus  $y = l[e_{\alpha_1} + (p-1)e_{\alpha_{1+k'}}] \in K_S$ . Now since  $y = l[e_{\alpha_1} + (p-1)e_{\alpha_{1+k'}}]$  is a scalar multiple of an arbitrary generator  $e_{\alpha_1} + (p-1)e_{\alpha_{1+k'}}$  of  $K$  and  $K_S$  is a group it follows that  $K_S$  contains  $K$ , that is  $K \subseteq K_S$ . However  $S \subseteq N$  implies that  $K_S \subseteq K$ . Hence  $K = K_S$  which completes the proof of part(i) for case (a).

- (b) Now let  $g$  contain both cycles of length  $s = pk + k'$  and cycles of length  $s = pk$ . Let  $g_i = (\alpha_1, \alpha_2, \dots, \alpha_{pk+k'})$  be one of the cycles of length  $s = pk + k'$  and  $g'_i = (\alpha'_1, \alpha'_2, \dots, \alpha'_{pk})$  be one of the cycles of length  $s = pk$ . It is clear that the elements  $(p-1)l \left( \sum_{j=1}^{k'} e_{\alpha_j} + \sum_{i=k'+1}^{pk} e_{\alpha'_i} \right)$  and  $(p-1)l \sum_{j=k'+1}^{pk+k'} e_{\alpha_j}$  both lie in  $S$  and that  $\phi_g[(p-1)l \sum_{j=1}^{k'} e_{\alpha_j}] + \phi_g[(p-1)l \sum_{j=k'+1}^{pk+k'} e_{\alpha_j}] = 0$ . Now since  $pk - k' = p(k-1) + (p-k')$ , we have as for case (a) above, the element given by

$$\begin{aligned} & \phi_g \left[ (p-1)l \left( \sum_{j=1}^{k'} e_{\alpha_j} + \sum_{i=k'+1}^{pk} e_{\alpha'_i} \right) \right] + \phi_g \left[ (p-1)l \sum_{j=k'+1}^{pk+k'} e_{\alpha_j} \right] \\ &= \phi_g \left[ (p-1)l \sum_{i=k'+1}^{pk} e_{\alpha'_i} \right] = l[e_{\alpha'_1} + (p-1)e_{\alpha'_{1+u}}] \end{aligned}$$

lies in  $K_S$ , for  $u \leq p - k' \leq p - 1$  and each  $l \in \{1, 2, \dots, p - 1\}$ . Now since the element  $l[e_{\alpha'_1} + (p-1)e_{\alpha'_{1+u}}]$  is a scalar multiple of a generator  $e_{\alpha'_1} + (p-1)e_{\alpha'_{1+u}}$  of  $K$  we have  $K \subseteq K_S$  as in (a) above. However since  $S \subseteq N$  it is the case that  $K_S \subseteq K$ . Hence  $K = K_S$  which completes the proof of part(i) for case (b).

- (ii) Now for the proof of part (ii), we show that if  $g$  contains only cycles of length  $s = pk$  and  $g'_i = (\alpha'_1, \alpha'_2, \dots, \alpha'_{pk})$  is one of them, then  $S$  does not contain an element  $d$  such that

$$\phi_g(d) = l[e_{\alpha'_1} + (p-1)e_{\alpha'_{1+r}}], \quad r \leq p-1.$$

Since  $l \sum_{i=1}^{pk} e_{\alpha'_i} \in S$  we have  $l \sum_{i=1}^r e_{\alpha'_i}$  and  $(p-1)l \sum_{i=r+1}^{pk} e_{\alpha'_i}$  are not in  $S$  for  $r \leq p-1$ . But by statement (7.1), for  $l \in \{1, 2, \dots, p-1\}$  the elements  $l \sum_{i=1}^r e_{\alpha'_i}$  and  $(p-1)l \sum_{i=r+1}^s e_{\alpha'_i}$  are the only ones such that  $\phi_g(d) = l[e_{\alpha'_1} + (p-1)e_{\alpha'_{1+r}}]$ ,  $r \leq p-1$ . Therefore  $S$  does not contain  $d$  such that  $\phi_g(d) = l[e_{\alpha'_1} + (p-1)e_{\alpha'_{1+r}}]$ ,  $r \leq p-1$ ,  $l \in \{1, 2, \dots, p-1\}$ . Hence

$$\left\{ l[e_{\alpha'_1} + (p-1)e_{\alpha'_{1+r}}] \mid l, r = 1, 2, \dots, p-1 \right\} \subseteq K \setminus K_S,$$

for each cycle  $g'_i$  of  $g$ .  $\square$

**Examples 7.1.2** In the following for each group  $B_S(p, n)$  and for a given  $g \in S_n$ , we give the groups  $K$  and  $K_S$ .

- (a) For the group  $B_S(2, 4)$  and for  $g = (12)(34) \in S_4$ , we obtain that

$$K = \langle e_1 + e_2, e_3 + e_4 \rangle$$

while

$$K_S = \langle e_1 + e_2 + e_3 + e_4 \rangle$$

does not contain any of the elements  $e_1 + e_2$  and  $e_3 + e_4$ . However for  $g = (123) \in S_4$  we obtain that

$$K = K_S = \langle e_1 + e_2, e_2 + e_3 \rangle .$$

(b) For the group  $B_S(2, 6)$  and for  $g = (12)(34)(56) \in S_6$ , we obtain that

$$K = \langle e_1 + e_2, e_3 + e_4, e_5 + e_6 \rangle$$

while

$$K_S = \langle e_1 + e_2 + e_3 + e_4, e_1 + e_2 + e_5 + e_6 \rangle$$

does not contain any of the elements in the set  $\{e_1 + e_2, e_3 + e_4, e_5 + e_6\}$ .

(c) For the group  $B_S(3, 3)$  and for  $g = (123) \in S_3$ , we obtain that

$$K = \langle e_1 + 2e_2, e_1 + 2e_3 \rangle$$

while

$$K_S = \langle e_1 + e_2 + e_3 \rangle$$

does not contain any of the elements in the set  $\{l(e_1 + 2e_2), l'(e_1 + 2e_3) \mid l, l' = 1, 2\}$ .

(d) For the group  $B_S(3, 6)$  and for  $g = (123)(456) \in S_6$ , we obtain that

$$K = \langle e_1 + 2e_2, e_1 + 2e_3, e_4 + 2e_5, e_4 + 2e_6 \rangle ,$$

while  $K_S$  does not contain any of the elements in the set

$$\{l(e_1 + 2e_2), l'(e_1 + 2e_3), l''(e_4 + 2e_5), l'''(e_4 + 2e_6) \mid l, l', l'', l''' = 1, 2\}.$$

Similarly for  $g = (123456) \in S_6$ , we obtain

$$K = \langle e_1 + 2e_2, e_1 + 2e_3, e_1 + 2e_4, e_1 + 2e_5, e_1 + 2e_6 \rangle ,$$

while  $K_S$  does not contain any of the elements in the set

$$\{l(e_1 + 2e_2), l'(e_1 + 2e_3), \mid l, l' = 1, 2\}.$$

The following result gives an explicit relationship between the groups  $S$ ,  $K$  and  $K_S$ .

**Proposition 7.1.3** *The group  $S/K_S$  is an extension of  $K/K_S$  by  $S/K$ , where*

$$|K/K_S| = \frac{|N/S|}{[C_N(g) : C_S(g)]} = p \text{ or } 1.$$

**Proof.** Define the map  $h : S/K_S \rightarrow S/K$  by

$$h(a + K_S) = \begin{cases} a + K & \text{if } a \in S \setminus K \\ K & \text{if } a \in K \end{cases}.$$

It is clear that  $h$  is an epimorphism with  $\text{Ker}(h) = K/K_S$ . Hence

$$(S/K_S)/(K/K_S) \cong S/K,$$

which implies that  $S/K_S$  is an extension of  $K/K_S$  by  $S/K$ . Further we have  $\text{Ker}(\phi_g) = \{d \in N \mid d - d^g = 0\} = C_N(g)$  and  $N/C_N(g) \cong K$ . Similarly by considering  $\phi_g$  restricted to  $S$ , we have  $S/C_S(g) \cong K_S$ . Since  $|N/S| = p$ , we have

$$|K/K_S| = \frac{|N/S|}{[C_N(g) : C_S(g)]} = p \text{ or } 1.$$

□

**Remark 7.1.4** *Following from Proposition 7.1.1, we have that*

(i) *if  $\lambda_s \neq 0$  for some  $s \neq pk$ ,  $k \in \mathbb{N} \cup \{0\}$ , then  $S/K_S = S/K < N/K$ . Since an orbit of the action of  $S_n$  on  $S$  is an orbit of the action of  $S_n$  on  $N$ , it follows that an orbit of the action of  $C_G(g)$  on  $S/K_S = S/K$  is an orbit of the action of  $C_G(g)$  on  $N/K$ .*

(ii) *if  $\lambda_s = 0$  for all  $s \neq pk$ ,  $k \in \mathbb{N} \cup \{0\}$ , then  $K_S \subset K$  and by Proposition 7.1.3 we have*

$$K/K_S \cong \mathbb{Z}_p.$$

*Thus  $K/K_S = \langle a + K_S \rangle$ , where  $a \in K \setminus K_S$ . Further for any  $a, b \in K \setminus K_S$ , where  $b \neq la$ , we have  $la + K_S = l'b + K_S$  for some  $l, l' \in \mathbb{Z}_p$ . We claim that each coset in  $K/K_S$  is  $C_G(g)$ -invariant, that is for  $u = la + K_S, l = 1, 2, \dots, p-1$  and  $h \in C_G(g)$  we have  $u^h = u$ . Since  $a \in K \subseteq S$  and  $\phi_g(a) \in K_S$ , we choose  $h = g$ .*

Then we have  $\phi_h(a) = a - a^h \in K_S$ . Thus  $a^h + K_S = a + K_S$  for some  $h \in C_G(g)$ . Now let  $u = la + K_S$ . Then  $u^h = (la)^h + K_S = l(a)^h + K_S = la + K_S = u$ , as claimed.

Let  $a \in S \setminus K$ . We consider what happens when the coset  $a + K$  is lifted to  $S/K_S$  in the following.

**Proposition 7.1.5** *Let  $a \in S \setminus K$  such that  $a + K$  lifts to a number of cosets in  $S/K_S$ . If  $a_1 + K_S$  and  $a_2 + K_S$  are any two such cosets, then  $a_1^h + K_S = a_2 + K_S$  for some  $h \in C_G(g)$ .*

**Proof.** If  $g$  is a cycle of length  $n$ , then  $S = K$  and  $a + K = K$  which is dealt with in Remark 7.1.4 part(ii) above. Now let  $g$  contain the cycles  $g_\alpha = (\alpha_1 \alpha_2 \cdots \alpha_s)$  and  $g_\beta = (\beta_1 \beta_2 \cdots \beta_{s'})$ . We show that  $a_1^h - a_2 \in K_S$  for some  $h \in C_G(g)$ . Let us take  $a_1, a_2 \in S \setminus K$  such that  $a_1 - a_2 \in K \setminus K_S$ . In this case we have

$$a_1 = \sum_{i=1}^s \sum_{j=1}^{s'} l_{ij} [e_{\alpha_i} + (p-1)e_{\beta_j}], \quad a_2 = \sum_{i=1}^s \sum_{j=1}^{s'} m_{ij} [e_{\alpha_i} + (p-1)e_{\beta_j}],$$

where  $l_{ij}, m_{ij} \in \mathbb{Z}_p$ , giving

$$\begin{aligned} a_1 - a_2 &= \left[ \sum_{j=1}^{s'} (l_{1j} - m_{1j}) \right] e_{\alpha_1} + \left[ \sum_{j=1}^{s'} (l_{uj} - m_{uj}) \right] e_{\alpha_u} \\ &+ \sum_{i(\neq u)=2}^s \left[ \sum_{j=1}^{s'} (l_{ij} - m_{ij}) \right] e_{\alpha_i} + \sum_{j=1}^{s'} \left[ \sum_{i=1}^s [m_{ij} + (p-1)l_{ij}] \right] e_{\beta_j}. \end{aligned}$$

But by Proposition 7.1.1 and Remark 7.1.4, if  $a_1 - a_2 \in K \setminus K_S$ , then

$$a_1 - a_2 = l [e_{\alpha_1} + (p-1)e_{\alpha_u}], \quad (7.2)$$

for some  $u \in \{2, 3, \dots, p-1\}$  and some  $l \in \{1, 2, \dots, p-1\}$ . Now from relation (7.2) and the above full expansion of  $a_1 - a_2$ , the coefficients of  $e_{\alpha_1}$  and  $e_{\alpha_u}$  are respectively  $\sum_{j=1}^{s'} (l_{1j} - m_{1j}) = l$  and  $\sum_{j=1}^{s'} (l_{uj} - m_{uj}) = l(p-1)$ , while the coefficients of the rest of the terms are essentially zero, that is,  $\sum_{j=1}^{s'} (l_{ij} - m_{ij}) = vp$ ,  $v \in \mathbb{Z}$  for  $e_{\alpha_i}$ ,  $i \neq 1, u$  and

$$\sum_{i=1}^s (p-1)(l_{ij} - m_{ij}) = v'p, \quad v' \in \mathbb{Z}, \quad (7.3)$$

for  $e_{\beta_j}$ , the later which yields  $\sum_{i=1}^s l_{ij} = \sum_{i=1}^s m_{ij}, \text{ mod } p$ .

Now we choose  $h \in C_G(g)$  such that  $h$  moves  $e_{\beta_j}$ 's but not  $e_{\alpha_i}$ 's, for example we take  $h$  containing the cycle  $g_\beta = (\beta_1 \beta_2 \cdots \beta_{s'})$  but not the cycle  $g_\alpha$ . With such an  $h$  and for  $u = 2, 3, \dots, p-1$ , we have

$$\begin{aligned} a_1^h - a_2 &= \left[ \sum_{j=1}^{s'} (l_{1j} - m_{1j}) \right] e_{\alpha_1} + \left[ \sum_{j=1}^{s'} (l_{uj} - m_{uj}) \right] e_{\alpha_u} \\ &\quad + \sum_{i(\neq u)=2}^s \left[ \sum_{j=1}^{s'} (l_{ij} - m_{ij}) \right] e_{\alpha_i} + \sum_{j=1}^{s'} \left[ \sum_{i=1}^s [m_{ij} e_{\beta_j} + (p-1)l_{ij}] e_{\beta_{h(j)}} \right] \\ &= \left[ \sum_{j=1}^{s'} (l_{1j} - m_{1j}) \right] e_{\alpha_1} + \left[ \sum_{j=1}^{s'} (l_{uj} - m_{uj}) \right] e_{\alpha_u} \\ &\quad + \sum_{i(\neq u)=2}^s \left[ \sum_{j=1}^{s'} (l_{ij} - m_{ij}) \right] e_{\alpha_i} + \sum_{j=1}^{s'} \left[ \sum_{i=1}^s m_{ij} e_{\beta_j} + (p-1) \sum_{i=1}^s l_{ij} e_{\beta_{j+1}} \right] \\ &= l[e_{\alpha_1} + (p-1)e_{\alpha_u}] + \sum_{j=1}^{s'} l_j [e_{\beta_j} + (p-1)e_{\beta_{j+1}}], \end{aligned}$$

where  $l_j = \sum_{i=1}^s m_{ij} = \sum_{i=1}^s l_{ij}$  as above and  $e_{\beta_{s'+1}} = e_{\beta_1}$ . Now the elements  $l[e_{\alpha_1} + (p-1)e_{\alpha_u}]$ ,  $u = 2, 3, \dots, p-1$ ,  $l = 1, 2, \dots, p-1$ , belong to  $K$ . Also since  $e_{\beta_j} + (p-1)e_{\beta_{j+1}}$  is a generator of  $K$ , the elements  $\sum_{j=1}^{s'} l_j [e_{\beta_j} + (p-1)e_{\beta_{j+1}}]$  belong to  $K$ . Therefore  $a_1^h - a_2$  belongs to  $K$ . However the only elements of  $K$  which are not in  $K_S$  are  $l[e_{\alpha_1} + (p-1)e_{\alpha_u}]$ ,  $u = 2, 3, \dots, p-1$ ,  $l = 1, 2, \dots, p-1$ . We conclude that  $a_1^h - a_2 \in K_S$ .  $\square$

We now give a result that describes the  $p$ -sets of partition  $[\lambda]$  which index the columns of the Fischer-Clifford matrices of  $B_S(p, n)$  but which also index the columns of the Fischer-Clifford matrices of  $B(p, n)$ . For  $g \in S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \cdots n^{\lambda_n})$ , let  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n})$  denote a Fischer-Clifford matrix of  $B_S(p, n)$ .

**Theorem 7.1.6** *Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \cdots n^{\lambda_n})$ . Let  $\beta$  be a type-zero  $p$ -set of partition  $[\lambda]$ . Then  $\beta$  indexes a column of the Fischer-Clifford matrix  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \cdots n^{\lambda_n})$  of  $B_S(p, n)$ . Furthermore*

- (i) if  $\lambda_s \neq 0$  for some  $s \neq pk$ ,  $k \in \mathbb{N} \cup \{0\}$ , then the columns of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  are indexed by type-zero  $p$ -sets of partition  $[\lambda]$  and are in 1-1 correspondence with the columns of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  that are indexed by type-zero  $p$ -sets of partition  $[\lambda]$ ,
- (ii) if  $\lambda_s = 0$  for all  $s \neq pk$ ,  $k \in \mathbb{N} \cup \{0\}$ , then apart from those columns described in (i) above,  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  has an additional  $p - 1$  columns arising from the first column of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .

**Proof.**

- (i) Let  $\beta$  be a type-zero  $p$ -set of partition  $[\lambda]$ . Then with  $\sum_{i=1}^p \sum_{s=1}^n (i - 1)a_{is} \equiv 0 \pmod{p}$ , we have

$$\sum_{i=1}^p \sum_{s=1}^n k' h a_{is} + \sum_{i=1}^p \sum_{s=1}^n (i - 1)a_{is} = \sum_{i=1}^p \sum_{s=1}^n (pk' + (i - 1))a_{is} \equiv 0 \pmod{p}$$

for some  $k' \in \mathbb{N} \cup \{0\}$ . However

$$\sum_{i=1}^p \sum_{s=1}^n (pk' + (i - 1))a_{is}$$

is the sum of the powers associated with an element  $\bar{g} = dg$  of  $B(p, n)$ . Now with elements of  $N = \mathbb{Z}_p^n$  written additively, the above implies that the element  $dg$  of  $B(p, n)$  of type  $\beta$  with  $d = (l_1, l_2, \dots, l_n)$  satisfies  $\sum_{s=1}^n l_s \equiv 0 \pmod{p}$ . This implies  $d \in S$ , and such are the elements of  $N$  that form the group  $B_S(p, n)$  with  $S_n$ . It follows that  $\beta$  indexes a column of the Fischer-Clifford matrix  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of  $B_S(p, n)$ .

- (ii) If  $\lambda_s = 0$  for all  $s \neq pk$ ,  $k \in \mathbb{N} \cup \{0\}$ , then by Proposition 7.1.1, we have  $\{l(e_{\alpha_1} + (p - 1)e_{\alpha_{1+r}}) \mid l, r = 1, 2, \dots, p - 1\} \subseteq K \setminus K_S$  for each cycle  $g_\alpha$  of  $g$ , that is the identity coset  $K$  in  $S/K$  lifts to  $p$  cosets given by  $l(e_{\alpha_1} + (p - 1)e_{\alpha_2}) + K_S, l \in \mathbb{Z}_p$ , in  $S/K_S$ . Now by Proposition 7.1.3, each coset in  $K/K_S$  is  $C_G(g)$ -invariant, so that we obtain  $p$  orbits instead of one from the action of  $C_G(g)$  on  $K/K_S$ . Furthermore by Proposition 7.1.5 all the cosets to which the coset  $a + K$ ,  $a \in S \setminus K$ , lifts in  $S/K_S$  form one orbit under the action of  $C_G(g)$ . Thus we obtain that the action of  $C_G(g)$  on  $S/K_S$  has  $p - 1$  more orbits

than the action of  $C_G(g)$  on  $S/K$ . Hence from part (i) of this result, we obtain that  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  has an additional  $p - 1$  columns arising from the first column of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .  $\square$

We will see later that parts of the  $p$  columns of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  arising from the first column of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  are identical. Therefore we refer to these  $p$  columns as identical columns of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .

**Corollary 7.1.7** *Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots n^{\lambda_n})$ . Let  $\beta = ((a_{11}, a_{12}, \dots, a_{1n}), (a_{21}, a_{22}, \dots, a_{2n}))$  such that  $\sum_{s=1}^n a_{2s}$  is even. Then  $\beta$  indexes a column of the Fischer-Clifford matrix  $F_S(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of  $B_S(2, n)$ . Furthermore*

- (i) *if  $\lambda_s \neq 0$  for some odd  $s$ , then each column of the Fischer-Clifford matrix  $F_S(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  is indexed by a 2-set of partition  $[\lambda]$  such that  $\sum_{s=1}^n a_{2s}$  is even,*
- (ii) *if  $\lambda_s = 0$  for all odd  $s$ , then apart from those columns described in (i) above,  $F_S(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  has one additional column arising from the first column of  $F(2, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .*

**Proof.** The proof follows from Theorem 7.1.6, taking  $p=2$ .  $\square$

Following from Theorem 7.1.6, we will say that a type-zero  $p$ -set of partition  $[\lambda]$  indexes a conjugacy class of  $B_S(p, n)$ . Furthermore if  $g$  is of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  where  $\lambda_s = 0$  for all  $s \neq pk, k \in \mathbb{N} \cup \{0\}$ , then  $B_S(p, n)$  has an additional  $p - 1$  conjugacy classes, each indexed by the type-zero  $p$ -set of partition  $[\lambda]$  that indexes the first column of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  but with subscripts  $1, 2, \dots, p-1$ . However the conjugacy class of  $B_S(p, n)$  corresponding to the orbit containing  $K_S$  in the action of  $C_G(g)$  on  $K/K_S$  will be indexed by the same type-zero  $p$ -set of partition  $[\lambda]$  but with subscript 0. By type  $Typ(\bar{g})$  of  $\bar{g} = dg$  in  $B_S(p, n)$  we will mean the type of  $\bar{g} = dg$  in  $B(p, n)$ .

**Proposition 7.1.8** *Let  $g \in S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . Let  $\bar{g} = dg$  be an element of  $B_S(p, n)$  of type  $(a_{is})$ , a type-zero  $p$  set of partition  $[\lambda]$ .*

- (i) *If  $\lambda_s \neq 0$  for some  $s \neq pk$ , then two elements  $\bar{g}$  and  $\bar{g}'$  of  $B_S(p, n)$  are conjugate if and only if  $Typ(\bar{g}) = Typ(\bar{g}')$ .*

(ii) Let  $\lambda_s = 0$  for all  $s \neq pk$ .

(a) If  $a_{is} \neq 0$  for some  $i \neq 1$ , then two elements  $\bar{g}$  and  $\bar{g}'$  of  $B_S(p, n)$  are conjugate if and only if  $\text{Typ}(\bar{g}) = \text{Typ}(\bar{g}')$ .

(b) If  $a_{is} = 0$  for all  $i \neq 1$ , then two elements  $\bar{g} = dg$  and  $\bar{g}' = d'g$  of  $B_S(p, n)$  of the same type are conjugate if and only if  $d$  and  $d'$  are sums of  $p \times l$   $e_s$ 's,  $l \in \mathbb{Z}_p$ . In this case for each  $l \in \mathbb{Z}_p$  there is a conjugacy class with representative  $dg = (la)g$  where  $a \in K \setminus K_S$ .

**Proof.**

(i) The proof follows from Proposition 3.1.3 and Theorem 7.1.6.

(ii) Further if  $g \in S_n$  is of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  where  $\lambda_s = 0$  for all  $s \neq pk$ , we know that the elements of a class of  $B(p, n)$  of type  $(a_{is})$  where  $a_{is} = 0$  for all  $i \neq 1$  breaks into  $p$  conjugacy classes of the same type  $(a_{is})$  where  $a_{is} = 0$  for all  $i \neq 1$  in the group  $B_S(p, n)$ . Since these classes are of the same type  $(a_{is})$  where  $a_{is} = 0$  for all  $i \neq 1$  it follows that for each element  $dg$  in these classes, the sum of the  $e_s$ 's in  $d$  is congruent to  $pl$ , where  $l = 0, 1, 2, \dots, p-1$ . Thus the representative elements of these classes are given by  $dg = (la)g = l(e_{\alpha_1} + (p-1)e_{\alpha_2})g$ , where  $l \in \mathbb{Z}_p$ .  $\square$

**Example 7.1.9** We consider the cases  $m = 2$  and  $m = 3$  respectively.

(i) We take the class of  $S_6$  of type  $(2^3)$  with representative  $g = (1\ 2)(3\ 4)(5\ 6)$ . It is clear that the type-zero 2-sets of partition  $[2^3]$  are  $(3, 0)$  and  $(1, 2)$ . We see from Tables 6.1 and 7.1 that the type  $(1, 2)$  indexes one class of  $B_S(2, 6)$  of size 360 as in  $B(2, 6)$ . However the  $(2^3)$ -class of  $B(2, 6)$  of type  $(3, 0)$  and size 120 splits into  $p = 2$  classes of  $B_S(2, 6)$  of types  $(3, 0)_0$ ,  $(3, 0)_1$  and size 60 each, with representatives  $g$  and  $(e_1 + e_2)g$ , respectively.

(ii) We take the class of  $S_3$  of type  $(3^1)$  with representative  $g = (1\ 2\ 3)$ . Here  $(1, 0, 0)$  is the only type-zero 3-set of partition  $[3^1]$ . We see from Table 7.7 that the  $(3^1)$ -class of  $B(3, 3)$  of type  $(1, 0, 0)$  and size 18 in  $B(3, 3)$  splits into  $p = 3$  classes of  $B_S(3, 3)$  of types  $(1, 0, 0)_0$ ,  $(1, 0, 0)_1$ ,  $(1, 0, 0)_2$  and size 6 each, with representatives  $g$ ,  $(e_1 + 2e_2)g$  and  $2(e_1 + 2e_2)g = (2e_1 + e_2)g$ , respectively.

**Proposition 7.1.10** *Let  $\bar{g}$  be an element of  $B_S(p, n)$  of type*

$$(1^{a_{11}} 2^{a_{12}} \dots n^{a_{1n}}, 1^{a_{21}} 2^{a_{22}} \dots n^{a_{2n}}, \dots, 1^{a_{p1}} 2^{a_{p2}} \dots n^{a_{pn}}),$$

or  $(a_{is})$ , a type-zero  $p$  set of partition  $[\lambda]$ . Then

(i) *the order of the centralizer of  $\bar{g}$  in  $B_S(p, n)$  is given by*

$$(a) \quad \frac{1}{p} \left( \prod_{i=1}^p \prod_{s=1}^n a_{is}! (ps)^{a_{is}} \right),$$

*if  $a_{is} \neq 0$  for some  $i \neq 1$ ,*

$$(b) \quad \prod_{s=1}^n \lambda_s! (ps)^{\lambda_s},$$

*if  $a_{is} = 0$  for all  $i \neq 1$ , that is the order of each centralizer in  $B_S(p, n)$  of the elements corresponding to the  $p$  identical columns of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  is equal to the order of the centralizer at the first column of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .*

(ii) *The order of an element  $\bar{g}$  of  $B_S(p, n)$ , is the least common multiple of the products  $s \times r$  for all positive  $a_{is}$  in a type-zero  $p$ -set of partition  $[\lambda]$ , where  $r$  is the order of the cycle-product  $c$  of a cycle of  $\bar{g}$  of length  $s$ . Further the orders of the elements  $\bar{g}$  in  $B_S(p, n)$  corresponding to the  $p$  identical columns of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  are equal to the order of the elements corresponding to the first column of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .*

**Proof.**

(i) (a) Since we have  $|B_S(p, n)| = \frac{1}{p}|B(p, n)| = p^{n-1}n!$  and the sizes of the conjugacy classes of  $B_S(p, n)$  remain as in  $B(p, n)$  by construction, the orders of the centralizers of  $B_S(p, n)$  must be divided by  $p$ . The formula

$$\frac{1}{p} \left( \prod_{i=1}^p \prod_{s=1}^n a_{is}! (ps)^{a_{is}} \right),$$

follows from Proposition 3.1.7. The condition  $a_{is} \neq 0$  for some  $i \neq 1$  refers to all types  $(a_{is})$  in this case the type-zero  $p$ -sets of partition  $[\lambda]$  that are different from the one indexing the first column of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .

- (b) The proof follows from Theorem 7.1.6 part(ii) since we are considering centralizers of elements coming from the same conjugacy class of  $B(p, n)$  of type  $(a_{is})$ , where  $a_{is} = 0$  for all  $i \neq 1$ , that is the type-zero  $p$ -set of partition  $[\lambda]$  indexing the first column of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .
- (ii) The proof follows from Theorem 7.1.6 since the elements of  $B_S(p, n)$  are elements of  $B(p, n)$  of the same orders. Further the elements corresponding to the  $p$  identical columns of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  come from the same conjugacy class of  $B(p, n)$  of type  $(a_{is})$  where  $a_{is} = 0$  for all  $i \neq 1$  and corresponding to the first column of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ , hence the order of each of these elements must be the same.  $\square$

Now we discuss the rows of the Fischer-Clifford matrices of  $B_S(p, n)$ , whose indices as we know must correspond to the inertia factor groups of  $B_S(p, n)$ . From Remark 3.3.14, the inertia factor groups of  $B_S(p, n)$  are mostly young subgroups of  $S_n$  except possibly one. Indeed if  $p \mid n$ , then one of the inertia factor groups of  $B_S(p, n)$  is the wreath product  $S_{\frac{n}{p}} \text{Wr} C_p$  (or  $(S_{\frac{n}{p}})^p : C_p$ ). This follows from Theorem 3.3.13 and the presence in  $A(n, p)$  of  $(k, k, k, \dots, k)$ ,  $k = \frac{n}{p}$ . While the character tables of the young subgroups are easily obtainable from GAP, we can construct the character table of  $(S_{\frac{n}{p}})^p : C_p$  using the following code,

`Display(CharacterTable(WreathProduct(SymmetricGroup( $\frac{n}{p}$ ),  $C_p$ )));`

where  $C_p$  is the cyclic group of order  $p$ . We also note that the character table of  $(S_{\frac{n}{p}})^p : C_p$  can be constructed using example ex16.c of a computer algebra system called SYMMETRICA [19]. Since  $(S_{\frac{n}{p}})^p : C_p$  is a split extension, its character table can also be constructed using the technique of Fischer-Clifford matrices. For another method for constructing the character table of the groups  $(S_{\frac{n}{2}})^2 : C_2$ , see [4]. We also have in particular the following result.

**Proposition 7.1.11** *The number of conjugacy classes of  $(S_{\frac{n}{2}})^2 : C_2$  is given by*

$$\frac{c(c+3)}{2},$$

where  $c$  is the number of conjugacy classes of  $S_{\frac{n}{2}}$ .

**Proof.** See [4].  $\square$

It is clear that each representative of a set of equivalent  $p$ -sets of partition  $[\lambda]$  corresponds to a young subgroup which is an inertia factor group. Therefore each representative of a set of equivalent  $p$ -sets of partition  $[\lambda]$  indexes a row of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . However corresponding to the inertia factor group  $(S_{\frac{n}{p}})^p : C_p$  of  $B_S(p, n)$ , we see in Proposition 7.1.12 below that the rows of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  are indexed by either a self equivalent  $p$ -set of partition  $[\lambda]$  or powers  $\xi^l$ ,  $1 \leq l \leq p - 1$  of a primitive  $p^{\text{th}}$  root  $\xi$  of unity.

**Proposition 7.1.12** *For each conjugacy class of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ , we choose the entries in the rows of  $F(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  indexed by each representative of the set of equivalent  $p$ -sets of partition  $[\lambda]$  and columns indexed by type-zero  $p$ -sets of partition  $[\lambda]$ , to form some of the rows of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . Further if  $p \mid n$ , we divide by  $p$  all the entries of a row of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  indexed by a self equivalent  $p$ -set of partition  $[\lambda]$  and columns indexed by type-zero  $p$ -sets of partition  $[\lambda]$ , to form a row of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . Furthermore*

- (i) *if  $\lambda_s \neq 0$  for some  $s \neq pk$ ,  $k \in \mathbb{N} \cup \{0\}$ , then the rows described above form the Fischer-Clifford matrix  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ ,*
- (ii) *if  $\lambda_s = 0$  for all  $s \neq pk$ ,  $k \in \mathbb{N} \cup \{0\}$ , in addition to the rows and columns described above, we introduce  $p - 1$  columns equal to column one above and introduce  $p - 1$  rows equal to the non-identity irreducible characters of the cyclic group  $C_p$  multiplied by  $p^{\dim(S/K)}$  respectively in the  $p$  identical columns and zeros elsewhere. The additional rows correspond to the inertia factor group  $(S_{\frac{n}{p}})^p : C_p$ .*

**Proof.** Since  $B_S(p, n)$  is a subgroup of the  $B(p, n)$ , to construct  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  we only need to choose the entries in the necessary rows and columns of the Fischer-Clifford matrix  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . The proof of the introductory part of the result follows from Theorem 7.1.6 and the choice of the inertia factor groups in Remark 3.3.14.

Now let  $p \mid n$ . Then there is one  $p$ -composition of  $n$  given by  $(k, k, k, \dots, k)$  where  $k = \frac{n}{p}$ . However we generally require  $p$  equivalent  $p$ -sets of partition  $[\lambda]$  for a row of

$F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ , but in this case there is only one self equivalent  $p$ -set of partition  $[\lambda]$ . Therefore dividing by  $p$ , all the entries of a row of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  indexed by a self equivalent  $p$ -set of partition  $[\lambda]$  is necessary. It is now clear that a self equivalent  $p$ -set of partition  $[\lambda]$  indexes a row of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  corresponding to the inertia factor group  $(S_{\frac{n}{p}})^p : C_p$ .

- (i) The proof follows from part(i) of Theorem 7.1.6.
- (ii) The proof follows from part (ii) of Theorem 7.1.6 and part (ii) of Proposition 7.1.8. Now applying part (i) of Proposition 1.5.2 we have that the Fischer-Clifford matrix at  $g \in S_n$  is the matrix of orbit sums of  $C_{\bar{g}}$  acting on the rows of the character table of  $S/K_S$ , which includes the character table of the cyclic group  $K/K_s \cong Z_p$ . Since  $p$  is prime, the groups  $S = p^{n-1}$  and  $S/K$  are finite dimensional vector spaces over  $Z_p$ , for which dimension of  $S/K$  is valid. Applying the orthogonality relations to the identical columns we obtain  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  as required. We also note that in this case  $p \mid n$ . Therefore as seen above we have that the additional  $p - 1$  rows of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  correspond to the group  $(S_{\frac{n}{p}})^p : C_p$ .  $\square$

We have now developed enough material to construct the Fischer-Clifford matrices of  $B_S(p, n)$ . We give an example in the following.

**Examples 7.1.13** *We give an application of Proposition 7.1.12 as well as the use of orthogonality relations to construct a Fischer-Clifford matrix of  $B_S(2, 6)$ .*

- (a) *We construct the Fischer-Clifford matrix  $F_S(2, 2^3)$  from the Fischer-Clifford matrix*

$$F(2, 2^3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

*given in Table 5.1. Using Corollary 7.1.7 the 2-sets of partition  $[2^3]$  where  $a_{21} + a_{22}$  is even, are  $\binom{3}{0}$  and  $\binom{1}{2}$ . These type-zero 2-sets of partition  $[2^3]$  index columns 1 and 3 of  $F(2, 2^3)$ , and choosing these columns results in the resultant matrix*

$$\begin{pmatrix} 1 & 1 \\ 3 & -1 \\ 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

We also have that the representatives of the sets of equivalent 2-sets of partition  $[2^3]$  are  $\binom{3}{0}$  and  $\binom{2}{1}$ . These representatives of the sets of equivalent 2-sets of partition  $[2^3]$  index rows 1 and 2 of  $F(2, 2^3)$ , and choosing the corresponding rows from the resultant matrix given above gives the following partial matrix of  $F_S(2, 2^3)$ :

$$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}.$$

Since for type  $(2^3)$  we have  $\lambda_s = 0$  for  $s \neq 2k$ ,  $k \in \mathbb{N} \cup \{0\}$ , we apply Proposition 7.1.12, to complete the columns and rows of  $F_S(2, 2^3)$ . Thus we repeat column 1 of the above partial Fischer-Clifford matrix and also introduce a third row with the entries  $(1) \times 2^{\dim(S/K)}$ ,  $(-1) \times 2^{\dim(S/K)}$  respectively in the identical columns and zeros elsewhere, where 1 and -1 are entries of the non-identity irreducible character of  $C_2$ . Now from  $S = 2^5$  we have  $\dim(S) = 5$ . By Example 7.1.2 (b)  $K = \text{Im}(\phi_g) = \langle e_1 + e_2, e_3 + e_4, e_5 + e_6 \rangle$ , so that  $\dim(K) = 3$ . Thus  $\dim(S/K) = 5 - 3 = 2$ . We obtain the third row of  $F_S(2, 2^3)$  as  $(2^2, -2^2, 0)$ . Hence

$$F_S(2, 2^3) = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & -1 \\ 4 & -4 & 0 \end{pmatrix}.$$

- (b) We note that we can calculate the entries 4, -4, 0 of the third row of  $F_S(2, 2^3)$  using the usual properties of Fischer-Clifford matrices as follows. Suppose firstly we use Corollary 7.1.7 (or Theorem 7.1.6) to obtain some entries of  $F_S(2, 2^3)$ . Then as in (a) above we obtain the following partial Fischer-Clifford matrix of  $F_S(2, 2^3)$ .

$$\begin{array}{c} 48 \\ 16 \\ 12 \end{array} \begin{pmatrix} 384 & 384 & 64 \\ 1 & 1 & 1 \\ 3 & 3 & -1 \\ x & y & 0 \end{pmatrix},$$

where 384, 384, 64 are the orders of centralizers of the  $(2^3)$ -conjugacy classes of  $B_S(2, 6)$  and 48, 16, 12 are the orders of centralizers in the respective inertia factor groups involved. Then using the second orthogonality relation for column 1 of the partial Fischer-Clifford matrix of  $F_S(2, 2^3)$ , we obtain the equation  $48(1) + 16(9) + 12(x^2) = 384$ . This equation has the solutions  $x = \pm 4$ , which are the two non-zero entries of the third row of the Fischer-Clifford matrix  $F_S(2, 2^3)$ .

**Remark 7.1.14** Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  where  $\lambda_s = 0$  for all  $s \neq mk$ ,  $k \in \mathbb{N}$ . We may use Theorem 7.1.6 to construct a partial Fischer-Clifford matrix of  $F_S(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . To determine  $\dim(S/K)$  referred to in Proposition 7.1.12 part(ii), we use the second orthogonality relations for the columns of the partial Fischer-Clifford matrix of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . This gives equations whose solutions are of form  $\xi^k p^r$ , where  $r = \dim(S/K)$  and  $0 \leq k \leq p-1$ .

**Definition 7.1.15** In the rest of this chapter

- (i) a column of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  indexed by a type-zero  $p$ -set of partition  $[\lambda]$ , will be called **type-zero column**,
- (ii) a **permissible row index** for  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  will mean either a representative of the set of equivalent  $p$ -sets of partition  $[\lambda]$ , a self-equivalent  $p$ -set of partition  $[\lambda]$  or with slight abuse of terminology primitive  $p^{\text{th}}$  roots  $\xi^k$ ,  $1 \leq k \leq p-1$ , of unity. Further a row of a Fischer-Clifford matrix of  $B(p, n)$  corresponding to a permissible row index will be called a **permissible row**.

**Proposition 7.1.16** *Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . We obtain a Fischer-Clifford matrix  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of  $B_S(p, n)$  by*

- (i) *selecting the type-zero columns of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ ,*
- (ii) *selecting the rows corresponding to permissible rows of  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ , in the resulting matrix in (i) above,*
- (iii) *dividing by  $p$  all the entries of a row, corresponding to a row indexed by a self equivalent  $p$ -set of partition  $[\lambda]$  in  $F(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ , in the resulting matrix in (ii) above,*
- (iv) *for each conjugacy class of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ , where  $\lambda_s = 0$  for all  $s \neq pk$ ,  $k \in \mathbb{N}$ , we repeat  $p - 1$  times the first column of the resulting matrix in (ii) above and introduce  $p - 1$  rows equal to the non-identity characters of  $C_p$  multiplied by  $p^{\dim(S/K)}$  respectively in the  $p$  identical columns and zeros elsewhere.*

**Proof.** On applying Theorem 7.1.6, Proposition 7.1.12 and Definition 7.1.15 the result follows.  $\square$

For reference purposes, the identical columns mentioned in Proposition 7.1.16 will each be indexed by the  $p$ -set of partition  $[\lambda]$  indexing the first column of the partial Fischer-Clifford matrix but with subscripts. However the additional rows described in Proposition 7.1.16 will be indexed by primitive  $p^{\text{th}}$  roots  $\xi^k$ ,  $1 \leq k \leq p - 1$ , of unity. We are now in a position to state a theorem for the combinatorial method for constructing the Fischer-Clifford matrices of the groups  $B_S(p, n)$ .

**Theorem 7.1.17** *Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .*

- (i) *Let  $\beta = ((a_{11}, a_{12}, \dots, a_{1n}), (a_{21}, a_{22}, \dots, a_{2n}), \dots, (a_{p1}, a_{p2}, \dots, a_{pn}))$  be a type-zero  $p$ -set of partition  $[\lambda]$ .*
- (1) *Let  $\delta = ((k_{11}, k_{12}, \dots, k_{1n}), (k_{21}, k_{22}, \dots, k_{2n}), \dots, (k_{p1}, k_{p2}, \dots, k_{pn}))$  be a representative of the set of equivalent  $p$ -sets of partition  $[\lambda]$ . An entry of the Fischer-Clifford matrix  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  which is in a column indexed by  $\beta$  and in a row indexed by  $\delta$  is given by*

$$f_{\bar{\lambda}_0}[\beta, \delta] = \sum_{M(\beta, \delta, n)} \left( \prod_{s=1}^n \prod_{i=1}^p \left( \begin{matrix} a_{is} \\ R_{a_i, \lambda_s} \end{matrix} \right) \xi^{\sum_{i=1}^p \sum_{j=1}^p (i-1)(j-1) (\sum_{s=1}^n r_{ij}^s)} \right)$$

where  $\sum_{i=1}^p a_{is} = \sum_{j=1}^p k_{js} = \lambda_s$  and

- (a)  $\xi$  is a  $p$ -th root of unity,
  - (b)  $M(\beta, \delta, n)$  is a set of magic matrices,
  - (c)  $R_{a_i, \lambda_s}$  is the  $i^{\text{th}}$  row of a partial matrix at position  $s$  of a magic matrix,
  - (d)  $\binom{a_{is}}{r_{i1}^s \cdots r_{ip}^s}$  is a multi-nomial coefficient.
- (2) Let  $\delta = ((k_{11}, k_{12}, \dots, k_{1n}), (k_{21}, k_{22}, \dots, k_{2n}), \dots, (k_{p1}, k_{p2}, \dots, k_{pn}))$  be a self equivalent  $p$ -set of partition  $[\lambda]$ . An entry of  $F_S(p, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  which is in a column indexed by  $\beta$  and in a row indexed by  $\delta$ , is given by  $\frac{f_{\bar{\lambda}_0}[\beta, \delta]}{p}$ , where  $f_{\bar{\lambda}_0}[\beta, \delta]$  is as defined in (1) above.

- (ii) For each conjugacy class of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ , where  $\lambda_s = 0$  for all  $s \neq mk, k \in \mathbb{N}$ , we repeat  $p - 1$  times the first column of the resulting matrix in (i) above and introduce  $p - 1$  rows equal to the non-identity characters of  $C_p$  multiplied by  $p^{\dim(S/K)}$  respectively in the  $p$  identical columns and zeros elsewhere.

**Proof.** The formulae in Theorems 4.3.1 and 4.3.3, for constructing the Fischer-Clifford matrices of  $B(p, n)$  are adapted to  $B_S(p, n)$  by considering only the type-zero  $p$ -sets of partitions of  $n$  and the permissible row indices. The proof is complete by Theorem 7.1.6, Proposition 7.1.12, Definition 7.1.15 and Proposition 7.1.16.  $\square$

We will now construct the Fischer-Clifford matrices and character tables of the groups  $B_S(2, 6)$ ,  $B_S(3, 3)$  and  $B_S(3, 5)$  in the following sections.

## 7.2 The Group $B_S(2, 6)$

In this section we construct the Fischer-Clifford matrices of  $B_S(2, 6)$  and use these to construct its character table. The group  $B_S(2, 6)$  is an affine subgroup of the symplectic group  $SP(6, 2)$  (see [47], [12]). In [47], Mpono has used the properties of Fischer-Clifford matrices given in Section 1.5.2 to construct the Fischer-Clifford matrices of the group  $B_S(2, 6)$ . However here we use Theorem 7.1.17 to construct the Fischer-Clifford matrices of  $B_S(2, 6)$ . We note that most of the entries of the Fischer-Clifford matrices of  $B_S(2, 6)$  have been calculated while constructing the Fischer-Clifford matrices of  $B(2, 6)$  in Chapters 4 and 5. Thus we only need to use Proposition

7.1.16 and the Fischer-Clifford matrices of  $B(2, 6)$  given in Table 5.1 to construct the Fischer-Clifford matrices of  $B_S(2, 6)$ , this is done in Subsection 7.2.1. The Fischer-Clifford matrices of  $B_S(2, 6)$  will be used to construct the character table of  $B_S(2, 6)$  in Subsections 7.2.2 and 7.2.3.

### 7.2.1 The Fischer-Clifford Matrices of $B_S(2, 6)$

We now use Proposition 7.1.16 and the Fischer-Clifford matrices of  $B(2, 6)$  given in Table 5.1 to construct the Fischer-Clifford matrices of  $B_S(2, 6)$ . For each conjugacy class of  $S_6$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$ , we construct the Fischer-Clifford matrix  $F_S(2, 1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$ . However as examples we will construct the Fischer-Clifford matrices of  $B_S(2, 6)$  given by  $F_S(2, 1^6)$  and  $F_S(2, 1^4 2)$  only.

#### Examples 7.2.1

- (a) Using Proposition 7.1.16 we determine the Fischer-Clifford matrix  $F_S(2, 1^6)$  from the Fischer-Clifford matrix  $F(2, 1^6)$  of the group  $B(2, 6)$ .

$$F(2, 1^6) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 2 & 0 & -2 & -4 & -6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & -20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & -4 & 2 & 0 & -2 & 4 & -6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

given in Table 5.1. From  $\lambda_1 = 6$  and  $p = 2$ , we have the following 2-sets of partition  $[1^6]$ :

$$\begin{pmatrix} 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix},$$

from which we choose the following type-zero 2-sets of partition  $[1^6]$  which index the columns of  $F_S(2, 1^6)$ :

$$\begin{pmatrix} 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix}.$$

The columns 1, 3, 5 and 7 of  $F(2, 1^6)$ , indexed by the above type-zero 2-sets of partition  $[1^6]$ , are respectively selected. We produce the following resultant matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & 2 & -2 & -6 \\ 15 & -1 & -1 & 15 \\ 20 & -4 & 4 & -20 \\ 15 & -1 & -1 & 15 \\ 6 & 2 & -2 & -6 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Now since  $2 \mid 6$  the action of  $S_6$  on  $\text{Irr}(2^5)$  produces  $\lfloor \frac{7}{2} \rfloor = 4$  orbits. Three of these orbits are of lengths  $\binom{6}{6-k_1 \ k_1}$ , where  $k_1 \in \{0, 1, 2\}$ , and the other one is of length  $\frac{1}{2} \binom{6}{3 \ 3}$ . Thus we choose the following representatives of sets of equivalent 2-sets of partition  $[1^6]$ , which index the rows of  $F_S(2, 1^6)$ :

$$\begin{pmatrix} 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Using the resultant matrix above, the rows corresponding to the above permissible row indices are selected (compare Tables 5.1 and 7.3). Further since  $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$  is a self-equivalent 2-set of partition  $[1^6]$ , we divide by  $p = 2$  the entries of the row indexed by  $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$  to provide a fourth row. Thus we obtain that

$$F_S(2, 1^6) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & 2 & -2 & -6 \\ 15 & -1 & -1 & 15 \\ 10 & -2 & 2 & -10 \end{pmatrix}.$$

- (b) We construct the Fischer-Clifford matrix  $F_S(2, 1^4 2^1)$  from the Fischer-Clifford matrix  $F(2, 1^4 2)$  of the group  $B(2, 6)$ .

$$F(2, 1^4 2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 & 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 & 6 & 0 & -2 & 0 & 6 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 4 & -2 & 0 & 2 & -4 & 4 & -2 & 0 & 2 & -4 \\ 4 & 2 & 0 & -2 & -4 & -4 & -2 & 0 & 2 & 4 \\ 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\ 6 & 0 & -2 & 0 & 6 & -6 & 0 & 2 & 0 & -6 \\ 4 & -2 & 0 & 2 & -4 & -4 & 2 & 0 & -2 & 4 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

given in Table 5.1. Using Proposition 7.1.16 we determine the Fischer-Clifford matrix  $F_S(2, 1^4 2)$ , from the Fischer-Clifford matrix  $F(2, 1^4 2)$ . From  $\lambda_1 = 4$ ,  $\lambda_2 = 1$  and  $p = 2$ , we have the following 2-sets of partition  $[1^4 2]$ :

$$\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \\ \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix},$$

from which we choose the following type-zero 2-sets of partition  $[1^4 2]$  which index the columns of  $F_S(2, 1^4 2)$ :

$$\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

The columns 1, 3, 5, 7 and 9 of  $F(2, 1^4 2)$ , indexed by the above type-zero 2-sets of partition  $[1^4 2]$ , are respectively selected. We produce the following resultant matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 0 & -4 & 2 & -2 \\ 6 & -2 & 6 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 \\ 4 & 0 & -4 & -2 & 2 \\ 4 & 0 & -4 & -2 & 2 \\ 1 & 1 & 1 & -1 & -1 \\ 6 & -2 & 6 & 0 & 0 \\ 4 & 0 & -4 & 2 & -2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We also select the following representatives of sets of equivalent 2-sets of partition  $[1^4 2]$  which index the rows of  $F_S(2, 1^4 2)$ :

$$\begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the resultant matrix, the rows corresponding to the above permissible row indices are selected (compare Tables 5.1 and 7.3). Thus we obtain that

$$F_S(2, 1^4 2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 0 & -4 & 2 & -2 \\ 6 & -2 & 6 & 0 & 0 \\ 4 & 0 & -4 & -2 & 2 \\ 1 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

This is an example of a case where there is no self equivalent 2-set of partition  $[1^4 2]$ . So there is no row where division by  $p = 2$  is done.

- (c) Earlier in Examples 7.1.13, we constructed the Fischer-Clifford matrix  $F_S(2, 2^3)$  of  $B_S(2, 6)$  as an example of a case of a class of  $S_6$  whose cycle type contains no cycle of odd length. We showed that

$$F_S(2, 2^3) = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & -1 \\ 4 & -4 & 0 \end{pmatrix}.$$

The above was done for each conjugacy class of  $S_6$  to give the Fischer-Clifford matrices of  $B_S(2, 6)$  given in Table 7.3. These are the same Fischer-Clifford matrices of  $B_S(2, 6)$  as those found in [41], [47]. The 37 conjugacy classes of  $B_S(2, 6)$ , including the respective orders of centralizers in  $B_S(2, 6)$ , are given in Table 7.1. We construct the character table of  $B_S(2, 6)$  in sections 7.2.3 and 7.2.2.

Table 7.1: Conjugacy classes of  $B_S(2, 6)$

$[g]_{S_6}$	type-zero 2-sets of partition $[\lambda]$	$(1^{a_{11}}2^{a_{12}} \dots 6^{a_{16}}, 1^{a_{21}}2^{a_{22}} \dots 6^{a_{26}})$	$\bar{g}$	$ C_G(\bar{g}) $	$ \bar{g} $
$(1^6) = 1a$	$(6, 0)$ $(4, 2)$ $(2, 4)$ $(0, 6)$	$(1^6, 1^0)$ $(1^4, 1^2)$ $(1^2, 1^4)$ $(1^0, 1^6)$	1A 2A 2B 2C	23040 1536 1536 23040	1 15 15 1
$(1^4 2) = 2a$	$((4, 1), (0, 0))$ $((2, 1), (2, 0))$ $((0, 1), (4, 0))$ $((3, 0), (1, 1))$ $((1, 0), (3, 1))$	$(1^4 2^1, 1^0 2^0)$ $(1^2 2^1, 1^2 2^0)$ $(1^0 2^1, 1^4 2^0)$ $(1^3 2^0, 1^1 2^1)$ $(1^1 2^0, 1^3 2^1)$	2D 2E 2F 4A 4B	768 128 768 192 192	30 180 30 120 120
$(1^2 2^2) = 2b$	$((2, 2), (0, 0))$ $((0, 2), (2, 0))$ $((1, 1), (1, 1))$ $((2, 0), (0, 2))$ $((0, 0), (2, 2))$	$(1^2 2^2, 1^0 2^0)$ $(1^0 2^2, 1^2 2^0)$ $(1^1 2^1, 1^1 2^1)$ $(1^2 2^0, 1^0 2^2)$ $(1^0 2^0, 1^2 2^2)$	2G 2H 4C 4D 4E	128 128 32 128 128	180 180 720 180 180
$(2^3) = 2c$	$(3, 0)_0$ $(3, 0)_1$ $(1, 2)$	$(2^3, 2^0)_0$ $(2^3, 2^0)_1$ $(2^1, 2^2)$	2I 2J 4F	384 384 64	60 60 360
$(1^3 3) = 3a$	$((3, 1), (0, 0))$ $((1, 1), (2, 0))$ $((2, 0), (1, 1))$ $((0, 0), (3, 1))$	$(1^3 3^1, 1^0 3^0)$ $(1^1 3^1, 1^2 3^0)$ $(1^2 3^0, 1^1 3^1)$ $(1^0 3^0, 1^3 3^1)$	3A 6A 6B 6C	144 48 48 144	160 480 480 160
$(123) = 6a$	$((1, 1, 1), (0, 0, 0))$ $((0, 0, 1), (1, 1, 0))$ $((0, 1, 0), (1, 0, 1))$ $((1, 0, 0), (0, 1, 1))$	$(1^1 2^1 3^1, 1^0 2^0 3^0)$ $(1^0 2^0 3^1, 1^1 2^1 3^0)$ $(1^0 2^1 3^0, 1^1 2^0 3^1)$ $(1^1 2^0 3^0, 1^0 2^1 3^1)$	6D 12A 6E 12B	24 24 24 24	960 960 960 960
$(3^2) = 3b$	$(2, 0)$ $(0, 2)$	$(3^2, 3^0)$ $(3^0, 3^2)$	3B 6F	36 36	640 640
$(1^2 4) = 4a$	$((2, 1), (0, 0))$ $((0, 1), (2, 0))$ $((1, 0), (1, 1))$	$(1^2 4^1, 1^0 4^0)$ $(1^0 4^1, 1^2 4^0)$ $(1^1 4^0, 1^1 4^1)$	4G 4H 8A	32 32 16	720 720 1440
$(24) = 4b$	$((1, 1), (0, 0))_0$ $((1, 1), (0, 0))_1$ $((0, 0), (1, 1))$	$(2^1 4^1, 2^0 4^0)_0$ $(2^1 4^1, 2^0 4^0)_1$ $(2^0 4^0, 2^1 4^1)$	4I 4J 8B	32 32 16	720 720 1440
$(15) = 5a$	$((1, 1), (0, 0))$ $((0, 0), (1, 1))$	$(1^1 5^1, 1^0 5^0)$ $(1^0 5^0, 1^1 5^1)$	5A 10A	10 10	2304 2304
$(6) = 6b$	$(1, 0)_0$ $(1, 0)_1$	$(6^1, 6^0)_0$ $(6^1, 6^0)_1$	6G 6H	12 12	1920 1920

Table 7.2: Permissible row indices for  $B_S(2, 6)$

$[g]_{S_6}$	permissible row indices	type (row)
$(1^6) = 1a$	(6, 0) (5, 1) (4, 2) (3, 3)	$(1^6, 1^0)$ $(1^5, 1^1)$ $(1^4, 1^2)$ $(1^3, 1^3)$
$(1^4 2) = 2a$	$((4, 1), (0, 0))$ $((3, 1), (1, 0))$ $((2, 1), (2, 0))$ $((1, 1), (3, 0))$ $((4, 0), (0, 1))$	$(1^4 2^1, 1^0 2^0)$ $(1^3 2^1, 1^1 2^0)$ $(1^2 2^1, 1^2 2^0)$ $(1^1 2^1, 1^3 2^0)$ $(1^4 2^0, 1^0 2^1)$
$(1^2 2^2) = 2b$	$((2, 2), (0, 0))$ $((1, 2), (1, 0))$ $((0, 2), (2, 0))$ $((2, 1), (0, 1))$ $((1, 1), (1, 1))$	$(1^2 2^2, 1^0 2^0)$ $(1^1 2^1, 1^1 2^0)$ $(1^0 2^2, 1^2 2^0)$ $(1^2 2^1, 1^0 2^1)$ $(1^1 2^1, 1^1 2^1)$
$(2^3) = 2c$	(3, 0) (2, 1) $\xi = -1$	$(2^3, 2^0)$ $(2^2, 2^1)$ -
$(1^3 3) = 3a$	$((3, 1), (0, 0))$ $((2, 1), (1, 0))$ $((1, 1), (2, 0))$ $((3, 0), (0, 1))$	$(1^3 3^1, 1^0 3^0)$ $(1^2 3^1, 1^1 3^0)$ $(1^1 3^1, 1^2 3^0)$
$(123) = 6a$	$((1, 1, 1), (0, 0, 0))$ $((0, 1, 1), (1, 0, 0))$ $((1, 0, 1), (0, 1, 0))$ $((0, 0, 1), (1, 1, 0))$	$(1^1 2^1 3^1, 1^0 2^0 3^0)$ $(1^0 2^1 3^1, 1^1 2^0 3^0)$ $(1^1 2^0 3^1, 1^0 2^1 3^0)$ $(1^0 2^0 3^1, 1^1 2^1 3^0)$
$(3^2) = 3b$	(2, 0) (1, 1)	$(3^2, 3^0)$ $(3^1, 3^1)$
$(1^2 4) = 4a$	$((2, 1), (0, 0))$ $((1, 1), (1, 0))$ $((0, 1), (2, 0))$	$(1^2 4^1, 1^0 4^0)$ $(1^1 4^1, 1^1 4^0)$ $(1^0 4^1, 1^2 4^0)$
$(24) = 4b$	$((1, 1), (0, 0))$ $((0, 1), (1, 0))$ $\xi = -1$	$(2^1 4^1, 2^0 4^0)$ $(2^0 4^1, 2^1 4^0)$ -
$(15) = 5a$	$((1, 1), (0, 0))$ $((0, 1), (1, 0))$	$(1^1 5^1, 1^0 5^0)$ $(1^0 5^1, 1^0 5^1)$
$(6) = 6b$	(1, 0) $\xi = -1$	$(6^1, 6^0)$ -

Table 7.3: Fischer-Clifford matrices of  $B_S(2, 6)$

$F_S(2, 1^6) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & 2 & -2 & -6 \\ 15 & -1 & -1 & 15 \\ 10 & -2 & 2 & -10 \end{pmatrix}$	$F_S(2, 2^3) = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & -1 \\ 4 & -4 & 0 \end{pmatrix}$
$F_S(2, 1^4 2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 0 & -4 & 2 & -2 \\ 6 & -2 & 6 & 0 & 0 \\ 4 & 0 & -4 & -2 & 2 \\ 1 & 1 & 1 & -1 & -1 \end{pmatrix}$	$F_S(2, 6) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$F_S(2, 1^2 2^2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -2 & 0 & 2 & -2 \\ 1 & 1 & -1 & 1 & 1 \\ 2 & 2 & 0 & -2 & -2 \\ 2 & -2 & 0 & -2 & 2 \end{pmatrix}$	$F_S(2, 1^2 4) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 0 \\ 1 & 1 & -1 \end{pmatrix}$
$F_S(2, 1^3 3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & 1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & 1 & -1 & -1 \end{pmatrix}$	$F_S(2, 15) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$F_S(2, 123) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$	$F_S(2, 24) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}$
$F_S(2, 3^2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	

Table 7.4: Character table of  $(S_3)^2 : C_2$

Class	1a	2a	3a	2b	6a	3b	2c	4a	6b
Type	$(1^3, 1^3)$	$(1^3, 12)$	$(1^3, 3)$	$(12, 12)$	$(12, 3)$	$(3, 3)$	$(2, 2^2)$	$(2, 4)$	$(6)$
$\psi_1$	1	1	1	1	1	1	1	1	1
$\psi_2$	1	-1	1	1	-1	1	1	-1	1
$\psi_3$	1	1	1	1	1	1	-1	-1	-1
$\psi_4$	1	-1	1	1	-1	1	-1	1	-1
$\psi_5$	2	0	2	-2	0	2	0	0	0
$\psi_6$	4	0	-2	0	0	1	2	0	-1
$\psi_7$	4	0	-2	0	0	1	-2	0	1
$\psi_8$	4	-2	1	0	1	-2	0	0	0
$\psi_9$	4	2	1	0	-1	-2	0	0	0

Table 7.5: Fusion of inertia factor groups into  $S_6$

$S_6$	$S_5$	$S_4 \times S_2$	$(S_3)^2 : C_2$
$(1^6)$	$(1^5)$	$(1^4, 1^2)$	$(1^3, 1^3)$
$(1^4 2)$	$(1^3 2)$	$(1^2 2, 1^2), (1^4, 2)$	$(1^3, 12)$
$(1^2 2^2)$	$(12^2)$	$(1^2 2, 2), (2^2, 1^2)$	$(2, 12)$
$(2^3)$	-	$(2^2, 2)$	$(12, 2^2)$
$(1^3 3)$	$(1^2 3)$	$(13, 1^2)$	$(1^3, 3)$
$(123)$	$(23)$	$(13, 2)$	$(12, 3)$
$(3^2)$	-	-	$(3, 3)$
$(1^2 4)$	$(14)$	$(4, 1^2)$	-
$(24)$	-	$(4, 2)$	$(2, 4)$
$(15)$	$(5)$	-	-
$(6)$	-	-	$(6)$

### 7.2.2 The Inertia Factor Groups of $B_S(2, 6)$

We know from Theorem 3.2.7 that  $S_6$  acts on  $Irr(2^6)$  producing 7 orbits with the inertia factor groups  $S_6, S_5 \times S_1, S_4 \times S_2, S_3 \times S_3, S_2 \times S_4, S_5$  and  $S_6$ . By Remark 3.3.14 the sub-groups of  $S_6$  of indices 1, 6, 15, 10 are the required inertia factor groups of  $B_S(2, 6)$ . Hence the inertia factor groups of  $B_S(2, 6)$  are  $S_6, S_5 \times S_1, S_4 \times S_2$  and  $(S_3^2) : C_2$ . The character tables of these inertia factor groups are easily computed. The conjugacy classes of the inertia factor groups which fuse to a class of  $S_6$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$  are of type  $(1^{k_{11}} 2^{k_{12}} \dots 6^{k_{16}}, 1^{k_{21}} 2^{k_{22}} \dots 6^{k_{26}})$ , where  $\sum_{j=1}^2 k_{js} = \lambda_s$ . The fusion of the conjugacy classes of the inertia factor groups of  $B_S(2, 6)$  into  $S_6$  is given in Table 7.5 (also see [41], [47]). We use the inertia factor groups of  $B_S(2, 6)$  to construct the character table of  $B_S(2, 6)$  in the following section.

### 7.2.3 The Character Table of $B_S(2, 6)$

The character table of  $B_S(2, 6)$  is constructed by multiplying the partial character tables of the inertia factor groups of  $B_S(2, 6)$  by the corresponding rows of the Fischer-Clifford matrices in Table 7.3 according to the fusions given in Table 7.5. For example, from Table 7.5 we see that the conjugacy class  $(1^3, 12)$  of  $(S_3)^2 : C_2$  fuses to the class  $(1^4 2)$  of  $S_6$ . So we multiply the partial character table of  $(S_3)^2 : C_2$  at the element  $(1^3, 12)$  with row 4 of the Fischer-Clifford matrix  $F_S(2, 1^4 2)$  as follows:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 0 & -4 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -4 & -2 & 2 \\ -4 & 0 & 4 & 2 & -2 \\ 4 & 0 & -4 & -2 & 2 \\ -4 & 0 & 4 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & 0 & 8 & 4 & -4 \\ 8 & 0 & -8 & -4 & 4 \end{bmatrix},$$

which appears in block 4 of Table C.1 in the Appendix C under the class  $2a$  of  $S_6$ . This is done for each Fischer-Clifford matrix of  $B_S(2, 6)$  to give the full character table of  $B_S(2, 6)$  given in Tables C.1 to C.3 in the Appendix C. The orders of centralizers of  $B_S(2, 6)$  are listed in Table 7.1 and are not included in the character table of  $B_S(2, 6)$ . The correctness of the character table of  $B_S(2, 6)$  has been tested in GAP.

## 7.3 The Group $B_S(3, 3)$

In this section we construct the Fischer-Clifford matrices of  $B_S(3, 3)$  and use these to construct its character table. Let  $N = 3^3$ . Then  $S = 3^2 = \langle e_1 + 2e_2, e_1 + 2e_3 \rangle$  is a subgroup of  $N$ . Further  $S$  may be regarded as a vector space of dimension 2 over  $\mathbb{Z}_3$ . Explicitly  $S$  is a collection of all elements  $d = \sum_{s=1}^3 l_s e_s$  of  $N$  such that  $\sum_{s=1}^3 l_s \equiv 0 \pmod{3}$  or equivalently the union of the orbits of the action of  $S_3$  on  $N = 3^2$  of lengths  $\binom{3}{k_1 \ k_2 \ k_3}$ , where  $(k_1, k_2, k_3) \in A(3, 3)$ , such that  $\sum_{j=1}^3 (j-1)k_j \equiv 0 \pmod{3}$ . The group  $B_S(3, 3)$  is a split extension of  $S = 3^2$  by  $S_3$ , it exists by Theorem 3.3.5. In Subsection 7.3.1, we use Proposition 7.1.16 and the Fischer-Clifford matrices of  $B(3, 3)$  which can be constructed as shown in Chapters 4 and 5 to construct the

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Table 7.6: Fischer-Clifford matrices of  $B(3, 3)$ 

$$F(3, 1^3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & -\xi - 2\xi^2 & -2\xi - \xi^2 & \xi - \xi^2 & 0 & -\xi + \xi^2 & 3\xi & 2\xi + \xi^2 & \xi + 2\xi^2 & 3\xi^2 \\ 3 & -2\xi - \xi^2 & -\xi - 2\xi^2 & -\xi + \xi^2 & 0 & \xi - \xi^2 & 3\xi^2 & \xi + 2\xi^2 & 2\xi + \xi^2 & 3\xi \\ 3 & \xi - \xi^2 & -\xi + \xi^2 & 2\xi + \xi^2 & 0 & \xi + 2\xi^2 & 3\xi^2 & -2\xi - \xi^2 & -\xi - 2\xi^2 & 3\xi \\ 6 & 0 & 0 & 0 & -3 & 0 & 6 & 0 & 0 & 6 \\ 3 & -\xi + \xi^2 & \xi - \xi^2 & \xi + 2\xi^2 & 0 & 2\xi + \xi^2 & 3\xi & -\xi - 2\xi^2 & -2\xi - \xi^2 & 3\xi^2 \\ 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & 1 \\ 3 & 2\xi + \xi^2 & \xi + 2\xi^2 & -2\xi - \xi^2 & 0 & -\xi - 2\xi^2 & 3\xi & -\xi + \xi^2 & \xi - \xi^2 & 3\xi^2 \\ 3 & \xi + 2\xi^2 & 2\xi + \xi^2 & -\xi - 2\xi^2 & 0 & -2\xi - \xi^2 & 3\xi^2 & \xi - \xi^2 & -\xi + \xi^2 & 3\xi \\ 1 & \xi^2 & \xi & \xi & 1 & \xi^2 & 1 & \xi^2 & \xi & 1 \end{pmatrix}$$

$$F(3, 12) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \xi & \xi^2 & 1 & \xi & \xi^2 & 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi & 1 & \xi^2 & \xi & 1 & \xi^2 & \xi \\ 1 & 1 & 1 & \xi & \xi & \xi & \xi^2 & \xi^2 & \xi^2 \\ 1 & \xi & \xi^2 & \xi & \xi^2 & 1 & \xi^2 & 1 & \xi \\ 1 & \xi^2 & \xi & \xi & 1 & \xi^2 & \xi^2 & \xi & 1 \\ 1 & 1 & 1 & \xi^2 & \xi^2 & \xi^2 & \xi & \xi & \xi \\ 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & \xi & \xi^2 & 1 \\ 1 & \xi^2 & \xi & \xi^2 & \xi & 1 & \xi & 1 & \xi^2 \end{pmatrix}$$

$$F(3, 3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}$$

Fischer-Clifford matrices of  $B_S(3, 3)$ . The Fischer-Clifford matrices of  $B_S(3, 3)$  will be used to construct the character table of  $B_S(3, 3)$  in Subsections 7.3.2 and 7.3.3.

### 7.3.1 The Fischer-Clifford Matrices of $B_S(3, 3)$

In this subsection we use Proposition 7.1.16 and the Fischer-Clifford matrices of  $B(3, 3)$  given in Table 7.6 to construct the Fischer-clifford matrices of the group  $B_S(3, 3)$ . For each class of  $S_3$  of type  $(1^{\lambda_1}2^{\lambda_2}3^{\lambda_3})$  we construct a Fischer-Clifford matrix  $F_S(3, 1^{\lambda_1}2^{\lambda_2}3^{\lambda_3})$ . However as examples we construct the Fischer-Clifford matrices of  $B_S(3, 3)$  given by  $F_S(3, 1^3)$  and  $F_S(3, 3)$  only.

**Examples 7.3.1**

- (a) We construct the Fischer-Clifford matrix  $F_S(3, 1^3)$  from the Fischer-Clifford matrix  $F(3, 1^3)$  given in Table 7.6. For the Fischer-Clifford matrix  $F_S(3, 1^3)$  we have  $\lambda_1 = 3$  and  $p = 3$ . From the ten 3-sets of partition  $[1^3]$  we select the following four type-zero 3-sets of partition  $[1^3]$  which index the columns of  $F_S(3, 1^3)$ :

$$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

Thus the matrix  $F_S(3, 1^3)$  is of order 4. Since  $3 \mid 3$  the action of  $S_3$  on  $Irr(3^2)$  produces  $\lceil \frac{10}{3} \rceil = 4$  orbits. Three of these orbits are of lengths  $\binom{5}{k_1 k_2 k_3}$ , where  $(k_1, k_2, k_3) \in \{(3, 0, 0), (2, 1, 0), (2, 0, 1)\}$  and one of length  $\frac{1}{3} \binom{5}{1 1 1}$ . We also select the following four representatives of sets of equivalent 3-sets of partition  $[1^3]$  which index the rows of  $F_S(3, 1^3)$ :

$$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Now we select the entries in the columns and rows of  $F(3, 1^3)$  indexed by the type-zero 3-sets of partition  $[1^3]$  and permissible row indices respectively as found above. We produce the following resultant matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 3\xi & 3\xi^2 \\ 3 & 0 & 3\xi^2 & 3\xi \\ 6 & -3 & 6 & 6 \end{pmatrix}.$$

Further since  $(1, 1, 1)$  is a self equivalent 3-set of partition  $[1^3]$ , we divide the entries of the row corresponding to  $(1, 1, 1)$  in the resultant matrix above by  $p = 3$  to get a fourth row of  $F_S(3, 1^3)$ . This gives the Fischer-Clifford matrix

$$F_S(3, 1^3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 3\xi & 3\xi^2 \\ 3 & 0 & 3\xi^2 & 3\xi \\ 2 & -1 & 2 & 2 \end{pmatrix}$$

given in Table 7.9.

- (b) We construct the Fischer-Clifford matrix  $F_S(3, 3)$  from the Fischer-Clifford matrix  $F(3, 3)$  of the group  $B(3, 3)$

$$F(3, 3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}$$

given in Table 7.6. For the Fischer-Clifford matrix  $F_S(3, 3)$  we have  $\lambda_1 = 1$  and  $p = 3$ . From the three 3-sets of partition  $[3]$  we have only one candidate for a type-zero 3-set of partition  $[3]$ , namely  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  which indexes a column of  $F_S(3, 3)$ . We also select  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  as a representative of the set of equivalent 3-sets of partition  $[3]$  which indexes a row of  $F_S(3, 3)$ .

Now selecting the entry in the column and row of  $F(3, 3)$  indexed by the type-zero 3-set of partition  $[3]$  and permissible row index respectively as found above we obtain the partial matrix (1) of  $F_S(3, 3)$ . But since  $p = 3$  divides the length of the 3-cycle  $(123)$ , by Theorem 7.1.6 we repeat twice the column with this entry and introduce two extra rows given by the non-zero rows of the character table of  $C_3$  multiplied by  $3^{\dim(S/K)}$ . It can easily be seen from example 7.1.2(c) that for  $g = (123)$  we have  $S = K$  so that  $3^{\dim(S/K)} = 1$ . Thus the matrix  $F_S(3, 3)$  is of order 3 and is given by

$$F_S(3, 3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}.$$

Table 7.7: Conjugacy classes of  $B_S(3, 3)$

$[\mathfrak{g}]_{S_3}$	type-zero 3-sets of partition $[\lambda]$	$(1^{a_1} 1^{a_2} 2^{a_3} 3^{a_4}, 1^{a_5} 2^{a_6} 2^2 3^{a_7}, 1^{a_8} 3^1 2^{a_9} 3^{a_{10}})$	$\bar{g}$	$ C_G(\bar{g}) $	$ \bar{g} $
$(1^3) = 1a$	$(3, 0, 0)$	$(1^3, 1^0, 1^0)$	1A	54	1
	$(1, 1, 1)$	$(1^1, 1^1, 1^1)$	3A	9	6
	$(0, 3, 0)$	$(1^0, 1^3, 1^0)$	3B	54	1
	$(0, 0, 3)$	$(1^0, 1^0, 1^3)$	3C	54	1
$(12) = 2a$	$((1, 1), (0, 0), (0, 0))$	$(1^1 2^1, 1^0 2^0, 1^0 2^0)$	2A	6	9
	$((0, 0), (0, 1), (1, 0))$	$(1^0 2^0, 1^0 2^1, 1^1 2^0)$	6A	6	9
	$((0, 0), (1, 0), (0, 1))$	$(1^0 2^0, 1^1 2^0, 1^0 2^1)$	6B	6	9
$(3) = 3a$	$(1, 0, 0)_0$	$(3^1 3^0 3^0)_1$	3F	9	6
	$(1, 0, 0)_1$	$(3^1 3^0 3^0)_2$	3G	9	6
	$(1, 0, 0)_2$	$(3^1 3^0 3^0)_3$	3H	9	6

Table 7.8: Permissible row indices for  $B_S(3, 3)$

$[\mathfrak{g}]_{S_3}$	permissible row indices	type (row)
$(1^3) = 1a$	$(3, 0, 0)$	$(1^3, 1^0, 1^0)$
	$(2, 1, 0)$	$(1^2, 1^1, 1^0)$
	$(2, 0, 1)$	$(1^2, 1^0, 1^1)$
	$(1, 1, 1)$	$(1^1, 1^1, 1^1)$
$(12) = 2a$	$((1, 1), (0, 0), (0, 0))$	$(1^1 2^1, 1^0 2^0, 1^0 2^0)$
	$((0, 1), (1, 0), (0, 0))$	$(1^0 2^1, 1^1 2^0, 1^0 2^0)$
	$((0, 1), (0, 0), (1, 0))$	$(1^0 2^1, 1^0 2^0, 1^1 2^0)$
$(3) = 3a$	$(1, 0, 0)$	$(3^1 3^0 3^0)$
	$\xi$	---
	$\xi^2$	---

This was also done for the conjugacy class of  $S_3$  of type (12) to give the Fischer-Clifford matrices of  $B_S(3, 3)$  given in Table 7.9. We have also used the conjugacy classes of  $B_S(3, 3)$  and the properties of Fischer-Clifford matrices to construct the Fischer-Clifford matrices of  $B_S(3, 3)$ . It was found that the above Fischer-Clifford matrices are indeed the Fischer-Clifford matrices of  $B_S(3, 3)$ . The 10 conjugacy classes of  $B_S(3, 3)$ , including the respective orders of centralizers of  $B_S(3, 3)$ , are given in Table 7.7.

Table 7.9: Fischer-Clifford matrices of  $B_S(3, 3)$

$$F_S(3, 1^3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 3\xi & 3\xi^2 \\ 3 & 0 & 3\xi^2 & 3\xi \\ 2 & -1 & 2 & 2 \end{pmatrix}$$

$$F_S(3, 12) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}$$

$$F_S(3, 3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}$$

Table 7.10: Fusion of inertia factor groups into  $S_3$

$S_3$	$S_2$	$S_1^2 : C_3$
$(1^3)$	$(1^2)$	(1)
$(12)$	$(12)$	-
$(3)$	-	$(3), (3)'$

### 7.3.2 The Inertia Factor Groups of $B_S(3, 3)$

We can choose the inertia factor groups of  $B_S(3, 3)$  from the inertia factor groups of  $B(3, 3)$ . However by Remark 3.3.14 we obtain that the inertia factor groups of  $B_S(3, 3)$  are  $S_3$ ,  $S_2$  and  $(S_1)^2 : C_3$ . The character tables of these inertia factor groups are easily computed in GAP. And the conjugacy classes of the inertia factor groups which fuse to a class of  $S_3$  of type  $(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3})$  are of type

$$(1^{k_{11}} 2^{k_{12}} 3^{k_{13}}, 1^{k_{21}} 2^{k_{22}} 3^{k_{23}}, 1^{k_{31}} 2^{k_{32}} 3^{k_{33}}),$$

where  $\sum_{j=1}^3 k_{js} = \lambda_s$ . The fusion of the conjugacy classes of the inertia factor groups into  $S_3$  is given in Table 7.10. We use the inertia factor groups of  $B_S(3, 3)$  to construct the character table of  $B_S(3, 3)$  in the following section.

### 7.3.3 The Character Table of $B_S(3, 3)$

The character table of  $B_S(3, 3)$  is constructed by multiplying the partial character tables of the inertia factor groups of  $B_S(3, 3)$  by the corresponding rows of the Fischer-Clifford matrices in Table 7.9) according to the fusions given in Table 7.10. For example, from Table 7.10, we see that the conjugacy class (3) of  $(S_1)^2 : C_3$  fuses to the class (123) of  $S_3$ . But since there are two classes of elements of  $(S_1)^2 : C_3 \cong C_3$  fusing to the class 3a of  $S_3$ , we multiply the partial character table of  $(S_1)^2 : C_3$  at the classes (3) and (3)' with row 2 and 3 of the Fischer-Clifford matrix  $F_S(3, 3)$  as follows:

$$\begin{bmatrix} 1 & 1 \\ \xi & \xi^2 \\ \xi^2 & \xi \end{bmatrix} \begin{bmatrix} 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & \xi + \xi^2 & 2 \\ -1 & 2 & \xi + \xi^2 \end{bmatrix},$$

which appears in block 3 of Table D.1 in the Appendix D under the class 3a of  $S_3$ . This is done for each Fischer-Clifford matrix of  $B_S(3, 3)$  to give the full character table of  $B_S(3, 3)$  given in Table D.1 in the Appendix D. The orders of centralizers of  $B_S(3, 3)$  are listed in Table 7.7 and are not included in the character table of  $B_S(3, 3)$ . The correctness of the character table of  $B_S(3, 3)$  has been tested in GAP.

## 7.4 The Group $B_S(3, 5)$

In this section we construct the Fischer-Clifford matrices of  $B_S(3, 3)$  and use these to construct its character table. Let  $N = 3^5$ . Then  $S = 3^4 = \langle e_1 + 2e_{s+1} \mid s = 1, 2, 3, 4 \rangle$  is a subgroup of  $N$ . Further  $S$  may be regarded as a vector space of dimension 4 over  $\mathbb{Z}_3$ . Explicitly  $S$  is a collection of all elements  $d = \sum_{s=1}^5 l_s e_s$  of  $N$  such that  $\sum_{s=1}^5 l_s \equiv 0 \pmod{3}$  or the union of the orbits of the action of  $S_5$  on  $N = 3^5$  of lengths  $\binom{5}{k_1 k_2 k_3}$ , where  $(k_1, k_2, k_3) \in A(5, 3)$ , such that  $\sum_{j=1}^3 (j-1)k_j \equiv 0 \pmod{3}$ . The group  $B_S(3, 5)$  is a split extension of  $S = 3^4$  by  $S_5$ , it exists by Theorem 3.3.5. Here we use Proposition 7.1.16 and the Fischer-Clifford matrices of  $B(3, 5)$  constructed in Chapters 4 and 5 to construct the Fischer-Clifford matrices of  $B_S(3, 5)$ . The Fischer-Clifford matrices of  $B_S(3, 5)$  will be used to construct the character table of  $B_S(3, 5)$  in Subsections 7.4.2 and 7.4.3.

### 7.4.1 The Fischer-Clifford Matrices of $B_S(3, 5)$

In this subsection we use Proposition 7.1.16 and the Fischer-Clifford matrices of  $B(3, 5)$  constructed in Chapters 4 and 5 to construct the Fischer-clifford matrices of the group  $B_S(3, 5)$ . For each class of  $S_5$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 5^{\lambda_5})$  we construct the Fischer-Clifford matrix  $F_S(3, 1^{\lambda_1} 2^{\lambda_2} \dots 5^{\lambda_5})$ . However as examples we construct the Fischer-Clifford matrices of  $B_S(3, 5)$  given by  $F_S(3, 1^5)$  and  $F_S(3, 1^3 2)$  only.

#### Examples 7.4.1

- (a) We construct the Fischer-Clifford matrix  $F_S(3, 1^5)$  from the Fischer-Clifford matrix  $F(3, 1^5)$  given in Table 5.2. For the Fischer-Clifford matrix  $F_S(3, 1^5)$  we have  $\lambda_1 = 5$  and  $p = 3$ . From the 3-sets of partition  $[1^5]$  given in Chapter 4 we select the following seven type-zero 3-sets of partition  $[1^5]$  which index the columns of  $F_S(3, 1^5)$ :

$$\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}.$$

Thus the matrix  $F_S(3, 1^5)$  is of order 7. Since  $3 \nmid 5$  the action of  $S_5$  on  $\text{Irr}(3^4)$  produces  $\lceil \frac{21}{3} \rceil = 7$  orbits all of lengths  $\binom{5}{k_1 \ k_2 \ k_3}$ . We also select the following seven representatives of sets of equivalent 3-sets of partition  $[1^5]$  which index the rows of  $F_S(3, 1^5)$ :

$$\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

Now selecting the entries in the columns and rows of  $F(3, 1^5)$  indexed by the type-zero 3-sets of partition  $[1^5]$  and permissible row indices respectively as found above we obtain the Fischer-Clifford matrix  $F_S(3, 1^5)$  given in Table 7.13.

- (b) We construct the Fischer-Clifford matrix  $F_S(3, 1^3 2)$  from the Fischer-Clifford matrix  $F(3, 1^3 2)$  given in Tables 5.3 to 5.5. For the Fischer-Clifford matrix  $F_S(3, 1^3 2)$

we have  $\lambda_1 = 3, \lambda_2 = 1$  and  $p = 3$ . From the 3-sets of partition  $[1^3 2]$  given in Chapter 4 we choose the following ten type-zero 3-sets of partition  $[1^3 2]$  which index the columns of  $F_S(3, 1^3 2)$ :

$$\begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

Thus the matrix  $F_S(3, 1^3 2)$  is of order 10. We also select the following ten representatives of sets of equivalent 3-sets of partition  $[1^3 2]$  which index the rows of  $F_S(3, 1^3 2)$ :

$$\begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \end{pmatrix}.$$

Now selecting the entries in the columns and rows of  $F(3, 1^3 2)$  indexed by the type-zero 3-sets of partition  $[1^3 2]$  and permissible row indices respectively as found above we obtain the Fischer-Clifford matrix  $F_S(3, 1^3 2)$  given in Table 7.13.

This was done for each conjugacy classes of  $S_5$  to give the Fischer-Clifford matrices of  $B_S(3, 5)$  given in Table 7.13. The 36 conjugacy classes of  $B_S(3, 5)$ , including the respective orders of centralizers of  $B_S(3, 5)$ , are given in Table 7.11.

Table 7.11: Conjugacy classes of  $B_S(3, 5)$

$[g]_{S_5}$	type-zero 3-sets of partition $[\lambda]$	$(1^{a_{11}}2^{a_{12}} \dots 5^{a_{15}}, 1^{a_{21}}2^{a_{22}} \dots 5^{a_{25}}, 1^{a_{31}}2^{a_{32}} \dots 5^{a_{35}})$	$\bar{g}$	$ C_G(\bar{g}) $	$ \bar{g} $
$(1^5) = 1a$	(5, 0, 0) (3, 1, 1) (2, 3, 0) (2, 0, 3) (1, 2, 2) (0, 4, 1) (0, 1, 4)	$(1^5, 1^0, 1^0)$ $(1^3, 1^1, 1^1)$ $(1^2, 1^3, 1^0)$ $(1^2, 1^0, 1^3)$ $(1^1, 1^2, 1^2)$ $(1^0, 1^4, 1^1)$ $(1^0, 1^1, 1^4)$	1A 3B 3C 3D 3E 3F 3G	9720 486 972 972 324 1944 1944	1 20 10 10 30 5 5
$(1^3 2) = 2a$	$((3, 1), (0, 0), (0, 0))$ $((1, 1), (1, 0), (1, 0))$ $((0, 1), (3, 0), (0, 0))$ $((0, 1), (0, 0), (3, 0))$ $((2, 0), (0, 1), (1, 0))$ $((1, 0), (2, 1), (0, 0))$ $((0, 0), (1, 1), (2, 0))$ $((2, 0), (1, 0), (0, 1))$ $((1, 0), (0, 0), (2, 1))$ $((0, 0), (2, 0), (1, 1))$	$(1^3 2^1, 1^0 2^0, 1^0 2^0)$ $(1^1 2^1, 1^1 2^0, 1^1 2^0)$ $(1^0 2^1, 1^3 2^0, 1^0 2^0)$ $(1^0 2^1, 1^0 2^0, 1^3 2^0)$ $(1^2 2^0, 1^0 2^1, 1^1 2^0)$ $(1^1 2^0, 1^2 2^1, 1^0 2^0)$ $(1^0 2^0, 1^1 2^1, 1^2 2^0)$ $(1^2 2^0, 1^1 2^0, 1^0 2^1)$ $(1^1 2^0, 1^0 2^0, 1^2 2^1)$ $(1^0 2^0, 1^2 2^0, 1^1 2^1)$	2A 6A 6B 6C 6D 6E 6F 6G 6H 6I	324 54 324 324 108 108 108 108 108 108	30 180 30 30 90 90 90 90 90 90
$(1^1 2^2) = 2b$	$((1, 2), (0, 0), (0, 0))$ $((0, 1), (0, 1), (1, 0))$ $((0, 1), (1, 0), (0, 1))$ $((0, 0), (1, 2), (0, 0))$ $((1, 0), (0, 1), (0, 1))$ $((0, 0), (0, 0), (1, 2))$	$(1^1 2^2, 1^0 2^0, 1^0 2^0)$ $(1^0 2^1, 1^0 2^1, 1^1 2^0)$ $(1^0 2^1, 1^1 2^0, 1^0 2^1)$ $(1^0 2^0, 1^1 2^2, 1^0 2^0)$ $(1^1 2^0, 1^0 2^1, 1^0 2^1)$ $(1^0 2^0, 1^0 2^0, 1^1 2^2)$	2B 6J 6K 6L 6M 6N	72 36 36 72 36 72	135 270 270 135 270 135
$(1^2 3) = 3a$	$((2, 1), (0, 0), (0, 0))$ $((0, 1), (1, 0), (1, 0))$ $((1, 0), (0, 1), (1, 0))$ $((0, 0), (2, 1), (0, 0))$ $((1, 0), (1, 0), (0, 1))$ $((0, 0), (0, 0), (2, 1))$	$(1^2 3^1, 1^0 3^0, 1^0 3^0)$ $(1^0 3^1, 1^1 3^0, 1^1 3^0)$ $(1^1 3^0, 1^0 3^1, 1^1 3^0)$ $(1^0 3^0, 1^2 3^1, 1^0 3^0)$ $(1^1 3^0, 1^1 3^0, 1^0 3^1)$ $(1^0 3^0, 1^0 3^0, 1^2 3^1)$	3A 9A 9B 9C 9D 9E	54 27 27 54 27 54	180 360 360 180 360 180
$(23) = 6a$	$((1, 1), (0, 0), (0, 0))$ $((0, 0), (0, 1), (1, 0))$ $((0, 0), (1, 0), (0, 1))$	$(2^1 3^1, 2^0 3^0, 2^0 3^0)$ $(2^0 3^0, 2^0 3^1, 2^1 3^0)$ $(2^0 3^0, 2^1 3^0, 2^0 3^1)$	6O 18A 18B	18 18 18	540 540 540
$(14) = 4a$	$((1, 1), (0, 0), (0, 0))$ $((0, 0), (0, 1), (1, 0))$ $((0, 0), (1, 0), (0, 1))$	$(1^1 4^1, 1^0 4^0, 1^0 4^0)$ $(1^0 4^0, 1^0 4^1, 1^1 4^0)$ $(1^0 4^0, 1^1 4^0, 1^0 4^1)$	4A 12A 12B	12 12 12	810 810 810
$(5) = 5a$	(1, 0, 0)	$(5^1, 5^0, 5^0)$	5A	5	1944

Table 7.12: Permissible row indices for  $B_S(3, 5)$

$[g]_{S_5}$	permissible row indices	type (row)
$(1^5) = 1a$	(5, 0, 0) (4, 1, 0) (4, 0, 1) (3, 2, 0) (3, 1, 1) (3, 0, 2) (2, 2, 1)	$(1^5, 1^0, 1^0)$ $(1^4, 1^1, 1^0)$ $(1^4, 1^0, 1^1)$ $(1^3, 1^2, 1^0)$ $(1^3, 1^1, 1^1)$ $(1^3, 1^0, 1^2)$ $(1^2, 1^2, 1^1)$
$(1^3 2) = 2a$	$((3, 1), (0, 0), (0, 0))$ $((2, 1), (1, 0), (0, 0))$ $((2, 1), (0, 0), (1, 0))$ $((1, 1), (2, 0), (0, 0))$ $((1, 1), (1, 0), (1, 0))$ $((1, 1), (0, 0), (2, 0))$ $((0, 1), (3, 0), (0, 0))$ $((0, 1), (2, 0), (1, 0))$ $((0, 1), (1, 0), (2, 0))$ $((0, 1), (0, 0), (3, 0))$	$(1^3 2^1, 1^0 2^0, 1^0 2^0)$ $(1^2 2^1, 1^1 2^0, 1^0 2^0)$ $(1^2 2^1, 1^0 2^0, 1^1 2^0)$ $(1^1 2^1, 1^2 2^0, 1^0 2^0)$ $(1^1 2^1, 1^1 2^0, 1^1 2^0)$ $(1^1 2^1, 1^0 2^0, 1^2 2^0)$ $(1^0 2^1, 1^3 2^0, 1^0 2^0)$ $(1^0 2^1, 1^2 2^0, 1^1 2^0)$ $(1^0 2^1, 1^1 2^0, 1^2 2^0)$ $(1^0 2^1, 1^0 2^0, 1^3 2^0)$
$(1^1 2^2) = 2b$	$((1, 2), (0, 0), (0, 0))$ $((0, 2), (1, 0), (0, 0))$ $((0, 2), (0, 0), (1, 0))$ $((1, 1), (0, 1), (0, 0))$ $((0, 1), (1, 1), (0, 0))$ $((0, 1), (0, 1), (1, 0))$	$(1^1 2^2, 1^0 2^0, 1^0 2^0)$ $(1^0 2^2, 1^1 2^0, 1^0 2^0)$ $(1^0 2^2, 1^0 2^0, 1^1 2^0)$ $(1^1 2^1, 1^0 2^1, 1^0 2^0)$ $(1^0 2^1, 1^1 2^1, 1^0 2^0)$ $(1^0 2^1, 1^0 2^1, 1^1 2^0)$
$(1^2 3) = 3a$	$((2, 1), (0, 0), (0, 0))$ $((1, 1), (1, 0), (0, 0))$ $((1, 1), (0, 0), (1, 0))$ $((0, 1), (2, 1), (0, 0))$ $((0, 1), (1, 0), (1, 0))$ $((0, 1), (0, 0), (2, 0))$	$(1^2 3^1, 1^0 3^0, 1^0 3^0)$ $(1^1 3^1, 1^1 3^0, 1^0 3^0)$ $(1^1 3^1, 1^0 3^0, 1^1 3^0)$ $(1^0 3^1, 1^2 3^1, 1^0 3^0)$ $(1^0 3^1, 1^1 3^0, 1^1 3^0)$ $(1^0 3^1, 1^0 3^0, 1^2 3^0)$
$(23) = 6a$	$((1, 1), (0, 0), (0, 0))$ $((0, 1), (1, 0), (0, 0))$ $((0, 1), (0, 0), (1, 0))$	$(2^1 3^1, 2^1 3^0, 2^0 3^0)$ $(2^1 3^0, 2^1 3^1, 2^0 3^0)$ $(2^1 3^0, 2^1 3^0, 2^1 3^0)$
$(14) = 4a$	$((1, 1), (0, 0), (0, 0))$ $((0, 1), (1, 0), (0, 0))$ $((0, 1), (0, 0), (1, 0))$	$(1^1 4^1, 1^0 4^0, 1^0 4^0)$ $(1^0 4^1, 1^1 4^0, 1^0 4^0)$ $(1^0 4^1, 1^0 4^0, 1^1 4^0)$
$(5) = 5a$	(1, 0, 0)	$(5^1, 5^0, 5^0)$

Table 7.13: Fischer-Clifford matrices of  $B_S(3, 5)$ 

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$$F_S(3, 1^5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 2 & \xi - 2\xi^2 & -2\xi + \xi^2 & -1 & 4\xi + \xi^2 & \xi + 4\xi^2 & \\ 5 & 2 & -2\xi + \xi^2 & \xi - 2\xi^2 & -1 & \xi + 4\xi^2 & 4\xi + \xi^2 & \\ 10 & 1 & 5\xi + 2\xi^2 & 2\xi + 5\xi^2 & 1 & -4\xi + 2\xi^2 & 2\xi - 4\xi^2 & \\ 20 & -1 & 2 & 2 & -4 & 8 & 8 & \\ 10 & 1 & 2\xi + 5\xi^2 & 5\xi + 2\xi^2 & 1 & 2\xi - 4\xi^2 & -4\xi + 2\xi^2 & \\ 30 & -6 & 3 & 3 & 3 & -6 & -6 & \end{pmatrix}$$

$$F_S(3, 1^3 2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 3\xi & 3\xi^2 & -2\xi - \xi^2 & \xi - \xi^2 & \xi + 2\xi^2 & -\xi - 2\xi^2 & -\xi + \xi^2 & 2\xi + \xi^2 \\ 3 & 0 & 3\xi^2 & 3\xi & -\xi - 2\xi^2 & -\xi + \xi^2 & 2\xi + \xi^2 & -2\xi - \xi^2 & \xi - \xi^2 & \xi + 2\xi^2 \\ 3 & 0 & 3\xi^2 & 3\xi & -\xi + \xi^2 & 2\xi + \xi^2 & -\xi - 2\xi^2 & \xi - \xi^2 & \xi + 2\xi^2 & -2\xi - \xi^2 \\ 6 & -3 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3\xi & 3\xi^2 & \xi - \xi^2 & \xi + 2\xi^2 & -2\xi - \xi^2 & -\xi + \xi^2 & 2\xi + \xi^2 & -\xi - 2\xi^2 \\ 1 & 1 & 1 & 1 & \xi^2 & \xi^2 & \xi^2 & \xi & \xi & \xi \\ 3 & 0 & 3\xi & 3\xi^2 & \xi + 2\xi^2 & -2\xi - \xi^2 & \xi - \xi^2 & 2\xi + \xi^2 & -\xi - 2\xi^2 & -\xi + \xi^2 \\ 3 & 0 & 3\xi^2 & 3\xi & 2\xi + \xi^2 & -\xi - 2\xi^2 & -\xi + \xi^2 & \xi + 2\xi^2 & -2\xi - \xi^2 & \xi - \xi^2 \\ 1 & 1 & 1 & 1 & \xi & \xi & \xi & \xi^2 & \xi^2 & \xi^2 \end{pmatrix}$$

$$F_S(3, 1^2 3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & -\xi & 2\xi & -\xi^2 & 2\xi^2 \\ 2 & -1 & -\xi^2 & 2\xi^2 & -\xi & 2\xi \\ 1 & 1 & \xi^2 & \xi^2 & \xi & \xi \\ 2 & -1 & -1 & 2 & -1 & 2 \\ 1 & 1 & \xi & \xi & \xi^2 & \xi^2 \end{pmatrix}$$

$$F_S(3, 12^2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \xi^2 & \xi & \xi & 1 & \xi^2 \\ 1 & \xi & \xi^2 & \xi^2 & 1 & \xi \\ 2 & -\xi^2 & -\xi & 2\xi & -1 & 2\xi^2 \\ 2 & -\xi & -\xi^2 & 2\xi^2 & -1 & 2\xi \\ 2 & -1 & -1 & 2 & -1 & 2 \end{pmatrix}$$

$$F_S(3, 23) = F_S(3, 14) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi^2 & \xi \\ 1 & \xi & \xi^2 \end{pmatrix}$$

$$F_S(3, 5) = \begin{pmatrix} 1 \end{pmatrix}$$


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### 7.4.2 The Inertia Factor Groups of $B_S(3, 5)$

We can choose the inertia factor groups of  $B_S(3, 5)$  from the inertia factor groups of  $B(3, 5)$ . However by Remark 3.3.14 we obtain that the distinct inertia factor groups of  $B_S(3, 5)$  are  $S_5, S_4, S_3 \times S_2, S_2 \times S_3, S_2 \times S_2$  and  $S_3$ . The character tables of these inertia factor groups are easily computed. And the conjugacy classes of the inertia factor groups which fuse to a class of  $S_5$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 5^{\lambda_5})$  are of type

$$(1^{k_{11}} 2^{k_{12}} \dots 5^{k_{15}}, 1^{k_{21}} 2^{k_{22}} \dots 5^{k_{25}}, 1^{k_{31}} 2^{k_{32}} \dots 5^{k_{35}}),$$

where  $\sum_{j=1}^3 k_{js} = \lambda_s$ . The fusion of the conjugacy classes of the inertia factor groups into  $S_5$  is given in Table 6.9. We use the inertia factor groups of  $B_S(3, 5)$  to construct the character table of  $B_S(3, 5)$  in the following section.

### 7.4.3 The Character Table of $B_S(3, 5)$

The character table of  $B_S(3, 5)$  is constructed by multiplying the partial character tables of the inertia factor groups of  $B_S(3, 5)$  by the corresponding rows of the Fischer-Clifford matrices in Table 7.13 according to the fusions given in Table 6.9). For example, from Table 6.9, we see that the conjugacy class  $(3, 1^2)$  of  $S_3 \times S_2$  fuses to the class  $(1^2 3)$  of  $S_5$ . So we multiply the partial character table of  $S_3 \times S_2$  at the element  $(3, 1^2)$  with row 4 of the Fischer-Clifford matrix  $F_S(3, 1^2 3)$  as follows:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \xi^2 & \xi^2 & \xi & \xi \end{bmatrix} = \begin{bmatrix} 1 & 1 & \xi^2 & \xi^2 & \xi & \xi \\ 1 & 1 & \xi^2 & \xi^2 & \xi & \xi \\ 1 & 1 & \xi^2 & \xi^2 & \xi & \xi \\ 1 & 1 & \xi^2 & \xi^2 & \xi & \xi \\ -1 & -1 & -\xi^2 & -\xi^2 & -\xi & -\xi \\ -1 & -1 & -\xi^2 & -\xi^2 & -\xi & -\xi \end{bmatrix},$$

which appears in block 4 of Table E.3 in the Appendix E under the class 3a of  $S_5$ . This is done for each Fischer-Clifford matrix of  $B_S(3, 5)$  to give the full character table of  $B_S(3, 5)$  given in Tables E.1 and E.3 in the Appendix E. The orders of centralizers of  $B_S(3, 5)$  are listed in table 7.11 and are not included in the character table of  $B_S(3, 5)$ . The correctness of the character table of  $B_S(3, 5)$  has been tested in GAP.

## Chapter 8

# The Group $B_Q(m, n)$

As described in Section 3.3, the group  $B_Q(m, n)$  is a split extension of  $Q = \overline{\mathbb{Z}_m^n}$  by the symmetric group  $S_n$  (see Theorem 3.3.6). In Section 8.1, we describe the group  $B_Q(m, n)$  explicitly and give some results about its conjugacy classes. The group  $B_Q(m, n)$  may occur as a constituent of other important groups, such as simple groups, therefore we need to construct its character table. In this chapter we develop a method for constructing the Fischer-Clifford matrices of the group  $B_Q(m, n)$ .

In [35] List has presented a method for constructing the Fischer-Clifford matrices of group extensions of a faithful irreducible constituent of the elementary abelian group  $2^n$  by symmetric groups. This is done by choosing entries from appropriate rows and columns of the Fischer-Clifford matrices of  $B(2, n)$ . In Section 8.2 we adapt the combinatorial method for constructing the Fischer-Clifford matrices of  $B(m, n)$  to construct the Fischer-Clifford matrices of the groups  $B_Q(m, n)$ . In Sections 8.3 and 8.4, we apply the results obtained in Sections 8.1 and 8.2 to construct the Fischer-Clifford matrices and character tables of the groups  $B_Q(2, 6)$  and  $B_Q(3, 5)$  respectively. We note that with the exception of Corollary 8.2.11, Remark 8.2.12 and Proposition 8.2.13, which are valid only when  $p$  is prime, most of the results in this chapter are true for a general  $m$ .

### 8.1 Some Preliminary Results

In this section we discuss some results which we use in Section 8.2 for the construction of the Fischer-Clifford matrices of  $B_Q(m, n)$ . The group  $B_Q(m, n)$  may be considered

as the quotient group  $B(m, n)/M$ . Let  $N = \mathbb{Z}_m^n$  and  $M = \langle \sum_{s=1}^n e_s \rangle$  as in Chapter 3. Then we have the following result.

**Theorem 8.1.1** Let  $Q = \overline{\mathbb{Z}_m^n}$  and  $M$  as above.

- (i)  $M \triangleleft B(m, n)$ .
- (ii)  $B_Q(m, n) = B(m, n)/M$ .

**Proof.** We take  $M$  and  $N$  as subgroups of  $B(m, n)$  with the operation multiplication.

- (i) Let  $x \in M$  and  $dg \in B(m, n)$ , where  $d \in N$  and  $g \in S_n$ . Then  $dgx(dg)^{-1} = dx^gd^{-1} = x^g \in M$ , since  $N$  is abelian and  $M$  is  $S_n$ -invariant. So  $M \triangleleft B(m, n)$ .
- (ii) We show that the groups  $B_Q(m, n)$  and  $B(m, n)/M$  are equal and that the operations on the two groups are also equal. Let  $x \in B_Q(m, n)$ . Then  $x = (dM)g$  and  $(dM)g = d(gM) \in B(m, n)/M$ , since  $M$  is normal in  $B(m, n)$ . Similarly if  $x = (dg)M \in B(m, n)/M$  we have  $x = d(gM) = d(Mg) = (dM)g \in B_Q(m, n)$ . Hence  $B_Q(m, n) = B(m, n)/M$ .

Let  $(d_1M)g_1, (d_2M)g_2 \in B_Q(m, n)$  and  $(d_1g_1M), (d_2g_2M) \in B(m, n)/M$  then

$$\begin{aligned}
 (d_1M)g_1(d_2M)g_2 &= (d_1M)(d_2M)^{g_1}g_1g_2 \\
 &= (d_1M)(d_2^{g_1}M)g_1g_2 \\
 &= (d_1d_2^{g_1}M)g_1g_2 \\
 &= d_1d_2^{g_1}M(g_1g_2) \\
 &= d_1d_2^{g_1}(g_1g_2)M \\
 &= (d_1g_1)(d_2g_2)M \\
 &= (d_1g_1)M(d_2g_2)M. \quad \square
 \end{aligned}$$

Thus if necessary we may refer to the group  $B(m, n)/M$  in place of the group  $B_Q(m, n)$ . For example the element  $(d + M)g$  of  $B_Q(m, n)$  when  $N$  is written additively, may be taken as the element  $dgM$  of  $B(m, n)/M$ .

**Proposition 8.1.2** *Let  $d_1g_1, d_2g_2 \in B(m, n)$ , where  $d_1, d_2 \in N$  and  $g_1, g_2 \in S_n$ . If  $g_1$  and  $g_2$  are not conjugate in  $S_n$ , then*

$$(i) \quad (d_1 + M)g_1 \neq (d_2 + M)g_2.$$

$$(ii) \quad (d_1 + M)g_1 \text{ and } (d_2 + M)g_2 \text{ are not conjugate in } B_Q(m, n).$$

**Proof.**

(i) Suppose that  $(d_1 + M)g_1 = (d_2 + M)g_2$ , that is  $d_1g_1M = d_2g_2M$ . Then

$$\begin{aligned} d_1g_1M = d_2g_2M &\Rightarrow (d_1g_1)(d_2g_2)^{-1} \in M \\ &\Rightarrow d_1(g_1g_2^{-1})d_2^{-1} \in M \\ &\Rightarrow d_1(d_2^{-1})^{(g_1g_2^{-1})}(g_1g_2^{-1}) \in M, \\ &\Rightarrow g_1g_2^{-1} \in N \cap S_n \\ &\Rightarrow g_1g_2^{-1} = 1_N \\ &\Rightarrow g_1 = g_2, \end{aligned}$$

that is  $g_1$  is conjugate to  $g_2$ , which is a contradiction. It follows that  $(d_1 + M)g_1 \neq (d_2 + M)g_2$ .

(ii) Suppose that  $(d_1 + M)g_1$  and  $(d_2 + M)g_2$  are conjugate, that is  $(d_1g_1M)^{dgM} = d_2g_2M$  for some  $dgM \in B(m, n)/M$ , where  $d \in N$  and  $g \in S_n$ . We have  $(d_1g_1)^{dg}(d_2g_2)^{-1} \in M$  which implies that  $d_1^{dg}g_1^{dg}g_2^{-1}d_2^{-1} \in M$ , so that  $d_1^{dg}(dg)(g_1)(g^{-1}d^{-1})(g_2^{-1}d_2^{-1}) \in M$ . Now

$$\begin{aligned} d_1^{dg}(dg)(g_1)(g^{-1}d^{-1})(g_2^{-1}d_2^{-1}) \in M &\Rightarrow (d_1^{dg}d)(gg_1g^{-1})(d^{-1})(g_2^{-1}d_2^{-1}) \in M \\ &\Rightarrow (d_1^{dg}d)(d^{-1})^{(gg_1g^{-1})}(gg_1g^{-1})(g_2^{-1}d_2^{-1}) \in M \\ &\Rightarrow [(d_1^{dg}d)(d^{-1})^{g_1^g}](g_1^g)(g_2^{-1}d_2^{-1}) \in M \\ &\Rightarrow [(d_1^{dg}d)(d^{-1})^{g_1^g}](g_1^gg_2^{-1})(d_2^{-1}) \in M \\ &\Rightarrow [(d_1^{dg}d)(d^{-1})^{g_1^g}][(d_2^{-1})^{(g_1^gg_2^{-1})}](g_1^gg_2^{-1}) \in M \\ &\Rightarrow g_1^gg_2^{-1} \in N \cap S_n \\ &\Rightarrow g_1^gg_2^{-1} = 1_N, \\ &\Rightarrow g_1^g = g_2, \end{aligned}$$

that is  $g_1$  is conjugate to  $g_2$  in  $S_n$ , which is a contradiction. It follows that  $(d_1 + M)g_1$  is not conjugate to  $(d_2 + M)g_2$ .  $\square$

**Proposition 8.1.3** *Let  $H$  be a normal subgroup of a group  $G$ . Let  $a, b, g \in G$ .*

- (i) *If  $a$  and  $b$  are conjugate in  $G$ , then  $aH$  and  $bH$  are conjugate in  $G/H$ .*
- (ii) *Further,  $ab^{-1} \in H$  if and only if  $agH = bgH$ .*

**Proof.** The proof is trivial.  $\square$

**Corollary 8.1.4** *Let  $dg, d'g \in B(m, n)$ , where  $d, d' \in N$  and  $g \in S_n$ .*

- (i) *If  $dg$  and  $d'g$  are conjugate in  $B(m, n)$ , then  $(d + M)g$  and  $(d' + M)g$  are conjugate in  $B_Q(m, n)$ .*
- (ii) *Further,  $d - d' \in M$  if and only if  $(d + M)g = (d' + M)g$ .*

**Proof.**

- (i) Since  $M$  is normal in  $B(m, n)$ , by part (i) of Proposition 8.1.3, if  $dg$  is conjugate to  $d'g$  in  $B(m, n)$  then  $dgM$  is conjugate to  $d'gM$  in  $B(m, n)/M$ . Now with the correspondence between  $dgM \in B(m, n)/M$  and  $(d + M)g \in B_Q(m, n)$  given by Theorem 8.1.1, it follows that  $(d + M)g$  is conjugate to  $(d' + M)g$  in  $B_Q(m, n)$ .
- (ii) Since  $M$  is normal in  $B(m, n)$  by part (ii) of Proposition 8.1.3, we have  $dd'^{-1} \in M$  if and only if  $dgM = d'gM$  in  $B(m, n)/M$ . Now by Theorem 8.1.1 we have  $d - d' \in M$  if and only if  $(d + M)g = (d' + M)g$  in  $B_Q(m, n)$  as required.  $\square$

## 8.2 Fischer-Clifford Matrices of $B_Q(m, n)$

In this section, we adapt the combinatorial method for constructing the Fischer-Clifford matrices of  $B(m, n)$  described in Chapter 4 to construct the Fischer-Clifford matrices of the group  $B_Q(m, n)$ , where  $Q = \overline{\mathbb{Z}_m^n}$  is a quotient of  $N = \mathbb{Z}_m^n$ . In sequel, we present some results which describe the construction of the Fischer-Clifford matrices of  $B_Q(m, n)$ , from those of  $B(m, n)$ . For  $g \in S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ , let  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  denote the Fischer-Clifford matrix of  $B_Q(m, n)$ .

**Proposition 8.2.1** *Let  $d, d' \in N$  such that  $d - v = d'$  where  $v = \sum_{s=1}^n l e_s \in M$ , for some  $l \in \mathbb{Z}_m$ . Let  $dg$  and  $d'g$  be class representatives of two distinct conjugacy classes of  $B(m, n)$  respectively indexed by  $\beta$  and  $\mu$ , where  $\beta$  and  $\mu$  are  $m$ -sets of partition  $[\lambda]$  indexing columns of a Fischer-Clifford matrix  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_m})$ , corresponding to a conjugacy class of  $B(m, n)$ . Then  $\beta$  and  $\mu$  index the same column in  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_m})$ , corresponding to a conjugacy class of  $B_Q(m, n)$  with class representative  $(d + M)g = (d' + M)g$ .*

**Proof.** We have  $d - v = d'$  implies  $(d - v) + M = d' + M$ , and hence  $d + M = d' + M$  since  $v \in M$ . Now since  $d + M = d' + M$  we have  $(d + M)g = (d' + M)g$ . Thus  $(d + M)g = (d' + M)g$  is a representative of a conjugacy class in  $B_Q(m, n)$  indexed by both  $\beta$  and  $\mu$ . It follows that  $\beta$  and  $\mu$  index the same column in  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_m})$ .  $\square$

**Remark 8.2.2** *We note that the statement  $d - v = d'$  in Proposition 8.2.1 may be replaced by  $d + v = d'$  since the elements  $(d - v) + M$  and  $(d + v) + M$  are equal in  $B_Q(m, n)$ . Further taking quotients of  $B(m, n)$  by  $M$  is equivalent to calculating for each  $d \in N$ , the elements  $d + v$  for all  $v \in M$ , and then multiplying these by representatives of conjugacy classes of  $S_n$ .*

Let  $(d' + M)g = (d + M)g$ , where  $g \in S_n$ ,  $d, d' \in N$ . Then it is clear from Remark 8.2.2 that the type of  $d'g$  may be determined by considering the type of  $(d + v)g$ . Thus we have the following definition.

**Definition 8.2.3** *Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots n^{\lambda_n})$ . Let  $\mu = (b_{is})$  and  $\beta = (a_{is})$ ,  $i = 1, 2, \dots, m$ ,  $s = 1, 2, \dots, n$ , be  $m$ -sets of partition  $[\lambda]$ . Then  $\beta$  and  $\mu$  are said to be  **$g$ -equivalent** if  $\text{Typ}(dg) = \beta$  and  $\text{Typ}((d + v)g) = \mu$  for some  $d \in N$  and  $v \in M$ . However if  $\text{Typ}(dg) = \text{Typ}((d + v)g) = \beta$ , then  $\beta$  is said to be **self  $g$ -equivalent**. For  $m = 2$ , we note that  $g$ -equivalence of 2-sets of partition  $[\lambda]$  is the same as odd equivalence of 2-sets of partition  $[\lambda]$ .*

It follows from Proposition 8.1.4 that some conjugacy classes of  $B(m, n)$  produce one conjugacy class of  $B(m, n)/M$ , and by Theorem 8.1.1 this also happens to these conjugacy classes of  $B(m, n)$  in  $B_Q(m, n)$ . For this reason we will take the type  $\text{Typ}((d + M)g)$  of  $(d + M)g \in B_Q(m, n)$  to be the type  $\text{Typ}(dg)$  of  $dg$ , where  $dg$  is a representative of one of the conjugacy classes of  $B(m, n)$  producing the one conjugacy

class of  $B_Q(m, n)$  with representative  $(d + M)g$ . The following result describes the  $m$ -sets of partition  $[\lambda]$  which index the columns of the Fischer-Clifford matrices of  $B_Q(m, n)$  and therefore the types for the conjugacy classes of  $B_Q(m, n)$ .

**Theorem 8.2.4** *Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots n^{\lambda_n})$ . Let  $\beta$  be a representative of the set of  $g$ -equivalent  $m$ -sets of partition  $[\lambda]$ . Then  $\beta$  (or equivalently the set of  $g$ -equivalent  $m$ -sets of partition  $[\lambda]$ ) indexes a column of the Fischer-Clifford matrix  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of  $B_Q(m, n)$ . Furthermore*

- (i) *if  $\lambda_s \neq 0$  for some  $s \neq mk$ ,  $k \in \mathbb{N} \cup \{0\}$ , then the columns of  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  are indexed by representatives of sets of  $g$ -equivalent  $m$ -sets of partition  $[\lambda]$  and are in correspondence with the columns of  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  that are indexed by  $g$ -equivalent  $m$ -sets of partition  $[\lambda]$ ,*
- (ii) *if  $\lambda_s = 0$  for all  $s \neq mk$ ,  $k \in \mathbb{N} \cup \{0\}$ , then the columns of  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  are indexed by  $m$ -sets of partition  $[\lambda]$  and are in 1-1 correspondence with the columns of  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .*

**Proof.**

- (i) Let  $\mu = (b_{is})$  be  $g$ -equivalent to  $\beta = (a_{is})$ ,  $i = 1, 2, \dots, m$ ,  $s = 1, 2, \dots, n$ . Fix  $i$ . Then by definition of  $g$ -equivalence of  $m$ -sets of partition  $[\lambda]$ , we have  $b_{i's} = a_{is}$  for  $s \neq mk$ , where  $i' = (i + j)(\text{mod } m)$ ,  $j \in \mathbb{N} \cup \{0\}$  and  $b_{is} = a_{is}$  for  $s = mk$ . But by Proposition 8.2.1,  $j$  is the residue  $\text{mod } m$  of the sum of the powers of an  $m^{\text{th}}$  root of unity due to adding  $v = \sum_{z=1}^n l e_z$  such that  $d + v = d'$ , and  $dg$  and  $d'g$  are representatives of the conjugacy classes of  $B(m, n)$  indexed by  $\beta$  and  $\mu$  respectively. We just need to show that the addition of  $v$  to  $d$  does not change the residue  $\text{mod } m$  in  $d'$  for cycles of  $dg$  of length  $s = mk$  but changes for cycles of length  $s \neq mk$ . Without loss of generalization, let  $\bar{g} = dg$  contain a cycle  $\bar{g}_1 = d_1 g_1$  of length  $s \neq mk$  and another cycle  $\bar{g}_2 = d_2 g_2$  of length  $s = mk$ , each cycle such that for its associated powers  $k_{ij}$  we have  $\sum_{j=1}^s k_{ij} \equiv (i-1)(\text{mod } m)$ . We see that for each cycle  $d_1 g_1$  such that for its associated powers  $k_{ij}$  we have  $\sum_{j=1}^s k_{ij}$  is equal to say  $t$ , the addition of  $v = \sum_{z=1}^n l e_z$  to  $d_1$  yields a cycle of the same length  $s \neq mk$  such that for its associated powers  $k_{ij}$  we have  $\sum_{j=1}^s k_{ij}$  is equal to  $t + sl = t + (mk + k')l$  for some positive integer  $k' < m$ . This is so since the power  $l$  is added  $s = mk + k'$  times to  $t$  for the length  $s = mk + k'$ . Now

$t + (mk + k')l$  is congruent, mod  $m$ , to some  $k'' \neq (i - 1) \in \mathbb{Z}_m$ . This re-arranges cyclically the  $a_{is}$  at  $s \neq mk$  of  $\beta$  such that  $a_{is}$  becomes  $a_{i's}$  with  $i' = i + sl$  to give  $\mu$ . Thus  $b_{i's} = a_{is}$ . But for the cycle  $d_2g_2$  such that for its associated powers  $k_{ij}$  we have  $\sum_{j=1}^s k_{ij}$  is equal to say  $u$  and congruent to  $(i - 1)(\text{mod } m)$ , the addition of  $v = \sum_{z=1}^n le_z$  to  $d_2$  yields a cycle of the same length  $s = mk$  such that for its associated powers  $k_{ij}$  we have  $\sum_{j=1}^s k_{ij}$  is equal to  $u + mkl$ , which still is congruent to  $(i - 1)(\text{mod } m)$ . Further since in the type of  $dg \in B(m, n)$  we talk about cycles of  $dg$ , the  $m$ -set of partition  $[\lambda]$  indexing a class of  $B(m, n)$  containing  $dg$  must be expressed in terms of cycles of  $g$ , the  $m$ -sets of partition  $[\lambda]$  indexing classes of  $B_Q(m, n)$  are said to be  $g$ -equivalent (see Example 8.2.7 below).

- (ii) If  $\lambda_s = 0$  for all  $s \neq mk$  in  $[\lambda]$ , then there are no sets of  $g$ -equivalent  $m$ -sets of partition  $[\lambda]$ . Now with only the cycles of length  $s = mk$ , by the last part of the proof of part(i) of this theorem, for each cycle  $d_2g_2$  of length  $s = mk$ , the addition of  $v = \sum_{z=1}^n le_z$  to  $d_2$  yields a cycle of the same length  $s = mk$  such that for its associated powers  $k_{ij}$  we have  $\sum_{j=1}^s k_{ij}$  is equal to  $u + mkl$  which is congruent to  $(i - 1)(\text{mod } m)$ , that is  $d - d' \notin M$  or equivalently  $d + v \neq d'$ . Thus for any two columns of  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  corresponding to class representatives  $dg$  and  $d'g$ , we have  $(d + M)g \neq (d' + M)g$ . It follows that for  $g$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  where  $\lambda_s = 0$  for all  $s \neq mk$  in  $[\lambda]$ , each  $m$ -set of partition  $[\lambda]$  indexing a column of  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  indexes a column of  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .  $\square$

**Remark 8.2.5** We note that a set of  $g$ -equivalent  $m$ -sets of partition  $[\lambda]$  may contain only a self  $g$ -equivalent  $m$ -set of partition  $[\lambda]$  considered as its own representative. In such a case the self  $g$ -equivalent  $m$ -set of partition  $[\lambda]$  indexes a unique column of  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . Further in the case where  $\lambda_s = 0$  for all  $s \neq mk$  in  $[\lambda]$ , each  $m$ -set of partition  $[\lambda]$  indexes a unique column of  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .

**Examples 8.2.6** We give the  $m$ -sets of partition  $[\lambda]$  indexing the columns of the Fischer-Clifford matrices  $F_Q(2, 1^4 2)$  and  $F_Q(2, 2^1 4^1)$  of  $B_Q(2, 6)$ .

- (a) We note that  $\beta = ((0, 0), (4, 1))$  is a type for a conjugacy class of  $B(2, 6)$  coming from the class  $(1^4 2)$  of  $S_6$ . Now the elements of the class of  $B(2, 6)$  indexed by  $\beta$  contain  $a_{11} = 0$  cycles of length 1 each with no negative sign,  $a_{12} = 0$  cycles

of length 2 each with no negative sign,  $a_{21} = 4$  cycles of length 1 each with 1 negative sign and  $a_{22} = 1$  cycle of length 2 each with 1 negative sign. Let  $g = (5\ 6) \in S_6$ , it follows that  $dg$  with cycle notation

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix} \begin{pmatrix} 5\ 6 \\ -6\ 5 \end{pmatrix} = \sum_{s=1}^5 e_s g$$

is an element of the class of  $B(2, 6)$  indexed by  $\beta$ . Now we construct  $\mu$  from  $\beta$ , satisfying the hypothesis of Theorem 8.2.4. We require that in

$\mu = ((b_{11}, b_{12}), (b_{21}, b_{22}))$ , the condition  $b_{11} = a_{21}$  and  $b_{21} = a_{11}$  at  $s = 1$ ,  $b_{12} = a_{12}$  and  $b_{22} = a_{22}$  at  $s = 2$  must be satisfied. Thus  $b_{11} = 4$  and  $b_{21} = 0$  at  $s = 1$  while  $b_{12} = 0$  and  $b_{22} = 1$  at  $s = 2$  with  $\mu = ((4, 0), (0, 1))$ . Now we produce a class representative  $d'g$  of the class of  $B(2, 6)$  indexed by  $\mu$  which is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 5\ 6 \\ -6\ 5 \end{pmatrix} = \begin{pmatrix} 5\ 6 \\ -6\ 5 \end{pmatrix} = e_5 g.$$

In this case we see that  $d + d' = (\sum_{s=1}^5 e_s) + e_5 = \sum_{s=1}^4 e_s \notin M$ . This is not what we wish to obtain, however due to the non-unique choice of the number of negative signs *mod* 2 in the definition of the type  $\mu$  (see Definition 3.1.2) we must have considered a conjugate  $e_5 g$  of the necessary element  $d''g = e_6 g$ , where

$$\begin{pmatrix} 5\ 6 \\ 6\ -5 \end{pmatrix} M = e_6 g M.$$

Now  $d''$  is such that  $d + d'' = \sum_{s=1}^6 e_s$ . Alternatively it can be shown easily that

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix} \begin{pmatrix} 5\ 6 \\ -6\ 5 \end{pmatrix} M = \begin{pmatrix} 5\ 6 \\ 6\ -5 \end{pmatrix} M,$$

that is  $(\sum_{s=1}^5 e_s)gM = e_6 gM$  in which subtraction of  $\sum_{s=1}^6 e_s$  from  $d$  of  $dgM$  yields  $d''$  of  $d''gM$  in  $B_Q(2, 6)$ . Thus the two classes of  $B(2, 6)$  with class representatives  $(\sum_{s=1}^5 e_s)g$  and  $e_6 g$  produce one class with representative  $(\sum_{s=1}^5 e_s)gM = e_6 gM$  in  $B_Q(2, 6)$ . Further we note that the cycle  $(5\ 6)$  of length 2 retains the negative sign after subtracting  $\sum_{s=1}^6 e_s$  from  $d$  of  $dgM$ .

- (b) We consider the class of  $S_6$  of type  $(2^1 4^1)$ . Here none of the 2-sets of partition  $[2^1 4^1]$  are  $g$ -equivalent. By part (ii) of Theorem 8.2.4, all the 2-sets of partition  $[2^1 4^1]$ , that is

$$\begin{pmatrix} 1\ 1 \\ 0\ 0 \end{pmatrix}, \begin{pmatrix} 0\ 1 \\ 1\ 0 \end{pmatrix}, \begin{pmatrix} 1\ 0 \\ 0\ 1 \end{pmatrix}, \begin{pmatrix} 0\ 0 \\ 1\ 1 \end{pmatrix},$$

index the columns of  $F_Q(2, 2^1 4^1)$ .

**Example 8.2.7**

We note that in Examples 8.2.6, we may have used the definition of odd-equivalent 2-sets of partitions  $[1^4 2]$  and  $[2^1 4^1]$  to identify the indices for the columns of the Fischer-Clifford matrices  $F_Q(2, 1^4 2)$  and  $F_Q(2, 2^1 4^1)$  respectively. However we show that in the case of the Fischer-Clifford matrix  $F_Q(3, 1^1 2^2)$  where  $m = 3 (\neq 2)$ , we instead need to use the definition of  $g$ -equivalent 3-sets of partition  $[1^1 2^2]$ . We take say  $g = (2\ 3)(4\ 5)$  to be the representative of the class of  $S_5$  of type  $(1^1 2^2)$  and consider the class of  $B_Q(3, 5)$  of type  $((1, 2), (0, 0), (0, 0))$  also with representative  $g$ . To determine the 3-sets of partition  $[1^1 2^2]$  indexing the classes of  $B(3, 5)$  which produce a class of  $B_Q(3, 5)$  of type  $((1, 2), (0, 0), (0, 0))$ , we obtain the elements  $d + v$ , where  $v \in M = \langle l \sum_{z=1}^5 e_z \mid l \in \mathbb{Z}_3 \rangle$  and  $d = 1_N$  of  $g$ . Doing this we obtain the elements  $g$ ,  $\sum_{z=1}^5 e_z g$ ,  $2 \sum_{z=1}^5 e_z g$ , which are the representatives of the classes of  $B(3, 5)$  of types  $((1, 2), (0, 0), (0, 0))$ ,  $((0, 0), (1, 0), (0, 2))$  and  $((0, 0), (0, 2), (1, 0))$  respectively. These types are  $g$ -equivalent (not really odd equivalent). The conjugacy class of  $B(3, 5)$  of types equal to these three  $(2\ 3)(4\ 5)$ -equivalent 3-sets of partition  $[1^1 2^2]$  produce a class of  $B_Q(3, 5)$  of type  $((1, 2), (0, 0), (0, 0))$  and representative  $gM = \sum_{z=1}^5 e_z gM = 2 \sum_{z=1}^5 e_z gM$ . This is also true from the fact that the only classes of  $B_Q(3, 5)$  arising from the class of  $S_5$  of type  $(1^1 2^2)$  and containing elements of order 2 are those classes with representatives  $gM$ ,  $\sum_{z=1}^5 e_z gM$  and  $2 \sum_{z=1}^5 e_z gM$ .

Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots n^{\lambda_n})$ . Following from Theorem 8.2.4 and Remark 8.2.5, we will say that

- (i) when  $\lambda_s \neq 0$  for some  $s \neq mk$ , a representative (or equivalently the set) of  $g$ -equivalent  $m$ -sets of partition  $[\lambda]$  indexes a conjugacy class of  $B_Q(m, n)$ .
- (ii) when  $\lambda_s = 0$  for all  $s \neq mk$ , each  $m$ -set of partition  $[\lambda]$  indexes a conjugacy class of  $B_Q(m, n)$ .

**Proposition 8.2.8** *Two elements  $\bar{q}$  and  $\bar{q}'$  are conjugate in  $B_Q(m, n)$  if and only if  $Typ(\bar{q}) = Typ(\bar{q}')$ .*

**Proof.** The proof follows from Proposition 3.1.3, Theorem 8.2.4 and Remark 8.2.5.  $\square$ .

**Proposition 8.2.9** *Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} 3^{\lambda_3} \dots n^{\lambda_n})$ . Let  $\bar{q} = (d + M)g$  be an element of  $B_Q(m, n)$  of type  $\beta = (a_{is})$  or*

$$(1^{a_{11}} 2^{a_{12}} \dots n^{a_{1n}}, 1^{a_{21}} 2^{a_{22}} \dots n^{a_{2n}}, \dots, 1^{a_{p1}} 2^{a_{p2}} \dots n^{a_{pn}}).$$

(i) *If  $\lambda_s \neq 0$  for some  $s \neq mk$ , then the order of the centralizer of  $\bar{q} = (d + M)g$  in  $B_Q(m, n)$  is given by*

$$m^{1-r} \prod_{i=1}^m \prod_{s=1}^n (ms)^{a_{is}} (a_{is}!) \quad ,$$

*where either  $r = 1$  when  $\beta$  is a self  $g$ -equivalent  $m$ -set of partition  $[\lambda]$  or  $r = 2$  when  $\beta$  a representative of  $m$   $g$ -equivalent  $m$ -sets of partition  $[\lambda]$ .*

(ii) *If  $\lambda_s = 0$  for all  $s \neq mk$ , then the order of the centralizer of  $\bar{q} = (d + M)g$  in  $B_Q(m, n)$  is given by*

$$\prod_{i=1}^m \prod_{s=1}^n (ms)^{a_{is}} (a_{is}!) \quad .$$

**Proof.**

- (i) It is clear that in a set of  $g$ -equivalent  $m$ -sets of partition  $[\lambda]$  there is either one element or  $m$  elements. It follows from Theorem 8.2.4 that one or  $m$  conjugacy classes of  $B(m, n)$  of the same size produce a conjugacy class of  $B_Q(m, n)$  of the same size. Thus to obtain the orders of the centralizers in  $B_Q(m, n)$  we must divide the orders of the centralizers in  $B(m, n)$  by one or  $m$  accordingly. In the formula for the orders of centralizers in  $B_Q(m, n)$ , this division by one or  $m$  is taken care of by taking  $r = 1$  or  $r = 2$  respectively.
- (ii) This clearly follows from Theorem 8.2.4 as in this case each  $m$ -set of partition  $[\lambda]$  indexes a conjugacy class of  $B_Q(m, n)$  as in  $B(m, n)$ .  $\square$

The order of  $\bar{q} = (d + M)g$  in  $B_Q(m, n)$  will be calculated in different ways on a case by case basis. Thus we have several results that may be used to calculate the order of  $(d + M)g$  in  $B_Q(m, n)$ .

**Proposition 8.2.10** *Let  $\bar{g} = (d + M)g = dgM \in B_Q(m, n)$ ,  $\bar{g} = dg \in B(m, n)$  where  $g \in S_n$ . Let  $o(g) = u$ ,  $o(dg) = l$  and  $o((d + M)g) = k$ . Then*

(i)  $k \mid l$ .

(ii)  $u \mid k$ .

**Proof.**

(i) Since  $(dgM)^l = (dg)^l M = M$ ,  $o(dgM)$  divides  $l$ , that is  $k \mid l$ .

(ii) We have

$$\begin{aligned} [(dg)M]^k &= (dg)^k M \\ &= dd^g d^{g^2} d^{g^3} \dots d^{g^{k-1}} g^k M \\ &= M, \end{aligned}$$

which implies that  $dd^g d^{g^2} d^{g^3} \dots d^{g^{k-1}} g^k \in M$ . Now for some  $d' \in M$ ,

$$\begin{aligned} d' = dd^g d^{g^2} d^{g^3} \dots d^{g^{k-1}} g^k \in M &\Rightarrow g^k = (dd^g d^{g^2} d^{g^3} \dots d^{g^{k-1}})^{-1} d' \in N \\ &\Rightarrow g^k \in N \\ &\Rightarrow g^k \in N \cap S_n \\ &\Rightarrow g^k = 1_N \\ &\Rightarrow o(g) \mid k, \end{aligned}$$

that is  $u \mid k$ .  $\square$

In the following results we take  $m = p$  to be prime. Note that in this situation  $N$ ,  $M$  and  $Q$  will be elementary abelian  $p$ -groups.

**Corollary 8.2.11** *Let  $\bar{g} = dg \in B(p, n)$  where  $d \in N$  and  $g \in S_n$ . Let  $o(g) = u$ . Then either  $o(dg) = u$  or  $o(dg) = pu$ .*

**Proof.** Since  $p$  is prime, we have that  $B(p, n)$  is an extension of elementary abelian  $p$ -group  $N$ . Now by Theorem 1.2.14, the result follows.  $\square$

**Remark 8.2.12** Similarly applying Theorem 1.2.14 to  $B_Q(m, n)$ , we have that  $o((d + M)g) = u$  or  $o((d + M)g) = pu$ , where  $g \in S_n$  and  $u = o(g)$ . However  $o((d + M)g) \mid o(dg)$  by Proposition 8.2.10 part (i). In the case of  $o(dg) = u$ , then  $o((d + M)g) \mid u$  and since  $u = o(g)$  we have  $u \mid o((d + M)g)$  by Proposition 8.2.10 part (iii). Therefore  $o((d + M)g) = u$ .

If  $o(dg) = pu$ , then  $o((d + M)g) \in \{u, pu\}$  as was mentioned at the beginning of the Remark.

**Proposition 8.2.13** Let  $g \in S_n$  and  $o(g) = u$ . Let  $(d + M)g \in B_Q(p, n)$ .

- (i) If  $dd^g d^{g^2} d^{g^3} \dots d^{g^{u-1}} g^u \in M$ , then  $o(d + M)g = o(g) = u$ .
- (ii) If  $dd^g d^{g^2} d^{g^3} \dots d^{g^{u-1}} g^u \notin M$ , then  $o(d + M)g = pu$ .

**Proof.**

$$\begin{aligned} [(d + M)g]^u &= (dgM)^u \\ &= (dg)^u M \\ &= dd^g d^{g^2} d^{g^3} \dots d^{g^{u-1}} g^u M. \end{aligned}$$

- (i) In this case by (8.1) we have  $((d + M)g)^u = M$ , which implies that  $o((d + M)g) \mid u$ . Since by part (iii) of Proposition 8.2.10 we have  $u \mid o((d + M)g)$ , we deduce that  $o((d + M)g) = u = o(g)$ .
- (ii) Since  $dd^g d^{g^2} d^{g^3} \dots d^{g^{k-1}} g^k \notin M$ , we have  $dd^g d^{g^2} d^{g^3} \dots d^{g^{k-1}} g^k M \neq M$ , that is  $[(d + M)g]^u \neq M$ , and hence  $o((d + M)g) = u$ . Thus  $o((d + M)g) = pu$ .  $\square$

It is clear from the fore-going results that the determination of the orders of the elements of  $B_Q(m, n)$  is not so conclusive, it requires the analysis of both the structures of  $d + M \in Q$  and  $g \in S_n$ . We note that in [47] Mpono has used coset analysis to develop computer programmes in CAYLEY which are used for the computation of conjugacy classes and the orders of the class representatives of the group extension  $\overline{G} = N:G$  where  $N$  is an elementary abelian  $p$ -group for a prime  $p$  on which a linear group  $G$  acts. These programmes can similarly be applied to the extension  $B_Q(m, n)$  by considering a subgroup of a linear group which is isomorphic to  $S_n$ .

We now discuss the indices of the rows of  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  in terms of the indices of the rows of  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .

**Proposition 8.2.14** *The rows of  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  are indexed by even  $m$ -sets of partition  $[\lambda]$ .*

**Proof.** From the action of  $S_n$  on  $\text{Irr}(Q)$  in Theorem 3.3.25 and Remark 3.3.26 the inertia factor groups of  $B_Q(m, n)$  are the group  $S_{k_1} \times S_{k_2} \dots \times S_{k_m}$ , where  $(k_1, k_2, \dots, k_m) \in A(n, m)$ , such that  $\sum_{j=1}^m (j-1)k_j \equiv 0 \pmod{m}$ . Thus for each class of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  these inertia factor groups correspond to those

$$\delta = ((k_{11}, k_{12}, \dots, k_{1n}), (k_{21}, k_{22}, \dots, k_{2n}), \dots, (k_{m1}, k_{m2}, \dots, k_{mn}))$$

where  $\sum_{j=1}^m k_{js} = \lambda_s, \sum_{s=1}^n s k_{js} = k_j$ , and such that  $\sum_{j=1}^m \sum_{s=1}^n (j-1) s k_{js} \equiv 0 \pmod{m}$ . By Definition 2.2.2 such  $\delta$  are called even  $m$ -sets of partition  $[\lambda]$ .  $\square$

**Remark 8.2.15** *We note from Proposition 8.2.14 that if  $g$  is of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  where  $\lambda_s = 0$  for all  $s \neq mk$ , every  $m$ -set of partition  $[\lambda]$  satisfies the formula  $\sum_{j=1}^m \sum_{s=1}^n (j-1) s k_{js} \equiv 0 \pmod{m}$  with  $s = mk$  and is therefore even. This is as expected for in this case as we have seen above, all the  $m$ -sets of partition  $[\lambda]$  index the columns of  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ . It is now clear that in this case we have  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}) = F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .*

**Definition 8.2.16** *In the rest of this chapter, a **permissible column index** will mean a representative of a set of  $g$ -equivalent  $m$ -sets of partition  $[\lambda]$ . A column of  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  indexed by a permissible column index will be called a **permissible column**. Similarly a row of  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  indexed by an even  $m$ -set of partition  $[\lambda]$  will be called an **even row**, and the corresponding inertia factor group will be called an **even inertia factor group**.*

**Proposition 8.2.17** *Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .*

(i) *If  $\lambda_s \neq 0$  for some  $s \neq mk$ , then we obtain the Fischer-Clifford matrix  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  of  $B_Q(m, n)$  by*

- (a) *selecting permissible columns of  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  and*
- (b) *selecting even rows of the resultant matrix in (i) above.*

(ii) *If  $\lambda_s = 0$  for all  $s \neq mk$ , then  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}) = F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .*

**Proof.** Since the characters of  $B(m, n)$  are characters of  $B_Q(m, n)$  with  $M$  in their kernel, to obtain  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  we only need to select the entries of  $F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  corresponding to the necessary  $m$ -sets of partitions  $[\lambda]$  as described by Theorem 8.2.4 and Proposition 8.2.14. On using Definition 8.2.16, the result follows.  $\square$

We are now in a position to state a combinatorial method for constructing the Fischer-Clifford matrices of  $B_Q(m, n)$ .

**Theorem 8.2.18** *Let  $g$  be an element of  $S_n$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .*

(i) *Let  $\lambda_s \neq 0$  for some  $s \neq mk$  and  $\beta = (a_{is})$  be a permissible column index and  $\delta = (k_{is})$  be an even  $m$ -set of partition  $[\lambda]$ . An entry of the Fischer-Clifford matrix  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  which is in a column indexed by  $\beta$  and in a row indexed by  $\delta$  is given by*

$$f_{\lambda_0}[\beta, \delta] = \sum_{M(\beta, \delta, n)} \left( \prod_{s=1}^n \prod_{i=1}^m \begin{pmatrix} a_{is} \\ R_{a_i, \lambda_s} \end{pmatrix} \xi^{\sum_{i=1}^m \sum_{j=1}^m (i-1)(j-1) (\sum_{s=1}^n r_{ij}^s)} \right)$$

where  $\sum_{i=1}^m a_{is} = \sum_{j=1}^m k_{js} = \lambda_s$  and

- (a)  $\xi$  is an  $m^{\text{th}}$  root of unity,
- (b)  $M(\beta, \delta, n)$  is a set of magic matrices,
- (c)  $R_{a_i, \lambda_s}$  is the  $i^{\text{th}}$  row of a partial matrix at position  $s$  of a magic matrix in  $M(\beta, \delta, n)$ ,
- (d)  $\begin{pmatrix} a_{is} \\ r_{i1}^s \dots r_{im}^s \end{pmatrix}$  is a multi-nomial coefficient.

(ii) *Let  $\lambda_s = 0$  for all  $s \neq mk$ . Then  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}) = F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$ .*

**Proof.** The formulae in Theorems 4.3.1 and 4.3.3 for constructing the Fischer-Clifford matrices of  $B(m, n)$  are adapted to  $B_Q(m, n)$  by considering permissible column indices and even  $m$ -sets of partition  $[\lambda]$  respectively. The proof follows from Theorem 8.2.4, Definition 8.2.16 and Propositions 8.2.14 and 8.2.17.  $\square$

We will now apply the results above to construct the Fischer-Clifford matrices of the groups  $B_Q(2, 6)$  and  $B_Q(3, 5)$  as examples in the following sections.

### 8.3 The Group $B_Q(2, 6)$

In this section we construct the Fischer-Clifford matrices of  $B_Q(2, 6)$  and use these to construct its character table. Let  $N = 2^6$ ,  $M = \langle \sum_{s=1}^6 e_s \rangle$  and  $Q = 2^5 = \langle e_s + M \mid s = 1, 2, 3, 4, 5 \rangle$ , the quotient of  $N$  by  $M$ . Then  $Q$  may be regarded as a vector space of dimension 5 over  $\mathbb{Z}_2$ . The group  $B_Q(2, 6)$  is a split extension of  $Q$  by  $S_6$  (see Theorem 3.3.6). Explicitly  $B_Q(2, 6)$  is a Quotient of  $B(2, 6)$  by  $M$ . In Subsection 8.3.1, we use Proposition 8.2.17 and the Fischer-Clifford matrices of  $B(2, 6)$  given in Table 5.1 to construct the Fischer-Clifford matrices of  $B_Q(2, 6)$ . In Subsections 8.3.2 and 8.3.3, the Fischer-Clifford matrices of  $B_Q(2, 6)$  will be used to construct the character table of  $B_Q(2, 6)$ .

#### 8.3.1 The Fischer-Clifford Matrices of $B_Q(2, 6)$

We now construct the Fischer-Clifford matrices of  $B_Q(2, 6)$  from those of  $B(2, 6)$ . For each conjugacy class of  $S_6$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$ , we construct the Fischer-Clifford matrix  $F_Q(2, 1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$ .

##### Examples 8.3.1

We note from the results in Section 8.2 that the Fischer-Clifford matrices of  $B_Q(2, 6)$  will depend on the type  $(1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$  of the conjugacy class of  $S_n$ , that is being considered. Therefore by Proposition 8.2.17 part(ii), for the conjugacy classes of  $S_n$  of type such that  $\lambda_s = 0$  for all  $s \neq mk$ , we have  $F_Q(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n}) = F(m, 1^{\lambda_1} 2^{\lambda_2} \dots n^{\lambda_n})$  and these are listed in Table 8.1

However for those conjugacy classes of  $S_n$  of type such that  $\lambda_s \neq 0$  for some  $s \neq mk$ , we use part (i) of Proposition 8.2.17. We construct the Fischer-Clifford matrices  $F_Q(2, 1^6)$  and  $F_Q(2, 1^4 2)$  below.

- (a) By Proposition 8.2.17 and Definition 8.2.16, we determine the Fischer-Clifford matrix  $F_Q(2, 1^6)$ , from the Fischer-Clifford matrix  $F(2, 1^6)$ . From  $\lambda_1 = 6$  and  $m = 2$ , we obtain the following 2-sets of partition  $[1^6]$ :

$$\begin{pmatrix} 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix},$$

Table 8.1:  $B(2, 6)$  Fischer-Clifford matrices of  $B_Q(2, 6)$ 

$$\begin{aligned}
 F_Q(2, 2^3) = F(2, 2^3) &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
 F_Q(2, 6) = F(2, 6) &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
 F_Q(2, 2^4) = F(2, 2^4) &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}
 \end{aligned}$$

from which we choose the following four permissible column indices:

$$\begin{pmatrix} 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Hence the Fischer-Clifford matrix  $F_Q(2, 1^6)$  is of order 4. From the seven 2-sets of partition  $[1^6]$  we also select the following four even 2-sets of partition  $[1^6]$ :

$$\begin{pmatrix} 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix}.$$

Now the entries in the columns and rows of  $F(2, 1^6)$  respectively corresponding to the above four permissible column indices and four even 2-sets of partition  $[1^6]$  are selected (compare Tables 5.1 and 8.4) to give

$$F_Q(2, 1^6) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 15 & 5 & -1 & -3 \\ 15 & -5 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

- (b) By Proposition 8.2.17 and Definition 8.2.16, we determine the Fischer-Clifford matrix  $F_Q(2, 1^4 2)$  from the Fischer-Clifford matrix  $F(2, 1^4 2)$ . From  $\lambda_1 = 4$ ,  $\lambda_2 = 1$  and  $m = 2$ , we obtain the following ten 2-sets of partition  $[1^4 2]$ :

$$\begin{aligned} & \begin{pmatrix} 41 \\ 00 \end{pmatrix}, \begin{pmatrix} 31 \\ 10 \end{pmatrix}, \begin{pmatrix} 21 \\ 20 \end{pmatrix}, \begin{pmatrix} 11 \\ 30 \end{pmatrix}, \begin{pmatrix} 01 \\ 40 \end{pmatrix}, \\ & \begin{pmatrix} 40 \\ 01 \end{pmatrix}, \begin{pmatrix} 30 \\ 11 \end{pmatrix}, \begin{pmatrix} 20 \\ 21 \end{pmatrix}, \begin{pmatrix} 10 \\ 31 \end{pmatrix}, \begin{pmatrix} 00 \\ 41 \end{pmatrix}, \end{aligned}$$

from which we choose the following six permissible column indices:

$$\begin{pmatrix} 41 \\ 00 \end{pmatrix}, \begin{pmatrix} 31 \\ 10 \end{pmatrix}, \begin{pmatrix} 21 \\ 20 \end{pmatrix}, \begin{pmatrix} 40 \\ 01 \end{pmatrix}, \begin{pmatrix} 30 \\ 11 \end{pmatrix}, \begin{pmatrix} 20 \\ 21 \end{pmatrix}.$$

Hence the matrix  $F_Q(2, 1^4 2)$  is of order 6. From the ten 2-sets of partition  $[1^4 2]$ , we also select the following six even 2-sets of partition  $[1^4 2]$ :

$$\begin{pmatrix} 41 \\ 00 \end{pmatrix}, \begin{pmatrix} 21 \\ 20 \end{pmatrix}, \begin{pmatrix} 01 \\ 40 \end{pmatrix}, \begin{pmatrix} 40 \\ 01 \end{pmatrix}, \begin{pmatrix} 20 \\ 21 \end{pmatrix}, \begin{pmatrix} 00 \\ 41 \end{pmatrix}.$$

Now the entries in the columns and rows of  $F(2, 1^4 2)$  respectively corresponding to the above six permissible column indices and six even 2-sets of partition  $[1^4 2]$  are selected (compare Tables 5.1 and 8.4) to give

$$F_Q(2, 1^4 2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 0 & -2 & 6 & 0 & -2 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 6 & 0 & -2 & -6 & 0 & 2 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

The above was done for each conjugacy classes of  $S_6$  to give the Fischer-Clifford matrices of  $B_Q(2, 6)$ . The Fischer-Clifford matrices of  $B_Q(2, 6)$  are given in Table 8.4.

The forty two conjugacy classes of  $B_Q(2, 6)$ , including the respective orders of centralizers in  $B_Q(2, 6)$  are given in Table 8.2. In the following sections we use the Fischer-Clifford matrices in Table 8.4 to construct the character table of  $B_Q(2, 6)$ , which is given in Tables F.1 to F.4 of Appendix F.

Table 8.2: Conjugacy classes of  $B_Q(2, 6)$

$[\mathfrak{g}]_{S_6}$	permissible column indices	$(1^{a_{11}} 2^{a_{12}} \dots 6^{a_{16}}, 1^{a_{21}} 2^{a_{22}} \dots 6^{a_{26}})$	$\bar{g}$	$ C_G(\bar{g}) $	$  \bar{g}  $
$(1^6) = 1a$	(6, 0)	$(1^6, 1^0)$	1A	23040	1
	(5, 1)	$(1^5, 1^1)$	2A	3840	6
	(4, 2)	$(1^4, 1^2)$	2B	1536	15
	(3, 3)	$(1^3, 1^3)$	2C	2304	10
$(1^4 2) = 2a$	$((4, 1), (0, 0))$	$(1^4 2^1, 1^0 2^0)$	2D	768	30
	$((3, 1), (1, 0))$	$(1^3 2^1, 1^1 2^0)$	2E	192	120
	$((2, 1), (2, 0))$	$(1^2 2^1, 1^2 2^0)$	2F	256	90
	$((4, 0), (0, 1))$	$(1^4 2^0, 1^0 2^1)$	4A	768	30
	$((3, 0), (1, 1))$	$(1^3 2^0, 1^1 2^1)$	4B	192	120
	$((2, 0), (2, 1))$	$(1^2 2^0, 1^2 2^1)$	4C	256	90
$(1^2 2^2) = 2b$	$((2, 2), (0, 0))$	$(1^2 2^2, 1^0 2^0)$	2G	128	180
	$((1, 2), (1, 0))$	$(1^1 2^2, 1^1 2^0)$	2H	128	180
	$((2, 1), (0, 1))$	$(1^2 2^1, 1^0 2^1)$	4D	64	360
	$((1, 1), (1, 1))$	$(1^1 2^1, 1^1 2^1)$	4E	64	360
	$((2, 0), (0, 2))$	$(1^2 2^0, 1^0 2^2)$	4F	128	180
	$((1, 0), (1, 2))$	$(1^1 2^0, 1^1 2^2)$	4G	128	180
$(2^3) = 2c$	(3, 0)	$(2^3, 2^0)$	2I	384	60
	(2, 1)	$(2^2, 2^1)$	4H	128	180
	(1, 2)	$(2^1, 2^2)$	4I	128	180
	(0, 3)	$(2^0, 2^3)$	2J	384	60
$(1^3 3) = 3a$	$((3, 1), (0, 0))$	$(1^3 3^1, 1^0 3^0)$	3A	144	160
	$((2, 1), (1, 0))$	$(1^2 3^1, 1^1 3^0)$	6A	48	480
	$((1, 1), (2, 0))$	$(1^1 3^1, 1^2 3^0)$	6B	48	480
	$((0, 1), (3, 0))$	$(1^0 3^1, 1^3 3^0)$	6C	144	160
$(123) = 6a$	$((1, 1, 1), (0, 0, 0))$	$(1^1 2^1 3^1, 1^0 2^0 3^0)$	6D	24	960
	$((0, 1, 1), (1, 0, 0))$	$(1^0 2^1 3^1, 1^1 2^0 3^0)$	6E	24	960
	$((1, 0, 1), (0, 1, 0))$	$(1^1 2^0 3^1, 1^0 2^1 3^0)$	12A	24	960
	$((0, 0, 1), (1, 1, 0))$	$(1^0 2^0 3^1, 1^1 2^1 3^0)$	12B	24	960
$(3^2) = 3b$	(2, 0)	$(3^2, 3^0)$	3B	36	640
	(1, 1)	$(3^1, 3^1)$	6F	36	640
$(1^2 4) = 4a$	$((2, 1), (0, 0))$	$(1^2 4^1, 1^0 4^0)$	4J	32	720
	$((1, 1), (1, 0))$	$(1^1 4^1, 1^1 4^0)$	4K	32	720
	$((2, 0), (0, 1))$	$(1^2 4^0, 1^0 4^1)$	8A	32	720
	$((1, 0), (1, 1))$	$(1^1 4^0, 1^1 4^1)$	8B	32	720
$(24) = 4b$	$((1, 1), (0, 0))$	$(2^1 4^1, 2^0 4^0)$	4L	32	720
	$((0, 1), (1, 0))$	$(2^0 4^1, 2^1 4^0)$	4M	32	720
	$((1, 0), (0, 1))$	$(2^1 4^0, 2^0 4^1)$	8C	32	720
	$((0, 0), (1, 1))$	$(2^0 4^0, 2^1 4^1)$	8D	32	720
$(15) = 5a$	$((1, 1), (0, 0))$	$(1^1 5^1, 1^0 5^0)$	5A	10	2304
	$((0, 1), (1, 0))$	$(1^0 5^1, 1^1 5^0)$	10A	10	2304
$(6) = 6b$	(1, 0)	$(6^1, 6^0)$	6G	12	1920
	(0, 1)	$(6^0, 6^1)$	6H	12	1920

Table 8.3: Even 2-sets of partition  $[\lambda]$

$[\mathfrak{g}]_{S_6}$	even 2-sets of partition $[\lambda]$	type (row)
$(1^6) = 1a$	(6, 0) (4, 2) (2, 4) (0, 6)	$(1^6, 1^0)$ $(1^4, 1^2)$ $(1^2, 1^4)$ $(1^0, 1^6)$
$(1^4 2) = 2a$	$((4, 1), (0, 0))$ $((2, 1), (2, 0))$ $((0, 1), (4, 0))$ $((4, 0), (0, 1))$ $((2, 0), (2, 1))$ $((0, 0), (4, 1))$	$(1^4 2^1, 1^{0 2 0})$ $(1^2 2^1, 1^2 2^0)$ $(1^0 2^1, 1^4 2^0)$ $(1^4 2^0, 1^0 2^1)$ $(1^2 2^0, 1^2 2^1)$ $(1^0 2^0, 1^4 2^1)$
$(1^2 2^2) = 2b$	$((2, 2), (0, 0))$ $((0, 2), (2, 0))$ $((2, 1), (0, 1))$ $((0, 1), (2, 1))$ $((2, 0), (0, 2))$ $((0, 0), (2, 2))$	$(1^2 2^2, 1^0 2^0)$ $(1^0 2^2, 1^2 2^0)$ $(1^2 2^1, 1^0 2^1)$ $(1^0 2^1, 1^2 2^1)$ $(1^2 2^0, 1^0 2^2)$ $(1^0 2^0, 1^2 2^2)$
$(2^3) = 2c$	(3, 0) (2, 1) (1, 2) (0, 3)	$(2^3, 2^0)$ $(2^2, 2^1)$ $(2^1, 2^2)$ $(2^0, 2^3)$
$(1^3 3) = 3a$	$((3, 1), (0, 0))$ $((1, 1), (2, 0))$ $((2, 0), (1, 1))$ $((0, 0), (3, 1))$	$(1^3 3^1, 1^0 3^0)$ $(1^1 3^1, 1^2 3^0)$ $(1^2 3^0, 1^1 3^1)$ $(1^0 3^0, 1^3 3^1)$
$(123) = 6a$	$((1, 1, 1), (0, 0, 0))$ $((1, 0, 1), (0, 1, 0))$ $((0, 1, 0), (1, 0, 1))$ $((0, 0, 0), (1, 1, 1))$	$(1^1 2^1 3^1, 1^0 2^0 3^0)$ $(1^1 2^0 3^1, 1^0 2^1 3^0)$ $(1^0 2^1 3^0, 1^1 2^0 3^1)$ $(1^0 2^0 3^0, 1^1 2^1 3^1)$
$(3^2) = 3b$	(2, 0) (0, 2)	$(3^2, 3^0)$ $(3^0, 3^2)$
$(1^2 4) = 4a$	$((2, 1), (0, 0))$ $((0, 1), (2, 0))$ $((2, 0), (0, 1))$ $((0, 0), (2, 1))$	$(1^2 4^1, 1^0 4^0)$ $(1^0 4^1, 1^2 4^0)$ $(1^2 4^0, 1^0 4^1)$ $(1^0 4^0, 1^2 4^1)$
$(24) = 4b$	$((1, 1), (0, 0))$ $((0, 1), (1, 0))$ $((1, 0), (0, 1))$ $((0, 0), (1, 1))$	$(2^1 4^1, 2^0 4^0)$ $(2^0 4^1, 2^1 4^0)$ $(2^1 4^0, 2^0 4^1)$ $(2^0 4^0, 2^1 4^1)$
$(15) = 5a$	$((1, 1), (0, 0))$ $((0, 0), (1, 1))$	$(1^1 5^1, 1^0 5^0)$ $(1^0 5^0, 1^1 5^1)$
$(6) = 6b$	(1, 0) (0, 1)	$(6^1, 6^0)$ $(6^0, 6^1)$

Table 8.4: Fischer-Clifford matrices of  $B_Q(2, 6)$ 

$F_Q(2, 1^6) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 15 & 5 & -1 & -3 \\ 15 & -5 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix}$	$F_Q(2, 2^3) = F(2, 2^3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix}$
$F_Q(2, 1^4 2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 0 & -2 & 6 & 0 & -2 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 6 & 0 & -2 & -6 & 0 & 2 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}$	$F_Q(2, 6) = F(2, 6) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$F_Q(2, 1^2 2^2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 2 & 2 & 0 & 0 & -2 & -2 \\ 2 & -2 & 0 & 0 & -2 & 2 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix}$	$F_Q(2, 3^2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$F_Q(2, 1^3 3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 3 \\ 3 & 1 & -1 & -3 \\ 1 & -1 & 1 & -1 \end{pmatrix}$	$F_Q(2, 1^2 4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$
$F_Q(2, 1 2 3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$	$F_Q(2, 2 4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$
$F_Q(2, 1 5) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	

### 8.3.2 The Inertia Factor Groups of $B_Q(2, 6)$

We know that  $S_6$  acts on  $Irr(2^6)$  producing seven orbits with the inertia factor groups  $S_6, S_5 \times S_1, S_4 \times S_2, S_3 \times S_3, S_2 \times S_4, S_5$  and  $S_6$ . Now from Corollary 3.3.27, the inertia factor groups of  $B_Q(2, 6)$  are given by  $S_{k_1} \times S_{k_2}$ , where  $k_2$  is even. Hence the even inertia factor groups of  $B_Q(2, 6)$  are  $S_6, S_4 \times S_2, S_2 \times S_4$  and  $S_6$ . The character tables of these inertia factor groups are easily computed. The conjugacy classes of the inertia factor groups  $S_{k_1} \times S_{k_2}$  which fuse to a class of  $S_6$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 6^{\lambda_6})$  are of type  $(1^{k_{11}} 2^{k_{12}} \dots 6^{k_{16}}, 1^{k_{21}} 2^{k_{22}} \dots 6^{k_{26}})$ , where  $\sum_{j=1}^2 k_{js} = \lambda_s$ . The fusion of the conjugacy classes of the inertia factor groups into  $S_6$  is given in Table 6.4. We use the inertia factor groups of  $B_Q(2, 6)$  to construct the character table of  $B_Q(2, 6)$  in the following section.

### 8.3.3 The Character Table of $B_Q(2, 6)$

The character table of  $B_Q(2, 6)$  is constructed by multiplying the partial character tables of the inertia factor groups by the corresponding rows of the Fischer-Clifford matrices in Table 8.4 according to the fusions given in Table 6.4. For example, from Table 6.4 we see that the conjugacy classes  $(1^2 2, 1^2)$  and  $(1^4, 2)$  of  $S_4 \times S_2$  fuse to the class  $(1^4 2)$  of  $S_6$ . So we multiply the partial character table of  $S_4 \times S_2$  at the elements  $(1^2 2, 1^2)$  and  $(1^4, 2)$  with the rows 2 and 3 of the Fischer-Clifford matrix  $F_Q(2, 1^4 2)$  as follows:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \\ 2 & 0 \\ -2 & 0 \\ -3 & -1 \\ -3 & 1 \\ 3 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 6 & 0 & -2 & 6 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 1 & -1 & 5 & -1 & -3 \\ 5 & -1 & -3 & 7 & 1 & -1 \\ -5 & 1 & 3 & -7 & -1 & 1 \\ -7 & -1 & 1 & -5 & 1 & 3 \\ 2 & 2 & 2 & -2 & -2 & -2 \\ -2 & -2 & -2 & 2 & 2 & 2 \\ -9 & -3 & -1 & -3 & 3 & 5 \\ 3 & -3 & -5 & 9 & 3 & 1 \\ -3 & 3 & 5 & -9 & -3 & -1 \\ 9 & 3 & 1 & 3 & -3 & -5 \end{bmatrix},$$

which appears in block 2 of Table F.1 in the Appendix F under the class  $2a$  of  $S_6$ . This is done for each Fischer-Clifford matrix to give the full character table of  $B_Q(2, 6)$  given in Tables F.1 to F.4 in the Appendix F. The orders of centralizers of  $B_Q(2, 6)$

are listed in table 8.2 and are not included in the character table of  $B_Q(2, 6)$ . The correctness of the character table of  $B_Q(2, 6)$  has been tested in GAP.

## 8.4 The Group $B_Q(3, 5)$

In this section we construct the Fischer-Clifford matrices of  $B_Q(3, 5)$  and use these to construct its character table. Let  $N = 3^5$ ,  $M = \langle \sum_{s=1}^5 e_s \rangle$  and  $Q = 3^4 = \langle e_s + M \mid s = 1, 2, 3, 4 \rangle$ , the quotient of  $N$  by  $M$ . Then  $Q$  may be regarded as a vector space of dimension 4 over  $\mathbb{Z}_3$ . The group  $B_Q(3, 5)$  is a split extension of  $Q$  by  $S_5$  (see Theorem 3.3.6). Explicitly  $B_Q(3, 5)$  is a Quotient of  $B(3, 5)$  by  $M$ . In Subsection 8.4.1, we use Proposition 8.2.17 and the Fischer-Clifford matrices of  $B(3, 5)$  given in Tables 5.2 to 5.4 to construct the Fischer-Clifford matrices of  $B_Q(3, 5)$ . The Fischer-Clifford matrices of  $B_Q(3, 5)$  will be used to construct the character table of  $B_Q(3, 5)$  in Subsections 8.4.2 and 8.4.3.

### 8.4.1 The Fischer-Clifford Matrices of Group $B_Q(3, 5)$

In this subsection we construct the Fischer-Clifford matrices of the group  $B_Q(3, 5)$ , the split extension of  $Q = 3^4$  by  $S_5$ , as a second example on the groups  $B_Q(m, n)$ . We determine the Fischer-Clifford matrices of  $B_Q(3, 5)$  from those of  $B(3, 5)$ . The Fischer-Clifford matrices of  $B(3, 5)$  are given in Chapter 6. For each class of  $S_5$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 5^{\lambda_5})$  we construct the Fischer-Clifford matrix  $F_Q(3, 1^{\lambda_1} 2^{\lambda_2} \dots 5^{\lambda_5})$ .

#### Examples 8.4.1

Since  $3 \nmid 5$ , all the conjugacy classes of  $S_5$  are of type such that  $\lambda_s \neq 0$  for some  $s \neq 3k$ . Thus we use part (i) of Proposition 8.2.17, to construct only the Fischer-Clifford matrices  $F_Q(3, 1^5)$  and  $F_Q(3, 1^3 2)$  of  $B_Q(3, 5)$ .

- (a) For the Fischer-Clifford matrix  $F_Q(3, 1^5)$  we have  $\lambda_1 = 5$  and  $m = 3$ . From the twenty one 3-sets of partition  $[1^5]$  we choose the following seven permissible column indices:

$$\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

Thus the matrix  $F_Q(3, 1^5)$  is of order 7. From the twenty one 3-sets of partition  $[1^5]$ , we also choose the following seven even 3-sets of partition  $[1^5]$ , which index the rows of  $F_Q(3, 1^5)$ :

$$\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}.$$

The entries in the columns and rows of  $F(3, 1^5)$  respectively corresponding to the above seven permissible column indices and seven even 3-sets of partition  $[1^5]$  are selected (compare Tables 8.7 and 5.2). We obtain that

$$F_Q(3, 1^5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 20 & 8 & 8 & 2 & -1 & 2 & -4 \\ 10 & 2\xi - 4\xi^2 & -4\xi + 2\xi^2 & 5\xi + 2\xi^2 & 1 & 2\xi + 5\xi^2 & 1 \\ 10 & -4\xi + 2\xi^2 & 2\xi - 4\xi^2 & 2\xi + 5\xi^2 & 1 & 5\xi + 2\xi^2 & 1 \\ 30 & -6 & -6 & 3 & -6 & 3 & 3 \\ 5 & 4\xi + \xi^2 & \xi + 4\xi^2 & -2\xi + \xi^2 & 2 & \xi - 2\xi^2 & -1 \\ 5 & \xi + 4\xi^2 & 4\xi + \xi^2 & \xi - 2\xi^2 & 2 & -2\xi + \xi^2 & -1 \end{pmatrix}$$

which is given in Table 8.7.

- (b) For the Fischer-Clifford matrix  $F_S(3, 1^3 2)$  we have  $\lambda_1 = 3, \lambda_2 = 1$  and  $m = 3$ . From the thirty 3-sets of partition  $[1^3 2]$  we choose the following ten permissible column indices:

$$\begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \end{pmatrix}.$$

Thus the Fischer-Clifford matrix  $F_Q(3, 1^3 2)$  is of order 10. Similarly from the thirty 3-sets of partition  $[1^3 2]$  we choose the following ten even 3-sets of partition  $[1^3 2]$ , which index the rows of  $F_Q(3, 1^3 2)$ :

$$\begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The entries in the columns and rows of  $F(3, 1^3 2)$  respectively corresponding to the above ten permissible column indices and ten even 3-sets of partition  $[1^3 2]$  are selected (compare Tables 8.7 and 5.5). We obtain that

$$F_Q(3, 1^3 2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 0 & 0 & 0 & -3 & 0 & 6 & 0 & 0 & 6 \\ 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & 1 \\ 1 & \xi^2 & \xi & \xi & 1 & \xi^2 & 1 & \xi^2 & \xi & 1 \\ 3 & -\xi - 2\xi^2 & -2\xi - \xi^2 & \xi - \xi^2 & 0 & -\xi + \xi^2 & 3\xi & 2\xi + \xi^2 & \xi + 2\xi^2 & 3\xi^2 \\ 3 & -\xi + \xi^2 & \xi - \xi^2 & \xi + 2\xi^2 & 0 & 2\xi + \xi^2 & 3\xi & -\xi - 2\xi^2 & -2\xi - \xi^2 & 3\xi^2 \\ 3 & 2\xi + \xi^2 & \xi + 2\xi^2 & -2\xi - \xi^2 & 0 & -\xi - 2\xi^2 & 3\xi & -\xi + \xi^2 & \xi - \xi^2 & 3\xi^2 \\ 3 & -2\xi - \xi^2 & -\xi - 2\xi^2 & -\xi + \xi^2 & 0 & \xi - \xi^2 & 3\xi^2 & \xi + 2\xi^2 & 2\xi + \xi^2 & 3\xi \\ 3 & \xi - \xi^2 & -\xi + \xi^2 & 2\xi + \xi^2 & 0 & \xi + 2\xi^2 & 3\xi^2 & -2\xi - \xi^2 & -\xi - 2\xi^2 & 3\xi \\ 3 & \xi + 2\xi^2 & 2\xi + \xi^2 & -\xi - 2\xi^2 & 0 & -2\xi - \xi^2 & 3\xi^2 & \xi - \xi^2 & -\xi + \xi^2 & 3\xi \end{pmatrix}$$

which is given in Table 8.7.

The above was done for all the rest of the conjugacy classes of  $S_5$  to give the Fischer-Clifford matrices of  $B_Q(3, 5)$ . The Fischer-Clifford matrices of  $B_Q(3, 5)$  are given in Table 8.7. The thirty six conjugacy classes of  $B_Q(3, 5)$ , including the respective orders of centralizers of  $B_Q(3, 5)$  are given in Table 8.5. In the following sections we use the Fischer-Clifford matrices given in Table 8.7 to construct the character table of  $B_Q(3, 5)$  given in the Appendix G.

Table 8.5: Conjugacy classes of  $B_Q(3, 5)$

$[\mathfrak{g}]_{S_n}$	permissible column indices	$(1^{a_{11}} 2^{a_{12}} \dots 5^{a_{15}}, 1^{a_{21}} 2^{a_{22}} \dots 5^{a_{25}}, 1^{a_{31}} 2^{a_{32}} \dots 5^{a_{35}})$	$\bar{g}$	$ C_G(\bar{g}) $	$  \bar{g}  $
$(1^5) = 1a$	(5, 0, 0)	$(1^5, 1^0, 1^0)$	1A	9720	1
	(4, 1, 0)	$(1^4, 1^1, 1^0)$	3A	1944	5
	(4, 0, 1)	$(1^4, 1^0, 1^1)$	3B	1944	5
	(3, 2, 0)	$(1^3, 1^2, 1^0)$	3C	972	10
	(3, 1, 1)	$(1^3, 1^1, 1^1)$	3D	486	20
	(3, 0, 2)	$(1^3, 1^0, 1^2)$	3E	972	10
	(2, 2, 1)	$(1^2, 1^2, 1^1)$	3F	324	30
	$(1^3 2) = 2a$	$((3, 1), (0, 0), (0, 0))$	$(1^3 2^1, 1^0 2^0, 1^0 2^0)$	2A	324
$((2, 1), (1, 0), (0, 0))$		$(1^2 2^1, 1^1 2^0, 1^0 2^0)$	6A	108	90
$((2, 1), (0, 0), (1, 0))$		$(1^2 2^1, 1^0 2^0, 1^1 2^0)$	6B	108	90
$((1, 1), (2, 0), (0, 0))$		$(1^1 2^1, 1^2 2^0, 1^0 2^0)$	6C	108	90
$((1, 1), (1, 0), (1, 0))$		$(1^1 2^1, 1^1 2^0, 1^1 2^0)$	6D	54	180
$((1, 1), (0, 0), (2, 0))$		$(1^1 2^1, 1^0 2^0, 1^2 2^0)$	6E	108	90
$((0, 1), (3, 0), (0, 0))$		$(1^0 2^1, 1^3 2^0, 1^0 2^0)$	6F	324	30
$((0, 1), (2, 0), (1, 0))$		$(1^0 2^1, 1^2 2^0, 1^1 2^0)$	6G	108	90
$((0, 1), (1, 0), (2, 0))$		$(1^0 2^1, 1^1 2^0, 1^2 2^0)$	6H	108	90
$((0, 1), (0, 0), (3, 0))$		$(1^0 2^1, 1^0 2^0, 1^3 2^0)$	6I	324	30
$(1^1 2^2) = 2b$		$((1, 2), (0, 0), (0, 0))$	$(1^1 2^2, 1^0 2^0, 1^0 2^0)$	2B	72
	$((0, 2), (1, 0), (0, 0))$	$(1^0 2^2, 1^1 2^0, 1^0 2^0)$	6J	72	135
	$((0, 2), (0, 0), (1, 0))$	$(1^0 2^2, 1^0 2^0, 1^1 2^0)$	6K	72	135
	$((1, 1), (0, 1), (0, 0))$	$(1^1 2^1, 1^0 2^1, 1^0 2^0)$	6L	36	270
	$((0, 1), (1, 1), (0, 0))$	$(1^0 2^1, 1^1 2^1, 1^0 2^0)$	6M	36	270
	$((0, 1), (0, 1), (1, 0))$	$(1^0 2^1, 1^0 2^1, 1^1 2^0)$	6N	36	270
	$(1^2 3) = 3a$	$((2, 1), (0, 0), (0, 0))$	$(1^2 3^1, 1^0 3^0, 1^0 3^0)$	3G	54
$((1, 1), (1, 0), (0, 0))$		$(1^1 3^1, 1^1 3^0, 1^0 3^0)$	3H	27	360
$((2, 0), (0, 1), (0, 0))$		$(1^2 3^0, 1^0 3^1, 1^0 3^0)$	9A	54	180
$((1, 0), (1, 1), (0, 0))$		$(1^1 3^0, 1^1 3^1, 1^0 3^0)$	9B	27	360
$((2, 0), (0, 0), (0, 1))$		$(1^2 3^0, 1^0 3^0, 1^0 3^1)$	9C	54	180
$((1, 0), (1, 0), (0, 1))$		$(1^1 3^0, 1^1 3^0, 1^0 3^1)$	9D	27	360
$(23) = 6a$		$((1, 1), (0, 0), (0, 0))$	$(2^1 3^1, 2^0 3^0, 2^0 3^0)$	6O	18
	$((1, 0), (0, 1), (0, 0))$	$(2^1 3^0, 2^0 3^1, 2^0 3^0)$	18A	18	540
	$((1, 0), (0, 0), (0, 1))$	$(2^1 3^0, 2^0 3^0, 2^0 3^1)$	18B	18	540
$(14) = 4a$	$((1, 1), (0, 0), (0, 0))$	$(1^1 4^1, 1^0 4^0, 1^0 4^0)$	4A	12	810
	$((0, 1), (1, 0), (0, 0))$	$(1^0 4^1, 1^1 4^0, 1^0 4^0)$	12A	12	810
	$((0, 1), (0, 0), (1, 0))$	$(1^0 4^1, 1^0 4^0, 1^1 4^0)$	12B	12	810
$(5) = 5a$	(1, 0, 0)	$(5^1, 5^0, 5^0)$	5A	5	1944

Table 8.6: Even 3-sets of partition  $[\lambda]$

$[g]_{S_3}$	even 3-sets of partition $[\lambda]$	type (row)
$(1^5) = 1a$	(5, 0, 0) (3, 1, 1) (2, 3, 0) (2, 0, 3) (1, 2, 2) (0, 4, 1) (0, 1, 4)	$(1^5, 1^0, 1^0)$ $(1^3, 1^1, 1^1)$ $(1^2, 1^3, 1^0)$ $(1^2, 1^0, 1^3)$ $(1^1, 1^2, 1^2)$ $(1^0, 1^4, 1^1)$ $(1^0, 1^1, 1^4)$
$(1^3 2) = 2a$	$((3, 1), (0, 0), (0, 0))$ $((1, 1), (1, 0), (1, 0))$ $((0, 1), (3, 0), (0, 0))$ $((0, 1), (0, 0), (3, 0))$ $((2, 0), (1, 1), (0, 0))$ $((1, 0), (0, 1), (2, 0))$ $((0, 0), (2, 1), (1, 0))$ $((2, 0), (0, 0), (1, 1))$ $((1, 0), (2, 0), (0, 1))$ $((0, 0), (1, 0), (2, 0))$	$(1^3 2^1, 1^0 2^0, 1^0 2^0)$ $(1^1 2^1, 1^1 2^0, 1^1 2^0)$ $(1^0 2^1, 1^3 2^0, 1^0 2^0)$ $(1^0 2^1, 1^0 2^0, 1^3 2^0)$ $(1^2 2^0, 1^1 2^1, 1^0 2^0)$ $(1^1 2^0, 1^0 2^1, 1^2 2^0)$ $(1^0 2^0, 1^2 2^1, 1^1 2^0)$ $(1^2 2^0, 1^0 2^0, 1^1 2^1)$ $(1^1 2^0, 1^2 2^0, 1^0 2^1)$ $(1^0 2^0, 1^1 2^0, 1^2 2^0)$
$(1^1 2^2) = 2b$	$((1, 2), (0, 0), (0, 0))$ $((1, 0), (0, 1), (0, 1))$ $((0, 1), (1, 1), (0, 0))$ $((0, 0), (1, 0), (0, 2))$ $((0, 1), (0, 0), (1, 1))$ $((0, 0), (0, 2), (1, 0))$	$(1^1 2^2, 1^0 2^0, 1^0 2^0)$ $(1^1 2^0, 1^0 2^1, 1^0 2^1)$ $(1^0 2^1, 1^1 2^1, 1^0 2^0)$ $(1^0 2^0, 1^1 2^0, 1^0 2^2)$ $(1^0 2^1, 1^0 2^0, 1^1 2^1)$ $(1^0 2^0, 1^0 2^2, 1^1 2^0)$
$(1^2 3) = 3a$	$((2, 1), (0, 0), (0, 0))$ $((0, 1), (1, 0), (1, 0))$ $((2, 0), (0, 1), (0, 0))$ $((0, 0), (1, 1), (1, 0))$ $((2, 0), (0, 0), (0, 1))$ $((0, 0), (1, 0), (1, 1))$	$(1^2 3^1, 1^0 3^0, 1^0 3^0)$ $(1^0 3^1, 1^1 3^0, 1^1 3^0)$ $(1^2 3^0, 1^0 3^1, 1^0 3^0)$ $(1^0 3^0, 1^1 3^1, 1^1 3^0)$ $(1^2 3^0, 1^0 3^0, 1^0 3^1)$ $(1^0 3^0, 1^1 3^0, 1^1 3^1)$
$(2 3) = 6a$	$((1, 1), (0, 0), (0, 0))$ $((1, 0), (0, 1), (0, 0))$ $((1, 0), (0, 0), (0, 1))$	$(2^1 3^1, 2^0 3^0, 2^0 3^0)$ $(2^1 3^0, 2^0 3^1, 2^0 3^0)$ $(2^1 3^0, 2^0 3^0, 2^0 3^1)$
$(1 4) = 4a$	$((1, 1), (0, 0), (0, 0))$ $((0, 0), (0, 1), (1, 0))$ $((0, 0), (1, 0), (0, 1))$	$(1^1 4^1, 1^0 4^0, 1^0 4^0)$ $(1^0 4^0, 1^0 4^1, 1^1 4^0)$ $(1^0 4^0, 1^1 4^0, 1^0 4^1)$
$(5) = 5a$	(1, 0, 0)	$(5^1, 5^0, 5^0)$

Table 8.7: Fischer-Clifford matrices of  $B_Q(3, 5)$

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$$F_Q(3, 1^5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 20 & 8 & 8 & 2 & -1 & 2 & -4 \\ 10 & 2\xi - 4\xi^2 & -4\xi + 2\xi^2 & 5\xi + 2\xi^2 & 1 & 2\xi + 5\xi^2 & 1 \\ 10 & -4\xi + 2\xi^2 & 2\xi - 4\xi^2 & 2\xi + 5\xi^2 & 1 & 5\xi + 2\xi^2 & 1 \\ 30 & -6 & -6 & 3 & -6 & 3 & 3 \\ 5 & 4\xi + \xi^2 & \xi + 4\xi^2 & -2\xi + \xi^2 & 2 & \xi - 2\xi^2 & -1 \\ 5 & \xi + 4\xi^2 & 4\xi + \xi^2 & \xi - 2\xi^2 & 2 & -2\xi + \xi^2 & -1 \end{pmatrix}$$

$$F_Q(3, 1^3 2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 0 & 0 & 0 & -3 & 0 & 6 & 0 & 0 & 6 \\ 1 & \xi & \xi^2 & \xi^2 & 1 & \xi & 1 & \xi & \xi^2 & 1 \\ 1 & \xi^2 & \xi & \xi & 1 & \xi^2 & 1 & \xi^2 & \xi & 1 \\ 3 & -\xi - 2\xi^2 & -2\xi - \xi^2 & \xi - \xi^2 & 0 & -\xi + \xi^2 & 3\xi & 2\xi + \xi^2 & \xi + 2\xi^2 & 3\xi^2 \\ 3 & -\xi + \xi^2 & \xi - \xi^2 & \xi + 2\xi^2 & 0 & 2\xi + \xi^2 & 3\xi & -\xi - 2\xi^2 & -2\xi - \xi^2 & 3\xi^2 \\ 3 & 2\xi + \xi^2 & \xi + 2\xi^2 & -2\xi - \xi^2 & 0 & -\xi - 2\xi^2 & 3\xi & -\xi + \xi^2 & \xi - \xi^2 & 3\xi^2 \\ 3 & -2\xi - \xi^2 & -\xi - 2\xi^2 & -\xi + \xi^2 & 0 & \xi - \xi^2 & 3\xi^2 & \xi + 2\xi^2 & 2\xi + \xi^2 & 3\xi \\ 3 & \xi - \xi^2 & -\xi + \xi^2 & 2\xi + \xi^2 & 0 & \xi + 2\xi^2 & 3\xi^2 & -2\xi - \xi^2 & -\xi - 2\xi^2 & 3\xi \\ 3 & \xi + 2\xi^2 & 2\xi + \xi^2 & -\xi - 2\xi^2 & 0 & -2\xi - \xi^2 & 3\xi^2 & \xi - \xi^2 & -\xi + \xi^2 & 3\xi \end{pmatrix}$$

$$F_Q(3, 1^2 3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 2 & -1 & 2 & -1 \\ 1 & 1 & \xi & \xi & \xi^2 & \xi^2 \\ 2 & -1 & 2\xi & -\xi & 2\xi^2 & -\xi^2 \\ 1 & 1 & \xi^2 & \xi^2 & \xi & \xi \\ 2 & -1 & 2\xi^2 & -\xi^2 & 2\xi & -\xi \end{pmatrix}$$

$$F_Q(3, 12^2) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2\xi & 2\xi^2 & -\xi^2 & -1 & -\xi \\ 2 & 2\xi^2 & 2\xi & -\xi & -1 & -\xi^2 \\ 1 & \xi^2 & \xi & \xi & 1 & \xi^2 \\ 2 & 2 & 2 & -1 & -1 & -1 \\ 1 & \xi & \xi^2 & \xi^2 & 1 & \xi \end{pmatrix}$$

$$F_Q(3, 23) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi^2 & \xi \\ 1 & \xi & \xi^2 \end{pmatrix}$$

$$F_Q(3, 14) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{pmatrix}$$

$$F_Q(3, 5) = \begin{pmatrix} 1 \end{pmatrix}$$


---

### 8.4.2 The Inertia Factor Groups of $B_Q(3, 5)$

We know that  $S_5$  acts on  $\text{Irr}(3^5)$  producing twenty one orbits with the distinct inertia factor groups  $S_5, S_4, S_3 \times S_2, S_2 \times S_3, S_2 \times S_2$  and  $S_3$ . Now from Theorem 3.3.25 and Remark 3.3.26, the distinct even inertia factor groups of  $B_Q(3, 5)$  are  $S_5, S_3, S_2 \times S_3, S_2 \times S_2$  and  $S_4$ . The character tables of these inertia factor groups are easily computed. The conjugacy classes of the inertia factor groups  $S_{k_1} \times S_{k_2}$  which fuse to a class of  $S_5$  of type  $(1^{\lambda_1} 2^{\lambda_2} \dots 5^{\lambda_5})$  are of type  $(1^{k_{11}} 2^{k_{12}} \dots 5^{k_{15}}, 1^{k_{21}} 2^{k_{22}} \dots 5^{k_{25}}, 1^{k_{31}} 2^{k_{32}} \dots 5^{k_{35}})$ , where  $\sum_{j=1}^3 k_{js} = \lambda_s$ . The fusion of the conjugacy classes of the inertia factor groups into  $S_5$  is given in Table 6.9. We use the inertia factor groups of  $B_Q(3, 5)$  to construct the character table of  $B_Q(3, 5)$  in the following section.

### 8.4.3 The Character Table of $B_Q(3, 5)$

The character table of  $B_Q(3, 5)$  is constructed by multiplying the partial character tables of the inertia factor groups for example  $S_3$ , by the corresponding rows of the Fischer-Clifford matrices in Table 8.7 according to the fusions given in Table 6.9). For example, from Table 6.9 we see that the conjugacy class (12) of  $S_3$  fuses to the class  $(1^3 2)$  of  $S_5$ . So we multiply the partial character table of  $S_3$  at the element (12) with row 2 of the Fischer-Clifford matrix  $F_Q(3, 1^3 2)$  as follows:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 & 0 & -3 & 0 & 6 & 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 0 & -3 & 0 & 6 & 0 & 0 & 6 \\ -6 & 0 & 0 & 0 & 3 & 0 & -6 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which appears in block 2 of Table G.2 in the Appendix G under the class 2a of  $S_5$ . This is done for each Fischer-Clifford matrix of  $B_Q(3, 5)$  to give the full character table of  $B_Q(3, 5)$  given in Tables G.1 to G.3 in the Appendix G. The orders of centralizers of  $B_Q(3, 5)$  are listed in Table 8.5 and are not included in the character table of  $B_Q(3, 5)$ . The correctness of the character table of  $B_Q(3, 5)$  has been tested in GAP.

**Remark 8.4.2** We note that by Theorem 8.1.1 we can determine the character table of  $B_Q(m, n)$  from that of  $B(m, n)$ .

## Appendix A

### Character Table of $B(2, 6)$







Table A.4: Character table of  $B(2,6)$  (continued)

[g]	3b			4a					4b				5a				6b		
[g]	3D	6J	6K	4O	8A	4P	8B	4Q	8C	4R	8D	8E	4S	5a	10A	10B	5B	6L	12E
X1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	-1	-1
X3	2	2	2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0
X4	2	2	2	1	1	1	1	1	1	-1	-1	-1	-1	0	0	0	0	0	0
X5	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	1	1
X6	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	0	0	0	0	-1	-1
X7	0	0	0	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	0	0
X8	0	0	0	-1	-1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	0	0
X9	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1
X10	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
X11	-2	-2	-2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0
X12	0	0	0	2	0	-2	2	0	-2	0	0	0	0	1	-1	1	-1	0	0
X13	0	0	0	-2	0	2	-2	0	2	0	0	0	0	1	-1	1	-1	0	0
X14	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	-1	1	0	0
X15	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1	-1	1	0	0
X16	0	0	0	-2	0	2	-2	0	2	0	0	0	0	0	0	0	0	0	0
X17	0	0	0	2	0	-2	2	0	-2	0	0	0	0	0	0	0	0	0	0
X18	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1	0	0
X19	0	0	0	1	-1	1	1	-1	1	1	-1	1	-1	0	0	0	0	0	0
X20	0	0	0	-1	1	-1	-1	1	-1	-1	1	-1	1	0	0	0	0	0	0
X21	0	0	0	1	-1	1	1	-1	1	-1	1	-1	1	0	0	0	0	0	0
X22	0	0	0	-1	1	-1	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0
X23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X25	0	0	0	1	-1	1	1	-1	1	-1	1	-1	1	0	0	0	0	0	0
X26	0	0	0	-1	1	-1	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0
X27	0	0	0	1	-1	1	1	-1	1	1	-1	1	-1	0	0	0	0	0	0
X28	0	0	0	-1	1	-1	-1	1	-1	-1	1	-1	1	0	0	0	0	0	0
X29	2	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X30	2	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X31	2	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X32	2	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X33	-2	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X34	-2	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X35	-2	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X36	-2	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X37	2	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X38	0	0	0	1	1	1	-1	-1	-1	1	1	-1	-1	0	0	0	0	0	0
X39	0	0	0	1	1	1	-1	-1	-1	-1	-1	1	1	0	0	0	0	0	0
X40	0	0	0	-1	-1	-1	1	1	1	1	1	-1	-1	0	0	0	0	0	0
X41	0	0	0	-1	-1	-1	1	1	1	-1	-1	1	1	0	0	0	0	0	0
X42	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X43	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X44	0	0	0	1	1	1	-1	-1	-1	-1	-1	1	1	0	0	0	0	0	0
X45	0	0	0	1	1	1	-1	-1	-1	1	1	-1	-1	0	0	0	0	0	0
X46	0	0	0	-1	-1	-1	1	1	1	1	1	-1	-1	0	0	0	0	0	0
X47	0	0	0	-1	-1	-1	1	1	1	-1	-1	1	1	0	0	0	0	0	0
X48	0	0	0	2	0	-2	-2	0	2	0	0	0	0	1	1	-1	-1	0	0
X49	0	0	0	-2	0	2	2	0	-2	0	0	0	0	1	1	-1	-1	0	0
X50	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	1	1	0	0
X51	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	1	1	0	0
X52	0	0	0	-2	0	2	2	0	-2	0	0	0	0	0	0	0	0	0	0
X53	0	0	0	2	0	-2	-2	0	2	0	0	0	0	0	0	0	0	0	0
X54	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1	0	0
X55	1	-1	1	1	-1	1	-1	1	-1	1	-1	-1	1	1	-1	-1	1	1	-1
X56	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	1	-1	-1	1	-1	1
X57	2	-2	2	-1	1	-1	1	-1	1	-1	1	1	-1	0	0	0	0	0	0
X58	2	-2	2	1	-1	1	-1	1	-1	-1	1	1	-1	0	0	0	0	0	0
X59	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	0	0	0	0	1	-1
X60	-1	1	-1	1	-1	1	-1	1	-1	-1	1	1	-1	0	0	0	0	-1	1
X61	0	0	0	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	1	-1	0	0
X62	0	0	0	-1	1	-1	1	-1	1	1	-1	-1	1	-1	1	1	-1	0	0
X63	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	1
X64	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	-1
X65	-2	2	-2	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1	0	0

## Appendix B

### Character Table of $B(3, 5)$

Table B.1: Character table of  $B(3,5)$

$[g]$	1a												
$[\bar{g}]$	1A	3A	3B	3C	3D	3E	3F	3G	3H	3I	3J	3K	3L
X1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	1	1	1	1	1	1	1	1	1	1	1	1
X3	4	4	4	4	4	4	4	4	4	4	4	4	4
X4	4	4	4	4	4	4	4	4	4	4	4	4	4
X5	5	5	5	5	5	5	5	5	5	5	5	5	5
X6	5	5	5	5	5	5	5	5	5	5	5	5	5
X7	6	6	6	6	6	6	6	6	6	6	6	6	6
X8	5	$-3\xi-4\omega$	$-4\xi-3\omega$	$-4\xi-3\omega$	$-\xi-3\omega$	2	$-3\xi-\omega$	$-\xi$	$\omega$	$-\xi$	$-2\xi+\omega$	$3\xi-\omega$	$2\xi$
X9	5	$-3\xi-4\omega$	$-4\xi-3\omega$	$-4\xi-3\omega$	$-\xi-3\omega$	2	$-3\xi-\omega$	$-\xi$	$\omega$	$-\xi$	$-2\xi+\omega$	$3\xi-\omega$	$2\xi$
X10	10	$-6\xi-8\omega$	$-8\xi-6\omega$	$-8\xi-6\omega$	$-2\xi-6\omega$	4	$-6\xi-2\omega$	$-2\xi$	$2\omega$	$-2\xi$	$-4\xi+2\omega$	$6\xi-2\omega$	$4\xi$
X11	15	$-9\xi-12\omega$	$-12\xi-9\omega$	$-12\xi-9\omega$	$-3\xi-9\omega$	6	$-9\xi-3\omega$	$-3\xi$	$3\omega$	$-3\xi$	$-6\xi+3\omega$	$9\xi-3\omega$	$6\xi$
X12	15	$-9\xi-12\omega$	$-12\xi-9\omega$	$-12\xi-9\omega$	$-3\xi-9\omega$	6	$-9\xi-3\omega$	$-3\xi$	$3\omega$	$-3\xi$	$-6\xi+3\omega$	$9\xi-3\omega$	$6\xi$
X13	5	$-4\xi-3\omega$	$-3\xi-4\omega$	$-3\xi-4\omega$	$-3\xi-\omega$	2	$-\xi-3\omega$	$-2\xi+\omega$	$-\xi$	$-\omega$	$\xi-2\omega$	$-\xi+3\omega$	$2\omega$
X14	5	$-4\xi-3\omega$	$-3\xi-4\omega$	$-3\xi-4\omega$	$-3\xi-\omega$	2	$-\xi-3\omega$	$-2\xi+\omega$	$-\xi$	$-\omega$	$\xi-2\omega$	$-\xi+3\omega$	$2\omega$
X15	10	$-8\xi-6\omega$	$-6\xi-8\omega$	$-6\xi-8\omega$	$-6\xi-2\omega$	4	$-4\xi-2\omega$	$-4\xi+2\omega$	$-2\xi$	$-2\omega$	$-2\xi-4\omega$	$-2\xi+6\omega$	$4\omega$
X16	15	$-12\xi-9\omega$	$-9\xi-12\omega$	$-9\xi-12\omega$	$-9\xi-3\omega$	6	$-3\xi-9\omega$	$-6\xi+3\omega$	$-3\xi$	$-3\omega$	$3\xi-6\omega$	$-3\xi+9\omega$	$6\omega$
X17	15	$-12\xi-9\omega$	$-9\xi-12\omega$	$-9\xi-12\omega$	$-9\xi-3\omega$	6	$-3\xi-9\omega$	$-6\xi+3\omega$	$-3\xi$	$-3\omega$	$3\xi-6\omega$	$-3\xi+9\omega$	$6\omega$
X18	10	$-2\xi-6\omega$	$-6\xi-2\omega$	$-6\xi-2\omega$	$3\xi-2\omega$	1	$-2\xi+3\omega$	$5\xi+2\omega$	$\xi$	$\omega$	$2\xi+5\omega$	$4\xi+6\omega$	$\omega$
X19	10	$-2\xi-6\omega$	$-6\xi-2\omega$	$-6\xi-2\omega$	$3\xi-2\omega$	1	$-2\xi+3\omega$	$5\xi+2\omega$	$\xi$	$\omega$	$2\xi+5\omega$	$4\xi+6\omega$	$\omega$
X20	10	$-2\xi-6\omega$	$-6\xi-2\omega$	$-6\xi-2\omega$	$3\xi-2\omega$	1	$-2\xi+3\omega$	$5\xi+2\omega$	$\xi$	$\omega$	$2\xi+5\omega$	$4\xi+6\omega$	$\omega$
X21	10	$-2\xi-6\omega$	$-6\xi-2\omega$	$-6\xi-2\omega$	$3\xi-2\omega$	1	$-2\xi+3\omega$	$5\xi+2\omega$	$\xi$	$\omega$	$2\xi+5\omega$	$4\xi+6\omega$	$\omega$
X22	20	$-4\xi-12\omega$	$-12\xi-4\omega$	$-12\xi-4\omega$	$6\xi-4\omega$	2	$-4\xi+6\omega$	$10\xi+4\omega$	$2\xi$	$2\omega$	$4\xi+10\omega$	$8\xi+12\omega$	$2\omega$
X23	20	$-4\xi-12\omega$	$-12\xi-4\omega$	$-12\xi-4\omega$	$6\xi-4\omega$	2	$-4\xi+6\omega$	$10\xi+4\omega$	$2\xi$	$2\omega$	$4\xi+10\omega$	$8\xi+12\omega$	$2\omega$
X24	20	8	8	8	2	-1	2	2	-4	-4	2	8	-1
X25	20	8	8	8	2	-1	2	2	-4	-4	2	8	-1
X26	40	16	16	16	4	-2	4	4	-8	-8	4	16	-2
X27	10	$-6\xi-2\omega$	$-2\xi-6\omega$	$-2\xi-6\omega$	$-2\xi+3\omega$	1	$3\xi-2\omega$	$2\xi+5\omega$	$\omega$	$\xi$	$5\xi+2\omega$	$6\xi+4\omega$	$\xi$
X28	10	$-6\xi-2\omega$	$-2\xi-6\omega$	$-2\xi-6\omega$	$-2\xi+3\omega$	1	$3\xi-2\omega$	$2\xi+5\omega$	$\omega$	$\xi$	$5\xi+2\omega$	$6\xi+4\omega$	$\xi$
X29	10	$-6\xi-2\omega$	$-2\xi-6\omega$	$-2\xi-6\omega$	$-2\xi+3\omega$	1	$3\xi-2\omega$	$2\xi+5\omega$	$\omega$	$\xi$	$5\xi+2\omega$	$6\xi+4\omega$	$\xi$
X30	10	$-6\xi-2\omega$	$-2\xi-6\omega$	$-2\xi-6\omega$	$-2\xi+3\omega$	1	$3\xi-2\omega$	$2\xi+5\omega$	$\omega$	$\xi$	$5\xi+2\omega$	$6\xi+4\omega$	$\xi$
X31	20	$-12\xi-4\omega$	$-4\xi-12\omega$	$-4\xi-12\omega$	$-4\xi+6\omega$	2	$6\xi-4\omega$	$4\xi+10\omega$	$2\omega$	$2\xi$	$10\xi+4\omega$	$12\xi+8\omega$	$2\xi$
X32	20	$-12\xi-4\omega$	$-4\xi-12\omega$	$-4\xi-12\omega$	$-4\xi+6\omega$	2	$6\xi-4\omega$	$4\xi+10\omega$	$2\omega$	$2\xi$	$10\xi+4\omega$	$12\xi+8\omega$	$2\xi$
X33	10	$2\xi-4\omega$	$-4\xi+2\omega$	$-4\xi+2\omega$	$5\xi+2\omega$	1	$2\xi+5\omega$	$2\xi+5\omega$	1	1	$5\xi+2\omega$	$-4\xi+2\omega$	1
X34	10	$2\xi-4\omega$	$-4\xi+2\omega$	$-4\xi+2\omega$	$5\xi+2\omega$	1	$2\xi+5\omega$	$2\xi+5\omega$	1	1	$5\xi+2\omega$	$-4\xi+2\omega$	1
X35	10	$2\xi-4\omega$	$-4\xi+2\omega$	$-4\xi+2\omega$	$5\xi+2\omega$	1	$2\xi+5\omega$	$2\xi+5\omega$	1	1	$5\xi+2\omega$	$-4\xi+2\omega$	1
X36	10	$2\xi-4\omega$	$-4\xi+2\omega$	$-4\xi+2\omega$	$5\xi+2\omega$	1	$2\xi+5\omega$	$2\xi+5\omega$	1	1	$5\xi+2\omega$	$-4\xi+2\omega$	1
X37	20	$4\xi-8\omega$	$-8\xi+4\omega$	$-8\xi+4\omega$	$10\xi+4\omega$	2	$4\xi+10\omega$	$4\xi+10\omega$	2	2	$10\xi+4\omega$	$-8\xi+4\omega$	2
X38	20	$4\xi-8\omega$	$-8\xi+4\omega$	$-8\xi+4\omega$	$10\xi+4\omega$	2	$4\xi+10\omega$	$4\xi+10\omega$	2	2	$10\xi+4\omega$	$-8\xi+4\omega$	2
X39	30	$-6\xi$	$-6\xi$	$-6\xi$	$3\xi$	-6	$3\omega$	$3\omega$	3	$3\xi$	3	$-6\omega$	$-6\xi$
X40	30	$-6\xi$	$-6\xi$	$-6\xi$	$3\xi$	-6	$3\omega$	$3\omega$	3	$3\xi$	3	$-6\omega$	$-6\xi$
X41	30	$-6\xi$	$-6\xi$	$-6\xi$	$3\xi$	-6	$3\omega$	$3\omega$	3	$3\xi$	3	$-6\omega$	$-6\xi$
X42	30	$-6\xi$	$-6\xi$	$-6\xi$	$3\xi$	-6	$3\omega$	$3\omega$	3	$3\xi$	3	$-6\omega$	$-6\xi$
X43	30	$-6\xi$	$-6\xi$	$-6\xi$	$3\xi$	-6	$3\omega$	$3\omega$	3	$3\xi$	3	$-6\xi$	$-6\omega$
X44	30	$-6\xi$	$-6\xi$	$-6\xi$	$3\xi$	-6	$3\omega$	$3\omega$	3	$3\xi$	3	$-6\xi$	$-6\omega$
X45	30	$-6\xi$	$-6\xi$	$-6\xi$	$3\xi$	-6	$3\omega$	$3\omega$	3	$3\xi$	3	$-6\xi$	$-6\omega$
X46	30	$-6\xi$	$-6\xi$	$-6\xi$	$3\xi$	-6	$3\omega$	$3\omega$	3	$3\xi$	3	$-6\xi$	$-6\omega$
X47	10	$-4\xi+2\omega$	$2\xi-4\omega$	$2\xi-4\omega$	$2\xi+5\omega$	1	$5\xi+2\omega$	$5\xi+2\omega$	1	1	$2\xi+5\omega$	$2\xi-4\omega$	1
X48	10	$-4\xi+2\omega$	$2\xi-4\omega$	$2\xi-4\omega$	$2\xi+5\omega$	1	$5\xi+2\omega$	$5\xi+2\omega$	1	1	$2\xi+5\omega$	$2\xi-4\omega$	1
X49	10	$-4\xi+2\omega$	$2\xi-4\omega$	$2\xi-4\omega$	$2\xi+5\omega$	1	$5\xi+2\omega$	$5\xi+2\omega$	1	1	$2\xi+5\omega$	$2\xi-4\omega$	1
X50	10	$-4\xi+2\omega$	$2\xi-4\omega$	$2\xi-4\omega$	$2\xi+5\omega$	1	$5\xi+2\omega$	$5\xi+2\omega$	1	1	$2\xi+5\omega$	$2\xi-4\omega$	1
X51	20	$-8\xi+4\omega$	$4\xi-8\omega$	$4\xi-8\omega$	$4\xi+10\omega$	2	$10\xi+4\omega$	$10\xi+4\omega$	2	2	$4\xi+10\omega$	$4\xi-8\omega$	2
X52	20	$-8\xi+4\omega$	$4\xi-8\omega$	$4\xi-8\omega$	$4\xi+10\omega$	2	$10\xi+4\omega$	$10\xi+4\omega$	2	2	$4\xi+10\omega$	$4\xi-8\omega$	2
X53	5	$3\xi-\omega$	$-\xi+3\omega$	$-\xi+3\omega$	$2\xi+3\omega$	2	$3\xi+2\omega$	$-2\xi+\omega$	$-\omega$	$-\xi$	$\xi-2\omega$	$-3\xi-4\omega$	$2\xi$
X54	5	$3\xi-\omega$	$-\xi+3\omega$	$-\xi+3\omega$	$2\xi+3\omega$	2	$3\xi+2\omega$	$-2\xi+\omega$	$-\omega$	$-\xi$	$\xi-2\omega$	$-3\xi-4\omega$	$2\xi$
X55	10	$6\xi-2\omega$	$-2\xi+6\omega$	$-2\xi+6\omega$	$4\xi+6\omega$	4	$6\xi+4\omega$	$-4\xi+2\omega$	$-2\omega$	$-2\xi$	$2\xi-4\omega$	$-6\xi-8\omega$	$4\xi$
X56	15	$9\xi-3\omega$	$-3\xi+9\omega$	$-3\xi+9\omega$	$6\xi+9\omega$	6	$9\xi+6\omega$	$-6\xi+3\omega$	$-3\omega$	$-3\xi$	$3\xi-6\omega$	$-9\xi-12\omega$	$6\xi$
X57	15	$9\xi-3\omega$	$-3\xi+9\omega$	$-3\xi+9\omega$	$6\xi+9\omega$	6	$9\xi+6\omega$	$-6\xi+3\omega$	$-3\omega$	$-3\xi$	$3\xi-6\omega$	$-9\xi-12\omega$	$6\xi$

$$\xi^3 = 1, \omega = \xi^2$$

Table B.2: Character table of  $B(3,5)$  (continued)

$[g]$	1a							
$[\bar{g}]$	3M	3N	3O	3P	3Q	3R	3S	3T
$\chi_1$	1	1	1	1	1	1	1	1
$\chi_2$	1	1	1	1	1	1	1	1
$\chi_3$	4	4	4	4	4	4	4	4
$\chi_4$	4	4	4	4	4	4	4	4
$\chi_5$	5	5	5	5	5	5	5	5
$\chi_6$	5	5	5	5	5	5	5	5
$\chi_7$	6	6	6	6	6	6	6	6
$\chi_8$	$2\omega$	$-\xi+3\omega$	$5\xi$	$4\xi+\omega$	$3\xi+2\omega$	$2\xi+3\omega$	$\xi+4\omega$	$5\omega$
$\chi_9$	$2\omega$	$-\xi+3\omega$	$5\xi$	$4\xi+\omega$	$3\xi+2\omega$	$2\xi+3\omega$	$\xi+4\omega$	$5\omega$
$\chi_{10}$	$4\omega$	$-2\xi+6\omega$	$10\xi$	$8\xi+2\omega$	$6\xi+4\omega$	$4\xi+6\omega$	$2\xi+8\omega$	$10\omega$
$\chi_{11}$	$6\omega$	$-3\xi+9\omega$	$15\xi$	$12\xi+3\omega$	$9\xi+6\omega$	$6\xi+9\omega$	$3\xi+12\omega$	$15\omega$
$\chi_{12}$	$6\omega$	$-3\xi+9\omega$	$15\xi$	$12\xi+3\omega$	$9\xi+6\omega$	$6\xi+9\omega$	$3\xi+12\omega$	$15\omega$
$\chi_{13}$	$2\xi$	$3\xi-\omega$	$5\omega$	$\xi+4\omega$	$2\xi+3\omega$	$3\xi+2\omega$	$4\xi+\omega$	$5\xi$
$\chi_{14}$	$2\xi$	$3\xi-\omega$	$5\omega$	$\xi+4\omega$	$2\xi+3\omega$	$3\xi+2\omega$	$4\xi+\omega$	$5\xi$
$\chi_{15}$	$4\xi$	$6\xi-2\omega$	$10\omega$	$2\xi+8\omega$	$4\xi+6\omega$	$6\xi+4\omega$	$8\xi+2\omega$	$10\xi$
$\chi_{16}$	$6\xi$	$9\xi-3\omega$	$15\omega$	$3\xi+12\omega$	$6\xi+9\omega$	$9\xi+6\omega$	$12\xi+3\omega$	$15\xi$
$\chi_{17}$	$6\xi$	$9\xi-3\omega$	$15\omega$	$3\xi+12\omega$	$6\xi+9\omega$	$9\xi+6\omega$	$12\xi+3\omega$	$15\xi$
$\chi_{18}$	$\xi$	$6\xi+4\omega$	$10\omega$	$-4\xi+2\omega$	$-5\xi-3\omega$	$-3\xi-5\omega$	$2\xi-4\omega$	$10\xi$
$\chi_{19}$	$\xi$	$6\xi+4\omega$	$10\omega$	$-4\xi+2\omega$	$-5\xi-3\omega$	$-3\xi-5\omega$	$2\xi-4\omega$	$10\xi$
$\chi_{20}$	$\xi$	$6\xi+4\omega$	$10\omega$	$-4\xi+2\omega$	$-5\xi-3\omega$	$-3\xi-5\omega$	$2\xi-4\omega$	$10\xi$
$\chi_{21}$	$\xi$	$6\xi+4\omega$	$10\omega$	$-4\xi+2\omega$	$-5\xi-3\omega$	$-3\xi-5\omega$	$2\xi-4\omega$	$10\xi$
$\chi_{22}$	$2\xi$	$12\xi+8\omega$	$20\omega$	$-8\xi+4\omega$	$-10\xi-6\omega$	$-6\xi-10\omega$	$4\xi-8\omega$	$20\xi$
$\chi_{23}$	$2\xi$	$12\xi+8\omega$	$20\omega$	$-8\xi+4\omega$	$-10\xi-6\omega$	$-6\xi-10\omega$	$4\xi-8\omega$	$20\xi$
$\chi_{24}$	-1	8	20	8	2	2	8	20
$\chi_{25}$	-1	8	20	8	2	2	8	20
$\chi_{26}$	-2	16	40	16	4	4	16	40
$\chi_{27}$	$\omega$	$4\xi+6\omega$	$10\xi$	$\xi-4\omega$	$-3\xi-5\omega$	5	$-4\xi+2\omega$	$10\omega$
$\chi_{28}$	$\omega$	$4\xi+6\omega$	$10\xi$	$\xi-4\omega$	$-3\xi-5\omega$	5	$-4\xi+2\omega$	$10\omega$
$\chi_{29}$	$\omega$	$4\xi+6\omega$	$10\xi$	$\xi-4\omega$	$-3\xi-5\omega$	5	$-4\xi+2\omega$	$10\omega$
$\chi_{30}$	$\omega$	$4\xi+6\omega$	$10\xi$	$\xi-4\omega$	$-3\xi-5\omega$	5	$-4\xi+2\omega$	$10\omega$
$\chi_{31}$	$2\omega$	$8\xi+12\omega$	$20\xi$	$2\xi-8\omega$	$-6\xi-10\omega$	10	$-8\xi+4\omega$	$20\omega$
$\chi_{32}$	$2\omega$	$8\xi+12\omega$	$20\xi$	$2\xi-8\omega$	$-6\xi-10\omega$	10	$-8\xi+4\omega$	$20\omega$
$\chi_{33}$	1	$2\xi-4\omega$	10	$2\xi-4\omega$	$5\xi+2\omega$	$2\xi+5\omega$	$-4\xi+2\omega$	10
$\chi_{34}$	1	$2\xi-4\omega$	10	$2\xi-4\omega$	$5\xi+2\omega$	$2\xi+5\omega$	$-4\xi+2\omega$	10
$\chi_{35}$	1	$2\xi-4\omega$	10	$2\xi-4\omega$	$5\xi+2\omega$	$2\xi+5\omega$	$-4\xi+2\omega$	10
$\chi_{36}$	1	$2\xi-4\omega$	10	$2\xi-4\omega$	$5\xi+2\omega$	$2\xi+5\omega$	$-4\xi+2\omega$	10
$\chi_{37}$	2	$4\xi-8\omega$	20	$4\xi-8\omega$	$10\xi+4\omega$	$4\xi+10\omega$	$-8\xi+4\omega$	20
$\chi_{38}$	2	$4\xi-8\omega$	20	$4\xi-8\omega$	$10\xi+4\omega$	$4\xi+10\omega$	$-8\xi+4\omega$	20
$\chi_{39}$	$-6\omega$	-6\xi	$30\xi$	-6	$3\omega$	$3\xi$	-6	$30\omega$
$\chi_{40}$	$-6\omega$	-6\xi	$30\xi$	-6	$3\omega$	$3\xi$	-6	$30\omega$
$\chi_{41}$	$-6\omega$	-6\xi	$30\xi$	-6	$3\omega$	$3\xi$	-6	$30\omega$
$\chi_{42}$	$-6\omega$	-6\xi	$30\xi$	-6	$3\omega$	$3\xi$	-6	$30\omega$
$\chi_{43}$	-6\xi	-6\omega	$30\omega$	-6	$3\xi$	$3\omega$	-6	$30\xi$
$\chi_{44}$	-6\xi	-6\omega	$30\omega$	-6	$3\xi$	$3\omega$	-6	$30\xi$
$\chi_{45}$	-6\xi	-6\omega	$30\omega$	-6	$3\xi$	$3\omega$	-6	$30\xi$
$\chi_{46}$	-6\xi	-6\omega	$30\omega$	-6	$3\xi$	$3\omega$	-6	$30\xi$
$\chi_{47}$	1	$-4\xi+2\omega$	10	$-4\xi+2\omega$	$2\xi+5\omega$	$5\xi+2\omega$	$2\xi-4\omega$	10
$\chi_{48}$	1	$-4\xi+2\omega$	10	$-4\xi+2\omega$	$2\xi+5\omega$	$5\xi+2\omega$	$2\xi-4\omega$	10
$\chi_{49}$	1	$-4\xi+2\omega$	10	$-4\xi+2\omega$	$2\xi+5\omega$	$5\xi+2\omega$	$2\xi-4\omega$	10
$\chi_{50}$	1	$-4\xi+2\omega$	10	$-4\xi+2\omega$	$2\xi+5\omega$	$5\xi+2\omega$	$2\xi-4\omega$	10
$\chi_{51}$	2	$-8\xi+4\omega$	20	$-8\xi+4\omega$	$4\xi+10\omega$	$10\xi+4\omega$	$4\xi-8\omega$	20
$\chi_{52}$	2	$-8\xi+4\omega$	20	$-8\xi+4\omega$	$4\xi+10\omega$	$10\xi+4\omega$	$4\xi-8\omega$	20
$\chi_{53}$	$2\omega$	$-4\xi-3\omega$	$5\xi$	$\xi+4\omega$	$-3\xi-\omega$	$-\xi-3\omega$	$4\xi+\omega$	$5\omega$
$\chi_{54}$	$2\omega$	$-4\xi-3\omega$	$5\xi$	$\xi+4\omega$	$-3\xi-\omega$	$-\xi-3\omega$	$4\xi+\omega$	$5\omega$
$\chi_{55}$	$4\omega$	$-8\xi-6\omega$	$10\xi$	$2\xi+8\omega$	$-6\xi-2\omega$	$-2\xi-6\omega$	$8\xi+2\omega$	$10\omega$
$\chi_{56}$	$6\omega$	$-12\xi-9\omega$	$15\xi$	$3\xi+12\omega$	$-9\xi-3\omega$	$-3\xi-9\omega$	$12\xi+3\omega$	$15\omega$
$\chi_{57}$	$6\omega$	$-12\xi-9\omega$	$15\xi$	$3\xi+12\omega$	$-9\xi-3\omega$	$-3\xi-9\omega$	$12\xi+3\omega$	$15\omega$

$$\xi^3 = 1, \omega = \xi^2$$

Table B.3: Character table of  $B(3, 5)$  (continued)

$[g]$	$1a$												
$[\bar{g}]$	1A	3A	3B	3C	3D	3E	3F	3G	3H	3I	3J	3K	3L
$\chi_{58}$	20	$8\xi$	$8\omega$	$2\omega$	-1	$2\xi$	2	$-4\xi$	$-4\omega$	2	$8\xi$	$-\omega$	-4
$\chi_{59}$	20	$8\xi$	$8\omega$	$2\omega$	-1	$2\xi$	2	$-4\xi$	$-4\omega$	2	$8\xi$	$-\omega$	-4
$\chi_{60}$	40	$16\xi$	$16\omega$	$4\omega$	-2	$4\xi$	4	$-8\xi$	$-8\omega$	4	$16\xi$	$-2\omega$	-8
$\chi_{61}$	30	-6	-6	3	-6	3	3	3	3	3	-6	-6	3
$\chi_{62}$	30	-6	-6	3	-6	3	3	3	3	3	-6	-6	3
$\chi_{63}$	30	-6	-6	3	-6	3	3	3	3	3	-6	-6	3
$\chi_{64}$	30	-6	-6	3	-6	3	3	3	3	3	-6	-6	3
$\chi_{65}$	20	$8\omega$	$8\xi$	$2\xi$	-1	$2\omega$	2	$-4\omega$	$-4\xi$	2	$8\omega$	$-\xi$	-4
$\chi_{66}$	20	$8\omega$	$8\xi$	$2\xi$	-1	$2\omega$	2	$-4\omega$	$-4\xi$	2	$8\omega$	$-\xi$	-4
$\chi_{67}$	40	$16\omega$	$16\xi$	$4\xi$	-2	$4\omega$	4	$-8\omega$	$-8\xi$	4	$16\omega$	$-2\xi$	-8
$\chi_{68}$	5	$-\xi+3\omega$	$3\xi-\omega$	$3\xi+2\omega$	2	$2\xi+3\omega$	$\xi-2\omega$	$-\xi$	$\omega$	$-2\xi+\omega$	$4\xi-3\omega$	$2\omega$	-1
$\chi_{69}$	5	$-\xi+3\omega$	$3\xi-\omega$	$3\xi+2\omega$	2	$2\xi+3\omega$	$\xi-2\omega$	$-\xi$	$\omega$	$-2\xi+\omega$	$4\xi-3\omega$	$2\omega$	-1
$\chi_{70}$	10	$-2\xi+6\omega$	$6\xi-2\omega$	$6\xi+4\omega$	4	$4\xi+6\omega$	$2\xi-4\omega$	$-2\xi$	$2\omega$	$-4\xi+2\omega$	$8\xi-6\omega$	$4\omega$	-2
$\chi_{71}$	15	$-3\xi+9\omega$	$9\xi-3\omega$	$9\xi+6\omega$	6	$6\xi+9\omega$	$3\xi-6\omega$	$-3\xi$	$3\omega$	$-6\xi+3\omega$	$12\xi-9\omega$	$6\omega$	-3
$\chi_{72}$	15	$-3\xi+9\omega$	$9\xi-3\omega$	$9\xi+6\omega$	6	$6\xi+9\omega$	$3\xi-6\omega$	$-3\xi$	$3\omega$	$-6\xi+3\omega$	$12\xi-9\omega$	$6\omega$	-3
$\chi_{73}$	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1
$\chi_{74}$	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1
$\chi_{75}$	4	$4\xi$	$4\omega$	$4\omega$	4	$4\xi$	4	$4\xi$	$4\omega$	4	$4\xi$	$4\omega$	4
$\chi_{76}$	4	$4\xi$	$4\omega$	$4\omega$	4	$4\xi$	4	$4\xi$	$4\omega$	4	$4\xi$	$4\omega$	4
$\chi_{77}$	5	$5\xi$	$5\omega$	$5\omega$	5	$5\xi$	5	$5\xi$	$5\omega$	5	$5\xi$	$5\omega$	5
$\chi_{78}$	5	$5\xi$	$5\omega$	$5\omega$	5	$5\xi$	5	$5\xi$	$5\omega$	5	$5\xi$	$5\omega$	5
$\chi_{79}$	6	$6\xi$	$6\omega$	$6\omega$	6	$6\xi$	6	$6\xi$	$6\omega$	6	$6\xi$	$6\omega$	6
$\chi_{80}$	5	$4\xi+\omega$	$\xi+4\omega$	$-2\xi+\omega$	2	$-2\omega$	$\xi-2\omega$	-1	-1	$-2\xi+\omega$	$\xi+4\omega$	2	-1
$\chi_{81}$	5	$4\xi+\omega$	$\xi+4\omega$	$-2\xi+\omega$	2	$-2\omega$	$\xi-2\omega$	-1	-1	$-2\xi+\omega$	$\xi+4\omega$	2	-1
$\chi_{82}$	10	$8\xi+2\omega$	$2\xi+8\omega$	$-4\xi+2\omega$	4	$-4\omega$	$2\xi-4\omega$	-2	-2	$-4\xi+2\omega$	$2\xi+8\omega$	4	-2
$\chi_{83}$	15	$12\xi+3\omega$	$3\xi+12\omega$	$-6\xi+3\omega$	6	$-6\omega$	$3\xi-6\omega$	-3	-3	$-6\xi+3\omega$	$3\xi+12\omega$	6	-3
$\chi_{84}$	15	$12\xi+3\omega$	$3\xi+12\omega$	$-6\xi+3\omega$	6	$-6\omega$	$3\xi-6\omega$	-3	-3	$-6\xi+3\omega$	$3\xi+12\omega$	6	-3
$\chi_{85}$	10	$6\xi+4\omega$	$4\xi+6\omega$	$-5\xi-3\omega$	1	$-3\xi-5\omega$	$5\xi+2\omega$	$\omega$	$\xi$	$2\xi+5\omega$	$-6\xi-2\omega$	$\xi$	1
$\chi_{86}$	10	$6\xi+4\omega$	$4\xi+6\omega$	$-5\xi-3\omega$	1	$-3\xi-5\omega$	$5\xi+2\omega$	$\omega$	$\xi$	$2\xi+5\omega$	$-6\xi-2\omega$	$\xi$	1
$\chi_{87}$	10	$6\xi+4\omega$	$4\xi+6\omega$	$-5\xi-3\omega$	1	$-3\xi-5\omega$	$5\xi+2\omega$	$\omega$	$\xi$	$2\xi+5\omega$	$-6\xi-2\omega$	$\xi$	1
$\chi_{88}$	10	$6\xi+4\omega$	$4\xi+6\omega$	$-5\xi-3\omega$	1	$-3\xi-5\omega$	$5\xi+2\omega$	$\omega$	$\xi$	$2\xi+5\omega$	$-6\xi-2\omega$	$\xi$	1
$\chi_{89}$	20	$12\xi+8\omega$	$8\xi+12\omega$	$-10\xi-6\omega$	2	$-6\xi-10\omega$	$10\xi+4\omega$	$2\omega$	$2\xi$	$4\xi+10\omega$	$-12\xi-4\omega$	$2\xi$	2
$\chi_{90}$	20	$12\xi+8\omega$	$8\xi+12\omega$	$-10\xi-6\omega$	2	$-6\xi-10\omega$	$10\xi+4\omega$	$2\omega$	$2\xi$	$4\xi+10\omega$	$-12\xi-4\omega$	$2\xi$	2
$\chi_{91}$	10	$4\xi+6\omega$	$6\xi+4\omega$	$-3\xi-5\omega$	1	$-5\xi-3\omega$	$2\xi+5\omega$	$\xi$	$\omega$	$5\xi+2\omega$	$-2\xi-6\omega$	$\omega$	1
$\chi_{92}$	10	$4\xi+6\omega$	$6\xi+4\omega$	$-3\xi-5\omega$	1	$-5\xi-3\omega$	$2\xi+5\omega$	$\xi$	$\omega$	$5\xi+2\omega$	$-2\xi-6\omega$	$\omega$	1
$\chi_{93}$	10	$4\xi+6\omega$	$6\xi+4\omega$	$-3\xi-5\omega$	1	$-5\xi-3\omega$	$2\xi+5\omega$	$\xi$	$\omega$	$5\xi+2\omega$	$-2\xi-6\omega$	$\omega$	1
$\chi_{94}$	10	$4\xi+6\omega$	$6\xi+4\omega$	$-3\xi-5\omega$	1	$-5\xi-3\omega$	$2\xi+5\omega$	$\xi$	$\omega$	$5\xi+2\omega$	$-2\xi-6\omega$	$\omega$	1
$\chi_{95}$	20	$8\xi+12\omega$	$12\xi+8\omega$	$-6\xi-10\omega$	2	$-10\xi-6\omega$	$4\xi+10\omega$	$2\xi$	$2\omega$	$10\xi+4\omega$	$-4\xi-12\omega$	$2\omega$	2
$\chi_{96}$	20	$8\xi+12\omega$	$12\xi+8\omega$	$-6\xi-10\omega$	2	$-10\xi-6\omega$	$4\xi+10\omega$	$2\xi$	$2\omega$	$10\xi+4\omega$	$-4\xi-12\omega$	$2\omega$	2
$\chi_{97}$	5	$\xi+4\omega$	$4\xi+\omega$	$\xi-2\omega$	2	$-2\xi+\omega$	$-2\xi+\omega$	-1	-1	$\xi-2\omega$	$4\xi+\omega$	2	-1
$\chi_{98}$	5	$\xi+4\omega$	$4\xi+\omega$	$\xi-2\omega$	2	$-2\xi+\omega$	$-2\xi+\omega$	-1	-1	$\xi-2\omega$	$4\xi+\omega$	2	-1
$\chi_{99}$	10	$2\xi+8\omega$	$8\xi+2\omega$	$2\xi-4\omega$	4	$-4\xi+2\omega$	$-4\xi+2\omega$	-2	-2	$2\xi-4\omega$	$8\xi+2\omega$	4	-2
$\chi_{100}$	15	$3\xi+12\omega$	$12\xi+3\omega$	$3\xi-6\omega$	6	$-6\xi+3\omega$	$-6\xi+3\omega$	-3	-3	$3\xi-6\omega$	$12\xi+3\omega$	6	-3
$\chi_{101}$	15	$3\xi+12\omega$	$12\xi+3\omega$	$3\xi-6\omega$	6	$-6\xi+3\omega$	$-6\xi+3\omega$	-3	-3	$3\xi-6\omega$	$12\xi+3\omega$	6	-3
$\chi_{102}$	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1
$\chi_{103}$	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1
$\chi_{104}$	4	$4\omega$	$4\xi$	$4\xi$	4	$4\omega$	4	$4\omega$	$4\xi$	4	$4\omega$	$4\xi$	4
$\chi_{105}$	4	$4\omega$	$4\xi$	$4\xi$	4	$4\omega$	4	$4\omega$	$4\xi$	4	$4\omega$	$4\xi$	4
$\chi_{106}$	5	$5\omega$	$5\xi$	$5\xi$	5	$5\omega$	5	$5\omega$	$5\xi$	5	$5\omega$	$5\xi$	5
$\chi_{107}$	5	$5\omega$	$5\xi$	$5\xi$	5	$5\omega$	5	$5\omega$	$5\xi$	5	$5\omega$	$5\xi$	5
$\chi_{108}$	6	$6\omega$	$6\xi$	$6\xi$	6	$6\omega$	6	$6\omega$	$6\xi$	6	$6\omega$	$6\xi$	6

$$\xi^3 = 1, \omega = \xi^2$$

Table B.4: Character table of  $B(3,5)$  (continued)

$[g]$	$1a$							
$[\bar{g}]$	3M	3N	3O	3P	3Q	3R	3S	3T
X58	$-\xi$	$8\omega$	$20\omega$	8	$2\xi$	$2\omega$	8	$20\xi$
X59	$-\xi$	$8\omega$	$20\omega$	8	$2\xi$	$2\omega$	8	$20\xi$
X60	$-2\xi$	$16\omega$	$40\omega$	16	$4\xi$	$4\omega$	16	$40\xi$
X61	-6	-6	30	-6	3	3	-6	30
X62	-6	-6	30	-6	3	3	-6	30
X63	-6	-6	30	-6	3	3	-6	30
X64	-6	-6	30	-6	3	3	-6	30
X65	$-\omega$	$8\xi$	$20\xi$	8	$2\omega$	$2\xi$	8	$20\omega$
X66	$-\omega$	$8\xi$	$20\xi$	8	$2\omega$	$2\xi$	8	$20\omega$
X67	$-2\omega$	$16\xi$	$40\xi$	16	$4\omega$	$4\xi$	16	$40\omega$
X68	$2\xi$	$-3\xi-4\omega$	$5\omega$	$4\xi+\omega$	$-\xi-3\omega$	$-3\xi-\omega$	$\xi+4\omega$	$5\xi$
X69	$2\xi$	$-3\xi-4\omega$	$5\omega$	$4\xi+\omega$	$-\xi-3\omega$	$-3\xi-\omega$	$\xi+4\omega$	$5\xi$
X70	$4\xi$	$-6\xi-8\omega$	$10\omega$	$8\xi+2\omega$	$-2\xi-6\omega$	$-6\xi-2\omega$	$2\xi+8\omega$	$10\xi$
X71	$6\xi$	$-9\xi-12\omega$	$15\omega$	$12\xi+3\omega$	$-3\xi-9\omega$	$-9\xi-3\omega$	$3\xi+12\omega$	$15\xi$
X72	$6\xi$	$-9\xi-12\omega$	$15\omega$	$12\xi+3\omega$	$-3\xi-9\omega$	$-9\xi-3\omega$	$3\xi+12\omega$	$15\xi$
X73	$\xi$	$\omega$	$\omega$	1	$\xi$	$\omega$	1	$\xi$
X74	$\xi$	$\omega$	$\omega$	1	$\xi$	$\omega$	1	$\xi$
X75	$4\xi$	$4\omega$	$4\omega$	4	$4\xi$	$4\omega$	4	$4\xi$
X76	$4\xi$	$4\omega$	$4\omega$	4	$4\xi$	$4\omega$	4	$4\xi$
X77	$5\xi$	$5\omega$	$5\omega$	5	$5\xi$	$5\omega$	5	$5\xi$
X78	$5\xi$	$5\omega$	$5\omega$	5	$5\xi$	$5\omega$	5	$5\xi$
X79	$6\xi$	$6\omega$	$6\omega$	6	$6\xi$	$6\omega$	6	$6\xi$
X80	2	$4\xi+\omega$	5	$4\xi+\omega$	$-2\xi+\omega$	$\xi-2\omega$	$\xi+4\omega$	5
X81	2	$4\xi+\omega$	5	$4\xi+\omega$	$-2\xi+\omega$	$\xi-2\omega$	$\xi+4\omega$	5
X82	4	$8\xi+2\omega$	10	$8\xi+2\omega$	$-4\xi+2\omega$	$2\xi-4\omega$	$2\xi+8\omega$	10
X83	6	$12\xi+3\omega$	15	$12\xi+3\omega$	$-6\xi+3\omega$	$3\xi-6\omega$	$3\xi+12\omega$	15
X84	6	$12\xi+3\omega$	15	$12\xi+3\omega$	$-6\xi+3\omega$	$3\xi-6\omega$	$3\xi+12\omega$	15
X85	$\omega$	$-2\xi-6\omega$	$10\xi$	$-4\xi+2\omega$	$3\xi-2\omega$	$-2\xi+3\omega$	$2\xi-4\omega$	$10\omega$
X86	$\omega$	$-2\xi-6\omega$	$10\xi$	$-4\xi+2\omega$	$3\xi-2\omega$	$-2\xi+3\omega$	$2\xi-4\omega$	$10\omega$
X87	$\omega$	$-2\xi-6\omega$	$10\xi$	$-4\xi+2\omega$	$3\xi-2\omega$	$-2\xi+3\omega$	$2\xi-4\omega$	$10\omega$
X88	$\omega$	$-2\xi-6\omega$	$10\xi$	$-4\xi+2\omega$	$3\xi-2\omega$	$-2\xi+3\omega$	$2\xi-4\omega$	$10\omega$
X89	$2\omega$	$-4\xi-12\omega$	$20\xi$	$-8\xi+4\omega$	$6\xi-4\omega$	$-4\xi+6\omega$	$4\xi-8\omega$	$20\omega$
X90	$2\omega$	$-4\xi-12\omega$	$20\xi$	$-8\xi+4\omega$	$6\xi-4\omega$	$-4\xi+6\omega$	$4\xi-8\omega$	$20\omega$
X91	$\xi$	$-6\xi-2\omega$	$10\omega$	$2\xi-4\omega$	$-2\xi+3\omega$	$3\xi-2\omega$	$-4\xi+2\omega$	$10\xi$
X92	$\xi$	$-6\xi-2\omega$	$10\omega$	$2\xi-4\omega$	$-2\xi+3\omega$	$3\xi-2\omega$	$-4\xi+2\omega$	$10\xi$
X93	$\xi$	$-6\xi-2\omega$	$10\omega$	$2\xi-4\omega$	$-2\xi+3\omega$	$3\xi-2\omega$	$-4\xi+2\omega$	$10\xi$
X94	$\xi$	$-6\xi-2\omega$	$10\omega$	$2\xi-4\omega$	$-2\xi+3\omega$	$3\xi-2\omega$	$-4\xi+2\omega$	$10\xi$
X95	$2\xi$	$-12\xi-4\omega$	$20\omega$	$4\xi-8\omega$	$-4\xi+6\omega$	$6\xi-4\omega$	$-8\xi+4\omega$	$20\xi$
X96	$2\xi$	$-12\xi-4\omega$	$20\omega$	$4\xi-8\omega$	$-4\xi+6\omega$	$6\xi-4\omega$	$-8\xi+4\omega$	$20\xi$
X97	2	$\xi+4\omega$	5	$\xi+4\omega$	$\xi-2\omega$	$-2\xi+\omega$	$4\xi+\omega$	5
X98	2	$\xi+4\omega$	5	$\xi+4\omega$	$\xi-2\omega$	$-2\xi+\omega$	$4\xi+\omega$	5
X99	4	$2\xi+8\omega$	10	$2\xi+8\omega$	$2\xi-4\omega$	$-4\xi+2\omega$	$8\xi+2\omega$	10
X100	6	$3\xi+12\omega$	15	$3\xi+12\omega$	$3\xi-6\omega$	$-6\xi+3\omega$	$12\xi+3\omega$	15
X101	6	$3\xi+12\omega$	15	$3\xi+12\omega$	$3\xi-6\omega$	$-6\xi+3\omega$	$12\xi+3\omega$	15
X102	$\omega$	$\xi$	$\xi$	1	$\omega$	$\xi$	1	$\omega$
X103	$\omega$	$\xi$	$\xi$	1	$\omega$	$\xi$	1	$\omega$
X104	$4\omega$	$4\xi$	$4\xi$	4	$4\omega$	$4\xi$	4	$4\omega$
X105	$4\omega$	$4\xi$	$4\xi$	4	$4\omega$	$4\xi$	4	$4\omega$
X106	$5\omega$	$5\xi$	$5\xi$	5	$5\omega$	$5\xi$	5	$5\omega$
X107	$5\omega$	$5\xi$	$5\xi$	5	$5\omega$	$5\xi$	5	$5\omega$
X108	$6\omega$	$6\xi$	$6\xi$	6	$6\omega$	$6\xi$	6	$6\omega$

$$\xi^3 = 1, \omega = \xi^2$$

Table B.5: Character table of  $B(3,5)$  (continued)

$[g]$	2a									
$[\bar{g}]$	2A	6A	6B	6C	6D	6E	6F	6G	6H	6I
$\chi_1$	1	1	1	1	1	1	1	1	1	1
$\chi_2$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
$\chi_3$	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
$\chi_4$	2	2	2	2	2	2	2	2	2	2
$\chi_5$	1	1	1	1	1	1	1	1	1	1
$\chi_6$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
$\chi_7$	0	0	0	0	0	0	0	0	0	0
$\chi_8$	3	$-\xi-2\omega$	$-2\xi-\omega$	$\xi-\omega$	0	$-\xi+\omega$	3\xi	$2\xi+\omega$	$\xi+2\omega$	3\omega
$\chi_9$	-3	$\xi+2\omega$	$2\xi+\omega$	$-\xi+\omega$	0	$\xi-\omega$	-3\xi	$-2\xi-\omega$	$-\xi-2\omega$	-3\omega
$\chi_{10}$	0	0	0	0	0	0	0	0	0	0
$\chi_{11}$	-3	$\xi+2\omega$	$2\xi+\omega$	$-\xi+\omega$	0	$\xi-\omega$	-3\xi	$-2\xi-\omega$	$-\xi-2\omega$	-3\omega
$\chi_{12}$	3	$-\xi-2\omega$	$-2\xi-\omega$	$\xi-\omega$	0	$-\xi+\omega$	3\xi	$2\xi+\omega$	$\xi+2\omega$	3\omega
$\chi_{13}$	3	$-2\xi-\omega$	$-\xi-2\omega$	$-\xi+\omega$	0	$\xi-\omega$	3\omega	$\xi+2\omega$	$2\xi+\omega$	3\xi
$\chi_{14}$	-3	$2\xi+\omega$	$\xi+2\omega$	$\xi-\omega$	0	$-\xi+\omega$	-3\omega	$-\xi-2\omega$	$-2\xi-\omega$	-3\xi
$\chi_{15}$	0	0	0	0	0	0	0	0	0	0
$\chi_{16}$	-3	$2\xi+\omega$	$\xi+2\omega$	$\xi-\omega$	0	$-\xi+\omega$	-3\omega	$-\xi-2\omega$	$-2\xi-\omega$	-3\xi
$\chi_{17}$	3	$-2\xi-\omega$	$-\xi-2\omega$	$-\xi+\omega$	0	$\xi-\omega$	3\omega	$\xi+2\omega$	$2\xi+\omega$	3\xi
$\chi_{18}$	4	$-2\omega$	$-2\xi$	$\xi$	1	$\omega$	$-\xi+2\omega$	$-3\xi-2\omega$	$-2\xi-3\omega$	$2\xi-\omega$
$\chi_{19}$	2	$2\xi$	$2\omega$	$3\xi+2\omega$	-1	$2\xi+3\omega$	$\xi+4\omega$	$-\xi$	$-\omega$	$4\xi+\omega$
$\chi_{20}$	-2	$-2\xi$	$-2\omega$	$-3\xi-2\omega$	1	$-2\xi-3\omega$	$-\xi-4\omega$	$\xi$	$\omega$	$-4\xi-\omega$
$\chi_{21}$	-4	$2\omega$	$2\xi$	$-\xi$	-1	$-\omega$	$\xi-2\omega$	$3\xi+2\omega$	$2\xi+3\omega$	$-2\xi+\omega$
$\chi_{22}$	-6	$-2\xi+2\omega$	$2\xi-2\omega$	$-4\xi-2\omega$	0	$-2\xi-4\omega$	-6\omega	$4\xi+2\omega$	$2\xi+4\omega$	-6\xi
$\chi_{23}$	6	$2\xi-2\omega$	$-2\xi+2\omega$	$4\xi+2\omega$	0	$2\xi+4\omega$	6\omega	$-4\xi-2\omega$	$-2\xi-4\omega$	6\xi
$\chi_{24}$	6	0	0	0	-3	0	6	0	0	6
$\chi_{25}$	-6	0	0	0	3	0	-6	0	0	-6
$\chi_{26}$	0	0	0	0	0	0	0	0	0	0
$\chi_{27}$	4	$-2\xi$	$-2\omega$	$\omega$	1	$\xi$	$2\xi-\omega$	$-2\xi-3\omega$	$-3\xi-2\omega$	$-\xi+2\omega$
$\chi_{28}$	2	$2\omega$	$2\xi$	$2\xi+3\omega$	-1	$3\xi+2\omega$	$4\xi+\omega$	$-\omega$	$-\xi$	$\xi+4\omega$
$\chi_{29}$	-2	$-2\omega$	$-2\xi$	$-2\xi-3\omega$	1	$-3\xi-2\omega$	$-4\xi-\omega$	$\omega$	$\xi$	$-\xi-4\omega$
$\chi_{30}$	-4	$2\xi$	$2\omega$	$-\omega$	-1	$-\xi$	$-2\xi+\omega$	$2\xi+3\omega$	$3\xi+2\omega$	$\xi-2\omega$
$\chi_{31}$	-6	$2\xi-2\omega$	$-2\xi+2\omega$	$-2\xi-4\omega$	0	$-4\xi-2\omega$	-6\xi	$2\xi+4\omega$	$4\xi+2\omega$	-6\omega
$\chi_{32}$	6	$-2\xi+2\omega$	$2\xi-2\omega$	$2\xi+4\omega$	0	$4\xi+2\omega$	6\xi	$-2\xi-4\omega$	$-4\xi-2\omega$	6\omega
$\chi_{33}$	4	$-2\omega$	$-2\xi$	$\xi$	1	$\omega$	$2\xi-\omega$	$3\xi+\omega$	$\xi+3\omega$	$-\xi+2\omega$
$\chi_{34}$	-2	$-2$	$-2$	$-\xi+2\omega$	1	$2\xi-\omega$	$-4\xi-\omega$	1	1	$-\xi-4\omega$
$\chi_{35}$	2	$2$	$2$	$\xi-2\omega$	-1	$-2\xi+\omega$	$4\xi+\omega$	-1	-1	$\xi+4\omega$
$\chi_{36}$	-4	$2\omega$	$2\xi$	$-\xi$	-1	$-\omega$	$-2\xi+\omega$	$-3\xi-\omega$	$-\xi-3\omega$	$\xi-2\omega$
$\chi_{37}$	-6	$2\xi+4\omega$	$4\xi+2\omega$	$-2\xi+2\omega$	0	$2\xi-2\omega$	-6\xi	$-4\xi-2\omega$	$-2\xi-4\omega$	-6\omega
$\chi_{38}$	6	$-2\xi-4\omega$	$-4\xi-2\omega$	$2\xi-2\omega$	0	$-2\xi+2\omega$	6\xi	$4\xi+2\omega$	$2\xi+4\omega$	6\omega
$\chi_{39}$	6	0	0	-3\xi	0	-3\omega	-3	3\omega	3\xi	-3
$\chi_{40}$	0	$4\xi+2\omega$	$2\xi+4\omega$	$-\xi-2\omega$	0	$-2\xi-\omega$	$3\xi-3\omega$	$-2\xi-\omega$	$-\xi-2\omega$	$-3\xi+3\omega$
$\chi_{41}$	0	$-4\xi-2\omega$	$-2\xi-4\omega$	$\xi+2\omega$	0	$2\xi+\omega$	$-3\xi+3\omega$	$2\xi+\omega$	$\xi+2\omega$	$3\xi-3\omega$
$\chi_{42}$	-6	0	0	3\xi	0	3\omega	3	-3\omega	-3\xi	3
$\chi_{43}$	6	0	0	-3\omega	0	-3\xi	-3	3\xi	3\omega	-3
$\chi_{44}$	0	$2\xi+4\omega$	$4\xi+2\omega$	$-2\xi-\omega$	0	$-\xi-2\omega$	$-3\xi+3\omega$	$-\xi-2\omega$	$-2\xi-\omega$	$3\xi-3\omega$
$\chi_{45}$	0	$-2\xi-4\omega$	$-4\xi-2\omega$	$2\xi+\omega$	0	$\xi+2\omega$	$3\xi-3\omega$	$\xi+2\omega$	$2\xi+\omega$	$-3\xi+3\omega$
$\chi_{46}$	-6	0	0	3\omega	0	3\xi	3	-3\xi	-3\omega	3
$\chi_{47}$	4	$-2\xi$	$-2\omega$	$\omega$	1	$\xi$	$-\xi+2\omega$	$\xi+3\omega$	$3\xi+\omega$	$2\xi-\omega$
$\chi_{48}$	-2	$-2$	$-2$	$2\xi-\omega$	1	$-\xi+2\omega$	$-\xi-4\omega$	1	1	$-4\xi-\omega$
$\chi_{49}$	2	$2$	$2$	$-2\xi+\omega$	-1	$\xi-2\omega$	$\xi+4\omega$	-1	-1	$4\xi+\omega$
$\chi_{50}$	-4	$2\xi$	$2\omega$	$-\omega$	-1	$-\xi$	$\xi-2\omega$	$-\xi-3\omega$	$-3\xi-\omega$	$-2\xi+\omega$
$\chi_{51}$	-6	$4\xi+2\omega$	$2\xi+4\omega$	$2\xi-2\omega$	0	$-2\xi+2\omega$	-6\omega	$-2\xi-4\omega$	$-4\xi-2\omega$	-6\xi
$\chi_{52}$	6	$-4\xi-2\omega$	$-2\xi-4\omega$	$-2\xi+2\omega$	0	$2\xi-2\omega$	6\omega	$2\xi+4\omega$	$4\xi+2\omega$	6\xi
$\chi_{53}$	3	$\xi-\omega$	$-\xi+\omega$	$2\xi+\omega$	0	$\xi+2\omega$	3\omega	$-2\xi-\omega$	$-\xi-2\omega$	3\xi
$\chi_{54}$	-3	$-\xi+\omega$	$\xi-\omega$	$-2\xi-\omega$	0	$-\xi-2\omega$	-3\omega	$2\xi+\omega$	$\xi+2\omega$	-3\xi
$\chi_{55}$	0	0	0	0	0	0	0	0	0	0
$\chi_{56}$	-3	$-\xi+\omega$	$\xi-\omega$	$-2\xi-\omega$	0	$-\xi-2\omega$	-3\omega	$2\xi+\omega$	$\xi+2\omega$	-3\xi
$\chi_{57}$	3	$\xi-\omega$	$-\xi+\omega$	$2\xi+\omega$	0	$\xi+2\omega$	3\omega	$-2\xi-\omega$	$-\xi-2\omega$	3\xi

$$\xi^3 = 1, \omega = \xi^2$$

Table B.6: Character table of  $B(3,5)$  (continued)

[g]	2a									
[g]	6J	6K	6L	6M	6N	6O	6P	6Q	6R	6S
X1	1	1	1	1	1	1	1	1	1	1
X2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
X3	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
X4	2	2	2	2	2	2	2	2	2	2
X5	1	1	1	1	1	1	1	1	1	1
X6	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
X7	0	0	0	0	0	0	0	0	0	0
X8	3	$-\xi-2\omega$	$-2\xi-\omega$	$\xi-\omega$	0	$-\xi+\omega$	3\xi	$2\xi+\omega$	$\xi+2\omega$	3\omega
X9	-3	$\xi+2\omega$	$2\xi+\omega$	$-\xi+\omega$	0	$\xi-\omega$	-3\xi	$-2\xi-\omega$	$-\xi-2\omega$	-3\omega
X10	0	0	0	0	0	0	0	0	0	0
X11	-3	$\xi+2\omega$	$2\xi+\omega$	$-\xi+\omega$	0	$\xi-\omega$	-3\xi	$-2\xi-\omega$	$-\xi-2\omega$	-3\omega
X12	3	$-\xi-2\omega$	$-2\xi-\omega$	$\xi-\omega$	0	$-\xi+\omega$	3\xi	$2\xi+\omega$	$\xi+2\omega$	3\omega
X13	3	$-2\xi-\omega$	$-\xi-2\omega$	$-\xi+\omega$	0	$\xi-\omega$	3\omega	$\xi+2\omega$	$2\xi+\omega$	3\xi
X14	-3	$2\xi+\omega$	$\xi+2\omega$	$\xi-\omega$	0	$-\xi+\omega$	-3\omega	$-\xi-2\omega$	$-2\xi-\omega$	-3\xi
X15	0	0	0	0	0	0	0	0	0	0
X16	-3	$2\xi+\omega$	$\xi+2\omega$	$\xi-\omega$	0	$-\xi+\omega$	-3\omega	$-\xi-2\omega$	$-2\xi-\omega$	-3\xi
X17	3	$-2\xi-\omega$	$-\xi-2\omega$	$-\xi+\omega$	0	$\xi-\omega$	3\omega	$\xi+2\omega$	$2\xi+\omega$	3\xi
X18	$-2\xi-3\omega$	$2\xi-\omega$	$\omega$	$3\xi+\omega$	$\xi$	-2	$\xi+3\omega$	1	-2\omega	4\xi
X19	$-4\xi-3\omega$	$-\omega$	$-2\xi+\omega$	-1	$-\xi$	2\omega	$-\xi+3\omega$	-3\xi-\omega	2	2\xi
X20	$4\xi+3\omega$	$\omega$	$2\xi-\omega$	1	$\xi$	-2\omega	$\xi-3\omega$	$3\xi+\omega$	-2	-2\xi
X21	$2\xi+3\omega$	$-2\xi+\omega$	$-\omega$	$-3\xi-\omega$	$-\xi$	2	$-\xi-3\omega$	-1	2\omega	-4\xi
X22	-6	$-2\xi+2\omega$	$2\xi-2\omega$	$-4\xi-2\omega$	0	$-2\xi-4\omega$	-6\omega	$4\xi+2\omega$	$2\xi+4\omega$	-6\xi
X23	6	$2\xi-2\omega$	$-2\xi+2\omega$	$4\xi+2\omega$	0	$2\xi+4\omega$	6\omega	$-4\xi-2\omega$	$-2\xi-4\omega$	6\xi
X24	6	0	0	0	-3	0	6	0	0	6
X25	-6	0	0	0	3	0	-6	0	0	-6
X26	0	0	0	0	0	0	0	0	0	0
X27	$-3\xi-2\omega$	$-\xi+2\omega$	$\xi$	$\xi+3\omega$	$\omega$	-2	$3\xi+\omega$	1	-2\xi	4\omega
X28	$-3\xi-4\omega$	$-\xi$	$\xi-2\omega$	-1	$-\omega$	2\xi	$3\xi-\omega$	$-\xi-3\omega$	2	2\omega
X29	$3\xi+4\omega$	$\xi$	$-\xi+2\omega$	1	$\omega$	-2\xi	$-3\xi+\omega$	$\xi+3\omega$	-2	-2\omega
X30	$3\xi+2\omega$	$\xi-2\omega$	$-\xi$	$-\xi-3\omega$	$-\omega$	2	$-3\xi-\omega$	-1	2\xi	-4\omega
X31	-6	$2\xi-2\omega$	$-2\xi+2\omega$	$-2\xi-4\omega$	0	$-4\xi-2\omega$	-6\xi	$2\xi+4\omega$	$4\xi+2\omega$	-6\omega
X32	6	$-2\xi+2\omega$	$2\xi-2\omega$	$2\xi+4\omega$	0	$4\xi+2\omega$	6\xi	$-2\xi-4\omega$	$-4\xi-2\omega$	6\omega
X33	$2\xi-\omega$	$3\xi+\omega$	$\xi$	$\xi+3\omega$	1	-2\omega	$-\xi+2\omega$	$\omega$	-2\xi	4
X34	$-4\xi-\omega$	1	$-\xi+2\omega$	1	1	-2	$-\xi-4\omega$	$2\xi-\omega$	-2	-2
X35	$4\xi+\omega$	-1	$\xi-2\omega$	-1	-1	2	$\xi+4\omega$	$-2\xi+\omega$	2	2
X36	$-2\xi+\omega$	$-3\xi-\omega$	$-\xi$	$-\xi-3\omega$	-1	2\omega	$\xi-2\omega$	$-\omega$	2\xi	-4
X37	-6\xi	$-4\xi-2\omega$	$-2\xi+2\omega$	$-2\xi-4\omega$	0	$2\xi+4\omega$	-6\omega	$2\xi-2\omega$	$4\xi+2\omega$	-6
X38	6\xi	$4\xi+2\omega$	$2\xi-2\omega$	$2\xi+4\omega$	0	$-2\xi-4\omega$	6\omega	$-2\xi+2\omega$	$-4\xi-2\omega$	6
X39	-3\omega	3\xi	-3	3	0	0	-3\omega	-3\xi	0	6\omega
X40	$-6\xi-3\omega$	$\xi+2\omega$	$-\xi+\omega$	$-\xi+\omega$	0	$-2\xi-4\omega$	$6\xi+3\omega$	$\xi+2\omega$	$2\xi-2\omega$	0
X41	$6\xi+3\omega$	$-\xi-2\omega$	$\xi-\omega$	$\xi-\omega$	0	$2\xi+4\omega$	$-6\xi-3\omega$	$-\xi-2\omega$	$-2\xi+2\omega$	0
X42	3\omega	-3\xi	3	-3	0	0	3\omega	3\xi	0	-6\omega
X43	-3\xi	3\omega	-3	3	0	0	-3\xi	-3\omega	0	6\xi
X44	$-3\xi-6\omega$	$2\xi+\omega$	$\xi-\omega$	$\xi-\omega$	0	$-4\xi-2\omega$	$3\xi+6\omega$	$2\xi+\omega$	$-2\xi+2\omega$	0
X45	$3\xi+6\omega$	$-2\xi-\omega$	$-\xi+\omega$	$-\xi+\omega$	0	$4\xi+2\omega$	$-3\xi-6\omega$	$-2\xi-\omega$	$2\xi-2\omega$	0
X46	3\xi	-3\omega	3	-3	0	0	3\xi	3\omega	0	-6\xi
X47	$-\xi+2\omega$	$\xi+3\omega$	$\omega$	$3\xi+\omega$	1	-2\xi	$2\xi-\omega$	$\xi$	-2\omega	4
X48	$-\xi-4\omega$	1	$2\xi-\omega$	1	1	-2	$-4\xi-\omega$	$-\xi+2\omega$	-2	-2
X49	$\xi+4\omega$	-1	$-2\xi+\omega$	-1	-1	2	$4\xi+\omega$	$\xi-2\omega$	2	2
X50	$\xi-2\omega$	$-\xi-3\omega$	$-\omega$	$-3\xi-\omega$	-1	2\xi	$-2\xi+\omega$	$-\xi$	2\omega	-4
X51	-6\omega	$-2\xi-4\omega$	$2\xi-2\omega$	$-4\xi-2\omega$	0	$4\xi+2\omega$	-6\xi	$-2\xi+2\omega$	$2\xi+4\omega$	-6
X52	6\omega	$2\xi+4\omega$	$-2\xi+2\omega$	$4\xi+2\omega$	0	$-4\xi-2\omega$	6\xi	$2\xi-2\omega$	$-2\xi-4\omega$	6
X53	3\xi	$\xi+2\omega$	$-\xi-2\omega$	$-\xi+\omega$	0	$-2\xi-\omega$	3	$\xi-\omega$	$2\xi+\omega$	3\omega
X54	-3\xi	$-\xi-2\omega$	$\xi+2\omega$	$\xi-\omega$	0	$2\xi+\omega$	-3	$-\xi+\omega$	$-2\xi-\omega$	-3\omega
X55	0	0	0	0	0	0	0	0	0	0
X56	-3\xi	$-\xi-2\omega$	$\xi+2\omega$	$\xi-\omega$	0	$2\xi+\omega$	-3	$-\xi+\omega$	$-2\xi-\omega$	-3\omega
X57	3\xi	$\xi+2\omega$	$-\xi-2\omega$	$-\xi+\omega$	0	$-2\xi-\omega$	3	$\xi-\omega$	$2\xi+\omega$	3\omega

$$\xi^3 = 1, \omega = \xi^2$$

Table B.7: Character table of  $B(3,5)$  (continued)

$[g]$	$2a$									
$[\bar{g}]$	$6T$	$6U$	$6V$	$6W$	$6X$	$6Y$	$6Z$	$6AB$	$6AC$	$6AD$
$\chi_1$	1	1	1	1	1	1	1	1	1	1
$\chi_2$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
$\chi_3$	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
$\chi_4$	2	2	2	2	2	2	2	2	2	2
$\chi_5$	1	1	1	1	1	1	1	1	1	1
$\chi_6$	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
$\chi_7$	0	0	0	0	0	0	0	0	0	0
$\chi_8$	3	$-\xi-2\omega$	$-2\xi-\omega$	$\xi-\omega$	0	$-\xi+\omega$	$3\xi$	$2\xi+\omega$	$\xi+2\omega$	$3\omega$
$\chi_9$	-3	$\xi+2\omega$	$2\xi+\omega$	$-\xi+\omega$	0	$\xi-\omega$	-3\xi	$-2\xi-\omega$	$-\xi-2\omega$	-3\omega
$\chi_{10}$	0	0	0	0	0	0	0	0	0	0
$\chi_{11}$	-3	$\xi+2\omega$	$2\xi+\omega$	$-\xi+\omega$	0	$\xi-\omega$	-3\xi	$-2\xi-\omega$	$-\xi-2\omega$	-3\omega
$\chi_{12}$	3	$-\xi-2\omega$	$-2\xi-\omega$	$\xi-\omega$	0	$-\xi+\omega$	$3\xi$	$2\xi+\omega$	$\xi+2\omega$	$3\omega$
$\chi_{13}$	3	$-2\xi-\omega$	$-\xi-2\omega$	$-\xi+\omega$	0	$\xi-\omega$	$3\omega$	$\xi+2\omega$	$2\xi+\omega$	$3\xi$
$\chi_{14}$	-3	$2\xi+\omega$	$\xi+2\omega$	$\xi-\omega$	0	$-\xi+\omega$	-3\omega	$-\xi-2\omega$	$-2\xi-\omega$	-3\xi
$\chi_{15}$	0	0	0	0	0	0	0	0	0	0
$\chi_{16}$	-3	$2\xi+\omega$	$\xi+2\omega$	$\xi-\omega$	0	$-\xi+\omega$	-3\omega	$-\xi-2\omega$	$-2\xi-\omega$	-3\xi
$\chi_{17}$	3	$-2\xi-\omega$	$-\xi-2\omega$	$-\xi+\omega$	0	$\xi-\omega$	$3\omega$	$\xi+2\omega$	$2\xi+\omega$	$3\xi$
$\chi_{18}$	$-3\xi-2\omega$	$\xi$	$-\xi+2\omega$	-2	$\omega$	$\xi+3\omega$	$4\omega$	-2\xi	1	$3\xi+\omega$
$\chi_{19}$	$-3\xi-4\omega$	$\xi-2\omega$	$-\xi$	$2\xi$	$-\omega$	-1	$2\omega$	2	$-\xi-3\omega$	$3\xi-\omega$
$\chi_{20}$	$3\xi+4\omega$	$-\xi+2\omega$	$\xi$	$-2\xi$	$\omega$	1	-2\omega	-2	$\xi+3\omega$	$-3\xi+\omega$
$\chi_{21}$	$3\xi+2\omega$	$-\xi$	$\xi-2\omega$	2	$-\omega$	$-\xi-3\omega$	-4\omega	$2\xi$	-1	$-3\xi-\omega$
$\chi_{22}$	-6	$-2\xi+2\omega$	$2\xi-2\omega$	$-4\xi-2\omega$	0	$-2\xi-4\omega$	-6\omega	$4\xi+2\omega$	$2\xi+4\omega$	-6\xi
$\chi_{23}$	6	$2\xi-2\omega$	$-2\xi+2\omega$	$4\xi+2\omega$	0	$2\xi+4\omega$	$6\omega$	$-4\xi-2\omega$	$-2\xi-4\omega$	$6\xi$
$\chi_{24}$	6	0	0	0	-3	0	6	0	0	6
$\chi_{25}$	-6	0	0	0	3	0	-6	0	0	-6
$\chi_{26}$	0	0	0	0	0	0	0	0	0	0
$\chi_{27}$	$-2\xi-3\omega$	$\omega$	$2\xi-\omega$	-2	$\xi$	$3\xi+\omega$	$4\xi$	-2\omega	1	$\xi+3\omega$
$\chi_{28}$	$-4\xi-3\omega$	$-2\xi+\omega$	$-\omega$	$2\omega$	$-\xi$	-1	$2\xi$	2	$-3\xi-\omega$	$-\xi+3\omega$
$\chi_{29}$	$4\xi+3\omega$	$2\xi-\omega$	$\omega$	$-2\omega$	$\xi$	1	-2\xi	-2	$3\xi+\omega$	$\xi-3\omega$
$\chi_{30}$	$2\xi+3\omega$	$-\omega$	$-2\xi+\omega$	2	$-\xi$	$-3\xi-\omega$	-4\xi	$2\omega$	-1	$-\xi-3\omega$
$\chi_{31}$	-6	$2\xi-2\omega$	$-2\xi+2\omega$	$-2\xi-4\omega$	0	$-4\xi-2\omega$	-6\xi	$2\xi+4\omega$	$4\xi+2\omega$	-6\omega
$\chi_{32}$	6	$-2\xi+2\omega$	$2\xi-2\omega$	$2\xi+4\omega$	0	$4\xi+2\omega$	$6\xi$	$-2\xi-4\omega$	$-4\xi-2\omega$	$6\omega$
$\chi_{33}$	$-\xi+2\omega$	$\omega$	$\xi+3\omega$	-2\xi	1	$3\xi+\omega$	4	-2\omega	$\xi$	$2\xi-\omega$
$\chi_{34}$	$-\xi-4\omega$	$2\xi-\omega$	1	-2	1	1	-2	-2	$-\xi+2\omega$	$-4\xi-\omega$
$\chi_{35}$	$\xi+4\omega$	$-2\xi+\omega$	-1	2	-1	-1	2	2	$\xi-2\omega$	$4\xi+\omega$
$\chi_{36}$	$\xi-2\omega$	$-\omega$	$-\xi-3\omega$	$2\xi$	-1	$-3\xi-\omega$	-4	$2\omega$	$-\xi$	$-2\xi+\omega$
$\chi_{37}$	-6\omega	$2\xi-2\omega$	$-2\xi-4\omega$	$4\xi+2\omega$	0	$-4\xi-2\omega$	-6	$2\xi+4\omega$	$-2\xi+2\omega$	-6\xi
$\chi_{38}$	6\omega	$-2\xi+2\omega$	$2\xi+4\omega$	$-4\xi-2\omega$	0	$4\xi+2\omega$	6	$-2\xi-4\omega$	$2\xi-2\omega$	$6\xi$
$\chi_{39}$	-3\xi	-3	$3\omega$	0	0	3	$6\xi$	0	-3\omega	-3\xi
$\chi_{40}$	$-3\xi-6\omega$	$\xi-\omega$	$2\xi+\omega$	$-4\xi-2\omega$	0	$\xi-\omega$	0	$-2\xi+2\omega$	$2\xi+\omega$	$3\xi+6\omega$
$\chi_{41}$	$3\xi+6\omega$	$-\xi+\omega$	$-2\xi-\omega$	$4\xi+2\omega$	0	$-\xi+\omega$	0	$2\xi-2\omega$	$-2\xi-\omega$	$-3\xi-6\omega$
$\chi_{42}$	$3\xi$	3	-3\omega	0	0	-3	-6\xi	0	$3\omega$	$3\xi$
$\chi_{43}$	-3\omega	-3	$3\xi$	0	0	3	$6\omega$	0	-3\xi	-3\omega
$\chi_{44}$	$-6\xi-3\omega$	$-\xi+\omega$	$\xi+2\omega$	$-2\xi-4\omega$	0	$-\xi+\omega$	0	$2\xi-2\omega$	$\xi+2\omega$	$6\xi+3\omega$
$\chi_{45}$	$6\xi+3\omega$	$\xi-\omega$	$-\xi-2\omega$	$2\xi+4\omega$	0	$\xi-\omega$	0	$-2\xi+2\omega$	$-\xi-2\omega$	$-6\xi-3\omega$
$\chi_{46}$	$3\omega$	3	-3\xi	0	0	-3	-6\omega	0	$3\xi$	$3\omega$
$\chi_{47}$	$2\xi-\omega$	$\xi$	$3\xi+\omega$	-2\omega	1	$\xi+3\omega$	4	-2\xi	$\omega$	$-\xi+2\omega$
$\chi_{48}$	$-4\xi-\omega$	$-\xi+2\omega$	1	-2	1	1	-2	-2	$2\xi-\omega$	$-\xi-4\omega$
$\chi_{49}$	$4\xi+\omega$	$\xi-2\omega$	-1	2	-1	-1	2	2	$-2\xi+\omega$	$\xi+4\omega$
$\chi_{50}$	$-2\xi+\omega$	$-\xi$	$-3\xi-\omega$	$2\omega$	-1	$-\xi-3\omega$	-4	$2\xi$	$-\omega$	$\xi-2\omega$
$\chi_{51}$	-6\xi	$-2\xi+2\omega$	$-4\xi-2\omega$	$2\xi+4\omega$	0	$-2\xi-4\omega$	-6	$4\xi+2\omega$	$2\xi-2\omega$	-6\omega
$\chi_{52}$	6\xi	$2\xi-2\omega$	$4\xi+2\omega$	$-2\xi-4\omega$	0	$2\xi+4\omega$	6	$-4\xi-2\omega$	$-2\xi+2\omega$	$6\omega$
$\chi_{53}$	$3\omega$	$-2\xi-\omega$	$2\xi+\omega$	$-\xi-2\omega$	0	$\xi-\omega$	$3\xi$	$\xi+2\omega$	$-\xi+\omega$	3
$\chi_{54}$	-3\omega	$2\xi+\omega$	$-2\xi-\omega$	$\xi+2\omega$	0	$-\xi+\omega$	-3\xi	$-\xi-2\omega$	$\xi-\omega$	-3
$\chi_{55}$	0	0	0	0	0	0	0	0	0	0
$\chi_{56}$	-3\omega	$2\xi+\omega$	$-2\xi-\omega$	$\xi+2\omega$	0	$-\xi+\omega$	-3\xi	$-\xi-2\omega$	$\xi-\omega$	-3
$\chi_{57}$	$3\omega$	$-2\xi-\omega$	$2\xi+\omega$	$-\xi-2\omega$	0	$\xi-\omega$	$3\xi$	$\xi+2\omega$	$-\xi+\omega$	3

$$\xi^3 = 1, \omega = \xi^2$$

Table B.8: Character table of  $B(3,5)$  (continued)

$[g]$	2a									
$[g]$	2A	6A	6B	6C	6D	6E	6F	6G	6H	6I
X58	6	0	0	0	-3	0	6	0	0	6
X59	-6	0	0	0	3	0	-6	0	0	-6
X60	0	0	0	0	0	0	0	0	0	0
X61	6	0	0	-3	0	-3	-3	3	3	-3
X62	0	$-2\xi+2\omega$	$2\xi-2\omega$	$-\xi+\omega$	0	$\xi-\omega$	$3\xi-3\omega$	$\xi-\omega$	$-\xi+\omega$	$-3\xi+3\omega$
X63	0	$2\xi-2\omega$	$-2\xi+2\omega$	$\xi-\omega$	0	$-\xi+\omega$	$-3\xi+3\omega$	$-\xi+\omega$	$\xi-\omega$	$3\xi-3\omega$
X64	-6	0	0	3	0	3	3	-3	-3	3
X65	6	0	0	0	-3	0	6	0	0	6
X66	-6	0	0	0	3	0	-6	0	0	-6
X67	0	0	0	0	0	0	0	0	0	0
X68	3	$-\xi+\omega$	$\xi-\omega$	$\xi+2\omega$	0	$2\xi+\omega$	$3\xi$	$-\xi-2\omega$	$-2\xi-\omega$	$3\omega$
X69	-3	$\xi-\omega$	$-\xi+\omega$	$-\xi-2\omega$	0	$-2\xi-\omega$	$-3\xi$	$\xi+2\omega$	$2\xi+\omega$	$-3\omega$
X70	0	0	0	0	0	0	0	0	0	0
X71	-3	$\xi-\omega$	$-\xi+\omega$	$-\xi-2\omega$	0	$-2\xi-\omega$	$-3\xi$	$\xi+2\omega$	$2\xi+\omega$	$-3\omega$
X72	3	$-\xi+\omega$	$\xi-\omega$	$\xi+2\omega$	0	$2\xi+\omega$	$3\xi$	$-\xi-2\omega$	$-2\xi-\omega$	$3\omega$
X73	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	1
X74	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	-1	$-\xi$	$-\omega$	-1
X75	-2	$-2\xi$	$-2\omega$	$-2\omega$	-2	$-2\xi$	-2	$-2\xi$	$-2\omega$	-2
X76	2	$2\xi$	$2\omega$	$2\omega$	2	$2\xi$	2	$2\xi$	$2\omega$	2
X77	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	1
X78	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	-1	$-\xi$	$-\omega$	-1
X79	0	0	0	0	0	0	0	0	0	0
X80	3	$2\xi+\omega$	$\xi+2\omega$	$-2\xi-\omega$	0	$-\xi-2\omega$	$3\xi$	$-\xi+\omega$	$\xi-\omega$	$3\omega$
X81	-3	$-2\xi-\omega$	$-\xi-2\omega$	$2\xi+\omega$	0	$\xi+2\omega$	$-3\xi$	$\xi-\omega$	$-\xi+\omega$	$-3\omega$
X82	0	0	0	0	0	0	0	0	0	0
X83	-3	$-2\xi-\omega$	$-\xi-2\omega$	$2\xi+\omega$	0	$\xi+2\omega$	$-3\xi$	$\xi-\omega$	$-\xi+\omega$	$-3\omega$
X84	3	$2\xi+\omega$	$\xi+2\omega$	$-2\xi-\omega$	0	$-\xi-2\omega$	$3\xi$	$-\xi+\omega$	$\xi-\omega$	$3\omega$
X85	4	-2	-2	1	1	1	$-\xi+2\omega$	$2\xi-\omega$	$-\xi+2\omega$	$2\xi-\omega$
X86	2	$2\omega$	$2\xi$	$-\xi-3\omega$	-1	$-3\xi-\omega$	$\xi+4\omega$	$-\omega$	$-\xi$	$4\xi+\omega$
X87	-2	$-2\omega$	$-2\xi$	$\xi+3\omega$	1	$3\xi+\omega$	$-\xi-4\omega$	$\omega$	$\xi$	$-4\xi-\omega$
X88	-4	2	2	-1	-1	-1	$\xi-2\omega$	$-2\xi+\omega$	$\xi-2\omega$	$-2\xi+\omega$
X89	-6	$-2\xi-4\omega$	$-4\xi-2\omega$	$2\xi+4\omega$	0	$4\xi+2\omega$	$-6\omega$	$-2\xi+2\omega$	$2\xi-2\omega$	$-6\xi$
X90	6	$2\xi+4\omega$	$4\xi+2\omega$	$-2\xi-4\omega$	0	$-4\xi-2\omega$	$6\omega$	$2\xi-2\omega$	$-2\xi+2\omega$	$6\xi$
X91	4	-2	-2	1	1	1	$2\xi-\omega$	$-\xi+2\omega$	$2\xi-\omega$	$-\xi+2\omega$
X92	-2	$-2\xi$	$-2\omega$	$3\xi+\omega$	1	$\xi+3\omega$	$-4\xi-\omega$	$\xi$	$\omega$	$-\xi-4\omega$
X93	2	$2\xi$	$2\omega$	$-3\xi-\omega$	-1	$-\xi-3\omega$	$4\xi+\omega$	$-\xi$	$-\omega$	$\xi+4\omega$
X94	-4	2	2	-1	-1	-1	$-2\xi+\omega$	$\xi-2\omega$	$-2\xi+\omega$	$\xi-2\omega$
X95	-6	$-4\xi-2\omega$	$-2\xi-4\omega$	$4\xi+2\omega$	0	$2\xi+4\omega$	$-6\xi$	$2\xi-2\omega$	$-2\xi+2\omega$	$-6\omega$
X96	6	$4\xi+2\omega$	$2\xi+4\omega$	$-4\xi-2\omega$	0	$-2\xi-4\omega$	$6\xi$	$-2\xi+2\omega$	$2\xi-2\omega$	$6\omega$
X97	3	$\xi+2\omega$	$2\xi+\omega$	$-\xi-2\omega$	0	$-2\xi-\omega$	$3\omega$	$\xi-\omega$	$-\xi+\omega$	$3\xi$
X98	-3	$-\xi-2\omega$	$-2\xi-\omega$	$\xi+2\omega$	0	$2\xi+\omega$	$-3\omega$	$-\xi+\omega$	$\xi-\omega$	$-3\xi$
X99	0	0	0	0	0	0	0	0	0	0
X100	-3	$-\xi-2\omega$	$-2\xi-\omega$	$\xi+2\omega$	0	$2\xi+\omega$	$-3\omega$	$-\xi+\omega$	$\xi-\omega$	$-3\xi$
X101	3	$\xi+2\omega$	$2\xi+\omega$	$-\xi-2\omega$	0	$-2\xi-\omega$	$3\omega$	$\xi-\omega$	$-\xi+\omega$	$3\xi$
X102	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	1
X103	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	-1	$-\omega$	$-\xi$	-1
X104	-2	$-2\omega$	$-2\xi$	$-2\xi$	-2	$-2\omega$	-2	$-2\omega$	$-2\xi$	-2
X105	2	$2\omega$	$2\xi$	$2\xi$	2	$2\omega$	2	$2\omega$	$2\xi$	2
X106	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	1
X107	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	-1	$-\omega$	$-\xi$	-1
X108	0	0	0	0	0	0	0	0	0	0

$$\xi^3 = 1, \omega = \xi^2$$

Table B.9: Character table of  $B(3,5)$  (continued)

$[g]$	<b>2a</b>									
$[\bar{g}]$	<b>6J</b>	<b>6K</b>	<b>6L</b>	<b>6M</b>	<b>6N</b>	<b>6O</b>	<b>6P</b>	<b>6Q</b>	<b>6R</b>	<b>6S</b>
$\chi_{58}$	6 $\xi$	0	0	0	-3 $\xi$	0	6 $\xi$	0	0	6 $\xi$
$\chi_{59}$	-6 $\xi$	0	0	0	3 $\xi$	0	-6 $\xi$	0	0	-6 $\xi$
$\chi_{60}$	0	0	0	0	0	0	0	0	0	0
$\chi_{61}$	-3	3	-3	3	0	0	-3	-3	0	6
$\chi_{62}$	3 $\xi$ -3 $\omega$	$\xi$ - $\omega$	- $\xi$ + $\omega$	- $\xi$ + $\omega$	0	-2 $\xi$ +2 $\omega$	-3 $\xi$ +3 $\omega$	$\xi$ - $\omega$	2 $\xi$ -2 $\omega$	0
$\chi_{63}$	-3 $\xi$ +3 $\omega$	- $\xi$ + $\omega$	$\xi$ - $\omega$	$\xi$ - $\omega$	0	2 $\xi$ -2 $\omega$	3 $\xi$ -3 $\omega$	- $\xi$ + $\omega$	-2 $\xi$ +2 $\omega$	0
$\chi_{64}$	3	-3	3	-3	0	0	3	3	0	-6
$\chi_{65}$	6 $\omega$	0	0	0	-3 $\omega$	0	6 $\omega$	0	0	6 $\omega$
$\chi_{66}$	-6 $\omega$	0	0	0	3 $\omega$	0	-6 $\omega$	0	0	-6 $\omega$
$\chi_{67}$	0	0	0	0	0	0	0	0	0	0
$\chi_{68}$	3 $\omega$	2 $\xi$ + $\omega$	-2 $\xi$ - $\omega$	$\xi$ - $\omega$	0	- $\xi$ -2 $\omega$	3	- $\xi$ + $\omega$	$\xi$ +2 $\omega$	3 $\xi$
$\chi_{69}$	-3 $\omega$	-2 $\xi$ - $\omega$	2 $\xi$ + $\omega$	- $\xi$ + $\omega$	0	$\xi$ +2 $\omega$	-3	$\xi$ - $\omega$	- $\xi$ -2 $\omega$	-3 $\xi$
$\chi_{70}$	0	0	0	0	0	0	0	0	0	0
$\chi_{71}$	-3 $\omega$	-2 $\xi$ - $\omega$	2 $\xi$ + $\omega$	- $\xi$ + $\omega$	0	$\xi$ +2 $\omega$	-3	$\xi$ - $\omega$	- $\xi$ -2 $\omega$	-3 $\xi$
$\chi_{72}$	3 $\omega$	2 $\xi$ + $\omega$	-2 $\xi$ - $\omega$	$\xi$ - $\omega$	0	- $\xi$ -2 $\omega$	3	- $\xi$ + $\omega$	$\xi$ +2 $\omega$	3 $\xi$
$\chi_{73}$	$\xi$	$\omega$	1	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\xi$
$\chi_{74}$	- $\xi$	- $\omega$	-1	-1	- $\xi$	- $\omega$	- $\xi$	- $\omega$	-1	- $\xi$
$\chi_{75}$	-2 $\xi$	-2 $\omega$	-2	-2	-2 $\xi$	-2 $\omega$	-2 $\xi$	-2 $\omega$	-2	-2 $\xi$
$\chi_{76}$	2 $\xi$	2 $\omega$	2	2	2 $\xi$	2 $\omega$	2 $\xi$	2 $\omega$	2	2 $\xi$
$\chi_{77}$	$\xi$	$\omega$	1	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\xi$
$\chi_{78}$	- $\xi$	- $\omega$	-1	-1	- $\xi$	- $\omega$	- $\xi$	- $\omega$	-1	- $\xi$
$\chi_{79}$	0	0	0	0	0	0	0	0	0	0
$\chi_{80}$	3 $\xi$	- $\xi$ + $\omega$	-2 $\xi$ - $\omega$	$\xi$ - $\omega$	0	2 $\xi$ + $\omega$	3 $\omega$	- $\xi$ -2 $\omega$	$\xi$ +2 $\omega$	3
$\chi_{81}$	-3 $\xi$	$\xi$ - $\omega$	2 $\xi$ + $\omega$	- $\xi$ + $\omega$	0	-2 $\xi$ - $\omega$	-3 $\omega$	$\xi$ +2 $\omega$	- $\xi$ -2 $\omega$	-3
$\chi_{82}$	0	0	0	0	0	0	0	0	0	0
$\chi_{83}$	-3 $\xi$	$\xi$ - $\omega$	2 $\xi$ + $\omega$	- $\xi$ + $\omega$	0	-2 $\xi$ - $\omega$	-3 $\omega$	$\xi$ +2 $\omega$	- $\xi$ -2 $\omega$	-3
$\chi_{84}$	3 $\xi$	- $\xi$ + $\omega$	-2 $\xi$ - $\omega$	$\xi$ - $\omega$	0	2 $\xi$ + $\omega$	3 $\omega$	- $\xi$ -2 $\omega$	$\xi$ +2 $\omega$	3
$\chi_{85}$	3 $\xi$ + $\omega$	-3 $\xi$ -2 $\omega$	$\omega$	3 $\xi$ + $\omega$	$\omega$	-2 $\omega$	-3 $\xi$ -2 $\omega$	$\omega$	-2 $\omega$	4 $\omega$
$\chi_{86}$	3 $\xi$ - $\omega$	- $\xi$	-2 $\xi$ + $\omega$	-1	- $\omega$	2 $\xi$	-3 $\xi$ -4 $\omega$	2 $\xi$ +3 $\omega$	2	2 $\omega$
$\chi_{87}$	-3 $\xi$ + $\omega$	$\xi$	2 $\xi$ - $\omega$	1	$\omega$	-2 $\xi$	3 $\xi$ +4 $\omega$	-2 $\xi$ -3 $\omega$	-2	-2 $\omega$
$\chi_{88}$	-3 $\xi$ - $\omega$	3 $\xi$ +2 $\omega$	- $\omega$	-3 $\xi$ - $\omega$	- $\omega$	2 $\omega$	3 $\xi$ +2 $\omega$	- $\omega$	2 $\omega$	-4 $\omega$
$\chi_{89}$	-6 $\xi$	4 $\xi$ +2 $\omega$	2 $\xi$ -2 $\omega$	-4 $\xi$ -2 $\omega$	0	-2 $\xi$ +2 $\omega$	-6	-2 $\xi$ -4 $\omega$	2 $\xi$ +4 $\omega$	-6 $\omega$
$\chi_{90}$	6 $\xi$	-4 $\xi$ -2 $\omega$	-2 $\xi$ +2 $\omega$	4 $\xi$ +2 $\omega$	0	2 $\xi$ -2 $\omega$	6	2 $\xi$ +4 $\omega$	-2 $\xi$ -4 $\omega$	6 $\omega$
$\chi_{91}$	$\xi$ +3 $\omega$	-2 $\xi$ -3 $\omega$	$\xi$	$\xi$ +3 $\omega$	$\xi$	-2 $\xi$	-2 $\xi$ -3 $\omega$	$\xi$	-2 $\xi$	4 $\xi$
$\chi_{92}$	$\xi$ -3 $\omega$	$\omega$	- $\xi$ +2 $\omega$	1	$\xi$	-2 $\omega$	4 $\xi$ +3 $\omega$	-3 $\xi$ -2 $\omega$	-2	-2 $\xi$
$\chi_{93}$	- $\xi$ +3 $\omega$	- $\omega$	$\xi$ -2 $\omega$	-1	- $\xi$	2 $\omega$	-4 $\xi$ -3 $\omega$	3 $\xi$ +2 $\omega$	2	2 $\xi$
$\chi_{94}$	- $\xi$ -3 $\omega$	2 $\xi$ +3 $\omega$	- $\xi$	- $\xi$ -3 $\omega$	- $\xi$	2 $\xi$	2 $\xi$ +3 $\omega$	- $\xi$	2 $\xi$	-4 $\xi$
$\chi_{95}$	-6 $\omega$	2 $\xi$ +4 $\omega$	-2 $\xi$ +2 $\omega$	-2 $\xi$ -4 $\omega$	0	2 $\xi$ -2 $\omega$	-6	-4 $\xi$ -2 $\omega$	4 $\xi$ +2 $\omega$	-6 $\xi$
$\chi_{96}$	6 $\omega$	-2 $\xi$ -4 $\omega$	2 $\xi$ -2 $\omega$	2 $\xi$ +4 $\omega$	0	-2 $\xi$ +2 $\omega$	6	4 $\xi$ +2 $\omega$	-4 $\xi$ -2 $\omega$	6 $\xi$
$\chi_{97}$	3 $\omega$	$\xi$ - $\omega$	- $\xi$ -2 $\omega$	- $\xi$ + $\omega$	0	$\xi$ +2 $\omega$	3 $\xi$	-2 $\xi$ - $\omega$	2 $\xi$ + $\omega$	3
$\chi_{98}$	-3 $\omega$	- $\xi$ + $\omega$	$\xi$ +2 $\omega$	$\xi$ - $\omega$	0	- $\xi$ -2 $\omega$	-3 $\xi$	2 $\xi$ + $\omega$	-2 $\xi$ - $\omega$	-3
$\chi_{99}$	0	0	0	0	0	0	0	0	0	0
$\chi_{100}$	-3 $\omega$	- $\xi$ + $\omega$	$\xi$ +2 $\omega$	$\xi$ - $\omega$	0	- $\xi$ -2 $\omega$	-3 $\xi$	2 $\xi$ + $\omega$	-2 $\xi$ - $\omega$	-3
$\chi_{101}$	3 $\omega$	$\xi$ - $\omega$	- $\xi$ -2 $\omega$	- $\xi$ + $\omega$	0	$\xi$ +2 $\omega$	3 $\xi$	-2 $\xi$ - $\omega$	2 $\xi$ + $\omega$	3
$\chi_{102}$	$\omega$	$\xi$	1	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\omega$
$\chi_{103}$	- $\omega$	- $\xi$	-1	-1	- $\omega$	- $\xi$	- $\omega$	- $\xi$	-1	- $\omega$
$\chi_{104}$	-2 $\omega$	-2 $\xi$	-2	-2	-2 $\omega$	-2 $\xi$	-2 $\omega$	-2 $\xi$	-2	-2 $\omega$
$\chi_{105}$	2 $\omega$	2 $\xi$	2	2	2 $\omega$	2 $\xi$	2 $\omega$	2 $\xi$	2	2 $\omega$
$\chi_{106}$	$\omega$	$\xi$	1	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\omega$
$\chi_{107}$	- $\omega$	- $\xi$	-1	-1	- $\omega$	- $\xi$	- $\omega$	- $\xi$	-1	- $\omega$
$\chi_{108}$	0	0	0	0	0	0	0	0	0	0

$$\xi^3 = 1, \omega = \xi^2$$

Table B.10: Character table of  $B(3, 5)$  (continued)

[g]	2a									
[g]	6T	6U	6V	6W	6X	6Y	6Z	6AB	6AC	6AD
X58	6 $\omega$	0	0	0	-3 $\omega$	0	6 $\omega$	0	0	6 $\omega$
X59	-6 $\omega$	0	0	0	3 $\omega$	0	-6 $\omega$	0	0	-6 $\omega$
X60	0	0	0	0	0	0	0	0	0	0
X61	-3	-3	3	0	0	3	6	0	-3	-3
X62	-3 $\xi$ +3 $\omega$	$\xi$ - $\omega$	$-\xi$ + $\omega$	2 $\xi$ -2 $\omega$	0	$\xi$ - $\omega$	0	-2 $\xi$ +2 $\omega$	$-\xi$ + $\omega$	3 $\xi$ -3 $\omega$
X63	3 $\xi$ -3 $\omega$	$-\xi$ + $\omega$	$\xi$ - $\omega$	-2 $\xi$ +2 $\omega$	0	$-\xi$ + $\omega$	0	2 $\xi$ -2 $\omega$	$\xi$ - $\omega$	-3 $\xi$ +3 $\omega$
X64	3	3	-3	0	0	-3	-6	0	3	3
X65	6 $\xi$	0	0	0	-3 $\xi$	0	6 $\xi$	0	0	6 $\xi$
X66	-6 $\xi$	0	0	0	3 $\xi$	0	-6 $\xi$	0	0	-6 $\xi$
X67	0	0	0	0	0	0	0	0	0	0
X68	3 $\xi$	$-\xi$ -2 $\omega$	$\xi$ +2 $\omega$	-2 $\xi$ - $\omega$	0	$-\xi$ + $\omega$	3 $\omega$	2 $\xi$ + $\omega$	$\xi$ - $\omega$	3
X69	-3 $\xi$	$\xi$ +2 $\omega$	$-\xi$ -2 $\omega$	2 $\xi$ + $\omega$	0	$\xi$ - $\omega$	-3 $\omega$	-2 $\xi$ - $\omega$	$-\xi$ + $\omega$	-3
X70	0	0	0	0	0	0	0	0	0	0
X71	-3 $\xi$	$\xi$ +2 $\omega$	$-\xi$ -2 $\omega$	2 $\xi$ + $\omega$	0	$\xi$ - $\omega$	-3 $\omega$	-2 $\xi$ - $\omega$	$-\xi$ + $\omega$	-3
X72	3 $\xi$	$-\xi$ -2 $\omega$	$\xi$ +2 $\omega$	-2 $\xi$ - $\omega$	0	$-\xi$ + $\omega$	3 $\omega$	2 $\xi$ + $\omega$	$\xi$ - $\omega$	3
X73	$\omega$	1	$\xi$	$\xi$	$\omega$	1	$\omega$	1	$\xi$	$\omega$
X74	- $\omega$	-1	$-\xi$	$-\xi$	- $\omega$	-1	- $\omega$	-1	$-\xi$	- $\omega$
X75	-2 $\omega$	-2	-2 $\xi$	-2 $\xi$	-2 $\omega$	-2	-2 $\omega$	-2	-2 $\xi$	-2 $\omega$
X76	2 $\omega$	2	2 $\xi$	2 $\xi$	2 $\omega$	2	2 $\omega$	2	2 $\xi$	2 $\omega$
X77	$\omega$	1	$\xi$	$\xi$	$\omega$	1	$\omega$	1	$\xi$	$\omega$
X78	- $\omega$	-1	$-\xi$	$-\xi$	- $\omega$	-1	- $\omega$	-1	$-\xi$	- $\omega$
X79	0	0	0	0	0	0	0	0	0	0
X80	3 $\omega$	$-\xi$ -2 $\omega$	$\xi$ - $\omega$	$\xi$ +2 $\omega$	0	$-\xi$ + $\omega$	3	2 $\xi$ + $\omega$	-2 $\xi$ - $\omega$	3 $\xi$
X81	-3 $\omega$	$\xi$ +2 $\omega$	$-\xi$ + $\omega$	$-\xi$ -2 $\omega$	0	$\xi$ - $\omega$	-3	-2 $\xi$ - $\omega$	2 $\xi$ + $\omega$	-3 $\xi$
X82	0	0	0	0	0	0	0	0	0	0
X83	-3 $\omega$	$\xi$ +2 $\omega$	$-\xi$ + $\omega$	$-\xi$ -2 $\omega$	0	$\xi$ - $\omega$	-3	-2 $\xi$ - $\omega$	2 $\xi$ + $\omega$	-3 $\xi$
X84	3 $\omega$	$-\xi$ -2 $\omega$	$\xi$ - $\omega$	$\xi$ +2 $\omega$	0	$-\xi$ + $\omega$	3	2 $\xi$ + $\omega$	-2 $\xi$ - $\omega$	3 $\xi$
X85	$\xi$ +3 $\omega$	$\xi$	-2 $\xi$ -3 $\omega$	-2 $\xi$	$\xi$	$\xi$ +3 $\omega$	4 $\xi$	-2 $\xi$	$\xi$	-2 $\xi$ -3 $\omega$
X86	$-\xi$ +3 $\omega$	$\xi$ -2 $\omega$	- $\omega$	2 $\omega$	$-\xi$	-1	2 $\xi$	2	3 $\xi$ +2 $\omega$	-4 $\xi$ -3 $\omega$
X87	$\xi$ -3 $\omega$	$-\xi$ +2 $\omega$	$\omega$	-2 $\omega$	$\xi$	1	-2 $\xi$	-2	-3 $\xi$ -2 $\omega$	4 $\xi$ +3 $\omega$
X88	$-\xi$ -3 $\omega$	$-\xi$	2 $\xi$ +3 $\omega$	2 $\xi$	$-\xi$	$-\xi$ -3 $\omega$	-4 $\xi$	2 $\xi$	$-\xi$	2 $\xi$ +3 $\omega$
X89	-6 $\omega$	-2 $\xi$ +2 $\omega$	2 $\xi$ +4 $\omega$	2 $\xi$ -2 $\omega$	0	-2 $\xi$ -4 $\omega$	-6 $\xi$	4 $\xi$ +2 $\omega$	-4 $\xi$ -2 $\omega$	-6
X90	6 $\omega$	2 $\xi$ -2 $\omega$	-2 $\xi$ -4 $\omega$	-2 $\xi$ +2 $\omega$	0	2 $\xi$ +4 $\omega$	6 $\xi$	-4 $\xi$ -2 $\omega$	4 $\xi$ +2 $\omega$	6
X91	3 $\xi$ + $\omega$	$\omega$	-3 $\xi$ -2 $\omega$	-2 $\omega$	$\omega$	3 $\xi$ + $\omega$	4 $\omega$	-2 $\omega$	$\omega$	-3 $\xi$ -2 $\omega$
X92	-3 $\xi$ + $\omega$	2 $\xi$ - $\omega$	$\xi$	-2 $\xi$	$\omega$	1	-2 $\omega$	-2	-2 $\xi$ -3 $\omega$	3 $\xi$ +4 $\omega$
X93	3 $\xi$ - $\omega$	-2 $\xi$ + $\omega$	$-\xi$	2 $\xi$	- $\omega$	-1	2 $\omega$	2	2 $\xi$ +3 $\omega$	-3 $\xi$ -4 $\omega$
X94	-3 $\xi$ - $\omega$	- $\omega$	3 $\xi$ +2 $\omega$	2 $\omega$	- $\omega$	-3 $\xi$ - $\omega$	-4 $\omega$	2 $\omega$	- $\omega$	3 $\xi$ +2 $\omega$
X95	-6 $\xi$	2 $\xi$ -2 $\omega$	4 $\xi$ +2 $\omega$	-2 $\xi$ +2 $\omega$	0	-4 $\xi$ -2 $\omega$	-6 $\omega$	2 $\xi$ +4 $\omega$	-2 $\xi$ -4 $\omega$	-6
X96	6 $\xi$	-2 $\xi$ +2 $\omega$	-4 $\xi$ -2 $\omega$	2 $\xi$ -2 $\omega$	0	4 $\xi$ +2 $\omega$	6 $\omega$	-2 $\xi$ -4 $\omega$	2 $\xi$ +4 $\omega$	6
X97	3 $\xi$	-2 $\xi$ - $\omega$	$-\xi$ + $\omega$	2 $\xi$ + $\omega$	0	$\xi$ - $\omega$	3	$\xi$ +2 $\omega$	$-\xi$ -2 $\omega$	3 $\omega$
X98	-3 $\xi$	2 $\xi$ + $\omega$	$\xi$ - $\omega$	-2 $\xi$ - $\omega$	0	$-\xi$ + $\omega$	-3	$-\xi$ -2 $\omega$	$\xi$ +2 $\omega$	-3 $\omega$
X99	0	0	0	0	0	0	0	0	0	0
X100	-3 $\xi$	2 $\xi$ + $\omega$	$\xi$ - $\omega$	-2 $\xi$ - $\omega$	0	$-\xi$ + $\omega$	-3	$-\xi$ -2 $\omega$	$\xi$ +2 $\omega$	-3 $\omega$
X101	3 $\xi$	-2 $\xi$ - $\omega$	$-\xi$ + $\omega$	2 $\xi$ + $\omega$	0	$\xi$ - $\omega$	3	$\xi$ +2 $\omega$	$-\xi$ -2 $\omega$	3 $\omega$
X102	$\xi$	1	$\omega$	$\omega$	$\xi$	1	$\xi$	1	$\omega$	$\xi$
X103	$-\xi$	-1	- $\omega$	- $\omega$	$-\xi$	-1	$-\xi$	-1	- $\omega$	$-\xi$
X104	-2 $\xi$	-2	-2 $\omega$	-2 $\omega$	-2 $\xi$	-2	-2 $\xi$	-2	-2 $\omega$	-2 $\xi$
X105	2 $\xi$	2	2 $\omega$	2 $\omega$	2 $\xi$	2	2 $\xi$	2	2 $\omega$	2 $\xi$
X106	$\xi$	1	$\omega$	$\omega$	$\xi$	1	$\xi$	1	$\omega$	$\xi$
X107	$-\xi$	-1	- $\omega$	- $\omega$	$-\xi$	-1	$-\xi$	-1	- $\omega$	$-\xi$
X108	0	0	0	0	0	0	0	0	0	0

$$\xi^3 = 1, \omega = \xi^2$$

Table B.11: Character table of  $B(3,5)$  (continued)

$[g]$	$3a$																	
$[g]$	3U	3V	3W	3X	3Y	3Z	9A	9B	9C	9D	9E	9F	9G	9H	9I	9J	9K	9L
X1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X5	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
X6	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
X7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X8	2	$-\omega$	$-\xi$	$2\xi$	-1	$2\omega$	2	$-\omega$	$-\xi$	$2\xi$	-1	$2\omega$	2	$-\omega$	$-\xi$	$2\xi$	-1	$2\omega$
X9	2	$-\omega$	$-\xi$	$2\xi$	-1	$2\omega$	2	$-\omega$	$-\xi$	$2\xi$	-1	$2\omega$	2	$-\omega$	$-\xi$	$2\xi$	-1	$2\omega$
X10	-2	$\omega$	$\xi$	$-2\xi$	1	$-2\omega$	-2	$\omega$	$\xi$	$-2\xi$	1	$-2\omega$	-2	$\omega$	$\xi$	$-2\xi$	1	$-2\omega$
X11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X13	2	$-\xi$	$-\omega$	$2\omega$	-1	$2\xi$	2	$-\xi$	$-\omega$	$2\omega$	-1	$2\xi$	2	$-\xi$	$-\omega$	$2\omega$	-1	$2\xi$
X14	2	$-\xi$	$-\omega$	$2\omega$	-1	$2\xi$	2	$-\xi$	$-\omega$	$2\omega$	-1	$2\xi$	2	$-\xi$	$-\omega$	$2\omega$	-1	$2\xi$
X15	-2	$\xi$	$\omega$	$-2\omega$	1	$-2\xi$	-2	$\xi$	$\omega$	$-2\omega$	1	$-2\xi$	-2	$\xi$	$\omega$	$-2\omega$	1	$-2\xi$
X16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X18	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\omega$	1	$\xi$
X19	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\omega$	1	$\xi$
X20	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\omega$	1	$\xi$
X21	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\omega$	1	$\xi$
X22	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$
X23	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$
X24	2	-1	-1	2	-1	2	2	-1	-1	2	-1	2	2	-1	-1	2	-1	2
X25	2	-1	-1	2	-1	2	2	-1	-1	2	-1	2	2	-1	-1	2	-1	2
X26	-2	1	1	-2	1	-2	-2	1	1	-2	1	-2	-2	1	1	-2	1	-2
X27	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\xi$	1	$\omega$
X28	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\xi$	1	$\omega$
X29	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\xi$	1	$\omega$
X30	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\xi$	1	$\omega$
X31	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$
X32	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$
X33	1	1	1	1	1	1	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$
X34	1	1	1	1	1	1	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$
X35	1	1	1	1	1	1	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$
X36	1	1	1	1	1	1	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$
X37	-1	-1	-1	-1	-1	-1	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$
X38	-1	-1	-1	-1	-1	-1	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$
X39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X41	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X42	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X43	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X44	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X45	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X46	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X47	1	1	1	1	1	1	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$
X48	1	1	1	1	1	1	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$
X49	1	1	1	1	1	1	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$
X50	1	1	1	1	1	1	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$
X51	-1	-1	-1	-1	-1	-1	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$
X52	-1	-1	-1	-1	-1	-1	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$
X53	2	$-\omega$	$-\xi$	$2\xi$	-1	$2\omega$	$2\xi$	-1	$-\omega$	$2\omega$	$-\xi$	2	$2\omega$	$-\xi$	-1	2	$-\omega$	$2\xi$
X54	2	$-\omega$	$-\xi$	$2\xi$	-1	$2\omega$	$2\xi$	-1	$-\omega$	$2\omega$	$-\xi$	2	$2\omega$	$-\xi$	-1	2	$-\omega$	$2\xi$
X55	-2	$\omega$	$\xi$	$-2\xi$	1	$-2\omega$	$-2\xi$	1	$\omega$	$-2\omega$	$\xi$	-2	$-2\omega$	$\xi$	1	-2	$\omega$	$-2\xi$
X56	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X57	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$$\xi^3 = 1, \omega = \xi^2$$

Table B.12: Character table of  $B(3,5)$  (continued)

$[g]$	3a																	
$[g]$	3U	3V	3W	3X	3Y	3Z	9A	9B	9C	9D	9E	9F	9G	9H	9I	9J	9K	9L
X58	2	- $\xi$	- $\omega$	2 $\omega$	-1	2 $\xi$	2 $\xi$	- $\omega$	-1	2	- $\xi$	2 $\omega$	2 $\omega$	-1	- $\xi$	2 $\xi$	- $\omega$	2
X59	2	- $\xi$	- $\omega$	2 $\omega$	-1	2 $\xi$	2 $\xi$	- $\omega$	-1	2	- $\xi$	2 $\omega$	2 $\omega$	-1	- $\xi$	2 $\xi$	- $\omega$	2
X60	-2	$\xi$	$\omega$	-2 $\omega$	1	-2 $\xi$	-2 $\xi$	$\omega$	1	-2	$\xi$	-2 $\omega$	-2 $\omega$	1	$\xi$	-2 $\xi$	$\omega$	-2
X61	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X62	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X63	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X65	2	- $\omega$	- $\xi$	2 $\xi$	-1	2 $\omega$	2 $\omega$	- $\xi$	-1	2	- $\omega$	2 $\xi$	2 $\xi$	-1	- $\omega$	2 $\omega$	- $\xi$	2
X66	2	- $\omega$	- $\xi$	2 $\xi$	-1	2 $\omega$	2 $\omega$	- $\xi$	-1	2	- $\omega$	2 $\xi$	2 $\xi$	-1	- $\omega$	2 $\omega$	- $\xi$	2
X67	-2	$\omega$	$\xi$	-2 $\xi$	1	-2 $\omega$	-2 $\omega$	$\xi$	1	-2	$\omega$	-2 $\xi$	-2 $\xi$	1	$\omega$	-2 $\omega$	$\xi$	-2
X68	2	- $\xi$	- $\omega$	2 $\omega$	-1	2 $\xi$	2 $\omega$	-1	- $\xi$	2 $\xi$	- $\omega$	2	2 $\xi$	- $\omega$	-1	2	- $\xi$	2 $\omega$
X69	2	- $\xi$	- $\omega$	2 $\omega$	-1	2 $\xi$	2 $\omega$	-1	- $\xi$	2 $\xi$	- $\omega$	2	2 $\xi$	- $\omega$	-1	2	- $\xi$	2 $\omega$
X70	-2	$\xi$	$\omega$	-2 $\omega$	1	-2 $\xi$	-2 $\omega$	1	$\xi$	-2	$\omega$	-2	-2 $\xi$	$\omega$	1	-2	$\xi$	-2 $\omega$
X71	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X72	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X73	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1
X74	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1
X75	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1
X76	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1
X77	-1	- $\xi$	- $\omega$	- $\omega$	-1	- $\xi$	- $\xi$	- $\omega$	-1	-1	- $\xi$	- $\omega$	- $\omega$	-1	- $\xi$	- $\xi$	- $\omega$	-1
X78	-1	- $\xi$	- $\omega$	- $\omega$	-1	- $\xi$	- $\xi$	- $\omega$	-1	-1	- $\xi$	- $\omega$	- $\omega$	-1	- $\xi$	- $\xi$	- $\omega$	-1
X79	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X80	2	-1	-1	2	-1	2	2 $\xi$	- $\xi$	- $\xi$	2 $\xi$	- $\xi$	2 $\xi$	2 $\omega$	- $\omega$	- $\omega$	2 $\omega$	- $\omega$	2 $\omega$
X81	2	-1	-1	2	-1	2	2 $\xi$	- $\xi$	- $\xi$	2 $\xi$	- $\xi$	2 $\xi$	2 $\omega$	- $\omega$	- $\omega$	2 $\omega$	- $\omega$	2 $\omega$
X82	-2	1	1	-2	1	-2	-2 $\xi$	$\xi$	$\xi$	-2 $\xi$	$\xi$	-2 $\xi$	-2 $\omega$	$\omega$	$\omega$	-2 $\omega$	$\omega$	-2 $\omega$
X83	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X84	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X85	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	$\omega$	$\xi$	1	1	$\omega$	$\xi$
X86	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	$\omega$	$\xi$	1	1	$\omega$	$\xi$
X87	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	$\omega$	$\xi$	1	1	$\omega$	$\xi$
X88	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	$\omega$	$\xi$	1	1	$\omega$	$\xi$
X89	-1	- $\omega$	- $\xi$	- $\xi$	-1	- $\omega$	- $\xi$	-1	- $\omega$	- $\omega$	- $\xi$	-1	- $\omega$	- $\xi$	-1	-1	- $\omega$	- $\xi$
X90	-1	- $\omega$	- $\xi$	- $\xi$	-1	- $\omega$	- $\xi$	-1	- $\omega$	- $\omega$	- $\xi$	-1	- $\omega$	- $\xi$	-1	-1	- $\omega$	- $\xi$
X91	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	$\xi$	$\omega$	1	1	$\xi$	$\omega$
X92	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	$\xi$	$\omega$	1	1	$\xi$	$\omega$
X93	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	$\xi$	$\omega$	1	1	$\xi$	$\omega$
X94	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	$\xi$	$\omega$	1	1	$\xi$	$\omega$
X95	-1	- $\xi$	- $\omega$	- $\omega$	-1	- $\xi$	- $\omega$	-1	- $\xi$	- $\xi$	- $\omega$	-1	- $\xi$	- $\omega$	-1	-1	- $\xi$	- $\omega$
X96	-1	- $\xi$	- $\omega$	- $\omega$	-1	- $\xi$	- $\omega$	-1	- $\xi$	- $\xi$	- $\omega$	-1	- $\xi$	- $\omega$	-1	-1	- $\xi$	- $\omega$
X97	2	-1	-1	2	-1	2	2 $\omega$	- $\omega$	- $\omega$	2 $\omega$	- $\omega$	2 $\omega$	2 $\xi$	- $\xi$	- $\xi$	2 $\xi$	- $\xi$	2 $\xi$
X98	2	-1	-1	2	-1	2	2 $\omega$	- $\omega$	- $\omega$	2 $\omega$	- $\omega$	2 $\omega$	2 $\xi$	- $\xi$	- $\xi$	2 $\xi$	- $\xi$	2 $\xi$
X99	-2	1	1	-2	1	-2	-2 $\omega$	$\omega$	$\omega$	-2 $\omega$	$\omega$	-2 $\omega$	-2 $\xi$	$\xi$	$\xi$	-2 $\xi$	$\xi$	-2 $\xi$
X100	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X101	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X102	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1
X103	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1
X104	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1
X105	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1
X106	-1	- $\omega$	- $\xi$	- $\xi$	-1	- $\omega$	- $\omega$	- $\xi$	-1	-1	- $\omega$	- $\xi$	- $\xi$	-1	- $\omega$	- $\omega$	- $\xi$	-1
X107	-1	- $\omega$	- $\xi$	- $\xi$	-1	- $\omega$	- $\omega$	- $\xi$	-1	-1	- $\omega$	- $\xi$	- $\xi$	-1	- $\omega$	- $\omega$	- $\xi$	-1
X108	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$$\xi^3 = 1, \omega = \xi^2$$

Table B.13: Character table of  $B(3,5)$  (continued)

$[g]$	2b																	
$[g]$	2B	6ae	6af	6ag	6ah	6ai	6aj	6ak	6al	6am	6an	6ao	6ap	6aq	6ar	6as	6at	6au
X1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X7	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
X8	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$
X9	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$
X10	2	$2\xi$	$2\omega$	2	$2\xi$	$2\omega$	2	$2\xi$	$2\omega$	2	$2\xi$	$2\omega$	2	$2\xi$	$2\omega$	2	$2\xi$	$2\omega$
X11	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$
X12	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$
X13	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$
X14	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$
X15	2	$2\omega$	$2\xi$	2	$2\omega$	$2\xi$	2	$2\omega$	$2\xi$	2	$2\omega$	$2\xi$	2	$2\omega$	$2\xi$	2	$2\omega$	$2\xi$
X16	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$
X17	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$
X18	2	2	2	$-\omega$	$-\omega$	$-\omega$	$-\xi$	$-\xi$	$-\xi$	$2\xi$	$2\xi$	$2\xi$	-1	-1	-1	$2\omega$	$2\omega$	$2\omega$
X19	-2	-2	-2	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$	$-2\xi$	$-2\xi$	$-2\xi$	1	1	1	$-2\omega$	$-2\omega$	$-2\omega$
X20	-2	-2	-2	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$	$-2\xi$	$-2\xi$	$-2\xi$	1	1	1	$-2\omega$	$-2\omega$	$-2\omega$
X21	2	2	2	$-\omega$	$-\omega$	$-\omega$	$-\xi$	$-\xi$	$-\xi$	$2\xi$	$2\xi$	$2\xi$	-1	-1	-1	$2\omega$	$2\omega$	$2\omega$
X22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X27	2	2	2	$-\xi$	$-\xi$	$-\xi$	$-\omega$	$-\omega$	$-\omega$	$2\omega$	$2\omega$	$2\omega$	-1	-1	-1	$2\xi$	$2\xi$	$2\xi$
X28	-2	-2	-2	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$	$-2\omega$	$-2\omega$	$-2\omega$	1	1	1	$-2\xi$	$-2\xi$	$-2\xi$
X29	-2	-2	-2	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$	$-2\omega$	$-2\omega$	$-2\omega$	1	1	1	$-2\xi$	$-2\xi$	$-2\xi$
X30	2	2	2	$-\xi$	$-\xi$	$-\xi$	$-\omega$	$-\omega$	$-\omega$	$2\omega$	$2\omega$	$2\omega$	-1	-1	-1	$2\xi$	$2\xi$	$2\xi$
X31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X33	2	$2\xi$	$2\omega$	$-\omega$	-1	$-\xi$	$-\xi$	$-\omega$	-1	$2\xi$	$2\omega$	2	-1	$-\xi$	$-\omega$	$2\omega$	2	$2\xi$
X34	-2	$-2\xi$	$-2\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	$-2\xi$	$-2\omega$	-2	1	$\xi$	$\omega$	$-2\omega$	-2	$-2\xi$
X35	-2	$-2\xi$	$-2\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	$-2\xi$	$-2\omega$	-2	1	$\xi$	$\omega$	$-2\omega$	-2	$-2\xi$
X36	2	$2\xi$	$2\omega$	$-\omega$	-1	$-\xi$	$-\xi$	$-\omega$	-1	$2\xi$	$2\omega$	2	-1	$-\xi$	$-\omega$	$2\omega$	2	$2\xi$
X37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X39	2	$2\omega$	$2\xi$	$-\omega$	$-\xi$	-1	$-\xi$	-1	$-\omega$	$2\xi$	2	$2\omega$	-1	$-\omega$	$-\xi$	$2\omega$	$2\xi$	2
X40	-2	$-2\omega$	$-2\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$	$-2\xi$	-2	$-2\omega$	1	$\omega$	$\xi$	$-2\omega$	$-2\xi$	-2
X41	-2	$-2\omega$	$-2\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$	$-2\xi$	-2	$-2\omega$	1	$\omega$	$\xi$	$-2\omega$	$-2\xi$	-2
X42	2	$2\omega$	$2\xi$	$-\omega$	$-\xi$	-1	$-\xi$	-1	$-\omega$	$2\xi$	2	$2\omega$	-1	$-\omega$	$-\xi$	$2\omega$	$2\xi$	2
X43	2	$2\xi$	$2\omega$	$-\xi$	$-\omega$	-1	$-\omega$	-1	$-\xi$	$2\omega$	2	$2\xi$	-1	$-\xi$	$-\omega$	$2\xi$	$2\omega$	2
X44	-2	$-2\xi$	$-2\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$	$-2\omega$	-2	$-2\xi$	1	$\xi$	$\omega$	$-2\xi$	$-2\omega$	-2
X45	-2	$-2\xi$	$-2\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$	$-2\omega$	-2	$-2\xi$	1	$\xi$	$\omega$	$-2\xi$	$-2\omega$	-2
X46	2	$2\xi$	$2\omega$	$-\xi$	$-\omega$	-1	$-\omega$	-1	$-\xi$	$2\omega$	2	$2\xi$	-1	$-\xi$	$-\omega$	$2\xi$	$2\omega$	2
X47	2	$2\omega$	$2\xi$	$-\xi$	-1	$-\omega$	$-\omega$	$-\xi$	-1	$2\omega$	$2\xi$	2	-1	$-\omega$	$-\xi$	$2\xi$	2	$2\omega$
X48	-2	$-2\omega$	$-2\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	$-2\omega$	$-2\xi$	-2	1	$\omega$	$\xi$	$-2\xi$	-2	$-2\omega$
X49	-2	$-2\omega$	$-2\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	$-2\omega$	$-2\xi$	-2	1	$\omega$	$\xi$	$-2\xi$	-2	$-2\omega$
X50	2	$2\omega$	$2\xi$	$-\xi$	-1	$-\omega$	$-\omega$	$-\xi$	-1	$2\omega$	$2\xi$	2	-1	$-\omega$	$-\xi$	$2\xi$	2	$2\omega$
X51	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X52	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X53	1	1	1	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	1	1	1	$\xi$	$\xi$	$\xi$
X54	1	1	1	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	$\omega$	1	1	1	$\xi$	$\xi$	$\xi$
X55	2	2	2	$2\xi$	$2\xi$	$2\xi$	$2\omega$	$2\omega$	$2\omega$	$2\omega$	$2\omega$	$2\omega$	2	2	2	$2\xi$	$2\xi$	$2\xi$
X56	-1	-1	-1	$-\xi$	$-\xi$	$-\xi$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	-1	-1	-1	$-\xi$	$-\xi$	$-\xi$
X57	-1	-1	-1	$-\xi$	$-\xi$	$-\xi$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	-1	-1	-1	$-\xi$	$-\xi$	$-\xi$

$$\xi^3 = 1, \omega = \xi^2$$

Table B.14: Character table of  $B(3,5)$  (continued)

[g]	2b																	
[ḡ]	2B	6ae	6af	6ag	6ah	6ai	6aj	6ak	6al	6am	6an	6ao	6ap	6aq	6ar	6as	6at	6au
X58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X59	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X60	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X61	2	2	2	-1	-1	-1	-1	-1	-1	2	2	2	-1	-1	-1	2	2	2
X62	-2	-2	-2	1	1	1	1	1	1	-2	-2	-2	1	1	1	-2	-2	-2
X63	-2	-2	-2	1	1	1	1	1	1	-2	-2	-2	1	1	1	-2	-2	-2
X64	2	2	2	-1	-1	-1	-1	-1	-1	2	2	2	-1	-1	-1	2	2	2
X65	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X66	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X67	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X68	1	1	1	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	1	1	1	$\omega$	$\omega$	$\omega$
X69	1	1	1	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	$\xi$	1	1	1	$\omega$	$\omega$	$\omega$
X70	2	2	2	$2\omega$	$2\omega$	$2\omega$	$2\xi$	$2\xi$	$2\xi$	$2\xi$	$2\xi$	$2\xi$	2	2	2	$2\omega$	$2\omega$	$2\omega$
X71	-1	-1	-1	$-\omega$	$-\omega$	$-\omega$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	-1	-1	-1	$-\omega$	$-\omega$	$-\omega$
X72	-1	-1	-1	$-\omega$	$-\omega$	$-\omega$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	$-\xi$	-1	-1	-1	$-\omega$	$-\omega$	$-\omega$
X73	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\xi$	$\omega$	1
X74	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\xi$	$\omega$	1
X75	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X76	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X77	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\xi$	$\omega$	1
X78	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$	$\omega$	1	$\xi$	1	$\xi$	$\omega$	$\xi$	$\omega$	1
X79	-2	$-2\xi$	$-2\omega$	$-2\xi$	$-2\omega$	-2	$-2\omega$	-2	$-2\xi$	$-2\omega$	-2	$-2\xi$	-2	$-2\xi$	$-2\omega$	$-2\xi$	$-2\omega$	-2
X80	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	$\omega$	$\xi$	1	1	$\omega$	$\xi$	$\xi$	1	$\omega$
X81	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1	$\omega$	$\xi$	1	1	$\omega$	$\xi$	$\xi$	1	$\omega$
X82	2	$2\omega$	$2\xi$	$2\xi$	2	$2\omega$	$2\omega$	$2\xi$	2	$2\omega$	$2\xi$	2	2	$2\omega$	$2\xi$	$2\xi$	2	$2\omega$
X83	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$
X84	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$
X85	2	$2\xi$	$2\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	2	$2\xi$	$2\omega$	-1	$-\xi$	$-\omega$	2	$2\xi$	$2\omega$
X86	-2	$-2\xi$	$-2\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$	-2	$-2\xi$	$-2\omega$	1	$\xi$	$\omega$	-2	$-2\xi$	$-2\omega$
X87	-2	$-2\xi$	$-2\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$	-2	$-2\xi$	$-2\omega$	1	$\xi$	$\omega$	-2	$-2\xi$	$-2\omega$
X88	2	$2\xi$	$2\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	2	$2\xi$	$2\omega$	-1	$-\xi$	$-\omega$	2	$2\xi$	$2\omega$
X89	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X91	2	$2\omega$	$2\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	2	$2\omega$	$2\xi$	-1	$-\omega$	$-\xi$	2	$2\omega$	$2\xi$
X92	-2	$-2\omega$	$-2\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$	-2	$-2\omega$	$-2\xi$	1	$\omega$	$\xi$	-2	$-2\omega$	$-2\xi$
X93	-2	$-2\omega$	$-2\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$	-2	$-2\omega$	$-2\xi$	1	$\omega$	$\xi$	-2	$-2\omega$	$-2\xi$
X94	2	$2\omega$	$2\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	2	$2\omega$	$2\xi$	-1	$-\omega$	$-\xi$	2	$2\omega$	$2\xi$
X95	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X96	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X97	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	$\xi$	$\omega$	1	1	$\xi$	$\omega$	$\omega$	1	$\xi$
X98	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1	$\xi$	$\omega$	1	1	$\xi$	$\omega$	$\omega$	1	$\xi$
X99	2	$2\xi$	$2\omega$	$2\omega$	2	$2\xi$	$2\xi$	$2\omega$	2	$2\xi$	$2\omega$	2	2	$2\xi$	$2\omega$	$2\omega$	2	$2\xi$
X100	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$
X101	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$
X102	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\omega$	$\xi$	1
X103	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\omega$	$\xi$	1
X104	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X105	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X106	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\omega$	$\xi$	1
X107	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$	$\xi$	1	$\omega$	1	$\omega$	$\xi$	$\omega$	$\xi$	1
X108	-2	$-2\omega$	$-2\xi$	$-2\omega$	$-2\xi$	-2	$-2\xi$	-2	$-2\omega$	$-2\xi$	-2	$-2\omega$	-2	$-2\omega$	$-2\xi$	$-2\omega$	$-2\xi$	-2

$$\xi^3 = 1, \omega = \xi^2$$

Table B.15: Character table of  $B(3,5)$  (continued)

$[g]$	4a								
	4A	12A	12B	12C	12D	12E	12F	12G	12H
X1	1	1	1	1	1	1	1	1	1
X2	-1	-1	-1	-1	-1	-1	-1	-1	-1
X3	0	0	0	0	0	0	0	0	0
X4	0	0	0	0	0	0	0	0	0
X5	-1	-1	-1	-1	-1	-1	-1	-1	-1
X6	1	1	1	1	1	1	1	1	1
X7	0	0	0	0	0	0	0	0	0
X8	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$
X9	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$
X10	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$
X11	0	0	0	0	0	0	0	0	0
X12	0	0	0	0	0	0	0	0	0
X13	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$
X14	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$
X15	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$
X16	0	0	0	0	0	0	0	0	0
X17	0	0	0	0	0	0	0	0	0
X18	0	0	0	0	0	0	0	0	0
X19	0	0	0	0	0	0	0	0	0
X20	0	0	0	0	0	0	0	0	0
X21	0	0	0	0	0	0	0	0	0
X22	0	0	0	0	0	0	0	0	0
X23	0	0	0	0	0	0	0	0	0
X24	0	0	0	0	0	0	0	0	0
X25	0	0	0	0	0	0	0	0	0
X26	0	0	0	0	0	0	0	0	0
X27	0	0	0	0	0	0	0	0	0
X28	0	0	0	0	0	0	0	0	0
X29	0	0	0	0	0	0	0	0	0
X30	0	0	0	0	0	0	0	0	0
X31	0	0	0	0	0	0	0	0	0
X32	0	0	0	0	0	0	0	0	0
X33	0	0	0	0	0	0	0	0	0
X34	0	0	0	0	0	0	0	0	0
X35	0	0	0	0	0	0	0	0	0
X36	0	0	0	0	0	0	0	0	0
X37	0	0	0	0	0	0	0	0	0
X38	0	0	0	0	0	0	0	0	0
X39	0	0	0	0	0	0	0	0	0
X40	0	0	0	0	0	0	0	0	0
X41	0	0	0	0	0	0	0	0	0
X42	0	0	0	0	0	0	0	0	0
X43	0	0	0	0	0	0	0	0	0
X44	0	0	0	0	0	0	0	0	0
X45	0	0	0	0	0	0	0	0	0
X46	0	0	0	0	0	0	0	0	0
X47	0	0	0	0	0	0	0	0	0
X48	0	0	0	0	0	0	0	0	0
X49	0	0	0	0	0	0	0	0	0
X50	0	0	0	0	0	0	0	0	0
X51	0	0	0	0	0	0	0	0	0
X52	0	0	0	0	0	0	0	0	0
X53	1	1	1	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$
X54	1	1	1	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$
X55	-1	-1	-1	$-\xi$	$-\xi$	$-\xi$	$-\omega$	$-\omega$	$-\omega$
X56	0	0	0	0	0	0	0	0	0
X57	0	0	0	0	0	0	0	0	0

$$\xi^3 = 1, \omega = \xi^2$$

Table B.16: Character table of  $B(3,5)$  (continued)

[g]	4a								
[g]	4A	12A	12B	12C	12D	12E	12F	12G	12H
X58	0	0	0	0	0	0	0	0	0
X59	0	0	0	0	0	0	0	0	0
X60	0	0	0	0	0	0	0	0	0
X61	0	0	0	0	0	0	0	0	0
X62	0	0	0	0	0	0	0	0	0
X63	0	0	0	0	0	0	0	0	0
X64	0	0	0	0	0	0	0	0	0
X65	0	0	0	0	0	0	0	0	0
X66	0	0	0	0	0	0	0	0	0
X67	0	0	0	0	0	0	0	0	0
X68	1	1	1	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$
X69	1	1	1	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$
X70	-1	-1	-1	$-\omega$	$-\omega$	$-\omega$	$-\xi$	$-\xi$	$-\xi$
X71	0	0	0	0	0	0	0	0	0
X72	0	0	0	0	0	0	0	0	0
X73	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$
X74	-1	$-\xi$	$-\omega$	$-\xi$	$-\omega$	-1	$-\omega$	-1	$-\xi$
X75	0	0	0	0	0	0	0	0	0
X76	0	0	0	0	0	0	0	0	0
X77	-1	$-\xi$	$-\omega$	$-\xi$	$-\omega$	-1	$-\omega$	-1	$-\xi$
X78	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$
X79	0	0	0	0	0	0	0	0	0
X80	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1
X81	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1
X82	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	$-\omega$	$-\xi$	-1
X83	0	0	0	0	0	0	0	0	0
X84	0	0	0	0	0	0	0	0	0
X85	0	0	0	0	0	0	0	0	0
X86	0	0	0	0	0	0	0	0	0
X87	0	0	0	0	0	0	0	0	0
X88	0	0	0	0	0	0	0	0	0
X89	0	0	0	0	0	0	0	0	0
X90	0	0	0	0	0	0	0	0	0
X91	0	0	0	0	0	0	0	0	0
X92	0	0	0	0	0	0	0	0	0
X93	0	0	0	0	0	0	0	0	0
X94	0	0	0	0	0	0	0	0	0
X95	0	0	0	0	0	0	0	0	0
X96	0	0	0	0	0	0	0	0	0
X97	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1
X98	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1
X99	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	$-\xi$	$-\omega$	-1
X100	0	0	0	0	0	0	0	0	0
X101	0	0	0	0	0	0	0	0	0
X102	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$
X103	-1	$-\omega$	$-\xi$	$-\omega$	$-\xi$	-1	$-\xi$	-1	$-\omega$
X104	0	0	0	0	0	0	0	0	0
X105	0	0	0	0	0	0	0	0	0
X106	-1	$-\omega$	$-\xi$	$-\omega$	$-\xi$	-1	$-\xi$	-1	$-\omega$
X107	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$
X108	0	0	0	0	0	0	0	0	0

$$\xi^3 = 1, \omega = \xi^2$$

Table B.17: Character table of  $B(3,5)$  (continued)

$[g]$	6a								
$[\bar{g}]$	6AV	6AW	6AX	18A	18B	18C	18D	18E	18F
X1	1	1	1	1	1	1	1	1	1
X2	-1	-1	-1	-1	-1	-1	-1	-1	-1
X3	1	1	1	1	1	1	1	1	1
X4	-1	-1	-1	-1	-1	-1	-1	-1	-1
X5	1	1	1	1	1	1	1	1	1
X6	-1	-1	-1	-1	-1	-1	-1	-1	-1
X7	0	0	0	0	0	0	0	0	0
X8	0	0	0	0	0	0	0	0	0
X9	0	0	0	0	0	0	0	0	0
X10	0	0	0	0	0	0	0	0	0
X11	0	0	0	0	0	0	0	0	0
X12	0	0	0	0	0	0	0	0	0
X13	0	0	0	0	0	0	0	0	0
X14	0	0	0	0	0	0	0	0	0
X15	0	0	0	0	0	0	0	0	0
X16	0	0	0	0	0	0	0	0	0
X17	0	0	0	0	0	0	0	0	0
X18	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$
X19	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$
X20	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$
X21	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$
X22	1	$\xi$	$\omega$	1	$\xi$	$\omega$	1	$\xi$	$\omega$
X23	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$	-1	$-\xi$	$-\omega$
X294	0	0	0	0	0	0	0	0	0
X25	0	0	0	0	0	0	0	0	0
X26	0	0	0	0	0	0	0	0	0
X27	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$
X28	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$
X29	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$
X30	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$
X31	1	$\omega$	$\xi$	1	$\omega$	$\xi$	1	$\omega$	$\xi$
X32	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$	-1	$-\omega$	$-\xi$
X33	1	1	1	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$
X34	-1	-1	-1	$-\xi$	$-\xi$	$-\xi$	$-\omega$	$-\omega$	$-\omega$
X35	1	1	1	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$
X36	-1	-1	-1	$-\xi$	$-\xi$	$-\xi$	$-\omega$	$-\omega$	$-\omega$
X37	1	1	1	$\xi$	$\xi$	$\xi$	$\omega$	$\omega$	$\omega$
X38	-1	-1	-1	$-\xi$	$-\xi$	$-\xi$	$-\omega$	$-\omega$	$-\omega$
X39	0	0	0	0	0	0	0	0	0
X40	0	0	0	0	0	0	0	0	0
X41	0	0	0	0	0	0	0	0	0
X42	0	0	0	0	0	0	0	0	0
X43	0	0	0	0	0	0	0	0	0
X44	0	0	0	0	0	0	0	0	0
X45	0	0	0	0	0	0	0	0	0
X46	0	0	0	0	0	0	0	0	0
X47	1	1	1	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$
X48	-1	-1	-1	$-\omega$	$-\omega$	$-\omega$	$-\xi$	$-\xi$	$-\xi$
X49	1	1	1	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$
X50	-1	-1	-1	$-\omega$	$-\omega$	$-\omega$	$-\xi$	$-\xi$	$-\xi$
X51	1	1	1	$\omega$	$\omega$	$\omega$	$\xi$	$\xi$	$\xi$
X52	-1	-1	-1	$-\omega$	$-\omega$	$-\omega$	$-\xi$	$-\xi$	$-\xi$
X53	0	0	0	0	0	0	0	0	0
X54	0	0	0	0	0	0	0	0	0
X55	0	0	0	0	0	0	0	0	0
X56	0	0	0	0	0	0	0	0	0
X57	0	0	0	0	0	0	0	0	0

$$\xi^3 = 1, \omega = \xi^2$$

Table B.18: Character table of  $B(3,5)$  (continued)

$[g]$	6a								
$[g]$	6AV	6AW	6AX	18A	18B	18C	18D	18E	18F
X58	0	0	0	0	0	0	0	0	0
X59	0	0	0	0	0	0	0	0	0
X60	0	0	0	0	0	0	0	0	0
X61	0	0	0	0	0	0	0	0	0
X62	0	0	0	0	0	0	0	0	0
X63	0	0	0	0	0	0	0	0	0
X64	0	0	0	0	0	0	0	0	0
X65	0	0	0	0	0	0	0	0	0
X66	0	0	0	0	0	0	0	0	0
X67	0	0	0	0	0	0	0	0	0
X68	0	0	0	0	0	0	0	0	0
X69	0	0	0	0	0	0	0	0	0
X70	0	0	0	0	0	0	0	0	0
X71	0	0	0	0	0	0	0	0	0
X72	0	0	0	0	0	0	0	0	0
X73	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$
X74	-1	$-\xi$	$-\omega$	$-\xi$	$-\omega$	-1	$-\omega$	-1	$-\xi$
X75	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$
X76	-1	$-\xi$	$-\omega$	$-\xi$	$-\omega$	-1	$-\omega$	-1	$-\xi$
X77	1	$\xi$	$\omega$	$\xi$	$\omega$	1	$\omega$	1	$\xi$
X78	-1	$-\xi$	$-\omega$	$-\xi$	$-\omega$	-1	$-\omega$	-1	$-\xi$
X79	0	0	0	0	0	0	0	0	0
X80	0	0	0	0	0	0	0	0	0
X81	0	0	0	0	0	0	0	0	0
X82	0	0	0	0	0	0	0	0	0
X83	0	0	0	0	0	0	0	0	0
X84	0	0	0	0	0	0	0	0	0
X85	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1
X86	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1
X87	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	$-\omega$	$-\xi$	-1
X88	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	$-\omega$	$-\xi$	-1
X89	1	$\omega$	$\xi$	$\xi$	1	$\omega$	$\omega$	$\xi$	1
X90	-1	$-\omega$	$-\xi$	$-\xi$	-1	$-\omega$	$-\omega$	$-\xi$	-1
X91	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1
X92	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	$-\xi$	$-\omega$	-1
X93	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1
X94	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	$-\xi$	$-\omega$	-1
X95	1	$\xi$	$\omega$	$\omega$	1	$\xi$	$\xi$	$\omega$	1
X96	-1	$-\xi$	$-\omega$	$-\omega$	-1	$-\xi$	$-\xi$	$-\omega$	-1
X97	0	0	0	0	0	0	0	0	0
X98	0	0	0	0	0	0	0	0	0
X99	0	0	0	0	0	0	0	0	0
X100	0	0	0	0	0	0	0	0	0
X101	0	0	0	0	0	0	0	0	0
X102	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$
X103	-1	$-\omega$	$-\xi$	$-\omega$	$-\xi$	-1	$-\xi$	-1	$-\omega$
X104	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$
X105	-1	$-\omega$	$-\xi$	$-\omega$	$-\xi$	-1	$-\xi$	-1	$-\omega$
X106	1	$\omega$	$\xi$	$\omega$	$\xi$	1	$\xi$	1	$\omega$
X107	-1	$-\omega$	$-\xi$	$-\omega$	$-\xi$	-1	$-\xi$	-1	$-\omega$
X108	0	0	0	0	0	0	0	0	0

$$\xi^3 = 1, \omega = \xi^2$$

Table B.19: Character table of  $B(3,5)$  (continued)

$[g]$	5a		
$[\bar{g}]$	5A	15A	15B
$\chi_1$	1	1	1
$\chi_2$	1	1	1
$\chi_3$	-1	-1	-1
$\chi_4$	-1	-1	-1
$\chi_5$	0	0	0
$\chi_6$	0	0	0
$\chi_7$	1	1	1
$\chi_8$	0	0	0
$\chi_9$	0	0	0
$\chi_{10}$	0	0	0
$\chi_{11}$	0	0	0
$\chi_{12}$	0	0	0
$\chi_{13}$	0	0	0
$\chi_{14}$	0	0	0
$\chi_{15}$	0	0	0
$\chi_{16}$	0	0	0
$\chi_{17}$	0	0	0
$\chi_{18}$	0	0	0
$\chi_{19}$	0	0	0
$\chi_{20}$	0	0	0
$\chi_{21}$	0	0	0
$\chi_{22}$	0	0	0
$\chi_{23}$	0	0	0
$\chi_{24}$	0	0	0
$\chi_{25}$	0	0	0
$\chi_{26}$	0	0	0
$\chi_{27}$	0	0	0
$\chi_{28}$	0	0	0
$\chi_{29}$	0	0	0
$\chi_{30}$	1	$\xi$	$\omega$
$\chi_{31}$	1	$\xi$	$\omega$
$\chi_{32}$	-1	$-\xi$	$-\omega$
$\chi_{33}$	-1	$-\xi$	$-\omega$
$\chi_{34}$	0	0	0
$\chi_{35}$	0	0	0
$\chi_{36}$	1	$\xi$	$\omega$
$\chi_{37}$	0	0	0
$\chi_{38}$	0	0	0
$\chi_{39}$	0	0	0
$\chi_{40}$	0	0	0
$\chi_{41}$	0	0	0
$\chi_{42}$	0	0	0
$\chi_{43}$	0	0	0
$\chi_{44}$	0	0	0
$\chi_{45}$	0	0	0
$\chi_{46}$	0	0	0
$\chi_{47}$	0	0	0
$\chi_{48}$	0	0	0
$\chi_{49}$	0	0	0
$\chi_{50}$	0	0	0
$\chi_{51}$	0	0	0
$\chi_{52}$	0	0	0
$\chi_{53}$	0	0	0
$\chi_{54}$	0	0	0
$\chi_{55}$	0	0	0
$\chi_{56}$	0	0	0
$\chi_{57}$	0	0	0

$\xi^3 = 1, \omega = \xi^2$

Table B.20: Character table of  $B(3,5)$  (continued)

$[g]$	5a		
$[\bar{g}]$	5A	15A	15B
X58	0	0	0
X59	0	0	0
X60	0	0	0
X61	0	0	0
X62	0	0	0
X63	0	0	0
X64	0	0	0
X65	0	0	0
X66	0	0	0
X67	0	0	0
X68	0	0	0
X69	0	0	0
X70	0	0	0
X71	0	0	0
X72	0	0	0
X73	0	0	0
X74	0	0	0
X75	0	0	0
X76	0	0	0
X77	0	0	0
X78	0	0	0
X79	0	0	0
X80	0	0	0
X81	0	0	0
X82	0	0	0
X83	0	0	0
X84	0	0	0
X85	0	0	0
X86	0	0	0
X87	0	0	0
X88	0	0	0
X89	0	0	0
X90	0	0	0
X91	0	0	0
X92	0	0	0
X93	0	0	0
X94	0	0	0
X95	0	0	0
X96	0	0	0
X97	0	0	0
X98	0	0	0
X99	0	0	0
X100	0	0	0
X101	0	0	0
X102	1	$\epsilon$	$\xi$
X103	1	$\epsilon^2$	$\xi$
X104	-1	$-\epsilon$	$-\xi$
X105	-1	$-\epsilon^2$	$-\xi$
X106	0	0	0
X107	0	0	0
X108	1	$\epsilon$	$\xi$

$$\xi^3 = 1, \omega = \xi^2$$

## Appendix C

### Character Table of $B_S(2, 6)$

Table C.1: Character table of  $B_S(2,6)$

$[g]$	1a				2a					2b				
$[\bar{g}]$	1A	2A	2B	2C	2D	2E	2F	4A	4B	2G	2H	4C	4D	4E
X1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	1
X3	5	5	5	5	1	1	1	1	1	1	1	1	1	1
X4	5	5	5	5	-1	-1	-1	-1	-1	1	1	1	1	1
X5	5	5	5	5	-3	-3	-3	-3	-3	1	1	1	1	1
X6	5	5	5	5	3	3	3	3	3	1	1	1	1	1
X7	9	9	9	9	-3	-3	-3	-3	-3	1	1	1	1	1
X8	9	9	9	9	3	3	3	3	3	1	1	1	1	1
X9	10	10	10	10	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
X10	10	10	10	10	2	2	2	2	2	-2	-2	-2	-2	-2
X11	16	16	16	16	0	0	0	0	0	0	0	0	0	0
X12	6	2	-2	-6	4	0	-4	2	-2	2	-2	0	2	-2
X13	6	2	-2	-6	-4	0	4	-2	2	2	-2	0	2	-2
X14	24	8	-8	-24	-8	0	8	-4	4	0	0	0	0	0
X15	24	8	-8	-24	8	0	-8	4	-4	0	0	0	0	0
X16	30	10	-10	-30	4	0	-4	2	-2	2	-2	0	2	-2
X17	30	10	-10	-30	-4	0	4	-2	2	2	-2	0	2	-2
X18	36	12	-12	-36	0	0	0	0	0	-4	4	0	-4	4
X19	15	-1	-1	15	7	-1	7	-1	-1	3	3	-1	-1	-1
X20	15	-1	-1	15	-5	3	-5	-1	-1	-1	-1	-1	3	3
X21	15	-1	-1	15	5	-3	5	1	1	-1	-1	-1	3	3
X22	15	-1	-1	15	-7	1	-7	1	1	3	3	-1	-1	-1
X23	30	-2	-2	30	2	2	2	-2	-2	2	2	-2	2	2
X24	30	-2	-2	30	-2	-2	-2	2	2	2	2	-2	2	2
X25	45	-3	-3	45	-9	-1	-9	3	3	1	1	1	-3	-3
X26	45	-3	-3	45	3	-5	3	3	3	-3	-3	1	1	1
X27	45	-3	-3	45	-3	5	-3	-3	-3	-3	-3	1	1	1
X28	45	-3	-3	45	9	1	9	-3	-3	1	1	1	-3	-3
X29	10	-2	2	-10	4	0	-4	-2	2	2	-2	0	-2	2
X30	10	-2	2	-10	-4	0	4	2	-2	2	-2	0	-2	2
X31	10	-2	2	-10	4	0	-4	-2	2	2	-2	0	-2	2
X32	10	-2	2	-10	-4	0	4	2	-2	2	-2	0	-2	2
X33	20	-4	4	-20	0	0	0	0	0	-4	4	0	4	-4
X34	40	-8	8	-40	0	0	0	0	0	0	0	0	0	0
X35	40	-8	8	-40	0	0	0	0	0	0	0	0	0	0
X36	40	-8	8	-40	-8	0	8	4	-4	0	0	0	0	0
X37	40	-8	8	-40	8	0	-8	-4	4	0	0	0	0	0

Table C.2: Character table of  $B_S(2,6)$  (continued)

$[g]$	2c			3a				6a			
$[\bar{g}]$	2I	2J	4F	3A	6A	6B	6C	6D	12A	6E	12B
X1	1	1	1	1	1	1	1	1	1	1	1
X2	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
X3	-3	-3	-3	-1	-1	-1	-1	1	1	1	1
X4	3	3	3	-1	-1	-1	-1	-1	-1	-1	-1
X5	1	1	1	2	2	2	2	0	0	0	0
X6	-1	-1	-1	2	2	2	2	0	0	0	0
X7	-3	-3	-3	0	0	0	0	0	0	0	0
X8	3	3	3	0	0	0	0	0	0	0	0
X9	2	2	2	1	1	1	1	1	1	1	1
X10	-2	-2	-2	1	1	1	1	-1	-1	-1	-1
X11	0	0	0	-2	-2	-2	-2	0	0	0	0
X12	0	0	0	3	-1	1	-3	1	-1	-1	1
X13	0	0	0	3	-1	1	-3	-1	1	1	-1
X14	0	0	0	3	-1	1	-3	1	-1	-1	1
X15	0	0	0	3	-1	1	-3	-1	1	1	-1
X16	0	0	0	-3	1	-1	3	1	-1	-1	1
X17	0	0	0	-3	1	-1	3	-1	1	1	-1
X18	0	0	0	0	0	0	0	0	0	0	0
X19	3	3	-1	3	-1	-1	3	1	-1	1	-1
X20	3	3	-1	3	-1	-1	3	1	-1	1	-1
X21	-3	-3	1	3	-1	-1	3	-1	1	-1	1
X22	-3	-3	1	3	-1	-1	3	-1	1	-1	1
X23	6	6	-2	-3	1	1	-3	-1	1	-1	1
X24	-6	-6	2	-3	1	1	-3	1	-1	1	-1
X25	3	3	-1	0	0	0	0	0	0	0	0
X26	3	3	-1	0	0	0	0	0	0	0	0
X27	-3	-3	1	0	0	0	0	0	0	0	0
X28	-3	-3	1	0	0	0	0	0	0	0	0
X29	-4	4	0	1	1	-1	-1	1	1	-1	-1
X30	-4	4	0	1	1	-1	-1	-1	-1	1	1
X31	4	-4	0	1	1	-1	-1	1	1	-1	-1
X32	4	-4	0	1	1	-1	-1	-1	-1	1	1
X33	0	0	0	2	2	-2	-2	0	0	0	0
X34	-8	8	0	-2	-2	2	2	0	0	0	0
X35	8	-8	0	-2	-2	2	2	0	0	0	0
X36	0	0	0	1	1	-1	-1	1	1	-1	-1
X37	0	0	0	1	1	-1	-1	-1	-1	1	1



## Appendix D

### Character Table of $B_S(3, 3)$

Table D.1: Character table of  $B_S(3,3)$

$[g]$	$1a \quad \xi^3 = 1$				$2a$			$3a$		
$[\bar{g}]$	1A	3A	3B	3C	2A	6A	6B	3A	3B	3C
$\chi_1$	1	1	1	1	1	1	1	1	1	1
$\chi_2$	1	1	1	1	-1	-1	-1	1	1	1
$\chi_3$	2	2	2	2	0	0	0	-1	-1	-1
$\chi_4$	3	0	$3\xi$	$3\xi^2$	1	$\xi$	$\xi^2$	0	0	0
$\chi_5$	3	0	$3\xi$	$3\xi^2$	-1	$-\xi$	$-\xi^2$	0	0	0
$\chi_6$	3	0	$3\xi^2$	$3\xi$	1	$\xi^2$	$\xi$	0	0	0
$\chi_7$	3	0	$3\xi^2$	$3\xi$	-1	$-\xi^2$	$-\xi$	0	0	0
$\chi_8$	2	-1	2	2	0	0	0	2	-1	-1
$\chi_9$	2	-1	2	2	0	0	0	-1	$\xi + \xi^2$	2
$\chi_{10}$	2	-1	2	2	0	0	0	-1	2	$\xi + \xi^2$

## Appendix E

### Character Table of $B_S(3, 5)$

Table E.1: Character table of  $B_S(3, 5)$

$[g]$	1a						
	1A	3A	3B	3C	3D	3E	3F
$\chi_1$	1	1	1	1	1	1	1
$\chi_2$	1	1	1	1	1	1	1
$\chi_3$	4	4	4	4	4	4	4
$\chi_4$	4	4	4	4	4	4	4
$\chi_5$	5	5	5	5	5	5	5
$\chi_6$	5	5	5	5	5	5	5
$\chi_7$	6	6	6	6	6	6	6
$\chi_8$	5	2	$\xi-2\omega$	$-2\xi+\omega$	-1	$4\xi+\omega$	$\xi+4\omega$
$\chi_9$	5	2	$\xi-2\omega$	$-2\xi+\omega$	-1	$4\xi+\omega$	$\xi+4\omega$
$\chi_{10}$	10	4	$2\xi-4\omega$	$-4\xi+2\omega$	-2	$8\xi+2\omega$	$2\xi+8\omega$
$\chi_{11}$	15	6	$3\xi-6\omega$	$-6\xi+3\omega$	-3	$12\xi+3\omega$	$3\xi+12\omega$
$\chi_{12}$	15	6	$3\xi-6\omega$	$-6\xi+3\omega$	-3	$12\xi+3\omega$	$3\xi+12\omega$
$\chi_{13}$	5	2	$-2\xi+\omega$	$\xi-2\omega$	-1	$\xi+4\omega$	$4\xi+\omega$
$\chi_{14}$	5	2	$-2\xi+\omega$	$\xi-2\omega$	-1	$\xi+4\omega$	$4\xi+\omega$
$\chi_{15}$	10	4	$-4\xi+2\omega$	$2\xi-4\omega$	-2	$2\xi+8\omega$	$8\xi+2\omega$
$\chi_{16}$	15	6	$-6\xi+3\omega$	$3\xi-6\omega$	-3	$3\xi+12\omega$	$12\xi+3\omega$
$\chi_{17}$	15	6	$-6\xi+3\omega$	$3\xi-6\omega$	-3	$3\xi+12\omega$	$12\xi+3\omega$
$\chi_{18}$	10	1	$5\xi+2\omega$	$2\xi+5\omega$	1	$-4\xi+2\omega$	$2\xi-4\omega$
$\chi_{19}$	10	1	$5\xi+2\omega$	$2\xi+5\omega$	1	$-4\xi+2\omega$	$2\xi-4\omega$
$\chi_{20}$	10	1	$5\xi+2\omega$	$2\xi+5\omega$	1	$-4\xi+2\omega$	$2\xi-4\omega$
$\chi_{21}$	10	1	$5\xi+2\omega$	$2\xi+5\omega$	1	$-4\xi+2\omega$	$2\xi-4\omega$
$\chi_{22}$	20	2	$10\xi+4\omega$	$4\xi+10\omega$	2	$-8\xi+4\omega$	$4\xi-8\omega$
$\chi_{23}$	20	2	$10\xi+4\omega$	$4\xi+10\omega$	2	$-8\xi+4\omega$	$4\xi-8\omega$
$\chi_{24}$	20	-1	2	2	-4	8	8
$\chi_{25}$	20	-1	2	2	-4	8	8
$\chi_{26}$	40	-2	4	4	-8	16	16
$\chi_{27}$	10	1	$2\xi+5\omega$	$5\xi+2\omega$	1	$2\xi-4\omega$	$-4\xi+2\omega$
$\chi_{28}$	10	1	$2\xi+5\omega$	$5\xi+2\omega$	1	$2\xi-4\omega$	$-4\xi+2\omega$
$\chi_{29}$	10	1	$2\xi+5\omega$	$5\xi+2\omega$	1	$2\xi-4\omega$	$-4\xi+2\omega$
$\chi_{30}$	10	1	$2\xi+5\omega$	$5\xi+2\omega$	1	$2\xi-4\omega$	$-4\xi+2\omega$
$\chi_{31}$	20	2	$4\xi+10\omega$	$10\xi+4\omega$	2	$4\xi-8\omega$	$-8\xi+4\omega$
$\chi_{32}$	20	2	$4\xi+10\omega$	$10\xi+4\omega$	2	$4\xi-8\omega$	$-8\xi+4\omega$
$\chi_{33}$	30	-6	3	3	3	-6	-6
$\chi_{34}$	30	-6	3	3	3	-6	-6
$\chi_{35}$	30	-6	3	3	3	-6	-6
$\chi_{36}$	30	-6	3	3	3	-6	-6

Table E.2: Character table of  $B_S(3,5)$  (continued)

$[g]$	$2a$									
$[g]$	2A	6A	6B	6C	6D	6E	6F	6G	6H	6I
X1	1	1	1	1	1	1	1	1	1	1
X2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
X3	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
X4	2	2	2	2	2	2	2	2	2	2
X5	1	1	1	1	1	1	1	1	1	1
X6	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
X7	0	0	0	0	0	0	0	0	0	0
X8	3	0	3 $\xi$	3 $\omega$	-2 $\xi$ - $\omega$	$\xi$ - $\omega$	$\xi$ +2 $\omega$	- $\xi$ -2 $\omega$	- $\xi$ + $\omega$	2 $\xi$ + $\omega$
X9	-3	0	-3 $\xi$	-3 $\omega$	2 $\xi$ + $\omega$	- $\xi$ + $\omega$	- $\xi$ -2 $\omega$	$\xi$ +2 $\omega$	$\xi$ - $\omega$	-2 $\xi$ - $\omega$
X10	0	0	0	0	0	0	0	0	0	0
X11	-3	0	-3 $\xi$	-3 $\omega$	2 $\xi$ + $\omega$	- $\xi$ + $\omega$	- $\xi$ -2 $\omega$	$\xi$ +2 $\omega$	$\xi$ - $\omega$	-2 $\xi$ - $\omega$
X12	3	0	3 $\xi$	3 $\omega$	-2 $\xi$ - $\omega$	$\xi$ - $\omega$	$\xi$ +2 $\omega$	- $\xi$ -2 $\omega$	- $\xi$ + $\omega$	2 $\xi$ + $\omega$
X13	3	0	3 $\omega$	3 $\xi$	- $\xi$ -2 $\omega$	- $\xi$ + $\omega$	2 $\xi$ + $\omega$	-2 $\xi$ - $\omega$	$\xi$ - $\omega$	$\xi$ +2 $\omega$
X14	-3	0	-3 $\omega$	-3 $\xi$	$\xi$ +2 $\omega$	$\xi$ - $\omega$	-2 $\xi$ - $\omega$	2 $\xi$ + $\omega$	- $\xi$ + $\omega$	- $\xi$ -2 $\omega$
X15	0	0	0	0	0	0	0	0	0	0
X16	-3	0	-3 $\omega$	-3 $\xi$	$\xi$ +2 $\omega$	$\xi$ - $\omega$	-2 $\xi$ - $\omega$	2 $\xi$ + $\omega$	- $\xi$ + $\omega$	- $\xi$ -2 $\omega$
X17	3	0	3 $\omega$	3 $\xi$	- $\xi$ -2 $\omega$	- $\xi$ + $\omega$	2 $\xi$ + $\omega$	-2 $\xi$ - $\omega$	$\xi$ - $\omega$	$\xi$ +2 $\omega$
X18	4	1	- $\xi$ +2 $\omega$	2 $\xi$ - $\omega$	$\omega$	3 $\xi$ + $\omega$	-2 $\omega$	$\xi$	$\xi$ +3 $\omega$	-2 $\xi$
X19	-2	1	- $\xi$ -4 $\omega$	-4 $\xi$ - $\omega$	2 $\xi$ - $\omega$	1	-2	- $\xi$ +2 $\omega$	1	-2
X20	2	-1	$\xi$ +4 $\omega$	4 $\xi$ + $\omega$	-2 $\xi$ + $\omega$	-1	2	$\xi$ -2 $\omega$	-1	2
X21	-4	-1	$\xi$ -2 $\omega$	-2 $\xi$ + $\omega$	- $\omega$	-3 $\xi$ - $\omega$	2 $\omega$	- $\xi$	- $\xi$ -3 $\omega$	2 $\xi$
X22	-2	-2	-2	-2	-2 $\xi$	-2 $\xi$	-2 $\xi$	-2 $\omega$	-2 $\omega$	-2 $\omega$
X23	2	2	2	2	2 $\xi$	2 $\xi$	2 $\xi$	2 $\omega$	2 $\omega$	2 $\omega$
X24	6	-3	6	6	0	0	0	0	0	0
X25	-6	3	-6	-6	0	0	0	0	0	0
X26	0	0	0	0	0	0	0	0	0	0
X27	4	1	2 $\xi$ - $\omega$	- $\xi$ +2 $\omega$	$\xi$	$\xi$ +3 $\omega$	-2 $\xi$	$\omega$	3 $\xi$ + $\omega$	-2 $\omega$
X28	-2	1	-4 $\xi$ - $\omega$	- $\xi$ -4 $\omega$	- $\xi$ +2 $\omega$	1	-2	2 $\xi$ - $\omega$	1	-2
X29	2	-1	4 $\xi$ + $\omega$	$\xi$ +4 $\omega$	$\xi$ -2 $\omega$	-1	2	-2 $\xi$ + $\omega$	-1	2
X30	-4	-1	-2 $\xi$ + $\omega$	$\xi$ -2 $\omega$	- $\xi$	- $\xi$ -3 $\omega$	2 $\xi$	- $\omega$	-3 $\xi$ - $\omega$	2 $\omega$
X31	-2	-2	-2	-2	-2 $\omega$	-2 $\omega$	-2 $\omega$	-2 $\xi$	-2 $\xi$	-2 $\xi$
X32	2	2	2	2	2 $\omega$	2 $\omega$	2 $\omega$	2 $\xi$	2 $\xi$	2 $\xi$
X33	6	0	-3	-3	-3	3	0	-3	3	0
X34	0	0	-3 $\xi$ +3 $\omega$	3 $\xi$ -3 $\omega$	$\xi$ - $\omega$	$\xi$ - $\omega$	-2 $\xi$ +2 $\omega$	- $\xi$ + $\omega$	- $\xi$ + $\omega$	2 $\xi$ -2 $\omega$
X35	0	0	3 $\xi$ -3 $\omega$	-3 $\xi$ +3 $\omega$	- $\xi$ + $\omega$	- $\xi$ + $\omega$	2 $\xi$ -2 $\omega$	$\xi$ - $\omega$	$\xi$ - $\omega$	-2 $\xi$ +2 $\omega$
X36	-6	0	3	3	3	-3	0	3	-3	0



## Appendix F

### Character Table of $B_Q(2, 6)$

Table F.1: Character table of  $B_Q(2, 6)$

$[g]$	1a				2a					
$[\bar{g}]$	1A	2A	2B	2C	2D	2E	2F	4A	4B	4C
X1	1	1	1	1	1	1	1	1	1	1
X2	1	1	1	1	-1	-1	-1	-1	-1	-1
X3	5	5	5	5	1	1	1	1	1	1
X4	5	5	5	5	-1	-1	-1	-1	-1	-1
X5	5	5	5	5	-3	-3	-3	-3	-3	-3
X6	5	5	5	5	3	3	3	3	3	3
X7	9	9	9	9	-3	-3	-3	-3	-3	-3
X8	9	9	9	9	3	3	3	3	3	3
X9	10	10	10	10	-2	-2	-2	-2	-2	-2
X10	10	10	10	10	2	2	2	2	2	2
X11	16	16	16	16	0	0	0	0	0	0
X12	15	5	-1	-3	7	1	-1	5	-1	-3
X13	15	5	-1	-3	-5	1	3	-7	-1	1
X14	15	5	-1	-3	5	-1	-3	7	1	-1
X15	15	5	-1	-3	-7	-1	1	-5	1	3
X16	30	10	-2	-6	2	2	2	-2	-2	-2
X17	30	10	-2	-6	-2	-2	-2	2	2	2
X18	45	15	-3	-9	-9	-3	-1	-3	3	5
X19	45	15	-3	-9	3	-3	-5	9	3	1
X20	45	15	-3	-9	-3	3	5	-9	-3	-1
X21	45	15	-3	-9	9	3	1	3	-3	-5
X22	15	-5	-1	3	7	-1	-1	-5	-1	3
X23	15	-5	-1	3	5	1	-3	-7	1	1
X24	15	-5	-1	3	-7	1	1	5	1	-3
X25	15	-5	-1	3	-5	-1	3	7	-1	-1
X26	30	-10	-2	6	2	-2	2	2	-2	2
X27	30	-10	-2	6	-2	2	-2	-2	2	-2
X28	45	-15	-3	9	-9	3	-1	3	3	-5
X29	45	-15	-3	9	-3	-3	5	9	-3	1
X30	45	-15	-3	9	3	3	-5	-9	3	-1
X31	45	-15	-3	9	9	-3	1	-3	-3	5
X32	1	-1	1	-1	1	-1	1	-1	1	-1
X33	1	-1	1	-1	-1	1	-1	1	-1	1
X34	5	-5	5	-5	1	-1	1	-1	1	-1
X35	5	-5	5	-5	-1	1	-1	1	-1	1
X36	5	-5	5	-5	-3	3	-3	3	-3	3
X37	5	-5	5	-5	3	-3	3	-3	3	-3
X38	9	-9	9	-9	-3	3	-3	3	-3	3
X39	9	-9	9	-9	3	-3	3	-3	3	-3
X40	10	-10	10	-10	-2	2	-2	2	-2	2
X41	10	-10	10	-10	2	-2	2	-2	2	-2
X42	16	-16	16	-16	0	0	0	0	0	0



Table F.3: Character table of  $B_Q(2, 6)$  (continued)

$[g]$	$3a$				$6a$			
$[g]$	3A	6A	6B	6C	6D	6E	12A	12B
X1	1	1	1	1	1	1	1	1
X2	1	1	1	1	-1	-1	-1	-1
X3	-1	-1	-1	-1	1	1	1	1
X4	-1	-1	-1	-1	-1	-1	-1	-1
X5	2	2	2	2	0	0	0	0
X6	2	2	2	2	0	0	0	0
X7	0	0	0	0	0	0	0	0
X8	0	0	0	0	0	0	0	0
X9	1	1	1	1	1	1	1	1
X10	1	1	1	1	-1	-1	-1	-1
X11	-2	-2	-2	-2	0	0	0	0
X12	3	-1	-1	3	1	1	-1	-1
X13	3	-1	-1	3	1	1	-1	-1
X14	3	-1	-1	3	-1	-1	1	1
X15	3	-1	-1	3	-1	-1	1	1
X16	-3	1	1	-3	-1	-1	1	1
X17	-3	1	1	-3	1	1	-1	-1
X18	0	0	0	0	0	0	0	0
X19	0	0	0	0	0	0	0	0
X20	0	0	0	0	0	0	0	0
X21	0	0	0	0	0	0	0	0
X22	3	1	-1	-3	1	-1	1	-1
X23	3	1	-1	-3	-1	1	-1	1
X24	3	1	-1	-3	-1	1	-1	1
X25	3	1	-1	-3	1	-1	1	-1
X26	-3	-1	1	3	-1	1	-1	1
X27	-3	-1	1	3	1	-1	1	-1
X28	0	0	0	0	0	0	0	0
X29	0	0	0	0	0	0	0	0
X30	0	0	0	0	0	0	0	0
X31	0	0	0	0	0	0	0	0
X32	1	-1	1	-1	1	-1	-1	1
X33	1	-1	1	-1	-1	1	1	-1
X34	-1	1	-1	1	1	-1	-1	1
X35	-1	1	-1	1	-1	1	1	-1
X36	2	-2	2	-2	0	0	0	0
X37	2	-2	2	-2	0	0	0	0
X38	0	0	0	0	0	0	0	0
X39	0	0	0	0	0	0	0	0
X40	1	-1	1	-1	1	-1	-1	1
X41	1	-1	1	-1	-1	1	1	-1
X42	-2	2	-2	2	0	0	0	0

Table F.4: Character table of  $B_Q(2,6)$  (continued)

$[g]$	3b		4a				4b				5a		6b	
$[\bar{g}]$	3B	6F	4J	4K	8A	8B	4L	4M	8C	8D	5A	10A	6G	6H
X1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1
X3	2	2	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0
X4	2	2	1	1	1	1	-1	-1	-1	-1	0	0	0	0
X5	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	1	1
X6	-1	-1	1	1	1	1	-1	-1	-1	-1	0	0	-1	-1
X7	0	0	1	1	1	1	1	1	1	1	-1	-1	0	0
X8	0	0	-1	-1	-1	-1	1	1	1	1	-1	-1	0	0
X9	1	1	0	0	0	0	0	0	0	0	0	0	-1	-1
X10	1	1	0	0	0	0	0	0	0	0	0	0	1	1
X11	-2	-2	0	0	0	0	0	0	0	0	1	1	0	0
X12	0	0	1	-1	1	-1	1	1	-1	-1	0	0	0	0
X13	0	0	-1	1	-1	1	-1	-1	1	1	0	0	0	0
X14	0	0	1	-1	1	-1	-1	-1	1	1	0	0	0	0
X15	0	0	-1	1	-1	1	1	1	-1	-1	0	0	0	0
X16	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X17	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X18	0	0	1	-1	1	-1	-1	-1	1	1	0	0	0	0
X19	0	0	-1	1	-1	1	1	1	-1	-1	0	0	0	0
X20	0	0	1	-1	1	-1	1	1	-1	-1	0	0	0	0
X21	0	0	-1	1	-1	1	-1	-1	1	1	0	0	0	0
X22	0	0	1	1	-1	-1	1	-1	1	-1	0	0	0	0
X23	0	0	1	1	-1	-1	1	-1	1	-1	0	0	0	0
X24	0	0	-1	-1	1	1	-1	1	-1	1	0	0	0	0
X25	0	0	-1	-1	1	1	-1	1	-1	1	0	0	0	0
X26	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X27	0	0	0	0	0	0	0	0	0	0	0	0	0	0
X28	0	0	1	1	-1	-1	-1	1	-1	1	0	0	0	0
X29	0	0	1	1	-1	-1	1	-1	1	-1	0	0	0	0
X30	0	0	-1	-1	1	1	1	-1	1	-1	0	0	0	0
X31	0	0	-1	-1	1	1	-1	1	-1	1	0	0	0	0
X32	1	-1	1	-1	-1	1	1	-1	-1	1	1	-1	1	-1
X33	1	-1	-1	1	1	-1	1	-1	-1	1	1	-1	-1	1
X34	2	-2	-1	1	1	-1	-1	1	1	-1	0	0	0	0
X35	2	-2	1	-1	-1	1	-1	1	1	-1	0	0	0	0
X36	-1	1	-1	1	1	-1	-1	1	1	-1	0	0	1	-1
X37	-1	1	1	-1	-1	1	-1	1	1	-1	0	0	-1	1
X38	0	0	1	-1	-1	1	1	-1	-1	1	-1	1	0	0
X39	0	0	-1	1	1	-1	1	-1	-1	1	-1	1	0	0
X40	1	-1	0	0	0	0	0	0	0	0	0	0	-1	1
X41	1	-1	0	0	0	0	0	0	0	0	0	0	1	-1
X42	-2	2	0	0	0	0	0	0	0	0	1	-1	0	0

## Appendix G

### Character Table of $B_Q(3, 5)$

Table G.1: Character table of  $B_Q(3, 5)$ 

$[g]$	1a						
$[\bar{g}]$	1A	3A	3B	3C	3D	3E	3F
$\chi_1$	1	1	1	1	1	1	1
$\chi_2$	1	1	1	1	1	1	1
$\chi_3$	4	4	4	4	4	4	4
$\chi_4$	4	4	4	4	4	4	4
$\chi_5$	5	5	5	5	5	5	5
$\chi_6$	5	5	5	5	5	5	5
$\chi_7$	6	6	6	6	6	6	6
$\chi_8$	20	8	8	2	-1	2	-4
$\chi_9$	20	8	8	2	-1	2	-4
$\chi_{10}$	40	16	16	4	-2	4	-8
$\chi_{11}$	10	$2\xi-4\omega$	$-4\xi+2\omega$	$5\xi+2\omega$	1	$2\xi+5\omega$	1
$\chi_{12}$	10	$2\xi-4\omega$	$-4\xi+2\omega$	$5\xi+2\omega$	1	$2\xi+5\omega$	1
$\chi_{13}$	10	$2\xi-4\omega$	$-4\xi+2\omega$	$5\xi+2\omega$	1	$2\xi+5\omega$	1
$\chi_{14}$	10	$2\xi-4\omega$	$-4\xi+2\omega$	$5\xi+2\omega$	1	$2\xi+5\omega$	1
$\chi_{15}$	20	$4\xi-8\omega$	$-8\xi+4\omega$	$10\xi+4\omega$	2	$4\xi+10\omega$	2
$\chi_{16}$	20	$4\xi-8\omega$	$-8\xi+4\omega$	$10\xi+4\omega$	2	$4\xi+10\omega$	2
$\chi_{17}$	10	$-4\xi+2\omega$	$2\xi-4\omega$	$2\xi+5\omega$	1	$5\xi+2\omega$	1
$\chi_{18}$	10	$-4\xi+2\omega$	$2\xi-4\omega$	$2\xi+5\omega$	1	$5\xi+2\omega$	1
$\chi_{19}$	10	$-4\xi+2\omega$	$2\xi-4\omega$	$2\xi+5\omega$	1	$5\xi+2\omega$	1
$\chi_{20}$	10	$-4\xi+2\omega$	$2\xi-4\omega$	$2\xi+5\omega$	1	$5\xi+2\omega$	1
$\chi_{21}$	20	$-8\xi+4\omega$	$4\xi-8\omega$	$4\xi+10\omega$	2	$10\xi+4\omega$	2
$\chi_{22}$	20	$-8\xi+4\omega$	$4\xi-8\omega$	$4\xi+10\omega$	2	$10\xi+4\omega$	2
$\chi_{23}$	30	-6	-6	3	-6	3	3
$\chi_{24}$	30	-6	-6	3	-6	3	3
$\chi_{25}$	30	-6	-6	3	-6	3	3
$\chi_{26}$	30	-6	-6	3	-6	3	3
$\chi_{27}$	5	$4\xi+\omega$	$\xi+4\omega$	$-2\xi+\omega$	2	$\xi-2\omega$	-1
$\chi_{28}$	5	$4\xi+\omega$	$\xi+4\omega$	$-2\xi+\omega$	2	$\xi-2\omega$	-1
$\chi_{29}$	10	$8\xi+2\omega$	$2\xi+8\omega$	$-4\xi+2\omega$	4	$2\xi-4\omega$	-2
$\chi_{30}$	15	$12\xi+3\omega$	$3\xi+12\omega$	$-6\xi+3\omega$	6	$3\xi-6\omega$	-3
$\chi_{31}$	15	$12\xi+3\omega$	$3\xi+12\omega$	$-6\xi+3\omega$	6	$3\xi-6\omega$	-3
$\chi_{32}$	5	$\xi+4\omega$	$4\xi+\omega$	$\xi-2\omega$	2	$-2\xi+\omega$	-1
$\chi_{33}$	5	$\xi+4\omega$	$4\xi+\omega$	$\xi-2\omega$	2	$-2\xi+\omega$	-1
$\chi_{34}$	10	$2\xi+8\omega$	$8\xi+2\omega$	$2\xi-4\omega$	4	$-4\xi+2\omega$	-2
$\chi_{35}$	15	$3\xi+12\omega$	$12\xi+3\omega$	$3\xi-6\omega$	6	$-6\xi+3\omega$	-3
$\chi_{36}$	15	$3\xi+12\omega$	$12\xi+3\omega$	$3\xi-6\omega$	6	$-6\xi+3\omega$	-3

Table G.2: Character table of  $B_Q(3,5)$  (continued)

$[g]$	$2a$									
$[g]$	2A	6A	6B	6C	6D	6E	6F	6G	6H	6I
X1	1	1	1	1	1	1	1	1	1	1
X2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
X3	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
X4	2	2	2	2	2	2	2	2	2	2
X5	1	1	1	1	1	1	1	1	1	1
X6	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
X7	0	0	0	0	0	0	0	0	0	0
X8	6	0	0	0	-3	0	6	0	0	6
X9	-6	0	0	0	3	0	-6	0	0	-6
X10	0	0	0	0	0	0	0	0	0	0
X11	4	-2 $\omega$	-2 $\xi$	$\xi$	1	$\omega$	2 $\xi$ - $\omega$	3 $\xi$ + $\omega$	$\xi$ +3 $\omega$	- $\xi$ +2 $\omega$
X12	2	2	2	$\xi$ -2 $\omega$	-1	-2 $\xi$ + $\omega$	4 $\xi$ + $\omega$	-1	-1	$\xi$ +4 $\omega$
X13	-2	-2	-2	- $\xi$ +2 $\omega$	1	2 $\xi$ - $\omega$	-4 $\xi$ - $\omega$	1	1	- $\xi$ -4 $\omega$
X14	-4	2 $\omega$	2 $\xi$	- $\xi$	-1	- $\omega$	-2 $\xi$ + $\omega$	-3 $\xi$ - $\omega$	- $\xi$ -3 $\omega$	$\xi$ -2 $\omega$
X15	-2	-2 $\xi$	-2 $\omega$	-2 $\omega$	-2	-2 $\xi$	-2	-2 $\xi$	-2 $\omega$	-2
X16	2	2 $\xi$	2 $\omega$	2 $\omega$	2	2 $\xi$	2	2 $\xi$	2 $\omega$	2
X17	4	-2 $\xi$	-2 $\omega$	$\omega$	1	$\xi$	- $\xi$ +2 $\omega$	$\xi$ +3 $\omega$	3 $\xi$ + $\omega$	2 $\xi$ - $\omega$
X18	2	2	2	-2 $\xi$ + $\omega$	-1	$\xi$ -2 $\omega$	$\xi$ +4 $\omega$	-1	-1	4 $\xi$ + $\omega$
X19	-2	-2	-2	2 $\xi$ - $\omega$	1	- $\xi$ +2 $\omega$	- $\xi$ -4 $\omega$	1	1	-4 $\xi$ - $\omega$
X20	-4	2 $\xi$	2 $\omega$	- $\omega$	-1	- $\xi$	$\xi$ -2 $\omega$	- $\xi$ -3 $\omega$	-3 $\xi$ - $\omega$	-2 $\xi$ + $\omega$
X21	-2	-2 $\omega$	-2 $\xi$	-2 $\xi$	-2	-2 $\omega$	-2	-2 $\omega$	-2 $\xi$	-2
X22	2	2 $\omega$	2 $\xi$	2 $\xi$	2	2 $\omega$	2	2 $\omega$	2 $\xi$	2
X23	6	0	0	-3	0	-3	-3	3	3	-3
X24	0	2 $\xi$ -2 $\omega$	-2 $\xi$ +2 $\omega$	$\xi$ - $\omega$	0	- $\xi$ + $\omega$	-3 $\xi$ +3 $\omega$	- $\xi$ + $\omega$	$\xi$ - $\omega$	3 $\xi$ -3 $\omega$
X25	0	-2 $\xi$ +2 $\omega$	2 $\xi$ -2 $\omega$	- $\xi$ + $\omega$	0	$\xi$ - $\omega$	3 $\xi$ -3 $\omega$	$\xi$ - $\omega$	- $\xi$ + $\omega$	-3 $\xi$ +3 $\omega$
X26	-6	0	0	3	0	3	3	-3	-3	3
X27	3	2 $\xi$ + $\omega$	$\xi$ +2 $\omega$	-2 $\xi$ - $\omega$	0	- $\xi$ -2 $\omega$	3 $\xi$	- $\xi$ + $\omega$	$\xi$ - $\omega$	3 $\omega$
X28	-3	-2 $\xi$ - $\omega$	- $\xi$ -2 $\omega$	2 $\xi$ + $\omega$	0	$\xi$ +2 $\omega$	-3 $\xi$	$\xi$ - $\omega$	- $\xi$ + $\omega$	-3 $\omega$
X29	0	0	0	0	0	0	0	0	0	0
X30	-3	-2 $\xi$ - $\omega$	- $\xi$ -2 $\omega$	2 $\xi$ + $\omega$	0	$\xi$ +2 $\omega$	-3 $\xi$	$\xi$ - $\omega$	- $\xi$ + $\omega$	-3 $\omega$
X31	3	2 $\xi$ + $\omega$	$\xi$ +2 $\omega$	-2 $\xi$ - $\omega$	0	- $\xi$ -2 $\omega$	3 $\xi$	- $\xi$ + $\omega$	$\xi$ - $\omega$	3 $\omega$
X32	3	$\xi$ +2 $\omega$	2 $\xi$ + $\omega$	- $\xi$ -2 $\omega$	0	-2 $\xi$ - $\omega$	3 $\omega$	$\xi$ - $\omega$	- $\xi$ + $\omega$	3 $\xi$
X33	-3	- $\xi$ -2 $\omega$	-2 $\xi$ - $\omega$	$\xi$ +2 $\omega$	0	2 $\xi$ + $\omega$	-3 $\omega$	- $\xi$ + $\omega$	$\xi$ - $\omega$	-3 $\xi$
X34	0	0	0	0	0	0	0	0	0	0
X35	-3	- $\xi$ -2 $\omega$	-2 $\xi$ - $\omega$	$\xi$ +2 $\omega$	0	2 $\xi$ + $\omega$	-3 $\omega$	- $\xi$ + $\omega$	$\xi$ - $\omega$	-3 $\xi$
X36	3	$\xi$ +2 $\omega$	2 $\xi$ + $\omega$	- $\xi$ -2 $\omega$	0	-2 $\xi$ - $\omega$	3 $\omega$	$\xi$ - $\omega$	- $\xi$ + $\omega$	3 $\xi$

Table G.3: Character table of  $B_Q(3,5)$  (continued)

[g]	2b						3a						6a			4a			5a
[ḡ]	2B	6J	6K	6L	6M	6N	3G	3H	9A	9B	9C	9D	6O	18A	18B	4A	12A	12B	5A
X1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
X3	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	-1
X4	0	0	0	0	0	0	1	1	1	1	1	1	-1	-1	-1	0	0	0	-1
X5	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	-1	-1	-1	0
X6	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	0
X7	-2	-2	-2	-2	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	1
X8	0	0	0	0	0	0	2	-1	2	-1	2	-1	0	0	0	0	0	0	0
X9	0	0	0	0	0	0	2	-1	2	-1	2	-1	0	0	0	0	0	0	0
X10	0	0	0	0	0	0	-2	1	-2	1	-2	1	0	0	0	0	0	0	0
X11	2	2ξ	2ω	-ω	-1	-ξ	1	1	ξ	ξ	ω	ω	1	ξ	ω	0	0	0	0
X12	-2	-2ξ	-2ω	ω	1	ξ	1	1	ξ	ξ	ω	ω	-1	-ξ	-ω	0	0	0	0
X13	-2	-2ξ	-2ω	ω	1	ξ	1	1	ξ	ξ	ω	ω	1	ξ	ω	0	0	0	0
X14	2	2ξ	2ω	-ω	-1	-ξ	1	1	ξ	ξ	ω	ω	-1	-ξ	-ω	0	0	0	0
X15	0	0	0	0	0	0	-1	-1	-ξ	-ξ	-ω	-ω	1	ξ	ω	0	0	0	0
X16	0	0	0	0	0	0	-1	-1	-ξ	-ξ	-ω	-ω	-1	-ξ	-ω	0	0	0	0
X17	2	2ω	2ξ	-ξ	-1	-ω	1	1	ω	ω	ξ	ξ	1	ω	ξ	0	0	0	0
X18	-2	-2ω	-2ξ	ξ	1	ω	1	1	ω	ω	ξ	ξ	-1	-ω	-ξ	0	0	0	0
X19	-2	-2ω	-2ξ	ξ	1	ω	1	1	ω	ω	ξ	ξ	1	ω	ξ	0	0	0	0
X20	2	2ω	2ξ	-ξ	-1	-ω	1	1	ω	ω	ξ	ξ	-1	-ω	-ξ	0	0	0	0
X21	0	0	0	0	0	0	-1	-1	-ω	-ω	-ξ	-ξ	1	ω	ξ	0	0	0	0
X22	0	0	0	0	0	0	-1	-1	-ω	-ω	-ξ	-ξ	-1	-ω	-ξ	0	0	0	0
X23	2	2	2	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
X24	-2	-2	-2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
X25	-2	-2	-2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
X26	2	2	2	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
X27	1	ω	ξ	ξ	1	ω	2	-1	2ξ	-ξ	2ω	-ω	0	0	0	1	ω	ξ	0
X28	1	ω	ξ	ξ	1	ω	2	-1	2ξ	-ξ	2ω	-ω	0	0	0	-1	-ω	-ξ	0
X29	2	2ω	2ξ	2ξ	2	2ω	-2	1	-2ξ	ξ	-2ω	ω	0	0	0	0	0	0	0
X30	-1	-ω	-ξ	-ξ	-1	-ω	0	0	0	0	0	0	0	0	0	1	ω	ξ	0
X31	-1	-ω	-ξ	-ξ	-1	-ω	0	0	0	0	0	0	0	0	0	-1	-ω	-ξ	0
X32	1	ξ	ω	ω	1	ξ	2	-1	2ω	-ω	2ξ	-ξ	0	0	0	1	ξ	ω	0
X33	1	ξ	ω	ω	1	ξ	2	-1	2ω	-ω	2ξ	-ξ	0	0	0	-1	-ξ	-ω	0
X34	2	2ξ	2ω	2ω	2	2ξ	-2	1	-2ω	ω	-2ξ	ξ	0	0	0	0	0	0	0
X35	-1	-ξ	-ω	-ω	-1	-ξ	0	0	0	0	0	0	0	0	0	1	ξ	ω	0
X36	-1	-ξ	-ω	-ω	-1	-ξ	0	0	0	0	0	0	0	0	0	-1	-ξ	-ω	0

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