Developing first-year mathematics student teachers' understanding of the concepts of the definite and the indefinite integrals and their link through the fundamental theorem of calculus: An action Research Project in Rwanda

By

Faustin Habineza

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DECLARATION

I, Faustin Habineza, declare that

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ABSTRACT

This thesis describes an Action Research project within the researcher's practice as a teacher educator in Rwanda. A teaching style informed by the Theory of Didactical Situations in Mathematics (Artigue, 1994; Brousseau, 1997; 2004; Douady, 1991) and by the Zone of Proximal Development (Gallimore & Tharp, 1990; Meira & Lerman, 2001; Rowlands, 2003; Vygotsky, 1978) was conducted with first-year mathematics student teachers in Rwanda. The aim of the teaching model was to develop the student teachers' understanding of the concepts of the definite and the indefinite integrals and their link through the fundamental theorem of calculus.

The findings of the analysis answer the research questions, on the one hand, of what concept images (Tall & Vinner, 1981; Vinner & Dreyfus, 1989) of the underlying concepts of integrals student teachers exhibit, and how the student teachers' concept images evolved during the teaching. On the other hand, the findings answer the research questions of what didactical situations are likely to further student teachers' understanding of the definite and the indefinite integrals and their link through the fundamental theorem of calculus; and finally they answer the question of what learning activities student teachers engage in when dealing with integrals and under what circumstances understanding is furthered.

An analysis of student teachers' responses expressed during semi-structured interviews organised at three different points of time - before, during, and after the teaching - shows that the student teachers' evoked concept images evolved significantly from pseudo-objects of the definite and the indefinite integrals to include almost all the underlying concept layers of the definite integral, namely, the partition, the product, the sum, and the limit of a sum, especially in the symbolical representation. However, only a limited evolution of the student teachers' understanding of the fundamental theorem of calculus was demonstrated after completion of the teaching.

With regard to the teaching methods, after analysis of the video recordings of the lessons, I identified nine main didactical episodes which occurred during the teaching. Interactions during these episodes contributed to the development of the student teachers' understanding of the concepts of the definite and the indefinite integrals and their link through the fundamental theorem of calculus. During these interactions, the student teachers were engaged in various cognitive processes which were purposefully framed by functions of communication, mainly the referential function, the expressive function, and the conative function. In these forms of communication, the conative function in which I asked questions and instructed the students to participate in interaction was predominant. The student teachers also reacted by using mainly the expressive and the referential functions to indicate what knowledge they were producing. In these exchanges between the teacher and the student teachers and among the student teachers themselves, two didactical episodes in which two student teachers overtly expressed their understanding have been observed. The analysis of these didactical episodes shows that the first student teacher's understanding has been triggered by a question that I addressed to the student after a long trial and error of searching for a mistake, whereas the second student's understanding was activated by an indicative answer given by another student to the question of the student who expressed the understanding. In the former case, the student exhibited what he had understood while in the latter case the student did not. This suggests that during interactions between a teacher and a student, asking questions further the student' s understanding more than providing him or her with the information to be learnt.

Finally, during this study, I gained the awareness that the teacher in a mathematics classroom has to have various decisional, organisational and managerial skills and adapt them to the circumstances that emerge during classroom activities and according to the evolution of the knowledge being learned. Also, the study showed me that in most of the time the student teachers were at the center of the activities which I organised in the classroom. Therefore, the teaching methods that I used during my teaching can assist in the process of changing from a teacher-centred style of teaching towards a student-centred style.

This study contributed to the field of mathematics education by providing a mathematical framework which can be used by other researchers to analyse students' understanding of integrals. This study also contributed in providing a model of teaching integrals and of researching a mathematics (integrals) classroom which indicates episodes in which understanding may occur. This study finally contributed to my professional development as a teacher educator and a researcher. I practiced the theory of didactical situation in mathematics. I experienced the implementation of some of its concepts such as the devolution, the adidactical situation, the institutionalization, and the didactical contract and how this can be broken by students (the case of Edmond). In this case of Edmond, I realised that my listening to students needs to be improved. As a researcher, I learnt a lot about theoretical frameworks, paradigms of study and analysis and interpretation of data. The theory of didactical situations in mathematics, the action research cyclical spiral, and the revised Bloom's Taxonomy will remain at my hand reach during my mathematics teacher educator career. However, there is still a need to improve in the analysis of data especially from the students' standpoint; that is, the analysis of the learning aspect needs to be more practiced and improved.

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ABBREVIATIONS

AMESA	Association for Mathematics Education of South Africa
Antider	Antiderivative
FTC	Fundamental theorem of calculus
FTC-VDI	Fundamental theorem of calculus – version of the derivative of the integral function
FTC-VID	Fundamental theorem of calculus – version of the integral of the derivative of a function
FTC-VEA	Fundamental theorem of calculus – version of the evaluation of area (the same as FTC-VID)
KIE	Kigali Institute of Education, Rwanda
MINEDUC	Ministry of Education, Science, Technology, and Scientific Research, Rwanda
Nb	Number
NCTM	National Council of Teachers of Mathematics
S	Student
SAARMSTE	Southern African Association for Research in Mathematics, Science, and Technology
SS	Some Students
SSL	All the students in a large group (the teacher not involved)
SST	The group of all the students and the teacher
SSW	All the students in the class
T	Teacher
tt	Turns
UKZN	University of KwaZulu-Natal, Republic of South Africa

CHAPTER 1: INTRODUCTION

1.1. Introduction

The idea of teaching and learning integrals using the theory of didactical situations in mathematics emerged during an in-service teacher training session that was organised by the Rwandan Ministry of Education, Science, Technology, and Scientific Research in 2003. The aim of the session was to train primary teachers to conceive problem-situations that could engage pupils in learning mathematics, science, and technology in meaningful ways. I was invited to the session as a resource person for the component of mathematics. During interactions, a teacher complained about not understanding integrals that he learnt during his pre-service teacher education. It must be noted that in Rwanda, integrals and integration are taught in grade 12 and thus are a necessary component of secondary teacher education.

As a lecturer of calculus in the Kigali Institute of Education which is in charge of training pre-service teacher educators, I felt that there was a need to intervene in pre-service teacher education in order to improve the teaching and learning of integrals. As the teachers seemed to like the use of problem-situations in the learning of mathematics, I felt that the same idea of problem-situations could be developed for the learning of integrals, at higher level of secondary schools and even at the university level.

Fortunately the following year of 2004, I was granted a bursary to pursue doctoral studies in the Republic of South Africa. I took this opportunity to undertake research on how I could improve the teaching and learning of integrals in meaningful ways in Rwanda.

I left my country Rwanda for the Republic of South Africa with the research problem of how to conceive problem-situations that could help students to improve their understanding of the concept of integral. From the outset, my supervisor advised me to read the Theory of Didactical Situations (Artigue, 1994; Brousseau, 1997, 2004; Douady, 1991; Laborde, 1994). Other preliminary readings during this period were about the concept of an individual's concept image (Tall & Vinner, 1981) and the analysis of students' understanding of some of the concepts of calculus using the notion of concept image (Bezuidenhout, 2001; Bezuidenhout, Human, & Olivier, 1998; Vinner & Dreyfus, 1989; Zandieh, 2000) and about the dual nature of mathematical conceptions (Sfard, 1991; Sfard & Linchevski, 1994). Finally, readings on research methodologies (Cohen, Manion, & Morrison, 2000; Guba & Lincoln, 1994) and discussions in doctoral seminars confirmed that my research was an action research project (Elliot, 1991; Kemmis, 1988; McNiff, 2000) which I had to implement in the classroom. I finalised the preparation of my research proposal in November 2004. In the following section, I present the research questions of my study.

1.2. Research questions

The readings above, further discussed in the Literature Review Chapter, led me to refine my research problem and to transform it into the following research questions.

Question 1: What basic ideas underlie the concepts of the definite and the indefinite integrals and their link through the fundamental theorem of calculus?

Question 2: What concept images of integrals do mathematics student teachers exhibit?

Question 3: On the basis of the conceptual analysis and the analysis of the student teachers' concept images, what didactical situations are likely to further mathematics student teachers' understanding of integrals?

Question 4: What learning activities do mathematics student teachers engage in when working on integrals and under what circumstances does understanding occur?

Question 5: How do student teachers' concept images evolve during the didactically engineered course on integrals?

The first question was about the underlying concepts of integrals and the fundamental theorem of calculus. The second question was about the concept images that the student teachers could evoke at any given time of interviews or during the communication in the classroom. The third question was about the teaching and learning situations or the didactical episodes during the teaching period. The fourth question was about the learning activities as a combination of cognitive processes and the mathematical objects related to integrals and to the fundamental theorem of calculus. The second part of the fourth question concerned the moments when the student teachers could express their understanding (Dreyfus, 1991) of a given concept. These moments are often popularly referred to as the 'aha moments'.

The fifth question of how the student teachers' concept images developed occasioned two interpretations. The first interpretation was to observe changes of the concept images over closer sequences of time. In this perspective, one way to answer the question was to use a detailed analysis of sentence by sentence of the transcript and to adopt for example the framework of the interactive flowcharts (Kieran, 2001; Ryve, 2006; Sfard, 2001) with a focus on the content of the text. With this approach, it would be difficult to get coherent conclusions of the whole teaching. The second interpretation was to observe the speech-forms which were used by the teacher and the student teachers during the communication in the classroom. This latter interpretation led to the analysis of speech events and verbal interactions (Gumperz, 1972; Hymes, 1972). After some time of trying the use of the former perspective, I decided to adopt this second perspective and to focus mainly on the observable changes in the student teachers' understanding.

Concerning the evolution of the concept images I decided to consider changes on a larger scale, namely, at the times of the interviews, and at the end of the third and of the fifteenth lessons. This latter perspective conforms to the holistic analysis which is more efficient than the atomistic approach to analysis of thinking and speech (Moll, 1990; Vygotsky, 1987). In the holistic analysis, my findings are more unified and more consistent with regard to my research questions. In the next section, I outline the research process that I followed to address my research questions.

1.3. Outline of the research process

To address the research questions mentioned above, I designed my research as an action research project that I implemented in the classroom while teaching integrals to first-year mathematics student teachers.

The actual research activities started in 2005. During the first semester of 2005, I conducted the first round of the interviews to collect the student teachers' prior concept images (Tall & Vinner, 1981). The questions of all the interviews are in appendix A. After analysis of the student teachers' concept images from the first round of the interviews, I prepared tasks that I would devolve to student teachers and a pedagogical sequence that I would follow to experiment the evolution of their concept images and consequently to evaluate the impact of my teaching approach. The pedagogical sequence that I prepared and the notes of the course that I handed out to the student teachers after the tenth lesson are included in appendix G.

The actual teaching on integrals was done in the second semester of the year 2005. Before I started the teaching, I held a second round of interviews to confirm the prior concept images of the first-year student teachers who were registered to the course of differential and integral calculus. During the teaching, an assistant would video record the fifteen lessons that constituted the course component involving integration, and after each lesson I collected the student teachers' worksheets. At the end of the course, I held a third round of interviews with student teachers to collect their posterior concept images. The teaching and the interviews were conducted in French which is the second language of the student teachers after Kinyarwanda (the language of Rwanda).

After the activities related to the teaching, I started the activities related to the analysis of the collected data. In the first semester of the year 2006, I transcribed and translated the interviews from French to English, and I started the analysis in English. In the second semester of the year 2006, I started the transcription and the translation of the video recordings of the communication in the classroom while refining the analysis of the interviews.

In the year 2007, I started the analysis of the communication in the classroom. In the first semester of the year 2007, I used an atomistic approach to analysis which I alluded to in the previous section but the analysis was unsuccessful. In the second semester of the same year 2007, I changed the perspective from the atomistic to the holistic analysis. This latter approach proved to be very powerful in providing coherent findings and improved conclusions.

During the year 2008, I finalised the writing of the thesis. The findings reported in this thesis result from a sustained implementation of the cognitive processes of analysing and structuring, evaluating and judging, quantifying, and comparing. These processes were applied simultaneously in the treatment of the data collected during the communication in the classroom and during the interviews. In the following section I outline the content of the chapters of the thesis.

1.4. Outline of chapters of the thesis

The thesis is composed of eight chapters. Briefly, in the first four chapters I present the generalities of the research. In the next three chapters I present the data and their analysis. In the last chapter, I give some conclusions and recommendations about the evolution of the student teachers' concept images of the definite and the indefinite integrals and of the fundamental theorem of calculus. Moreover, I present the model of teaching that I used during my teaching of integrals and the fundamental theorem of calculus; and finally, I end the last chapter by giving some recommendations for improving the teaching and learning of integrals in Rwanda.

In more details, in the first chapter I provide an overview of the thesis by presenting the motivation of the study, the research questions, and the outlines of the research process and of the content of the chapters. In the second chapter, I present a review of the literature about the teaching and the learning of integrals and of the fundamental theorem of calculus. In this chapter, I also present some literature about the concept images, the dual nature of mathematical conceptions, and about interactions in teaching and learning mathematics. In the third chapter I describe the theoretical framework of the thesis, and in the fourth chapter I present the methodology of the research and the research methods that I used during the research.

For the pool of chapters concerning the data, in the fifth chapter I present and analyse the data concerning the student teachers' concept images of integrals evoked during the first, the second, and the third round of the interviews. Moreover, I present and analyse the student teachers' concept images of the definite integral evoked after the third lesson. To end this chapter, I summarise the evolution of the student teachers' concept images of the definite integral and I give the evolution of the student teachers' concept images of the indefinite integral evoked during the three rounds of the interviews. In the sixth chapter, I present the data related to the teaching of integrals and the fundamental theorem of calculus and I analyse the data related to the understanding of the concept of the definite integral. In the seventh chapter, I present and analyse the student teachers' understanding of the fundamental theorem of calculus – though only the version of evaluation of area. In the eighth and concluding chapter, I offer some conclusions and recommendations about the evolution of the student teachers' evoked concept images of integrals. Moreover, I present an explanation of what I and the student teachers did during the teaching and learning in the classroom. Furthermore, I engage the relevance and limitations of my study and some critical reflections on frameworks, methodology and the theory of didactical situations in mathematics. To end this concluding chapter, I propose some recommendations that can contribute to improve the teaching and learning of integrals and the fundamental theorem of calculus in Rwanda.

CHAPTER 2: LITERATURE REVIEW

2.1. Introduction

During my literature reading I was more interested in studies within the domain of calculus education, especially the topic of integration. In this domain of calculus, many researchers have focused on teaching and learning of calculus in general (Artigue, 1991, 1996; Tall, 1992, 1996); other researchers in calculus have conducted studies on the teaching and learning of integration (Berger, 2006; Burn, 1999; Koepf & Ben-Israel, 1994; Orton, 1983b); and one study dealt with the teaching and learning of the fundamental theorem of calculus (Thompson, 1994). All these studies focused on student's difficulties in learning the mathematical concepts of the domain of calculus in order to propose some ways that can be used to help students to understand these concepts.

However, there are other aspects of mathematics education research that are worthwhile to consider in the teaching and learning of integration. Some relevant aspects concern the role of language and interactions in a mathematics classroom. In these perspectives, researches have revealed the crucial role of language in learning of mathematics (Austin & Howson, 1979; Douek, 2005; Meaney, 2005; Pimm, 1994; Rotman, 1988; Zevenbergen, 2000). Other researches have revealed the importance of interactions in the learning of mathematics in the classroom (Bartolini Bussi, 1994, 2005; Bauersfeld, 1992, 1994; Christiansen, 1996, 1997; Cobb, 1995; Cobb, Wood, Yackel, & McNeal, 1992; Cobb, Yackel, & Wood, 1992; Holton & Thomas, 2001; Lerman, 1996, 2000, 2001a; Meira & Lerman, 2001; Simon, 1995; Steffe & Thompson, 2000a; Voigt, 1994; Yackel & Cobb, 1996). These studies demonstrated various forms of interactions and of language that contribute to learning mathematics in the classroom. All these forms of interactions and of language contribute to students' understanding of mathematics and to alleviation of the students' difficulties in learning mathematics. Another important aspect related to maintaining interaction among students and teachers concerns the cognitive demands of the instructional tasks used in teachers' classrooms (Boston & Smith, 2009).

This chapter presents worthwhile literature that I read during this study. The literature covers national and international research in the domain of Mathematics Education related to the teaching and learning of calculus and of mathematics in general. After this introductory section, I present the literature about research in teaching and learning in the domain of Calculus in general. In the third section, I engage literature about teaching and learning of the definite integral and the indefinite integral, and in the fourth section I present literature about the teaching and learning of the fundamental theorem of calculus. Emphasis is put on research on teaching and learning of the definite and the indefinite integrals and of the fundamental theorem of calculus because they are the object of my study. In the fifth section, I present a review of the concept of the concept image as described by Tall and Vinner (1981) and in the sixth section I present the operational and structural conceptions as described by Sfard (1991). In the seventh section, I present a view of mathematics from linguistic perspective as described by Bloomfield (1935). And finally in the eighth section, I review some studies about verbal interactions in the classroom.

2.2. Research on teaching and learning in calculus

In the research on teaching and learning of calculus there are at least four categories of studies that deal with specific concepts of calculus. The first category concerns studies about the concept of a limit and the notion of infinity (Bezuidenhout, 2001; Cornu, 1991; Dubinsky, Weller, Mcdonald, & Brown, 2005a, 2005b; Sierpinska, 1987). The second category concerns studies about the concept of the derivative (Orton, 1983a; White & Mitchelmore, 1996; Zandieh, 2000). The third category regards studies about the concept of the integral. This category can be subdivided into three subcategories, namely the studies regarding the concept of the definite integral (Burn, 1999; Burt, Magnes, Schwarz, & Hartke, 2008; Orton, 1983b; Rösken & Rolka (2007); Schneider, 1991; Tarvainen, 2008), the concept of the indefinite integral (Koepf & Ben-Israel, 1994; Metaxas, 2007) , and the fundamental theorem of calculus (Thompson, 1994). The fourth category comprises studies that deal with the whole range of the key concepts of calculus (Artigue, 1991, 1996; Ferrini-Mundy & Gaudard, 1992; Juter, 2009; Tall, 1992, 1996).

However, the boundaries of these categories are not firmly fixed since a study of a given concept can touch on any other concept for the sake of the coherence among the concepts of calculus. Finally, as the concept of a function underlies almost all the concepts of calculus, it is important to mention here some of the various studies that deal specifically with the concept of a function (Aspinwal, Shaw, & Presmeg, 1997; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Vinner, 1983; Vinner & Dreyfus, 1989).

Most of the abovementioned studies present difficulties that students have with respect to the examined concepts and propose a variety of ways to overcome them in order to improve students' understanding of the concept under study. I give some examples of these difficulties from the third and fourth categories. I start with the fourth category and end with the third category as its subcategories are the concern of this thesis.

The examples from the fourth category that I detail here are from studies of Artigue (1991), Tall (1992), Ferrini-Mundy and Gaudard (1992), and Juter (2009). Artigue (1991) discussed students' conceptions with regard to elementary calculus, differentials, and processes of differentiation and integration. She presented difficulties which students face when they are dealing with these topics. Her results show that students exhibit their conceptions predominantly in the algebraic mode of procedure in contrast to geometric and graphic modes; she also found that students do not demonstrate a grasp of the meaning of limits and of approximations. Based on my personal experiences, I agree with her that a premature algebraic algorithmisation deprives students of the means to understand the underlying concepts that lead to the differential and integral processes. As she noted, students seem to concentrate on rules and correct answers instead of exerting activities that direct them to understand the relationships among concepts.

Amongst other difficulties of learning the basics of analysis/calculus that Artigue identified in the above study there are the sophisticated structure of the foundational objects such as sequences and functions, the obstacles caused by the conflicting of everyday meaning of some concept such as limit and infinity, the use of specific techniques such as upper and lower bounds, and the formalisation of definitions that conflict with students' everyday conception. Also falling within the research category addressing several calculus concepts is the paper of Artigue (1996). In this paper, Artigue went further in describing and categorising the above mentioned difficulties related to the teaching and learning of elementary analysis.

With regard to the teaching of integration, Artigue (1991) discussed the use of the approach called scientific debate. She said that this approach occurred in a context where student are encouraged to conjecture and debate ideas in groups, where arguments are proposed and addressed to other students rather than the teacher. In this perspective, Artigue presented the didactical engineering associated with this approach. She gave students the problem of finding the intensity of the force which is exerted between a point mass and thin bar. The researchers had hypothesised that if appropriately managed, the scientific debate of this first problem could help to solve the problem. One method was through visualising the bar as being made up of tiny slices, calculating the force, then refining and passing to the limit through integration. Artigue said that the vast majority of the students suggested a solution by conceiving the mass of the rod concentrated at the centre of gravity, which proves erroneous. Artigue also presented other alternative solutions that the students produced in the course of the experiments. Among other problem-situations that Artigue gave students during her didactical engineering there are situations about averages and problems using time and space variables in two and three dimensions. Artigue said that the effects of scientific debate within the overall instruction revealed improved understanding of integration.

On his side, Tall (1992) presented various types of difficulties which students have in Calculus. The main difficulties concern the notion of limit and the infinite processes. Other difficulties mentioned by Tall are the restricted mental images of functions, translation of real-world problems into Leibniz notation, and selection and use of appropriate representations. As noticed by Tall, these difficulties are mainly caused by cognitive conflicts due to multiple meanings of words such as limits and infinity. To these words, I add the word 'derivative' which may also (in some languages) be interpreted in the sense of parental relationships in which children can be seen as deriving or originating from their parents, and the word ‘integral’ in the sense of being an essential part of a whole thing.

The study of Ferrini-Mundy and Gaudard (1992) analysed the relationship between the secondary school calculus experience and the performance in first year college calculus. They found that students who had studied a full year of secondary school calculus, whether it be an Advanced Placement course or not, were more successful in the course than those who had either no calculus in secondary school, or only a brief introduction. Ferrini-Mundy and Gaudard indicated that the calculus reform that was then underway was to move college calculus teaching in a direction that is more conceptual, application-oriented, and technology intensive. They observed that the new approaches to calculus would encourage the use of small-group work, laboratory approaches, and more student-centred activities. The study of Ferrini-Mundy and Gaudard also raised the question of balancing procedural and conceptual aspects, taking into account the specialisation of the students who were engineers. This question raises the matter of the content and the scope in teaching calculus.

Finally in this category, Juter (2009) used the concept of concept image (see section 2.5, at page 25 in this chapter) to analyse students’ concept images before and after a ten weeks analysis course. The study mainly concerned the students’ conceptions of functions, limits, derivatives, integrals and continuity. Fifteen students aged 19 years or older were involved in the study. Questionnaires and interviews were used to collect data. In her analysis of the data, Juter found that at the beginning of the course, the students’ descriptions of the above mentioned concepts “were in some cases intuitive, some had focus on methods for problem solving and some were erroneous or not written at all” (pp. 80-81). Juter also found that a number of pre-conceptions endured the course. However, she reported that, after the course, mental representations were connected to images which supported the students’ understanding and some of the students’ descriptions included graphs and images resembling pictures used in the course. She also said that “pictures can cause confusion rather than insight” (p. 81). She illustrated this confusion by the fact that some students had the impression that limits are actually the limits determined by the endpoints of the interval (an upper and a lower bound) instead of the conceptions that are meant in the process of limiting. Juter also analysed the links that the students established among the mentioned concepts at stake in the study, namely, the limit, the derivative, the integral, and the continuity. From the responses, she created five categories that she used to analyse the links that the students established among the above concepts. These categories are valid link, meaning true relevant link revealing a core feature of the concept; invalid link and misconception, meaning untrue link due to a misconception of the concept; invalid link and counter perception, meaning untrue statements of the students; irrelevant link and no reason, meaning no actual motivation for the link is provided; and irrelevant link and no substance, meaning peripheral true link without substance relevant for the concept. Regarding the interpretation of these categories, Juter concluded that "a vast number of valid links means that the concept is well understood" (p. 83). Students with several invalid or irrelevant links have potentialities to improve their understanding. As Juter has noted "invalid links require rearranging of the concept image to become valid, whereas irrelevant links may be more easily turned to valid ones through added details and awareness" (p. 83). As stated, Juter, becoming aware of this changing process from irrelevant or invalid links to valid ones in an analysis course. This can help secondary student teachers "gain a deeper understanding of their, and their future students', processes of learning mathematics" (p. 83). It can also contribute to improvement of their process of teaching analysis. Juter's study can be understood as an action research project in a mathematics classroom.

Juter's study is related to my study as it deals, like mine, with the development of students' concept images after an analysis course. The concerned concepts were the function, the limit, the continuity, the derivative and the integral. This latter concept is the main concern of my study. Therefore, it was worth to review Juter's study in order to have ideas about how students link the abovementioned concepts.

In summing up, these general studies about the key concepts of calculus reveal that the students had difficulties in understanding these concepts due to emphasis put on procedural techniques and algorithmisation in algebraic mode in contrast to geometric and graphic modes (Artigue, 1991) and to conflicting meanings (Tall, 1992) of words used to express the concepts. In order to improve the situation, the researchers proposed the introduction of the geometric and the graphic modes in the teaching of the key concepts of calculus. They proposed a move from procedural teaching towards more conceptual teaching (Artigue, 1991; Ferrini-Mundy & Gaudard, 1992) and they also proposed the promotion of group work and the scientific debate approach (Artigue, 1991). Computers and appropriate software can help in the visualisation of the concepts. The graphic and dynamic possibilities offered by computers can give a cognitive base for the concepts of derivative and integral which can lead to later formalisations (Artigue, 1991). Finally, an action research project in a mathematics classroom can help secondary student teachers gain a deeper understanding of their processes of learning analysis (Juter, 2009). In the next section, I now turn to studies specifically related to integration.

2.3. Research on teaching and learning the definite and the indefinite integrals

Within the category of research on integration, some researchers have been interested in researching the teaching and the learning of the definite and the indefinite integrals. Some of these researchers are Orton (1983b), Schneider (1991), Koepf and Ben-Israel (1994), Burn (1999), Berger (2006), Rösken and Rolka (2007), and Tarvainen (2008). In what follows, I summarise their studies.

Orton (1983b) used clinical interviews to investigate the understanding of elementary calculus of 110 students in the age of 16 to 22 years. He analysed responses to tasks on integration and limit using a five-point scale from 0 to 4 to allow statistical analysis. He found that when students were asked whether the sequences could be used to obtain the exact area under a curve, only a few students felt the necessity to use the sequences. Moreover, using Donaldson's (as quoted in Orton, 1983b, p. 4) categorisation of errors to analyse students' understanding of integration, Orton found that students experienced difficulties related to the structural understanding of integration conceived as a limit of a sum (see next chapter). Orton said that many teachers have accepted these difficulties and have reacted to them in the following ways. Some chose a curriculum which avoids the introduction of calculus to non-specialists. Others introduced integration as a rule, that is, as inverse differentiation. And others tried to build up understanding of limits and relevant algebra, hoping that students would grasp what calculus is when they are introduced to it. However, Orton believed that rules without reasons can not be justified, and he suggested a curriculum that would be helpful to a subsequent conceptually oriented introduction of integration. This curriculum included the topic of limits and infinity, sequences and series, areas of irregular shapes by counting squares in order to allow discussion of approaching a limit from above and below, and the idea of approaching the area of a circle through reassembling sectors into approximation to a rectangle which provides opportunity for discussing limits. Orton evoked the use of calculators in numerical integration. Though there are some practical difficulties related to calculators, he said, they can ease informal, investigatory approaches to the area under curve. In his study, Orton found that students also experienced difficulties with elementary algebra. He said that these difficulties might be obscuring the fundamental ideas of calculus. He gave the example that a student can not make sense of the derivation of the idea of integration if the algebra involved in summing rectangular areas causes confusion. In his proposed curriculum, he added that certain formulas for summation of series such as $\sum_{i=1}^{n} i$, $\sum_{i=1}^{n} i^2$ and $\sum_{i=1}^{n} i^3$ are required if the approach to integration will attempt formal proofs even though these formulae add another example of dependence of calculus on algebra. To end his curriculum, Orton advised to introduce calculus by providing various illustrations including both diagrams and graphs. He said that many students interviewed in his study had problems with finding areas when the curve crossing an axis, or more generally in understanding the relationship between a definite integral and areas under the curve.

All these suggestions of Orton express the need for introducing integration from the conceptual perspective rather than the procedural aspect. Therefore, a thorough analysis is necessary to identify the underpinning concepts that can direct students to grasp concept of integral from its deepest roots as a limit of a sum. This analysis is the topic one section in chapter 3.

Another study related to integration is the one done by Schneider (1991). In this study "an epistemological obstacle caused by infinite subdivisions of surfaces and solids", Schneider analysed several mistakes that pupils of 15 to 18 years old made while calculating surfaces and solids. Among the identified misconceptions there was the one that she articulated as follows: *with a gathering of disjoined indivisibles always corresponds an addition of measures*. She interpreted these conceptions using the concept of epistemological obstacle of "heterogenea" which accounts for conceptions of mixing up magnitudes of distinct dimensions (solids with surfaces or surfaces with lines). This consists in unconscious and undue shifting in the pupil's mind between the field of magnitudes and the field of their measures.

She adopted the concept of *epistemological obstacle* introduced by Bachelard (as quoted by Schneider, 1991). Bachelard gave to this concept the following sense.

> *[…] it is in terms of obstacles that we must pose the problem of scientific knowledge*. It is not just a question of considering external obstacles, like the complexity and the transience of scientific phenomena, nor to lament the feebleness of the human sense. It is in the act of gaining knowledge itself, to know, intimately, what appear, as an inevitable result of functional necessity, to retard the speed of learning and cause cognitive difficulties. It is here that we may find the causes of stagnation and even regression, that we may perceive the reasons for inertia, which we call epistemological obstacles. (See Schneider, 1991; see also Cornu, 1991, p.158)

In this perspective of epistemological obstacles, Cornu (1991) said that it is possible to distinguish several different types of obstacles, namely, the *genetic and psychological obstacles* which occur as a result of personal development of the student, the *didactical obstacles* which occur because of the nature of the teaching and the teacher and finally the *epistemological obstacles* which occur because of the nature of the mathematical concepts themselves.

In her study, the main research instrument was constituted by a sequence of problems probing the pupils' mental representation about areas and volumes. With regard to integration which is my main concern in this review, Schneider identified two errors committed by the students.

The first error consists in deducing that areas of $\int_0^1 x^3\,dx$, $\int_0^1 x dx$ and $\int_0^1 x^2\,dx$ verify the relation $\int_0^1 x^3 dx / \int_0^1 x\,dx = \int_0^1 x^2\,dx$ because $x^3/x = x^2$. The second error consists in concluding that solids of revolution respectively generated by rotation around ox-axis, surfaces under $y = x^2$ and $y = 2x$ between the bounds 0 and 2, have volumes $\int_0^2 \pi x^4 dx$ and $\int_0^2 4\pi x^2 dx$ which verify the relation $\int_0^2 \pi x^4 dx / \int_0^2 4\pi x^2 dx = \int_0^2 x^2/4\,dx$ because their homologous sections have a ratio equal to $x^2/4$: $\pi x^4/4\pi x^2 = x^2/4$.

Concerning the fundamental theorem of calculus, $\frac{dS(x)}{dx} = f(x)$, and its proof, Schneider reported that some students proposed $f(x)$ as the limit of the difference $S(x+h) - S(x)$ when h tends to 0, where $S(x) = \int_a^x f(t)dt$. She reported that the explanation of the students was: As long as h becomes smaller, the strips become thinner and thinner until they are reduced to line segments $f(x)$. Schneider also reported a student's wondering of why $\int_a^a f(x)\,dx$ is not equal to $f(a)$. Schneider noted these errors also witness undue interference of perceiving magnitudes in the domain of measures. The thinking is alternating between the numerical and the geometrical representations.

To conclude, Schneider said that these errors demonstrate that a certain perception of magnitudes is mixed in the calculations of areas or volumes in unconscious or undue manner which, most often, leads to an error; and magnitudes of distinct dimensions are mixed in this perception (solids with surfaces or surfaces with line segments). The presence of these two errors is a manifestation of what Schneider called the obstacle of heterogeneity of dimensions.

In addition to the above mentioned students' errors, Schneider reported that the students she observed debated the question of knowing whether the area under $y = x^3$ between 0 and 1 is exactly or approximately $1/4$. She said that this debate originates in the mixing of magnitudes with measures and in the conceptions of the concept of limit in numerical and geometrical settings after partitioning the surface into rectangles. After reviewing this study, I agree that a well-prepared task leading to this debate can contribute to enriching students' concept images of integrals.

The above discussion explains how students met the obstacle of heterogeneity. In the third part of her study, she discussed whether the obstacle of heterogeneity of dimensions is really an epistemological obstacle. She positively concluded that it is, and advised that this concept has not to be "congealed". As it is a model of interpreting a certain reality, it needs to be considered as hypothetic and used as long as it satisfies its users' needs. If it does not provide the expected services from it, it will be replaced by another one if this can be found. Schneider's study provides an extension or a new way of understanding and explaining what epistemological obstacle is. However, her explanation of how students met the obstacle of heterogeneity is an interesting element to take into account in order to enrich the students' concept images of integrals.

Another study of the category of integration is the one done by Koepf and Ben-Israel (1994). After considering the definite integral as a limit of Riemann sums $\int_a^b f(x)dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k$, Koepf and Ben-Israel indicated that an indefinite integral can be defined either as an antiderivative or as a definite integral over a variable interval $\int_a^x f(t)dt$ (Koepf & Ben-Israel, 1994). They indicated that the latter can be demonstrated in the classroom using symbolical computation and they gave examples of functions t^m (for all real m), $\sin t$, $\cos t$, e^t , and $\ln t$. They stated that performing this symbolical computation in the classroom makes students gain a concrete understanding of the concept of the indefinite integral after which they can then compute other indefinite integrals using other means such as computer facilities or other integration techniques. Moreover, they said that using symbolic computation, the indefinite integrals becomes as elementary and palpable as the definite integral. They added that the teaching of indefinite integrals immediately after the definite integral becomes possible by computing indefinite integrals as definite integrals over variable intervals. (This is the approach that I tried during my teaching reported in this study.) Finally they claimed that definite and indefinite integrals can be taught independently of differentiation, which is then not considered as a prerequisite for teaching the indefinite integral.

In the same perspective, Burn (1999) referred to an approach called genetic method. This method uses historical insights and developments to locate steps in the teaching of integration. After two examples of the Greek method of exhaustion, Burn used historical developments from the 17^{th} and the 19^{th} centuries to propose conceptual steps which a student might take in a conventional first course in analysis up to the Riemann integral. According to Burn, the first step deals with parallel rectangular strips of equal width. He gave as examples the methods of Fermat and Roberval to find areas between the curves $y = x^2$ and $y = x^3$ and the x-axis and the logarithmic property of areas under the curve $y = \frac{1}{x}$. The second step that he proposed deals with parallel rectangular strips of unequal width. He gave as examples the geometrical dissection to $\int_a^\infty \frac{dx}{x^2}$, the geometrical dissection applied to $\int_1^a \frac{dx}{x}$. In this second step, he proposed to deal with the fundamental theorem of calculus, with the Newton/Leibniz assumptions. In the third step, he proposed to move from geometry to algebra. The topics to deal with in this step are the Cauchy definition of the integral, the Cauchy version of the fundamental theorem of calculus, the Riemann integral, and finally the upper and the lower sums. Burn said that this conventional instructional material using historical insights and historical problems can improve students' understanding of integration.

On her side, Berger (2006), based on Vygotsky's theory of concept formation analysed the development of understanding of the definition of an improper integral with an infinite integration limit $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$ for a continuous function f on the interval $[a, \infty)$. She argued that the notion of the functional usage of sign (the use of a mathematical sign prior to full understanding of the mathematical object it signifies), together with the construct of the pseudo-concept (the concept is not fully constructed by the student), can be used to explain how the divide between an individual's mathematical activities and the socially sanctioned mathematical definition is bridged. In this respect, she argued that idiosyncratic mathematical activities can be regarded as manifestations of complex thinking which, with socially regulation, can be transformed into pseudo-concepts and, after further activities and further social regulations, into concepts. From this point of view, the student's understanding of a mathematical concept is improved through this transformative process.

Another study linked to the definite integral is the one done by Rösken and Rolka (2007). Using the concepts of concept image and concept definition as described by Tall and Vinner (1981) (see section 2.5 at page 25 in this chapter), Rösken and Rolka analysed students' conceptual learning regarding the notion of definite integral. They used a comprehensive questionnaire to ascertain students' concept image and concept definition and the corresponding problem solving competence. Twenty four students in grade 12 of a German secondary school participated in the study. In their findings, they indicated that "definitions play a marginal role in students' learning whereby intuition inherent in concept images dominates the conceptual learning" (p. 181).

Regarding the question "what do you understand by the $\int_a^b f(x)dx$?", the responses of the students, during the pre-study, revealed two main aspects. Some of the students explain the symbol of the integral by evoking "the integral from a to b of the function f"; others referred to the area aspect by evoking "the area that the graph of f includes with the x-axis in the interval [a, b]". According to Rösken and Rolka, in the students' responses, "the explication of the symbol of integral refers more to the concept definition while the interpretation as area is more related to the concept image" (p. 192). They also found that there were some conflicting issues in the students' concept images. Some of these issues were the non distinction of the concepts of areas and integral, the connection of the symbol of integral to a specific type of function, and the restriction of the calculation of the area to a specific graph of function. They also concluded that intuitive ideas are an important and decisive part of a concept image. As they said, “not only mathematical knowledge but also other experiences have a considerable effect on problem solving” (p. 202). Finally, they found that the students’ evoked concept image was primarily restricted to techniques and procedures to the advantage of visualization. About the concepts of concept image and concept definition they said that they are “a powerful model to deeper analyse the construction of mathematical knowledge” (p. 203).

More recently, Tarvainen (2008) presented a way of forming integral expressions such as $\int_a^b f(x)dx$ by using differential or infinitesimal derivations. After having recognised that a rigorous justification of these informal derivations is a major pedagogical problem in calculus, Tarvainen presented checks by which one can verify that integral expressions obtained by differential derivations are correct. He said that the proposed checks are based on the intermediate and mean value theorems. His rigorous justification specifically addresses students who are used to rigorous mathematics. He started his justification by adopting the modelling point of view used in physics and engineering that “quantities are assumed to exist by physical and geometrical intuition and integrals are derived using physical and geometrical knowledge” (Tarvainen, 2008, p. 61). He announced and produced the proof of the following theorem. I will come back to the mathematical notion involved in this theorem in the next chapter of theoretical frameworks.

> Consider the differential procedure to set up integrals, where by, on an arbitrary, short interval $[x, x+dx]$, the accumulation of a quantity is approximately a product $f(x)dx$, where f is a continuous function defined on an interval $[a, b]$. Assume that (1) if the interval $[a, b]$ is divided into subintervals, the total accumulation of the quantity on the whole interval $[a, b]$ is the sum of the accumulations on the subinterval (an additivity assumption), (2) for every subinterval $[x, x+dx]$, there is an intermediate value $x^* \in [x, x+dx]$ such that the exact accumulation over the interval $[x, x+dx]$ is $f(x^*)dx$, then the

> accumulation of the quantity over the interval $[a,b]$ is the integral $\int_a^b f(x)dx$. (Tarvainen, 2008, p. 64)

In the proof that he provided for this theorem, Tarvainen used the formula of the limit of sum $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$ which he called a property of the definite integral of a continuous function. This theorem illustrates another view of defining the concept of the definite integral. In this version, however, the underlying concepts of partition/subdivision, product, sum, and limit are evoked in the same manner as they have been evoked by Koepf and Ben-Israel (1994) in their formula defining the definite integral.

In summing up, after having analysed researches related to teaching and learning the definite and the indefinite integrals I found that students have difficulties of various forms when they are learning calculus in general and integration in particular. In order to help them overcome these difficulties, several solutions have been suggested. Some of them are common to all the concepts of calculus. In this category of common solutions there are the ones of proceeding from geometry to algebra, using visualisation and computers, and introducing the scientific debate approach (Artigue, 1991, 1996; Tall, 1996). Regarding suggestions specific to the teaching and learning of integrals, Orton (1983b) proposed reorganising the curriculum in order to improve the students' understanding of integration. An observation by Koepf and Ben-Israel (1994) was that the teaching of indefinite integrals could be done immediately after the definite integral if indefinite integrals are considered as definite integrals over variable intervals. This approach could free the teaching of indefinite integrals from the differentiation – as antiderivatives - especially for simple functions. Some historical insights and developments could be very helpful in developing students' understanding of integration if the genetic method (Burn, 1999) is adopted. In the perspective of Vygotsky's theory of concept formation, I found that some literature supported that understanding of a mathematical concept involves a transformative process which is socially regulated to bridge individual mathematical activities and the socially accepted definition (Berger, 2006). Finally, there are researchers who concentrated on rigour in mathematics and contributed to various ways of introducing the concept of integral (Tarvainen, 2008).

It can be seen from these studies that there is no unique approach to teaching and learning integrals. Each researcher is trying to show the best way to bring the students understand integrals. Burn (1999) points out that the historical development of the teaching material can help students understand integrals better; Rösken and Rolka (2007) proposed to integrate intuitive ideas in helping students to develop their concept images since they found that not only mathematical knowledge but also other experiences have a considerable effect on problem solving. Tarvainen (2008) proposes the use of differentials to set integrals, while Koepf and Ben-Israel (1994) suggested freeing integrals from derivatives by using the limit of a sum to define definite integrals and considering the upper limit of the definite integral as a variable when dealing with indefinite integrals. The obstacle of heterogeneity as described by Schneider (1991) is important and deserves to be taken into account in order to enrich students' concept images of the definite integral. I will return to this analysis in the summary that I provide at the end of the next section, and in section 2.9.

On another side, Thompson (1994) has been interested in studying the teaching and learning of the fundamental theorem of calculus. I review this study in the next section.

2.4. Research on teaching and learning the fundamental theorem of calculus

Thompson (1994) analysed the operational understanding of the fundamental theorem of calculus using the notion of image as a way of describing how students think. He devised and implemented a teaching experiment with a group of nineteen students enrolled in a course on computers in teaching mathematics. Seventeen of the students have completed three semesters of the calculus with grade B or better and the other two had a grade of C. Seven of the students had taken the advanced calculus and four were taking the advanced course. Thus, it is clear that that the group under study was supposed already to have an appreciable understanding of the key concepts of calculus.

Notwithstanding that fact, Thompson found that these students had some difficulties understanding the fundamental theorem of calculus; he concluded that those difficulties are rooted in their images of function, in their fixing in an isolated manner of the variations of variables, and in a weak conceptual understanding of the average rate of change. Also, he found that notations also cause difficulties to students. To overcome these difficulties, he suggested that a great deal of image-building concerning accumulation, rate of change, and rate of accumulation must precede their coordination and synthesis into the fundamental theorem of calculus.

Thompson finally pointed out that notations also often cause difficulties to students. These notations symbolise many things at a time so that students may not see what the meaning of the symbols is in a given context. Are the symbols representing operations or the results of these operations? Thus, to elucidate and remove that potential confusion and increase students' understanding of symbols and the concepts they represent, there is a need to understand the operational and the structural meaning embedded in those symbols; Sfard (1991), among others, has clarified the concept of a function in this respect and Zandieh (2000) gave some clarifications for the concept of the derivative. However, there are no studies on developing students' understanding of the concepts of the definite and the indefinite integrals and their link through the fundamental theorem of calculus. This thesis will contribute to addressing that need of setting up strategies that could promote the understanding of the concepts related to integrals and to the fundamental theorem of calculus.

In his teaching experiment, Thompson (1994) adopted the intuitive development of the fundamental theorem as opposed to the procedural/instrumental development, and proposed the use of the mean value theorems for integrals and for derivatives in the proof of the fundamental theorem. He said that the mean value theorem for integrals allows a formal proof of the fundamental theorem to go smoothly since one can "substitute the average value of the integrated function for the integral of the function over increment in its argument" (Thompson, 1994, p. 271). Regarding the mean value for derivatives, he said that it supports a conceptualisation of what is going on, namely, “the accumulation (integral) of the multiplicatively-constructed quantity $f(t)dt$ is changing at an average rate of change that is equal to $f(t)$ for some t in $[x, x+\Delta x]$” (Thompson, 1994, p. 271).

In summary, educational researchers in the domain of calculus, especially in the domain of the teaching and learning of integrals and of the fundamental theorem of calculus, are joining their efforts to find ways to reduce students’ difficulties and to improve their conceptual understanding of integrals. Their efforts are mainly based on the introduction of integration by using the concept of the limit of sum. Moreover, some researchers proposed the use of computers and appropriate software in order to introduce other representations in addition to the symbolical (algebraic) representation. Computers and appropriate software can facilitate visualisation in graphical representation and can improve results in numerical representation.

With regard to the teaching material, researchers, such as Artigue (1991) based on the scientific debate approach and on the didactical engineering, Burn (1999) based on the genetic method and on historical insights and developments, and Orton (1983b) based on his experience, proposed content organisation which could lead to improving the teaching integration. All of them proposed instructional materials aiming at defining the definite integral as the limit of a sum.

Koepf and Ben-Israel (1994) proposed to teach indefinite integrals immediately after the definite integrals, computing them as definite integrals over variable intervals, for as many functions as sufficient to make the students get the point. This perspective of Koepf and Ben-Israel allows for the teaching of both definite and indefinite integrals independently of differentiation, at least for some uncomplicated functions. And Thompson (1994) adopted the intuitive development of the fundamental theorem of calculus instead of the procedural development and proposed the use of the mean value theorems for integrals and for derivatives in the formal proof of the fundamental theorem of calculus. All the propositions of the above researchers constitute a good starting point. In fact, there is still a need to refine their propositions regarding the content and the implementation of the teaching approaches such as the scientific debate approach in the actual classrooms of other contexts.

In addition, the abovementioned studies concentrated mainly on difficulties that students encountered when they were dealing with the subject-matter and how to overcome them. However, they did not elaborate further on interactions between the teachers and the students in order to reduce these difficulties. Moreover, the above mentioned studies did not trace the evolution of the students' understanding of the concept under consideration; neither did they trace the improvement of the students after a given period of time. But more importantly, not much has been done in investigating students' concept images (Tall & Vinner, 1981) of the integral concepts. The present thesis will contribute to fill this gap. In the next section, I present the notion of the concept image as described by Tall and Vinner (1981), as this informs the theoretical framework and underlies the analysis of the development of students' understanding.

2.5. Concept image and concept definition

According to Tall and Vinner (1981), an individual's *concept image* of a given concept is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). Tall and Vinner noted that the concept image of a concept is built up over years and changes "as the individual meets new stimuli and matures" (p. 152). Among the components of the definition of the individual's concept image, there are the mental pictures on the one hand, the associated properties and processes on the other hand. Vinner (1975) described a person's mental picture of a noun or noun phrases in terms of the set of all mental pictures of objects denoted by that noun or noun phrases. In fact, denoting the noun by C and the person by P, he indicated that "P's mental image of C will be defined as the set of all pictures that have ever been associated with C in P's mind, namely the set of all pictures of objects denoted by C in P's mind" (p. 339). Vinner (1983) specified that the word pictures "include any visual representation of the concept (even symbols)" (p. 293). Finally, Vinner and Dreyfus (1989) added to the previous description the fact that the mental picture "means any kind of representation – picture, symbolic form, diagram, graph, etc" (p. 356).

Furthermore, Tall and Vinner (1981) indicated that a *concept definition* includes words that an individual uses to describe that concept. They noticed that a concept definition can be formal or personal, according to whether the individual makes a statement accepted by the mathematics community or not. Moreover, they mentioned that an individual's concept definition is a part of the individual's concept image. This is confirmed by Vinner (1983) who indicated that definitions produced by individuals are a description of their concept images.

Finally, Tall and Vinner (1981) used the expression *evoked concept image* to indicate "the portion of the concept image which is activated at a particular time" (p. 152). This implies that the evoked concept image is subject not only to the circumstances in which the individual is prompted to demonstrate his or her concept image of the concept at stake but also to the time at and manner in which the prompt is made.

The concepts of the concept image and of the concept definition as described by Tall and Vinner (1981) correlate with the zone of proximal development (Vygotsky, 1978) in what concerns the process of maturation of "functions" that are in "embryonic state" (p. 86). In fact, in their description of a person's concept image, Tall and Vinner (1981) stated that the concept image is built up over years, changing as the individual meets new stimuli and matures. Thus, if one considers two or more different points in time, the person may evoke different concept images, some may conform to the formal concept image and others may differ. In the latter case, the person's concept image may be different from the cultural one because of the fact that the person was not prompted appropriately or that his or her concept image was not yet developed in a way to be conform to the cultural concept image. In this latter case, the person's concept image may still be in the process of maturing or in an embryonic state (Vygotsky, 1978); in this case, the evoked concept image can be said to be in the zone of proximal development.

However, the person's concept image may also depend on other factors such as the circumstances in which the concept image is being evoked.

In this perspective, Sfard (1991) contributed a lot in categorising conceptions that a person can evoke about a given mathematical object. In the next section I give the description of mathematical conceptions as provided by Sfard (1991).

2.6. Operational and structural conceptions of mathematical objects

According to Sfard (1991), an individual is said to have a structural conception of a mathematical concept when he or she conceives the mathematical concept as if it is an abstract object in a static way, whereas the individual is said to have an operational conception of the same concept when he or she focuses the thinking on the "processes, algorithms and actions" (p. 4) contained in the concept at stake. Sfard (1991), assuming as true that mathematical objects originate in operations, indicated that first a process is performed on already familiar objects, then an "idea of turning this process into an autonomous entity" (p. 18) emerges, and finally the new entity is seen as an integrated, object-like whole on which another process of higher level can be performed.

In addition, Sfard (1991) noticed that the dual nature of mathematical constructs (p. 5) can be observed in various types of representations. Using the example of function, Sfard (Ibid.) showed how some representations appear to be more susceptible to structural interpretation than others and how the algebraic representation is easily interpreted in both ways, that is, in the operational form and in the structural form. She noticed that in the algebraic representation, it is easy to interpret the process-object duality because of the dual meaning of the equality sign which can be regarded as a symbol of identity or as a command for executing the operations that appear at one of its sides. However, it is possible to go beyond the equality symbol and consider only a symbolical representation of a certain mathematical concept, for example, the symbol $\int_a^b f(x)dx$, as both a structural and an operational conception without it being necessary to mention the implied process which is symbolised by $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$. In this case, at a given time, two persons may see different aspect: One person can focus on the structural aspect and the other one can focus on the process aspect. This divergence of interpreting the symbols leads to misunderstanding between the two persons and is proposed to be a source of many difficulties that students have in mathematics (Tall, 1992; Thompson, 1994). When a person evokes the object aspect of a mathematical concept without being able to evoke the process aspect that underpins that object, the person is said to have “semantically debased or pseudo-structural conceptions” (Sfard & Linchevski, 1994, p. 220).

In the same vein, Zandieh (2000) describes a pseudo-structural conception as “an object with no internal structure” (p. 107). Thus, an individual is said to have a pseudo-structural conception when the object conception manifested by that individual does not include the objects of lower levels and does not refer to the processes that led to it. Zandieh (2000) used the term “pseudo object” (p. 107) instead of pseudo-structural conception used by Sfard and Linchevski (1994). I will adopt the term pseudo-object as explained by Zandieh (2000) as a conception of the mathematical object in which a person evokes only the structural aspect without evoking the underlying process.

In brief, in this section and the preceding one, I presented concepts that allude to ways of how mathematical objects are developed and communicated between and among novices and confirmed members of the mathematical community. These members need to communicate and interact through formal organised groups such as mathematics classrooms, or informal organised groups such as seminars and conferences. In the next section, I present the important role played by language in mathematics communication and development.

2.7. Language and mathematical activities

With regard to language, Ferdinand de Saussure (quoted in Gumperz, 1972) distinguished “speech (parole)” from “language (langue)” (p. 5). In this respect, “’speech’ refers to the actual sounds produced by speakers, while ‘language’ represents shared pattern” (p. 5).

These two concepts of speech and language are involved in re-creation of mathematics knowledge described by the paradigm of social constructivism (Cobb, 1995; Ernest, 1991; Simon, 1995). Concerning the role of speech and language, Ernest (1991) wrote:

> The social constructivist thesis is that objective knowledge of mathematics exists in and through the social world of human actions, interactions and rules, supported by individuals' subjective knowledge of mathematics (and language and social life), which need constant re-creation. (p. 83)

In this perspective of involvement of speech and language in the re-creation of mathematics, Bloomfield (1935) wrote:

> In connection with science, language is specialised in the direction of forms which successfully communicate handling responses and lend themselves to elaborate reshaping (calculation). To invent and to employ these forms is to carry on mathematics. …Mathematics appears as a science only so long as we believe that the mathematician is not creating speech-forms and discourses but exploring an unknown realm of 'concepts' or 'ideas'. (pp. 55-56)

From this perspective, language plays an important role since through language members of the mathematical community can evaluate whether mathematical activities are creating new concepts or reshaping the old ones. Therefore, in talking mathematics, speech occasions interactions which contribute to development of mathematical conceptions. In the next section I talk of verbal interactions in the classroom and their contribution to learning of mathematics.

2.8. Verbal interactions in the classroom

Many researchers and authors have been interested in observing and analysing verbal interaction in the classroom (Andersen & Andersen, 1982; Barker, 1982a, 1982b; Bellack, Kliebard, Hyman, & Smith, 1966; Cameron-Jones & Morrison, 1973; Cegala, 1982; De Landsheere, 1973; Freimuth, 1982; Friedrich, 1982; Malamah-Thomas, 1987; Nuthall & Church, 1973; Trowbridge, 1973; Wragg, 1973). I will focus on the study by Bellack et al. (1966) because I found their framework interesting to study teacher's moves in the classroom. In their study, they used quantitative methods to describe and analyse linguistic behaviour of teachers and students in high school social studies classes. Their subjects were 15 high schools teachers and 345 students in classes studying a unit on international trade. Their data were verbatim transcripts of tape recordings of four class lessons for each of the 15 classes. Therefore, they analysed 60 class lessons.

In their study of the teaching in classroom, Bellack et al. (1966) created a framework to analyse the teaching and learning in the classroom. Their framework to analyse verbal actions of students and of teachers comprises four categories, namely the categories of structuring, soliciting, responding, and reacting. They called these verbal actions "pedagogical moves" which they classified "in terms of the pedagogical functions they perform in the classroom discourse" (p. 4).

Bellack et al. (1966) described these pedagogical moves as follows. The structuring moves were moves that "serve to the pedagogical function of setting the context for the subsequent behavior by either launching or halting-excluding interaction between students and teachers" (p. 4). In these conditions the focus could be on attracting attention to a topic or to a problem to be discussed during the coming episode. The soliciting moves were moves that "serve to elicit a verbal response, to encourage persons addressed to attend to something, or to elicit a physical response" (p. 4). These moves comprise questions, commands, imperatives, and requests. The responding moves were in a reciprocal relationship to soliciting moves and their pedagogical function was to "fulfill the expectation of soliciting moves" (p. 4). The reacting moves were moves "occasioned by a structuring, soliciting, responding, or prior reacting move, but were not directly elicited by them" (p. 4). Their pedagogical function was "to modify (by clarifying, synthesising, or expanding) and /or to rate (positively or negatively) what had been said previously" (p. 4). Applying quantitative methods to this framework, Bellack et al. (ibid.) found that the teacher dominates the structuring moves at 86%, the soliciting moves at 86%, and the reacting moves at 81%. In contrast, they found that the teacher was responsible for only 12% of the responding moves. They concluded that "the control by the teacher of the three of the pedagogical moves leaves the pupil with a very limited role to play in the classroom discussions" (p. 47). The teaching styles in these classrooms were teacher-centred. However, other educators and researchers such as Alrø and Skovsmose (2002) and Zhang (2003) have contrasted this teacher-centred style by supporting the student-centred classrooms.

In his doctoral dissertation about verbal communication in the classroom in which he used quantitative methods, Bayer (as quoted in De Landsheere, 1973) used De Landsheere's (1973) system of classroom interaction analysis, the Bellack et al.'s (1966) system for the study of language in the classroom, and the Bloom's (1956) taxonomy to analyse cognitive processes, in the classroom in order to make a multidimensional analysis of verbal communication in the classroom. In his study, Bayer observed three periods for 15 elementary school teachers. The periods were about the teaching of arithmetic (discussion of a problem), reading (discussion of a text), and science, history or geography. For each period, Bayer audio taped three parts, namely the first five minutes, from the 25th to the 20th minute, and the last five minutes. Therefore, Bayer analysed 45 periods in total. Among the results of his study as reported by De Lansheere (1973), Bayer (ibidem) found that a five-minute random sample "was representative of the teacher's style of communication, irrespective of the subject taught or of the stage of any period considered" (p. 79). In his conclusion, Bayer said that his findings were characteristics of a teacher-centered style which was frequent in his country. This study is not up to date and cannot be generalised to all teaching styles. As noted above, this teacher-centred style has been challenged by other educators and researchers to support the student-centred style.

Moving from quantitative methods to qualitative studies of the communication in the classroom, researchers have highlighted the role of language in teaching and learning of mathematics (Austin & Howson, 1979; Douek, 2005; Meaney, 2005; Pimm, 1994; Rotman, 1988; Zevenbergen, 2000). In the same vein, other researchers have endeavoured to demonstrate the importance of interactions during the learning of mathematics in the classroom (Bartolini Bussi, 1994, 2005; Bauersfeld, 1992, 1994; Christiansen, 1996, 1997; Cobb, Wood et al., 1992; Cobb, Yackel et al., 1992; Holton & Thomas, 2001; Laborde, 1994; Simon, 1995; Voigt, 1994; Yackel & Cobb, 1996).

Bauersfeld (1992) discussed general possible teaching and learning strategies from of a social constructivist perspective. He supported his arguments by using examples from related case studies. Here, I give only the list of the possible strategies that he discussed. The reader interested in any of them can consult the reference. These strategies are flexible interpretation versus ritualised reading, discussion of solved tasks, learning by contrasting, underdetermined tasks, constructive geometry versus Euclidian approaches, understanding students versus training them, path setting in realistic situations, process versus product orientation. These strategies can help in the organisation of didactical situations in which the students are invited to participate in the constitution of the mathematics they have to learn.

From this perspective of social constructivism, Simon (1995) explored pedagogical implications of his theoretical constructivist perspectives during a teaching experiment. The analysis of his data led to the development of a model of teacher decision making with respect to mathematical tasks. His model is based on the coordination of the teacher's goals with regard to student learning and his responsibility to be attentive and responsive to the mathematical thinking of the students. In what he called the mathematics teaching cycle, he produced a diagram of the model constituted with three components, namely the teacher's knowledge, the hypothetical learning trajectory and the assessment of students' knowledge. In the trajectory component he put three elements, namely, the teacher's learning goal, the teacher's plan for learning activities, and the teacher's hypothesis of the learning process. He linked the component of hypothetical trajectory to the component of the assessment of students' knowledge by the activity of interactive constitution of classroom activities. However, such a model is only valid in the context in which it has been constituted and depends on the envisaged purpose (Maxwell, 1992). Moreover this model does not characterise the forms of the verbal interactions that constitute the interactive constitution of the classroom activities. Therefore, there is a gap to be filled in the modeling of the teacher decision-making with respect to mathematical tasks.

Cobb, Yackel et al. (1992) analysed one ten minute episode in which three seven year-old students engaged in a collaborative small group activity to explore the relationship between individual learning and group development. They suggested that “the individual and group development are interdependent and that they are related reflexively” (p. 119). They described this interdependence by saying that “the children’s individual mathematical activities were constrained by their participation in the interactive constitution of a taken-as-shared basis for mathematical activity” (p. 119) and that the “evolving basis for mathematical activity did not exist apart from and was interactively constituted as each child attempted to coordinate her mathematical activity with that of the other members of the group” (p. 119). To summarise, they said that “the children learned in classroom situations as they participated in the interactive constitution of the situation in which they learned” (p. 119). Referring to the theory of didactical situations (Brousseau, 1997), they characterised the one ten minutes episode as a situation of action during which the children learned as they interactively constituted situations of justification or validation which they ended by getting a consensus on one solution method.

Cobb, Wood et al. (1992) analysed two teachers’ behaviours and distinguished two classroom traditions. They called one tradition the school mathematics tradition and the other one the inquiry mathematics tradition. They said that, in the two traditions, the teacher was an authority in the classroom but the distinction between the teachers resided in the manner in which they expressed their institutionalised authority in action. In one classroom, the teacher acted as the sole validator of her students’ interpretations and solutions as she initiated them into the realm of mathematical instructions. They said that it was because the mathematical practices interactively constituted in this classroom could not be explained or justified that learning was synonymous with acting in accord with the teacher’s expectations. In the other class, the teacher and the children acted together as a community of validators. They said that children’s developing ability to assess the legitimacy of each others’ mathematical activity was closely related to the observation that the activity was intrinsically explainable and justifiable. These are words which invoke the cognitive processes of the category of ‘understand’ as described by

Anderson et al. (2001). Cobb, Wood et al. (ibid.) added that the children developed that ability as they participated in the interactive constitution of situations for explanation and justification under the teacher's guidance. They observed that "a primary way in which the teacher expressed her authority in action was by initiating and guiding the interactive constitution of these situations of explanation and justification and by redescribing certain aspects of the children's mathematical activity but not the others" (p. 594). The first part of this observation leads to ways of interactive constitution of what count as acceptable explanation and justification and to influence of sociomathematical norms on mathematical argumentation and learning opportunities, as described by Yackel and Cobb (1996). The second part of the observation invokes the phase of the institutionalisation described in the theory of didactical situation as described by Brousseau (1997). Cobb, Wood et al. (ibid.) continued by saying that the children were not obliged to act in the ways the teacher had in mind and that the teacher actively encouraged them to say how they actually interpreted and solved tasks. The concluded by saying that "the enculturation into the mathematical ways of knowing occurred as the teacher capitalised on their relatively autonomous constructions to guide the constitution of taken-as-shared mathematical meanings and practices" (p. 595). However, they did not give details of how to actively encourage the students or how to guide the constitution of taken-as-shared mathematical meanings and practices. More specifically, despite the advice they provided, the question of how the teacher has to interact or the characterisation of the forms of interactions remains unspecified.

Voigt (1994), using a microethnographical case study, observed an elementary teacher and first graders when they were negotiating mathematical meanings. In this regard, based on a symbolic interactionist view that interaction is more than a sequence of actions and reactions, Voigt said that "mathematical meanings are negotiated even if the participants do not explicitly argue from different points of view" (p. 281). In fact, there is a cycle of interpretation of each participant's action. According to Voigt (ibid.), "the participant of the interaction monitors his action in accordance with what he assumes to be the other participants' background understandings, expectations, etc." (p. 280).

Similarly, the other participants make sense of this action by adopting what they believe to be the actor's background understandings, intentions, etc". And then "the subsequent actions of the other participants are interpreted by the former actor with regard to his expectations and can prompt a reconsideration, and so on" (p. 281). This sequence of interpretations of the others' understanding in order to reconsider action leads to a continuous spiral of cycles of interactions among the participants.

In this context of the negotiation of mathematical meaning, Christiansen (1997) stated that there is an additional negotiation concerning the task since "meaning is always meaning in a particular context" (p. 1). She added that the negotiation of meaning presupposes a taken-to-be-shared understanding of the situation and that when several students are working together, a negotiation of the perspective from which the content should be addressed must be undertaken. This "taken-to-be-shared" is necessary to allow the continuation of the spiral of cycles of interactions that I evoked in the preceding paragraph.

Combining qualitative and quantitative analyses of interactions, Holton and Thomas (2001) analysed mathematical interactions that took place between the participants in a mathematics class. The analysed types of interactions were teacher-class, teacher-group, and student-student. The purpose of the analyses was to try to understand how mathematical learning and understanding might take place through talk and how this might be influenced by the task and other classroom variables. Holton and Thomas (ibid.) started their study by discussing the concept of scaffolding and who provide scaffolding for someone else's learning. Is it the teacher who has to provide scaffolding or can it be provided by a peer? They presented scaffolding as to refer to "the guidance and interactional support given by a tutor in the zone of proximal development" (Smith, as quoted in Holton & Thomas, 2001, p. 79). Moreover they evoked some means which are used in scaffolding. These means as identified by Cambourne are "focusing on a gap which the learner needs, extending – challenging or raising the ante, refocusing – encouraging clarification, and redirecting – offering new information" (Cambourne, as quoted in Holton & Thomas, 2001, p. 79). Their analysis of teacher-student interactions showed three types of teacher-student interactions, namely, interactions with the whole class, interactions with a group of students, and interactions with a single student. They found that learning to ask questions is as important as being able to answer them and that groups are the best place to practice asking scaffolding questions and that such practice should be nurtured by the teacher. Moreover they found that it is likely that understanding come more quickly and at a greater depth as result of the interactions between the two students that they were considering. In their conclusion, they said that using both qualitative and quantitative data they observed that learning and understanding take place in group situations and that these cognitive processes are catalysed by talk. They said that "much of what has been achieved in the dyads in the classrooms studied could not have been achieved as quickly, if at all, by the whole class teaching or by students working alone" (p. 102). However, in their analysis of time spent on classroom activities in some of the lessons, they found that in three of the four lessons, the group work activities were given less time than the whole class activities. This was unfortunate for the students because the group work were more likely to enhance their learning and understanding. Holton and Thomas (ibid.) also observed that an important role for teachers is to monitor the progress of the groups. And finally, they noted that the mathematical interactions that appeared to be of the highest quality took place between students in situations where they were presented with problems of a demanding nature. They summarised the ways in which the teachers analysed in their study could have improved the chances of the learning. These ways are providing problems of demanding nature, providing appropriate scaffolding by asking questions which stimulate discussion and require explanation, and providing more time for students to interact in groups and less time in whole class, teacher-driven situations, where the students were not stimulated to make any significant contribution. Clearly Holton and Thomas's ways of improving learning for students is in the inquiry traditions as described by Cobb, Wood, et al. (1992), summarised previously.

Also, the teacher described by Holton and Thomas (ibid.) has some of the qualities of expert teacher as identified by Hattie (2003). According to Hattie (ibid.), an expert teacher can identify essential representations of their subject, can guide learning trough classroom interactions, can monitor learning and provide feedback, can attend to affective attributes, and can influence students' outcomes. Therefore, in the line of Holton and Thomas (ibid.), expert teachers can provide problems of demanding nature because they have essential representation of their subjects and they can provide appropriate scaffolding by asking questions which stimulate discussion and require explanation.

In summing up, during mathematics classroom interactions, there is negotiation of mathematical meaning (Voigt, 1992) and there is also negotiation of the task (Christiansen, 1997) in order to get a taken-as-shared basis for further interactions. From an interactionist perspective, the participant of the interaction monitors his or her action in accordance with what he or she assumes to be the background understanding and expectation (Voigt, 1992) and episodes of justification or validation interactively constituted by children lead them to learning and to institutionalising one solution method (Cobb, Yackel et al., 1992). In his Mathematics Teaching Model, Simon (1995) proposed that teachers has to take into account their goals, their plans for learning activities, their hypothesis of learning process and their assessment of the students' knowledge. Finally, using both quantitative and qualitative data, Holton and Thomas (2001) observed that learning and understanding take place in group situations and that these processes are catalysed by talking. They said that during the group work, the teacher's role is to monitor the progress of the groups; and finally, they noted three ways that are likely to increase chances of learning. The first way is to provide to students problems of a demanding nature, that is, problems that are not too easy and not too difficult for them to start solving; the second way is to provide appropriate scaffolding by asking questions which stimulate discussion and require some explanation about the knowledge included in the task being dealt with; and the third way is to provide more time for students to interact in groups and less time in whole class teacher-driven situations.

All these studies demonstrated the role of interactions in the learning and the teaching of mathematics. However, the practical characterisation of the forms of these interactions is still to be elaborated. The study of Holton and Thomas has contributed to such characterisation by elaborating the form of asking questions to enhance learning and understanding. However, in accordance with Steffe and Thompson (2000b), learning how to interact with the students is still a central issue in any mathematics classroom. There is still needs to find more forms of interacting in order to make the teaching of mathematics more effective and to enhance learning and understanding for the students. In the current study I will join my efforts to those already engaged by others researchers to find how the teacher could efficiently interact with the students in the classroom in order to improve the students' understanding of mathematical objects under consideration.

2.9. Conclusion

In this chapter I presented a review of the literature about research on the teaching and learning of calculus, especially on the teaching and learning of integrals and the fundamental theorem of calculus. The studies that I reviewed include those of Artigue (1991, 1996), Ferrini-Mundy and Gaudard (1992), Juter (2009), Orton (1983b), Rösken and Rolka (2007), Schneider (1991), Tall (1996), and Thompson (1994). All these studies presented students' difficulties related to the learning of calculus in general, and to integrals and the fundamental theorem in particular. Based on these difficulties, the researchers proposed teaching content and approaches that may contribute to improve students' understanding of integrals and of the fundamental theorem of calculus. Their propositions in relation to the content centered on the introduction of topics that would lead to defining the definite integral as a limit of sum. Concerning the fundamental theorem of calculus, Thompson (1994) proposed the intuitive approach and the use of the mean value theorems for integrals and derivatives in the formal proof of the fundamental theorem of calculus. Thompson also suggested that a great deal of image-building regarding accumulation, rate of change, and rate of accumulation must precede their coordination and synthesis into the fundamental theorem of calculus. Moreover, the researchers proposed the use of computers and appropriate software to facilitate the introduction of visualisations of concepts and representations other than symbolical representation such as the graphical representation. The introduction of computers could also improve the use of the numerical representation. The studies of the researchers Koepf and Ben-Israel (1994), Burn (1999), Berger (2006), and Tarvainen (2008) also contributed to proposing alternative teaching material and ways of teaching it and of understanding idiosyncratic mathematical activities, in the case of Berger (2006). Juter (2009), Rösken and Rolka (2007) and Schneider (1991) proposed some elements of how the students' concept images can be more developed. The concept of obstacle of heterogeneity (Schneider, 1991) can contribute a lot in improving the students' understanding of integrals.

After that, I presented a review on the concept of concept image as described by Tall and Vinner (1981), on the operational and structural conceptions as described by Sfard (1991) and a view of mathematical activities from the linguistic perspective as presented by Bloomfield (1935).

Finally I presented studies about verbal interactions in the classroom including, among others, the studies of Cobb, Yackel et al. (1992), Voigt (1994), Christiansen (1997), and Holton and Thomas (2001). These studies on verbal interactions in the classroom exhibited the important role of interactions in promoting learning and understanding. Voigt (1994) and Christiansen (1997) focused on negotiation of mathematical meaning during interactions and Christiansen (1997) added that during the classroom interactions there is also the negotiation of the task. Cobb, Yackel et al. (1992) revealed the importance of interactions in the individual and group development during episodes of justification or validation. And finally, in their study, Holton and Thomas (2001) found that learning and understanding happen in group work and proposed three ways in which teachers can enhance learning and understanding for the students. These three ways are the provision of problems of a demanding nature, the provision of appropriate scaffolding by asking questions which stimulate discussion and require some explanation, and provision of more time for students to interact in groups and less time in whole class teacher-driven situations.

In reviewing the literature on teaching and learning calculus I found that researchers in this domain put emphasis on the students' difficulties and on the global ways of improving the situation. Most of these studies did not involve analyses of actual interactions that occurred during the classroom communication, especially the studies in relation to the teaching and learning of definite and indefinite integrals. Therefore there is a need to undertake research in that area.

In my research I attempted to join my effort to that of the previously quoted researchers in order to find ways of improving learning and understanding of the concepts of integrals and the fundamental theorem of calculus. Firstly, I described a mathematical framework composed of the basic concepts underlying integrals (see chapter three). This framework will help me to analyse the students' evoked concept images of integrals. Secondly, I analysed the communication in the classroom when I was teaching to students the basic concepts underlying integrals. More specifically, I analysed the effect of the various functions of language in helping students to understand integrals. Finally, I identified the concepts of the theory of didactical situation in mathematics which were playing in the background of my interactions with the students. Thus, seen globally, all these components of my study contribute to improve the teaching and learning of integrals and the fundamental theorem of calculus.

In the next section, I present, in details, the mathematical framework mentioned above and other frameworks related to teaching, functions of language, cognitive processes and understanding.

CHAPTER 3: THEORETICAL FRAMEWORKS

3.1. Introduction

In this chapter, I present the theoretical frameworks that I used throughout the study. These are the frameworks that I used to plan my teaching and the frameworks that I used to analyse my data. The frameworks related to the planning of my teaching are presented in sections two to four and some specific strategies related to design of the teaching are given in section five. The frameworks for analysing the data are given in the sections six to eight. Also, some elements described in section two have been used in the analysis of data. As I will summarise it at the end of this chapter in Figure 8 (section 3.9, p. 81), six theoretical frameworks were used to answer the five research questions mentioned in section 1.2 (p. 2). These theoretical frameworks are the concept of integral and the fundamental theorem of calculus, the theory of didactical situations in mathematics, the zone of proximal development, the episodes of the communication in the classroom, the functions of language in the classroom, and the cognitive processes. In the following sections, I describe each of them and at the same time, I show to which research question the framework at stake contributes to provide answer.

More precisely, the chapter is made up of nine sections. After this introductory section, the second section concerns the mathematical objects of integrals and of the fundamental theorem of calculus. These concepts are unpacked in a way which both guided my teaching and was used in determining the students' evoked concept images. In the third section I explain some concepts from the theory of didactical situations in mathematics. In the fourth section, I deal with the concept of the zone of proximal development. In section five, I present some specific teaching and learning strategies that I used in my teaching. In the sixth and the seventh sections I deal with the concepts in connection with the communication and the functions of language. In the eighth section, I engage the theoretical framework of cognitive processes. The ninth section visualizes, through Figure 8, connections between the research questions of the thesis and the theoretical frameworks and thereby concludes the chapter.

3.2. Integrals and the Fundamental Theorem of Calculus

The following text about integrals and the fundamental theorem of calculus contains the basic notions that I had planned to teach to first-year student teachers in the course entitled Differential and Integral Calculus in order to do my research about these student teachers' evoked concept images (Tall & Vinner, 1981) and their evolution. In the text, I have provided reference to textbooks and journal articles that I used in analyzing the concepts of the definite and the indefinite integrals in order to compile the teaching material. Based on this, I elaborate the framework that I used to analyse the student teachers' concept images evoked during different rounds of interviews and during the communication in the classroom (Chapter five). Thus, this mathematical text, besides informing my teaching, constitutes the basis of the framework that I used to analyse the data in order to answer three of my research questions related to concept images, namely, the first, the second, and the fifth question as presented in the first chapter of this thesis (see p. 2).

It is true that the four layers that I used in this study cannot tackle all the aspects of the concepts of the definite and the indefinite integral. Some of those aspects that my framework did not cater for concern the existence and uniqueness of the definite integral and other issues related to partitions of the interval or the surface. However, these aspects can be picked up on after the core ideas of the concepts have been introduced. As this study dealt with one cycle of an action research project, the next cycle could take into account of what has not been able to be dealt with using my four layers framework. This calls for an extended framework for future research.

In writing about integrals, authors of calculus textbooks distinguish the concept of the definite integrals from the concept of the indefinite integral. In some of these calculus textbooks, the indefinite integral is called the antiderivative or primitive. In the following two subsections, I present the conceptual analysis of the definite and the indefinite integrals, their properties, and theorems related to these concepts. In the third subsection, before I present the fundamental theorem of calculus which unifies the concepts of integral and derivative, I talk about a debate concerning which aspect constitutes the first or the second version of the FTC or which of them is the “true” FTC.

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3.2.1 Definite integral

Many authors of textbooks and researchers who deal with integration define the definite integral by using a formula that involves a limit of sum (Burn, 1999; Koepf & Ben-Israel, 1994; Smith & Minton, 2002; Stewart, 1998).

Smith and Minton (2002) give the following definition of the definite integral:

> The definite integral of f from a to b is $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$ for any function f defined on $[a, b]$ for which the limit exists and is the same for any choice of evaluation points $c_1, c_2, ..., c_n$. When the limit exists, we say that f is integrable on $[a, b]$. (p. 351)

Stewart (1998) defines the definite integral as follows:

> If f is a continuous function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0\ (=a), x_1, x_2, ..., x_n\ (=b)$ be the endpoints of these subintervals and we choose sample points $x_1^*, x_2^*, ..., x_n^*$ in these subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then, the definite integral of f from a to b is $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$. (p. 361)

This definition of the definite integral is similar to what Tarvainen (2008) announced as a theorem, discussed in the literature review (p. 20).

When I was dealing with the concept of the definite integral, I adapted these two definitions and defined the definite integral of any continuous function f over an interval $[a, b]$ by the formula $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ where $\Delta x = \frac{b-a}{n}$ and x_i any point in the i^{th} subinterval $[x_{i-1}, x_i]$ of the interval $[a, b]$ $(i = 1,2,3,..., n)$.

Concerning the existence and the uniqueness of the limit, since it is assumed that f is continuous, the limit in the definition above always exists and it has the same value regardless how the sample points x_i^* are chosen. If the sample points are taken to be the right-hand endpoints, then $x_i^* = x_i$ and the definition of the integral becomes $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$. If the sample points are taken to be the left-hand endpoints, then $x_i^* = x_{i-1}$ and the definition of the integral becomes $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$. Alternatively, the midpoint x_i^* of the subinterval or any other number between x_{i-1} and x_i could be chosen.

Finally, the limit in the definition above also exist if f has a finite number of removable or jump discontinuities. However, the limit does not exist if the function has infinite discontinuities in the interval [a,b]. The reader can find definitions and examples of various types of discontinuities of functions for example in Stewart (1998).

If I attempted to read out (verbalisation) the abovementioned formula given in the definition of the integral, I would say "the definite integral of f of x dx from a to b equals the limit, when n tends toward infinity, of the sum of f of x_i multiplied by delta x, i varying from 1 to n, where delta x equals b minus a divided by n."

I can say that the key words in this sentence are definite integral, limit, sum, multiplied (product), delta x, minus, and divided (implying partition or division).

Moreover, a closer analysis of the second component of this formula reveals four layers in the symbolical representation. The first layer comprises the symbol Δx representing the result of the operation of dividing $b-a$ by n and is noted by $\Delta x = \frac{b-a}{n}$; another element of this layer is the symbol $f(x_i)$ that represent the value of the function $f(x)$ at points x_i when i varies from i to n.

The second layer includes the operation of multiplying $f(x_i)$ and Δx. The result of this operation is represented by the symbol $A_i = f(x_i)\Delta x$ and corresponds to the i^{th} subarea.

As I said in chapter two (section 2.6, p. 27), Sfard (1991) pointed out that the symbolical representation can be interpreted as an operation or as an object. This duality of interpretation is explained by the dual meaning of the equality sign (=) which "can be regarded as a symbol of identity or as a command for executing the operations appearing at its right side" (Sfard, 1991, p. 6).
In my analysis, I will say that the student has evoked both interpretations if he or she evokes the two sides of the equality. If a student evokes only the right side of the equality sign, I will say that he or she only evoked the process aspect in the symbolical representation.

The third layer includes the operation of adding the subareas of the preceding layer. The result of this operation is represented by the symbol $S_n = \sum_{i}^{n} A_i$ or $S_n = \sum_{i=1}^{n} f(x_i)\Delta x$ and corresponds to the n^{th} approximate area.

The fourth layer includes the operation of limiting the preceding series. The result of this operation is represented by the symbol $\int_a^b f(x)dx = \lim_{n\to\infty} S_n$ or $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ and corresponds to the exact area of the surface under the curve of $f(x)$. Graphically, the symbol $S_n = \sum_{i=1}^{n} A_i$ represents the areas obtained using the right rectangles (over-sum) or the left rectangles (under-sum) (see diagrams in Appendix G, p. 360). This implies that these areas are the same.

In this thesis, I will refer to the first layer as the layer of the partition; the second layer as the layer of the product, the third layer as the layer of the sum, and the fourth layer as the layer of the limit. Finally, I will refer to the group of these layers as the underlying concepts of the concept of the definite integral.

When a person uses many symbols as in the formula mentioned above to explain what the definite integral is, I will say that the person is reasoning in *symbolical representation*. When the language used by a person to talk about the definite integral is predominantly from the geometry domain or the person uses graphs to talk about the definite integral, I will say that the person is reasoning in the *graphical representation*. When numbers are predominant in the person's discourse about the definite integral, I will say that the person is reasoning in *numerical representation*. When the person talks without supporting his or her speech with none of the three representations, I will say that the person is reasoning in *verbal representation*. I will use the following Figure 1 to analyse a student teacher's concept image (Tall & Vinner, 1981) of the definite integral likely to be evoked during the interviews and during the communication in the classroom.

	Representations			
	Verbal	Graphical	Numerical	Symbolical
Partition				
Product				
Sum				
Limit				

Figure 1. Outline of the underlying concepts of the definite integral

3.2.1.1. The underlying concepts of the definite integral in the symbolical representation

Figure 2 below summarises the underlying concepts of the definite integral in the symbolical representation as presented in the preceding section. The figure gives the processes and the objects (Sfard, 1991) related to the underlying concepts of the concept of the definite integral. These processes and objects represent the basic ideas related to the concept of the definite integral.

Process-object layers of the definite integral			
Names	Process-object layers	Process aspects	Resulting object aspects
Partition	$\Delta x = \frac{b-a}{n}$	$\frac{b-a}{n}$	Δx
Product	$A_i = f(x_i)\Delta x$	$f(x_i)\Delta x$	A_i
Sum	$S_n = \sum_{i=1}^{n} A_i$ or $S_n = \sum_{i=1}^{n} f(x_i)\Delta x$	$\sum_{i=1}^{n} A_i$ or $\sum_{i=1}^{n} f(x_i)\Delta x$	S_n
Limit	$L = \lim_{n\to\infty} S_n$ or $L = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	$\lim_{n\to\infty} S_n$ or $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	L
Definite integral	$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	$\int_a^b f(x)dx$

Figure 2. The underlying concepts of the definite integral in the symbolical representation

3.2.1.2. Contexts of the definite integral

A mathematical context is a text that comes before, after or with the text that expresses the mathematical concept under consideration. It can be that the context evoked by the speaker is appropriate or inappropriate with regards to the concept, or it can be that no context is evoked to accompany the concept that is being presented. In the case of the concept of the definite integral, there are a variety of appropriate contexts. Firstly, there are contexts of a mathematical nature: an area under a curve of a function, a volume of a solid with a certain base and height, a length of a curve of a function from one point to another, an area between two curves from one point to another, a volume of a solid of revolution, and so on. Secondly, the context can be within the domain of probability and statistics: the triad probability, probability density function and the random variable; the mean of a random variable with a given probability density function on a given interval. Thirdly, the following triads in physics constitute appropriate contexts for the definite integral: distance-velocity-time; velocity-acceleration-time; Energy-power-time; work-force-distance; and mass-density-length. Other contexts could be the population growth at a given rate in time from the domain of demography or the disintegration or the elimination of a substance at a given rate in time from the domain of chemistry and biology. In brief, an appropriate context for the definite integral involves a triad of a quantity to be determined, a quantity expressing a certain relation between variables, and an interval of the independent variable. More about contexts of the definite integrals can be found in Stewart (1998).

3.2.1.3. Properties of the definite integral

Although this study does not target the properties of the definite integral, it is worth recalling them here as they will appear in the classroom communication about the mathematical proof of the fundamental theorem of calculus. The reader interested in these properties can refer to textbooks (see for example, Smith & Minton, 2002). In certain textbooks that deal with the definite integral, these properties are stated as theorems.

Theorem 1 (Smith & Minton, 2002, pp. 356-357)

If f and g are integrable function on the interval $[a, b]$ and c is any constant, then the following properties are true.

1. $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$,
2. $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$,
3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ and

4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for any c in $[a, b]$.

The formulas below follow from the definition of the integral (Smith & Minton, 2002, p. 357):

1. For any integrable function f, if $a < b$, we have

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$ and

2. If $f(a)$ is defined, then we have $\int_a^a f(x)dx = 0$.

Theorem 2 (Integral Mean Value Theorem)

(Smith & Minton, 2002, p. 360)

If f is continuous on $[a, b]$, there is a number $c \in (a,b)$ for which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Another theorem that intervenes in the proof of the fundamental theorem of calculus is the Mean Value Theorem for the derivative.

Theorem 3 (Derivative Mean Value Theorem)

(Smith & Minton, 2002, p. 232)

Suppose that f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

3.2.2 Indefinite integral

Authors of calculus textbooks and researchers who deal with integration also define the indefinite integral of a given function f.

Smith and Minton (2002) define the indefinite integral as follows:

> Let F be any antiderivative of f. The indefinite integral of $f(x)$ (with respect to x), is defined by $\int f(x)dx = F(x) + c$ where c is any arbitrary constant (the constant of integration). (p. 324)

Stewart (1998) defines an indefinite integral of a function f as being the symbol $\int f(x)dx$ used to represent an antiderivative of the function f. He explains this idea by the symbols "$\int f(x)dx = F(x)$ means $F'(x) = f(x)$" (p. 374). Moreover, Stewart noted that the definite integral $\int_a^b f(x)dx$ is distinguishable from the indefinite integral $\int f(x)dx$ by the fact that a definite integral is a number whereas the indefinite integral is a function. Stewart finally noted that the connection between the two integral is given by the following relationship which is related to the fundamental theorem of calculus:

> If a function f is continuous on $[a, b]$, then $\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b$ (Stewart, 1998, p. 374).

Koepf and Ben-Israel (1994), on their side, noted that the indefinite integral of an integrable function f in an interval $[a, b]$ has two definitions. In the first definition Koepf and Ben-Israel described the "indefinite integral as an antiderivative (or a primitive), i.e. a function F satisfying the equality $F'(x) = f(x)$ at all points x in the interval $[a, b]$, with possible exception of a countable set" (p. 115). The function F being defined up to a constant called the constant of integration. In their second definition, they considered the indefinite integral as a "definite integral over a variable interval $F(x) = \int_a^x f(t)dt$ with the constant of integration determined by the lower endpoint a" (p. 115).

This second definition of the indefinite integral has been evoked also by Courant (1937) who defined the indefinite integral as "a function of the upper limit" (p. 110) and who insisted on the fact that it is an indefinite integral instead of being the indefinite integral. Courant, further, elaborated on the representation of an indefinite integral by the formula $F(x) = c + \int_a^x f(u)du = \int f(x)dx$ (p. 116). Here, the upper limit x, the lower limit a, and the additive constant c have been omitted and the letter x has been used for the variable of integration. In his comments, Courant said that this last change introduced some confusion and that it would have been more consistent if this change had been avoided "in order to prevent confusion with the upper limit x which is the independent variable in $F(x)$" (p. 116). Furthermore, Courant insisted on the indeterminacy feature of the indefinite integral that depends on the status of being variable of the upper or the lower limit of the integral. In this regard, Courant said that "in using the notation $\int f(x)dx$ we must not lose sight of the indeterminacy connected with it, i.e. the fact that that symbol always denotes *an* indefinite integral only" (p. 116, italics in original).

During my teaching I adopted the view of Koepf and Ben-Israel (1994) which considers the indefinite integral as a definite integral over a variable interval. I took this option because I had found that considering the indefinite integral in this manner offers a better way of understanding the FTC (Function version): If $A(x) = \int_a^x f(t)dt$ represents the area under the curve of $f(x)$. (See above about the omission of the limits of integration for the indefinite integral); then the derivative of the area function gives the function that delimitates that area: $\frac{dA(x)}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x)$ (*)

In terms of Stewart,

> The derivative of a definite integral with respect to its upper limit is the integrand evaluated at the upper limit. (Stewart, 1998, p. 386, note in margin)

The formula (*) is the FTC (Function version).

Thus, in my analysis of concept images related to the indefinite integral I used the aspects of pseudo-object, process, and object as in the case of the definite integral defined over a variable interval. In particular, I considered as variable the upper limit of the interval.

3.2.3 The Fundamental Theorem of Calculus

In many calculus textbooks, the fundamental theorem of calculus is presented in two versions. One version deals with the function aspect of the integral and the other version deals with its aspect of a number. Some textbooks take one version as the first version and the other one as the second version. The debate of which is the first and which is the second or which is the "'true' FTC" (Koepf & Ben-Israel, 1994, Note 2) will not be engaged in this thesis. I will only deal with the evocation of elements that constitute the one or the other version of the fundamental theorem of calculus. In analysing a person's concept image (Tall & Vinner, 1981), I will use the phrase "fundamental theorem of calculus – version of the derivative of the integral (FTC-VDI)" - to designate the version of fundamental theorem of calculus related to the function aspect, and I will use the phrase "fundamental theorem of calculus – version of the integral of the derivative (FTC-VID or FTC-VEA) – to designate the version related to the number aspect of the theorem.

Concerning the formulation of the theorem, I will refer to the formulations provided by Smith and Minton (2002).

Theorem (FTC-VID or FTC-VEA):

If f is continuous on $[a, b]$ and $F(x)$ is any antiderivative of f, then

$\int_a^b f(x)dx = F(b) - F(a)$. (Smith & Minton, 2002, p. 364)

Theorem (FTC-VDI):

If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$ on $[a, b]$. (Smith & Minton, 2002, p. 368)

A version of the mathematical proofs of these two theorems can be found in Smith and Minton (2002).

I included in this section the presentation of the fundamental theorem of calculus to complete the material that I used to develop students' understanding of integrals. The analysis of the students' understanding of the FTC will concern only the version of evaluation of area (FTC-VEA). In this regard, I will not use the concepts of pseudo-object, process, or object but rather I will resort to the concept of understanding as described by Dreyfus (1991). According to Dreyfus (1991), understanding is more than knowing or being skilled. Rather, it is an event, like a click or a spark, which occurs in the student's mind, more often, after a long sequence of learning activities.

I will return to this conception at the end of section 3.5 (p. 65). In the next section, I present icons that I will use to illustrate student teachers' evoked concept images.

3.2.4 Icons for illustration of student teachers' evoked concept images

During the analysis of the student teachers' evoked concept images of mathematical concepts, I used some icons to illustrate the evoked concept images in diagrams. The icons and the meanings that I assigned to them are given in Figure 3. The first icon is an empty circle; the second icon is a crossed circle; the third icon is a shaded circle; the fourth icon is a tree-like symbol; and the fifth is a tick symbol. I used these icons to illustrate the student teachers' mathematical conceptions (Sfard, 1991; Sfard & Linchevski, 1994; Zandieh, 2000) or the student teachers' concept images (Tall & Vinner, 1981) evoked during the interviews or after the third lesson of the teaching.

Icons	Definitions
○	Pseudo-object, that is to say, a mathematical object is correctly evoked without the underpinning operations
⊗	Mathematical process, that is to say, a mathematical operation is correctly evoked but the resulting object is not evoked
●	Both the mathematical process and the resulting object are correctly evoked
	The mathematical object or the process is incorrectly or imperfectly evoked; it is still in an embryonic state, or in the process of maturation (Vygotsky, 1978) and needs to be developed. The mathematical object or process is incomplete.
✓	A given context has been evoked by the interviewee

Figure 3. Icons to illustrate student teachers' evoked concept images

The icons of empty and shaded circles have been inspired by Zandieh (2000) but the other two icons, namely, the crossed circle and the tree-like symbol, are mine.

To sum up, the mathematical objects and the processes of the definite and indefinite integrals that I presented in the preceding subsections are likely to be evoked by the teacher during the teaching and learning of the concepts of the definite and the indefinite integrals and during the teaching and learning of the fundamental theorem of calculus. Moreover, these mathematical concepts are likely to be evoked by the student teachers during interviews and during the communication in the classroom. However, the actual mathematical objects evoked during the realisation of any curriculum and the learning of these objects depend "…on the social relationships, the communication system, which the teacher sets up" (Douglas Barnes, quoted in Cazden, 2001, p. 2). In summary, the framework for analysing students' concept images consists of layers, representations (figure 1), contexts, and process-object status (figure 3).

In the next section I introduce some of the concepts that informed my teaching of the mathematical objects presented above and the systems of communication that I set up during the lessons.

3.3. The Theory of Didactical Situations in Mathematics

My teaching and the systems of communication that I set up during my teaching were essentially informed by the Theory of Didactical Situations (TDS) in Mathematics (Brousseau, 1997, 2004). When I will be describing didactical situations that occurred during the communication in the classroom, I will use some of the concepts of this theory. These concepts will thus contribute to answering my third research question about the didactical situations that are likely to further the student teachers' understanding of the concept of the definite and the indefinite integrals and of the fundamental theorem of calculus. Thus, the TDS served both as a framework for the teaching and as an analytical framework.

The theory of didactical situations rests on many didactical concepts including the devolution, the adidactical situation, the didactical situation, the institutionalisation, the didactical contract, the didactical engineering, the milieu, and the adidactical milieu. The Theory of Didactical Situations in Mathematical has got many followers belonging to what is identified as 'the French school' (Artigue, 1994; Brousseau & Gibel, 2005; Douady, 1991; Flückiger, 2005; Hersant & Perrin-Glorian, 2005; Laborde, 1994; Laborde & Perrin-Glorian, 2005; Sensivy, Schubauer-Leoni, Mercier, Ligozat, & Perrot, 2005).

In this study I will adopt some of the didactical concepts as described by (Brousseau, 1997). In particular, I will use the concepts of devolution, adidactical situation, didactical situation, and institutionalisation. Brouseau (1997) described these concepts as follows.

D*evolution* is "the act by which the teacher makes the student accept the responsibility for an (adidactical) learning situation or for a problem, and accepts the consequences of this transfer of this responsibility" (p. 230). The devolution is a situation that is likely to occur at the beginning of a lesson or topic.

Another didactical concept described by Brousseau is the *adidactical situation*. This situation is located "between the moment the student accepts the problem as if it were her own and the moment when she produces her answer" (Brousseau, 1997, p. 30). During this period, “the teacher refrains from interfering and suggesting the knowledge that she wants to see appear” (ibid., p. 30). The purpose of the adidactical situation is to make the student produce his or her own answer which is driven only by the student’s understanding of the problem. The adidactical situation allows the student to get the meaning of the knowledge to be learnt when he or she is independently interacting with the subject matter using the acquired knowledge. The adidactical situation creates an occasion for students to use by themselves previous knowledge and to acquire additional one in situations simulating those that they will come across *“outside any teaching context and in the absence of any intentional direction”* (Brousseau, 1997, p. 30, italics is mine).

Thirdly, when the teacher starts “communicating information, questions, teaching methods, heuristics, etc” (ibid., p. 31), he or she becomes involved in “the system of interaction of the student with the problems” (ibid., p. 31). In this case, according to Brouseau (1997), the situation becomes the *didactical situation*. I will also use the phrase *interactive situation* as an alternative name for the didactical situation as used in this case. The purpose of changing the situation from adidactical to didactical situation is to make the didactical relation (which involves the teacher, the student and the subject matter) continue in the case the student is unable to solve the problem. The teacher facilitates the student’s progress, for example, by questioning or giving heuristics.

Finally, the situation of *institutionalisation* includes various actions that the teacher performs in order to endorse the responsibility of what happened in the classroom activities and to create ways of continuation of these classroom activities. Brousseau specified some of these actions that the teachers have to do during the situation of institutionalisation:

> They have to record what the students did, describe what went on and what was related to the knowledge in question, give a status to what happened in the class as a result of both the teacher’s and the students’ initiatives, take responsibility for an object of teaching and identify it, bring these products of knowledge closer to others (either cultural or linked with the curriculum), and indicate that they could be used again. (Brousseau, 1997, p. 236)

This situation of institutionalisation can occur during the lesson, as what can be called local institutionalisation (Douady, 1991), or at the end of the lesson.

I will also adopt definitions of other terms such as milieu, adidactical milieu, situation, problem-situation, didactical contract as they have been given by Brousseau (1997) or other followers of the theory of didactical situations in mathematics. According to Brousseau (1997), a *milieu* is “everything that acts on the student or that she acts on” (p. 9). On her side, Laborde (1994) specified that the milieu consists of “all elements of the environment of the task on which students can act and which gives them feedback of various kinds on what they are doing” (p. 156).

In relation to the milieu, Brousseau defined the *adidactical milieu* as “the image, within the didactical relationships, of the milieu which is ‘external’ to the teaching itself; that is to say, stripped of didactical intentions and presuppositions” (p. 229, italics in original). In this way, the adidactical milieu is about representation of future relationships between the real situations when the student will have finished his or her studies and what he or she is doing at the present moment in the classroom.

With regard to the situation and the problem-situation, Brouseau (1997) used the term *situation* to designate “the set of circumstances in which the student finds herself, the relationships that unify her with the *milieu*, the set of ‘givens’ that characterise an action or an evolution” (p. 214, italics in original). Brousseau also specified what a problem-situation is. He said that a problem-situation is a situation “that necessitates an ada[pta]tion, a response, by the student” (p. 214).

Concerning the term *didactical contract* Brousseau described it as follows:

> In a teaching situation, prepared and delivered by a teacher, the student generally has the task of solving the (mathematical) problem she is given, but access to this task is made through interpretation of the questions asked, the information provided and the constraints that have been imposed, which are all the constants in the teacher’s method of instruction. These (specific) habits of the teacher are expected by the student and the behaviour of the student is expected by the teacher; this is the didactical contract. (Brousseau, 1997, p. 225)

I consider this concept of didactical contract to be among the fundamental concepts of the theory of didactical situations in mathematics in that it serves as a justification of the organisation of situations of the devolution, the adidactical situation, the interactive situations, and the institutionalisation. As Brousseau (1997) said, the teacher, instead of communicating the knowledge, has to devolve a "good" problem from the student learns when the devolution has taken place. But if the student refuses or avoids the problem, or is unable to solve it, the teacher is obliged to help him or her. So, is tied a relation which determines, mainly in an implicit way, what either the teacher or the student has the responsibility to manage and be accountable with regard to each other. This relation constitutes the didactical contract which directs the teacher to organize the mentioned situations. The concept of didactical contract makes the complex process of teaching and learning be seen as a social action in which the participants engage their personal commitment.

To conclude this short overview of the theory of didactical situations in mathematics, I provide the global definition of a didactical situation as provided by Brousseau and I give some condition that a problem has to satisfy in order to be source of learning (Douady, 1991). According to Brousseau (1997), a situation is a set of circumstances in which the student finds herself; it includes also the relationships that unify the student with the milieu and it concerns the set of the "givens" that characterise an action or an evolution. A problem-situation is a situation that necessitates an adaptation or a response by the student. Using these notions, Brousseau gave the following definition of a didactical situation:

> A didactical situation is a situation in which there is a direct and an indirect manifestation of a will to teach – a teacher. In general, in a didactical situation one can identify at least one problem-situation and a didactical contract. (Brousseau, 1997, p. 214)

With regard to problems that are likely to be sources of learning, Douady (1991) set the following four conditions. Firstly, the statement, that is, the context and questions of the problem, has to have meaning for the pupils. Secondly, the pupils can not solve the problem completely for diverse reasons such as the procedure considered is too long, it causes errors, or it has to be used outside of its known field of validity; this condition concerns especially constraints that have to be included in the problems. Thirdly, the knowledge aimed at by the learning – content or method – is made up by tools adapted to the problem; and fourthly, the problem can be formulated in at least two different settings. This fourth condition concerns representations in which the problem has to be interpreted or translated from one representation to another in order to promote understanding. For example, a problem given in words is interpreted in graphical representation or a problem given in graphical representation is translated into symbolical representation. This representation or translation demonstrates the process of understanding. More concretely, during the third round of the interviews that I held with the students (see section 5.5, p. 159), it can be seen that some student teachers resorted to graphical representations instead of using only words to explain what they understood by definite integral.

When a problem satisfying these conditions is devolved to students, the teacher can expect the students to work independently without his or her intervention to communicate the knowledge he or she wants the students to produce. The students will be working in an adidactical situation. But when the teacher starts interacting with the students about the problem that he or she gives to them, the situation becomes didactical or interactive.

In the perspective of interactions in the classroom, I will adopt the term episode as described by Watson and Potter (quoted in Ervin-Tripp, 1972) to discern the didactical situations described by Brousseau (1997). The term episode is appropriate to analyse situations with more emphasis on interactions than on topic. In this respect, Watson and Potter used the term episode as:

> A unit of analysis which can terminate whenever there is a change in the major participants, the role system of the participants, the focus of attention, and the relationship toward the focus of attention" (Watson & Potter, quoted in Ervin-Tripp, 1972, p. 247).

Ervin-Tripp (1972) rationalised the need for this term:

> This term, rather than topic, was chosen to differentiate cases where a similar apparent topic might be within a person's experience, part of an ongoing activity, or an abstract referential category, as in a discussion" (p. 247).

The term episode as described in this quotation is also applicable to the case of discussion about a given mathematical object in a mathematics classroom. Therefore, combining the idea of didactical situations as described by Brousseau (1997) and the idea of episode, I prefer to use the phrase *didactical episode* instead of didactical situation.

I used the term didactical episode to differentiate didactical phases that occurred during my lessons while the same topic was addressed throughout many different phases. To differentiate the didactical episodes I used the criteria evoked in Watson and Potter's definition of episode quoted above. The distinction of didactical episodes assisted me in determining the durations of time that I spent treating a given topic since the students needed to be given sufficient time in order to understand the topic under study. I will use the same phrase of didactical episode to indicate an episode in which a student is interacting with a task. In this case, the student is in implicit interaction with the people who contributed to the conception of the task.

Some of these didactical episodes fell within the devolution, the adidactical situation, the interactive situation, and the institutionalisation. As it can be noticed, the adidactical situation and the interactive (didactical) situation are intertwined. A problem to be solved independently is devolved to the learner (adidactical situation). When the learner is unable to fulfil the task alone, the teacher or a more capable peer intervenes to guide the solving of the problem; thus the learner and the teacher or the capable peer enter in the interactive (didactical) situation. At a time the teacher or the capable peer leaves the learner and the learner solves next stages of the problem alone (adidactical situation) and later on the teacher or the more capable peer may intervene if help is required (didactical situation). That is, there is an alternation of the independent problem solving episode (adidactical situation) and the one of problem solving under the facilitation of the teacher or the more capable peer (didactical situation). In the context of interactions between the teacher and the students, the concept of the zone of proximal development, alluded to in the preceding lines, was of great help in justifying the occurrence of some episodes. In the next section, I present this concept.

3.4. The Zone of Proximal Development

In addition to the Theory of Didactical Situations, my teaching was informed by the concept of the Zone of Proximal Development (Vygotsky, 1978). Some of the ideas underpinning this concept helped me in the description of the communication in the classroom. Thus these ideas, also, contributed in answering my third research question about the didactical situations that are likely to further the student teachers' understanding of the concepts of the definite and the indefinite integrals and of the fundamental theorem of calculus.

The literature says that the concept of the zone of proximal development has been introduced by Vygotsky in a lecture in March 1933 (Meira & Lerman, 2001; van der Veer & Valsiner, 1991). Vygotsky defines the zone of proximal development as

> …the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. (Vygotsky, 1978, p. 86)

This concept of the zone of proximal development has some principles which clarify the relationship between learning and development. One of these principles is that learning creates the zone of proximal development. In this respect, Vygotsky said:

> We propose that an essential feature of learning is that it creates the zone of proximal development; that is, it awakens a variety of internal developmental processes that are able to operate only when the child is interacting with people in his environment and in cooperation with peers. Once these processes are internalised, they become part of the child's independent development achievement. (Vygotsky, 1978, p. 90)

Another principle concerns the intrapsychological development. Regarding this aspect, Vygotsky stated the following principle:

> Any function in the child's cultural development appears twice, or on two planes. First, it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological [intermental] category, and then within the child as an intrapsychological [intramental] category. This is equally true with regard to voluntary attention, logical memory, the formation of concepts, and the development of volition… [I]t goes without saying that internalisation transforms the process itself and changes its structure and functions. Social relations or relations among people genetically underlie all higher functions and their relationships. (Vygotsky, quoted in Wertsch & Bivens, 1993, p. 207, square brackets in original)

There is another principle that concerns instruction. In this respect, Vygotsky stated that the zone of proximal development is a defining feature of the relationship between instruction and development. He said:

> In school, the child receives instruction not in what he can do independently but in what he cannot yet do. He receives instruction in what is accessible to him in collaboration with, or under the guidance of, a teacher. This is a fundamental characteristic of instruction. Therefore, the zone of proximal development - which determines the domain of transitions that are accessible to the children - is a defining feature of the relationship between instruction and development. What lies in the zone of proximal development at one stage is realised and moves to the level of actual development at a second. In other words, what the child is able to do in collaboration today he will be able to do independently tomorrow. (Vygotsky, 1987, p. 211)

Finally, on the practical standpoint, Vygotsky said:

> It is important to determine the lower threshold of instruction…. It is equally important to determine the upper threshold of instruction. Productive instruction can occur only within the limits of these two thresholds. Only between these thresholds do we find the optimal period for instruction in a given subject. *The teacher must orient his work not on the yesterday's development in the child but on tomorrow's*. Only then will he be able to use instruction to bring out those processes of development that now lie in the zone of proximal development. (Vygotsky, 1987, p. 211, italics in original)

However, as demonstrated below, the concept of the ZPD has had many interpretations in the course of its history (Valsiner & van der Veer, 1993), especially when the concept was used in context of the teaching process. The following interpretations were closer to my teaching strategies. The interpretation of the zone of proximal development as co-

construction of future through bounded indeterminacy (Valsiner & van der Veer, 1993) as explained in the quote below, the zone of proximal development as a dialogical mediation (Wertsch & Bivens, 1993) between the teacher and the students, the teaching through the Zone of proximal development as assisting performance when assistance is required as stated by Tharp and Gallimore (as quoted in Gallimore & Tharp, 1990, pp. 183-184), and the zone of proximal development as a symbolic space for the emergence of diverse forms of communication about sign mediation (Meira & Lerman, 2001).

According to Valsiner and van der Veer (1993), Valsiner proposed a theoretical perspective that views development as organised by bounded indeterminacy. In this perspective,

> The psychological processes are viewed as developing by sets of interpersonal (and subsequently intrapersonal, semiotic) constraint systems that determine the direction of the nearest future development. These constraint systems are constantly reorganised by the coconstructive efforts of the developing person and his or her social others in particular environmental settings. (Valsiner & van der Veer, 1993, p. 56)

During my teaching I set up some constraint systems, in particular in the tasks that I devolved to the student teachers. The constraint such as “not to use the antiderivative” when I requested the student teachers to calculate the area below a curve of their choice contributed to orient the students in the direction likely to develop the underlying concepts of the concept of the definite integral. The constraints were reorganised in the second task when I request the student teachers to use not any curve but a parabola.

In addition to the constraints from the written tasks mentioned above, the mediational means such as the forms of language that I used and the active engagement of the student teachers contributed to the intermental and intramental functioning (Wertsch & Bivens, 1993) and consequently to the development of the student teachers’ understanding of the mathematical concepts. During my teaching, I made an effort to keep my discourse predominantly dialogic as opposed to predominantly univocal. In the latter case, according to Wertsch and Bivens (1993), the communication in intermental functioning is viewed in terms of the transmission of information while in the former case that communication is viewed in terms of texts that can serve as thinking devices, that is "as texts with which one is to come into dialogic contact by questioning, rejecting, appropriating, and forth" (p. 213).

Moreover, in this context of the zone of proximal development, I adopted the definition of the teaching as it has been conceptualised by Tharp and Gallimore (as quoted in Gallimore & Tharp, 1990, pp. 183-184) when they said that:

> ... Teaching consists of assisting performance through the Zone of Proximal Development. Teaching can be said to occur when assistance is offered at points in the ZPD at which performance requires assistance.

In addition to serve as a tool for conducting the teaching process, the ZPD can help in analysing the teaching process. This is based on the interpretation of the zone of proximal development as a symbolic space for the emergence of diverse forms of communication to bring about sign mediation (Meira & Lerman, 2001). This interpretation contributed to distinguishing didactical episodes that occurred during my teaching.

To sum up, in the context of the concept of the zone of proximal development, teacher and the student teachers together constructed the mathematical objects that were under study by using a sequence of systems of constraints that I imposed in order to orient the student teachers in the direction that I had intended (Valsiner & van der Veer, 1993); we used a dialogical mediation in which we came into contact by asking, rejecting, appropriating, etc (Wertsch & Bivens, 1993). At some times, during my teaching, I assisted the student teachers when I judged that assistance was needed. In so doing I helped the students to move forward in the co-constructed ZPD. Thus, I followed the view of Tharp and Gallimore (quoted in Gallimore & Tharp, 1990) concerning teaching through the ZPD. When I considered ZPD as a tool for analysing the teaching process I observed various forms of interactions that occurred during my communication with student teachers or among themselves (Meira & Lerman, 2001). In the next section, I present other teaching strategies that contributed to the development of the student teachers' understanding of the integrals.

3.5. Reconstructing mathematical proofs and other specific teaching strategies for understanding integrals

This section presents some specific strategies in teaching mathematics. Mason (1999) described many strategies that can be used to teach mathematics to College Students. Among these strategies there are the strategies of Student Generated Exercises, Say What You See, and Scrambled Proof. To implement the strategy of Student Generated Exercises, Mason (ibid.) proposed to ask students to make up and do their own questions. In so doing, the students "shift their attention from reacting to whatever question is put in front of them and becoming absorbed in particularities, to locating the structure of the questions" (p. 12). Hence, the students learn not only to solve the problems but also how to recognise and to pose problems. This strategy is very instructive especially for student teachers.

To practice the strategy of Say What You See, Mason (ibid.) proposed to "put up a diagram or a short worked examples, and ask students to announce what they can see" (p. 13). In so doing, the teacher can notice where the student who is talking puts emphasis or has difficulties. With the help of computers with specialised software in the domain under consideration, the teacher can generate diagrams on the computer, and ask students to manipulate and then to say what they are observing. On the side of the students, by listening to what others are saying, students can remember or discover some of the relationships that they had forgotten or that were not familiar to them previously.

With regard to the strategy of "scrambled Proof", Mason (ibid.) proposed to take an important proof and cut into statements, and then invite the students to try to reconstruct the proof from the pieces. This strategy makes students recognise the features of how a proof begins and how it ends. When students are invited to work in groups, this strategy makes them discuss and validate the right sequence of the statements of the proof.

These three strategies generate cognitive processes of the categories of analysing and of creating as described by Anderson et al.(2001); I will present these categories in section 3.8 (p. 79ff). In addition, the strategy of Scrambled Proof can generate the cognitive processes of validating, discussing, arguing, refuting, negating, agreeing, disagreeing, and the like. The strategy of Say What You See can generate the cognitive processes of observing, exploring, visualising, manipulating, and the like. Therefore, these three strategies helped me to organise didactical situations which were likely to improve the student teachers' concept images of the concepts that were under consideration.

In the analysis of the data, the cognitive processes embedded in these strategies contributed to the description of sequences of cognitive processes that led the student teachers to understand the concepts of the definite and the indefinite integrals and their link through the fundamental theorem of calculus.

With regard to student teachers' understanding, I also adopted the perspective of Dreyfus (1991) to whom understanding is more than knowing or being skilled. According to Dreyfus (ibid.), understanding is a process which occurs in the student's mind; it may be quick as a click of the mind or, more often, "based upon a long sequence of learning activities during which a great variety of mental processes occur and interact" (p. 25). However, as it has been alluded to in the preceding section about the Zone of Proximal Development, the communication between the teacher and the student teachers contributed to developing the student teachers' understanding. So, it is necessary to go into details about communication. In the next section, I present this aspect of communication in the classroom.

3.6. Episodes in the communication in the classroom

After the overarching ideas of the theory of didactical situations and of the zone of proximal development, there is a need to develop the framework for the analysis of the actual practice implemented in the classroom. The diagram below (Figure 4) displays such a framework confectioned after examination of the video recordings of the classroom communication. This framework, in conjunction with the framework about the theory of didactical situations in mathematics presented in section 3.3., contributes to answering the question about didactical situations that are likely to further the student teachers' understanding of the concept of the definite and the indefinite integrals and of the fundamental theorem of calculus.

Many authors and researchers have focused on communication in general (Baril & Guillet, 1996; Charles & Williame, 1988; Ervin-Tripp, 1972; Gumperz, 1972; Gumperz & Hymes, 1972; Hymes, 1972; Jakobson, 1981); many other authors and researchers focused on communication in the classroom (Barker, 1982a; Cazden, 2001; Cazden, John, & Hymes, 1972; De Landsheere, 1973; Horner & Gussow, 1972; Wells, 1999); and finally many other researchers focused on communication in the mathematics classroom (Adler, 2001; Christiansen, 1996, 1997; Douek, 2005; Kieran, 2001; Lerman, 1996, 2000, 2001a, 2001b, 2005; Meira & Lerman, 2001; Sfard, 2001; Vithal, 2003). In this thesis, I will combine ideas from each of the three perspectives to study the communication about the concept of the definite and of the indefinite integral and about the fundamental theorem of calculus in the classroom.

Barker (1982a) determines four levels of communication in the classroom, namely the intrapersonal communication, the interpersonal communication, the group communication and the cultural communication. Within these levels, Barker determined a variety of types of communication depending on the originator of the message and on the receiver. Among the factors of the Barker's (1982a) framework, the channels of the communication play an important role in the transmission of the message. In the case of the interpersonal and the group level, one of the channels may be sound waves or light waves. In the case of the cultural level, one of the channels may be books, articles, customs, written and unwritten laws. In the case of intrapersonal, the channels are neural pathways and smooth muscles. The types of communication in each level as presented by Barker are the following.

At the intrapersonal level, Barker distinguished two types:

$T-T$: Teacher addressing himself or herself

S_1-S_1: Any student addressing himself or herself

At the interpersonal level, Barker distinguished two types:

$T - S_1$: Teacher addressing any Student S1

$S_1 - S_2$: Any student S1 addressing any other student S2

At the group level, Baker distinguished three types:

$T - Ss$: Teacher addressing more than one student in a class

$Ss - T$: More than one student in a class (via a spokesperson) addressing the teacher

$S_1 - Ss$: One student addressing several students

At the cultural level, Barker distinguished two types:

$C - Ts$: Many persons and /or groups (authors, educators, legislators, etc.) addressing teachers

$C - Ss$: Many persons and /or groups (authors, educators, legislators, etc.) addressing students

On the basis of the theory of didactical situations in mathematics (Brousseau, 1997) and of the ideas of Ervin-Tripp (1972) about episodes, of Barker (1982) about the levels and types of the communication in the classroom, of Meira and Lerman (2001) about the zone of proximal development as a symbolic space in which various forms of communication can occur, and of Wertsch and Bivens (1993) about the mediational view of a verbal text, I identified the following types of episodes that are observable in the video recordings of my communication in the classroom (Figure 4). I used the interpretation given in this figure to provide an overview of each of the fifteen lessons that constituted the communication about the concepts of the definite and the indefinite integrals and about the fundamental theorem of calculus. Finally, these episodes assisted me in the determination of the durations of the phases that emerged during the lessons.

Episodes	Actions
T-SSd	Devolution
S1-C	Independent problem solving (any student S1 in communication with many ABSENT persons C through tasks; adidactical situation)
T-S1	Dialogic mediation of individual problem solving (the teacher T in dialogue with any student S1)
S1-SS1	Problem solving in small groups (collaboration of any student S1 with other students in any small group (SS1)
T-SS1	Dialogic mediation of problem solving in small group (the teacher T in dialogue with any small group SS1)
S1-SSL	Discussion of students' solutions in a large group SSL (the teacher is not involved in the discussion led by any student S1)
S1-SST	Presentation and discussion of student's solution in whole class in the presence of the teacher (any student S1 leads the discussion)
T-SSW	Discussion of solutions in whole class led by the teacher
T-SSi	Institutionalization

Figure 4. Episodes identified during the teaching

In order to complete the answer to the research question of determining the didactical situations which are likely to further the understanding of the concepts under study, I present in the next section the functions of the communication in the classroom.

3.7. Functions of language in the classroom

This framework related to functions of language also contributed to answering my third question related to didactical situations that are likely to further the understanding of the concepts of the definite and the indefinite integrals and of the fundamental theorem of calculus. It also contributed to answering part of the fourth research question about the circumstances in which understanding occurred. This framework, as the preceding one, is closer to the practice implemented in the communication in the classroom. It specifically concerns the interactions between the teacher and the students when talking about integrals and the fundamental theorem of calculus.

In the analysis of the data related to the communication, I proceeded to analysing the cognitive processes (See section 3.8, p. 79, for the framework of Anderson et al., 2001) and the mathematical objects (as presented in section 3. 2, p. 42) that were conveyed by the message communicated to the students. I especially focused on sentences expressed in the interrogative and in the imperative moods because these moods explicitly invite the addressee to participate in the narrated event (the topic under consideration) and to execute the action conveyed by the verb. In this section, I give details regarding the moods and the functions of language according to Jakobson (1981). The functions of language have been very useful when I was analysing the cognitive processes evoked by the student teachers or by the teacher. At this time, I focused on the conative function (Jakobson, ibid.) which characterises the connection between the action of learning and the student teachers as has been pointed out by Vinogradov quoted later in this section. However, during the analysis of the circumstances in which the student teachers exhibited their understanding, I focused on the expressive and the referential function (Jakobson, ibid.).

The study of classroom discourse in from the linguistics perspective has received attention from some classroom researchers. Cazden (2001) noted that “the study of classroom discourse is…a kind of *applied linguistics* – the study of language use in a given social setting” (p. 3, italics in original). In this respect, Cazden (2001) stated three core categories of functions of language that she named the propositional, the social, and the expressive functions. The propositional function corresponds to the communication of propositional information and it is also called the referential, cognitive, or ideational function; the social function corresponds to the establishment and maintenance of social relationships; and the expressive function corresponds to the expression of the speaker’s identity and attitudes.

In this domain of linguistics, Halliday (as in Lankshear & Knobel, 2004) also developed a social semiotic theory of communication based on three functions, namely, the ideational, the interpersonal, and the textual functions. Users and followers of Halliday's theory have interpreted these functions. Winsløw (2004) said that the ideational function concerns choices that determine the mathematical activity; the interpersonal function establishes the role, the status, and the identity of the participants in the communication; and the textual function realises the mode of discourse such as narrating, giving instructions, reasoning, and so on. According to Winsløw (2004), the study of the ideational function provides a more direct way of assessing the nature and construction of students' beliefs and images regarding mathematics than interviews or questionnaires. (On a more general level, Lankshear and Knobel (2004) said that the ideational function deals with the representation of experiences and things and the connection between them; it involves the sign-maker in making choices about how to interpret objects, interactions, relations between things and experiences; the interpersonal function deals with representing social relations between the producer of a sign and the receiver of the sign; and the textual function deals with coherence of texts as complexes of signs with the context for which they were produced.)

In addition to studying the ideational function, as noted by Winsløw (2004), in order to assess the nature and construction of students' beliefs and images related to mathematics, it is also important to study the interpersonal function in order to identify and characterise influences that other people, such as the teacher or other students, or other circumstances, have had on those beliefs and images constructed by a particular student. The conative function of language as described by Jakobson (1981) is a tool that can be used in such a study in relation to interpersonal influences on beliefs and images of mathematics by students and their teacher.

In the domain of linguistics, Bloomfield and Jakobson made contributions (Bloomfield, 1935, 1939; Jakobson, 1971, 1981) that I adopted in order to analyse verbal interactions that I had with students or that they had among themselves during a mathematics classroom.

Bloomfield (1939) pointed out that "language makes human behavior different from that of animals because it established a minute and accurate interaction between individuals" (p. 15). Once the interaction is established, either the hearer can respond using activities such as manipulation, mimicry, gesture, locomotion, observation, etc., or he or she can respond by addressing another speech to the first speaker or to other persons involved in the speech event.

Besides the establishment of interactions between persons, Jakobson (1981) described other functions of language so that he brought the number of functions of language to six. According to him, these six functions are anchored respectively in the 'factors' of addresser, addressee, context, message, contact, and code that constitute any speech event in any act of verbal communication. The addresser is the person who sends a message to the addressee who receives it; the context is constituted by elements of the environment and those that are actualised by the message; the contact is the physical channel and psychological connection that enable the addressor and the addressee to enter and stay in communication (for example the sound); the code is constituted by the set of signs that are, to some extent, common to the addressor and addressee; the message is constituted by the set of linguistic signs that are announced.

In addition to the factors of the communication, Jakobson (ibidem) determined six functions of the communication anchored in each of the factors. The six functions are: the referential (cognitive, or denotative) function which concerns the objective elements of the strict information included in the message, the expressive (or emotive) function that deals with the sender's judgments, impressions, and emotions on the content of the message, the phatic (or 'of contact') function in which the sender acts on the channel to establish, maintain or eventually interrupt the physical or psychological contact with the receiver, the conative function by which the sender attract the direct attraction of the receiver who has to feel concerned by the message, the poetic function that deals with the aesthetics of the message, and the meta-linguistic (glossing) function by which the sender defines the terms that he wants to be clear for the receiver or that the receiver ignores.

The following figure (Figure 5) summarises the description of the six functions of the communication as conceived by Jakobson (ibidem). To the factor of context corresponds the referential function; to the addresser corresponds the expressive or emotive function; to the addressee corresponds the conative function; to the contact corresponds the phatic function; to the code corresponds the metalingual or glossing function; and finally, to the message corresponds the poetic function.

The six functions of language are also applicable in the context of the communication in the classroom where speech events involving the teacher and the students occur in order to talk about a given topic. The theoretical framework of functions of language is therefore a tool that can help in characterising didactical situations in the classroom. In my study, as I have said, this framework of functions of language helped in answering the research question of what didactical situations are likely to further the student teachers' understanding of the definite and the indefinite integrals and of the fundamental theorem of calculus. It also helped in answering the research question of what learning activities student teachers engaged in and under what circumstances the understanding occurred.

Jakobson' s (1981) framework of functions of language is underpinned by the concept of "shifter" as described by Jakobson (1971) . According to Jakobson (1971), any verb is concerned with a narrated event. In this regard, Jakobson subdivided verbal categories (Gender, Number, Voice, Status, Aspect, Taxis, Person, Mood, Tense, and Evidential) into those which involve and those which do not involve the participants of the event (for more information about these categories, see Jakobson, 1971, pp. 133-143). Moreover, he added that categories that involve the participants may characterise either the participants themselves or their relation to the narrated event. Then he defined "Shifters" as those categories that characterise the narrated event and/or its participants with reference to the speech event or its participants. He then stated that the categories of Person (I, you, he, she, it, we, they), Mood (indicative, imperative, vocative), Tense (past, present, future), and Evidential were shifters while the rest of the categories were Non-shifters.

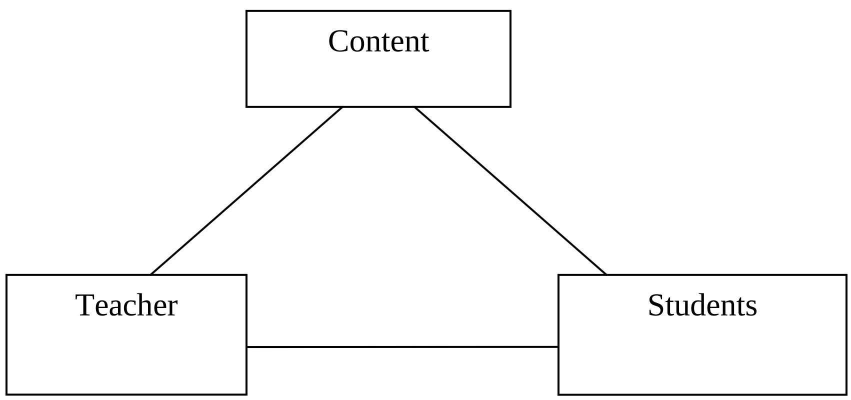
Functions	Factors	Descriptions and Indicators
Referential (or cognitive or denotative)	Context or referent	The referential function aims to orient to the referent. It includes all the information elements. It is the foundations of most of the messages. It is recognizable by the use of the third person pronoun (he, she, they,) and the neutral pronoun (it).
Expressive (or emotive)	Addresser	The expressive function allows the addresser to communicate his or her impressions, emotions, or judgments on the content of his or her massage. It is revealed through the pitch, the intonations, and the rhythm of the discourse. It is recognizable by the use of the first person (I, we), interjections, and the like.
Conative	Addressee	The conative function aims to draw direct attention of the addressee who must feel concerned by the message. It solicits precisely the addressee. It is recognizable by the use of the second person (you) and the use of the vocative, imperative, and interrogative moods. It invokes commands (or orders) and demands (or requests) to the addressee. It invites the addressee to participate in the event or to execute an action.
Phatic	Contact or Channel	The phatic function allows the establishment, the expansion, the maintenance, and the break down of the physical contact and the psychological connection with the addressee or the discontinuing of the communication. It allows verifying whether the message is received by the addressee or checking whether the channel works. The phatic function is recognizable by the use of empty words, words emptied of their actual meanings, politeness words, presence assuring words, solidarity, repetitions, and the like.
Metalingual (or Glossing)	Code	The glossing function allows defining or explaining terms that the addressee ignores. The function appears after expressions of the type 'what does it mean?', 'That means', 'what I mean is', and the like. The function is often instigated by questions of the interlocutor such as "I do not follow you - what do you mean?"
Poetic	Message	The poetic function is based on the combination of verbs and nouns selected by the addresser to communicate the topic at hand. The selection is made on the basis of equivalence, similarity and dissimilarity, synonymy and antonymy, while the combination, the built-up of the sequence, is based on contiguity.

Figure 5. Outline of the Functions of language according to Jakobson (1981)

Among the shifters, the categories of Person and of Mood involve participants of the narrated event with reference to the participants of the speech event. With regard to the category of Person, Jakobson (1971) said that the first person (I) signals the identity of a participant of the narrated event with the performer of the speech event whereas the second person (you) signals the identity with the actual or the potential undergoer of the speech event. In the context of the communication in the classroom the first person can designate either the teacher or any student and the second person can designate also either the teacher or any student depending on the standpoint taken by the analyst of the situation of the communication.

With regard to the mood, Jakobson (1971) said that the mood characterises the relation between the narrated event and its participants with reference to the participants of the speech event. In the same vein, Vinogradov pointed out that this category "reflects the speaker's view of the character of the connection between the action and the actor or the goal" (quoted in Jakobson, 1971, p. 135). Among the moods frequently used to communicate about a given topic to be taught and learned in the classroom, there are the indicative mood, the interrogative mood, and the imperative mood. The interrogative mood calls on a response from the addressee while the imperative mood calls for a participation in the narrated event. Moreover, according to Rotman (1988), the indicative mood, in its declarative aspect, conveys information from the speaker to the addressee while the interrogative asks for information. Again, in this linguistics perspective, Rotman (1988) indicated that the imperative mood has two aspects, namely the inclusive imperatives (Let's go) and the exclusive imperative (Go). According to him, inclusive commands "demand that the speaker and the hearer institute and inhabit a common world or that they share some specific argued conviction about an item in such world" while the exclusive commands "dictate that certain operations meaningful in an already shared world be executed" (p. 9). In mathematics, the former aspect of imperatives, as Rotman (1988) has noted, applies to verbs such as consider, define, prove, and their synonyms, while the latter aspect applies essentially to the mathematical actions denoted by all other verbs such as add, count, multiply, find, calculate, differentiate, integrate, and so on.

My use of Jakobson' s (1981) framework, on the one hand, consisted in identifying sentences whose verbs are expressed in the first, the second, or the third person in order to characterise the participants of the speech event. On the other hand, my use of this framework consisted in discerning the mood that were used, either the indicative mood, the interrogative, or the imperative mood, in order to characterise the relation between the teaching and learning process of the integrals and of the fundamental theorem of calculus and its participants - the teacher and the students - with reference to the participants of the classroom communication which were also the teacher and the students. This relation is, in fact, the classical triangular didactical relation, shown below, representing the interactions between the teacher and the students about a given content (Brousseau, 1997; Jonnaert & Vander Borght, 2003).



If I interpret Vinogradov quoted above, the actual didactical situations created in a 'living' classroom communication depend on the teacher's view of the character of the connection between the student and the action of learning the content under consideration. Here, only the teacher's view is concerned because it is the teacher who, at the first instance, ensures the connection between the student and the content.

This teacher's view, in the case of mathematics classes, is expressed by the moods that he or she uses to connect the students to the mathematical content. Therefore the moods used by a teacher characterise how he envisages connecting the student to the content that is to be learned: Is the teacher indicating the mathematical content (indicative mood)? Is the teacher ordering the students to execute some mathematical activities (Interrogative mood)? Is the teacher asking some questions to students (interrogative moods)?

In relation to the triangular didactical relationship between the teacher and the student when they are interacting about a given topic, my teaching model is pyramidal as shown in the diagram below (Figure 6). It is a more detailed model that I will use to explain and determine the didactical situations that are likely to further the students' understanding of a mathematical concept.

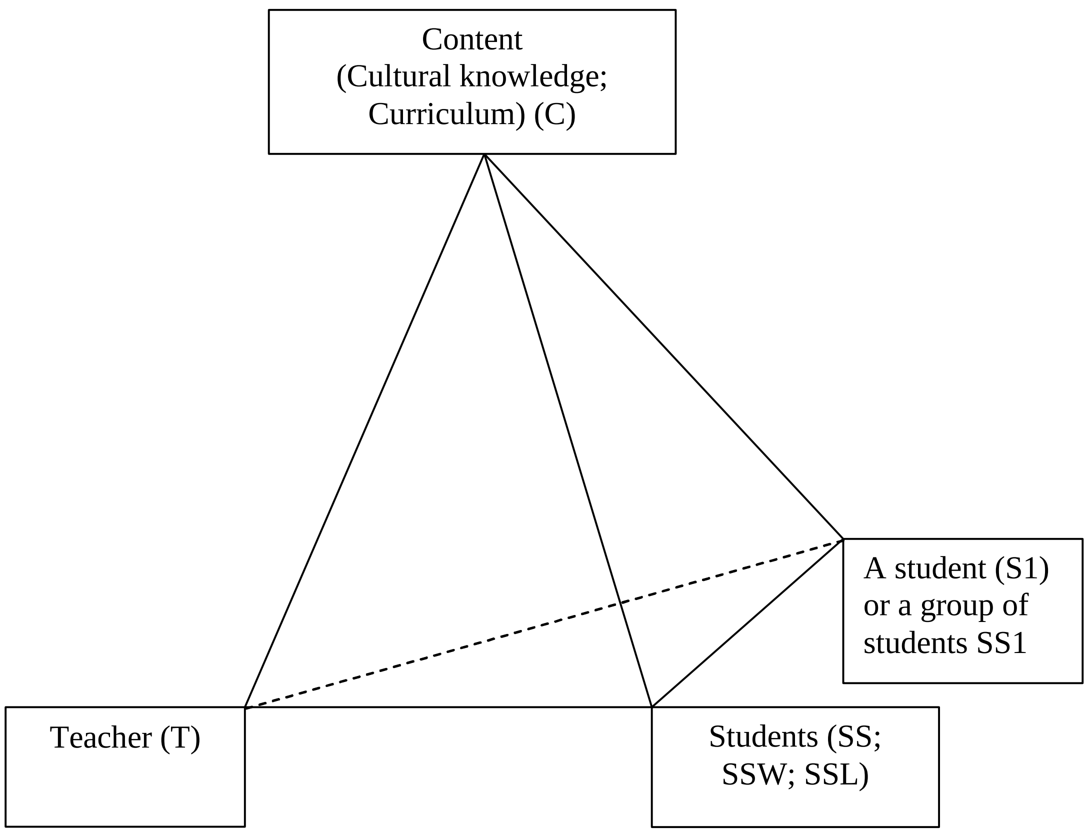


Figure 6. Pyramidal model of teaching

The arrows of the base of the pyramid represent the communicational level between the teacher and the students and among the students themselves. This base represents the moods that the speakers utilise to address their hearer. In accordance with the framework presented in section 3.6 above, these arrows correspond to the episodes discussed previously in Figure 4: The teacher addressing a student (T-S1); The teacher addressing a group of some students (T-SS1); A student addressing the whole class (S1-SSW) or addressing a large group (S1-SSL); The teacher addressing the whole class (T_SSW).

These significations will be the background of the description of the lessons and the calculation of the time spent in each episode of the interactions in order to visualise the importance that was given to the didactical situations under consideration.

In this study, the characterisation of the relationship between the narrated event and its participants with reference to the participants of the speech event will focus on the types of communication T-SSW, S1-SSL, and S1-SST. However, this characterisation can also be applied to interactions of the types T-S1, T-SS1, and S1-SS1.

This point marks the bifurcation point which differentiates my study from almost all the studies I discussed in the literature review. In fact, most of these studies analysed the relation between the narrated event and its participants without reference to the participants (especially the teacher). In reference to the model of Jakobson (1971) that I described above, they analyse the students’ positions vis-à-vis the content without reference to behavior of the teacher; the role played by the teacher in the process of developing the students’ produced mathematics which is analysed by the researchers is not given much attention. These researchers do not characterise the relationships between the teacher and the students that participated in the communication in the classroom. More concretely, they do not categorise the forms of the verbal interactions that occurred between the teacher and the students and among the students themselves. Most of them study the content of the interactions with less attention to the forms of these interactions. Therefore my study aims to contribute to filling this gap.

I used the model above to describe, interpret, and theorise the didactical situations that occurred during my teaching. In using the terms description, interpretation and theorisation I am referring to their definitions as provided by Maxwell (1992) while he was discussing the validity of an account in qualitative research. I give these definitions under the section of validity in the next chapter, methodology.

My pyramidal model is a decomposition of Brousseau' s system that brings into play "the teacher, the student, the student's immediate environment and the cultural *milieu*" (Brousseau, 1997, p. 56).

To complete the presentation of the frameworks that informed my teaching and those that I used to analyse my data, I present, in the following section, a framework for identifying the cognitive processes that are likely to occur in a classroom communication about integrals and about the fundamental theorem of calculus.

3.8. Learning activities and cognitive processes

As I said in the preceding section, the framework under consideration in this section is related to framework of Anderson et al. (2001) about cognitive processes. This framework is a revision of Bloom's (1956) taxonomy of educational objectives. This framework did not inform my teaching; it is an analytical framework for analyzing both the teaching and the learning.

Cognitive processes constitute an important component of learning activities. In what follows, I provide a description of some of the cognitive processes that are likely to occur in a mathematics classroom. The six categories of cognitive processes that I am presenting in Figure 7 below have been categorised by Anderson et al. (2001).

	Cognitive process dimension					
	Remember	Understand	Apply	Analyze	Evaluate	Create
Factual knowledge						
Conceptual knowledge						
Procedural knowledge						
Meta-cognitive knowledge						

Figure 7. Anderson et al. (2001) framework

More details about the cognitive process dimension can be found in Anderson et al (2001, pp. 63-92) while details about the knowledge dimension are provided in Anderson et al. (2001, pp. 38-62). Here, I give only a brief description of the categories and the names of the subcategories of each category.

The category of *remember* concerns the processes of retrieving relevant knowledge from long-term memory. This category has two subcategories, namely, recognising and recalling. The category of *understand* is about the processes of constructing meaning from instructional messages, including oral, written, and graphical communication. This category has seven subcategories, namely, interpreting, exemplifying, classifying, summarising, inferring, comparing, and explaining. The category of *apply* concerns the processes of carrying out or using a procedure in a given situation. It has two subcategories, namely, executing and implementing. The category of *analyse* deals with the processes of breaking down material into its constituent parts and determining how the parts relate to one another and to an overall structure or purpose. It has three subcategories, namely, differentiating, organising, and attributing. The category of *evaluate* concerns the processes of making judgments based on criteria and standards. It has two subcategories, namely, checking and critiquing. The category of *create* deals with the processes of putting elements together to form a coherent or functional whole; reorganising elements into a new pattern or structure. This category has three subcategories, namely generating, planning, and producing. As Anderson et al. (ibid.)

have said, the category of remember is closely related to retention whereas the other categories of understand, apply, analyse, evaluate, and create are related to furthering and transferring knowledge.

Additional categories of cognitive processes emerged from my data. These are the category of *validate* which deals with the processes of defending a position and the category of *explore* which concerns the processes of observing and carefully manipulating a material in order to find out more about it. It is different from the category of 'analyse' which concerns selection according to some criteria. The category of validate has two subcategories, namely, arguing and refuting. And finally, the category of explore has two subcategories visualising and manipulating. These categories of validate and explore are still to be refined.

3.9. Conclusion

In this chapter, I presented theoretical frameworks that I used to plan my teaching and to analyse my data. The first theoretical framework was about the mathematical objects related to integrals and to the fundamental theorem of calculus. The second theoretical framework was about the teaching methods. I presented some concepts from the theory of didactical situations in mathematics as described by Brousseau (1997), and the concept of the zone of proximal development as conceived by Vygotsky (1978) and some of its interpretations (Gallimore & Tharp, 1990; Meira & Lerman, 2001; Wertsch & Bivens, 1993). I also presented some teaching strategies as described by Mason (1999). The third framework was about the communication in the classroom and the functions of language. I presented the types and levels of communication in the classroom as described by Barker (1982a), and Jakobson's (1981) framework of functions of language. Finally, I presented the framework of cognitive processes as described by Anderson et al. (2001). As promised in the introduction of this chapter, I summarise, in Figure 8 below, which theories or concepts contributed to answering which research question. As it can be noticed from this figure, some of the theoretical frameworks helped to answer more than one research question and two research questions were answered by more than one theoretical framework. In the next chapter, I present the methodology and the research methods that I used during the study.

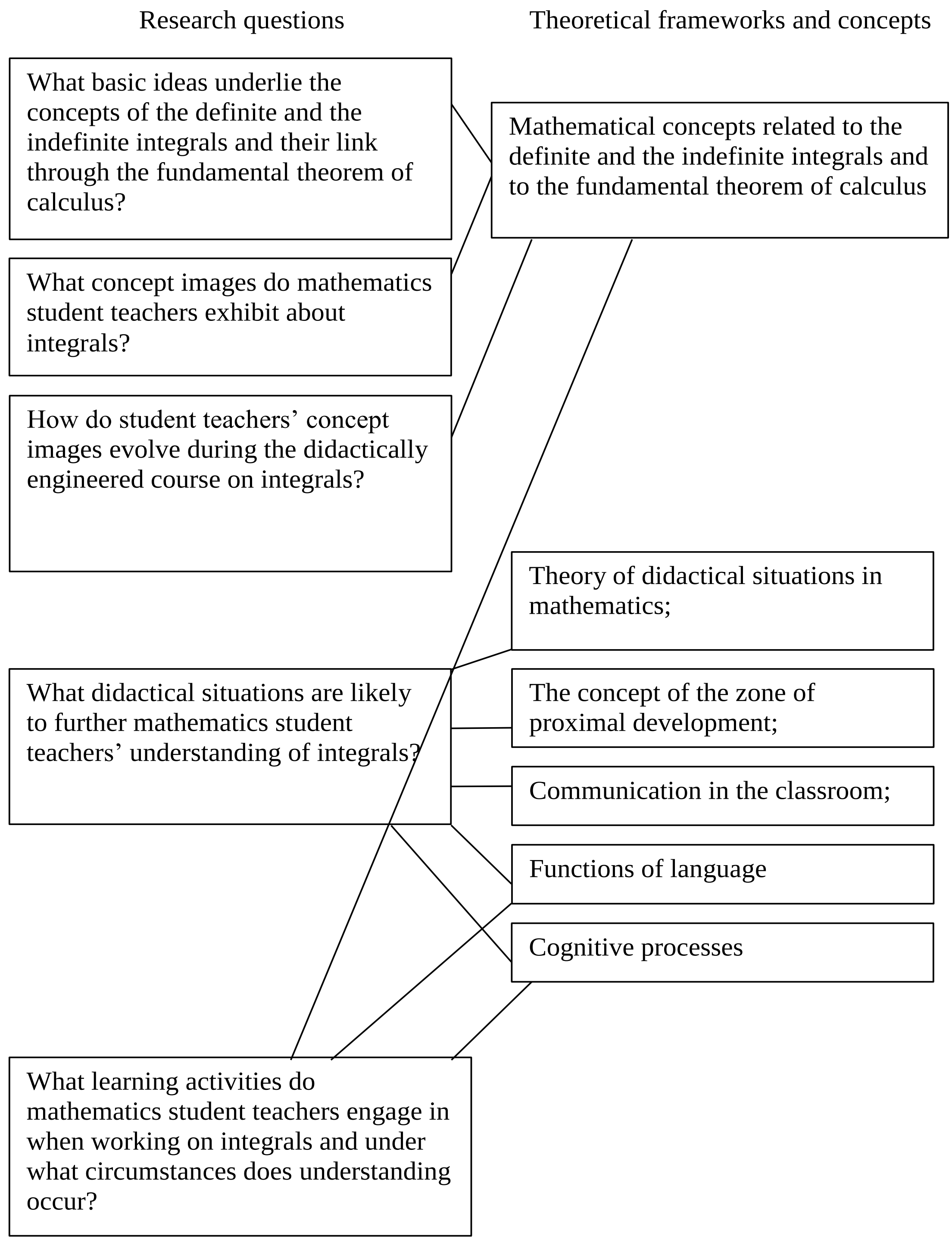


Figure 8. Correspondence between the theoretical frameworks and the research questions

Chapter 4: METHODOLOGY

4.1. Introduction

This chapter is composed of six sections. After this introductory section, the second and the third sections deal with the paradigm of the study. The fourth section presents some ideas about action research, teaching experiments, and didactical engineering. The fifth section concerns the research methods that I used to conduct the study. In this section I present the participants of the study, the methods and the procedures that I used for the collection of data and those that I used to analyse the data. In this same section, I discuss the validity and reliability of the data collection and data analysis and, finally, I present some elements related to ethics. In the sixth section, I conclude the chapter. To start, I present, in the next section, the paradigm of the study.

4.2. Research paradigms

Many authors have been interested in describing research paradigms (Cohen et al., 2000; Freebody, 2003; Guba & Lincoln, 1994; McNiff, 2000; Neuman, 2006). Guba and Lincoln (1994) define paradigms as sets of "basic belief systems based on ontological, epistemological, and methodological assumptions" (p. 107). They said that "the basic beliefs that define inquiry paradigms are summarised by responses given… to the three fundamental questions" (p. 108) related to ontology, epistemology and methodology. Those fundamental questions are "what is the form and the nature of reality and, therefore, what is there that can be known about it?", "what is the nature of the relationship between the knower or the would-be knower and what can be known?", "how the inquirer (would-be knower) go about finding out whatever he or she believes can be known?" (p. 108). Among the main paradigms described by the abovementioned authors, especially by Guba and Lincoln (1994), there are the paradigms of positivism, postpositivism, critical theory, and constructivism – often called interpretivism. Guba and Lincoln (Ibid.) described each of these paradigms in terms of answers they believed proponents of each paradigm would make to the three questions mentioned above. In the following four paragraphs, I summarise these expected answers as conceived by Guba and Lincoln (1994) for each paradigm.

For the paradigm of positivism, at the ontological level, the expected answer of its proponents is that of a naïve realism in which an apprehendable reality is assumed to exist, driven by immutable natural laws and mechanisms. Knowledge of the "way things are" is conventionally summarised in the form of time- and context-free generalisations, some of which take the form of cause-effect laws. At the epistemological level, the expected answer is that of dualism and objectivity. The investigator and the investigated object are assumed to be independent entities, and it is assumed that the investigator is capable of studying the object without influencing or being influenced by it. Finally, at the methodological level the answer is that of experimental and manipulative methods. Questions and/or hypotheses are stated in propositional form and subjected to empirical test to verify them.

For the paradigm of postpositivism, the expected answer at the ontological level is that of critical realism. Reality is assumed to exist but to be imperfectly apprehendable because of basically flawed human intellectual mechanisms and the fundamentally intractable nature of phenomena. At the epistemological level, the expected answer is that of modified dualism and objectivity. Dualism is largely abandoned as not possible to maintain, but objectivity remains a regulatory ideal; special emphasis is placed on external guardians of objectivity such as critical traditions (Do the findings fit with preexisting knowledge?) and critical community (such as editors, referees, and professional peers). At the methodological level, the expected answer is that of the modified experimental and manipulative methods. In this paradigm, the methodology redresses some of the problems noted above by doing inquiry in more natural settings, collecting more situational information, and reintroducing discovery as an element in inquiry, and, in the social sciences particularly, soliciting emic viewpoints to assist in determining the meanings and purposes that people ascribe to their actions. In this paradigm compared to the positivist position, there is an increase of utilisation of qualitative techniques.

For the critical theory paradigm, the expected ontology is historical realism. A reality is assumed to be apprehendable as it has been shaped by social, political, cultural, economic, ethnic, and gender factors, and then crystallised into a series of structures that are then taken as real, that is, natural and immutable. At the epistemological level, the expected answer is of transitional and subjectivist nature. The investigator and the investigated object are assumed to be interactively linked, with the values of the investigator influencing the inquiry. Findings are value mediated. At the methodological level, the answer is of dialogic and dialectical nature. The transactional nature of inquiry requires a dialogue between the investigator and the subjects of the inquiry; that dialogue must be dialectical in nature to transform ignorance and misapprehensions into more informed consciousness in order to see how the structures might be changed and to comprehend the actions required to effective change.

For the constructivism paradigm, at the ontological level, the expected answer is that of the relativism. Realities are apprehendable in the form of multiple, intangible mental constructions, socially and experientially based, local and specific in nature, and dependent for their form and content on the individual persons or groups holding the constructions. At the epistemological level, the expected answer is of transitional and subjectivist nature. The investigator and the object of investigation are assumed to be interactively linked so that the findings are literally created as the investigation proceeds. The conventional distinction between ontology and epistemology disappears, as in the case of critical theory. Finally, methodologies are hermeneutical and dialectical in nature. The variable and personal intramental nature of social constructions suggests that individual constructions can be elicited and refined only through interaction between and among investigator and respondents. The varying constructions are interpreted using conventional hermeneutical techniques, and are compared and contrasted through a dialectical interchange.

Guba and Lincoln (1994) indicated that, for them, the term critical theory denotes a set of several alternative paradigms including neo-Marxism, feminism, materialism, and participatory inquiry. Concerning the participatory inquiry, Mouton (2001) stated that participatory research is a study that involves "the subjects of the research (or research participants) as an integral part of the design" and that it uses "mainly qualitative methods in order to gain understanding and insight into life-worlds of the research participants" (p. 150). After this overview of research paradigms, I give in the next section, my answers to the three defining questions of a paradigm as mentioned above by Guba and Lincoln (1994).

4.3. The paradigm of this study

In this section, I present my assumptions related to the ontological, epistemological and methodological (Guba & Lincoln, 1994) aspects of my study. At the ontological level, the reality of the classroom communication is subject to the context in which it is being created. It is socially shaped and reshaped over time by the participants, namely, the teacher and the students. The types of communication in the classroom involve the teacher and the students. All the time that they meet for teaching and learning a given topic, the types of communication are likely to change at any time of the lesson, depending on the enthusiasm of the participants that are involved in interaction. The structures of the lessons are not static. They are structured according to the state of knowledge of the teacher and the students in the classroom. Therefore, the reality is apprehendable in the form of mental constructions, socially- and experienced- based, local and specific to the context and dependent for its form and content on individual constructions.

With regard to my status of researcher and my relationship with what could be known (Guba & Lincoln, 1994), that is, at the epistemological level, the object to be researched contains to some extent my influence as a person. The students' concept images observed during the study are not independent from my person since I contributed to their constructions through interactions with the students. This situation corroborates the observation made by Guba and Lincoln (1994) about the inseparability of the ontology and the epistemology in the constructivism paradigm. Therefore, researchers need to acknowledge that they are researching realities that they constructed themselves and that their observations are not absolutely exact because of the inevitably human influence.

At the methodological level, the way of knowing was hermeneutical. All along the period of the investigation, I took time to reflect on and interpret what happened and caused changes in the students' concept images of integrals. In sum, given my answers to the three defining questions of a paradigm as defined by Guba and Lincoln (1994), my research paradigm is constructivism.

From the beginning of my study, I identified it as an action research. In order to locate it in a well-known paradigm, I resorted to answer the three questions defining a paradigm as determined by Guba and Lincoln (1994). After answering the three defining questions I located my study in the paradigm of constructivism as I have just mentioned. However, I was somehow confused because in most of cases, action research is under the umbrella of the critical theory paradigm.

My confusion was removed after reading Lincoln (2001) where it is said that "there are several instances where action research and constructivism might be considered indistinguishable, either in theory or in practice" (p. 126). In addition to this statement, Lincoln (2001) elaborated on six ways in which these paradigms may be viewed as symmetrical. These ways include mandates for action, the press for social justice, the new covenant between researcher and researched, the relationships between researcher and academia, the new mandates for what constitutes ethical practices, and expanded epistemologies for mutual learning. After reading these six points which are among the joining points of action research and constructivism, my concerns about the paradigm of my study were tranquillised.

In the perspective of this paradigm, I describe in the next section three concepts that influenced methodologically my study.

4.4. Action research, teaching experiment, and didactical engineering

The concept of action research is becoming a powerful research paradigm (Altrichter, Kemmis, McTaggart, & Zuber-Skerritt, 1991; Elliot, 1991; Henning, 2004; McNiff, 2000, 2002; McNiff, Lomax, & Whitehead, 1996; McNiff & Whitehead, 2005; Mouton, 2001; Zuber-Skerritt, 1991).

Altrichter et al. (1991) gave comprehensive characteristics of a situation in which action research is said to be occurring. They said that such a situation is the one in which "people reflect and improve their *own* work and their *own* situations by tightly interlinking their reflection and action and also making their experience public not only to other participants but also to other persons interested in and concerned about the work and the situation" (p. 8, italics in originals). They stated seven additional conditions to characterise such a situation in which action research is said to be occurring. In brief, these conditions are the data-gathering by participants themselves (or with the help of others) in relation to their questions, the participation in decision-making, the power-sharing and the relative suspension of hierarchical ways of working towards industrial democracy, the collaboration among members of the group as a critical community, the self-reflection, self-evaluation and self-management by autonomous and responsible persons and groups, the learning progressively and publicly by doing and by making mistakes in a self-reflective spiral of planning, acting, observing, reflecting, replanning, etc., and finally, the reflection which supports the idea of the (self-)reflective practitioner.

More specifically in the context of education, Elliot (1991) showed that the main characteristic of action research is the reflection on the relationship in particular circumstances between the teaching process and the learning outcomes. In this reflection the fundamental aim, as Altrichter et al. (1991) have also stated, is to reflect and improve one's own practice in one's own situation.

Action research paradigm has been used and it continues to be used in many domains of social research including education (Elliot, 1991; Feldman & Minstrell, 2000;

Frankenstein, 1987; Lankshear & Knobel, 2004; McNiff & Whitehead, 2005; Shor, 1987; Steffe & Thompson, 2000b; Sutter, 2006; Wallerstein, 1987). In the domain of education, some researchers prefer to use the term 'teacher research' instead of action research as far as action research is done by a teacher in the classroom (Lankshear & Knobel, 2004; Sutter, 2006). However, some debates have arisen about who are teacher researchers. In this respect, I agree with Lankshear and Knobel (2004) that "teacher research can be done in classrooms, libraries, homes, communities and anywhere else where one can obtain, analyse and interpret information pertinent to one's vocation as a teacher" (p. 9).

With respect to the teacher research done in the classroom, Steffe and Thompson (2000b) announced that

> Teaching actions occur in a teaching experiment in the context of interacting with students. However, interaction is not taken as a given - learning how to interact with students is a central issue in any teaching experiment. … If the researchers knew ahead of time how to interact with the teaching experiments' students and what the outcomes of those interactions might be, there would be little reason for conducting a teaching experiment (p. 279).

Moreover, Steffe and Thompson (2000b) stated that "a primary purpose for using teaching experiment is for researchers to experience, firsthand, students' mathematical learning and reasoning" (p. 267).

Thus, there is a difference between 'teacher research' which is research done by the teacher, and a *teaching experiment*, though the latter can certainly carry a strong research component. A teaching experiment can be done by the teacher, which often would focus on effect of teaching actions but that generally implies looking into learning as well. A teaching experiment can be done by researchers who may want to focus on students' learning and reasoning only but who could also look at teaching actions in the light of learning. In my study, the primary purpose was to examine my teaching actions when I would be interacting with the students in the classroom and then to look deeper into the students' learning and thinking. My problem was almost similar to the one evoked by Steffe and Thompson (ibid.): How could I improve my ways of interacting with the students in order to help them learn mathematics meaningfully and not just memorise the mathematical formulas without knowing where they come from?

In this context of interactions between the teacher, the students and the mathematical knowledge, the theory of didactical situations (Brousseau, 1997) provided me with some elements of teaching methods that could help me deal with my problem. In this study, I utilised these teaching methods.

In addition to the concepts described by Brousseau (1997) that I presented in chapter three (p. 55ff), Artigue (1994) claims that the theory of didactical situations is based on a constructivist approach which operates on the principle that knowledge is constructed through adaptation to an environment of a problematic kind to the subject. She continued by stating that the theory aims to become a theory for controlling teaching situations in their relationships with the production of mathematical knowledge. She also said that the theory considers didactical systems made up of three mutually interacting components, namely, the teacher, the student, and the mathematical knowledge.

Another important concept often evoked in the context of the theory of didactical situations is the concept of *didactical engineering*. According to Artigue (1994) the term didactical engineering labels “the work involved in the preparation of teaching contents” (p. 29). Moreover, Artigue (ibid.) added that alongside the elaboration of the text of the knowledge under consideration, the term didactical engineering encompasses “the setting of this knowledge in situations that allow their learning to be managed in a controlled manner” (p. 29). According to Artigue (1994), there are two types of didactical engineering, namely the didactical engineering of research and the didactical engineering of production. On the one hand, Artigue said:

> Didactic engineering, seen as a research methodology, is, firstly, characterised by an experimental schema based on class[room] ‘didactic sequences’, by which we mean based on the design, the production, the observation, and the analysis of teaching sequences. Here classically two levels are distinguished, *micro-engineering* and *macro-engineering*, depending on the size of the didactic sequences involved in the research. (Artigue, as quoted in Flückiger, 2005, p. 61, italics and square brackets in original)

On the other hand, according to Artigue (1994) the didactical engineering of production is “more concerned with satisfying the classical conditions imposed on engineering work: effectiveness, power, adaptability to different contexts, and so forth” (p. 35).

Also, according to Artigue (1994), the label of didactical engineering is viewed as a means to approach two questions, namely “the question of the relationship between research and action on the teaching system” and “the question of the place assigned within research methodologies to ‘didactical performances’ in class” (p. 30). Moreover, she said that the route that the didactical engineering would take was determined by this double function.

In the case of my study, I used the two aspects of the didactical engineering simultaneously. In fact, I produced teaching materials and used them to teach the mathematical knowledge that they contained. However, I did not follow all the four phases of the didactical engineering as they are proposed by Artigue (1994). My didactical engineering was implemented simultaneously with its production. After the production and the implementation of the first part of the engineering (preparation of the problem containing the knowledge to be learned and the setting of it in the didactical situations), I used the outcomes of this implementation to produce the next part of the didactical engineering. In order to produce this part of the didactical engineering, I analysed the student teachers’ mathematical productions and I adjusted the content accordingly. Also, I took into account these productions in setting the new knowledge in the new situations of the next lesson. During the implementation of my didactical engineering I asked the student teachers to evaluate the teaching methods that I was using. As a consequence of their critiques I adjusted the organisation of the interactions in the classroom. Thus my didactical engineering contributed to the creation of a contextual reality to be researched in a cycle of actions of design, production, implementation, observation, analysis, evaluation, re-design.

In sum, my study involved aspects of the three concepts that I described in this section. The student teachers and I formed a speech community (Bloomfield, 1935; Hymes, 1972) in which we structured our way of teaching and learning integrals and the fundamental theorem of calculus. The abovementioned concepts also underpinned the methodology that I used to construct and research the researchable reality. This reality concentrated on “didactical systems built up around a teacher and his or her students, …, plunged in the global teaching system, and open … to the society in which the teaching system is located” (Artigue, 1994, p. 29).. In the next section I present the research methods that I used during the study.

4.5. Research Methods

4.5.1 Participants of the research

4.5.1.1. The teacher-Researcher

I am a male of 50 years old. I did my studies leading to B.Sc. (Mathematics – Physics) at the National University of Rwanda (Rwanda) from 1980 to 1983 and I did my studies leading to M.Sc. (Mathematics) at the University of Sherbrooke (Canada) from 1983 to 1987. After my studies, I worked in various areas of Education and of Administration in Rwanda. I taught at the National University of Rwanda as an assistant lecturer (two years); I worked at the Central Bank of Rwanda as an analyst-programmer (five years); I worked in a private enterprise as a human resources manager (two years); and I worked in the Rwandan Ministry of Education (seven years), first in the division of Educational Statistics and Planning and after that, in the division of Teacher Management and Development. Since year 2002, I have been a lecturer at Kigali Institute of Education which granted me a study leave to pursue my doctoral studies at the University of KwaZulu-Natal.

I was in Rwanda when the 1994 genocide was perpetrated in this country. Most of the human and material resources have been destroyed. After this 1994 genocide, my country made a great effort to reconstruct itself in all domains and reconstructing education was amongst its top priorities. Especially, my country reached an excellent position with regard to access in Primary and Secondary Education; even in Tertiary Education, the number of students is increasing at a satisfactory rate. However, with regard to quality in education (qualified teachers and availability of teaching-learning materials), my country has still to invest more efforts and means to fill the gap caused by the 1994 genocide. My situation as teacher-researcher is located in this context of post 1994 genocide reconstruction of education in Rwanda; also, as a father of three adolescents and having adopted an orphan, I would like them and all Rwandan children to benefit from a high quality education to which the outcomes of my research will have contributed.

The Government of Rwanda has managed to set up various strategies to fight against any genocide ideology. Among its main realisations, there are the creation of a commission in charge of fighting against any genocide ideology, a commission of national unity and reconciliation, and the instauration of a national identity which does not contain ethnic identification. All these strategies contribute to the creation of a unified Rwandan People. My belief which I share with many Rwandans is that Education constitutes a "true" path through which fighting against genocide ideology and constructing unity and reconciliation of Rwandans will be achieved.

4.5.1.2. The student teachers

Eleven student teachers whose ages range between 23 and 34 years, with a mean of 26 years, participated in the study. It was a convenient class of student teachers who were registered for their first year at the Institute. Nine were male and two were female.

It is not necessary to know the students' ethnic group. As I have said in the preceding subsection, the National Identity does not show the ethnic identification and the body appearance is very misleading. At my best knowledge, in the present Rwandan Government, the ethnic identification does not influence the orientation of students in various sections of study, such as Mathematics, Sciences, language, Social studies, engineering, and so on. Thus, it is useless to wonder about this information.

Eight of them had followed the option of Teacher Training during their secondary education and the other three had followed the option of Biology and Chemistry. All of them had been exposed to some notions of calculus during their secondary education. According to the national curriculum (Ministry of Education, 1999, 2000), they had learned some notions of integrals in grade twelve. In the next two subdivisions, I present lists of the topics of integration that they had dealt with during the secondary education and after that I present the list of topics of integration that they were expected to deal with in Kigali Institute of Education.

4.5.1.3. Lists of topics related to integration in Teacher Training option

According to the curriculum of the Teacher Training option (Ministry of Education, 2000, pp. 24-27), eight of the eleven student teachers had encountered the following topics:

Integration of functions

Integral functions
-Simple examples of integral functions and definition of integral function
-Relation between integrals of two functions over the same interval
-Existence of a unique integral
-The notation $\int f(x)dx$
-Properties of linearity
Integral as the inverse of differentiation
-Integration of constant functions
-Integration of power functions (x^r where $r \in Q \setminus \{-1\}$)
-Integration of trigonometric functions
Calculation of integrals
-Integral of sum of functions
-Integral of a product of functions
-Integral of polynomial functions

Integration of continuous function
-Definition of the integral of a continuous function over the interval $[a,b]$
-Use of the integration to find area of a plane surface
-Properties of integrals
-An integral is positive [‘Positivity’]: if $a \le b$ and $f \ge 0$, then $\int_a^b f(x)dx \ge 0$
-Linearity with respect to the function

$$\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

-Methods of integration
-Integration by decomposition,
-Integration by parts,
-Integration by change of variable)
-Application of integration
-Area of a surface limited by an arc of a curve, the x-axis and two lines parallel to the y-axis
-Area of surface limited by two arcs of curves and two lines parallel to the y-axis.

4.5.1.4. List of topics related to integration in Biology and Chemistry option

According to the curriculum of Biology and Chemistry option (Ministry of Education, 1999, pp. 35-37), three of the eleven student teachers had encountered the following topics:

Anti-derivatives

-Definition
-Properties
-Continuity of the anti-derivative of a function
-The set of the anti-derivatives of a function
-Immediate anti-derivatives
-Formulae which are necessary to know, which give anti-derivatives of function, which associate to x a constant function, the power functions x^n where $n \in Q \setminus \{-1\}$, or the trigonometric functions
-General methods of anti-differentiation
-Anti-differentiation of a sum of function a product of function by a constant (linearity property)
-Anti-differentiation by parts
-Anti-differentiation by change of variable
-Anti-differentiation by combination of the preceding methods
-Anti-differentiation of some classes of functions
-Rational functions
-General rule (without proof) based on the decomposition of a proper fraction into a sum of simple fractions
-Irrational functions
-Trigonometric functions (using trigonometric identities and change of variables)

Integral of a continuous function

-Definition
-Properties
-Linearity with respect to functions
-Permutation of limits and equal limits
-Additivity with respect to intervals (Chasles' relation)

-Positivity [if $a \leq b$ and $f(x) \geq 0$, then $\int_a^b f(x)dx \geq 0$]
-Mean value theorem
-Integral with upper limit variable
-Fundamental Theorem of Calculus
-Methods of integration
-Applications:
-Calculation of area of a surface
-Calculation of a volume of solid of revolution
-Calculation of lengths

4.5.1.5. List of topics related to integration at Kigali Institute of Education

According to the curriculum at the Kigali Institute of Education (Kigali Institute of Education, 2000, p. 17), all the student teachers were expected to study the following topics. I list all the content of the course that the student teachers had to study but the research concerned only the teaching-learning process of integration.

Functions
-Domains, images, pre-images, specification of functions
-Composition of functions, special and inverse functions; Injection, surjection and bijection

Differential calculus
-Limits, continuity, derivatives, rules for derivatives
-Rolle's Theorem, Lagrange's theorem, Taylor's formula, Maclaurin's formula, curves of functions

Integration
-Indefinite and definite integration
-Integration methods
-Exponential and logarithmic functions (their differentiation and integration)
-Mean value theorem
-Determination of area of a plane surface and surface of revolution
-Improper integrals
-Sequences and Series

After this presentation of the backgrounds of the two categories of the participants of the research, namely the teacher, who was at the same time the researcher, and the student teachers, I present the methods and the procedures that I used to collect the data.

4.5.2 The data collection methods

Considering my research questions, the data that were to be collected from the participants were the student teachers' concepts images, the didactical situations in the classroom as materialised by the communication between the teacher and the student teachers and among the student teachers themselves, and the understandings expressed by the student teachers. These are all spoken data. In this regard, Lankshear and Knober (2004) exposed various methods of collecting spoken data in qualitative teacher researcher. Among these methods there are interviewing and contextualised recording.

According to Lankshear and Knobel (2004) interviews are planned, prearranged interactions between one or more people, where one person is responsible for asking questions pertaining to a particular theme or topic of formal interest and the other (or others) are responsible for responding to these question (p. 198). Three types of interviews are structured, unstructured, and semi-structured interviews. As Lankshear and Knobel (2004) have stated, structured interviews aim to maximise comparisons across responses to questions of the interview; the key characteristic of such interviews is a pre-set list of questions asked in a fixed order and there are no deviations from the list, regardless of the response to the question asked. With regard to unstructured interviews, they do not have any pre-prepared lists of questions. The researcher begins the interview with a general idea of the topic to be discussed in mind, but lets the interviewee determine the direction of the interview and the ground covered in the discussion. Unstructured interviews aim to solicit as much information as possible without confining the respondent to particular themes or topics. Semi-structured interviews are between structured and unstructured interviews. They include a list of pre-prepared questions, but the teacher researcher uses these questions as guide only, and follows up on relevant comments made by the interviewee. Semi-structured interviews have the inconvenience that they can never be repeated in exactly the same way with each interviewee. However, like the unstructured interviews, they encourage elaboration of themes emerging in the course of the interview. The researcher can always compare different responses to the same questions, while at the same time remaining open to important but unforeseen information or points of discussion.

With regard to the contextualised recording of spoken data, Lankshear and Knobel (2004) said that the researcher focuses on part of an event or activity as it occurs by using recording devices, such as audio or video tape, to capture speech *in situ* in order to focus on language uses and language processes relevant to his or her research question. They said that contextualised spoken data occur naturally within a given setting like a classroom, staffroom, school reception area, and so on. They added that recording contextualised spoken data requires preserving as far as possible the complexity of and relationships between the interactions, activities and language uses that take place during the course of an event. Within educational research, much contextualised recording of spoken data occurs in classrooms. According to Lankshear and Knobel (ibid.), teacher-researchers record contextualised verbal interactions in classrooms in order to better understand the social functions and effects of language.

During my study, I used these two methods. I used the interviewing method, in particular semi-structured interviews, to collect student teachers' evoked concept images since these semi-structured interviews have advantages as exposed by Lankshear and Knobel (2004) and summarised above. I used the contextualised recording method, to collect data related to didactical situations and to student teachers' understanding exhibited during the communication in the classroom.

Lankshear and Knobel (2004) described what written data in qualitative teacher research are. They exposed two groups of written data. The first group comprises extant texts or documents and the second group includes written responses to open-ended questions in surveys and questionnaires, or journals that participants have produced on the request of the researcher. In this study, the written data that I used are written tasks that I conceived and devolved to the student teachers during the teaching, written summaries produced by the student teachers after the third lesson of the teaching, and recordings of what was written on the chalkboard. However, these tasks and the summaries have been motivated and prompted by the research (Lankshear & Knobel, 2004). Therefore, the written data that I used in my study do not fall in the category of extant texts or documents. I will consider them as spoken data produced during the communication in the classroom. In the next two sections, I present the implementation of these methods. I start by the implementation of the semi-structured interviews.

4.5.2.1. Semi-structured interviews

In order to prepare the teaching, I held the first round of the semi-structured interviews with student teachers five months before the teaching in order to identify their concept images about integrals. All the student teachers reported in this study participated individually in the semi-structured interviews at this time of the first round of the interviews.

Five months after the first round of the interviews, I started the actual teaching. I spent six hours on teaching functions, nine hours on teaching limits, fifteen hours on teaching derivatives, and thirty two hours forty minutes on teaching integrals and the fundamental theorem of calculus. As I said earlier, the research concerned only the part explicitly addressing integration.

Ten days before I started the actual teaching of the part of the course related to integration, I held the second round of the semi-structured interviews in order to seek for any eventual evolution of the student teachers' concept images of the definite integral.

At the end of the teaching, I held the third round of the semi-structured interviews in order to see what concept images that student teachers would exhibit at that period of time.

For all three rounds of the semi-structured interviews, I met with the student teacher one by one in a room that used to be a teacher's office and that was reorganised for the occasion of these interviews. In the interview room, student teachers were given a paper, a pencil, and a ruler in a case he or she would like to write and to draw. All the interviews were conducted in French and they were all recorded on audio tapes. Appendix A presents the questions I asked the student teachers during the three rounds of interviews.

4.5.2.2. Contextualised recording of spoken data in the classroom

The data related to the communication in the classroom have been collected through a sequence of fifteen lessons dealing with the topics of integration. . When I was teaching the concepts of integrals and the fundamental theorem of calculus, I hired a person to video record all the sequences of the teaching-learning process. During the teaching period, the hired person moved around the classroom to capture the situations from various angles. Sometimes she could video record a particular student when I was in interaction with this student. When a student was presenting his/her solution on the chalkboard or when I was expounding at the chalkboard, the video recorder was then oriented there. The intention was to record as many central events as possible.

4.5.2.3. The Communication in the Classroom about Integrals and the FTC

The communication about the concept of the definite integral was done during the first six lessons. During the following four lessons we communicated about the concept of the indefinite integral, and during the next four lessons, we communicated about the fundamental theorem and the student teachers did some exploration of integrals on computers using the Geometer's sketchpad software. The last lesson was devoted to the self-evaluation about their learning and the teaching methods.

In order to communicate about each of the mentioned topics, I first prepared tasks that I devolved to student teachers during the lessons. These tasks are reproduced in appendices B to G together with an indication of possible solutions. The first task that I devolved during the second lesson and the second task that I devolved during the third lesson are in appendix B. These tasks were about the calculation of areas under curves. The third task that I devolved to the student teachers in lesson seven is in appendix C. This third task was about the concept of the indefinite integral. Some of the students' solutions to this task are in appendix D. The fourth task is in appendix E and concerned the fundamental theorem of calculus - the function version. I devolved this fourth task to student teachers in the eleventh lesson. The fifth task is in appendix F and concerned the fundamental theorem of calculus – the number version. The sixth task is in appendix G and concerned the exploration of integrals on computers using the Geometer's Sketchpad Software. I devolved this sixth task during the thirteenth lesson.

In what follows, I give the list of the lessons, the topics that were to be spoken about during the specified lesson, and its estimated duration, based on the video recordings. In addition, I give in brief the material that I prepared and the strategies that I used to organise the communication in the classroom.

Lesson 1: **Time: 2h02**
Topic: What questions do students have about integrals?
Defining the general objective and the content of the course: In this lesson, students were brought to see that they could perform calculations but do not know the underlying reasoning; a cognitive conflict was created. The students took decision and made commitments about what they would like to know, namely, the underlying reasoning. They contributed to orient the content to be taught and learnt.
Material: Questions to test prior knowledge
Strategy: Questions-answers interactions;
Devolution of the problem;
Individual work, work in small groups and discussion in the whole class;

Lesson 2: **Time: 1h48**
Topic: Evaluation of area under a curve
Material: Task one about determining area under a curve, restricting students from using anti-derivatives
Strategy: Devolution of task one;
Problems generated by students themselves;
Working individually (adidactical situation);
Teacher's mediation to each student at seat place: Students were advised to use subdivisions and geometrical figures in order to evaluate areas.

Lesson 3: **Time: 2h00**
Topic: Evaluation of area under a curve
Material: Task two about area under a parabola
Strategy: Devolution of task two;

Working individually (adidactical situation);
Teacher's mediation to each student at seat place: Students were advised to use subdivisions and geometrical figures that they were able to evaluate areas.

Lesson 4: **Time: 2h51**
Topic: Evaluation of area under a curve (commencement of students' presentations)
Material: Rules of summation
Strategy: Devolution of the situation;
Management of interactions;
Students' presentations on the chalkboard in front of the whole class;

Lesson 5: **Time: 2h07**
Topic: Evaluation of area under a curve (continuation of students' presentations)
Material: Rules of summation
Strategy: Devolution of the situation;
Management of interactions;
Students' presentation on the chalkboard in front of the whole class;
Homework at the end of the lesson;

Lesson 6: **Time: 2h14**
Topic: Evaluation of area under a curve (end of students' presentations)
Self-Evaluation of students' learning and the teaching methods
Material: Definition of the definite integral and notations
Strategy: Each student presents what he or she has learnt up to then and gives comments about the teaching methods that are being used;
Management of interventions;
Institutionalisation of the definition of the definite integral;

Lesson 7: **Time: 1h45**
Topic: The indefinite integral and the rediscovery of the FTC
Material: Task three about the indefinite integrals and the rediscovery of the FTC
Possible solutions of the task
Strategy: Devolution of the task;
Working in small groups (adidactical situation);
Teacher mediation in small groups;

Lesson 8: **Time: 2h00**
Topic: The indefinite integral and the rediscovery of the FTC
Material: The general objective of the course
The formula of the definite integral
The formula of the indefinite integral
The rules of summation
Strategy: Reminding the general objective;
Reminding the formula of the definite integral and of the indefinite integral;
Reminding the rules of summation;
Working in small groups (students);

Teacher mediation in small groups;

Lesson 9: **Time: 2h25**
Topic: The indefinite integral and the rediscovery of the FTC (beginning of students' presentations)
Material: The same as in lesson 8
Strategy: Devolution of the situation;
Students' presentations on the chalkboard;
Management of interventions;
Homework about indefinite integrals;

Lesson 10: **Time: 2h00**
Topic: The indefinite integral and the rediscovery of the FTC (end of students' presentations)
Material: The same as in lesson 9
Handout on the definite integral, the Riemann sums, the indefinite integral and the statement of the FTC (see point 1 of appendix G)
Strategy: Devolution of the situation;
Students' presentations on the chalkboard;
Management of interventions;
Institutionalisation of the definite integral, the Riemann sums, the indefinite integral and the FTC;

Lesson 11: **Time: 3h41**
Topic: The proof of the FTC - function version (Task four)
Material: A set of unordered numbered slips of paper of the components of the proof of the FTC, part 1
Strategy: Devolution of the situation / distribution of the numbered slips of paper;
Reconstruction of a mathematical proof;
Individual work followed by presentations on the board and discussion in the whole class;
Management of interventions;
Frequent intervention of the teacher during the discussion because the strategy of reconstructing proof was new for the students;

Lesson 12: **Time: 3h00**
Topic: The proof of the FTC – function version (continuation and end of the reconstruction) and the proof of the FTC – number version (Task five)
Material: The same as in lesson 11 for the FTC, part 1 and
A set of unordered numbered slips of the components of the proof of the FTC – number version
Strategy: Devolution of the situation / distribution of the numbered slips of paper;
Reconstruction of a mathematical proof;
Individual work followed by presentations on the chalkboard and discussion in the whole class;

The teacher did not participate (he was out of the classroom) in the discussion, in order to test the students understanding of the process of reconstruction of the proof;
Institutionalisation of the proof of the FTC, part 1 and 2;

Lesson 13: **Time: 2h26**
Topic: Exploration of integrals with Geometer's Sketchpad: Building area and cumulative area (Task six)
Material: Computer and the Sketchpad files (Building area and cumulative area)
A set of instructions of the task
A set of unordered numbered slips of a proof of a property of the definite integral
Strategy: Observation on computer in small group (say what you see);
Writing individual report of observations;
Reconstructing the proof of one property of the definite integral as an exercise while waiting for their turn to observe on computer (one computer was available for the three small groups);

Lesson 14: **Time: 1h33**
Topic: Exploration of integral with sketchpad (students' presentations)
Material: Possible answers of the questions of the observations
Computer and sketchpad files for further observations
Strategy: Devolution of the situation;
Students present on the chalkboard in front of the whole class;
Management of interventions;
Institutionalisation of the possible results of explorations (writing on the board);

Lesson 15: **Time: 0h48**
Topic: Evaluation of learning and teaching methods
Material: Questions to be answered
Propositions of a plan of dates for interviews and exam
Strategy: Devolution of the situation;
Students present in front of the whole class;
Teacher to be prepared mentally to receive all kinds of critiques;
Teacher to be prepared to listen and to thank students for their comments and their contributions during the whole course;
Teacher thanks students and provides concluding comments.

After each lesson, I did some reflections (Elliot, 1991) and some activities in order to prepare the next lesson: Viewing the video recordings, reflecting on what happened in class, viewing students' scripts, reflecting on students' productions, deciding on what to do during the next lesson, preparing material for the next lesson, reflecting on my behaviour and deciding on my behaviour for the next lesson, and finally setting up a new plan for the next lesson taking into account all the results of the reflections and decisions.

4.5.2.4. Data capturing and data editing

The interviews and the classroom communication were transcribed in French and translated into English. The analysis presented in this thesis is based on these translations. The persons who served as transcribers and translators of the video recordings of the classroom communication are bilingual and close to the context in which I conducted the teaching. For the interviews, I transcribed and translated the tapes so that the texts cohere with what the student teachers said. During the analysis, I continually reviewed the videos recordings and, on some occasion, the tapes in order to minimise errors.

4.5.2.5. Validity and Reliability of the collected data

With regard to the validity of the data to produce what I aimed to find out through the data collection (Lankshear & Knobel, 2004), the communication in the classroom produced the expected results in terms of didactic situations (Brousseau, 1997, 2004) and interactions in the classroom communication (Barker, 1982a) between the teacher and the student teachers and among the student teachers. Still, I agree that the produced results are valid in the particular context in which the data have been collected.

The semi-structured interviews made the student teachers evoke their concept images of the mathematical concepts (Tall & Vinner, 1981) that they were asked to deal with. However, these concept images are highly dependable on the context in which the questions were embedded. In fact, depending on how the questions are asked or on which conditions or which milieu (Brousseau, 1997) surrounds the questions, the student teachers can dig deeper (Ryve, 2006) and may produce a response that reveals a different concept images.

4.5.3 Data analysis methods, Coding, and Validity and Reliability

4.5.3.1. Sociolinguistics and Classroom discourse analysis

Besides the description of methods of collecting spoken data, Lankshear and Knobel (2004) described methods of analysing spoken data in qualitative teacher research. They described, among others, the sociolinguistics and discourse analysis. They describe the former thusly: "sociolinguistic approaches to analysing spoken data focus on language as it is used in social interactions" (p. 280). They said, firstly, that when one analyses data sociolinguistically, one identifies elements of language use that indicate each speaker's social and cultural understandings and practices. Secondly, one identifies "things that get done in the world through language use, how they get done and with what consequences for which people, by looking at how language operates as a social practice within particular contexts" (p. 280). I used the first aspect of this perspective of sociolinguistics to establish comparison of the student teachers' evoked concept images and I used the second aspect to describe effects of functions of language in instigating student teachers' understanding.

With regard to discourse analysis, Lankshear and Knobel (2004), said that discourse analysis is a branch of sociolinguistics but it differs from other sociolinguistics approaches to analysing spoken text in two ways. Firstly, in discourse analysis the data analysis focuses primarily on the purpose and meaning of the overall text prior to employing more detailed forms of analysis of the types used in sociolinguistics, or in linguistics. Secondly, discourse analysis "deliberately draws attention to complex relationships in language use, social systems and social structures or institutions" (p. 290). In the domain of linguistics, I made recourse to Jakobson's (1981) framework for discourse analysis. I described this framework in the preceding chapter. I used this framework to classify functions of language that I and the student teachers used during our interactions in the classroom.

After the analysis of discourse by using Jakobson's (1981) framework, I quantified the findings using the concept of the density of communication as described by Bloomfield (1935). Bloomfield (1935) defined this concept in the following terms:

> Imagine a huge chart with a dot for every speaker in the community, and imagine that every time any speaker uttered a sentence, an arrow were drawn into the chart pointing from his dot to the dot representing each one of his hearers. At the end of a given period of time … this chart would show us the density of communication within the community (p. 46).

This concept of density of communication is important in that it can be used to "explain the likeness and the unlikeness between various speakers in the community, or, … to predict the degree of likeness for any two given speakers" (ibid. , p. 47), in the occurrence the teacher and the student teachers.

Using these frameworks and the theoretical frameworks regarding the theory of didactical situations and the concept of the zone of proximal development that I presented in the third chapter of this thesis, I described the lessons that I conducted with the student teachers. Moreover, combining these frameworks with Anderson et al.'s (2001) framework of cognitive processes and Dreyfus's (1991) view of the process of understanding, I analysed the cognitive processes that led some of the student teachers to exhibit their understanding after a long sequence of various mental processes (Dreyfus, 1991) and I described the circumstances in which these student teachers' understanding occurred. In addition to these overarching frameworks, I used the techniques of manifest coding and of latent coding (Neuman, 2006) during the actual analysis.

4.5.3.2. Manifest and latent coding

The manifest coding, as described by Neuman (2006) consists in "coding the visible, surface content in a text" (p. 325) while using latent coding (also called "semantic analysis" (Neuman, 2006) the researcher "looks for the underlying, implicit meaning in the content of a text" (p. 326). As I said previously, I resorted to these two ways of coding in all my analyses. During my analyses of a given part of a spoken text, I first looked for the visible surfaced content and then I looked for the underlying meaning in the content of the text.

4.5.3.3. Validity and reliability of the analysis of the data

All qualitative researchers agree that not all possible accounts of some individual, situation, phenomenon, text, institution, or program are equally useful, credible, or legitimate. The ways in which researchers make these discriminations do not pertain entirely to the internal coherence, elegance, or plausibility of the account itself, but often refer to the relationship between the account and the phenomena that the account is about. Maxwell (1992) connected validity to "this relationship between an account and something outside of that account whether this something is construed as objective reality, the constructions of actors, or a variety of other possible interpretations" (p. 283).

In my study that "something" that is invoked by Maxwell in his description of validity is the process of teaching and learning of integrals and the fundamental theorem of calculus. In my account of it, I emphasised the teaching process rather than the learning process because I wanted to learn how the teacher could interact with students so that the teaching and the learning become efficient. However, I am convinced that there are other possible accounts of the data that I presented in this study and that they are also valid from other perspectives. Therefore in my study, I do not claim that my account is absolute but rather I recognise that my account depends on the purposes, the circumstances, and the perspectives that I adopted during the analysis.

As Maxwell (ibid.) has added, there is not only one correct objective account of this realm outside of the account itself and

> As observers and interpreters of the world, we are inextricably part of it; we cannot step outside our own experience to obtain some observer-independent account of what we experience. Thus, it is always possible for there to be different, equally valid accounts from different perspectives. (p. 283)

This is in fact what characterises the paradigm of constructivism.

Concerning the actual analysis, as I said previously, I used the manifest coding which is reputable to be credible (Neuman, 2006). In the cases where the manifest coding was not likely to be useful, I applied the semantic coding (Neuman, 2006). However, in the actual analysis, both types of coding were used simultaneously when I was coding the evolution of student teachers' concept images and when I was determining the cognitive processes in which the student teachers were engaged in during the lessons. When the manifest coding and the semantic coding coincided, then there was not much need for further interpretation as the category was already determined. In that case the credibility was high (Neuman, 2006). In some cases where the manifest seemed to differ from the semantic, I resorted to interpretation of what the respondent could want to mean in the considered context.

I also used interpretation to delimitate episodes that emerged during the teaching and to classify the cognitive processes that were part of the learning activities. With regard to the cognitive processes, most of them were manifest in the interactions. During the analysis, I interpreted the cognitive processes from both the standpoints of the students and the teacher's intentions. In fact, in my study, I emphasised the analysis of the process of teaching more than the process of learning since my purpose was how I could improve my practice as a teacher. However, the credibility of my analysis is increased by the fact that I used evaluative frameworks to interpret what I had communicated to students (Cohen et al., 2000; Maxwell, 1992).

A final comment about validity concerns the language of the study. Even though I can understand and write English, it is my third language. Neither the persons who transcribed and translated the data are completely bilingual. Therefore, I agree that, despite the efforts that I deployed to check and to edit the data, some errors of translation from one language to another or in the interpretation of some of the data during the analysis, might have escaped my attention.

4.5.4 Ethical issues

To deal with ethical issues, firstly, I got the informed consent from my student teachers and thereafter I managed to protect and respect the privacy and the dignity of the student teachers (Lankshear & Knobel, 2004). During the lessons the participation to communicate his or her solution was voluntary. During the interviews, the student teacher selected a date which was convenient to him or her and could postpone it until he or she felt ready to come for the interview. To guarantee the anonymity of the student teachers, I used pseudonyms in the transcripts of the interviews and of the classroom communication and I used an anonymous style in reporting what happened in the methodology section and everywhere in the writing up of this thesis. Concerning confidentiality, I selected transcribers and translators who are aware of research ethics so that they can not divulgate what they saw and heard in the video recordings. After the transcriptions, I stored the tapes and the video recordings in a secured place. In the text of the thesis, I only described the situations verbally, without showing pictures from video recordings of the classroom episodes. Indeed, as Lerman (2001a) noted, despite that technology may allow insertion of video extracts in a text, ethical constraints limited me to produce such visualization.

4.6. Conclusion

In this chapter, I presented the paradigm of my study which is what has been referred to as constructivism (Guba & Lincoln, 1994) materialised by researching realities that I constructed in my classroom. Next, I presented research methods that I used to collect the data and to analyse them. In this respect, I first presented the participants of the research; then I presented the methods that I used to collect the data which were the interviewing and contextualised recording of spoken data (Lankshear & Knobel, 2004). After that, I presented the methods that I used to analyse the data, namely, the sociolinguistics and the discourse analysis (Lankshear & Knobel, 2004). Finally, I presented some validity and ethical issues related to my study. In the next chapter, I present the data and the analysis related to student teachers' evoked concept images of the concepts of the definite and the indefinite integrals.

CHAPTER 5: STUDENT TEACHERS' EVOKED CONCEPT IMAGES OF INTEGRALS

5.1. Introduction

This chapter presents the findings about the student teachers' concept images (Tall & Vinner, 1981) of integrals evoked at different periods of the research. The chapter is made up of eight sections. After this introduction, the second section presents the concept images that the student teachers evoked during the first round of the interviews conducted five months before the teaching. The third section presents the concept images that the student teachers evoked during the second round of the interviews conducted ten days before the teaching to verify an eventual change in the student teachers' concept images since they were more in the learning conditions. At the end of the third lesson of the course, I requested the student teachers to summarise in writing what they had learned until then. I collected their manuscripts and later on, I analysed them. The findings of the analysis in terms of concept images that the student teachers evoked at that time are presented in the fourth section. The fifth section presents the concept images that the student teachers evoked during the third round of the interviews conducted within the five days which followed the end of the teaching. The sixth section presents a summary of the evolution of the student teachers' concept images of the definite integral during the period of the six months surrounding the teaching of the course. The seventh section presents the findings related to the student teachers' concept images of the indefinite integral evoked during the three rounds of the interviews. And lastly, the eighth section concludes the chapter.

In order to shorten the chapter, I give detailed analysis of the concept images for only four student teachers, namely Bernard, Cassim, David, and Edmond. These detailed analyses illustrate the way in which I did the analysis. For the other student teachers I give only the results of the analysis.

The selection and the order of the four student teachers are based on preliminary findings of a quick analysis that I did on the student teachers' answers of the questions of the first round of the interviews. After this analysis, I classified the student teachers according to how well they had performed during the said interviews. From this quick analysis, Bernard had answered the questions of the first interview relatively well. He was followed by Cassim, then David and Edmond. All the student teachers were also classified according to this principle. To present the subsequent analyses, I chose to follow this order that I determined at that time of the first round of the interviews.

The findings of the analyses of the interviews and of the written productions after the third lesson are displayed in diagrams for each of the four abovementioned student teachers. However, at the end of the analysis of each round of the interviews and at the end of the analysis of the student teachers' productions after the third lesson, I give a synoptic diagram which summarises the findings of the analysis for all the eleven student teachers who participated in the study. In these synoptic diagrams, the alphabetic letters in capital, B, C, D, E, F, G, H, J, K, M, N, represent the initials of the pseudonyms of the said student teachers. In addition to diagrams summarising the findings of analyses, I produced tables from synoptic diagrams to present numbers of student teachers who evoked, either correctly or not, any of the process-object layers that underpin the concept of the definite integral. To end this introductory section, I explain the symbols that I will use to code the student teachers' conceptual understanding all along this chapter. These symbols will appear in diagrams in which I will be representing the findings of my analyses. The analytical meanings that I assigned to these symbols are given in Figure 3 Chapter 3 section 3.2.4 at page 53. Here I give specific examples to illustrate their use.

The following diagram is an example of the illustrative diagrams that I will produce. On this diagram, the reader can recognize symbols of empty circles (○), shaded circles (●), crossed circles (⊗) and tree-like ().

Process – Object Layers	Context: Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		●	○	○
Product			●	⊗
Sum		⊗		●
Limit		○ 🌳		

The above illustrative diagram represents the results of the analysis of an extract written by student teacher (Cassim, section 5.4.2). As an example, the shaded circle (●) in the cell intersecting the layer of partition and the column of graphical representation indicates that the student teacher evoked the process and the resulting object in the graphical representation when he wrote "one has first to subdivide the area into 3 parts" and "one obtains 3 small rectangles". An empty circle (○) is put in the cell intersection the layer of partition and the column of symbolical representation to indicate that the student teacher only evoked the resulting object or 'pseudoobject' $\Delta x = width$ while the process usually represented by the symbol $\frac{b-a}{n}$ is not evoked. A crossed circle (⊗) is put in cell intersecting the layer of product and the column of symbolical representation to indicate that the student correctly evoked only the process represented by $f(x_i)\Delta x$ but not the resulting object usually represented by A_i. An empty circle (○) and a tree-like symbol (🌳) are put in the cell intersecting the layer of limit and the column of graphical representation to indicate that the student teacher correctly evoked the pseudoobject of the layer of the limit in the graphical representation while the process aspect is incorrectly evoked and needs to be developed.

The symbols of empty and shaded circles are taken from Zandieh (2000) while the crossed circle and the tree-like symbols are my constructions. In the crossed circle, I recognize the cross symbol (x) which reminds me the operation of multiplication but in this thesis the cross circle stands for all operations performed on objects. Similarly, the tree symbol, in this thesis, represents a developing or a growing 'thing'.

After this explanation of the symbols that I will use in coding student teachers' conceptual understanding, I present, in the next section, the analysis of the student teachers' concept images evoked during the first round of the interviews.

5.2. Student teachers' concept images of the integrals evoked during the first round of interviews

This section presents the analysis of the interviews that I held with the student teachers in the first round of interviews. After detailed analysis of extracts of interviews with each of the student teachers, a diagram summarises the analysis in terms of various representations and mathematical conceptions of processes and objects or pseudoobjects, using the frameworks described in Chapter 3. In the next subsection, I analyse the concept images evoked by Bernard during the first round of the interviews.

5.2.1 Bernard's concept images of integrals evoked during the first interview

During the first interview, Bernard exhibited the following concept images of integrals in general and of the definite integral in particular.

Extract B1

1.	FH:	If you have a function f, what do you understand by the integral of this function?
2.	Bernard:	By the integral of a function f, one understands the inverse function of the given function.
3.	FH:	Ok. Now what do you understand by the definite integral?
4.	Bernard:	A definite integral is an integral which has limits, lower limit and upper limit.

In line 2, Bernard's statement about the integral is incorrect since he said that "by the integral of a function one understands the inverse of the given function". Taken in isolation, the phrase "inverse function of the given function" could make think of the inverse of a function (f^{-1}). However, in the context of the definition of the indefinite integral the phrase could signify that Bernard was trying to refer to the antiderivative. However, in this perspective, his idea is incomplete and none of the characteristics of the indefinite integral as an antiderivative are apparent. Therefore, to illustrate the emergence of the inversion process of the differentiation that Bernard attempt to evoke orally, I put a tree-like symbol () in the cell intersecting the layer of the antiderivative (indefinite integral) and the column of verbal representation (Figure 9).

In line 4, Bernard described the definite integral in terms of limits. Taken in isolation, the word "limit" could invoke two aspects of limit when the definite integral is under examination: the limit process and the limits of integration. However, the adjectives "lower" and "upper" that he added make it clear that it is the limits of integration that are concerned here. Such description of the definite integral refers to a part of the characteristics of the concept of the definite integral and not to the underlying concepts of the definite integral. To illustrate such a description of the definite integral which is still to be developed, I put a tree-like symbol () in the cell intersecting the layer of the definite integral and the column of the verbal representation. Possibly, the limits of integration are not linked to the concept of the definite integral but rather to symbolical calculation of the definite integral. But here, I have given Bernard's concept image the benefit of doubt.

Concepts and theorem	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Antiderivative (indefinite integral)				
Definite integral				
FTC (VEA)				
Area as application:				

Figure 9. Bernard's concept image evoked during the first interview

At this time of the first interview, Bernard's evoked concept image of the definite integral does not include any of the underlying concepts of the definite integral. Only the limits of integration are evoked to describe what the definite integral is. He attempted to engage the pseudoobject aspect of the antiderivative layer of the indefinite integral as described in figure 3 (Chapter 3, p. 54). Nothing about the FTC was evoked.

5.2.2 Cassim's concept image of the definite integral during the first round of the interviews

The following excerpt contains the concept image of the definite integral demonstrated by Cassim during the first round of the interviews.

Excerpt C1

5. FH: If one has a function f, what do you understand by the integral of this function?

6. Cassim: The integral of this function f of x it is the function opposed to the derivative of this function, that is, the integral of the function of the derivative is the integral of x, that is, for a function *f* I have to explain what the derivative is. The function *f* which is equal to x plus h, h being the increment of the function; it is the limit of x minus the limit of h which is a constant over h, given that h tends

to infinity (he wrote the symbols: $f(x)$, $f(x+h)'$, $\frac{\lim(x)-\lim(h)}{h}$, $h \to \infty$).

7. FH: Now, what do you understand by the definite integral?

8. Cassim: The definite integral is the integral whose limits are well defined, that is, there are an upper limit and a lower limit. The answer that one must have is that the definite integral must be a number well defined.

Like Bernard, Cassim attempt to refer to the antiderivative as he said, in line 6, that the integral is "the function opposed to the derivative of this function". Taken in isolation, the phrase "the function opposed to the derivative of this function" could lead to the opposed function in the sense of the symmetric function. In the perspective of integration this phrase could lead to the search for an antiderivative. However, Cassim does not expand on what the antiderivative is, but provides a symbolic explanation related the underpinning concepts of the derivative (ratio and limit as described in Zandieh (2000)). This answer made me conclude that Cassim did not evoke the layer of the antiderivative and the underlying concepts of integrals correctly. Therefore, to illustrate Cassim's idea about the integral evoked in line 6, I put a tree-like symbol () in the cell intersecting the layer of the antiderivative (indefinite integral) and the verbal representation (Figure 10) to indicate that Cassim incorrectly evoked the object aspect of the antiderivative in verbal representation.

Concepts and theorem	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Antiderivative (indefinite integral)				
Definite integral	○			
FTC (VEA)				
Area as application:				

Figure 10. Cassim's concept image evoked during the first interview

Concerning the definite integral, like Bernard, Cassim described it by referring to lower limit and upper limit of integration. But in addition to this, Cassim at the end of line 8 evoked the fact that the definite integral is a number when he said that "the answer that one must have is that the definite integral must be a number well defined". In this statement Cassim evoked an important aspect of the nature of the definite integral of being a number. However, Cassim did not provide the underlying concepts described in figure 2 (Chapter 3, p. 47) that lead to the number that he had evoked. Thus the aspect of the definite integral he evoked is of pseudoobject character in the sense that it lacks the underlying process (Sfard, 1991; Sfard & Linchevski, 1994). Therefore to illustrate Cassim's idea evoked in line 8, I put an empty circle (○) in the cell intersecting the layer of the definite integral and the column of the verbal representation.

In brief, at the time of the first interview, Cassim exhibited only the pseudoobject aspect of the concept of the definite integral in the verbal representation. He, incorrectly, attempted to exhibit the pseudoobject aspect of the antiderivative (indefinite integral) layer as described in Chapter 3, and nothing about the FTC was exhibited.

5.2.3 David's concept images of integrals during the first interview

During the first interview, David exhibited the following concept images.

Excerpt D1

9.	FH:	If one has a function f of x, what do you understand by the integral of this function?
10.	David:	The integral of this function can be understood as the search for the function that has been differentiated to find our function f of x.
11.	FH:	Ok. What do you understand by the definite integral?
12.	David:	One speaks of a definite integral when one has firstly found an indefinite integral and then one defines the integral by using limits of this integral, that is, our integral has two limits for example a and b (David writes $\int_a^b f(x)dx$) and we have upper limit and lower limit, then one defines this integral after having integrated indefinitely, one defines it by replacing upper limit in

> our found function by subtracting the lower limit also replacing it in our function and then one finds the definite integral.

In line 10, David expressed the idea of the process of the antidifferentiation as he said "the integral is the search for the function that has been differentiated to find our function f of x". In this statement David evoked the pseudoobject of the antiderivative. However, he did not evoke the underlying process as described in chapter 3. Therefore in the illustrative diagram, I put an empty circle (○) in the cell intersecting the row of the antiderivative (indefinite integral) and the column of the verbal representation (Figure 11).

In line 12, David evoked the definite integral when he said "one defines the integral by using limits of this integral, that is, our integral has two limits for example a and b" and when he wrote the symbol $\int_a^b f(x)dx$. However he did not provide any further explanation that makes use of the underlying concepts of the definite integral in that symbolical representation. The words "limits" used in the statement "one defines the integral by using limits of this integral, that is, our integral has two limits for example a and b" clearly evokes the limits of integration and not the limit process as described in figure 2 (Chapter 3, p. 47). Therefore I put tree-like symbol () in the cell intersecting the layer of the definite integral and the column of the symbolical representation to indicate that David incompletely evoked the object aspect of the definite integral in the symbolical representation.

Concepts and theorem	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Antiderivative (indefinite integral)	○			
Definite integral				
FTC (VEA)				
Area as application:				

Figure 11. David's concept image evoked during the first interview

Finally at the end of line 12, David made allusion to the fundamental theorem of calculus when he explained how to find the definite integral by replacing the lower limit and the upper limit in the indefinite integral. However the language used is imprecise and reflects a symbol-manipulation approach as he said "(…) one defines this integral after having integrated indefinitely, one defines it by replacing upper limit in our found function by subtracting the lower limit also replacing it in our function (…)". The phrase "one defines this integral" refers to the definite integral while "having integrated indefinitely" refers to the indefinite integral. Such inaccurate language is also used in the phrase "one defines the integral by using limits of this integral, that is, our integral has two limits for example a and b" at the beginning of line 12. Therefore, taking into account this imprecision about the FTC (VEA), I put a tree-like symbol () in the cell intersecting the layer of the FTC (VEA) and the column of verbal representation. Also, I put a tree-like symbol () in the cell intersecting the layer of the definite integral and the column of verbal representation to indicate that the verbal formulation used by David is incomplete and needs to be improved.

During the first interview, David evoked only a pseudoobject of the concept of the indefinite integral. He also attempted to use the FTC (VEA) to describe the definite integral, but in a way that appeared to be strongly focused on symbol manipulation.

5.2.4 Edmond's concept images of integrals during the first interview

During the first interview, Edmond exhibited the following concept images of integrals.

Except E1

13. FH: What do you understand by the integral of a given function f?
14. Edmond: The integral of a given function, in fact, if we have a function for example $f(x)=2x+1$ and if one integrates this function $2x+1$, then it is for what? It is for finding a function that gave birth to this resulting function. For example if we use this rule saying that it is the integral of two x dx plus the integral of dx, which is equal to two if one puts out two one remains with integral of x dx plus integral of dx. It is equal to x square over two plus x and here one simplifies and it remains x square plus x. You see here that it is a function that one differentiated in order to find this given function. That is actually the integral. It is to find the 'mother' function that one has to differentiate in order to find the given function.
15. FH: OK. Now what do you understand by the definite integral?
16. Edmond: The definite integral is an integral for which limits have been clarified. Thus there is an interval on which the pupil has to study this function in limits well defined, for example if one gave integral of sine squared of x dx and asked to study this function from zero to half pi [Edmond writes $\int_0^{\pi/2}\sin^2 x dx$]. Then one has first to find the integral like this $\int \sin^2 xdx$, that is, one has to integrate that trigonometric function $\sin^2 x$; If one does the calculations one will use formulae that say that the integral of a function f(x) over limits a and b dx is equal to that function over limits a and b; it is equal to f(b) minus f(a).

In line 14, Edmond evoked the process of antidifferentiation when he exemplified 'antidifferentiating'the function $f(x)=2x+1$. In the same line, he said that "it [the integral] is a function that one differentiated in order to find this given function". He used the words 'mother' and 'to give birth' to reinforce the inversion of the derivative. He combined the verbal and the symbolical representations to describe the antiderivative. Moreover, in line 16, Edmond evoked the pseudoobject of the indefinite integral when he wrote the symbol $\int \sin^2 xdx$. However, in his speech, he did not evoke any of the underlying concepts of the indefinite integral as described in

Chapter 3. To indicate that Edmond correctly evoked the pseudoobject aspects of the indefinite integral in verbal and in symbolical representations, I put empty circles (○) in the cells intersecting the layer of the antiderivative (indefinite integral) and the columns of the verbal and of the symbolical representations (Figure 12).

Concepts and theorem	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Antiderivative (indefinite integral)	○			○
Definite integral				
FTC (VEA)				
Area as application:				

Figure 12. Edmond's concept image evoked during the first interview

Furthermore, in line 16, like his classmates cited previously, Edmond described the definite integral in terms of the limits of integration. And like Bernard, he evoked the symbolical representation of the definite integral by writing the following symbol $\int_0^{\pi/2} \sin^2 x dx$ when he was elaborating on what the definite integral is. However he did not correctly evoked what the definite integral is and he did not evoke its underlying concepts. To illustrate Edmond's concept images, I put a tree-like symbol () in the cell intersecting the layer of the definite integral and the column of the symbolical representation to indicate that Edmond incompletely evoked the object of the definite integral in the symbolical representation.

Finally, at the end of the line 16, Edmond evoked the formula of the FTC (VEA) when he said "If one does the calculations one will use formulae that say that the integral of a function f(x) over limits a and b dx is equal to that function on limits a and b; it is equal to f(b) minus f(a)." Possibly, through these words, Edmond was referring to the following symbols which represent the FTC (VEA), $\int_a^b f(x)dx = \left[F(x)\right]_a^b$ or $\int_a^b f(x)dx = F(b) - F(a)$. However, the symbols were not evoked as symbols but rather in words. To illustrate this statement, I put a tree-like symbol () in the cell intersecting the layer of the FTC (VEA) and the verbal representation to indicate that Edmond incompletely evoked the FTC (VEA) in the verbal representation and a tree-like symbol () in the cell intersecting the row of the FTC (VEA) and the column of the symbolical representation to indicate that the formula of the FTC (VEA) was incompletely evoked.

Briefly, at the time of the first interview, Edmond evoked only pseudoobject aspects of the indefinite integral. He also evoked some aspects of calculating the definite integral and of the FTC (VEA); especially the formula of the FTC (VEA) was evoked, though somewhat incomplete.

5.2.5 Other student teachers' concept images of integrals during the first round of interviews

I applied the same analysis to the relevant extracts of the first interviews for each of the other student teachers. The concept images they evoked at that time of the first round of the interviews are summarised in the synoptic diagram that follows (Figure 13). The letters written in the columns represent the first letters of the student teachers' pseudonyms: B-Bernard, C-Cassim, D-David, E-Edmond, F-Ferdinand, G-Gerard, H-Honorate, J-Jimmy, K-Kevin, M-Merriam, and N-Norbert.

The table is subdivided into four parts corresponding to the four representations. The first column of each part states the mathematical concepts and the other columns indicate the status of the student teachers' concept images.

	B	C	D	E	F	G	H	J	K	M	N
Verbal representation											
Antiderivative (Indefinite integral)	🌳	🌳	O	O				🌳			
Definite integral	🌳	O	🌳			🌳		🌳			
FTC (VEA)			🌳	🌳							
Graphical representation											
Antiderivative (Indefinite integral)						🌳					
Definite integral											
FTC (VEA)											
Numerical representation											
Antiderivative (Indefinite integral)											
Definite integral											
FTC (VEA)											
Symbolical representation											
Antiderivative (Indefinite integral)				O			🌳		🌳	🌳	🌳
Definite integral			🌳	🌳	🌳	🌳			🌳	🌳	
FTC (VEA)				🌳					🌳		
Area as a context of application					✓	🌳	✓		✓		
	B	**C**	**D**	**E**	**F**	**G**	**H**	**J**	**K**	**M**	**N**

Figure 13. Synoptic diagram of student teachers' concept images evoked during the first round of the interviews

With regard to the underlying concepts of the definite integral in the symbolical representation, the following table (Table 1) constructed from the synoptic diagram presented above shows that none of the eleven (0/11/ student teachers correctly evoked the underpinning layers of the definite integral; only five out of eleven (5/11) of the student teachers incorrectly evoked the object of the definite integral; and the remain six out eleven of the student teachers did not evoke any symbol related to the definite integral.

Process-object layers	Correctly evoked (Nb)	Incorrectly evoked (Nb)	Not evoked (Nb)
Partition $\Delta x = \dfrac{b-a}{n}$	0	0	0
Product $A_i = f(x_i)\Delta x$	0	0	0
Sum $S_i = \sum_{i=1}^{n} f(x_i)\Delta x$	0	0	0
Limit $L = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	0	0	0
Definite integral (pseudo-object) $\int_a^b f(x)dx$	0	5	6
Definite integral $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	0	0	0

Table 1. Student teachers' concept images of the definite integral evoked during the first round of the interviews

With regard to the indefinite integral, figure 13 shows that only one student teacher (Edmond) correctly evoked the pseudoobject of the indefinite integral in the symbolical representation. Concerning the FTC (VEA), only two student teachers (Edmond and Kevin) evoked, though incorrectly, the FTC (VEA) in the symbolical representation.

In the verbal representation, two student teachers (David and Edmond) correctly evoked the pseudoobject of the indefinite integral; four student teachers incorrectly evoked the pseudoobject of the definite integral and five student teachers did not evoke anything related to integrals. Concerning the FTC (VEA) only two student teachers (David and Edmond) evoked it in the verbal representation, yet it was incomplete.

This was the situation of the student teachers' concept images five months before the teaching experiment. Ten days before the teaching I held a second round of interviews with the student teachers. I conducted this second round before the teaching of integration to identify any possible change in the student teachers' concept images after the teaching and learning of limits and derivatives. I was checking whether the learning of limits and of derivatives had occasioned any change or any recall of what integrals are. Moreover, I had suspected that after the first round of the interviews and during the first part of the course, some student teachers who might be highly interested in the subject matter could have consulted textbooks or their peers about what the definite or the indefinite integrals are. In the next section, I present the student teachers' concept images evoked during the second round of the interviews.

5.3. Student teachers' concept images of integrals evoked during the second round of the interviews

As in the previous section, I present the analysis of the student teachers' concept images of integrals. After the analysis I summarise in a synoptic diagram these student teachers' concept images. Using this synoptic diagram, I make a table (Table 2, p. 158) that presents the results of the analysis. I followed the same presentation of the findings to allow easy comparison. In the next subsection, I present the analysis of Bernard's concept images of integrals evoked during the second interview.

5.3.1 Bernard's concept images of integrals evoked during the second interview

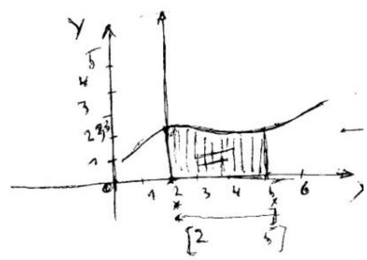
During the second interview that I held with student teachers in the middle of the course, after I had taught them the concept of function, the concept of a limit of a function, and the concept of the derivative, just some days before I started the component of integration, Bernard demonstrated the following concept images of integrals. Bernard's evoked concept images were wider than the ones evoked in the first interview. His concept images were different to the ones accepted by the wider mathematical community. However, they could constitute a starting point for the process of developing his understanding of integrals.

Extract B2

17. FH: OK. Now if you have to explain to your pupil what the integral is, what will you tell him?

18. Bernard: The integral, I will say that it is a value which is obtained from what is called a primitive [antiderivative]. This value is called integral, and this value is a limit actually. Thus the integral can be defined as a limit of the antiderivative;

19. Bernard: This value is calculated as a limit and can be sketched [Bernard sketches the following diagram]



…here we have axes and one puts scales and then one tries to associate a function. The function will be represented by a curve, for example like this. And concerning the integral as one says it is the value of a limit. We will take for example two points, the point 2 and the point 5. And then one tries to calculate the limit of this function at point 2. In fact as the function here is continuous, one will obtain the image of point 2 on the representative curve of the function and that limit will be equal to 2, 2.3 for example. Considering the point 5, we will have for example the value 2. As the limit is calculated firstly by calculating the left limit and then the right limit, for the point 2 we have the left limit and the right limit. As the two limits are the same the limit exists. In a similar manner for 5, going from the right to the left we have the value here. Now the values that one can read on the y-axis are the same as the values we have here. Thus at each point we have values. But as we considered points which are different, we have here an interval between 2 and 5. These values are here and this is a surface in fact.

20. Bernard: This [area of] surface is the value of limits and at the same time it expresses the value of the integral. Thus in fact the integral is a value similar to the limit.

21. Bernard: The only difference is that for the limit one has one single point but for integrals in most cases [one has] a surface which is limited by two vertical lines of course which are parallel to the y-axis, the curve, and the x-axis. It is this surface which is the integral, the value in fact.

22. Bernard: The sum, that is, it is the sum of limits of these points, by the function that has been considered here; when one adds these limits one obtains this surface.

In this extract, Bernard exhibited some ideas related to the underlying concepts of the definite integral but they are not yet clearly expressed and the two limit concepts are confused.

In line 18, Bernard described the integral as a value obtained from the antiderivative when he said “The integral, I will say that it is a value which is obtained from what is called primitive [antiderivative (indefinite integral)].” He was alluding to finding the definite integral by using the antiderivative. However he pursued his explanation about that value by saying that “that value is a limit” and he concluded by saying that “the integral can be defined as a limit of the antiderivative.” Here, Bernard evoked the layer of the antiderivative (indefinite integral) that is used to find the definite integral as a value. In his statement the word “limit” refers more likely to the limit process of a function at a point than to the limits of integral, if we take into account the explanation that he provided in line 19. However, he applied the limit process to the antiderivative instead of applying it to a sequence of sums of areas. Nonetheless, in the perspective of this latter aspect the idea of the limit process is emerging though still at the embryonic state. Therefore, to illustrate Bernard’s ideas expressed in line 18, I used two diagrams simultaneously (shown in Figure 14 at the end of this section). I put an empty circle (○) in the cell intersecting the layer of the antiderivative and the column of the verbal representation (see upper part of Figure 14) to indicate that Bernard evoked the pseudoobject aspect of the antiderivative (indefinite integral, primitive) in the verbal representation and I put a tree-like symbol ( ) in the cell intersecting the layer of limit and the column of verbal representation to indicate that the idea of describing the integral as a limit value is emerging, though still incorrect (see lower part of Figure 14).

This idea of a limit value expressed in line 18 connects the two diagrams that illustrate Bernard's concept image of the definite integral. As Bernard's concept images deepens, it is necessary to include the second diagram in figure 14. The evocation of 'imprecise' ideas constitutes the emergence of Bernard's zone of proximal development (ZPD), that is, a zone where ideas are not fully-fledged but are in the process of maturing; those ideas are in an embryonic state and may mature in the coming days (Vygotsky, 1978, p.86) depending on the milieu (Brousseau, 1997).

Moreover, from line 18 to line 22, Bernard attempted to use the graphical representation to explain what the integral is. He evoked the area of a surface as a value of limits. As noted previously, this word 'limit' is more likely to refer to the limit process than to the limits of integration. However, he was not clear what he was talking about as he noticed it himself in line 21 by saying that in the case of the limit there is one single point and in the case of the surface there are two points (2 and 5 in this case) for which one has to calculate the corresponding limit using the left limit and the right limit. Therefore I illustrate Bernard's ideas that are still in a fuzzy state by putting a tree-like symbol ( ) in the cell intersecting the layer of the limit and the graphical representation.

Finally, in line 22 he evoked the idea of a sum. But again, his idea of sum refers to the sum of limits. Therefore in the graphical representation, the ideas of limit and sum that Bernard expressed in these statements are in emerging state and they need to be developed so that they can express what the integral is in the graphical representation using the proper ideas of limit of sums. Similarly the idea of using the context of area under a curve as manifested in line 19 is in an embryonic state and needs to mature so that it can be used with assurance to explain what the integral is. Therefore to illustrate those emerging ideas expressed by Bernard I put in the corresponding cells a tree-like symbol ( ).

In brief, in this extract from the second interview, Bernard evoked some ideas of the underlying concepts of integrals. However, those ideas were still in an embryonic state and need to be developed as illustrated by many tree symbols in Figure 14 below.

Concepts and theorem	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Antiderivative (indefinite integral)	O			
Definite integral				
FTC (VEA)				
Area as application:				

Process – Object Layers	Context: area under a curve 			
	Representations			
	Verbal (oral and written words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition				
Product				
Sum				
Limit				

Figure 14. Bernard's concept image evoked during the second interview

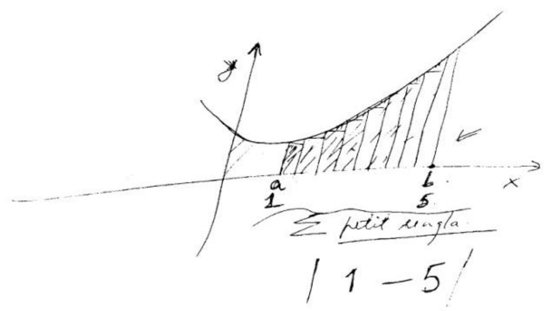
5.3.2 Cassim's concept images of integrals during the second interview

During the second interview, Cassim exhibited concept images of integrals presented in the following extract.

Excerpt C2

23. FH: If your pupil asks you to explain to him what an integral is what can you tell him?

24. Cassim: If my pupil asks me to explain to him what it is, the integral is…, suppose that we have here the axes [Cassim sketches the diagram below], given a function which cut the axes like this, then the integral from a to b. We will have here small rectangles which are cut in an infinity of rectangles, the integral of this function between a and b is the sum of these small rectangles. These small areas are also included [Cassim points to the diagram]. The sum of these small rectangles above is also included. That is what is called the integral of the function from a to b.



25. FH: If he asks you to explain what the definite integral is, what can you tell him?

26. Cassim: The definite integral, here we took the values a and b which are not well defined; if the limits are well defined, that is, for example from 1 to 5, the definite integral of the function will be the area which is delimited by the curve of the function and the axes but that area must be contained between 1 and 5.

In this extract, Cassim exhibited some ideas related to the underlying concepts of the definite integral. In line 24, Cassim sketched a curve in the plane and used the diagram to explain in the graphical representation what the integral is. He evoked the idea related to the layer of the partition when he said that “We will have here small rectangles which are cut in an infinity of rectangles.” and “These small areas also included”. Moreover he evoked the idea of the layer of sum when he said that “the integral of this function between a and b is the sum of these small rectangles” and “The sum of these little rectangles above also included". However, those ideas, like in the case of Bernard, are not yet precise enough to explain exactly what the integral is. There are still some imperfections to be clarified. Therefore I put a tree-like symbol () in the cells intersecting the layers of partition and of sum and the graphical representation (Figure 15) to indicate that the evoked ideas are still in the process of maturing. Moreover, I put the phrase "area under a curve" and the tick symbol (✓) to indicate that Cassim correctly evoked the context of area under a curve.

In line 26, Cassim referred to the limits of integration and combined them with the idea of area to explain what the definite integral is. At this stage he seemed to see the area as an application of the definite integral which leads me to assume that the reference to 'integral' in line 24 implied 'definite integral'. Thus he did not evoke a concept image of the indefinite integral.

Process – Object Layers	**Context: Area under a curve** ✓			
	Representations			
	Verbal (oral words)	**Graphical/Visual (geometrical figures)**	**Numerical (numerical application)**	**Symbolical (generalization)**
Partition				
Product				
Sum				
Limit				

Figure 15. Cassim's concept image evoked during the second interview

During this second interview, Cassim's concept images include some references to the underlying concepts of the definite integral. However they are still incomplete.

In this second interview, when compared to his concept images evoked in the first interview, Cassim's concept images of integrals moved from verbal representation to graphical representation and from the pseudoobject aspects to involving some underlying concepts.

5.3.3 David's concept images of integrals during the second interview

During the second interview, David exhibited the concept images of integrals presented in following extract.

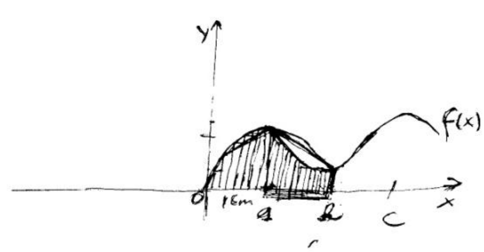
Except D2

27. FH: If your student asks you to explain to him what the integral is, what will you tell him?

28. David: To explain firstly, I will use the graphical diagram to show him what the integral is. By visualising, the pupil will understand what integral is and finally he will be back in calculations but knowing what he has to do and what he is finding.

29. FH: How will you do that visualisation?

30. David: For example I will draw axes on which there are graduations [scale], i.e. the "orthonormed" axes, we have for example values a, values b and even value c (see below the diagram drawn by David). When the curve…, say here the curve increases and reaches its maximum and decreases and reaches its minimum and increases again. In this case, on a given point, we have values here, considering the x-axis; it is possible to show him the integral of f(x) on these points here. So the integral is defined from zero to b, zero is value, zero of x and the b of x. Then the integral of this function is defined by this surface which between the curve and the x-axis.



31. FH: What will you tell him if he asks you what the definite integral is?
32. David: The definite integral is what I am showing. So if he asks me that question, I will show him how one defines that integral, how one can calculate this integral.
33. FH: How, how one calculates that?
34. David: You see here we will have rectangles, as stairs, always subdividing this part here. There are possibilities of subdividing, subdividing again and subdividing until values nearest to the infinity because if we do these rectangles, finally there will be some values that remain and these values are included in our area here (showing the diagram). So by subdividing again and making small rectangles, finally we will have to find an approximate value of the definite integral of our function here.
35. FH: When will we have the integral which is not the approximation?
36. David: I think that by subdividing until the infinity, we will have the integral.

In this excerpt, David exhibited some ideas related to the underlying concepts of the definite integral. In line 28, he started by evoking the use of a graphical diagram and in line 30 he elaborated on the diagram by drawing the graph of a function and by saying that the integral of the function from 0 to b is defined by the surface between the curve and the x-axis. David combined the limits of integration and the area of a surface to describe the integral. These characteristics are starting points for the description of the nature of the integral in the graphical representation. To illustrate this incomplete description of the integral provided by David, I put a tree-like symbol () in the cell intersecting the layer of the definite integral and the column of graphical representation (Figure 16). Moreover, to illustrate this last idea of integral as defined by the surface I put the word “area” followed by a tick symbol (✓) in the cell reserved to the application field of the definite integral. Finally, to illustrate the idea of drawing a curve to delimitate the surface I put the phrase “area under the curve” followed by the tick symbol (✓) to indicate that the context of area under a curve is correctly evoked.

In line 34, the process of subdividing is clearly evoked, and the resulting object of small rectangles in stairs also is evoked when he said “we will have rectangles, as stairs, always subdividing this part here" and "There are possibilities of subdividing, subdividing again and subdividing until values nearest to the infinity because if we do these rectangles, finally there will be some values that remain and these values are included in our area here (showing the diagram)". Moreover the conclusion he made by saying that "by subdividing again and making small rectangles, finally we will have to find an approximate value of the definite integral of our function" points to the layer of the process of limit. Consequently, I put a shaded circle (●) in the cell intersecting the layer of partition and the graphical representation to indicate that David correctly evoked the process and the resulting object of the partition layer in the graphical representation. By the end of line 34, David expressed an incorrect idea of finding an approximate value for the definite integral since the definite integral is the exact value of the area. However, if the idea was combined with the idea expressed in line 36 where David said "I think that by subdividing until infinity we will have the integral" that incorrect idea could develop toward the right idea of a limit value. Thus, he did not correctly exhibit the layer of limit. The idea of the limit process needs to be developed. Therefore I put a tree-like symbol () in the cell intersecting the layer of the limit and the graphical representation.

During the second interview, David's concept images include some references to the underlying concepts of the definite integral. As it is shown in the illustrative matrix below, David evoked a context and some ideas related to two of the four process-object layers underlying the definite integral in graphical/visual representation.

Concepts and theorem	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Antiderivative (indefinite integral)				
Definite integral				
FTC (VEA)				
Area as application: ✓				

Process – Object Layers	Context: Area under a curve ✓			
	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		●		
Product				
Sum				
Limit				

Figure 16. David's concept image evoked during the second interview

5.3.4 Edmond's concept images of integrals during the second interview

During the second interview, Edmond exhibited concept images of integrals presented in the following extract.

Excerpt E2

37. FH: If you had to explain to your student what an integral is, what could you tell him?

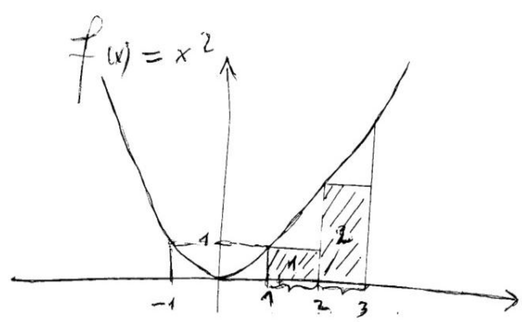
38. Edmond: For this indefinite integral, we said that it is a function that has been differentiated in order to find the given function.

39. FH: Now for the definite integral what could you tell him?

40. Edmond: For the definite integral, here we were given a function for example our function $f(x) = x^2$; but here it was specified, the limits in which we had to study this function were specified. For example from zero to one; then what one might do is to use the formula [he writes] $\int x^2 dx = \frac{x^3}{3} + C$, that is x cubed over three at the limits zero to one [he writes] $\left[\frac{x^3}{3}\right]_0^1$, we used the formula to evaluate this formula, that is this integral is equal to f of b minus f of a [he writes] $F(b) - F(a)$. That is we replaced these limits in the function. Suppose zero it is the limit a and here it is the limit b. Then we replace we find [he writes] $\frac{1}{3}\left[x^3\right]_0^1$ and we find [he writes] $\frac{1}{3}(1^3 - 0) = \frac{1}{3}u^2$. Here for the definite integral, we were asked to find the area of this function at the limits zero to one.

41. FH: What is the area of a function?

42. Edmond: Area of a function is the part delimited by this function and the abscissa axis. For example here if we take our function x^2, then if we draw the graph after having sketched the axes we put the units [he draws the following diagram].



43. Edmond: And after, for example here we have -1 and 1 and 1 and our function will pass from the coordinates (1, 1) and (-1, 1) and if we draw we will find a function like this [showing on the diagram]. The curve of our function has its concavity upward. Then here from zero to one, it is this part delimited by the curve and the x-axis from zero to one that is called the area. The area at the interval defined.

44. FH: How do you link the integral and the area?

45. Edmond: After having sketched the curve and delimited the area on a given interval, then to calculate the integral with the given limits, one has to integrate this function with its limits. That is, to find this area one has to integrate the function with the limits from zero to one.

46. FH: If you had to explain to your student, what will you tell to him about the meaning of this symbol $\int$?

47. Edmond: In fact, we were told that this symbol means the integral. In French, it is said, it is 'sha'; but in mathematics it means the 'integral'.

In this excerpt, in line 38, Edmond evoked the antiderivative (indefinite integral) in a verbal representation since he said "for the indefinite integral, we said it is a function that has been differentiated in order to find the given function". Moreover in line 40, he referred to limits of integration and to the indefinite integral to explain what the definite integral is but this time he evoked the symbolical representation as he wrote the symbol $\int x^2dx = \frac{x^3}{3} + C$ of the indefinite integral for the function $f(x) = x^2$. However he did not evoke any of the underlying concepts of the integral as described in figure 2 (p. 47). Therefore I put an empty circle (○) in the cells intersecting the layer of indefinite integral and the columns of the symbolical representation and the verbal representation to indicate that Edmond evoked the pseudoobject aspect of the indefinite integral in these representations (Figure 17).

In the same line 40, Edmond evoked the formula of the FTC (VEA) when he was trying to describe what the definite integral is, but he did not exhibit the ideas that underpin the FTC (VEA). Therefore I put an empty circle (○) in the cell intersecting the layer of the FTC (VEA) and the column of the symbolical representation to indicate that Edmond evoked the pseudoobject of the FTC (VEA) in symbolical representation.

Finally, Edmond evoked the context of area as a field of application of the definite integral as he said in line 40 that "for the definite integral we were asked to find the area of this function at the limits zero to one". In line 44, even though he sketched a curve to delimitate an area, Edmond evoked the area delimited as a field of application and not as a context to describe the definite integral. Therefore in the illustrative diagram, I put the word 'area' followed by the tick symbol (✓) in the cell reserved for the application field.

At this stage, Edmond did not evoke any of the underlying concepts of the definite integral as described in figure 2. The symbol $\int$ did not prompt him to evoke some of the underlying concepts of the integral.

Concepts and theorem	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Antiderivative (indefinite integral)				O
Definite integral				
FTC (VEA)				O
Area as application: ✓				

Figure 17. Edmond's concept image evoked during the second interview

During this interview, Edmond's concept images include pseudoobjects of the indefinite integral and the FTC (VEA) and an appropriate context for developing the concept of the definite integral. It appears that his concept images have not changed significantly since the first interview.

5.3.5 Other student teachers' concept images of integrals during the second round of the interviews

At this time of the second round of the interviews, the other student teachers exhibited the concept images shown in the diagrams below (Figure 18 and Figure 19). Diagram 18 shows that some of the student teachers evoked pseudoobjects of integrals and of the FTC (VEA) while diagram 19 shows that some of the student teachers also evoked the underlying concepts of the definite integral. However, these underlying concepts were still to be developed as they were not correctly evoked in relation to the context of the definite integral. In these diagrams, the tick symbol (✓) indicates that the student teacher evoked the context of area; the three types of circles and the tree-like symbol are used as I described in the theoretical framework and as in the analyses in the previous sections. The vertical reading of the diagrams gives the individual student teacher's concept images in each representation whereas the horizontal reading allows for a comparison of the student teachers' concept images.

The diagram in Figure 19 shows that during the second round of the interviews, nine out eleven student teachers evoked the context of area under a curve, a context that is appropriate for the concept of the definite integral; only five student teachers evoked some underlying layers of the concept of the definite integral in the graphical/visual representation; the other six student teachers did not evoke any of the underlying concepts of the definite integral. In these interviews, only two out of eleven student teachers did not evoke the phrase 'area under a curve' which is one starting point for developing the underlying concepts of the definite integral.

	B	C	D	E	F	G	H	J	K	M	N
Verbal representation											
Antiderivative (Indefinite integral)	○						○			○	○
Definite integral											○
FTC (VEA)											
Graphical representation											
Antiderivative (Indefinite integral)											
Definite integral											
FTC (VEA)											
Numerical representation											
Antiderivative (Indefinite integral)											
Definite integral											
FTC (VEA)											
Symbolical representation											
Antiderivative (Indefinite integral)				○			○				○
Definite integral											○
FTC (VEA)				○							
Area as application		✓	✓	✓	✓	✓					✓
	B	C	D	E	F	G	H	J	K	M	N

Figure 18. First synoptic diagram of student teachers' concept images evoked during the second round of the interviews

Student teachers	B	C	D	E	F	G	H	J	K	M	N
Area under a curve	🌳	✓	✓		🌳	🌳		🌳	🌳	🌳	🌳
Verbal representation											
Partition											
Product											
Sum											
Limit	🌳										
Graphical representation											
Partition		🌳	●					🌳	🌳		
Product											
Sum	🌳	🌳						🌳	🌳		
Limit	🌳		🌳								
Numerical representation											
Partition											
Product											
Sum											
Limit											
Symbolical representation											
Partition											
Product											
Sum											
Limit											
	B	C	D	E	F	G	H	J	K	M	N

Figure 19. Second synoptic diagram of student teachers' concept images evoked during the second round of the interviews

Concerning the underlying concepts of the definite integral, some of them were tentatively evoked in the graphical representation as shown in the above diagram (Figure 19). From this diagram, the numbers of the student teachers who attempted to evoke the underlying concepts of the definite integral in the graphical representation were four for the layer of partition, zero for the layer of product, four for the layer of sum, and two for the layer of limit. None of the underlying concept was evoked in the symbolical representation.

At the time of the second interviews held some few days before the starting of the teaching of the component of integration, the underlying concepts were emerging in the graphical representation as shown in the preceding synoptic diagram. As I have said previously, I conducted these interviews to verify whether the teaching and learning of the concepts of limit and of derivative would not have occasioned a change or a recall in the student teachers' concept images of integrals and I was also verifying whether some of the student teachers had consulted textbooks, peers, or other resources in order to improve their understanding of integrals. According to the curriculum they had followed in secondary schools, they had never learned the underlying concepts before they came to the Institute. It is, therefore, plausible that those who evoked some of the underlying layers in the graphical representation could have consulted textbooks.

After these interviews, I started the teaching of the component of integration. During the first lesson, I discussed with the student teachers their questions about integrals and we defined together the objective that would lead the whole course. During the second and the third lesson the student teachers worked on tasks about the evaluation of areas under curves without using antiderivatives. These tasks are presented in appendix B. At the end of the third lesson, I requested the student teachers to summarise in writing on their worksheets what they had learnt until then. In the next section, I present the analysis of their productions after this third lesson.

5.4. Student teachers' concept images evoked during the teaching after the lesson three

This section presents the analysis of the student teachers' productions at the end of the third lesson of the teaching. The findings of the analysis in terms of representations and of mathematical processes and their resulting objects are summarised in a diagram after the text that the student teachers produced at the end of the lesson.

The analysis of the student teachers' concept images evoked after the third lesson consists in identifying process aspects and their resulting objects in sentences written by each student teacher at that time. Detailed analysis will be presented only for four student teachers. For each of the student teachers, I will label the sentences that contain the process or the resulting object by sequential numbers to indicate which sentences I refer to, and after that I will present the illustrative diagrams. The detailed analysis of those sentences follows the same approach as the one that I used for analysing the spoken text produced during interviews.

At the end of the detailed analysis of the concept images for all the student teachers, I will present a synoptic table for all the concept images evoked by the student teachers. This synoptic table will be used to make comparison with the other concept images evoked by the student teachers during interviews. This comparison will allow me to conclude about the evolution of the student teachers' evoked concept images in symbolical representation at the end of the teaching. The student teachers' concept images are presented in alphabetical order of the pseudonyms. In the next subsection, I present the concept images evoked by Bernard.

5.4.1 Bernard's concept image of the definite integral evoked after lesson three

At the end of the third lesson, Bernard wrote the following text. I underlined and numbered sentences that I identified as relevant for determining the evoked concept images. The number of the sentence is preceded by the symbol #. The first digit of the number represents 'chapter 5' and the second digit is the 'sequential number' of the sentence.

> To find the area delimited by a parabola and the x-axis between fixed boundaries for example [0, 5], <u>one proceeds by the subdivision of the interval [2, 5] into 3 subintervals of 1 unity as base</u> [# 5.1]; the height of each rectangle is obtained in calculating f(x) for x comprised in each of the intervals; for example: x=2, x_n=5,

f(2.5), f(3.5), f(4.5); the obtained area is 9.6875 [# 5. 2]; the next subdivision gives 6 subintervals of 0.5 length each and one obtains 6 rectangles [# 5. 3]; the heights are obtained in the same way as the preceding subdivision in calculating the f(x) for x comprised in the subintervals: f(2.25), f(2.75), f(3.25), f(3.75), f(4.25), f(4.75). In multiplying the bases by the heights, we obtain the areas of rectangles [# 5. 4] and in adding those areas of rectangles, we will obtain 9.73439375[# 5. 5]. In repeating the same operation for the next subdivision one obtains 12 rectangles and the obtained area is 9.746092813. The more one subdivides many times, the area under the curve increases[# 5. 6]. This increase is due to the change of heights of rectangles. The heights vary with the number of subdivisions. The area tends to the exact value [# 5. 7]when one continues to subdivide.

In sentence[# 5.1], Bernard evoked the layer of partition. Reasoning in graphical representation, he evoked the process of subdividing the interval and the resulting object of the process, namely the subintervals. Therefore I put a shaded circle (●) in the cell intersecting the layer of partition and the column of graphical/visual representation (Figure 20). In the same sentence[# 5.1], Bernard evoked the resulting object of the process in numerical representation when he said "subintervals of 1 unity as base". Therefore I put an empty circle (○) in the cell intersecting the layer of partition and the column of numerical representation.

In sentence[# 5. 2], Bernard evoked the object of the layer of the product by giving only the result of the operation in numerical representation since he said "the obtained area is 9.6875". However the process that led to this result has not been evoked. Therefore I put an empty circle (○) in the cell intersecting the layer of product and the column of numerical representation.

The sentence [# 5.3] is about the layer of partition and leads to the same analysis as in sentence [# 5.1]. In sentence [# 5.4], Bernard evoked the layer of product in graphical representation since he said “in multiplying the bases by the heights, we obtain the areas of rectangles”. In this statement, Bernard evoked both the process of ‘multiplying’ and the resulting object of ‘areas’. Therefore I put a shaded circle (●) in the cell intersecting the layer of product and the column of graphical representation.

In sentence [# 5.5], Bernard evoked the layer of sum by saying “in adding those areas of rectangles, we will obtain 9.73439375”. He evoked the process of adding in the graphical representation as he said “in adding those areas” while he gave the resulting object in numerical representation “we obtain 9.73439375”. Therefore I put a crossed circle (⊗) in the cell intersecting the layer of sum and the column of graphical representation and an empty circle (○) in the cell intersecting the layer of sum and the column of numerical representation.

In sentences [# 5.6] and [# 5.7], Bernard evoked the layer of limit. He correctly evoked the resulting object of ‘exact value of the area’ in sentence [# 5.7] as he said “the area tends to the exact value when one continues to subdivide”. However, the process of limiting is not clearly evoked as a limit of a sequence of approximate areas. Therefore I put an empty circle (○) and a tree-like symbol () in the cell intersecting the layer of limit and the column of the graphical representation to indicate that Bernard evoked the pseudoobject of the layer of limit and the process in still in embryonic state and needs to be developed.

Process – Object Layers	Context: Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		●	○	
Product		●	○	
Sum		⊗	○	
Limit		○ 		

Figure 20. Bernard's concept images evoked after the third lesson

5.4.2 Cassim's concept image of the definite integral evoked after lesson three

At the end of the third lesson, Cassim wrote the following text:

> To find the area comprised between the curve and the x-axis from 0 to 3, <u>one has first to subdivide the area into 3 parts</u> [# 5. 8], that is to say, subdividing the interval into 3. <u>One obtains 3 small rectangles of 1 cm width</u> [# 5. 9]. The height of the 3 rectangles increases as the function increases. For the first subdivision, the width is 1 cm; the height are f(0.5)=0.125, f(1.5)=1.125, f(2.5)=3.125 because $f(x) = \frac{x^2}{2}$. <u>The obtained area is $1cm.(f(0.5) + f(1.5) + f(2.5))$ = $1cm.(0.125 + 1.125 + 3.125)$ = $4.375cm.1cm = 4.375cm^2$</u> [# 5. 10]. For the second subdivision, we obtain 6 rectangles whose heights increase as long as the function increases. The width is 0.5. The obtained area is 4.459125 cm^2 in proceeding in the same way as in the first subdivision. For the third subdivision, the width is 0.25cm and the number of rectangles is 12. The obtained area is 4.5322875cm^2. Generally one can say that <u>the area delimited by the curve and the x-axis is the</u>

sum of the areas of small rectangles $[\# 5.11]$ when one performs a subdivision of that area over the given interval. When one increases the subdivisions, the area tends to be exact $[\# 5.12]$: $Area = \sum_{i=1}^{n} h_i \Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$ $[\# 5.13]$, $h_i = f(x_i)$, $\Delta x = width$ $[\# 5.14]$

In sentences $[\# 5.8]$ and $[\# 5.9]$ and in formula $[\# 5.14]$, Cassim was dealing with the layer of partition. In sentence $[\# 5.8]$, Cassim evoked the process of subdividing in graphical representation when he said "one has first to subdivide the area into 3 parts". In sentence $[\# 5.9]$, Cassim evoked the resulting object of the process of subdividing in graphical representation as he said "One obtains 3 small rectangles". Therefore I put a shaded circle (●) in the cell intersecting the layer of partition and the column of graphical representation (Figure 21) to indicate that Cassim evoked both the process and the resulting object in graphical representation. In sentence $[\# 5.9]$, he also evoked the resulting object of the process of subdividing in numerical representation when he wrote "3 small rectangles "of 1 cm" width. However the process in numerical representation is not evoked. Therefore I put an empty circle (○) in the cell intersecting the layer of partition and the column of numerical representation. Finally in the formula $[\# 5.14]$, Cassim evoked only the resulting object $\Delta x = width$ while the process usually represented by the formula $\frac{b-a}{n}$ was not evoked at that time. Therefore I put an empty circle (○) in the cell intersecting the layer of partition and the column of symbolical representation.

In sentence $[\# 5.10]$ and in formula $[\# 5.13]$, Cassim dealt with the layer of product. In sentence $[\# 5.10]$, Cassim evoked the process and the resulting object in numerical representation when he wrote "The obtained area is (…) $4.375cm.1cm = 4.375cm^2$". The first part of the equality represents the process of multiplying while the second part of the equality represents the resulting object. Therefore I put a shaded circle (●) in the cell intersecting the layer of product and the column of numerical representation. In formula $[\# 5.13]$, cassim wrote $f(x_i)\Delta x$ which represents the process of multiplication in symbolical representation. However, he did not give any symbol for the resulting object. Therefore I put a crossed circle ($\otimes$) in the cell intersecting the layer of product and the column of symbolical representation to indicate that Cassim correctly evoked only the process but not the resulting object.

In sentence $[\# 5.11]$ and in formula $[\# 5.13]$, Cassim was dealing with the layer of sum. In sentence $[\# 5.11]$, he evoked the process of adding areas in graphical representation as he wrote "the area delimited by the curve and the x-axis is the sum of the areas of small rectangles". However, the resulting object of the total area is not evoked. Therefore, I put a crossed circle ($\otimes$) in the cell intersecting the layer of sum and the column of graphical representation to indicate that only the process has been evoked. In the formula $[\# 5.13]$, $Area = \sum_{i=1}^{n} h_i \Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$, the process aspect was evoked in the second part of the formula while the resulting object was evoked by the first part symbolised by "Area". Therefore I put a shaded circle (●) in the cell intersecting the layer of sum and the column of symbolical representation.

Finally, in the sentence $[\# 5.12]$, Cassim dealt with the layer of limit as he said "When one increases the subdivisions, the area tends to be exact". However, only the resulting object is correctly evoked. The process of limiting as a limiting of approximate areas is not correctly stated; it is still in embryonic state. Therefore I put an empty circle (○) and a tree-like symbol () in the cell intersecting the layer of limit and the column of graphical representation to indicate that he correctly evoked the pseudoobject of the layer of limit in graphical representation while the process aspect is incorrectly evoked and needs to be developed.

Process – Object Layers	Context: Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		●	○	○
Product			●	⊗
Sum		⊗		●
Limit		○ 		

Figure 21. Cassim's concept images evoked after the third lesson

5.4.3 David's concept image of the definite integral evoked after lesson three

At the end of the third lesson, David wrote the following text:

> In conclusion, one has understood the method of calculating area without using the antiderivatives and one can conclude that the <u>more one subdivides the more errors of calculation decrease</u> [# 5.15]. And finally we have admitted an effective method of calculating area <u>in subdividing intervals into small parts</u> [# 5.16]. The <u>real values of the area can be calculated by the addition of partial areas</u> [# 5.17] and by <u>comparing we find that the errors of calculations decrease</u> [# 5.18].

In sentence [# 5.15], David is dealing with the layer of subdivision and the layer of limit as he said "the more one subdivides the more errors of calculation decrease". This sentence represents the process aspect in the verbal representation. I, therefore, put a crossed circle (⊗) in the cell intersecting the layer of partition and the column of verbal representation. In sentence [# 5.16], David referred to the layer of subdivision in graphical representation as he said "in subdividing intervals into small parts". This phrase evoked the process of “subdividing intervals” and the resulting object of “small parts”. Therefore I put a shaded circle (●) in the cell intersecting the layer of partition and the column of graphical representation (Figure 22) to indicate that David evoked both the process and the resulting object of the layer of partition.

In sentence [# 5.17], David evoked the layer of sum in the graphical representation as he said “the real values of the area can be calculated by the addition of partial areas”. He evoked the process aspect when he said “can be calculated by the addition of partial areas”. However the resulting object is not correctly evoked as he said “the real values of the area”. This phrase is likely to refer to the exact area below the curve. Therefore I put a crossed circle ($\otimes$) and a tree-like symbol () in the cell intersecting the layer of sum and the column of graphical representation to indicate that David correctly evoked the process of adding but the resulting aspect is not correctly stated and is still to be developed.

In sentence [# 5.18], David again evoked the layer of limit as he said “by comparing we find that the errors of calculations decrease”. This phrase correctly evokes the process aspect of the limit layer and the phrase ‘the errors of calculation’ could lead to the resulting aspect. However, the resulting object which is the ‘exact area’ is not stated. Therefore, I put a crossed circle ($\otimes$) and a tree-like symbol () in the cell intersecting the layer of limit and the column of graphical representation to indicate that Cassim correctly evoked the process aspect of the layer of limit but that the resulting object was not evoked.

Process – Object Layers	Context:			
	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition	⊗	●		
Product				
Sum		⊗ 		
Limit		 ⊗		

Figure 22. David's concept images evoked after the third lesson

5.4.4 Edmond's concept image of the definite integral evoked after lesson three

At the end of the third lesson, Edmond wrote the following:

[Edmond wrote a lot of symbolic text. I give here key lines of what he wrote]

$\underline{B_1 = 1\,cm}$, $\underline{B_2 = \frac{1}{2}\,cm}$, $\underline{B_3 = \frac{1}{4}\,cm}$, …, $\underline{B_{10} = \frac{1}{2^9}\,cm}$, $\underline{B_n = \frac{1}{2^{n-1}}\,cm}$ [# 5.19].

To calculate <u>the total area, one has to add areas of rectangles calculated in multiplying the base (l) by the height (L)</u> [# 5.20]. For example: $\underline{Area1\,(a_1) = B_1 \,.\, h_1 = 10\,cm^2}$, $\underline{a_2 = B_2 \,.\, h_2 = 10{,}2925\,cm^2}$, $\underline{a_3 = B_3 \,.\, h_3 = 10{,}....cm^2}$, … $\underline{a_n = B_n \,.\, h_n = total\ area}$ [# 5.21].

<u>As the base is divided by 2 at each subdivision</u> $B_2 = \frac{B_1}{2}$, $B_3 = \frac{B_1}{4}$, $B_4 = \frac{B_1}{8}$, …, $B_n = \frac{B_1}{2^{n-1}}$ [# 5.22], (…), the total area is $base1 \,.\, h_1 + \frac{base1}{2} \,.\, h_2 + \frac{base1}{4} \,.\, h_3 + ...$

As I took $base1 = 2\ cm$, then the total area is $\underline{A = 2\ cm\ .\sum_{n=1}^{\infty}\frac{h_1}{2^{n-1}} + \frac{h_2}{2^{n-1}} + \frac{h_3}{2^{n-1}} + ...}$ $[\#5.23]$. $A = 2\ cm\ .\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}H$. The sum of height equals $4\ x\ 2\ cm = 8\ cm = 2^3$, $A = 2cm.\sum_{n=1}^{\infty}\frac{1.8\ cm}{2^{n-1}}$, $A = 16\ cm^2.\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$, $(\dots)$ $\underline{Total\ Area = \lim_{n\to\infty}\frac{2\ .\ 2^3}{2^{n-1}} = \lim_{n\to\infty}\frac{2^4}{2^{n-1}}}$ $[\#5.24]$.

In formulas $[\#5.19]$ and $[\#5.22]$, Edmond evoked the layer of partition in symbolical representation since he tried to generalise the formula of the partition.

$\underline{B_1 = 1\ cm}$, $\underline{B_2 = \frac{1}{2}\ cm}$, $\underline{B_3 = \frac{1}{4}\ cm}$, …, $\underline{B_{10} = \frac{1}{2^9}\ cm}$, $\underline{B_n = \frac{1}{2^{n-1}}\ cm}$. $[\#5.19]$

$B_2 = \frac{B_1}{2}$, $B_3 = \frac{B_1}{4}$, $B_4 = \frac{B_1}{8}$, …, $B_n = \frac{B_1}{2^{n-1}}$ $[\#5.22]$. However his formulas were not correctly generalised. Therefore I put tree-like symbol () in the cell intersecting the layer of partition and the column of symbolical representation (Figure 23) to indicate that the formula is not correctly evoked and is still to be developed. In formulas $[\#5.19]$ and $[\#5.22]$, Edmond used numbers to represent the resulting object of the formulas. For that I put an empty circle (○) in the cell intersecting the layer of partition and the numerical representation to indicate that Edmond evoked the resulting object without evoking the process aspect in numerical representation.

In sentence $[\#5.20]$, Edmond was dealing with the layer of product in graphical representation as he said "areas of rectangles calculated in multiplying the base (l) by the height (L)". The process aspect was evoked when he said "in multiplying the base (l) by the height (L)" while the resulting objects was evoked by the phrase "areas of rectangles calculated". Therefore I put a shade circle (●) in the cell intersecting the layer of product and the column of graphical representation.

Moreover in formulas $[\# 5.21]$, Edmond evoked the layer of product symbolically. The process aspect was evoked in symbols while the resulting object was given in numbers. When he tried to generalise, he evoked the total area. The formula shows some confusion and needed to be clarified. Therefore I put a tree-like symbol () in the cell intersecting the layer of product and the column of symbolical representation to indicate that the formula is incorrectly evoked and is still to be developed. Moreover I put an empty circle ($\bigcirc$) in the cell intersecting the layer of product and the column of numerical representation to indicate the fact that Edmond evoked the resulting object of the layer in numerical representation. For example: $\underline{Area1\,(a_1) = B_1\ \ h_1 = 10\,cm^2}$, $\underline{a_2 = B_2\ \ h_2 = 10{,}2925\,cm^2}$, $\underline{a_3 = B_3\,.h_3 = 10,....cm^2}$, $\ldots \underline{a_n = B_n\ h_n = total\ \ area}$ $[\# 5.21]$.

In sentence $[\# 5.20]$ and in formula $[\# 5.23]$, Edmond dealt with the layer of sum first in graphical representation when he said "to calculate the total area, one has to add areas of rectangles" and then in the symbolical representation when he wrote the formula $\underline{A = 2\,cm \sum_{n=1}^{\infty} \frac{h_1}{2^{n-1}} + \frac{h_2}{2^{n-1}} + \frac{h_3}{2^{n-1}} + ...}$ in line $[\# 5.23]$. In the graphical representation, the process aspect is evoked by the phrase "to calculate the total area, one has to add areas of rectangles" in line #5.20 and the resulting object was not correctly evoked. The phrase "the total area" that he evoked refers rather to "the exact area below the curve". Therefore I put a crossed circle ($\otimes$) in the cell intersecting the layer of sum and the column of graphical representation. Also, I put a tree-like symbol () in the cell intersecting the layer of sum and the column of symbolical representation to indicate that the incorrect formula he used in line #5.23 needs to be developed.

In the formula $[\# 5.24]$, Edmond was dealing with the layer of limit in symbolical representation. However, the formula $\underline{Total\ \ Area = \lim_{n\to\infty} \frac{2\,.\,2^3}{2^{n-1}} = \lim_{n\to\infty} \frac{2^4}{2^{n-1}}}$ $[\# 5.24]$ that he evoked was not the correct one usually used in the context of integration and needed to be developed. Therefore I put a tree-like symbol () to indicate that the evoked formula of the limit is incorrect and needs to be developed. As it can be seen through the symbolic text written by Edmond, his evoked concept image reflects all the underlying layers of the definite integral though they are not correctly evoked. These layers are the partition, the product, the sum, and the limit.

Process – Object Layers	Context: Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		⊗	○	
Product		●	○	
Sum		⊗		
Limit				

Figure 23. Edmond's concept images evoked after the third lesson

5.4.5 Other student teachers' concept images of the definite integral evoked after lesson three and the synoptic diagram

At the end of the third lesson, all the student teachers produced summaries of what they had learned until then. I analysed their productions as in the preceding cases of Bernard, Cassim, David, and Edmond. The following diagram summarises the findings of the analysis (Figure 24). The alphabetical letters and the icons have the same meanings as in the previous presentations.

Student teachers	B	C	D	E	F	G	H	J	K	M	N
Area under a curve	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Verbal representation											
Partition			⊗								
Product											
Sum											
Limit											
Graphical representation											
Partition	●	○	●	⊗	●	●	●	⊗	●	●	⊗ 🌳
Product	●		⊗ 🌳	●						○	
Sum	⊗	⊗	🌳	⊗	🌳	🌳	⊗				🌳
Limit	○ 🌳	○ 🌳	⊗ 🌳		○ 🌳		○ 🌳	🌳	○ 🌳		🌳
Numerical representation											
Partition	○	○		○		🌳	○	○			🌳
Product	○	●		○				○		○	⊗
Sum	○					🌳	○				●
Limit											
Symbolical representation											
Partition		○		🌳						🌳	
Product		⊗		🌳						🌳	
Sum		●		🌳						🌳	
Limit				🌳							
	B	C	D	E	F	G	H	J	K	M	N

Figure 24. Synoptic diagram of student teachers' concept images evoked after the third lesson

This synoptic diagram shows that at the end of the third lesson, almost all the student teachers evoked the underlying concepts of the definite integral in the graphical and in the numerical representations. However, most of them had not yet evoked these underlying concepts in the symbolical representation. Only one student teacher (Cassim) correctly evoked the symbols representing most of the underlying concepts of the concept of definite integral. The symbols used by the two others were not correctly used to represent the underlying concepts of the definite integral.

From the synoptic diagram above, I confectioned the following table (Table 2) which presents the numbers of the student teachers who evoked the various underpinning concepts of the definite integral in the symbolical representation.

Layers	Correctly evoked (numbers)	Incorrectly evoked (numbers)	Not evoked (numbers)
Partition $\Delta x = \dfrac{b-a}{n}$	○ (1)	2	8
Product $A_i = f(x_i)\Delta x$	⊗ (1)	2	8
Sum $S_i = \sum_{i=1}^{n} f(x_i)\Delta x$	⊗ (1) ● (1)	1	8
Limit $L = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	0	1	10
Definite integral (pseudo-object) $\int_a^b f(x)dx$	0	0	11
Definite integral $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	0	0	11

Table 2. Student teachers' concept images of the definite integral evoked after lesson three

Only one of the student teachers correctly evoked some aspects of the underlying concepts and only two of them incorrectly evoked some of the aspects of the underlying concepts. The rest of the student teachers did not evoke any of the aspects of the layer under consideration.

After the third lesson of the teaching, the graphical and the numerical representations were strongly evoked by the majority of the student teachers. This was the situation at the end of the third lesson. Comparing this to the results from the previous interviews, it is evident that the students have engaged the various elements (layers) of the concept of integral. The fact that they worked with graphical representations and related numerical calculations, in a guided discovery/reinvention process, is likely to explain why the graphical and numerical representations are more strongly evoked than the symbolic representation.

For the remaining of the teaching, I decided to change the teaching strategies in order to help the student teachers move forward their concept images. For the next three lessons that dealt with the evaluation of areas under curves without using antiderivatives, I requested the student teachers to communicate and discuss their solutions to the tasks. Then from the seventh lesson up to the fifteenth, I continued the teaching about the other topics, namely the indefinite integral and the fundamental theorem of calculus. In chapter six, I will say more about the communication in the classroom. At the end of the teaching, I held a third round of interviews. The next section presents the results of the analysis of what the student teachers evoked about the definite integral during the third round of the interviews.

5.5. Student teachers' concept images of integrals evoked during the third round of the interviews

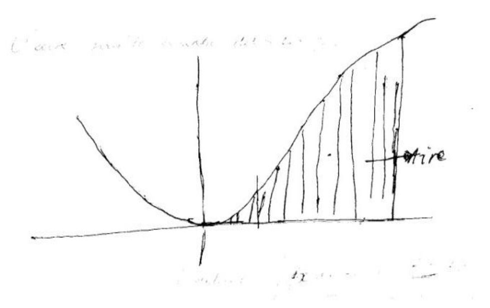
This section presents the analysis of the student teachers' concept images evoked during the third round of the interviews. Like in the previous section, after the analysis of each student teacher's evoked concept images, I present a diagram summarising the analysis in terms of representations and mathematical conceptions of processes and their resulting objects, and at the end I present a synoptic diagram including all the student teachers' concept images. From this synoptic diagram, I produce a table (Table 3) showing the numbers of student teachers who evoked the various aspects of the underlying concepts of the definite integral in symbolical representation. The presentation of the analysis of the student teachers' concept images follows the alphabetical order of their pseudonyms. In the next subsection I analyse the concept image evoked by Bernard.

5.5.1 Bernard's concept images of integrals evoked during the third interview

After some warm-up questions regarding his potential career of teaching, Bernard exhibited the following concept image of the definite integral.

Extract B3

48. FH: And if you were a teacher and your pupil asked you to explain to him what the integral is; what will you tell him?
49. Bernard: I will tell him that the integral is defined as the area which is under the curve and that area is delimited by the axis x and thus it is the area which is delimited by the x-axis and the curve.
50. FH: And how will you explain him that it is the area under the curve?
51. Bernard: By sketching a diagram [Bernard sketches the following diagram].



Here I choose a curve; a curve that has a form of a parabola that passes by the point zero. Then considering the part which is shaded, it is that part that I will consider as being the area under the curve.

52. FH: And how will you tell him that the integral is the area?
53. Bernard: In view of that, this part seems not to have any form with regard to geometric figures; I will explain that this part can be subdivided into geometric figures for which one can calculate area. Those figures are rectangles. Then calculating area of each rectangle and summing these areas, one obtains area under the curve; the area being the summation of the product of heights and bases of these rectangles….and from here, I will explain the two

types of integrals, the indefinite and the definite integrals. [Bernard explained the indefinite integral]

54. FH: And for the definite integral?

55. Bernard: For the definite integral, one has also the elongated S always with the same function; here we have the lower limit and the upper limit. This will be equal to the limit of the summation of f of x_i multiplied by the variation when n tends to infinity. [Bernard writes $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$]. Here concerning this definite integral, one obtains a value because one fixes the limits…the boundaries, the lower limit and the upper limit. In this case the area will be delimited by the curve, the x-axis and the two lines which are determined by the boundaries, thus the vertical lines that pass by the limits.

In line 49, Bernard defined the integral as an area that is below a curve since he said "the integral is defined as the area which is under the curve and that area is delimited by the axis x". In this statement, Bernard sees the area as an application domain for the integral. However in line 51 he elaborated his idea about the area under a curve by sketching a diagram and a curve of a parabola and by shading the surface below the curve. From this, I consider that Bernard has evoked area in two perspectives, first as an application domain of the integral and then as a context to explain the integral. Therefore in the illustrative diagram of Bernard's concept image I put the word "area" followed by a tick symbol (✓) in the cell reserved to the application and I put also the phrase "area under a curve" followed by a tick symbol (✓) in the cell reserved to context to indicate that Bernard's ideas about area are clearly evoked in the two perspectives as far as integrals are concerned (Figure 25).

My question in line 48 concerned integrals in general. Bernard first answered this general question of what the integral is. After some explanation about the link of the integral to the area (lines 49, 51, 53) and also about what the indefinite integral is, I asked the question about the definite integral in line 54. It is in answering this question that Bernard explained (line 55) what the definite integral is.

Compared to the extracts of the two previous interviews with Bernard, this extract from the third round of the interviews contains more ideas related to the underlying concepts of the concept of definite integral.

In line 53, while explaining how the integral is related to the area under a curve, Bernard evoked the layers of partition, of product, and of sum in graphical representation. Firstly, Bernard evoked the layer of partition. In fact, he talked about subdividing the area under a curve into geometric figures of which one can calculate the area and he continued by saying that those figures are rectangles ("this part can be subdivided into geometric figures for which one can calculate area. Those figures are rectangles"). The first part of the sentence ("this part can be subdivided into geometrical figures") evokes the process of subdividing for the partition layer and the second sentence evokes the resulting object (rectangles) of the partition layer. Therefore I put a shaded circle (●) in the cell intersecting the layer of the partition and the column of the graphical representation to indicate that Bernard evoked both the process and the object aspects of the layer of the partition in the graphical/visual representation.

Secondly, in the same line 53, Bernard evoked the layers of product and of sum. In fact he said: "Then calculating area of each rectangle and summing these areas, one obtains area under the curve; the area being the summation of product of heights and bases of these rectangles". The process of multiplying is evoked in the phrase "calculating area of each rectangle" and in the phrase "product of heights and bases". The resulting object is evoked by the phrase "area of each rectangle" at the beginning of his sentence. Therefore, I put a shaded circle (●) in the cell intersecting the layer of the product and the column of the graphical/visual representation to indicate that Bernard evoked both the process and the object aspects of the layer of product.

Concerning the layer of sum also evoked in line 53 as mentioned previously, the process of adding is evoked by the phrase "summing these areas". The object of the process of adding areas in the context of the definite integral context is the total approximate area.

Bernard evoked the total area under the curve ("one obtains area under the curve"); he did not evoke the total approximate area. His idea about the resulting object of the process of adding is inexact because it is disassociated from the limiting process. Therefore, I put a crossed circle ($\otimes$) in the cell intersecting the layer of the sum and the graphical/visual representation to indicate that Bernard correctly evoked the process aspect without evoking the correct resulting object of the layer of sum in the graphical/visual representation. The layer of limit in the graphical has not been evoked. Therefore the corresponding cells remain empty.

My question in line 54 followed the answer that Bernard had just given to the question of what the indefinite integral is. I was asking him to pursue by explaining what the definite integral is. His explanation then turned into the symbolical representation and in line 55 he evoked the following formula $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$.

While he was writing this formula he said "These will be equal to the limit of the summation of f of x_i multiplied by the variation when n tends to infinity". Furthermore, he linked the formula to the idea of the definite integral as a value as he said "concerning this definite integral, one obtains a value because one fixes the limits … the boundaries, the lower limit and the upper limit". Therefore he has linked the diagram of the concept of the definite integral with the diagram of the underlying concepts. Hence Bernard displayed an extended understanding of the concept of the definite integral.

Through the formula $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$, Bernard exhibited some ideas of the underlying concepts of the definite integral in the symbolical representation. Firstly, the layer of partition is represented by the symbol Δx (he calls it "the variation"). This symbol Δx stands for the object of the partition layer. The process aspect symbolised by $\frac{x_n - x_0}{n}$ is not evoked. So I put an empty circle ($\bigcirc$) in the cell intersecting the layer of partition and the column of symbolical representation to indicate that, at this time of the interview, Bernard evoked only the pseudoobject of the layer of partition in the symbolical representation. The circle would have been shaded if Bernard had evoked the equation $\Delta x = \frac{x_n - x_0}{n}$. Secondly, the layer of product is represented by the symbol $f(x_i)\Delta x$ (he said "f of x_i multiplied by the variation"). This symbol $f(x_i).\Delta x$ stands for the process of multiplying. The resulting object symbolised for example by A_i to represent the area is not evoked. So I put a crossed circle ($\otimes$) in the cell intersecting the layer of product and the symbolical representation to indicate that at this time of the interview, Bernard evoked only the process aspect of the layer of product in symbolical representation. The circle would have been shaded if Bernard had evoked the equation $A_i = f(x_i)\Delta x$. Thirdly, the layer of sum is represented by the symbol $\sum_{i=1}^{n} f(x_i)\Delta x$ (he said "…summation of f of x_i multiplied by the variation"). This symbol $\sum_{i=1}^{n} f(x_i)\Delta x$ stands for the process of adding areas. The resulting object symbolised for example by S_n is not evoked. So, I put a crossed circle ($\otimes$) in the cell intersecting the row of the layer of sum and the symbolical representation to indicate that Bernard evoked only the process aspect of the layer of sum in symbolical representation. The circle would have been shaded if Bernard had evoked the equation $S_n = \sum_{i=1}^{n} f(x_i)\Delta x$ which also specify the resulting object S_n. I will come back to these difficulties of coding using this framework in the last section of this chapter (conclusion and discussion).

Fourthly, the layer of limit is represented by the symbol $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$. The limiting process is symbolised by $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ and the resulting object of the limiting process is symbolised by $\int_a^b f(x)dx$ and Bernard linked the two aspects (the process and the value) by writing the corresponding equation (he said "This will be equal to the limit of the summation of f of x_i multiplied by the variation when n tends to infinity") and by saying that "concerning this definite integral, one obtains a value because one fixes the limits … the boundaries, the lower limit and the upper limit". Therefore I put a shaded circle (●) in the cell intersecting the layer of the limit and the column of the symbolical representation to indicate that Bernard correctly evoked both the process and the object aspects of the layer of the limit in the symbolical representation.

Concepts and theorem	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Antiderivative (indefinite integral)				
Definite integral				
FTC (VEA)				
Area as application ✓				

Process – Object Layers	Context: Area under a curve ✓			
	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		●		○
Product		●		⊗
Sum		⊗		⊗
Limit				●

Figure 25. Bernard's concept images evoked during the third interview

During the third interview Bernard's concept image of the definite integral included many of the underlying process-object layers and the mathematical context of area under a curve and the area as an application domain for the integral. As it is shown on the illustrative diagram, Bernard referred to all the four underlying process-object layers in the symbolical representation and to three of them in the graphical/visual representation. Bernard's concept image changed significantly toward the formal concept image of the definite integral. In the symbolical representation, he gave the formula containing all the layers of the definition of the concept of the definite integral. His confusion with the limit from interview 2 seems to have disappeared. In line 55, he used the phrase "limit of a summation" instead of the "sum of limit" that he used in line 22 of the second interview. Bernard has now associated the appropriate meaning of 'limit' to the mathematical context of the definite integral. In the graphical/visual representation, the layers of partition and of product are firmly developed, while the layer of the sum is still to be developed since the resulting object of the process is not evoked. In the symbolical representation, only the layer of the limit is evoked in its two aspects of process and object; for the other layers of the partition, the product, and the sum, at least one of the two aspects is not evoked during this interview. Therefore there was still a need to develop the missing aspects in the coming days.

I can say that the teaching experiment was very beneficial to Bernard because the ideas that were classified before the teaching to be in an emerging state or in a process of maturing have now been developed into clear processes and objects. In chapter six, I will return to how Bernard and the other students engaged the concepts during the teaching experiment itself and how they helped each other or how the teacher guided and assisted them in order to include the underlying concepts of the definite integral into their concept images. This collaboration among student teachers will be explicit especially in lesson five and in lesson twelve during the episodes in which the "aha moments" of Edmond and of Bernard occurred.

In summing up, the illustration of Bernard's concept images of the definite integral through the three rounds of the interviews shows that his concept image of the definite integral has evolved during the period of six months that the research lasted. His concept image moved from not evoking any of the underlying elements of the formal concept image of the definite integral to including all the four underlying concepts of the formal concept image in the symbolical representation and three of them in the graphical/visual representations, within the mathematical context of area under a curve.

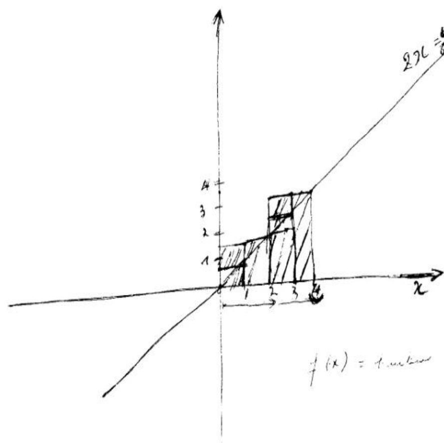
However, Bernard did not evoke all the process-object layers in all representations. In fact he did not evoke any layer in the numerical representation. Neither did he evoke all the aspects of all the process-object layers that he evoked in symbolical representation or in graphical/visual representation; especially he did not display the object aspects of the layer of product and of the layer of sum in symbolical representation and he did not evoke the correct object aspect of the layer of sum in graphical representation. He did not evoke those aspects even in verbal representation. Finally, he only evoked one context for this concept. At this stage, his concept image of the definite integral can be said to be fairly strong. However, to improve Bernard's concept images, and other student teachers' concept images as well, the teacher should readjust and enrich the tasks so that they lead the student teachers to see also the resulting object aspects (such as the sequences) in the symbolical representation. This requirement is needed for student teachers because they should be more aware of all the aspects of the concepts in order to be able to explain them to their students.

5.5.2 Cassim's concept images of integrals during the third interview

The following excerpt shows Cassim's concept images of integrals evoked during the third interview.

Excerpt C3

56.	FH:	Will you teach after your studies at KIE?
57.	Cassim:	Yes of course.
58.	FH:	Now, if you were a teacher and your student asks you to explain to him what the integral is, what will you tell him?
59.	Cassim:	… I will show him what the integral is by using the graphical diagram. Suppose that we have a function. Firstly I start by a linear function $y = 2x$ [Cassim sketches the following diagram];



and to explain what the integral is, I will tell him that it [the integral] is the area that is comprised between the curve or the graph of the function $2x$ and the abscissa axis x. This area will be comprised between two points on the x-axis and the smallest of these points is called the lower limit whereas the biggest is called the upper limit. Here is the graduation of the axes. For example, to find the integral of the same function from 0 to 1 or simply let's say the shaded area.

60. Cassim: Firstly, one proceeds to the subdivision of the whole area. One starts the subdivision firstly in two, and starting from that one can draw two rectangles given that the base of the two rectangles is the same, we will have two rectangles with the same base but with different heights.

61. Cassim: After having found the areas of the two rectangles, one will proceed also to subdivide the two rectangles into other small rectangles. And one will obtain four rectangles like these. Also these four rectangles will have the same base but different heights. As before one will calculate the areas of the four rectangles given that the areas of these rectangles have been determined before. Also one will subdivide the four rectangles into eight rectangles and will calculate the area following the same procedure.

62. Cassim: After that, one will notice, when he will have done the biggest subdivision possible, that the area above the curve of the graph tends to decrease towards to the curve (showing the curve).

63. Cassim: This will have as consequence that the limit as n tends to the infinity, n being the number of subdivision or the number of the rectangles found, of the sum of f of x_i, f of x_i being the height here for example here at 3, f of 3 is by projecting the image of 3 to the function on the y-axis, the interval comprised between 0 and the image of the point 3 is what is called the height of this rectangle and from then,

64. Cassim: I will tell him that the integral or to find the area delimited by this function and the x-axis is equal to the limit when n tends to the infinity of summation of $f(x_i)$ i.e. height times delta x given that delta x is equal to the greater limit xn minus x0 the lower limit divided by the number of subdivision of the small rectangles found. Simply it will be equal to the limit when n tends to the infinity of sum of $f(x_i)$ times xn minus xo over n.

65. Cassim: This is equal to the integral or the sum from zero to four, in our case, of 2x dx. The area is equal (Cassim writes the following formula)

$$Area = \lim_{n\to\infty}\sum f(x_i).\Delta x = \lim_{n\to\infty}\sum f(x_i).\frac{x_n - x_0}{n} = \int_0^4 2xdx.$$

Concerning this symbol the integral from zero to four of 2x dx (Cassim wrote the symbol) $\int_0^4 2xdx$, one can take the opportunity to explain this 2x and the derivative. That is, this function is the derivative of another function and the relation that exist with the function whose derivative is this one is what is called integration.

This extract from the third round of interviews evoked more references related to the underlying concepts of the definite integral. In fact, in line 59, Cassim introduced his reasoning in graphical representation and announced the mathematical context of area under a curve. He evoked that when he said: “I will show him the integral by using the graphical diagram”. Before he continued his explanation, Cassim also evoked the area as an application domain of the integral since he said: “I will tell him that it [the integral] is the area that is comprised between the curve or the graph of the function 2x and the abscissa axis x”. He emphasised this idea by evoking the limits of integration as he said “this area will be comprised between two points on the x-axis and the smallest of these points is called the lower limit whereas the biggest is called the upper limit”. Therefore I put the word ‘area’ followed by a tick symbol (✓) in the cell reserved to the application to indicate that Cassim consider the area as an application domain for the integral. Moreover, I put the phrase “area under the curve” followed by a tick symbol (✓) in the cell reserved to context to illustrate the idea that Cassim evoked the mathematical context of area under a curve by which he explained what the integral is (Figure 26).

In addition, in line 60, Cassim exhibited the process-object layer of partition in the graphical/visual representation. The process of subdividing is evoked when he said “one proceeds to the subdivision of the whole area". The result of the process is evoked when he sad: "we will have rectangles with the same base but with different heights". Therefore, I put a shaded circle (●) in the cell intersecting the process-object of the layer of the partition and the column of graphical/visual representation.

Moreover, in line 61, Cassim exhibited the product layer. Both the process and the object of this layer are included in the phrases "after having found the areas of the four rectangles", "as before one calculate the areas of the four rectangles" and "…one will calculate the area following the same procedure". How to calculate area is the process and the area is the resulting object of the process. So I put a shaded circle (●) in the box intersecting the layer of product and the column of the graphical/visual representation.

Furthermore, in line 62, Cassim evoked the layer of limit. The process of limiting was evoked when he said: "area above the curve of the graph tends to decrease towards the curve". However the resulting object - the exact area under the curve - is not evoked. He said "decrease towards the curve" instead of saying "towards the exact area". Therefore I put a crossed circle ($\otimes$) in the cell intersecting the row of the layer of limit and the column of the graphical/visual representation.

Finally, in line 62 to 64, Cassim transformed his graphical reasoning into the symbolical representation. In fact, in line 62, he said: "This will have as consequence that the limit as n tends to the infinity, n being the number of subdivision or the number of the rectangles found, of the sum of f of x_i, f of x_i being the height here, for example here at 3, the f of 3 is by projecting the image of 3 to the function on the y-axis …". This transformation ended when Cassim said: "it will be equal to the limit when n tends to the infinity of sum of f of x_i times xn minus x0 over n" (line 63). In line 65, he wrote the following formula $Area = \lim_{n\to\infty}\sum f(x_i)\Delta x = \lim_{n\to\infty}\sum f(x_i)\frac{x_n - x_0}{n} = \int_0^4 2x dx$. By writing this formula, Cassim evoked the layers of partition, of product, of sum, and of limit in symbolical representation. Concerning the layer of partition, the process of subdivision is manifested when Cassim wrote by the symbol $\frac{x_n - x_0}{n}$ in the second member of the second equality replacing the symbol Δx used in the first equality. The resulting object that is usually manifested by the symbol Δx is evoked in the second member of the first equality. Therefore I put a shaded circle (●) in the cell intersecting the row of the layer of partition and the column of the symbolical representation. Concerning the layer of product, Cassim evoked the process of multiplying when he wrote the symbol $f(x_i)\frac{x_n - x_0}{n}$. However, the resulting object usually symbolised by A_i is not evoked. Therefore I put a crossed circle ($\otimes$) in the cell intersecting the row of the layer of product and the column of symbolical representation to indicate that symbolically, Cassim only evoked the process of the product layer. Concerning the layer of sum, Cassim only evoked the process of adding when he wrote the symbol $\sum f(x_i)\frac{x_n - x_0}{n}$. However, the resulting object usually symbolised by S_n is not evoked. So I put a crossed circle ($\otimes$) in the cell intersecting the layer of sum and the column of symbolical representation to indicate that symbolically, Cassim only evoked the process aspect of the layer of sum. Concerning the limit layer, Cassim evoked both the process and the object aspect aspects when he wrote the formula $\lim_{n\to\infty}\sum f(x_i)\frac{x_n - x_0}{n} = \int_0^4 2x\,dx$. The process aspect is symbolised by the signs $\lim_{n\to\infty}\sum f(x_i)\frac{x_n - x_0}{n}$ and the resulting aspect is symbolised by the signs $\int_0^4 2x\,dx$. Therefore I put a shaded circle (●) in the cell intersecting the layer of limit and the column of the symbolical representation. Finally in line 65, Cassim made allusion to the indefinite integral in terms of the antiderivative when he said "this function is the derivative of another function and the relation that exists with the function whose derivative is this one is what is called integration". Therefore I put an empty circle (○) in the cell intersecting the row of antiderivative (indefinite integral) and the column of verbal representation to indicate that Cassim evoked the pseudo-object of the indefinite integral in the verbal representation.

Concepts and theorem	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Antiderivative (indefinite integral)	○			
Definite integral				
FTC (VEA)				
Area as application ✓				

Process – Object Layers	Context: Area under a curve ✓			
	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		●		●
Product		●		⊗
Sum				⊗
Limit		⊗		●

Figure 26. Cassim's concept images evoked during the third interview

During the third interview, Cassim’s concept images include many references to the underlying concepts of the concept of the definite integral. After having evoked an appropriate context for developing the concept of the definite integral he evoked the underlying concepts in graphical/visual representation and in symbolical representation. As it is shown in the illustrative diagram, he evoked three of the four layers in the graphical visual representation and he evoked all the four layers in the symbolical representation.

Summing up, the illustration of Cassim's concept images of the definite integral through the three rounds of the interviews shows that his evoked concept images have evolved during the period of six months, moving from having only one pseudoobject in verbal representation to including most of the underlying concepts of the formal concept image in the symbolical and in the graphical/visual representations, within the mathematical context of area under a curve.

Cassim did not evoke all the process-object layers in all representations. He did not evoke any of the process-object layers in numerical representation. Neither did he evoke all the aspects of all the process-object layers that he evoked in symbolical or in graphical/visual representation; especially the object aspect of the layers of product and of sum in symbolical representation and the object aspect of the layer of limit in graphical/visual representation are not evoked. He did not evoke them even in verbal representation. And he only activated one context, namely that of area under a curve.

His concept images can be said to be fairly strong.

5.5.3 David's concept images of integrals during the third interview

During the third interview held after the teaching the course, David exhibited the concept images of integrals presented in the following extract.

Excerpt D3

66.	FH:	Will you teach after your studies at KIE?
67.	David:	Yes, because I am following the training of teachers. So, probably I will teach.
68.	FH:	If you were a teacher and your pupil asks you to explain him, what the integral is, what will you tell him?
69.	David:	For the integral I will explain that the integral is the sum of areas, if here there is a curve and if one wants to calculate here are the graduations [sketching a graph in the plane; see the diagram below]. These graduations are done under the curve. Hence, one determines really the integral of the function on any limits and this diagram shows really that it is the sum, it is the area

comprised between the limits of interval that are fixed, i.e. the lower limit and the upper limit;

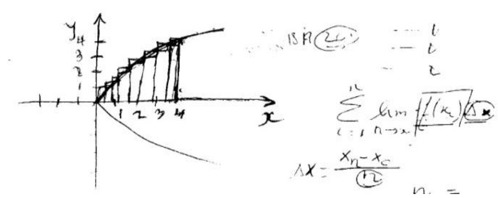
70. David: Finally I will show that the integral is the calculation of areas and this area found by subdividing is the sum of these subdivisions and the sum of these subdivisions gives the integral.

71. FH: How will you show him that it is an area?

72. David: Let one have a curve, thus a function, always when we talk of a function, there are axes, one construct the graph. We fix an upper limit equals to 4 and a lower limit equals to 0. Thus there is the area which is under the curve with respect to axes.

73. David: Then I will show him that one can do graduations, one subdivides this area until infinity to find the sum with respect to the variable.

74. David: When one says the variation, this variation of x which will help me to explain well to the pupil, I will show him that this variation helps us, the more the upper value tends to 0, the more the subdivisions tend to give us the actual value of the area.



Thus doing subdivisions, for the first subdivision, the pupil will find a value, in this value there are some errors, for the second variation the pupil will find a value, which has errors but not the same as those found before, the third subdivision, I will subdivide until I will show him that more we tend to infinity more we will find an actual value of area.

75. FH: How will he or she know that it is the area?

76. David: He will know that it is the area because I will use this variation of x [David writes Δx] and I will also use the image of our area on the y-axis,

77. David: I will show that it is a rectangle, thus doing subdivision one obtains rectangles, and we find area of these rectangles, and the more we tend to infinity, these errors that we have on the curve, under the curve, tend to decrease while approaching to the curve. Then the pupil will find that it is area because we tried to use geometrical figures that helped us to calculate area.

78. FH: How will the pupil find that it is the area?

79. David: Because, it is the sum of limits [David writes $\sum_{i=1}^{n} \lim_{n\to\infty} f(x_i)\Delta x$]; when we say limits, it means we will have limited, these heights here on the diagram; thus there are some limits; and these limits are different because the height here is different to the height here

and different to the height here, I will show him that one will have to find the limits. You see that these limits have to be found with respect to the function f(x); which is a factor of the variation of x [David writes Δx].

80. David: If it is a factor of the variation of x it means that it is the height times the base. Thus it is like this that one can show to the pupil and what if the pupil asks me what the variation of x is. I will tell him that for the variation of x, the more we subdivide the more the base changes. In this case, we have the upper limit minus the lower limit, because when subdividing one fixes the upper limit and the lower limit, in this case there is the upper limit minus the lower limit over the number of subdivisions [David writes $\frac{x_n - x_0}{n}$]. The number of subdivisions, if for example we subdivided 100 times, in this case the number of subdivisions n will be 100. That is how I can explain to the pupil how to find the integral without always using the formulae or the theories but practically under the curve there is area which is bounded by the lower limit and the upper limit.

81. David: And the more we subdivide to infinity we make many subdivisions to decrease these areas that exceed the curve. In that case, after having explained the variation of x, and also after having explained this limit of the function here, in this case the pupil will have to understand that it is the area because you see that the variation of the x is here whereas f(x) is this height here.

82. FH: Will you tell him that the integral is equal to what?

83. David: I will tell him that the integral is equal to the sum. If we have the integral of the function f(x) dx for an upper limit for example 4 and here zero [David writes $\int_0^4 f(x)dx$], I will tell him that this integral is equal to the sum for i which is equal to 1 and varies to n , … it is the first subdivision, n subdivisions, thus many subdivisions, to this limit, it tends to, the number of subdivisions tends to infinity, I will show him that it is equal to the sum of these subdivisions [David writes $\sum_{i=1}^{n} \lim_{n\to\infty} f(x_i)\Delta x$]. When we say the sum of these subdivisions, you see, here also one has to show these subdivisions, those limits. In this case, you see, the x_i will be equal to xo the lower limit plus i times the variation of x (David writes $x_i = x_0 + i\Delta x$). You see, there is always the multiplication. Thus it is in search of the area but if we have the limit as n tends to infinity, thus you see that we will suppress the error of this area that exceeds the curve, or when we subdivide for example in the other sense, we will find values which do not exceed but which do not reach to the curve as one sees in the diagram. Thus, one can subdivide starting say from twenty

subdivisions, making twenty subdivisions, twenty one, twenty two, twenty three, twenty four, one will find values which exceed the curve but by subdividing into nineteen, eighteen, like this, one will find values which… errors are suppressed from the curve. In this case one does subdivisions and subdivisions, finally one suppresses these errors and finally one find the actual value.

84. FH: If your pupil asks you to explain to him/her what the definite integral is, what will you tell him/her?

85. David: The definite integral is what I was saying, even if it has not been specified, the integral is defined because one has specified the lower limit and the upper limit. If it is specified it is the definite integral.

In this extract from the third round of the interviews, David's concept image includes more references to the underlying concepts of the concept of definite integral. In fact, in line 69, David evoked an appropriate context for developing the concept of the definite integral when he sketched a graph in the plane while saying: "if here there is a curve and if one wants to calculate here are the graduations. These graduations are done under the curve". So I put the phrase 'area under a curve' followed by a tick symbol (✓) in the cell reserved to context in the illustrative diagram (Figure 27) to indicate that David evoked the appropriate context of area under a curve. The sketching of and references to a graph led me to consider the remainder of David's reasoning in the graphical/visual representation.

In lines 70, David evoked the layer of partition when he said: "this area found by subdividing is the sum of these subdivisions and the sum of these subdivisions gives the integral". Later in line 77, David said: "doing subdivision, one finds rectangles". In these phrases, David evoked the process of subdividing when he mentioned the words 'by subdividing' and 'doing subdivisions'. David also evoked the resulting object by the phrase 'one obtains rectangles'. Thus, I put a shaded circle (●) in the cell intersecting the partition layer and the graphical/visual representation.

Moreover, in line 70 and in line 73, David evoked the layer of sum when he said: "this area found by subdividing is the sum of these subdivisions and the sum of these subdivisions gives the integral" (line 70). The words sum evoked by David in this sentence alludes to the process of adding of the underlying sum layer of the concept of the definite integral. However, the resulting object which is the total approximate area is not evoked. For this reason I put a crossed circle (⊗) in the cell intersecting the row of layer of sum and the graphical/visual representation.

The process of adding was evoked in line 73 when he said that "one subdivides this area until infinity to find the sum with respect to the variable". This shows that David has some confusion about the layer of sum. In fact he is confusing the process of sum involved in the underlying sum layer of the definite integral with the sum of limits of a real-valued function as it will be seen in line 79.

In line 74, David evoked the layer of limit in the graphical/visual representation when he said "the more the upper value [the value of area above the curve] tends to zero, the more the subdivisions tend to give the actual [exact] value of the area". The limiting process is evoked by the first part of the sentence when he said: "the more the upper value [the value of area above the curve] tends to zero, the more the subdivisions tend to give". The resulting object of the limiting process is the actual [exact] value of the area as mentioned at the end of the sentence when he said "to give the actual [exact] area". Consequently, I put a shaded circle (●) in the cell intersecting the row of the limit layer and the column of the graphical/visual representation to indicate that David evoked both the process and the object aspect of the layer of limit in graphical/visual representation.

Again, in line 77, besides evoking the partition and the limit layers, David evoked the process aspect of the product layer when he said that "…we tried to use geometrical figures that helped us to calculate area". To "calculate area" could invoke simultaneously the operation of multiplying and the resulting object of area. However, the process of 'multiplying' was not precisely evoked; only the area which is the resulting object was evoked. For that reason I put an empty circle (○) in the cell intersecting the row of the product layer and the column of the graphical/visual representation.

Finally, from line 79 to line 83, David switched to symbolical representation and evoked all the underlying layers of partition, of product, of sum, and of limit. Firstly, some confusion is to be noticed concerning the evoked terms of sum and limit with regard the underlying layer of sum and the underlying layer of limit. In line 79 and in line 83 he simultaneously evoked the sum and the limit layers as he said and wrote $\sum_{i=1}^{n} \lim_{n\to\infty} f(x_i)\Delta x$. As it is seen, he inverted the layers in writing and he said "it is the sum of limits" (line 79). He is possibly confusing the concept of limit of a real valued function with the limit of sum involved in the concept of the definite integral. This interpretation is supported by the fact that he said: "these limits have to be found with respect to the function f(x)" (line 79) and: "this limit of the function here" (line 81). Therefore, because of this confusion around the terms "sum" and "limit", the cells corresponding to the intersection of the layers of sum and of limit with the column of the symbolical representation will contain tree-like symbols to indicate that the processes of sum and of limit are in the process of maturing. Therefore I put a tree-like symbol () in the cell intersection the layer of sum and the symbolical representation. (I am aware that the student appears to have a mis-conception and that I have previously used the tree-like symbol to indicate incomplete conceptions. I discuss this issue at the end of this chapter, see p. 205.) Also I put a tree-like symbol () in the cell intersecting the layer of the limit and the symbolical representation to illustrate that embryonic state of the process of limit. In the limit layer the tree-like symbol will be side by side with an empty circle (○) that indicates that David correctly evoked the pseudoobject of the layer of limit when he wrote the symbol $\int_0^4 f(x)dx$ in line 83.

Secondly, in line 79 and 80, David evoked the layer of partition. In fact the object Δx of the layer of partition is evoked when he wrote the symbol Δx and when he said: "for the variation of x, the more one subdivides, the more the base changes". The process aspect was evoked when David said: "In this case we have the upper limit minus the lower limit, because subdividing one fixes the upper limit and the lower limit, in this case there is the upper limit minus the lower limit over the number of subdivisions" (line 80) and when he wrote the symbol $\frac{x_n - x_0}{n}$. This formula invokes the process of subdividing the interval $[x_0, x_n]$ into subintervals that are the bases of length Δx. Therefore, I put a shaded circle (●) in the cell intersecting the row of the layer of partition and the column of symbolical representation to indicate that David evoked both the process and the object aspect of the layer of partition in symbolical representation.

Thirdly, in lines 79, 80, and 83, David evoked the layer of product. The process of the product layer is evoked when he wrote the symbol $f(x_i)\Delta x$ in the formula $\sum_{i=1}^{n} \lim_{n\to\infty} f(x_i)\Delta x$ and said “you see these limits have to be found with respect to the function $f(x)$ but which is a factor of the variation of x, Δx” (line 79); he continued by explaining what he understood by the factor of the variation of x. He said: “it means that it is the height times the base” (line 80); and he said “you see there is always the multiplication; thus it is the search of the area” (line 83). However the resulting object of partial areas usually symbolised by A_i is not evoked. Therefore, I put a crossed circle ($\otimes$) in the cell intersecting the row of the layer of the product and the symbolical representation.

Fourthly, in line 83, David evoked the layer of sum and the layer of limit. However, as already mentioned previously, he is confusing the sums and the limits involved in the underlying concepts of the concept of the definite integral with the sums and the limits involved in the ‘ordinary’ real valued functions. However, he correctly evoked the resulting object of the process of limiting when he noted the symbol $\int_0^4 f(x)dx$. Therefore I put an empty circle (○) in the cell intersecting the layer of limit and the column of symbolical representation; the process aspect of this layer is not correctly evoked and thus needs to be developed. Therefore, I put a tree-like symbol () in the cell intersecting the layer of limit and the column of symbolical representation. .

Concepts and theorem	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Antiderivative (indefinite integral)				
Definite integral				
FTC (VEA)				
Area as application: ✓				

Process – Object Layers	Context: Area under a curve ✓			
	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		●		●
Product		○		⊗
Sum		⊗		
Limit		●		○ 

Figure 27. David's concept images evoked during the third interview

During the third interview, David‘s concept images include more references to the underlying layers of the concept of the definite integral. As it is shown in the illustrative diagram, David referred correctly to three of the four sub-layers in the symbolical representation and all of the four sub-layers in the graphical/visual representation, and the context is appropriate. However, David exhibited confusions about the sum layer and the limit layer with regard to the process aspects especially in the symbolical representation.

Summing up, the illustration of David's concept images of the definite integral through the three rounds of the interviews shows that his evoked concept images have evolved during the period of six months. In fact, the graphical/visual representation includes most of the elements of the formal concept image within the mathematical context of area under a curve. In the symbolical representation the concept image moved slightly from having one pseudoobject in the limit layer to include the partition and the product layers.

However, in the symbolical representation confusions are still visible in the limit layer and the sum layer in which David is still reasoning in the context of limit and sum of limits of an ordinary function of a real variable. Moreover, David did not evoke all the process-object layers in all representations. In fact he did not evoke any of the process-object in numerical representation. Neither did he evoke all the aspects of all the process-object layers that he evoked in the symbolical or in the graphical/visual representation; especially the process aspect of the layer of product and the object aspect of the layer of sum in the graphical/visual representation and both the process and object aspects of the layer of sum and the process aspect of the layer of limit in the symbolical representation. He did not evoke these aspects in verbal representation either.

At the end of the course, David had not yet succeeded in making clear the difference between the limit of sum in the integral's context and the limit of sum in the function context. So further tasks are needed to bring him see this difference.

David's concept images can be said to be moderate with some strong elements.

5.5.4 Edmond's concept images of integrals during the third interview

During the third interview, Edmond exhibited the following concept images of integrals.

Excerpt E3

86.	FH:	Will you teach when you finish your studies in KIE?
87.	Edmond:	Actually because I followed a training that leads to pedagogic acts, I will be a teacher. But possibly I can also orient myself in other career related to education.
88.	FH:	As what for example?
89.	Edmond:	For example in inspection of schools and probably in politics also.

90. FH: How in politics, in what politics?

91. Edmond: In the basic institutions, as a Major of district or sector.

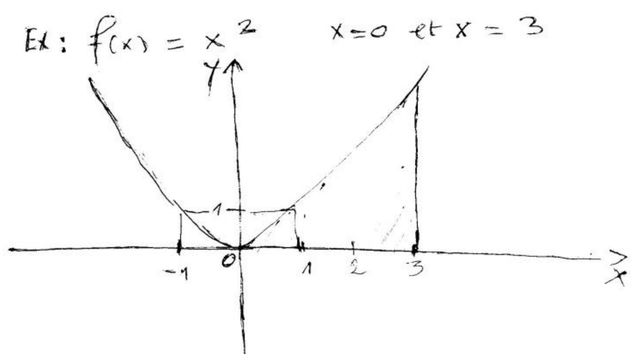
92. FH: Ok. And if you were oriented in education as a teacher for example and your pupil asks you to explain him or her, what integral is, what can you tell him?

93. Edmond: After having acquired knowledge about integral, I can explain the integral as follows.

94. FH: Explain then to your pupil so that he understands what integral is.

95. Edmond: Actually to explain the integral, the integral is in fact, it is a function, for example given a function for example a function f of x which is equal to x square (Edmond writes the function $f(x) = x^2$).

96. Edmond: Then, I can sketch this function from -1 and 1 (see the following diagram).Then suppose that we are asked to calculate the integral of this function delimited by the axes abscissa x=0 and x=3. Then I delimit my function here, three [writing in the diagram]

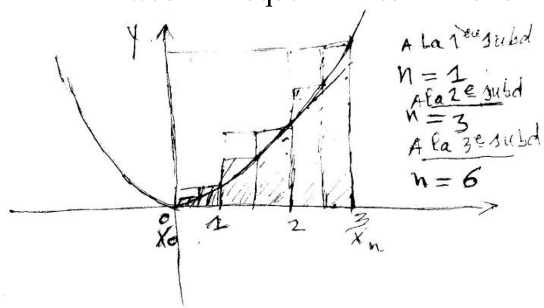


and we are asked to calculate the integral of this function from 0 to 3. 0 is the lower limit and 3 is the upper limit.

97. Edmond: Then in explaining what the integral is, I can tell the pupils that the integral is the area of this part of the curve delimited by these two limits. This is actually the meaning of the integral. Then in explaining how to find this integral, this area,

98. Edmond: first of all, I will do the subdivisions, I show to them how one does subdivisions, I acquired my knowledge on three subdivisions; I saw how one can do subdivisions. There are subdivisions, by starting on area which is above the curve, but there are also other subdivisions starting by the area under the curve and subdividing in increasing until all the area under the curve has been covered. There is also another subdivision of the middle. I can show that. Suppose that I have my function from zero to three (see the diagram below). I can say firstly the first subdivision starting by the rectangles above so that the area under the curve is covered. For the first subdivision, I have only one rectangle. And for the second subdivision we have here 1 and 2, and here for the second subdivision I can subdivide in this way

here, you see that I have 3 rectangles, and here you see the number of rectangles, as one goes along, the number of rectangles increases, this number of rectangles corresponds to n, thus n being the number of rectangles as one goes along. At the first subdivision n was equal to one ($n = 1$),



at the second subdivision, I saw that $n = 3$, at the third subdivision I will have $n = 6$;

99. Edmond: you see that as long as I subdivide, the number of rectangles increases and the area tends to the area under the curve. Then this shows me that, if for example we subdivide until we obtain for example 27 rectangles; this will give us the impression that n is 27. Then in explaining I will tell to them that as long as one subdivides, as long as one obtains many rectangles actually, one tends to this area here, the area under the curve; then one will obtain this area under the curve when? When one will have the rectangles that one can not enumerate that is many, many, and many rectangles. Thus if we have many rectangles then we will have n which is largely high, then one will have this area if one has an infinite number of rectangles; that is n tends to the infinity ($n \to \infty$). Thus one will have this area under the curve.

100. Edmond: Then I say that to calculate this area exactly, one will calculate the area under the curve, the total area or the exact area is equal to the limit of the sum of f(x_i) delta x_i going from 1 to n when n tends to infinity [Edmond writes $Total\ area\ (exact) = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$]. Then you see that there are some concepts that one has to explain here in our formula. There is f(x_i), what is f(x_i), you see that each abscissa has its image on the ordinate axis and this image corresponds to the height. Actually f(x_i) is the height. f(x_i) is the height that ones obtain as long as one subdivides, whereas delta x (Δx) is the base, the lowest that will be reached to have the exact area. Then how to calculate this x_i? This x_i will be calculated as follow: x_i will be the lower limit x_0 plus i times Δx ($x_i = x_0 + i\Delta x$). I do not know if I can also explain this?

101. FH: What is Δx?

102. Edmond: Delta x being the upper limit that I can designate as x_n here we have x_0 the lower limit, x_n minus x_0 divided by the number of rectangles that we will obtain which is symbolised by n as we said (while writing $\Delta x = \frac{x_n - x_0}{n}$). Here we have (1) and (2), in replacing (2) in (1), we have $x_i = x_0 + i(\frac{x_n - x_0}{n})$. Then if we took as an example the function $f(x) = x^2$ from the lower limit 0 and the upper limit 3, then $x_i = \frac{3i}{n}$.

103. FH: When will you talk about the integral then?

104. Edmond: After having calculated this area. Because I said that the integral corresponds to the area under this curve delimited by the limits. The exact area is equal to the limit of the sum i from 1 till n f of three over n i times dx which is three over n when n tends to infinity (Edmond wrote $Exact\ area = \lim_{n\to\infty}\sum_{i=1}^{n} f(\frac{3}{n}i).\frac{3}{n}$). Because $f(x)$ is equal to x^2, $f(\frac{3}{n}i)$ I will replace and I find $Exact\ area = \lim_{n\to\infty}\sum_{i=1}^{n}(\frac{3}{n}i)^2.\frac{3}{n}$. [Calculations continue up to the end]. The exact area is equal to twenty seven over three. It will depend on the unit used ($Exact\ area = \frac{27}{3}u^2$). Then this area that I obtain corresponds to the integral of our function x^2 from 0 to 3 $Exact\ area = \frac{27}{3}u^2 = \int_0^3 x^2 dx$.

In lines 87, 89, 91, Edmond was referring to a practice that often occurs in Rwanda. Teachers are, at some occasions, recruited to occupy some posts in local administration as Mayors of districts or Executive Secretaries of Sectors or Cells. Because these posts are relatively better paying than posts in teaching, student teachers envision leaving the teaching profession to join these posts.

Also this mobility of teachers can be explained by a lack of other trained personnel to occupy the posts that Edmond mentioned in his answers. In fact, during the 1994 genocide that occurred in Rwanda, most of qualified personnel has been killed or crossed the country to exile. Despite the efforts made by the Rwandan Government, there is still a gap to be filled in order to satisfy the whole socio-politico-economic system. So, the answers of Edmond during the warm-up of the third interview can be located also in this national context.

All the ethnic groups are involved in the system without any discrimination. As I said I chapter four when I was describing the student teachers as participants in this research, there is no need to identify them according to their ethnic groups because these are not part of recruitment criteria.

To fight against this mobility of teachers, the authorities in charge of Teachers Development and Management System continue to elaborate policies that improve the education of teachers and to retain them in the teaching profession. They propose new curricula that are more profession-oriented and set up various ways to provide incentives to teachers. Some of these are the creation of Teacher's Cooperatives and annual increase of 3% of their salaries.

The rest of this extract from the third round of interviews includes more references to the underlying concepts of the concept of the definite integral. In fact, in line 96, Edmond used a graphical/visual representation by sketching the graph of the function he gave as an example in line 95. He said: "I can sketch this function from -1 to 1". Later on he said: the integral is the area of this part of the curve delimited by these two limits". Because of these statements, I put the phrase 'area under a curve' followed by the tick symbol (✓) in the cell reserved to the illustrative diagram (Figure 28) to indicate that Edmond evoked an appropriate context for developing the concept of the definite integral, and I considered his next reasoning in the graphical/visual representation.

In line 98, Edmond evoked the layer of partition in graphical/visual representation. The process aspect is manifested by the evocation of the word 'do the subdivisions,' 'subdividing,' 'I subdivide' throughout the text. In addition, those words are used coherently with its meaning regarding the definite integral. In the same line 98, Edmond evoked the word 'rectangles' as results of subdividing operation. Therefore I put a shaded circle (●) in the cell intersecting the layer of partition and the column of graphical representation.

Moreover, in line 99, Edmond was invoking the layer of limit in the graphical/visual representation. The process of limiting is expressed when he says “as long as I subdivide, the number of rectangles increases and the area tends to the area under the curve”. The result of the process of limiting is evoked in the last part of the following Edmond’s statement: “as long as one obtains many rectangles actually, one tends to this area here, the area under the curve”. The phrase ‘area under the curve’ stands for the ‘exact area’ evoked at the beginning of line 100 when Edmond said: “Then I say that to calculate this area exactly, one will calculate the area under the curve, the total area or the exact area”. Consequently, I put a shaded circle (●) in the cell intersecting the row of the layer of limit and the column of the graphical representation.

Finally, in line 100, Edmond switched to the symbolical representation and evoked all the four underlying layers of the concept of the definite integral. In fact, he evoked all the process-object layers of the definite integral when he said: “the total area or the exact area is equal to the limit of the sum of f of s delta x I going from 1 to n when n tends to infinity” and when he wrote the formula $Total\ area\ (exact) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$. Firstly, concerning the layer of partition, the object aspect is evoked when Edmond wrote the symbol Δx and the process aspect symbolised by $\frac{x_n - x_0}{n}$ is evoked in line 101 when Edmond wrote the symbol $\Delta x = \frac{x_n - x_0}{n}$. Therefore I put shaded circle (●) in the cell intersecting the layer of partition and the column of the symbolical representation to indicate that Edmond evoked symbolically both the process and the object aspects of the layer of partition.

Secondly, the layer of product is represented by the symbol $f(x_i)\Delta x$. This symbol $f(x_i)\Delta x$ stands for the process of multiplying. The resulting object usually symbolised by A_i to represent the area is not evoked. So I put a crossed circle ($\otimes$) in the cell intersecting the layer of product and the symbolical representation to indicate that Edmond only evoked the process aspect of the layer of product in the symbolical representation. The circle would have been shaded if Edmond had evoked the equation $A_i = f(x_i)\Delta x$. Thirdly, the layer of sum is evoked by the symbol $\sum_{i=1}^{n} f(x_i)\Delta x$. This symbol $\sum_{i=1}^{n} f(x_i)\Delta x$ stands for the process of adding areas. The resulting object usually symbolised by S_n is not evoked. So, I put a crossed circle ($\otimes$) in the cell intersecting the row of the layer of sum and the symbolical representation to indicate that Edmond symbolically evoked only the process aspect of the layer of sum. The circle would have been shaded if Edmond had evoked the equation $S_n = \sum_{i=1}^{n} f(x_i)\Delta x$. Fourthly, the layer of limit is usually symbolised by the equation $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$. The limiting process is symbolised by $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$ and the resulting object of the limiting process is symbolised by $\int_a^b f(x)dx$. Edmond evoked the process-object layer of limit by writing the following formulas. The first formula evokes the process aspect of the layer of limit and the last formula evokes the object aspect of the layer of limit. The second formula provides the connection between the other two formulas. $Exact\ area = \lim_{n\to\infty}\sum_{i=1}^{n}(\frac{3}{n}i)^2\frac{3}{n}$, $Exact\ area = \frac{27}{3}u^2$, and $\frac{27}{3}u^2 = \int_0^3 x^2dx$. Therefore I put a shaded circle (●) in the cell intersecting the row of layer of limit and the column of the symbolical representation.

Process – Object Layers	Context: Area under a curve ✓			
	Representations			
	Verbal (oral words)	Graphical/Visual (geometrical figures)	Numerical (numerical application)	Symbolical (generalization)
Partition		●		●
Product				⊗
Sum				⊗
Limit		●		●

Figure 28. Edmond's concept images evoked during the third interview

During the third round of interviews Edmond's concept image included more references to the underlying concepts of the concept of the definite integral. As it is shown in the illustrative diagram, Edmond referred to all the four sub layers in the symbolical representation and to two of the four sub layers in graphical representation. And he evoked the appropriate context of area under a curve, but no other contexts.

In summing up, Edmond's concept image evolved during the three rounds of the interviews from a pseudo-object of the definite integral evoked during the first round to involve an appropriate context and most of the underlying concepts in the symbolical representation and two of them in graphical/visual representation during the third round. The symbolical representation seems to be his preferable representation.

However, Edmond did not evoke all the process-object layers in all representations. In fact, he did not evoke the process-object layers in numerical representation. Neither did he evoke all the aspects of all the process-object layers evoked in symbolical or in graphical/visual representation; especially the object aspects of the layer of product and of the layer of sum in symbolical representation. He did not evoke those aspects in verbal representation.

At the end of the course, his concept image is fairly strong. However, his concept images may have been better if the tasks used in the classroom had involved more constraints leading the student teachers to the process and object aspects of sequences in symbolical representation.

5.5.5 Other student teachers' concept images of integrals during the third round of the interviews

All the student teachers who attended the course participated in the third round of the interviews. As in the cases of Bernard, Cassim, David, and Edmond, I analysed what they said during this third round of the interviews. The following diagram summarises the concept images that they exhibited (Figure 29). This synoptic diagram shows that almost all the student teachers evoked the context of area under a curve and that they evoked almost all the underlying concepts of the definite integral. All the student teachers except Kevin evoked almost all the underlying concept in the symbolical representation and they evoked some of them in the graphical/visual representation.

The synoptic diagram shows that the Ferdinand's evoked concept image differs from the other student teachers' evoked concept images as he evoked all the underlying concepts of the definite integral in the numerical representation.

Student teachers	B	C	D	E	F	G	H	J	K	M	N
Area under a curve	✓	✓	✓	✓	✓	✓	✓	✓	🌳	✓	✓
Verbal representation											
Partition											
Product											
Sum											
Limit											
Graphical representation											
Partition	●	●	●	●	●	●	●	●	⊗	●	●
Product	●	●	○		●		●				
Sum	⊗ 🌳		⊗		●		⊗		⊗		⊗
Limit		⊗	●	●	●	○		●			●
Numerical representation											
Partition					●						
Product					●						
Sum					●						
Limit					⊗						
Symbolical representation											
Partition	○	●	●	●	●	●	●	●		●	○
Product	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗		⊗	⊗
Sum	⊗	⊗	🌳	⊗	⊗	⊗	⊗	⊗		⊗	⊗
Limit	●	●	○ 🌳	●	⊗	●	●	●		●	⊗
	B	C	D	E	F	G	H	J	K	M	N

Figure 29. Synoptic diagram of student teachers' concept images evoked during the third round of the interviews

In comparison with the synoptic diagrams of the previous sections, the diagram illustrates the evolution of student teachers' concept images over the period of time when the interviews were administrated and also over the period of teaching. In this respect, the student teachers' evoked concept images moved from nothing or from the pseudoobject of the concept of the definite integral in symbolical representation to include most of the underlying concepts of the definite integral in the graphical and the symbolical representations. However, two outlying cases can be identified. The first case concerns Kevin who evoked a weak concept image at the end of the teaching and the second case concerns Ferdinand who did evoke some of the underlying concepts of the definite integral in numerical representation while other student teachers exhibited their concept images only in graphical and in symbolical representations. In the perspective of constructivism, the existence of these cases illustrate that students will construct different concept images even after participating in the same classroom situations over an extended period of time.

With regard to the underlying concepts of the definite integral, the table below (Table 3), produced from the synoptic diagram above, shows the numbers of the student teachers who evoked or not, in symbolical representation, the underlying concepts of the definite integral during the third round of the interviews. As shown in the table, most of the student teachers correctly evoked the underlying concepts of the definite integral in the symbolical representation. Eight student teachers correctly evoked both aspects of the layer of the partition. Ten student teachers evoked only the process aspect of the layer of the product. Nine student teachers evoked the process aspect of the layer of the sum. And finally, one student teacher (David) incorrectly evoked the process aspect of the layer of the limit and one student teacher (Kevin) did not evoke any of the layers in the symbolical representation.

Process-object layers	Correctly evoked (number)	Incorrectly evoked (number)	Not evoked (number)
Partition $\Delta x = \frac{b-a}{n}$	● (8) ○ (2)	0	1
Product $A_i = f(x_i)\Delta x$	⊗ (10)	0	1
Sum $S_i = \sum_{i=1}^{n} f(x_i)\Delta x$	⊗ (9)	1	1
Limit $L = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	⊗ (9)	1	1
Definite integral (pseudo-object) $\int_a^b f(x)dx$	○ (1)	0	0
Definite integral $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	● (7) ⊗ (2) ○ (1)	0	1

Table 3. Student teachers' concept images of the definite integral evoked during the third interview

5.6. Evolution of the student teachers’ concept images of the concept of the definite integral

The following table (Table 4) summarises the findings of the analyses of the student teachers’ concept images and shows the evolution of these student teachers’ concept images during the six months period of teaching. From this table, it is noticeable that the student teachers’ concept images evolved at each round of the interviews. The figures in brackets in the columns of interviews 1, Lesson 3, and Interviews 3 represent the numbers of the student teachers who correctly evoked, in the symbolical representation, the process-object layers during the first round of the interviews, the third lesson, and the third round of the interviews. The figures in brackets in column interviews 2 represent the number of student teachers who evoked the layer in the graphical or in symbolical representations as indicated in the table.

During the first round of the interviews, none (0) of the student teachers correctly evoked the pseudoobject of the definite integral in symbolical representation. During the second round of the interviews, only one student teacher correctly evoked the pseudoobject of the definite integral in symbolical representation, other student teachers were moving toward the underlying concepts which they incorrectly evoked in graphical representation. After the third lesson only one student teacher correctly evoked in symbolical representation some of the underlying concepts of the definite integral.

Process-object layers	Interviews 1	Interviews 2	Lesson 3	Interviews 3
Partition $\Delta x = \frac{b-a}{n}$		 **(4) (graphical)**	○ **(1)**	● **(8)** ○ **(2)** **Nothing evoked (1)**
Product $A_i = f(x_i)\Delta x$			⊗ **(1)**	⊗ **(10)** **Nothing evoked (1)**
Sum $S_n = \sum_{i=1}^{n} f(x_i)\Delta x$		 **(4) (graphical)**	● **(1)** ⊗ **(1)**	⊗ **(9)** **Incorrectly evoked or nothing (2)**
Limit $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$	○ **(0)**	○ **(1) (symbolical)**  **(2) (graphical)**		● **(7)** ⊗ **(2)** ○ **(1)** **Nothing evoked (1)**

Table 4. Evolution of the student teachers' concept images in the period of six months surrounding the teaching

During the third round of the interviews, eight student teachers correctly evoked both the process and the resulting object of the layer of the partition layer while two student teachers only correctly evoked the pseudoobject of this layer and one student teacher did not evoke this layer. For the intermediate layers of product and of sum only the process aspect was correctly evoked by most of the student teachers (ten and nine respectively). Concerning the layer of the limit, seven student teachers correctly evoked both the process and the resulting aspects, two student teachers evoked the process aspect only, one student teacher evoked the pseudoobject, and one student teacher did not evoke anything of the layer.

The student teachers' concept images of the definite integral kept changing from the pseudoobject incorrectly evoked during the first round of the interviews towards the inclusion of all the underlying concepts during the third round of the interviews held at the end of the course. At this time of the third round of the interviews, most of the student teachers correctly evoked the formula linking the symbol of the definite integral with the symbol of the limit of sum.

From this situation, I can assert that the teaching-learning process made the student teachers learn new mathematical knowledge. However, not all the aspects of the targeted mathematical knowledge were evoked by the student teachers. Even during the classroom activities these aspects have not been evoked. Therefore the tasks used in the classroom are to be revisited to also include all these aspects.

5.7. Student teachers' concept images of the concept of the indefinite integral

An analysis similar to the one that I did for the case of the concept of the definite integral, consisting in identifying processes and their resulting objects, shows that the student teachers' concept images of the concept of the indefinite integral also evolved during the teaching and the learning that I conducted with them.

Using extracts from Bernard's, Cassim's, and David's interviews, I exemplify this evolution of the student teachers' concept images. The selection and the order of the student teachers is the same as the one that I evoked in the introduction of this chapter; that is, the student teachers were ordered following their performance during the first round of the interviews. For each of these three student teachers, I analyse successively the three interviews that I held with him.

During the first round of the interviews, Bernard, when he was answering to my question about his understanding of the indefinite integral ("And what do you understand by the indefinite integral?"), said:

> An integral is indefinite in the contrary case when there are no limits [of integration]; that is, the two limits [of integration] are not specified.

Bernard gave this answer after he had answered the question related to the definite integral. That is why he was referring to the "contrary case" of no limits [of integration] or limits [of integration] not specified. During this first interview, the answers of Bernard were in verbal representation and incomplete as an explanation of an indefinite integral. None of the underlying concepts of the indefinite integral was evoked.

During the second round of the interviews, Bernard said:

> Concerning the indefinite integral one can take the example of the case where we have a single value. One considers the symbol of the primitive [antiderivative] of f of x dx without limits [of integration]. [Bernard wrote the symbol $\int f(x)dx$]. Then one can do what is called "primitivation" [antidifferentiation], which is the inverse of the derivative function and one obtains always a function, but for the definite integral one obtains the value.

In this extract, Bernard evoked the pseudo-object of the indefinite integral in the symbolical representation when he wrote the symbol $\int f(x)dx$. This aspect of pseudo-object is also confirmed by the fact that Bernard evoked the process of antidifferentiation which does not involve any of the underlying concepts of the indefinite integral.

During the third round of the interviews, Bernard said:

> ……As a definition of the indefinite integral, it is always the area under curve; thus the area which is delimited by the curve and the x-axis but this time the limits [of integration] are variables. This gives us the indefinite integral; for the definite integral, one has limits [of integration] which are fixed.

When Bernard was answering to the question of "How do you write that?" he said:

> To note the two types of integrals, for example for the indefinite integral, we have the summation, the elongated S, of the function like this [Bernard writes $\int f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$].

The first extract from the third interview shows that Bernard's concept images include the context of area under the curve. In the second extract, Bernard evoked the two aspects of the underlying layer of limit. The process aspect is given by the second side of the equality sign (=) $\int f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$ while the object aspect is given by its first side.

However, Bernard did not evoke any of the other underlying layers of the indefinite integral considered as a definite integral with the upper limit variable. Those other underlying layers are the partition, the product, and the sum.

Nevertheless, an examination of the three interviews show that Bernard' concept images of the indefinite integral had evolved from words to the pseudo-object, $\int f(x)dx$, and then to process-object of the layer of the limit in the symbolical representation, $\int f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$.

Cassim, for his part, during the first round of the interviews, said:

> It [the indefinite integral] is an integral whose limits [of integration] are not well defined; that is, for a function f of x, the integral of f of x dx [Cassim wrote $\int f(x)dx$], the answer has to be found in values not fixed; that is, in values x, y, z whose values are not well defined.

In this extract from the first interview, Cassim evoked the pseudo-object of an indefinite integral in symbolical representation, $\int f(x)dx$, which he explained imprecisely in terms of "values x, y, z whose values are not well defined".

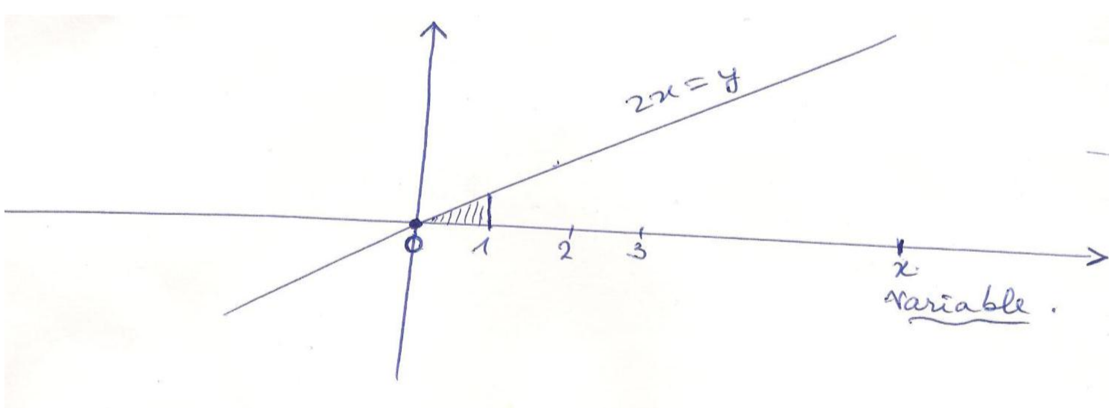
During the second round of the interviews, Cassim said:

> Concerning the indefinite integral, it is all this area (showing a diagram drawn in answering a question about the definite integral, see section 5.3.2, page 121 CRef). The area contained between the axes of x and y and delimited by curve of the function.

In this extract from the second interview, Cassim evoked only the context of "area delimited by [under] a curve".

During the third round of the interviews, Cassim said:

> Concerning the indefinite integral, I will tell him to proceed like this: [Cassim drew the following diagram that he used in his explanation].



Firstly, as we will have determined the integrals using the definition which says that the integral of 2x dx will be equal to the sum of x …it is the limit as n tends to the infinity. [Cassim wrote $\int 2xdx = \lim_{n\to\infty}\sum(2i\Delta x)\frac{\Delta x}{n}$ and continued the process after having introduced the formula $\sum i = \frac{n^2}{2}+\frac{n}{2}$ and at the end he wrote $\lim_{n\to\infty} x^2 = x^2$]

Cassim continued:

But we return back to explain what is indefinite integral. If we have zero and the same function $y = 2x$ and the delimitation 1, 2, 3, and by noting x a value which is variable, x variable, we will have this diagram. If we use the formula seen previously, one can have, if the lower limit is 0 and the x variable and one try to find the area; one will have that, given that $\Delta x = \frac{x-0}{n}$, the area from here to there (showing the diagram) will be equal, as we have seen, to the limit as n tends to the infinity, we found that this limit is equal to x^2 (Cassim wrote $\lim_{n\to\infty}\sum_{i=1}^{n} i(\frac{x}{n}).\frac{x}{n} = x^2$). When one increases $x_0 = 0$ for example to one $x_0 = 1$, and x_n remaining as x, this will give us, that is, to determine the area from one to x it suffices to use the area seen previously as the limit as n tends to infinity: $\lim_{n\to\infty}\sum_{i=1}^{n} i(\frac{x}{n})(\frac{x}{n})$ - $\lim_{n\to\infty}\sum_{i=1}^{n} i(\frac{1}{n})(\frac{1}{n})$. [He continued the procedure up to the end and wrote] the area from 1 to x or $\int_1^x 2xdx = \int_0^x 2xdx - \int_0^1 2xdx = x^2 - 1$. When x_0 is increased to two he wrote $\int_2^x 2xdx = \int_0^x 2xdx - \int_0^2 2xdx$.

Cassim concluded:

In conclusion, to determine the integral from zero to a variable value which x, the indefinite integral is the integral of a function when the limits are not fixed, i.e. when the limits are variable as we have explained. For example in our case we had $\int_0^x 2xdx = x^2$. When these two limits are not fixed, one has to subtract something which is considered as a constant, hence one can write as follow: $\int_0^x 2xdx = x^2 - c$ = $x^2 + (-c)$ = $x^2 + |c|$. This is the indefinite integral.

In this excerpt from the third interview, Cassim evoked some of the underlying concepts of an indefinite integral. First, he evoked the graphical representation when he drew the diagram given above. He, then, switched to the symbolical representation to evoke both the process and the object aspects of the underlying layer of the limit when he wrote the formula $\int 2xdx = \lim_{n\to\infty}\sum(2i\Delta x)\frac{\Delta x}{n}$. The object aspect was given in the left side of the equality symbol while the process aspect was given in the right side of the equality. However, the process aspect was not correctly evoked (the limits within which the partition was to be made are not clearly evoked and indices of the symbol $\sum$ are missing).

In the middle part of the extract, Cassim evoked both the process and the object of the layer of the partition in symbolical representation when he wrote the symbol $\Delta x = \frac{x-0}{n}$. The process aspect is given by the right side of the equality sign (=) and the object aspect is given by its left side. Then, cassim continued to perform the procedural operations related to the limit layer.

In his conclusion, Cassim evoked the pseudo-object of the indefinite integral. In fact, he recalled the formula usually used to determine the antiderivative $\int_0^x 2xdx = x^2 - c$

In sum, Cassim's evoked concept images of the indefinite integral, conceived as a definite integral with upper limit variable, evolved from the pseudoobject to include some of the underlying concepts of the indefinite integral, namely the layers of partition and limit.

As for the previous student teachers, David's concept images of the indefinite integral evolved, though slightly. During the first interview, David said:

> The indefinite integral is the integral that one can not find the value; that is, one integrates a function and one finds a function which has been differentiated to find *f*

> of *x,* but one has unspecified values; that is, the integral does not use limits [of integration]; that is, the integral does not have limits [of integration]

In this extract, David verbally referred to the pseudoobject of an indefinite integral seen as antiderivative ("one finds a function which has been differentiated to find f of x, but one has unspecified values").

During the second interview, David said:

> The indefinite integral is the integral where for example, there are constants and parts of the function; that is, there is a part that contains the variables and another part that contains constants. The sum, when we say the sum, here, to find the indefinite integral, we do the sum of all the areas of all these rectangles, after the indefinite integral is when the constant, when we have the sum of functions which compose our big function F(x). Finally we will have a constant function, an independent term, and I will show him that there is no possibility to determine that constant because we add this constant because the constant which has been differentiated also has found zero as answer and that zero can not be visualized. So, we can say that this integral is indefinite because we can not calculate the area without sometimes visualizing on the curve of the function.

In this extract, David verbally evoked the pseudoobject of the indefinite integral when he said in the second line "that is, there is a part that contains the variables and another part that contains constants". He evoked layer of sum in the graphical representation when he said "The sum, when we say the sum, here, to find the indefinite integral, we do the sum of all the areas of all these rectangles". The process aspect is evoked when he said "we do the sum of all the areas of all these rectangles". However the object aspect is not articulated. This graphical representation is then abandoned and David think about the sum of functions as he said "after the indefinite integral is when the constant, when we have the sum of functions which compose our big function F(x)". So, the graphical representation was not yet well installed. David also alluded to the graphical representation and to the area under a curve at the end of the extract when he said "we can say that this integral is indefinite because we can not calculate the area without sometimes visualizing on the curve of the function".

When David was answering the question "You said that it is a function plus a constant, how do you obtain that function, the indefinite integral is a function plus a constant, how do you obtain that function?" he said:

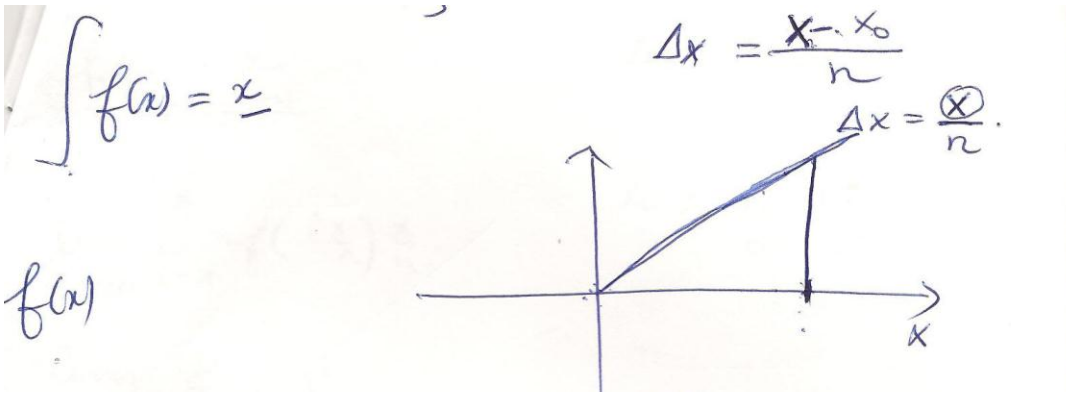
> In fact, what I said is when you have variables and the independent term that has a variable, say a variable in x exponent zero, in this case we will have one which does not change anything in the multiplication, in any case when calculating, we do calculations; if it is the indefinite integral we will not have the limits of the integral. If we give an example, if we have $x^2 + x + 5$, the indefinite integral is the sum of the integrals of each term of our sum here. Therefore, we will have integral $\int x^2 dx$ plus integral $\int xdx$ plus integral $\int 5dx$ and we know that the integral of a constant can be put out and we will have integral of identity function and by using the formula $\int x^n dx = \frac{x^{n+1}}{n+1} + C$. This constant does not appear in the function to integrate. But if we add the constant which is unknown and it is an assumption because the constant is there but disappear because when differentiating we have found that the derivative of the constant is zero.

In this second part of the extract from the second interview, David evoked in symbolical representation the pseudoobject of the indefinite integral when he wrote the symbols of indefinite integral such as $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

During the third interview, David, answering the question "Now if he asks you what the indefinite integral is?" said:

> In this case, I will tell him that it is the integral where the upper limit is specified. That is, the upper limit is not specified and one has the lower limit but for the area, one has not specified the delimitation of that area. Thus one subdivides, and finds area until to infinity, where the curve…
> Let say that he asks me the indefinite integral of the function say that we have [David wrote $f(x) = x$ and then he added the integral symbol to make the symbol $\int f(x) = x$, see aside the following diagram], it is indefinite, that is why we have not put limits [of integration] here; the limits are not put here because the integral is indefinite. The function is there but the integral is not defined. Because even the variation, what we call the variation of x varies from x only, thus…say we have our function here (David drew the following diagram); that is, here, it is x. that is, it is all values of x which is here but which is not specified; that is, it is all x, all values of x, minus the lower limit which can be specified over n subdivision, because always we subdivide (David

wrote $\Delta x = \frac{x - x_0}{n}$). If we have then from zero, then the variation of x will be x over n ($\Delta x = \frac{x}{n}$). This x will help me to explain the indefinite integral, that is, the upper limit is not specified; it is for this reason instead of writing x_n here, we precise that there is any value defined from x which is considered as the upper limit.



If it is indefinite, that is, one has not specified the lower limit, one asked to find the area under the curve without specifying the interval. In this case we will have… if pupil…, if one teaches the integrals, the essential is to show firstly the definite integral and then after the indefinite integral because the explanation comes from the definite integral to explain the indefinite. When we studied the integrals we started by the indefinite integrals and finally we did the definite. But we will have to define the thing and continue with the definition of it.

In this extract, David evoked the layer of the partition in the symbolical representation when he wrote the symbol $\Delta x = \frac{x_n - x_0}{n}$. The process aspect of the layer is represented by the right side of the equality while the left side represents the object aspect of the layer.

David also tried to use the graphical representation as he mentioned "but for the area, one has not specified the delimitation of that area. Thus one subdivides, and finds area until to infinity". In this representation, he alluded to the layer of partition when he said "one subdivides" and to the layer of limit when he said "and finds area until to infinity".

Compared to the previous interviews, David's concept images of the indefinite integral slightly evolved from the pseudoobject to include one of the underlying layers, namely the layer of partition as noticed above. Also David tried to include the layers of partition and limit in graphical representation.

This way of analysing was applied to extracts from all the student teachers' interviews about the concept of indefinite integral. In summary, I found that, during the first round of the interviews that I held with the student teachers before the teaching, only two student teachers evoked the pseudoobject of the concept of the indefinite integral, $\int f(x)dx$, and the other nine did not evoke any symbol related to the indefinite integral when they were answering the question related to the concept of the indefinite integral. During the second round of the interviews held also before the teaching, four student teachers evoked the pseudoobject of the concept of the indefinite integral, $\int f(x)dx$, and the other seven did not evoke any symbol related to the indefinite integral when they were answering to the question related to the indefinite integral. During the third round of interviews that I held with the student teachers after the course, two student teachers evoked the pseudoobject of the concept of the indefinite integral $\int f(x)dx$, five student teachers evoked the process aspect $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$, and four student teachers evoked both the underlying process and the resulting object $\int f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$. Therefore the course that I conducted with the student teachers contributed to the evolution of their concept images of the concept of the indefinite integral.

Like in the case of the definite integral, the student teachers' concept images of the indefinite integral also evolved during the teaching-learning process. However, not all the aspects have been evoked by the student teachers during interviews. Some of these aspects, as in the case of the definite integral, have not been taught during the classroom activities. Therefore a revisiting of the tasks used during the teaching-learning process is needed so that they include the steps leading to the missing aspects of the concepts under examination. Also, I experienced some difficulties during the use of the analytical framework, especially with regard to the concept of indefinite integral for which answers of the student teachers during interviews were not enough elaborated. Thus, for example, the use of icons was hard to apply. I will say more about them in the following section, ‘conclusion and discussion’.

5.8. Conclusion and discussion

In this chapter, I presented the findings of the analysis of the student teachers’ concept images of integrals in various representations. These concept images have been evoked during three rounds of interviews and during the communication in the classroom at the end of the third lesson. The first round of the interviews was conducted five months before the teaching of the course and the second round of the interviews was conducted ten days before the teaching of the part of the course related to integration. The third round of the interviews was conducted after the course. The findings of the analysis of the student teachers’ concept images show that these concept images evolved during the period of the research from evoking the pseudoobject (Sfard, 1991; Sfard & Linchevski, 1994) to evoking almost all the underlying concepts of the definite integral as shown in Table 4. The same trend of evolution is also observable for the concept of the indefinite integral for which 4 of the 11 student teachers evoked both the process aspect and the resulting object in symbolical representation after the course whereas only 2 of the 11 student teachers had succeeded to only evoke the pseudoobject during the first round of the interviews held before the course. Finally, as it can be noticed from the first and the second round of the interviews, the student teachers’ concept images developed before the teaching of integrals started. This might have been caused by the fact that at the beginning of the research, I told them that I was doing research on their concept images and that there would be other interviews. So, during the period between the first and the second interviews, some of the student teachers might have revised what they had learnt in their secondary schools and others might have consulted textbooks to improve their answers during the subsequent interviews. This of course produces its own validity issues; it will never be possible to say to what extent the students’ concept images would have developed in a similar respect without the presence of interviews.

In the same vein, it is worth continuing this discussion about the findings, the analytical framework, and the icons used to illustrate the student teachers' concept images. During the analysis, some difficulties related to the use of icons arose when the process and the resulting object were independently evoked, that is, when they were not linked. The coding of this situation became more complicated when one aspect was incorrectly evoked as in the case of David in the third interview; in this case, the pseudoobject was correctly evoked and later on the process aspect also was evoked but incorrectly. To illustrate this situation, I used two icons in the same cell intersecting the layer of limit and the column of symbolical representation. It seemed difficult to find a superseding icon to represent this case.

Other difficulty was around the use of the tree-like icon (). Originally, it was introduced because it was found that only noting what correct concept images were evoked gave an incomplete illustration of the students' knowledge. Throughout the analysis, I used this tree-like icon to illustrate an incorrectly evoked aspect; I also used it to represent any imperfectly evoked aspect; and I used this icon to illustrate an aspect which was in the process of developing. The appreciation of these three cases was difficult to maintain all along the analysis and the use of the tree-like icon caused a high risk of inconsistence. It would have been better to use different icons for each case.

The coding in verbal representation, especially during the third round of interviews, has not been used because the words were accompanied by other representations. The words were used to state the formula or the geometrical figures. In this case, I coded in the symbolical or the graphical representations instead of coding in the verbal representation. This is one reason that in the verbal representation there is no illustrative icon.

The coding of the processes of product and of sum in the symbolical representation during the third round of the interviews can raise some doubts. The student teachers did not explain deeply these processes and they did not evoke their resulting objects. I illustrate them on the basis of their appearance in the formula evoked by the student teachers. During the teaching, these layers of product and of sum had not been largely elaborated. Therefore, the tasks that the student teachers dealt with need to be revised in order to also instigate and reinforce the learning of these intermediate layers.

About the indefinite integral defined as a definite integral whose upper limit is variable, the aspect of the variability of the upper limit is not apparent in the second part of formula $\int_a^x f(t)dt = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$. This situation may make debatable the conclusion that the student teacher correctly evoked both the process and the object aspects of the indefinite integral in the symbolical representation if no other explanation is provided.

Left aside these difficulties of coding which can be improved (though I recognize that any attempt to create overviews of students' concept images are bound to have limitations), the analytical framework works well in analysing the teaching-learning outcomes especially when the pseudoobjects or the processes and their resulting objects are correctly evoked by the student teachers. Beyond being an analytical tool, the framework can also serve to plan teaching materials and to evaluate textbooks which contributed to the development of the student teachers' concept images of integrals jointly with the interactions between the teacher and the student teachers.

In the next chapter, I analyse what happened in the classroom that made the student teachers develop their concept images in a way that they included the layers of the underlying concepts of the integrals.

CHAPTER 6: COMMUNICATION IN THE CLASSROOM

6.1. Introduction

In this chapter, I present data and findings of analysis of the lessons that constituted my teaching about the definite and the indefinite integrals and about the fundamental theorem of calculus. The chapter is made up of six sections. After the introduction, I present the didactical episodes that occurred during the first lesson and their durations on the basis of the video recordings. I used the term episode in the sense described by Watson and Potter (quoted in Ervin-Tripp, 1972) as a unit of analysis which can terminate whenever there is a change in the major participants, the role system of participants, the focus of attention, and the relationship toward the focus of attention (see p. 59). I resorted to each of these characteristics to determine episodes that emerged during the lessons. In the third section, I present the didactical episodes that occurred during the second lesson and their durations. In the fourth section, I summarise the episodes that occurred during all the lessons (Table 7). I end the fourth section by discussing which didactical situations are likely to further the student teachers' understanding of the concepts of the definite and the indefinite integrals and the fundamental theorem of calculus. In section five, I narrate a sequence of cognitive processes that led Edmond to develop his concept images of the concept of the definite integral and to express his understanding during the fifth lesson. In this section, I also analyse the role played by the functions of language used in the episode in which learning took place. To end the fifth section, I give my point of view about the use of functions of language and their effects in relation to the style of teaching that I adopted during my lessons. The sixth section concludes the chapter. In the following section, I present the episodes that occurred during the first lesson.

6.2. Didactical episodes during lesson one

This section deals with the first lesson which was planned to last two hours. The topic of the lesson was to identify previous knowledge of the student teachers and questions that student teachers had about integrals and to jointly define the objective of the course –

aiming at being a devolution (p. 55), which will be discussed in later sections. Their previous knowledge and questions were to be used in designing tasks which would help students to construct new knowledge. The tasks were required to satisfy Douady's (1991) conditions of problems to be sources of learning as presented in section 3.3.

6.2.1 Identification of previous experiences of the student teachers

To begin the first lesson of the course, I started by asking student teachers to recall what they had learned in their secondary education. I addressed the question to the whole class and then I called out Cassim to say what he remembered about integrals. Each student teacher was expected to exhibit his or her prior knowledge about integrals. The exhibited prior knowledge served as the beginning point to harmonise the goals of the participants and to jointly define a shared objective. Some of the student teachers' answers to my question are found in the following extracts of the transcription of lesson one (pp. 1-3, turns 1-32). The communication during this 12 minutes episode was in the interpersonal mode T-S1 (p. 69).

1. Teacher: ((Teacher distributes papers and places tape recorder and says)): Let us start the second part of our course concerning integration. When we consider the program of secondary schools we realise that you had learnt integrals. Can anybody among you tell us what he remembers about integrals? Cassim, what do you remember about the integrals?

In his answer, Cassim evoked the idea of sum (line 2). Unfortunately, I interrupted him because I had in my mind that the student teachers had not learned integrals in terms of the underlying concepts of partition/subdivision, product, sum, and limit during their secondary education. My presumptions were erroneous since some other student teachers also evoked the sum in their responses related to what they remembered at this time of the first lesson. After my interruption, Cassim said that he had seen formulae that are used in integration (line 4). He exemplified some of these formulae (lines 6 and 8).

2. Cassim: The integral is the sum …((teacher interrupts))
3. Teacher: What have you seen in secondary school?

4. Cassim: ((Cassim continues)) We have seen the formulae utilised to integrate.
5. Teacher: Like what for example?
6. Cassim: We have seen for example what I remember, I remember that the integrals, a function for example x to the power of n, this equals to n x to the power of n minus one and this n x to the power of n minus one, it is what we call integral.
7. Teacher: What else have you seen? Do you remember what you have seen in secondary school?
8. Cassim: Another part that I remember is the integration, the integration but given the limits, which means the surface limited by the line of the curve of the function, and the interval that equals to position a for example a b. Then by replacing the surface with x with the upper limit minus the surface by replacing x with the lower limit the result is obtained in units of surface.

The next turn was for David to recall what he learnt in secondary education. He evoked the indefinite integrals and the definite integrals which allowed him to calculate the area of the surface comprised between limits and the graph of the function.

9. Teacher: Can somebody else tell us what he learnt in secondary school?
10. David: Apart from the integrals, we have seen how we can get the indefinite integrals and definite integrals and, with those indefinite integrals we do calculations to integrate a function and also to get the other function. However, to get the definite integrals we have to integrate a function, and by integrating a function we get its indefinite integral, but because that indefinite integral has limits, an upper limit and a lower limit that obliged us to calculate properly the surface comprised between those two limits in relation to the graph of the function.

After David, it was the turn of Edmond to recall what he learnt in secondary education. Edmond also stated that they have seen the indefinite and the definite integrals. He also referred to the antiderivatives as an alternative name for indefinite integrals and he remembered applying formulae to calculate the area of a surface within given limits by replacing in the function the limits.

11. Teacher: So, what did you see in secondary school?
12. Edmond: Also we have seen as he said, there are definite integrals and indefinite integrals. The indefinite integrals commonly identified as primitives [antiderivatives], these integrals allowed us to calculate the function that we had derived to obtain that function

there, and then there we used the formulae. For the function, for the definite integrals they give the integrals and we integrate those integrals, after the integration of those integrals we calculate the surface within the given limits. But also they would give the graph of the function and we delimit that graph with limits, and they ask to determine the marked surface within the limits and then to calculate that surface. We would at first integrate and apply the formula, by replacing in the function the upper limit minus the lower limit.

After Edmond, Ferdinand took his turn. In his speech, Ferdinand evoked the idea of sum since he said that the integral sign means sum. He also evoked the word "delta x" which may recall the partition idea, and finally he evoked the process aspect of the product as he said "that is the sum of delta x multiplied by f of x". Like Cassim, Ferdinand evoked some aspects of the underlying concepts of the definite integral during the first lesson.

13. Teacher: Yes, somebody else, what did you learn in secondary school?
14. Ferdinand: In secondary school we learnt for example if they speak about the expression integral of a derived function for example integral of f of x derived in terms of x, our function has three parts. The sign integral which means sum f of x, which means the variation, that is the images, how the images vary and delta x. Thus, if we take delta x multiplied by f of x, we get the surface. With that sum, that is the sum of delta x multiplied by f of x. This justifies that the integrals justify the space of the surface.

Then Norbert, in his turn, recalled the antiderivative function. However, in the beginning of his sentence it was not clear whether he was referring to what Ferdinand had said concerning the sum, the product, and delta x as he said "apart from that". It appears that Norbert focused on the indefinite integral in his statement.

15. Teacher: What did you learn in secondary school apart from that?
16. Norbert: Apart from that, by integrating a function we get a function which is primitive [antiderivative], that is, the function that we had from the beginning.

Student teachers continued to take their turns. In his turn, Gerard remembered that he had learnt indefinite and definite integrals. He did not give any more details.

Kevin, on his side, remembered having learnt definite integrals as integrals that have limits, and indefinite integrals as integrals that do not have limits.

The same question of recall was asked to Jimmy, Merriam, Bernard and Honorate. Jimmy evoked the indefinite integrals in terms of not having limits and definite integrals in terms of having upper and lower limits. Jimmy also evoked the application of the definite integral in terms of applying a formula to get the area of a surface. Merriam said that she had learned the definition of the integral and its utility but she did not provide any detail. Bernard evoked the contexts in which the integral is used. In addition to the area of a surface, Bernard also evoked solids of revolution, such as cones.

17. Bernard: From the function we have seen how after having calculated the surface delimited by the representative curve of the function and the axis of x as well as y, which is the axis of ordinates. Thus, when we have a surface delimited by the representative curve of the function with the two verticals axes as well as the axis of x, we obtain a surface. Then after having obtained that surface we have seen that if we make that surface turn on the axis of x for example, we obtain a solid of revolution. Among those solids we can obtain the cones for example in the case where we have the triangle. By making our triangle turn on the axis of x, we obtain the cone. Thus, for the construction of solids of revolution we can do it from the integrals.

On her side, Honorate evoked the definite integrals as having limits while the indefinite integrals do not have limits. She also evoked the integral as an antiderivative – interestingly referring to it as “a function of a derivative”, potentially expandable into an operator view on integrals. She evoked the techniques of integration such as integration by parts and integration of power function. She evoked the idea of sum as she said “This sign of integration means the sum”.

18. Honorate: We have seen the definite integrals and indefinite integrals. The definite integrals have limits but the indefinite integrals don’t have limits. Normally the definition of the integral is a primitive [antiderivative], a function of a derivative. We have learnt also the ways of integrating by part, for example the integral of x to the power of n d of x and dx, that is, the differentiation of times equal to n x to the power of n minus one. This sign of integration

> means the sum, the sum to integrate the integrated. Thus, to calculate the surface using a function, on a graph, we have two limits which allow us to find that surface which is here in the middle. With the use of those limits we have the integral of a b, f of x which is equal to a primitive [antiderivative] which can help us to find that surface.

In sum, during this episode which was devoted to the exhibition of prior knowledge, only Cassim, Ferdinand, and Honorate exhibited some ideas related to the underlying concepts of the definite integral. Regretfully, I did not engage their reference to 'sum' further in this lesson; perhaps the devolution would have taken a different turn if I had.

To further determine the students' previous knowledge, I gave them an integration task.

6.2.2 Independent problem solving

At the 12th minute of the lesson, the student teachers were given a function to integrate. I asked them to do it individually.

> 19. Teacher: Ok, we see that each of you have seen integrals. Now this time around, you are going to complete your knowledge by adding the significance of the integral and the meaning of what an integral is. For the meaning let us start with a little example, we give ourselves the function y equals x squared represented by this graph here, we are asked to find the area of the surface delimited by the function y equals square going from one, x equals one until x equals two. Let us do it individually; try to find the area of this surface. ((pause)) It is about finding the surface delimited by the curve between one and two. ((He gives time to students to do the exercise. After some times the teacher started moving from seat to seat to monitor what the student teachers were doing)).
>
> (Transcription of lesson one, p. 3, turn 33)

From this line 19, a debate can be engaged about what would have happened if I had not said that "It is about finding the surface delimited by the curve between one and two".

After this introduction by the teacher, the communication mode was of the S1-C type (cf. p. 69) and the episode lasted eight minutes. After this episode, I invited Jimmy to present his solution to the exercise on the chalkboard.

6.2.3 Jimmy's presentation of his solution

After the independent problem solving of the previous episode, I requested that a student teacher went to do the exercise on the chalkboard. At the 20th minute of the lesson, Jimmy went to the chalkboard to do the integration. He used the antiderivative and arrived at the correct numerical answer.

This episode was in the S1-C mode of communication and lasted four minutes. I then wanted the students to realize that their understanding of the integral lacked conceptual underpinning.

6.2.4 Problematisation of the relationship between area, definite integral, indefinite integral, and antiderivative

At the 24th minute, I started to 'problematise' the relationship between the area, the definite integral, the indefinite integral, and the antiderivative. I asked questions to student teachers and they answered in turns. The discussion was very productive of various ideas. The following extracts from the transcription of lesson one illustrate conceptions evoked by some of the student teachers during the discussions. When I became assured that none of the student teachers could conceptually evoke the relation connecting area, definite integral, indefinite integral, and antiderivative, I decided to stop the discussion and to invite the student teachers to form small groups and to formulate questions they had about integrals (see the end of line 29). This intervention constituted the turning point as it confirmed the question which the student teachers had not succeeded to answer ("But then how does it give the area of the surface?"). It also gave a promise that the student teachers would be helped and supported by the teacher ("This is a point to which we will come back again"). Thus, the turning point was both in terms of constituting a devolution and in terms of affirming the didactical contract between teacher and students.

20. Teacher: What is the function x cubed over three called ((referring to the result of the integration in the previous task))? The function that you got, x cubed over three? Others!
21. Merriam: It is a primitive function
22. Teacher: A primitive function, what then a function has to do with surfaces? Why do you have to get the primitives in order to get the areas?
23. David: Because it means that it is the initial function, there is a function which has been derived in order to get the function to integrate, so we have to come back to the initial f of x to find the surface because there we have in the plan there, we have our curve f of x we must then come back to an initial function.

The discussions continued between the teacher and the student teachers. The following are extracts of the student teachers' speeches. Notice that the utterances do not follow each other immediately. I only reproduced here utterances of some of the student teachers who alluded to some of the underlying layers of the definite integral, namely, the partition, the product, the sum, or the limit, or who evoked the context of area. As said above, I ended the discussions (line 29) and invited the student teachers to think deeply about the relationship between areas and integrals and to evoke questions they might have about integrals.

24. Edmond: Here we have seen that this area, to determine this area we have to do the sum after having subdivided that surface into small triangles, then this sum equals to the limit, to the limit of f of x when x is tending toward zero. We have then a derivable function admitting a limit whereby we must calculate that derived function because the surface is the sum and a sum is obtained after the calculation of the limit.
25. Cassim: To find the surface of a function, we have first of all to integrate in order to get the initial function or the primitive as the name indicates, to be able to arrive at the surface swept by the same primitive function and the axis of x. To me the reason which caused us to find the primitive is because we would not find the surface swept by a derived function, but we must find the surface through the primitive function.
26. Bernard: …Speaking about the definite integral, we understand directly that it is a primitive which has limits. (…) Then if we have the

area, the area which is the entire sum of the values, the values of the limits, which means the sum of values obtained by the limits….

27. Ferdinand: To me, what I see as the relationship between the indefinite integral and the definite integral, if we use for example the mathematical inclusion I see that the definite integral is included in the indefinite integral because the indefinite integral comprises the entire surface delimited by the entire function in relation to the axis of x (…).

28. Honorate: The definite integral is included in the indefinite integral because we have first to calculate the indefinite integral so that we can calculate by using the limits of the definite integral. If, by calculating the indefinite integral we get the primitive, this primitive helps us thanks to limits to get the area.

29. Teacher: Ok. Does anybody have other ideas about this? How are these three elements related? ... Ok. I wanted to reinforce what I said when we have a function we can find the primitive by doing the reverse operation which is also called the integral; but again we must understand what represent the object, the result of this reverse operation. What do we obtain? We obtain a function, as you have said; when we differentiate it we obtain the function from which we had started. But then how does it give the area of the surface? This is a point to which we will return. As there are other interventions (….) you are going to break into groups and in your groups you will write which problem we have to deal with. What is the problem you have to solve in our class? In your groups you are going to formulate the problem that you have concerning integrals. Is it OK?

(Transcription of lesson One, pp. 6-9)

In this episode, the mode of communication was of T-SSW type and the episode lasted thirty eight minutes. After this episode, the student teachers got into small groups and formulated their questions about integrals.

6.2.5 Formulation of the student teachers' questions about integrals

At the 42nd minute, I requested the student teachers to form small groups and to formulate questions they had about the integrals.

30. Teacher: Get into your groups and formulate the question that we have to deal with in this course…in a way that you will be able to …Listen to instructions, in fact what you are going … to

formulate the question and then in your groups; afterwards a representative will go to the chalkboard to present his question and then we are gong to find a common question. And each one of us will be obliged to get in fact the solution of that question. Ok, let us go on now; let us try to formulate the question we are dealing with.

31. Jimmy: … other questions of calculation of integrals?

32. Teacher: Yes, discuss among you and find the question which needs clarification in fact in the course of integration. What question do you have?

(Transcription of lesson one, pp. 9, lines 114-116)

The mode of the remainder of the communication was at the group level in the S1-SS1 type (the teacher did not engage the students during this group work) and the episode lasted eight minutes.

6.2.6 Presentation of the identified problems at the chalkboard by representatives of the groups

At the 50th minute, representatives of the groups presented the questions at the chalkboard. David went to the board to present questions from the first group.

33. David: ((he writes on the board the questions from his group)). Why do we have to get the primitive function in order to arrive at the definite integral ($\int_b^a f(x)dx$ and $F(a)-F(b)=?$)? There is a second one: What is the relationship which exists between the formula of integration and that of derivation? The relationship that exists between the formula of derivation and that of integration? Thus, what we need to know when we say for example integral x to the power of n dx by formula we say that this is equal to x to the power of n plus one over n plus n ($\int x^n dx = \frac{x^{n+1}}{n+1}$). Concerning the derivative if we take x to the power of n derivative it equals n to the power of n minus one multiply by x derivative, the x derivative equals to one, let's say that here we put u, here so the u derivative ($\left(u^n\right)' = nu^{n-1}u'$). So, we want to know the relationship which exists between this formula here and this one. Third question then the question that we have, we want the purpose of integrals in human activity. Meaning, what is the utility of studying the integrals; it is then the

object of integrals in the domains of human activities. Therefore if we want to say something about it, such as how the integrals serve for example in the agriculture or to the farmers? Or to the truck drivers, that is our question. Fourth, can we get anything else apart from the surface by integrating a function? In any case we have four questions and we want to know something about them.

34. Teacher: Ok, Yes!

35. Ferdinand: I can reinforce the second question, for example there, here if we are given the integral of x to the power of n delta x ((writes $\int x^n dx$)), why can we not calculate that integral by using for example the logarithm of x to the power of n ($(\log x)^n$)? Why do we have to calculate this way? What pushed people to follow that schema?

(Transcription of lesson one, lines 119-121)

Then, Edmond went to the chalkboard to present the questions from his group, shown in the extract below from the transcription of lesson one. Among the questions identified by his group, the third was related to the calculation of the limit of a sum. In the middle of his presentation, Edmond said "The third question, …, let us say that here we have a function f of x in fact to calculate this surface (drawing at the chalkboard), is there any way we can subdivide this surface and after the subdivision of this surface we calculate the entire surface by making the sum of this, there? Thus, our problem is: How do we calculate the limit of the sum of those subdivisions of the surface delimited by the limits of a given function?" From this presentation, this question of how to calculate the limit of a sum became the center of Edmond's presentation. It was the first time that the idea of the limit of the sum was clearly evoked in the classroom, though Edmond had previously referred loosely to these ideas (line 24 on p. 214). Where did the idea come from? My assumption is that Edmond had read the idea in a textbook or he had heard it from his previous teachers. But the latter case is unlikely to occur because in that case he would have remembered roughly how that limit is calculated. Indeed, in lesson 10 Bernard and Edmond acknowledged that they consulted books of analysis (Transcription of lesson 10, p. 1, lines 10-18).

36. Teacher: next

37. Edmond: Concerning integrals, the questions that we have are four. We have a question, you see that to find the surface of a function f of

> x delimited by the limits a b (he wrote $\int_a^b f(x)dx$), here what do we do? Here we have first of all to make the integration of that function (he wrote on the chalkboard $\int f(x)dx = F(x) + C$). After the integration of that function we have to replace in that function that we got, we will replace, suppose that it is the function that we obtain on a certain constant and we will replace in that function the limits by using the formula (he wrote at the chalkboard the formula $\left[F(x)\right]_a^b$). The first question that we have is: Why do we have to make use of the primitive in order to arrive at the definite integral? That is our question. Second, we have seen that we can calculate the surface of a function by integration. Can we calculate the volumes as well? That is our question. How can we calculate the volumes by using the integrals? That is the subdivision. The second subdivision of this question: If we have a surface and we make that surface turn around the axis (drawings at the chalkboard), can't we calculate also that surface that, let's say that we have a surface and we make that surface, so, we are asking ourselves, the surface of revolution of that function, how can we calculate the surface of revolution? Each time by using the integrals. This is the second question. The third question, here as we have seen that to calculate, let's say that here we have a function f of x in fact to calculate this surface (drawing at the chalkboard), is there any way we can subdivide this surface and after the subdivision of this surface we calculate the entire surface by making the sum of this there? Thus, our question is: How do we calculate the limit of the sum of those subdivisions of the surface delimited by the limits of a given function? This is in fact the third question. The fourth question that we have: What are the applications and the utility of integrals? Here we want to know if in the domain of physics and in the domain of the mathematics or technology, where are the integrals. Because every time that we hear that when pilots are piloting airplanes they make use of integrals, in the computers on the levers we see the integrals, also in the very complicated and scientific calculators we see integrals. How does it work and what is their utility in the scientific domain? These are in fact our questions for which we want to know the answers.
>
> (Transcription of lesson one, pp. 10-11, line 122-123)

Finally, Bernard went to the chalkboard to present the questions from the third group. His presentation brought in the question of why study integrals. Then he focused on the logical sequence of the topics of calculus (limits, derivatives, antiderivatives, and integrals) and he insisted, on behalf of his group, on learning more about the utility of integrals in everyday life and on their application in other scientific domains such as Physics and Chemistry, and Topography. He also wanted to know the interpretations of the definitions of the concepts of integrals. And at the end, he evoked the importance of the graphical reasoning and explanation. Here, Bernard and his group started feeling the need to understand, as they want to involve the graphical representation and to be able to explain what they learnt. They want to avoid 'memorisation' and they reject the reasoning based only on 'theories' since Bernard, in his last question, wishes "to reinforce the graphical reasoning instead of reasoning uniquely, theoretically, and simple theories without being able to explain".

38. Bernard: We have identified almost seven questions. The first question is identical to the one of the former group and concerns the reason of studying integrals. Why do we need to study the integrals? Secondly, we have identified the following question: The logical succession between the limits, the derivatives, the primitives and the integrals. Because we have noticed that some of us we had studied integrals, limits and derivatives but only in insufficient measures. Thirdly, we want to go further in learning integrals in daily life. What is the utility of integrals in daily life? Fourthly, we have: From the definitions, what are the interpretations of those concepts, concepts of integrals? Lastly, we have the following question, the relationship between integrals and other courses like geography, physics, chemistry and I think even topography. The sixth question is to reinforce the graphical reasoning instead of reasoning uniquely, theoretically, and simple theories without being able to explain.

(Transcription of lesson one, pp. 11, lines 125)

In this episode, the communication was of S1-SST type and the episode lasted eighteen minutes.

6.2.7 Formulation of the common question to be answered during the course

At the 68th minute, after the presentations of the questions identified by the groups, I organised a whole-class discussion to determine the common question that would lead our course. This served to jointly determine the objective of the course.

39. Teacher: In fact, the common question which will guide your group after all, you have to formulate, what is the key question that guides, which can facilitate the understanding of all that? Which one among the three presentations from what you have said, from what they have said, from what they have? What do you think is the key question?

(Transcription of lesson one, p.12, lines 126)

Each student teacher gave his or her point of view. For example, for Merriam, the common question was the application and the utility of integrals whereas for David, the common question was that of knowing the relationship between antiderivatives and indefinite integrals.

40. Merriam: The most common question, I see that, it is the application and the utility of integrals
41. Teacher: Ok, others!
42. David: We have seen that among the questions that we have exposed, the groups converge on, ask themselves some, even if those questions are not formulated in the same way, but you can see that the questions are the same. There is for example the question of knowing the relationship between the primitive functions [antiderivatives] and indefinite integrals.

(Transcription of lesson one, p.12, lines 127-129)

The discussion continued between me and the student teachers trying to harmonise my goals and theirs. The crucial intervention was done by Edmond when he said that the key question was “how do we interpret theoretically, graphically and practically the relationship between integrals and derivatives” (Transcription of lesson one). At the end, we concluded by agreeing that the key question suggested by Edmond would lead our course. This conclusion was a devolution for later lessons. The student teachers had defined a question for which each one of them had the responsibility to find the answer.

In this episode, the mode of communication was of the type T-SSw and it lasted thirty four minutes and was the last of the lesson. In the next section, I summarise the didactical episodes that occurred during the first lesson.

6.2.8 Synopsis of the didactical episodes in lesson one

The duration of the didactical episodes described in the previous subsections is displayed in the following table (Table 5). This table shows that more than half of the time, I was leading the activities of the classroom since the T-SSw mode of verbal interaction was predominant. This mode of communication cumulates 60% of the time spent in the didactic episodes of the lesson. Moreover, the table shows that there has not been an explicit devolution of a specific problem at the beginning of the lesson, but the questions to the students served to direct their attention to the need for more knowledge. In fact the lesson was designed to formulate the key questions to be dealt with during the course. The devolution which was done in this lesson was rather the formulation of the questions to be answered during the lessons that would follow. Finally, the table shows that there has been no institutionalisation. Actually, the course was at the beginning phase and there was not yet any substantial treatment of subject matter.

In brief, the first lesson served as devolution for the whole learning situation: The student teachers were made responsible for the key question they jointly formulated.

As the table below shows, there had not been any dialogic mediation when the student teachers were working in small groups and they did not discuss their solutions in the absence of the teacher.

Episodes	Actions	Duration (minutes)	Percentage (%)
T-SSd	Devolution	0	0
S1-C	Adidactical situation	12	10
T-S1	Dialogic mediation of individual problem solving	12	10
S1-SS1	Problem solving in small groups	8	6
T-SS1	Dialogic mediation of problem solving in small group	0	0
S1-SS_L	Discussion of students' solutions in a large group (teacher not involved)	0	0
S1-SST	Presentation and discussion of student's solution in whole class (a student leads the discussion)	18	15
T-SSw	Discussion in whole class led by the teacher	72	60
T-SSi	Institutionalization	0	0
Total		122 (2h02)	101

(Note: The total of percentages may not add to 100% because of rounding.)

Table 5. Duration of didactic episodes of the first lesson

In summing up, in this section related to lesson one I described the didactical episodes that emerged during the lesson and I calculated their durations. This description provided the base for the determination of the didactical situations which are likely to further the understanding of integrals. At the end of the description, it appears that the didactical episode of the T-SSW type, which lasted longer than all the other didactical episodes, contributed to developing the student teachers' understanding of integrals because the student teachers were guided to formulate the key question that would guide their learning of integrals. In the next section I describe the didactical episodes that occurred in the second lesson and how they moved forward the student teachers' understanding of integrals.

6.3. Didactical episodes during lesson two

This section deals with the second lesson that was planned to last two hours. The topic of the lesson was to evaluate areas under curves; through an adidactical situation, tasks related to this topic guided students towards the conceptual levels comprising the notion of the definite integral. This second lesson was the first of a five-series of lessons in which the concept of the definite integral was taught and learnt. In this section, I present in details the didactical episodes that emerged during the lesson and I calculate their durations on the basis of the video recordings of the lesson. In the next subsection, I describe the starting episode in which I devolved the first task to the student teachers. The content of this first task is attached in appendix B.

6.3.1 Devolution of task one

The devolution of the first task was the first episode of the second lesson. During this episode, I addressed the student teachers in order to tell them what they were requested to do in order to answer the key question they formulated during the first lesson and I distributed task one to them. Two student teachers reacted to my speech by asking for more clarification about the task. I responded to their concerns by requesting them to try to carry out the task and by promising them that the purpose of the task was in relation to the question formulated in the first lesson.

42. Teacher: Let's do some exercises. Eeeh besides. Eeeh ((he makes noise with the fingers)) you have to give back all the sheets upon which you have worked. In fact, I will ask you to give them back so that I can help you on the basis of what you have done. This work is useful for you and for the future generation; effectively because I

would want that one learns; eeh! How can we modernise the teaching of mathematics and all the learning system of mathematics? That is to say, that you have a big role to play in the process of the learning of mathematics. Eeeh! Just in considering the exchanges that we had... Eeeh…according to our last talk of last time, it was clear that the main common question was …? ((He wanted to have the answer from the class. He bows down to pick up pieces of chalk)).

43. Teacher: Do you remember the question?
44. The class: No
45. Teacher: What is the question? Which one?
46. Jimmy: How to interpret theoretically and practically the relationship between integrals and derivatives.
47. Teacher: ((He writes the title down on the blackboard)) How to interpret theoretically and practically the relationships between integrals and derivatives.
48. Teacher: So, this is the question of our class. At the end of this course, each one will be able to interpret theoretically and practically the relationship between integrals and derivatives. The tasks that will be presented to us, effectively, will allow us to progress towards the answer of this question. Effectively, your active participation in the completion of the task will allow us to understand the questions that you expressed at the time of our exchanges, even the key question. Today, we will begin our lesson by the calculation of the areas.

(Transcription of lesson two, p.1, lines 1-7)

After this introductory presentation, I distributed task one (see Appendix B, p. 321). I read the task out loud. During this communication of the task, I invited the student teachers to perform actions by stating the questions in the exclusive imperative mood (“give an example of a surface limited by a curve and the axis of the abscissas x…” and “determine the elements that you can modify in the question…”). Also, I guided the student teachers by telling them to base their reasoning on the geometrical ideas that they had expressed during the first lesson when we were exchanging ideas about integrals. However, and most importantly, I imposed the constraint of not using antiderivatives. This constraint made the task unfamiliar to the student teachers so that some of them, namely, Edmond and Jimmy, reacted to it (lines 55 and 59).

49. Teacher: ((Teacher writes on the blackboard while speaking)) the calculation of the surface areas… eeeh… The calculation of the

areas under the curves. And for that, we have a problem to resolve in relation to that matter. This problem, I distribute it to you ((Teacher distributes the papers to the students))

50. Teacher: and then, ok.
51. Teacher: Who is absent?
52. The class: Norbert is absent.
53. Teacher: Will he return? Ah! Therefore, the problem is the following one: I will read it out loud and you will follow. At the time of our exchanges during our first lesson about integrals, one could see that everybody tried to define the integral by drawing a curve and some geometric surfaces under this curve and between the boundaries that you fixed. Now, based on these geometrical ideas that you have showed, first, give an example of a plane surface limited by a curve and the axis of the abscissas x where the boundaries are fixed, for which you determine the area *without using antiderivatives*. Second question. Determine the elements that you can modify in the question, in the first question, to have other problems about areas that you will resolve by using the method that you defined in response to the first question.
54. Teacher: ok there?

(Transcription of lesson two, pp.1-2, lines 8-12)

Two student teachers, Edmond and Jimmy, reacted to my talk and to the task. The following extract shows Edmond's reaction.

55. Edmond: Me, I have a question.
56. Teacher: You can read it out loud.
57. Edmond: Give an example of a plane surface limited by a curve and the axis of the abscissas x where the boundaries are fixed, for which you determine the area without using antiderivatives.
58. Teacher: There; therefore you have to give an example of a surface in the plane that you attempted to modify the other time and then you have to determine this area. Is there anyone who still has a problem in relation to this subject?

(Transcription of lesson two, p. 2, lines 13-16)

When Edmond called out to react to the task ("Me I have a question", line 55), I did not allow him to say his question. In fact, I did not want to interact with him because this was supposed to be an adidactical situation. To avoid these interactions, I requested Edmond to read out loud the task. He read it and I confirmed what he had read without any further comments.

Jimmy also reacted (line 59). I explained to him what he had to do and finally I used the inclusive imperative mood ("Let us do the exercise") and I assured him that after the exercise would have been done, we would see what would have happened ("then we will see what will happen", line 61). After my answer to Jimmy's reaction, the student teachers accepted to start the execution of the task. The student teachers accepted the responsibility of the subsequent learning episode. But their acceptance also reflects a degree of trust that the student teachers have in their teacher that, later on, he will guide their learning; this is in fact connected to the didactical contract. Here, the didactical contract was operating in the background.

59. Jimmy: Sorry! The problem is that here one said that… It is difficult to find the area without using antiderivatives. I do not understand. I do not know if it is possible!

60. Teacher: Eeeh! Exactly; you will try and we will see. I give you an example…, in any case when you were explaining you drew, then you drew figures and then you made strips and you went to the antiderivative. And now what did you skip? You will give yourselves an example and then…. Let us start by doing the exercise and then we will see what will happen.

((The students start on the exercises quietly; each one is concentrated on his or her paper while the teacher is distributing supplementary papers to them. After the distribution the teacher sat in a corner of the classroom.))

(Transcription of lesson two, p. 2, lines 17-18)

The communication in this episode was of the mode T-SSd and lasted seven minutes. In the next subdivision, I describe the episode of independent problem solving.

6.3.2 Adidactical situation: Independent problem solving

At the 7th minute, the student teachers started to work independently on the task. I distributed additional squared sheets in case the student teachers needed to draw graphs or diagrams. After this, I sat in a corner of the classroom and let the student teachers work independently on the task.

In this episode, the communication mode was S1-C and the episode lasted 13 minutes. In the next subdivision, I describe the dialogic mediation of the problem solving.

6.3.3 Dialogic mediation of individual problem solving (didactical situation)

At the 20th minute, after the episode of the independent work in the adidactical situation, I started moving from desk to desk to monitor what the student teachers were doing. At the 24th minute, I approached Edmond and conversed with him as follow. I mainly asked questions about what Edmond was doing (lines 62, 64, 66, 70, and 72) and encouraged him to engage with the task (lines 68 and 74). I insisted that he find the numerical values (line 66) because I wanted him to understand the intermediate steps (product and sum) related to calculation of the total area in numerical representation before he arrived at the generalisation (use of symbols). As they were student teachers, I wanted them to understand all the steps so that they can explain them to their students when they will become teachers.

On his side Edmond expressed what he did. The interrogative form that I used made him express his personal work since he used the first person (I) to answer the questions (lines 61, 62, 65, 67, 73). Especially, he told me that he used the method of the plane rectangles and that he did subdivision. The idea of subdivision emerged during the description of the problems that the student teachers had about integrals during the first lesson, as seen in the previous section. The idea was brought in by the group for which Edmond was the representative. As I said earlier, Edmond might have read this idea in a textbook since here also he expressed, in line 61, the name of the method that he used ("I used the method of the plane rectangles").

61. Edmond: I used the method of the planes, the method of the plane rectangles. I subdivided and then I called this sum S1, the first sum, and the sum of this area. And I again subdivided.
62. Teacher: Where did you write it? Speak up!

63. Edmond: I subdivided one here, a half here and three half here, and then I drew other rectangles so that I can have this area here and then this area also will be included in the other area.
64. Teacher: And how much did you get?
65. Edmond: I called that second area S2.
66. Teacher: How much did you have get?
67. Edmond: The numerical value? The numerical value, I have not found.
68. Teacher: This was the question. It is necessary to find the numerical value, it is necessary to find the numerical value because the area it is a number and not the "S's"
69. Edmond: It is a numerical value but, since here one does subdivisions infinitely, certainly one cannot find the numerical values without using the preceding areas.
70. Teacher: Why?
71. Edmond: Since, therefore the subdivision is not limited, except if one calculates the limit.
72. Teacher: And why can one not calculate the limit then?
73. Edmond: That is what I would want to do here. I would want to progress so that I calculate the limit.
74. Teacher: Ok. Carry on.

((Teacher leaves Edmond and goes towards Merriam.))

(Transcription of lesson two, p. 3, lines 35-48)

In line 69 and 71, Edmond used the neutral pronouns (it, one) to refer to the mathematics that I had evoked in line 68 and to defend his method. Edmond used the third person (it, one) to put the responsibility of his choice away from him and unto the general mathematics community. He did want to take the entire responsibility of what he was doing.

After 45 minutes 30 seconds, I had the following conversations with Bernard, then with Honorate, Edmond, and Merriam. During the conversations, I mainly asked questions about what they were doing. Bernard showed me the functions that he used, namely the linear function in line 75, and the parabola x^2 in line 79. He observed that in the case of the parabola there were some differences between lengths of consecutive rectangles and that there was a need to subdivide until infinity (end of line 79). I asked for the result of the first subdivision because I wanted him to start the construction of a numerical sequence of sum of areas in order to get the idea of the limit. It is here that I wanted the student teachers to construct the processes of multiplying and of adding in numerical representation before the generalisation. But the student teachers wanted to generalise before getting into the details underlying the generalisation. They ignored that mathematicians reached generalisation after observations of many various results of elementary operations.

Also, the learning task did not succeed in constraining the student teachers to master these processes and their corresponding results specifically and numerically before generalising. This gap in the task is noticeable in the analysis of the student teachers' concept images presented in chapter 5. It may be a result of the work in the previous lesson; the students were aware that this needed to be generalized in order to answer their question. In retrospect, I could have stated that the purpose was to learn from a few examples first, to make sure the generalizations were correct; but I was constrained by making the situation be adidactical.

75. Bernard: Here, one has considered a linear function. Here, I considered a function which has the same equation and the same slope as the first one. There nothing changed
76. Teacher: Was that a modification?
77. Bernard: No. It does not change anything. It was a matter of verification.
78. Teacher: Oh! Ok.
79. Bernard: But here I took another function equals to x^2 that gives me a parabola. I noticed some changes. The procedure remains the same for these rectangles; but concerning the increase, it is not the same. For example according to the 2, we have the same subdivision here on the x-axis but in regard to the lengths, we have a big difference between the two lengths of the consecutive rectangles. Concerning the triangles one has not even rectangular triangles as in the case of a linear function; in this case, we have rectangular triangles that are identical but here one notices that the triangles are not identical. So to find the area located under this curve it is preferable to divide until infinity because in doing simple subdivision here we have for example a triangle that is not at all rectangular.
80. Teacher: Which area have you found for the first subdivision?
81. Bernard: The first subdivision, I have not yet arrived there, I only did the subdivision.
82. Teacher: Ok.

(Transcription of lesson two, p. 7, lines 105-109)

Sometimes, I constrained student teachers to follow my preconceived plan (lines 76 and 80). In fact I was in a dilemma in this teaching. On the one side, I wanted the student teachers to produce their own mathematics; on the other side, I wanted them to produce mathematics which is closer to the formal mathematics and to produce this in a limited time frame. The latter was more transparent in the questions that I asked to student teachers during my teaching.

After 47th minutes 30 seconds, I approached Honorate and I proceeded as for the previous student teachers by asking questions.

83. Teacher: Let us go! Here.
84. Honorate: I took a function f(x) = x+1 and I look for the curve and I found this area here. I noticed that this is a rectangular triangle.
85. Teacher: Where is the curve? Show it to me?
86. Honorate: I looked for the curve and I noticed that is a rectangular triangle. Basing on these unities -1 to 4, here it is 5 cm and the height is 5 cm and then I looked for the area of the rectangular triangle; it is the base multiply by the height over of 2. And concerning the second question I modified the boundaries. I used plus 1 and then I had the function x-1, and basing on this function, I saw that my curve has changed. Then I was verifying and calculating this area.
87. Teacher: Ok. Carry on.

(Transcription of lesson two, p. 8, lines 110-113)

At the 49th minute, I returned to Edmond and I had the following conversation. I proceeded by asking questions (lines 88, 90, 92, 94, 96, 100, and 104). In response to my questions, Edmond described what he did. He used the first person (I) to express his personal method (lines 89, 93, 95, 103) and in line 105, he accepted to calculate the area corresponding to the first subdivision since I had insisted on finding this area in line 105. As I said earlier, I insisted on the calculation of numerical values in order to bring the student teachers to construct the intermediate layers of product and of sum before they arrive to the calculation of the limit. As I said previously, because they are student teachers, I would like them to be able to explain also these intermediate layers in all representations. At this time they were still at the level of the subdivision.

88. Teacher: What did you find for your first subdivision?

89. Edmond: I did subdivision until n. And for the second one I modified the boundaries.
90. Teacher: Justly; what value did you find?
91. Edmond: The value is not numerical, it is numerical but in the form of limits.
92. Teacher: Exactly, in relation to this subdivision that you have found?
93. Edmond: I have found that to calculate the whole area after having subdivided until n subdivisions, I found that it is this.
94. Teacher: Where are the n subdivisions? Are these the n subdivisions or they are equal to how much?
95. Edmond: I looked for. Look! I did the first subdivision …
96. Teacher: What is the area corresponding to the first subdivision?
97. Edmond: To the first subdivision?
98. Teacher: Yes.
99. Edmond: It is this.
100. Teacher: Justly; can you not calculate it?
101. Edmond: The calculation?
102. Teacher: Yes.
103. Edmond: I did not calculate it. I have considered this as it was S1.
104. Teacher: Yes. But S1, you see that it is theoretical. Now it is necessary to find the numbers because here you see you have numbers, can you not calculate them? In relation to the first subdivision, what is the area corresponding to the first subdivision?
105. Edmond: I am going to calculate it.

(Transcription of lesson two, p. 8, lines 114-125)

As I said previously, during my interactions, the consideration of what they had produced escaped to my attention and sometimes I interrupted the student teachers (lines 95 -96) to make them follow my preconceived plan. After some times of resistance, student teachers surrendered to the didactical contract and did what I proposed (line 105)

Notice that in line 104, I did not provide any argument about why I forced Edmond to find numbers and calculate areas corresponding to the first division. He trusted my judgment because of the devolution of the learning situation which had been effective. The student teachers had accepted the responsibility for their learning. However, there is another interpretation to that. It can be argued that Edmond accepted to pursue the learning because of the power relations of the classroom. The didactical contract could constrain Edmond to do what I said. According to the didactical contract, students are required to obey instructions made by the teacher; unless they break it.

At the 50th minute, I directed myself towards Merriam to check what she had done and I requested her to find more examples of questions.

At the 51st minute, I addressed the whole class to provide some supplementary information. However the main activity during the episode was to communicate with every student teacher individually in order to monitor his or her progress and direct the students to perform numerical calculations for specific cases. I wanted them to find as many examples as possible so that they could reach the generalisation *after* having passed through the steps of the product and of the sum related to area.

106. Teacher: ((addressing the whole class)) Do not limit yourselves to two problems only. I did not say one or two but several problems. Do you remember our objective? It is there (showing on the chalkboard). If you limit yourselves there you will not arrive. You have to find several problems, several problems to make you feel exactly that it is not only one problem. One has several problems. Why do you limit yourself and, there, we have seen the method that becomes a special case it is necessary to make a generalisation in order to understand. It is not only one particular case or just one or two special cases; it is necessary to work toward the generalisation. It is at that time that we will understand it. Our objective, we have it there: how to interpret that? One can not interpret it only in the linearity, it is necessary also to resolve other problems and find the general method that can be used; a general method that can be used without, without using the antiderivative.

(Transcription of lesson two, p. 9, line 131)

Based on what the student teachers had produced, I advised them not to limit themselves to one example but to do several examples so that they can arrive to the generalisation ("if you limit yourselves there you will not arrive").

The message "it is necessary to make a generalization in order to understand. It is not only one particular case or just one or two special cases; it is necessary to work toward the generalization" can be contradictory to the students. In fact, it asked the students to work toward the generalization and at the same time to produce several examples before they generalise.

These kinds of moves and conversations to monitor the students' progress and direct them towards my desired processes and goals continued until the end of the lesson. By the end of the lesson, at the 107th minute, I requested the student teachers to hand in their scripts so that I could analyse their productions in order to prepare the next lesson. In these episodes, the mode of the communication was of the interpersonal type T-S1 during which I dialogically mediated the problem solving with each of the student teacher at his or her seat place. When I was talking with one student, the other ten students were working on their own. However, I had started interfering in the students' productions by my questioning. The intentions to teach which had been hidden in the preceding situation had now re-appeared in the face of the students. The situation was no longer adidactical but it had become didactical. The duration of these episodes totalised 88 minutes. In the next subsection, I summarise in a table the duration of the abovementioned didactical episodes.

6.3.4 Synopsis of the didactical episodes

The following table (Table 6) summarises the duration of the didactic episodes identified in the second lesson. In this lesson, I identified three kinds of didactical episodes which were clearly apparent.

	Actions	Duration (minutes)	Percentage (%)
T-SSd	Devolution	7	6
S1-C	Adidactical situation	13	12
T-S1	Dialogic mediation of individual problem solving (productions were influenced by questions of the teacher)	88	81
S1-SS1	Problem solving in small groups	0	0
T-SS1	Dialogic mediation of problem solving in small group	0	0
S1-SS_L	Discussion of students' solutions in a large group (teacher not involved)	0	0
S1-SST	Presentation and discussion of student's solution in whole class (a student leads the discussion)	0	0
T-SSw	Discussion in whole class led by the teacher	0	0
T-SSi	Institutionalization	0	0
Total		108 (1h48)	99

(Note: The total of percentages may not add to 100% because of rounding.)

Table 6. Distribution of time spent in episodes of lesson two

The devolution phase spanned 6% of the time, the independent problem solving in adidactical situation lasted 12%, and the dialogic mediation of individual problem solving 81%. Therefore, the predominant mode of communication in the second lesson was the T-S1 mode where dialogic mediation of problem solving was engaged and maintained between me and each of the student teachers. As I said in the preceding subsection, in these episodes, the students' productions were no longer independent (S1-C) because my intentions to teach were no longer hidden. I have started interfering in the students' work while in the S1-C situations, teachers hide their intentions to teach; they act as if they were *absent* in the students' work. In the current episodes, the student teachers' productions were no longer independent. Students' works were influenced by the questions that I posed to them and by my conversations with other student teachers.

So, the adidactical situation was only in the communication of the S1-C mode where the student teachers were only dialoguing with the task before my mediation through questions. In the next section, I summarise the didactical episodes that occurred during the rest of the lessons and I provide a synoptic table summarising their durations.

6.4. Didactical episodes during the rest of the lessons

6.4.1 Overview of the rest of the lessons

This section includes a brief description of the rest of the lessons and presents a synoptic table summarising the duration of all the didactical episodes.

Following the same format that I used for lesson two, I analysed the other thirteen lessons. The third lesson was a continuation of the second lesson. For the third lesson, I modified the task that the student had started to solve. Instead of each student solving his or her own task, I asked the whole class to find the area under a parabola so that it became easier to control their progress. In fact, I had noticed that it was difficult to control and help each student teacher if each one was using his or her own curve. The structure of this lesson was similar to the structure of the second lesson, as it will be seen in the synoptic diagram of the didactical episodes. I started the lesson by an episode of the devolution, followed by the episode of independent problem solving, then I mediated the individual problem solving at the seat place of the student. This form of mediation continued until the moment that I requested the student teachers to summarise in writing what they had learned until then. The summaries of the student teachers have been analysed and presented in chapter five (Table 2, p. 158).

During lesson four, five and six, some of the student teachers accepted to present their solutions to task two and some discussions were engaged during these presentations. I will return later to lesson six in which an ‘aha moment’ was observed during the discussion of a student teacher’s solution. In this didactical episode, the communication was of the type S1-SST in which a student teacher was leading the discussion.

During lessons seven and eight, I devolved the third task about the indefinite integral (see Appendix C, p. 322)

During lesson nine and ten, some of the student teachers presented their solution to task three (Appendix D, p. 323) and discussions were held about these solutions. Two groups (Group 1 and Group 2) had performed well while the other groups were still having difficulties to get the symbolical representation of the area.

In lessons eleven and twelve, I devolved to the student teachers tasks related to the proof of the fundamental theorem of calculus (Appendices E and F). These tasks were inspired by the gambit of scrambled proof as proposed by Mason (1999). During lesson twelve, when student teachers were discussing the reconstruction of the mathematical proof of the fundamental theorem of calculus (version of evaluation of areas) an ‘aha moment’ was also observed. I will return to this episode of the type S1-SSL later. In this episode, I did not take part in the discussions.

During the thirteenth lesson, I devolved to the student teachers the sixth task related to the building of areas and the cumulative area. Task six is reproduced in appendix F. In order to facilitate the student teachers' work with this task, I brought into the classroom a computer so that the student teachers could observe and explore integrals and the fundamental theorem of calculus. I used the Geometer' Sketchpad software.

During the fourteenth lesson, the student teachers presented and discussed their observations made about integrals and the fundamental theorem using the computer. At the end of this lesson, I institutionalised the student teachers' mathematics produced during discussions and observations assisted by the computer. Finally, I devoted the fifteenth lesson to the self-evaluation.

6.4.2 Synoptic table of the didactical episodes

The following table 7 summarises the didactical episodes and their durations during all the fifteen lessons. The table shows that an episode of devolution (T-SSd) appeared at the beginning of each lesson; the table also shows that an adidactical situation of students' independent problem solving (S1-C) appeared in lessons one to six, and nine to fourteen; group work (S1-SS1) appeared in lesson one, seven, and eight; mediation of independent problem solving (T-S1) appeared in lessons one, two, three, and eleven; mediation of problem solving in groups (T-SS1) appeared in lesson seven, eight, and thirteen; problem solving in a large group in which the teacher did not participate (S1-SSL) appeared in lesson twelve; whole class discussion led by a student (S1-SST) appeared in lesson one, four, five, six, nine, ten, eleven, fourteen, and fifteen; episodes of discussion in a whole class led by the teacher (T-SSw) appeared in lesson one, six, and eleven; and finally, an episode of institutionalisation (T-SSi) appeared in lesson five, six, nine, ten, twelve, fourteen, and fifteen.

Considering the whole table, episodes of discussion in a whole group led by a student (S1-SST) was the most frequent. It had a total duration of 583 minutes out of the 1960 minutes that comprised the whole course; that is, the episodes of S1-SST type lasted 30% of the duration of the course. The second most common was the episodes of independent problem solving that lasted 424 minutes; that is, 22% of the time that the course lasted.

Therefore, the student teachers were given enough time to talk and collaborate overtly and also to think in order to link their prior knowledge to what they were learning.

Moreover, from this table, it is noticeable that the student teachers were given time to interact in episodes driven by one of them. These episodes are those in which interactions were of the types S1-SS1, S1-SSL, and S1-SST. These episodes cumulated 38% of the time that the course lasted. The episodes T-SSd in lesson four, five, six, nine, ten, and fourteen were devolutions of situations of communication and discussion of student teachers' solutions.

Such conditions are in accordance with one of the three ways evoked by Holton and Thomas (2001) that the teacher can use to enhance chances for students to learn. In the same vein, I satisfied the second way evoked by Holton and Thomas (2001) to foster learning by the fact of asking student teachers questions that required some explanation. Some examples of such questions can be found in the episodes in which interactions were of the types T-S1 as shown in the description of the second lesson, in the subsection 6.3.3, when I was interacting with Edmond in lines 62 to 75 and with Bernard in lines 76 to 83. Therefore, during the whole course, the student teachers were given sufficient occasions for learning and understanding the concepts that were under study. It is in the episodes of interactions of the types S1-SST and S1-SSL that understanding (Dreyfus, 1991) has been observed. In the next section, I give further explanation and unpack circumstances that led the student teachers to develop their concept images.

Episodes	T-SSd	S1-C	T-S1	S1-SS1	T-SS1	S1-SSL	S1-SST	T-SSw	T-SSi	Total by lesson
Lesson 1	0	12	12	8	0	0	18	72	0	122 (2h02)
Lesson 2	7	13	88	0	0	0	0	0	0	108 (1h48)
Lesson 3	4	16	100	0	0	0	0	0	0	120 (2h00)
Lesson 4	2	80	0	0	0	0	89	0	0	171 (2h51)
Lesson 5	2	20	0	0	0	0	92	0	13	127 (2h07)
Lesson 6	3	27	0	0	0	0	64	33	7	134 (2h14)
Lesson 7	13	0	0	17	75	0	0	0	0	105 (1h45)
Lesson 8	2	0	0	28	90	0	0	0	0	120 (2h00)
Lesson 9	3	49	0	0	0	0	83	0	10	145 (2h25)
Lesson 10	10	10	0	0	0	0	88	0	12	120 (2h00)
Lesson 11	10	55	45	0	0	0	49	62	0	221 (3h41)
Lesson 12	3	70	0	0	0	105	0	0	2	180 (3h00)
Lesson 13	4	48	0	0	94	0	0	0	0	146 (2h26)
Lesson 14	3	24	0	0	0	0	62	0	4	93 (1h33)
Lesson 15	2	0	0	0	0	0	38	0	8	48 (0h48)
Total all lessons	68	424	245	53	259	105	583	167	56	1960 (32h40)

Table 7. Time spent in various episodes during all lessons

6.5. The development of Edmond's concept images of the definite integral

I selected Edmond because he was the one who participated most actively in determining the objective which led the teaching. Moreover, he was the one who most break the didactical contract. In fact, he resisted to my instructions during interactions and continued his way of solving the problem until he produced an acceptable solution that was institutionalized for the whole class.

6.5.1 A sequence of various cognitive processes

As Dreyfus (1991) has stated, understanding is more than knowing or being skilled; He said that understanding is a process that occurs in the student's mind - as "a click of the mind" - after "a long sequence of learning activities during which a great variety of mental processes occur and interact" (p. 25). During my teaching, such an understanding occurred in the case of Edmond in lesson five after a sequence of cognitive processes initiated from the first interview that I held with him five months before the teaching of the course. In this subsection, I narrate the journey that led to Edmond's aha moment. All the cognitive processes evoked in this narrative refer to the taxonomy of Anderson et al. (2001) – see Figure 7, p. 79ff. I looked at the verbs used both the interviewer and by Edmond; in fact, this framework can be used to analyse the teaching and the learning.

During the first interview, Edmond was engaged in the process of understanding (Anderson et al., 2001) when I asked to him to express his understanding of the definite integral. The question of the semi-structured interview was about what Edmond understood by the definite integral of a given function f (see appendix A). This question contained the cognitive process of understanding which was the first in the sequence of cognitive processes that the student teachers went through during the period of the research, including the period of the teaching.

During the first round of the interview (see section 5.2.4, p. 122), Edmond engaged in the cognitive process of *exemplifying*, a subcategory of the category of understand, when he

said "…if we have a function, *for example*, $f(x) = 2x+1$"; "……*for example*, if we use this rule saying that……"; "….*for example*, if one gave integral of sinus square…".
He engaged also in the cognitive process of *using*, a subcategory of the category of of apply, when he said "…if we *use* this rule saying that …"; "one will *use* the formula that says that …".

At this time of the first interview, Edmond evoked the pseudoobject (Sfard, 1991; Sfard & Linchevski, 1994) of the definite integral as it is shown in Chapter 5, subsection 5.2.4 (Figure 12). This figure shows that in the symbolical representation, Edmond evoked the pseudoobjects of the definite and of the indefinite integrals.

During the second interview (see subsection 5.3.4., p. 137), Edmond engaged in the cognitive process of *exemplifying*, a subcategory of the category of understand, when he said "… here we were given a function for example our function $f(x) = x^2$", "….For example from zero to one…"; ""For example here it takes our function x^2 …"

He also engaged in the cognitive process of *using*, a subcategory of the category of apply when he said "….then what one must do is to *use* the formula $\int x^2 dx = \frac{x^3}{3} + C$ …"; "…we *used* the formula to evaluate this formula".
Finally he engaged in the cognitive process of *translating* from symbolical to graphical representation, a subcategory of the category of understand, when he said "…here if we *take our function* x^2, then if we *draw the graph* after having sketched the axes…"

As in the first interview, Edmond again exhibited the pseudoobject of the definite integral as it is shown in Chapter 5, subsection 5.3.4 (Figure 17). This figure also reflects that, in symbolical representation, Edmond evoked the pseudoobjects of the indefinite integral $\int x^2 dx = \frac{x^3}{3} + C$ and of the fundamental theorem of calculus $\frac{1}{3}\left[x^3\right]_0^1$ or $\frac{1}{3}(1^3 - 0) = \frac{1}{3}u^2$.

During the third interview (see subsection 5.5.4, p. 181), Edmond engaged in the process of *exemplifying*, a subcategory of the category of understand when he said "*for example*, given a function f of x which is equal to x square ($f(x) = x^2$)…"

He also engaged in the cognitive process of *translating* from symbolical to graphical representation, a subcategory of the category of understand, when he said "*I can sketch this function* from -1 to 1".

He engaged in the cognitive process of *explaining*, a subcategory of the category of understand when he said "when *in explaining* what integral is, I can tell the pupils…"; "then you see that there are some concepts that one has *to explain* here in our formula".

Finally he engaged in the cognitive process of *calculating*, a subcategory of the category apply when he said "then I say that *to calculate* this area exactly…one will *calculate*…the exact area is equal to the limit of the sum… $Total\ area\,(exact) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$".

During the first lesson, Edmond was engaged in a variety of cognitive processes. In the first instance, the objective of the lesson was to identify questions that student teachers had about integrals and to identify the key question to lead the course. Therefore, during the lesson, I was expecting a synthesis of the goals of the participants. However, during the subsequent lessons, Edmond always showed a monitoring (Voigt, 1994) of his own goal even though it has been incorporated in the common goal. This is manifested in his tentative to break the didactical contract, for example in lesson described previously (see for example, Extract from line 88 to line 105).

At the beginning of the first lesson, when I asked the student to tell what they remember about integrals from secondary schools, Edmond was engaged in the process of remembering, especially in the recalling process. The question that I addressed to student teachers was "Can anybody tell us what he remember about integrals learned in secondary schools?” At this time, his answer exhibited what he did about integrals. In the extract below, he evoked verbally the definite and the indefinite integrals and also the determination of area of surface by applying the formula that invokes the fundamental theorem of calculus (“at first integrate and apply the formula, by replacing in the function the upper limit minus the lower limit”). At this time, he was engaged in the cognitive process of recalling the knowledge he had learnt which is of the category of Remember and the cognitive process of executing a procedure to find an area of a surface which is in the category of Apply.

107. Edmond: Also we have seen as he said, there are definite integrals and indefinite integrals. The indefinite integrals commonly identified as primitives, these integrals allowed us to calculate the function that we had derived to obtain that function there, and then there we used the formulae. For the function, for the definite integrals they give the integrals and we integrate those integrals, after the integration of those integrals we calculate the surface with the given limit. But also they would give the graph of the function and we delimit that graph with the limit, and they ask [us] to determine the marked surface delimited by the limits and then to calculate that surface. We would at first integrate and apply the formula, by replacing in the function the upper limit minus the lower limit.

(Transcription of lesson one, p. 2, line 12)

In this answer, Edmond evoked the pseudoobject of the definite integral and of the indefinite integral. He also alluded to the application of the fundamental theorem of calculus when he said that they integrated and then they applied “the formula by replacing in the function the upper limit minus the lower limit”.

After the process of recalling of the category of Remember, Edmond was directed to engage in the process of explaining of the category of Understand when I asked the class to explain what a primitive function has to do with areas and why do they have to get primitives in order to get areas. To this question, Edmond gave the following answer. He evoked the sum, the subdivision, and the limit.

108. Edmond: Here we have seen that this area, to determine this area we have to do the sum, the sum after having subdivided that surface into small triangles, then this sum equals to the limit, to the limit of f of x when x is tending towards zero. We then have a derivable limit at one point, admitting a limit whereby we must calculate that differentiated function because the surface is a sum and a sum is obtained after the calculation of the limit.

(Transcription of lesson one, p. 5, line 77)

This answer was surprising to me, since I was fairly certain that the students had not engaged the underlying concepts of the integral in their secondary schooling and in the two first interviews he did not evoke any of these underlying concepts. He alluded to the underlying concepts of the definite integral since he talked of the subdivision of the surface, the sum, and even the limit. However the limit he evoked did not correspond to the context of the definite integral as he said "…the limit of f of x when x is tending towards zero". This phrase evokes the limit in the context of function. Therefore, only some of the underlying concepts of the definite integral that Edmond evoked at this time were related to integrals, though in embryonic state.

So, it is plausible that, in the above extract, Edmond was engaged in the cognitive process of executing, a subcategory of the category of apply since he said "to determine this area we have *to do the sum*, the sum after having subdivided that surface into small triangles".

During the discussion about the key question the student teachers had about integrals, Edmond was engaged in the process of interpreting, a subcategory of the category of Understand. He was the one to suggest the question which has been approved by the class as the key question and which contained the cognitive process of interpreting (How do we interpret theoretically, graphically, and practically the relationship between integrals and derivatives, Transcription of lesson one, p. 17, line 209).

Other cognitive processes in which Edmond was oriented to engage in were given in task one (appendix B) devolved to all student teachers during lesson two.

109. Teacher: (…) Now while basing on these geometrical ideas that you have showed, first give an example of a surface limited by a curve and the axis of the abscissas x where the limits are fixed for which you determine the area without using the primitive. Second question; determine the elements that you can modify in the question, in the first question, to have other problems on areas that you will resolve by using the method that you defined in response to the first question.

(Transcription of lesson two, p. 2, line 11)

Working with the first part of task one, Edmond was oriented to engage in the process of understand/exemplifying ("give an example of a surface limited by a curve and the axis of the abscissas x where the boundaries are fixed"). This can also be seen as a recall of those examples.

In the second part of task one, Edmond was oriented to engage in the process of analysing/selecting since I told him to determine elements that he could modify in the first question in order to have other problems on areas to solve using the method that they would have defined in answering the first question ("determine the elements that you can modify in the question ... to have other problems on areas that you will resolve by using the method that you defined in response to the first question"). While it is possible to engage this question as understanding or applying, Edmond clearly worked with the prospect of generalizing in mind, making it a strong example of analysing.

In the task one, Edmond also was oriented to engage in the process of creating/producing a method of solving the task ("determine the area without using the primitive").

And, in the last part of the task, Edmond was engaged in the process of apply/using ("…that you will resolve by using the method that you defined in response to the first question").

During the third lesson, Edmond was oriented to engage in the same cognitive processes as the ones previously intended to because for the second task that I devolved to the student teachers in the third lesson, I only specified the curve under which the area should be calculated. I told them to use the parabolic function so that I could compare their progress.

During the fourth to the sixth lesson, some intended cognitive processes were devolved to the student teachers. The following table (Table 8) summarises the range of the intended cognitive processes that were components of the tasks and the learning situations (communication and discussion of their solutions) of the student teachers during the six lessons in which they dealt with the concept of the definite integral.

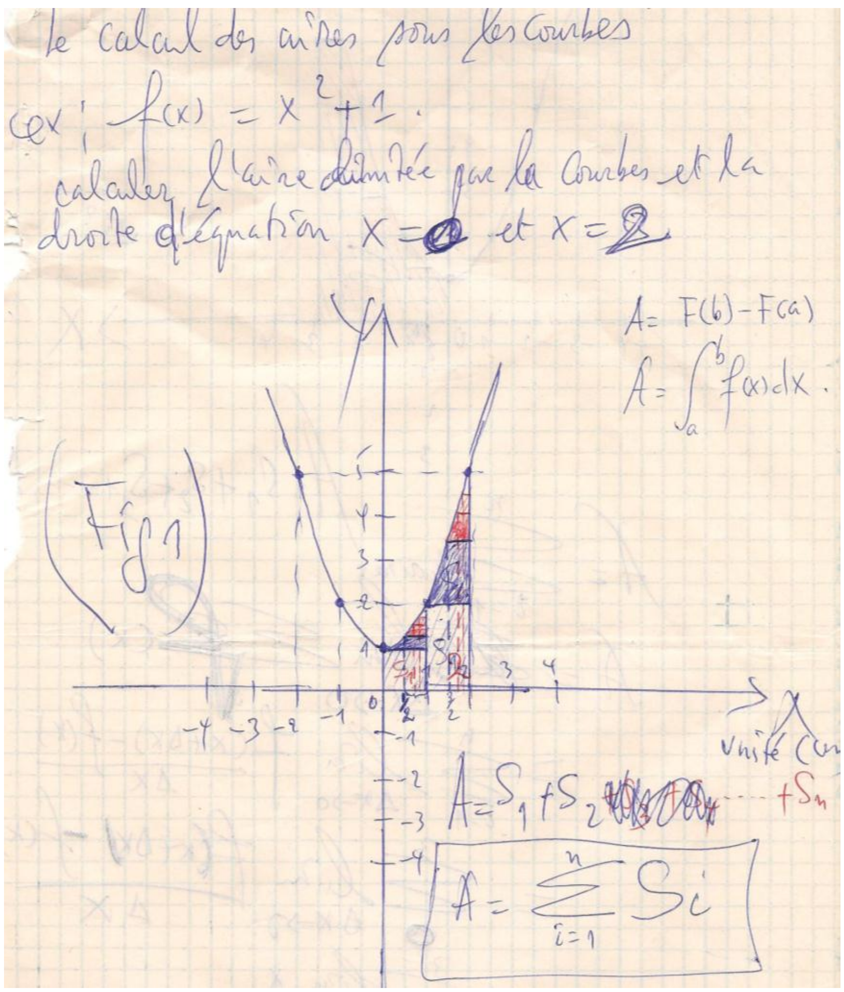
	Cognitive processes							
	Remember	Understand	Apply	Analyze	Evaluate	Create	Validate	Explore
Lesson 1	3	2	2	3	0	2	1	0
Lesson 2	2	4	3	1	0	2	0	0
Lesson 3	0	0	2	0	0	0	0	0
Lesson 4	0	2	1	0	0	0	0	0
Lesson 5	1	0	0	0	0	0	0	0
Lesson 6	1	0	0	0	3	0	0	0
Total L1 to L6	7	8	8	4	3	4	1	0
(%)	20	23	23	11	9	11	3	0

Table 8. Synopsis of intended cognitive processes in which student teachers were to engage in for the learning of the definite integral

In the table, the absolute figures correspond to the frequency of the cognitive processes determined by counting the evoked verbs from the collection of verbs which constitute that process category. After the determination of the frequency of each process category, I calculated their percentages. Thus, as shown in the table, The intended processes of the remember category were proposed at a rate of 20% to allow Edmond (and other student teachers) to connect his prior knowledge to the new one and the intended processes of the other categories (80%) were proposed to help move forward Edmond's concept images integrals.

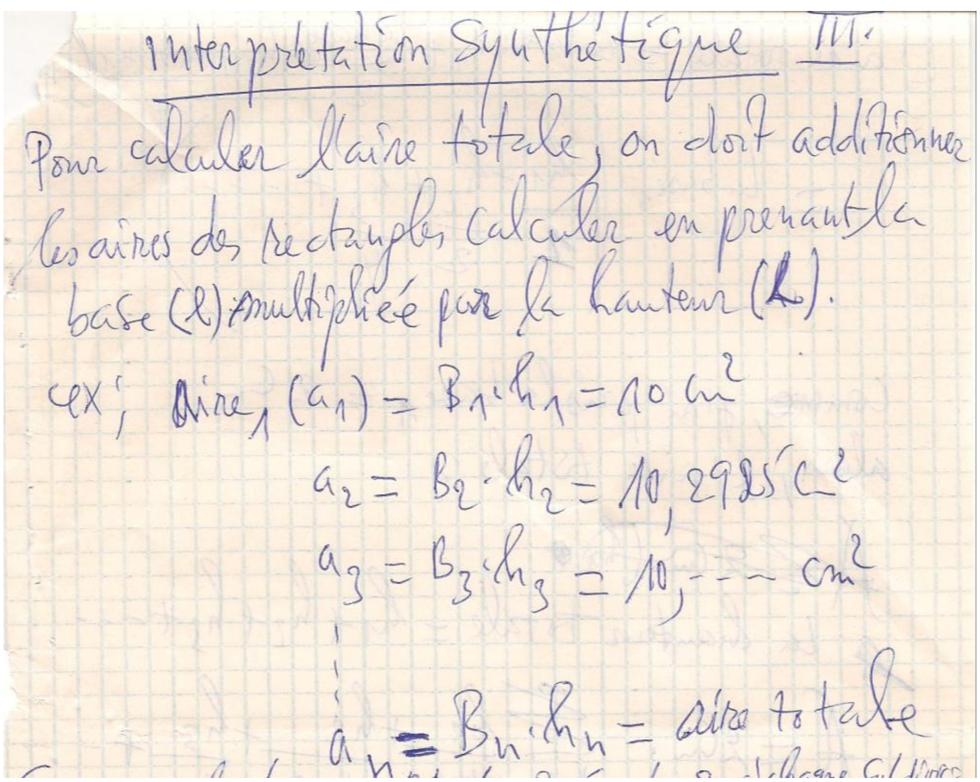
These intended cognitive processes contributed to Edmond (and other student teachers) engage in some cognitive processes and then develop his (their) concept images of integrals.

The following scripts show some of these cognitive process which Edmond engaged in. In the script below, taken from lesson two, Edmond engaged in the cognitive process of *translating*, a subcategory of the category of understand. He translated from the symbolical representation ($f(x)=x^2+1$) to graphical representation (see diagram) and then to symbolical representation ($A=S_1+S_2+...+S_n;\ A=\sum_{i=1}^{n} S_i$).



At the same time in this script, Edmond engaged in the process of exemplifying when he wrote "ex : $f(x) = x^2 + 1$" and in the process of recalling, a subcategory of the category of remember when he wrote the formulas "$A = F(b) - F(a)$" and "$A = \int_a^b f(x)dx$" which represent the Fundamental theorem of calculus in the version of evaluating areas using antiderivatives. But in task, the use of antiderivatives was not allowed ("without using antiderivatives"). So, Edmond was engaged in various cognitive processes during the second lesson.

In the third lesson, Edmond also engaged in a variety of cognitive processes. The title of his script (*synthetical interpretation*) shows that he engaged in the processes of *interpreting* and *synthesising* which are subcategories of the category understand. In interpreting he was translation from graphical representation to symbolical representation via numerical representation. In synthesizing, he was summarizing what he had done so far. He also engaged in the cognitive process of *exemplifying* when he wrote the abbreviation "*Ex*:" followed by a list of formulas.


interprétation Synthétique III.

Pour calculer l'aire totale, on doit additionner les aires des rectangles calculer en prenant la base (l) multipliée par la hauteur (h).

ex: Aire$_1$ $(a_1) = B_1 \cdot h_1 = 10\ cm^2$

$a_2 = B_2 \cdot h_2 = 10,2925\ cm^2$

$a_3 = B_3 \cdot h_3 = 10,---\ cm^2$

$a_n = B_n \cdot h_n =$ aire totale

Therefore, appropriate intended cognitive processes, like the one mentioned above, contribute to conceiving learning activities that are likely to further the understanding of integrals.

The Edmond's scripts displayed above show how he was productive. This productivity was caused by the interactivity of the cognitive processes which he was engaged in. These cognitive processes mostly were induced by the intended cognitive processes evoked in the tasks which were devolved to the student teachers.

However, the cognitive processes alone may not produce the expected effect of understanding. In order to be effective, they need to be integrated in an organised purposeful system of communication. To this effect, in the task that I devolved to the student teachers I used the exclusive imperative mood ("Give an example"; "determine the elements") as opposed to the inclusive imperative moods ("Let us give an example"; "Let us determine the elements"). The exclusive imperative directed the student teachers to execute the operations facilitating the cognitive process without expecting that I could intervene before they had done anything. With the use of the exclusive imperative I had selected for myself "the role of controller" (Berry, quoted in Rotman, 1988, p. 8) of what the student teachers would have done. This control, however, was not done silently; I had to communicate explicitly with the student teachers. The way of communicating that I adopted during this control played an essential role in the development of the student teachers' concept images. In the next subsection, I outline the role of the functions of oral communication during my teaching.

6.5.2 The role of functions of communication in the classroom

In addition to providing a great variety of cognitive processes to the student teachers, I used a non-traditional way of communicating in my classroom. My previous way of teaching was mostly characterised by my 'attempted' transmission of the knowledge to be learned; but in the current case, I more frequently used the interrogative and the encouragement forms than the transmissive or the indicative forms. However, my listening to students was still limited during the classroom interactions. During my dialogues with the students I did not construct much on their productions but rather I forced them to orient themselves to my intended knowledge. The following extract of my first dialogue with Edmond exemplifies these observations.

110. Edmond: I used the method of the planes, the method of the plane rectangles. I subdivided and then I called this sum S1, the first sum, and the sum of this area. And I again subdivided.
111. Teacher: Where did you write it? Speak loudly!
112. Edmond: I subdivided one here, a half here and three half here, and then I draw other rectangles so that I can have this area here and then this area also will be included in the other area.
113. Teacher: And how much did you get?
114. Edmond: I called that second area S2.
115. Teacher: How much did you get?
116. Edmond: The numerical value! The numerical value, I have not found.
117. Teacher: This was the question. It is necessary to find the numerical value, it is necessary to find the numerical value because the area it is a number and not the "S's"
118. Edmond: It is a numerical value but, since here one does subdivisions infinitely, certainly one cannot find the numerical values without using the preceding areas.
119. Teacher: Why?
120. Edmond: Since, therefore the subdivision is not limited, except if one calculates the limit.
121. Teacher: And why one can not calculate the limit then?
122. Edmond: That is what I would want to do here. I would want to progress so that I calculate the limit.
123. Teacher: Ok. Carry on.

(Transcription of lesson two, p. 3, lines 35-48)

From this extract, it can be said that:

Edmond is set on calculating the exact area. That is why he does not calculate approximate areas. He has a very clear goal for himself, namely to develop a way of calculating the exact area and generalize this. In a way, he is driven by the question formulated in the first lesson. Thus, he is in the most exemplary way working in an adidactical situation; that is also what the task invited. However, the teacher is 'over-ruling' the instructions of the task by insisting on numerical values for the series of approximate areas. The teacher does not explain why that is necessary, and does not wait until the students have struggled with finding limits.

When I approached Edmond, he started to tell me what he had done in line 110. Here, he evoked the layer of subdivision and of sum. Then in line 111, I asked him a question of where he had written what he was telling me. Instead of showing where he wrote what he did, he pursued telling me what he did. I again asked him another question in line 113 about what value he found for the area. Instead of telling me the number, he told me the name of the surface. In line 115, I repeated the question. My questions focusing on the numerical value surprises him and he expressed in line 116 that he had not calculated that numerical value. (In fact Edmond preferred working in symbolical representation than in numerical representation as one can see it in his summary after the third lesson (Chapter five, subsection 5.4.4)). As he had agreed, in line 116, that he did not calculate the numerical value, I then used the referential function, in line 117, to refer him to the context of the task; I indicated him that that was the question and that “it is necessary to find the numerical value because the area is a number and not the ‘S’s’”.

In this first part of this episode, I and Edmond were not thinking in the same direction. He was still in the adidactical situation pursuing his personal productions while me I wanted to direct him to my preconceived mathematical knowledge in the didactical situation. I wanted to push him to the underlying layers of the integral in the numerical representation before the generalisation in the symbolical representation. Moreover, in my head, I was pressed by the time which was running quickly and the pace of getting to the underlying layers was slow. These circumstances forced me to, at some occasions, to take the control of the situation.

Edmond tried, in line 118, to resist my indication of what he would have done by arguing that since one had done subdivisions infinitely, one could not find the numerical values without using the preceding areas. I took that opportunity to ask him a question, in line 119, of explaining why that was not possible. In the first part of his intervention in line 120, Edmond gave the argument that the subdivision is unlimited; but in the second part of his intervention, he changed his mind by evoking a possible exception as said “except if one calculates the limit”. In line 121, I used this opening to step back and introduce a new question implicitly requesting him to calculate the limit as I said “And why one can not calculate limit then?” This question made Edmond accept to continue the independent work as he said, in line 122, “I would want to progress so that I calculate the limit”.

In this second part of this episode, I abandoned the control of the situation to embrace the productions of Edmond since I asked him to explain his position and I let him continue in his direction of calculating the limit after doing subdivisions. So, I think that in classroom situation, some circumstances, such time, pace, and content to cover, can influence teachers’ decisions to let students continue working in the adidactical situation or to bring them in the didactical situation.

In this extract, I used the conative function (interrogative mood) and the referential function of the oral communication to direct Edmond to change the direction but he resisted. The cognitive processes evoked in this extract were of the category of creating his method that he could apply (to find the numerical values or to calculate the limit). During the whole extract, I used the referential function once (in line 117) where I used the third person or the neutral pronoun to refer to the context of the activity (Winsløw, 2004); I used the conative function (interrogative mood) five times (lines 111, 113, 115, 119, and 121) to bring Edmond to take route that can help him to get progressively to the method required by the task; and I used the phatic function to encourage him while I was interrupting the communication with him in line 123. On his side, Edmond used the expressive function (the first person, I) five times (lines 110, 112, 114, 116, 122). He also used the referential function in lines 118 and 120 to talk about the context of the task of finding numerical values and of calculating the limit. Therefore during this dialogue, the predominant function on my side was the conative function while on the side of the Edmond the predominant function was the expressive function.

In this way of communicating, I tried to bring Edmond to take another direction because I was under pressure of time and slow pace. Edmond refused to abandon his way of working. I let him continue in his way of thinking.

During the whole course, I made an effort to use the conative function (interrogative, imperative, and vocative forms) more frequently than the expressive function or the referential function. In so doing, I was expecting to give the student teachers opportunity to express their understanding of the concepts and develop their own concept images. This extract and the extract related to the exhibition of Edmond's understanding presented in the next subsection exemplify such effort to give the student teachers occasions to talk about their solutions and their conceptions.

Briefly, in order to guide Edmond towards seeing quickly the knowledge included in the task, I adopted a language that uses the conative function (interrogative mood) as in lines 111, 113, 115, 119, and 121. I also used the referential function (the third person of verbs and the neutral pronoun, it) in line 117 to indicate to him what he needed to do. Edmond expressed what he did in his attempt to find the solution of the task. He used the expressive function (the first person, and the personal pronoun, I) to say what he did in lines 110, 112, and 114. He evoked the subdivision and the sum. But he had not yet produced sufficient details to satisfy my intentions with the task. After my question which was oriented in his thinking in line 121, he accepted (in line 122), to continue the solving of the problem by calculating the limit.

As I said previously, Edmond did not change his way of thinking; he continued in the same direction. He broke the didactical contract refusing to leave the adidactical situation. My intentions during the interactions were not followed by the student. However, this was beneficial for him because, it allowed him to pursue the learning as it will be seen later.

The third lesson had the same structure as the second lesson. In fact, in this lesson, the student teachers had to deal with a modified version of task one that I described above (the modified task is in appendix B). The difference between the two tasks is that in the modified task, I specified to student teachers to determine the area under a parabola.

At the end of the lesson, I requested the student teachers to summarise what they had learnt until then. An analysis of their production is presented in chapter 5, subsection 5.4 and Figure 23 presents findings of the analysis of Edmond's productions. The student teachers were still calculating areas of rectangles and triangles.

Seeing that the progress was slow and that the concept images were still in need of development, I decided that from lesson four, I would let the student teachers communicate and discuss their solutions. During lesson four, Cassim volunteered to present his solution and the student teachers discussed some of its elements. After Cassim, it was Edmond's turn. Edmond presented his symbolic solution (he had broken the didactical contract by refusing to go into the numerical representation), and during the presentation he brought in the formula for the limit of the sum to determine the exact area under a parabola. However, he became stuck in his calculation of the limit because he was unaware of some formula related to the rules of summation. I assisted him by providing the formula of the series $\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$. He pursued his calculation and he obtained the exact area by using the limit of a sum.

After Edmond had made such a great effort to calculate the limit of a sum, I judged that the student teachers had had all the elements necessary to make the connection between integrals and the underlying concepts.

Edmond's way of solving the task made the class take a leap forward. The tasks and the participation of the students contributed a lot to the achievement of this step of understanding. The verbal communication also played a certain role. The position of not communicating the knowledge but rather to ask questions or give instructions (conative function) retained the student teachers on the track of the learning until the understanding.

However, from the task and from student teachers' difficulties of answering some questions regarding what interpretation they could give to the area found by Edmond using the limit of a sum, I judged that it was time to assist them directly. I used the definition of teaching in the zone of proximal development as provided by Tharp and Gallimore (see framework in chapter 3) that "…Teaching can be said to occur when assistance is offered at points in the ZPD at which performance requires assistance."

So, drawing from Edmond's presentation, I communicated to the class the following formula that links the limit of a sum to the definite integral $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$. The student teachers noted the formula in their notebooks. However, it was noticeable that they were just noting it as something new but the understanding was not yet reached. For some of the student teachers, their concept images analysed in chapter five reflect this observation. To improve their understanding, I decided to continue the course with additional student teachers' presentations in lesson five.

The first presentation was done by Ferdinand. He presented his solution but when he reached at the formula $A = \lim_{n\to\infty} B_n \sum_{i=1}^{n} x_0{}^2 + 2x_0 i\Delta x + i^2\Delta^2 x$, he was unable to go further. Edmond voluntarily decided to continue the presentation. He started by writing on the chalkboard the formula $\Delta x = x_n - x_0$ and $\Delta x = 3-0$. In the context of integrals, the formula was wrong because the correct formula is written as $\Delta x = \frac{x_n - x_0}{n}$ or $\Delta x = \frac{3-0}{n}$. But I let him continue his calculation until he noticed that there was a mistake and then corrected it. In so doing, I was thinking that letting him discover the mistake would make him understand better the layer of partition (see theoretical framework in chapter three). This formula constituted a cognitive conflict because in some other circumstances (for example, in differentiation), Edmond's notation could be used. At the 96^{th} minute, he noticed that the calculation he was doing was not correct and he started looking for the origins of the mistake. At the 98^{th} minute, other students, especially Jimmy and then Bernard intervened but the mistake was not immediately found. The excerpt presented in the next section shows in what circumstances Edmond succeeded to identify and to correct the mistake.

6.5.3 Edmond's understanding of the layer of partition

This understanding occurred during the episode of the type S1-SST in which Edmond was leading the communication between him and the group of the other student teachers and the teacher. After Edmond and some of his colleagues had made some efforts looking for the mistake, I asked him the question "where is the mistake?" (line 129) in order to make him continue the effort of thinking and looking for the mistake instead of showing him where the mistake was. When I realised that he was still stuck, I called for the other student teachers to participate (line 130). Bernard volunteered to join Edmond in front of the class and collaborate in searching for the mistake. In lines 131 and 133 Bernard used the referential function to indicate to Edmond where the error might possibly be. When I realised that their effort was unsuccessful, I asked Edmond questions about the meanings of the symbols x and Δx, in line 134. These questions seemed to show Edmond where to look and made him remember what Δx was, and in line 135 he used various functions of language to communicate his understanding. Firstly, he used the expressive function to express his emotion of understanding as a 'flash in the mind' ("Ah yes I see"). Secondly, he used the referential function to indicate what he had understood ("The error is here" and the formulas that he wrote $\Delta x = \frac{x_n - x_0}{n}$ and $\Delta x = \frac{3-0}{n}$). He made the necessary corrections in the rest of the calculations and finally, he again used the referential function to indicate the meaning of the symbols Δx ("It is the minimal difference").

129. Teacher: Where is the mistake? (break of two seconds)

130. Teacher: Others?

131. Bernard: Maybe the error is here. It was necessary to keep delta x as delta x squared.

132. Edmond: That is B_n ((a symbol of the base of a rectangle of the n^th^ subdivision)).

133. Bernard: It must be here.

((Bernard and Edmond both think))

134. Teacher: What is x in delta x [Δx]? Delta x [Δx] is equal to what?
135. Edmond: Ah yes, I see! The error is here. There is an error here! ((he corrects the formulas of the partition $\Delta x = \frac{x_n - x_0}{n}$ and $\Delta x = \frac{3-0}{n}$ and he says with slicing gesture)) It is the minimal difference ((he makes the correction to the other symbols and continues the calculation up to the end)) ((The class applauds))

136. Teacher: I think that it is ok on that side. I think that there is a need for some formulas on series that can help us in the future. We will note these formulas. We will stop here and we will resume on Monday. Then here are the formulas. ((Teacher writes the formulas on the chalkboard))

(Transcription of lesson five, p. 12, lines 231-238, DVD of lesson #5 [99:00-103:07])

In this extract, I used the conative function (the interrogative and the vocative moods) three times, namely in lines 129, 130, and 134. I used the expressive function in line 136 to express my thinking and my feelings. The student teacher Bernard used the referential function in line 131 and in line 133. Edmond used the referential function in line 132 and he used the expressive function and the referential function in line 135. As in this previous extract, I mainly used the conative function to involve Edmond in a deep thinking while Edmond used the expressive function to express his understanding - as a ‘flash in the mind’ (“Ah yes I see”) - and the referential function to indicate what he had understood (“it is the minimal difference”).

At the same time, Edmond pointed up to support his understanding and he moved to the left side of the chalkboard to make all the corrections in the formulas he had written at the beginning of his intervention. He changed the formulas $\Delta x = x_n - x_0$ and $\Delta x = 3-0$ into the formula $\Delta x = \frac{x_n - x_0}{n} = \frac{3-0}{n} = \frac{3}{n}$. To convince the other students and the teacher that he understood what Δx was, he accompanied his statement with a hand gesture of slicing. Then, after the corrections related to Δx, Edmond continued the calculations and arrived at the final answer. At the end of the presentation, the class applauded. This manifests that the student teachers agreed with what Edmond had done and said. The claps were also a symbol of thanking Edmond for his effort in order to get the right solution of the task.

After the applause, I used the expressive function to address the whole class in order to express my impressions on what Edmond had done at the chalkboard by saying "I think it is ok on that side" (line 136). My feedback was conveying an evaluative message to confirm that what Edmond has been writing on the chalkboard was correct and it, implicitly, invited the student teachers to retain the knowledge produced by Edmond.

From this time, the student teachers started using the formula of the limit of a sum in their calculation of areas under curves. They had acquired new knowledge in their mathematical scheme. The concept images they evoked during the third round of the interviews, which are presented in chapter 5 (section 5.5) and summarised in 28 in the case of Edmond and in Figure 29 and Table 3 for all the other student teachers, came mainly from this presentation done by Edmond and from the communication of the formulas that I made after lesson four.

The presentation of Edmond made other student teachers rectify where they have made mistakes and complete where they were still working on. So, this way of teaching which makes students work first on the tasks, and then present and discuss their solutions in order to reach acceptable knowledge that the teacher can institutionalise, contributes to improve students' understanding of integrals as showed by students' concept images evoked during the third interviews.

The questions that I asked to the student teachers put the responsibility for making sense of the symbols they were using, in the occurrence the symbol of the layer of the partition $\Delta x = \frac{x_n - x_0}{n}$, onto them. The questions also contributed to push the student teachers to identify domains which they need interrogate further, such as the calculation of the limit of a sum. However, some aspects of the mathematical knowledge needed to be institutionalised by the teacher in order to give a sign post for the learning process. This was the case of the institutionalisation of the formula $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$. Later on, when the student teachers' understanding was checked (during the third round of interviews), they demonstrated good understanding by providing explanation for the knowledge acquired through the teacher's questioning while there was no explanation for the knowledge directly communicated by the teacher.

The cognitive processes which they were engaged in during the learning process (translating from one representation to another, exemplifying, explaining, synthesising, interpreting, summarizing, generalising, etc.) contributed to making sense of what they were learning and to improve their understanding of this knowledge.

As shown in Table 3 seven student teachers correctly evoked both the process and the resulting object of the layer of the limit $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$, two only correctly evoked the process aspect $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$, one only correctly evoked the pseudoobject $\int_a^b f(x)dx$, and one did not evoke anything related to layer of the limit.

However, explanations were only provided for the layer of partition for which the student teachers took time to work 'hands-on'; the other layers were simply evoked without more explanation, especially the layers of product and sum.

The understanding (Dreyfus, 1991) exhibited by Edmond stems from his great participation in the shaping of the activities of the course. Edmond was the representative of the group who produced the question of how to calculate the limit of the sum and he was the one who proposed the key problem - how to interpret theoretically and practically the relationship between integrals and derivatives - that served as the objective of the course. During his subsequent interventions, be it in the dialogues that I held with him, or in the presentations of his solutions to tasks, he showed that he was led by the question of how to calculate the limit of the sum evoked by his group during the first lesson.

Therefore, in order to increase students' chances to reach understanding (Dreyfus, 1991), the teacher should, at least, make them participate in the definition of the objective of the course, expose them to a wide range of cognitive processes during the teaching, adopt a discourse that use the conative function more frequently than the expressive function or the referential function, and finally, allow enough amount of time to the solving of tasks and to the presentation and discussion of their solutions.

The tasks have to be sources of learning (Douady, 1991). In addition to the intended knowledge, they need to contain intended cognitive processes that will induce actually cognitive processes which the students will engage in during the adidactical situation (Brousseau, 1997). If the devolution has taken place, then they will break the didactical contract, like Edmond, and their learning will increase.

6.5.4 The role of the conative function

According to Lewis's (1891; Reprint 1979) dictionary, the word "conative" originates from a Latin word "conari" which means "to undertake, endeavour, attempt, try, venture, seek, aim, make an effort, begin, make trial of" (p. 169). Jakobson (1981) used "conative" to distinguish the function of language that involves the addressee thanks to the use of the imperative, the interrogative, or the vocative forms of interacting. The conative function makes the addressee feels concerned by the message and then he or she makes an effort to execute the action that is commanded by the verb in the case of the imperative, to provide the information requested in the case of a question, or to be ready to communicate with the addresser in the case of the vocative form.

Interpreted against this background, the emphasis of the conative function, accompanied with a reduction of the expressive and the referential functions on the side of the teacher, can help any teacher who would like to move from a teacher-centered teaching style to a student-centered teaching style. My teaching described here was an attempt to undertake such a move from the teacher-centered style of teaching to the student-centered style of teaching. The Committee on Undergraduate Science Education (as quoted by Zhang, 2003) described both the teacher-centered and the student-centered teaching strategies as follows: In the teacher-centered style of teaching, the teacher is the focal point of all activity; he or she is the main source of the knowledge and the authoritative expert; he or she plays a leading role and transfers information. In this teaching style, the focus is mainly on facts and skills and rarely on the relationship between them. In the contrary, in the student-centered style of teaching, the focus is on the student and on his or her cognitive development. The teacher's goal becomes that of helping the students grasp the development of knowledge as a process rather than a product. The focus of the classroom activities turns into the student-centered process of inquiry itself and not on the products of inquiry. However, this remains an ideal. Constraints, such time and content to be covered, can force the teacher to deviate this focus and see what contributions can be made by a teacher-centred style to complement the student-centred style.

During my teaching, in order to put into practice strategies leading to a student-centered style, I used the conative function in a number of ways. I used the imperative forms in my tasks to make the student teachers execute operations (Rotman, 1988) conveyed by the cognitive processes. During the dialogues with the student teachers I used the interrogative form to request the student teachers to express their understanding. Sometimes during the dialogues with the student teachers, I used the referential function to help the student teachers move forward. My main target was to involve as many student teachers as possible in the solving of the tasks and consequently in the development of their concept images of the concepts that were being learned. In brief, I wanted the student teachers to change from being passive receivers of knowledge to co-creators of knowledge. As I said previously, I acknowledge that this is an ideal. In my teaching, I did not achieve this target all the time; for example at some occasions during interactions, I attempted to bring Edmond follow my preconceived plan instead of to listen attentively him and embrace his productions. However, a student-centred style is a good target for teaching that wants students to make sense of what integral is.

6.6. Conclusion

In this chapter, I presented didactical episodes that occurred during the lessons that constituted my teaching about the concepts of the definite and the indefinite integrals and about the fundamental theorem of calculus. During my teaching, I observed a student teacher's understanding as a 'click in the mind' in the episode of the type S1-SST of lesson five in which a student teacher, Edmond, was leading the communication between him and the group of the rest of the student teachers and the teacher.

Next, I narrated a sequence of cognitive processes that are, among others, the causes of the abovementioned student teacher's understanding. I noticed that in order to increase students' chances to reach understanding, the teacher should, at least, make them participate in the definition of the objective of the course, expose them to a wide range of cognitive processes during the teaching, adopt a discourse that use the conative function more frequently than the expressive function or the referential function, and finally, allow enough time to the solving of tasks and to the presentation and discussion of their solutions.

Finally, with regard to functions of language and their effects in relation to the style of teaching that I adopted during my lessons, I observed that I used the conative function more than the referential and the expressive functions. This way of communicating can help in prompting students' understanding as in the case of Edmond's understanding of the formula of the partition $\Delta x = \frac{x_n - x_0}{n}$.

In the next chapter, I present the student teachers' understanding of the fundamental theorem of calculus.

CHAPTER 7: STUDENT TEACHERS' UNDERSTANDING OF THE FUNDAMENTAL THEOREM OF CALCULUS

7.1. Introduction

In this chapter, I present the student teachers' concept images related to the fundamental theorem of calculus and the process of their development. This chapter is made up of four sections. After this introduction, I present the student teachers' understanding of the fundamental theorem of calculus during the final lesson (lesson fifteen) in a self-evaluation episode. In the third section, I present a sequence of cognitive processes that made the student teachers develop such an understanding. In this section, I also describe the circumstances in which Bernard exhibited his understanding (Dreyfus, 1991). The fourth section concludes the chapter.

7.2. Student teachers' understanding of the fundamental theorem of calculus

During the final lesson of the course, I requested the student teachers to evaluate what they had learnt. Each student teacher went in front of the class to tell his and her classmates what he or she had learnt.

During the self-evaluation, two student teachers, namely, Ferdinand and Bernard evoked the fundamental theorem of calculus. Ferdinand said "(…) concerning the relationship between integrals and derivatives, as the computer showed us, we were able to find that relationship that the [area of the] surface can be obtained from the function that was differentiated but also that can be obtained through the primitive function". In this sentence, Ferdinand was evoking the fundamental theorem of calculus in its version of the evaluation of area of a surface and there was no apparent difference from what was known at the start of the teaching.

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Bernard on his side said "(…) the relationship between integrals and derivatives is very clear at the moment; the derivative of the integral gives the function corresponding to that integral". In this sentence Bernard was evoking the fundamental theorem of calculus in its version of the derivative of the integral function.

Four other student teachers referred to the relationship between derivatives and integrals but they did not evoke that relationship. Those are Cassim, Gerard, Edmond and Honorate. Cassim said "with the help of the computer I have just discovered the relationship that exists between the derivatives and the integrals in theory". Gerard said "There is a relationship between the derivatives and the integrals". Edmond said "Another point is the relationship between integrals and derivatives. I have seen it theoretically and also practically we have seen that relationship (pointing to the computer)". Honorate said "Concerning the relationship between integrals and derivatives, I observed it on computer with the help of the function that you have given to us".

Other four student teachers did not refer to the relationship between integrals and derivatives during their self-evaluation and one of the student teacher was absent during the episode of the self-evaluation. The following table (Table 9) gives the numbers of student teachers who evoked, alluded, or did not evoke the fundamental theorem of calculus during the last lesson of the course.

Theorem	Evoked (number)	Alluded to (number)	Not evoked (number)
FTC (Relationship between integrals and derivatives)	2	4	5

Table 9. Number of student teachers who evoked the FTC

This understanding differs from the understanding displayed during the interviews presented in chapter 5. During those interviews some of the student teachers evoked only the formulae without alluding to the relationship between integrals and derivatives. So, at the end of the course, the student teachers' understanding of the fundamental theorem of calculus had not developed since only few student teachers could evoke it or allude to it. In the next section, I analyse how this development of understanding of the fundamental theorem of calculus came out.

7.3. The development of the student teachers' understanding of the fundamental theorem of calculus

7.3.1 A sequence of cognitive processes

The student teachers' understanding of the FTC started developing from the first lesson when the student teachers set as the objective of the course "how to interpret theoretically and practically the relationship between integrals and derivatives". This objective was a precursor to make the student teachers engage in cognitive processes leading to the development of the fundamental theorem of calculus, first through developing the concept of the definite integral as a limit of a sum, next through generalizing this to a function – the indefinite integral, thirdly by exploring the relationship between this as an operator and differentiation, and finally by proving both that these operators are each other inverse and the application of this to calculation of area. Even before the course, the student teachers were oriented to engage in the process of explaining the relationship between areas and primitive functions when I asked them the following question during the second round of the interviews: "If your pupil asks you to explain to him or to her in what the primitives [anti-derivatives] have to do with area, what will you tell him or her?" This same question was also asked during the first lesson.

Moreover, from lesson seven to lesson fourteen the student teachers were engaged in a variety of cognitive processes as shown in the following Table 10. In this table, the categories of the cognitive processes are determined using the framework of Anderson et al. (2001) and the numbers correspond to the frequencies of the categories. They are obtained by counting the frequencies of the evoked verbs representing the concerned category. The categories of Validate and Explore are still to be refined since they are my creation, added to the framework. I combined the teaching–learning process of the fundamental theorem of calculus with the process of teaching and learning the indefinite integral because I had thought when I was preparing the task three on indefinite integral (appendix C) that the fundamental theorem of calculus could be well understood by comparing the formula of the area with the function that delimited that area [$\frac{d}{dx}(A(x)) = f(x)$].

Processes	Remember	Understand	Apply	Analyze	Evaluate	Create	Validate	Explore
Lesson 7	1	3	1	1	0	1	0	0
Lesson 8	1	0	0	0	0	0	0	0
Lesson 9	0	0	0	0	0	0	0	0
Lesson 10	1	0	0	0	0	0	0	0
Lesson 11	0	0	0	3	0	0	2	0
Lesson 12	0	0	0	1	0	0	1	0
Lesson 13	0	0	0	1	0	1	1	3
Lesson 14	1	1	0	0	0	0	0	0
Total L 7 to L 14	4	4	1	6	0	2	4	3
%	17	17	4	25	0	8	17	13

Table 10. Intended Cognitive processes from lesson seven to lesson fourteen

From lesson seven to lesson ten, the student teachers were dealing with task three related to indefinite integrals and to the fundamental theorem of calculus (appendix C). During the seventh lesson, they had to engage in the following cognitive processes: apply/use (instruction a.1: "use the lower limit x=0……"), understand/generalise (instruction a.2: "vary …. until you are able to do a generalisation of the formula of area"), understand/compare (instruction a.3: compare the found formula of the area with the given function…."), and analyse/select (instruction b: 'choose a function of another kind….”). In the table above, we see that the cognitive processes of the categories of validate and explore in relation to the FTC started to appear from lesson eleven to lesson fourteen. During these lessons, the student teachers were dealing with the reconstruction of mathematical proofs of the fundamental theorem of calculus and they were exploring integrals by the help of computer using the Geometer's Sketchpad software. The corresponding tasks with the intended cognitive processes mentioned in table 10 above are attached in appendices E, F, and G. The analysis of the cognitive processes to engage in through these tasks follows the example of analysis of cognitive processes in task three. In the next subsection, I present the case of the development of Bernard's understanding of the fundamental theorem of calculus.

7.3.2 Bernard's understanding during interactions about the proof of the fundamental theorem of calculus (version of evaluation of area)

Besides the sequence of intended cognitive processes mentioned in Table 10 and the amount of time that I allocated to the teaching and learning of the fundamental theorem of calculus (see Table 7), the systems of communication that I adopted in the classroom also contributed to the development of the understanding of the fundamental theorem of calculus.

During lesson eight in which the student teachers were dealing with the third question of task three (Appendix C), namely the question of comparing the formula of the area with the function that delimited that area, the student teachers did not succeed in completing the comparison on their own. I had to intervene by reminding them to think in the direction of the objective of the course. It is after this reminder that the student teachers succeeded in determining the derivative of the formula of the area and to notice that derivative corresponded to the given function that delimited that area. The student teachers were assisted in their efforts to find the relationship between integrals and derivatives. This teacher's action is in accordance with Tharp and Gallimore (quoted in

Gallimore & Tharp, 1990) that "teaching can be said to occur when assistance is offered at points in the ZPD at which performance requires assistance." (p. 184)

Subsequent to this assistance, an understanding (Dreyfus, 1991) about the fundamental theorem of calculus was observed during lesson twelve in the episode of the type S1-SSL. The student teachers were reconstructing the mathematical proof of the fundamental theorem of calculus (version of the evaluation of area: $\int_a^b f(x)dx = F(b) - F(a)$).

137. Jimmy: Therefore, the number that is final is four. Number four says that this is equal to F(b) – F(a) what needed to be proved, since that last quantity is a constant.
138. Bernard: A question. On number four we have "equal to F(b) – F(a) what needed to be proved, since that last quantity is a constant", what is the expression that could come before F(b) – F(a)?
139. Ferdinand: The expression is "the integral of f(x) dx, what needed to be proved".
140. Cassim: What is equal?

((Three students point to the blackboard to show the expression))

141. Bernard: Ah, I see!

((Cassim nods his head to show his agreement and claps his hands; then the whole class applauds.))

((The students relax and continue to exchange comments about the course.))
((Bernard tell them to wait (probably for the teacher who is not present in the classroom))

((Teacher enters the classroom and Bernard submits to him the copies from students.))

In this extract from the episode of the type S1-SSL from which I was absent, the understanding (Dreyfus, 1991) of Bernard in line 141 ("Ah, I see!") is triggered by the referential function (the verb of the sentence is in the third person) used by Ferdinand in line 139 ("The expression is the integral of f(x) dx what needed to be proved!") and by the pointing to the chalkboard done by the three student teachers to show the expression asked by Bernard in line 138 ("what is the expression that could come before F(b) –

F(a)?"). In line 140, Cassim also asked a question to the class related to the one Bernard had asked ("what is equal?"). After Cassim's question, the referential function used by Ferdinand was reinforced by the other three student teachers who pointed to the chalkboard to show Bernard and Cassim the expression evoked by Ferdinand. This situation which led to Bernard's understanding is different from the one furthering Edmond's understanding described in Chapter 6, subsection 6. 5. 3 (line 136 of the extract) which was triggered by the conative function (interrogative mood) that I used as I was participating in the situation. In contrary to Edmond's situation, Bernard did not exhibit what he had understood.

Also, the nods of Cassim indicate that he had understood 'something' after the three student teachers had pointed to the chalkboard to show the expression requested by Bernard. But neither he nor Cassim said what they had understood. In the case of Edmond's understanding, what is understood is exhibited whereas in the case of Bernard and Cassim it is not. This can be presumed to be linked to the "triggers" of the understanding: The conative function makes the student exhibit his or her understanding whereas the referential function does not. Therefore, if all things are kept the same, the use of the conative function (the interrogative mood) can be said to be more effective to make student teacher exhibit understanding than the referential function.

7.4. Conclusion

In this chapter, I presented student teachers' understanding of the fundamental theorem of calculus which has been demonstrated during final lesson fifteen in a self-evaluation episode. The student teachers' understanding of the fundamental theorem of calculus has not evolved significantly during the course. Much effort has been engaged to improve the understanding of integrals; however, I did not keep the same trend of efforts for the fundamental theorem of calculus. I think that having combined its learning with the one of the indefinite integral reduced the student teachers' attentiveness to it. Student teachers were too much absorbed to the calculation of the limit of the sum (see some examples in appendix D) and did not notice the importance of differentiating the formula representing the area as an integral with the upper limit variable.

Only two student teachers correctly evoked the fundamental theorem of calculus. In this chapter, I also presented a sequence of cognitive processes that led to Bernard's understanding during interactions about the proof of the fundamental theorem of calculus. During these interactions, Bernard's understanding has been prompted by an interaction using the referential function. This case is different from the case of Edmond's understanding which has been prompted by an interaction using the conative function. In the latter case, what is understood is exhibited whereas in the former case, it is not.

However, this analysis did not allow me to draw very convincing arguments on how student teachers learnt the FTC. Further analysis about the understanding of the FTC will be done later in the development of my research and teaching career.

Chapter 8: CONCLUSIONS AND RECOMMENDATIONS

In this chapter, I present conclusions of and recommendations from my study. The chapter is made up of six sections. In the first section, I present conclusions about the evolution of the student teachers' concept images of the definite and the indefinite integrals. In the second section I present conclusions about the evolution of the student teachers' understanding of the fundamental theorem of calculus. In the third section, I propose a model of teaching and learning integrals and the fundamental theorem of calculus. In the fourth section, I address the relevance and limitations of my study and I provide some reflections on frameworks. In the fifth section, I provide further critical reflections and recommendations on methodology and the theory of didactical situations in mathematics. And finally, in the sixth section, I state some recommendations which can assist in improving the teaching and learning of integrals and the fundamental theorem of calculus in Rwanda.

8.1. Student teachers' evoked concept images of the concepts of the definite and the indefinite integrals and their evolution

As I have shown in chapter 5, the student teachers' concept images of the definite integral evolved as a result of the teaching methods that I used during the teaching process. Table 1 shows the student teachers' concept images of the definite integral evoked during the first round of interviews, held with them before the teaching, while Table 3 shows the student teachers' concept images of the definite integral evoked during the third round of interviews, held with them after the teaching. A comparison of these two tables shows that the student teachers' evoked concept images, which were dominated by the pseudoobject (Sfard, 1991, Sfard & Linchevski, 1994) of the definite integral at the first round of the interviews, evolved during the teaching. In fact, at the third round of the interviews, almost all the student teachers evoked a concept image that included almost all the underlying concepts of the definite integral, namely the concepts of the partition/subdivision, the product, the sum, and the limit. Table 4 summarises the said evolution. It was a good achievement for these student teachers.

With regard to the concept of the indefinite integral, an evolution of the student teachers' concept images was observed since some of them evoked, especially in symbolical representation, some of the underlying concepts included in the formula of the limit of a sum. These types of concept images stem from the fact that during the teaching I had adopted the possibility of teaching indefinite integrals immediately after the definite integrals by computing them as definite integrals over variable intervals as proposed by Koepf and Ben-Israel (1994).

8.2. Student teachers' understanding of the fundamental theorem of calculus

An analysis of the student teachers' understanding of the fundamental theorem of calculus demonstrated during the last lesson of the teaching (Table 9) shows that there have been minor improvements in the student teachers' understanding of the fundamental theorem of calculus. I think that the cause of this weak result may have originated in the fact that I did not organise a specific task related to the understanding of the theorem and the student teachers spent too much effort in calculating the limit of a sum related to indefinite integral. This process reduced their attentiveness to the derivative of the formula representing the area. In fact, during the teaching, I combined in the same task the understanding of the theorem with the understanding of the indefinite integral. This combination might have had some disadvantages to the understanding of the theorem. I think that, when they were working on the task, the student teachers focused more on the indefinite integral than on the fundamental theorem of calculus. If this is the case, adjustments are to be made in the future in order to improve the student teachers' understanding of the fundamental theorem of calculus. Another case to consider in explaining the weak findings observed in the student teachers' understanding of the fundamental theorem of calculus might be the fact that I did not manage to evoke enough of their understanding. Here also adjustments are needed to improve my research approach so that better results are achieved.

8.3. A model for teaching and learning integrals and the fundamental theorem of calculus

According to Maxwell (1992), a theoretical understanding differs from a descriptive and an interpretive understanding by two factors. Firstly, the theoretical understanding represents the degree of abstraction of the account in question from the immediate physical and mental phenomena studied. Moreover, the theoretical understanding explicitly addresses the theoretical constructions that the researcher brings to or develops during the study. Secondly, the theoretical understanding incorporates participants' concepts and theories, but its purpose goes beyond a simple description of these participants' perspectives. The theoretical understanding leads to elaboration of a theory as an explanation, involving a description or an interpretation, of the phenomena being studied. Finally, Maxwell (1992) pointed out that a theory comprises two components, namely, "the concepts or categories that the theory employs, and the relationships that are thought to exist among these concepts" (p. 291).

In the line of this conception of what a theory is, I present the following model of teaching and learning the concepts of the definite and the indefinite integrals and the fundamental theorem of calculus.

As I said in chapter three, my theoretical model extends the triangular didactical relationship (Brousseau, 1997; Jonnaert & Vander Borght, 2003) to a pyramidal didactical relationship (Figure 6). In what follows, I explain my teaching model and I indicate the lessons during which I applied some of the elements of the model.

Before the actual teaching in the classroom occurs, the teacher (T) has to prepare problems or tasks on the basis of appropriate selected cognitive processes (verbs) to be combined with the topics (mathematical objects) to be learned. Knowing appropriate cognitive processes helps in the preparation of problems or tasks that are sources of learning (Douady, 1991). Table 8 in chapter six and Table 10 in chapter seven give categories of such appropriate cognitive processes. Five of these categories, namely the categories of remember, understand, apply, analyse, evaluate, and create, have been described by (Anderson et al., 2001) and I described two other supplementary categories that can help in preparing problems or tasks related to understanding the proofs of the fundamental theorem of calculus. These are the category of validate and of explore. The problems or tasks that are designed to promote learning had to take into account the students' prior concept images of the concepts to be learned so that the students connect the new knowledge to their previous one in a "'healthy' developmental chain" (Sfard & Linchevski, 1994, p. 221). Therefore, problems or tasks that are potential sources of learning (Douady, 1991) are the first material that a teacher has to have in order to use the pyramidal model. In addition to containing the content to be learnt, the task must include intended cognitive processes which will induce the actual cognitive processes that the students will engage in. Also, the teacher has to have in his or her background some concepts from the theory of didactical situations in mathematics (Brousseau, 1997). The following is an elaboration on the implementation of the model.

The actual implementation of the pyramidal model requires a sequence of steps; firstly, the teacher (T) devolved the problem to the whole class (SS). The use of the imperative and the interrogative forms help in making the student teachers accept to start solving the problem. This type of communication is at the level of the group where the teacher addresses the whole class (T-SSd). This phase appeared in each of my lessons in order to devolve the problems or the situations of communication and discussion of solutions as shown in Table 7. This phase is similar to Brousseau's (1997) phase of devolution.

Secondly, the teacher (T) gives time to students (S1) to solve the problem independently. Each student works on the solving of the problem and thinks about the knowledge to be learned. The communication in this episode is at the level of communicating with many absent mathematicians through the task (S1-C). This episode can also be organized in small group (S1-SS1 or SS1-C). The independence in problem solving is sustained by the imperative or the interrogative moods (Rotman, 1988) that the teacher uses while he or she is preparing the written tasks to be devolved to the students. The exclusive imperative mood is more likely to make students endeavour independently. This phase appeared clearly in the second and the third lesson described in chapter six, and in the twelfth and thirteenth lessons described in chapter eight as shown in Table 7. It also appeared in other lessons especially at the beginning of a presentation of a student's solution before any other participant called out to ask a question. This episode corresponds to Brousseau's (1997) adidactical situation.

Thirdly, after some times of independent work, the teacher moves from desk to desk to mediate the problem solving with each student. The teacher uses the vocative or the interrogative forms to establish the contact with the student at his or her desk and to maintain the interactions. In these episodes, the communication is at the level of the interpersonal between the teacher and an individual student (T-S1); the teacher, through responses from the student, can identify whether there is some misinterpretation of the learning activity and then he or she can make necessary adjustments. Here a debate can be raised about the idea that when the teacher is interacting with one student, the others continue working alone. It is true that the other students continue working alone, but the teacher's intentions to teach is no longer hidden. The teacher is trying to help; but as it has been noted in the case of Edmond, if the devolution has taken place, students will not leave the adidactical situation and will break the didactical contract. As Brousseau (1997) has also noted, this breaking of the didactical contract is a good "thing" for learning (see the case of Edmond).

This step was frequent in the second and the third lessons described in chapter six, and in the seventh and the eighth lessons described in chapter eight. Table 7 displays such episodes where the phases of T-S1 occurred. This phase corresponds to the Brousseau's (1997) situation where the teacher is "involved in the game with the system of interaction of the student with the problem she gives her" (p. 31).

Fourthly, when the teacher (T) notices that the students have progressed sufficiently, the teacher invites students (SS) to volunteer to present their solutions. In these episodes, the communication is at the level of the group (T-SS, or T-SSW). However, I classified this phase in the category T-SSd because it was a phase of the devolution of situation of communication and discussion of the student teachers' solutions as I noted previously about Table 7 in chapter 6 (section 6.4.2, p. 237). As it is shown in Table 7, this type of communication appeared in lessons four, five, six, nine, ten, and fourteen. In these episodes, the teacher established the rules that the students would have to follow in order to ask questions to their peer who would be presenting his or her solution when there was something they did not understand and they sought more clarification. In these episodes the teacher used the imperative, the interrogative and the vocative moods to solicit directly the students to accept to present their solutions. Table 7 shows that this phase was frequent in lessons four, five, six, nine, ten, and fourteen.

When a volunteer decides to do his or her presentation, a new phase starts. This phase can include the communicational type of S1-C (presentation of the student's solution before any other participant intervenes) and the communicational type of S1-SST (discussion of the solution after a student or the teacher has asked a question or has made any other intervention). The vocative and the interrogative moods are used by the participants to establish the interaction. In this phase, the teacher can provide assistance if it is required to move the situation forward (Gallimore & Tharp, 1990). During my teaching, it is in an episode of this nature that I took the opportunity to provide one rule of summation, $\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$, and the formula linking the limit of a sum to the definite integral, $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$, in lesson four. Moreover, during this phase, moments of understanding as "a click of the mind" (Dreyfus, 1991) can be observed as in students' learning. In fact, it is in episode of the type S1-SST that Edmond's understanding described above was observed while he was talking about the meaning of the symbol Δx of the partition of the interval $[a, b]$ ("Ah, yes, I see! … It is the minimal difference").

Fifthly, at the end of a certain period of teaching, when the teacher feels that the students have made progress in learning, that is, that they have produced some mathematics that need to be given status (Brousseau, 1997; Douady, 1991), then the teacher facilitates institutionalisation of these students' productions. These episodes constitute the phase of institutionalisation. I classified this episode in the communicational type at the level of the group (T-SSi). As shown in Table 7, during my teaching, the phase of the institutionalisation occured at the end of lessons five, six, ten, and fourteen.

Alternatively, the phase S1-C (adidactical situation) may be replaced by the phase SS1-C where the students solve the problem in collaboration in small groups; in this situation, the communication is of the type S1-SS1. And the phase T-S1 may then be replaced by the phase T-SS1 where the teacher (T) mediates the solving of the problem in small groups. These episodes happened in lessons seven, eight, and thirteen, as shown in Table 7.

In order to further the promotion of the collaboration in groups, the teacher can organise a discussion of students' solution in a large group with the teacher absent. This situation is represented by the communicational mode of S1-SSL. During my teaching, I organised such a situation in lesson twelve. The indicative, interrogative, vocative moods are all used by the students during their discussion. In this phase, moments of understanding as a click of the mind (Dreyfus, 1991), which is a demonstration of learning, can be observed. In fact, it is in this phase, that Bernard's and Cassim's learnings were noticed. The circumstances are described in chapter 7, subsection 7.3.2.

All these episodes occur when 'things' are running smoothly. But this is not always the case. Some times students are not able to solve the devolved problem in an acceptable manner. In this case, the teacher needs to step in, in order to avoid an unplanned rupture of the didactical relationship. When the students are not able to solve the problem, the didactical relationship has to "continue at all costs" (Brousseau, 1997, p. 32). At this moment, the teacher has to intervene in the solving of problem in the whole class mode. I categorised this communicational mode as of the type T-SSW. During my teaching, this case happened in lesson eleven when the students had difficulties with reconstructing the proof of the FTC-VDI. After dialogue between the teacher and the students about the difficulties encountered by the students, we decided to reconstruct the proof together. During the collaboration about the reconstruction of the proof of the FTC-VDI, the students were learning the ways of reconstructing a proof of a given theorem (Mason, 1999). In the next lesson, I organised a similar task related to the reconstruction of the proof of the FTC-VEA. The students first reconstructed the proof individually and, after that, they discussed their solutions in a large group in the communicational mode of S1-SSL. It is in this discussion related to the reconstruction of the proof of the FTC-VEA that the above mentioned learnings of Bernard and of Cassim were observed.

In summing up, the model is based on the verbal interactions between the teacher and the students and among the students themselves. The teacher strives to make the students play an important role in the whole process of teaching and learning. To that, the teacher, firstly, makes the students interact with the tasks independently.

Secondly, after some times of independent work, the teacher mediates the independent work. The ideal should be to listen to the students and help them to connect their new knowledge to the previous one. But here there is a risk that the teacher wants to bring the students see the teacher's mathematics or the formal one which is disconnected from the previous knowledge of the students. In this case, the understanding becomes difficult for the students to achieve. So, listening and embracing students' productions are necessary in order to ensure that students connect new knowledge to the previous one.

Thirdly the teacher allows the students to communicate and discuss the solutions to the task. And finally, the teacher institutionalises some of the students' mathematical productions which have been accepted by the class.

Apart from the episodes of devolution which can be planned before lessons, the other episodes emerge (Meira & Lerman, 2001) when the teacher or any student asks a question (interrogative mood), gives an order (imperative mood), or calls out (vocative mood). The verbal interactions are maintained and nourished when the speech is relayed (Bloomfield, 1939) between the participants. However, the speech relay is not automatic. The teacher must be alert and vigilant in order to identify threats of rupture of the communication and then remove them and assure the continuation of the relay by calling on other students to intervene, by asking questions, or by helping the student who is presenting his or her solution. The emergence and the maintenance of relayed interactions occurred in the lessons when the student teachers had to present and discuss their solutions. Specifically, these relayed interactions emerged and were maintained during the episodes where the communication was of the types S1-SST and S1-SSL. The episodes of the type S1-SST occurred in the fourth, the fifth, the ninth, the tenth, and the fourteenth lessons while the S1-SSL type occurred in the twelfth lesson. These interactions contributed to the development of the student teachers' concept images since it was in these types of interactions that the 'aha' moments of understanding were observed. Edmond's learning was observed in the episode of the type S1-SST during lesson five and the Bernard's and Cassim's learnings were observed in the episode of the type S1-SSL during lesson twelve.

In conclusion, during the teaching and the learning of integrals and the fundamental theorem of calculus, nine kinds of didactical situations are likely to happen. These didactical situations are those in which the communication is of the type T-SSd, S1-C or SS1-C, T-S1 or T-SS1, S1-SST, T-SSW, S1-SSL, and T-SSi. Learnings were observed in didactical situations of the types S1-SST and S1-SSL.

Complementing the communicational organisation, the teacher needs to master the mathematical objects to be learned and taught. Then he or she will be able to recognise and to bring the student's productions closer to the cultural knowledge or to the curricular content during the T-SSi mode of communication. The student teachers' evoked concept images are the results of the guidance and assistance that I and some of the students provided to the student teachers who participated in the communication in classroom.

They are also the results of the commitment of the student teachers to execute directives and to answer questions that were addressed to them.

Also, the teacher needs to select appropriate cognitive processes (Anderson et al., 2001; Bloom et al., 1956) when he or she is preparing learning activities or tasks that are likely to further students' learning and understanding. In order to help the students develop conceptual understanding, that is, to establish relationships between the underlying concepts of the definite integral, students have to be engaged in cognitive processes selected from the categories of understand, apply, analyse, evaluate, and create. With regard to the development of the understanding of the fundamental theorem of calculus, in addition to the cognitive processes already mentioned, it is necessary to adjoin additional processes of the categories of validate and of explore. The category of remember also is useful since it allows students to build new knowledge basing on previous one in order to assure a non-broken developmental chain (Sfard, 1991; Sfard & Linchevski, 1994) of a given mathematical concept. Therefore, in addition to knowing the mathematical concept, knowing which intended cognitive processes to promote when preparing learning activities is an important aspect that is likely to contribute to developing students' understanding of the concepts of the definite and the indefinite integrals and of the fundamental theorem of calculus.

The above-described model helped me to understand how I could move from a teacher-centered style of teaching towards a student-centered style of teaching. In this latter style, students are solicited and allowed to speak and to express their understanding of the concepts under consideration so that they contribute to the creation of opportunities for teaching and for learning (Hewitt, 2005) and consequently they contribute to developing their concept images of the considered concepts.

In this transition from the teacher-centered style of teaching to the student-centered style, the functions of communication played an important role. The conative function which is recognisable by the use of the interrogative, the imperative, and the vocative forms played the essential role of involving the student teachers in the interactions. During these interactions, the student teachers exhibited their understanding (Dreyfus, 1991) and consequently they contributed to the development of their concept images of the topic which was being learned. Other functions of the communication, such as the phatic function contributed to the maintenance of the interactions once established. I used this function to support the student teacher who was working on a problem by listening to him or her and assuring him or her of my presence through empty words such as “eeuh”, “huum”, and “ok”. I also used this phatic function to break the contact in order to leave the student teacher to continue his or her work independently through messages such as “ok carry on!”. In this process, I reduced the use of the expressive and the referential functions when I was communicating with any student teacher. Rather, I increased the use of the conative function (interrogative, imperative, vocative forms) in order to stimulate the student teachers to use the expressive and the referential functions by which they exhibit their understanding of the topic under treatment, as in the case of Edmond. In brief, the model required a great commitment to changing the traditional way of lecturing in order to promote a student-centered way of teaching ideally based essentially on listening to what the students were saying and on encouraging them to speak up and to discuss their propositions. I used the word ‘ideally’ because I experienced many difficulties in maintaining my listening to students. I also know that time and the extent of the content that needs to be covered may result in teachers deviating from this task of listening.

However, the said model has some inconveniences. Firstly, the application of the model is time-consuming. In fact, the time estimated and displayed on the time table was not always respected especially when I wanted each student to reach a certain degree of knowing or when I was striving for closure of the topic. During my teaching, I exceeded the planned time of the time table in lessons four, five, six, and eleven. Some student teachers complained about the exceeding of the time displayed on the time table during the self-evaluation, and when the allocated time was exceeded, some of them seemed not to follow alertly the rest of the lesson. Therefore, it is important to balance the benefit of exceeding the allocated time and if necessary to explain to the students why the lesson is extended; otherwise the teacher’s target will not be reached during this exceeding time.

Finally, another inconvenience associated with the issue of time and the slow pace is the issue of content coverage. The content covered when the pyramidal model is used is relatively less compared to the content covered when lecturing. However, as Dreyfus (1991) has noted when he was discussing the process of learning by discovering, efficiency should be judged “not only in terms of topics covered but also in terms of depth of understanding” (p. 40). Such a debate, as Dreyfus (ibid.) has also noted, raises the issue of how the teacher must balance the emphasis between the two main components of mathematical learning activities, namely the cognitive processes and the mathematical objects, so that students learn the mathematical objects in speedy, meaningful ways. As a contribution to this debate, I propose the pyramidal model as an intermediate solution because it allows latitude to the teacher so that he or she can manage all these issues according to the context in which the model is being implemented.

In conclusion, the pyramidal model allowed me to help the student teachers understand the concepts of the definite and the indefinite integrals and the fundamental theorem of calculus. The didactical situations likely to further students’ understanding were those in which the students interacted in groups among themselves and with the teacher while they were presenting, discussing, and explaining their solutions of problems. Such interactions were of the types S1-SST and S1-SSL.

Moreover, as Dreyfus (1991) has noted, the students understood the concepts after a long sequence of various cognitive processes in which the students were engaged. These cognitive processes were part of learning activities that I had set up in order to help the students to learn. In these learning activities, I had to vary the range of the cognitive processes which should include the processes of the categories of remember, understand, apply, analyse, evaluate, and create as described by Anderson et al. (2001). Other categories of cognitive processes, such as the categories of validate and of explore, emerged from the corresponding circumstances which I decided to create for the students.

Finally, the cognitive processes were framed by a purposeful form of communication. In this regard, I used functions of language with more emphasis on the conative function (vocative, imperative, and interrogative moods). This conative function has the great role of inviting the students to participate actively in solving problems and in discussing their solutions. In addition to establish interactions, the conative function also serves to maintain them during discussion in which a student is requested, by the teacher or by other students, to provide explanation of what he or she has done. In the perspectives of Cobb, Yackel et al. (1992) and of Holton and Thomas (2001), in creating situations in which the students provide some explanation or validation, the teacher increases chances of the students to learn and to understand. Of course, the creation of such conditions depends greatly on the teacher's commitment to involve the students in the process of learning and to move from the teacher-centered style toward the student-centered one. This was my case; I was committed to change my way of teaching and make it more effective in making student teachers understand conceptually the topics under consideration.

At the end of this study, I feel having gained sufficient tools that will allow me to adopt the student-centered style of teaching. However, I need to improve on my listening to students and help them to make connections based on their actual productions and not on my preconceived knowledge. Taking the example of the communication with Edmond, I need to reduce the power of forcing students to see what I have planned and to let them pursue their directions which ends by providing good learning outcomes. This is in the perspective of supporting students who succeed to break the didactical contract and remain in the adidactical situation. In the next section I discuss the relevance of the study and its limitations.

8.4. Relevance and limitations of the study

The study allowed me to learn how to organise a teaching and learning process of the concepts of the definite and the indefinite integrals and of the fundamental theorem of calculus. During that process, I took advantage of the theory of didactical situations in mathematics (Brousseau, 1997) and of the concept of the zone of proximal development (Vygotsky, 1978) to engage the complexity of the process. Moreover, during the study, I learned how to analyse interactions which occurred during that teaching and learning process and which led to the student teachers' understanding (Dreyfus, 1991). In this regard, the categories of cognitive processes as described by Anderson et al. (2001) and the functions of language as described by Jakobson (1981) assisted me during the process of analysing the itinerary that some of the student teachers followed to arrive at the understanding of the concepts that were under consideration. These tools of cognitive processes and of functions of language can be used to explain, to plan, to predict, and to evaluate verbal interactions in any mathematics classroom. In fact a certain sequence of cognitive processes framed by a purposeful use of functions of language is likely to lead to students' understanding of the concept that is being learned.

Moreover, the use of the tools mentioned above contributed to improving my practice as a mathematics teacher and to initiating me to do research in mathematics education. Using these tools, I am able to analyse the cognitive processes which the students were engaged in when they were executing tasks and the functions of language that I mainly used during my teaching. At the same time, I can analyse and identify whether the students participated in their learning and demonstrated their understanding by using functions of language. Thereby, I can determine which didactical situations are likely to further students' understanding of the topic under consideration.

As this study was my first work in relation to analysis of situations to be proposed to students both from an epistemologico-mathematical point of view and from an adidactical-didactical point of view, other works in the same perspective will be undertaken later on to deepen these points of view. With more serious analysis of the situations some of the results will be anticipated, in contrary to what has not been in this study, and this will contribute to improve organisation of more efficient didactical situations.

My study is essentially based on the analysis of the communication during the episodes in which understanding occurred. However, the communication is at the basis of other episodes that emerged during the teaching. Therefore, similar studies can be undertaken in order to analyse the communication in these episodes. The findings of such studies can contribute to producing more indicators to the general behaviours of the teacher and of the students during the whole process of teaching and learning the concepts of integrals and identifying what can be done to reduce the time that I spent during my teaching.

Furthermore, this study is relevant to the mathematics education research community. The framework of mathematical processes and objects that I produced in this study can be used by other researchers to analyse students' concept images at a given time of the teaching of integrals; it can also be used to trace the evolution of the students' concept images through a certain period of teaching a course on integrals.

However, my study has some limitations at this level. The study was set to deeply assess qualitatively the student teachers' concept images of integrals and their evolution. In such a qualitative study, interviews with student teachers or carefully selected tasks can contribute to get insightful data. The researcher has to choose the methods and techniques according to many parameters surrounding the study and make the choice from the outset of the research project. In my case, I choose interviews, especially the semi-structured interviews, that I think offer more freedom for interviewees to express their ideas than tasks would. When students are performing tasks, I think that some activities, such as being forced to write down answers, can limit exhibition of some elements of their concept images. However, if the option of using tasks as a way of collecting additional data on students' concept images of integrals is taken into account from the conception of the study, tasks can be useful and their answers can enrich the analyses and enhance the recommendations of the study. In my study, the option of tasks was not chosen and there were no pre- and post test scores to be compared; the study was explicitly qualitative.

My mathematical framework, together with other analyses of learning integrals, can also be used to evaluate mathematics textbooks dealing with teaching and learning integrals conceptually and supporting student-centred style. A textbook written according to the layers of this framework should include the following sections. The first section can deal with tasks involving the layer of partition; the second with the layer of product; the third with the layer of sum; and the fourth with the layer of the limit of a sum. After these sections, the textbook should then provide the definition of the integral as the limit of a sum. From this stage the textbook can continue with a section of properties of integrals. The properties can be proved using the definition of integral as a limit of a sum. The textbook then can continue with the techniques of integrals and finally provides some applications of integrals. In order to support the student-centred style, the intended cognitive processes should be highlighted at the beginning of each section.

Another relevance of the study to the mathematics education research community and to the planner of teacher education programme concerns the framework of the cognitive processes. The combination of the cognitive processes with the mathematical entities in terms of processes and objects conceptions constitute a strong foundation of conceiving learning activities/tasks that promote learning with understanding instead of rote learning. However, this result is not forcefully reached. Not all the students demonstrate understanding at the same time. Some of them resort to rote learning despite the effort made by the teacher to promote learning with understanding. As reflected in my study some of the students evoked the formulas without providing any explanation. However, the ideal is that once the teacher masters how to combine the cognitive processes and the mathematical entities, then he or she will able to conceive tasks that are more likely to promote learning with understanding rather than rote learning. The framework of cognitive processes is easy to use but it requires the researcher to know a wide range of synonyms in order to be able to classify each verb in the corresponding category of cognitive processes.

The framework of functions of language helped me to analyse my classroom verbal interactions with the students. These functions of language are especially helpful in identifying the extent to which the students are involved in the process of teaching and learning. They serve to differentiate whether the teaching style is teacher-centered or student-centered. They allow the researcher to answer the question whether the teacher indicates to students the knowledge to be learned or whether he or she directs the students to engage in some cognitive processes to discover the knowledge by themselves while he or she only facilitates the involvement of the students.

The approach of analysing the form of the speech that I adopted in my study did not show the content of the mathematics that the students produced *in situ*. However, it showed some of the circumstances in which the learning and understanding occurred. Here, I call upon other researchers to combine the two aspects of the form and the content of the speech to trace, at small scales, the evolution of the students' learning and understanding of the topic under consideration and identify complementary circumstances that can also contribute to further integrals' learning and understanding. Despite these weaknesses, the framework of functions of language is interesting for a classroom teacher who wants to analyse and to detect the effect of his or her language on the students' behaviour during the actual teaching and learning process.

This section on the relevance and limitations of my study provided some reflections of my theoretical frameworks. These reflections will be continued in the next section in which I reflect on action research and the theory of didactical situations in mathematics.

8.5. Further reflections on methodology and the theory of didactical situations in mathematics

In addition to the reflections on theoretical frameworks presented in the preceding section, I give further critical reflections on methodology and the theory of didactical situations in mathematics.

The action research that I adopted during my study helped me to improve my teaching practice and to develop my research skills. I succeeded to move from lecturing to student-centered teaching. This move was essentially due to the students' engagement in the cognitive processes that I embedded in the learning activities. Also, during this study, I gained awareness that the teacher in a mathematics classroom has to have various decisional, organisational and managerial skills and adapt them to the circumstances that emerge during classroom activities and according to the evolution of the knowledge being learned. During my action research, for example, I was expected to decide when to change from the adidactical situation to the interactive one or to invite students to present their solutions to tasks in front of the class.

In addition to improving my practice during my action research, I was expected to produce knowledge about teaching and learning integrals. In this aspect of research, I produced a mathematical framework of teaching and learning integrals; I supported the importance of engaging students in a variety of appropriate cognitive processes when preparing tasks that would be likely to further learning of integrals. I reinforced that the teacher should also adopt an appropriate language – predominantly conative- to make the students work in a sustained independency without frequently resorting to the help of the teacher.

However, the implementation of action research is not without difficulties. The action requires the teacher to be committed to perform the cycle of plan/design – implement/act – observe/evaluate - reflect – re-plan a modified or a changed plan. This commitment engages the teacher in an organised way of working and monitoring and in a vigilant decision-making process. The research aspect requires the teacher-researcher to constantly observe in every detail what is happening in the classroom. However, to coordinate the two aspects of action and research does not succeed all the time. At some occasions, one aspect suffers in favour of the other. In my case, I acknowledge that the action aspect got more attention than the research aspect during my teaching. However, there is not such a clear-cut line between the two aspects because the reflections during the cycle of action are part of the research aspect. So, there is no need to see the two aspects of action research as separate; rather it is necessary to see it as a bridge that links research and action or theory and practice (Raymond & Leinenbach, 2000).

The implementation of the teaching and the collection of data went as planned. However, the analysis of the data was difficult to handle as the collection of data was going along. Most of the time, I was too absorbed in the preparation and the implementation of the teaching and the collection of data. Insufficient time was allocated to the analysis of relationships among events at a regular basis during the collection of data. I waited until the end of the transcription of the classroom communication to do a proper analysis of the events. I think this situation is linked to the paradigm of the study in which I concentrated too much effort on action during the implementation of the teaching and not sufficient time on the research aspect, as I said above.

Concerning action research in education, Zeichner (2001) presented five major traditions of educational action research in English-speaking countries. These traditions are the USA tradition which developed out from the work of Kurt Lewis, the British teacher-as-researcher movement supported by academics such as Laurence Stenhouse and John Elliot, the Australia participatory action research movement supported by the work of Stephen Kemmis and Robin McTaggart, the contemporary teacher researcher movement in North America developed since the 1980s by teachers supported by university colleagues and subject matter associations, and the recent tradition of self-study research by college and university educators who research into their own practice. Zeichner also presented two aspects of viewing educational action research: first as professional development and then as knowledge production.

Seen from the standpoint of professional development, Zeichner stated some conditions under which educational action research led to “a movement towards more learner-centred instruction and improvements of student learning” (p. 279). In my study, I made the same observation that during my teaching, I moved towards the learner-centred teaching style and student teachers’ concept images were improved.

Seen from the standpoint of knowledge production, Zeichner stated that “there is an increased acceptance of action research as a legitimate form of inquiry that can potentially inform practitioners, decision-makers, researchers and teachers educators” (p.

279). However, Zeichner also acknowledged that there had been a controversy about educational action research as legitimate educational inquiry and on the value of the produced knowledge. On this point, the proponents of educational action research need to intensify studies and to publish their findings so that the opponents become convinced of the value of the knowledge produced through action research. Moreover, the proponents of educational need to set up standards for evaluating the produced knowledge in order to avoid that their studies are evaluated according to the standards applied to traditional interpretive research. In so doing, the proponents of educational action research will keep their research paradigm growing and being more accepted.

In this perspective, I summarise the model that I followed during my study for researching my classroom. The model can inspire other interested researchers in classroom action research about teaching and learning integrals. Having in mind the description of a situation in which action research is said to be occurring as provided by Altricher at al. (1991), the characterisation of action research made by Elliot (1991), the pyramidal model that I presented previously, and the critical reflections above about action research, I propose the following model (Figure 30) for researching the teaching and learning of integrals and of the fundamental theorem of calculus.

The first big box of the model represents the teacher’s work at the beginning of a lesson (devolution) and at the end of the lesson or at the end of a sequence of lessons in which the students will have dealt with a certain problem (institutionalisation). The second big box of the model represents episodes of full engagement of the students in the learning process. During these episodes the researcher can observe moments of learning and understanding (Dreyfus, 1991), especially in the episodes of the types S1-SST and S1-SSL in which student are expressing their mathematical ideas. The third big box represents the outcomes of the teaching-learning process in terms of students’ concept images. After the evaluation of these students’ concept images at the end of a cycle, the cycle can restart in order to improve the process.

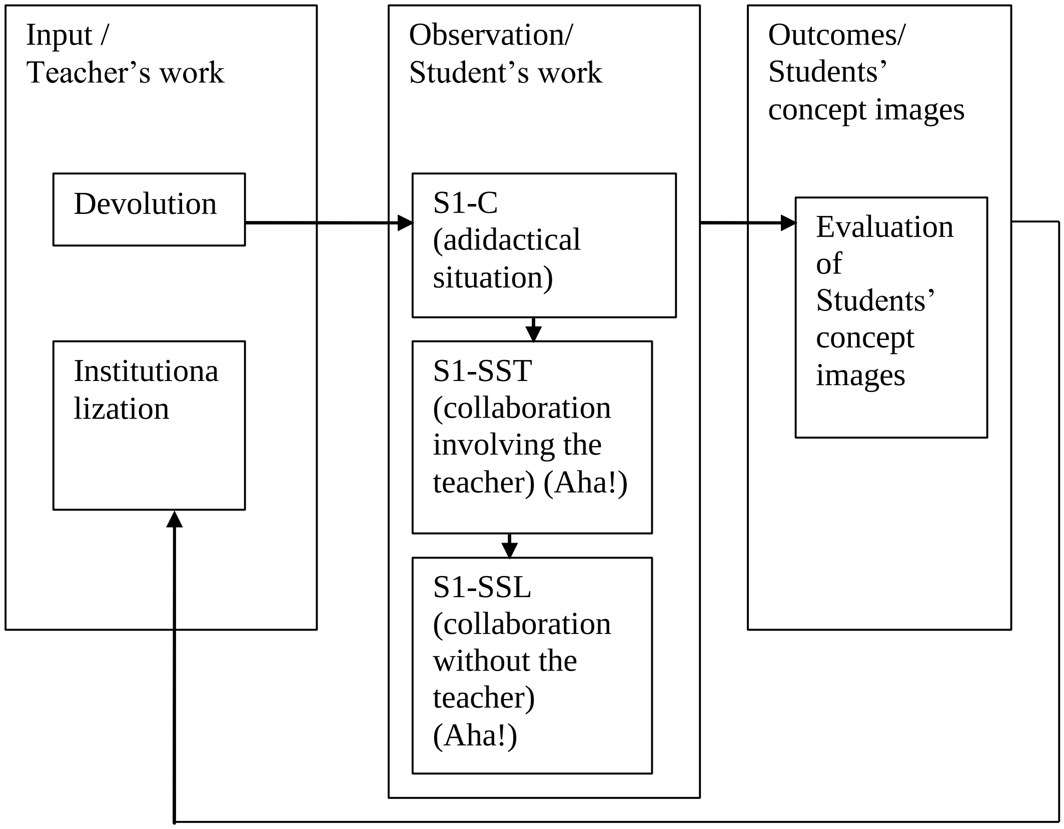


Figure 30. A model of researching the classroom

To end these critical reflections, I provide my point of view on the theory of didactical situations in mathematics conceived by Brousseau (1997). This theory was the cornerstone of my study. In the introductory chapter of this thesis, I said that I undertook this study having in mind the objective of being able to conceive problem-situations that could help students learn integrals with understanding. I said also that after many meetings with my supervisor, she advised me to read this theory. The theory provided me with many concepts such as devolution, adidactical situation, didactical situation, institutionalisation, didactical contract, problem-situation, etc. I tried to adapt these concepts during my teaching. My point of view is that the adaptation was successful. However, the idea of breaking down the didactical contract made me feel sad when I was obliged to stop the communication with a student to make him or her work further. I also noticed that some of the students were frustrated by this rupture of didactical contract. Fortunately, sometimes later, I had to return to the student to resume the communication.

At the long last, the students became used to this breaking down and resuming the didactical contract; but it is worth noticing it caused an inconvenience in the relationship between the teacher and the student which might have had a negative impact in the beginning of the process. Despite this inconvenience, the theory provides concepts which can contribute to improve the process of learning and teaching integrals. Moreover, the breaking down of the didactical contract was essential in the process of developing the student teachers' concept images of integrals. As pointed out by Voigt (1994), using his or her background knowledge of the teacher's supposed emotion coupled with his or her emotion of being frustrated by the breaking down of the didactical contract, the student teacher "might search for more advanced ways to interpret and to solve the problem at hand" (p. 281). Thus, the student teachers' concept images were pushed forward through their own initiatives.

Of course, in order to reach better results, the theory of didactical situations in mathematics needs to be associated with other frameworks such as the cognitive framework for the preparation of effective tasks and problems and the functions of language for conducting a classroom communication which makes the students more active, participatory, and responsible. In particular, better directives for how and when to challenge students would be useful.

8.6. Reflections of the impact of the framework used in the research project and its results

As presented in chapter 3, I developed a mathematical framework made of some underlying layers of the concept of the integral (Figure 3). All my research activities such as the teaching experiment, the collection of data (interviews and classroom communication), the analysis and interpretation of data, and the conclusions and recommendations of the study, were mainly led by this mathematical framework. The use of this predetermined framework had advantages and disadvantages. On the side of advantages, it helped in framing the study in order to maintain its focus and avoid dispersion of efforts in the many complexities of mathematical classroom.

On the side of disadvantages, I acknowledge the risk of the framework becoming limited and not covering various other aspects that might emerge in the classroom or left out consciously or not by the a priori analysis of the concept of the integral. However, since the study was of the nature of an action research in which I only considered one cycle for this thesis, the elements that have not been included in my framework could be considered and be used in an ulterior research project of this kind. Then, the findings presented in this thesis could serve as reference or be extended in this future research project.

Such research project can be very beneficial in the Rwandan context in which my study was also conducted. The present study showed how underlying layers of the integral can be introduced in Rwandan mathematical classrooms; the new action research project can provide some additional deepening ideas for the improvement of students' understanding of the concept of integral. Here, I am thinking about the deepening of the layers of product and sum and the revealing of the epistemological obstacle of heterogeneity of dimensions (Schneider, 1991). In the next section, I present an epistemological reflection on the concept of the integral.

8.7. An epistemological reflection on the concept of integral

The development of the concept of integral

The concept of integral can be traced from the ancient Greek. According to Burn (1999), the initial motivation for most of the early development of integration was the measurement of areas bounded by curves. Integration was used in the calculation of areas of regular or irregular geometrical figures but it was more useful in the latter case because in the former its usefulness could be unperceived.

Burn (ibid.) pursued by saying that at the times of Euclid (300 BC), the areas of squares were seen in the ratio of squares on their sides and this comparison extended to rectangles, parallelograms and triangles and hence to polygons. The problem was to know whether this proportion could extend to circles. Euclid attempted this problem by studying the areas of regular polygons inscribed in and circumscribed about the circle. This reasoning was a precursor of the lower and upper sums that were used in the later development of integration. Archimedes (250 BC) showed that the area of a segment of a parabola was 4/3 times the area of the largest triangle which could be drawn in the segment. This approach was called the quadrature of the parabola because it determined a square with area equal to the required parabolic segment. In this approach, we can trace the limiting argument used Archimedes. In the 17th Century (1636), Fermat and Roberval found areas between the curves $y = x^2$ and $y = x^3$ and the x-axis. They used methods involving subdividing the area into rectangular strips, parallel to the y-axis, each of width a/n and gave a solution referring to the use of the limiting process. Their methods referred to the Greek method of exhaustion. Later on in 1640, while using a geometrical subdivision to find $\int_a^\infty \frac{dx}{x^2}$, Fermat referred to Archimedes' method of inscribed and circumscribed polygons.

In the 18th century, the conditions of integrability of any real function which had been initiated by Newton in 1687, was pursued by Riemann in 1854 and Darboux in 1875 and others followers. In this regard, the Newton's (1687) generalisation of the method of Fermat and Roberval is adequate to establish the integrability of any real monotonic function, f, whether continuous or not, because it establishes the arbitrary closeness of upper and lower sums:

> If the domain of such a function $[a, b]$ is divided into n equal segments, the difference between an upper sum (of areas of minimal circumscribing rectangles) and a lower sum (of areas of maximum inscribed rectangles) is $(f(b) - f(a))\frac{b-a}{n}$. This difference tends to 0 as n tends to infinity, as in the Greek method of exhaustion. The upper sums are greater than the area being

> sought and the lower sums are less than the area being sought. Since the upper and the lower sums are arbitrary close, the integral is well defined. (Newton, as quoted by Burn, 1999, p. 16)

At this stage of the history of integration, we can identify the underlying layers of the concept of integral as described in chapter 3 of this thesis. These are the layers of subdivisions and of sum of partial areas (the layer of product is understood in the calculation of area as a product of width and length); also we can see that the limiting process have been used throughout the developed of integration as long as the geometrical context in which the concept of integral was born is concerned.

The discovery of the fundamental theorem of calculus

In the meantime, as noted by Burn (1999), during the time of Newton, independent methods of computing the slopes of tangents and areas under curves revealed that for polynomials, these two kinds of computation were algebraic inverses of each other. In search for a justification of such situation, Newton in 1666, thanks to his dynamic view of curves formed by a moving point, saw that the rate of change of the area under a curve was given by the y-coordinate at the point of change. In modern notation, Newton's discovery is written as $\frac{dA(x)}{dx} = f(x)$, where $A(x)$ is the "moving" area delimited by the curve $y = f(x)$ and the x-axis.

Also in the modern notation of $\int_a^b f(x)dx$, The integral sign , $\int$, recalls the Leibniz' symbol S for sum in the definition of integral as a limit of sum (as in this thesis) while the d in dx is Leibniz' initial for difference.

Up to this seventeenth century, the concept of integral was led by a geometrical context. Since then, a new era of symbolisation in the treatment of integral started.

From Geometry to Algebra

As reported by Burn (1999), the fundamental theorem was so fruitful that in the eighteenth century, integration became synonymous with anti-differentiation. Integrals were taken to be indefinite integrals (Courant, 1935) and definite integrals were evaluated by finding the difference between two values of an anti-derivative.

Throughout the eighteenth century, because of the power of the fundamental theorem, integration was regarded as the opposite of differentiation. Functions were more considered than curves. This was a manifestation of a shift from geometry to algebra.

The leader in this matter was Cauchy (as quoted in Burn, 1999). In his *Résumé des leçons sur le Calcul infinitesimal* written in 1823, Cauchy said that the first concern was to establish the existence of the integrals before studying their properties. It is reported that in his Leçon 21, he considered a function continuous on a closed interval $[a, b]$ and worked from an arbitrary subdivision of the interval into n pieces: $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$

He then examined the sum

$$S = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2) + \cdots + f(x_{n-1})(x_n - x_{n-1})$$

At the end, he conceived a limit of S as the subdivision becomes finer and the greatest of the $(x_{i+1} - x_i)$ tends to 0. He then established this limit as the definition of the integral. This definition of Cauchy's is a generalisation of the earlier Greek ideas. In Cauchy's definition, we also recognise the layer of subdivision, product and sum in the symbolical representation.

As presented by Burn (1999), Riemann in 1854 examined functions which were defined on a closed interval [a, b], but without continuity as a precondition. Riemann considered an arbitrary subdivision of the interval into n pieces: $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$.

He also examined the sum

$$S = f(t_0)(x_1 - x_0) + f(t_1)(x_2 - x_1) + f(t_2)(x_3 - x_2) + \cdots + f(t_{n-1})(x_n - x_{n-1})$$

where $x_i \leq t_i \leq x_{i+1}$and said that the function f was integrable when S has a limit as $\max(x_{i+1} - x_i) \to 0$.

In Riemann's definition, we also recognise all the four layers underlying the concept of integral as described in chapter 3 of this thesis. The layers of subdivision, product, and sum are expressed in symbolical representation, but the symbol of the limit layer is not explicitly given.

Regarding the uniqueness of the integral, Burn (1999) said that the Darboux sums provide an appropriate method for dealing with it. Darboux said that a function which is bounded on an interval has both a least upper bound and a greatest lower bound on that interval. Symbolically, if $f(x)$ is bound on $[x_i - x_{i+i}]$, it has a least upper bound M_i on that interval and a greatest lower bound m_i, and $m_i(x_{i+1} - x_i) \leq f(t_i)(x_{i-1} - x_i) \leq M_i(x_{i+1} - x_i)$ for all t_i such that $x_i \leq t_i \leq x_{i+1}$. Adding the *n* inequalities related to the parts of S as in the Riemann's definition above, we get

$$\sum_{i=0}^{i=n-1} m_i\,(x_{i+1} - x_i) = a\ lower\ sum \leq S \leq an\ upper\ sum = \sum_{i=0}^{i=n-1} M_i\,(x_{i+1} - x_i)$$

When the subdivision gets finer, the lower sum increases and the upper sum decreases. But any lower sum can be shown to be less than any upper sum by uniting their subdivisions. If the lower sums and the upper sums can get arbitrary close, they define a unique limit, which is the integral. In this definition of integral which uses the Darboux's sums, we recognise also the layers of subdivision, product, sum, and limit. Also, in this definition the symbol of the limit is not explicitly evoked.

As we can notice, the eighteenth century emphasised the mathematical rigour by insisting on the existence and the uniqueness of the integral.

In conclusion, through all the centuries, it is recognisable that the concept of integral has been initiated and defined in terms of the layers of subdivision, product, sum and limit, be it in geometry or in algebra. However, in the seventeenth century, the discovery of the fundamental theorem introduced another way of dealing with integral by considering integration as an anti-differentiation. The definite integral became a tool to evaluate areas using the fundamental theorem by finding the difference between two values of an anti-derivative at the bounds of the interval of integration. At some extent, the fundamental theorem removed the burden of using series in the calculation of definite integrals.

The use of the fundamental theorem is still in force in many educational systems where integrals are part of their curricula. The Rwandan educational system is not an exception to this way of dealing with integrals. My study envisages introducing a change in dealing with integrals in Rwandan educational system, at least in Colleges and Universities in charge of teacher education. I would like that student teachers understand the underlying layers of the concept of integral, by using the context of area in which the concept of integral was born. Also, the mathematical knowledge that I gained during my study, for example the knowledge related to the obstacle of heterogeneity of dimensions (Schneider, 1991), should also be part of topics to be learned in connection with the understanding of integrals, as it can contribute to extension of the student teachers' concept images of integrals.

Until now, in many educational systems that I know, in teaching and learning of integration, the emphasis is put on integrals as anti-derivatives and on definite integrals as a means to evaluate areas by finding the difference between two values of an anti-derivative (see the first part of the fundamental theorem of calculus as described in chapter 3 of this thesis). To introduce the change that I am envisaging, it will require additional efforts and time for the teacher who is eager to introduce the conceptual understanding of integral. It is during this effort of creating relationships among the underlying layers that the epistemological obstacle of heterogeneity of dimensions evoked by Schneider (1991) may occur. So the teacher will have to take this into account if it occurs.

In my study, I did not notice whether the obstacle of heterogeneity of dimensions were evoked by the student teachers (Schneider's paper was published only as I completed the study). What I am sure is that they did not publicly discuss this obstacle and my study was not oriented to that perspective.

Considering all the above mentioned developments of the concepts of the definite and the indefinite integrals and the eminent discovery of the fundamental theorem of calculus, various orientations of teaching integrals have been followed. Some educational systems emphasized the geometrical aspects others emphasized the symbolical development of the indefinite integrals. In all Rwandan schools from grade 12, it is this latter aspect of integral which has been emphasized. My study was about to develop teaching strategies that could further the conceptual understanding of integrals. My mathematical framework in this study emphasized only the basic ideas up to the definition of the concept of integrals. The other mathematical aspects beyond the definition were not catered for in my study and they can be tackled in a direction different from what I had in mind. Also, there is need to develop a new framework or an extended framework of the one that I used in this study. These frameworks may include topics regarding conditions of existence and uniqueness, properties of integrals, techniques of integration and applications of integrals. Such a study could be undertaken later on in the reinforcement of my research and teaching career.

In the next section, I reflect on the journey of my study and the resulting thesis.

8.8. Connection, consistence and complementarity between the theoretical frameworks

As it has been noticed by Cobb, P., Yackel, E., Wood, T., and McNeal, B. (1992), Brousseau's (1997) theory of didactical situations, as described in chapter 3, section 3.3 of this thesis, provides "a theoretical framework that can guide the development of didactical experiments which take account of both the specific mathematical tasks posed to students and the social settings within which students should, ideally, attempt to complete these tasks" (p. 577). In this perspective, the theory of didactical situations can be linked to the zone of proximal development (ZPD).

The definition of the zone of proximal development has been given in chapter 3, section 3.4 of this thesis. The ZPD is "the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers." (Vygotsky, 1978, p. 86)

As it transpires in its definition, the ZPD gives more importance to the learner when it comes to the actual development level. However the adult guidance or the collaboration with more capable peers is also considered as important when it comes to the potential development. Rowlands (2003) provides more precisions about the ZPD and how it should be understood. The ZPD is a concept and method. According to Rowlands (Ibidem) the ZPD is a relational concept between the two concepts actual development and potential development. Rowlands (Ibidem) adds that "Vygotsky (1978, 1987) spoke of that relation as one of instruction or mediation" (p. 160) and saw the ZPD "as an essential method in understanding the dynamics between instruction and development" (p. 161).

In this methodological perspective, Vygotsky (1978) said that his method might be called an "experimental-developmental method. This method calls the experimenter "to intervene in some developmental process in order to observe how such intervention changes it. The primary motivation for doing this is to observe genetic process. The method is called experimental-development in the sense that it artificially provokes a process of psychological development". (Vygotsky, as quoted by Rowlands, 2003, p.163).

This method is based on the principle of double stimulation:

> A problem is set and the child is observed in solving it. Then a ready method is set and the child is observed in applying it. An […] executive instrument is

> constructed from these two observations, thus enabling the researcher to mediate between the task and its fulfillment. (Rowlands, 2003, p. 163)

From the two preceding quotations, a parallel can be drawn between the ZPD and the theory of didactical situation. The episode of "a problem is set and the child is observed in solving it" can be compared to the "adidactical situation" organised by the teacher after he/she has devolved the problem to students. The episode of "Then a ready method is set and the child is observed in applying it" can be related to the "interactive situation or the didactical situation".

The adidactical situation and the didactical situation are intertwined as it has been noted at the end of the section 3.3 in which I presented the theory of didactical situations in mathematics.

The phase "An executive instrument is constructed from these two observations, thus enabling the researcher to mediate between the task and its fulfilment" shows the elaboration of theories of learning after the collection of data during the two observations.

Thus, both the two theoretical frameworks are linked and complement each other and are consistent in educational research setting, especially in classroom context. They are both of them based on experiments in searching and explaining how students learn.

The other theoretical frameworks used in my study, namely, the functions of language and the cognitive processes, served to confection the explanation of how learning occurred during my experiment. So they are also complement to two above mentioned theoretical frameworks.

Ontologically, within the frameworks of the theory of didactical situations and of the ZPD, the reality to be investigated is local and specific constructed by the researcher in order to observe the impact of his/her intervention on the developmental process.

In both the theory of didactical situations and the ZPD as method, the researcher observes the situations constructed in the classroom by the teacher or the teacher-researcher. The teacher or the teacher researcher intervenes in the construction of the reality be investigated. Based on the results observed after the independent solving problem (adidactical situation) the teacher may decide to adjust the task in order to enable the learner to pursue the cognitive development. He/she may also decide to interact with the learner by providing prompts and hints (interactive/didactical situation) that help the learner to further his/her development. Therefore, the reality to be investigated is created and continually recreated by the investigator. This implies that the two frameworks are consistence with the research paradigm of constructivism, which is the paradigm of my study.

On the basis of the observed data, the analysis of the learner's behaviours (cognitive processes and functions of language) in completing the task enables the researcher to provide explanation of the observed development and to construct the "executive instrument" that enables him or her to mediate the task and its fulfilment. In sum, a study with many complexities, in the occurrence the researching of mathematics classroom needs various complementary connected theoretical frameworks in order to handle many of its complexities.

8.9. My Journey to this thesis

The journey to this thesis started in March 2004 as I said in its introductory chapter. As a teacher of mathematics in a Pre-service Teacher Education Institute, my aim was to be able to help mathematics teachers to organize efficient didactical situations in mathematics. Seven years after, my belief is that I am confident in having reached my aim. After reading many documents in the domain of Teaching and Learning Mathematics, I feel I am well equipped with enough tools that can help me in dealing with my job of mathematics teacher educator. Documents about the nature of mathematics (Sfard, 1991), Bloom's taxonomy (Anderson et al., 2001), the concept image and the concept definition (Tall & Vinner, 1981), the theory of didactical situations in mathematics (Brousseau, 1997), the functions of language (Jackobson, 1971) in the classroom, the zone of proximal development (Vygotsky, 1978) and various research methodologies will be at my disposal all along the rest of my teaching and research career.

My journey was characterised by new ‘learnings’. As my background was pure mathematics, I had to work hard to move towards my new domain of mathematics education. During the journey, I needed to integrate the educational aspects in my former mathematics and to see both aspects in any mathematical text that I or my students produced. In this regard, the framework about cognitive processes (Anderson et al., 2001) contributed a lot to make me able to link my mathematics with education. In any mathematical task, I am able to see the educational aspect and the mathematical aspect based on the identification of the intended cognitive process and the nature of the involved mathematics. In students’ productions, in addition to the produced mathematics, I can identify the realized cognitive processes.

Another new and important learning during this study is to be able to see both actions of teaching and learning when I and my students are in the classroom. Actually, before this research, I was in the teacher-centred tradition of teaching, whereas after the study I moved towards a learner-centred style which gives to the students time to play their role in learning. The fact of having been for so long in the teacher-centred tradition caused me difficulty during the first half period of my research because my analysis of the data was focused only on the teacher’s action. It is only later in the second half of the period of my research that I changed the focus and saw also the work done by the students. In addition to the mathematics they produced, I started considering the cognitive processes they realised during their learning. It is only after the inclusion of this focus of learning that the thesis started taking shape.

The rest of the time, since mid 2008, the main activity was the writing of thesis. Here I can mention that I worked hard to write in English, which is my third language. However, with the support of my supervisor and other friends, I achieved this requirement. In ending this section, I signal a last difficulty related to lack of funds which also disturbed me during my journey. By the second semester 2008, my study bursary was over. I had to leave South Africa and continue working from Rwanda, my home country. I continued communicating with my supervisor via the Internet until the end of the study and the writing of this thesis. The study was fruitful because I gained interesting new knowledge which is very helpful for my professional career of mathematics teacher educator and researcher in mathematics education. As have noted Reason and Marshal (2001), the research project was at my heart to sustain me over the several years required.

After the submission of the thesis, I had to wait for the comments of the Examiners. When I received them, my stress was released and I started to do the proposed corrections and revisions. Their comments helped me to improve the quality of my study. Actually, in response to them, I added more critical reflections on the frameworks and their connections and I provided more specific recommendations to mathematics teachers. Their suggestions related to language editing also contributed to make the thesis more understandable. For all of this, I thank all the Examiners and I acknowledge their valuable contributions to this thesis.

In the next section, I conclude this chapter by providing some specific and practical recommendations that can help to improve the teaching and learning of integrals and mathematics in Rwanda.

8.10. Recommendations about teaching and learning integrals and mathematics in Rwanda

The following list provides some practical recommendations related to improvement of teaching and learning of integrals and mathematics in Rwanda.

The following recommendations are addressed to mathematics teachers.

1. The teacher in a mathematics classroom has to have various decisional, organizational and managerial skills and adapt them to the circumstances that emerge during classroom activities and according to the evolution of the knowledge being learned.

2. In addition to including the content to be learnt, the mathematics teacher should prepare a task or a problem that includes intended cognitive processes which will induce

the actual cognitive processes in which the students will be engaged. The intended cognitive processes should aim at the high level cognitive processes; that is, the categories of analyse, evaluate, and create.

3. The mathematics teachers should have in their backgrounds some knowledge of the theory of didactical situations in mathematics in order to manage mathematics classrooms involving consciously planned adidactic situations.

4. In order to maintain interactions during a mathematics classroom, the mathematics teacher should ensure the continuation of the communication among the students by calling out students to intervene, asking questions, or helping by questioning the student who is presenting his or her solution to a task.

5. In a mathematics classroom communication, the mathematics teacher should use the interrogative, the imperative, and the vocative forms, which are language forms that specifically involve the students in interactions or instruct them to perform an activity.

6. The mathematics teachers should use the interrogative or the imperative forms when they are assessing the learning of their students. These forms of language instruct the students to perform action imbedded in the cognitive process intended by teachers when they are preparing the assessment.

7. In order to deal efficiently with the above recommendations, the mathematics teacher should be conversant with Bloom's taxonomy, especially its revised version as presented by Anderson et al. (2001).

The following recommendations are addressed to Kigali Institute of Education which is in charge of Pre-Service Teacher Education in Rwanda.

1. To extend the content of the course entitled Special Methodology or Didactics of Mathematics in order to include the Theory of Didactical Situations (Brousseau, 1997,

2004), the concept of the Zone of Proximal Development (Vygotsky, 1978), and some ideas of Mason (1999).

3. To increase attentiveness of teachers of mathematics to types and levels of communication (Barker, 1982a) and to functions of language in the classroom (Cazden, 2001; Cazden et al., 1972; Jakobson, 1981).

4. To introduce the revised version of Bloom's taxonomy in pre-service teacher education in order to draw the attentiveness to the two components of learning activities: the cognitive processes (verbs) and the mathematical objects (nouns, complements of the verbs) (Anderson, 2005; Anderson et al., 2001).

5. To increase awareness of mathematics teachers to recognise when and how students express their understanding of a given concept (Dreyfus, 1991).

6. To organise an in-service teacher training system that will assist in-service mathematics teachers in the use of the theory of didactical situations and of the concept of the zone of proximal development in their mathematics classrooms.

7. To carry out studies on verbal interactions in different mathematics classrooms in order to propose improvement of the teaching and learning of mathematics.

8. To initiate mathematics student teachers to research methodologies in Mathematics Education.

9. To establish a specific entity of Educational Research in Mathematics, Science, and Technology which will be in charge of research activities in the field of Mathematics, Science, and Technology Education in Rwanda.

10. To create a section of Mathematics and Science Education in Pre-service Teacher Education of Kigali Institute of Education

11. It is encouraging that the Government of Rwanda continues to develop policies that professionalise and retain teachers in the teaching. The Government should limit transfers of teachers to local administration but the Government could make teachers participate in various consultative organs without leaving their profession. This could be another way of valuing the teaching profession and reconstructing the socio-politico-economic system destroyed by the 1994 genocide.

GENERAL CONCLUSION

Developing student teachers' understanding of the concepts of the definite and the indefinite integrals and their link through the fundamental theorem of calculus turned into the setting up of a set of systems of communication during my teaching of these concepts. The systems of communication that I put in place were informed by the theory of didactical situations (Brousseau, 1997, 2004), the concept of zone of proximal development (Vygotsky, 1978), and some specific teaching strategies elaborated by Mason (1999).

In order to analyse the impact of these systems of communication during my teaching, I held two rounds of interviews with the student teachers at different periods of time before the teaching and I held a third round of interviews after the teaching. Also, during the teaching, I requested the student teachers to evaluate their learning after the third lesson and in the fifteenth lesson. The concept images that the student teachers evoked during the interviews and during the third and the fifteenth lessons are presented in details in chapter five and in chapter seven. The analyses showed that student teachers' evoked concept images evolved significantly in the cases of the definite integral (Table 4) and of the indefinite integrals. They evolved from pseudo-object of the definite integral to include almost all the four underlying concepts, namely, the partition, the product, the sum, and the limit of a sum. Concerning the evolution of student teachers' understanding of the fundamental theorem of calculus, the analysis showed that there has been minor evolution (Table 9). All these analyses about concept images of integrals and the understanding of the FTC provided answers to my first research question about the basic ideas underlying the definite and the indefinite integrals and the fundamental theorem of calculus. They also provided answers to my second and my fifth research questions about the student teachers' concept images and their evolution.

In chapter six, I presented a didactical situation in which Edmond exhibited his learning and understanding as described by Dreyfus (1991). Also, in chapter seven I presented a situation in which Bernard exhibited his understanding related to the FTC. These learning and understandings occurred after students' involvement in a long sequence of various cognitive processes framed by a purposeful set of functions of language. The cognitive processes associated with the mathematical objects were key components in conceiving learning activities that were likely to promote learning and understanding. This observation answers my fourth research question about learning activities which the student teachers engaged in when working on integrals.

Basing on the evoked concept images and on the situations in which Edmond and Bernard exhibited their learning and understandings, I can say that the teaching methods that I used were effective in promoting student teachers' development and understanding of the concepts that I was teaching to them. The didactical situations in which the communication is of the types S1-SST and S1-SSL are likely to produce observable student teachers' learning and understandings of the mathematical concept that is being learnt. These cases answer my third research question about didactical situations that are likely to further understanding and my fourth research question about the circumstances under which understanding occurs.

In conclusion, the tools that I used for organising my teaching are recommendable to a teacher who would like to change from a teacher-centred teaching style to the student-centred teaching style in order to help students learn with understanding. Also, the analytical tools from famous authors such as (Dreyfus, 1991; Jakobson, 1981; Sfard, 1991; Sfard & Linchevski, 1994; Tall & Vinner, 1981) and the mathematical framework that I developed in this thesis to analyse the effectiveness of my teaching can be used to analyse a teaching-learning process about the concepts of the definite and the indefinite integrals and of the fundamental theorem of calculus if the researcher has decided to use similar methods to the ones that I used. Finally, the recommendations that I made in this study can help in improving the teaching and learning of integrals in Rwanda if conditions are similar to those in my study.

REFERENCES

Adler, J. (2001). *Teaching Mathematics in Multilingual Classrooms*. Dordrecht: Kluwer Academic Publishers.

Alrø, H., & Skovsmose, O. (2002). *Dialogue and Learning in Mathematics Education: Intention, Reflection, Critique*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Altrichter, H., Kemmis, S., McTaggart, R., & Zuber-Skerritt, O. (1991). Defining, Confining or Refining Action Research? In O. Zuber-Skerritt (Ed.), *Action Research for Change and Development* (pp. 3-9). Aldershot: Avebury.

Andersen, P., & Andersen, J. (1982). Nonverbal Immediacy in Instruction. In L. L. Barker (Ed.), *Communication in the Classroom* (pp. 98-120). Englewood Cliffs, N.J.: Prentice-Hall.

Anderson, L. W. (2005). Objectives, Evaluation, and the Improvement of Education. *Studies in Educational Evaluation, 31*, 102-113.

Anderson, L. W., Krathwohl, D. R., Airasian, P. W., Cruikshank, K. A., Mayer, R. E., Pintrich, P. R., et al. (Eds.). (2001). *A Taxonomy for Learning, Teaching, and Assessing: A Revision of Bloom's Taxonomy of Educational Objectives* (Abridged ed.). New York: Addison Wesley Longman.

Artigue, M. (1991). Analysis. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 167-198). Dordrecht: Kluwer Academic Publishers.

Artigue, M. (1994). Didactical Engineering as a Framework for the Conception of Teaching Products. In R. Biehler, R. W. Scholz, R. Sträßer & B. Winkelmann (Eds.), *Didactics of Mathematics as a Scientific Discipline* (pp. 27-39). Dordrecht: Kluwer Academic Publishers.

Artigue, M. (1996). *Teaching and Learning Elementary Analysis.* Paper presented at the 8th International Congress on Mathematical Education, Sevilla, Spain.

Aspinwal, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrallable Mental Imagery: Graphical Connections between a Function and its Derivative. *Educational Studies in Mathematics, 33*, 301-317.

Austin, J. L., & Howson, A. G. (1979). Language and Mathematical Education. *Educational Studies in Mathematics, 10*, 161-197.

Baril, D., & Guillet, J. (1996). *Techniques d'Expression Ecrite et Orale*. Paris: Dalloz.

Barker, L. L. (1982a). An Introduction to Classroom Communication. In L. L. Barker (Ed.), *Communication in the Classroom* (pp. 1-15). Englewood Cliffs, N.J.: Prentice-Hall.

Barker, L. L. (Ed.). (1982b). *Communication in the Classroom*. Englewood Cliffs, N.J.: Prentice-Hall.

Bartolini Bussi, M. G. (1994). Theoretical and Empirical Approaches to Classroom Interaction. In R. Biehler, R. W. Scholz, R. Sträßer & B. Winkelmann (Eds.), *Didactics of Mathematics as a Scientific Discipline* (pp. 121-132). Dordrecht: Kluwer Academic Publishers.

Bartolini Bussi, M. G. (2005). When Classroom Situation is the Unit of Analysis: The Potential Impact on Research in Mathematics Education. *Educational Studies in Mathematics, 59*, 299-311.

Bauersfeld, H. (1992). Classroom Cultures from a Social Constructivist's Perspective. *Educational Studies in Mathematics, 23*(5), 467-481.

Bauersfeld, H. (1994). Theoretical Perspectives on Interaction in the Mathematics Classroom. In R. Biehler, R. W. Scholz, R. Sträßer & B. Winkelmann (Eds.), *Didactics of Mathematics as a Scientific Discipline* (pp. 133-146). Dordrecht: Kluwer Academic Publishers.

Bellack, A. A., Kliebard, H. M., Hyman, R. T., & Smith, F. L. (1966). *The Language of the Classroom*. Columbia: Teachers College Press.

Berger, M. (2006). Making Mathematical Meaning: From Preconcepts to Pseudoconcepts to Concepts. *Pythagoras, 63*, 14-21.

Bezuidenhout, J. (2001). Limits and Continuity: Some Conceptions of First Year Students. *International Journal of Mathematical Education in Science and Technology, 32*(4), 487-500.

Bezuidenhout, J., Human, P., & Olivier, A. (1998). *Some Misconceptions Underlying First Year Students' Understanding of Average Rate and Average Value*. Paper presented at the 22nd conference of international Group for the Psychology of Mathematics Education, Stellenbosch, South Africa.

Bloom, B. S. (Ed.), Engelhart, M. D., Furst, E. J., Hill, W. H., & Krathwohl, D. R. (1956). *Taxonomy of Educational Objectives: Handbook 1: Cognitive Domain*. New York: David McKay Company.

Bloomfield, L. (1935). *Language*. London: Allen & Unwin.

Bloomfield, L. (1939). Linguistic Aspects of Science. *International Encyclopedia of Unified Science, 1*(4), 1-59.

Boston, M. D., & Smith, M. S. (2009). Transforming Secondary Mathematics Teaching: Increasing the Cognitive Demands of Instructional Tasks used in Teachers' Classrooms. *Journal for Research in Mathematics Education, 40*(2), 119-156.Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics, 23*, 247-285.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics* (Nicolas Balacheff, Martin Cooper, Rosamund Sutherland & Virginia Warfield, Trans.). Dordrecht: Kluwer Academic Publishers.

Brousseau, G. (2004). *Théorie des Situations Didactiques.* Dijon-Quétigny: La pensée sauvage.

Brousseau, G., & Gibel, P. (2005). Didactical Handling of Students' Reasoning Processes in Problem Solving Situations. *Educational Studies in Mathematics, 59*, 13-58.Burn, R. P. (1999). Integration, a Genetic Introduction. *Nordisk matematikkdidaktikk, 1*, 7-27.

Burt, D., Magnes, J., Schwarz, G., & Hartke, J. (2008). Discovering Integration Through a Physical Phenomenon. *Primus, 18*(3), 283-290.

Cameron-Jones, M., & Morrison, A. (1973). Teachers' Assessments of their Pupils. In G. Chanan (Ed.), *Towards a Science of Teaching* (pp. 26-59). Windsor, Berks: National Foundation for Educational Research.

Cauchy, A. L. (1823). *Résumé des Leçons sur le Calcul Infinitésimal*. Paris : Ellipses

Cazden, C. B. (2001). *Classroom Discourse: The Language of Teaching and Learning* (2nd ed.). Portsmouth: Heinemann.

Cazden, C. B., John, V. P., & Hymes, D. (Eds.). (1972). *Functions of Language in the Classroom*. New York: Teachers College Press.

Cegala, D. J. (1982). The Role and Assessment of Prerequisite Behaviors in Communication Instruction. In L. L. Barker (Ed.), *Communication in the Classroom* (pp. 16-38). Englewood Cliffs, N.J.: Prentice-Hall.

Charles, R., & Williame, C. (1988). *La Communication Orale*. Paris: Nathan.

Christiansen, I. M. (1996). *Mathematical Modelling in High School: From Idea to Practice.* Revised Doctoral Dissertation, Aalborg University, Denmark.

Christiansen, I. M. (1997). When Negotiation of Meaning is also Negotiation of Task. *Educational Studies in Mathematics, 34*, 1-25.

Cobb, P. (1995). Where is the Mind? A Coordination of Sociocultural and Cognitive Constructivist Perspectives. In C. T. Fosnot (Ed.), *Constructivism: Theory, Perspectives, and Practice* (2nd ed., pp. 39-57). New York: Teachers College, Columbia University.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of Classroom Mathematics Traditions: An Interactional Analysis. *American Educational Research Journal*, *29*(3), 573-604.

Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and Learning in Mathematics Classroom Situations. *Educational Studies in Mathematics, 23*, 99-122.

Cohen, L., Manion, L., & Morrison, K. (2000). *Research Methods in Education* (5th ed.). London: Routledge Falmer.

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153-166). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Courant, R. (1937). *Differential and Integral Calculus* (2nd ed.). London: Blackie & Son.

De Landsheere, G. (1973). Analysis of Verbal Interaction in the Classroom. In G. Chanan (Ed.), *Towards a Science of Teaching* (pp. 60-84). Windsor, Berks: National Foundation for Educational Research.

Douady, R. (1991). Tool, Object, Setting, Windows: Elements for Analysing and Constructing Didactical Situations in Mathematics. In A. J. Bishop, S. Mellin-Olsen & J. Van Dormolen (Eds.), *Mathematical Knowledge: Its Growth Through Teaching* (pp. 109-130). Dordrecht: Kluwer Academic Publishers.

Douek, N. (2005). Communication in the Mathematics Classroom: Argumentation and Development of Mathematical Knowledge. In A. Chronaki & I. M. Christiansen (Eds.), *Challenging Perspectives on Mathematics Classroom Communication* (pp. 145-172). Greenwich: Information Age Publishing.

Dreyfus, T. (1991). Advanced Mathematical Thinking Processes. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 25-41). Dordrecht: Kluwer Academic Publishers.

Dubinsky, E., Weller, K., Mcdonald, M. A., & Brown, A. (2005a). Some Historical Issues and Paradoxes regarding the Concept of Infinity: An APOS-based Analysis: Part 1. *Educational Studies in Mathematics, 58*, 335-359.

Dubinsky, E., Weller, K., Mcdonald, M. A., & Brown, A. (2005b). Some Historical Issues and Paradoxes regarding the Concept of Infinity: An APOS Analysis: Part 2. *Educational Studies in Mathematics, 60*, 253-266.

Elliot, J. (1991). *Action Research for Educational Change*. Buckingham: Open University Press.

Ernest, P. (1991). *The philosophy of Mathematics Education*. London: The Falmer Press.
Ervin-Tripp, S. (1972). On Sociolinguistic Rules: Alternation and Co-occurence. In J. J. Gumperz & D. Hymes (Eds.), *Directions in Sociolinguistics: The Ethnography of Communication* (pp. 213-250). New York: Holt, Rinehart and Winston.
Feldman, A., & Minstrell, J. (2000). Action Research as a Research Methodology for the Study of the Teaching and Learning of Science. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 429-455). London: Lawrence Erlbaum Associates.
Ferrini-Mundy, J., & Gaudard, M. (1992). Secondary School Calculus: Preparation or Pitfall in the Study of College Calculus. *Journal for Research in Mathematics Education, 23*(1), 56-71.
Flückiger, A. (2005). Macro-Situation and Numerical Knowledge Building: The Role of Pupils' Didactic Memory in Classroom Interations. *Educational Studies in Mathematics, 59*, 59-84.
Frankenstein, M. (1987). Critical Mathematics Education: An Application of Paulo Freire's Epistemology. In I. Shor (Ed.), *Freire for the Classroom: A Sourcebook for Liberatory Teaching* (pp. 180-210). Portsmouth: Boynton/Cook Publishers.
Freebody, P. (2003). *Qualitative Research in Education: Interaction and Practice*. London: SAGE Publications.
Freimuth, V. S. (1982). Communication Apprehension in the Classroom. In L. L. Barker (Ed.), *Communication in the Classroom* (pp. 121-145). Englewood Cliffs, N.J.: Prentice-Hall.
Friedrich, G. W. (1982). Classroom Interaction. In L. L. Barker (Ed.), *Communication in the Classroom* (pp. 55-76). Englewood, Cliffs, N.J.: Prentice-Hall.
Gallimore, R., & Tharp, R. (1990). Teaching Mind in Society: Teaching, Schooling, and Literate Discourse. In L. C. Moll (Ed.), *Vygotsky and Education: Instructional Implications and Applications of Sociohistorical Psychology* (pp. 175-205). New York: Cambridge University Press.
Guba, E. G., & Lincoln, Y. S. (1994). Competing Paradigms in Qualitative Research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of Qualitative Research*. Thousand Oaks: SAGE Publications.
Gumperz, J. J. (1972). Introduction. In J. J. Gumperz & D. Hymes (Eds.), *Directions in Sociolinguistics: The Ethnography of Communication* (pp. 1-25). New York: Holt, Rinehart and Winston.
Gumperz, J. J., & Hymes, D. (Eds.). (1972). *Directions in Sociolinguistics: The Ethnography of Communication*. New York: Holt, Rinehart and Winston.
Hattie, J. (2003). *Teachers make a Difference: What is the research evidence?* Paper presented at the Australian Council for Educational Research Annual Conference on Building Teacher Quality.
Henning, E. (2004). *Finding Your Way in Qualitative Research*. Pretoria: Van Schaik Publishers.
Hersant, M., & Perrin-Glorian, M.-J. (2005). Characterization of an Ordinary Teaching Practice with the Help of the Theory of Didactic Situations. *Educational Studies in Mathematics, 59*, 113-151.
Hewitt, D. (2005). Conflicts and Harmonies among Different Aspects of Mathematical Activity. In A. Chronaki & I. M. Christiansen (Eds.), *Challenging Perspectives on*

Mathematics Classroom Communication (pp. 205-233). Greenwich: Information Age Publishing.

Holton, D., & Thomas, G. (2001). Mathematical Interactions and their Influence on Learning. In D. Clarke (Ed.), *Perspectives on Practice and Meaning in Mathematics and Science Classrooms* (pp. 75-104). Dordrecht: Kluwer Academic Publishers.

Horner, V. M., & Gussow, J. D. (1972). John and Marry: A Pilot Study in Linguisitic Ecology. In C. B. Cazden, V. P. John & D. Hymes (Eds.), *Functions of Language in the Classroom* (pp. 155-194). New York: Teachers College Press.

Hymes, D. (1972). Models of the Interaction of Language and Social Life. In J. J. Gumperz & D. Hymes (Eds.), *Directions in Sociolinguistics: The Ethnography of Communication* (pp. 35-71). New York: Holt, Rinehart and Winston.

Jakobson, R. (1971). Shifters, Verbal Categories, and the Russian Verb. In S. Rudy (Ed.), *Roman Jakobson: Selected Writings Volume II: Word and Language* (pp. 130-147). The Hague: Mouton.

Jakobson, R. (1981). Linguistics and Poetics. In S. Rudy (Ed.), *Roman Jakobson: Selected Writings Volume III: Poetry of Grammar and Grammar of Poetry* (pp. 18-51). The Hague: Mouton.

Jonnaert, P., & Vander Borght, C. (2003). *Créer des Conditions d'Apprentissage: Un Cadre de Référence Socioconstructiviste pour une Formation didactique des Enseignants* (2nd ed.). Bruxelles: De Boeck & Larcier.

Juter, K. (2009). Development of Students' Concept Images in Analysis. *Nordic Studies in Mathematics Education, 14*(4), 65-87.

Kemmis, S. (1988). Action Research. In I. J. Keeves (Ed.), *Educational Research, Methodology, and Measurment: An International Handbook* (pp. 49-53). Oxford: Pergamon.

Key Curriculum Press. (2002). *Exploring Calculus with the Geometer's SketchPad (Compact Disc)*. Emeryville, California: Key Curriculum Press.

Kieran, C. (2001). The Mathematical Discourse of 13-Year-Old Partnered Problem Solving and its Relation to the Mathematics that Emerges. *Educational Studies in Mathematics*, *46*(1), 187-228.

Kigali Institute of Education. (2000). *Courses for Integrated Science Department*. Kigali, Rwanda: Department of Integrated Science Education.

Koepf, W., & Ben-Israel, A. (1994). The Definite Nature of Indefinite Integrals. *The International DERIVE Journal*, *1*(1), 115-131.

Laborde, C. (1994). Working in Small Groups: A Learning Situation? In R. Biehler, R. W. Scholz, R. Sträßer & B. Winkelmann (Eds.), *Didactics of Mathematics as a Scientific Discipline*, (pp. 147-158). Dordrecht: Kluwer Academic Publishers.

Laborde, C., & Perrin-Glorian, M.-J. (2005). Introduction: Teaching Situations as Object of Research: Empirical Studies within Theoretical Perspectives. *Educational Studies in Mathematics*, *59*, 1-12.

Lankshear, C., & Knobel, M. (2004). *A Handbook for Teacher Research: From Design to Implementation*. Berkshire: Open University Press.

Lerman, S. (1996). Intersubjectivity in Mathematics Learning: A Challenge to the Radical Constructivist Paradigm? *Journal for Research in Mathematics Education*, *27*(2), 133-150.

Lerman, S. (2000). A Case of Interpretations of *Social*: A response to Steffe and Thompson. *Journal for Research in Mathematics Education, 31*(2), 210-227.

Lerman, S. (2001a). Accounting for Accounts of Learning Mathematics: Reading the ZPD in Videos and Transcripts. In D. Clarke (Ed.), *Perspectives on Practice and Meaning in Mathematics and Science Classrooms* (pp. 53-71). Dordrecht: Kluwer Academic Publishers.

Lerman, S. (2001b). Cultural, Discursive Psychology: A Sociocultural Approach to Studying the Teaching and Learning of Mathematics. *Educational Studies in Mathematics, 46*, 87-113.

Lerman, S. (2005). The Linguistic Turn in Mathematics Education Research. In A. Chronaki & I. M. Christiansen (Eds.), *Challenging Perspectives on Mathematics Classroom Communication* (pp. 349-357). Greenwich: Information Age Publishing.

Lewis, C. T. (1891; Reprint 1979). *An Elementary Latin Dictionary*. Oxford: Oxford University Press.

Lincoln, Y. S. (2001). Engaging Sympathies: Relationships between Action Research and Social Constructivism. In P. Reason & H. Bradbury (Eds.), *Handbook of Action Research: Participative Inquiry & Practice* (pp. 124-132). London: Sage Publications.

Malamah-Thomas, A. (1987). *Classroom Interaction*. Oxford: Oxford University Press.

Mason, J. H. (1999). *Teaching Mathematics to College Students*.Unpublished manuscript, Milton Keynes, UK.

Maxwell, J. A. (1992). Understanding and Validity in Qualitative Research. *Harvard Educational Review*, *62*(3), 279-300.

McNiff, J. (2000). *Action Research in Organisations*. London: Routledge.

McNiff, J. (2002). *Action Research: Principles and Practice* (2nd ed.). London: Routledge Falmer.

McNiff, J., Lomax, P., & Whitehead, J. (1996). *You and Your Action Research Project*. London: Routledge.

McNiff , J., & Whitehead, J. (2005). *Action Research for Teachers: A Practical Guide*. London: David Fulton Publishers.

Meaney, T. (2005). Mathematics as Text. In A. Chronaki & I. M. Christiansen (Eds.), *Challenging Perspectives on Mathematics Classroom Communication* (pp. 109-141). Greenwich: Information Age Publishing.

Meira, L., & Lerman, S. (2001). *The Zone of Proximal Development as a Symbolic Space* (Social Science Research Papers No. 13). London: South Bank University.

Metaxas, N. (2007). Difficulties on Understanding the indefinite Integral. *The Korea Society of Educational Studies in Mathematics, Part 3*, 265-272.

Ministry of Education. (1999). *Mathematics Programme for Advanced Level: Bio-Chemistry Section*. Kigali, Rwanda: National Curriculum Development Centre.

Ministry of Education. (2000). *Mathematics Programme for Advanced Level: Teacher Training Section*. Kigali, Rwanda: National Curriculum Development Centre.

Moll, L. C. (1990). Introduction. In L. C. Moll (Ed.), *Vygotsky and Education: Instructional Implications and Applications of Sociohistorical Psychology* (pp. 1-27). Cambridge: Cambridge University Press.

Mouton, J. (2001). *How to Succeed in Your Master's & Doctoral Studies: A South African Guide and Resource Book*. Pretoria: Van Schaik Publishers.

Neuman, W. L. (2006). *Social Research Methods: Qualitative and Quantitative Approaches* (6th ed.). Boston: Pearson Education, Inc.

Nuthall, G., & Church, J. (1973). Experimental Studies of Teaching Behaviour. In G. Chanan (Ed.), *Towards a Science of Teaching* (pp. 9-25). Windsor, Berks: National Foundation for Educational Research.

Orton, A. (1983a). Students' Understanding of Differentiation. *Educational Studies in Mathematics*, *14*, 235-250.

Orton, A. (1983b). Students' Understanding of Integration. *Educational Studies in Mathematics*, *14*, 1-18.

Pimm, D. (1994). Mathematics Classroom Language: Form, Function and Force. In R. Biehler, R. W. Scholz, R. Sträßer & B. Winkelmann (Eds.), *Didactics of Mathematics as a Scientific Discipline* (pp. 159-169). Dordrecht: Kluwer Academic Publishers.

Reason, P., & Marshall, J. (2001). On Working with Graduate Research Students. In P. Reason & H. Bradbury (Eds.), *Handbook of Action Research: Participative Inquiry & Practice* (pp. 413-419). London: SAGE Publications.

Rösken, B., & Rolka, K. (2007). Integrating Intuition: The Role of Concept Image and Concept Definition for Students' Learning of Integral Calculus. *The Montana Mathematics Enthousiast, Monograph 3*, 181-204.

Rotman, B. (1988). Toward a Semiotics of Mathematics. *Semiotica, 72*(1/2), 1-35.

Rowlands, S. (2003). Vygotsky and the ZPD: Have We Got it Right? In S. Pope & O. Mcnamara (Eds.), *Research in Mathematics Education* (Vol. 5, pp. 155-170). London: British Society for Research into Learning Mathematics.

Ryve, A. (2006). Making Explicit the Analysis of Students' Mathematical Discourses - Revisiting a Newly Developed Methodological Framework. *Educational Studies in Mathematics, 62*, 191-209.

Schneider, M. (1991). Un obstacle épistémologique soulevé par des "découpages infinis" des surfaces et des solides. *Recherches en Didactiques des Mathématiques, 11*(2.3), 241-294.

Sensivy, G., Schubauer-Leoni, M.-L., Mercier, A., Ligozat, F., & Perrot, G. (2005). An Attempt to Model the Teacher's Action in the Mathematics Class. *Educational Studies in Mathematics*, *59*, 153-181.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the same Coin. *Educational Studies in Mathematics, 22*, 1-36.

Sfard, A. (2001). There is More to Discourse Than Meets the Ears: Looking at Thinking as Communicating to Learn More about Mathematical Learning. *Educational Studies in Mathematics*, *46*(1/3), 13-57.

Sfard, A., & Linchevski, L. (1994). The Gains and the Pitfalls of Reification: The Case of Algebra. *Educational Studies in Mathematics*, *26*, 191-228.

Shor, I. (Ed.). (1987). *Freire for the Classroom: A Sourcebook for Liberatory Teaching*. Portsmouth: Boynton/Cook Publishers.

Sierpinska, A. (1987). Humanities Students and Epistemological Obstacles related to Limits. *Educational Studies in Mathematics*, *18*, 371-397.

Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education, 26*, 114-145.

Smith, R. T., & Minton, R. B. (2002). *Calculus: Single Variable* (2nd ed.). New York: McGraw-Hill.

Steffe, L. P., & Thompson, P. W. (2000a). Interaction or Intersubjectivity? A reply to Lerman. *Journal for Research in Mathematics Education, 31*(2), 191-209.

Steffe, L. P., & Thompson, P. W. (2000b). Teaching Experiment Methodology: Underlying Principles and Essential Elements. In A. E. Kelley & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 267-306). London: Lawrence Erlbaum Associates.

Stewart, J. (1998). *Calculus: Concepts and Contexts*. Pacific Grove: Brooks/Cole Publishing Company.

Sutter, W. N. (2006). *Introduction to Educational Research: A Critical Thinking Approach*. Thousand Oaks: SAGE Publications.

Tall, D. (1992). *Students' Difficulties in Calculus.* Paper presented at the Plenary Presentation in Working Group 3, ICME, Québec, Canada.

Tall, D. (1996). Functions and Calculus. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International Handbook of Mathematics Education* (Vol. 4, pp. 289-325). Dordrecht: Kluwer Academic Publishers.

Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Reference to Limits and Continuity. *Educational Studies in Mathematics, 12*, 151-169.

Tarvainen, K. (2008). Justifying Differential Derivations when Setting up Definite Integrals. *International Journal of Mathematical Education in Science and Technology, 39*(1), 61-68.

Thompson, P. W. (1994). Images of Rate and Operational Understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*, 229-274.

Trowbridge, N. (1973). Teacher Self-Concept and Teaching Style. In G. Chanan (Ed.), *Towards a Science of Teaching* (pp. 135-141). Windsor, Berks: National Foundation for Educational Research.

Valsiner, J., & van der Veer, R. (1993). The Encoding of Distance: The Concept of the Zone of Proximal Development and its Interpretations. In R. R. Cocking & K. A. Renninger (Eds.), *The Development and Meaning of Psychological Distance* (pp. 35-62). Hillsdale, NJ: Lawrence Erlbaum Associates.

van der Veer, R., & Valsiner, J. (1991). *Understanding Vygotsky: A Quest for Synthesis*. Oxford: Blackwell.

Vinner, S. (1975). The Naive Platonic Approach as a Teaching Strategy in Arithmetics. *Educational Studies in Mathematics, 6*, 339-350.

Vinner, S. (1983). Concept Definition, Concept Image and the Notion of Function. *International Journal of Mathematical Education in Science and Technology 14*(3), 239-305.

Vinner, S., & Dreyfus, T. (1989). Images and Definitions for the Concept of Function. *Journal for Research in Mathematics Education, 20*(4), 356-366.

Vithal, R. (2003). *In Search of a Pedagogy of Conflict and Dialogue for Mathematics Education*. Dordrecht: Kuwer Academic Publishers.

Voigt, J. (1994). Negotiation of Mathematical Meaning and Learning Mathematics. *Educational Studies in Mathematics, 26*, 275-298.

Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, Ma: Harvard University Press.

Vygotsky, L. S. (1987). Thinking and Speech. In R. W. Rieber & A. S. Carton (Eds.), *The Collected Works of L.S. Vygotsky. Vol 1: Problems of General Psychology* (pp. 39-285). New York: Plenum Press.

Wallerstein, N. (1987). Problem-Posing Education: Freire's Method for Transformation. In I. Shor (Ed.), *Freire for the Classroom: A sourcebook for Liberatory Teaching* (pp. 33-44). Portsmouth: Boynton/Cook Publishers.

Wells, G. (1999). *Dialogic Inquiry: Toward a Sociocultural Practice and Theory of Education*. Cambridge: Cambridge University Press.

Wertsch, J. V., & Bivens, J. A. (1993). The Social Origins of Individual Mental Functioning: Alternatives and Perspectives. In R. R. Cocking & K. A. Renninger (Eds.), *The Development and Meaning of Psychological Distance* (pp. 203-218). Hillsdale, NJ: Lawrence Erlbaum Associates.

White, P., & Mitchelmore, M. (1996). Conceptual Knowledge in Introductory Calculus. *Journal for Research in Mathematics Education, 27*, 79-95.

Winsløw, C. (2004). Semiotics as an Analytic Tool for the Didactics of Mathematics. *Nordic Studies in Mathematics Education, 2*, 81-99.

Wragg, E. C. (1973). A Study of Student Teachers in the Classroom. In G. Chanan (Ed.), *Towards a Science of Teaching* (pp. 85-134). Windsor, Berks: National Foundation for Educational Research.

Yackel, E., & Cobb, P. (1996). Sociomathematical Norms, Argumentation, and Autonomy in Mathematics. *Journal for Research in Mathematics Education, 27*, 458-477.

Zandieh, M. J. (2000). A Theoretical Framework for Analyzing Student Understanding of the Concept of Derivative. *CBMS Issues in Mathematics Education, 8*, 103-127.

Zeichner, K. (2001). Educational Action Research. In P. Reason & H. Bradbury (Eds.), *Handbook of Action Research: Participative Inquiry and Practice* (pp. 273-283). London: Sage Publications.

Zevenbergen, R. (2000). Cracking the Code of Mathematics Classrooms: School Success as a Function of Linguistic, Social, and Cultural Background. In J. Boaler (Ed.), *Multiple Perspectives on Mathematics Teaching and Learning* (pp. 201-223). Westport, CT: Ablex Publishing.

Zhang, B. (2003). Using Student-centred Teaching Strategies in Calculus. *The China Papers, 2*, 100-103.

Zuber-Skerritt, O. (Ed.). (1991). *Action Research for Change and Development*. Aldershot: Avebury.

APPENDIX A: Questions for the 1st, 2nd, and 3rd interviews

Questions for the first round of the interviews

1. If *f* is any given function, what do you understand by the integral of the function *f*?
2. What do you understand by the definite integral of *f*?
3. What do you understand by the indefinite integral of *f*?
4. Does any relationship exist between the definite integrals and the indefinite integrals?
5. Does any difference exist between the definite integral and the indefinite integral?
6. Can you evaluate the following derivative?

$$\frac{d}{dx}\int_0^x t^2 dt$$

7. Do you know what the use of integrals is?

All these questions have been asked during the first interviews. However, during the viewing of the recordings, I noticed that the students' answers of these questions were not significant and I decided not to consider them for transcription.

Questions for the second round of the interviews

Warming up questions
What programme did you follow in your secondary school?
How many hours per week how did you have for mathematics?
In which year did you study derivatives?
In which year did you study integrals?
What did you study in integrals?
Did you teach before you came to study at KIE?

Questions

1. If your pupil asks you to explain to him or to her what the integral is, what will you tell him or her?

2. If your pupil asks you to explain to him or to her what the definite integral is, what will you tell him or her?

3. If your pupil asks you to explain to him or to her what the indefinite integral is, what will you tell him or her?

4. If your pupil asks you to explain to him or to her what the difference between the definite integral and the indefinite integral is, what will you tell him or her?

5. And if your pupil asks to explain to him or to her what the resemblance (similarity) between the definite integral and the indefinite integral, what will you tell him or her?

6. If your pupil asks your to explain to him or to her in what the primitives (antiderivatives) have to do with area (or integrals), what will you tell him or her?

7. If your pupil asks what the utility of integral is, what will you tell him or her?

Questions for the third round of the interviews

Warming questions
How are you?
How are the courses going?
Did you teach before you came to study at KIE?
Will you teach after you finish your study at KIE?

Questions

1. If you were a teacher and your pupil asked to explain to him or to her what the integral is, what would you tell him or her?

2. If your pupil asked you to explain to him or to her what the definite integral is, what would you tell him or her?

3. If your pupil asked you to explain to him or to her what the indefinite integral is, what would tell him or her?

4. If your pupil asked the difference between the definite integral and the indefinite integral, what would you tell him or her?

5. And if he or she asked you the resemblance (similarity, relation) between the definite integral and the indefinite integral, what would tell him or her?

6. If your pupil asked the relationship between the integrals and the derivatives, what would tell him or her?

7. And if he or she asked you in what the primitives (antiderivatives) have to do with integrals, what would you tell him or her

8. If your pupil asked you what the utility of integrals is, what would you tell him or her? Explain how.

APPENDIX B: Task one and Task two: finding areas under curves

Task 1: Finding areas under curves

1. Give an example of a plane surface bounded by a curve and the x-axis with fixed limits and determine its area without using the anti-derivative;

2. Determine which elements of the graphic in question 1 you can modify to get other problems about areas. Solve these problems using the method developed when you were answering question 1.

Task 2: Finding area under parabolas

Find the area bounded by a parabola and the x-axis within fixed limits (without using the anti-derivative)

Possible solutions to Task No 1 and Task No 2:

Details about the procedure of using the limit of sum to find the area under a continuous positive function can be found in Stewart (1998, pp. 350-357), in Smith & Minton (2002, pp. 350-355), and in Courant (1937, pp. 82-84). In these textbooks, one can also find some formulas of series of numbers.

Possible answers were obtained through the use of the limit of Riemann sums for the Left Rectangles, Middle Rectangles, and Right Rectangles.

APPENDIX C: TASK THREE: Finding indefinite integrals

Task 3: Finding indefinite integrals:

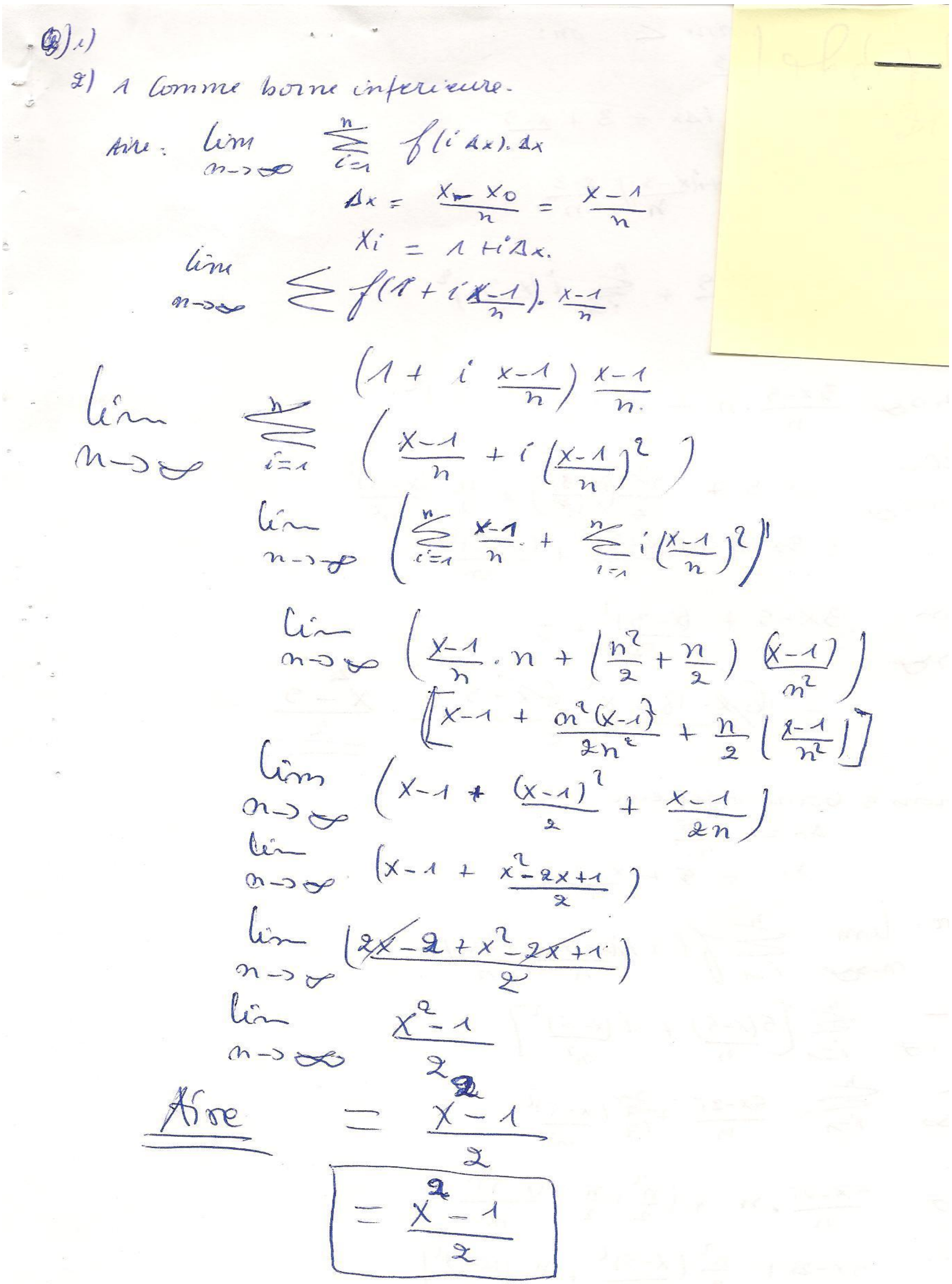
A. Each of you has received a type of function of those you engaged in the previous lessons (either constant, linear, or parabola); evaluate the area it delimits when the limits are variable.
1. Firstly, use the lower limit x = 0 and the upper limit on the positive x-axis.
2. Secondly, vary the lower limit in R_+ until you are able to do a generalisation of the formulation of the area.
3. Thirdly, compare the found formula of the area with the given function and express this comparison in your own words; find other variants [alternatives] of comparison.

B. Choose a function of another kind and answer the sub questions 1, 2, and 3 as in question A above.

Possible solution to Task No 3:

Detailed information is found in Courant (1937, pp. 109-113 and pp. 117-118) and in Clements et al. (2002, p. 133). Using the notion of the definite integral that has been engaged in the previous lessons, it was possible to manipulate algebraic operations and get a formula which represents the area under the given curve. The differentiation of the formula of the area leads to the initial function. So it was necessary to change conceptions of the upper bound x. That is, first to conceive it as a fixed unknown number (Sfard, 1994, p. 203) and then as a variable (Ibid., p. 203). Possible answers, resulting from algebraic manipulations, are given in point 3 of appendix G.

APPENDIX D: Some of the students' productions to task three related to indefinite integral


Q)1)

2) 1 comme borne inferieure.

Aire: $\lim_{n\to\infty} \sum_{i=1}^{n} f(i\Delta x).\Delta x$

$\Delta x = \frac{x - x_0}{n} = \frac{x-1}{n}$

$x_i = 1 + i\Delta x.$

$\lim_{n\to\infty} \sum f(1 + i\frac{x-1}{n}).\frac{x-1}{n}$

$\lim_{n\to\infty} \sum_{i=1}^{n} (1 + i\frac{x-1}{n})\frac{x-1}{n}$

$(\frac{x-1}{n} + i(\frac{x-1}{n})^2)$

$\lim_{n\to\infty} (\sum_{i=1}^{n} \frac{x-1}{n} + \sum_{i=1}^{n} i(\frac{x-1}{n})^2)$

$\lim_{n\to\infty} (\frac{x-1}{n}.n + (\frac{n^2}{2} + \frac{n}{2})\frac{(x-1)^2}{n^2})$

$[x-1 + \frac{n^2(x-1)^2}{2n^2} + \frac{n}{2}(\frac{x-1}{n^2})]$

$\lim_{n\to\infty} (x-1 + \frac{(x-1)^2}{2} + \frac{x-1}{2n})$

$\lim_{n\to\infty} (x-1 + \frac{x^2-2x+1}{2})$

$\lim_{n\to\infty} (\frac{2x-2+x^2-2x+1}{2})$

$\lim_{n\to\infty} \frac{x^2-1}{2}$

Aire $= \frac{x-1}{2}$

$= \frac{x^2-1}{2}$

3) prenons 3 comme borne < on :

$\lim\limits_{n-} \quad \Delta x = \frac{x-3}{n}$

$x_i = 3 + i\Delta x = 3 + \frac{x-3}{n}$

Aire. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f\left(3 + i\frac{x-3}{n}\right) \frac{x-3}{n}$

$\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{3x-9}{n} + \sum\limits_{i=1}^{n} i\left(\frac{x-3}{n}\right)^2$

$\lim\limits_{n\to\infty} \frac{3x-9}{n} \cdot n + \left(\frac{n^2}{2} + \frac{n}{2}\right)\left(\frac{x-3}{n}\right)^2.$

$\lim\limits_{n\to\infty} 3x-9 + \frac{n^2}{2}\left(\frac{(x-3)^2}{n^2}\right) + \frac{n}{2}\frac{(x-3)^2}{n^2}$

$= 3x-9 + \frac{(x-3)^2}{2} + \frac{(x-3)^2}{2n}.$

$\lim\limits_{n\to\infty} 3x-9 + \frac{(x-3)^2}{2} + 0$

$= \frac{6x - 18 + x^2 - 6x + 9}{2} = \frac{x^2 - 9}{2}$

4. prenons 5 borne inferieur

$\Delta x = \frac{x-5}{n}.$

$x_i = 5 + \frac{x-5}{n}.$

Aire : $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f\left(5 + i\frac{x-5}{n}\right)\left(\frac{x-5}{n}\right)$

$\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left[\frac{5(x-5)}{n} + i\frac{(x-5)^2}{n^2}\right]$

$\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{5x-25}{n} + \sum\limits_{i=1}^{n} i\frac{(x-5)^2}{n^2}$

$\lim\limits_{n\to\infty} \frac{5x-25}{n} \cdot n + \left(\frac{n^2}{2} + \frac{n}{2}\right)\frac{(x-5)^2}{n^2}$

$\lim\limits_{n\to\infty} 5x-25 + \frac{n^2}{2}\frac{(x-5)^2}{n^2} + \frac{n}{2}\left(\frac{(x-5)^2}{n^2}\right)$

$5x-25 + \frac{(x-5)^2}{2} + \frac{(x-5)^2}{2n}.$

$\lim_{x \to \infty} \frac{10x - 50 + x^2 - 10x + 25}{2} = \frac{x^2 - 25}{2}$.

Généralisation

Si nous appellons a borne inferieur.
la formule devient.

① $Aire = \frac{x^2 - a^2}{2}$.

Car : 1) $Aire = \frac{x^2}{2}$ borne $= 0$
2. " $\frac{x^2 - 1}{2}$ " $= 1$
3 " $\frac{x^2 - 9}{2}$ " 3
4 " $\frac{x^2 - 25}{2}$ " 5

la formule de l'aire est égale au carré de la formule de la fonction donnée moins le carré de la borne inferieur divisé par deux.

2) les variantes de la comparaison sont les autres f(x)
- F(x) constante
- F(x) parabolique
- F(x) exponentielle

F(x) constante : $Y = a$.

$\frac{x^2 - a^2}{2}$ et $\frac{X}{2}$

$\frac{x^2 - a^2}{2}$ nous donne l'aire et $\frac{x^2}{2}$ ~~$x = a$~~ $X = Y$ donne une droite

3). ② La derivée de la formule de l'aire donne en fonction de x donne la formule de la f(x)

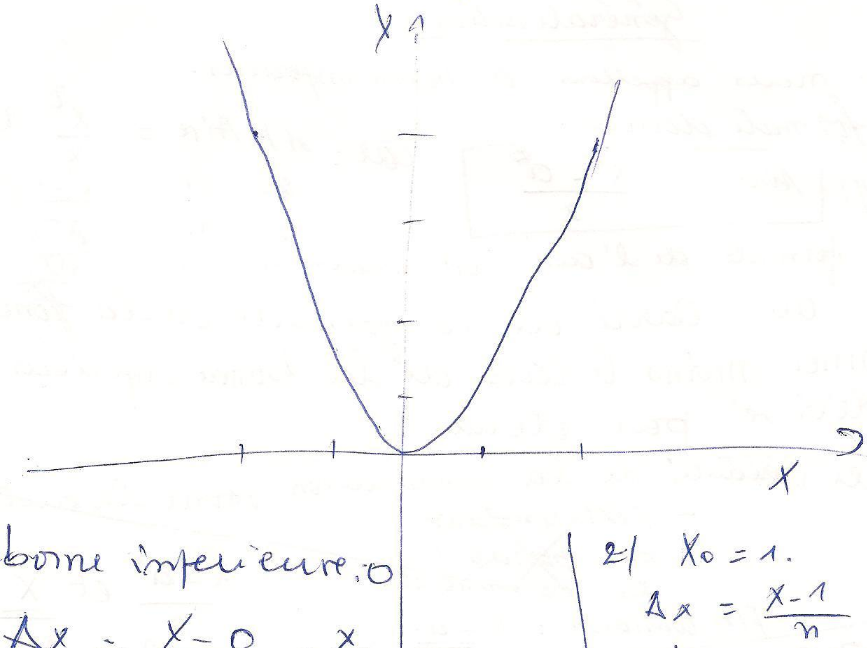
$F'(A) = f(x)$

3)

$F(A)$ = formule de l'aire.
$f(x)$ = formule de la f(x)

b)

b) une parabole



1) borne inferieure : 0

$\Delta x = \frac{X-0}{n} = \frac{x}{n}.$

$i\Delta x = x_0 + i\frac{x}{n} = \frac{ix}{n}$

Alors $\lim_{n\to\infty} \sum_{i=1}^{n} f\left(i\frac{x}{n}\right)^2 \frac{x}{n}$

$\sum_{i=1}^{n} i^2 \frac{x^3}{n^3}.$

$\lim_{n\to\infty} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) \frac{x^3}{n^3}.$

$= \frac{x^3}{3} + \frac{1x^3}{2n} + \frac{x^3}{6n^2}.$

$\boxed{= \frac{x^3}{3}}$

2) $X_0 = 1.$

$\Delta x = \frac{X-1}{n}$

$i\Delta x = 1 + i\left(\frac{x-1}{n}\right).$

$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + i\frac{x-1}{n}\right)^2 \left(\frac{x-1}{n}\right)$

$\lim_{n\to\infty} \sum_{i=1} \left(\frac{x-1}{n} + i^2 \frac{(x-1)^3}{n^3}\right.$

$\frac{x-1}{n}.n + \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\frac{x-1}{n^3}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + i\,\frac{x-1}{n}\right)^2 \left(\frac{x-1}{n}\right)$$

$$\sum_{i=1}^{n} \left(1 + 2i\,\frac{x-1}{n} + i^2\frac{(x-1)^2}{n^2}\right)\left(\frac{x-1}{n}\right)$$

$$\frac{x-1}{n} + \left(2i\,\frac{x-1}{n}\right)\left(\frac{x-1}{n}\right) + i^2\,\frac{(x-1)^2}{n^2}\,\frac{(x-1)}{n}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n} \frac{x-1}{n} + 2i\,\frac{(x-1)^2}{n^2} + i^2\,\frac{(x-1)^3}{n^3}$$

~~$n\cdot\frac{x-1}{n} + \frac{2n^2}{4}$~~

$$n\,\frac{x-1}{n} + 2\left(\frac{n^2}{2}+\frac{n}{2}\right)\left(\frac{(x-1)^2}{n^2}\right) + \left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\left(\frac{(x-1)^3}{n^3}\right)$$

$$x-1 + \frac{2n^2(x-1)^2}{2n^2} + \frac{2n}{2}\frac{(x-1)^2}{n^2} + \frac{n^3}{3}\frac{(x-1)^3}{n^3} + \frac{n^2}{2}\frac{(x-1)^3}{3n^3} + \frac{n}{6n^3}\frac{(x-1)^3}{3n^3}$$

$$x-1+(x-1)^2+\frac{(x-1)^2}{n} + \frac{(x-1)^3}{3} + \frac{(x-1)^3}{3n} + \frac{(x-1)^3}{6n^2}$$

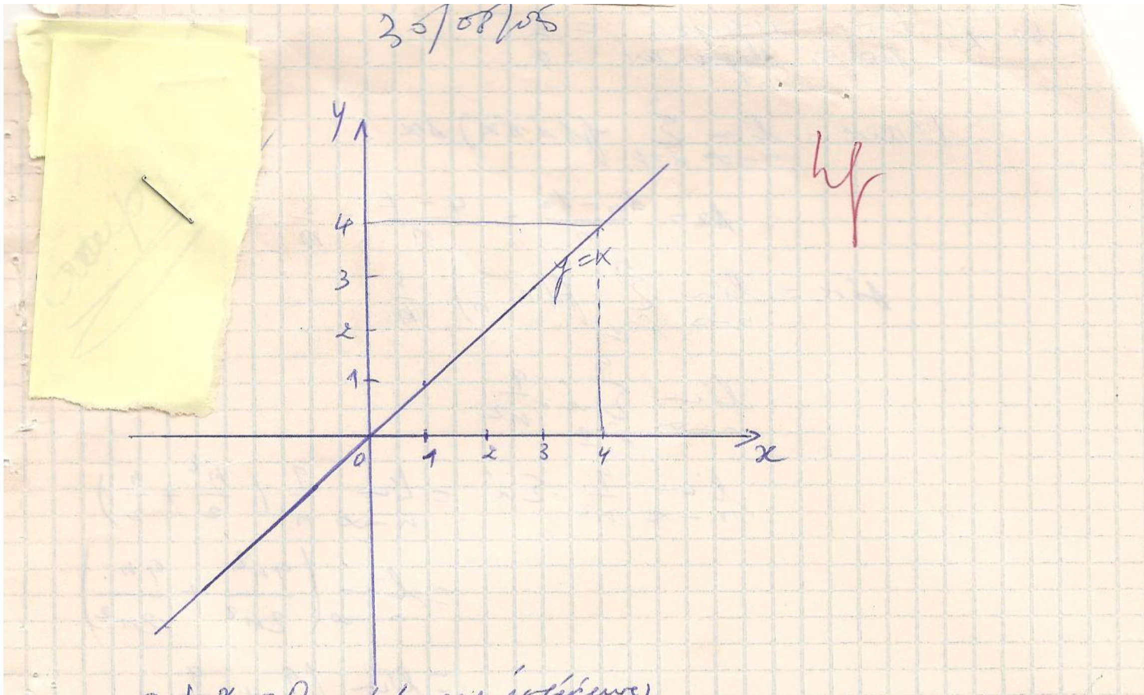
$$\boxed{x-1+(x-1)^2+\frac{(x-1)^3}{3}}$$

$$x-1+(x^2-2x+1)+\left(\frac{x^3-2x^2-x^2+1}{3}\right)$$

$$\frac{3x-3+3x^2-6x+3+x^3-3x^2-2x+1}{3} = \frac{x^3+2x^2-2x+1}{3}$$

$$\frac{3x-3+x^3-3x^2+3x-1+3x^2-6x+3}{3} = \boxed{\frac{x^3-1}{3}}$$

Production of Group 2



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4

a.) $x_0 = 0$ (borne inférieure)

$x_n = 4$ (bornes supérieures

$$L'aire = \int_0^4 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(i\Delta_n x)\cdot\Delta_n x$$

$$\Delta x = \frac{x_n - x_0}{n} = \frac{4-0}{n} = \frac{4}{n}$$

$$Aire = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(i\frac{4}{n}\right)\cdot\frac{4}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16}{n^2} i$$

$$\lim_{n\to\infty} \frac{16}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right)$$

$$\left(\frac{16n^2}{2n^2} + \frac{16n}{2n^2}\right)_0$$

$$\lim_{n\to\infty} = 8$$

4 2. Borne inférieur = 1.
borne supérieur = 4.

$$l'aire = \lim_{n \to \infty} \sum_{i=1}^{n} f(i \Delta x)\Delta x$$

$$\Delta x = \frac{x_n - x_0}{n} = \frac{4-1}{n} = \frac{3}{n}.$$

$$Aire = \lim_{n \to} \sum_{i=1}^{n} f(i\frac{3}{n}) \cdot \frac{3}{n}$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} i \cdot \frac{9}{n^2}$$

$$\lim_{n \to \infty} \frac{9}{n^2} \cdot \sum i = \lim_{n \to \infty} \frac{9}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right)$$

$$= \lim_{n \to \infty}\left(\frac{9n^2}{2n^2} + \frac{9n}{2n^2}\right)$$

$$= \lim_{n \to \infty}\left(\frac{9}{2} + \frac{9}{2n}\right)$$

$$= \frac{9}{2} = 4{,}5$$

borne inférieur = 2.
borne supérieur = 4.

$$\Delta x = \frac{x_n - x_0}{n} = \frac{2}{n}$$

$$l'aire = \lim_{n \to \infty} \sum_{i=1}^{n} f(i\frac{2}{n})\frac{2}{n}$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(i \cdot \frac{4}{n^2})$$

$$\lim_{n \to \infty} \frac{4}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right)$$

$$\lim_{n \to \infty} \left(\frac{4n^2}{2n^2} + \frac{4n}{2n^2}\right)$$

$$\lim_{n \to \infty} = 2.$$

Borne inférieur = 3, supérieur = 4.

$$\Delta x = \frac{1}{n}$$

$$i\left(\frac{1}{n^2}\right) \quad \frac{1}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) = \left(\frac{n^2}{2n^2} - \frac{n}{2n^2}\right)$$

$f(x) = x$

Ap. P $x = 0$ comme borne inférieur.

1. Aire $= \lim_{n \to \infty} \sum_{i=1}^{n} f(i \Delta x) \Delta x$ $\quad \Delta x = \frac{x-0}{n} = \frac{x}{n}$

$= \lim_{n \to \infty} \sum_{i=1}^{n} i \Delta x \cdot \Delta x$

$= \lim_{n \to \infty} \sum_{i=1}^{n} i \cdot \frac{x}{n} \cdot \frac{x}{n}$

$= \lim_{n \to \infty} \sum_{i=1}^{n} i \frac{x^2}{n^2}$

$= \lim_{n \to \infty} \left(\frac{n^2}{2} + \frac{n}{2} \right) \frac{x^2}{n^2}$

$= \lim_{n \to \infty} \left(\frac{n^2 x^2}{2n^2} + \frac{n x^2}{2n^2} \right)$

$= \lim_{n \to \infty} \left(\frac{x^2}{2} + \frac{x^2}{2n} \right)$ $\to 0$

$\boxed{= \frac{x^2}{2} + 0}$ 1ère formule de l'aire.

2. En prenant 1 comme borne inférieur.

$\Delta x = \frac{x-1}{n}$

$x_i = 1 + i \Delta x$ $\quad x_i = x_0 + i \Delta x$

Aire $= \lim_{n \to \infty} \sum_{i=1}^{n} f(1 + i \Delta x) \Delta x$.

$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + i \frac{x-1}{n} \right) \frac{x-1}{n}$

$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{x-1}{n} + i \frac{(x-1)^2}{n^2} \right)$

$= \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{x-1}{n} + \sum_{i=1}^{n} i \frac{(x-1)^2}{n^2} \right)$

$$\boxed{\sum_{i=1}^{n} c = nc}$$

$$\lim_{n\to\infty}\left(\frac{x-1}{n}\cdot n + \left(\frac{n^2}{2}+\frac{n}{2}\right)\frac{(x-1)^2}{n^2}\right)$$

$$\lim_{n\to\infty}\left((x-1) + \frac{n^2(x-1)^2}{2n^2} + \frac{n(x-1)^2}{2n^2}\right)$$

$$\lim_{n\to\infty}\left((x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^2}{2n}\right) \quad \searrow 0$$

$$\boxed{= (x-1) + \frac{(x-1)^2}{2}} \quad \text{Formule de l'aire.}$$

$$= \frac{x^2-1}{2}$$

en prenant $x_0 = 3$

$$\Delta x = \frac{x-3}{n} \qquad x_i = 3 + i\Delta x = 3 + i\,\frac{x-3}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(3 + i\,\frac{x-3}{n}\right)\frac{x-3}{n}$$

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{3(x-3)}{n} + \sum_{i=1}^{n}\left(\frac{n^2}{2}+\frac{n}{2}\right)\cdot\frac{x-3}{n}\right)\frac{x-3}{n}$$

$$\lim_{n\to\infty}\left(3(x-3) + \frac{n^2(x-3)^2}{2n^2} + \frac{n(x-3)^2}{2n^2}\right)$$

$$\lim_{n\to\infty}\left(3(x-3) + \frac{(x-3)^2}{2} + \frac{(x-3)^2}{2n}\right) \quad \searrow 0$$

$$\boxed{= 3(x-3) + \frac{(x-3)^2}{2}} = \frac{2\cdot 3(x-3)}{2} + \frac{x^2-6x+9}{2}$$

$$= \frac{6x - 18 + x^2 - 6x + 9}{2}$$

$$= \frac{x^2-9}{2} = \boxed{\frac{x^2-9}{2}}$$

En prenant 5 où borne inférieure

$\Delta x = \frac{x-5}{n}$ $x_i = 5 + i\,\frac{x-5}{n}$

Aire $\lim_{n\to\infty} \sum_{i=1}^{n} f\left(5 + i\,\frac{x-5}{n}\right)\frac{x-5}{n}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5(x-5)}{n} + \left(\frac{n^2}{2} + \frac{n}{2}\right)\frac{(x-5)^2}{n^2} \right)$$

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{5(x-5)}{n} + \sum_{i=1} \frac{n^2(x-5)^2}{2n^2} + \frac{n(x-5)^2}{2n^2} \right)$$

$$\lim_{n\to\infty} \left(5(x-5) + \frac{(x-5)^2}{2} + \frac{(x-5)^2}{2n^2} \right) \quad \frac{(x-5)^2}{2n^2} \to 0$$

$$= 5(x-5) + \frac{(x-5)^2}{2} = \frac{2.5(x-5) + (x-5)^2}{2}$$

$$= \frac{10x - 50 + x^2 - 10x + 25}{2}$$

$$= \boxed{\frac{x^2 - 25}{2}}$$

En prenant a comme borne inférieur,
l'aire sera $= \frac{x^2 - a^2}{2}$

3.1 Comparaison de $\frac{x^2 - a^2}{2}$ et $f(x) = x$ $\Rightarrow \frac{x^2}{2} - \frac{a^2}{2}$ et x

3.C* La formule de l'aire trouvée est égale au carré de la formule de la fonction donnée moins le carré de la borne inférieure le tout diviser par deux.

Les variantes de la comparaisons:
- bornes inférieures
- les exposants (degré)
- dénominateur (diviseurs).

- F. type de fonctions
  - Parabole . exponentielle
  - Cte -

B) En prenant une fonction – cte
Upe. parabole,
– exponentielle
– logarithmique,

• ~~L'aire a été trouvée à partir de la fonction~~
* La formule de l'aire s'obtient à partir de ~~l'aire~~ celle de la fonction donnée.

$$\left(\frac{x^2-a^2}{2}\right)' = \frac{(x^2-a^2)'.2-(2)'(x^2-a^2)}{2^2}$$

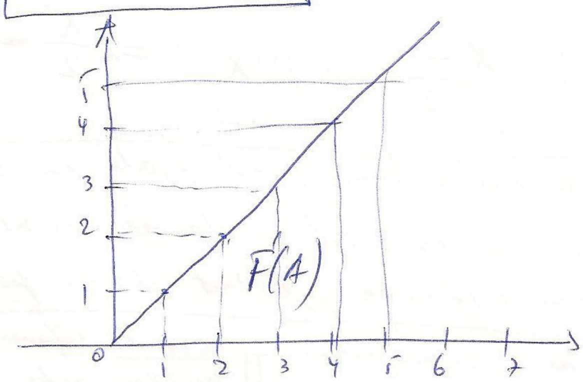
$$= \frac{2x \cdot 2 - 0.(x^2-a^2)}{2^2} = \frac{4x}{4} = x$$

* La dérivée de ~~l'aire~~ la formule de l'aire en fct de x donne la formule de la fonction de départ.

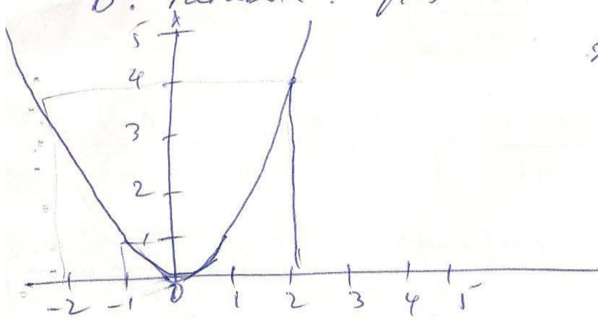
$$\left(\frac{x^2-a^2}{2}\right)' = x. \quad F'(A) = f(x)$$

En dérivant la formule de l'aire on obtient la formule de la fonction donnée.

$F'(A) = f(x)$ avec $F(A)$: l'aire
$f(x)$: fonction de départ.



B. Parabole : $f(x) = x^2$.



$x_0 = 0$

$\Delta x = \frac{X - 0}{n} = \frac{X}{n}$

$x_i = (0 + i\Delta x) = i \cdot \frac{X}{n}$

$Aire = \lim_{n \to \infty} \sum_{i=1}^{n} f(i \frac{x}{n}) \frac{x}{n} = \lim_{n \to \infty} \sum_{i=1}^{n} (i \frac{X}{n})^2 \cdot \frac{X}{n}$

$\lim_{n \to \infty} \sum_{i=1}^{n} i^2 \frac{X^3}{n^3} =$

$\lim_{n \to \infty} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) \frac{X^3}{n^3}$

$\lim_{n \to \infty} \left(\frac{n^3 X^3}{3n^3} + \frac{n^2 X^3}{2n^3} + \frac{n X^3}{6n^3}\right)$

$\lim_{n \to \infty} \left(\frac{X^3}{3} + \frac{X^3}{2n} + \frac{X^3}{6n^2}\right)$

$\searrow 0 \quad \searrow 0$

$= \boxed{\frac{X^3}{3}}$

Pour $X_0 = 1$ $\quad \Delta x = \frac{X-1}{n}$

$x_i = 1 + i\frac{X-1}{n} \quad \lim_{n \to \infty} \sum_{i=1}^{n} f(1 + i\frac{X-1}{n}) \frac{X-1}{n}$

$Aire = \lim_{n \to \infty} \sum_{i=1}^{n} (1 + i\frac{X-1}{n})^2 \cdot \frac{X-1}{n}$

$\lim_{n \to \infty} \sum_{i=1}^{n}$ ~~$(\frac{X-1}{n} + i\frac{(X-1)^3}{n^3}$~~

$\left(1^2 + 2i\frac{X-1}{n} + i^2\frac{(X-1)^2}{n^2}\right) \frac{X-1}{n}$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{x-1}{n}+2i\frac{(x-1)^2}{n^2}+i^2\frac{(x-1)^3}{n^3}\right)$$

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{x-1}{n}+2\left(\frac{n^2}{2}+\frac{n}{2}\right)\frac{(x-1)^2}{n^2}+\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\frac{(x-1)^3}{n^3}\right)$$

$$\lim_{n\to\infty}\left(\frac{x-1}{n}\cdot n+\frac{2n^2(x-1)^2}{2n^2}+\frac{2n(x-1)^2}{2n^2}+\frac{n^3(x-1)^3}{3n^3}+\frac{n^2(x-1)^3}{2n^3}+\frac{n(x-1)^3}{6n^3}\right)$$

$$\lim_{n\to\infty}\left((x-1)+(x-1)^2+\frac{(x-1)^2}{n}+\frac{(x-1)^3}{3}+\frac{(x-1)^3}{2n}+\frac{(x-1)^3}{6n^2}\right)$$

$\frac{(x-1)^2}{n}\to 0$, $\frac{(x-1)^3}{2n}\to 0$, $\frac{(x-1)^3}{6n^2}\to 0$

$$\boxed{=(x-1)+(x-1)^2+\frac{(x-1)^3}{3}}$$

$(x-1)+(x^2-2x+1)+\frac{(x-1)^3}{3}$

$x-1+x^2-2x+1+\frac{(x-1)^3}{3}$

x^2-x

$$(x-1)+(x^2-2x+1)+\frac{(x^2-2x+1)(x-1)}{3}=(x-1)+(x^2-2x+1)+*$$

$$(x-1)+(x^2-2x+1)+\frac{x^3-2x^2+x-x^2+2x-1}{3}$$

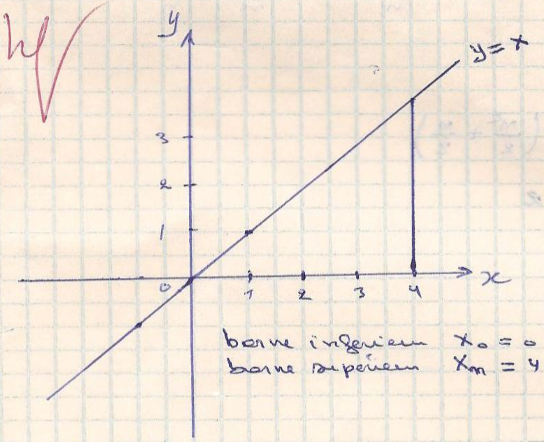
$$\frac{(3x-3)+3x^2-6x+3+x^3-2x^2+x-x^2+2x-1}{3}$$

$$\frac{3x-3+3x^2-6x+3+x^3-2x^2+x-x^2+2x-1}{3}$$

$$\frac{x^3+3x^2-2x^2-x^2+3x-6x+x+2x-3+3-1}{3}$$

$$\frac{x^3-1}{3}$$

Production of Group 3



borne inferieur $x_0 = 0$

borne superieur $X_n = 4$

les unité sont mesure en cm

a/ 1. $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum f(x_i)\,\Delta x$.

$= \lim_{n\to\infty} \sum_{i=1}^{n} f(i\,\Delta x)\,\Delta x$ c.a.d $\Delta x = \frac{X_n - X_0}{n} = \frac{4-0}{n} = \frac{4}{n}$

$= \lim_{n\to\infty} f\left(i\frac{4}{n}\right)\frac{4}{n}$

$= \lim_{n\to\infty} f\left(\frac{16}{n^2} i\right)$

$= \lim_{n\to\infty} \left(\frac{16}{n^2}\right)\left(\frac{n^2}{2} + \frac{n}{2}\right)$

$= \lim_{n\to\infty} \frac{16n^2}{2n^2} + \frac{16n}{2n^2}$

$= \lim_{n\to\infty} \frac{16}{2} + \frac{16}{2n}$.

$= 8 + 0$

$= 8\ cm^2$.

2. Borne inf. 1 Borne sup 4 $\Delta x = \frac{4-1}{n} = \frac{3}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\; i \cdot \frac{9}{n^2}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right)$

$= \lim_{n\to\infty} \sum \frac{9n^2}{2n^2} + \frac{9n}{2n^2}$

$= \frac{9}{2} = 4{,}5$

* borne inf. 2
  borne sup 4

$\lim_{n \to \infty} \sum_{i=1}^{n} f(i \Delta x) \Delta x$ $\Delta x = \frac{x_n - x_0}{n} = \frac{4-2}{n} = \frac{2}{n}$

$\lim_{n \to \infty} f\left(i \frac{2}{n}\right) \frac{2}{n}$

$\lim_{n \to \infty} \frac{4}{n^2} i \Rightarrow \lim_{n \to \infty} \frac{4}{n^2} \left(\frac{n^2}{2} + \frac{n}{2}\right)$

$\lim_{n \to \infty} \frac{4n^2}{2n^2} + \frac{4n}{2n^2} \Rightarrow 2$

* borne inf. 3
  borne sup 4

$\lim_{n \to \infty} \sum_{i=1}^{n} f(i \Delta x) \Delta x$

$\lim_{n \to \infty} \sum_{i=1}^{n}$

Bornes inf. = 0
Borne sup = X

Ainsi $= \lim_{n \to \infty} \sum_{i=1}^{n} f(i \Delta x) \Delta x$ $\qquad \Delta x = \frac{X-0}{n} = \frac{X}{n}$

$\lim_{n \to \infty} \sum_{i=1}^{n} f(i \frac{X}{n}) \frac{X}{n}$ $\qquad i = \frac{n^2}{2} + \frac{n}{2}$

$\lim \sum \frac{X^2}{n^2} i$

$\lim \sum \frac{X^2 n^2}{2n} + \frac{X^2 n}{2n} \Rightarrow \frac{X^2}{2}$

En prenant 1 born inf.
X borne sup. $\qquad \Delta x = \frac{X-1}{n}$

$\lim \sum_{i=1}^{n} f(i \Delta x) \Delta x$

$\lim \sum \left(\frac{X-1}{n}\right)^2 i$

$\lim \sum \frac{(X-1)^2}{n^2} i \Rightarrow \lim \sum \frac{(X-1)^2 n^2}{2n^2} + \frac{(X-1)^2 n}{2n^2}$

$\lim \frac{(X-1)^2}{2} + \frac{(X-1)^2}{2n}$

$\Rightarrow \boxed{\frac{(X-1)^2}{2}}$

$$\text{Aire} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\, \Delta x \qquad \Delta x = \frac{x-1}{n}$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_0 + i\,\Delta x) \frac{x-1}{n} \qquad x_i = x_0 + i\,\Delta x$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} f\left[1 + \left(\frac{x-1}{n}\right) i\right] \frac{x-1}{n} \qquad x^2 - 2x + 1 = (x-1)^2$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{x-1}{n} + \frac{(x-1)^2 i}{n^2}$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{x-1}{n} + \frac{(x-1)^2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right)$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{x-1}{n} + \frac{(x-1)^2 n^2}{2n^2} + \frac{(x-1)^2 n}{2n^2}$$

$$\frac{(x-1)^2}{n} + \frac{(x^2 - 2x + 1) n^2}{2n^2} + \frac{(x^2 - 2x + 1) n}{2n^2}$$

$$\frac{x-1}{n} + \frac{n^2x^2 - 2n^2x + n^2}{2n^2} + \frac{x^2 - 2x + 1}{2n}$$

$f(x) = x$

1. Aire $\lim_{n\to\infty} \sum_{n\to\infty}^{n} f(i\Delta x)\Delta x$ $\qquad \Delta x = \frac{x-0}{n} = \frac{x}{n}$

$= \lim_{n\to\infty} \sum^{n} i\,\Delta x \cdot \Delta x$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{x^2}{n^2} i$

$= \lim_{n\to\infty} \frac{x^2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right)$

$= \lim_{n\to\infty} \frac{x^2 n^2}{2n^2} + \frac{x^2 n}{2n^2}$

$= \boxed{\frac{x^2}{2}}$

2. 1 Comme borne inferieure. $\qquad \Delta x = \frac{x-1}{n}$

$x_i = 1 + i\Delta x.$

$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

$\lim_{n\to\infty} \sum_{i=1}^{n} (1 + i\Delta x)\Delta x$

$\lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + \left(\frac{x-1}{n}\right) i\right] \frac{x-1}{n}$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x-1}{n} + \frac{(x-1)^2}{n^2} i$

$\lim \sum_{i=1}^{n} \frac{x-1}{n} + \sum_{i=1}^{n} \frac{(x-1)^2}{n^2} i$

$\sum_{i=1}^{n} c = nc$

$\lim_{n\to\infty} n\frac{x-1}{n} + \frac{(x-1)^2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right)$

$\lim\ x - 1 + \frac{(x-1)^2 n^2}{2n^2} + \frac{(x-1)^2 n}{2n^2}$

$\boxed{x - 1 + \frac{(x-1)^2}{2}} = \frac{2x - 2 + (x-1)(x-1)}{2} = \frac{2x - 2 + x^2 - x - x + 1}{2}$

$= \boxed{\frac{x^2 - 1}{2}}$

Expo En prenant $x_0 = 3$

$$\Delta x = \frac{x-3}{n}$$

$$x_i = x_0 + i\Delta x = 3 + \frac{x-3}{n}i$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_0 + i\Delta x)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(3 + \frac{x-3}{n}i\right)\frac{x-3}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3(x-3)}{n} + \frac{(x-3)^2}{n^2}i$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3x-9}{n} + \sum_{i=1}^{n}\frac{(x-3)^2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right)$$

$$\lim_{n\to\infty} \frac{n(3x-9)}{n} + \frac{(x-3)^2 n^2}{2n^2} + \frac{(x-3)^2 n}{2n^2}$$

$$\lim_{n\to\infty} 3x-9 + \frac{(x-3)^2}{2}$$

$$3x-9 + \frac{(x-3)(x-3)}{2} = 3x-9 + \frac{x^2-3x-3x+9}{2}$$

$$3x-9 + \frac{x^2-6x+9}{2} = \frac{6x-18+x^2-6x+9}{2} = \boxed{\frac{x^2-9}{2}}$$

En prenant $x_0 = 5$

$$\Delta x = \frac{x-5}{n}$$

$$x_i = 5 + \frac{x-5}{n}i$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(5 + \frac{x-5}{n}i\right)\frac{x-5}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5(x-5)}{n} + \frac{(x-5)^2}{n^2}i$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5(x-5)}{n} + \sum\frac{(x-5)^2}{n^2}i$$

$$\lim_{n\to\infty} \frac{n(5x-25)}{n} + \frac{(x-5)^2}{n^2}\left(\frac{n^2}{2}+\frac{n}{2}\right)$$

$$= \lim_{n\to\infty} 5x-25 + \frac{(x-5)^2 n^2}{2n^2} + \frac{(x-5)^2 n}{2n^2}$$

$$= 5x-25 + \frac{(x-5)^2}{2}$$

$$= \frac{10x-50+x^2-10x+25}{2}$$

$$= \boxed{\frac{x^2-25}{2}}$$

Conclusion : en comprant la formule de l'aire trouve on voit que il ya 2 comme denominateur tjrs X^2 comme la borne superieur au carré epuis la borne inferieur est tjrs au carré

la formule générale $\frac{X^2-a^2}{2}$.

3) Comparaison de $\frac{x^2-a^2}{2}$ et $y=x$

la formule de l'aire trouve est égale au carré de la formule de la fonction donnée moins le carré de la borne inferieure le tout diviser par deux.

Variétés composantes : - bornes inferieur
- les exposant.
- le denominateur.

$$\left(\frac{x^2-a}{2}\right)' = \frac{(x^2-a^2)'2 - 2'(x^2-a^2)}{2^2} = \frac{2x\cdot 2 - 0(x^2-a^2)}{4} = \frac{4x}{4} = x$$

le derive de la formule de l'aire on obtient la formule de la fonction donnée

En derivant la formule de l'aire on obtient la formule de la fct

$\boxed{F(A) = f(x)}$ avec $F(A)$ = la aire
$f(x)$ = fct de départ.

b) Parabole : $f(x) = x^2$.

$\Delta x = \frac{x - 0}{n} = \frac{x}{n}$

$x_i = 0 + i\,\Delta x = i\,\frac{x}{n}$

Aire $\lim_{n \to \infty} f(i\,\Delta x)^2\,\Delta x$

$= \lim_{n \to \infty} \sum_{i=1}^{n} i^2 \left(\frac{x}{n}\right)^2 \cdot \frac{x}{n}$

$= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{x^3}{n^3} i^2$

$= \lim_{n \to \infty} \frac{x^3}{n^3} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right)$

$= \lim_{n \to \infty} \left(\frac{n^3 x^3}{3n^3} + \frac{n^2 x^3}{2n^3} + \frac{n x^3}{6n^3} \right)$

$= \lim_{n \to \infty} \frac{x^3}{3} + \frac{x^3}{2n} + \frac{x^3}{6n^2}$

$= \frac{x^3}{3}$.

En prenant $x_0 = 1$,

$\Delta x = \frac{x-1}{n}$

$x_i = 1 + i\,\frac{x-1}{n}$

d'aire : $\lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + i\,\frac{x-1}{n}\right)^2 \frac{x-1}{n} \Rightarrow \left[1^2 + 2i\,\frac{x-1}{n} + \left(\frac{x-1}{n}\right)^2 i^2\right] \frac{x-1}{n}$

$\lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{x-1}{n}\right) + \left(\frac{x-1}{n}\right)^2 2i + \frac{(x-1)^3}{n^3} i^2$.

$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{x-1}{n} + \frac{(x-1)^2}{n^2} \cdot 2\left(\frac{n^2}{2} + \frac{n}{2}\right) + \frac{(x-1)^3}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$

$= \lim_{n \to \infty} n \cdot \frac{x-1}{n} + \frac{2(x-1)^2 n^2}{2n^2} + \frac{2(x-1)^2 n}{2n^2} + \frac{(x-1)^3 n^3}{3n^3} + \frac{(x-1)^3 n^2}{2n^3} + \frac{(x-1)^3 n}{6n^3}$

$= \lim_{n \to \infty} x - 1 + \frac{2(x-1)^2}{2} + \underbrace{\frac{(2x-2)^2}{2n}}_{0} + \frac{(x-1)^3}{3} + \underbrace{\frac{(x-1)^3}{2n}}_{0} + \underbrace{\frac{(x-1)^3}{6n^2}}_{0}$

$= x - 1 + \frac{2(x-1)^2}{2} + \frac{(x-1)^3}{3}$

$= (x-1) + (x-1)^2 + \frac{(x-1)^3}{3}$

$$= (x-1) + (x-1)(x-1) + \frac{(x-1)^3}{3}$$

$$= (x-1) + x^2 - 2x + 1 + \frac{(x-1)(x^2-2x+1)}{3}$$

$$= (x-1) + x^2 - 2x + 1 + \frac{x^3 - 2x^2 + x - x^2 + 2x - 1}{3}$$

$$= \frac{3x - 3 + 3x^2 - 6x + 3 + x^3 - 2x^2 + x - x^2 + 2x - 1}{3} = \frac{3x - 6x + x + 2x + x^2 + x^3 - 1}{3}$$

$$= \frac{x^3 + x^2 - 1 - x^2}{3} = \boxed{\frac{x^3 - 1}{3}}.$$

En prenant $x_0 = 3$

$$\Delta x = \frac{x-3}{n}$$

$$x_i = 3 + i\frac{x-3}{n}$$

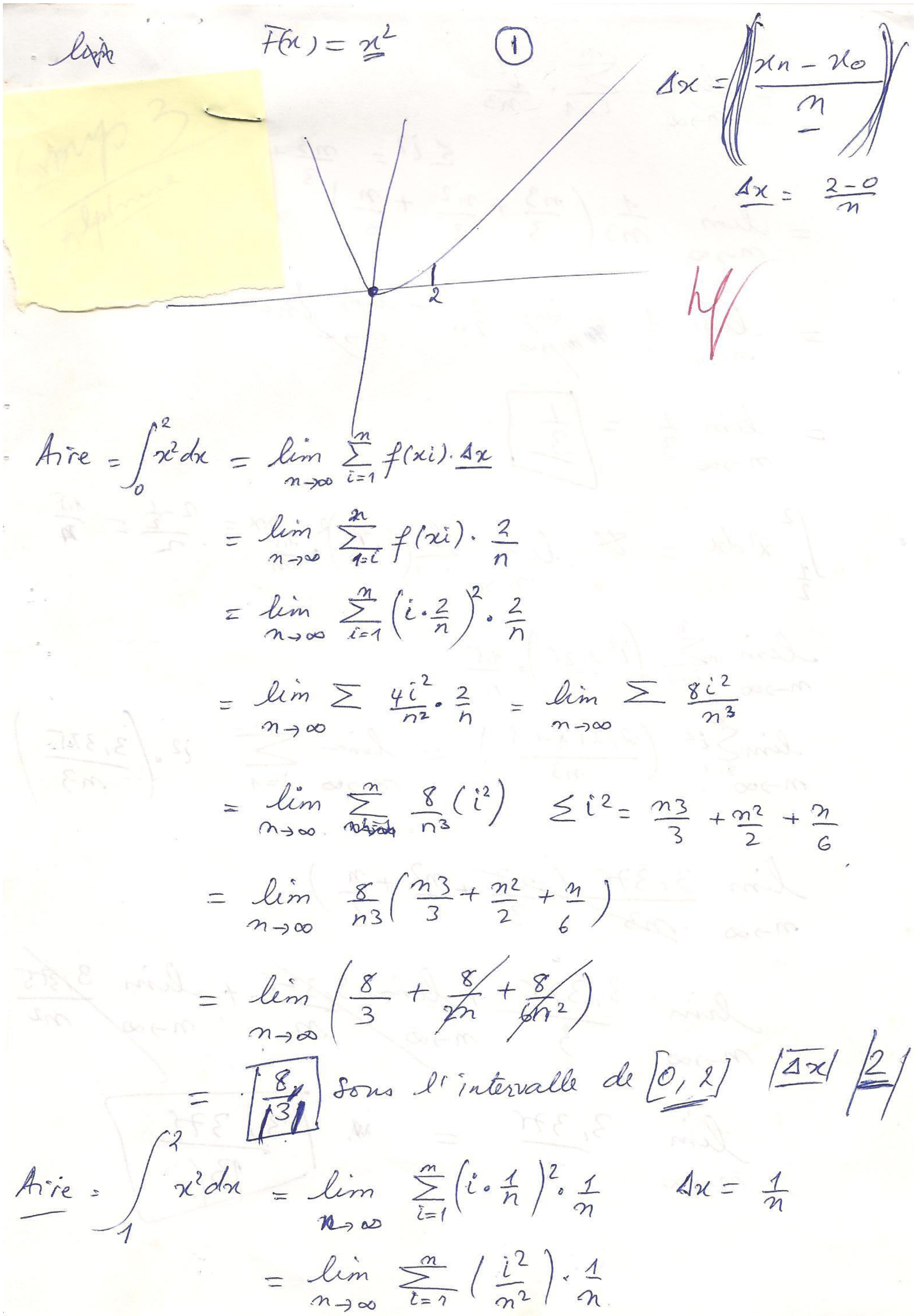
$$\text{l'aire} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(3 + i\frac{x-3}{n}\right)^2 \frac{x-3}{n}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(3^2 + 6i\frac{x-3}{n} + \frac{(x-3)^2}{n^2}\right) \frac{x-3}{n}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} 9\left(\frac{x-3}{n}\right) +$$

Productions of Group 4



$F(x) = x^2$ ①

$\Delta x = \left(\frac{x_n - x_0}{n}\right)$

$\Delta x = \frac{2-0}{n}$

$$\text{Aire} = \int_0^2 x^2 dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \frac{2}{n}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(i \cdot \frac{2}{n}\right)^2 \cdot \frac{2}{n}$$

$$= \lim_{n \to \infty} \sum \frac{4i^2}{n^2} \cdot \frac{2}{n} = \lim_{n \to \infty} \sum \frac{8i^2}{n^3}$$

$$= \lim_{n \to \infty} \sum^{n} \frac{8}{n^3}(i^2) \qquad \sum i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$= \lim_{n \to \infty} \frac{8}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \lim_{n \to \infty} \left(\frac{8}{3} + \frac{8}{2n} + \frac{8}{6n^2}\right)$$

$= \boxed{\frac{8}{3}}$ sous l'intervalle de $[0, 2]$ $|\Delta x|$ $|2|$

$$\text{Aire} = \int_1^2 x^2 dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left(i \cdot \frac{1}{n}\right)^2 \cdot \frac{1}{n} \qquad \Delta x = \frac{1}{n}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{i^2}{n^2}\right) \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$$

$$\sum i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \lim_{n\to\infty} \frac{1}{3} + \lim_{n\to\infty} \frac{1}{2n} + \lim_{n\to\infty} \frac{1}{6n^2}$$

$$= \lim_{n\to\infty} \frac{1}{3} = \boxed{\frac{1}{3}}$$

$$\int_{\frac{1}{2}}^{2} x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(i \cdot \frac{1{,}5}{n}\right)^2 \cdot \frac{1{,}5}{n} \quad \Delta x = \frac{2-\frac{1}{2}}{n} = \frac{1{,}5}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(i^2 \frac{2{,}25}{n^2}\right) \cdot \frac{1{,}5}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} i^2 \left(\frac{2{,}25 \times 1{,}5}{n^3}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} i^2 \cdot \left(\frac{3{,}375}{n^3}\right)$$

$$\lim_{n\to\infty} \frac{3{,}375}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$\lim_{n\to\infty} \frac{3{,}375}{3} + \lim_{n\to\infty} \frac{3{,}375}{n} + \lim_{n\to\infty} \frac{3{,}375}{n^2}$$

$$\lim_{n\to\infty} \frac{3{,}375}{3} = \boxed{\frac{3{,}375}{3}}$$

(2)

$$\int_{3/2}^{2} x^2 dx = \lim_{n \to 0} \sum_{i=1}^{n} \left(i \cdot \frac{0,5}{n}\right)^2 \cdot \frac{0,5}{n} \qquad \Delta x = \frac{0,5}{1}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} i^2 \cdot \frac{0,125}{n^2} \cdot \frac{0,5}{n}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} i^2 \cdot \frac{0,125}{n^3}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{0,125}{n^3} i^2$$

$$= \lim_{n \to \infty} \frac{0,125}{n^3} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \lim_{n \to \infty} \frac{0,125}{3} + \lim_{n \to \infty} \frac{0,125}{2n} + \lim_{n \to \infty} \frac{0,125}{6n^2}$$

$$= \lim_{n \to \infty} \frac{0,125}{3} = \boxed{\frac{0,125}{3}}$$

NJ

Généralisation

- Toute les airs trouves ont un dénominateur identique (qui est 3)
- Pour ce qui est du numérateur, on a constaté que
  - Dans le cas 1 : $\Delta x = 2$ $\boxed{2-0}=2$
    Le numérateur $8 = (2)^3$
  - Dans le cas 2 : $\Delta x = 1,5$ $2-0,5=1,5$
    Le numérateur est $(1,5)^3 = 3,375$
  - Dans le cas 3 : $\Delta x = 1$ $-1+2=\Delta x$
    Le numérateur est $(1)^3 = 1$

$\Delta x = \frac{5-1}{n} = \frac{4}{n}$

$$\int_1^5 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(i\frac{4}{n}\right)^2 \cdot \frac{4}{n}$$ (3)

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{16\,i^2}{n^2}\right)\frac{4}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{64\,i^2}{n^3}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{64}{n^3}\, i^2$$

$\frac{x_n - x_0}{n}$

$$= \lim_{n\to\infty} \frac{64}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \lim_{n\to\infty} \frac{64 n^3}{3 n^3} + \lim \frac{64 n^2}{2 n^3} + \lim_{n\to\infty} \frac{64 n}{6 n^3}$$

$$\lim_{n\to\infty} \frac{64}{3} = \boxed{\frac{64}{3}}$$

$$\int_2^5 x^2\,dx$$

Généralisation: $\int x^2 dx = \frac{(\Delta x)^3}{3} \cdot n$

the [illegible] Δx $= \frac{(\Delta x \cdot n)^3}{3}$

Δx: h $\quad \frac{x_n - x_0}{n}$

Subdivision

$\frac{(\Delta x \cdot n)^3}{3}$ $\quad \frac{2-1}{n} = \left(\frac{1}{n} \cdot n\right)^3$

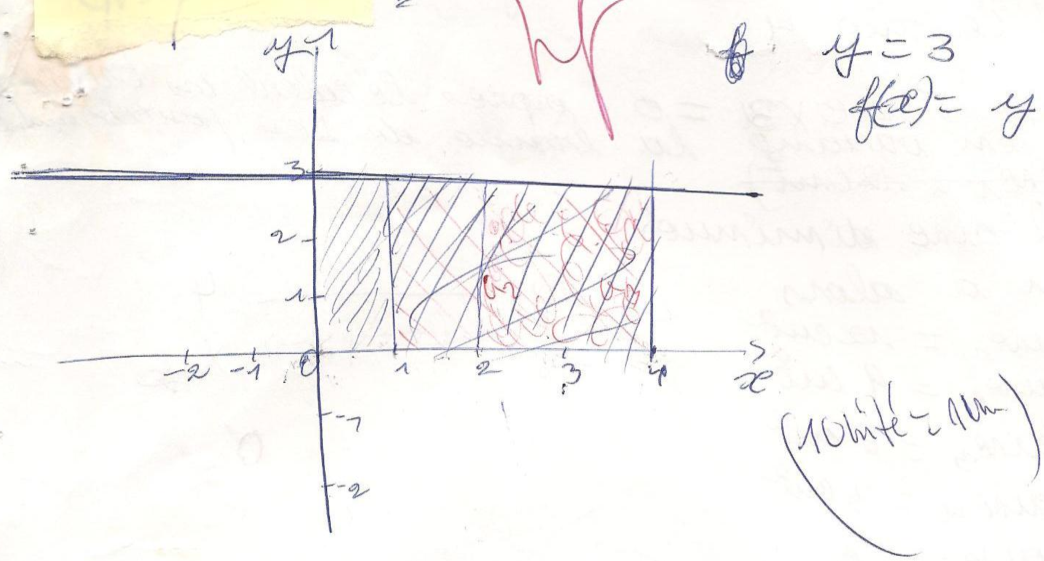
$\frac{x_n - x_0}{n} \cdot n$

Productions of Group 5

30/8/05

$y = 3$

$f(x) = y = 3$



borne inférieure = $x_0 = 0$

" supérieure = $x = 4$

~~La surfa~~

L'aire$_1$ = $4 \times 3 = 12\,cm^2$

Prenons x_0 comme 1. et la borne supérieure ne change pas. $(x = 4)$

L'aire$_2$ = $3 \times 3 = 9\,cm^2$

- Prenons encore x_0 comme 2 et la borne supérieure ne change pas $(x = 4)$

L'aire$_3$ = $2 \times 3 = 6\,cm^2$

- Prenons encore x_0 comme 3 et la borne est inchangeable.

L'aire$_4$ = $1 \times 3 = 3\,cm^2$

Prenons la bornes inférieure
Comme 4.

L'aire 5 = 0 x 3 = 0 après le calcul on trouve que en variant la borne de 1 cm positivement
(~~L'aire 1 = 12 cm²~~)
L' l'aire diminue.

On a alors
L'aire$_1$ = 12 cm²
L'aire$_2$ = 9 cm²
L'aire$_3$ = 6 cm²
L'aire$_4$ = 3 cm²
L'aire$_5$ = 0

0 ——— 4
X_0 ———→ X_n
0

3.($x_n - x_0$)

Si x_0 ~~to~~ → X_n ⟹ l'aire exacte → limite de la fonction, lorsque $x_0 \to X_n$

L'aire exacte $= \lim_{x_0 \to X_n} f(x).(X_n - X_0)$ $X_n - X_0 = \Delta X$

f(x) $X_n - x_0$ $f(x) = 3$
$X_n = 4$
$x_0 = 0$

$3(4-0) = 12\ cm^2$

$3(4-1) = 9\ cm^2$

$x_0 \to x_n$
$x_0 - x_n \to 0$
$\Rightarrow x_n - x_0 \to 0$
$\Delta x \to 0$

L'aire exacte $= \lim_{\Delta x \to 0} f(x).\Delta x$ (1)

- 2 -

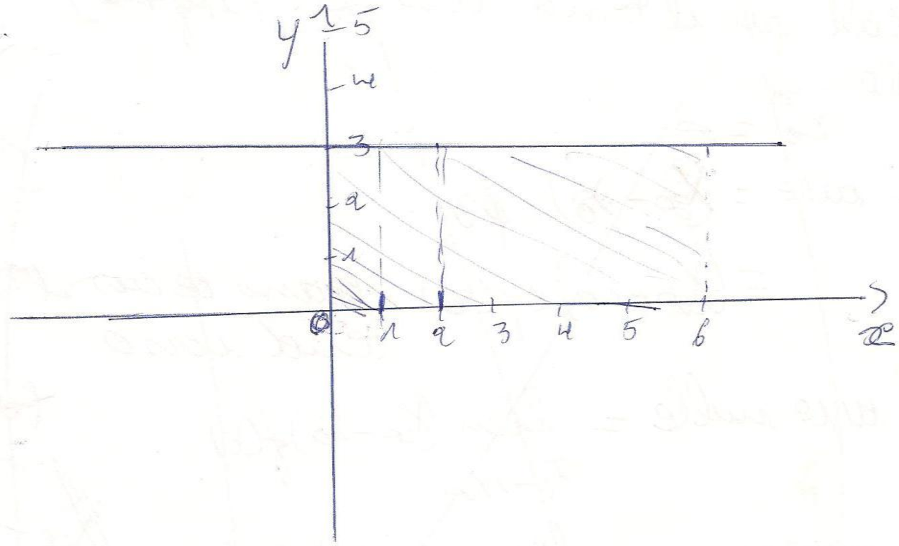
Comme $\Delta x \to 0$ $\lim\limits_{\Delta x \to 0} f(x) = \lim\limits_{\Delta x \to 0} f(x+\Delta x)$ (2)

On va remplace ~~dans~~ (2) dans (1)

On aura $\lim\limits_{\Delta x \to 0} f(x+\Delta x)\cdot\Delta x$

~~$\Rightarrow \lim f$~~

Donnons un autre exemple avant de generaliser.



Après avoir tracée la courbe $y = 3$ calculons les aires de 0 à 6 quand x_0 varie positivement

Si $x_0 = 0$

L'aire$_1$ = $6 \times 3 = 18\ cm^2$ $= (6-0)\times 3 = 18\ cm^2$ ($x_6 - x_0$)

$x_0 = 1$

L'aire$_2$ = $(6-1)\times 3 = 15\ cm^2$

$x_0 = 2$

L'aire$_3$ = $(6-2)\times 3 = 12\ cm^2$

$x_0 = 3$

L'aire$_4$ = $(6-3)\times 3 = 9\ cm^2$

• pour trouver l'aire exacte entre $x=6$ et $x=0$

$$A = (x_6 - x_0) \cdot f(x) = 6 \times 3 = 18\,cm^2$$

• Si on varie x_0 à 1,

L'aire exacte $= (x_6 - x_1) \cdot f(x) + (x_1 - x_0) \cdot f(x)$

• Si on fait varier x_0 à 2,

L'aire exacte $= (x_6 - x_2) \cdot f(x) + (x_2 - x_0) \cdot f(x)$ (a)

• Si on fait varier x_0 à 4,

L'aire $= (x_6 - x_4) \cdot f(x) + (x_4 - x_0) \cdot f(x)$

⋮

• Si on fait varier x_0 à 6,

L'aire exacte entre x_0 et x_6 $= (x_6 - x_6) \cdot f(x) + (x_6 - x_0) \cdot f(x)$

$= 0 + (x_6 - x_0) \cdot f(x)$.

supposons que la borne supérieure est indéfini dans l'intervalle $[0, \infty[$

Si $x_0 = 2 \Rightarrow A = (x_5 + \Delta x - x_2) f(x) +$

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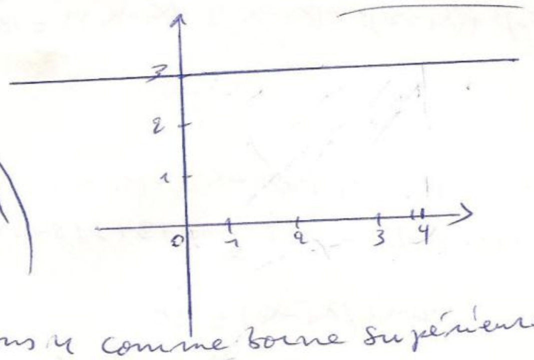
– 1 –

(Fig 1)



Prenons 4 comme borne supérieure et 0 comme borne inférieure.

$Aire_1 = (X_4 - X_0)3 = 4 \times 3 = 12\ u^2 \Rightarrow 3x - 3 \cdot 0 = 12\ cm^2$

Varions X_0 positivement jusqu'à 1

$A_2 = (X_4 - X_1)3 = (4-1)3 = 9\ u^2 \Rightarrow 3 \cdot x - 3 \cdot 1 = 12 - 3 = 9$

Varions X_0 jusqu'à 2

$Aire_3 = (X_4 - X_2)3 = (4-2)3 = 6\ u^2 = 3x - 3 \cdot 2 = 12 - 6 = 6$

Varions X_0 jusqu'à 3

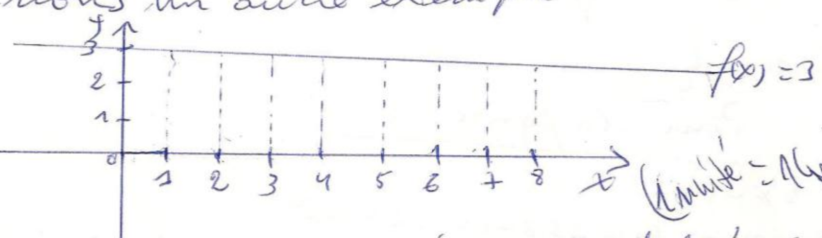
$Aire_4 = (X_4 - X_3)3 = (4-3)3 = 3\ u^2\ 3 \cdot x - 3 \cdot 3 = 12 - 9 = 3$

Varions X_0 jusqu'à 4

$Aire_5 = (X_4 - X_4)3 = 0\ u^2 \quad 3x - 3 \cdot 4 = 12 - 12 = 0$

Prenons un autre exemple

(Fig 2)



Prenons la borne superieur = 8 et la borne inférieure $= X_0 = 0$ $Aire_1 = (8 - 0)3 = 24\ cm^2$

Varions la borne $X_0 = 1$

$Aire_2\ (X_8 - X_1)3 + (X_1 - X_0)3 = 21 + 3 = 24\ cm^2$

Varions $X_0 = 2$

$Aire_3 : (X_8 - X_2)3 + (X_1 - X_0)3 + (X_1 - X_0)3 = 18 + 3 + 3 = 24\ cm^2$

Varions $X_0 = 3$

$Aire_3 : (X_8 - X_3)3 + (X_1 - X_0)3 + (X_1 - X_0)3 + (X_1 - X_0)3 = 15 + 3 + 3 + 3 = 24\ cm^2$

si $x_0 = 4$

$Aire_4 = (x_8 - x_4)3 + (x_1 - x_0)3 + (x_1 - x_0)3 + (x_1 - x_0)3 + (x_1 - x_0)3 = 12 + 3 + 3 + 3 + 3 = 24$

⋮

$x_0 = 8$,

$Aire_8\ (x_8 - x_8)3 + (x_1 - x_0)3 + (x_1 - x_0)3 + (x_1 - x_0)3 + (x_1 - x_0)3 + (x_1 - x_0)3 + (x_1 - x_0)3$
$+ (x_1 - x_0)3 + (x_1 - x_0)3 = 0 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 24\,cm^2$

Pour $X_0 = 0$ et $X = 8$ Aire $= \sum_{i=1}^{8} (x_1 - x_0)3$ avec $(x_1 - x_0) = \Delta x$

Aire $= \sum_{i=1}^{8} f(x)\Delta x$ $3 = f(x)$ $\sum^{8} 3 = \sum_{i=1}^{8} f(x_i)$

Supposons que on ignore la borne supérieure et que notre borne varie indéfiniment.
alors

$Aire = (x_\infty - x_\infty)3 + (x_1 - x_0)3 + (x_1 - x_0)3 + \dots$ ~~$+n(x_1 - x_0)3$~~ $+ n(x_1 - x_0)3$

$= \Delta x \cdot f(x) + \Delta x \cdot f(x) + \dots\dots\dots + n\Delta x f(x)$

Faustin: Pour $x_0 = 1$,
‖ $3X - 3 \cdot 1$, est-ce que ça a un rapport avec ce que vous êtes en train de faire?

Pour la figure 2

$3X - 3X_0$

si $x_0 = 0 \Rightarrow$ Aire $= 3.8 - 3.0 = 24\,cm^2$
si $x_0 = 1 \Rightarrow$ " $= 3.8 - 3.1 = 21\,cm^2$
si $x_0 = 2 \Rightarrow$ Aire $= 3.8 - 3.2 = 18\,cm^2$
si $x_0 = 3 \Rightarrow$ Aire $= 3.8 - 3.3 = 15\,cm^2$

⋮

$x_0 = 8 \Rightarrow$ Aire $= 3.8 - 3.8 = 0$

$3.x - 3.x_0$ $\quad$ $Cx_0 : 1 \longrightarrow n$ $\quad$ n = nombre indéfini $\quad$ -3-

i) $f(x).x - f(x)\Delta x = f(x).x + (-f(x).\Delta_1 x) = f(x).x + C_1$

A chaque variation positive l'aire diminue de $f(x)\Delta x$

x correspond à la base et est variable

$= f(x).x + C$

ii) $f(x).x - f(x)\Delta_2 x = f(x).x + C_2$

iii) $f(x).x - f(x)\Delta_3 x = f(x).x + C_3$

in) $= f(x).x - f(x).\Delta_n x = f(x).x + C_n$

$C_1 = C_2 = C_3 = \ldots\ldots = C_n$ car : $\Delta_1 x = \Delta_2 x = \Delta_3 x = \ldots \Delta_n x$

$C = |C|$

| L'aire $= f(x).x + C$ |

Donc L'aire $3x + C$

3. Comparaison.

- l'aire trouvée $= 3x + C$ } L'aire $= f(x).x + C$ (x étant une variable)

$f(x) = 3$

. La fonction est donnée par la dérivée de l'aire

$(3x + C)' = (3x)' + C'$

$= 3 + 0$

$= 3$

Comme l'aire correspond à l'intégrale de la fonction, et que la dérivée de cette intégrale égale

APPENDIX E: TASK FOUR: Mathematical proof of the fundamental theorem of calculus (function version)

Task four: Reconstruction of the proof of the FTC - VDI

The separated pieces of the proof to be reconstructed follow:

1. where we switched the limits of integration according to equation xx and combined the integral according to theorem 1 (iv).

2. $\frac{1}{h}\int_x^{x+h} f(t)dt = f(c) \qquad (4)$
for some number c between x and x+h.

3. $= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)dt + \int_x^a f(t)dt\right]$

4. Finally, since c is between x and x+h, we have that $c \to x,$ as $h \to 0$.

5. Using the definition we have

6. as required.

7. $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt = \lim_{h\to 0} f(c) = f(x)$

8. Since f is continuous, it follows from 3 and 4 that
$F'(x) =$

9. $F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$

10. $= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt \qquad (3)$

11. Looking very carefully to (3), you may recognise it as the limit of the average value of f(x) on the interval $[x, x+h]$ (if h>0). By the integral mean value theorem, we have

12. $= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right]$

Possible solutions

The solution of task No 4 is given by Smith and Minton (2002, p. 368).

For the FTC (function version) Smith and Minton gave the following order:

5, 9, 12, 3, 10, 1, 11, 2, 4, 8, 7, 6. However, variations are acceptable.

APPENDIX F: TASK FIVE: Mathematical proof of the fundamental theorem of calculus (version of evaluation of area)

Task five: Reconstruction of the proof of the FTC - VEA

The separated pieces of the proof to be reconstructed follow:

1. You recognise this last expression as a Riemann sum for f on $[a,b]$.

2. $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x$ (2)
for some $c_i \in (x_{i-1}, x_i)$.

3. $= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$. (1)

4. $= F(b) - F(a)$
as desired, since this last quantity is a constant.

5. $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty} [F(b) - F(a)]$

6. $= [\, F(x_1) - F(x_0)\,] + [\, F(x_2) - F(x_1)\,] + \ldots$

7. Working backward, note that by virtue of the cancellations, we can write
$F(b) - F(a) = F(x_n) - F(x_0)$

8. Since F is an antiderivative of f, F is differentiable on (a,b) and continuous on $[a,b]$. By the theorem of the Mean value theorem, we have for each $i = 1,2,...,n,$ that

9. $+ [\, F(x_n) - F(x_{n-1})\,]$

10. First, we partition the interval $[a,b]$: $a = x_0 < x_1 < x_2 .... < x_n = b$, where $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for $i = 1,2,...,n$.

11. Thus, owing to 1 and 2, we have,

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(c_i)\Delta x \quad (3)$$

12. Taking the limit of both sides of (4), as $n \to \infty$, we find that

Possible solutions

The solution is given by Smith and Minton (2002, p. 365)

For the FTC (Number version) they gave the following order:

10, 7, 6, 9, 3, 8, 2, 11, 1, 12, 5, 4

APPENDIX G: TASK SIX: Computer-based exploration of integrals and the fundamental theorem of calculus

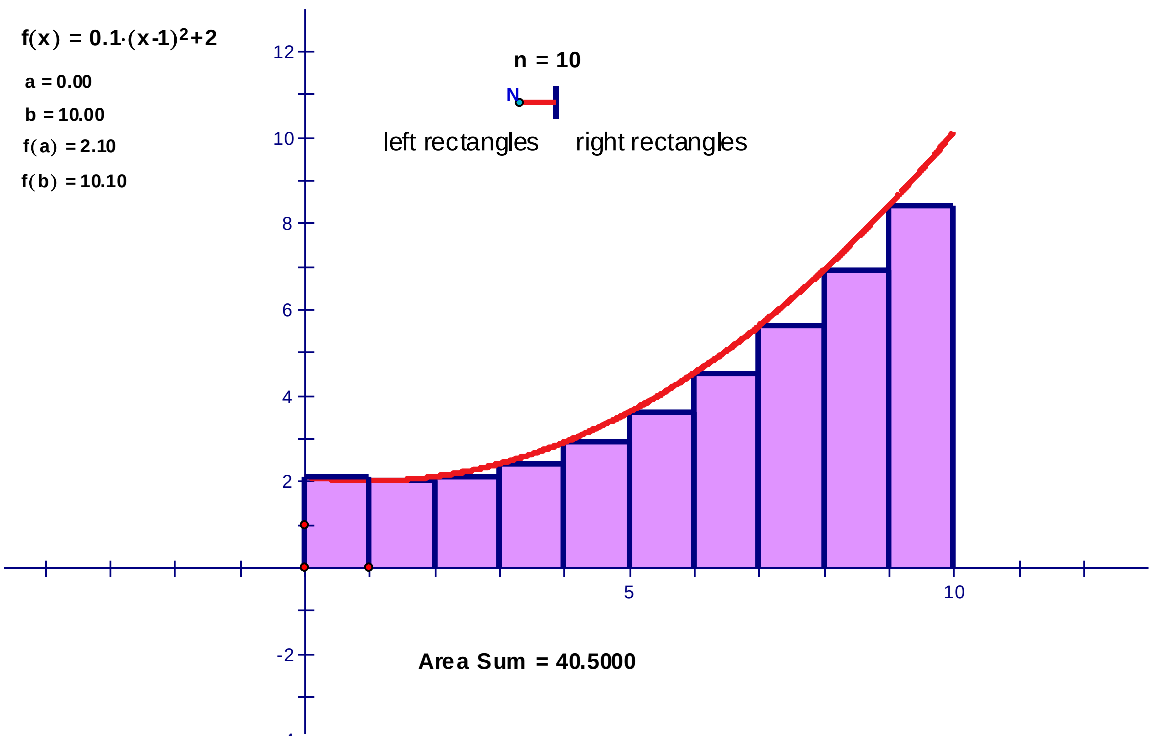
Task six: Computer-based exploration of integrals with Geometer's SketchPad (Key Curriculum Press, 2002):

Task 6a: Building area (sheet 1):

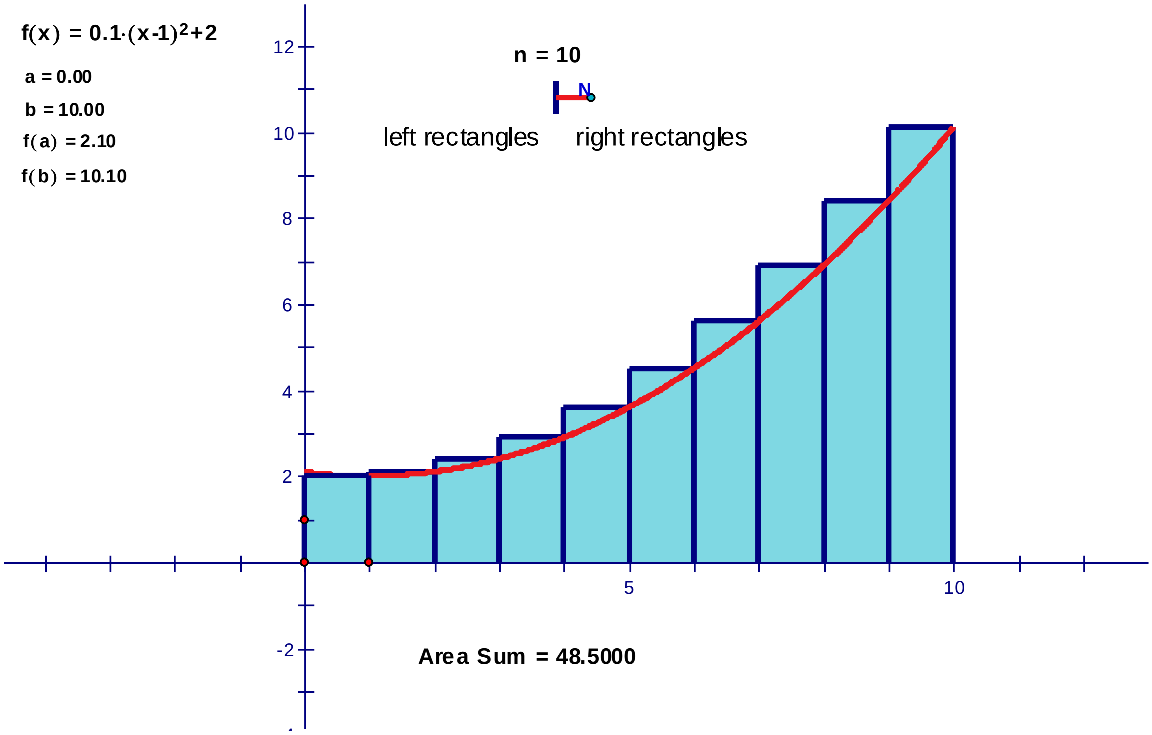
a. Vary the number of subdivisions to the left
b. Vary the number of subdivisions to the right
c. Describe what you are observing

The following diagrams are samples of what could be observed on the screen of a computer (Key Curriculum Press, 2002, Electronic version):

a. Varying number (n) of subdivisions to the left (here n=10)



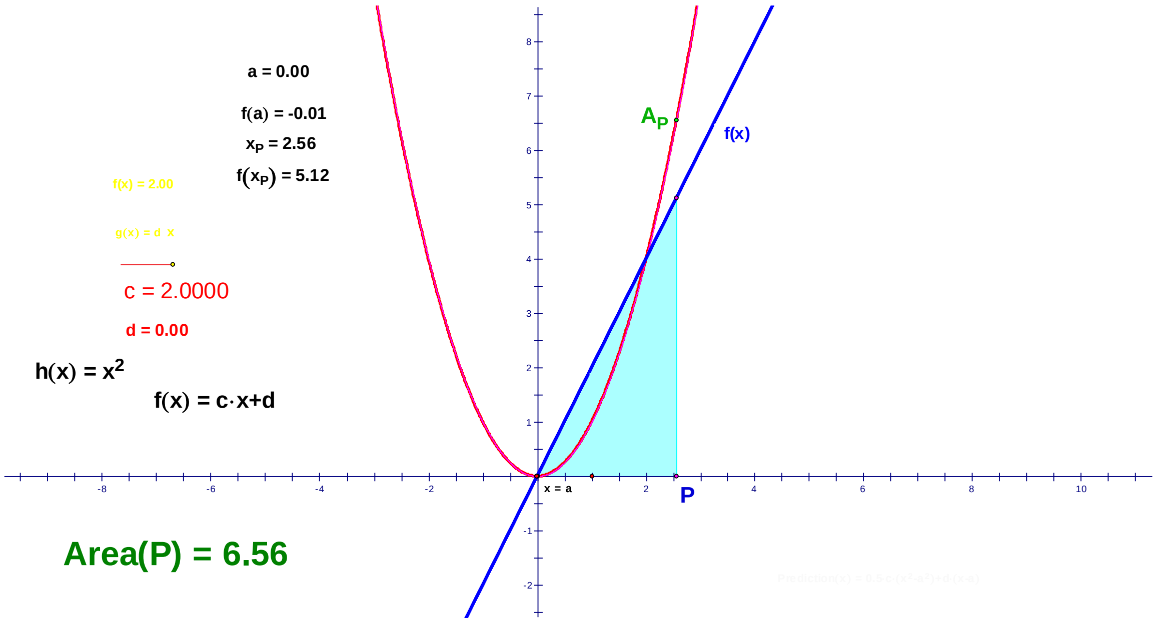
b. Varying number (n) of subdivisions to the right (here n=10)



Task 6b: Cumulative area (sheet 5):

a. Move the starting point from x=0 to x=1, to x=2, to x=3, to x=4, and so on.
b. In your own words, explain what you understand by the point Ap, its move and the curve that carries it, when the point x=a is moved from 0 to 1, and then to 2, and then to 3, and so on. Give details of your answer.

As in the preceding task, the following diagram is a sample of what could be observed on the screen of a computer (Key Curriculum Press, 2002, Electronic version):



As there was only one computer in the classroom, the students were given a proof to reconstruct when they were waiting for their turn to explore integrals on the screen of the computer. The small pieces to assemble in order to reconstruct the proof follow.

1. $= c\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x + d\lim_{n\to\infty}\sum_{i=1}^{n} g(c_i)\Delta x$

2. according to the rules of summation and to the fact that f and g are integrable

3. according to properties of limits

4. $= \lim_{n\to\infty}[c\sum_{i=1}^{n} f(c_i)\Delta x + d\sum_{i=1}^{n} g(c_i)\Delta x]$

5. By definition of integral, we have

6. $\int_a^b [cf(x) + dg(x)]dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$

7. obtained by referring again to the definition of integral.

8. $= c\int_a^b f(x)dx + d\int_a^b g(x)dx$

9. Let us prove the parts (i), (ii) and (iii) all at once by showing that for any constant c and d,

10. as it was desired.

11. = $\lim_{n\to\infty}\sum_{i=1}^{n}[cf(c_i)+dg(c_i)]\Delta x$

12. $\int_a^b [cf(x)+dg(x)]dx$

Possible order:

9, 6, 5, 12, 11, 4, 2, 1, 3, 8, 7, 10

APPENDIX H: Institutionalised and pre-prepared subject-matter including possible answers of task three

1. INSTITUTIONALISED SUBJECT-MATTER AFTER LESSON TEN

1.1. THE DEFINITION OF THE DEFINITE INTEGRAL

The definite integral of a function f(x) between x = a and x = b gives the area delimited by that function and the x-axis above the interval [a, b].

This definite integral is symbolized by

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i).\Delta x \qquad (1)$$

where $\Delta x = \dfrac{b-a}{n} = \dfrac{x_n - x_0}{n}$ and $f(x_i)$ are heights of rectangles of base Δx. These heights are drawn on $x_0, x_1, x_2, ..., x_n$ which are values taken on the interval [a, b].

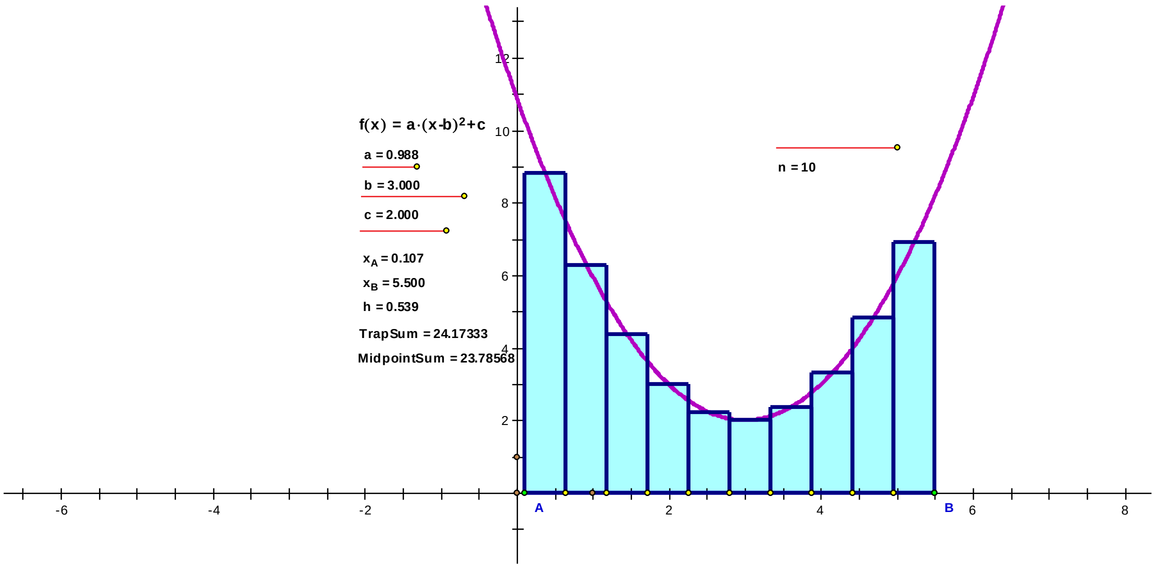
When the limit in (1) exists, then we say that the function f(x) is *integrable* on [a, b].

The sums $\sum_{i=1}^{n} f(x_i).\Delta x$ are commonly called the *Riemann sums*.

In the Riemann sum, the Greek letter $\sum$ indicates a sum, same thing as the elongated "S", designated by $\int$; this latter sign is used as the integral sign and designates the limit of a sum. The points a and b are respectively called *lower* and *upper limits of integration* and indicate the endpoints of the interval on which integration is being done. Those points are implicitly indicated in the Riemann sums. The dx in the integral corresponds to the variation Δx in the Riemann sums and indicates the *variable of integration*. Notice that the choice of the letter used to represent the variable of integration does not have any effect since the value of the definite integral is a number and not a function of the variable. For this reason, the variable of integration is called a *dummy variable* (or a *figurative variable*). The function f(x) is called the *integrand*.

The choice of the points $x_0, x_1, x_2, ..., x_n$ on the left, on the right, or on the middle of the subintervals of the interval leads to the method (approach) of finding the area using left rectangles, right rectangles, or middle rectangles. That says the heights (or lengths) of rectangles are traced from points on the left (left rectangles), the right (right rectangles) or the middle (middle rectangles) of the subintervals.

The diagrams of left rectangles and right rectangles have been shown in Appendix F. The next diagram shows the middle rectangles (Key Curriculum Press, 2002, Electronic version).



1.2. THE DEFINITION OF THE INDEFINITE INTEGRAL

The indefinite integral, $\int f(x)dx$, gives the area bounded by the function f(x) and the x-axis when the limits of integration are not fixed (i.e. when the limits of integration are variable). Symbolically, the indefinite integral is represented as follow

$$\int f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \qquad (2)$$

where $\Delta x = \dfrac{x-a}{n}$ with *a* as the lower limit and *x* as the upper limit and $f(x_i)$ the heights of rectangles of base Δx.

When the limit in (2) exists, the indefinite integral is written as follows:

$$\int f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = F(x) - F(a) = F(x) + C \qquad (3)$$

Where *F(x)* is a function of variable *x* and *F(a)* is a constant obtained by replacing the variable *x* in the function *F(x)* by the value *a*.

In the second equality of the equation (3), the constant C is a constant that depends on the starting point (lower limit) a and is equivalent to $-F(a)$.

When the variable x in the indefinite integral is replaced by a fixed point b on the x-axis, we obtain the definite integral $\int_a^b f(x)dx$. Reporting this in formula (3), we obtain

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (4)$$

which is the area bounded by the curve of *f(x)* and the x-axis between the limits *x=a* and *x=b*.

On the other side when one differentiates each member of the equality

$\int f(x)dx = F(x) - F(a)$, one finds that

$$\left(\int_a^x f(t)dt\right)' = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = F'(x) = f(x). \qquad (5)$$

These results (4) and (5) have been retained by mathematicians to constitute what they called the fundamental theorem of calculus. This fundamental theorem of calculus is enunciated in two parts.

1.3. THE FUNDAMENTAL THEOREM OF CALCULUS

(Part I)

If f is continuous on [a, b] and F(x) is a function such that $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

(Part II)

If f is continuous on [a, b] and $F(x) = \int_a^x f(t)dt$, then

$F'(x) = f(x)$ on [a, b]

(or $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$).

2. OTHER PRE-PREPARED SUBJECT-MATTER

In addition to the preparation of the content about the definitions of integrals and the fundamental theorem of calculus, I prepared the followings properties of integrals.

Theorem 1: (Properties of integrals):

If f and g are integrable on [a, b] and c is any constant, then the following are true.

i. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$,

ii. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$,

iii. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ and

iv. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for any c in [a, b]

(These properties are demonstrated using the definition of integral)

v. For any integrable function f, if $a \neq b$, we have

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

(This can be demonstrated using the definition of integral)

vi. if *f(a)* is defined, then $\int_a^a f(x)dx = 0$ (in this case actually $\Delta x = 0$)

Average value of a function

$$f_{ave} = \lim_{n\to\infty}\left[\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x\right] = \frac{1}{b-a}\int_a^b f(x)dx$$

Theorem 2 (Integral Mean Value Theorem):

If f is continuous on [a, b], there is a number $c \in (a,b)$ for which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Theorem 3 (Derivative Mean Value Theorem):

(This theorem is useful for the demonstration of the FTC I)

Suppose that f is continuous on the interval [a, b] and differentiable on the interval (a, b). Then there exists a number $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Proof of the FTC I (Smith and Mouton, 2002, p.365)

First, we partition [a,b]:

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b, \text{ where } x_i - x_{i-1} = \Delta x = \frac{b-a}{n}, \text{ for } i = 1,2,3,\ldots,n.$$

Working backward and making necessary cancellation, one can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + \ldots + [F(x_n) - F(x_{n-1})] \end{aligned}$$

$$= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \qquad (1)$$

Since F is an antiderivative of f, F is differentiable on (a,b) and continuous on [a, b]. By the Mean Value Theorem, we have for each $i = 1,2,3,...,n$, that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x \qquad (2)$$

for some $c_i \in (x_{i-1}, x_i)$. Thus from (1) and (2), we have

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(c_i)\Delta x \qquad (3)$$

One should recognise this last expression as a Riemann sum for f on [a, b]. Taking the limit of both sides of (3), as $n \to \infty$, one find that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty} [F(b) - F(a)] = F(b) - F(a)$$

as desired since this last quantity is a constant.

Proof of the FTC II (Smith and Mouton, 2002, p. 368)

Using the definition of derivative, we have

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^{h} f(t)dt\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt + \int_x^{a} f(t)dt\right] = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt \quad (1)$$

where we switched the limits of integration according to theorem 1 (v) and combined the integrals according to theorem 1 (iv).

Looking at the last term in (1), one may recognise it as the limit of the average value of *f(x)* on the interval [*x*, *x*+*h*] (if h>0).

By the Integral Mean Value Theorem, we have

$$\frac{1}{h}\int_x^{x+h} f(t)dt = f(c) \quad (2)$$

for some number c between *x* and *x+h*. Finally, since c is between x and x+h, we have that $c \to x$, as $h \to 0$. Since f is continuous, it follows from (1) and (2) that

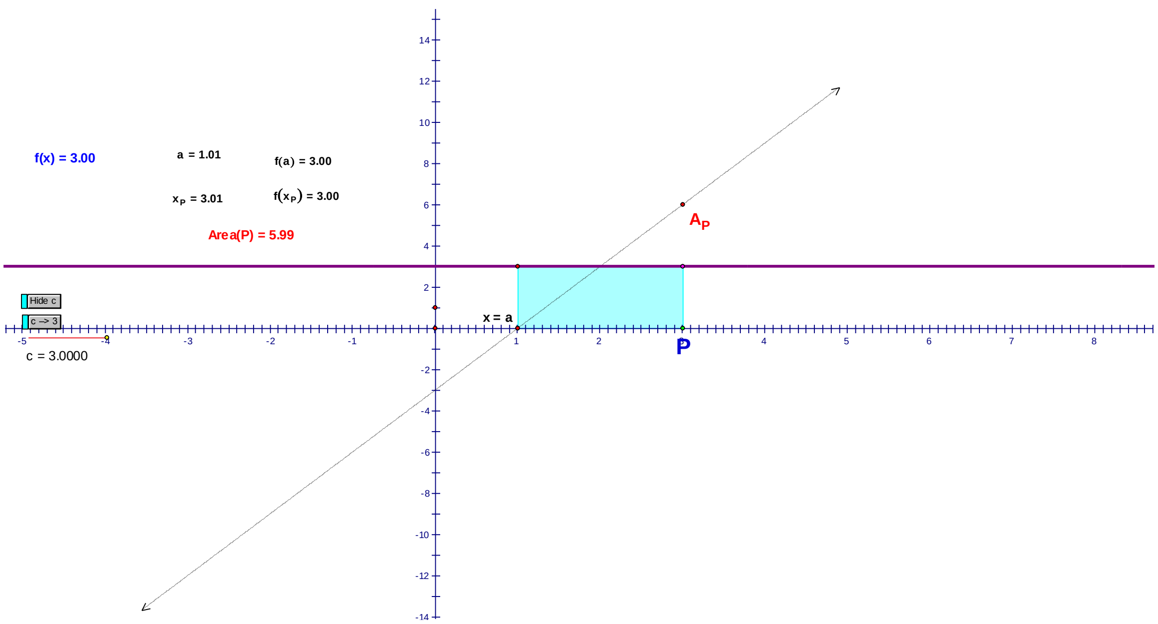
$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)dt = \lim_{h\to 0} f(c) = f(x)$$

as desired.

3. POSSIBLE ANSWERS FOR TASK THREE (LESSON SEVEN)

3.1. CONSTANT FUNCTION

$y = f(x) = 3$



(Key Curriculum Press, 2002, Electronic version)

1. Question 1: Starting point $x_0 = 0$

The area is equal to $\int_0^x 3dx = \lim_{n\to\infty} B_n \sum_{i=1}^{n} f(x_i) = \lim_{n\to\infty} \frac{x}{n}\sum_{i=1}^{n} 3 = \frac{x}{n}3n = 3x$

2. Question 2:

*Starting point $x_0 = 1$

$$\int_1^x 3dt = \lim_{n\to\infty} \frac{x-1}{n}\sum_{i=1}^{n} 3 = \lim_{n\to\infty} \frac{x-1}{n}3n = (x-1)3 = 3x-3$$

* Staring point $x_0 = 2$

$$\int_2^x 3dt = \lim_{n\to\infty} \frac{x-2}{n}\sum_{i=1}^{n} 3 = \lim_{n\to\infty} \frac{x-2}{n}3n = (x-2)3 = 3x-6$$

* Staring point $x_0 = 3$

$$\int_3^x 3dt = \lim_{n\to\infty} \frac{x-3}{n}\sum_{i=1}^{n} 3 = \lim_{n\to\infty} \frac{x-3}{n} 3n = (x-3)3 = 3x-9$$

* Staring point $x_0 = a$

$$\int_a^x 3dt = \lim_{n\to\infty} \frac{x-a}{n}\sum_{i=1}^{n} 3 = \lim_{n\to\infty} \frac{x-a}{n} 3n = (x-a)3 = 3x-3a = F(x)-F(a) = F(x)+C$$

3. Question 3:

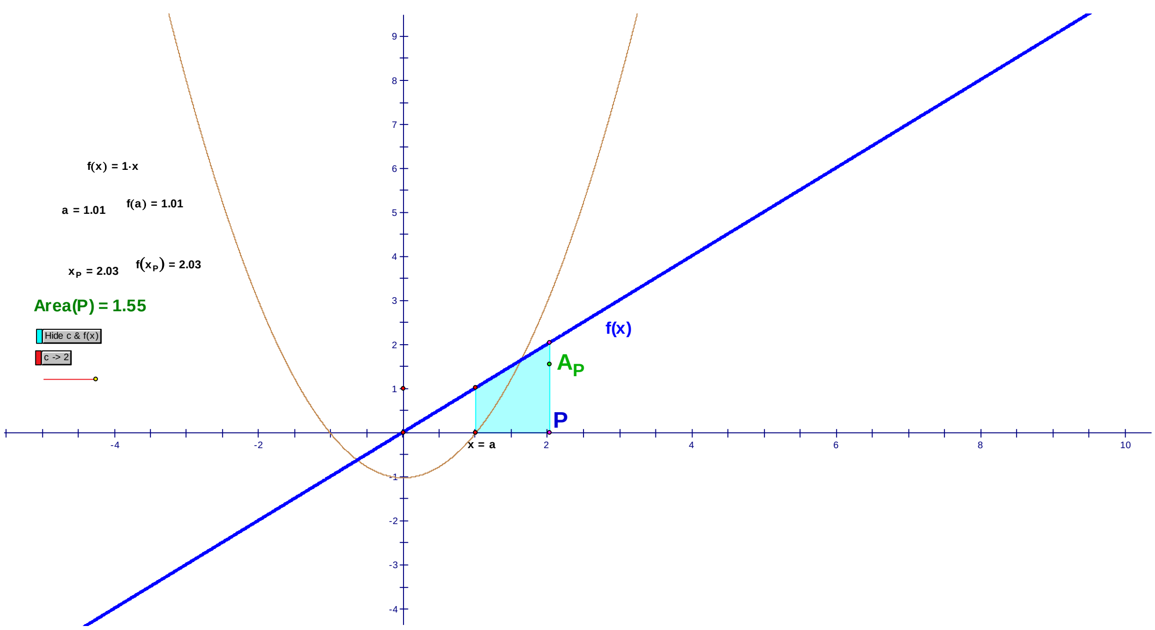
$$\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}(3x-3a) = 3 = f(x) = F'(x).$$

The derivative of the area bounded by the function f is f itself. The area F is the anti-derivative of f: $F'(x) = f(x)$

Thus, $\int_a^b F'(x)dx = F(b) - F(a)$.

3.2. LINEAR FUNCTION

$y = f(x) = x$



(Key Curriculum Press, 2002, Electronic version)

1. Question 1: Starting point $x_0 = 0$

The area is equal to $\int_0^x t dt = \lim_{n\to\infty} B_n \sum_{i=1}^{n} f(x_i) = \lim_{n\to\infty} \frac{x-0}{n} \sum_{i=1}^{n} (i\frac{x}{n}) = \lim_{n\to\infty} \frac{x}{n} \sum_{i=1}^{n} i\frac{x}{n} =$
$\lim_{n\to\infty} \frac{x^2}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{x^2}{n^2} (\frac{n^2}{2} + \frac{n}{2}) = \lim_{n\to\infty} (\frac{x^2}{2} + \frac{x^2}{2n}) = \frac{x^2}{2}$

2. Question 2:

*Starting point $x_0 = 1$

$\int_1^x t dt = \lim_{n\to\infty} \frac{x-1}{n} \sum_{i=1}^{n} f(1 + i(\frac{x-1}{n})) =$
$\lim_{n\to\infty} \frac{x-1}{n} \sum_{i=1}^{n} (1 + i(\frac{x-1}{n})) = \lim_{n\to\infty} (\frac{x-1}{n} \sum_{i=1}^{n} 1 + \frac{x-1}{n} \sum_{i=1}^{n} i(\frac{x-1}{n}))$
$= \lim_{n\to\infty} (\frac{x-1}{n} n + \frac{(x-1)^2}{n^2} \sum_{i=1}^{n} i) = (x-1) + \lim_{n\to\infty} \frac{(x-1)^2}{n^2} [\frac{n^2}{2} + \frac{n}{2}]$
$= (x-1) + \frac{(x-2)^2}{2} = x - 1 + \frac{x^2 - 2x + 1}{2} = \frac{2x - 2 + x^2 - 2x + 1}{2} = \frac{x^2}{2} - \frac{1}{2}$

* Starting point $x_0 = 2$

$\int_2^x t dt = \lim_{n\to\infty} (\frac{x-2}{n} \sum_{i=1}^{n} f(2 + i\frac{x-2}{n})) = \lim_{n\to\infty} (\frac{x-2}{n} \sum_{i=1}^{n} (2 + i\frac{x-2}{n})) =$
$\lim_{n\to\infty} \frac{x-2}{n} \sum_{i=1}^{n} 2 + \lim_{n\to\infty} \frac{x-2}{n} \sum_{i=1}^{n} i\frac{x-2}{n} = \lim_{n\to\infty} \frac{x-2}{n} 2n + \lim_{n\to\infty} (\frac{x-2}{n})^2 \sum_{i=1}^{n} i$

$(x-2)2 + \lim_{n\to\infty} \frac{(x-2)^2}{n^2} (\frac{n^2}{2} + \frac{n}{2}) = 2x - 4 + \frac{(x-2)^2}{2} = \frac{4x - 8 + x^2 - 4x + 4}{2}$
$= \frac{x^2}{2} - \frac{4}{2} = F(x) - F(2)$

* Starting point $x_0 = 3$

$\int_3^x t dt = \lim_{n\to\infty} \frac{x-3}{n} \sum_{i=1}^{n} f(3 + i\frac{x-3}{n}) = \lim_{n\to\infty} \frac{x-3}{n} \sum_{i=1}^{n} (3 + i\frac{x-3}{n})$
$= \lim_{n\to\infty} \frac{x-3}{n} \sum_{i=1}^{n} 3 + \lim_{n\to\infty} \frac{x-3}{n} \sum_{i=1}^{n} i\frac{x-3}{n} = \lim_{n\to\infty} \frac{x-3}{n} 3n + \lim_{n\to\infty} (\frac{x-3}{n})^2 \sum_{i=1}^{n} i$
$= (x-3)3 + \lim_{n\to\infty} \frac{(x-3)^2}{n^2} (\frac{n^2}{2} + \frac{n}{2}) = 3x - 9 + \frac{(x-3)^2}{2} = \frac{6x - 18 + x^2 - 6x + 9}{2} = \frac{x^2}{2} - \frac{9}{2}$
$= F(x) - F(3)$

* Starting point $x_0 = a$

$$\int_a^x tdt = \lim_{n\to\infty}\frac{x-a}{n}\sum_{i=1}^{n} f(a+i\frac{x-a}{n}) = \lim_{n\to\infty}\frac{x-a}{n}\sum_{i=1}^{n}(a+i\frac{x-a}{n})$$

$$= \lim_{n\to\infty}\frac{x-a}{n}\sum_{i=1}^{n}a + \lim_{n\to\infty}\frac{x-a}{n}\sum_{i=1}^{n}i\frac{x-a}{n} = \lim_{n\to\infty}\frac{x-a}{n}na + \lim_{n\to\infty}(\frac{x-a}{n})^2\sum_{i=1}^{n}i$$

$$= (x-a)a + \lim_{n\to\infty}(\frac{x-a}{n})^2(\frac{n^2}{2}+\frac{n}{2}) = ax-a^2+\frac{(x-a)^2}{2} = \frac{2ax-2a^2+x^2-2ax+a^2}{2}$$

$$= \frac{x^2}{2}-\frac{a^2}{2} = F(x)-F(a)$$

Thus $\int_a^x f(t)dt = F(x)-F(a) = F(x)+C$

3a). For a linear function f, we have $\int f(x)dx = F(x)-F(a) = F(x)+C$

Therefore $\frac{dF}{dx} = f(x)$ or $F'(x) = f(x)$

3b). The derivative of the area function delimited by the function f is this function itself.

3.3. QUADRATIC FUNCTION

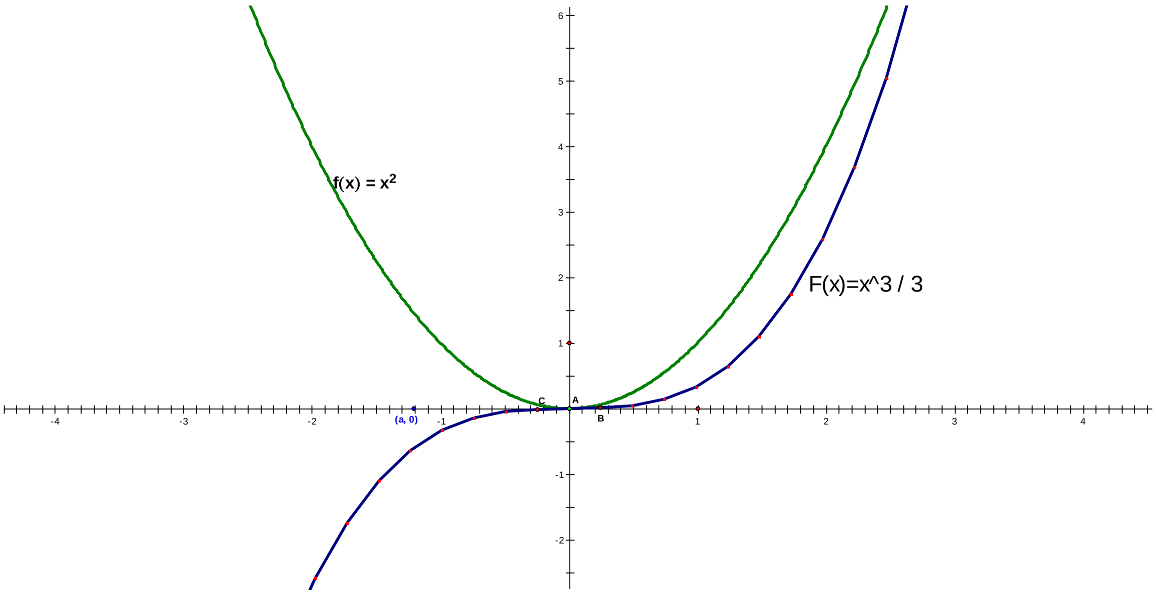
$$y = f(x) = x^2$$

1. Question 1: Starting point $x_0 = 0$

The area is equal to

$$\int_0^x t^2dt = \lim_{n\to\infty}B_n\sum_{i=1}^{n} f(x_i) = \lim_{n\to\infty}\frac{x}{n}\sum_{i=1}^{n} f(xi) = \lim_{n\to\infty}\frac{x}{n}\sum_{i=1}^{n}(i\frac{x}{n})^2 = \lim_{n\to\infty}\frac{x}{n}\sum_{i=1}^{n}i^2(\frac{x}{n})^2$$

$$= \lim_{n\to\infty}(\frac{x}{n})^3\sum_{i=1}^{n}i^2 = \lim_{n\to\infty}\frac{x^3}{n^3}(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}) = \frac{x^3}{3}$$



(Key Curriculum Press, 2002, Electronic version)

2. Question 2:

*Starting point $x_0 = 1$

$$\int_1^x t^2 dt = \lim_{n\to\infty} \frac{x-1}{n} \sum_{i=1}^{n} (1 + i\frac{x-1}{n})^2 = \lim_{n\to\infty} \frac{x-1}{n} \sum_{i=1}^{n} (1 + 2i\frac{x-1}{n} + i^2 (\frac{x-1}{n})^2$$

$$= \lim_{n\to\infty} \frac{x-1}{n} \sum_{i=1}^{n} 1 + \lim_{n\to\infty} \frac{x-1}{n} 2 \frac{x-1}{n} \sum_{i=1}^{n} i + \lim_{n\to\infty} \frac{x-1}{n} (\frac{x-1}{n})^2 \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{x-1}{n} n + \lim_{n\to\infty} (\frac{x-1}{n})^2 2 \sum_{i=1}^{n} i + \lim_{n\to\infty} (\frac{x-1}{n})^3 \sum_{i=1}^{n} i^2$$

$$= x - 1 + \lim_{n\to\infty} 2(\frac{x-1}{n})^2 (\frac{n^2}{2} + \frac{n}{2}) + \lim_{n\to\infty} (\frac{x-1}{n})^3 (\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6})$$

$$= (x-1) + 2\frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

$$= x - 1 + x^2 - 2x + 1 + \frac{x^3 - 3x^2 + 3x - 1}{3}$$

$$= \frac{3x - 3 + 3x^2 - 6x + 3 + x^3 - 3x^2 + 3x - 1}{3} = \frac{x^3}{3} - \frac{1}{3} = F(x) - F(1)$$

* Staring point $x_0 = 2$

$$\int_2^x t^2 dt = \lim_{n\to\infty} \frac{x-2}{n} \sum_{i=1}^{n} (2 + i(\frac{x-2}{n}))^2 = \lim_{n\to\infty} \frac{x-2}{n} \sum_{i=1}^{n} (4 + 4i\frac{x-2}{n} + i^2 (\frac{x-2}{n})^2)$$

$$= \lim_{n\to\infty} \frac{x-2}{n}[\sum_{i=1}^{n} 4 + 4\frac{x-2}{n}\sum_{i=1}^{n} i + (\frac{x-2}{n})^2 \sum_{i=1}^{n} i^2]$$

$$= \lim_{n\to\infty} \frac{x-2}{n}\sum_{i=1}^{n} 4 + \lim_{n\to\infty} 4(\frac{x-2}{n})^2 \sum_{i=1}^{n} i + \lim_{n\to\infty} (\frac{x-2}{n})^3 \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{x-2}{n} 4n + \lim_{n\to\infty} 4(\frac{x-2}{n})^2 (\frac{n^2}{2} + \frac{n}{2}) + \lim_{n\to\infty} (\frac{x-2}{n})^3 (\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{2})$$

$$= (x-2)4 + \frac{4}{2}(x-2)^2 + \frac{(x-2)^3}{3}$$

$$= 4x - 8 + 2x2 - 8x + 8 + \frac{x^3 - 6x^2 + 12x - 8}{3}$$

$$= \frac{12x - 24 + 6x^2 - 24x + 24 + x^3 - 6x^2 + 12x - 8}{3}$$

$$= \frac{x^3 - 8}{3} = \frac{x^3}{3} - \frac{8}{3} = \frac{x^3}{3} - \frac{2^3}{3} = F(x) - F(2) = F(x) + C$$

3. Therefore, $\frac{d}{dx}\int_a^x f(t)dt = f(x)$, i.e. $F'(x) = f(x)$

In words, the derivative of area delimited by the function is the function itself.

* Staring point $x_0 = 3$

$$\int_3^x t^2 dt = \lim_{n\to\infty} \frac{x-3}{n}\sum_{i=1}^{n}(3 + i\frac{x-3}{n})^2 = \lim_{n\to\infty} \frac{x-3}{n}\sum_{i=1}^{n}[9 + 6i\frac{x-3}{n} + i^2(\frac{x-3}{n})^2]$$

$$= \lim_{n\to\infty} \frac{x-3}{n}\sum_{i=1}^{n} 9 + \lim_{n\to\infty} 6(\frac{x-3}{n})^2 \sum_{i=1}^{n} i + \lim_{n\to\infty} (\frac{x-3}{n})^3 \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{x-3}{n} 9n + \lim_{n\to\infty} 6(\frac{x-3}{n})^2 (\frac{n^2}{2} + \frac{n}{2}) + \lim_{n\to\infty} (\frac{x-3}{n})^3 (\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6})$$

$$= (x-3)9 + 6\frac{(x-3)^2}{2} + \frac{(x-3)^3}{3}$$

$$= 9(x-3) + 3(x^2 - 6x + 9) + \frac{x^3 - 9x^2 + 27x - 27}{3}$$

$$= 9x - 27 + 3x^2 - 18x + 27 + \frac{x^3 - 9x^2 + 27x - 27}{3}$$

$$= \frac{27x - 81 + 9x^2 - 54x + 81 + x^3 - 9x^2 + 27x - 27}{3} = \frac{x^3 - 27}{3} = \frac{x^3}{3} - \frac{3^3}{3} = F(x) - F(3) =$$

$$F(x) + C$$

* Staring point $x_0 = a$

$$\int_a^x t^2 dt = \lim_{n\to\infty}\frac{x-a}{n}\sum_{i=1}^{n}(a+i\frac{x-a}{n})^2 = \lim_{n\to\infty}\frac{x-a}{n}\sum_{i=1}^{n}[a^2+2ai\frac{x-a}{n}+i^2(\frac{x-a}{n})^2]$$

$$=\lim_{n\to\infty}\frac{x-a}{n}\sum_{i=1}^{n}a^2 + \lim_{n\to\infty}2a(\frac{x-a}{n})^2\sum_{i=1}^{n}i + \lim_{n\to\infty}(\frac{x-a}{n})^3\sum_{i=1}^{n}i^2$$

$$=\lim_{n\to\infty}\frac{x-a}{n}na^2 + \lim_{n\to\infty}2a(\frac{x-a}{n})^2(\frac{n^2}{2}+\frac{n}{2}) + \lim_{n\to\infty}(\frac{x-a}{n})^3(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{2})$$

$$= (x-a)a^2 + 2a\frac{(x-a)^2}{2} + \frac{(x-a)^3}{3}$$

$$= a^2(x-a) + a(x^2-2ax+a^2) + \frac{x^3-3ax^2+3a^2x-a^3}{3}$$

$$= a^2x - a^3 + ax^2 - 2a^2x + a^3 + \frac{x^3-3ax^2+3a^2x-a^3}{3}$$

$$= \frac{3a^2x-3a^2+3ax^2-6a^2x+3a^2+x^3-3ax^2+3a^2x-a^3}{3}$$

$$= \frac{x^3-a^3}{3} = \frac{x^3}{3}-\frac{a^3}{3} = F(x)-F(a) = F(x)+C$$

3. Question3:

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}(3x-3a) = 3 = f(x) = F'(x).$$

The derivative of the area bounded by the function f is f itself. The area F is the anti-derivative of f: $F'(x) = f(x)$

Thus, $\int_a^b F'(x)dx = F(b)-F(a)$.

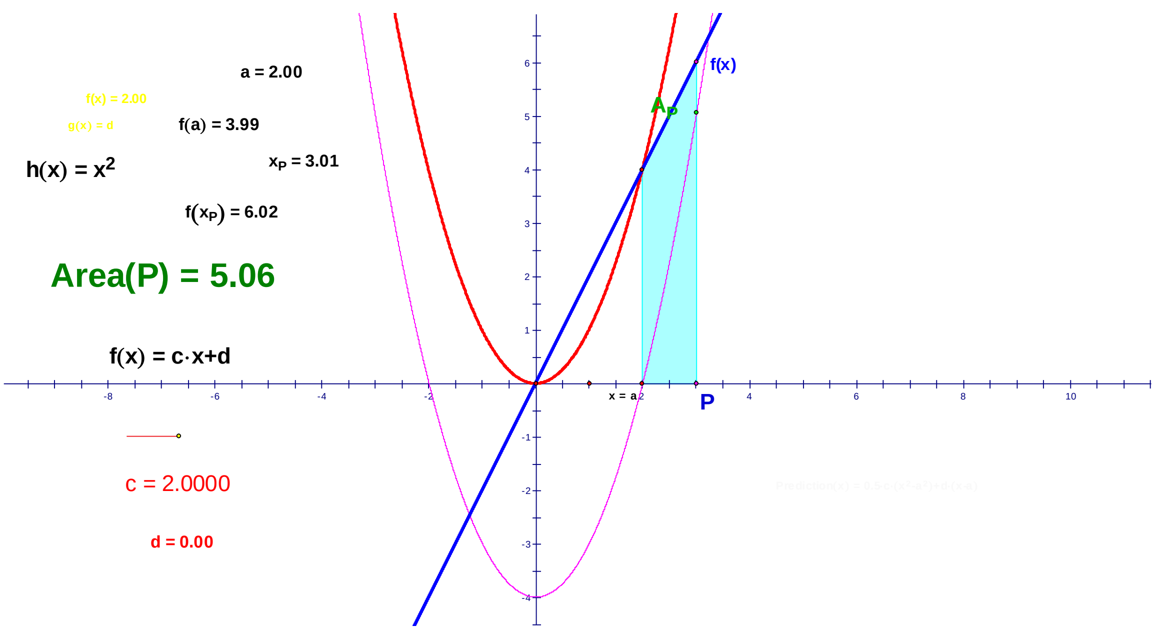
In conclusion the integral $\int_a^x f(x)dx = F(x)-F(a)$ is a primitive of f (x), or f(x) is the derivative of the integral.

4. PEDAGOGICAL SEQUENCE FOR A COMPUTER-BASED EXPLORATION OF INTEGRALS AND THE FTC: USE OF A TABLE METHOD AND GEOMETER'S SKETCHPAD (GSP)

The following is the pre-prepared pedagogical sequence that I used to make the students explore integrals with the help of computer and Geometer's sketchpad software. The task is based on the diagram below adapted from Key Curriculum Press (2002).

1) Use the illustrative example of f(x) = 2x to find another way of getting area under curve without using Riemann method
2) Students must think about how we emphasise area as a function and area as a difference value
3) Structure of task:
   - Fix f(x)
   - Choose x_a and then vary x_p
   - Then choose other options for x_a

4) Illustrative example f(x)= 2x, with $x_a = 2$, $x_p = 3$



(Key Curriculum Press, 2000, Electronic version)

5) Table showing cumulative area from x_a=2 to given x-value A(x)

From $x_a = 2$	to $x_p = 3$	$x_p = 4$	$x_p = 5$	$x_p = 6$	X_p= k	Any x
Cum area A(x)	5	12	21	32	K^2-4	

6) Need to think about A-notation:
   - Opportunity for students to come up with their own notation
   - Need to incorporate the starting value and end value in the notation
7) May want other values for cumulative area beyond those in the table, e.g. Fractions between whole numbers
8) Need to think about introducing "any x" idea and using x rather than a parameter such as k. This may help the transition to the idea of a function.

9) Students find formula from table $A(x) = x^2 - 4$
   - May not see pattern easily, prompt with link to perfect squares
     - easiest option with perfect squares is (perfect square) - 4
     - some might look at perfect square that lies below the value of A(x) as follows:

5 = 4 + 1
12 = 9 + 3
21 = 16 + 5
32 = 25 + 7

It is possible to get to equation but fairly complicated. Key issues: relate to x-value, notice that consecutive odd numbers are being added, get expression for odd numbers in relation to x.

x	**A(x)**	**Expression with perfect square**	**Linking to term in sequence**	**Linking all to x** Term number is x-2
3	5	4 + 1	$(x-1)^2 + T1$	$(x-1)^2 + 2(x-2) - 1$
4	12	9 + 3	$(x-1)^2 + T2$	$= x^2 - 2x + 1 + 2x -$
5	21	16 + 5	$(x-1)^2 + T3$	$4 - 1$
6	32	25 + 7	$(x-1)^2 + T4$	$= x^2 - 4$

10) Question to ask students: "What is the relationship between f (x) and A(x)?"
    - Want them to see that A'(x) = f (x)
11) Assume students already know how to find anti-derivative, F(x), of power functions
    - This implies that some work has been done on derivatives and anti-derivatives and students know that the anti-derivative is not unique
    - Students must determine an anti-derivative of f(x) ($F(x) = x^2$)
    - It must be clear that we are working with ONE POSSIBLE anti-derivative – the simplest one
12) Question to students: "What is the relationship between anti-derivative and cumulative area?" (perhaps avoid F(x) and A(x)?)
    - Want them to see that anti-derivative + constant = cumulative area, that is,
      F(x) – 4 = A(x), e.g. F(3) = 9 but A(3) = 5, so the difference is 4
      F(3) – 4 = A(3) etc.
13) Shift to graphic representation

- Draw graph of F(x) and A(x) on paper
- Make link between 2 graphs – vertical shift is the value of the constant

Question: "How have we moved graph of F(x) to get graph of A(x)?"
OR
Draw only F(x) and move to get graph represented by A(x)
OR
Draw both, trace one and move traced graph onto other graph.

- Must see link between F(x) – 4 and vertical shift downwards
- Must see that cumulative area is different from anti-derivative

14) Using graphs to confirm and predict

Question to students: "How can we use the graph of A(x) to check the area between 2 and 5? To predict area between 2 and 6"

- Here they must see area as a particular value of a function (A(x))

Question to students: "How can we use the graph of F(x) to check the area between 2 and 5, to predict area between 2 and 6, to predict area between 2 and 4.5?"

- How they must see area as difference value
- Vertical difference between function values (of F(x))

15) Repeat all of the above for $x_a = 3$
16) Students need to see link between what is subtracted, the value of x_a and the function F(x), vertical shift of F(x)

17) Then students could choose their own values of x_a,

- Must avoid negative values at this stage - will be dealt with later
- Could use any value, including 0 or 1. But these could be considered "special cases" since 0 hides the "subtraction component" and 1 hides the fact that the constant must be squared. Perhaps tutor should encourage some students to try 0 and 1, and then get whole class to explore whether these contradict the pattern/conjectures or whether they are consistent.

18) Question to students: "If you wanted to find the area from x = 1.5 to x = 6, how would you do it using only the graphs?"
19) Focus on varying the value of x_a to get pattern of $A(x_p) = F(x_p) – x_a^2 = F(x_p) – F(x_a)$
20) Students must accept that $F(x_p) – F(x_a)$ gives the cumulative area before they move on.
21) Must now shift back to irregular functions

APPENDIX I: Ethical clearance certificate

UNIVERSITY OF
KWAZULU-NATAL

RESEARCH OFFICE (GOVAN MBEKI CENTRE)
WESTVILLE CAMPUS
TELEPHONE NO.: 031 – 2603587
EMAIL : ximbap@ukzn.ac.za

14 SEPTEMBER 2005

MR. F HABINEZA (204520800)
EDUCATION

Dear Mr. Habineza

ETHICAL CLEARANCE APPROVAL NUMBER : HSS/05105A

I wish to confirm that ethical clearance has been granted for the following project:

> **"Developing the understanding of the concepts of definite and indefinite integrals and their link through the fundamental theorem of calculus of first year mathematics student teachers in Rwanda"**

Yours faithfully

....

MS. PHUMELELE XIMBA
RESEARCH OFFICE

PS: The following general condition is applicable to all projects that have been granted ethical clearance:

> **THE RELEVANT AUTHORITIES SHOULD BE CONTACTED IN ORDER TO OBTAIN THE NECESSARY APPROVAL SHOULD THE RESEARCH INVOLVE UTILIZATION OF SPACE AND/OR FACILITIES AT OTHER INSTITUTIONS/ORGANISATIONS. WHERE QUESTIONNAIRES ARE USED IN THE PROJECT, THE RESEARCHER SHOULD ENSURE THAT THE QUESTIONNAIRE INCLUDES A SECTION AT THE END WHICH SHOULD BE COMPLETED BY THE PARTICIPANT (PRIOR TO THE COMPLETION OF THE QUESTIONNAIRE) INDICATING THAT HE/SHE WAS INFORMED OF THE NATURE AND PURPOSE OF THE PROJECT AND THAT THE INFORMATION GIVEN WILL BE KEPT CONFIDENTIAL.**

cc. Faculty Officer
cc. Supervisor (I M Christiansen)

APPENDIX J: Other ethical documents

1. CONSENT INFORM

Je, soussigné, Etudiant au Département des Sciences Intégrées du KIGALI INSTITUTE OF EDUCATION (RWANDA), autorise Monsieur HABINEZA Faustin à collecter les données sur mon apprentissage dans le cours de Calcul Différentiel et Intégral (IS 102) et de les utiliser dans ses recherches sur le dévelopement de l'image conceptuelle des intégrales.

Ces données seront strictement confidentielles et ne seront utilisées que pour des fins de recherche et d'évaluation formative. Toute référence à un individu sera anonyme.

Noms	Prénoms	Signature	Date

2. AUTORISATION OF KIGALI INSTITUTE OF EDUCATION (next page)

Kigali Institute of Education
Faculty of Sciences
Department of Integrated Science

EDUCATION FOR DEVELOPMENT

To : The Rector, KIE.

From: Acting Head of Department (ISE).

Date: 17th December 2004

RE : Faustin Habineza is back for research and lectures

This is to inform you that Faustin Habineza, a lecturer in the Integrated Science Department is back for research in connection with his PhD. The department took the opportunity and discussed with him and he is ready to teach some Maths courses during the academic year 2005 while carrying out his research.
Courses that the mentioned lecturer will cover are the following:

1. **IS 102: Calculus I**
2. **IS 108: Calculus II**
3. **IS 304: Analytical Geometry I**
4. **IS 401: Ordinary Differential Equations**
5. **IS 209: Ordinary and Partial Differential Equations.**

The Integrated Science Department takes this opportunity to request that Habineza recovers during the next academic year some allowances that have been deducted from his salary when he went for further studies.

Kind regards.

Casimir M Karasira. CMKaras.

Acting HOD ISE.

CC:

- **Academic Vice-Rector**
- **Administrative and finance Vice-rector**
- **Academic registrar**
- **Chef du personnel**
- **Dean, Faculty of Science**
- ✓ **Habineza Faustin**