UNIVERSITY OF KWAZULU-NATAL

DIFFERENTIAL EQUATIONS FOR RELATIVISTIC RADIATING STARS

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Submitted in fulfilment of the academic requirements for the degree of Doctor of Philosophy in Science to the School of Mathematics, Statistics and Computer Science
College of Agriculture, Engineering and Science
University of KwaZulu-Natal

Durban

As the candidate's supervisors, we have approved this dissertation for submission.

Signed:	Prof. S D Maharaj	November 2013
Signed:	Prof. K S Govinder	November 2013

Abstract

We consider radiating spherical stars in general relativity when they are conformally flat, geodesic with shear, and accelerating, expanding and shearing. We study the junction conditions relating the pressure to the heat flux at the boundary of the star in each case. The boundary conditions are nonlinear partial differential equations in the metric functions. We transform the governing equations to ordinary differential equations using the geometric method of Lie. The Lie symmetry generators that leave the equations invariant are identified, and we generate the optimal system in each case. Each element of the optimal system is used to reduce the partial differential equations to ordinary differential equations which are further analyzed. As a result, particular solutions to the junction conditions are presented for all types of radiating stars. New exact solutions, which are group invariant under the action of Lie point infinitesimal symmetries, are found. Our solutions contain families of traveling wave solutions, self-similar variables, and other forms with different combinations of the spacetime variables. The gravitational potentials are given in terms of elementary functions, and the line elements can be given explicitly in all cases. We show that the Friedmann dust model is regained as a special case in particular solutions. We can connect our results to earlier investigations and we show explicitly that our models are generalizations. Some of our solutions satisfy a linear equation of state. We also regain previously obtained solutions for the Euclidean star as a special case in our accelerating model. Our results highlight the importance of Lie symmetries of differential equations for problems arising in relativistic astrophysics.

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I, Gezahegn Zewdie Abebe

Student Number: 210556524

declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

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Declaration 2- Publications

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this thesis (include publications in preparation, submitted, in press and published and give details of contributions of each author to experimental work and writing of each publication)

Publication 1.

Abebe G Z, Govinder K S and Maharaj S D, Lie symmetries for a conformally flat radiating star, *Int. J. Theor. Phys.* **52**, 3244 (2013).

(There were regular meetings between myself and my supervisors to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisors.)

Publication 2.

Abebe G Z, Maharaj S D and Govinder K S, Geodesic models generated by Lie symmetries, *Gen. Relativ. Gravit.* **46**, 1650 (2014a).

(There were regular meetings between myself and my supervisors to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisors.)

Publication 3.

Abebe G Z, Maharaj S D and Govinder K S, Generalized Euclidean stars with equation of state, *Gen. Relativ. Gravit.* under review (2014b).

(There were regular meetings between myself and my supervisors to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisors.)

Student Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Gezahegn Zewdie Abebe

November 2013

To

my late father Zewdie Abebe

Acknowledgments

I would like to express my sincere gratitude to the following people and bodies for their help, in diverse ways, towards the completion of this work.

- Firstly, I want to thank the Almighty God for His mercies and faithfulness before and throughout my studies at the University of KwaZulu-Natal (UKZN). All praise, glory and honour be to His holy name forever.
- My astute supervisors, Professor Sunil Maharaj and Professor Kesh Govinder for their support, comments, suggestions, advice, valuable guidance, constructive criticism and encouragement throughout the duration of this study.
- The staff of School of Mathematics, Statistics and Computer Science in general.
- Atsede Muleta from Gondar University for her continuous encouragement.
- My family members in Ethiopia especially my mother Erkale Tesema and my sister Samirawit Zewdie for their advice and continuing encouragement.
- My friends Sifatu Alemu and Muhaba Adege for their indescribable encouragement through out my study.
- Dr. Berhanu Bekele from Addis Ababa University for his continuous advice and encouragement.
- The National Research Foundation (NRF) and the University of KwaZulu-Natal for financial assistance in covering fees and subsistence.

Thank you!

Contents

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Chapter 1

Introduction

Albert Einstein extended the special theory of relativity to the general theory of relativity by incorporating the effects of gravitation in a four-dimensional manifold. This theory is well expressed in the language of differential geometry. General relativity describes the gravitational behaviour of gravitating matter accurately in relativistic astrophysics and cosmology. Results from the theory of general relativity are consistent with the observational and experimental results (Davies 1989, Will 1981). Globally the spacetime geometry of general relativity is not flat as in special relativity. However, locally, general relativity resembles special relativity. The Riemann tensor determines the spacetime geometry in general relativity which is curved and the matter tensor determines the matter content of the spacetime. The matter tensor, also called the energy momentum tensor, couples with Einstein tensor. The coupling of the Einstein tensor and the energy momentum tensor introduces a system of highly nonlinear partial differential equations. These equations are called the Einstein field equations. Malcolm (2004), Sean (1997), Stephani (2004) and Wald (1984) discuss the field equations in detail. Results from general relativity are very important for predicting and describing physical phenomena such as bending of light, gravitational lensing, black hole formation, gravitational red shift, the cosmic microwave background, gravitational waves, stellar stability, surface luminosity, relaxational effects, causal temperature profiles, particle production at the stellar surface, the cosmic censorship hypothesis of Penrose, gravitational collapse, etc.

Schwarzchild (1916a, 1916b) obtained exact solutions for the exterior and interior spacetimes, respectively. They are the first exact solutions for Einstein's field equations. The exterior and the interior solutions smoothly match at boundary of the star that may be helpful in modeling processes of stars with small energy density variation. Radiating stars emit radiation and null particles may be transported out to the exterior environment that should not be neglected. Since the exterior spacetime is not vacuum, it can best be described by the Vaidya (1951) solution. The smooth matching of the exterior Vaidya solution to the interior spacetimes of radiating stars give very important statements called the junction conditions. The boundary condition relating the pressure with the heat flux was formulated by Santos (1985) for a shear-free radiating star and later extended by Glass (1989) and Maharaj and Govender (2000) to incorporate the effect of shear. De Olivier *et al.* (1985) proposed the model of slow gravitational collapse of an interior, initially static configuration. Raychaudhuri (1955) pointed out that the slowest possible gravitational collapse arises in shear-free spacetimes

Relativistic radiating matter in conformally flat spacetimes were studied by Som and Santos (1981), Maiti (1982), Sanyal and Ray (1984), Modak (1984), Deng (1989) and Deng and Mannheim (1990). Ivanov (2012) analyzed shear-free perfect fluid spheres with heat flow containing as a special case the condition of conformal flatness. Spherical gravitational collapse were studied analytically by Grammenos and Kolassis (1992) assuming conformal flatness and anisotropy in the pressure due to neutrino flow. Herrera *et al.* (2004) proposed a conformally flat relativistic model without solving the junction condition exactly. Subsequently, Maharaj and Govender

(2005) and Herrera *et al.* (2006) generated exact classes of solutions by solving the junction condition directly in terms of elementary functions. Misthry *et al.* (2008) generated other classes of solution with vanishing shear by transforming the junction condition equation to an Abel equation. These solutions have vital applications in determining the gravitational behaviour of stars.

The first exact solution with the dissipative effects, satisfying the boundary conditions, was obtained by Kolassis *et al.* (1988) for a shear-free radiating star when fluid particles travel in geodesic motion. This exact solution has been widely used in investigating the physical features of radiating stars. The physical investigations include modeling of radiating gravitational collapse in spherical geometry with neutrino flux by Grammenos and Kolassis (1992) and analyzing astrophysical processes with heat flux which was demonstrated by Tomimura and Nunes (1993). The temperature in casual thermodynamics for particles traveling in geodesic motion produce higher central values than the Eckart as shown theory by Govender *et al.* (1998). A comprehensive treatment of the geodesic condition was undertaken by Thirukkanesh and Maharaj (2009). Govender and Thirukkanesh (2009) extended the model to include a nonvanishing cosmological constant. The Friedman dust solution is regained in the absence of heat flow in these treatments. These studies were accomplished under the assumption of geodesic motion with vanishing shear. Herrera *et al.* (2002) studied in general anisotropic fluids based on the geodesic condition.

Naidu *et al.* (2006) obtained the first exact solution for a shearing radiating star when fluid particles are in geodesic motion. Rajah and Maharaj (2008) extended this result and obtained new classes of solutions by transforming the junction condition and solving it. These extended classes of solutions are nonsingular at the origin. Thirukkanesh and Maharaj (2010) generated two classes of exact solutions that contain all previously known models as special cases. Chan (1997) studied configurations for accelerating, shearing and expanding models that are initially static and then collapse.

Chan (2000, 2001, 2003) and Pinheiro and Chan (2008, 2010) studied the luminosity, viscous effects and other physical features in the presence of shear. A numerical approach has been applied in the analysis of several investigations (Chan 2000, 2001, 2003 and Pinheiro and Chan 2008, 2010). Euclidean stars in general relativity may be modeled with nonvanishing shear, expansion and acceleration. For Euclidean stars the areal radius and proper radius are equal. Particular shearing solutions were found by Herrera and Santos (2010), Govender *et al.* (2010) and Govinder and Govender (2012). The general case of an accelerating and expanding model with shear was considered by Thirukkanesh *et al.* (2012). Their solutions were obtained by transforming the junction condition into linear, Bernoulli and inhomogeneous Riccati equations. Most recently Govender *et al.* (2013) studied the effect of shear in a radiating star undergoing dissipative collapse for a particular solution in the model of Thirukkanesh *et al.* (2012).

In this thesis we apply the extended Lie theory of differential equations to integrate the junction condition in different classes of relativistic radiating stars. The idea of continuous transformation group for differential equations was introduced by Sophus Lie. The theory of Lie groups has important applications in mathematics, engineering and physics. Construction of group invariant solutions for partial differential equations is one of the most important applications of the Lie theory of symmetries (Bluman and Kumei 1989, Bluman and Cole 1974, Hill 1992, Hydon 2000, Ibragimov 1994, Miwa 2000, Olver 1993). Our aim in this thesis is to solve the partial differential equations which govern the gravitational behaviour of a radiating star at the stellar surface in relativistic astrophysics. The Lie theory of differential equations is used to study the boundary conditions and to generate exact solutions to the field equations. This approach has been utilized very successfully in determining solutions to the Einstein field equations (Govinder *et al.* 1995, Govinder and Hansraj 2012, Kweyama *et al.* 2011, Hansraj *et al.* 2005, Leach and Govinder 1996, Msomi *et al.* 2010, 2011, 2012).

The boundary condition in Euclidean stars has been solved by Govinder and Govender (2012) with the help of a particular Lie symmetry. Equations that have previously been intractable or solved in an *ad hoc* manner may be systematically analyzed via this group theoretic method. Our analysis in this thesis is the first comprehensive treatment of the boundary condition for a radiating star using a symmetry approach.

A kth order differential equation of m dependent, $u = (u^1, u^2, \cdots, u^m)$, and n independent, $x = (x^1, x^2, \cdots, x^n)$, variables of the form

$$E(x, u, \partial u, \cdots, \partial^k u) = 0, \tag{1.1}$$

may admit a generator of the form

$$G = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^\mu(x,u)\frac{\partial}{\partial u^\mu}, \tag{1.2}$$

where ξ_i and η^μ are infinitesimal generator functions ($i = 1, 2, \cdots, n$ and $\mu = 1, 2, \cdots, m$) provided that

$$G^{[k]}E\big|_{E=0} = 0, \tag{1.3}$$

where $G^{[k]}$ is the kth prolongation of the symmetry G. The kth prolongation of the generator (1.2) is given as

$$\begin{aligned} G^{[k]} &= \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta^\mu(x,u)\frac{\partial}{\partial u^\mu} + \eta_i^{(1)\mu}(x,u,\partial u)\frac{\partial}{\partial u_i^\mu} + \cdots \\ &\quad + \eta_{i_1 i_2 \ldots i_k}^{(k)\mu}(x, u, \partial u, \ldots, \partial^\mu u)\frac{\partial}{\partial u^\mu_{i_1 i_2 \ldots i_k}}, \end{aligned} \tag{1.4}$$

with $k = 2, 3, \ldots$. The process is algorithmic and supported by different computer software packages like SYM and PROGRAM LIE (Dimas and Tsoubelis 2005, Head 1993).

Any group invariant solution obtained via any linear combination or individual symmetries can be transformed to the group invariant solution obtained by the symmetries in the optimal system. The optimal system of symmetries can be obtained by applying the action of adjoint representation to the general linear combination of the symmetries. The symmetries in the optimal system will be helpful to generate distinct group invariant solutions. The adjoint representation is given by summing up the Lie series as

$$\begin{aligned} \mathrm{Ad}\left(\exp\left(\epsilon G\right)\right) H &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \left(\mathrm{Ad}(G)\right)^n H, \\ &= H - \epsilon\left[G, H\right] + \frac{\epsilon^2}{2}\left[G, \left[G, H\right]\right] - \frac{\epsilon^3}{3!}\left[G, \left[G, \left[G, H\right]\right]\right] + \cdots, \end{aligned} \tag{1.5}$$

for symmetries G and H where $[,]$ is known as the Lie bracket augmented by a bilinear composition law (Bluman *et al.* 2010, Bluman and Anco 2002, Olver 1993, Pfeifer 2003).

This thesis contains:

- Chapter one: Introduction
- Chapter two: This chapter is based on the paper Abebe *et al.* (2013). We study the junction condition relating the radial pressure with the heat flux at the boundary of a relativistic radiating conformally flat star. The condition is highly nonlinear partial differential equation in one dependent function and two

independent variables. It is difficult to integrate the boundary condition using traditional methods. We obtain the Lie symmetries admitted by the boundary condition and generate the optimal system. We consider all symmetries in the optimal system to reduce the master equation to ordinary differential equations. We also consider a symmetry which is not in the optimal system as we could not integrate all the reduced ordinary differential equations. By transforming back to the original variables we present new solutions for the master equation. Our classes of solutions contain separable functions and self-similar variables in the gravitational potentials.

- Chapter three: The paper Abebe *et al.* (2014a) is the basis of chapter three. Here we consider an expanding and shearing radiating star when the fluid particles are traveling in geodesic motion. In particular, we study the junction condition relating the radial pressure to the heat flux at the boundary. Unlike chapter two, the presence of shear changes the nature of the governing equation and makes it more difficult to integrate. However with the help of Lie symmetry generators we transform the governing highly nonlinear partial differential equation into a system of ordinary differential equations. By further analyzing the ordinary differential equations and transforming back to the original variables we demonstrate classes of new and previously obtained solutions. The Friedmann dust model is also regained as a special case in the absence of heat flux in the relevant limit. Particular solutions obtained in this chapter have a form of traveling wave and others are expressed in terms of a self-similar variable.

- Chapter four: The paper Abebe *et al.* (2014b) is the main component of this thesis in chapter four. In this chapter, we study the junction condition of an accelerating, expanding and shearing relativistic radiating star. The junction condition at the boundary of the star is highly nonlinear partial differential equation in the metric functions. In this case, as the effects of shear and acceleration are included, the resulting governing equation is more difficult to integrate. We apply the Lie symmetry approach as in the previous chapters. We obtain the Lie symmetry generators that leave the equation for the junction condition invariant, and find the Lie algebra corresponding to the optimal system of the symmetries. The symmetries in the optimal system allow us to transform the boundary condition to ordinary differential equations. The various cases for which the resulting systems of equations can be solved are identified. For each of these cases the boundary condition is integrated and the gravitational potentials are found explicitly. A particular Lie generator produces a class of models which contains Euclidean stars as a special case. Our generalized Euclidean star model satisfies a linear equation of state in general. By considering a particular example we study the weak, dominant and strong energy conditions.

- Chapter five: Conclusion.

Chapter 2

Lie symmetries for a conformally flat radiating star

2.1 Introduction

We consider a conformally flat relativistic spherically symmetric radiating star. The assumption of conformal flatness is often made to integrate the Einstein field equations. Particular models discussing physical features of radiating spacetimes have been generated by Deng (1989), Deng and Mannheim (1990), Maiti (1982), Modak (1984), Sanyal and Ray (1984) and Som and Santos (1981). The treatment of Ivanov (2012) is a global analysis of shear-free perfect fluid spheres with heat flow containing as a special case the condition of conformal flatness. Analytical models of radiating spherical gravitational collapse were studied by Grammenos and Kolassis (1992) assuming conformal flatness and anisotropy in the pressure due to neutrino flow. With a vanishing Weyl tensor, Herrera *et al.* (2004) proposed a conformally flat relativistic model without solving the junction condition exactly. For this model Herrera *et al.* (2006) and Maharaj and Govender (2005) subsequently generated exact classes of solutions by solving the junction condition directly in terms of elementary functions. Misthry *et al.* (2008) generated other classes of solution with vanishing shear by transforming the junction condition equation to an Abel equation. These new conformally flat solutions are useful in determining the gravitational behaviour of stars.

The main objective of this chapter is to generate exact solutions for the equation governing the boundary condition of a conformally flat radiating star. The Lie theory of differential equations is used to study the boundary conditions and to generate exact solutions to the field equations.

In §2.2 we briefly discuss the conformally flat spacetime and present the junction condition for a radiating star. This equation is a highly nonlinear partial differential equation and difficult to solve directly using traditional methods. Therefore we utilize a geometric approach to find solutions. In §2.3 we obtain the Lie point symmetries for the junction boundary condition. Particular symmetries or any linear combination may help us to obtain group invariant solutions. We transform the boundary condition to an ordinary differential equation for each symmetry in the optimal system and exact solutions are found. In §2.4 we find solutions invariant under elements of the optimal system. New models are generated as a result. Another new model, this time invariant under the combination of symmetries $bG_1 + G_3$, is given in §2.5. An analysis of the physical features indicates the model is reasonable close to the centre. In §2.6 we make concluding remarks.

2.2 The model

We consider the particular case of spherically symmetric, shear-free spacetimes which are conformally flat when modeling the interior of a relativistic star. In this case there exists coordinates $(x^a) = (t, r, \theta, \phi)$ for which the line element may be expressed in the form

$$ds^2 = B^2 \left(-dt^2 + dr^2 + r^2 d\Omega^2\right), \tag{2.1}$$

where the metric function B is a function of t and r and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$.

The energy momentum tensor is given by

$$T_{ab} = (\mu + p)\, u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{2.2}$$

where μ is the density, p is the isotropic pressure, q_a is the heat flux, and π_{ab} is the anisotropic stress. The stress tensor is

$$\pi_{ab} = \left(p_{\parallel} - p_{\perp}\right)\left(n_a n_b - \frac{1}{3}h_{ab}\right), \tag{2.3}$$

where $p_{\parallel}$ is the radial pressure, $p_{\perp}$ is the tangential pressure and $\mathbf{n}$ is a unit radial vector. The isotropic pressure

$$p = \frac{1}{3}\left(p_{\parallel} + 2p_{\perp}\right),$$

relates the radial and the tangential pressures. The fluid four-velocity $\mathbf{u}$ is comoving and is given by

$$u^a = \frac{1}{B}\delta^a_0,$$

The heat flow vector $\mathbf{q}$ takes the form

$$q^a = (0, q, 0, 0),$$

since $q^a u_a = 0$ and the heat is assumed to flow in the radial direction. The kinematical quantities for the line element (2.1) are given by

$$\dot{u}^a = \left(0, \frac{B_r}{B^3}, 0, 0\right), \tag{2.4a}$$

$$\Theta = 3\frac{B_t}{B^2}, \tag{2.4b}$$

where $\dot{u}^a$ is the four-acceleration vector and Θ is the expansion scalar.

The Einstein field equations for the interior matter distribution become

$$\mu = 3\frac{B_t^2}{B^4} - \frac{1}{B^2}\left(2\frac{B_{rr}}{B} - \frac{B_r^2}{B^2} + \frac{4B_r}{rB}\right), \tag{2.5a}$$

$$p_{\parallel} = \frac{1}{B^2}\left(-2\frac{B_{tt}}{B} + \frac{B_t^2}{B^2} + 3\frac{B_r^2}{B^2} + \frac{4}{r}\frac{B_r}{B}\right), \tag{2.5b}$$

$$p_{\perp} = -2\frac{B_{tt}}{B^3} + \frac{B_t^2}{B^4} + \frac{2}{r}\frac{B_r}{B^3} - \frac{B_r^2}{B^4} + 2\frac{B_{rr}}{B^3}, \tag{2.5c}$$

$$q = -\frac{2}{B^3}\left(-\frac{B_{rt}}{B} + 2\frac{B_r B_t}{B^2}\right), \tag{2.5d}$$

for the line element (2.1). Equations (2.5) describe the gravitational interactions in the interior of a conformally flat star with heat flux and anisotropic pressure.

The boundary of a relativistic radiating star divides the entire spacetime into two distinct regions: the interior spacetime and the exterior spacetime. The exterior radiating spacetime

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right)dv^2 - 2dvdR + R^2 d\Omega^2, \tag{2.6}$$

where $m(v)$ denotes the mass of the star as measured by an observer at infinity, was first derived by Vaidya (1951). The spacetime is the unique spherically symmetric solution of the Einstein field equations for radially directed coherent radiation in the form of a null fluid. The interior spacetime (2.1) has to be matched along the boundary of the star to this exterior Vaidya spacetime.

The matching of the line elements (2.1) and (2.6), and the matching of the extrinsic curvature are necessary at the surface of the star. This matching leads to the following junction conditions

$$Bdt = \left[\left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{\frac{1}{2}} dv\right]_\Sigma, \tag{2.7a}$$

$$(rB)_\Sigma = R_\Sigma, \tag{2.7b}$$

$$m(v) = \left[\frac{r^3}{2}\left(\frac{B_t^2}{B} - \frac{B_r^2}{B}\right) - r^2 B_r\right]_\Sigma, \tag{2.7c}$$

$$(p_\parallel)_\Sigma = (Bq)_\Sigma, \tag{2.7d}$$

where Σ is the hypersurface that defines the boundary of the radiating sphere. The junction conditions (2.7) were completed by Santos (1985). The particular junction condition

$$(p_\parallel)_\Sigma = (Bq)_\Sigma, \tag{2.8}$$

is an additional differential equation that has to be solved together with the interior field equations (2.5) to complete the model of a relativistic radiating star. This is a nonlinear differential equation which has to be integrated on the boundary Σ of the star. By substituting equations (2.5b) and (2.5d) into (2.8) we have

$$2rBB_{rt} + 2rBB_{tt} - 4rB_rB_t - rB_t^2 - 3rB_r^2 - 4BB_r = 0, \tag{2.9}$$

at the boundary of a conformally flat star. Equation (2.9) is the master equation that governs the evolution of the model. We will attempt to integrate equation (2.9) using the Lie theory of extended groups applied to differential equations.

2.3 Lie symmetry analysis

We use the Lie analysis in an attempt to find new solutions to (2.9). We note that, except for Govinder and Govender (2012), no other attempt to apply symmetry analysis to the junction condition has been attempted. We know that an nth order differential equation

$$E\left(r, t, B, B_r, B_t, B_{rr}, B_{rt}, B_{tt}, \dots\right) = 0, \tag{2.10}$$

where $B = B(r,t)$, admits a Lie point symmetry

$$G = \xi_1\left(r, t, B\right)\frac{\partial}{\partial r} + \xi_2\left(r, t, B\right)\frac{\partial}{\partial t} + \eta\left(r, t, B\right)\frac{\partial}{\partial B}, \tag{2.11}$$

provided that

$$G^{[n]}E\big|_{E=0} = 0, \tag{2.12}$$

$[G_i, G_j]$	G_1	G_2	G_3
G_1	0	G_1	0
G_2	$-G_1$	0	0
G_3	0	0	0

Table 2.1: Commutation table for symmetries in (2.13)

where $G^{[n]}$ is the nth extension of the symmetry G in (2.11) (Olver 1993, Bluman 2010). The method is algorithmic and can be computed by using various software packages. Utilising `PROGRAM LIE` (Head 1993), we can demonstrate that (2.9) admits the following Lie point symmetries:

$$G_1 = \frac{\partial}{\partial t}, \tag{2.13a}$$

$$G_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \tag{2.13b}$$

$$G_3 = B\frac{\partial}{\partial B}, \tag{2.13c}$$

with the nonzero Lie bracket relationship $[G_1, G_2] = G_1$. The commutation table for symmetries in (2.13) is given in Table 2.1.

2.3.1 Optimal system

Given that (2.9) has the three symmetries (2.13), note that we can generate group invariant solutions using each symmetry in turn, or taking any linear combination of symmetries. Taking all possible combinations into account is not helpful. We proceed in a systematic manner by considering a subspace of this vector space. We utilize the subalgebraic structure of the symmetries (2.13) of the equation (2.9) to generate an optimal system of one-dimensional subgroups. Such an optimal system of subgroups is constructed by classifying the orbits of the infinitesimal adjoint representation of the Lie group on its related Lie algebra; this is achieved by using its infinitesimal generators. All group invariant solutions can be transformed to those obtained via this optimal system (Olver 1993).

The process is algorithmic. To determine the inequivalent subalgebras we begin with the following nonzero vector

$$G = a_1 G_1 + a_2 G_2 + a_3 G_3. \tag{2.14}$$

We try to remove as many of the coefficients, a_i of G, as possible through judicious applications of adjoint maps to G. The adjoint representation for symmetries in (2.13) is given in Table 2.2. As a result, we have

$$G_1 = \frac{\partial}{\partial t}, \tag{2.15a}$$

$$G_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \tag{2.15b}$$

$$aG_2 + G_3 = a\left(t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}\right) + B\frac{\partial}{\partial B}, \tag{2.15c}$$

Ad	G_1	G_2	G_3
G_1	G_1	$-\epsilon G_1 + G_2$	G_3
G_2	$e^{\epsilon} G_1$	G_2	G_3
G_3	G_1	G_2	G_3

Table 2.2: Adjoint representation table for symmetries in (2.13)

are the subalgebra of the symmetries in (2.13).

2.4 Solutions via symmetries in the optimal system

Using the generator

$$G_1 = \frac{\partial}{\partial t}, \tag{2.16}$$

we determine the invariants from the invariant surface condition

$$\frac{dt}{1} = \frac{dr}{0} = \frac{dB}{0}. \tag{2.17}$$

We obtain the invariants r and

$$B = y(r), \tag{2.18}$$

for the generator G_1. With this transformation equation (2.9) is reduced to

$$3ry' + 4y = 0. \tag{2.19}$$

Equation (2.19) is a first order, separable ordinary differential equation with solution

$$y = \frac{1}{r^{4/3}} \quad \Rightarrow \quad B = \frac{1}{r^{4/3}}, \tag{2.20}$$

where the constant of integration is taken to be unity. Since the gravitational potential B in (2.20) is independent of time, this solution cannot be applied to a radiating star.

The symmetry $aG_2 + G_3$ transforms the master equation to

$$\begin{aligned} &2a^2 (x-1)\, yy'' + 2a\,(1 + a - 3x - 2ax)\, yy' + a^2 \left(1 - 4x + 3x^2\right) y'^2 \\ &+ (3 + 4a)\, y^2 = 0, \end{aligned} \tag{2.21}$$

which is a second order nonlinear ordinary differential equation. This equation is quite difficult to solve since it has no symmetry in general for further reduction.

The invariants of G_2 are given by

$$x = \frac{t}{r}, \tag{2.22a}$$

$$B = y(x). \tag{2.22b}$$

For this transformation equation (2.9) is reduced to

$$(2x-2)\,yy'' + (2-4x)\,yy' + \left(1-4x+3x^2\right) y'^2 = 0. \tag{2.23}$$

Equation (2.23) is a second order nonlinear ordinary differential equation. The integration of (2.23) is not easy to complete using traditional methods. However, using the computer package Mathematica (Wolfram 2008) we find the solution

$$y = c_2 \exp\left(\int_1^x \frac{8e^{2z}\,(z-1)}{c_1 + 3e^{2z} - 10ze^{2z} + 6z^2e^{2z}}dz\right). \tag{2.24}$$

We can simplify (2.24) for particular parameter values by setting $c_1 = 0$ and $c_2 = 1$. Noting that (2.9) is invariant under scalings of B, we obtain the particular solution

$$B = \left(\sqrt{7} + 5 - 6\frac{t}{r}\right)^{\frac{14-2\sqrt{7}}{21}} \left(\sqrt{7} - 5 + 6\frac{t}{r}\right)^{\frac{14+2\sqrt{7}}{21}}, \tag{2.25}$$

for the master equation (2.9).

We emphasize that the result (2.25) is a new exact solution to the boundary condition (2.8) for a conformally radiating star with a shear-free matter distribution. It is not contained in any of the classes of solution found in previous investigations. The elementary form of the solution in (2.25) will assist in studying the physical features of a conformally flat radiating star.

For the solution (2.25) the line element (2.1) becomes

$$ds^2 = \left[\left(\sqrt{7} + 5 - 6\frac{t}{r}\right)^{\frac{28-4\sqrt{7}}{21}} \left(\sqrt{7} - 5 + 6\frac{t}{r}\right)^{\frac{28+4\sqrt{7}}{21}}\right]\left(-dt^2 + dr^2 + r^2 d\Omega^2\right). \tag{2.26}$$

The kinematical quantities for the line element (2.26) are given by

$$\dot{u}^a = \left(0, \frac{2r^3(r-t)t\left(5+\sqrt{7}-6\frac{t}{r}\right)^{\frac{4}{3\sqrt{7}}}\left(\frac{12(5r-3t)t}{r^2}-18\right)^{2/3}}{9\left(3r^2-10rt+6t^2\right)^3\left(\sqrt{7}-5+6\frac{t}{r}\right)^{\frac{4}{3\sqrt{7}}}}, 0, 0\right), \quad (2.27a)$$

$$\Theta = \frac{4\sqrt[3]{6}(r-t)\left(5+\sqrt{7}-6\frac{t}{r}\right)^{\frac{2}{3\sqrt{7}}}}{\left(\sqrt{7}-5+6\frac{t}{r}\right)^{\frac{2}{3\sqrt{7}}}\left(3-2(5r-3t)\frac{t}{r^2}\right)^{2/3}\left(3r^2-10rt+6t^2\right)}. \quad (2.27b)$$

We note that both the acceleration and expansion grow smaller with increasing time. The spacetime approaches asymptotic flatness.

The matter variables are given by

$$\mu = \frac{8\left(5+\sqrt{7}-6\frac{t}{r}\right)^{\frac{4}{3\sqrt{7}}}\left(-5+\sqrt{7}+6\frac{t}{r}\right)^{-\frac{4}{3\sqrt{7}}}\left(12r^4-24r^3t+15r^2t^2-4rt^3+2t^4\right)}{3\left(18-\frac{12(5r-3t)t}{r^2}\right)^{1/3}\left(3r^2-10rt+6t^2\right)^3}, \quad (2.28a)$$

$$p_{\|} = \frac{8r\left(5+\sqrt{7}-6\frac{t}{r}\right)^{\frac{4}{3\sqrt{7}}}\left(-5+\sqrt{7}+6\frac{t}{r}\right)^{-\frac{4}{3\sqrt{7}}}\left(3r^3+2r^2t-12rt^2+8t^3\right)}{3\left(18-\frac{12(5r-3t)t}{r^2}\right)^{1/3}\left(3r^2-10rt+6t^2\right)^3}, \quad (2.28b)$$

$$p_{\perp} = \frac{4\sqrt[3]{4}(r-t)\left(5+\sqrt{7}-6\frac{t}{r}\right)^{\frac{4}{3\sqrt{7}}}\left(-5+\sqrt{7}+6\frac{t}{r}\right)^{-\frac{4}{3\sqrt{7}}}\left(3r^3-4r^2t+8rt^2-4t^3\right)}{3\left(9-\frac{6(5r-3t)t}{r^2}\right)^{1/3}\left(3r^2-10rt+6t^2\right)^3}, \quad (2.28c)$$

$$q = \frac{4r^3\left(5+\sqrt{7}-6\frac{t}{r}\right)^{\frac{2}{\sqrt{7}}}\left(-5+\sqrt{7}+6\frac{t}{r}\right)^{-\frac{2}{\sqrt{7}}}\left(3r^3+2r^2t-12rt^2+8t^3\right)}{9\left(3r^2-10rt+6t^2\right)^4}, \quad (2.28d)$$

for the metric (2.26).

We remark that this new solution is given in terms of a self-similar variable $x = t/r$. The appearance of the self-similar variable implies the existence of a homothetic Killing vector. In shearing spherically symmetric spacetimes a homothetic vector was found by Wagh and Govinder (2006). The full conformal geometry of both shear-free and shearing spacetimes was completed by Moopanar and Maharaj (2010, 2013), respectively.

2.5 Invariance under $bG_1 + G_3$

Here we consider the combination $bG_1 + G_3$ which is not in the optimal system. This approach is taken as we were not able to solve all the equations obtained via the optimal system. It turns out that this is the best combination that yields a solution. In the symmetry

$$bG_1 + G_3 = b\frac{\partial}{\partial t} + B\frac{\partial}{\partial B}, \tag{2.29}$$

the constant b is nonzero and arbitrary. For the symmetry (2.29), we determine the invariants r and

$$B = \exp\left(\frac{t}{b}\right) y(r). \tag{2.30}$$

Using this transformation, equation (2.9) is reduced to the first order ordinary differential equation

$$3b^2 r y'^2 + 2b(2b + r)yy' - ry^2 = 0. \tag{2.31}$$

This is a highly nonlinear equation. However, equation (2.31) can be integrated to give two special solutions

$$y = \frac{\sqrt[3]{b+2r+2f(r)}}{\exp\left(\frac{r-2f(r)}{3b}\right)\sqrt[3]{[2b+r+2f(r)]^2}}, \tag{2.32a}$$

$$y = \frac{\sqrt[3]{(2b+r+2f(r))^2}}{r^{4/3}\exp\left(\frac{r+2f(r)}{3b}\right)\left(\sqrt[3]{b+2r+2f(r)}\right)}, \tag{2.32b}$$

where the constants of integration are set to unity and $f(r)=\sqrt{b^2+br+r^2}$. Hence we have found particular solutions to (2.9) of the form

$$B = \exp\left(\frac{3t-r+2f(r)}{3b}\right)\left(\frac{b+2r+2f(r)}{[2b+r+2f(r)]^2}\right)^{1/3}, \tag{2.33a}$$

$$B = \exp\left(\frac{3t-r-2f(r)}{3b}\right)\left(\frac{[2b+r+2f(r)]^2}{r^4\,[b+2r+2f(r)]}\right)^{1/3}, \tag{2.33b}$$

Note that in the above solutions $b \neq 0$ or we would simply have $B=B(r,t)$ in (2.30).

We have found two new solutions to the boundary condition (2.8) for a radiating star. The solutions (2.33a) and (2.33b) have been generated using invariance under bG_1+G_3 which is not in the optimal system. The metric function B is separable in the variables t and r in this class of solutions. We point out the fact that the new solutions are given in terms of elementary functions and this will help in the analysis of the physical features of a stellar model. The solution (2.33a) has the desirable feature of being regular at the stellar centre but the heat flux has the form

$$q = -\frac{2re^{\frac{r-2f(r)-3t}{b}}\left(2b+r+2f(r)\right)}{b^2\left(b+2\left(r+f(r)\right)\right)}. \tag{2.34}$$

The heat flux is always negative for positive b which implies inflow of energy across the boundary of the star. For a realistic model of radiating body the heat flow should be outwards to the exterior. This example suggests that even though Lie analysis does provide new solutions to the boundary condition, a careful analysis of the physical features is still necessary. The solution (2.33b) has a singularity at $r = 0$ and can only be applied in regions away from the stellar centre. Close to the centre another solution has to used; solution (2.33b) should be used as part of core-envelope model for radiating star. Solution (2.33b) has several desirable features which become clear in our physical analysis for regions away from the singularity at the centre of the radiating star. This realistic solution may be helpful in describing the interior spacetime of a radiating star in conformally flat spacetimes.

For the solution (2.33b) the line element (2.1) becomes

$$\begin{aligned} ds^2 &= \left[\exp\left(\frac{3t-r-2f(r)}{3b}\right)\left(\frac{[2b+r+2f(r)]^2}{r^4\,[b+2r+2f(r)]}\right)^{1/3}\right]^2 \\ &\quad \times\left(-dt^2+dr^2+r^2d\Omega^2\right), \end{aligned} \tag{2.35}$$

where $0 \leq t \leq \infty$. The kinematical quantities are given by

$$\begin{aligned} \dot{u}^a &= \Bigg(0, \frac{e^{\frac{2(r+2f(r)-3t)}{3b}}r^{5/3}}{-bf(r)\left(2b+r+2f(r)\right)^{7/3}\left(b+2\left(r+f(r)\right)\right)^{1/3}}\left(8b^4+6r^3\left(r+f(r)\right)\right. \\ &\quad \left. +4b^3\left(5r+2f(r)\right)+3br^2\left(6r+5f(r)\right)+2b^2r\left(13r+8f(r)\right)\right),0,0\Bigg), \end{aligned} \tag{2.36a}$$

$$\Theta = \frac{3e^{\frac{r+2f(r)-3t}{3b}}r^{4/3}\left(b+2\left(r+f(r)\right)\right)^{1/3}}{b\left(2b+r+2f(r)\right)^{2/3}}, \tag{2.36b}$$

for the line element (2.35). The acceleration and the expansion decrease for increasing time and the spacetime becomes asymptotically flat.

The matter variables become

$$\begin{aligned}
\mu &= \frac{2e^{\frac{2(r+2f(r)-3t)}{3b}}r^{2/3}}{b^2 f(r)\left(2b+r+2f(r)\right)^{10/3}\left(b+2\left(r+f(r)\right)\right)^{4/3}}\big(32b^7+72r^6\left(r+f(r)\right) \\
&\quad +16b^6\left(9r+2f(r)\right)+36br^5\left(9r+8f(r)\right)+25b^4r^2\left(31r+14f(r)\right)+18b^3r^3 \\
&\quad \times\left(51r+31f(r)\right)+9b^2r^4\left(78r+59f(r)\right)+2b^5r\left(213r+64f(r)\right)\big)\,, \qquad (2.37a)
\end{aligned}$$

$$\begin{aligned}
p_{\parallel} &= \frac{6e^{\frac{2(r+2f(r)-3t)}{3b}}r^{5/3}}{b^2\left(2b+r+2f(r)\right)^{10/3}\left(b+2\left(r+f(r)\right)\right)^{4/3}}\big(32b^5+24r^4\left(r+f(r)\right) \\
&\quad +16b^4\left(7r+2f(r)\right)+12br^3\left(8r+7f(r)\right)+6b^3r\left(31r+16f(r)\right) \\
&\quad +3b^2r^2\left(59r+42f(r)\right)\big)\,, \qquad (2.37b)
\end{aligned}$$

$$\begin{aligned}
p_{\perp} &= \frac{6e^{\frac{2(r+2f(r)-3t)}{3b}}r^{2/3}}{f(r)\left(2b+r+2f(r)\right)^{16/3}\left(b+2\left(r+f(r)\right)\right)^{7/3}}\big(1024b^8+648r^7\left(r+f(r)\right) \\
&\quad +5376b^6r\left(3r+f(r)\right)+256b^7\left(23r+4f(r)\right)+324br^6\left(13r+12f(r)\right) \\
&\quad +81b^2r^5\left(160r+133f(r)\right)+81b^3r^4\left(303r+220f(r)\right)+16b^5r^2 \\
&\quad \times\left(1709r+816f(r)\right)+4b^4r^3\left(7807r+4748f(r)\right)\big)\,, \qquad (2.37c)
\end{aligned}$$

$$q = \frac{2}{9b^2}e^{\frac{r+2f(r)-3t}{b}}r\left(2b^2+2r\left(r+f(r)\right)-b\left(r+2f(r)\right)\right)\,, \qquad (2.37d)$$

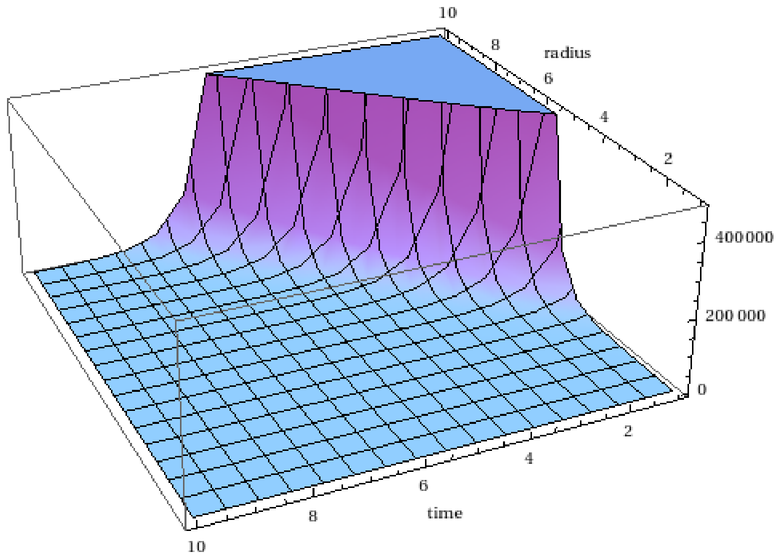

Figure 2.1: Energy density

for the metric (2.35). The quantities μ, $p_{\|}$, $p_{\perp}$ and q are regular in the interior of the relativistic star. They remain continuous and well behaved in regions of spacetime surrounding the stellar core. This feature is illustrated in Fig. 2.1–2.4. Fig. 2.1 is a plot for energy density, Fig. 2.2 represents the radial pressure $p_{\|}$, Fig 2.3 is a plot for the tangential pressure $p_{\perp}$, and Fig 2.4 illustrates the heat flow q. The properties for μ, $p_{\|}$ and $p_{\perp}$ are similar. These are desirable features and point to a physically viable model. We note that the heat flux decreases for increasing time. This implies that the star is radiating away energy as it approaches a static limit.

2.6 Discussion

We considered a relativistic radiating star in conformally flat spacetimes. We studied the junction condition which relates the radial pressure to the heat flux which is the master equation. We demonstrated that this equation admits three Lie point symme-

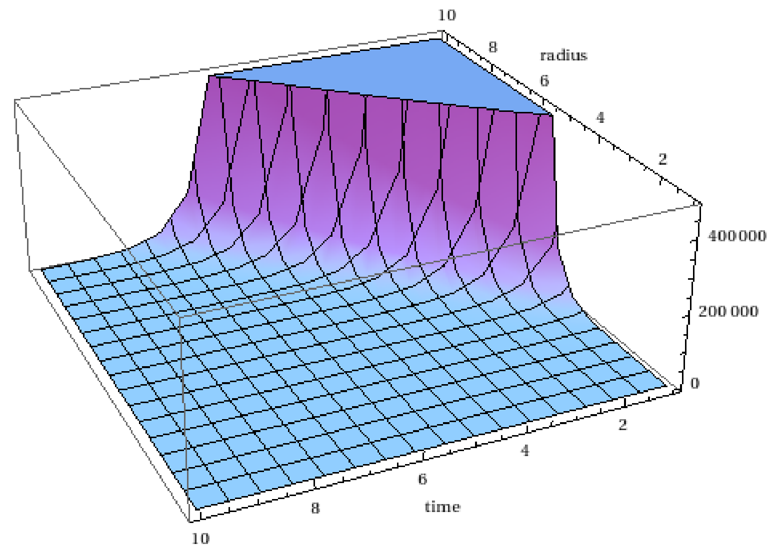

Figure 2.2: Radial pressure

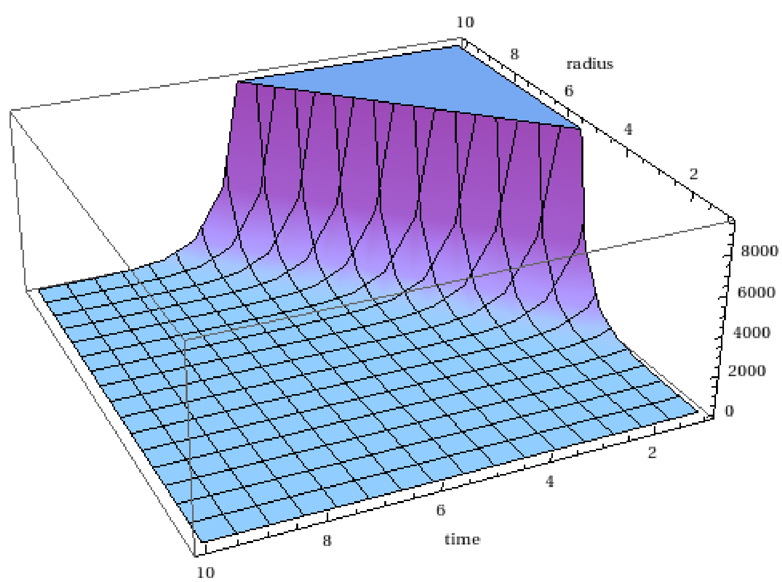

Figure 2.3: Tangential Pressure

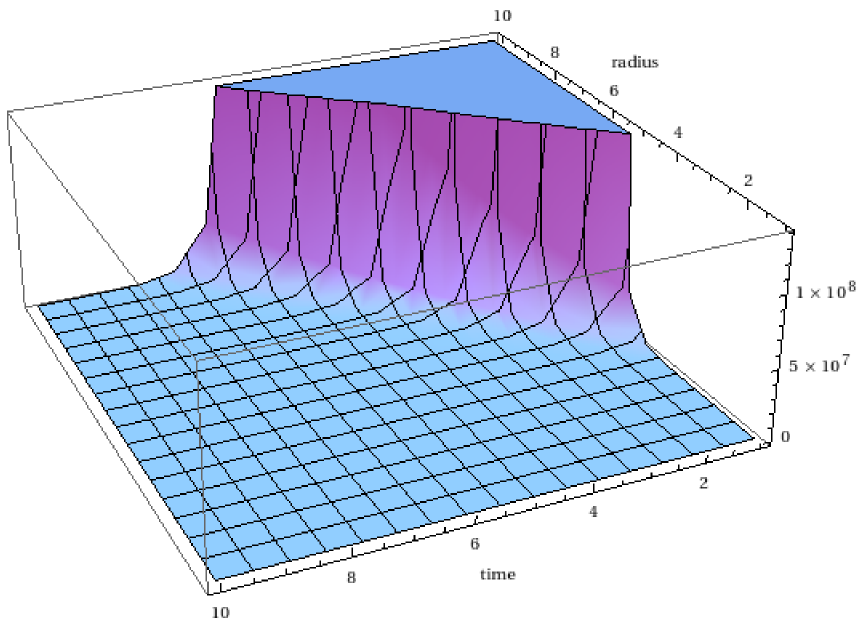

Figure 2.4: Heat flux

tries and obtained an optimal system. This was used to reduce the governing highly nonlinear partial differential equation to ordinary differential equations. We also used a symmetry combination which was not in the optimal system. By solving the reduced ordinary differential equations and transforming to the original variables we obtained new exact solutions for the master equation. We believe that the solutions obtained in this chapter are not contained in the literature. Two classes of models are of particular interest. The first class depends on the self similar variable t/r. In the second class the metric function is separable in the spacetime variables t and r. Two particular metrics could be identified in the second class. The first metric is regular at the centre but the heat flow is inwardly directed. The second metric is not regular at the centre but the heat flow is outwardly directed. Clearly the Lie analysis of differential equations is a useful technique in generating exact solutions to the boundary condition. However a subsequent study of the physical features remains necessary. For our example the matter variables are regular in a spacetime region at least close to the centre. The heat flux, acceleration and expansion are decreasing functions for large time. In our example observe that we obtain the relationship

$$p_{\|} = \mu\lambda, \quad \lambda = \frac{3rf(r)}{5b^2 + 8br + 8r^2 - 4bf(r) - 5rf(r)}, \tag{2.38}$$

which relates the $p_{\|}$ and μ. Thus the ratio $\frac{p_{\|}}{\mu}$ is independent of time. This property essentially arises from the separability of the metric (2.33b).

Chapter 3

Geodesic models generated by Lie symmetries

3.1 Introduction

In general relativity, the assumption, that the fluid particles travel in geodesic motion in radiating stars is physically reasonable. Several authors have studied a spherically symmetric relativistic radiating star with shear when the fluid particles are traveling in a geodesic motion using *ad hoc* methods. The first exact solution for a relativistic radiating star with shear was obtained by Naidu *et al.* (2008). Their treatment has been generalized by Rajah and Maharaj (2008) by transforming the junction condition to a Riccati equation. Thirukkanesh and Maharaj (2010) generated two classes of exact solutions that contain all previously known models as special cases. All the above mentioned treatments were modeled for a radiating stars when the fluid particles are traveling in a geodesic motion and regained the Friedmann dust model in the absence of heat flux in the relevant limit. The presence of acceleration changes the nature of the model.

Generating exact solutions to the boundary condition of a shearing radiating star in geodesic motion, using the symmetry approach, is the main objective of this chapter. In the past the Lie theory of differential equations has been used to great effect in solving the Einstein field equations in cosmology. With the help of the Lie symmetry theory of differential equations Govinder and Govender (2012) obtained an exact solution for the boundary condition for an Euclidean star. We believe that their treatment has been the first group theoretic approach to solve the boundary condition. Later Abebe *et al.* (2013) generated two classes of exact solutions for a conformally flat model by solving the junction condition exactly in the presence of anisotropic pressures using Lie symmetries. We expect that the application of the Lie symmetry analysis to a shearing and expanding radiating star when the fluid particles are in geodesic motion is likely to provide new insights.

We briefly introduce the shearing and expanding model, when the fluid particles are in geodesic motion, and present the junction condition in §3.2. We use a geometric approach to generate exact solutions. In §3.3 we find the Lie point symmetries that are admitted by the boundary condition. The optimal system for the symmetries is found to which all group invariant solutions can be transformed. The junction condition is transformed into an ordinary differential equation for each symmetry in §3.4, §3.5 and §3.6. By analysing the relevant ordinary differential equations, solutions are found to the boundary condition. We show the connection to known results and obtain limiting metrics. We make some concluding remarks in §3.7.

3.2 The model

The line element for the interior spacetime of a radiating star in geodesic motion with shear and expansion can be written as

$$ds^2 = -dt^2 + B^2 dr^2 + Y^2 d\Omega^2, \tag{3.1}$$

where $Y = Y(r,t)$ and $B = B(r,t)$ are the metric functions and $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. The fluid four-velocity $\mathbf{u}$ is comoving and is given by

$$u^a = \delta^a_0,$$

and the energy momentum has the form

$$T_{ab} = (\mu + p)\, u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{3.2}$$

where μ is the energy density of the fluid, p is the isotropic pressure, q_a is the heat flux and π_{ab} is the stress tensor. The heat flow vector $\mathbf{q}$ takes the form

$$q^a = (0, q, 0, 0),$$

since $q^a u_a = 0$ and the heat is assumed to flow in the radial direction. The stress tensor has the form

$$\pi_{ab} = \left(p_{\|} - p_{\perp}\right)\left(n_a n_b - \frac{1}{2} h_{ab}\right),$$

where $p_{\|}$ is radial pressure, $p_{\perp}$ is tangential pressure, $\mathbf{n}$ is a unit radial vector given by $n^a = \frac{1}{B}\delta^a_1$, and h_{ab} is the projection tensor. The isotropic pressure is given by

$$p = \frac{1}{3}\left(p_{\|} + 2p_{\perp}\right),$$

in terms of the radial pressure, $p_{\|}$, and the tangential pressure, $p_{\perp}$.

The four-acceleration $\dot{u}^a$, the expansion scalar Θ, and the magnitude of the shear scalar σ are given by

$$\dot{u}^a = 0, \tag{3.3a}$$

$$\Theta = 2\frac{Y_t}{Y} + \frac{B_t}{B}, \tag{3.3b}$$

$$\sigma = \frac{1}{3}\left(\frac{Y_t}{Y} - \frac{B_t}{B}\right), \tag{3.3c}$$

respectively for the line element (3.1). The Einstein field equations for the interior matter distribution become

$$\mu = 2\frac{B_t}{B}\frac{Y_t}{Y} + \frac{1}{Y^2} + \frac{Y_t^2}{Y^2} - \frac{1}{B^2}\left(2\frac{Y_{rr}}{Y} + \frac{Y_r^2}{Y^2} - 2\frac{B_r}{B}\frac{Y_r}{Y}\right), \tag{3.4a}$$

$$p_{\parallel} = -2\frac{Y_{tt}}{Y} - \frac{Y_t^2}{Y^2} + \frac{1}{B^2}\frac{Y_r^2}{Y^2} - \frac{1}{Y^2}, \tag{3.4b}$$

$$p_{\perp} = \frac{1}{B^2}\left(-\frac{B_r}{B}\frac{Y_r}{Y} + \frac{Y_{rr}}{Y}\right) - \left(\frac{B_{tt}}{B} + \frac{B_t}{B}\frac{Y_t}{Y} + \frac{Y_{tt}}{Y}\right), \tag{3.4c}$$

$$q = -\frac{2}{B^2}\left(-\frac{Y_{rt}}{Y} + \frac{B_t}{B}\frac{Y_r}{Y}\right), \tag{3.4d}$$

for the line element (3.1) and matter distribution (3.2). The subscripts stand for partial derivatives with respect to the independent variables t and r. The equations (3.4) describe the gravitational interactions in the interior of a geodesic shearing and expanding spherically symmetric star with heat flux and anisotropic pressure.

The exterior spacetime, describing the region outside the stellar boundary, is described by the Vaidya metric

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right) dv^2 - 2dvdR + R^2 d\Omega^2. \tag{3.5}$$

The matching of the exterior spacetime (3.5) with the interior spacetime (3.1) leads to the following set of junction conditions for the radiating star with shear in geodesic motion

$$dt = \left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{\frac{1}{2}} dv, \tag{3.6a}$$

$$(Y)_\Sigma = R_\Sigma(v), \tag{3.6b}$$

$$m(v) = \left[\frac{Y}{2}\left(1 + Y_t^2 - \frac{Y_r^2}{B}\right)\right]_\Sigma, \tag{3.6c}$$

$$(p_\|)_\Sigma = (Bq)_\Sigma, \tag{3.6d}$$

where Σ is the hypersurface that defines the boundary of the radiating sphere.

The junction condition (3.6d) was established by Santos (1985) for the first time in the case of shear-free spacetimes. Later it was extended by Glass (1989) for spacetimes with nonzero shear. From (3.4b), (3.4d) and (3.6d) we have

$$2B^2 Y Y_{tt} + B^2 Y_t^2 - Y_r^2 + 2BYY_{rt} - 2B_t Y Y_r + B^2 = 0, \tag{3.7}$$

at the boundary. Equation (3.7) is the fundamental differential equation governing the evolution of a shearing relativistic radiating star in geodesic motion. It is a complicated nonlinear partial differential equation and difficult to solve in general. We will attempt to integrate (3.7) to complete the model using the Lie theory of extended groups applied to differential equations.

3.3 Lie symmetry analysis

As mentioned earlier, the Lie symmetry method has been successfully used to generate exact solutions in General Relativity. In this chapter we attempt to find solutions for the boundary condition of an expanding and shearing radiating star when the fluid particles are traveling in geodesic motion using the Lie symmetry approach.

An nth order differential equation

$$F\left(r, t, B, Y, B_r, Y_r, B_t, Y_t, B_{rr}, Y_{rr}, B_{rt}, Y_{rt}, B_{tt}, Y_{tt}, \dots\right) = 0, \tag{3.8}$$

where $B = B(r,t)$ and $Y = Y(r,t)$, admits a Lie point symmetry of the form

$$G = \xi_1\left(r, t, B, Y\right)\frac{\partial}{\partial r} + \xi_2\left(r, t, B, Y\right)\frac{\partial}{\partial t} + \eta_1\left(r, t, B, Y\right)\frac{\partial}{\partial B} + \eta_2\left(r, t, B, Y\right)\frac{\partial}{\partial Y}, \tag{3.9}$$

provided that

$$G^{[n]}F\Big|_{F=0} = 0, \tag{3.10}$$

where $G^{[n]}$ is the nth prolongation of the symmetry G (Olver 1993, Bluman 2010). The process is algorithmic and so can be implemented by computer algebraic packages. Using `PROGRAM LIE` (Head 1993), we find that the junction condition (3.7) admits the infinite-dimensional set of symmetries

$[G_i, G_j]$	G_1	G_2	G_3
G_1	0	0	G_1
G_2	0	0	0
G_3	$-G_1$	0	0

Table 3.1: Commutation table for symmetries in (3.11)

$$G_1 = \frac{\partial}{\partial t}, \tag{3.11a}$$

$$G_2 = -Bf'(r)\frac{\partial}{\partial B} + f(r)\frac{\partial}{\partial r}, \tag{3.11b}$$

$$G_3 = B\frac{\partial}{\partial B} + Y\frac{\partial}{\partial Y} + t\frac{\partial}{\partial t}, \tag{3.11c}$$

where $f(r)$ is an arbitrary function of r. These symmetries tell us that (3.7) admits translational invariance (in t) as well as scaling invariance in B and r (together) as well as B, Y and t (together). When $f(r)$ is constant we have translational invariance in r as well. The appearance of translational invariance in both independent variables indicates that traveling wave solutions may exist. The generators (3.11) permit us to obtain the commutation table of symmetries (see Table 3.1).

Each symmetry and any linear combination of the symmetries may be helpful in solving (3.7). Any group invariant solution obtained by using these symmetries can be transformed to the group invariant solution obtained by the symmetries in the optimal system which is a subalgebra of the symmetries in (3.11). To obtain the subalgebra of the symmetries we begin with the nonzero vector

Ad	G_1	G_2	G_3
G_1	G_1	G_2	$-\epsilon G_1 + G_3$
G_2	G_1	G_2	G_3
G_3	$e^{\epsilon} G_1$	G_2	G_3

Table 3.2: Adjoint representation table for symmetries in (3.11)

$$G = a_1 G_1 + a_2 G_2 + a_3 G_3, \tag{3.12}$$

removing the coefficients a_i of G in a systematic way using applications of adjoint maps to G . The adjoint representation for symmetries in (3.11) is given in Table 3.2. Referring to Table 3.2 we obtain

$$G_1 = \frac{\partial}{\partial t}, \tag{3.13a}$$

$$aG_1 + G_2 = a\frac{\partial}{\partial t} - Bf'(r)\frac{\partial}{\partial B} + f(r)\frac{\partial}{\partial r}, \tag{3.13b}$$

$$aG_2 + G_3 = (1 - af'(r))B\frac{\partial}{\partial B} + Y\frac{\partial}{\partial Y} + t\frac{\partial}{\partial t} + af(r)\frac{\partial}{\partial r}, \tag{3.13c}$$

which is the optimal set of one-dimensional subalgebras of the symmetries in (3.11).

3.4 Invariance under G_1

Using the generator

$$G_1 = \frac{\partial}{\partial t}, \tag{3.14}$$

we determine the invariants from the surface condition

$$\frac{dt}{1} = \frac{dr}{0} = \frac{dB}{0} = \frac{dY}{0}. \tag{3.15}$$

We obtain the invariants r and

$$B = h(r), \quad Y = g(r), \tag{3.16}$$

for the generator G_1. Thus the gravitational potentials are static for the generator G_1 and the heat flux (3.4d) must vanish. We do not pursue this case further as the star is not radiating.

3.5 Invariance under $aG_1 + G_2$

The generator

$$aG_1 + G_2 = a\frac{\partial}{\partial t} - Bf'(r)\frac{\partial}{\partial B} + f(r)\frac{\partial}{\partial r}, \tag{3.17}$$

yields the surface condition

$$\frac{dt}{a} = \frac{dr}{f(r)} = \frac{dY}{0} = -\frac{dB}{Bf'(r)}, \tag{3.18}$$

from which we determine the invariants as

$$x = \int \frac{dr}{f(r)} - \frac{t}{a} \equiv \bar{f}(r) - \frac{t}{a}, \tag{3.19a}$$

$$B = \frac{h(x)}{f(r)}, \tag{3.19b}$$

$$Y = g(x), \tag{3.19c}$$

for the generator $aG_1 + G_2$. When $f(r) = 1$, our independent variable becomes

$$x = r - \frac{1}{a}t, \tag{3.20}$$

and so we are in the realm of traveling waves solutions, with wave speed equal to $1/a$.

Using the transformation (3.19) equation (3.7) becomes

$$2ag'gh' - 2agg''h + \left(a^2 + (2gg'' + g'^2)\right)h^2 = a^2 g'^2, \tag{3.21}$$

which is a Riccati equation in h. It is difficult to integrate (3.21) in general. We can find particular solutions for h by specifying the functional form of g.

We note solutions to the boundary condition with dependence on the variable x given in (3.19a) have not been found previously for particles traveling in geodesic motion in the stellar interior. The geodesic model of Thirukkanesh and Maharaj (2010), containing earlier models, has the potential

$$Y = [R_1(r)t + R_2(r)]^\alpha, \tag{3.22}$$

where $\alpha = 2/3, 1$. The functional dependence in (3.19c) is different from that in (3.22) since $g(x)$ is arbitrary. It is only in special cases, for particular forms of $R_1(r)$ and $R_2(r)$, that (3.19c) can be brought into the form (3.22). Therefore the classes of solution corresponding to (3.19) are new. This is not surprising since the Thirukkanesh and Maharaj (2010) models were generated using a method that transforms (3.7) to a first order separable equation. In our case we are seeking group invariant solutions to the second order differential equation (3.7) using the Lie theory of differential equations. We achieve this by restricting the coefficients in (3.21) which generate forms for the function g.

3.5.1 Case I: $2gg'' + g'^2 = 0$

We set

$$2gg'' + g'^2 = 0, \tag{3.23}$$

Then (3.23) can be integrated to give

$$g(x) = (b + cx)^{2/3}, \tag{3.24}$$

where b and c are arbitrary constants of integration. On substituting (3.24) into (3.21), we have

$$12c(b+cx)h' + 4c^2h + 9a(b+cx)^{2/3}h^2 = 4ac^2, \tag{3.25}$$

which is a simpler Riccati equation in h. On integration we obtain

$$h(x) = \frac{2c}{3(b+cx)^{1/3}} \left(\frac{\exp\left[\frac{3a}{c}(b+cx)^{1/3} + d\right] - 1}{\exp\left[\frac{3a}{c}(b+cx)^{1/3} + d\right] + 1} \right), \tag{3.26}$$

where d is an arbitrary constant of integration.

The gravitational functions are given by

$$B = \frac{2}{3} \frac{c}{f(r)\left(b + c\left(\bar{f}(r) - \frac{t}{a}\right)\right)^{1/3}} \left(\frac{\exp\left[\frac{3a}{c}\left(b + c\left(\bar{f}(r) - \frac{t}{a}\right)\right)^{1/3} + d\right] - 1}{\exp\left[\frac{3a}{c}\left(b + c\left(\bar{f}(r) - \frac{t}{a}\right)\right)^{1/3} + d\right] + 1} \right), \tag{3.27a}$$

$$Y = \left(b + c\left(\bar{f}(r) - \frac{t}{a}\right)\right)^{2/3}, \tag{3.27b}$$

which satisfy the boundary condition (3.7). The line element becomes

$$\begin{aligned} ds^2 = & -dt^2 + \frac{4}{9}\left[\frac{c}{f(r)\left(b + c\left(\bar{f}(r) - \frac{t}{a}\right)\right)^{1/3}} \right. \\ & \left. \times \left(\frac{\exp\left[\frac{3a}{c}\left(b + c\left(\bar{f}(r) - \frac{t}{a}\right)\right)^{1/3} + d\right] - 1}{\exp\left[\frac{3a}{c}\left(b + c\left(\bar{f}(r) - \frac{t}{a}\right)\right)^{1/3} + d\right] + 1} \right)\right]^2 dr^2 \\ & + \left(b + c\left(\bar{f}(r) - \frac{t}{a}\right)\right)^{4/3} d\Omega^2. \end{aligned} \tag{3.28}$$

We believe that this is a new solution for geodesic motion.

The functional dependence on the spacetime variables t and r in the metric functions is that of a traveling wave. This becomes clearer if we select a particular form of the function $f(r)$. We consider the special case $f(r) = 1$, $a = -1$, $b = 0$, $c = 1$, $d = 0$. Then we obtain the line element

$$ds^2 = -dt^2 + \frac{4}{9}\left[\frac{1}{(r+t)^{1/3}}\left(\frac{1-\exp\left(3\,(r+t)^{1/3}\right)}{1+\exp\left(3\,(r+t)^{1/3}\right)}\right)\right]^2 dr^2 \quad + [r+t]^{4/3}\, d\Omega^2, \quad (3.29)$$

which has an explicit traveling wave solution form. (Observe that the special case (3.29) arises essentially since the generator (3.17) has the reduced form $aG_1 + G_2 = -\frac{\partial}{\partial t} + \frac{\partial}{\partial r}$.) The spacetime is well-behaved as translational invariance under t removes the singularity that would occur at $t = r = 0$.

3.5.2 Case II: $2agg'' = 0$

If we set

$$2agg'' = 0, \quad (3.30)$$

then a simple integration gives

$$g(x) = bx + c, \quad (3.31)$$

Equation (3.21) becomes

$$2ab(c+bx)h' + \left(a^2+b^2\right)h^2 = a^2b^2, \tag{3.32}$$

which is also a Riccati equation in h. This can be integrated to give

$$h(x) = \frac{ab}{\sqrt{a^2+b^2}}\left(\frac{d\,(c+bx)^{\frac{\sqrt{a^2+b^2}}{b}} - 1}{d\,(c+bx)^{\frac{\sqrt{a^2+b^2}}{b}} + 1}\right), \tag{3.33}$$

where d is a nonzero constant.

The gravitational potentials have the form

$$B = \frac{ab}{f(r)\sqrt{a^2+b^2}}\left(\frac{d\left(c+b\left(\bar{f}(r)-\frac{t}{a}\right)\right)^{\frac{\sqrt{a^2+b^2}}{b}} - 1}{d\left(c+b\left(\bar{f}(r)-\frac{t}{a}\right)\right)^{\frac{\sqrt{a^2+b^2}}{b}} + 1}\right), \tag{3.34a}$$

$$Y = c + b\left(\bar{f}(r) - \frac{t}{a}\right), \tag{3.34b}$$

which is a solution for the boundary condition (3.7). The line element becomes

$$\begin{aligned} ds^2 &= -dt^2 + \left[\frac{ab}{f(r)\sqrt{a^2+b^2}}\left(\frac{d\left(c+b\left(\bar{f}(r)-\frac{t}{a}\right)\right)^{\frac{\sqrt{a^2+b^2}}{b}} - 1}{d\left(c+b\left(\bar{f}(r)-\frac{t}{a}\right)\right)^{\frac{\sqrt{a^2+b^2}}{b}} + 1}\right)\right]^2 dr^2 \\ &\quad + \left[c + b\left(\bar{f}(r) - \frac{t}{a}\right)\right]^2 d\Omega^2. \end{aligned} \tag{3.35}$$

We believe that this solution is new.

A simple form can be regained from (3.35). We set $a = -1$, $b = 1$, $d = 1$, $c = 0$ and $f(r) = 1$ to obtain

$$ds^2 = -dt^2 + \frac{1}{2}\left[\left(\frac{[r+t]^{\sqrt{2}}-1}{[r+t]^{\sqrt{2}}+1}\right)\right]^2 dr^2 + [r+t]^2\, d\Omega^2, \tag{3.36}$$

in terms of elementary functions. The time translational symmetry ensures that this spacetime is also well-behaved. Again this particular metric has the corresponding reduced Lie generator $aG_1 + G_2 = -\frac{\partial}{\partial t} + \frac{\partial}{\partial r}$ as in §3.5.1.

3.5.3 Case III: $a^2 + 2gg'' + g'^2 = 0$

We have also considered the case

$$a^2 + 2gg'' + g'^2 = 0, \tag{3.37}$$

The condition (3.37) can be integrated to give

$$\pm\frac{b}{a^3}\arctan\left(a\sqrt{\frac{g(x)}{b-a^2g(x)}}\right) \mp \frac{\sqrt{bg(x)-a^2g(x)^2}}{a^2} = x + c. \tag{3.38}$$

This an implicit solution involving the function $g(x)$. For this case (3.21) becomes

$$2g'gh' - 2gg''h = ag'^2, \tag{3.39}$$

which is a linear equation in h. This equation may be integrated to obtain

$$h(x) = g'\left(\frac{a}{2}\int\frac{dx}{g} + d\right), \tag{3.40}$$

where d is a constant of integration, and (3.40) gives h in terms of g explicitly. The gravitational potentials therefore are

$$B = g'\left(\frac{a}{2}\int\frac{dx}{g}+d\right), \tag{3.41a}$$

$$Y = g, \tag{3.41b}$$

which are functions of $x = \bar{f}(r) - \frac{t}{a}$, and g is given by (3.38). The line element is

$$ds^2 = -dt^2 + \left[g'\left(\frac{a}{2}\int\frac{dx}{g}+d\right)\right]^2 dr^2 + g^2 d\Omega^2, \tag{3.42}$$

which is written in terms of the function g only.

It is possible to rewrite the solution given above parametrically. We introduce a new parameter u for convenience. Then (3.38) can be represented as

$$x = \frac{b}{a^3}\left(u - \frac{1}{2}\sin(2u)\right) - c, \tag{3.43a}$$

$$g = \frac{b}{2a^2}\left(1 - \cos(2u)\right), \tag{3.43b}$$

in a parametric representation. In this case the line element (3.42) becomes

$$ds^2 = -dt^2 + \left[\frac{b}{2a^2}\left(1-\cos(2u)\right)\right]^2 \left\{\left[\frac{b}{a^2}\left(d - \frac{a^2}{b}\cot(x)\right)\sin(2x)\right]^2 dr^2 + d\Omega^2\right\}, \tag{3.44}$$

where the relationship between the u and x is given by (3.43a). If we set the arbitrary constant of integration $d=0$ then the line element (3.44) becomes

$$ds^2 = -dt^2 + \frac{b^2}{a^4}\sin^4(u)\left[4\cos^4(x)dr^2 + d\Omega^2\right], \tag{3.45}$$

which has simpler form.

3.6 Invariance under $aG_2 + G_3$

By using the generator

$$aG_2 + G_3 = (1 - af'(r))B\frac{\partial}{\partial B} + Y\frac{\partial}{\partial Y} + t\frac{\partial}{\partial t} + af(r)\frac{\partial}{\partial r}, \tag{3.46}$$

we find the invariants from the invariant surface condition

$$\frac{dt}{t} = \frac{dr}{af(r)} = \frac{dB}{(1 - af'(r))B} = \frac{dY}{Y}, \tag{3.47}$$

which are given by

$$x = \frac{t}{\exp\left(\int \frac{dr}{af(r)}\right)} \equiv \frac{t}{\exp\left(\tilde{f}(r)\right)}, \tag{3.48a}$$

$$B = h(x)\frac{\exp\left(\int \frac{dr}{af(r)}\right)}{f(r)} = h(x)\frac{\exp\left(\tilde{f}(r)\right)}{f(r)}, \tag{3.48b}$$

$$Y = g(x)t, \tag{3.48c}$$

for the symmetry $aG_2 + G_3$. We observe that the new independent variable x has a self-similar form. This is particularly evident when $f(r) = r/a$ as then we have

$$x = \frac{t}{r}. \tag{3.49}$$

With transformation (3.48) equation (3.7) becomes

$$2ax^3 g' g h' - 2ax^2 g\left(2g' + xg''\right)h + a^2\left(1 + g^2 + x^2 g'^2 + 2xg\left(3g' + xg''\right)\right)h^2 = x^4 g'. \tag{3.50}$$

Equation (3.50) is a different Riccati equation in h from those considered previously. As in §3.5 we can find group invariant solutions to (3.50) by restricting the coefficients which produce functional forms for the function g.

3.6.1 Case I: $g^2 + x^2 g'^2 + 2xg\left(3g' + xg''\right) = 0$

In this case we have

$$g^2 + x^2 g'^2 + 2xg\left(3g' + xg''\right) = 0 \tag{3.51}$$

To integrate we set

$$p(x) = (xg(x))^{3/2} \tag{3.52}$$

Then (3.51) becomes

$$p''(x) = 0 \tag{3.53}$$

with solution

$$p(x) = b + cx \tag{3.54}$$

and so (3.51) has the general solution

$$g(x) = \frac{(b + cx)^{2/3}}{x} \tag{3.55}$$

where b and c are arbitrary constants of integration. The transformation (3.52) was picked out due to the fact that (3.51) possesses eight Lie point symmetries and so is linearisable to the free particle equation.

On substituting equation (3.55) into (3.50) we have

$$6a\left(3b^2 + 4bcx + c^2x^2\right)h' - 4ac^2xh - 9a^2(b+cx)^{2/3}h^2 = -\left(9b^2 + 6bcx + c^2x^2\right), \tag{3.56}$$

which is also a Riccati equation in h. This can be integrated to give

$$h(x) \;=\; \frac{(3b + cx)}{3a(b + cx)^{1/3}}\left(\frac{1 - k + (k+1)\exp\left(\frac{3}{c}(b + cx)^{1/3}\right)}{1 - k - (k+1)\exp\left(\frac{3}{c}(b + cx)^{1/3}\right)}\right), \tag{3.57}$$

where k is an arbitrary constant of integration.

The gravitational potentials become

$$
\begin{aligned}
B &= \frac{\exp\left(\tilde{f}(r)\right)\left(3b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)}{3af(r)\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{1/3}} \\
&\quad \times \left(\frac{1-k+(k+1)\exp\left(\frac{3}{c}\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{1/3}\right)}{1-k-(k+1)\exp\left(\frac{3}{c}\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{1/3}\right)}\right),
\end{aligned}
\tag{3.58a}
$$

$$
Y = \exp\left(\tilde{f}(r)\right)\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{2/3}, \tag{3.58b}
$$

which is a particular solution for the master equation. The line element is

$$
\begin{aligned}
ds^2 &= -dt^2 + \left[\frac{\exp\left(\tilde{f}(r)\right)\left(3b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)}{3af(r)\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{1/3}}\right. \\
&\quad \left. \times \left(\frac{1-k+(k+1)\exp\left(\frac{3}{c}\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{1/3}\right)}{1-k-(k+1)\exp\left(\frac{3}{c}\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{1/3}\right)}\right)\right]^2 dr^2 \\
&\quad + \left[\exp\left(\tilde{f}(r)\right)\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{2/3}\right]^2 d\Omega^2.
\end{aligned}
\tag{3.59}
$$

We believe that the metric (3.59) has not been found before. Note that the gravitational potentials are given in terms of elementary functions. There is simplification in the form of the line elements for particular values of the parameter k. We consider these cases below.

$k = \pm 1$

If we set the arbitrary constant of integration $k = \pm 1$ in equation (3.59) then we obtain the metric

$$\begin{aligned} ds^2 &= -dt^2 + \left[\frac{\exp\left(\tilde{f}(r)\right)\left(3b + \frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)}{3af(r)\left(b + \frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{1/3}} \right]^2 dr^2 \\ &\quad + \exp\left(2\tilde{f}(r)\right)\left(b + \frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{4/3} d\Omega^2, \end{aligned} \tag{3.60}$$

which is a simple form.

If we set $\exp\left(\frac{1}{3}\tilde{f}(r)\right) = r$, $c = 1$ and $b = 0$ in (3.60) then we have

$$ds^2 = -dt^2 + t^{4/3}\left[dr^2 + r^2 d\Omega^2\right], \tag{3.61}$$

which is the Friedmann dust model in the absence of heat flux. This is a desirable feature as the Friedmann dust model also arises as a special case in the analyses of Kolasis *et al.* (1988), Naidu *et al.* (2006), Rajah and Maharaj (2008) and Thirukkanesh and Maharaj (2009, 2010).

$k = 0$

The case $k = 0$ in equation (3.59) is also of physical interest. This case contains earlier investigations of geodesic configurations with shear experiencing gravitational collapse. Hence the group invariant approach followed in this chapter does regain other physically viable models. If we set $\exp\left(\frac{1}{3}\tilde{f}(r)\right) = \beta(r)$, $b = 0$, $c = 1$ and utilize a translation of time ($t \to t + \alpha$) then (3.59) becomes

$$ds^2 = -dt^2 + (t+\alpha)^{4/3}\left[\beta'(r)^2\left(\frac{1+\exp\left(\frac{3(t+\alpha)^{1/3}}{\beta(r)}\right)}{1-\exp\left(\frac{3(t+\alpha)^{1/3}}{\beta(r)}\right)}\right)^2 dr^2 + \beta(r)^2 d\Omega^2\right]. \quad (3.62)$$

This solution is related to the first category of the Rajah and Maharaj (2008) models. In the Rajah and Maharaj (2008) model there is an additional function of integration which is absent in (3.62). When this quantity is set to be unity then (3.62) is exactly the same as the Rajah-Maharaj metric. Observe that if we further set $\beta(r) = r$ and $\alpha = 0$ then the line element (3.62) has the form

$$ds^2 = -dt^2 + t^{4/3}\left[\left(\frac{1+\exp\left(\frac{3t^{1/3}}{r}\right)}{1-\exp\left(\frac{3t^{1/3}}{r}\right)}\right)^2 dr^2 + r^2 d\Omega^2\right]. \quad (3.63)$$

This solution was first obtained by Naidu *et al.* (2006) for a shearing radiating star in geodesic motion. (Note that the arbitrary function of integration in the Rajah-Maharaj metric has to be set to be unity in Naidu *et al.* (2006) to regain (3.63). Thus, in the solution space of (3.7), their set of solutions overlap with our set of solutions.) They analysed heat dissipation and pressure anisotropy and showed that this was a realistic description of matter configuration undergoing gravitational collapse.

For sufficiently large values of t the expression $\left(\frac{1+\exp\left(\frac{3t^{1/3}}{r}\right)}{1-\exp\left(\frac{3t^{1/3}}{r}\right)}\right)^2$ approaches unity and the line element (3.63) is approximately

$$ds^2 \approx -dt^2 + t^{4/3}\left[dr^2 + r^2 d\Omega^2\right], \tag{3.64}$$

which is the limiting Friedmann dust model in the absence of heat flux. In the Rajah and Maharaj (2008) model the Friedmann dust model is regained exactly when a function of integration is set to be zero. In the case of the metric (3.63) the dust model only arises approximately.

If we set $\exp\left(\tilde{f}(r)\right) = r$, $b = 0$, $k = 0$ and $c = 1$ in (3.59) then we have

$$ds^2 = -dt^2 + \left(\frac{t}{r}\right)^{4/3}\left[\frac{1}{9}\left(\frac{1+\exp\left[3\left(\frac{t}{r}\right)^{1/3}\right]}{1-\exp\left[3\left(\frac{t}{r}\right)^{1/3}\right]}\right)^2 dr^2 + r^2 d\Omega^2\right], \tag{3.65}$$

in terms of the self-similar variable t/r. The self-similar variable indicates the existence of a homothetic Killing vector. Wagh and Govinder (2006) found a homothetic vector in shearing spherically symmetric spacetimes. Recently, Abebe *et al.* (2013) obtained new models for a conformally flat radiating star in which a particular class contains the self-similar variable.

3.6.2 Case II: $2g' + xg'' = 0$

If we set

$$2g' + xg'' = 0, \tag{3.66}$$

then we obtain the function

$$g(x) = \frac{b}{x} + c, \tag{3.67}$$

where b and c are arbitrary constants of integration. On substituting equation (3.67) into (3.50) we have

$$2ab(b+cx)h' - a^2\left(1+c^2\right)h^2 = -b^2, \tag{3.68}$$

which is a Riccati equation in h. This can be integrated to give

$$h(x) = \frac{b}{a\sqrt{1+c^2}}\left(\frac{1-k\left(b+cx\right)^{\frac{\sqrt{1+c^2}}{c}}}{1+k\left(b+cx\right)^{\frac{\sqrt{1+c^2}}{c}}}\right), \tag{3.69}$$

where k is a nonzero constant.

The gravitational potentials have the form

$$\begin{aligned} B &= \frac{b\exp\left(\tilde{f}(r)\right)}{a\sqrt{1+c^2}f(r)}\left(\frac{1-k\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{\frac{\sqrt{1+c^2}}{c}}}{1+k\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{\frac{\sqrt{1+c^2}}{c}}}\right), & \text{(3.70a)} \\ Y &= b\exp\left(\tilde{f}(r)\right)+ct, & \text{(3.70b)} \end{aligned}$$

which is a particular solution for the master equation (3.7). The metric is given by

$$\begin{aligned} ds^2 &= -dt^2 + \left[\frac{b\exp\left(\tilde{f}(r)\right)}{a\sqrt{1+c^2}f(r)}\left(\frac{1-k\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{\frac{\sqrt{1+c^2}}{c}}}{1+k\left(b+\frac{ct}{\exp\left(\tilde{f}(r)\right)}\right)^{\frac{\sqrt{1+c^2}}{c}}}\right)\right]^2 dr^2 \\ &\quad + \left[b\exp\left(\tilde{f}(r)\right)+ct\right]^2 d\Omega^2, \end{aligned} \tag{3.71}$$

which is another group invariant model.

If we set $\exp\left(\tilde{f}(r)\right) = r$, $a = 1/\sqrt{2}$, $b = 1$, $c = 1$ and $k = 1$ then the line element (3.71) becomes

$$ds^2 = -dt^2 + \left(\frac{1 - \left(1 + \frac{t}{r}\right)^{\sqrt{2}}}{1 + \left(1 + \frac{t}{r}\right)^{\sqrt{2}}}\right)^2 dr^2 + r^2 \left(1 + \frac{t}{r}\right)^2 d\Omega^2, \tag{3.72}$$

which has a simple form.

3.7 Discussion

We considered a shearing and expanding relativistic radiating star when the fluid particles are in geodesic motion. This model was analysed with the Lie infinitesimal generators applicable to differential equations. We studied in particular the junction condition which relates the radial pressure to the heat flux. Three Lie point symmetries admitted by this equation were found and an optimal system was obtained. The symmetries were used to reduce the governing highly nonlinear partial differential equation to ordinary differential equations. By solving the reduced ordinary differential equations, and transforming to the original variables, we obtained exact solutions for the master equation. It was particularly pleasing to observe that we were able to provide families of traveling wave solutions as well as families of self-similar solutions. Both types of solutions have be found to have great application in a variety of areas of Mathematical Physics (Ablowitz and Clarkson 1991, Sachdev 2000).

Our classes of solutions contain new and previously obtained solutions. We regained the Friedmann dust model as a special case of one family of solutions. In addition, the connection to the previous models of Naidu *et al.* (2006) and Rajah and

Maharaj (2008) was shown. Therefore we have demonstrated that the Lie method is a useful tool in modelling gravitational behaviour in collapse. The physical features of the models generated here will be studied in greater detail in the future. In particular, the traveling wave collapse and self-similar collapse should be of great interest.

This approach has allowed us to solve the rather complicated equation (3.7). In its original form, this equation is very difficult to solve. We used appropriate group invariants to reduce the equation to Riccati ODEs. While this allowed us to make some progress, the resulting equations did not yield to the standard approaches for solving Riccati equations. However, by making simplifying assumptions, we *were* able to solve the equations. It is remarkable that the simplifications also ensured that the standard techniques could be applied. In fact, the simplified Riccati equations could then be transformed into second order linear equations with constant coefficients! This is a elegant happenstance and completely unexpected. None of this would have been revealed if not for the Lie symmetry approach.

Chapter 4

Generalized Euclidean stars

4.1 Introduction

Models of relativistic radiating stars with expansion, shear and acceleration are the most general in spherically symmetric spacetimes. Again, the junction condition formulated by Santos (1985) is crucial to obtain exact models. Our aim in this chapter is to provide new exact solutions to the boundary condition of relativistic radiating stars with expansion, shear and acceleration. The presence of acceleration and the effect of shear change the nature of the resulting junction condition equation. A systematic study of these models was initiated by Chan (1997) for a configuration that is initially static and then collapses. In several analyses Chan (2000, 2001, 2003) and Pinheiro and Chan (2008, 2010) studied the luminosity, viscous effects and other physical features in the presence of shear. The physical analysis in the treatments (Chan 2000, 2001, 2003 and Pinheiro and Chan 2008, 2010) were accomplished using a numerical approach. Recently Thirukkanesh *et al.* (2012) obtained exact solutions for an accelerating and expanding model with shear by transforming the junction condition into linear, Bernoulli and inhomogeneous Riccati equations. Herrera and Santos (1997) studied the physical features in general when anisotropy is present. Euclidean stars may be modeled with nonvanishing shear. Such stars were studied by Govender *et al.* (2010), Govinder and Govender (2012) and Herrera and Santos (2010). The effect of shear in a radiating star undergoing dissipative collapse has been recently studied by Govender *et al.* (2013).

We present the junction condition for an accelerating, expanding and shearing radiating star in §4.2. This is a highly nonlinear partial differential equation in the metric functions. We obtain the Lie point symmetries for the junction condition and generate the optimal system of these symmetries. Using a particular symmetry in the optimal system, we transform the boundary condition to ordinary differential equations and group invariant solutions are obtained in §4.3. We find new classes of exact solutions to the boundary condition and regain known solutions. We show that the new solutions found satisfy a barotropic equation of state in §4.4. In §4.5 we consider the physical features and analyze the energy conditions in a particular example. The remaining cases of the optimal system for which the boundary condition can be integrated are listed in §4.6. In §4.7 we make concluding remarks.

4.2 The model

We consider the most general form of a spherically symmetric radiating star with acceleration, expansion and shear. The line element for the interior spacetime of such a star is given by

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.1}$$

where the metric functions A, B and Y are functions of the coordinate radius r and the temporal time variable t. The fluid four-velocity $\mathbf{u}$ is comoving and is given by $u^a = \frac{1}{A}\delta^a_0$. The kinematical quantities, the acceleration $\dot{u}^a$, the expansion scalar Θ, and the magnitude of the shear scalar σ, are give by

$$\dot{u}^a = \left(0, \frac{A_r}{AB^2}, 0, 0\right), \tag{4.2a}$$

$$\Theta = \frac{1}{A}\left(\frac{B_t}{B} + 2\frac{Y_t}{Y}\right), \tag{4.2b}$$

$$\sigma = -\frac{1}{3A}\left(\frac{B_t}{B} - \frac{Y_t}{Y}\right), \tag{4.2c}$$

where the subscripts denote differentiation with respect to r and t. The energy momentum tensor for the shearing model has the form

$$T_{ab} = (\mu + p)\, u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab}, \tag{4.3}$$

where μ is the density, p is the isotropic pressure, q_a is the heat flux, and π_{ab} is the anisotropic stress. The quantity $\mathbf{u}$ is the fluid four-velocity which satisfies $u^a u_a = -1$ and $u^a = \frac{1}{A}\delta^a_0$. The stress tensor is given by

$$\pi_{ab} = \left(p_\parallel - p_\perp\right)\left(n_a n_b - \frac{1}{3}h_{ab}\right). \tag{4.4}$$

We have introduced the radial pressure $p_\perp$, the tangential pressure $p_\perp$, the projection tensor h_{ab} and a unit radial vector $\mathbf{n}$ given by $n^a = \frac{1}{B}\delta^a_1$. The radial and the tangential pressures give the isotropic pressure $p = \frac{1}{3}\left(p_\parallel + 2p_\perp\right)$. Since the heat must flow in the radial direction the heat flow vector $\mathbf{q}$ may be written as

$$q^a = (0, Bq, 0, 0), \tag{4.5}$$

and $q^a u_a = 0$.

The Einstein field equations for the interior of the star have the form

$$\mu = \frac{2}{A^2}\frac{B_t}{B}\frac{Y_t}{Y} + \frac{1}{Y^2} + \frac{1}{A^2}\frac{Y_t^2}{Y^2} - \frac{1}{B^2}\left(2\frac{Y_{rr}}{Y} + \frac{Y_r^2}{Y^2} - 2\frac{B_r}{B}\frac{Y_r}{Y}\right), \tag{4.6a}$$

$$p_\parallel = \frac{1}{A^2}\left(-2\frac{Y_{tt}}{Y} - \frac{Y_t^2}{Y^2} + 2\frac{A_t}{A}\frac{Y_t}{Y}\right) + \frac{1}{B^2}\left(\frac{Y_r^2}{Y^2} + 2\frac{A_r}{A}\frac{Y_r}{Y}\right) - \frac{1}{Y^2}, \tag{4.6b}$$

$$\begin{aligned} p_\perp = & -\frac{1}{A^2}\left(\frac{B_{tt}}{B} - \frac{A_t}{A}\frac{B_t}{B} + \frac{B_t}{B}\frac{Y_t}{Y} - \frac{A_t}{A}\frac{Y_t}{Y} + \frac{Y_{tt}}{Y}\right) \\ & + \frac{1}{B^2}\left(\frac{A_{rr}}{A} - \frac{A_r}{A}\frac{B_r}{B} + \frac{A_r}{A}\frac{Y_r}{Y} - \frac{B_r}{B}\frac{Y_r}{Y} + \frac{Y_{rr}}{Y}\right), \end{aligned} \tag{4.6c}$$

$$q = -\frac{2}{AB}\left(-\frac{Y_{rt}}{Y} + \frac{B_t}{B}\frac{Y_r}{Y} + \frac{A_r}{A}\frac{Y_t}{Y}\right), \tag{4.6d}$$

for the metric (4.1). The matter variables μ, $p_\parallel$, $p_\perp$ and q can be determined explicitly once the potential functions A, B and Y are known. Equations (4.6) describe the gravitational interactions in the interior of an accelerating, expanding and shearing star with heat flux and anisotropic pressure.

The surface of a spherically symmetric radiating star is the boundary between the interior and the exterior spacetimes. The interior spacetime (4.1) has to be matched at the surface of the star to the exterior Vaidya spacetime

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right)dv^2 - 2dvdR + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.7}$$

In (4.7) the function $m(v)$ is the mass of the star at infinity. The metrics (4.1) and (4.7) have to be matched at the boundary of the star. The matching of the metrics and the extrinsic curvature at the surface of the star give the junction conditions

$$Adt = \left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{\frac{1}{2}} dv, \tag{4.8a}$$

$$(Y)_\Sigma = R_\Sigma(v), \tag{4.8b}$$

$$m(v) = \left[\frac{Y}{2}\left(1 + \frac{Y_t^2}{A^2} - \frac{Y_r^2}{B}\right)\right]_\Sigma, \tag{4.8c}$$

$$(p_\|)_\Sigma = (q)_\Sigma, \tag{4.8d}$$

at the hypersurface Σ of the radiating sphere. The junction condition (4.8d), for shear-free spacetimes was first found by Santos (1985), and, later extended to shearing spacetimes by Glass (1989). Equation (4.8d), together with (4.6b) and (4.6d), leads to the junction condition equation

$$2AB^2YY_{tt} + AB^2Y_t^2 - 2B^2YA_tY_t - 2ABYA_rY_t + 2A^2BYY_{rt} - 2A^2YA_rY_r$$

$$-2A^2YB_tY_r - A^3Y_r^2 + A^3B^2 = 0, \tag{4.9}$$

at the boundary of the star. Equation (4.9) determines the gravitational behaviour of the radiating anisotropic star with nonzero shear, acceleration and expansion. We need to solve (4.9) exactly to complete the model. This equation is a highly nonlinear partial differential equation and difficult to solve directly. Therefore we undertake a group theoretic analysis of (4.9) in order to find useful solutions.

A differential equation of order k

$$F\left(r, t, A, B, Y, A_r, B_r, Y_r, A_t, B_t, Y_t, A_{rr}, B_{rr}, Y_{rr}, A_{rt}, B_{rt}, Y_{rt}, A_{tt}, B_{tt}, Y_{tt}, \dots\right) = 0, \tag{4.10}$$

where $A = A(t,r)$, $B = B(r,t)$ and $Y = Y(r,t)$ admits a Lie point symmetry of the form

$$G = \xi_1 \frac{\partial}{\partial r} + \xi_2 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial A} + \eta_2 \frac{\partial}{\partial B} + \eta_3 \frac{\partial}{\partial Y}, \tag{4.11}$$

where ξ_i and η_j are functions of r, t, A, B and Y ($i = 1,2$ and $j = 1,2,3$), provided that

$$G^{[k]} F\big|_{F=0} = 0, \tag{4.12}$$

where $G^{[k]}$ is the kth prolongation of the symmetry G. The process is algorithmic. Using the computer software package PROGRAM LIE (Head 1993) we find that (4.9) possesses the symmetries

$[G_i, G_j]$	G_1	G_2	G_3
G_1	0	G_2	$-G_1$
G_2	$-G_2$	0	G_2
G_3	G_1	$-G_2$	0

Table 4.1: Commutation table for symmetries in (4.13)

$$G_1 = A\beta'(t)\frac{\partial}{\partial A} - \beta(t)\frac{\partial}{\partial t}, \tag{4.13a}$$

$$G_2 = B\alpha'(r)\frac{\partial}{\partial B} - \alpha(r)\frac{\partial}{\partial r}, \tag{4.13b}$$

$$G_3 = A\frac{\partial}{\partial A} + B\frac{\partial}{\partial B} + Y\frac{\partial}{\partial Y}, \tag{4.13c}$$

where $\beta(t)$ and $\alpha(r)$ are nonzero arbitrary functions of t and r respectively. This reveals that equation (4.9) is invariant under scalings of A and t together, B and r together, and A, B and Y together. Note that, since $\beta(t)$ and $\alpha(r)$ are arbitrary functions, they mask the expected invariance under translations in t and r separately (which is obtained by setting those functions to constants). These symmetries will be used to generate group invariant solutions. The commutation table for the generators (4.13) by Table 4.1.

Group invariant solutions obtained by using any linear combination of the individual symmetries in (4.13) may be transformed to the symmetries in the optimal system by applying action of the adjoint (Olver 1993). The adjoint representation for symmetries in (4.13) is given in table 4.2. We determine an optimal system of (4.13) to be

Ad	G_1	G_2	G_3
G_1	G_1	$e^{-\epsilon}G_2$	$\epsilon G_1 + G_3$
G_2	$G_1 + \epsilon G_2$	G_2	$-\epsilon G_2 + G_3$
G_3	$e^{-\epsilon}G_1$	$e^{\epsilon}G_2$	G_3

Table 4.2: Adjoint representation table for symmetries in (4.13)

$$G_1 = A\beta'(t)\frac{\partial}{\partial A} - \beta(t)\frac{\partial}{\partial t}, \tag{4.14a}$$

$$aG_1 + G_3 = (a\beta'(t)+1)A\frac{\partial}{\partial A} + B\frac{\partial}{\partial B} + Y\frac{\partial}{\partial Y} - a\beta(t)\frac{\partial}{\partial t}, \tag{4.14b}$$

$$G_2 - bG_1 = -bA\beta'(t)\frac{\partial}{\partial A} + B\alpha'(r)\frac{\partial}{\partial B} - \alpha(r)\frac{\partial}{\partial r} + b\beta(t)\frac{\partial}{\partial t}. \tag{4.14c}$$

4.3 Generalized Euclidean stars

The symmetries in (4.14) may be applied to reduce the governing partial differential equation to ordinary differential equations using the infinitesimal generators obtained via the Lie approach (Olver 1993, Bluman 2010). A number of different cases are possible which lead to exact solutions of the boundary condition (4.9). In this section we present only the most interesting case which has physically applicable features. The remaining integrable cases are included in §4.6.

The invariants of the particular generator

$$G_2 - bG_1 = -bA\beta'(t)\frac{\partial}{\partial A} + B\alpha'(r)\frac{\partial}{\partial B} - \alpha(r)\frac{\partial}{\partial r} + b\beta(t)\frac{\partial}{\partial t}, \qquad (4.15)$$

can be found from the system

$$\frac{dt}{b\beta(t)} = \frac{dr}{-\alpha(r)} = \frac{dA}{-bA\beta'(t)} = \frac{dB}{B\alpha'(r)} = \frac{dY}{0} \qquad (4.16)$$

to be

$$x = \int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}, \qquad (4.17a)$$

$$A = \frac{f(x)}{\beta(t)}, \qquad (4.17b)$$

$$B = \frac{g(x)}{\alpha(r)}, \qquad (4.17c)$$

$$Y = h(x). \qquad (4.17d)$$

We note that solutions obtained via the symmetry (4.15) contain traveling waves. This can be seen explicitly if we set $\beta(t) = \alpha(r) = 1$. Then the independent variable (4.17a) becomes

$$x = \frac{1}{b}t + r, \qquad (4.18)$$

with wave speed $1/b$. This fact arises since the generator (4.15) gives the reduced form

$$G_2 - bG_1 = b\frac{\partial}{\partial t} - \frac{\partial}{\partial r}, \tag{4.19}$$

in this case.

For the transformation (4.17), equation (4.9) reduces to

$$g' + \left(\frac{f'}{f} - \frac{h''}{h'}\right) g - \left(\frac{b^2 f^3 - 2hf'h' + f\left(h'^2 + 2hh''\right)}{2bf^2hh'}\right) g^2 + \left(bf' + \frac{bfh'}{2h}\right) = 0, \tag{4.20}$$

which is a nonlinear equation in the functions f, g and h. We have expressed (4.20) in a form which can be interpreted as a Riccati equation in g. In §4.6 we undertake a more detailed analysis of this equation. Here we note that the equation can be simplified if we assume

$$f = cg, \tag{4.21}$$

where c is an arbitrary constant. Then (4.20) becomes

$$g' + \left(\frac{(cb-1)h'^2 - 2hh''}{2(cb+1)hh'}\right) g - \left(\frac{c^2b^2}{2(cb+1)^2hh'}\right) g^3 = 0, \tag{4.22}$$

which is a Bernoulli equation in g of degree three. Even though h is an unknown function we can integrate (4.22) to obtain

$$g(x) = \frac{h^{\frac{1-cb}{2(cb+1)}} h'^{\frac{1}{cb+1}}}{\left(d - \frac{c^2b^2}{(cb+1)^2} \int_1^x h(z)^{\frac{-2cb}{cb+1}} h'(z)^{\frac{1-cb}{cb+1}} \, dz\right)^{1/2}}, \tag{4.23}$$

where d is a constant of integration. Thus we have found an exact solution to the boundary condition (4.9). The solution is expressible in terms of the function h which is arbitrary. Thus the potential functions become

$$A = \frac{c h(x)^{\frac{1-cb}{2(cb+1)}} h'(x)^{\frac{1}{cb+1}}}{\beta(t)\left(d - \frac{c^2 b^2}{(cb+1)^2} \int_1^x h(z)^{\frac{-2cb}{cb+1}} h'(z)^{\frac{1-cb}{cb+1}}\, dz\right)^{1/2}}, \tag{4.24a}$$

$$B = \frac{\beta(t)}{c\alpha(r)} A, \tag{4.24b}$$

$$Y = h(x), \tag{4.24c}$$

where h is a function of $x = \int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}$. This solution is expressed in terms of arbitrary constants and the arbitrary functions $\alpha(r)$, $\beta(t)$ and $h(x)$ so that we can generate infinitely many solutions to the boundary condition for particular choices.

For physical applications we need to show that the integrals in (4.24) may be written in closed form. This is possible for particular choices of the function $h(x)$. As a first example we set

$$h(x) = \exp\left(e + kx\right), \tag{4.25}$$

where e and k are constants. Then the potentials (4.24) become

$$A = \frac{ck^{\frac{1}{1+cb}} \exp\left(\frac{(3-cb)(e+kx)}{2+2cb}\right)}{\beta(t)\left(d - \frac{c^2b^2}{(1+cb)(3cb-1)k^{\frac{2cb}{1+cb}}}\left[\exp\left(\frac{(e+k)(1-3cb)}{1+cb}\right) - \exp\left(\frac{(e+kx)(1-3cb)}{1+cb}\right)\right]\right)^{1/2}}, \tag{4.26a}$$

$$B = \frac{\beta(t)}{c\alpha(r)}A \tag{4.26b}$$

$$Y = \exp(e+kx), \tag{4.26c}$$

and the integration has been completed. This model is expanding, accelerating and shearing. It is interesting to observe that if we set $d = \frac{c^2b^2\exp\left(\frac{(1-3cb)(e+k)}{1+cb}\right)}{(1+cb)(3cb-1)k^{\frac{2cb}{1+cb}}}$ then the shear vanishes.

As a second example we set

$$h(x) = (e+kx)^n, \tag{4.27}$$

and the integration in (4.24) can be performed. Then the potential functions have the form

$$A = \frac{c(kn)^{\frac{1}{1+cb}}(e+kx)^{\frac{3n-cbn-2}{2+2cb}}}{\beta(t)\left(d + \frac{c^2b^2n\left((e+kx)^{\frac{cb(2-3n)+n}{1+cb}} - (e+k)^{\frac{cb(2-3n)+n}{1+cb}}\right)}{(1+cb)(-n+cb(-2+3n))(kn)^{\frac{2cb}{1+cb}}}\right)^{1/2}}, \tag{4.28a}$$

$$B = \frac{\beta(t)}{c\alpha(r)}A, \tag{4.28b}$$

$$Y = (e+kx)^n. \tag{4.28c}$$

The solution (4.28) has interesting features which we will consider later. For now, we note that this model has nonvanishing expansion, acceleration and shear.

It is interesting to note that our solution contains those of Euclidean stars. In Euclidean stars the areal and proper radii are equal; this approach of modeling stars in general relativity with shear was developed by Herrera and Santos (2010). Govender *et al.* (2010) found other particular Euclidean solutions.

The metric (4.24) will yield the Euclidean star formulation provided $B = Y_r$. This yields the condition

$$h' = \frac{h^{\frac{1-cb}{2(cb+1)}} h'^{\frac{1}{cb+1}}}{\left(d - \frac{c^2 b^2}{(cb+1)^2} \int_1^x h(z)^{\frac{-2cb}{cb+1}} h'(z)^{\frac{1-cb}{cb+1}} \, dz\right)^{1/2}}, \tag{4.29}$$

If we now take (4.28) and set

$$b = 1, \quad e = 0, \quad k^n = \tilde{k}, \quad d = \frac{c^2 \tilde{k}^{\frac{1-3c}{1+c}} n^{\frac{1-c}{1+c}}}{(1+c)(3cn - 2c - n)}, \quad c = \frac{1 - n \pm \sqrt{1 - 4n + 3n^2}}{2(n-1)}, \tag{4.30}$$

then we obtain

$$A = \frac{x^{n-1}}{\beta(t)}, \tag{4.31a}$$

$$B = \tilde{k} n \frac{x^{n-1}}{\alpha(r)}, \tag{4.31b}$$

$$Y = \tilde{k} x^n, \quad \text{where } x = \int \frac{dt}{\beta(t)} + \int \frac{dr}{\alpha(r)}. \tag{4.31c}$$

This particular solution satisfies the master equation (4.9) provided that the constants n and $\tilde{k}$ are related via

$$\tilde{k}^2 n^3 - 2\tilde{k}n^2 + 2n(\tilde{k} - 1) + 2 = 0. \tag{4.32}$$

The solution (4.31) was previously obtained by Govinder and Govender (2012) in their study of the junction condition of an Euclidean star.

As the particular Euclidean star model (4.31) is contained in the more general class of solutions (4.24) we label the gravitational potentials in (4.24) as generalized Euclidean stars. We note that there may be other special parameter values for which h can be found explicitly which will correspond to other Euclidean stars.

4.4 Equation of state

We now study the physical features of the new generalized Euclidean model (4.24) that contains previously obtained solutions for Euclidean stars as a special case.

The kinematical quantities, the acceleration, the expansion scale and the magnitude of the shear scalar become

$$\dot{u}^a = \left(0, \frac{\alpha(r)}{2(1+cb)^3 h'}\left[\frac{c^2 b^2 (1+cb)}{h} - \frac{\varphi(x)}{h^{\frac{2}{1+cb}} h'^{\frac{2}{1+cb}}}\left((cb-1)h'^2 - 2hh''\right)\right], 0, 0\right), \tag{4.33a}$$

$$\Theta = \frac{c^2 b^2 (1+cb) h h'^{\frac{2+cb}{1+cb}} + \varphi(x) h^{\frac{2cb}{1+cb}} h'^{\frac{cb}{1+cb}} \left((5+3cb)h'^2 + 2hh''\right)}{2cb(1+cb)^2 \sqrt{\varphi(x)} h^{\frac{7+5cb}{2+2cb}} h'^2}, \tag{4.33b}$$

$$\sigma = \frac{\varphi(x) h^{\frac{2cb}{1+cb}} h'^{\frac{cb}{1+cb}} \left((1+3cb)h'^2 - 2hh''\right) - c^2 b^2 (1+cb) h h'^{\frac{2+cb}{1+cb}}}{6cb(1+cb)^2 \sqrt{\varphi(x)} h^{\frac{7+5cb}{2+2cb}} h'^2}, \tag{4.33c}$$

respectively, where we have set $\varphi(x) \equiv (1+cb)^2 d - c^2 b^2 \int_1^x h(z)^{-\frac{2cb}{1+cb}} h'(z)^{\frac{1-cb}{1+cb}}\, dz$. It is clear that these quantities are nonzero in general.

The matter variables (4.6) become

$$\mu = \frac{2\left(\left(1-b^3c^3\right)\varphi(x)h^{\frac{bc-1}{1+bc}}\ \left(h'^2+hh''\right)+b^2c^2(1+bc)(1+bc(1+bc))h'^{\frac{2}{1+bc}}\right)}{b^2c^2(1+bc)^3h^2h'^{\frac{2}{1+bc}}}, \tag{4.34a}$$

$$p_{\|} = q,$$

$$= \frac{2\left((bc-1)\varphi(x)h^{\frac{bc-1}{1+bc}}\ \left(h'^2+hh''\right)-b^2c^2(1+bc)h'^{\frac{2}{1+bc}}\right)}{bc(1+bc)^3h^2h'^{\frac{2}{1+bc}}}, \tag{4.34b}$$

$$\begin{aligned} p_{\perp} = {} & \frac{(bc-1)}{2b^2c^2(1+bc)^2\varphi(x)h^{\frac{3bc+5}{1+bc}}h'^{\frac{2bc+4}{1+bc}}}\left[b^4c^4(1+bc)h^{\frac{4}{1+bc}}h'^{\frac{4}{1+bc}}\right. \\ & -b^2c^2\varphi(x)h^{\frac{bc+3}{1+bc}}h'^{\frac{2}{1+bc}}\left(2bch'^2+(bc-1)hh''\right)+\varphi(x)^2h^2\left((bc-1)h'^4\right. \\ & \left.\left.+(3+bc)hh'^2h''-2h^2h''^2+2h^2h'h^{(3)}\right)\right], \end{aligned} \tag{4.34c}$$

for the potentials (4.24). For nonzero tangential pressure $p_{\perp}$ we must have $bc \neq 1$. If we set $bc = 1$ then $p_{\perp} = 0$ and the heat flux becomes negative.

We observe from (4.34) that the relationship

$$p_{\|}(\mu) = \lambda\mu, \quad \lambda = -\frac{cb}{c^2b^2+cb+1}, \tag{4.35}$$

is satisfied. Hence the generalized Euclidean star model (4.24) always satisfies a linear barotropic equation of state. This is a result independent of the analytic form of the arbitrary function $h(x)$. The Lie theory of differential equations has produced a family of exact solutions for general relativistic stellar models which is characterized by an equation of state. If we use the forms (4.31) of the Govinder and Govender (2012) particular solution then it is easy to show that

$$p_{\|}(\mu) = \lambda\mu, \quad \lambda = \frac{\tilde{k}^2 n^3 - 2n + 2}{\tilde{k}^2 n^2 (2 - 3n)}, \tag{4.36}$$

so that (4.35) is satisfied. Consequently their solution is a particular case of our more general case. The thermodynamic properties and other physical features of (4.36) point to a physical reasonable model. In particular the causal temperature is higher than that of the noncausal temperature in the core of the star.

4.5 Energy conditions

It is necessary to choose a particular form of the metric to study further physical features. Consequently we set $b = 1$, $e = 0$, $k^n = \tilde{k}$, $d = \frac{c^2 \tilde{k}^{\frac{1-3c}{1+c}} n^{\frac{1-c}{1+c}}}{(1+c)(3cn-2c-n)}$ and $c = -3$ in (4.28). This yields the kinematical quantities

$$\dot{u}^a = \left(0, \frac{9(n-1)\,\alpha(r)}{4\tilde{k}^2 n(5n-3)x^{2n-1}}, 0, 0\right), \tag{4.37a}$$

$$\Theta = \frac{(1-3n)}{2\tilde{k}\sqrt{n(5n-3)}x^n}, \tag{4.37b}$$

$$\sigma = -\frac{1}{6\tilde{k}\sqrt{n(5n-3)}x^n}. \tag{4.37c}$$

The kinematical quantities are well defined if $n > 3/5$.

The dynamical quantities have the form

$$\mu = \frac{7(n-1)}{2\tilde{k}^2(5n-3)x^{2n}}, \tag{4.38a}$$

$$p_{\|} = q,$$

$$= \frac{3(n-1)}{\tilde{k}^2(5n-3)x^{2n}}, \tag{4.38b}$$

$$p_{\perp} = \frac{2(n-1)^2}{\tilde{k}^2 n(5n-3)x^{2n}}, \tag{4.38c}$$

in our example. From (4.38) we note that $\mu > 0$, $p_{\|} > 0$, $p_{\perp} > 0$ for all $n > 1$.

The energy conditions for a matter distribution with isotropic pressures in the presence of heat flux were defined by Kolassis *et al.* (1988). This was extended by Chan (2003) for anisotropic pressures with heat flow. We follow the approach of Chan (2003) when evaluating the energy conditions. For our example we evaluate the quantities

$$|\mu + p_{\|}| - 2|q| = \frac{2(n-1)}{\tilde{k}^2(5n-3)x^{2n}}, \tag{4.39}$$

$$\mu - p_{\|} + 2p_{\perp} + \sqrt{(\mu + p_{\|})^2 - 4q^2} = \frac{2(n-1)(5n-2)}{\tilde{k}^2 n(5n-3)x^{2n}}. \tag{4.40}$$

We observe that these quantities are nonnegative if $n \geq 1$. In addition we have the following:

(i) weak energy conditions:

$$\begin{aligned} E_{\text{wec}} &= \mu - p_{\parallel} + \sqrt{(\mu + p_{\parallel})^2 - 4q^2} \\ &= \frac{6(n-1)}{\tilde{k}^2(5n-3)x^{2n}}, \end{aligned} \tag{4.41}$$

(ii) dominant energy conditions:

$$\begin{aligned} E_{\text{dec}}^{(1)} &= \mu - p_{\parallel} \\ &= \frac{2(n-1)}{\tilde{k}^2(5n-3)x^{2n}}, \end{aligned} \tag{4.42}$$

$$\begin{aligned} E_{\text{dec}}^{(2)} &= \mu - p_{\parallel} - 2p_{\perp} + \sqrt{(\mu + p_{\parallel})^2 - 4q^2} \\ &= \frac{2(n-1)(2+n)}{\tilde{k}^2 n(5n-3)x^{2n}}, \end{aligned} \tag{4.43}$$

(iii) strong energy conditions:

$$\begin{aligned} E_{\text{sec}} &= 2p_{\perp} + \sqrt{(\mu + p_{\parallel})^2 - 4q^2} \\ &= \frac{4(n-1)(2n-1)}{\tilde{k}^2 n(5n-3)x^{2n}}. \end{aligned} \tag{4.44}$$

We observe that $E_{\text{wec}} \geq 0$, $E_{\text{dec}}^{(1)} \geq 0$, $E_{\text{dec}}^{(2)} \geq 0$, and $E_{\text{sec}} \geq 0$ when $n \geq 1$. Hence the weak, dominant and strong energy conditions are satisfied in this example. This indicates that the matter distribution in generalized Euclidean star is physically reasonable. Note that Govinder and Govender (2012) also showed that their Euclidean star model (4.31), which is a special case of our solution (4.24), satisfies these conditions when $\tilde{k} = -2/9, n = 3$ and $x = -(c_1 r^2 + c_2 t^2)$ where c_1 and c_2 are positive constants.

4.6 Other solutions

In the section we consider the Lie symmetries (4.14) and the various cases for which the boundary condition (4.9) can be integrated. The case of generalized Euclidean stars was considered in §4.3. The other integrable cases are given below.

4.6.1 Static solution: Generator G_1

Using the generator

$$G_1 = A\beta'(t)\frac{\partial}{\partial A} - \beta(t)\frac{\partial}{\partial t}, \tag{4.45}$$

we determine the surface conditions

$$\frac{dt}{-\beta(t)} = \frac{dr}{0} = \frac{dA}{A\beta'(t)} = \frac{dB}{0} = \frac{dY}{0}. \tag{4.46}$$

The invariants are given by r and

$$A = \frac{f(r)}{\beta(t)}, \tag{4.47a}$$

$$B = g(r), \tag{4.47b}$$

$$Y = h(r). \tag{4.47c}$$

Note that for the solution corresponding to (4.47) the gravitational potentials become static and the star is not radiating.

4.6.2 Shear-free solution: Generator $aG_1 + G_3$

Using the generator

$$aG_1 + G_3 = (a\beta'(t) + 1)A\frac{\partial}{\partial A} + B\frac{\partial}{\partial B} + Y\frac{\partial}{\partial Y} - a\beta(t)\frac{\partial}{\partial t}, \tag{4.48}$$

we determine the surface conditions

$$\frac{dt}{-a\beta(t)} = \frac{dr}{0} = \frac{dA}{(1 + a\beta'(t))A} = \frac{dB}{B} = \frac{dY}{Y}. \tag{4.49}$$

The invariants are given by r and

$$A = \frac{f(r)}{\beta(t)\exp\left(\int \frac{dt}{a\beta(t)}\right)}, \tag{4.50a}$$

$$B = \frac{g(r)}{\exp\left(\int \frac{dt}{a\beta(t)}\right)}, \tag{4.50b}$$

$$Y = \frac{h(r)}{\exp\left(\int \frac{dt}{a\beta(t)}\right)}. \tag{4.50c}$$

In this case the shear vanishes.

4.6.3 Generator $G_2 - bG_1$: Linear equation

If we set

$$\frac{b^2 f^3 - 2hf'h' + f\left(h'^2 + 2hh''\right)}{2bf^2hh'} = 0, \tag{4.51}$$

in (4.20) then we integrate to find

$$f(x) = \frac{\sqrt{h}h'}{\sqrt{m - b^2 h}}, \tag{4.52}$$

where m is a constant. Substituting (4.52) into (4.20) we have

$$g' + \left(\frac{mh'}{2mh - 2b^2h^2}\right) g - \left(\frac{b\left(\left(b^2h - 2m\right)h'^2 + 2h\left(b^2h - m\right)h''\right)}{2\sqrt{h}\left(m - b^2h\right)^{3/2}}\right) = 0, \tag{4.53}$$

which is now linear in g. Equation (4.53) can be integrated to give

$$g(x)=\sqrt{\frac{m-b^2h}{h}}\left(\int_1^x \frac{b\left[\left(b^2h(z)-2m\right)h'(z)^2+2h(z)\left(b^2h(z)-m\right)h''(z)\right]}{2\left(m-b^2h(z)\right)^2}\,dz+n\right), \tag{4.54}$$

where n is arbitrary constant.

The potential functions can be written as

$$A = \frac{h'(x)}{\beta(t)}\sqrt{\frac{h(x)}{m-b^2h(x)}}, \tag{4.55a}$$

$$\begin{aligned} B = {} & \frac{1}{\alpha(r)}\sqrt{\frac{m-b^2h(x)}{h(x)}} \\ & \times\left(\int_1^x \frac{b\left[\left(b^2h(z)-2m\right)h'(z)^2+2h(z)\left(b^2h(z)-m\right)h''(z)\right]}{2\left(m-b^2h(z)\right)^2}\,dz+n\right), \end{aligned} \tag{4.55b}$$

$$Y = h(x), \tag{4.55c}$$

which is another particular solution to the master equation (4.9). Note that h is an arbitrary function of $x=\int\frac{dt}{b\beta(t)}+\int\frac{dr}{\alpha(r)}$ in this solution.

4.6.4 Generator G_2-bG_1: Bernoulli equation

If we set

$$h(x)=\frac{c}{f^2}, \tag{4.56}$$

where c is arbitrary, (4.20) reduces to the Bernoulli equation

$$g' + \left(\frac{4f'}{f} - \frac{f''}{f'}\right) g + \left(\frac{b^2 f^8 + 20c^2 f'^2 - 4c^2 f f''}{4bc^2 f^2 f'}\right) g^2 = 0. \tag{4.57}$$

Equation (4.57) can be integrated in general to give

$$g(x) = \frac{f'}{f^4 \left(\int_1^x \frac{b^2 f(z)^8 + 20c^2 f'(z)^2 - 4c^2 f(z) f''(z)}{4bc^2 f(z)^6}\, dz + w\right)}, \tag{4.58}$$

where w is an arbitrary constant.

Then the potential functions have the form

$$A = \frac{f(x)}{\beta(t)}, \tag{4.59a}$$

$$B = \frac{f'(x)}{\alpha(r) f^4(x) \left(\int_1^x \frac{b^2 f(z)^8 + 20c^2 f'(z)^2 - 4c^2 f(z) f''(z)}{4bc^2 f(z)^6}\, dz + w\right)}, \tag{4.59b}$$

$$Y = \frac{c}{f^2(x)}, \tag{4.59c}$$

which is a new solution to the master equation (4.9). Observe that f is arbitrary function of $x = \int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}$ for this solution.

4.6.5 Generator $G_2 - bG_1$: Riccati equation

While (4.20) is a Riccati equation, we observe that setting

$$f(x) = ch', \tag{4.60}$$

where c is constant, simplifies it considerably to

$$g' - \left(\frac{1+b^2c^2}{2bch}\right) g^2 + \left(\frac{bch'^2}{2h} + bch''\right) = 0. \tag{4.61}$$

We still cannot solve (4.61) in general. However it can be integrated for some particular functional forms of h. If we set h to be a simple quadratic function

$$h(x) = x^2, \tag{4.62}$$

then (4.61) gives

$$g(x) = \left(\frac{bcx}{1+b^2c^2}\right)\left[1 - \sqrt{9+8b^2c^2}\left(\frac{kx^{\sqrt{9+8b^2c^2}} - 1}{kx^{\sqrt{9+8b^2c^2}} + 1}\right)\right], \tag{4.63}$$

where k is a constant. Then the potential functions have the form

$$A = \frac{2c}{\beta(t)}\left(\int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}\right), \tag{4.64a}$$

$$\begin{aligned} B = {} & \left(\frac{bc\left(\int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}\right)}{(1+b^2c^2)\,\alpha(r)}\right) \\ & \times \left[1 - \sqrt{9+8b^2c^2}\left(\frac{k\left(\int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}\right)^{\sqrt{9+8b^2c^2}} - 1}{k\left(\int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}\right)^{\sqrt{9+8b^2c^2}} + 1}\right)\right], \end{aligned} \tag{4.64b}$$

$$Y = \left(\int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}\right)^2. \tag{4.64c}$$

This is another new solution for the master equation (4.9).

The solution (4.64) does not obey an equation of state in general. However when we set the constant $k = 0$, then (4.64) becomes

$$A = \frac{2c}{\beta(t)}\left(\int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}\right), \tag{4.65a}$$

$$B = \frac{bc\left(1+\sqrt{9+8b^2c^2}\right)}{\left(1+b^2c^2\right)\alpha(r)}\left(\int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}\right), \tag{4.65b}$$

$$Y = \left(\int \frac{dt}{b\beta(t)} + \int \frac{dr}{\alpha(r)}\right)^2. \tag{4.65c}$$

The special case (4.65) obeys the barotropic equation of state

$$p_{\|}(\mu) = \lambda\mu, \tag{4.66}$$

where $\lambda = \frac{b^2c^2\left(3-\sqrt{9+8b^2c^2}\right)-3\sqrt{9+8b^2c^2}+5}{2(7+5b^2c^2+b^4c^4)}$.

4.7 Discussion

We have shown that the Lie symmetry approach to differential equations is a useful tool that can assist in the search of exact solutions to the junction condition relating the radial pressure with the heat flux at the boundary of relativistic radiating star. Using the Lie approach we generated several new exact solutions to the boundary condition equation. Our solutions regain previously known solutions for the Euclidean star as a special case. The model of Govinder and Govender (2012) arises as a special case;

our analysis shows that their results are part of a more general class invariant under the action of Lie symmetry generators. A noteworthy feature of our new generalized Euclidean star models is that they obey a linear barotropic equation of state in general. Most of the radiating stellar models found in the past do not share this feature. The analysis of the weak, dominant and strong energy conditions in the example shows that the matter distribution is physically acceptable.

Chapter 5

Conclusion

The objective of this thesis was to generate exact solutions to the boundary condition of relativistic radiating stars when they are conformally flat, geodesic with shear, and accelerating, expanding and shearing. The boundary condition is a highly nonlinear partial differential equation in the metric functions. Integrating this equation is very difficult using traditional methods. We applied the extended Lie theory of differential equations. The Lie symmetries admitted by the governing partial differential equations in three cases were obtained and the optimal systems were generated. The symmetries in the optimal system were used to transform the partial differential equation into respective system of ordinary differential equations that have to be further analyzed. By solving the ordinary differential equations and transforming back to the original variables we generated exact solutions to the boundary condition. Our classes of exact solutions include new and previously obtained models. It is very interesting to observe that some previously obtained solutions are just special cases of our more general solutions which were obtained using Lie symmetry transformations. We also considered the physical features of our solutions. Some of our solutions obey a linear barotropic equation of state.

We now provide an overview for the main results in this thesis:

- In chapter two, the junction condition relating the radial with the heat flux at the boundary of a conformally flat radiating star was studied in detail. We found that this equation admits three Lie point symmetries and obtained an optimal system which was used to reduce the governing highly nonlinear partial differential equation to ordinary differential equations. We also used a symmetry combination which was not in the optimal system. By solving the reduced ordinary differential equations and transforming to the original variables we obtained new exact solutions for the master equation. We believe that the solutions obtained in this chapter are not contained in the literature. Two classes of models are of particular interest. The first class depends on the self similar variable t/r. In the second class the metric function is separable in the spacetime variables t and r. Two particular metrics could be identified in the second class. The first metric is regular at the centre but the heat flow is inwardly directed. The second metric is not regular at the centre but the heat flow is outwardly directed. For this example the matter variables are regular in a spacetime region at least close to the centre. The heat flux, acceleration and expansion are decreasing functions for large time.

- The objective of chapter three was to generate exact solutions for the junction condition that relates the radial pressure with the heat flux at the boundary of a shearing and expanding star when the fluid particles travel in geodesic motion. This was achieved by applying the Lie theory for differential equation. We presented the Lie symmetries for the junction condition. The optimal system for those symmetries were also obtained. With help of the symmetries in the optimal system, we transformed the master equation into ordinary differential equations. By solving the ordinary differential equations and transforming into the original variables, we generate several classes of exact solutions for the master equation. The solutions obtained in this chapter include previously obtained results as a special case. Interesting, some of our solutions have a form of a traveling wave and others can be written in terms of a self-similar variable.

- In chapter four, we studied the boundary condition for an accelerating, expanding and shearing radiating star which is a highly nonlinear partial differential equation. As the boundary condition depends on three dependent functions it is difficult to integrate using traditional methods. As in the previous chapters, we determine the Lie point symmetries admitted by the governing equation and reduced the master equation into ordinary differential equations that need to be further analyzed. These equations are of the Riccati type. We transform the master Riccati equation into a Bernoulli, linear and simpler Riccati equations that needed to be solved. By transforming into the original variables we present a new solution for the boundary condition by making assumption relating two unknown functions. The solution obtained in this chapter contains the previously obtained solution of an Euclidean star as a special case. Other forms are also solved to give a static solution and shear-free solution for the boundary condition when transformed to the original variables. We also demonstrate that several other classes of models are possible. We also show that our generalized Euclidean models obey a linear barotropic equation of state in general. We demonstrate that a particular example of our solution satisfies the weak, strong and dominant conditions. This implies that the matter distribution for the generalized Euclidean stars is physically acceptable.

In this thesis we showed that the Lie symmetry theory of differential equations is a useful method to generate exact solutions to the boundary condition in relativistic astrophysics. Studying the physical features of the solutions obtained via Lie symmetries is very important and this will be the object for future research. Note that we utilised the computer software package Mathematica (Wolfram 2008) for some of the integrations and to verify the correctness of all solutions.

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