Nonclassical solutions of hyperbolic conservation laws

A dissertation submitted to School of Mathematics, Statistics and Computer Science in fulfillment of the requirements for the degree of Masters in applied mathematics
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By

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Abstract

This dissertation studies the nonclassical shock waves which appears as limits of certain type diffusive-dispersive regularisation to hyperbolic of conservation laws. Such shocks occur very often when the flux function lacks the convexity especially when the initial conditions for Riemann problem belong to different region of convexity. They have negative entropy dissipation. They do not verify the classical Oleinik entropy criterion. The cubic function is taken as a flux function. The existence and uniqueness of such shock waves are studied. They are constructed as limits of traveling-wave solutions for diffusive-dispersive regularisation. A kinetic relation is introduced to choose a unique nonclassical solution to the Riemann problem.

The numerical simulations are investigated using a transport-equilibrium scheme to enable computing the nonclassical solution at the discrete level of kinetic function. The method is composed of an equilibrium step containing the kinetic relation at any nonclassical shock and a transport step advancing the discontinuity with time.

Declaration

I declare that this dissertation presents my original work and effort. It was carried out under the supervision of Dr. Jean Medard TCHOUKOUEGNO NGNOTCHOUYE in the School of Mathematics, Statistics and Computer Sciences, University of KwaZulu-Natal, Pietermaritzburg Campus.

It has not been submitted in any form to any university or institution of higher learning for any degree or qualification. Where use has been made of the work of others it is duly acknowledged.

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Dedication

This dissertation is dedicated to my late father Fiagan and my mum Hungnewo.

Acknowledgment

First of all, I am grateful to Almighty God who kept me safe and for his inspiration throughout this studies. I also thank my entire family, especially my brothers and sisters for their prayer support.

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Contents

Chapter 1

Introduction

This dissertation deals with the theory and the numerical computation of nonclassical solutions of hyperbolic conservation laws in the form

$$u_t + f(u)_x = 0, \quad u_0(x), \quad x \in \mathbb{R}, \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is called the flux function, $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is called the conserved variable, $x \in \mathbb{R}$ is the space variable, $t \in [0, \infty)$ is time, $u_0(x)$ is the initial condition. We are interested generally in the solution of the Riemann problem where the initial condition $u_0(x)$ is of heaviside type, possessing a single jump at the origin. For such Riemann problems, the weak solution is a superposition of fixed states, separated by the so called Lax curves, provided an entropy condition is satisfied. However, when the initial data do not satisfy the entropy condition, one can still find a solution and it is refered to as a nonclassical shock. A nonclassical system of conservation law is a system of conservation laws for which the solution comprises a nonclassical shock. Nonclassical systems of conservation laws arise in macroscopic models for crowd dynamics [1] and some models of magnetohydrodynamics [2].

In general, the weak solution of the Riemann problem with left and right states u^- and u^+ associated with a conservation law is not unique. Uniqueness is restored by requiring that the states u^+ and u^- satisfy an entropy condition that can be either the Lax inequality or the Oleinik entropy condition. These two conditions are obviously true when the flux function is convex. For nonconvex flux, these entropy conditions may fail and one can still construct a unique weak solution provided that the states u^+ and u^- satisfy a kinetic relation in the form $u^+ = \mathcal{K}(u^-)$. Just like for classical shock where the entropic solution is obtained as a limit of a diffusive regularisation, nonclassical solutions can also be obtained as a limit of a regularised equation with a diffusive and dispersive term. The numerical solution of nonclassical system of conservation laws is done in this work using the transport equilibrium scheme of Chalons [3]. The choice of this scheme is because it computes accurately nonclassical shock front using the known kinetic relation.

The nonclassical solution of hyperbolic conservation laws has been introduced first by LeFloch [4]. The existence and the uniqueness is studied by Hayes and LeFloch [5–7] by considering diffusive-dispersive regularization. They showed that the limit value given by diffusive-dispersive regularization and many similar continuous or discrete models verify the single entropy inequality. They showed that when the flux is convex, the entropy inequality select a unique weak solution of (1.1). However when the flux lacks convexity or concavity, this is no longer true and there is room for an additional selection criterion. Jacobs, McKinney and Shearer [8] and then Hayes and LeFloch [5] showed that limits of diffusive-dispersive regularizations depend on sign of the diffusive-dispersive's parameters later called ε and δ. The limits do not coincide with the classical entropy solutions of Kruzkov-Volpert's theory for which is the problem (1.1) has a unique classical entropy solution [9,10]. For our case we use the same regularisation by focusing on the case where $\varepsilon > 0$ and $\delta > 0$ for cubic type as flux function. In fact the case $\varepsilon > 0$ and $\delta < 0$ gives the Lax shocks (classical shocks) [5,8]. We used the kinetic relation function of propagation speed to restore the uniqueness by seting it to be equal to entropy dissipation.

The numerical computation of nonclassical solution of conservation laws is studied by Hages et al [5] where they compared the Beam-Warming and Lax-Wendroff schemes. They realised that the Beam-Warming scheme produces the non-classical shocks while no such shocks are obtained with the Lax-Wendroff scheme. The results obtained rely critically on the sign of the dispersion coefficient and the type of function under consideration. LeFloch and Mohammadian [11] also computed numerically the nonclassical solutions of conservation laws using high-order finite difference method. The main point of the study was to numerically determine kinetic functions associated with corresponding scheme. These approximations of the kinetic relation help to evaluate the ability of a scheme for computing nonclassical shocks.

The rest of this dissertation is organised as follows; in Chapter 2 we present some generalities on classical solution of systems of the conservation laws. Here an existence and uniqueness result of the entropic (in the sense of Lax) solution of Riemann problems is presented and it appears the solution is a justaposition of fixed states separated by Lax curves. These Lax curves can either be rarefaction waves, or shock waves or contact discontinuities waves. The solution can be computed numerically using the finite volume method. This method, whose semi-discrete form is conservative ensure that the discontinuous exact solution is captured accurately with the correct velocity of the shock front.

Chapter 3 is concerned with the theory of nonclassical solutions. The focus is on the scalar case. To restore uniqueness for such problem the states across a nonclassical shock are required to satisfy a so-called kinetic relation. This relation is expressed in terms of a kinetic function which in turn is constructed using the geometrical properties of the flux function. It is important to note that nonclassical shock appears in general for problem for which the flux function is not convex or not concave.

Chapter 4 deals with a numerical method for the solution of nonclassical system of conservation laws. This method called transport-equilibrium method consist of an equilibrium step where a conservative scheme is modified so as to introduce an equilibrium at each interface where there is a nonclassical shock and the transport step aims at propagating the corresponding discontinuity. The algorithm is illustrated on many examples among which the cubic flux that is of concave-convex type. Finally we present a conclusion and an outlook in Chapter 5.

Chapter 2

Preliminaries on systems of conservation laws

In this chapter we present basic results and concepts on system of conservation laws which arise naturally in applications as conservation of mass, momentum and energy. Most of the results presented in this chapter are taken from [12–14]. The interested reader may consult [13] for the analytical results and [15–19] for the numerical results on conservation laws in general.

We will first present results on scalar conservation laws and later consider the general case of systems of conservation laws.

2.1 Scalar conservation laws

A scalar conservation law in one dimension is a first order partial differential equation in the form

$$u_t + f(u)_x = 0, \tag{2.1}$$

where $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is called the conserved quantity, $x \in \mathbb{R}$ the space variable, $t \in [0, \infty)$ time, and $f : \mathbb{R} \to \mathbb{R}$ is the flux function.

The equation (2.1) is a suitable model for transport problems as illustrated below. Consider a fluid of density u flowing in a pipe. Let $x = a_1$ and $x = a_2$ be the locations of two cross sections of the pipe. By integrating the equation (2.1) over the interval $[a_1, a_2] \subset \mathbb{R}$, we find

$$\frac{d}{dt}\int_{a_1}^{a_2} u(t,x)dx = \int_{a_1}^{a_2} u_t(t,x)dx = -\int_{a_1}^{a_2} f(u(t,x))_x dx, \tag{2.2}$$

$$= f(u(t,a_1)) - f(u(t,a_2)), \tag{2.3}$$

$$= [influx \quad at \quad a_1] - [outflux \quad at \quad a_2]. \tag{2.4}$$

Hence the rate of change of u in the sections of the pipe limited by a cross section $x = a_1$ and $x = a_2$ depends only on the influx at $x = a_1$ and the outflux at $x = a_2$.

An example of such a problem is traffic flow for which u represent the number of vehicles per kilometer and the flux f, in the LWR model [20, 21] has the form

$$f(u) = [v(u)u]_x,$$

where $v = v(u)$ is the average velocity of the cars. If u_{max} is the maximum density of traffic and v_{max} the maximum velocity of cars, the velocity is linearly decreasing from v_{max} when the density is zero to zero when the density is u_{max}. When the density of cars is zero, drivers will drive at the speed v_{max} which in practice is the speed limit in the area. When the cars are in a bumper to bumper situation (density close to u_{max}), the drivers are driving with a speed close to zero.

Concepts of solution

A smooth solution of (2.1) is a function u which satisfies (2.1) at every $(t,x) \in [0,\infty) \times \mathbb{R}$. Smooth solution can be found using the method of characteristics. Indeed for smooth solution u the equation (2.1) can be written in the equivalent quasilinear form

$$u_t + f'(u)u_x = 0. \tag{2.5}$$

Consider an initial condition $u(0,x) = u_0(x)$. The idea of the method of characteristics consists of looking for the solution along a curve $x = x(t)$, called a characteristic curve so that the function $u = u(t, x(t))$ satisfies equation (2.5). The total derivative of u is then

$$\frac{du}{dt} = u_t(t,x(t)) + \frac{dx}{dt}(t)u_x((t,x(t)) = 0. \tag{2.6}$$

Comparing (2.5) and (2.6) we see that the characteristics must satisfy

$$\frac{dx}{dt} = f'(u), \quad \frac{du}{dt} = 0, \quad u(0,x) = u_0(x). \tag{2.7}$$

Solving this system of ordinary differential equation provides a solution for the partial differential equation (2.5).

For example consider the Cauchy problem for the advection equation

$$u_t + \alpha u_x = 0 \quad u(0,x) = u_0(x), \tag{2.8}$$

where $\alpha \in \mathbb{R}$ is given. The smooth solution of (2.8) is obtained by the method of characteristics. Indeed the characteristics emanating from $(0,\xi)$ satisfy

$$\begin{aligned} \tfrac{dx}{dt} &= \alpha \quad x = \xi \text{ at } t = 0, \\ \tfrac{du}{dt} &= 0 \quad u = u_0(\xi) \text{ at } t = 0. \end{aligned}$$

Solving this system of equations and eliminating ξ, the solution of the advection equation is found in the traveling waves form

$$u(t,x) = u_0(x - \alpha t). \tag{2.9}$$

As another example, we consider the Cauchy problem for the inviscid Burger's equation given by

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad u(0,x) = -sinx. \tag{2.10}$$

The smooth solution are found using the quasilinear form $u_t + uu_x = 0$. The characteristics starting at $(0,\xi)$ satisfy

$$\begin{aligned} \tfrac{dx}{dt} &= u \quad x = \xi \text{ at } t = 0, \\ \tfrac{du}{dt} &= 0 \quad u = -\sin(\xi) \text{ at } t = 0. \end{aligned}$$

Solving these equations, we obtain

$$u = -sin(\xi), \quad x = \xi - tsin(\xi).$$

The characteristics through $(0, -\frac{\pi}{2})$ is found by $x = -\frac{\pi}{2} + t$ while the characteristics through $(0, \frac{\pi}{2})$ is found by $x = \frac{\pi}{2} - t$. The two characteristics intersect at time $t = T^* = \pi$ as illustrated in 2.1. This leads to a multivalued solution which is not desirable, hence we aim at finding a weak solutions.

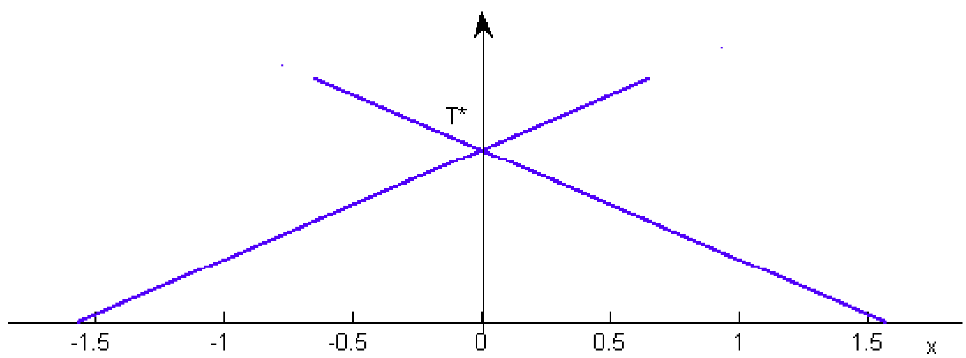


Figure 2.1: Two crossing characteristics of the Burger's equation.

2.1.1 Weak solution

Definition 2.1.1. Assume that the flux function f is $\mathcal{C}^1$. A measurable function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is weak solution of the system of conservation (2.1) if
$u : [0, \infty) \times \mathbb{R} \mapsto \mathbb{R}^n$ is continuous as function from $[0, \infty)$ into L^1_{loc} and for all $\varphi \in \mathcal{C}^1$ with compact support we have

$$\int_0^\infty \lim \left\{ \int [u\varphi_t + f(u)\varphi_x] dxdt \right\} = 0. \tag{2.11}$$

A characterization of weak solutions is given in the following lemma.

Lemma 2.1.1. The piece-wise constant function

$$u(t, x) = \begin{cases} u^- & if \quad x < st, \\ u^+ & if \quad x > st. \end{cases} \tag{2.12}$$

is weak solution of the conservation laws (2.1) if and only if

$$f(u^+) - f(u^-) = s(u^+ - u^-), \tag{2.13}$$

where u^-, $u^+ \in \mathbb{R}^n$, $s \in \mathbb{R}$.

Proof. 1. Let v be a test function that is a C^1 function with compact support in Λ^- and (Λ^+), where

$$\Lambda^+ = \Lambda \cap \{x > st\}, \quad \Lambda^- = \Lambda \cap \{x < st\}$$

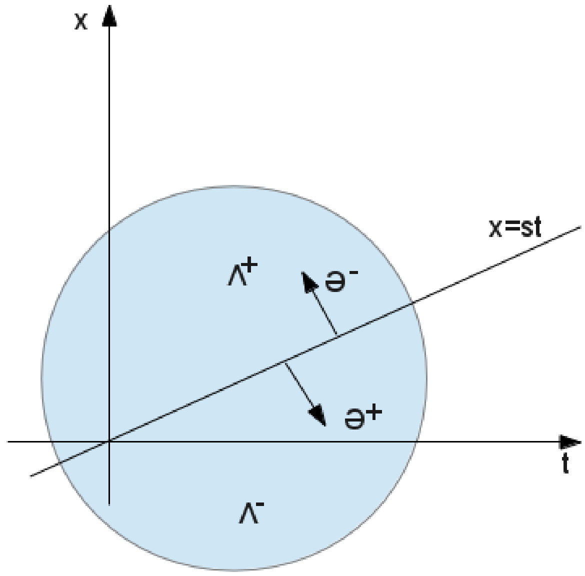


Figure 2.2: Rankine Hugoniot diagramm

Let n be the vector field such that

$$n = [v, f(v)]\varphi = [v\varphi, f(v)\varphi]$$

and

$$\begin{aligned} div(n) &= [v\varphi]_t + [f(v)\varphi]_x \\ &= v\varphi_t + f(v)\varphi_x \\ &= v\varphi_t + f(v)\varphi_x \end{aligned}$$

The equation (2.11) becomes

$$\int\int_{\Lambda^+ \cup \Lambda^-} div(n) dxdt = 0. \tag{2.14}$$

From the divergence theorem and the fact that $\varphi = 0$ on the boundary $\partial\Lambda$,

$$\int\int_{\Lambda^+\cup\Lambda^-} div(n)dxdt = 0 = \int_{\partial\Lambda^+} \partial^+ \cdot nds + \int_{\partial\Lambda^-} \partial,^- \cdot nds \tag{2.15}$$

where ∂^-, ∂^+ are the external normal vector to the domain Λ^-, Λ^+ respectively and ds the differential of the arc-length along the line $x = st$ defined by

$$\partial^+ ds = (s, -1)dt, \quad \partial^- ds = (-s, 1)dt.$$

It follows from (2.15) that

$$\int_{\partial\Lambda^+} \partial^+ \cdot nds + \int_{\partial\Lambda^-} \partial^- \cdot nds = \int [su^+ - f(u^+)]\varphi(t, st)dt + \int [-su^- + f(u^-)]\varphi(t, st)dt, \tag{2.16}$$

Finally

$$\int_{\partial\Lambda^+} \partial^+ \cdot nds + \int_{\partial\Lambda^-} \partial^- \cdot nds = \int [s(u^+ - u^-) - (f(u^+) - f(u^-))]\varphi(t, st)dt, \tag{2.17}$$

Therefore

$$\int [s(u^+ - u^-) - (f(u^+) - f(u^-))]\varphi(t, st)dt = 0. \tag{2.18}$$

The equation (2.18) is true $\forall\ \varphi \in \mathcal{C}_c^1$. This implies lemma (2.1.1).

2. Conversely let

$$\gamma_0 = \{(\xi(t), t)/t \in [0, T] = I\} \subset supp\varphi \quad supp\varphi = \Lambda^+ \cup \gamma_0 \cup \Lambda^-.$$

and

$$\xi'(t) = \frac{[f(u)]}{[u]}(\xi(t), t) \quad \forall \quad t \in [0, T].$$

We have

$$[f(u)](\xi(t), t) - [u]\xi'(t) = 0. \tag{2.19}$$

By integrating (2.19) with respect to t we have

$$\int_I \{[f(u)](\xi(t), t) - [u](\xi(t), t)\xi'(t)\}\varphi(\xi(t), t)dt = 0. \tag{2.20}$$

Let

$$v_e = (v_1, v_2) = (1, -\xi'(t)).$$

It follows that

$$\int_I \{[f(u)]v_1 + [u]v_2\}\varphi dt = 0 = \int_{\gamma_0} \{f(u^+)v_1 + u^+v_2\}\varphi dt - \int_{\gamma_0} \{f(u^-)v_1 + u^-v_2\}\varphi dt. \tag{2.21}$$

We will have

$$\int_{\gamma_0} \{f(u^+)v_1 + u^+v_2\}\varphi dt = \int_{\partial\Lambda_+} \{f(u^+)v_1 + u^+v_2\}\varphi dt. \tag{2.22}$$

By using the divergence theorem

$$\int_{\partial\Lambda_+} \{f(u^+)v_1 + u^+v_2\}\varphi dt = \int\int_{\Lambda_+} \{u\varphi_t + f(u)\varphi_x\}\varphi dxdt + \int\int_{\Lambda_+} \{u_t + f(u)_x\}\varphi dxdt. \tag{2.23}$$

But since our solution is classical outside of the shock we have

$$\int\int_{\Lambda_+} \{u_t + f(u)_x\}\varphi dxdt = 0. \tag{2.24}$$

Therefore we have

$$\int_{\gamma_0} \{f(u^+)v_1 + u^+v_2\}\varphi dt = \int\int_{\Lambda_+} \{u\varphi_t + f(u)\varphi_x\}\varphi dxdt. \tag{2.25}$$

Similarly we have

$$\int_I \{[f(u)]v_1 + [u]v_2\}\varphi dt = 0 = \tag{2.26}$$

$$-\int_{\gamma_0} \{f(u^-)v_1 + u^-v_2\}\varphi dt = \int\int_{\Lambda_-} \{u\varphi_t + f(u)\varphi_x\}\varphi dxdt. \tag{2.27}$$

Putting (2.26) and (2.25) together equation (2.21) becomes

$$\int_I \{[f(u)]v_1 + [u]v_2\}\varphi dt = 0 = \int\int_{\Lambda_+\cup\Lambda_-} \{u\varphi_t + f(u)\varphi_x\}\varphi dxdt. \tag{2.28}$$

We have proved therefore the converse. □

The formula (2.13) is called Rankine-Hugoniot condition. In the scalar case we can solve for s

to get

$$s = \frac{f(u^+) - f(u^-)}{u^+ - u^-} \tag{2.29}$$

called the shock speed and can be interpreted geometrically as the slope of the secant line passing throught the point $(u^-, f(u^-)$ and $(u^+, f(u^+))$ on the graph of flux function f.

Weak solution of system of conservation laws are not unique. Indeed consider the Riemann problem for the Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad u(0,x) = \begin{cases} 1 & if \quad x \geqslant 0, \\ 0 & if \quad x < 0. \end{cases} \tag{2.30}$$

For $0 < \rho < 1$,

$$u_\rho(t,x) = \begin{cases} 0 & if \quad x < \frac{\rho t}{2}, \\ \rho & if \quad \frac{\rho t}{2} < x < \frac{(1+\rho)t}{2}, \\ 1 & if \quad x \geq \frac{(1+\rho)t}{2}. \end{cases} \tag{2.31}$$

is a weak solution.

In fact the piecewise constant function u_ρ satisfies the equation outside of the jumps and the Rankine Hugoniot condition holds on the two lines of discontinuity $\{x = \frac{\rho t}{2}\}$ and $\{x = \frac{(1+\rho)t}{2}\}$. Hence the problem (2.30) has an infinite number of solutions.

To single out the physically relevant solution, we need some admissibility conditions.

(1) **Vanishing viscosity approach**: This approach aims to look at the solutions of the conservation law (2.1) as the limit of a sequence of solution of the viscous model [13]

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \tag{2.32}$$

as $\varepsilon \to 0$. In fact for general $n \times n$ systems, the major problem resides in etablishing the compactness of the approximating sequence [22]. We realise that $u^\varepsilon(t,x)$ solves (2.32) if and only if $u^\varepsilon(t,x) = u(t/\varepsilon, x/\varepsilon)$ for some function u which verifies

$$u_t + J(u)u_x = u_{xx}, \tag{2.33}$$

where $J(u)$ is Jacobian matrix of $f(u)$. In the analysis of vanishing viscosity approach the key step is to derive a priori estimates on the stability of solutions of (2.33). The solution are now taken from the viscous traveling profiles solutions of the form

$$u(t,x) = U(x - \lambda t), \tag{2.34}$$

where $\lambda \in \mathbb{R}$ and the function U must verifies the second order ODE

$$U'' = (J(U) - \lambda)U'. \tag{2.35}$$

In this new approach, the profile $u(.)$ of a viscous solution is viewed locally as a superposition of viscous traveling waves [22]. The scalar cases are fully obtained in [23, 24].

(2) **Lax inequality**: A jump in the solution between two states u_- and u_+, satisfying the Rankine Hugoniot condition, is admissible in the Lax sense, or satisfies the Lax entropy inequality if

$$f'(u^-) \geqslant s \geqslant f'(u^+). \tag{2.36}$$

Example

Consider the following Initial value problem for the Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad u(0,x) = \begin{cases} 0 & if \quad x < 0, \\ 2 & if \quad 0 < x < 1, \\ 1 & if \quad x > 1. \end{cases} \tag{2.37}$$

We aim at finding the unique entropy solution for all time $t \geqslant 0$. The initial condition has two jumps, see Figure 2.3, one located at $x = 0$ and the other at $x = 1$. For the left jump, $u^- = 0$, $u^+ = 2$ and the shock speed is found as $s = 1$. We see that $f'(u^-) = 0 < s$ and the Lax inequalities are not satisfied. Consequently, the entropy solution emanating from $x = 0$ is a rarefaction wave. One can show in a similar way that the Lax inequalities hold for the right jump. Hence the solution emanating from $x = 1$ is a shock wave. The characteristics satisfy

$$\frac{dx}{dt} = \begin{cases} 0 \quad if \quad \xi < 0, \\ 2 \quad if \quad 0 < \xi < 1, \\ 1 \quad if \quad \xi > 1. \end{cases}$$

Hence those emanating from $(0, \xi)$ satisfy

$$x = \begin{cases} \xi \quad if \quad \xi < 0, \\ 2t + \xi \quad if \quad 0 < \xi < 1, \\ t + \xi \quad if \quad \xi > 1. \end{cases}$$

The rarefaction wave is bounded above by the curve $x = 2t$. The shock position satisfies

$$\frac{dx}{dt} = \frac{3}{2}, \quad x(0) = 1.$$

The solution is found by

$$x = \frac{3}{2}t + 1.$$

The rarefaction wave crosses the shock wave when

$$2t = \frac{3}{2}t + 1,$$

giving $t = 2$ and $x(2) = 4$. Consequently, the entropy solution of the problem for $0 \leq t \leq 2$ is given by

$$u(t, x) = \begin{cases} 0 \quad if \quad x < 0, \\ \frac{x}{t} \quad if \quad 0 < x < 2t, \\ 2 \quad if \quad 2t < x < \frac{3t}{2} + 1, \\ 1 \quad if \quad x > \frac{3t}{2} + 1. \end{cases} \tag{2.38}$$

After the interaction, a shock emerges with its speed given by the Rankine-Hugoniot condition

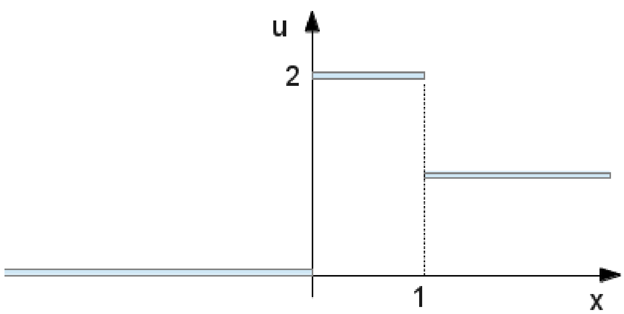
$$\frac{dx}{dt} = \frac{x}{2t} + \frac{1}{2}, \quad x(2) = 4. \tag{2.39}$$

Solving (2.39) gives the shock position

$$x(t) = \sqrt{t} + t + 2 - \sqrt{2}.$$

The entropy solution of the problem (2.37) for $t > 2$ is therefore given by

$$u(t,x) = \begin{cases} 0 \quad if \quad x < 0, \\ \frac{x}{t} \quad if \quad 0 < x < \sqrt{t} + t + 2 - \sqrt{2}, \\ 1 \quad if \quad x > \sqrt{t} + t + 2 - \sqrt{2}. \end{cases} \tag{2.40}$$



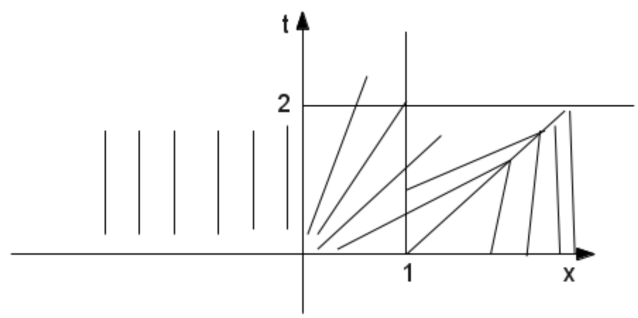


Figure 2.3: Initial condition (left) and characteristics propagation (right) for the example

2.2 System of conservation laws

A system of $n \times n$ conservation laws in one dimension is an equation of the form

$$\partial_t u + \partial_x f(u) = 0, \tag{2.41}$$

where $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}^n,\ f : \mathbb{R}^n \to \mathbb{R}^n$.

For smooth solutions the equation (2.41) is equivalent to the quasilinear form

$$u_t + J(u)u_x = 0, \tag{2.42}$$

where $J(u) = Df(u)$ is the Jacobian matrix of the flux f at the point u.

Definition 2.2.1. A system of conservation laws is said to be strictly hyperbolic if the Jacobian matrix $J(u)$ has n real, distinct eigenvalues.

For strictly hyperbolic systems of conservation laws, the eigenvalues can be sorted as

$$\lambda_1(u) < \lambda_2(u) < ..... < \lambda_n(u).$$

For these eigenvalues, we can find basis $r_i(u)$ and $l_j(u),\ i,\ j = 1, 2..., n.$ of right and left eigenvectors respectively with the normalisation

$$l_j(u).r_i(u) = \delta^i_j, \tag{2.43}$$

where δ^i_j is the Kronecker delta defined by

$$\delta^i_j = \begin{cases} 1 \quad if \quad i = j, \\ 0 \quad if \quad i \neq j. \end{cases} \tag{2.44}$$

Definition 2.2.2. i).The i-th field is genuinely nonlinear if for all u

$$D\lambda_i(u) \cdot r_i(u) > 0.$$

ii).The i-th field is linearly degenerate if for all u

$$D\lambda_i(u) \cdot r_i(u) = 0,$$

where D denotes the derivative with respect to the conserved variable u.

2.2.1 Linear system

Consider the Cauchy problem for a linear system of conservation laws

$$u_t + Au_x = 0, \quad u(0,x) = \bar{u}(x), \tag{2.45}$$

where A is an $n \times n$ constant matrix with real and distinct eigenvalues λ_i, and corresponding left and right eigenvectors r_i and $l_j,\ i = 1, \dots, n$. For smooth solutions, we can write the components of the vector u in the basis of right eigenvectors as

$$u_j = l_j \cdot u.$$

Multiplying (2.45) on the left by l_j, gives

$$(u_j)_t + \lambda_j (u_j)_x = 0; \quad u_j(0,x) = \bar{u}_j(x),\ j = 1, \dots, n. \tag{2.46}$$

(2.46) is scalar advection equation whose solution is given by

$$u_j(t,x) = \bar{u}_j(x - \lambda_j t).$$

The solution of the Cauchy problem is then

$$u(t,x) = \sum_{j=1}^{n} \bar{u}_j(x - \alpha_j t) r_j. \tag{2.47}$$

For the Riemann problem (2.45), the initial condition $\bar{u}(x)$ satisfies

$$\bar{u}(x) = \begin{cases} u^- & if \quad x < 0, \\ u^+ & if \quad x > 0. \end{cases} \tag{2.48}$$

We can write the jump in the initial data $u^+ - u^-$ as linear combination of the right eigenvectors of A

$$u^+ - u^- = \sum_{j=1}^{n} \vartheta_j r_j.$$

where ϑ_j, $j = 1, ...n$ are scalars. We construct the intermediate states

$$\psi_j = u^- + \sum_{i \le j} \vartheta_i r_i, \quad j = 0, ...n.$$

so that the difference

$$\psi_j - \psi_{j-1} = \sum_{i \le j} \vartheta_i r_i - \sum_{i \le j-1} \vartheta_i r_i = \vartheta_j r_j$$

is a j-eigenvector of A. The corresponding solution of (2.45) with $\bar{u}$ as in (2.48) is

$$u(t,x) = \begin{cases} \psi_0 = u^- \quad if \quad \frac{x}{t} < \lambda_1, \\ ..... \\ \psi_j \quad for \quad \lambda_j < \frac{x}{t} < \lambda_{j+1}, \\ ..... \\ \psi_n = u^+ \quad if \quad \frac{x}{t} > \lambda_n. \end{cases} \tag{2.49}$$

2.2.2 Admissibility condition

As for the scalar case, the weak solution of system of conservation laws are not unique. Further admissibility conditions that aim to single out the physically relevant solution are presented below.

Entropy inequality

Definition 2.2.3. A continuously differentiable function $\mathcal{U} : \mathbb{R}^n \mapsto \mathbb{R}$ is called an entropy for the system of conservation laws (2.1) with entropy flux $\mathcal{F} : \mathbb{R}^n \mapsto \mathbb{R}$ if

$$D\mathcal{U}(u)Df(u) = D\mathcal{F}(u), \quad \forall u \in \mathbb{R}^n, \tag{2.50}$$

holds.

The pair $(\mathcal{U}, \mathcal{F})$ is called an entropy-entropy flux pair for the system of conservation laws (2.1).

A weak solution u of a system of conservation laws satisfy an entropy inequality if

$$\mathcal{U}(u)_t + \mathcal{F}(u)_x \leqslant 0, \tag{2.51}$$

holds, for all entropy-entropy flux pair $(\mathcal{U}, \mathcal{F})$.

Finding an entropy-entropy flux pair is easy when $n = 2$. Indeed in this case (2.50) is a system of 2 partial differential equations. But when $n \geq 3$ the system (2.50) is overdetermined. In fact for $n \times n$ systems the equations (2.50) can be regarded as a first order differential system of n equations for two scalar functions variables $\mathcal{U}$, $\mathcal{F}$. Note that when we have an entropy-entropy flux pair $(\mathcal{U}, \mathcal{F})$, then for a smooth solution u of the conservation laws the associated image $\mathcal{U}(u)$, satisfies

$$\mathcal{U}(u)_t + \mathcal{F}(u)_x = 0, \tag{2.52}$$

holds. Indeed

$$\mathcal{U}(u)_t + \mathcal{F}(u)_x = D\mathcal{U}(u)u_t + D\mathcal{F}(u)u_x = D\mathcal{U}(u)(-Df(u)u_x) + D\mathcal{F}(u)u_x = 0. \tag{2.53}$$

For Burger's equation for which the flux function is $f(u) = \frac{u^2}{2}$, an entropy-entropy flux pair is found as

$$\mathcal{U}(u) = u^3, \quad \mathcal{F}(u) = \frac{3u^4}{4}.$$

In fact we have from the equation (2.50) $\mathcal{U}'(u)f'(u) = \mathcal{F}'(u)$. Since $f'(u) = u$, we let $\mathcal{U}(u) = u^3$, and then $\mathcal{F}'(u) = 3u^3$. Integrating gives $\mathcal{F}(u) = \frac{3u^4}{4}$. The function

$$u(t,x) = \begin{cases} 1 & if \quad x < \frac{t}{2}, \\ 0 & if \quad x \geqslant \frac{t}{2}, \end{cases} \tag{2.54}$$

is a discontinuous weak entropy solution for the Burger's equation. Indeed, at the point of jump, the following entropy inequality is satisfied:

$$\mathcal{F}(u^+) - \mathcal{F}(u^-) < s[\mathcal{U}(u^+) - \mathcal{U}(u^-)].$$

This is because here $s = \frac{1}{2}$, $u^- = 1$, $u^+ = 0$, and then $\mathcal{F}(u^+) - \mathcal{F}(u^-) = -\frac{3}{4}$ and $\mathcal{U}(u^+) - \mathcal{U}(u^-) = 1$.

Lax inequality

A shock solution of the system of conservation laws of the form

$$u(t,x) = \begin{cases} u^- & if \quad x < s_i t, \\ u^+ & if \quad x > s_i t, \end{cases} \tag{2.55}$$

satisfies the Lax inequalities if

$$\lambda_i(u^-) \geqslant s_i \geqslant \lambda_i(u^+), \tag{2.56}$$

where s_i is the shock speed given by Rankine-Hugoniot condition and $\lambda_i(u)$ is the i-th eigenvalue of the Jacobian matrix of the flux function.

2.2.3 Riemann problem for system of conservation laws

We now present the general procedure for the construction of the solution of the Riemann problem for a nonlinear system of conservation laws

$$\begin{cases} u_t + f(u)_x = 0, \\ u_0(x) = \begin{cases} u^- & if \quad x < 0, \\ u^+ & if \quad x > 0. \end{cases} \end{cases} \tag{2.57}$$

Shock and Rarefaction waves

Assume the system of equation (2.57) is strictly hyperbolic with each field either genuinely nonlinear or linear degenerate. Let $\lambda_i = \lambda_i(u)$ and $r_i = r_i(u)$ be an eigenvalue and the corresponding eigenvectors of Jacobian matrix $J(u)$ of f.

Definition 2.2.4. The i-rarefaction curve is the integral curve of the vector field r_i passing through $\overline{u} \in \mathbb{R}^n$.

$$\frac{du}{ds} = r_i(u), \quad u(0) = \overline{u}. \tag{2.58}$$

We denote the solution of (2.58) by

$$s \mapsto R_i(s)(\overline{u}). \tag{2.59}$$

Definition 2.2.5. For a given state $\overline{u} \in \mathbb{R}$, the i-shock curve through $\overline{u}$ is the set of solutions of the Rankine-Hugoniot conditions

$$f(u) - f(\overline{u}) = s(u - \overline{u}), \tag{2.60}$$

We denote the solution of (2.60) by

$$s \mapsto S_i(s)(\overline{u}). \tag{2.61}$$

Note that the equation (2.60) is a system of n equations in $n+1$ unknown hence the solution set describe a curve.

Contact discontinuity

When the i-th field is linearly degenerate (see Definition 2.2.2), the shock and the rarefaction curve coincide and are called contact discontinuity curve.
The existence of solutions to the Riemann problem is given by the following theorem.

Theorem 2.2.1. Assume the system of equation (2.57) is strictly hyperbolic with each field either genuinely nonlinear or linear degenerate. For $\| u^+ - u^- \|$ sufficiently small, there exists a unique entropy (in the sense of Lax) solution to the Riemann problem (2.57). The solution comprises m+1 constant states $u^- = u_0,\ u_1,\ \ldots..,u_{m-1},\ u_m = u^+$. When the i-th characteristic field is linearly degenerate u_i is joined to u_{i-1} by an i-contact discontinuity, while when the i-characteristic field is genuinely nonlinear u_i is joined to u_{i-1} by either an i-(Lax) rarefaction or an i-(Lax) shock.

A proof of this theorem can be found in [13].

Example: Consider the Riemann problem for the conservation laws

$$u_t + f(u)_x = 0 \tag{2.62}$$

with the initial condition

$$u_0(x) = \begin{cases} u^- & if \quad x < 0, \\ u^+ & if \quad x > 0. \end{cases} \tag{2.63}$$

where

$$u = \begin{pmatrix} v \\ \phi \end{pmatrix} \quad \text{and} \quad f(u) = \begin{pmatrix} \frac{v^2}{2} + \phi \\ v\phi \end{pmatrix}.$$

The Jacobian matrix of the flux function is

$$A = Df(v, \phi) = \begin{pmatrix} v & 1 \\ \phi & v \end{pmatrix}, \tag{2.64}$$

1. The eigenvalues are found as solution of the quadratic equation

$$\begin{vmatrix} v - \lambda & 1 \\ \phi & v - \lambda \end{vmatrix} = 0 \Rightarrow (v - \lambda)^2 - \phi = 0. \tag{2.65}$$

   The solution is

$$\lambda_1 = \lambda_1(v, \phi) = v - \sqrt{\phi} \;\text{ and }\; \lambda_2 = \lambda_2(v, \phi) = v + \sqrt{\phi}.$$

   We have

$$D\lambda_1.r_1 = \left(1, -\frac{1}{2\sqrt{\phi}}\right)\begin{pmatrix} 1 \\ -\sqrt{\phi} \end{pmatrix} = 1 + \frac{1}{2} = \frac{3}{2}, \tag{2.66}$$

$$D\lambda_2.r_2 = \left(1, \frac{1}{2\sqrt{\phi}}\right)\begin{pmatrix} 1 \\ \sqrt{\phi} \end{pmatrix} = 1 + \frac{1}{2} = \frac{3}{2}. \tag{2.67}$$

   Hence the two fields are both genuinely nonlinear.

2. To get the shock curves we do the following. For a fixed state $(\overline{v}, \overline{\phi})$, the Rankine-Hugoniot jump conditions reads

$$f(v, \phi) - f(\overline{v}, \overline{\phi}) = s[(v, \phi)^t - (\overline{v}, \overline{\phi})^t],$$

   This implies

$$\begin{cases} \frac{v^2}{2} - \frac{\overline{v}^2}{2} + \phi - \overline{\phi} = s(v - \overline{v}) \\ v\phi - \overline{v}\overline{\phi} = s(\phi - \overline{\phi}). \end{cases} \tag{2.68}$$

In (2.68), s is the shock speed. Our strategy is to solve (2.68) for ϕ and s in terms of v. From the first equation in (2.68), we get

$$\phi = \overline{\phi} - \frac{1}{2}(v^2 - \overline{v}^2) + s(v - \overline{v}). \tag{2.69}$$

Substitute in the second equation and after simplification arrive at the following quadratic in s,

$$2s^2 - (3v + \overline{v})s + v\overline{v} + v^2 - 2\overline{\phi} = 0. \tag{2.70}$$

Solving we find

$$\begin{cases} s_1 = s_1(v; \overline{v}, \overline{\phi}) = \frac{1}{4}(3v + \overline{v}) - \frac{1}{4}\sqrt{(v - \overline{v})^2 + 16\overline{\phi}}, \\ s_2 = s_2(v; \overline{v}, \overline{\phi}) = \frac{1}{4}(3v + \overline{v}) + \frac{1}{4}\sqrt{(v - \overline{v})^2 + 16\overline{\phi}}. \end{cases} \tag{2.71}$$

When $v = \overline{v}$ we get $s_1 = \lambda_1$ and $s_2 = \lambda_2$. Substituting in (2.69) we get

$$\begin{cases} S_1 = \overline{\phi} + \frac{1}{4}(v - \overline{v})((v - \overline{v}) - \sqrt{(v - \overline{v})^2 + 16\overline{\phi}}, \\ S_2 = \overline{\phi} + \frac{1}{4}(v - \overline{v})((v - \overline{v}) + \sqrt{(v - \overline{v})^2 + 16\overline{\phi}}. \end{cases} \tag{2.72}$$

3. For a given left state $(\overline{v}, \overline{\phi})$, the state that can be connected to $(\overline{v}, \overline{\phi})$ to the right through a 1-shock wave must satisfy the Lax inequality

$$\lambda_1(\overline{v}, \overline{\phi}) \geqslant s_1(v; \overline{v}, \overline{\phi})$$

or

$$\overline{v} - \sqrt{\overline{\phi}} \geqslant \frac{1}{4}(3v + \overline{v}) - \frac{1}{4}\sqrt{(v - \overline{v})^2 + 16\overline{\phi}}$$

Solving this inequality, we find $v \leqslant \overline{v}$. Similarly, for a given right state $(\overline{v}, \overline{\phi})$, the state that can be connected to $(\overline{v}, \overline{\phi})$ to the left through a 1-shock waves must satisfy the Lax inequality

$$s_1(v; \overline{v}, \overline{\phi}) \geqslant \lambda_1(\overline{v}, \overline{\phi})$$

or

$$\frac{1}{4}(3v+\overline{v}) - \frac{1}{4}\sqrt{(v-\overline{v})^2+16\overline{\phi}} \geqslant \overline{v} - \sqrt{\overline{\phi}}$$

and the solution is easily found to be $v \geqslant \overline{v}$. Repeating the same approach for the 2-shock, we find that the forward 2-shock is admissible if $v \geqslant \overline{v}$ and the backward 2-shock is admissible in the sense of Lax if $v \leqslant \overline{v}$.

4. The rarefaction curves are found as follows

$$\frac{d}{dt}\begin{pmatrix} u \\ \phi \end{pmatrix} = \frac{2}{3} r_{1,2}(v,\phi) = \begin{pmatrix} 1 \\ \pm\sqrt{\phi} \end{pmatrix}, \begin{pmatrix} v \\ \phi \end{pmatrix}(0) = \begin{pmatrix} \overline{v} \\ \overline{\phi} \end{pmatrix}. \tag{2.73}$$

For the 1-rarefaction curve, this system of equations reads

$$\begin{cases} \frac{dv}{dt} = \frac{2}{3} \quad v(0) = \overline{v}, \\ \frac{d\phi}{dt} = -\frac{2}{3}\sqrt{\phi} \quad \phi(0) = \overline{\phi}. \end{cases} \tag{2.74}$$

Solving this coupled system of ODEs gives

$$v = \frac{2}{3}t + \overline{v} \quad \text{and} \quad \phi = \left(\sqrt{\overline{\phi}} - \frac{1}{3}t\right)^2. \tag{2.75}$$

Eliminating t, we finally get for the 1-rarefaction curve

$$R_1(v;\overline{v},\overline{\phi}) = \left(\sqrt{\overline{\phi}} - \frac{1}{2}(v-\overline{v})\right)^2, \tag{2.76}$$

and for the 2-rarefaction curve

$$R_2(v;\overline{v},\overline{\phi}) = \left(\sqrt{\overline{\phi}} + \frac{1}{2}(v-\overline{v})\right)^2, \tag{2.77}$$

In summary the forward (+) and backward (-) Lax curves are found as

$$L_1^+(v;\overline{v},\overline{\phi}) = \begin{cases} R_1(v;\overline{v},\overline{\phi}) = \left(\sqrt{\overline{\phi}} - \frac{1}{2}(v-\overline{v})\right)^2 \quad if \quad v > \overline{v}, \\ S_1 = \overline{\phi} + \frac{1}{4}(v-\overline{v})\left\{(v-\overline{v}) - \sqrt{(v-\overline{v})^2+16\overline{\phi}}\right\} \quad if \quad v \leqslant \overline{v}; \end{cases} \tag{2.78}$$

$$L_1^-(v;\overline{v},\overline{\phi}) = \begin{cases} R_1(v;\overline{v},\overline{\phi}) = (\sqrt{\overline{\phi}} - \frac{1}{2}(v-\overline{v}))^2 \quad if \quad v \leqslant \overline{v}, \\ S_1 = \overline{\phi} + \frac{1}{4}(v-\overline{v}) \left\{ (v-\overline{v}) - \sqrt{(v-\overline{v})^2 + 16\overline{\phi}} \right\} \quad if \quad v > \overline{v}; \end{cases} \tag{2.79}$$

$$L_2^+(v;\overline{v},\overline{\phi}) = \begin{cases} R_2(v;\overline{v},\overline{\phi}) = (\sqrt{\overline{\phi}} + \frac{1}{2}(v-\overline{v}))^2 \quad if \quad v < \overline{v}, \\ S_2 = \overline{\phi} + \frac{1}{4}(v-\overline{v}) \left\{ (v-\overline{v}) + \sqrt{(v-\overline{v})^2 + 16\overline{\phi}} \right\} \quad if \quad v \geqslant \overline{v}; \end{cases} \tag{2.80}$$

$$L_2^-(v;\overline{v},\overline{\phi}) = \begin{cases} R_2(v;\overline{v},\overline{\phi}) = (\sqrt{\overline{\phi}} + \frac{1}{2}(v-\overline{v}))^2 \quad if \quad v \geqslant \overline{v}, \\ S_2 = \overline{\phi} + \frac{1}{4}(v-\overline{v}) \left\{ (v-\overline{v}) + \sqrt{(v-\overline{v})^2 + 16\overline{\phi}} \right\} \quad if \quad v < \overline{v}. \end{cases} \tag{2.81}$$

The shock speeds are found as

$$\begin{cases} s_1 = s_1(v;\overline{v},\overline{\phi}) = \frac{1}{4}(3v+\overline{v}) - \frac{1}{4}\sqrt{(v-\overline{v})^2 + 16\overline{\phi}}, \\ s_2 = s_2(v;\overline{v},\overline{\phi}) = \frac{1}{4}(3v+\overline{v}) + \frac{1}{4}\sqrt{(v-\overline{v})^2 + 16\overline{\phi}}. \end{cases} \tag{2.82}$$

2.3 Finite volume scheme for conservation laws

We are now interested in the numerical solution of the Cauchy problem for a system of conservation laws

$$u_t + f(u)_x = 0, \quad t > 0, \quad u(0,x) = \overline{u}(x) \quad \text{for} \quad x \in [a,b]. \tag{2.83}$$

We subdivise the spatial domain $[a,b]$ into points $x_i = a + i\Delta x_i$ where $i = 0, 1, ..., N$. We define the mid point $x_{i+1/2} = \frac{x_i + x_{i+1}}{2}$ and space width $\Delta x_i = x_i - x_{i-1}$. We also define the cell average

$$u_i = \frac{1}{\triangle x_i} \int_{I_i} u(x,t)dx, \tag{2.84}$$

where $I_i = [x_{i-1/2}; x_{i+1/2})$, $i = 1, ...N$. Integrating (2.83) over I_i gives

$$\frac{du_i}{dt} = \frac{1}{\triangle x}[f(u(x_{i-1/2},t)) - f(u(x_{i+1/2},t))]. \tag{2.85}$$

To complete the definition of the numerical scheme we need to find the correct approximations of the term $f(u(x_{i+1/2}, t))$ which define the numerical flux function.

We denote

$$F_{i+1/2} \approx f(u(x_{i+1/2}, t)) = \mathcal{F}(u_i, u_{i+1}). \tag{2.86}$$

where the function $\mathcal{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is called the numerical flux function. It may depend on more argument depending on the order of the scheme [25–27]. For first order scheme, only two arguments are necessary. The numerical flux is required to be consistent with the continuous flux in the sense that

$$\mathcal{F}(u, u) = f(u) \tag{2.87}$$

With approximation (2.86) and using (2.85), we arrive at the semi discrete numerical scheme

$$\frac{du}{dt} = \frac{1}{\triangle x}[F_{i-1/2} - F_{i+1/2}]. \tag{2.88}$$

The scheme of the form (2.88) are called conservative scheme. For convergence to the exact solution, the scheme must be conservative and consistent [18, 25, 26].

2.3.1 Example of numerical scheme

Here we are going to present some simple examples of useful schemes [16–19].

1. Upwind scheme: It is defined by the numerical flux

$$F_{i+1/2} = \begin{cases} f(u_i) \quad if \quad f'(u_i) \geqslant 0, \\ f(u_{i+1}) \quad if \quad f'(u_{i+1}) \leqslant 0. \end{cases} \tag{2.89}$$

2. Lax-Friedrichs scheme: It is based in central differencing and is very stable and the numerical flux is

$$F_{i+1/2} = \frac{1}{2}[u_{i-1}^n + u_{i+1}^n - \alpha(f(u_{i+1}^n) - f(u_i^n))], \tag{2.90}$$

where $\alpha = \max_u |f'(u)|$.

3. Local Lax-Friedrichs scheme: Its numerical flux is defined by

$$F_{i+1/2} = \frac{1}{2}[f(u_i) + f(u_{i+1}) - \alpha_{i+1/2}(u_{i+1} - u_i)], \tag{2.91}$$

where $\alpha_{i+1/2} = \max(u_i, u_{i+1})|f'(u)|$

4. Richtmyer two-step Lax-Wendroff method: The numerical flux which corresponds to that is

$$F_{i+1/2} = \frac{1}{2}[f(u_i) + f(u_{i+1}) - \alpha f'(u_{i+1/2})(f(u_{i+1}) - f(u_i))]. \tag{2.92}$$

5. MacCormarck's method: The corresponding numerical flux is

$$F_{i+1/2} = \frac{1}{2}[f(u_i) + f(u_i - \alpha(f(u_i) - f(u_{i-1}))]. \tag{2.93}$$

2.3.2 Time discretisation and the CFL condition

The complete description of the numerical solution of the conservation laws (2.1) is done by integrating with respect to time the semi-discrete scheme (2.88). The semi-discrete scheme (2.88) can be written in the general form $du/dt = L(u)$. Integrating $\frac{du}{dt} = L(u)$ using the forward Euler scheme gives

$$u^{n+1} = u^n + \Delta t L(u^n), \tag{2.94}$$

This scheme is stable provided that the time step Δt in from (2.94) is under the CFL condition

$$\frac{\Delta t \times \max \lambda_i}{\Delta x} \leq 1, \tag{2.95}$$

and λ_i are the eigenvalues of the Jacobian matrix J of the flux function f.

An m-stage (SSP) Runge Kutta method for the solution of $\frac{du}{dt} = L(u)$ takes the form

$$u^{(0)} = u^n, \tag{2.96}$$

$$u^{(i)} = \sum_{k=0}^{i-1} [\alpha_{i,k} u^{(k)} + \Delta t \beta_{i,k} L(u^{(k)})], \quad \alpha_{i,k} \geq 0 \quad i = 1, ..m, \tag{2.97}$$

$$u^{(n+1)} = u^m. \tag{2.98}$$

For the consistency we should have

$$\sum_{k=0}^{i-1} \alpha_{i,k} = 1.$$

We have to point out that when $\beta_{i,k}$ is negative, $\beta_{i,k}\widehat{L}(u^{(k)})$ is used instead of $\beta_{i,k}L(u^{(k)})$ where $\widehat{L}$ represents the approximation of the same spatial derivative as L. This change of sign of $\beta_{i,k}$ preserves the strong stability property $\| u^{n+1} \| \leq \| u^n \|$ for the first order Euler scheme, solved backward in time

$$u^{n+1} = u^n - \Delta t \widehat{L}(u^n), \tag{2.99}$$

which is obtained by resolving the negative version of the system of conservation laws

$$u_t - f(u)_x = 0. \tag{2.100}$$

Definition 2.3.1. A finite volume scheme for a first-order equation is stable if the iterates remain bounded as the grid is refined

For CFL coefficient $c = 1$ and taking (m, p) as m-stage pth order method (Optimal SSP Runge-Kutta) we have, with $\beta_{i,k} \geqslant 0$:

(1) SSPRK(2,2): An optimal second order scheme is

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n), \\ u^{(n+1)} &= \frac{1}{2}u^n + \frac{1}{2}u^{(1)} + \frac{1}{2}\Delta t L(u^{(1)}). \end{aligned}$$

(2) SSPRK(3,3): An optimal third order is

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n), \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}), \\ u^{(n+1)} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}). \end{aligned}$$

We have to point out that the idea behind SSP methods is to assume that the first order time discretization of the process of lines ODE is strongly stable under a certain norm, when the time step Δt is suitably restricted, and then try to find a higher order time discretization (Runge-Kutta) that maintains strong stability for the same norm, perhaps under a different time step restriction.

Example 1: Linear advection equation

We take a linear advection equation with periodic boundary data

$$u_t + u_x = 0, \quad u(0,x) = u_0(x), \quad u(0,t) = u(1,t). \tag{2.101}$$

As initial data $u_0(x)$ we take a combination of the smooth $cos(\frac{\pi x}{4})$ and double step function.

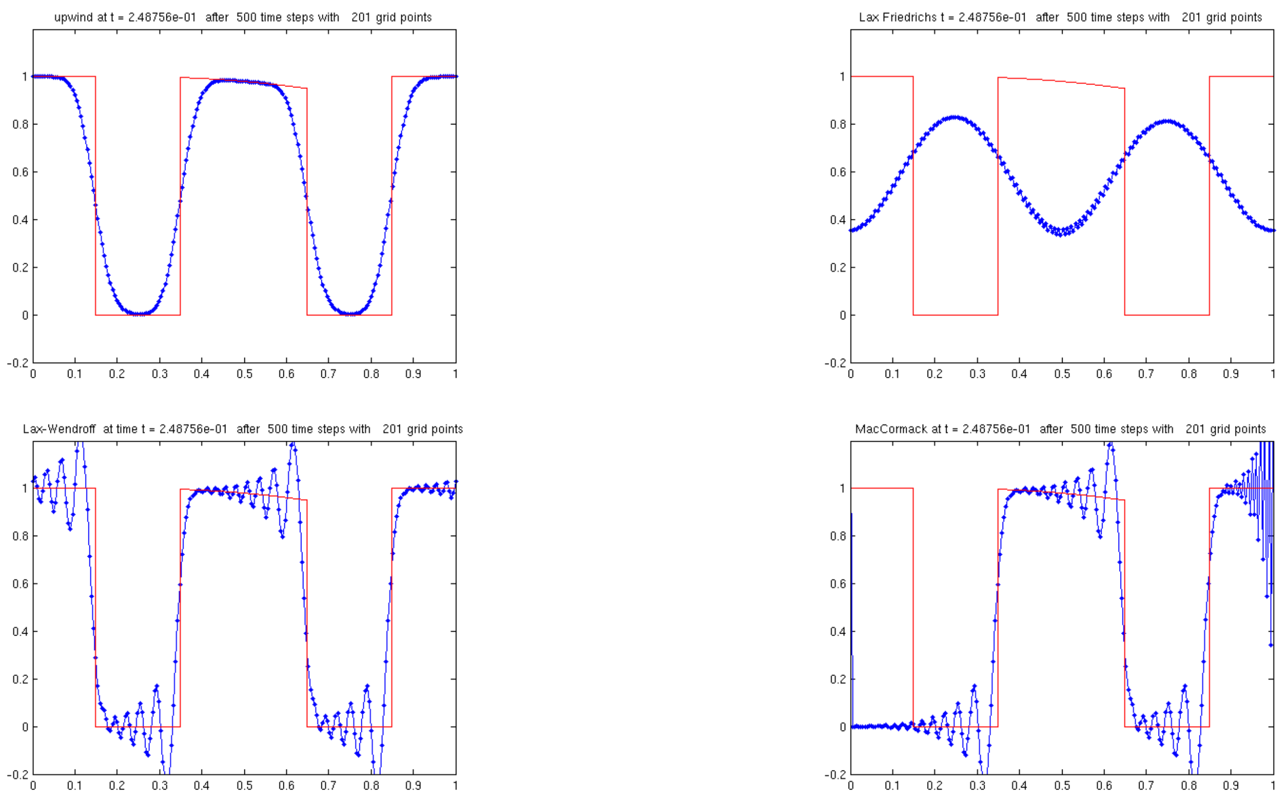


Figure 2.4: Approximate solutions at t=1 of the linear advection equation

Figure 2.4 shows approximate solutions at $t = 1$ which is computed by the four schemes (upwind, Lax-Friedrichs, Lax-Wendroff and MacCormarck schemes) on a grid with 200 nodes using a time-step restriction $\Delta t = 0.1\Delta x$. We see that the two first schemes (upwind and Lax-Friedrichs) smear both part and the discontinuity path of the advected profile. The second-order schemes (Lax-Wendroff and MacCormarck), on the other hand, preserve the smooth profile quite accurately with oscillations around the discontinuities.

Example 2: Dam break problem

The dam break problem [28] is generated by the homogeneous one dimensional shallow water equations:

$$U_t + f(U)_x = 0 \text{ where } U = (h, hu)^t \text{ and } f(U) = (h, hu^2 + g\frac{h^2}{2})$$

where h represents water height, $x \in [0, L]$ for the ideal case of a flat and frictionless channel of unit width and rectangular cross section, with the initial conditions

$$u(x,0) = 0,\;\; h(x,0) = \begin{cases} h_L \; if\, x \leqslant \frac{L}{2}, \\ h_R \; if\;\, x > \frac{L}{2}. \end{cases} \tag{2.102}$$

In the example, we presented two ratios of initial water depths $h_L/h_R = 10$ and $h_L/h_R = 100$ and we took as time $t = 10$, 20 and space interval $\Delta x = 0.5$.

The first example use the following initial conditions

$$u = 0,\; h = \begin{cases} 1 \;\, if \;\; x < 1000, \\ 0.1 \;\, if \;\; x \geqslant 1000, \end{cases}$$

and the second example use the following initial conditions

$$u = 0,\; h = \begin{cases} 50 \;\, if \;\; x < 1000, \\ 0.5 \;\, if \;\; x \geqslant 1000. \end{cases}$$

All the example are computed in the $cfl = 0.45$ condition at the time $T = 10$ and $T = 20$ with grid points $N = 4000$ using Lax-Friedrichs scheme. The numerical results are presented in the figures 2.5 and 2.6

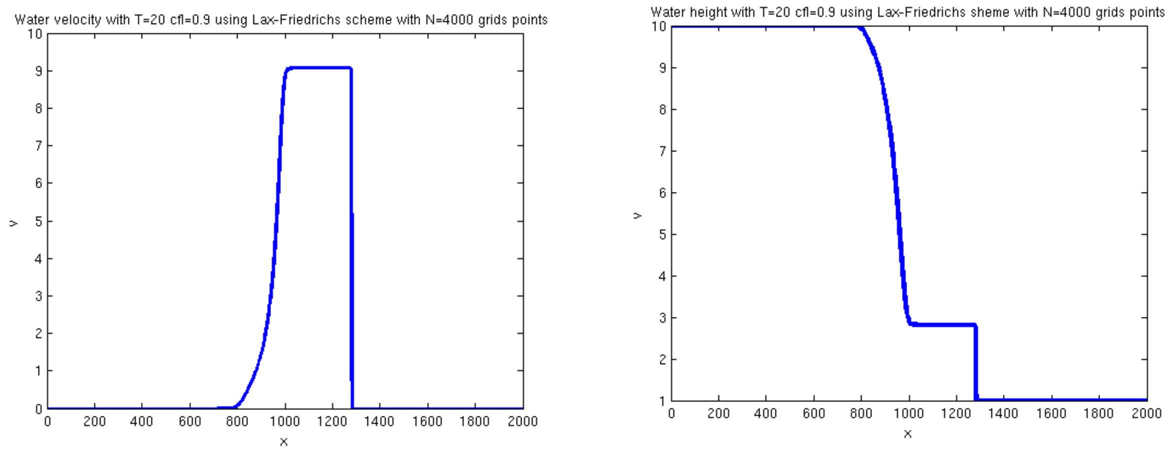


Figure 2.5: Water velocity (left) and water height (right) for the dam break problem computed at time $T = 20s$.

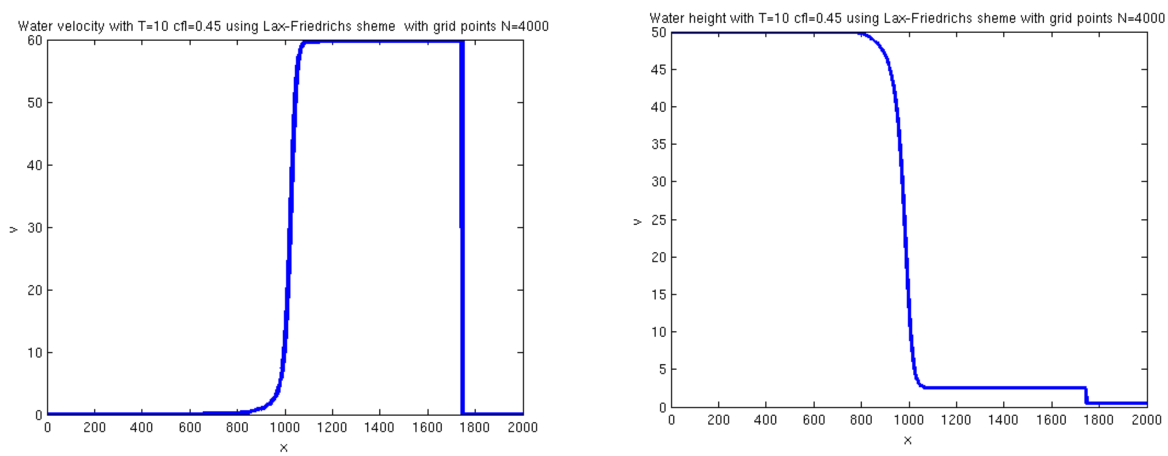


Figure 2.6: Water velocity (left) and water height (right) for the dam break problem computed at time $T = 10s$.

The numerical implementation Figures 2.5 and 2.6 show that our solution agree with the analytical solution [29] which is a rarefaction wave moving to the left and a shock wave moving to the right. Our results are similar to those obtained in the papers [28] where they use Lax-Wendroff scheme.

Example 3: Euler equation of gas dynamics: The shock tube problem

We present the "shock tube problem" of gas dynamics [15] which is governed by the Euler equations in the form

$$\rho_t + (\rho u)_x = 0, \tag{2.103}$$

$$(\rho u)_t + (\rho u^2 + p)_x = 0, \tag{2.104}$$

$$E_t + (u(E + p))_x = 0. \tag{2.105}$$

where ρ the density, u the velocity, p the pressure, E the energy with additional equation called equation of state of a polytropic gas:

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2. \tag{2.106}$$

with $\gamma = 1.4$.

Indeed the shock tube problem is described as the tube filled with gas initially separeted by membrane into two compartments. The gas has higher pressure and density in one half of the tube than in the other half with zero velocity everywhere. We allow the gas to flow when we remove at $t = 0$ the membrane expecting the motion to have lower pressure. We are solving the Riemann problem computed at $t = 1$

$$q_t + f(q)_x = 0, \quad \begin{cases} (\rho_l, u_l, p_l) = (3, 0, 10), \ x < 0, \\ (\rho_r, u_r, p_r) = (1, 0, 1), \ x > 0. \end{cases}$$

where $q = (\rho, \rho u, E)^t$ and we use the $CFL = 0.05$ and respectively the grids points 1000 and 2000.

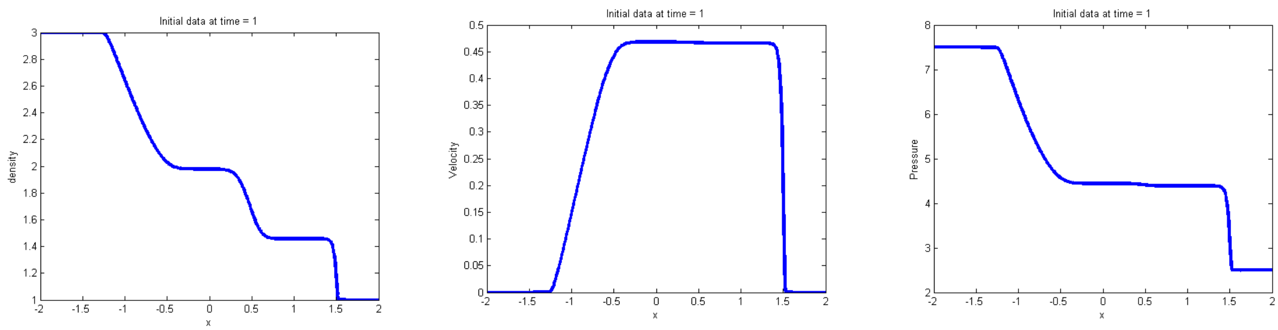


Figure 2.7: Gas density (left), gas velocity (middle), gas pressure (right) for the tube shock problem computed at time $T = 1s$ with grid points 1000.

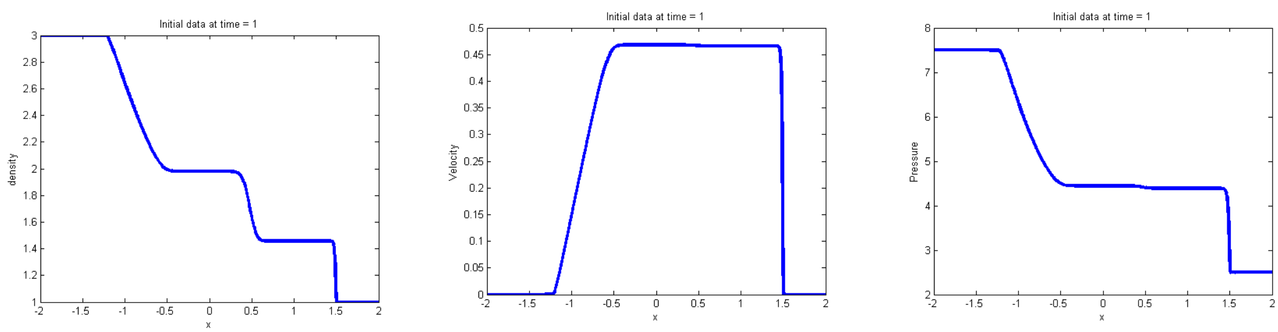


Figure 2.8: Gas density (left), gas velocity (middle), gas pressure (right) for the tube shock problem computed at time $T = 1s$ with grid points 2000.

The numerical implementation in Figure 2.7 and 2.8 agree with the one implemented in [15] which the shock waves propagate into the region of lower pressure followed by contact discontinuity and a rarefaction waves.

Chapter 3

Nonclassical solution of hyperbolic conservation laws

3.1 Introduction

This chapter deals with the theory of nonclassical solution of scalar conservation laws. We will be concerned by the solution of the Cauchy problem

$$u_t + f(u)_x = 0, \ \ u(0, x) = u_0(x), \tag{3.1}$$

where the conserved variable u, the flux f and the initial data u_0 are as in Chapter 2. In the study of the Cauchy problem, we will start by studying the Riemann problem where the initial condition u_0 is of the Heaviside type between two given states u^- and u^+. In general, the solution of the Riemann problem is a set of fixed states separated by the so-called Lax curves (see Section 2.2.3). In the classical setting, the uniqueness of the weak solution is restored by using entropy admissibility conditions such as the Lax inequality (2.56) or the Oleinik entropy condition. For some cases with non-convex flux, at the point of jump of the weak solution, neither the Lax inequalities nor the Oleinik entropy condition are satisfied. This gives rise to the so-called nonclassical shock. A nonclassical solution of the Riemann problem for a system of conservation laws is a solution containing a nonclassical shock. For such solution, uniqueness is restored by using the so-called kinetic function which requires the existence of a kinetic function $\mathcal{K}$ such that $u^+ = \mathcal{K}(u^-)$ across any jump in the solution profile.

Hayes and Leflock [5,6,30] proved the existence of nonclassical solution of conservation law in the scalar case using a regularisation that is both diffusive and dispersive. The nonclassical solution is obtained as the diffusion and dispersion coefficients tend to zero. LeFloch and Mishra [2] considered a nonlinear system of conservation laws arising in ideal magnetohydrodynamics. They showed that the initial value problem for this model may lead to solutions exhibiting nonclassical shock waves. They determined the associated kinetic function characterizing the dynamics of undercompressive shocks driven by resistivity and Hall effects. The study of nonclassical solutions of conservation laws using numerical methods has been done by Chalons [3] using a transport equilibrium scheme and by Kurganov and Petrova [31] using particle methods. Abeyratne et al [32] proposed a finite difference scheme with controled dissipation property. To accurately capture the numerical solution, front tracking should be used and a nucleation condition need to be included in order to classify any discontinuity in the numerical solution as classical or nonclassical shock [33, 34].

The rest of this chapter is organised as follows. In Section 3.2, we recall the construction of the classical solution when the flux function is not uniformly convex and the Oleinik entropy condition is satisfied. Then in Section 3.3 we investigate the entropy dissipation followed by the section 3.4 which talked about kinetic relation. Finally we investigate in section 3.5 the existence of nonclassical solution as limit of traveling waves.

3.2 Oleinik Entropy Condition

In this section we consider the conservation law (3.1) when the flux function f is not uniformly convex. When the flux is convex, the Lax inequalities (2.56) is enough to ensure the uniqueness of the solution of the Riemann problem between two states u^- and u^+. When the flux function is not uniformly convex, the Lax inequalities are not enough to guarantee the uniqueness of the solution. In this case, we require a stronger condition

$$\frac{f(u)-f(u^-)}{u-u^-} \geq \frac{f(u^+)-f(u^-)}{u^+-u^-}. \tag{3.2}$$

This condition is called the Oleinik entropy condition. The following elementary lemma gives conditions for which the Oleinik entropy condition is satisfied.

Lemma 3.2.1. Consider the Riemann problem for (3.1) with data u^- and u^+ and such that the flux function satisfies either of the following two conditions.

(i) $u^- > u^+$ and the chord connecting $(u^-, f(u^-))$ and $(u^+, f(u^+))$ lies above the graph of $f(u)$,

(ii) $u^- < u^+$ and the chord connecting $(u^-, f(u^-))$ and $(u^+, f(u^+))$ lies below the graph of $f(u)$,

Then the Oleinik entropy condition is satisfied and the Riemann problem possess a unique solution given by

$$u(t,x) = \begin{cases} u^-, & x < st, \\ u^+, & x > st, \end{cases} \tag{3.3}$$

where

$$s = \frac{f(u^+) - f(u^-)}{u^+ - u^-}.$$

Now if neither of the conditions (i) and (ii) in Lemma3.2.1 is satisfied as in Figure 3.1, for $u^- > u^+$, the other case being similar, we construct the two states u_2 and u_3 as shown

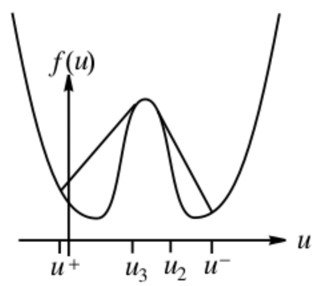


Figure 3.1: A non convex flux function

in the Figure 3.1 where the chord connecting $(u^-, f(u^-))$ and $(u_2, f(u_2))$ and the chord connecting $(u^+, f(u^+))$ and $(u_3, f(u_3))$ are tangent to the graph of f. The Oleinik entropy condition (3.2) and therefore the Lax inequalities are satisfied between u^+ and u_3 and between u_2 and u^-. The solution of the Riemann problem is two shocks between u^+ and u_3 and between u_2 and u^- and we put a rarefaction wave from u_2 to u_3. This gives the explicit solution

$$u(t,x) = \begin{cases} u^-, \ x/t < f'(u_2), \\ G(x/t), \ f'(u_2) < x/t < f'(u_3), \\ u^+, \ x/t > f'(u_3), \end{cases} \tag{3.4}$$

where $G = (f')^{-1}$.

3.3 Entropy dissipation function

Consider a conservation law in the form (3.1) possessing an entropy-entropy flux pair $(\mathcal{U}, \mathcal{F})$. We have the following definition

Definition 3.3.1. (i) The entropy dissipation function $\mathcal{E}$ is defined as

$$\mathcal{E} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \tag{3.5}$$

$$(u, v) \mapsto -s[\mathcal{U}(v) - \mathcal{U}(u)] + \mathcal{F}(v) - \mathcal{F}(u), \tag{3.6}$$

where $s = \frac{f(v)-f(u)}{v-u}$.

(ii) The tangent function $\mathcal{T}$ is defined by

$$\mathcal{T} : \mathbb{R} \to \mathbb{R} \text{ such that} f'(\mathcal{T}(u)) = \frac{f(u) - f(\mathcal{T}(u))}{u - \mathcal{T}(u)},\ u \neq \mathcal{T}(u),\ \mathcal{T}(0) = 0.$$

(iii) The zero entropy dissipation $\mathcal{E}^0$ function is a function $\mathcal{E}^0 : \mathbb{R} \mapsto \mathbb{R}$ which satisfies

$$\mathcal{E}(u, \mathcal{E}^0(u)) = 0 \text{ with } \mathcal{E}^0(u) \neq u \text{ for } u \neq 0.$$

3.3.1 Example

Consider the conservation law

$$u_t + (u^3 + 2u)_x = 0,$$

where the flux the function is $f(u) = u^3 + 2u$. An entropy-entropy flux pair for this equation is given by

$$\mathcal{U} = u^2 \text{ and } \mathcal{F}(u) = 6u^3 + 4u.$$

The entropy dissipation function here is defined as

$$\mathcal{E}(u, v) = -(v^3 - u^3 + 2v - 2u)(v + u) + 6v^3 - 6u^3 + 4v - 4u,$$

which simplifies to

$$\mathcal{E}(u, v) = (u - v)\left(u^3 + v^3 + 2u^2v + 2uv^2 - 6u^2 - 6v^2 - 6uv + 2u + 2v - 4\right).$$

The tangent function satisfies

$$f'(\mathcal{T}(u)) = \frac{f(u) - f(\mathcal{T}(u))}{u - \mathcal{T}(u)} \text{ and } u \neq \mathcal{T}(u).$$

Since $f'(u) = 3u^2 + 2$, it follows that

$$3[\mathcal{T}(u)]^2 + 2 = \frac{u^3 + 2u - [\mathcal{T}(u)]^3 - 2[\mathcal{T}(u)]}{u - [\mathcal{T}(u)]},$$

Therefore $\mathcal{T}(u)$ satisfies the following equation

$$2[\mathcal{T}(u)]^3 - 3u[\mathcal{T}(u)]^2 + u^3 = 0. \tag{3.7}$$

Solving (3.7) gives

$$\mathcal{T}(u) = u,\ \mathcal{T}(u) = -\frac{u}{2}.$$

Since the flux function is concave-convex, then $\mathcal{T}$ is monotone decreasing hence we discard the values $\mathcal{T}(u) = u$ which is double roots, to get that the tangent function here is

$$\mathcal{T}(u) = -\frac{u}{2}.$$

The zero entropy dissipation function is obtained by solving

$$(\mathcal{E}^0(u) - u)[\mathcal{E}^0(u)^3 + (2u-6)\mathcal{E}^0(u)^2 + (2u^2 - 6u + 2)\mathcal{E}^0(u) + u^3 - 6u^2 + 2u - 4] = 0.$$

Since $\mathcal{E}^0(u) \neq u$, It follows that

$$\mathcal{E}^0(u)^3 + (2u-6)\mathcal{E}^0(u)^2 + (2u^2 - 6u + 2)\mathcal{E}^0(u) + u^3 - 6u^2 + 2u - 4 = 0. \tag{3.8}$$

This equation (3.8) has some real solution whose complicated expressions are not included in this dissertation.

The concepts presented in Definition 3.3.1 allows us to solve the Riemann problem. We have the following lemma.

Lemma 3.3.1. A shock wave of the form

$$u(t,x) = \begin{cases} u^- & x < st, \\ u^+ & x > st. \end{cases} \tag{3.9}$$

satisfies the single entropy inequality (2.51) if and only if

$$u^+ \in \begin{cases} [\mathcal{E}^0(u^-), u^-], & u^- \geq 0, \\ [u^-, \mathcal{E}^0(u^-)], & u^- \leq 0. \end{cases} \tag{3.10}$$

The proof of this Lemma 3.3.1 can be found in [4]. We can now define the concept of nonclassical shock.

Definition 3.3.2. Among the shocks satisfying the Lemma 3.3.1 there are some verifying the Oleinik entropy relation (3.2) (therefore Lax inequalities), we called them Classical shock or Lax shock. They belong to the following set

$$u^+ \in \begin{cases} [\mathcal{T}(u^-), u^-], & u^- \geq 0, \\ [u^-, \mathcal{T}(u^-)], & u^- \leq 0. \end{cases} \tag{3.11}$$

On the hand those which verifying Lemma 3.3.1 but violate the Oleinik entropy relation(3.2), they are called nonclassical shocks. They belong to the following set

$$u^+ \in \begin{cases} [\mathcal{E}^0(u^-), \mathcal{T}(u^-)), & u^- \geq 0, \\ [\mathcal{T}(u^-), \mathcal{E}^0(u^-)], & u^- \leq 0. \end{cases} \tag{3.12}$$

We denote the inverse of tangent function $\mathcal{T}^{-1}$, zero entropy dissipation function $[\mathcal{E}^0]^{-1}$. The entropy dissipation function has the following property.

Theorem 3.3.2 (Entropy dissipation for concave-convex flux). For a given left-hand state $u > 0$ the function $\mathcal{E}(u, .)$ is monotone decreasing in $(-\infty, \mathcal{T}(u)]$ and monotone increasing in $[\mathcal{T}(u), +\infty)$.

A proof of this theorem can be found in [4]

As consequence of this theorem there exists some value $\mathcal{E}^0(u)$ satisfying

$$\mathcal{E}(u, \mathcal{E}^0(u)) = 0, \;\; \mathcal{E}^0(u) \in (\mathcal{T}^{-1}(u), \mathcal{T}(u)).$$

The function $\mathcal{E}^0(u) : \mathbb{R} \to \mathbb{R}$ is monotone decreasing function with

$$\mathcal{E}^0(\mathcal{E}^0(u)) = u, \ \partial_u \mathcal{E}^0(u) < 0. \tag{3.13}$$

The zero entropy dissipation $\mathcal{E}^0$ enable one to determine the critical limit for the range of the kinetic functions. To extend the definition of $\mathcal{E}^0$ we introduce the function ρ such that for $u^- \neq u^+$ and $u^- \neq \rho(u^+, u^-)$ we have

$$\frac{f(\rho(u^+, u^-)) - f(u^-)}{\rho(u^+, u^-) - u_-} = \frac{f(u^+) - f(u^-)}{u^+ - u^-}. \tag{3.14}$$

We can also define its companion $\mathcal{E}_c^0 : \mathbb{R} \to \mathbb{R}$ satisfying

$$\frac{f(u) - f(\mathcal{E}_c^0(u))}{f(u) - \mathcal{E}_c^0(u)} = \frac{f(u) - f(\mathcal{E}^0(u))}{f(u) - \mathcal{E}^0(u)}, \tag{3.15}$$

so that the points $(\mathcal{E}^0(u), f(\mathcal{E}^0(u))),\ (\mathcal{E}_c^0(u), f(\mathcal{E}_c^0(u))),\ (u, f(u))$ are aligned.

The rarefaction set to the Riemann problem with cubic function are found by

$$\mathcal{R}(u^-) = \begin{cases} [u^-, \infty), & u^- > 0, \\ (-\infty, \infty), & u^- = 0, \\ (-\infty, u^-], & u^- < 0. \end{cases}$$

If the initial data of the Riemann problem belongs to same region of convexity or concavity the Riemann solution is always classical otherwise the Riemann solution is nonclassical. In the rest of the section the focus will be on concave-convex or convex concave function for which

$$uf''(u) > 0, \text{ when } u \neq 0,\ f'''(u), \text{ and } \lim_{|u| \to +\infty} f'(u) = +\infty, \tag{3.16}$$

$$uf''(u) < 0, \text{ when } u \neq 0,\ f'''(u), \text{ and } \lim_{|u| \to +\infty} f'(u) = -\infty. \tag{3.17}$$

and explain how to construct nonclassical entropy solutions of the Riemann problem (2.57). The prototype of interest is the cubic flux $f(u) = u^3 + au$ presenting a single inflection point.

The following theorem [4] give the solution to Riemann problem for concave-convex case and convex-concave case.

Theorem 3.3.3. Assume that the flux function in (3.1) is concave-convex. Consider the Riemann problem with initial data u^- and u^+. Then the solution of the Riemann problem has the following form in the class of piecewise smooth functions, for definiteness $u^- \geq 0$,

(i) If $u^+ \geq u^-$, the solution is unique and consists of a rarefaction connecting continuously u^- to u^+.

(ii) If $u^+ \in [\mathcal{E}_c^0(u^-), u^-)$, the solution is unique and consists of a classical shock connecting u^- to u^+.

(iii) If $u^+ \in [\mathcal{E}^0(u^+), \mathcal{E}_c^0(u^-))$, there exist infinitely many solutions, consisting of a nonclassical shock connecting u^- to some intermediate state u_m followed by

  (i) a classical shock if $u_m < \rho(u^-, u^+)$

  (ii) or a rarefaction if $u_m \geq u^+$. The values $u^+ \in [\mathcal{T}(u^+), \mathcal{E}_c^0(u^+)]$ can also be attained with a single classical shock.

(iii) If $u^+ \leq \mathcal{E}^0(u^-)$, there exist infinitely many solutions, consisting of a nonclassical shock connecting u^- to some intermediate state $u_m \in [\mathcal{E}^0(u^-), \mathcal{T}(u^-)]$ followed by a rarefaction connecting continuously to u^+.

Proof. We know that from the theorem 2.2.1

(1) A shock connecting a state u^- to a state $u^+ < u^-$ cannot be followed by another shock or by a rarefaction.

(2) A rarefaction cannot be followed by a shock but a rarefaction can always be continued by attaching to it another rarefaction.

We realise that a rarefaction wave can be added after a right-contact wave due to the fact that the left-hand of the rarefaction waves has a quicker speed than or equal to the shock speed. To have a unique solution we made the following observations

(1) When u^- is connected to u^+ by a shock waves, after no other wave can be added except when $u+ = \mathcal{E}_c^0(u^-)$ due to the fact that the shock is then a right-contact and can be followed with a rarefaction which preserve that the solution is monotone.

(2) When u^- is connected to u^+ by a rarefaction waves, after no other wave can be added except another rarefaction.

So the case (i) and (ii) are solved. Therefore we conclude that a Riemann solution contains at most two elementary waves (shock waves and rarefaction) which lead to uniqueness of result. For the case (iii) we just need to see that if a shock waves and a rarefactions as (i) and (ii) are the only admissible solutions. It is well know that two shock waves can be combined only when the smaller speed of the right-hand one is greater or equal to the largest speed of the left-hand wave. We will have

(1) u^- connected to $u^+ \in [\mathcal{E}^0(u^-), \mathcal{T}(u^-))$ by a nonclassical shock can be followed only by a shock connecting to a value $u_m \in [u^+, \rho(u^-, u^+))$ or else by a rarefaction to $u_m \leqslant u^+$. In fact for each state $u_m \in [u^+, \rho(u^-, u^+))$ is associated with a classical shock which propagates with the speed $s(u_m, u^+)$. This leads $s(u_m, u^+) \geqslant s(u^-, u^+)$, where $s(u_m, u^+)$ and $s(u^-, u^+)$ are obtained by Rankine-Hugoniot relation between u_m and u^+ after u^- and u^+. These states u_m, u^- and u^+ are therefore attainable by just adding a classical shock after the nonclassical one. Moreover a state $u_m \in [\mathcal{E}_c^0(u^+), \mathcal{E}^0(u^+))$ cannot be reached by adding a second shock after the non-classical because $\mathcal{E}^0(u^+) = u^-$. It follows that any shock which connects u^+ to some state $u_m > \mathcal{E}_c^0(u^+)$ travels with a smaller speed($s(u^+, u_m) < s(u^-, u^+)$). We have finally the states $u_m < u^+$ cannot be reached due to the fact that they are associated with rarefactions which travel faster than the nonclassical shock.

(2) After a classical shock leaving from a state u^- and reaching u^+, no other wave can be added except when $u^+ = \mathcal{T}(u^-)$ and, in that case, a rarefaction only can follow the classical shock. Indeed from the theorem 2.2.1 a classical shock cannot be added after another classical shock, nor a rarefaction except when $u^+ = \mathcal{T}(u^-)$. It follows by taking into Consideration a nonclassical shock emanating from u^+ and reaching u_m. Assuming that $u^+ < 0$. For the nonclassical shock to be admissible one needs $u_m \leqslant \mathcal{E}^0(u^+)$, but we should order the speeds, $s(u^+, u_m) > s(u^-, u^+)$, it follows thus $u_m > u^-$. The condition (3.13) combined with the fact that $\mathcal{E}^0$ is monotone, and the inequality $u^+ > \mathcal{E}^0(u^-)$ we find also $u^- = \mathcal{E}^0(\mathcal{E}^0(u^-)) \geqslant \mathcal{E}^0(u^+) \geqslant u_m$, which lead to a contradiction because we can never have both $u_m > u^-$ and $u^- \geqslant u_m$ at the same time. It follows that we can have more than two waves (shock waves and rarefactions waves). We have proved the point (iii)

□

For the convex-concave case we have

Theorem 3.3.4. Suppose that f is a convex-concave flux function(3.17) with a given Riemann initial conditions u^- and u^+ satisfying the entropy condition (2.51), then the Riemann problem admits the following solutions in the class of piecewise function with $u^- \geq 0$:

(1) If $u^+ \geq u^-$, the solution is unique and consists of a classical shock wave connecting u^- to u^+.

(2) If $u^+ \in [0, u^-)$, the solution is unique and consists of a rarefaction wave connecting monotonically u^- to u^+.

(3) If $u^+ \in [\mathcal{T}^{-1}(u^-), 0)$, there are infinitely many solutions, consisting of a rarefaction wave connecting u^- to some intermediate state u_m with $0 \leq u_m \leq [\mathcal{E}^0]^{-1} \leq u^+$, followed by a classical or nonclassical shock connecting to u^+.

(4) If $u^+ \in (-\infty, \mathcal{E}^0(u^-))$, the solution may contain a classical shock connecting u^- to some state $u_m > u^-$, followed with a classical or nonclassical shock connecting to u^+. This happens when there exists u_m satisfying $\rho(u_m, u^+) < u^- < u_m < [\mathcal{E}^0]^{-1}(u^+)$.

(5) Finally, if $u^+ \in (\infty, [\mathcal{T}]^{-1}(u^-)]$, there exists a solution connecting u^- to $u+$ by a classical shock wave.

The proof can be found in [4]. The convex-concave case is treated similar to concave-convex case.

Alternatively to the entropy dissipation function, one can also construct nonclassical solutions of conservation laws using the so-called kinetic function.

3.4 Kinetic relation

In this part the focus will be in kinetic relation. In fact the kinetic relation enable us to choose the physically relevant nonclassical solution of Riemann problem (3.1) with initial data u^+, u^-.

Definition 3.4.1. i)A kinetic function $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ is a monotone decreasing function and Liptschitz continuous mapping

$$\begin{cases} \mathcal{E}^0(u) < \mathcal{K}(u) < \mathcal{T}(u), & u > 0, \\ \mathcal{T}(u) < \mathcal{K}(u) < \mathcal{E}^0(u), & u < 0, \end{cases} \tag{3.18}$$

with contraction property

$$|\mathcal{K}(\mathcal{K}(u))| < |u|, \quad u \neq 0.$$

ii) We call companion $\mathcal{K}_c$ of a kinetic function $\mathcal{K}$, any function that satisfies

$$\mathcal{K}_c : \mathbb{R} \to \mathbb{R} \tag{3.19}$$

$$u \mapsto \begin{cases} \mathcal{K}_c(u) = \mathcal{K}(u) & \text{if} \quad \mathcal{K}(u) = \mathcal{T}(u), \\ \mathcal{K}_c(u) \neq \mathcal{K}(u) \neq u & \text{if} \quad \frac{f(u)-f(\mathcal{K}_c(u))}{f(u)-\mathcal{K}_c(u)} = \frac{f(u)-f(\mathcal{T}(u))}{f(u)-\mathcal{T}(u)}, u \neq 0. \end{cases} \tag{3.20}$$

and for which

$$\begin{cases} \mathcal{T}(u) < \mathcal{K}_c(u) < \mathcal{E}^0(u), & u > 0, \\ \mathcal{E}^0(u) < \mathcal{K}_c(u) < \mathcal{T}(u), & u < 0. \end{cases}$$

Definition 3.4.2. Under the assumptions of the Theoreom 3.3.3. A weak solution of (3.1) in the class of piecewise smooth functions is called a nonclassical entropy solution (associated with the kinetic function $\mathcal{K}$) if any nonclassical shock connecting two states u^- and u^+ satisfies the kinetic relation

$$u^+ = \mathcal{K}(u^-). \tag{3.21}$$

We can now attempt to point out all the shock waves which are admissible (all classical shock and nonclassical) connecting u^- to u^+. For concave-convex case [4] we have

Theorem 3.4.1. Under the hypothesis of the Theorem 3.3.3. The Riemann problem admits an unique nonclassical entropy solution in the class of piecewise functions, given as follows when $u^- > 0$

(i) If $u^+ \geqslant u^-$, the solution is a rarefaction connecting u^- to u^+.

(ii) If $u^+ \in [\mathcal{K}(u^-), u^-)$, the solution is a classical shock waves connecting u^- to u^+.

(iii) If $u^+ \in [\mathcal{K}(u^-), \mathcal{T}(u^-))$, the solution consists of a nonclassical shock connecting u^- to $\mathcal{K}(u^-)$ followed by a classical shock.

(iv) If $u^+ \leqslant \mathcal{K}(u^-)$, the solution consists of a nonclassical shock connecting u^- to $\mathcal{K}(u^-)$ followed by a rarefaction connecting to u^-.

When $u^- \leqslant 0$ we have

(i) If $u^+ \leqslant u^-$, the solution is a rarefaction connecting u^- to u^+.

(ii) If $u^+ \in [u^-, \mathcal{K}(u^-))$, the solution is a classical shock waves connecting u^- to u^+.

(iii) If $u^+ \in (\mathcal{T}(u^-), \mathcal{K}(u^-))$, the solution consists of a nonclassical shock connecting u^- to $\mathcal{K}(u^-)$ followed by a classical shock connecting $\mathcal{K}(u^-)$ to u^+.

(iv) If $u^+ \geqslant \mathcal{K}(u^-)$, the solution consists of a nonclassical shock connecting u^- to $\mathcal{K}(u^-)$ followed by a rarefaction connecting to u^+.

The proof of this theorem is similar to the 3.3.3. For convex-concave case [4] we have

Theorem 3.4.2. Under the hypothesis of the theorem 3.3.4. The Riemann problem admits an unique nonclassical entropy solution in the class of piecewise function, given as follows when $u^- > 0$

(i) If $u^+ \geqslant u^-$, the solution is unique and consists of a classical shock wave connecting u^- to u^+.

(ii) If $u^+ \in [0, u^-)$, the solution is unique and consists of a rarefaction waves monotonically connecting u^- to u^+.

(iii) If $u^+ \in [\mathcal{K}(u^-), 0)$, the solution contains a rarefaction wave connecting u^- to $u_m = \mathcal{K}(u^+)$, followed with a nonclassical shock connecting to u^+.

(iv) If $u^+ \leq \mathcal{K}(u^-)$, the solution contains

- if $u^- > \rho([\mathcal{K}]^{-1}(u^+), u^+)$, classical shock connecting u^- to $u_m = \mathcal{K}]^{-1}(u^+)$ followed by nonclassical shock connecting u_m to u^+.
- if $u^- \leqslant \rho([\mathcal{K}]^{-1}(u^+), u^+)$ a single classical shock connected u^+ to u^-.

when $u^- \leqslant 0$

(i) If $u^+ \leqslant u^-$, the solution is classical shock connecting u^- to u^+.

(ii) If $u^+ \in (u^-, 0]$, the solution is a rarefaction wave connecting u^- to u^+.

(iii) If $u^+ \in (0, \mathcal{K}(u^-))$, the solution contains a rarefaction waves connecting u^- to $[\mathcal{K}]^{-1}(u^+)$ followed by a nonclassical shock connecting $[\mathcal{K}]^{-1}(u^+)$ to u^+.

(iv) If $u^+ \geqslant \mathcal{K}(u^-)$, the solution consists of a nonclassical shock connecting u^- to $\mathcal{K}(u^-)$ followed by a rarefaction connecting to u^+.

(iv) If $u^+ \geqslant \mathcal{K}(u^-)$, the solution contains

- if $u^- < \rho([\mathcal{K}]^{-1}(u^+), u^+)$, classical shock connecting u^- to $[\mathcal{K}]^{-1}(u^+)$ followed by nonclassical shock connecting $\mathcal{K}]^{-1}(u^+)$ to u^+.
- if $u^- \geqslant \rho([\mathcal{K}]^{-1}(u^+), u^+)$ a single classical shock connected u^- to u^+.

3.4.1 Selection Rule

For any given initial u^-, u^+ data for the Riemann problem each u^- is associated to a corresponding set $\mathcal{S}_{nc}$ [35] such that

$$\begin{cases} \mathcal{S}_{nc} \subset \{u < \mathcal{K}_c(u), \quad u^- > 0\}, \\ \mathcal{S}_{nc} \subset \{u > \mathcal{K}_c(u), \quad u^- < 0\}. \end{cases} \tag{3.22}$$

Let the set $\mathcal{S}_{nc}(0) = \emptyset$. We have

(i) If $u^+ \in \mathcal{S}_{nc}(u^-)$, then the solution is nonclassical otherwise

(ii) the solution is classical.

For instance by letting $\mathcal{S}_{nc}(u^-) = \emptyset$, then the classical solution is picked out for all u^+.

Another simple way to select the solution is to introduce the nucleation condition which is to define the set $\mathcal{S}_{nc}$ through a threshold. Therefore we look at the Lipshitz continuous nucleation threshold function $N : \mathbb{R} \to \mathbb{R}$ with condition

$$\begin{cases} \mathcal{T}(u) < N(u) < \mathcal{K}_c(u), \quad u > 0, \\ \mathcal{K}_c(u) < N(u) < \mathcal{T}(u), \quad u < 0. \end{cases} \tag{3.23}$$

Finally we specify the nonclassical set

$$\mathcal{S}_{nc}(u^-) = \begin{cases} [N(u), +\infty), u^- < 0, \\ (-\infty, N(u)], u^- > 0. \end{cases} \tag{3.24}$$

3.5 The existence of nonclassical solution, limit of traveling waves

In this section, just as for classical shock where the entropic solution is found as a limit of a diffusive regularisation, nonclassical solutions can also be found as a limit of a regularised equation with a diffusive and dispersive term. We consider therefore a diffusive and dispersive regularisation of a conservation law with a cubic flux function

$$u_t + (u^3)_x = \varepsilon u_{xx} + \delta u_{xxx}, \tag{3.25}$$

where $\varepsilon > 0$ and δ are real parameters.

The case where $\varepsilon > 0$ and $\delta < 0$ (which gives a classical solution), is solved by the following theorem.

Theorem 3.5.1. Assume $\varepsilon > 0$ and $\delta < 0$. Let u satisfies the equation (3.25) having traveling waves solutions of the form

$$u = u(x - st), \tag{3.26}$$

where s propagation speed. Let

$$u(x,t) = \begin{cases} u^- & \text{if} \quad x < st, \\ u^+ & \text{if} \quad x > st, \end{cases} \tag{3.27}$$

be the shocks waves solution of the equation

$$u_t + (u^3)_x = 0, \tag{3.28}$$

such that in the limit, the traveling waves solutions (3.26) approach as shock waves solutions of the equation (3.28). Therefore there is a trajectory for the second order derivative of the equation

$$-s(u - u^-) + u^3 - (u^3)^- - \varepsilon u^{'} = \delta u^{''}, \tag{3.29}$$

for which the boundary conditions

$$u(+\infty) = u^+, \quad u(-\infty) = u^-, \quad u^{'}(+\infty) = 0, \tag{3.30}$$

$$u^{'}(-\infty) = 0, \quad u^{''}(+\infty) = 0, \quad u^{''}(-\infty) = 0, \tag{3.31}$$

hold if and only if (3.27) is classical shock.

The proof of this theorem 3.5.1 can be found in [8].

Remark 3.5.1. Note that the motivation for studying the equation (3.25) and (3.28) is refered to chapter 1 paragraph 3.

Now we focus in the case where $\varepsilon > 0$ and $\delta > 0$. This case is more interesting because there is admissible shocks that violate the Lax entropy condition. Let us assume that for given two states u^- and u^+ such that $u^-u^+ < 0$ and

$$w(\xi) = u(x,t), \quad \xi = \frac{x - st}{\sqrt{\delta}}, \quad w(-\infty) = u^-, w(-\infty) = u^+, \tag{3.32}$$

$$w^{'}(-\infty) = 0, \quad w^{'}(+\infty) = 0, \quad w^{''}(+\infty) = 0, \quad w^{''}(-\infty) = 0. \tag{3.33}$$

With the new change of variable we have

$$u_t = \frac{du}{d\xi}\frac{d\xi}{dt} = -w'\frac{s}{\sqrt{\delta}}, \quad (u^3)_x = \frac{3w^2w'}{\sqrt{\delta}}, \quad u_x = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{w'}{\sqrt{\delta}}, \quad u_{xx} = \frac{w''}{\delta}, \quad u_{xxx} = \frac{w'''}{\delta^{\frac{3}{2}}},$$

So the equation (3.25) becomes

$$-w'\frac{s}{\sqrt{\delta}} + \frac{3w^2w'}{\sqrt{\delta}} = \varepsilon\frac{w''}{\delta} + \delta\frac{w'''}{\delta^{\frac{3}{2}}}, \tag{3.34}$$

It follows

$$-sw' + 3w^2w' = \varepsilon\frac{w''}{\sqrt{\delta}} + w''', \tag{3.35}$$

Finally we have

$$-sw' + (w^3)' = \varepsilon\frac{w''}{\sqrt{\delta}} + w'''. \tag{3.36}$$

Let us rewrite the equation (3.36) with $\sigma = \frac{\varepsilon}{\sqrt{\delta}}$. We have

$$-sw' + (w^3)' = \sigma w'' + w'''. \tag{3.37}$$

The equation (3.37) is conservative [5] and taking in account the balance of diffusive and dispersive effect with σ fixed and limit ε, $\delta \to 0$, we have

$$-sw' + (w^3)' = \sigma w'' + w''' = 0, \tag{3.38}$$

By integrating once we have

$$-sw + (w^3) + c = \sigma w' + w'' = l(w), \tag{3.39}$$

where c is constant of integration and l is function in w. We will have

$$\begin{cases} l(w) = -sw + (w^3) + c, \\ w'' = -\sigma w' + l(w). \end{cases} \tag{3.40}$$

Also when $\delta \to 0$, we have from (3.36)

$$w'' = 0 \Longrightarrow w' = k(\mu).$$

where k is integration function in μ. Let $k(\mu) = \mu$. Therefore we have

$$\begin{cases} w' = \mu, \\ \mu' = -\sigma\mu + l(w), \end{cases} \tag{3.41}$$

For getting the constant c in the equation

$$l(w) = c - sw + (w^3), \tag{3.42}$$

the boundary conditions are used. Indeed

$$\lim_{|\xi|\to\infty} \mu' = \lim_{|\xi|\to\infty} w'' = 0 = c - su_l + u_l^3,$$

Therefore using equation (3.39), we have to make it easier to follow he argument,

$$c = su^- - [u^-]^3 \quad \text{and} \quad l(w) = su^- - [u^-]^3 - sw + w^3. \tag{3.43}$$

The equilibrium point are the point of the system of equation

$$\begin{cases} w' = \mu = 0, \\ \mu' = -\sigma\mu + l(w) = 0, \end{cases} \tag{3.44}$$

It follows that

$$l(w) = 0 = su^- - [u^-]^3 - sw + w^3.$$

Let E_q be the set of equilibrium points. We have

$$E_q = \{(w, \mu) = (\overline{w}, 0) | l(w) = 0\}. \tag{3.45}$$

The following proposition is used to get the equilibrium point.

Proposition 3.5.1. *1. Given a state $u^- > 0$, the set $\Lambda(u^-)$ consisting of all states u^+ that can be achieved through a diffusive traveling wave taking the values u^- and u^+ at the left and the right ends respectively, is given by*

$$\Lambda(u^-) = \left[-\frac{1}{2}u^-, u^-\right).$$

2. For a state given $u^- > 0$, The solution of Riemann problem with initial data u^-, u^+ satisfies

(i) a rarefaction waves if $u^+ \geq u^-$,

(ii) a shock waves, if $u^+ \in \Lambda(u^-)$,

(iii) a shock waves with associated rarefaction if $u^+ \notin \Lambda(u^-)$.

A proof can be found in [5].

Coming back to the determination of equilibrium point we consider the initial data u^-, u^+ such that

$$u^- > 0 \quad \text{and} \quad u^+ \notin \Lambda(u^-) \quad \text{means} \quad u^+ \neq -\frac{1}{2}u^-.$$

$l(w)$ is a polynomial of degree 3, so we have at most 3 roots which we denote by

$$w_l^* = u^- > w_m^* > w_r^* \quad \text{and} \quad w_l^* = u^- \quad \text{due to} \quad l(u^-) = 0.$$

Since $l(u^-) = 0$, we rewrite $l(w)$

$$w^3 - sw + su^- - [u^-]^3 = (w - u^-)(aw^2 + bw + c) = aw^3 + (b - au^-)w^2 + (c - au^-)w - cu^-$$

where a, b, c are reals coefficient. By identification we get

$$a = 1, \quad b = u^-, \quad c = -s + [u^-]^2$$

It follows that

$$w_l^* = u^- \quad \text{and} \quad w_m^*, w_r^*$$

are roots of the equation

$$w^2 + u^- w + [u^-]^2 - s = 0. \tag{3.46}$$

Therefore we determine s as follows

$$s = w^2 + u^- w + [u^-]^2. \tag{3.47}$$

Also in the polynomial $l(w)$, the coefficient of degree 2 is zero ($l(w)$ no quadratic term [12]) therefore the roots w_l^*, w_m^*, w_r^* verify

$$w_l^* + w_m^* + w_r^* = 0. \tag{3.48}$$

In the papers [8], Jacob, Mc Kinney and M. Shealter proved that there is a trajectory which passed through the point w_l^*, w_r^* in the form

$$\mu(\xi) = \frac{1}{\sqrt{2}}(w(\xi) - w_l^*)(w(\xi) - w_r^*), \tag{3.49}$$

Indeed they sought an invariant parabola through the point w_l^*, w_R^* in the form

$$\mu(\xi) = c(w(\xi) - w_l^*)(w(\xi) - w_r^*), \tag{3.50}$$

where c is constant. Let us determine c

$$\frac{d\mu}{dw} = c(2w - w_l^* - w_r^*). \tag{3.51}$$

We have will by multiplying (3.50) and (3.51) side by side we have

$$\mu\frac{d\mu}{dw} = c^2(2w - w_l^* - w_R^*)(w(\xi) - w_l^*)(w(\xi) - w_r^*). \tag{3.52}$$

But using equation (3.41) the gradient along μ is

$$\mu\frac{d\mu}{dw} = -\sigma\mu + (w(\xi) - w_l^*)(w(\xi) - w_r^*)(w - w_m^*). \tag{3.53}$$

By using (3.52) and (3.53) we get

$$c^2(2w - w_l^* - w_r^*) = -c\sigma + w - w_m^*. \tag{3.54}$$

By identification of the coefficient of w, It follows that

$$2c^2 = 1 \Longrightarrow c = \frac{1}{\sqrt{2}} \quad \text{or} \quad c = -\frac{1}{\sqrt{2}}.$$

Let decrease w from w_l^* to w_r^* and $\mu = w' < 0$ [8]; therefore we choose the positive roots $c = \frac{1}{\sqrt{2}}$. The function w reach the value 0 by continuity. Let $w(\xi) = 0$. From the equation (3.54) by replacing c with its value, we have

$$\frac{1}{2}(w_l^* + w_r^*) = \frac{\sigma}{\sqrt{2}} + w_m^*, \tag{3.55}$$

If follows

$$w_l^* + w_r^* = \sqrt{2}\sigma + 2w_m^*. \tag{3.56}$$

By putting the equation (3.56) in the equation (3.48) we have

$$w_m^* = -\frac{\sqrt{2}}{3}\sigma \quad \text{and} \quad w_r^* = -w_l^* + \frac{\sqrt{2}}{3}\sigma = -u^- + \frac{\sqrt{2}}{3}\sigma. \tag{3.57}$$

Finally by replacing w_r^*, w_m^* with its values in the equation (3.46) we get the shock speed

$$s = [u^-]^2 - [u^-]\frac{\sqrt{2}}{3}\sigma + \frac{2}{9}\sigma^2. \tag{3.58}$$

So the set E_q is finally defined by

$$E_q = \left\{ (u^-, 0), \quad \left(-\frac{\sqrt{2}}{3}\sigma, 0\right), \quad \left(-w_l^* + \frac{\sqrt{2}}{3}\sigma, 0\right) \right\}.$$

The Jacobian matrix J of the system of equation (3.41) is

$$J = \begin{pmatrix} 0 & 1 \\ l'(w) & -\sigma \end{pmatrix}$$

The eigenvalues in λ are solution of the equation

$$\lambda^2 + \lambda\sigma + s - 3(w^*)^2 = 0, \tag{3.59}$$

where w^* can be w_l^* or w_r^* or w_m^*. The discriminant of the equation (3.59) is

$$\Delta = \sigma^2 + 4\left[3(w^*)^2 - s\right] = \sigma^2 + 4l'(w^*),$$

(i) Since the coefficient of the polynomial $l(w)$ of degree 3 is positive, we have $l'(w^*) > 0$ at the exterior of the equilibria w_l^* and w_r^* $(w_l^* > w_m^* > w_r^*)$. Therefore we have $\Delta > 0$ since

$$\sqrt{\sigma^2 + 4l'(w^*} > \sigma$$

due to the fact that $l'(w^*) > 0$ and

$$\sigma^2 + 4l'(w^*) > \sigma^2.$$

Finally we get

$$\lambda_1(w^*) = \frac{1}{2}\left(-\sigma - \sqrt{\sigma^2 + 4l'(w^*}\right) = -\frac{1}{2}\left[\frac{4l'(w^*)}{-\sigma + \sqrt{\sigma^2 + 4l'(w^*)}}\right], \tag{3.60}$$

$$\lambda_2(w^*) = \frac{1}{2}\left(-\sigma + \sqrt{\sigma^2 + 4l'(w^*)}\right) = -\frac{1}{2}\left[\frac{4l'(w^*)}{-\sigma - \sqrt{\sigma^2 + 4l'(w^*)}}\right]. \tag{3.61}$$

It follows that $\lambda_1 > 0$ and $\lambda_2 < 0$. Finally we conclude that the state w_l^* and w_r^* are saddle points.

(ii) At the equilibrium point w_m^*, we have $l'(w_m^*) < 0$ and therefore the equilibrium point w_m^* is stable (due to $\sigma > 0$ and $\lambda_1(w^*)$ and $\lambda_2(w^*)$ have non-zero reals parts) and is either a node if $\Delta \geq 0$ (two negative real eigenvalues), or a spiral if $\Delta > 0$ (two complex conjugate eigenvalues).

In summary, the right state w_r^* is only found from the left state w_l^* (3.57). From the paper [8], we notice that the trajectory which connected two saddle point w_l^* and w_r^* must satisfied

$$w_r^* < w_m^*. \tag{3.62}$$

It follows that

$$-w_l^* + \frac{\sqrt{2}}{3}\sigma < -\frac{\sqrt{2}}{3}\sigma. \tag{3.63}$$

Therefore

$$w_l^* > \frac{2\sqrt{2}}{3}\sigma. \tag{3.64}$$

But if $w_r^* < w_m^*$ is not satisfied the trajectory is a connection between unstable point(saddle) and stable point (node/spiral).

We realize also that the traveling waves which pass through w_r^* and w_l^* is not quicker than the speed of propagation in both side of w_r^* and w_l^*. In fact $l'(w) = 3w^2 - s$ in both side of w_r^* and w_l^*. As a result the saddle-saddle trajectory turns into undercompressive shock when we take the limit of ε with $\delta \to 0$. This shock doesn't verify the Lax entropy criterion and the Oleinik entropy criterion. We can attempt to state the following proposition

Theorem 3.5.2. Assume $\sigma > 0$. Therefore there is a trajectory from a saddle point equilibrium (w_l^*, o)

to saddle point (w_r^*, o) for the system (3.41) if and only

$$w_l^* > \frac{2\sqrt{2}}{3}\sigma \text{ and } s = [u^-]^2 - \frac{\sqrt{2}}{3}\sigma[u^-] + \frac{2}{9}\sigma^2. \tag{3.65}$$

From above proposition we conclude if $w_l^* < \frac{2\sqrt{2}}{3}\sigma$ we get the classical solution which is described from the proposition (3.5.1).

When $w_l^* > \frac{2\sqrt{2}}{3}\sigma$,

(i) The solution is a rarefaction wave if $u^+ \geq u^-$,

(ii) A classical shock waves if $-\frac{\sqrt{2}}{3}\sigma \leq u^+ \leq u^-$,

(iii) We have, if $-u^- + \frac{\sqrt{2}}{3}\sigma \leq u^+ \leq -\frac{\sqrt{2}}{3}\sigma$, nonclassical shock from u^- to $w_r^* = -u^- + \frac{\sqrt{2}}{3}\sigma$ followed by a classical shock connecting to u^+,

(iv) If $u^+ \leq -u^- + \frac{\sqrt{2}}{3}\sigma$ we have slow nonclassical shock waves from u^- to $-u^- + \frac{\sqrt{2}}{3}\sigma$ went along by a rarefaction waves connecting to u^+.

3.5.1 Kinetic relation derived from traveling waves

For definiteness consider the entropy, entropy-flux pair $(\mathcal{U}(u), \mathcal{F}(u)) = (\frac{u^2}{2}, \frac{3u^4}{4})$ which is linked to the flux $f(u) = u^3$. The results can be generalised for the family of entropies $\mathcal{U}(u) = \frac{u^{2n}}{2n}$ [5]. For nonclassical solution we require the entropy dissipation function $\mathcal{E}$ to be equal to the kinetic function $\mathcal{K}$. Let u^-, u^+ be given states and v_m such that $u^- > v_m > u^+$ to be determined. The Rankine-Hugoniot condition gives the speed of propagation of the shock between u^- and u^+ as

$$s = [u^+]^2 + u^+u^- + [u^-]^2. \tag{3.66}$$

The entropy dissipation function is given as

$$\mathcal{E}(u^-, v_m) = \frac{1}{4}(v_m - u^-)^2([v_m]^2 - [u^-]^2). \tag{3.67}$$

To get all the admissible entropy solution the entropy dissipation function should be negative implying

$$[v_m]^2 \leq [u^-]^2, \tag{3.68}$$

since $(v_m - u^-)^2 > 0$. Let

$$z = \frac{s}{[u^-]^2}, \quad a = \frac{v_m}{u^-}, \tag{3.69}$$

Note that a should be negative since $u^- v_m < 0$ for a given $u^- > 0$. Combining (3.66) and (3.69) gives

$$z = a^2 + a + 1. \tag{3.70}$$

Solving (3.70) gives

$$a = \frac{-1 - \sqrt{4z - 3}}{2}, \tag{3.71}$$

which is valid for $z \geq \frac{3}{4}$. Brian T. Hages and Phillipe Lefloch [5] proved that to stay in the allowable region for two waves nonclassical solution we must have $z \leq 1$ so that the admissibility condition (3.68) for entropy dissipation must be satisfied. In order to select a unique nonclassical solution we introduce the Kinetic function $\mathcal{K}$. For nonclassical solution we require the entropy dissipation to be equal to kinetic function which is a function of speed of propagation

$$\mathcal{E}(u^-, v_m) = \mathcal{K}(s). \tag{3.72}$$

For our purpose we rescale the Kinetic function by the relation

$$\mathcal{K}(s) = [u^-]^4 \phi(z, u^-), \quad z \in \left[\frac{3}{4}, 1\right]. \tag{3.73}$$

Proposition 3.5.2. *Assume that the Kinetic function $\mathcal{K}(s)$ is a smooth function defined for $s \in [0, +\infty)$ and for all $u^- > 0$ satisfies the conditions*

$$(i)\ \mathcal{K}(s) < 0, \quad s \in (0, +\infty), \tag{3.74}$$

$$(ii)\ \mathcal{K}(0) = 0, \tag{3.75}$$

$$(iii)\ \mathcal{K}(s) \geqslant -\frac{3s^2}{4}, \quad s \in (0, +\infty). \tag{3.76}$$

Consider two initial states $u^- > 0$ and $u^+ < 0$. Then, in the family of two-wave solutions generated by u^- and u^+, there exists a unique solution consisting of a nonclassical shock from u^- to an intermediate state denoted by $v_m(u^-) \in (u^-, \frac{u^-}{2})$ with wave speed denoted by $s \in (0, +\infty)$ and a classical wave from $v_m(u^-)$ to u^+ such that the kinetic relation (3.72) and (3.73) holds for the nonclassical wave. The classical wave is a shock wave if $u^+ > v_m(u^-)$ and a rarefaction wave if $u^+ \leqslant v_m(u^-)$.

Proof. We just need to prove that the kinetic relation ϕ according to (3.73) select a unique value z everywhere in the range $[\frac{3}{4},1]$ leading to prove that the kinetic relation $\mathcal{K}$ and the entropy dissipation has one intersection point in the range $[\frac{3}{4},1]$ with respect to z. Indeed by considering $a=\frac{v_m}{u^-}$,

$$p(a)=\mathcal{E}(u^-,v_m)=\frac{1}{4}(v_m-u^-)^2([v_m]^2-[u^-]^2)=\frac{[u^-]^4}{4}(a-1)^2(a^2-1)=\frac{[u^-]^4}{4}\phi(z,a), \qquad (3.77)$$

Firstly

$$p\prime(a)=\frac{[u^-]^4}{2}(a-1)^2(2a+1). \qquad (3.78)$$

p is decreasing when $-1\leqslant a<-\frac{1}{2}$. Therefore since $a=\frac{-1-\sqrt{4z-3}}{2}$ is decreasing function with respect to z, by replacing with its values in the range $-1<a<-\frac{1}{2}$, we get $\frac{3}{4}\leqslant z<1$.
Secondly by using chain rule,

$$[p(a(z))]'=a'(z)\times p'(a)[z], \qquad (3.79)$$

Since a and p are decreasing function with respect to z in the range $\frac{3}{4}\leqslant z<1$, we have therefore their composition

$$[p(a(z))]=\frac{[u^-]^4}{4}\left\{z^2+z\left(3+2\sqrt{4z-3}\right)-\frac{3}{2}\sqrt{4z-3}-\frac{9}{2}\right\} \qquad (3.80)$$

is strictly increasing. Since p with respect to z is an increasing function, any decreasing function ϕ with respect to z over the interval $(\frac{3}{4},1)$ crosses the graph of p exactly in one point. It follows that

$$\frac{[u^-]^4}{4}\lim_{z\longrightarrow\frac{3}{4}^+}\phi(z,s)\geqslant\mathcal{E}\left(a\left(\frac{3}{4}\right)\right)=-\left[\frac{3}{4}\right]^3\left[u^-\right]^4, \qquad (3.81)$$

$$\frac{[u^-]^4}{4}\lim_{z\longrightarrow1^-}\phi(z,s)\leq\mathcal{E}(a(1))=0. \qquad (3.82)$$

□

We have to point out that proposition 3.5.2 shows only the existence of the nonclassical solution but cannot select it. It does select the classical solution in

$$u^+=-u^--v_m(u^-)\in\left(-\frac{u^-}{2},0\right),\ \text{or } z(u^-)=\frac{3}{4}\text{ and } u^+\leq-\frac{u^-}{2}.$$

In proposition 3.5.2

$$v_m(u^-)\in\left(u^-,\frac{u^-}{2}\right)\Rightarrow -u^--v_m\in\left(-\frac{u^-}{2},0\right).$$

We found that $-u^- - v_m(u^-) < 0$ since $u^- > 0$.

By using an exact kinetic relation, we recuperate the solutions found as limits of vanishing viscosity-dispersion approximations. Indeed in the same spirit Jacobs, MCKinney, and Shearer [8] proved the existence of nonclassical solution using traveling waves. We recall their results for our analysis by defining a normalized parameter

$$\nu = \frac{\sigma}{u^-}, \tag{3.83}$$

and putting a normalized parameter (3.83) in the shock speed we have

$$s = [u^-]^2 - \frac{\sqrt{2}}{3}\sigma[u^-] + \frac{2}{9}\sigma^2 = [u^-]^2 \left(1 - \frac{\sqrt{2}}{3}\nu + \frac{2}{9}\nu^2\right). \tag{3.84}$$

We proved the following theorem which reproduces the unique solution to the Riemann problem for diffusive-dispersive regularizations for the cubic scalar equation(3.25), through the use of a specific kinetic relation and nucleation criterion.

Theorem 3.5.3. The unique solution to the Riemann problem for diffusive-dispersive regularizations (3.25) is equivalent to nonclassical shocks for the cubic scalar equation (3.28) which satisfy the kinetic relation for ϕ having the explicit dependence

$$\mathcal{K}(s) = [u^-]^4\phi(z) = p[a(z)], \tag{3.85}$$

where $p[a(z)]$ is defined by (3.80) and s is defined by (3.84). The nonclassical shocks must also satisfy the nucleation criteria

$$u^+ < -\sqrt{2}\sigma/3, \tag{3.86}$$

$$u^- > 2\sqrt{2}\sigma/3. \tag{3.87}$$

Proof. We have

$$z = \frac{s}{[u^-]^2} = 1 - \frac{\sqrt{2}}{3}\nu + \frac{2}{9}\nu^2,$$

by using (3.69). In the amount $p[a(z)]$ we have

$$\sqrt{4z-3} = \sqrt{1 - \frac{4}{3}\sqrt{2}\nu + \frac{8}{9}} = \sqrt{\left\{\frac{2}{3}\sqrt{2}\nu - 1\right\}^2} = 1 - \frac{2}{3}\sqrt{2}\nu, \tag{3.88}$$

since $u^- > 2\sqrt{2}\sigma/3$. Indeed from the assumption of the theorem 3.5.3 and using (3.83) we have

$$u^- > 2\sqrt{2}\sigma/3 \Longrightarrow 1 > 2\sqrt{2}\nu/3.$$

By using (3.88) in $p[a(z)]$, It follows that

$$\mathcal{K}(s) = p[a(z)] = [u^-]^4 \left(\frac{\nu^4}{81} - \frac{\sqrt{2}}{9}\nu^3 + \frac{2}{3}\nu^2 - \frac{2\sqrt{2}}{3}\nu \right). \tag{3.89}$$

Let $v_m = w_r^* = -u^- + \frac{\sqrt{2}}{3}\sigma$ from (3.57). The entropy dissipation (3.67)

$$\mathcal{E}(u^-, w_r^*) = \frac{1}{4}\left(-u^- + \frac{\sqrt{2}}{3} - u^- \right)^2 \left(\left[u^- + \frac{\sqrt{2}}{3} \right]^2 - [u^-]^2 \right).$$

It follows that

$$\mathcal{E}(u^-, w_r^*) = \left\{ \frac{\sigma^4}{81} - \frac{\sqrt{2}}{9}u^-\sigma^3 + \frac{2}{3}[u^-]^2\sigma^2 - \frac{2\sqrt{2}}{3}[u^-]^3\sigma \right\}, \tag{3.90}$$

By putting the normalized ν (3.83) in (3.90) we get

$$\mathcal{E}(u^-, w_r^*) = [u^-]^4 \left(\frac{\nu^4}{81} - \frac{\sqrt{2}}{9}\nu^3 + \frac{2}{3}\nu^2 - \frac{2\sqrt{2}}{3}\nu \right). \tag{3.91}$$

According to the proposition 3.5.2, the unique value selected by kinetic relation (3.72) is

$$v_m = w_r^* = -u^- + \frac{\sqrt{2}}{3}\sigma. \tag{3.92}$$

$\square$

Chapter 4

Transport Equilibrium schemes for computing nonclassical solutions of systems of conservation laws

4.1 Introduction

The numerical computation of nonclassial solution of systems of conservation laws is a challenging problem and a very active field of research. The main difficulty resides in the approximation of the kinetic function $\mathcal{K}$ at the discrete level. There are two main approaches to the computation of nonclassical shocks. In the first approach, one uses a regularisation of the problem with a diffusive and dispersive terms in order to approximate the kinetic function. In the second approach, armed with the full knowledge of the kinetic function, ones tracks the nonclassical shocks and resolve them accurately by the means of the kinetic relation, that justify why the approach is called sharp interface approach. The transport equilibrium schemes, first developed by Chalons [3] falls in the second approach. The method consists of an equilibrium step which incorporate the kinetic relation at any nonclassical shock and a transport step which advances the discontinuity with time. The method resolves accurately nonclassical shocks with the correct shock position and shock velocity. The drawback of the method is that it requires knowledge of the kinetic function and the underlying Riemann solution. This makes the method not suitable for any complex application.

4.2 Presentation of the algorithm

We present the transport equilibrium scheme for the computation of nonclassical solutions of a conservation law in the form

$$u_t + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0. \tag{4.1}$$

where the notations are as in the previous chapter. We assume that the the conservation law has a kinetic relation $\mathcal{K}$ are defined as in (3.4.1) and the tangent function $\mathcal{T}$ as in (3.3.1). We discretise the space and time by means of the grids $x_{i+1/2} = i\Delta x,\ \ i \in \mathbb{Z}$ and $t_n = n\Delta t$ where Δx and Δt is the space step and time step, respectively. The method seek to find a piecewise constant approximation v_i^n of $v(x_i, t_n)$ at any time t_n in the interval $I_i = [x_{i-1/2}; x_{i+1/2})$. In the finite volume framework, the method solves at each grid interface $x_{i+1/2}$ a Riemann problem for the conservation laws with left and right data v_{i-1}^n and v_i^n, respectively. Nonclassical shock appears in general when the left and right states belongs to two different regions on concavity of the flux function f. We assume that the flux function is concave convex or convex-concave as is described in (3.16) and (3.17). We recall the following properties of concave-convex flux

$$uf''(u) > 0, \text{ when } u \neq 0,\ f'''(u), \text{ and } \lim_{|u|\to+\infty} f'(u) = +\infty, \tag{4.2}$$

or convex-concave, that is they satisfy

$$uf''(u) < 0, \text{ when } u \neq 0,\ f'''(u), \text{ and } \lim_{|u|\to+\infty} f'(u) = -\infty. \tag{4.3}$$

The typical example of interest for us will be the cubic flux function $f(u) = u^3$. By keeping this in mind, let us insert two subsets $\mathcal{C}$ and $\mathcal{N}$ made of all the pairs $(u^-, u^+) \in \mathbb{R}^2$ with $u^+u^- < 0$ and such that the Riemann solution with initial data u^+ and u^- satisfying the single entropy inequality (2.51) and Kinetic relation (3.21) is respectively classical and nonclassical. We have therefore

$$\mathcal{C} = \begin{cases} \{(u^-, u^+), u^+u^- < 0 \,|\, u^+u^- \geqslant u^-\mathcal{T}(u^-)\} \text{ if } f \text{ satisfies (3.16)} \\ \{(u^-, u^+), u^+u^- < 0 \,|\, u^+u^- \leqslant u^-\mathcal{K}(u^-) \text{ and } [u^-]^2 \leqslant u^-\rho(\mathcal{K}^{-1}(u^+), u^+)\} \text{ if } f \text{ satisfies (3.17)}\ , \end{cases} \tag{4.4}$$

and

$$\mathcal{N} = \begin{cases} \{(u^-, u^+), u^+u^- < 0 \,|u^+u^- < u^-\mathcal{T}(u^-)\} \text{ if } f \text{ satisfies (3.16)} \\ \{(u^-, u^+), u^+u^- < 0 \,|u^+u^- > u^-\mathcal{T}(u^-) \quad \text{if } f \text{ satisfies (3.17)} \quad \textit{or} \\ \{u^+u^- \leqslant u^-\mathcal{K}(u^-), [u^-]^2 \leqslant u^-\rho(\mathcal{K}^{-1}(u^+), u^+)\} \text{ if } f \text{ satisfies (3.17).} \end{cases} \tag{4.5}$$

Notice that $\mathcal{C} = \emptyset$ if f sastifies (4.2) and $u^-\mathcal{K}(u^-) \geqslant 0$. The Riemann solution combined with a pair (u^-, u^+) in $\mathcal{C}$ (when is it not empty) is always a classical shock connecting u^- to u^+ while if (u^-, u^+) belongs to $\mathcal{N}$, the Riemann solution is nonclassical [3] apart from if $u^+ = \mathcal{K}(u^-)$.

Concerning the solutions remaining always either in $\mathbb{R}_-$ or $\mathbb{R}_+$, we choose on a numerical flux function h consistent with the flux function f defined in (3.1) and consider the following 3-point explicit conservative scheme

$$v_i^{n+1} = v_i^n - \lambda[h_{i+1/2} - h_{i-1/2}], \; i \in \mathbb{Z}, \tag{4.6}$$

with $\lambda = \frac{\Delta t}{\Delta x}$ defined under the CFL restriction

$$\lambda \max |f\prime(u)| \leqslant \frac{1}{2} \tag{4.7}$$

and $h_{i+1/2} = h(v_i^n, v_{i+1}^n)$ for all $\in \mathbb{Z}$ and $h(u,u) = f(u) \, \forall u \in \mathbb{R}$. The objective is to figure out how to transform such a conservative scheme (4.6) in order to rightly capture all the solutions of the Riemann problem with a given initial data u^+ and u^- verifying the single entropy inequality (2.51) and Kinetic relation (3.21) that is including those associated with the case $u^+u^- < 0$ in initial data.

The algorithm comprises two main steps. We have an Equilibrium step and a Transport step. In the Equilibrium step, we propose to modify any given consistent and conservative scheme so we can put at stationary some admissible discontinuities. Then, the transport step aims at diffusing these discontinuities [3].

(1) When Riemann initial data is such that $u^+u^- < 0$ and the corresponding solution is simply a shock wave which can be classical or nonclassical from u^- to u^+ give rise to spurious values distinct from u^- to u^+ if we used the conservative scheme (4.6).

In fact looking at the following natural discretization of the initial data

$$v_i^0 = \begin{cases} u^- \; i \leqslant 0, \\ u^+ \; i \geqslant 1, \end{cases} \tag{4.8}$$

It follows that by using the conservative scheme (4.6)

$$v_i^1 = \begin{cases} u^- - \lambda[h(u^-, u^-) - h(u^-, u^-)] = u^-, \; i \leqslant -1, \\ u^- - \lambda[h(u^-, u^+) - h(u^-, u^-)], \; i = 0, \\ u^+ - \lambda[h(u^+, u^+) - h(u^-, u^+)], \; i = 1, \\ u^+ - \lambda[h(u^+, u^+) - h(u^+, u^+)] = u^+, \; i \geqslant 2, \end{cases} \tag{4.9}$$

with $v_0^1 \notin \{u^-, u^+\}$ and $v_1^1 \notin \{u^-, u^+\}$. The aim is to keep a sharp interface between u^- to u^+ propagating at speed s given by Rankine-Hugoniot conditions(2.13) $s = s(u^-, u^+)$

To reach this goal, we propose to use the following nonconservative formula

$$v_i^{n+1} = v_i^n - \lambda[h_{i+1/2}^a - h_{i-1/2}^b], \; i \in \mathbb{Z}, \tag{4.10}$$

where the numerical fluxes $h_{i+1/2}^a = h_{i+1/2}^a(v_i^n, v_{i+1}^n)$ and $h_{i+1/2}^b = h^b(v_i^n, v_{i+1}^n)$ are defined as follows

$$h_{i+1/2}^a(v, v) = h_{i+1/2}^b(v, v) = h(v, v),$$

for all v. It follows that if $(u^-, u^+) \in \mathcal{C}$ we set

$$h_{i+1/2}^a(u^-, u^+) = h(u^-, u^-), \;\; h_{i+1/2}^b(u^-, u^+) = h(u^+, u^+) \text{ for } i \in \mathbb{Z}, \tag{4.11}$$

which is enough to avoid the nondesired intermediate values

$$\begin{cases} v_0^1 = u^- - \lambda[h_{1/2}^a(u^-, u^+) - h_{-1/2}^b(u^-, u^-)] = u^- - \lambda[h(u^-, u^-) - h(u^-, u^-)] = u^-, \\ v_1^1 = u^+ - \lambda[h_{3/2}^a(u^+, u^+) - h_{1/2}^b(u^-, u^+)] = u^+ - \lambda[h(u^+, u^+) - h(u^+, u^+)] = u^+. \end{cases} \tag{4.12}$$

In the same manner, if $(u^-, u^+) \in \mathcal{N}$ we set

$$h_{i+1/2}^a(u^-, u^+) = h(u^-, \mathcal{K}^{-1}(u^+)), \;\; h_{i+1/2}^b(u^-, u^+) = h(\mathcal{K}(u^-), u^+) \text{ for } i \in \mathbb{Z}, \tag{4.13}$$

we have the same result as (4.12).

The new consideration (4.10) of the scheme (4.6) enable to take off nondesired values and make stationary the initial discretization (4.8). Furthermore when it should be moving at speed s a transport step must be added in the algorithm.

(2) The second motivation involve the circumstances where $(u^-, u^+) \in \mathcal{N}$ but $u^+ \neq \mathcal{K}(u^-)$. Our aims is to force the numerical scheme to create such a nonclassical discontinuity $(u^+ = \mathcal{K}(u^-))$

The algorithm presented above can be summarised in two main steps.

(1) The transport step: aims to make stationary some of admissible shocks with Riemann problem with initial data u^+ and u^- satisfying the single entropy inequality (2.51) and Kinetic relation (3.21). We consider the new update formula (4.10) of the scheme (4.6)

$$h^a_{i+1/2} = h^a_{i+1/2}(v_i^n, v_{i+1}^n) = \begin{cases} h(v_i^n, v_i^n) \ \text{ if } \ (v_i^n, v_{i+1}^n) \in \mathcal{C}, \\ h(v_i^n, \mathcal{K}^{-1}(v_{i+1}^n)) \ \text{ if } \ (v_i^n, v_{i+1}^n) \in \mathcal{N}, \\ h_{i+1/2} = h(v_i^n, v_{i+1}^n) \ \text{ otherwise}, \end{cases} \tag{4.14}$$

and

$$h^b_{i+1/2} = h^b_{i+1/2}(v_i^n, v_{i+1}^n) = \begin{cases} h(v_i^n, v_i^n) \ \text{ if } \ (v_i^n, v_{i+1}^n) \in \mathcal{C}, \\ h(\mathcal{K}(v_i^n), v_{i+1}^n) \ \text{ if } \ (v_i^n, v_{i+1}^n) \in \mathcal{N}, \\ h_{i+1/2} = h(v_i^n, v_{i+1}^n) \ \text{ otherwise}. \end{cases} \tag{4.15}$$

(2) the equilibrium step takes into account the transport of the solution got at intermediate time t^{n+1-} with the speed $s(u^-, u^+)$ defined at the each interface $x_{i+1/2}$ by

$$s_{i+1/2}(v_i^n, v_{i+1}^n) = \begin{cases} s(v_i^{n+1-}, v_{i+1}^{n+1-}) \ \text{ if } \ (v_i^n, v_{i+1}^n) \in \mathcal{C} \cup \mathcal{N}, \\ 0 \ \text{ otherwise}, \end{cases} \tag{4.16}$$

and find solution to transport equation with the speed $s_{i+1/2}$. For getting a new approximation v_i^{n+1} at time $t^{n+1} = t^n + \Delta t$ we select randomly on interval $[x_{i-1/2}, x_{i-1/2})$ a value in the juxtaposition of these Riemann solutions at time Δt. Furthermore given a random sequence k_n

within $(0,1)$ we set

$$v_i^{n+1} = \begin{cases} v_{i-1}^{n+1-} & \text{if } k_{n+1} \in (0, \lambda s_{i-1/2}^+), \\ v_i^{n+1-} & \text{if } k_{n+1} \in [\lambda s_{i-1/2}^+, 1 + \lambda s_{i+1/2}^-], \\ v_{i+1}^{n+1-} & \text{if } k_{n+1} \in [1 + \lambda s_{i+1/2}^-, 1). \end{cases} \tag{4.17}$$

where $s_{i+1/2}^+ = \max(s_{i+1/2}, 0)$, $s_{i+1/2}^- = \min(s_{i+1/2}, 0)$.

4.3 Convergence of the algorithm

The algorithm presented in the previous section has a non conservative numerical flux function. In general for a numerical scheme for the solution of system of conservation law to converge, the scheme should be consistent, in the sense that the numerical flux reduces to the continuous flux when its argument are equal, and conservative. Convergence problems are then an issue for the presented algorithm. It turns out that the algorithm is convergent for the class of classical and nonclassical shock solutions as presented in the following theorem.

Theorem 4.3.1. Under the CFL condition (4.7) the scheme described by (4.14), (4.2) (4.16), (4.17) with a Rieman problem satisfying the single entropy inequality (2.51) and Kinetic relation (3.21) is convergent in the following sense

(1) Constant state: Assume that $v = v_{i-1}^n = v_i^n = v_{i+1}^n$, then $v_i^{n+1} = v$

(2) Classical or nonclassical shock (region of different convexity or concavity): Let u^- and u^+ be two constant states such that $u^-u^+ < 0$ and that can be connected by an admissible classical shock or nonclassical shock. Assume that $v_i^0 = u^-$ if $i \leq 0$ and $v_i^0 = u^+$ if $i \geq 0$. Then the scheme described by (4.14), (4.2) (4.16), (4.17) converges to the solution of Riemann problem satisfying the single entropy inequality (2.51) and Kinetic relation (3.21) given by $v(x,t) = u^+$ if $x < s(u^-, u^+)t$ and $u(x,t) = u^+$ otherwise. In particular, we have $v_i^n \in \{u^-, u^+\}$ $\forall i \in \mathbb{Z}$ and $n \in \mathbb{N}$ so that the discontinuity is pointed.

(3) Classical solution (same region of convexity or concavity): In this case by assuming v_{i-1}^n, v_i^n v_{i+1}^n are either all non positive or negative gives probable convergence depending on (4.14), (4.2), (4.16), (4.17) since v_{i+1}^n coincide with usual conservative scheme (4.6).

The proof can be found [3]

4.4 Numerical experiments

To experiment the above algorithm (transport-equilibrium scheme) we choose the following test cubic $f(u) = u^3$ in the concave-convex case (the case convex-concave is similar by considering $-f(u)$). The corresponding entropy entropy-flux is $(\mathcal{U}, \mathcal{F}) = (u^2, \frac{3}{2}u^4)$. We respectively find the tangent function, the zero entropy dissipation function, the kinetic function, the intermerdiate function ρ

$$\mathcal{T}(u) = -\frac{u}{2},\ \ \mathcal{E}^0(u) = -u,\ \ \mathcal{K} = -\frac{3}{4}u,\ \ \rho(u,v) = -u - v.$$

We choose the numerical flux Local Lax-freidrichs scheme h by

$$h(u,v) = \frac{1}{2}(f(u) + f(v) - k(u,v)(v-u)),\ \ k(u,v) = \max(|f'|(u), |f'|(v)).$$

We compare the result of the transport equilibrium scheme with the Local Lax-freidrichs scheme.

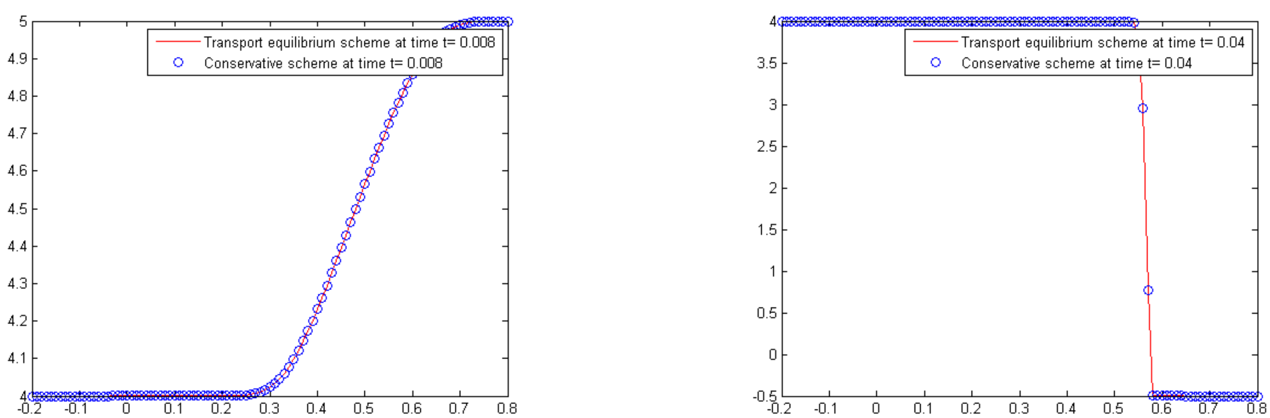


Figure 4.1: Solution of the Riemann problem for the cubic flux with the intial data $u^- = 4$, $u^+ = 5$ (left) and $u^- = 4$, $u^+ = -0.5$ (right).

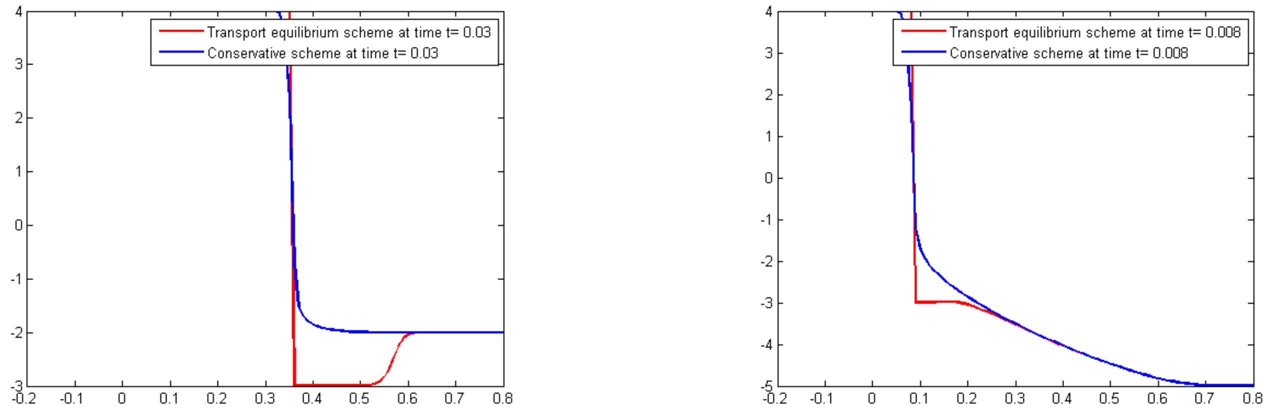


Figure 4.2: Solution of the Riemann problem for the cubic flux with the intial data $u^- = 4$, $u^+ = -2$ (left) and $u^- = 4$, $u^+ = -5$ (right).

The numerical result in the Figures 4.1 and 4.2 compare well with the results found in the paper [3]. The solutions contain shocks as well nonclassical shock and rarefactions which move from the left to the right. In the Figure 4.1 the transport equilibrium scheme and the Local Lax-freidrichs scheme capture the classical solution and both coincide while in the case where the solution contain a nonclassical shocks (Figure 4.2), the standard numerical schemes fail to resolve the discontinuity.

Chapter 5

Conclusion

This dissertation aim was to study the theory and the numerical approximation of nonclassical solution of hyperbolic conservation laws. The construction of the solution of the Riemann problem was done using a kinetic function that depends on the geometrical properties of the flux function. For the numerical approximation we used a numerical scheme called transport-equilibrium scheme that tracked the nonclassical front and resolve it accurately using the kinetic function. A drawback of this work is it focuses more on the scalar case and more precisely on the cubic flux function. It will be interesting to see how the results found here can be extended to systems of conservation laws in general. For such case a precise definition of the kinetic function needs to be investigated.

For further work the application of nonclassical solution in the model of crowd dynamic will be studied where the flux function possess two inflection points.

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