ASPECTS OF FUNCTIONAL VARIATIONS OF DOMINATION IN GRAPHS

by

Laura Marie Harris

Submitted in fulfillment of the
requirements for the degree of
Doctor of Philosophy
in the
School of Mathematics, Statistics and Information Technology
University of Natal
Pietermaritzburg.

November 2003

Abstract

Let $G = (V, E)$ be a graph. For any real valued function $f : V \rightarrow \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The weight of f is defined as $f(V)$. A signed k-subdominating function (signed kSF) of G is defined as a function $f : V \rightarrow \{-1, 1\}$ such that $f(N[v]) \geq 1$ for at least k vertices of G, where $N[v]$ denotes the closed neighborhood of v. The signed k-subdomination number of a graph G, denoted by $\gamma_{ks}^{-11}(G)$, is equal to $\min\{f(V) \mid f \text{ is a signed } kSF \text{ of } G\}$. If instead of the range $\{-1, 1\}$, we require the range $\{-1, 0, 1\}$, then we obtain the concept of a minus k-subdominating function. Its associated parameter, called the minus k-subdomination number of G, is denoted by $\gamma_{ks}^{-101}(G)$.

A total signed dominating function (signed $TkSF$) of G is defined as a function $f : V \rightarrow \{-1, 1\}$ such that $f(N(v)) \geq 1$ for at least k vertices of G, where $N(v)$ denotes the open neighborhood of v. The total signed k-subdomination number of a graph G, denoted by $\gamma_{tks}^{-11}(G)$, is equal to $\min\{f(V) \mid f \text{ is a signed } TkSF \text{ of } G\}$. If instead of the range $\{-1, 1\}$, we require the range $\{-1, 0, 1\}$, then we obtain the concept of a total minus k-subdominating function. Its associated parameter, called the total minus k-subdomination number of G, is denoted by $\gamma_{tks}^{-101}(G)$.

In Chapter 2, we survey recent results on signed and minus k-subdomination in graphs.

In Chapter 3, we compute the signed and minus k-subdomination numbers for certain complete multipartite graphs and their complements, generalizing results due to Holm [30].

In Chapter 4, we give a lower bound on the total signed k-subdomination number in terms of the minimum degree, maximum degree and the order of the graph. A lower bound in terms of the degree sequence is also given. We then compute the

total signed k-subdomination number of a cycle, and present a characterization of graphs G with equal total signed k-subdomination and total signed ℓ-subdomination numbers. Finally, we establish a sharp upper bound on the total signed k-subdomination number of a tree in terms of its order n and k where $1 \leq k < n$, and characterize trees attaining these bounds for certain values of k. For this purpose, we first establish the total signed k-subdomination number of simple structures, including paths and spiders.

In Chapter 5, we show that the decision problem corresponding to the computation of the total minus domination number of a graph is **NP**-complete, even when restricted to bipartite graphs or chordal graphs. For a fixed k, we show that the decision problem corresponding to determining whether a graph has a total minus domination function of weight at most k may be **NP**-complete, even when restricted to bipartite or chordal graphs. Also in Chapter 5, linear time algorithms for computing $\gamma_{tns}^{-11}(T)$ and $\gamma_{tns}^{-101}(T)$ for an arbitrary tree T are presented, where $n = n(T)$.

In Chapter 6, we present cubic time algorithms to compute $\gamma_{tks}^{-11}(T)$ and $\gamma_{tks}^{-101}(T)$ for a tree T. We show that the decision problem corresponding to the computation of $\gamma_{tks}^{-11}(G)$ is **NP**-complete, and that the decision problem corresponding to the computation of $\gamma_{tks}^{-101}(T)$ is **NP**-complete, even for bipartite graphs. In addition, we present cubic time algorithms to compute $\gamma_{ks}^{-11}(T)$ and $\gamma_{ks}^{-101}(T)$ for a tree T, solving problems appearing in [25].

For Jim and Michael

Preface

The work described in this thesis was carried out under the supervision and direction of Professor Michael A. Henning, School of Mathematics, Statistics and Information Technology, University of Natal, Pietermaritzburg and Professor Johannes H. Hattingh, Department of Mathematics and Statistics, Georgia State University, Atlanta, Georgia, USA, from May 1999 to September 2003.

The thesis represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:

Laura Marie Harris

Professor Michael A. Henning (Supervisor)

Professor Johannes H. Hattingh (Co-Supervisor)

Acknowledgments

I wish to thank ...

Johan Hattingh, whose confidence in me allowed me to pursue a dream. I will always appreciate his guidance, encouragement and kindness. I am honored to have him as an advisor.

Michael Henning, whose help in an uncertain time made all of this possible. He has my deepest gratitude for all of his assistance. He is truly a pleasure to know.

Wayne Goddard, whose assistance and generosity provided the funding for this research.

Elna Ungerer, who helped begin this process. I thoroughly enjoyed working with her.

Robert Parris, a true friend. He assisted in many ways on this project.

Jim, my daily strength and support. Words cannot express my appreciation for everything he means to me.

Michael, the miracle in my life. He is an angel and the answer to my prayers. There is nothing I would rather be than his mother.

Mom and Dad. I honestly give thanks every day for them and the home they made for us. There is no way to repay the gifts they have given me. I pray that I will demonstrate my gratitude through the home I help build for Michael.

John, who has always represented more than a brother and a friend. Words cannot adequately describe my admiration and love for him. I am so proud to be his sister.

Priscilla, who has been a source of encouragement and friendship from the first time we met. I will always be grateful that she and her loving family are a part of my life.

Sophie and Will, my beautiful niece and nephew. They bring joy to the lives of all who know them and strengthen my hope in the future.

Ann and Bill, who raised two fine sons. I hope to do as well with Michael.

No word of thanks would be complete without remembering my grandparents. They were four very different people and each one had the ability to make me feel infinitely precious. I miss them more than I can say.

Contents

Chapter 1

INTRODUCTION

In the first section of this chapter, we define the necessary concepts that will be used throughout the thesis. Then, in Section 1.2, we give a brief overview of the history of domination theory and define the necessary domination concepts that will be used.

1.1 Preliminary definitions

A *graph* G is a finite nonempty set of objects called *vertices* (the singular is *vertex*) together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$ (or V if no confusion is likely), while the *edge set* of G is denoted by $E(G)$ (or E). The number of vertices in $V(G)$ is denoted by $n(G)$ which is also known as the *order* of the graph G, while the number of edges in $E(G)$ is denoted by $m(G)$. A graph G is *trivial* if $n(G) = 1$ and *non-trivial* if $n(G) \geq 2$. For a graph G, if $n(G) = n$ and $m(G) = m$, then G is called a (n, m)-graph. Unless otherwise specified, the symbols n and m (or $n(G)$ and $m(G)$) will be reserved exclusively for the order and number of edges, respectively, of a graph G. By $G = (V, E)$ we will imply the graph G with vertex set V and edge set E.

The edge $e = uv$ is said to *join* the vertices u and v. If $e = uv$ is an edge of G, then u and v are *adjacent vertices*, while u and e are *incident* as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*.

The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ and such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G.

The *degree of a vertex* v in a graph G is the number of edges of G incident with v and is denoted by $\deg(v)$. If the graph G with respect to which the degree is considered is not clear from the context, we shall write $\deg_G(v)$ to denote such a degree. The *minimum degree* of a vertex in G is denoted by $\delta(G)$ and the *maximum degree* by $\Delta(G)$. If there is a vertex $v \in V(G)$ such that $\deg(v) = 0$, then v is called an *isolated vertex*, and if $\deg(v) = 1$, then v is called an *end-vertex* of G. A vertex is called *odd* or *even* depending on whether its degree is odd or even.

The vertex adjacent to an end-vertex is called a *remote vertex* of G. A graph is *regular of degree* r if for each vertex v of G, $\deg(v) = r$; such graphs are also called *r-regular*. A graph is *complete* if every two of its vertices are adjacent. A complete (n, m)-graph is therefore a regular graph of degree $n - 1$ having $m = \frac{n(n-1)}{2}$ edges; we denote this graph by K_n. The complement $\overline{K_n}$ of K_n has n vertices and no edges and is referred to as the *empty graph* of order n.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, then we write $H \subseteq G$. If U is a nonempty subset of the vertex set $V(G)$ of a graph G, then the *subgraph $G[U]$ of G induced by U* is the graph with vertex set U and whose edge set consists of all those edges of G incident with two elements of U.

Let u and v be (not necessarily distinct) vertices of a graph G. A *u-v walk* of G is a finite, alternating sequence $u = v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n = v$ of vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = v_{i-1}v_i$ for $i = 1, 2, \ldots, n$. The number n (the number of occurrences of edges) is called the *length* of the walk. A *trivial walk* contains no edges. Often only the vertices of a walk are indicated since the edges present are then evident. A u-v walk is *closed* or *open* depending on whether $u = v$ or $u \neq v$. A *u-v trail* is a u-v walk in which no edge is repeated, while a *u-v path* is a u-v walk in which no vertex is repeated. A nontrivial closed trail of a graph G is referred to as a *circuit* of G, and a circuit $v_1, v_2, \ldots, v_n, v_1$ ($n \geq 3$) whose n vertices are distinct is called a *cycle*. A graph of order n that is a path (or a cycle) is denoted by P_n (or C_n), respectively. Therefore, $P_n : v_1, v_2, \ldots, v_n$ indicates a path of length $n-1$ on the vertices $v_1, v_2, \ldots, v_n$, while C_n indicates a cycle of length n on the same vertices.

A graph G is *connected* if for every pair of distinct vertices $u, v \in V(G)$ there exists a path between u and v in G, and *disconnected* if it is not connected. The relation 'is connected to' is an equivalence relation on the vertex set of every graph G. Each subgraph induced by the vertices in a resulting equivalence class is called a *connected component* or simply a *component* of G.

A *tree* is a connected graph which has no cycles. A *directed tree* is an asymmetric digraph whose underlying graph is a tree [7]. A directed tree T is called a *rooted tree* if there exists a vertex r of T, called the *root*, such that for every vertex v of T, there is an r-v path in T. If T is a rooted tree, then it is customary to draw T with root r at the top, say level 0, the vertices adjacent to r are placed one level below, at level 1, and any vertex adjacent to a vertex at level 1 is at level 2, etc. More formally, a vertex x in a rooted tree with root r is at level i if and only if the r-x path in T has length i. Let T be a rooted tree. If a vertex v of T is adjacent to u and u lies in the level below v, then u is called a *child* of v, and v is the *parent* of

u. An end-vertex of a tree, or a vertex with no children, is also called a *leaf* of the tree. The set of all of the leaves of a tree T is denoted by $L(T)$. A vertex that is adjacent to a leaf is called a *support vertex*.

A *spider* is a tree with at most one vertex of degree greater than two. If a spider T is a path, we call one of the end-vertices of T the *head* of the spider; otherwise, we call the vertex of maximum degree the *head* of the spider. The paths emanating from the head of the spider we call the *legs* of the spider. An *even spider* is a spider with all legs of even length. In particular, a path of even length (and therefore odd order) is an even spider.

If $U \subseteq V(G)$ then $G - U$ indicates the subgraph of G induced by the vertices of $V(G) - U$. If $F \subseteq E(G)$ then $G - F$ indicates the subgraph of G with vertex set equal to that of G and edge set consisting of all edges in $E(G) - F$. Suppose G_1 and G_2 are two graphs with disjoint vertex sets. Then the *union* $G = G_1 \cup G_2$ has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. A graph G is *isomorphic* to a graph H, denoted $G \cong H$, if there exists a one-to-one mapping ϕ, called an *isomorphism*, from $V(G)$ onto $V(H)$ such that ϕ preserves adjacency, that is $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$.

A graph G is *r-partite*, $r \geq 1$, if it is possible to partition V into r subsets $V_1, V_2, \ldots, V_r$ (called *partite sets*) such that every element of E joins a vertex of V_i to a vertex of $V_j, i \neq j$. If G is a 1-partite graph of order n, then $G \cong \overline{K_n}$. For $r = 2$, such graphs are called *bipartite graphs*, and where the specification of r is of no significance, an r-partite graph is also referred to as a *multipartite graph*. A *complete r-partite graph* G is an r-partite graph with partite sets $V_1, V_2, \ldots, V_r$ having the added property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$. If $|V_i| = n_i$, then this graph is denoted by $K(n_1, n_2, \ldots, n_r)$. (The order of the numbers $n_1, n_2, \ldots, n_r$ is not important.) A complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$, is denoted by $K(m, n)$ or $K_{m,n}$. The graph $K_{1,n-1}$ is called a *star* and its order is n; in this case, the vertex in the partite set containing only one vertex is referred to as the *center* of the star. The *split graph*, denoted by $S_{m,n}$, is defined as the complete multipartite graph with one partite set of cardinality m and n partite sets of cardinality one.

The following definitions and explanations concerning the theory of **NP**-completeness may be found in [12]. The theory of **NP**-completeness provides many straightforward techniques for proving that a given problem is "just as hard" as some other problems that are known to be **NP**-hard and have been confounding experts for years. The knowledge that a certain problem is **NP**-complete provides valuable information on approaching the problem. In short, the primary application of the theory of **NP**-completeness is to assist algorithm designers on deciding whether a particular problem is difficult or not. A problem is *intractable* if there is no polynomial time algorithm that can solve it. The principal technique used for demonstrating that two problems are related is that of transforming one to the other. Such a transformation provides the means for converting any algorithm that solves the second problem into a corresponding algorithm for solving the first problem. The class **NP** of decision problems (problems whose solutions are either "yes" or "no") is a class of problems that can be solved in polynomial time by a nondeterministic computer. The equivalence class consisting of the "hardest" problems in **NP** is known as the class of **NP**-*complete problems.* However, problems outside of **NP** may also be hard. A problem which is at least as hard as the **NP**-complete problems, is called **NP**-*hard.*

The honor of being the "first" **NP**-complete problem goes to a decision problem from Boolean logic known as the **SATISFIABILITY problem** (**SAT**, for short). Let $U = \{u_1, u_2, \ldots, u_m\}$ be a set of Boolean variables. A *truth assignment* for U is a function $t : U \rightarrow \{T, F\}$. If $t(u) = T$ we say that u is "true" under t, and if $t(u) = F$ we say that u is "false". If u is a variable in U, then u and $\overline{u}$ are *literals* over u. The literal $\overline{u}$ is true if and only if the variable u is false.

A *clause* over U is a set of literals over U, such as $\{u_1, \overline{u}_3, u_8\}$. A collection C of clauses over U is *satisfiable* if and only if there exists some truth assignment for U that simultaneously satisfies all the clauses in C.

For any further information on the subject of **NP**-completeness the reader is referred to [12].

1.2 Domination in graphs

The earliest ideas of dominating sets seem to date back to the origin of the game of chess in India over 400 years ago, in which one studies sets of chess pieces which cover or dominate various opposing pieces or various squares of the chessboard. There are many other examples of dominating sets.

One such example is the situation of a prison in which prisoners must be monitored by guards at all times. The assumption here is that the cells are set up in such a way that a guard in one cell could monitor the prisoners in the adjacent cells. The issues of budget and personnel dictate to have as few guards as possible. The question then becomes: what is the minimum number of guards necessary to monitor or "dominate" all the cells? The graph that represents this situation is one in which the vertices represent the cells and the edges indicate which cells are adjacent to each other. The question stated above can be answered by finding the domination number of the associated graph.

The concept of *total domination* can be illustrated by assuming that there is a suspicion of misconduct among the guards. Thus, not only does each prisoner need to be monitored by a guard, but each guard must be in the view of another guard. To find the minimum number of guards necessary in this situation, we need to find the total domination number of the associated graph.

Finally, the topic of domination was given formal mathematical definition with the publications of books by Berge [2] in 1958 and Ore [32] in 1962. Ore first referred to the *domination number* of a graph, while Berge used the term *coefficient of extremal stability*. Until 1977 relatively little work was done on this topic until Cockayne and Hedetniemi published a survey paper [5] of the results that had been obtained up to that time. Since that time, many papers on domination and variations of domination have been published. The vast literature on this subject has been surveyed and detailed in two books by Haynes et al. [24, 25].

One common definition of domination is given in terms of sets of vertices. Let G be a graph and D a set of vertices such that every vertex in G is in D or adjacent to at least one vertex in D. Then D is called a *dominating set* of G, and the smallest cardinality of such a dominating set of G is defined as the *domination number* of G, denoted by $\gamma(G)$.

A functional definition of domination is also possible and allows for several interesting variations. Let $G = (V, E)$ be a graph and let v be a vertex in V. The *open neighborhood* of v is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u|uv \in E\}$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. For a set S of vertices, we define the open neighborhood, $N(S)$, of S as $\cup_{v \in S} N(v)$, and the closed neighborhood, $N[S]$, of S as $N(S) \cup S$. The sets $N_G(v)$ (or $N_G[v]$) will respectively denote the open (or closed) neighborhood of v with respect to the graph G. For any real valued function $f : V \to \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The *weight* of f is defined as $f(V)$. We say that $f : V \to \{0, 1\}$ is a *dominating function* of G if $f(N[v]) \geq 1$ for all $v \in V$. Since the characteristic function of a dominating set is also a dominating function, $\gamma(G)$ can equivalently be defined as min$\{f(V) \mid f$ is a dominating function of $G\}$.

Changing the range of f, defined above, to $\{-1, 1\}$ or $\{-1, 0, 1\}$ subject to the same restrictions on f as above, lead to *signed domination* [11] and *minus domination* [10] in graphs.

More specifically, a *minus dominating function* is defined in [10] as a function $f : V \to \{-1, 0, 1\}$ such that $f(N[v]) \geq 1$ for every $v \in V$. The *minus domination number* of a graph G is $\gamma^-(G) = \min\{f(V) \mid f$ is a minus dominating function of $G\}$.

A *signed dominating function* is defined in [11] as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for every $v \in V$. The *signed domination number* of a graph G is $\gamma_s(G) = \min\{f(V) \mid f$ is a signed dominating function of $G\}$.

A *majority dominating function* of G is defined in [4] as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for at least half the vertices v in V. The *majority domination number* of a graph G is $\gamma_{\mathrm{maj}}(G) = \min\{f(V) \mid f$ is a majority dominating function of $G\}$.

Let $k \in \mathbf{Z}^+$ such that $1 \leq k \leq |V|$. More generally, a *signed k-subdominating function* for G, or *signed kSF*, is defined in [8] as a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for at least k vertices of G. The *signed k-subdomination number* of a graph G, denoted by $\gamma_{ks}^{-11}(G)$, is equal to $\min\{f(V) \mid f$ is a signed kSF of $G\}$. In the special cases where $k = |V|$ and $k = \lceil\frac{|V|}{2}\rceil$, $\gamma_{ks}^{-11}(G)$ is respectively the signed domination number and the majority domination number.

A *minus k-subdominating function* for G, or *minus kSF*, is defined in [3] as a function $f : V \to \{-1, 0, 1\}$ such that $f(N[v]) \geq 1$ for at least k vertices of G. The *minus k-subdomination number* of a graph G, denoted by $\gamma_{ks}^{-101}(G)$, is equal to $\min\{f(V) \mid f$ is a minus kSF of $G\}$. In the special case where $k = |V|$, $\gamma_{ks}^{-101}(G)$ is the minus domination number.

If every vertex of a graph is adjacent to some vertex of a set S, then S is called a *total dominating set* of G. For $\delta(G) \geq 1$, the *total domination number*, denoted by $\gamma_t(G)$, is defined as the minimum cardinality of a total dominating set of G.

Alternatively, $f : V \to \{0, 1\}$ is a *total dominating function* of G if $f(N(v)) \geq 1$ for all $v \in V$. Since the characteristic function of a total dominating set is also a total dominating function, $\gamma_t(G)$ can equivalently be defined as $\min\{f(V) \mid f$ is a total dominating function of $G\}$.

An analogous theory for total k-subdominating functions arise when instead of using the "closed" neighborhood in the definition of a k-subdominating function, we use the "open" neighborhood. A *total signed k-subdominating function* for G, or *signed TkSF*, is defined as a function $f : V \to \{-1, 1\}$ such that $f(N(v)) \geq 1$ for at least k vertices of G. The *total signed k-subdomination number* of a graph G, denoted by $\gamma_{tks}^{-11}(G)$, is equal to $\min\{f(V) \mid f$ is a signed $TkSF$ of $G\}$. In the special case where $k = |V|$, $\gamma_{tks}^{-11}(G)$ is the total signed domination number studied in [27, 37]. Similarly, a *total minus k-subdominating function* for G, or *minus TkSF*, is defined as a function $f : V \to \{-1, 0, 1\}$ such that $f(N(v)) \geq 1$ for at least k vertices of G. The *total minus k-subdomination number* of a graph G, denoted by $\gamma_{tks}^{-101}(G)$, is equal to $\min\{f(V) \mid f$ is a minus $TkSF$ of $G\}$.

The motivation for studying the signed (minus, respectively) total k-subdomination number is rich and varied from a modeling perspective. For example, by assigning the values -1 or $+1$ (as well as 0, in the case for minus) to the vertices of a graph we can model networks of people or organizations in which global decisions must be made. We assume that each individual has one vote and that each individual has an initial opinion. We assign $+1$ to vertices (individuals) which have a positive opinion and -1 to vertices which have a negative opinion (as well as 0, in the case of minus, to vertices which have a neutral opinion). We also assume, however, that an individual's vote is affected by the opinions of neighboring individuals. In particular, each individual gives equal weight to the opinions of neighboring individuals (thus individuals of high degree have greater "influence"). A voter votes 'aye' if there are more vertices in its (open) neighborhood with positive opinion than with negative opinion, otherwise the vote is 'nay'. We seek an assignment of opinions that guarantee at least k vertices voting aye. We call such an assignment of opinions a k-positive assignment. Among all k-positive assignments of opinions, we are interested primarily in the minimum number of vertices (individuals) who have a positive opinion. The signed (minus, respectively) total k-subdomination number is the minimum possible sum of all opinions, -1 for a negative opinion and $+1$ for a positive opinion, in a k-positive assignment of opinions. The signed (minus, respectively) total k-subdomination number represents, therefore, the minimum number of individuals which can have positive opinions and in doing so force at least k individuals to vote aye.

Suppose we are given real valued functions $f, g : V \to \mathbf{R}$. Then $g < f$ if and only if $g(v) \leq f(v)$ for every $v \in V$ and $g(w) < f(w)$ for at least one $w \in V$.

If f is a kSF (total kSF, respectively) of G, then the *set of covered vertices* of f is denoted by C_f and defined by $C_f = \{v \in V | f(N[v]) \geq 1\}$ ($C_f = \{v \in V | f(N(v)) \geq 1\}$, respectively). If $v \in C_f$, we say v is *covered* by f; otherwise it is *uncovered* by f.

In Chapter 2, we survey recent results on signed and minus k-subdomination in graphs.

In Chapter 3, we compute the signed and minus k-subdomination numbers for certain complete multipartite graphs and their complements, generalizing results due to Holm [30].

In Chapter 4, we give a lower bound on the total signed k-subdomination number in terms of the minimum degree, maximum degree and the order of the graph. A lower bound in terms of the degree sequence is also given. We then compute the total signed k-subdomination number of a cycle, and present a characterization of graphs G with equal total signed k-subdomination and total signed ℓ-subdomination numbers. Finally, we establish a sharp upper bound on the total signed k-subdomination number of a tree in terms of its order n and k where $1 \leq k < n$, and characterize trees attaining these bounds for certain values of k. For this purpose, we first establish the total signed k-subdomination number of simple structures, including paths and spiders.

In Chapter 5, we show that the decision problem corresponding to the computation of the total minus domination number of a graph is **NP**-complete, even when restricted to bipartite graphs or chordal graphs. For a fixed k, we show that the decision problem corresponding to determining whether a graph has a total minus domination function of weight at most k may be **NP**-complete, even when restricted to bipartite or chordal graphs. Also in Chapter 5, linear time algorithms for computing $\gamma_{tns}^{-11}(T)$ and $\gamma_{tns}^{-101}(T)$ for an arbitrary tree T are presented, where $n = n(T)$.

In Chapter 6, we present cubic time algorithms to compute $\gamma_{tks}^{-11}(T)$ and $\gamma_{tks}^{-101}(T)$ for a tree T. We show that the decision problem corresponding to the computation of $\gamma_{tks}^{-11}(G)$ is **NP**-complete, and that the decision problem corresponding to the computation of $\gamma_{tks}^{-101}(G)$ is **NP**-complete, even for bipartite graphs. In addition, we present cubic time algorithms to compute $\gamma_{ks}^{-11}(T)$ and $\gamma_{ks}^{-101}(T)$ for a tree T, solving problems appearing in [25].

Chapter 2

A LITERATURE SURVEY

In this chapter, we survey the literature on majority domination, signed k-subdomination, and minus k-subdomination in graphs.

2.1 Majority Domination in Classes of Graphs

The majority domination numbers of the following classes of graphs have been determined.

Theorem 2.1 [4] For $n \geq 1$,

$$\gamma_{\text{maj}}(K_n) = \begin{cases} 1 \; if \; n \; is \; odd \\ 2 \; if \; n \; is \; even \end{cases}.$$

Theorem 2.2 [4] For $n \geq 2$,

$$\gamma_{\text{maj}}(K_{1,n-1}) = \begin{cases} 1 \; if \; n \; is \; odd \\ 2 \; if \; n \; is \; even \end{cases}.$$

Theorem 2.3 [4] For $t \geq s \geq 2$,

$$\gamma_{\text{maj}}(K_{s,t}) = \begin{cases} 2 - t \ if \ s \ is \ even \\ 3 - t \ if \ s \ is \ odd \end{cases}.$$

Theorem 2.4 [4] For $t \geq s \geq 1$,

$$\gamma_{\text{maj}}(K_s \cup K_t) = \begin{cases} 1 - s \ if \ t \ is \ odd \\ 2 - s \ if \ t \ is \ even \end{cases}.$$

The majority domination number has also been determined for paths and cycles. This will be addressed in Section 2.2, where we look at the first generalization of majority domination in graphs.

Holm [30] determines the majority domination number of the following complete multipartite graphs.

Theorem 2.5 [30] For integers $m > n \geq 1$,

$$\gamma_{\text{maj}}(K_{\underbrace{1,\ldots,1}_{m},n}) = \begin{cases} 1 & \text{if } n+m \text{ is odd} \\ 2 & \text{if } n+m \text{ is even} \end{cases}.$$

Theorem 2.6 [30] For integers $m \geq 2$ and $n \geq 3$,

$$\gamma_{\text{maj}}(K_{\underbrace{n,\ldots,n}_{m}}) = \begin{cases} 2-n & \text{if } m \text{ and } n \text{ are even} \\ 3-n & \text{if } m \text{ is even and } n \text{ is odd} \\ 4-n & \text{if } m \text{ is odd} \end{cases}.$$

Theorem 2.7 [30] $\gamma_{\text{maj}}(K_{\underbrace{2,\ldots,2}_{m}}) = 0$.

The following result involves the disjoint union of a complete graph and a graph G.

Theorem 2.8 [30] Suppose that $n > m \geq 1$. If G is a graph of order m and $H = K_n \cup G$, then

$$\gamma_{\text{maj}}(H) = \begin{cases} 1 - m & \text{if } n \text{ is odd} \\ 2 - m & \text{if } n \text{ is even} \end{cases}.$$

Theorem 2.9 [30] For integers $m > n \geq 1$,

$$\gamma_{\text{maj}}(\overline{K}_m \cup K_n) = \begin{cases} 1 - n & \text{if } m \text{ and } n \text{ are odd} \\ 2 - n & \text{if } m \text{ is even} \\ 3 - n & \text{if } n \text{ is even and } m \text{ is odd} \end{cases}.$$

Holm also determines the majority domination number of the union of m complete graphs of order n.

Theorem 2.10 [30] For integers $m > 2$ and $n \geq 2$,

$$\gamma_{\text{maj}}(\bigcup_{i=1}^{m} K_n) = \begin{cases} \lceil m/2 \rceil - n\lfloor m/2 \rfloor & \text{if } n \text{ is odd} \\ 2\lceil m/2 \rceil - n\lfloor m/2 \rfloor & \text{if } n \text{ is even} \end{cases}.$$

2.2 Signed k-subdomination in Graphs

In this section, we survey some recent results concerning signed k-subdomination in graphs.

Let f be a signed kSF of $G = (V, E)$. Let $P_f = \{v \in V \mid f(v) = 1\}$ and $B_f = \{v \in V \mid f(N[v]) \in \{1, 2\}\}$. For $A, B \subseteq V$, we say *A dominates B*, denoted by $A \succ B$, if for each $b \in B, N[b] \cap A \neq \emptyset$.

Theorem 2.11 [8] A signed kSF f is minimal if and only if for each k-subset K of $C_f, K \cap B_f \succ P_f$.

Let $\gamma(n,k)$ be the minimum value of $\gamma_{ks}^{-11}(T)$ taken over all trees T of order n ($n \geq k$) and $\mathcal{S}(n,k)$ be the set of such trees T with $\gamma_{ks}^{-11}(T) = \gamma(n,k)$. Further, let $\sigma(T)$ be the degree sum of all vertices of T with degree at least three and define $\mathcal{T}(n,k) = \{T \in \mathcal{S}(n,k) \mid \sigma(T) \text{ is minimum}\}$.

Theorem 2.12 [8] For any n, $\mathcal{S}(n,k) = \{P_n\}$.

Theorem 2.13 [8] For $n \geq 2$ and $1 \leq k \leq n$,

$$\gamma_{ks}^{-11}(P_n) = 2\lfloor (2k+4)/3 \rfloor - n.$$

Note that if $k = \lceil \frac{n}{2} \rceil$, then we obtain the following result.

Theorem 2.14 [8] For $n \geq 3$,

$$\gamma_{\mathrm{maj}}(P_n) = \begin{cases} -2\lfloor \frac{n-4}{6} \rfloor & \text{for } n \text{ even} \\ 1 - 2\lfloor \frac{n-3}{6} \rfloor & \text{for } n \text{ odd} \end{cases}.$$

Using Theorems 2.12 and 2.13, Cockayne and Mynhardt established the following result.

Theorem 2.15 [8] If T is a tree of order $n \geq 2$ and k is an integer such that $1 \leq k \leq n$, then

$$\gamma_{ks}^{-11}(T) \geq 2\lfloor (2k+4)/3 \rfloor - n$$

with equality for $T = P_n$.

This result sheds new light on the following result.

Theorem 2.16 [11] Let T be a tree of order $n \geq 2$. Then $\gamma_s(T) \geq \frac{n+4}{3}$ with equality if and only if T is a path on $3j+2$ vertices, for $j \geq 0$.

Let $n \geq 2$ be an integer and let k be an integer such that $1 \leq k \leq n$. Trees T of order n for which $\gamma_{ks}^{-11}(T) = 2\lfloor(2k+4)/3\rfloor - n$ were recently characterized by Hattingh and Ungerer. The statement of this result is rather intricate and the reader is therefore referred to [22] for the details.

The *comet* $C_{s,t}$, where s and t are positive integers, denotes the tree obtained by identifying the center of the star $K_{1,s}$ with an end-vertex of P_t, the path of order t. So $C_{s,1} \cong K_{1,s}$ and $C_{1,p-1} \cong P_p$. Beineke and Henning [1] computed $\gamma_{ks}^{-11}(C_{s,t})$ for $k = s+t$ and for $k = \lceil \frac{s+t}{2} \rceil + 1$. Hattingh and Ungerer extended their result as follows.

Theorem 2.17 [23] Let n, s and t be positive integers such that $n = s + t$ and let $G = C_{s,t}$. If $s, t \geq 2$, then

$$\gamma_{ks}^{-11}(G) = \begin{cases} 2\lfloor(2k+4)/3\rfloor - n & \text{if } k \leq t-1 \\ 2(k - \lceil \frac{t}{3} \rceil + 2) - n & \text{if } t \leq k \text{ and } \; (k \leq t + \lfloor \frac{s}{2} \rfloor - 2,\ t \equiv 0 \pmod 3 \text{ or} \\ & \qquad\qquad\quad\;\; k \leq t + \lfloor \frac{s}{2} \rfloor,\ t \equiv 1 \pmod 3 \text{ or} \\ & \qquad\qquad\quad\;\; k \leq t + \lfloor \frac{s}{2} \rfloor - 1,\ t \equiv 2 \pmod 3)) \\ 2(k - \lceil \frac{t}{3} \rceil + 1) - n & \text{otherwise.} \end{cases}$$

The value of $\gamma_{ks}^{-11}(C_n)$ is calculated in [19].

Theorem 2.18 [19] If $n \geq 3$ and $1 \leq k \leq n-1$, then

$$\gamma_{ks}^{-11}(C_n) = \begin{cases} \frac{n-2}{3} & \text{if } k = n-1 \text{ and } k \equiv 1 \pmod 3 \\ 2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise.} \end{cases}$$

This result generalizes the following.

Theorem 2.19 [4] If $n \geq 3$, then

$$\gamma_{\mathrm{maj}}(C_n) = \gamma_{\mathrm{maj}}(P_n).$$

Theorem 2.20 [19] If $n \geq 3$ and $1 \leq k \leq n$, then for every r-regular ($r \geq 2$) graph G of order n,

$$\gamma_{ks}^{-11}(G) \geq \begin{cases} k\frac{r+3}{r+1} - n & r \text{ odd} \\ \\ k\frac{r+2}{r+1} - n & \text{for } r \text{ even,} \end{cases}$$

and these bounds are best possible.

The following result combines results due to Dunbar, Henning, Hedetniemi and Slater (r even) [11] and Henning and Slater (r odd) [29].

Theorem 2.21 *For every r-regular ($r \geq 2$) graph G of order n,*

$$\gamma_s(G) \geq \begin{cases} \frac{2n}{r+1} & r \text{ odd} \\ \\ \frac{n}{r+1} & r \text{ even.} \end{cases}$$

Theorem 2.22 [36] For every cubic graph G of order n, $\gamma_{\text{maj}}(G) \geq -\frac{n}{4}$.

Since $\gamma_{\text{maj}}(2K_4) = -2 = -\frac{8}{4}$, this bound is best possible.

Theorem 2.23 [26] For every r-regular ($r \geq 2$) graph $G = (V, E)$ of order n,

$$\gamma_{\text{maj}}(G) \geq \begin{cases} \left(\frac{1-r}{2(r+1)}\right) n & r \text{ odd} \\ \\ \left(\frac{-r}{2(r+1)}\right) n & r \text{ even,} \end{cases}$$

and these bounds are best possible.

Note that, if $k = n$ in the statement of Theorem 2.20, then we obtain the result of Theorem 2.21, and, if $k = \lceil \frac{n}{2} \rceil$, then we obtain the result of Theorem 2.23.

Theorem 2.24 [4] For any connected graph G of order n,

$$\gamma_{\mathrm{maj}}(G) \leq \begin{cases} 1 \; if \; n \; is \; odd \\ 2 \; if \; n \; is \; even \end{cases}.$$

Obviously, if f is a majority dominating function of $G = (V, E)$, then f is a signed kSF for each $k \leq \lceil |V|/2 \rceil$. Hence we have the following corollary.

Corollary 1 [8] For any connected graph G of order n and integer $k \leq \lceil \frac{n}{2} \rceil$,

$$\gamma_{ks}^{-11}(G) \leq \begin{cases} 1 \; if \; n \; is \; odd \\ 2 \; if \; n \; is \; even \end{cases}.$$

That this bound is sharp can be seen by noting that $\gamma_{ks}^{-11}(K_{2t+1}) = 1$ and $\gamma_{ks}^{-11}(K_{2t}) = 2$ for each $k \leq t + 1$.

In [8], this bound is improved for trees and extended to an upper bound for $\gamma_{ks}^{-11}(T)$ for all $k \in \{1, \ldots, n\}$.

Theorem 2.25 [8] For any tree T of order n and integer $k \in \{1, \ldots, n\}$, $\gamma_{ks}^{-11}(T) \leq 2(k+1) - n$.

That this bound is exact for trees of order n when $k \leq \frac{1}{2}n$ follows easily since $\gamma_{ks}^{-11}(K_{1,n-1}) = 2(k+1) - n$ if $k \leq \frac{1}{2}n$. The following result, initially formulated as a conjecture by Cockayne and Mynhardt [8], was recently settled independently by Chang et al. and Kang et al.

Theorem 2.26 [6, 31] For any tree T of order n and any k with $\frac{1}{2}n < k \leq n, \gamma_{ks}^{-11}(T) \leq 2k - n$.

The following conjecture of Cockayne and Mynhardt is shown to be false in [28] for the special case when $k = \lceil \frac{n+1}{2} \rceil$.

Conjecture 2.27 [8] For any connected graph G of order n and any k with $\frac{1}{2}n < k \leq n, \gamma_{ks}^{-11}(G) \leq 2k - n$.

The remainder of Cockayne and Mynhardt's paper [8] is devoted to determining conditions on k such that $\gamma_{ks}^{-11}(T) \leq 2k - n$ for certain classes of trees of order n.

Theorem 2.28 [31] For any connected graph G of order n and any k with $\frac{n}{2} < k \leq n$, $\gamma_{ks}^{-11}(G) \leq 2\lceil \frac{k}{n-k+1} \rceil (n - k + 1) - n$.

In [6], Chang et al. give a lower bound for the signed k-subdomination number of a graph in terms of its order and degree sequence.

Theorem 2.29 [6] If G is a graph of order n with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$, then $\gamma_{ks}^{-11}(G) \geq -n + \frac{2}{d_n+1} \sum_{j=1}^{k} \lceil \frac{d_j+2}{2} \rceil$.

Lastly, consider the decision problem

PARTIAL SIGNED DOMINATING FUNCTION (PSDF)
INSTANCE: A graph G, positive rational number $r \leq 1$ (in its simplest form) and an integer ℓ.
QUESTION: Is there a function $f : V(G) \to \{-1, 1\}$ of weight ℓ or less for G such that $|C_f| \geq r|V(G)|$?

Hattingh, Henning and Ungerer [19] showed that **PSDF** is **NP**-complete by describing a polynomial transformation from the following problem [12]:

PLANAR 4-REGULAR DOMINATING SET
INSTANCE: A planar 4-regular graph $G = (V, E)$ and a positive integer $k \leq \frac{|V|}{2}$.
QUESTION: Is there a dominating set of cardinality k or less for G?

If $r = 1$, then **PSDF** is the **NP**-complete problem **SIGNED DOMINATION** [18]. Hence, we also assume that $0 < r < 1$.

Theorem 2.30 [19] The decision problem **PSDF** is **NP**-complete.

This result generalizes the corresponding result of [4].

2.3 Minus k-subdomination in Graphs

In this section, we survey some recent results concerning minus k-subdomination in graphs.

Let f be a minus kSF for the graph $G = (V, E)$. We use three sets for such an f:

$$\begin{aligned} B_f &= \{v \in V | f[v] = 1\}, \\ P_f &= \{v \in V | f(v) \geq 0\} \\ \text{and } C_f &= \{v \in V | f[v] \geq 1\}. \end{aligned}$$

As before, a vertex $v \in C_f$ is *covered* by f; all other vertices are *uncovered* by f. Note that $B_f \subseteq C_f$.

Theorem 2.31 [3] A minus kSF f is minimal if and only if for each k-subset K of C_f we have $B_f \cap K \succ P_f$.

Theorem 2.32 [3] If $n \geq 2$ and $1 \leq k \leq n-1$, then $\gamma_{ks}^{-101}(P_n) = \lceil \frac{k}{3} \rceil + k - n + 1$.

The following result is proved in [10].

Theorem 2.33 [10] For the path P_n, $\gamma_{ns}^{-101}(P_n) = \lceil \frac{n}{3} \rceil$.

Hattingh and Ungerer [21] established the following result.

Theorem 2.34 [21] If T is a tree of order $n \geq 2$ and k is an integer such that $1 \leq k \leq n-1$, then

$$\gamma_{ks}^{-101}(T) \geq k - n + 2.$$

Moreover, this bound is best possible.

However, trees which achieve the lower bound were not characterized in [21]. The following result solves this problem.

Theorem 2.35 [20] Let $n \geq 2$ and let $1 \leq k \leq n-1$ be an integer. Then, for a tree T of order n, $\gamma_{ks}^{-101}(T) = k - n + 2$ if and only if one of the following holds.

1. T has a vertex v adjacent to at least k end-vertices.
2. T has a vertex v with $\deg(v) = k$ and at least $k-1$ neighbors of v are end-vertices.
3. T has two adjacent vertices u and v with $\deg(u) + \deg(v) = k + 2$ such that u and v together are adjacent to at least $k-2$ end-vertices.
4. T has a vertex w of degree three and two of the neighbors of w together are adjacent to exactly $k-3$ other vertices, all of which are end-vertices.

This result supplements the following result of [10].

Theorem 2.36 [10] If T is a tree, then $\gamma_{ns}^{-101}(T) \geq 1$. Furthermore, equality holds if and only if T is a star.

The value of $\gamma_{ks}^{-101}(G)$, where G is a comet, is calculated in [23].

Theorem 2.37 [23] Let n, s and t be positive integers such that $n = s + t$, let k be an integer such that $1 \leq k \leq n-1$ and let $G = C_{s,t}$. If $t \geq 2$ and $s \geq 2$, then

$$\gamma_{ks}^{-101}(G) = \begin{cases} k - n + 2 & \text{if } 1 \leq k \leq s \\ \lceil \frac{k-s+1}{3} \rceil + k - n + 1 & \text{if } s + 1 \leq k \leq n. \end{cases}$$

Note that $\gamma^{-}(C_{s,t}) = \lceil \frac{t+1}{3} \rceil$, where s and t are positive integers.

The value of $\gamma_{ks}^{-101}(C_n)$ is calculated in [21].

Theorem 2.38 [21] If $n \geq 3$ and $1 \leq k \leq n-1$, then

$$\gamma_{ks}^{-101}(C_n) = \begin{cases} \lceil \frac{n-2}{3} \rceil & \text{if } k = n-1 \text{ and } (k \equiv 0 \text{ or } k \equiv 1 \pmod 3) \\ 2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise.} \end{cases}$$

This result supplements the following result.

Theorem 2.39 [10] If $n \geq 3$, then $\gamma_{ns}^{-101}(C_n) = \lceil \frac{n}{3} \rceil$.

Lastly, consider the decision problem

PARTIAL MINUS DOMINATING FUNCTION (PMDF)

INSTANCE: A graph G, positive rational number $r \leq 1$ (in its simplest form) and an integer ℓ.

QUESTION: Is there a function $f : V(G) \to \{-1, 0, 1\}$ of weight ℓ or less for G such that $|C_f| \geq r|V(G)|$?

Hattingh, McRae and Ungerer [20] showed that **PMDF** is **NP**-complete by describing a polynomial transformation from the following **NP**-complete problem [12]:

EXACT COVER BY 3-SETS (X3C)

INSTANCE: A set $X = \{x_1, \ldots, x_{3q}\}$ and a set $C = \{C_1, \ldots, C_t\}$ where $C_j \subseteq X$ and $|C_j| = 3$ for $j = 1, \ldots, t$.

QUESTION: Does C have a pairwise disjoint q-subset of C whose union is X (i.e. an exact cover)?

If $r = 1$, then **PMDF** is the **NP**-complete problem **MINUS DOMINATING FUNCTION** [9]. Hence, we also assume that $r < 1$.

Theorem 2.40 [20] **PMDF** is **NP**-complete, even for bipartite graphs.

Chapter 3

COMPLETE MULTIPARTITE GRAPHS

3.1 Introduction

We devote this chapter to the signed and minus k-subdomination numbers of certain complete multipartite graphs and their complements. In Section 3.2, we compute the values of $\gamma_{ks}^{-11}(K_n)$ and $\gamma_{ks}^{-101}(K_n)$. In Section 3.3, we determine the values of $\gamma_{ks}^{-11}(K_{m,n})$ and $\gamma_{ks}^{-101}(K_{m,n})$, where $K_{m,n}$ is the complete bipartite graph with partite sets V_m and V_n, where $|V_m| = m$ and $|V_n| = n$. In Section 3.4, we compute the values of $\gamma_{ks}^{-11}(K_m \cup K_n)$ and $\gamma_{ks}^{-101}(K_m \cup K_n)$. Note that $K_m \cup K_n = \overline{K}_{m,n}$. In Section 3.5, we determine $\gamma_{ks}^{-11}(S_{m,n})$ and $\gamma_{ks}^{-101}(S_{m,n})$. The results of this chapter have been published in [17].

The notation that we will need in this chapter is as follows. Suppose f is a signed or a minus kSF of a graph G. The set of *positive vertices* (*zero vertices*, *negative vertices*, respectively) is defined as $P_f = \{v|f(v) = 1\}$ ($Z_f = \{v|f(v) = 0\}$, $N_f = \{v|f(v) = -1\}$, respectively). Note that f is a signed kSF, then $Z_f = \emptyset$.

For $v \in V(G)$, we will abbreviate $f(N[v])$ as $f[v]$.

3.2 Complete graphs

In this section we compute the values of $\gamma_{ks}^{-11}(G)$ and $\gamma_{ks}^{-101}(G)$ for a complete graph G.

Proposition 3.1 *If $1 \leq k \leq n$, then*

$$\gamma_{ks}^{-11}(K_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}.$$

Proof. Since $k \geq 1$, there is a vertex $v \in V(K_n) \cap C_f$. If n is odd, $1 \leq f[v] = f(V(K_n))$. If n is even then $1 \leq f[v] = f(V(K_n))$ implies that $|P_f| \geq \frac{n}{2} + 1$. Thus, $f(V(K_n)) \geq (\frac{n}{2} + 1) + (-1)(\frac{n}{2} - 1) = 2$.
Let $U \subseteq V(K_n)$ such that $|U| = \lfloor \frac{n}{2} \rfloor + 1$. The function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with

$$f(V(K_n)) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}. \quad \diamondsuit$$

Proposition 3.2 *If $1 \leq k \leq n$, then $\gamma_{ks}^{-101}(K_n) = 1$.*

Proof. Since $k \geq 1$, there is a vertex $v \in V(K_n) \cap C_f$. Thus, $1 \leq f[v] = f(V(K_n))$. The function f that assigns 1 to exactly one vertex of K_n and 0 to all other vertices of K_n is a kSF of K_n such that $f(V(K_n)) = 1$. $\diamondsuit$

3.3 Complete bipartite graphs

In this section we compute the value of $\gamma_{ks}^{-11}(G)$ and $\gamma_{ks}^{-101}(G)$ for complete bipartite graphs G. The partite sets of $K_{m,n}$ will be denoted by V_m and V_n, respectively.

Proposition 3.3 *Suppose $n \geq m \geq 2$. If $1 \leq k \leq n$, then*

$$\gamma_{ks}^{-11}(K_{m,n}) = \begin{cases} 3-n & \textit{if } m \textit{ is odd} \\ 2-n & \textit{if } m \textit{ is even} \end{cases}$$

Proof. Before proceeding further, we prove the following claim:

Claim. If $v \in V_n \cap C_f$, then $f(V) \geq \begin{cases} 3-n & \text{if } m \text{ is odd} \\ 2-n & \text{if } m \text{ is even} \end{cases}$.

Proof. On the one hand, if $f(v) = -1$, then

$$\begin{aligned} f(V) &= f(V_m) + f(V_n) \\ &\geq (\lceil \tfrac{m}{2} \rceil - \lfloor \tfrac{m}{2} \rfloor + 2) + (-1)n \\ &= \begin{cases} 3-n & \text{if } m \text{ is odd} \\ 2-n & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

On the other hand, if $f(v) = 1$, then

$$\begin{aligned} f(V) &= f(V_m) + f(V_n) \\ &\geq (\lceil \tfrac{m}{2} \rceil - \lfloor \tfrac{m}{2} \rfloor) + (-1)(n-1) + 1 \\ &= \begin{cases} 3-n & \text{if } m \text{ is odd} \\ 2-n & \text{if } m \text{ is even.} \end{cases} \quad \diamondsuit \end{aligned}$$

Since $k \geq 1$, there is a covered vertex, say v. If $v \in V_n$, then

$$f(V) \geq \begin{cases} 3-n & \text{if } m \text{ is odd} \\ 2-n & \text{if } m \text{ is even} \end{cases}$$

Note that

$$\begin{cases} \min\{3-m,3-n\} & \text{if } n \text{ is odd and } m \text{ is odd} \\ \min\{2-m,3-n\} & \text{if } n \text{ is even and } m \text{ is odd} \\ \min\{3-m,2-n\} & \text{if } n \text{ is odd and } m \text{ is even} \\ \min\{2-m,2-n\} & \text{if } n \text{ is even and } m \text{ is even} \end{cases}$$

$$= \begin{cases} 3-n & \text{if } n \text{ is odd and } m \text{ is odd} \\ 3-n & \text{if } n \text{ is even and } m \text{ is odd} \\ 2-n & \text{if } n \text{ is odd and } m \text{ is even} \\ 2-n & \text{if } n \text{ is even and } m \text{ is even} \end{cases}.$$

If $v \in V_m$, then, by the above,

$$\begin{aligned} f(V) &\geq \begin{cases} 3-m & \text{if } n \text{ is odd} \\ 2-m & \text{if } n \text{ is even} \end{cases} \\ &\geq \begin{cases} 3-n & \text{if } m \text{ is odd} \\ 2-n & \text{if } m \text{ is even} \end{cases}. \end{aligned}$$

Let $U \subseteq V_m$ such that $|U| = \lceil \frac{m}{2} \rceil + 1$. The function g defined by

$$g(v) = \begin{cases} 1 & \text{if } v \in U \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with

$$g(V) = \begin{cases} 3-n & \text{if } m \text{ is odd} \\ 2-n & \text{if } m \text{ is even} \end{cases}.$$

The result now follows. $\diamondsuit$

Lemma 3.4 *If $v \in V_m \cap C_f$ and $f(v) = -1$, then $f(V_n) \geq \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor + 2$. Moreover, if $f(V_n) = \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor + 2$, then $|V_n \cap P_f| = \lceil \frac{n}{2} \rceil + 1$.*

Proof. Since $f(v) = -1$ and $f[v] \geq 1$, then $|V_n \cap P_f| = \lceil \frac{n}{2} \rceil + 1$. Thus, $f(V_n) \geq (\lceil \frac{n}{2} \rceil + 1) - (\lfloor \frac{n}{2} \rfloor - 1) = \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor + 2$. $\diamondsuit$

Lemma 3.5 *If $v \in V_m \cap C_f$ and $f(v) = 1$, then $f(V_n) \geq \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor$. Moreover, if $f(V_n) = \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor$, then $|V_n \cap P_f| = \lceil \frac{n}{2} \rceil$.*

Proof. Since $f(v) = 1$ and $f[v] \geq 1$, then $|V_n \cap P_f| = \lceil \frac{n}{2} \rceil$. Thus, $f(V_n) \geq \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor$. $\diamondsuit$

We henceforth assume $k > n$, $V_m \cap C_f \neq \emptyset$ and $V_n \cap C_f \neq \emptyset$. Let $u \in V_m \cap C_f$ and let $v \in V_n \cap C_f$.

Proposition 3.6 *Suppose $n \geq m \geq 2$. If $n < k \leq \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$, then*

$$\gamma_{ks}^{-11}(K_{m,n}) = \begin{cases} 0 & \textit{if } m \textit{ and } n \textit{ are even} \\ 1 & \textit{if } m \textit{ and } n \textit{ have different parities} \\ 2 & \textit{if } m \textit{ and } n \textit{ are odd} \end{cases}.$$

Proof. By Lemmas 3.4 and 3.5,

$$\begin{aligned} f(V) &= f(V_m) + f(V_n) \\ &\geq \lceil \tfrac{n}{2} \rceil - \lfloor \tfrac{n}{2} \rfloor + \lceil \tfrac{m}{2} \rceil - \lfloor \tfrac{m}{2} \rfloor \\ &= \begin{cases} 0 & \text{if } m \text{ and } n \text{ are even} \\ 1 & \text{if } m \text{ and } n \text{ have different parities} \\ 2 & \text{if } m \text{ and } n \text{ are odd} \end{cases}. \end{aligned}$$

Let $U_1 \subseteq V_m$ such that $|U_1| = \lceil \frac{m}{2} \rceil$ and let $U_2 \subseteq V_n$ such that $|U_2| = \lceil \frac{n}{2} \rceil$. The function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U_1 \cup U_2 \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with

$$f(V) = \begin{cases} 0 & \text{if } m \text{ and } n \text{ are even} \\ 1 & \text{if } m \text{ and } n \text{ have different parities} \\ 2 & \text{if } m \text{ and } n \text{ are odd} \end{cases}.$$

The result now follows. $\diamondsuit$

Lemma 3.7 *If $f(u) = f(v) = 1$, then*

$$f(V) \geq \begin{cases} 0 & \textit{if } m \textit{ and } n \textit{ are even} \\ 1 & \textit{if } m \textit{ and } n \textit{ have different parities} \\ 2 & \textit{if } m \textit{ and } n \textit{ are odd} \end{cases}.$$

Moreover, if equality holds, then $k \leq \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$.

Proof. By Lemma 3.5,

$$\begin{aligned} f(V) &\geq \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil - \left\lfloor \frac{m}{2} \right\rfloor \\ &= \begin{cases} 0 & \text{if } m \text{ and } n \text{ are even} \\ 1 & \text{if } m \text{ and } n \text{ have different parities} \\ 2 & \text{if } m \text{ and } n \text{ are odd} \end{cases}. \end{aligned}$$

If $f(V)$ equals this lower bound, then $|V_n \cap P_f| = \lceil \frac{n}{2} \rceil$ and $|V_m \cap P_f| = \lceil \frac{m}{2} \rceil$. Then $N_f \cap C_f = \emptyset$, so $k \leq |C_f| \leq \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$. $\diamondsuit$

Lemma 3.8 *If $f(u) = 1$ and $f(v) = -1$, then*

$$f(V) \geq \begin{cases} 2 & \textit{if } m \textit{ and } n \textit{ are even} \\ 3 & \textit{if } m \textit{ and } n \textit{ have different parities} \\ 4 & \textit{if } m \textit{ and } n \textit{ are odd} \end{cases}.$$

Moreover, if equality holds, then $k \leq n + \lceil \frac{m}{2} \rceil + 1$.

Proof. By Lemmas 3.4 and 3.5,

$$\begin{aligned} f(V) &\geq \left\lceil\frac{n}{2}\right\rceil - \left\lfloor\frac{n}{2}\right\rfloor + \left\lceil\frac{m}{2}\right\rceil - \left\lfloor\frac{m}{2}\right\rfloor + 2 \\ &= \begin{cases} 2 & \text{if } m \text{ and } n \text{ are even} \\ 3 & \text{if } m \text{ and } n \text{ have different parities} \\ 4 & \text{if } m \text{ and } n \text{ are odd} \end{cases}. \end{aligned}$$

If $f(V)$ equals this lower bound, then $|V_n \cap P_f| = \lceil\frac{n}{2}\rceil$ and $|V_m \cap P_f| = \lceil\frac{m}{2}\rceil + 1$. Then $(N_f \cap V_m) \cap C_f = \emptyset$, so $k \leq |C_f| \leq \lceil\frac{m}{2}\rceil + 1 + n$. $\diamondsuit$

Similarly one may prove

Lemma 3.9 *If $f(u) = -1$ and $f(v) = 1$, then*

$$f(V) \geq \begin{cases} 2 & \textit{if } m \textit{ and } n \textit{ are even} \\ 3 & \textit{if } m \textit{ and } n \textit{ have different parities} \\ 4 & \textit{if } m \textit{ and } n \textit{ are odd} \end{cases}.$$

Moreover, if equality holds, then $k \leq m + \lceil\frac{n}{2}\rceil + 1$.

Lemma 3.10 *If $f(u) = -1$ and $f(v) = -1$, then*

$$f(V) \geq \begin{cases} 4 & \textit{if } m \textit{ and } n \textit{ are even} \\ 5 & \textit{if } m \textit{ and } n \textit{ have different parities} \\ 6 & \textit{if } m \textit{ and } n \textit{ are odd} \end{cases}.$$

Proof. By Lemma 3.4, $f(V) \geq \lceil\frac{n}{2}\rceil - \lfloor\frac{n}{2}\rfloor + 2 + \lceil\frac{m}{2}\rceil - \lfloor\frac{m}{2}\rfloor + 2 =$

$$\begin{cases} 4 & \text{if } m \text{ and } n \text{ are even} \\ 5 & \text{if } m \text{ and } n \text{ have different parities} \\ 6 & \text{if } m \text{ and } n \text{ are odd} \end{cases}. \diamondsuit$$

Proposition 3.11 *Suppose* $n \geq m \geq 2$. *If* $\max\{\lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil + 1, n+1\} \leq k \leq n + \lceil \frac{m}{2} \rceil + 1$, *then*

$$\gamma_{ks}^{-11}(K_{m,n}) = \begin{cases} 2 & \text{if } m \text{ and } n \text{ are even} \\ 3 & \text{if } m \text{ and } n \text{ have different parities} \\ 4 & \text{if } m \text{ and } n \text{ are odd} \end{cases}.$$

Proof. Since $k \geq \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil + 1$, Lemmas 3.5 and 3.7 imply that if $f(u) = f(v) = 1$, then

$$f(V) \geq \begin{cases} 2 & \text{if } m \text{ and } n \text{ are even} \\ 3 & \text{if } m \text{ and } n \text{ have different parities} \\ 4 & \text{if } m \text{ and } n \text{ are odd} \end{cases}.$$

If $f(u) = 1$ and $f(v) = -1$, $f(u) = -1$ and $f(v) = 1$ or $f(u) = f(v) = -1$, the result follows from Lemmas 3.8, 3.9 and 3.10.

Let $U_1 \subseteq V_m$ such that $|U_1| = \lceil \frac{m}{2} \rceil + 1$ and let $U_2 \subseteq V_n$ such that $|U_2| = \lceil \frac{n}{2} \rceil$. The function f defined by $f(v) = \begin{cases} 1 & \text{if } v \in U_1 \cup U_2 \\ -1 & \text{otherwise} \end{cases}$ is a kSF with

$$f(V) = \begin{cases} 2 & \text{if } m \text{ and } n \text{ are even} \\ 3 & \text{if } m \text{ and } n \text{ have different parities} \\ 4 & \text{if } m \text{ and } n \text{ are odd} \end{cases}. \quad \diamondsuit$$

Similarly,

Proposition 3.12 *Suppose* $n \geq m \geq 2$. *If* $\max\{\lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil + 1, m+1\} \leq k \leq m + \lceil \frac{n}{2} \rceil + 1$, *then*

$$\gamma_{ks}^{-11}(K_{m,n}) = \begin{cases} 2 & \text{if } m \text{ and } n \text{ are even} \\ 3 & \text{if } m \text{ and } n \text{ have different parities} \\ 4 & \text{if } m \text{ and } n \text{ are odd} \end{cases}.$$

Proposition 3.13 *Suppose $n \geq m \geq 2$. If $\max\{n + \lceil \frac{m}{2} \rceil + 1, m + \lceil \frac{n}{2} \rceil + 1\} + 1 \leq k \leq m+n$, then*

$$\gamma_{ks}^{-11}(K_{m,n}) = \begin{cases} 4 & \text{if } m \text{ and } n \text{ are even} \\ 5 & \text{if } m \text{ and } n \text{ have different parities} \\ 6 & \text{if } m \text{ and } n \text{ are odd} \end{cases}.$$

Proof. Since $k \geq \max\{n + \lceil \frac{m}{2} \rceil + 1, m + \lceil \frac{n}{2} \rceil + 1\} + 1$, Lemmas 3.4 and 3.5 imply that if $f(u) = 1$ and $f(v) = -1$ or if $f(u) = -1$ and $f(v) = 1$, then

$$f(V) \geq \begin{cases} 4 & \text{if } m \text{ and } n \text{ are even} \\ 5 & \text{if } m \text{ and } n \text{ have different parities} \\ 6 & \text{if } m \text{ and } n \text{ are odd} \end{cases}.$$

If $f(u) = f(v) = -1$, the result follows from Lemma 3.10. If $f(u) = f(v) = 1$, then we may assume that $C_f \cap N_f = \emptyset$, or else we have a previous case. This fact and Lemma 3.5 imply $|V_n \cap P_f| = \lceil \frac{n}{2} \rceil$ and $|V_m \cap P_f| = \lceil \frac{m}{2} \rceil$. Thus, $f(V) = \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor + \lceil \frac{m}{2} \rceil - \lfloor \frac{m}{2} \rfloor$. Lemma 3.7 implies that $k \leq \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$, which is a contradiction.

Let $U_1 \subseteq V_m$ such that $|U_1| = \lceil \frac{m}{2} \rceil + 1$ and let $U_2 \subseteq V_n$ such that $|U_2| = \lceil \frac{n}{2} \rceil + 1$. The function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U_1 \cup U_2 \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with

$$f(V) = \begin{cases} 4 & \text{if } m \text{ and } n \text{ are even} \\ 5 & \text{if } m \text{ and } n \text{ have different parities} \\ 6 & \text{if } m \text{ and } n \text{ are odd} \end{cases}.$$

The result now follows. ◇

Proposition 3.14 *Let $n \geq 1$ be an integer. Then*

$$\gamma_{ks}^{-11}(K_{1,n}) = \begin{cases} 2k-n+1 & \text{if } k \leq \lceil \frac{n}{2} \rceil \\ 2k-n-1 & \text{if } k \geq \lceil \frac{n}{2} \rceil + 1 \end{cases}.$$

Proof. Let $V(K_{1,n}) = \{v, v_1, \ldots, v_n\}$ with v being the central vertex. Let $V' = \{v_1, \ldots, v_n\}$.

Case 1. $k = 1$. If $v_i \in C_f$, then $f(v) = f(v_i) = 1$, and $f(V) \geq 2 + (n-1)(-1) = 3 - n$. If $V' \cap C_f = \emptyset$, then $v \in C_f$, and

$$f(V) \geq 1 + \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{otherwise} \end{cases}.$$

Thus, $f(V) \geq 3 - n = 2 \cdot 1 - n + 1 = 2k - n + 1$. The function f defined by $f(v) = f(v_1) = 1$ and $f(v) = -1$ otherwise is a kSF with $f(V) = 2k - n + 1$.

Case 2. $2 \leq k$.

Since $k \geq 2$, $f(v) = 1$.

Case 2.1 $k \leq \lceil \frac{n}{2} \rceil - 1$.

If $v \notin C_f$, then $f(V) \geq 1 + k + (n-k)(-1) = 2k - n + 1$. Suppose, therefore, $v \in C_f$. Then $|P_f \cap V'| \geq \lceil \frac{n}{2} \rceil$. Assume $\{v_1, \ldots, v_{\lceil \frac{n}{2} \rceil}\} \subseteq P_f$. Then the function f^* defined by $f^*(v) = 1$ for all $v \in P_f - \{v_{\lceil \frac{n}{2} \rceil}\}$ and $f^*(v) = -1$ otherwise is a kSF of $K_{1,n}$ such that $f^*(V) < f(V)$, which is a contradiction. Thus, $f(V) \geq 2k - n + 1$. The function f defined by $f(v) = 1$, $f(v_i) = 1$ for $i = 1, \ldots, k$ and $f(v) = -1$ otherwise is a kSF of $K_{1,n}$ such that $f(V) = 2k - n + 1$.

Case 2.2 $k = \lceil \frac{n}{2} \rceil$.

If $v \notin C_f$, then $f(V) \geq 2k - n + 1$. Suppose, therefore, that $v \in C_f$. Then

$$\begin{aligned} f(V) &\geq 1+\lceil \tfrac{n}{2} \rceil - \lfloor \tfrac{n}{2} \rfloor \\ &= \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{otherwise} \end{cases} \\ &= 2\lceil \tfrac{n}{2} \rceil - n + 1 \\ &= 2k - n + 1. \end{aligned}$$

The function f defined by $f(v) = 1$, $f(v_i) = 1$ for $i = 1, \ldots, k$ and $f(v) = -1$ otherwise is a kSF of $K_{1,n}$ such that $f(V) = 2k - n + 1$.

Case 2.3 $k \geq \lceil \frac{n}{2} \rceil + 1$.

If $v \notin C_f$, then $f(V) \geq 2k - n + 1$. Suppose, therefore, that $v \in C_f$. Then

$$f(V) \geq 1 + \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{otherwise} \end{cases}.$$

Also, $f(V) \geq k + (n - k + 1)(-1) = 2k - n - 1$. Thus,

$$\begin{aligned} f(V) &\geq \begin{cases} \max\{2, 2k-n-1\} & \text{if } n \text{ is odd} \\ \max\{1, 2k-n-1\} & \text{otherwise} \end{cases} \\ &= 2k - n - 1. \end{aligned}$$

Thus, $f(V) \geq 2k - n - 1$. The function f defined by $f(v) = 1$, $f(v_i) = 1$ for $i = 1, \ldots, k-1$ and $f(v) = -1$ otherwise is a kSF of $K_{1,n}$ such that $f(V) = 2k-n-1$. The result follows. $\diamondsuit$

The value of $\gamma_{ks}^{-11}(K_{m,n})$ is completely determined by Propositions 3.3, 3.6, 3.11, 3.12, 3.13 and 3.14. This result generalizes a result of [4].

We now turn our attention to the computation of $\gamma_{ks}^{-101}(K_{m,n})$.

Lemma 3.15 *If $v \in V_m \cap C_f$, then*

$$f(V_n) \geq 1 - f(v).$$

Proof. Since $1 \leq f[v] = f(v) + f(V_n)$, it follows that $f(V_n) \geq 1 - f(v)$. $\diamondsuit$

Proposition 3.16 *Suppose $n \geq m \geq 2$. If $1 \leq k \leq n$, then*

$$\gamma_{ks}^{-101}(K_{m,n}) = 2 - n.$$

Proof. Before proceeding further, we prove

Claim 1 *If $v \in V_n \cap C_f$, then $f(V) \geq 2 - n$.*

Proof. If $f(v) = -1$, then, by Lemma 3.15, $f(V_m) \geq 2$. Thus, $f(V) = f(V_m) + f(V_n) \geq 2 + (-1)n = 2 - n$. If $f(v) = 0$, then, by Lemma 3.15, $f(V_m) \geq 1$. Thus, $f(V) = f(V_m) + f(V_n) \geq 1 + 0 + (-1)(n-1) = 2 - n$. If $f(v) = 1$, then, by Lemma 3.15, $f(V_m) \geq 0$. Thus, $f(V) = f(V_m) + f(V_n) \geq 0 + 1 + (-1)(n-1) = 2 - n$. $\diamondsuit$

Since $k \geq 1$, there is a covered vertex, say v. On the one hand, if $v \in V_n$, then $f(V) \geq 2 - n$. On the other hand, if $v \in V_m$, then $f(V) \geq 2 - m \geq 2 - n$.

Let $u \in V_m$ and let $U \subseteq V_m - \{u\}$ such that $|U| = \lceil \frac{m}{2} \rceil$. If m is odd, the function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U \\ 0 & \text{if } v = u \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with $f(V) = 2 - n$. If m is even, then the function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U \cup \{u\} \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with $f(V) = 2 - n$. The result follows. $\diamondsuit$

We henceforth assume $k > n$, $V_m \cap C_f \neq \emptyset$ and $V_n \cap C_f \neq \emptyset$. Let $u \in V_m \cap C_f$ and let $v \in V_n \cap C_f$.

Proposition 3.17 *Suppose $n \geq m \geq 2$. If $n < k \leq n + \lceil \frac{m}{2} \rceil$, then*

$$\gamma_{ks}^{-101}(K_{m,n}) = 1.$$

Proof. If $f(u) \leq 0$, then, by Lemma 3.15, $f(V) = f(V_m) + f(V_n) \geq 0 + 1 = 1$. Similarly, if $f(v) \leq 0$, then $f(V) = f(V_m) + f(V_n) \geq 1 + 0 = 1$. We assume therefore that $f(u) = f(v) = 1$. Let $V_n' = V_n - \{v\}$. Let $V_m' = V_m - \{u\}$.

Before proceeding further, we prove

Claim 2 *$f(V_n') \geq 0$ or $f(V_m') \geq 0$.*

Proof. Suppose, to the contrary, that $f(V_n') < 0$ and $f(V_m') < 0$. Since $1 \leq f[u] = f(u) + f(v) + f(V_n') = 1 + 1 + f(V_n')$, we have $f(V_n') \geq -1$, whence $f(V_n') = -1$ and $f(V_n) = 0$. Similarly, $f(V_m) = 0$. Let $\ell = |V_n \cap P_f|$. Then $0 = f(V_n) \geq \ell + (n - \ell)(-1) = 2\ell - n$, so that $\ell \leq \frac{n}{2}$. Thus, $|V_n \cap P_f| \leq \frac{n}{2}$, and, similarly, $|V_m \cap P_f| \leq \frac{m}{2}$. Hence, $|P_f| \leq \frac{n}{2} + \frac{m}{2} \leq \frac{n}{2} + \frac{n}{2} = n$.

If $x \in C_f$, then, without loss of generality, we may assume that $x \in V_m$, so that $1 \leq f[x] = f(x) + f(V_n) = f(x) \leq 1$, which implies that $x \in P_f$. Thus, $C_f \subseteq P_f$.

We conclude that $n + 1 \leq k \leq |C_f| \leq |P_f| \leq n$, which is a contradiction, and the claim follows. $\diamondsuit$

So either $f(V_n') \geq 0$ or $f(V_m') \geq 0$. If $f(V_n') \geq 0$, then $f(V) = f(v) + f(u) + f(V_n') + f(V_m') \geq 1 + 1 + 0 - 1 = 1$. Similarly, if $f(V_m') \geq 0$, then $f(V) \geq 1$. Thus, $f(V) \geq 1$.

Let $u \in V_m$ and let $U \subseteq V_m - \{u\}$ such that $|U| = \lceil \frac{m}{2} \rceil$. If m is odd, the function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U \\ 0 & \text{if } v \in V_n \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with $f(V) = 1$. If m is even, then the function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U \\ 0 & \text{if } v \in V_n \cup \{u\} \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with $f(V) = 1$. The result follows. $\diamondsuit$

Proposition 3.18 *Suppose $n \geq m \geq 2$. If $n + \lceil \frac{m}{2} \rceil + 1 \leq k \leq n + m$, then*

$$\gamma_{ks}^{-101}(K_{m,n}) = 2.$$

Proof. Let f be a kSF of $K_{m,n}$ with $f(V) = \gamma_{ks}^{-101}(K_{m,n})$ such that $|Z_f|$ is maximized. By Lemma 3.15, $f(V_m) \geq 0$ and $f(V_n) \geq 0$.

Suppose $f(V_n) = 0$. If $|V_m \cap C_f| \leq \lceil \frac{m}{2} \rceil$, then $k \leq |C_f| \leq n + \lceil \frac{m}{2} \rceil$, which is a contradiction. Thus, $|V_m \cap C_f| \geq \lceil \frac{m}{2} \rceil + 1$. Let $x \in V_m \cap C_f$. Then, since $1 \leq f[x] = f(x) + f(V_n) = f(x) \leq 1$, we have $f(x) = 1$. Thus, $V_m \cap C_f \subseteq V_m \cap P_f$, so that $|V_m \cap P_f| \geq |V_m \cap C_f| \geq \lceil \frac{m}{2} \rceil + 1$. Hence, $f(V) = f(V_m) + f(V_n) \geq (\lceil \frac{m}{2} \rceil + 1) - (\lfloor \frac{m}{2} \rfloor - 1) + 0 = \lceil \frac{m}{2} \rceil - \lfloor \frac{m}{2} \rfloor + 2 \geq 2$, as desired.

Assume, therefore, that $f(V_n) \geq 1$. If $f(V_n) \geq 2$, then $f(V) = f(V_m) + f(V_n) \geq 0 + 2 \geq 2$, as required. Thus, assume $f(V_n) = 1$. We are done if $f(V_m) \geq 1$, and so we assume that $f(V_m) = 0$.

We show that $V_m \subseteq Z_f$. For suppose to the contrary that $x \in V_m \cap P_f$ and $y \in V_m \cap N_f$. Define $f^* : V \to \{-1, 0, 1\}$ by $f^*(z) = 0$ if $z \in \{x, y\}$ and $f^*(z) = f(z)$

if $z \notin \{x, y\}$. Then f^* is a kSF of $K_{m,n}$ with $f^*(V) = f(V)$ and $|Z_{f^*}| > |Z_f|$, which contradicts the choice of f.

We conclude that $V_m \subseteq C_f$, so that $|V_n \cap C_f| \geq k - m \geq n + \lceil \frac{m}{2} \rceil + 1 - m = n - \lfloor \frac{m}{2} \rfloor + 1$.

Let $y \in V_n \cap C_f$. Then, since $1 \leq f[y] = f(y) + f(V_m) = f(y) \leq 1$, we have $f(y) = 1$. Thus, $V_n \cap C_f \subseteq V_n \cap P_f$, so that $|V_n \cap P_f| \geq |V_n \cap C_f| \geq n - \lfloor \frac{m}{2} \rfloor + 1$. Hence, $f(V_n) \geq (n - \lfloor \frac{m}{2} \rfloor + 1) - (n - (n - \lfloor \frac{m}{2} \rfloor + 1)) = (n - \lfloor \frac{m}{2} \rfloor + 1) - (\lfloor \frac{m}{2} \rfloor - 1) = n - 2\lfloor \frac{m}{2} \rfloor + 2 \geq m - 2\lfloor \frac{m}{2} \rfloor + 2 \geq 2$, a contradiction.

Let $u \in V_m$ and let $v \in V_n$. The function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u, v\} \\ 0 & \text{otherwise} \end{cases}$$

is a kSF of $K_{m,n}$ such that $f(V) = 2$. The result now follows. $\diamondsuit$

Hattingh and Ungerer [21] obtained the following lower bound on $\gamma_{ks}^{-101}(T)$ for a tree T.

Proposition 3.19 *If T is a tree of order $n \geq 2$ and $1 \leq k \leq n - 1$, then*

$$\gamma_{ks}^{-101}(T) \geq k - n + 2.$$

As a consequence we obtain

Proposition 3.20 *If $n \geq 1$ is an integer and $k \leq n$, then*

$$\gamma_{ks}^{-101}(K_{1,n}) = k - n + 1.$$

Proof. By Proposition 3.19, $\gamma_{ks}^{-101}(K_{1,n}) \geq k - (n+1) + 2 = k - n + 1$. The function f that assigns 1 to the central vertex, 0 to k leaves and -1 to the remaining leaves of $K_{1,n}$ is a kSF with $f(V) = k - n + 1$. The result follows. $\diamondsuit$

Proposition 3.21 *If $n \geq 1$ is an integer, then*

$$\gamma_{(n+1)s}^{-101}(K_{1,n}) = 1.$$

Proof. If v is the central vertex of $K_{1,n}$, then $f(V) = f[v] \geq 1$. The function f that assigns 1 to v and 0 to the remaining vertices of $K_{1,n}$ is a kSF of $K_{1,n}$ with $f(V) = 1$. The result follows. $\diamondsuit$

The value of $\gamma_{ks}^{-101}(K_{m,n})$ is completely determined by Propositions 3.16, 3.17, 3.18, 3.20, 3.21.

3.4 The disjoint union of two complete graphs

In this section we compute the values of $\gamma_{ks}^{-11}(G)$ and $\gamma_{ks}^{-101}(G)$ where G is the disjoint union of two complete graphs. Let $V_m = V(K_m)$ and $V_n = V(K_n)$.

Proposition 3.22 *If $1 \leq k \leq m \leq n$, then*

$$\gamma_{ks}^{-11}(K_m \cup K_n) = \begin{cases} 1-n & \text{if } m \text{ is odd} \\ 2-n & \text{if } m \text{ is even} \end{cases}.$$

Proof. Since $k \geq 1$, there is a covered vertex, say $v \in V_m \cup V_n$. If $v \in V_m$, then by Proposition 3.1, $f(V_m) \geq 1$ if m is odd and $f(V_m) \geq 2$ if m is even. Thus, if m is odd, $f(V_m \cup V_n) = f(V_m) + f(V_n) \geq 1 + (-1)n = 1 - n$. On the other hand, if m is even, $f(V_m \cup V_n) = f(V_m) + f(V_n) \geq 2 + (-1)n = 2 - n$. Similarly it can be shown that if $v \in V_n$, then $f(V_m \cup V_n) \geq (-1)m + 1 = 1 - m$ if n is odd and $f(V_m \cup V_n) \geq (-1)m + 2 = 2 - m$ if n is even.

Note that

$$\begin{cases} \min\{1-m, 1-n\} & \text{if } n \text{ is odd and } m \text{ is odd} \\ \min\{2-m, 1-n\} & \text{if } n \text{ is even and } m \text{ is odd} \\ \min\{1-m, 2-n\} & \text{if } n \text{ is odd and } m \text{ is even} \\ \min\{2-m, 2-n\} & \text{if } n \text{ is even and } m \text{ is even} \end{cases}$$

$$= \begin{cases} 1-n & \text{if } n \text{ is odd and } m \text{ is odd} \\ 1-n & \text{if } n \text{ is even and } m \text{ is odd} \\ 2-n & \text{if } n \text{ is odd and } m \text{ is even} \\ 2-n & \text{if } n \text{ is even and } m \text{ is even} \end{cases}.$$

Thus, $f(V_m \cup V_n) \geq \begin{cases} 1-n & \text{if } m \text{ is odd} \\ 2-n & \text{if } m \text{ is even} \end{cases}.$

Let $U \subseteq V_m$ such that $|U| = \lfloor \frac{m}{2} \rfloor + 1$. The function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with

$$f(V_m \cup V_n) = \begin{cases} 1-n & \text{if } m \text{ is odd} \\ 2-n & \text{if } m \text{ is even} \end{cases}. \ \diamond$$

Proposition 3.23 *Suppose $n \geq m \geq 1$. If $m < k \leq n$ then*

$$\gamma_{ks}^{-11}(K_m \cup K_n) = \begin{cases} 1-m & \textit{if } n \textit{ is odd} \\ 2-m & \textit{if } n \textit{ is even} \end{cases}.$$

Proof. Since $k > m$, there is a covered vertex, $v \in V_n$. By Proposition 3.1, $f(V_n) \geq 1$ if n is odd and $f(V_n) \geq 2$ if n is even. Thus, if n is odd,

$f(V_m \cup V_n) = f(V_m) + f(V_n) \geq (-1)m + 1 = 1 - m$. On the other hand, if n is even, $f(V_m \cup V_n) = f(V_m) + f(V_n) \geq (-1)m + 2 = 2 - m$.

Let $U \subseteq V_n$ such that $|U| = \lfloor \frac{n}{2} \rfloor + 1$. The function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U \\ -1 & \text{otherwise} \end{cases}$$

is a kSF such that

$$f(V_m \cup V_n) = \begin{cases} 1 - m & \text{if } n \text{ is odd} \\ 2 - m & \text{if } n \text{ is even} \end{cases} . \diamondsuit$$

Proposition 3.24 *Suppose $n \geq m \geq 1$. If $n < k \leq n + m$ then*

$$\gamma_{ks}^{-11}(K_m \cup K_n) = \begin{cases} 2 & \textit{if } m \textit{ and } n \textit{ are odd} \\ 3 & \textit{if } m \textit{ and } n \textit{ have different parities} \\ 4 & \textit{if } m \textit{ and } n \textit{ are even} \end{cases} .$$

Proof. Since $k \geq n$, there is a covered vertex, $v \in V_m$ and a covered vertex, $u \in V_n$. By Proposition 3.1,

$$f(V_m) \geq \begin{cases} 1 & \text{if } m \text{ is odd} \\ 2 & \text{if } m \text{ is even} \end{cases}$$

and

$$f(V_n) \geq \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases} .$$

Thus,

$$f(V_m \cup V_n) \geq \begin{cases} 2 & \text{if } m \text{ and } n \text{ are odd} \\ 3 & \text{if } m \text{ and } n \text{ have different parities} \\ 4 & \text{if } m \text{ and } n \text{ are even} \end{cases} .$$

Let $U_1 \subseteq V_m$ such that $|U_1| = \lfloor \frac{m}{2} \rfloor + 1$. Let $U_2 \subseteq V_n$ such that $|U_2| = \lfloor \frac{n}{2} \rfloor + 1$. The function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U_1 \cup U_2 \\ -1 & \text{otherwise} \end{cases}$$

is a kSF such that

$$f(V_m \cup V_n) = \begin{cases} 2 & \text{if } m \text{ and } n \text{ are odd} \\ 3 & \text{if } m \text{ and } n \text{ have different parities} \\ 4 & \text{if } m \text{ and } n \text{ are even} \end{cases} . \diamondsuit$$

These results generalize a result of [4].

We now turn our attention to computing $\gamma_{ks}^{-101}(K_m \cup K_n)$.

Proposition 3.25 *If $1 \leq k \leq m \leq n$, then*

$$\gamma_{ks}^{-101}(K_m \cup K_n) = 1 - n.$$

Proof. Since $k \geq 1$, $C_f \neq \emptyset$, say $v \in C_f$. If $v \in V_m$, then by Proposition 3.2, $f(V_m) \geq 1$. Thus, $f(V_m \cup V_n) = f(V_m) + f(V_n) \geq 1 + (-1)n = 1 - n$. Similarly, if $v \in V_n$, then $f(V_m \cup V_n) \geq 1 - m \geq 1 - n$. Let $u \in V_m$. The function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v = u \\ 0 & \text{if } v \in V_m - \{u\} \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with $f(V_m \cup V_n) = 1 - n$. $\diamondsuit$

Proposition 3.26 *If $1 \leq m < k \leq n$, then*

$$\gamma_{ks}^{-101}(K_m \cup K_n) = 1 - m.$$

Proof. Since $k > m$, there is a vertex v, say, such that $v \in V_n \cap C_f$. As before, $f(V_n) \geq 1$ and $f(V_m \cup V_n) \geq 1 - m$. Let $u \in V_n$. The function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v = u \\ 0 & \text{if } v \in V_n - \{u\} \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with $f(V_m \cup V_n) = 1 - m$. ◇

Proposition 3.27 *Suppose $n \geq m \geq 1$. If $n < k \leq n + m$ then*

$$\gamma_{ks}^{-101}(K_m \cup K_n) = 2.$$

Proof. Since $k > n$, there is a covered vertex, $v \in V_m$ and a covered vertex, $u \in V_n$. By Proposition 3.2, $f(V_m) \geq 1$ and $f(V_n) \geq 1$, so that $f(V_m \cup V_n) \geq 2$. Let $u_1 \in V_m$ and $u_2 \in V_n$. The function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in \{u_1, u_2\} \\ 0 & \text{otherwise} \end{cases}$$

is a kSF with $f(V_m \cup V_n) = 2$. ◇

3.5 Split graphs

In this section we compute $\gamma_{ks}^{-11}(G)$ and $\gamma_{ks}^{-101}(G)$ of a split graph G. The set V_n will denote the vertices in the clique and the set V_m will denote the set of independent vertices. Since $S_{1,n}$ is the complete graph of order $n+1$, we may assume that $m \geq 2$.

Lemma 3.28 *If $V_n \cap C_f \neq \emptyset$, then*

$$f(V(S_{m,n})) \geq \begin{cases} 2 & \textit{if } m + n \textit{ is even} \\ 1 & \textit{if } m + n \textit{ is odd} \end{cases}.$$

Proof. If $v \in V_n \cap C_f$, then $1 \le f[v] = f(V(S_{m,n}))$, which implies that $|P_f \cap (V_m \cup V_n)| \ge \lfloor \frac{m+n}{2} \rfloor + 1$. Therefore, if $m+n$ is even, then $f(V(S_{m,n})) \ge (\frac{m+n}{2} + 1) - (\frac{m+n}{2} - 1) = 2$. If $m+n$ is odd, then $f(V(S_{m,n})) \ge (\lfloor \frac{m+n}{2} \rfloor + 1) - (\lceil \frac{m+n}{2} \rceil - 1) = 1$. $\diamondsuit$

Proposition 3.29 *Suppose $n \ge 1$ and $m \ge 1$. If $1 \le k \le m$, then*

$$\gamma_{ks}^{-11}(S_{m,n}) = \begin{cases} 2-m & \text{if } n \text{ is even} \\ 3-m & \text{if } n \text{ is odd} \end{cases}.$$

Proof. Since $k \ge 1$, there is a covered vertex, v (say). On the one hand, if $v \in V_n$, then, by Lemma 3.28,

$$f(V(S_{m,n})) \ge \begin{cases} 2 & \text{if } m+n \text{ is even} \\ 1 & \text{if } m+n \text{ is odd} \end{cases}.$$

On the other hand, suppose $v \in V_m$. If $f(v) = -1$, then $f[v] = -1 + f(V_n) \ge 1$, and so $f(V_n) \ge 2$, which implies that $f(V(S_{m,n})) = f(V_m) + f(V_n) \ge 2 - m$. Moreover, if n is odd, then $|P_f \cap V_n| \ge \lceil \frac{n}{2} \rceil + 1$ and $f(V_n) \ge (\lceil \frac{n}{2} \rceil + 1) - (\lfloor \frac{n}{2} \rfloor - 1) = 3$. Therefore $f(V(S_{m,n})) = f(V_m) + f(V_n) \ge 3 - m$. If $f(v) = 1$, then $f[v] = 1 + f(V_n) \ge 1$, and so $f(V_n) \ge 0$, which implies that $f(V(S_{m,n})) = f(V_m) + f(V_n) \ge 1 - (m-1) + 0 = 2 - m$. Moreover, if n is odd, $|P_f \cap V_n| \ge \lceil \frac{n}{2} \rceil$ and $f(V_n) \ge \lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor = 1$. Thus, $f(V(S_{m,n})) = f(V_m) + f(V_n) \ge 1 - (m-1) + 1 = 3 - m$.

We conclude that if $v \in V_m$,

$$\begin{aligned} f(V(S_{m,n})) &= f(V_m) + f(V_n) \\ &\ge \begin{cases} 3-m & \text{if } n \text{ is odd} \\ 2-m & \text{if } n \text{ is even} \end{cases}. \end{aligned}$$

Notice if $m \geq 2$, then $2 - m < 3 - m < 1 < 2$. Therefore

$$f(V(S_{m,n})) \geq \begin{cases} 2 - m & \text{if } n \text{ is even} \\ 3 - m & \text{if } n \text{ is odd} \end{cases}.$$

Let $U \subseteq V_n$ such that $|U| = \lceil \frac{n}{2} \rceil + 1$. The function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with

$$f(V(S_{m,n})) = \begin{cases} 3 - m & \text{if } n \text{ is odd} \\ 2 - m & \text{if } n \text{ is even} \end{cases}. \ \diamondsuit$$

Proposition 3.30 *Suppose $n \geq 1$ and $m \geq 1$. If $m < k \leq n + m$ then*

$$\gamma_{ks}^{-11}(S_{m,n}) = \begin{cases} 2 & \textit{if } m + n \textit{ is even} \\ 1 & \textit{if } m + n \textit{ is odd} \end{cases}.$$

Proof. Since $k > m$, there is a covered vertex $v \in V_n$. By Lemma 3.28,

$$f(V(S_{m,n})) \geq \begin{cases} 2 & \text{if } m + n \text{ is even} \\ 1 & \text{if } m + n \text{ is odd} \end{cases}.$$

Let $U_1 \subseteq V_n$ be such that $|U_1| = \lceil \frac{n}{2} \rceil + 1$ and let $U_2 \subseteq V_m$ such that

$$|U_2| = \begin{cases} \lceil \frac{m}{2} \rceil & \text{if } m \text{ and } n \text{ are both even} \\ \lceil \frac{m}{2} \rceil - 1 & \text{otherwise} \end{cases}.$$

Then the function f defined by

$$f(v) = \begin{cases} 1 & \text{if } v \in U_1 \cup U_2 \\ -1 & \text{otherwise} \end{cases}$$

is a kSF with

$$f(V(S_{m,n})) = \begin{cases} 2 & \text{if } m+n \text{ is even} \\ 1 & \text{if } m+n \text{ is odd} \end{cases} . \diamondsuit$$

We now turn our attention to computing $\gamma_{ks}^{-101}(G)$ of a split graph G.

Proposition 3.31 *Suppose $n \geq 1$ and $m \geq 2$. If $1 \leq k \leq m$ then*

$$\gamma_{ks}^{-101}(S_{m,n}) = 2 - m.$$

Proof. Since $k \geq 1$, there is a covered vertex, $v \in V_m \cup V_n$. If $v \in V_n$, then $1 \leq f[v] = f(V(S_{m,n}))$. Now suppose $v \in V_m$. If $f(v) = -1$, then $f(V_n) \geq 2$ and $f(V(S_{m,n})) = f(V_m) + f(V_n) \geq 2 - m$. If $f(v) = 0$, then $f(V_n) \geq 1$ and $f(V(S_{m,n})) = f(V_m) + f(V_n) \geq 0 - (m-1) + 1 = 2 - m$. If $f(v) = 1$, then $f(V_n) \geq 0$ and $f(V(S_{m,n})) = f(V_m) + f(V_n) \geq 0 + 1 - (m-1) = 2 - m$. Since $m \geq 2$, $2 - m \leq 1$. Hence, $f(V(S_{m,n})) \geq 2 - m$.

The function f that assigns 1 to exactly two vertices of V_n, 0 to the remaining vertices of V_n and -1 to all vertices of V_m is a kSF with $f(V(S_{m,n})) = 2 - m$. $\diamondsuit$

Proposition 3.32 *Suppose $n \geq 1$ and $m \geq 2$. If $m < k \leq n + m$ then*

$$\gamma_{ks}^{-101}(S_{m,n}) = 1.$$

Proof. Since $k > m$, there is a covered vertex $v \in V_n$. Thus, $1 \leq f[v] = f(V(S_{m,n}))$. The function that assigns 1 to exactly one vertex of V_n and 0 to all remaining vertices of $S_{m,n}$ is a kSF with $f(V(S_{m,n})) = 1$. $\diamondsuit$

Chapter 4

TOTAL SIGNED k-SUBDOMINATION NUMBERS

4.1 Introduction

In this chapter, we focus on total signed k-subdomination, or, as it will not cause any confusion, total k-subdomination. In Section 4.2, we give a lower bound on the total k-subdomination number in terms of the minimum degree, maximum degree and the order of the graph. A lower bound in terms of the degree sequence is also given. In Section 4.3, we compute the total k-subdomination number of a cycle. In Section 4.4, we present a characterization of graphs G with equal total k-subdomination and total ℓ-subdomination numbers. In the final section, we establish a sharp upper bound on the total k-subdomination number of a tree in terms of its order n and k where $1 \leq k < n$. Moreover, we characterize trees attaining these bounds for certain values of k. For this purpose, we first establish the total k-subdomination number of simple structures, including paths and spiders.

4.2 Lower bounds

Our aim in this section is to give lower bounds on the total k-subdomination number of a graph. We first establish such a lower bound in terms of its minimum degree, maximum degree and its order. The second lower bound is in terms of the degree sequence of the graph. We begin with the following observation. (For a vertex $v \in V$, we denote $f(N(v))$ by $f[v]$.)

Observation 4.1 *Let f be a TkSF of G and let $v \in C_f$. If $\deg v$ is even, then $f[v] \geq 2$, while if $\deg v$ is odd, then $f[v] \geq 1$.*

Theorem 4.2 *Let $G = (V, E)$ be a graph of order n with minimum degree δ and maximum degree Δ. For $1 \leq k \leq n$, let f be a $\gamma_{tks}^{-11}(G)$-function, and let ℓ denote the number of vertices with even degree in C_f. Then,*

$$\gamma_{tks}^{-11}(G) \geq \frac{2k(1+\Delta) + \delta n - 3n\Delta + 2\ell}{\Delta + \delta}.$$

Proof. We consider the sum $N = \sum \sum f(u)$, where the outer sum is over all $v \in V$ and the inner sum is over all $u \in N(v)$. This sum counts the value $f(u)$ exactly $\deg u$ times for each $u \in V$, so $N = \sum(\deg u) \cdot f(u)$, over

all $u \in V$. Let V_{even} denote the set of all vertices with even degree in C_f. Then, by Observation 4.1, $N = \sum f[v]$ over all $v \in V$ satisfies

$$\begin{aligned}
N &= \sum_{v\in V_{even}} f[v] + \sum_{v\in C_f - V_{even}} f[v] + \sum_{v\notin C_f} f[v] \\
&\geq 2\ell + |C_f| - \ell + (n - |C_f|)(-\Delta) \\
&= \ell + |C_f|(1+\Delta) - n\Delta \\
&\geq \ell + k(1+\Delta) - n\Delta. \qquad (4.1)
\end{aligned}$$

Let P and M be the sets of those vertices in G which are assigned under f the values $+1$ and -1, respectively. Then, $\gamma_{tks}^{-11}(G) = f(V) = |P| - |M| = n - 2|M|$. We now write V as the disjoint union of six sets. Let $P = P_\Delta \cup P_\delta \cup P_\lambda$ where P_Δ and P_δ are sets of all vertices of P with degree equal to Δ and δ, respectively, and P_λ contains all other vertices in P, if any. Let $M = M_\Delta \cup M_\delta \cup M_\lambda$ where M_Δ, M_δ, and M_λ are defined similarly. Further, for $i \in \{\Delta, \delta, \lambda\}$, let V_i be defined by $V_i = P_i \cup M_i$. Thus, $n = |V_\Delta| + |V_\delta| + |V_\lambda|$.

If $u \in V_\lambda$, then $\delta + 1 \leq \deg u \leq \Delta - 1$. Therefore, writing the sum in (4.1) as the sum of six summations and replacing $f(u)$ with the corresponding value of 1 or -1 yields

$$\sum_{u\in P_\Delta} \Delta + \sum_{x\in P_\delta} \delta + \sum_{x\in P_\lambda} (\Delta - 1) - \sum_{x\in M_\Delta} \Delta - \sum_{x\in M_\delta} \delta - \sum_{x\in M_\lambda} (\delta + 1) \geq \ell + k(1+\Delta) - n\Delta.$$

Replacing $|P_i|$ with $|V_i| - |M_i|$ for $i \in \{\Delta, \delta, \lambda\}$, yields

$$\begin{aligned}
&\Delta|V_\Delta| + \delta|V_\delta| + (\Delta - 1)|V_\lambda| - 2\Delta|M_\Delta| - 2\delta|M_\delta| - (\Delta + \delta)|M_\lambda| \\
\geq\ & \ell + k(1+\Delta) - n\Delta. \qquad (4.2)
\end{aligned}$$

We now simplify the left hand side of (4.2) as follows. Replacing $|V_\delta|$ with $|P_\delta|+|M_\delta|$, and $|M_\delta| + |M_\lambda|$ with $|M| - |M_\Delta|$, we have

$$\delta|V_\delta| - 2\delta|M_\delta| - \delta|M_\lambda| = \delta|P_\delta| - \delta|M_\delta| - \delta|M_\lambda| = \delta|P_\delta| - \delta|M| + \delta|M_\Delta|. \quad (4.3)$$

Further, replacing $|V_\Delta|$ with $n - |V_\delta| - |V_\lambda|$, we have

$$\begin{aligned} & \Delta|V_\Delta| + \Delta|V_\lambda| - 2\Delta|M_\Delta| - \Delta|M_\lambda| \\ = \quad & n\Delta - \Delta|V_\delta| - 2\Delta|M_\Delta| - \Delta|M_\lambda| \\ = \quad & n\Delta - \Delta|P_\delta| - \Delta|M| - \Delta|M_\Delta|. \end{aligned} \quad (4.4)$$

Using (4.3) and (4.4), the left hand side of (4.2) can be written as

$$n\Delta - |V_\lambda| - (\Delta - \delta)|P_\delta| - (\Delta + \delta)|M| - (\Delta - \delta)|M_\Delta|.$$

Thus (4.2) becomes

$$\begin{aligned} 2n\Delta - k(1+\Delta) - \ell &\geq |V_\lambda| + (\Delta - \delta)|P_\delta| + (\Delta + \delta)|M| + (\Delta - \delta)|M_\Delta| \\ &\geq (\Delta + \delta)|M|. \end{aligned} \quad (4.5)$$

Hence, since $\gamma_{tks}^{-11}(G) = n - 2|M|$, it follows from (4.5) that

$$\gamma_{tks}^{-11}(G) \geq n - 2\left(\frac{2n\Delta - k(1+\Delta) - \ell}{\Delta + \delta}\right) = \frac{2k(1+\Delta) + \delta n - 3n\Delta + 2\ell}{\Delta + \delta},$$

as desired. $\diamondsuit$

The next result gives a lower bound on the total k-subdomination number of a graph in terms of its degree sequence.

Theorem 4.3 *Let $G = (V, E)$ be a graph of order n where the degrees d_i of vertices v_i satisfy $d_1 \leq d_2 \leq \cdots \leq d_n$, let f be a $\gamma_{tks}^{-11}(G)$-function, and let ℓ denote the number of vertices of even degree in C_f. Then,*

$$\gamma_{tks}^{-11}(G) \geq \left(\frac{\ell + k + \sum_{i=1}^{k} d_i}{d_n} \right) - n.$$

Proof. Let f be a $\gamma_{tks}^{-11}(G)$-function. Let V_{even} denote the set of all vertices with even degree in C_f. Let $g\colon V \to \{0,1\}$ be the function defined by $g(v) = (f(v)+1)/2$ for all vertices $v \in V$. We consider the sum $N = \sum\sum g(u)$, where the outer sum is over all $v \in C_f$ and the inner sum is over all $u \in N(v)$. Then,

$$\begin{aligned} N &= \sum_{v\in C_f}\sum_{u\in N(v)} \frac{1}{2}(f(u)+1) = \sum_{v\in C_f} \frac{1}{2}(f[v] + \deg v) = \frac{1}{2}(\sum_{v\in C_f} f[v] + \sum_{v\in C_f} \deg v) \\ &\geq \frac{1}{2}(\sum_{i=1}^{k} d_i + \sum_{v\in V_{even}} \deg v + \sum_{v\in C_f - V_{even}} \deg v) \geq \frac{1}{2}(2\ell + |C_f| - \ell + \sum_{i=1}^{k} d_i) \\ &\geq \frac{1}{2}(\ell + k + \sum_{i=1}^{k} d_i). \end{aligned}$$

On the other hand,

$$N \leq \sum_{v\in V}\sum_{u\in N(v)} g(u) = \sum_{v\in V}(\deg v)\cdot g(v) \leq d_n\, g(V),$$

and so

$$g(V) \geq \frac{(\ell + k + \sum_{i=1}^{k} d_i)}{2d_n}$$

The desired result now follows since $\gamma_{tks}^{-11}(G) = f(V) = 2g(V) - n$. $\diamondsuit$

As an immediate consequence of Theorem 4.2 or Theorem 4.3, we have the following result.

Corollary 4.4 *For $r \geq 1$, if G is an r-regular graph of order n, then*

$$\gamma_{tks}^{-11}(G) \geq \begin{cases} k\left(\dfrac{r+1}{r}\right) - n & \text{if } r \text{ is odd} \\ k\left(\dfrac{r+2}{r}\right) - n & \text{if } r \text{ is even} \end{cases}$$

Corollary 4.5 *If G is a graph of order n, size m and maximum degree Δ, then*

$$\gamma_{tks}^{-11}(G) \geq k - 2n + \frac{k+2m}{\Delta}.$$

Proof. Let the degrees d_i of the vertices of G satisfy $d_1 \leq d_2 \leq \cdots \leq d_n = \Delta$. It follows from Theorem 4.3 that

$$\begin{aligned} \gamma_{tks}^{-11}(G) &\geq \frac{1}{\Delta}\left(k + \sum_{i=1}^{k} d_i\right) - n \\ &= \frac{1}{\Delta}\left(k + 2m - \sum_{i=k+1}^{n} d_i\right) - n \\ &\geq \frac{1}{\Delta}(k + 2m - (n-k)\Delta) - n \\ &= k - 2n + \frac{k+2m}{\Delta}. \qquad \diamondsuit \end{aligned}$$

4.3 Cycles

Our aim in this section is to determine the total k-subdomination number of a cycle. As a special case of Corollary 4.4, we have that $\gamma_{tks}^{-11}(C_n) \geq 2k - n$. If $k \in \{n/2, n\}$, we show this lower bound is sharp. We shall prove:

Proposition 4.6 *For $n \geq 3$ and $1 \leq k \leq n$,*

$$\gamma_{tks}^{-11}(C_n) = \begin{cases} 2k - n & \text{if } k \in \{n/2, n\} \\ \\ 2k + 2 - n & \text{otherwise.} \end{cases}$$

Proof. We show first that $\gamma_{tks}^{-11}(C_n) \geq 2k + 2 - n$ except when $k = n/2$ or $k = n$, in which case $\gamma_{tks}^{-11}(C_n) = 2k - n$. Let f be a $\gamma_{tks}^{-11}(C_n)$-function. Let $M = \{v \in V(C_n) \mid f(v) = -1\}$ and $P = \{v \in V(C_n) \mid f(v) = +1\}$. Note that, since $k \geq 1$, $P \neq \emptyset$. Let $M_c = C_f \cap M$, $P_c = C_f \cap P$, $M_{uc} = M - M_c$ and $P_{uc} = P - P_c$. Let $H = G[M_c \cup P]$, i.e., H is the subgraph of G induced by $M_c \cup P$ where $G = C_n$. The two vertices adjacent to a vertex in M_c are in P_{uc}, while the two vertices adjacent to a vertex in P_c are in P. It follows that

$$2m(G[P]) = \sum_{v \in P} \deg_{G[P]} v \geq \sum_{v \in P_c} \deg_{G[P]} v = 2|P_c|,$$

whence $m(G[P]) \geq |P_c|$. Thus $m(H) = 2|M_c| + m(G[P]) \geq 2|M_c| + |P_c|$. Further if $m(G[P]) = |P_c|$, then $\deg_{G[P]}(v) = 0$ for all $v \in P_{uc}$ and, since C_n is connected and none of the vertices in P_c are adjacent to any of the vertices of $M \cup P_{uc}$, either $V = P_c$ or $P_c = \emptyset$. So, if $m(G[P]) = |P_c|$, either $V = P_c$ or $P = P_{uc}$ and $m(G[P]) = 0$.

Case 1. $M_{uc} = \emptyset$. Then $H \cong C_n$, so $|M_c| + |P| = m(H) \geq 2|M_c| + |P_c|$. Thus, $|P| \geq |M_c| + |P_c| = |C_f| \geq k$ and so $\gamma_{tks}^{-11}(C_n) \geq 2k - n$. If we have strict inequality in any of the above inequalities or if $|C_f| \geq k+1$, then $|P| \geq k+1$, and $\gamma_{tks}^{-11}(C_n) = 2|P| - n \geq 2(k+1) - n = 2k+2-n$. Hence, suppose we have equality throughout in the above inequalities and $|M_c| + |P_c| = k$. Then, by our remarks above, either $V = P_c$, in which case $|P_c| = k = n$, or $P_c = \emptyset$, in which case $|M_c| = k$ and $n = m(H) = 2|M_c| = 2k$ and so $k = n/2$.

Case 2. $M_{uc} \neq \emptyset$. In this case H consists of a disjoint union of $\ell \geq 1$ paths. Then, $|M_c|+|P|-\ell = m(H) \geq 2|M_c|+|P_c|$. Thus, $|P| \geq |M_c|+|P_c|+\ell \geq |C_f|+1 \geq k+1$, and so $\gamma_{tks}^{-11}(C_n) \geq 2(k+1) - n = 2k+2-n$.

We have shown that $\gamma_{tks}^{-11}(C_n) \geq 2(k+1) - n = 2k+2-n$ except when $k = n/2$ or $k = n$, in which case $\gamma_{tks}^{-11}(C_n) \geq 2k - n$. We now show that $\gamma_{tks}^{-11}(C_n) \leq 2k - n$ if $k = n/2$ or $k = n$ and that $\gamma_{tks}^{-11}(C_n) \leq 2k+2-n$ otherwise. For this purpose, we denote the vertex set of the cycle C_n by $\{0, 1, \ldots, n-1\}$. We now define a function $f(V(C_n)) \to \{-1, 1\}$ as follows:

For $1 \leq k < n/2$, let $f(v_i) = 1$ if $i \in \{0, 2, \ldots, 2k\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2(k+1) - n$, and $\{v_1, v_3, \ldots, v_{2k-1}\} \subseteq C_f$, so that $|C_f| \geq k$.

For $k = n/2$, let $f(v_i) = 1$ if i is even and $f(v_i) = -1$ otherwise. Then, $f(V) = 0$ and $\{v_1, v_3, \ldots, v_{n-1}\} \subseteq C_f$, so that $|C_f| \geq k$.

For $(n+2)/2 \leq k \leq n-1$ and n even, let $f(v_i) = 1$ if i is even or $i \in \{1, 3, \ldots, 2k-n+1\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2|P| - n = 2k-n+2$, and $\{v_1, v_3, \ldots, v_{n-1}\} \cup \{v_2, v_4, \ldots, v_{2k-n}\} \subseteq C_f$ so that $|C_f| \geq n/2 + (k - n/2) = k$.

For $(n+1)/2 \leq k \leq n-1$ and n odd, let $f(v_i) = 1$ if i is even or $i \in \{1, 3, \ldots, 2k-n\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2|P| - n = 2k - n + 2$, and $\{v_1, v_3, \ldots, v_{n-2}\} \cup \{v_0, v_2, \ldots, v_{2k-n-1}\} \subseteq C_f$ so that $|C_f| \geq (n-1)/2 + (2k - n + 1)/2 = k$.

For $k = n$, the function that assigns 1 to every vertex of the cycle is the desired function.

In all the above cases, f is a $TkSF$ of C_n. Thus, $\gamma_{tks}^{-11}(C_n) \leq f(V) = 2k - n$ if $k = n/2$ or $k = n$, while $\gamma_{tks}^{-11}(C_n) \leq f(V) = 2k + 2 - n$ otherwise. $\diamondsuit$

4.4 Graphs with equal total k- and ℓ-subdomination numbers

Our aim in this section is to give a characterization of graphs G with equal total k-subdomination and total ℓ-subdomination numbers where $1 \leq k < \ell \leq |V(G)|$. Our proof is along similar lines to that presented in [34].

Theorem 4.7 *Let $G = (V, E)$ be a graph of order n and let $1 \leq k < \ell \leq n$ be integers. Then $\gamma_{tks}^{-11}(G) = \gamma_{t\ell s}^{-11}(G)$ if and only if there exists a partition (P, M) of V for which*

1. $|N(x) \cap P| - |N(x) \cap M| \geq 1$ *for at least ℓ of the vertices of G, and*

2. *for any $P' \subseteq P$ and any $M' \subseteq M$ satisfying $|P'| > |M'|$, we have* $|\{x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| > n - k$.

Proof. Suppose $\gamma_{tks}^{-11}(G) = \gamma_{t\ell s}^{-11}(G)$. Let f be a $TlSF$ of G such that $f(V) = \gamma_{tks}^{-11}(G) = \gamma_{t\ell s}^{-11}(G)$. Let $P = \{x \in V \mid f(x) = 1\}$ and $M = \{x \in V \mid f(x) = -1\}$. Then (P, M) constitutes a partition of V. For each $x \in C_f$, we have $f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1$. Since $|C_f| \geq \ell$, Condition (1) holds.

To verify that Condition (2) holds, consider any $P' \subseteq P$ and $M' \subseteq M$ such that $|P'| > |M'|$. Let $X = (P \setminus P') \cup M'$ and $Y = (M \setminus M') \cup P'$. Define a function

$g : V \to \{-1, 1\}$ as follows: $g(x) = 1$ for every $x \in X$ and $g(x) = -1$ for every $x \in Y$. Then $g(V) = |X| - |Y| = (|P| - |P'| + |M'|) - (|M| - |M'| + |P'|) = |P| - |M| - 2(|P'| - |M'|) < |P| - |M| = f(V) = \gamma_{tks}^{-11}(G)$. Thus, g is not a $TkSF$ of G, and so $|C_g| < k$. Consequently,

$$|\{x \in V \mid g[x] \leq 0\}| = |V - C_g| = n - |C_g| > n - k. \tag{4.6}$$

Note that

$$\begin{aligned}
g[x] &= |N(x) \cap X| - |N(x) \cap Y| \\
&= |N(x) \cap ((P \setminus P') \cup M')| - |N(x) \cap ((M \setminus M') \cup P')| \\
&= |N(x) \cap (P \setminus P')| + |N(x) \cap M'| - |N(x) \cap (M \setminus M')| \\
&\quad -|N(x) \cap P'| \\
&= |N(x) \cap P| - |N(x) \cap P'| + |N(x) \cap M'| - |N(x) \cap M| \\
&\quad +|N(x) \cap M'| - |N(x) \cap P'| \\
&= |N(x) \cap P| - |N(x) \cap M| - 2(|N(x) \cap P'| - |N(x) \cap M'|). \qquad (4.7)
\end{aligned}$$

Combining (4.6) and (4.7), we obtain Condition 2.

For the sufficiency, suppose there is a partition (P, M) of V such that Conditions (1) and (2) hold. Define a function $f\colon V \to \{-1, 1\}$ as follows: $f(x) = 1$ for every $x \in P$ and $f(x) = -1$ for every $x \in M$. Then $f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1$ for at least ℓ vertices of G (by Condition (1)). Thus, f is a $T\ell SF$ of G, so that $\gamma_{t\ell s}^{-11}(G) \leq |P| - |M|$.

We now show that $\gamma_{tks}^{-11}(G) \geq |P| - |M|$: Suppose, to the contrary, $\gamma_{tks}^{-11}(G) < |P| - |M|$. Let g be a $TkSF$ of G such that $\gamma_{tks}^{-11}(G) = g(V)$. Let $X = \{x \in V \mid$

Chapter 4

TOTAL SIGNED k-SUBDOMINATION NUMBERS

4.1 Introduction

In this chapter, we focus on total signed k-subdomination, or, as it will not cause any confusion, total k-subdomination. In Section 4.2, we give a lower bound on the total k-subdomination number in terms of the minimum degree, maximum degree and the order of the graph. A lower bound in terms of the degree sequence is also given. In Section 4.3, we compute the total k-subdomination number of a cycle. In Section 4.4, we present a characterization of graphs G with equal total k-subdomination and total ℓ-subdomination numbers. In the final section, we establish a sharp upper bound on the total k-subdomination number of a tree in terms of its order n and k where $1 \leq k < n$. Moreover, we characterize trees attaining these bounds for certain values of k. For this purpose, we first establish the total k-subdomination number of simple structures, including paths and spiders.

4.2 Lower bounds

Our aim in this section is to give lower bounds on the total k-subdomination number of a graph. We first establish such a lower bound in terms of its minimum degree, maximum degree and its order. The second lower bound is in terms of the degree sequence of the graph. We begin with the following observation. (For a vertex $v \in V$, we denote $f(N(v))$ by $f[v]$.)

Observation 4.1 *Let f be a TkSF of G and let $v \in C_f$. If $\deg v$ is even, then $f[v] \geq 2$, while if $\deg v$ is odd, then $f[v] \geq 1$.*

Theorem 4.2 *Let $G = (V, E)$ be a graph of order n with minimum degree δ and maximum degree Δ. For $1 \leq k \leq n$, let f be a $\gamma_{tks}^{-11}(G)$-function, and let ℓ denote the number of vertices with even degree in C_f. Then,*

$$\gamma_{tks}^{-11}(G) \geq \frac{2k(1+\Delta) + \delta n - 3n\Delta + 2\ell}{\Delta + \delta}.$$

Proof. We consider the sum $N = \sum \sum f(u)$, where the outer sum is over all $v \in V$ and the inner sum is over all $u \in N(v)$. This sum counts the value $f(u)$ exactly $\deg u$ times for each $u \in V$, so $N = \sum (\deg u) \cdot f(u)$, over

all $u \in V$. Let V_{even} denote the set of all vertices with even degree in C_f. Then, by Observation 4.1, $N = \sum f[v]$ over all $v \in V$ satisfies

$$\begin{aligned} N &= \sum_{v \in V_{even}} f[v] + \sum_{v \in C_f - V_{even}} f[v] + \sum_{v \notin C_f} f[v] \\ &\geq 2\ell + |C_f| - \ell + (n - |C_f|)(-\Delta) \\ &= \ell + |C_f|(1 + \Delta) - n\Delta \\ &\geq \ell + k(1 + \Delta) - n\Delta. \end{aligned} \tag{4.1}$$

Let P and M be the sets of those vertices in G which are assigned under f the values $+1$ and -1, respectively. Then, $\gamma_{tks}^{-11}(G) = f(V) = |P| - |M| = n - 2|M|$. We now write V as the disjoint union of six sets. Let $P = P_\Delta \cup P_\delta \cup P_\lambda$ where P_Δ and P_δ are sets of all vertices of P with degree equal to Δ and δ, respectively, and P_λ contains all other vertices in P, if any. Let $M = M_\Delta \cup M_\delta \cup M_\lambda$ where M_Δ, M_δ, and M_λ are defined similarly. Further, for $i \in \{\Delta, \delta, \lambda\}$, let V_i be defined by $V_i = P_i \cup M_i$. Thus, $n = |V_\Delta| + |V_\delta| + |V_\lambda|$.

If $u \in V_\lambda$, then $\delta + 1 \leq \deg u \leq \Delta - 1$. Therefore, writing the sum in (4.1) as the sum of six summations and replacing $f(u)$ with the corresponding value of 1 or -1 yields

$$\sum_{u \in P_\Delta} \Delta + \sum_{x \in P_\delta} \delta + \sum_{x \in P_\lambda} (\Delta - 1) - \sum_{x \in M_\Delta} \Delta - \sum_{x \in M_\delta} \delta - \sum_{x \in M_\lambda} (\delta + 1) \geq \ell + k(1 + \Delta) - n\Delta.$$

Replacing $|P_i|$ with $|V_i| - |M_i|$ for $i \in \{\Delta, \delta, \lambda\}$, yields

$$\begin{aligned} &\Delta|V_\Delta| + \delta|V_\delta| + (\Delta - 1)|V_\lambda| - 2\Delta|M_\Delta| - 2\delta|M_\delta| - (\Delta + \delta)|M_\lambda| \\ \geq\ &\ell + k(1 + \Delta) - n\Delta. \end{aligned} \tag{4.2}$$

We now simplify the left hand side of (4.2) as follows. Replacing $|V_\delta|$ with $|P_\delta|+|M_\delta|$, and $|M_\delta|+|M_\lambda|$ with $|M|-|M_\Delta|$, we have

$$\delta|V_\delta| - 2\delta|M_\delta| - \delta|M_\lambda| = \delta|P_\delta| - \delta|M_\delta| - \delta|M_\lambda| = \delta|P_\delta| - \delta|M| + \delta|M_\Delta|. \quad (4.3)$$

Further, replacing $|V_\Delta|$ with $n-|V_\delta|-|V_\lambda|$, we have

$$\begin{aligned} & \Delta|V_\Delta| + \Delta|V_\lambda| - 2\Delta|M_\Delta| - \Delta|M_\lambda| \\ = \quad & n\Delta - \Delta|V_\delta| - 2\Delta|M_\Delta| - \Delta|M_\lambda| \\ = \quad & n\Delta - \Delta|P_\delta| - \Delta|M| - \Delta|M_\Delta|. \qquad (4.4) \end{aligned}$$

Using (4.3) and (4.4), the left hand side of (4.2) can be written as

$$n\Delta - |V_\lambda| - (\Delta-\delta)|P_\delta| - (\Delta+\delta)|M| - (\Delta-\delta)|M_\Delta|.$$

Thus (4.2) becomes

$$\begin{aligned} 2n\Delta - k(1+\Delta) - \ell \quad & \geq \quad |V_\lambda| + (\Delta-\delta)|P_\delta| + (\Delta+\delta)|M| + (\Delta-\delta)|M_\Delta| \\ & \geq \quad (\Delta+\delta)|M|. \qquad (4.5) \end{aligned}$$

Hence, since $\gamma_{tks}^{-11}(G) = n - 2|M|$, it follows from (4.5) that

$$\gamma_{tks}^{-11}(G) \geq n - 2\left(\frac{2n\Delta - k(1+\Delta) - \ell}{\Delta+\delta}\right) = \frac{2k(1+\Delta) + \delta n - 3n\Delta + 2\ell}{\Delta+\delta},$$

as desired. $\diamondsuit$

The next result gives a lower bound on the total k-subdomination number of a graph in terms of its degree sequence.

Theorem 4.3 *Let $G = (V, E)$ be a graph of order n where the degrees d_i of vertices v_i satisfy $d_1 \leq d_2 \leq \cdots \leq d_n$, let f be a $\gamma_{tks}^{-11}(G)$-function, and let ℓ denote the number of vertices of even degree in C_f. Then,*

$$\gamma_{tks}^{-11}(G) \geq \left(\frac{\ell + k + \sum_{i=1}^{k} d_i}{d_n}\right) - n.$$

Proof. Let f be a $\gamma_{tks}^{-11}(G)$-function. Let V_{even} denote the set of all vertices with even degree in C_f. Let $g\colon V \to \{0,1\}$ be the function defined by $g(v) = (f(v)+1)/2$ for all vertices $v \in V$. We consider the sum $N = \sum\sum g(u)$, where the outer sum is over all $v \in C_f$ and the inner sum is over all $u \in N(v)$. Then,

$$\begin{aligned} N &= \sum_{v\in C_f}\sum_{u\in N(v)} \frac{1}{2}(f(u)+1) = \sum_{v\in C_f} \frac{1}{2}(f[v] + \deg v) = \frac{1}{2}(\sum_{v\in C_f} f[v] + \sum_{v\in C_f} \deg v) \\ &\geq \frac{1}{2}(\sum_{i=1}^{k} d_i + \sum_{v\in V_{even}} \deg v + \sum_{v\in C_f - V_{even}} \deg v) \geq \frac{1}{2}(2\ell + |C_f| - \ell + \sum_{i=1}^{k} d_i) \\ &\geq \frac{1}{2}(\ell + k + \sum_{i=1}^{k} d_i). \end{aligned}$$

On the other hand,

$$N \leq \sum_{v\in V}\sum_{u\in N(v)} g(u) = \sum_{v\in V}(\deg v)\cdot g(v) \leq d_n\, g(V),$$

and so

$$g(V) \geq \frac{(\ell + k + \sum_{i=1}^{k} d_i)}{2d_n}.$$

The desired result now follows since $\gamma_{tks}^{-11}(G) = f(V) = 2g(V) - n$. $\diamondsuit$

As an immediate consequence of Theorem 4.2 or Theorem 4.3, we have the following result.

Corollary 4.4 *For $r \geq 1$, if G is an r-regular graph of order n, then*

$$\gamma_{tks}^{-11}(G) \geq \begin{cases} k\left(\dfrac{r+1}{r}\right) - n & \text{if } r \text{ is odd} \\[2ex] k\left(\dfrac{r+2}{r}\right) - n & \text{if } r \text{ is even} \end{cases}$$

Corollary 4.5 *If G is a graph of order n, size m and maximum degree Δ, then*

$$\gamma_{tks}^{-11}(G) \geq k - 2n + \frac{k+2m}{\Delta}.$$

Proof. Let the degrees d_i of the vertices of G satisfy $d_1 \leq d_2 \leq \cdots \leq d_n = \Delta$. It follows from Theorem 4.3 that

$$\begin{aligned} \gamma_{tks}^{-11}(G) &\geq \frac{1}{\Delta}\left(k + \sum_{i=1}^{k} d_i\right) - n \\ &= \frac{1}{\Delta}\left(k + 2m - \sum_{i=k+1}^{n} d_i\right) - n \\ &\geq \frac{1}{\Delta}(k + 2m - (n-k)\Delta) - n \\ &= k - 2n + \frac{k+2m}{\Delta}. \qquad \diamond \end{aligned}$$

4.3 Cycles

Our aim in this section is to determine the total k-subdomination number of a cycle. As a special case of Corollary 4.4, we have that $\gamma_{tks}^{-11}(C_n) \geq 2k - n$. If $k \in \{n/2, n\}$, we show this lower bound is sharp. We shall prove:

Proposition 4.6 *For $n \geq 3$ and $1 \leq k \leq n$,*

$$\gamma_{tks}^{-11}(C_n) = \begin{cases} 2k - n & \text{if } k \in \{n/2, n\} \\ \\ 2k + 2 - n & \text{otherwise.} \end{cases}$$

Proof. We show first that $\gamma_{tks}^{-11}(C_n) \geq 2k + 2 - n$ except when $k = n/2$ or $k = n$, in which case $\gamma_{tks}^{-11}(C_n) = 2k - n$. Let f be a $\gamma_{tks}^{-11}(C_n)$-function. Let $M = \{v \in V(C_n) \mid f(v) = -1\}$ and $P = \{v \in V(C_n) \mid f(v) = +1\}$. Note that, since $k \geq 1$, $P \neq \emptyset$. Let $M_c = C_f \cap M$, $P_c = C_f \cap P$, $M_{uc} = M - M_c$ and $P_{uc} = P - P_c$. Let $H = G[M_c \cup P]$, i.e., H is the subgraph of G induced by $M_c \cup P$ where $G = C_n$. The two vertices adjacent to a vertex in M_c are in P_{uc}, while the two vertices adjacent to a vertex in P_c are in P. It follows that

$$2m(G[P]) = \sum_{v \in P} \deg_{G[P]} v \geq \sum_{v \in P_c} \deg_{G[P]} v = 2|P_c|,$$

whence $m(G[P]) \geq |P_c|$. Thus $m(H) = 2|M_c| + m(G[P]) \geq 2|M_c| + |P_c|$. Further if $m(G[P]) = |P_c|$, then $\deg_{G[P]}(v) = 0$ for all $v \in P_{uc}$ and, since C_n is connected and none of the vertices in P_c are adjacent to any of the vertices of $M \cup P_{uc}$, either $V = P_c$ or $P_c = \emptyset$. So, if $m(G[P]) = |P_c|$, either $V = P_c$ or $P = P_{uc}$ and $m(G[P]) = 0$.

Case 1. $M_{uc} = \emptyset$. Then $H \cong C_n$, so $|M_c| + |P| = m(H) \geq 2|M_c| + |P_c|$. Thus, $|P| \geq |M_c| + |P_c| = |C_f| \geq k$ and so $\gamma_{tks}^{-11}(C_n) \geq 2k - n$. If we have strict inequality in any of the above inequalities or if $|C_f| \geq k+1$, then $|P| \geq k+1$, and $\gamma_{tks}^{-11}(C_n) = 2|P| - n \geq 2(k+1) - n = 2k+2-n$. Hence, suppose we have equality throughout in the above inequalities and $|M_c| + |P_c| = k$. Then, by our remarks above, either $V = P_c$, in which case $|P_c| = k = n$, or $P_c = \emptyset$, in which case $|M_c| = k$ and $n = m(H) = 2|M_c| = 2k$ and so $k = n/2$.

Case 2. $M_{uc} \neq \emptyset$. In this case H consists of a disjoint union of $\ell \geq 1$ paths. Then, $|M_c| + |P| - \ell = m(H) \geq 2|M_c| + |P_c|$. Thus, $|P| \geq |M_c| + |P_c| + \ell \geq |C_f| + 1 \geq k+1$, and so $\gamma_{tks}^{-11}(C_n) \geq 2(k+1) - n = 2k+2-n$.

We have shown that $\gamma_{tks}^{-11}(C_n) \geq 2(k+1) - n = 2k+2-n$ except when $k = n/2$ or $k = n$, in which case $\gamma_{tks}^{-11}(C_n) \geq 2k - n$. We now show that $\gamma_{tks}^{-11}(C_n) \leq 2k - n$ if $k = n/2$ or $k = n$ and that $\gamma_{tks}^{-11}(C_n) \leq 2k+2-n$ otherwise. For this purpose, we denote the vertex set of the cycle C_n by $\{0, 1, \ldots, n-1\}$. We now define a function $f(V(C_n)) \to \{-1, 1\}$ as follows:

For $1 \leq k < n/2$, let $f(v_i) = 1$ if $i \in \{0, 2, \ldots, 2k\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2(k+1) - n$, and $\{v_1, v_3, \ldots, v_{2k-1}\} \subseteq C_f$, so that $|C_f| \geq k$.

For $k = n/2$, let $f(v_i) = 1$ if i is even and $f(v_i) = -1$ otherwise. Then, $f(V) = 0$ and $\{v_1, v_3, \ldots, v_{n-1}\} \subseteq C_f$, so that $|C_f| \geq k$.

For $(n+2)/2 \leq k \leq n-1$ and n even, let $f(v_i) = 1$ if i is even or $i \in \{1, 3, \ldots, 2k-n+1\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2|P| - n = 2k - n + 2$, and $\{v_1, v_3, \ldots, v_{n-1}\} \cup \{v_2, v_4, \ldots, v_{2k-n}\} \subseteq C_f$ so that $|C_f| \geq n/2 + (k - n/2) = k$.

For $(n+1)/2 \leq k \leq n-1$ and n odd, let $f(v_i) = 1$ if i is even or $i \in \{1, 3, \ldots, 2k-n\}$ and $f(v_i) = -1$ otherwise. Then, $f(V) = 2|P| - n = 2k - n + 2$, and $\{v_1, v_3, \ldots, v_{n-2}\} \cup \{v_0, v_2, \ldots, v_{2k-n-1}\} \subseteq C_f$ so that $|C_f| \geq (n-1)/2 + (2k - n + 1)/2 = k$.

For $k = n$, the function that assigns 1 to every vertex of the cycle is the desired function.

In all the above cases, f is a $TkSF$ of C_n. Thus, $\gamma_{tks}^{-11}(C_n) \leq f(V) = 2k - n$ if $k = n/2$ or $k = n$, while $\gamma_{tks}^{-11}(C_n) \leq f(V) = 2k + 2 - n$ otherwise. $\diamondsuit$

4.4 Graphs with equal total k- and ℓ-subdomination numbers

Our aim in this section is to give a characterization of graphs G with equal total k-subdomination and total ℓ-subdomination numbers where $1 \leq k < \ell \leq |V(G)|$. Our proof is along similar lines to that presented in [34].

Theorem 4.7 *Let $G = (V, E)$ be a graph of order n and let $1 \leq k < \ell \leq n$ be integers. Then $\gamma_{tks}^{-11}(G) = \gamma_{t\ell s}^{-11}(G)$ if and only if there exists a partition (P, M) of V for which*

1. $|N(x) \cap P| - |N(x) \cap M| \geq 1$ *for at least ℓ of the vertices of G, and*
2. *for any $P' \subseteq P$ and any $M' \subseteq M$ satisfying $|P'| > |M'|$, we have* $|\{x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| > n - k$.

Proof. Suppose $\gamma_{tks}^{-11}(G) = \gamma_{t\ell s}^{-11}(G)$. Let f be a $TlSF$ of G such that $f(V) = \gamma_{tks}^{-11}(G) = \gamma_{t\ell s}^{-11}(G)$. Let $P = \{x \in V \mid f(x) = 1\}$ and $M = \{x \in V \mid f(x) = -1\}$. Then (P, M) constitutes a partition of V. For each $x \in C_f$, we have $f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1$. Since $|C_f| \geq \ell$, Condition (1) holds.

To verify that Condition (2) holds, consider any $P' \subseteq P$ and $M' \subseteq M$ such that $|P'| > |M'|$. Let $X = (P \setminus P') \cup M'$ and $Y = (M \setminus M') \cup P'$. Define a function

$g : V \to \{-1,1\}$ as follows: $g(x) = 1$ for every $x \in X$ and $g(x) = -1$ for every $x \in Y$. Then $g(V) = |X| - |Y| = (|P| - |P'| + |M'|) - (|M| - |M'| + |P'|) = |P| - |M| - 2(|P'| - |M'|) < |P| - |M| = f(V) = \gamma_{tks}^{-11}(G)$. Thus, g is not a $TkSF$ of G, and so $|C_g| < k$. Consequently,

$$|\{x \in V \mid g[x] \leq 0\}| = |V - C_g| = n - |C_g| > n - k. \tag{4.6}$$

Note that

$$\begin{aligned}
g[x] &= |N(x) \cap X| - |N(x) \cap Y| \\
&= |N(x) \cap ((P \setminus P') \cup M')| - |N(x) \cap ((M \setminus M') \cup P')| \\
&= |N(x) \cap (P \setminus P')| + |N(x) \cap M'| - |N(x) \cap (M \setminus M')| \\
&\qquad -|N(x) \cap P'| \\
&= |N(x) \cap P| - |N(x) \cap P'| + |N(x) \cap M'| - |N(x) \cap M| \\
&\qquad +|N(x) \cap M'| - |N(x) \cap P'| \\
&= |N(x) \cap P| - |N(x) \cap M| - 2(|N(x) \cap P'| - |N(x) \cap M'|). \qquad (4.7)
\end{aligned}$$

Combining (4.6) and (4.7), we obtain Condition 2.

For the sufficiency, suppose there is a partition (P, M) of V such that Conditions (1) and (2) hold. Define a function $f \colon V \to \{-1,1\}$ as follows: $f(x) = 1$ for every $x \in P$ and $f(x) = -1$ for every $x \in M$. Then $f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1$ for at least ℓ vertices of G (by Condition (1)). Thus, f is a $T\ell SF$ of G, so that $\gamma_{t\ell s}^{-11}(G) \leq |P| - |M|$.

We now show that $\gamma_{tks}^{-11}(G) \geq |P| - |M|$: Suppose, to the contrary, $\gamma_{tks}^{-11}(G) < |P| - |M|$. Let g be a $TkSF$ of G such that $\gamma_{tks}^{-11}(G) = g(V)$. Let $X = \{x \in V \mid$

$g(x) = 1\}$ and $Y = \{x \in V \mid g(x) = -1\}$. Let $P' = P \setminus X$ and $M' = M \setminus Y$. Then $P' \subseteq P$, $M' \subseteq M$, $X = (P \setminus P') \cup M'$ and $Y = (M \setminus M') \cup P'$. Moreover, $|P|-|M|+2(|M'|-|P'|) = |P|-|P'|+|M'|-|M|+|M'|-|P'| = |X|-|Y| = \gamma_{tks}^{-11}(G) < |P|-|M|$, so that $|P'| > |M'|$. By Condition (2), $|V - C_g| = |\{x \in V \mid g[x] \leq 0\}| = |\{x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| > n-k$. Thus, $|C_g| < k$, contradicting the fact that g is $TkSF$ of G. Hence, $\gamma_{tks}^{-11}(G) \geq |P| - |M|$.

We conclude that $|P| - |M| \leq \gamma_{tks}^{-11}(G) \leq \gamma_{tls}^{-11}(G) \leq |P| - |M|$, so that $\gamma_{tks}^{-11}(G) = \gamma_{tls}^{-11}(G)$. $\diamond$

Theorem 4.8 *Let $G = (V, E)$ be a graph of order n and let $1 \leq k \leq n$ be integers. Then $\gamma_{tks}^{-11}(G) = a$ if and only if there exists a partition (P, M) of V for which*

1. $|N(x) \cap P| - |N(x) \cap M| \geq 1$ *for at least k of the vertices of G,*
2. $|P| - |M| = a$, *and*
3. *for any $P' \subseteq P$ and any $M' \subseteq M$ satisfying $|P'| > |M'|$, we have*

   $|\{x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| > n - k$.

Proof. Suppose $\gamma_{tks}^{-11}(G) = a$. Let f be a $TkSF$ of G such that $f(V) = \gamma_{tks}^{-11}(G) = a$. Let $P = \{x \in V \mid f(x) = 1\}$ and $M = \{x \in V \mid f(x) = -1\}$. Conditions (1) and (3) follows as in the proof of Theorem 4.7. Moreover, $f(V) = |P| - |M|$, so Condition (2) holds.

For the sufficiency, suppose there is a partition (P, M) of V such that Conditions (1), (2) and (3) hold. Define a function $f\colon V \to \{-1, 1\}$ as follows: $f(x) = 1$ for every $x \in P$ and $f(x) = -1$ for every $x \in M$. Then $f[x] = |N(x) \cap P| - |N(x) \cap M| \geq 1$ for at least k vertices of G (by Condition (1)). Thus, f is a $TkSF$ of G, so that $\gamma_{tks}^{-11}(G) \leq |P| - |M| = a$ (by Condition (2)). As in the proof of Theorem 4.7,

$\gamma_{ks}(G) \geq |P| - |M|$. Hence, $|P| - |M| \leq \gamma_{ks}(G) \leq |P| - |M| = a$, so that $\gamma_{ks}(G) = a$. $\diamondsuit$

4.5 Trees

An *opinion function* on G is a function $f: V \to \{-1, +1\}$; $f(v)$ is the *opinion* of the vertex v. For an opinion function f, we say that a vertex v *votes aye* if $f[v] \geq 1$ and *nay* otherwise. Thus, $\gamma_{tks}^{-11}(G) = \min\{f(V) \mid f$ is an opinion function of G in which at least k vertices vote aye $\}$.

By giving a positive opinion to the center of a star of order $n \geq 3$ and negative opinions to all the leaves we obtain a $TkSF$ of the star. Thus

Proposition 4.9 *For $n \geq 3$ and $1 \leq k < n$, $\gamma_{tks}^{-11}(K_{1,n-1}) = 2 - n$.*

Hence the total k-subdomination number of a tree can be arbitrarily large negative if k is less than the order of the tree.

When $k = n$, the total k-subdomination number is the total signed domination number. In [27], lower and upper bounds on the total signed domination number of a tree in terms of its order are given and the trees attaining these bounds are characterized.

Theorem 4.10 [27] If T is a tree of order $n \geq 2$, then

$$2 \leq \gamma_{tns}^{-11}(T) \leq n.$$

Furthermore, $\gamma_{tns}^{-11}(T) = 2$ if and only if every vertex $v \in V(T) - L(T)$ has odd degree and is adjacent to at least $(\deg v - 1)/2$ leaves, while $\gamma_{tns}^{-11}(T) = n$ if and only if every vertex of T is a support vertex or is adjacent to a vertex of degree 2.

Our aim in this section is to establish a sharp upper bound on the total k-subdomination number of a tree in terms of its order n and k when $1 \leq k < n$, and to characterize trees attaining these bounds for certain values of k. For this purpose, we first establish the total k-subdomination number of simple structures, including paths and spiders.

4.5.1 Paths

In this subsection, we establish the total k-subdomination number of a path. We begin with the following lemma.

Lemma 4.11 *For $n \geq 3$ and $1 \leq k < n$, there exists a $\gamma_{tks}^{-11}(P_n)$-function that assigns to one of its leaves a negative opinion and to its neighbor a positive opinion.*

Proof. Let T be the path $v_1, v_2, \ldots, v_n$ and let f be a $\gamma_{tks}^{-11}(T)$-function. Let i be the smallest subscript such that $f(v_i) = -1$. If $i \geq 2$, then the function obtained from f by interchanging the values of v_1 and v_i is an opinion function having the same weight as f and with at least as many vertices voting aye as under f. Hence, we can choose f so that $f(v_1) = -1$. Now let j be the smallest subscript such that $f(v_j) = 1$. If $j \geq 3$, then the function obtained from f by interchanging the values of v_2 and v_j is an opinion function having the same weight as f and with at least as many vertices voting aye as under f. Hence, we can choose f so that $f(v_2) = 1$. $\diamond$

Proposition 4.12 *For $n \geq 2$ and $1 \leq k \leq n$,*

$$\gamma_{tks}^{-11}(P_n) = \begin{cases} -1 & \textit{if } k = \frac{1}{2}(n+1) \\ \\ 2k - n & \textit{otherwise.} \end{cases}$$

Proof. We proceed by induction on the order $n \geq 2$ of a path P_n. If $n = 2$, then $\gamma_{tks}^{-11}(P_2) = 2k - n$ for $k = 1$ or $k = 2$. Suppose $n = 3$. If $k = 3$, then $\gamma_{tks}^{-11}(P_3) = 3 = 2k - n$, while for $1 \leq k \leq 2$, $\gamma_{tks}^{-11}(P_3) = -1$ by Proposition 4.9 and the desired result follows. This proves the base cases when $n = 2$ or $n = 3$.

Suppose that $n \geq 4$ and that for every nontrivial path $P_{n'}$ of order $n' < n$, and any integer k' with $1 \leq k' \leq n'$, $\gamma_{tks}^{-11}(P_{n'}) \leq -1$ if $k' = (n'+1)/2$ and $\gamma_{tks}^{-11}(P_{n'}) \leq 2k'-n'$ otherwise. Let T be a path P_n of order n. Let u be a leaf of T and let v be the vertex adjacent to u.

If $k = 1$, then giving a positive opinion to v and negative opinions to all other vertices of T we obtain a $TkSF$ of T of weight $2 - n$. Since $\gamma_{tks}^{-11}(G) \geq 2 - n$ for all graphs G with no isolated vertex, $\gamma_{tks}^{-11}(P_n) = 2 - n = 2k - n$. Hence we may assume $k \geq 2$. Furthermore, if $k = n$, then the result follows from Theorem 4.10. Hence we may assume that $k < n$. Let $T' = T - u - v$. Then, T' is a path of order $n' = n - 2$. Let $k' = k - 1$. Since $2 \leq k \leq n - 1$, it follows that $1 \leq k' \leq n'$.

Let f' be a $\gamma_{tk's}^{-11}(T')$-function. Let $f\colon V(T) \to \{-1, 1\}$ be the function defined by $f(w) = f'(w)$ if $w \in V(T')$, $f(v) = 1$ and $f(u) = -1$. Every vertex that votes aye in T' also votes aye in T, while u votes aye in T. Hence at least $k' + 1 = k$ vertices of T vote aye, and so f is a $TkSF$ of T. Thus, $\gamma_{tks}^{-11}(T) \leq f(V(T)) = f'(V(T')) = \gamma_{tk's}^{-11}(T')$. On the other hand, by Lemma 4.11 there exists a $\gamma_{tks}^{-11}(T)$-function g that assigns to u a negative opinion and to v a positive opinion. Let g' be the restriction of g to $V(T')$. Then, g' is a $Tk'sF$ of T'. Thus, $\gamma_{tk's}^{-11}(T') \leq g'(V(T')) = g(V(T)) = \gamma_{tks}^{-11}(T)$. Consequently, $\gamma_{tks}^{-11}(T) = \gamma_{tk's}^{-11}(T')$.

Suppose $k' = (n' + 1)/2$. Then, $k = (n + 1)/2$ and by the inductive hypothesis, $f'(V(T')) = -1$, and so $\gamma_{tks}^{-11}(T) = f'(V(T')) = -1$. Suppose $k' \neq (n' + 1)/2$. Then, $k \neq (n + 1)/2$ and by the inductive hypothesis, $f'(V(T')) = 2k' - n' = 2k - n$, and so $\gamma_{tks}^{-11}(T) = f'(V(T')) = 2k - n$. $\diamondsuit$

4.5.2 Spiders

In this subsection, we establish the total k-subdomination number of an even spider. We begin with the following lemma.

Lemma 4.13 *Let T be an even spider of (odd) order $n \geq 3$. For $(n+1)/2 \leq k < n$, there exists a $\gamma_{tks}^{-11}(T)$-function that assigns to a leaf at maximum distance from the head of T a negative opinion and to its neighbor a positive opinion.*

Proof. If T is a path, then the result follows from Lemma 4.11. Hence we may assume that $\Delta(T) \geq 3$. Let v be the head of the spider and let f be a $\gamma_{tks}^{-11}(T)$-function. Let x be a leaf at maximum distance from v. For two distinct vertices a and b of T, we denote by $f_{a,b}$ the function obtained from f by interchanging the values of a and b and leaving the values of all other vertices unchanged.

Suppose $f(x) = 1$. We construct a new opinion function g having the same weight as f and with at least as many vertices voting aye as under f but with $g(x) = -1$. If some vertex on the v–x path has a negative opinion, then let w be such a vertex at maximum distance from v (possibly, $v = w$) and take $g = f_{w,x}$. On the other hand, suppose every vertex on the v–x path has a positive opinion. Since $k < n$, at least one vertex of T has a negative opinion. Hence there exists a leaf z of T such that the v–z path contains at least one vertex with a negative opinion. Let $v, v_1, v_2, \ldots, v_{2r} = z$ denote the v–z path. If $f(v_i) = -1$ for some i with i odd, then take $g = f_{v_j,x}$ where j is the largest odd integer such that $f(v_j) = -1$. On the other hand, if $f(v_i) = 1$ for all odd i, then take $g = f_{v_i,x}$ where i is the smallest (even) integer such that $f(v_i) = -1$. Hence we can choose f so that $f(x) = -1$.

Let w be the vertex adjacent to x. Suppose $f(w) = -1$. We construct a new opinion function h having the same weight as f and with at least as many vertices voting aye as under h but with $h(w) = 1$ and $h(x) = -1$. If every vertex different from v that is at even distance from v has a negative opinion, then every vertex at odd distance from v votes nay. Further, since x votes nay, this would imply that $k \leq (n-1)/2$, a contradiction. Hence $f(d) = 1$ for at least one vertex $d \neq v$ at even distance from v. We can now take $h = f_{d,w}$ where d is a vertex at maximum even distance from v with $f(d) = 1$. Hence we can choose f so that $f(w) = 1$ and $f(x) = -1$, as desired. $\diamondsuit$

Proposition 4.14 *Let T be an even spider of (odd) order $n \geq 3$. For $(n+1)/2 \leq k \leq n$,*

$$\gamma_{tks}^{-11}(T) = \begin{cases} -1 & \text{if } k = \frac{1}{2}(n+1) \\ \\ 2k-n & \text{otherwise,} \end{cases}$$

Proof. We proceed by induction on the order n of the even spider T. If T is a path, then the desired result follows from Proposition 4.12. In particular, this proves the base cases when $n = 3$ and $n = 5$. Suppose that $n \geq 7$ (and so, $k \geq 4$) and that for every even spider T' of (odd) order $n' < n$, and any integer k' with $(n'+1)/2 \leq k' \leq n'$, $\gamma_{tks}^{-11}(T') = -1$ if $k' = (n'+1)/2$ and $\gamma_{tks}^{-11}(T') = 2k' - n'$ otherwise. Let T be an even spider of order n. We may assume $\Delta(T) \geq 3$, for otherwise the result follows from Proposition 4.12. Let v be the head of T. Let x be a leaf at maximum distance from v in T and let w be the support vertex adjacent to x.

If $k = n$, then the result follows from Theorem 4.10. Hence we may assume that $k \leq n-1$. Let $T' = T - w - x$. Then, T' is an even spider of order $n' = n-2$. Let $k' = k-1$, and so $(n'+1)/2 \leq k' \leq n'$.

Any $Tk'sF$ of T' can be extended to a $TkSF$ of T by assigning a positive opinion to the vertex w and a negative opinion to the vertex x. It follows that $\gamma_{tks}^{-11}(T) \leq \gamma_{tk's}^{-11}(T')$. On the other hand, by Lemma 4.13 there exists a $\gamma_{tks}^{-11}(T)$-function g that assigns to x a negative opinion and to w a positive opinion. Let g'

be the restriction of g to $V(T')$. If $d(v,x) \geq 4$, then g' is a $Tk'sF$ of T'. Suppose $d(v,x) = 2$. Then, T is an even spider with every leg of length 2. Since $k \geq (n+1)/2$, we can choose g so that every neighbor of v has a positive opinion and still $g(x) = -1$ (if some neighbor a of v has a negative opinion, then there must be a leaf b with a positive opinion, and we can simply take $g^* = g_{a,b}$). Hence, once again g' is a $Tk'sF$ of T'. Thus, $\gamma_{tk's}^{-11}(T') \leq g'(V(T')) = g(V(T)) = \gamma_{tks}^{-11}(T)$. Consequently, $\gamma_{tks}^{-11}(T) = \gamma_{tk's}^{-11}(T')$.

Suppose $k' = (n'+1)/2$. Then, $k = (n+1)/2$ and by the inductive hypothesis, $\gamma_{tk's}^{-11}(T') = -1$, and so $\gamma_{tks}^{-11}(T) = -1$. Suppose $k' \geq (n'+3)/2$. Then, $k \geq (n+3)/2$ and by the inductive hypothesis, $\gamma_{tk's}^{-11}(T') = 2k' - n' = 2k - n$, and so $\gamma_{tks}^{-11}(T) = 2k - n$. $\diamondsuit$

4.5.3 Upper Bounds

We now present an upper bound on the total k-subdomination number of a tree in terms of its order and k.

Theorem 4.15 *For any tree T of order $n \geq 2$, and any integer k with $1 \leq k \leq n$,*

$$\gamma_{tks}^{-11}(T) \leq \begin{cases} -1 & \text{if } k = \frac{1}{2}(n+1) \\ \\ 2k - n & \text{otherwise.} \end{cases}$$

and these bounds are sharp.

Proof. We proceed by induction on the order $n \geq 2$ of a tree T. If $n \in \{2,3\}$, then $T = P_n$ and the result follows from Proposition 4.12. This proves the base cases when $n = 2$ or $n = 3$.

Suppose that $n \geq 4$ and that for every nontrivial tree T' of order $n' < n$, and any integer k' with $1 \leq k' \leq n' - 1$, $\gamma_{tks}^{-11}(T') \leq -1$ if $k' = (n' + 1)/2$ and $\gamma_{tks}^{-11}(T') \leq 2k' - n'$ otherwise. Let T be a tree of order n.

If T is a star, then, by Proposition 4.9, $\gamma_{tks}^{-11}(T) = 2 - n < -1$. Thus, $\gamma_{tks}^{-11}(T) = 2k - n$ if $k = 1$, while $\gamma_{tks}^{-11}(T) < 2k - n$ if $2 \leq k \leq n$. Hence the desired result follows if T is a star. Thus we may assume that $\operatorname{diam}(T) \geq 3$.

If $k = n$, then, by Theorem 4.10, $\gamma_{tks}^{-11}(T) \leq n = 2k - n$. Hence we may assume $k < n$.

Let T be rooted at a leaf r of a longest path. Let v be a vertex at distance $\operatorname{diam}(T)-1$ from r on a longest path starting at r, and let w be the parent of v. Let $|N(v) - \{w\}| = m$. Then, $m \geq 1$. If $k \leq m$, then giving a positive opinion to v and negative opinions to all the other vertices we obtain a $TkSF$ of T of weight $2 - n$, and the desired result follows. Hence we may assume $k > m$.

Let $T' = T - (N[v] - \{w\})$. Then, T' has order $n' = n - m - 1$. Since $\operatorname{diam}(T) \geq 3$, $n' \geq 2$. Let $k' = k - m$. Since $m + 1 \leq k \leq n - 1$, we have $1 \leq k' \leq n'$. Let f' be a $\gamma_{tk's}^{-11}(T')$-function. Let $f\colon V(T) \to \{-1, 1\}$ be the function defined by $f(w) = f'(w)$ if $w \in V(T')$, $f(v) = 1$ and $f(u) = -1$ for every child of v. Every vertex that votes aye in T' also votes aye in T, while each child of v votes aye in T. Hence at least $k' + m = k$ vertices of T vote aye, and so f is a $TkSF$ of T. Thus, $\gamma_{tks}^{-11}(T) \leq f(V(T)) = f'(V(T')) + 1 - m$.

Suppose $k' = (n' + 1)/2$. Then, $k = (n + m)/2$. By the inductive hypothesis, $\gamma_{tk's}^{-11}(T') \leq -1$, and so $\gamma_{tks}^{-11}(T) \leq -m$. Thus if $m = 1$, then $k = (n + 1)/2$ and $\gamma_{tks}^{-11}(T) \leq -1$, while if $m \geq 2$, then $k \geq (n + 2)/2$ and $\gamma_{tks}^{-11}(T) \leq -2 < 2 \leq 2k - n$. In any event, the result follows.

On the other hand, suppose $k' \neq (n' + 1)/2$. By the inductive hypothesis, $\gamma_{tk's}^{-11}(T') \leq 2k' - n' = 2k - n + 1 - m$, and so $\gamma_{tks}^{-11}(T) \leq 2k - n + 2(1 - m)$.

Suppose $k = (n+1)/2$. Then, $k' = (n' - m + 2)/2$. Since $k' \neq (n'+1)/2$, it follows that $m \geq 2$, and so $\gamma^{-11}_{tks}(T) \leq 2k - n + 2(1-m) \leq -1$. Suppose $k \neq (n+1)/2$. Then, since $m \geq 1$, $\gamma^{-11}_{tks}(T) \leq 2k - n$. Once again, the desired result follows.

That the bounds are sharp, follows from Proposition 4.12. ◇

As an immediate consequence of Theorem 4.8, we have the following result.

Corollary 4.16 *Let $T = (V, E)$ be a tree of order n and let $1 \leq k \leq n$ be an integer. Then, $\gamma^{-11}_{tks}(T) = 2k - n$ if and only if there exists a partition (P, M) of V for which*

1. $|N(x) \cap P| - |N(x) \cap M| \geq 1$ *for at least k of the vertices of T,*
2. $|P| - |M| = 2k - n$, *and*
3. *for any $P' \subseteq P$ and any $M' \subseteq M$ satisfying $|P'| > |M'|$, we have* $|\{x \in V \mid 2(|N(x) \cap P'| - |N(x) \cap M'|) \geq |N(x) \cap P| - |N(x) \cap M|\}| > n - k$.

4.5.4 Extremal Trees

The trees of order n with maximum total k-subdomination number when $k = n$ are characterized in Theorem 4.10. Our aim in this subsection is to characterize those trees of order n achieving the maximum possible total k-subdomination number when (i) n is even and $n/2 \leq k \leq n/2 + 3$ and when (ii) n is odd and $(n+3)/2 \leq k \leq (n+5)/2$, i.e., we characterize those trees achieving equality in the upper bounds in Theorem 4.15 for $k = n/2$ and $n/2 + 1 \leq k \leq n/2 + 3$. We first consider the case when n is even.

Theorem 4.17 *For any tree T of even order $n \geq 2$, and any integer k with $n/2 \leq k \leq n/2 + 3$, $\gamma^{-11}_{tks}(T) = 2k - n$ if and only if T is a path.*

Proof. The sufficiency follows from Proposition 4.12. To prove the necessity, we proceed by induction on the order n of a tree T, where $n \geq 2$ is even, $n/2 \leq k \leq n$, $k = n/2 + i$ where $0 \leq i \leq 3$, and $\gamma_{tks}^{-11}(T) = 2k - n$. If $n = 2$, then $T = P_2$. If $n = 4$, then $k \in \{2, 3, 4\}$. If $T = K_{1,3}$, then it follows from Theorem 4.10 (if $k = 4$) or Proposition 4.9 (if $k \in \{2, 3\}$) that $\gamma_{tks}^{-11}(T) < 2k - n$, a contradiction. Hence, $T = P_4$. Thus if $n \in \{2, 4\}$, then T is a path. This proves the base cases when $n = 2$ and $n = 4$.

Suppose that $n \geq 4$ is even and that for every nontrivial tree T' of even order $n' < n$, and any integer k' with $n'/2 \leq k' \leq n'$ and $k' = n'/2 + i$ where $0 \leq i \leq 3$, that if $\gamma_{tk's}^{-11}(T') = 2k' - n'$, then T' is a path. Let T be a tree of order n.

If $k = n$, then since $k \leq (n + 6)/2$, $n = k = 6$, and so it follows from Theorem 4.10 that $\gamma_{tks}^{-11}(T) \leq n < 2k - n$ unless $T = P_6$. Hence the desired result follows if $k = n$. Thus we may assume $k < n$. In particular, if $k = n/2 + 3$, then $n \geq 8$.

Following the notation used in paragraph 5 and 6 of the proof of Theorem 4.15, $\gamma_{tks}^{-11}(T) \leq f(V(T)) = f'(V(T')) + 1 - m$. If $k' = (n' + 1)/2$, then $k = (n + m)/2$. By Theorem 4.15, $\gamma_{tk's}^{-11}(T') \leq -1$, and so $\gamma_{tks}^{-11}(T) \leq -m$, a contradiction. Hence, $k' \neq (n' + 1)/2$. By Theorem 4.15, $\gamma_{tk's}^{-11}(T') \leq 2k' - n' = 2k - n + 1 - m$, and so $\gamma_{tks}^{-11}(T) \leq 2k - n + 2(1 - m)$. If $m \geq 2$, then $\gamma_{tks}^{-11}(T) \leq 2k - n + 2(1 - m) \leq 2(k - 1) - n$, a contradiction. Hence, $m = 1$, and so $k' = k - m = k - 1$, $n' = n - 2$ and $\gamma_{tks}^{-11}(T) \leq f'(V(T')) = \gamma_{tk's}^{-11}(T')$. Furthermore, n' is even and $n'/2 \leq k' \leq n'$ and $k' = n'/2 + i$ where $0 \leq i \leq 3$.

By Theorem 4.15, $\gamma_{tk's}^{-11}(T') \leq 2k' - n'$. If $\gamma_{tk's}^{-11}(T') \leq 2(k' - 1) - n'$, then $\gamma_{tks}^{-11}(T) \leq 2(k - 1) - n$, a contradiction. Hence, $\gamma_{tk's}^{-11}(T') = 2k' - n'$. Applying the inductive hypothesis to T', T' is a path. Let u denote the child of v.

Suppose w is neither a leaf nor a support vertex of T'. Let v' be the child of w different from v, and let u' be the child of v'. Assign a positive opinion to w and its two children and to all vertices of degree 2 at even distance from w. Assign a negative opinion to all remaining vertices. For $k = n/2 + 1$ this is a $TkSF$. If $k = n/2$, reassign to w a negative opinion. If $k = n/2 + 2$, reassign to u a positive opinion, while if $k = n/2 + 3$, reassign to each of u and u' a positive opinion. In all cases, this produces a $TkSF$ of weight $2(k-1) - n$, and so $\gamma_{tks}^{-11}(T) \leq 2(k-1) - n$, a contradiction. Hence, w is either a leaf or a support vertex of T'.

Suppose that w is a support vertex of T'. Let v' be the child of w different from v. Assign a positive opinion to w and to all vertices different from v' whose distance from w in T is odd. Assign a negative opinion to all remaining vertices. For $k = n/2 + 1$ this is a $TkSF$. If $k = n/2$, reassign to w a negative opinion. If $k = n/2 + 2$, reassign to u a positive opinion, while if $k = n/2 + 3$, reassign to each of the two vertices at distance 2 from w positive opinion. In all cases, this produces a $TkSF$ of weight $2(k-1) - n$, and so $\gamma_{tks}^{-11}(T) \leq 2(k-1) - n$, a contradiction. Thus, w is a leaf in T', whence T is a path of even order. $\diamondsuit$

Theorem 4.17 is not true for $n/2 + 4 \leq k \leq n - 1$. For example, let $\ell \geq 2$ be an integer and let T be a spider of order $n = 2(\ell + 3)$ with three legs, two of length 2 and one of length $2\ell + 1$. Then for $n/2 + 4 \leq k \leq n-1$, $\gamma_{tks}^{-11}(T) = 2k - n$. Thus if $\gamma_{tks}^{-11}(T) = 2k - n$ and $n/2 + 4 \leq k \leq n - 1$, then T is not necessarily a path.

Next we characterize those trees of order n achieving the maximum possible total k-subdomination number when n is odd and $(n+3)/2 \leq k \leq (n+5)/2$.

Theorem 4.18 *For any tree T of odd order $n \geq 3$, and for any integer k with $(n+3)/2 \leq k \leq (n+5)/2$, $\gamma_{tks}^{-11}(T) = 2k - n$ if and only if T is an even spider.*

Proof. The sufficiency follows from Proposition 4.14. To prove the necessity, we proceed by induction on the order n of a tree T, where $n \geq 3$ is odd, $k = (n+1)/2 + i$

where $1 \leq i \leq 2$ and $\gamma_{tks}^{-11}(T) = 2k - n$. If $n = 3$, then $T = P_3$. If $n = 5$, then $k \in \{4, 5\}$. If T is not a path, then it is easy to check that $\gamma_{tks}^{-11}(T) < 2k - n$, a contradiction. Hence, $T = P_5$. Thus if $n \in \{3, 5\}$, then T is an even spider. This proves the base cases when $n = 3$ and $n = 5$.

Suppose that $n \geq 7$ is odd and that for every tree T' of odd order n', where $3 \leq n' < n$, and for $k' = (n' + 1)/2 + i$ where $1 \leq i \leq 2$, that if $\gamma_{tk's}^{-11}(T') = 2k' - n'$, then T' is an even spider. Let T be a tree of order n with $\gamma_{tks}^{-11}(T) = 2k - n$, where $k = (n + 1)/2 + i$ and $1 \leq i \leq 2$. Since $n \geq 7$, $k < n$. If T is a star, then by Proposition 4.9, $\gamma_{tks}^{-11}(T) = 2 - n < -1$, a contradiction. Hence, $\text{diam}\,(T) \geq 3$.

Following the notation used in the proof of Theorem 4.15, let T be rooted at a leaf r of a longest path, let v be a vertex at distance $\text{diam}(T) - 1$ from r on a longest path starting at r and let w be the parent of v. Let $|N(v) - \{w\}| = m$. If $m \geq 2$, then proceeding as in the proof of Theorem 4.17, $\gamma_{tks}^{-11}(T) < 2k - n$, a contradiction. Hence, $m = 1$. Let u be the child of v. Then, T' has odd order $n' = n - 2$. Let $k' = k - 1$, and so $k' = (n' + 1)/2 + i$. By Theorem 4.15, $\gamma_{tk's}^{-11}(T') \leq 2k' - n'$.

If $\gamma_{tk's}^{-11}(T') < 2k' - n'$, then any $\gamma_{tk's}^{-11}(T')$-function can be extended to a TKSF of T by assigning a positive opinion to v and a negative opinion to u, whence $\gamma_{tks}^{-11}(T) < 2k - n$, a contradiction. Hence, $\gamma_{tk's}^{-11}(T') = 2k' - n'$. Applying the inductive hypothesis to T', T' is an even spider.

Let g be the opinion function that assigns to the head of T' and all vertices at even distance from the head in T' a negative opinion and to all other vertices a positive opinion. Then the head of T' and all vertices at even distance from the head in T' vote aye under g, while all other vertices vote nay. Hence, $(n' + 1)/2 = (n - 1)/2$ vertices of T' vote aye under g. Further, $g(V(T')) = -1$, and all leaves of T' have a negative opinion under g while all support vertices of T' have a positive opinion under g.

Suppose w is a support vertex of T'. Let v' be the child of w in T'. If $k = (n+3)/2$, then the function g can be extended to a $TkSF$ of T by assigning to both u and v a positive opinion. If $k = (n+5)/2$, then the function g can be extended to a $TkSF$ of T by assigning to both u and v a positive opinion and by reassigning to v' a positive opinion. In both cases, we produce a $TkSF$ of weight $2(k-1)-n$, a contradiction. Hence w is a leaf of T' or at distance 2 from a leaf in T'. It follows that if w is a leaf of T' or if w is the head of T', then T is an even spider. Hence we may assume that w is at distance 2 from a leaf in T' but that w is neither a leaf of T' nor the head of T'.

Let x be the head of T'. By assumption, $x \neq w$. Suppose that T' is not a path. Then, x has degree at least 3. Since T' is an even spider, $d(x, w)$ is even. Let P denote the x–w path and let h be the opinion function of T defined as follows: let $h(v) = 1$ and $h(u) = -1$, let $h(y) = -g(y)$ for all vertices $y \in V(P)$, and let $h(y) = g(y)$ for all remaining vertices $y \in V(T') - V(P)$. Then, $h(V(T)) = g(V(T)) + 2 = 1$. If $y \in V(T')$ and $y \notin V(P) - \{w, x\}$ and y votes aye in T' under g, then y also votes aye in T under h. If $y \in V(P) - \{w, x\}$ and y is at odd distance from x, then y votes nay in T' under g but aye in T under h. On the other hand, if $y \in V(P) - \{w, x\}$ and y is at even distance from x, then y votes aye in T' under g but nay in T under h. Since $|V(P)|$ is odd, it follows that the number of vertices in $V(T')$ that vote aye in T is one more than the number of vertices in $V(T')$ that vote aye in T', i.e., there are $(n'+3)/2 = (n+1)/2$ vertices in $V(T')$ that vote aye in T under h. Since u votes aye in T, there are therefore $(n+3)/2$ vertices in $V(T)$ that vote aye in T. Hence if $k = (n+3)/2$, then h is a $TkSF$ of T, whence $\gamma_{tks}^{-11}(T) \leq h(V(T)) = 1 < 2k - n$, a contradiction. If $k = (n+5)/2$, then reassigning a positive opinion to the leaf in T' at distance 2 from w produces a $TkSF$ of T of weight $2(k-1) - n$, a contradiction. Hence, T' is a path, and so T is an even spider (with w as its head). $\diamondsuit$

Theorem 4.18 is not true for $(n+7)/2 \leq k \leq n-1$. For example, let $\ell \geq 3$ be an integer and let T be a tree of order $n = 2\ell + 3$ obtained from an even spider with ℓ legs each of length 2 by adding a path of length 2 to one of the support vertices of the spider. Then, for $(n+7)/2 \leq k \leq n-1$, $\gamma_{tks}^{-11}(T) = 2k - n$. Thus, if $\gamma_{tks}^{-11}(T) = 2k - n$ and $(n+7)/2 \leq k \leq n-1$, then T is not necessarily an even spider.

Chapter 5

COMPLEXITY OF TOTAL MINUS AND SIGNED DOMINATION

5.1 Introduction

This chapter is devoted to complexity issues of total minus and total signed domination. In Section 5.2, we discuss the complexity of the decision problems corresponding to the computation of $\gamma_t^{-101}(G)$ and $\gamma_t^{-11}(G)$ of a graph, where $\gamma_t^{-101}(G) = \gamma_{tns}^{-101}(G)$, $\gamma_t^{-11}(G) = \gamma_{tns}^{-11}(G)$ and $n = n(G)$. In Sections 5.3 and 5.4, we present linear algorithms for finding $\gamma_t^{-101}(T)$ and $\gamma_t^{-11}(T)$ of a nontrivial tree T. A total minus dominating function will be abbreviated by $TMDF$, while a total signed dominating function will be abbreviated by $TSDF$.

5.2 Complexity issues

In this section we discuss complexity issues regarding the computation of $\gamma_t^{-101}(G)$ and $\gamma_t^{-11}(G)$ for a graph G.

The following decision problem corresponding to the computation of the total domination number is known to be **NP**-complete, even when restricted to bipartite graphs or chordal graphs [33].

TOTAL DOMINATION (TD)

Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

Question: Does G have a total dominating set of cardinality k or less?

We will demonstrate a polynomial time reduction from this problem to the following decision problem:

TOTAL MINUS DOMINATION (TMD)

Instance: A graph $H = (V, E)$ and a positive integer $\ell \leq |V|$.

Question: Does H have a $TMDF$ of weight ℓ or less?

Theorem 5.1 TMD *is* **NP**-*complete, even when restricted to bipartite or chordal graphs.*

Proof. It is obvious that **TMD** is a member of **NP** since we can, in polynomial time, guess a function $f : V \to \{-1, 0, 1\}$ and verify that f has weight at most ℓ and is a $TMDF$. We next show how a polynomial time algorithm for **TMD** could be used to solve **TD**. Given a graph $G = (V, E)$ and a positive integer k, construct the graph H by adding to each vertex v_i of G a path of length four, consisting of the consecutive vertices v_i, w_i, x_i, y_i and z_i. It is easy to see that the graph H can be constructed in polynomial time, and that if G is a bipartite or chordal graph, then so too is H.

Lemma 5.2 $\gamma_t^{-101}(H) = \gamma_t(H) = \gamma_t(G) + 2|V(G)|$.

Proof. Let $v_i \in V(G)$ and let f be a $\gamma_t^{-101}(H)$-function. Since $N(z_i) = \{y_i\}$ and $f(N(z_i)) \geq 1$, we have $f(y_i) = 1$. Also, $1 \leq f(N(y_i)) = f(z_i) + f(x_i)$, so that $f(z_i) \geq 0$ and $f(x_i) \geq 0$. Similarly, using the facts that $1 \leq f(N(x_i))$ and $1 \leq f(N(w_i))$, we have $f(w_i) \geq 0$ and $f(v_i) \geq 0$.

Thus, $\mathrm{Im}(f) \subseteq \{0, 1\}$, and so f is a TDF of H. Consequently, $\gamma_t(H) \leq f(V(H)) = \gamma_t^{-101}(H)$. On the other hand, if S is a $\gamma_t(H)$-set, then the characteristic function h of S is a $TMDF$ of H, so $\gamma_t^{-101}(H) \leq h(V(H)) = \gamma_t(H)$. Consequently, $\gamma_t^{-101}(H) = \gamma_t(H)$.

Let $n = |V(G)|$ and let S be a $\gamma_t(G)$-set. Then $S \cup \bigcup_{i=1}^{n} \{x_i, y_i\}$ is a total dominating set of H. Thus, $\gamma_t(H) \leq \gamma_t(G) + 2n$.

To see that the reverse inequality holds, let S be a $\gamma_t(H)$-set for which $|S \cap (\bigcup_{i=1}^{n} \{w_i, x_i, y_i, z_i\})|$ is minimized.

We may assume, without loss of generality, $z_i \notin S$ and $\{x_i, y_i\} \subseteq S$. For suppose $z_i \in S$. It follows $y_i \in S$. If $x_i \in S$, then $S - \{z_i\}$ is a total dominating set, contradicting the minimality of S. Thus, $x_i \notin S$, and $S' = (S - \{z_i\}) \cup \{x_i\}$ is a $\gamma_t(H)$-set such that $z_i \notin S'$ and $\{x_i, y_i\} \subseteq S'$.

We next show that $w_i \notin S$ for all $1 \leq i \leq n$. For suppose, to the contrary, $w_i \in S$ for some $1 \leq i \leq n$. Since $S - \{w_i\}$ is not a total dominating set, v_i is uniquely (open) dominated by w_i. Let v_j be any vertex adjacent to v_i. Then $v_j \notin S$. If $v_i \in S$, then $S' = (S - \{w_i\}) \cup \{v_j\}$ is a $\gamma_t(H)$-set with $|S' \cap (\bigcup_{i=1}^{n} \{w_i, x_i, y_i, z_i\})| < |S \cap (\bigcup_{i=1}^{n} \{w_i, x_i, y_i, z_i\})|$, which is a contradiction. We may, therefore, assume $v_i \notin S$. If v_j is dominated by some vertex $v_\ell \in S$, then $(S - \{w_i\}) \cup \{v_j\}$ is a $\gamma_t(H)$-set, contradicting our choice of S, as before. Thus, v_j must be uniquely dominated by w_j. But then $(S - \{w_i, w_j\}) \cup \{v_i, v_j\}$ is a $\gamma_t(H)$-set, again contradicting our choice of S.

Since $w_i \notin S$ for all $1 \leq i \leq n$, $S - \cup_{i=1}^{n}\{x_i, y_i\}$ is a total dominating set of G, so $\gamma_t(G) \leq |S| - 2n = \gamma_t(H) - 2n$. It now follows that $\gamma_t(H) = \gamma_t(G) + 2|V(G)|$. $\diamondsuit$

Lemma 5.2 implies that if we let $\ell = k + 2|V(G)|$, then $\gamma_t(G) \leq k$ if and only if $\gamma_t^{-101}(H) \leq \ell$, and our proof is complete. $\diamondsuit$

Problem **TD** is polynomial for fixed k. To see this, let $G = (V, E)$ be a graph with $|V| = n$. If $k \geq n$, then V is a total dominating set of G of cardinality at most k. On the other hand, if $k < n$, then consider all the r-subsets of V, where $r = 1, \ldots, k$. There are $\sum_{r=1}^{k} \binom{n}{r}$ of these subsets, which is bounded above by the polynomial $\sum_{r=1}^{k} n^r$. It takes a polynomial amount of time to verify that a set is or is not a total dominating set. These remarks show that it takes a polynomial amount of time to verify whether G has a total dominating set of cardinality at most k when k is fixed. Hence for fixed k, **TD** $\in$ **P**.

In contrast, we now show that for a fixed k, **TMD** may be **NP**-complete. To see this, we will demonstrate a polynomial time reduction of **TMD** to the following decision problem.

ZERO TOTAL MINUS DOMINATION (ZTMD)

Instance: A graph $G = (V, E)$.

Question: Does G have a $TMDF$ of weight at most 0?

Theorem 5.3 **ZTMD** *is* **NP**-*complete, even when restricted to bipartite or chordal graphs.*

Proof. It is obvious that **ZTMD** is a member of **NP** since we can, in polynomial time, guess at a function $f : V(G) \to \{-1, 0, 1\}$ and verify that f has weight at most 0 and is a $TMDF$.

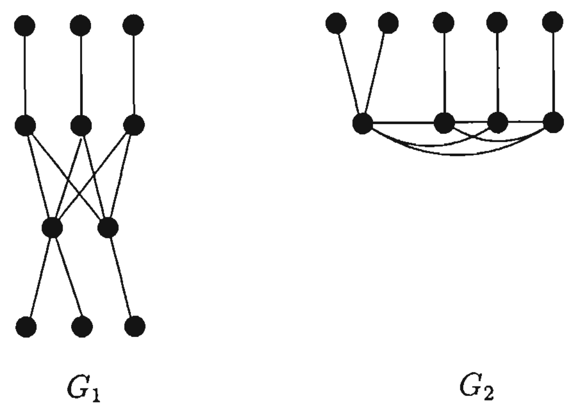

Figure 5.1:

We next show how a polynomial time algorithm for **ZTMD** could be used to solve **TMD** in polynomial time. Before proceeding further, we use the above figure to prove the following helpful result.

Lemma 5.4 $\gamma_t^{-101}(G_i) = \gamma_t^{-11}(G_i) = -1$ *for* $i = 1, 2$.

Proof. Suppose f is a $\gamma_t^{-101}(G_i)$-function ($\gamma_t^{-11}(G_i)$-function, respectively). Every vertex adjacent to an end-vertex must receive 1 under f, since otherwise that end-vertex would not have an open neighborhood sum of at least 1 under f. If any end-vertex has a value other than -1 assigned to it by f, we may reassign -1 to it and the resulting function will still be a $TMDF$ ($TSDF$, respectively) of G_i, which is a contradiction. Thus, each end-vertex of G_i is assigned -1 by f. It now follows that $\gamma_t^{-101}(G_i) = \gamma_t^{-11}(G_i) = -1$. $\diamondsuit$

Note that G_1 is bipartite, while G_2 is chordal.

Given a graph $H = (V, E)$ and a positive integer ℓ, let $J_1 = H \cup \bigcup_{j=1}^{\ell} H_{1,j}$, where $H_{1,j} \cong G_1$ for $j = 1, \ldots, \ell$ ($J_2 = H \cup \bigcup_{j=1}^{\ell} H_{2,j}$, where $H_{2,j} \cong G_2$ for $j = 1, \ldots, \ell$, respectively). It is clear that J_1 (J_2, respectively) can be constructed in polynomial time. Note that if H is bipartite (chordal, respectively), then so too is J_1 (J_2, respectively).

We now show that $\gamma_t^{-101}(H) \leq \ell$ if and only if $\gamma_t^{-101}(J_i) \leq 0$ for $i = 1, 2$. Let $1 \leq i \leq 2$. Suppose first $\gamma_t^{-101}(H) \leq \ell$ and f is a $\gamma_t^{-101}(H)$-function. Let f_j be any TMDF of weight -1 for $H_{i,j}$ for $j = 1, \ldots, \ell$. Define $g : V(G) \to \{-1, 0, 1\}$ where $G = J_i$ by $g(x) = f_j(x)$ if $x \in V(H_{i,j})$, $(j = 1, \ldots, \ell)$, while $g(x) = f(x)$ for $x \in V(H)$. Then g is a $TMDF$ of G of weight $\gamma_t^{-101}(H) + \ell(-1) \leq \ell - \ell = 0$. Conversely, suppose $\gamma_t^{-101}(J_i) \leq 0$ and g is a $\gamma_t^{-101}(J_i)$-function. Let f be the restriction of g on $V(H)$ and let f_j be the restriction of g on $V(H_{i,j})$ for $j = 1, \ldots, \ell$. Then $\gamma_t^{-101}(H) + \ell(-1) = \gamma_t^{-101}(H) + \sum_{j=1}^{\ell} \gamma_t^{-101}(H_{i,j}) \leq f(V(H)) + \sum_{j=1}^{\ell} f_j(V(H_{i,j})) = g(V(J_i)) = \gamma_t^{-101}(J_i) \leq 0$, so that $\gamma_t^{-101}(H) \leq \ell$. $\diamondsuit$

Henning [27] showed that the following decision problem is **NP**-complete.

TOTAL SIGNED DOMINATION (TSD)

Instance: A graph $H = (V, E)$ and a positive integer $\ell \leq |V|$.

Question: Does H have a $TSDF$ of weight ℓ or less?

Theorem 5.5 **TSD** *is* **NP**-*complete, even when restricted to bipartite or chordal graphs.*

As before, by using Lemma 5.4, one may show that the following decision problem is **NP**-complete, even for bipartite and chordal graphs.

ZERO TOTAL SIGNED DOMINATION (ZTSD)

Instance: A graph $G = (V, E)$.

Question: Does G have a $TSDF$ of weight at most 0?

5.3 A linear algorithm for $\gamma_t^{-101}(T)$

Next we present a linear algorithm for finding a $\gamma_t^{-101}(T)$-function in a nontrivial tree T. The variable *OpenSum* denotes the sum of the values assigned to the open neighborhood of v.

Algorithm: **TOTAL MINUS DOMINATION(TMD).** *Given a nontrivial tree T on n vertices, root the tree T and label the vertices of T from 1 to n so that $label(w) > label(y)$ if the level of vertex w is less than the level of vertex y. Note the root of T will be labeled n.*

```
for i ← 1 to n do
    f(i) ← −1
for i ← 1 to n do
begin
      1.  if vertex i is a leaf and i < n
          then begin
                      OpenSum ← 1
                      f(parent(i)) ← 1
               end
               else OpenSum ← f(N(i))
      2.  if i < n
          then while (OpenSum < 1) and (f(parent(i)) < 1) do
               begin
                      parent(i) ← f(parent(i)) + 1
                      OpenSum ← OpenSum + 1
               end
      3.  while OpenSum < 1 do
          begin
               Choose a child of i, say v, for which f(v) < 1
```

while $(OpenSum < 1)$ **and** $(f(v) < 1)$ **do**
begin
$f(v) \leftarrow f(v) + 1$
$OpenSum \leftarrow OpenSum + 1$
end
end

Theorem 5.6 *Algorithm* **TMD** *produces a* $\gamma_t^{-101}(T)$*-function in a nontrivial tree* T.

Proof. Let $T = (V, E)$ be a nontrivial tree of order n and let f be the function produced by the Algorithm **TMD**. Then $f : V \to \{-1, 0, 1\}$. For convenience, the variable *OpenSum* which was used by Algorithm **TMD** when it considered the vertex v, will be denoted by $OpenSum(v)$.

Lemma 5.7 *The function* f *produced by Algorithm* **TMD** *is a* $TMDF$.

Proof. First consider the case when v is a leaf. The algorithm assigns, in Step 1, the value 1 to the parent of v, and since values are never decreased by the algorithm, the open neighborhood sum of v is at least one.

Next consider the case when v is not a leaf. If $OpenSum(v) \geq 1$, we are done. If not, then Steps 2 and 3 of the algorithm increase the value of vertices in the open neighborhood of v such that $OpenSum(v) \geq 1$, as required. $\diamondsuit$

To show that the function f obtained by Algorithm **TMD** is a $\gamma_t^{-101}(T)$-function, let g be any $\gamma_t^{-101}(T)$-function for the rooted tree T. If $f \neq g$, then we will show that g can be transformed into a new $\gamma_t^{-101}(T)$-function g' that will differ from f in fewer values than g did. This process will continue until $f = g'$. Suppose, then, that $f \neq g$. Let v be the lowest labeled vertex for which $f(v) \neq g(v)$. Then *all* descendants of v are assigned the same value under g as under f.

Lemma 5.8 *If $g(v) < f(v)$, then the initial value assigned to the vertex v was increased in Step 3 of Algorithm* **TMD**.

Proof. Suppose the value of v was increased in Step 1. Then v is the parent of some leaf, say u. Since $g(v) < f(v)$, we have $g(v) \leq 0$. But then $g(N(u)) = g(v) \leq 0$, contradicting the fact that g is a $TMDF$ of T.

Suppose the value of v was increased in Step 2. This occurred when the algorithm was processing a vertex, say u, whose parent is v. Then $f(N(u)) \leq 1$ and $g(N(u)) = g(N(u)-\{v\})+g(v) = f(N(u)-\{v\})+g(v) = f(N(u))-f(v)+g(v) < f(N(u)) \leq 1$, which contradicts the fact that g is a $TMDF$ for T. $\diamondsuit$

Lemma 5.9 *If $g(v) < f(v)$, then the function g' defined by $g'(u) = f(u)$ if $u \in N(parent(v))$ and $g'(u) = g(u)$ if $u \notin N(parent(v))$ is a $\gamma_t^{-101}(T)$-function that differs from f in fewer values than does g.*

Proof. By Lemma 5.8, the initial value of v is increased in Step 3 of Algorithm **TMD**, which occurs when the parent of v was being processed. Let w be the parent of v. So g' is defined by $g'(u) = f(u)$ if $u \in N(w)$ and $g'(u) = g(u)$ for all remaining vertices in V.

The algorithm ensures that $f(N(w)) = 1$. Also, since g is a $TMDF$ of T, $f(N(w)) = 1 \leq g(N(w))$. Furthermore, $g'(V) = g'(V - N(w)) + g'(N(w)) = g(V - N(w)) + f(N(w)) \leq g(V - N(w)) + g(N(w)) = g(V)$. Thus, $g'(V) \leq g(V)$.

Since all the descendants of w, other than its children, have the same values under g as under f, $g'(N(u)) = f(N(u))$ if $u = w$ or if u is a descendant of w, other than a child of w. Moreover, since the value of v was increased in Step 3, then, if w had a parent, its value was either already 1 or otherwise it was increased to 1 in Step 2. Thus, $g'(N(u)) \geq g(N(u))$ for all vertices u different from w or a descendant of w, other than a child of w. Thus, since f and g are $TMDF$s of T, so too is g'. Since

$g'(V) \leq g(V)$, g' is a $\gamma_t^{-101}(T)$-function of T that differs from f in fewer values than does g. $\diamondsuit$

We now consider the case where $f(v) < g(v)$. We will need the following result.

Lemma 5.10 *A TMDF on a graph $G = (V, E)$ is minimal if and only if for every vertex $v \in V$ with $f(v) \in \{0, 1\}$, there exists a vertex $u \in N(v)$ with $f(N(u)) = 1$.*

Proof. Let f be a minimal $TMDF$ of G. Suppose there is a vertex $v \in V$ with $f(v) \in \{0, 1\}$ and $f(N(u)) \geq 2$ for every vertex $u \in N(v)$. Define a function $g : V \to \{-1, 0, 1\}$ by $g(v) = f(v) - 1$ and $g(w) = f(w)$ for all $w \neq v$. Thus $g(N(w)) = f(N(w)) \geq 1$ for all $w \notin N(v)$ and $g(N(w)) = f(N(w)) - 1 \geq 1$ for all $w \in N(v)$. So g is a $TMDF$ with $g < f$, contradicting the minimality of f.

Conversely, let f be a $TMDF$ such that for every vertex $v \in V$ with $f(v) \in \{0, 1\}$, there exists a vertex $u \in N(v)$ with $f(N(u)) = 1$. Suppose f is not minimal. Then there exists a $TMDF$ g with $g < f$. Thus, $g(w) \leq f(w)$ for all $w \in V$ and there exists a vertex $v \in V$ such that $g(v) < f(v)$. Therefore $f(v) \in \{0, 1\}$ and by the assumption there is a vertex $u \in N(v)$ with $f(N(u)) = 1$. So $g(N(u)) \leq f(N(u)) - 1 = 0$, which contradicts the fact that g is a $TMDF$. $\diamondsuit$

If the vertex v is the root then $f(V) < g(V) = \gamma_t^{-101}(T)$ which is a contradiction. Thus, we may assume that v is not the root of T.

Since the labeling at each level is arbitrary, if any vertex x at the same level as v has $g(x) < f(x)$, we can proceed as before to find a $TMDF$ g' that agrees with f in more values than g does. Thus we may assume that every vertex x at the same level as v has $f(x) \leq g(x)$.

Since $f(v) < g(v)$, we know that $g(x) \in \{0, 1\}$. By Lemma 5.10, there must be a vertex $x \in N(v)$ such that $g(N(x)) = 1$. Let w be the parent of v and u be the parent of w. If $f(u) \leq g(u)$, then $f(N(x)) = f(N(x) - \{v\}) + f(v) \leq$

$g(N(x) - \{v\}) + g(v) - 1 = g(N(x)) - 1 = 0$, which contradicts the fact that f is a $TMDF$.

Thus $f(u) > g(u)$. Suppose $f(u) = g(u)+r$ and $f(v) = g(v) - s$ where $r, s \in \{1, 2\}$. Define $g' : V \to \{-1, 0, 1\}$ as follows: $g'(y) = g(y)$ for all vertices $y \in V - \{u, v\}$,

$$g'(u) = \begin{cases} f(u) - 1 & \text{if } r = 2 \text{ and } s = 1 \\ f(u) & \text{otherwise} \end{cases}$$

and

$$g'(v) = \begin{cases} f(v) + 1 & \text{if } r = 1 \text{ and } s = 2 \\ f(v) & \text{otherwise} \end{cases} .$$

Then

$$\begin{aligned} g'(u) &= \begin{cases} f(u) - 1 & \text{if } r = 2 \text{ and } s = 1 \\ f(u) & \text{otherwise} \end{cases} \\ &= \begin{cases} g(u) + r - 1 & \text{if } r = 2 \text{ and } s = 1 \\ g(u) + r & \text{otherwise} \end{cases} \\ &\geq g(u) + 1. \end{aligned}$$

It follows that the only vertex with possibly a smaller value under g' than under g is v. For each child x of v, we have $g'(N(x)) = g'(N(x) - \{v\}) + g'(v) \geq f(N(x) - \{v\}) + f(v) = f(N(x)) \geq 1$.

Furthermore,

$$g'(u) + g'(v) = \begin{cases} f(u) + f(v) + 1 & \text{if } r = 1 \text{ and } s = 2 \\ f(u) - 1 + f(v) & \text{if } r = 2 \text{ and } s = 1 \\ f(u) + f(v) & \text{otherwise} \end{cases}$$

$$= \begin{cases} (g(u)+1)+(g(v)-2)+1 & \text{if } r=1 \text{ and } s=2 \\ (g(u)+2)-1+(g(v)-1) & \text{if } r=2 \text{ and } s=1 \\ g(u)+g(v) & \text{otherwise} \end{cases}$$

$$= g(u)+g(v).$$

Thus, $g'(N(w)) = g'(N(w)-\{u,v\})+g'(u)+g'(v) = g(N(w)-\{u,v\})+g(u)+g(v) = g(N(w)) \geq 1$ and $g'(V) = g'(V-\{u,v\})+g'(u)+g'(v) = g(V-\{u,v\})+g(u)+g(v) = g(V)$. This shows that g' is a $\gamma_t^{-101}(T)$-function which differs from f in fewer values than does g. $\diamondsuit$

5.4 A linear algorithm for $\gamma_t^{-11}(T)$

In our final section, we present a linear algorithm for finding a minimum total signed dominating function in a nontrivial tree T. The algorithm roots the tree T and associates various variables with the vertices of T as it proceeds. For any vertex v, the variable $MinSum$ denotes the miminum possible sum of values that may be assigned to the open neighborhood of v. So $MinSum = 1$ or 2 depending on whether v has odd or even degree, respectively. The variable *OpenSum* denotes the sum of the values assigned to the open neighborhood of v.

Algorithm: **TOTAL SIGNED DOMINATION (TSD)**. *Given a nontrivial tree T on n vertices, root the tree T and relabel the vertices of T from 1 to n so that label(w) > label(y) if the level of vertex w is less than the level of vertex y. Note the root of T will be labeled n.*

for $i \leftarrow 1$ to n do

  $f(i) \leftarrow -1$

for $i \leftarrow 1$ to n do

```
begin
    1.   deg i ← degree of the vertex i in T
    2.   if deg i is odd
         then MinSum ← 1
         else MinSum ← 2
    3.   if vertex i is a leaf and i < n
         then begin
                     OpenSum ← 1
    3.1.             f(parent(i)) ← 1
              end
         else OpenSum ← f(N(i))
    4.   if OpenSum < MinSum
         then begin
                     if i < n and f(parent(i)) = −1
                     then begin
    4.1.                         f(parent(i) = 1
                                 OpenSum ← OpenSum + 2;
                          end
                     while OpenSum < MinSum do
                     begin
    4.2.                    increase the value of one of the children of i;
                            OpenSum ← OpenSum + 2
                     end
              end
end
```

We now verify the validity of Algorithm **TSD**.

Theorem 5.11 *Algorithm* **TSD** *produces a* $\gamma_t^{-11}(T)$*-function in a nontrivial tree* T.

Proof. Let $T = (V, E)$ be a nontrivial tree of order n, and let f be the function produced by Algorithm **TSD**. Then $f : V \to \{-1, 1\}$. For convenience, the variables $MinSum$ and $OpenSum$, which were used by Algorithm **TSD** when it considered the vertex v, will be denoted by $MinSum(v)$ and $OpenSum(v)$, respectively.

Lemma 5.12 *The function f produced by Algorithm* **TSD** *is a TSDF for T.*

Proof. First consider the case when v is a leaf. The algorithm assigns, in Step 3, the value 1 to the parent of v, and since values are never decreased by the algorithm, the open neighborhood sum of v is at least one.

Next consider the case when v is not a leaf. If $OpenSum(v) \geq MinSum(v) \geq 1$, we are done. If not, then Step 4 of the algorithm increases the value of vertices in the open neighborhood of v such that $OpenSum(v) \geq MinSum(v) \geq 1$, as required. $\diamondsuit$

To show that the $TSDF$ f obtained by Algorithm **TSD** is minimum, let g be any $\gamma_t^{-11}(T)$-function for the rooted tree T. If $f \neq g$, then we will show that g can be transformed into a new $\gamma_t^{-11}(T)$-function g' that will differ from f in fewer values than g did. This process will continue until $f = g'$. Suppose, then, that $f \neq g$. Let v be the lowest labeled vertex for which $f(v) \neq g(v)$. Then *all* descendants of v are assigned the same value under g as under f.

Lemma 5.13 *If $g(v) < f(v)$, then the initial value assigned to the vertex v was increased in Step 4.2 of Algorithm* **TSD**.

Proof. Suppose the value of v was increased in Step 3.1. Then v is the parent of some leaf, say u. But then $g(N(u)) = g(v) = -1$, contradicting the fact that g is a $TSDF$ of T.

Suppose the value of v was increased in Step 4.1. This occurred when the algorithm was processing a vertex, say u, whose parent is v. Then $f(N(u)) = MinSum(u) \leq 2$

and $g(N(u)) = g(N(u) - \{v\}) + g(v) = f(N(u) - \{v\}) - 1 = f(N(u)) - f(v) - 1 = f(N(u)) - 2 \leq 0$, which is a contradiction.

Thus, the value of v was increased in Step 4.2. of Algorithm **TSD**. $\diamondsuit$

Lemma 5.14 *If $g(v) < f(v)$, then the function g' defined by $g'(u) = f(u)$ if $u \in N(parent(v))$ and $g'(u) = g(u)$ if $u \notin N(parent(v))$ is a $\gamma_t^{-11}(T)$-function of T that differs from f in fewer values than does g.*

Proof. By Lemma 5.13, the initial value assigned to the vertex v was increased in Step 4.2 of Algorithm **TSD** and this occurs when the parent of v was being processed. Let w be the parent of v. Thus g' is defined by $g'(u) = f(u)$ if $u \in N(w)$ and $g'(u) = g(u)$ for all remaining vertices u in V.

Then $f(N(w)) = MinSum(w)$. If $\deg w$ is even, then $MinSum(w) = 2$, so $g(N(w)) \geq 2 = MinSum(w) = f(N(w))$. If $\deg w$ is odd, then $g(N(w)) \geq 1 = MinSum(w) = f(N(w))$. Hence, $f(N(w)) \leq g(N(w))$. Furthermore, $g'(V) = g'(V - N(w)) + g'(N(w)) = g(V - N(w)) + f(N(w)) \leq g(V - N(w)) + g(N(w)) = g(V)$. Since all the descendants of w, other than its children, have the same values under g as under f, $g'(N(u)) = f(N(u))$ if $u = w$ or if u is a descendant of w, other than a child of w. Moreover, since the value of v was increased in Step 4.2, then, if w had a parent, its value was either already 1 or otherwise it was increased to 1 in Step 4.1. Thus, $f(parent(w)) = 1$, so that $g'(N(u)) \geq g(N(u))$ for all vertices u different from w or a descendant of w, other than a child of w. Thus, since f and g are $TSDF$s of T, so too is g'. Since $g'(V) \leq g(V)$, g' is a $\gamma_t^{-11}(T)$-function of T that differs from f in fewer values than does g. $\diamondsuit$

It remains for us to consider the case where $f(v) < g(v)$. We will need the following result from [27].

Lemma 5.15 *A $TSDF$ f on a graph $G = (V, E)$ is minimal if and only if for every vertex $v \in V$ with $f(v) = 1$, there exists a vertex $u \in N(v)$ with $f(N(u)) \in \{1, 2\}$.*

Here the vertex v is not the root of T, for otherwise $f(V) < g(V) = \gamma_t^{-11}(T)$, which is impossible. Since the labeling of the vertices was arbitrary at each level, if any vertex x at the same level as v has $g(x) < f(x)$, we can proceed as before to find a $TSDF$ g' that agrees with f in more values than under g. So we may assume in what follows that every vertex x at the same level as v has $f(x) \leq g(x)$.

Since $f(v) < g(v)$, it follows that $f(v) = -1$ and $g(v) = 1$. By the minimality of g (cf. Lemma 5.15), there exists a vertex $x \in N(v)$ such that $g(N(x)) \in \{1, 2\}$. Let w be the parent of v and let u be the parent of w. If $f(u) \leq g(u)$, then $f(N(x)) = f(N(x) - \{v\}) + f(v) \leq g(N(x) - \{v\}) + g(v) - 2 = g(N(x)) - 2 \leq 0$, which is a contradiction.

Hence $f(u) > g(u)$, i.e., $f(u) = 1$ and $g(u) = -1$. Define a function $g' : V \rightarrow \{-1, 1\}$ by $g'(y) = g(y)$ if $y \in V - \{v, u\}$, $g'(v) = -1$ and $g'(u) = 1$. Note that $f(v) = g'(v) = -1$ and $f(u) = g'(u) = 1$. The only vertices whose neighborhood sums are decremented under g' are the children of v. However, these open neighborhood sums under g' are at least as large as under f. Thus, since g and f are $TSDF$s, so too is g'. Furthermore, $g'(V) = g(V)$, so that g' is a $\gamma_t^{-11}(T)$-function which differs from f in fewer values than does g. $\diamondsuit$

Chapter 6

COMPLEXITY OF TOTAL k-SUBDOMINATION

6.1 Introduction

In this chapter, we focus on the algorithmic complexity of k-subdomination. In Section 6.2, we show that the decision problem corresponding to the computation of the total signed k-subdomination number is **NP**-complete. In Section 6.3, we present a cubic time algorithm to compute the total signed k-subdomination number of a tree. In Section 6.4, we discuss an algorithm that appears in [35]. The algorithm is omitted and the complexity analysis seems to be incorrect. We correct this by providing a detailed cubic time algorithm to compute $\gamma_{ks}^{-11}(T)$ of a tree T. In Section 6.5, we show that the decision problem corresponding to the computation of the total minus k-subdomination number is **NP**-complete, even for bipartite graphs. In Section 6.6, we present a cubic time algorithm to compute the total minus k-subdomination number of a tree. Finally, in Section 6.7, we provide a cubic time algorithm to compute $\gamma_{ks}^{-101}(T)$ of a tree T.

6.2 Complexity of total signed domination

In this section, we show that the decision problem

TOTAL SIGNED SUBDOMINATING FUNCTION (TSSF)

INSTANCE: A graph $G = (V, E)$, positive integers c, d such that $\gcd(c, d) = 1$ and $0 < \frac{c}{d} \leq 1$ and an integer t.

QUESTION: Is there a total signed subdominating function f such that $f(V) \leq t$ and $|C_f| \geq \lceil \frac{c|V|}{d} \rceil$?

is **NP**-complete by describing a polynomial transformation from the following problem:

TOTAL DOMINATING SET, RESTRICTED TO 4-REGULAR GRAPHS (TDS)

INSTANCE: A 4-regular graph $G = (V, E)$ and a positive integer $k \leq \frac{|V|}{2}$.

QUESTION: Is there a total dominating set of cardinality k or less for G?

We first show that **TDS** is **NP**-complete by describing a polynomial transformation from the decision problem **DOMINATING SET**.

DOMINATING SET, RESTRICTED TO PLANAR CUBIC GRAPHS (DS)

INSTANCE: A planar cubic graph $G = (V, E)$ and a positive integer $k \leq \frac{|V|}{2}$.

QUESTION: Is there a total dominating set of cardinality k or less for G?

Starting with the graph G, take two copies of the vertex set of G (which will be independent sets), and join a vertex to all vertices in the other copy that are in its closed neighborhood in G. The resulting graph has total domination number equal to twice the domination number of G. This construction transforms a cubic graph into a 4-regular graph. Since **DS** is **NP**-complete [12], **TDS** is **NP**-complete.

If $\frac{c}{d} = 1$, then **TSSF** is the **NP**-complete problem **TOTAL SIGNED**

DOMINATION (see [13], [27], and Theorem 5.5). Hence, we also assume that $0 < \frac{c}{d} < 1$. For convenience, we set $q = \frac{c}{d}$, and denote $\min\{f(V(G)) \mid f$ is a total signed subdominating function with $|C_f| \geq \lceil q|V(G)|\rceil\}$ by $\gamma_q(G)$.

We will need the following lemma.

Lemma 6.1 *If c, d, p are positive integers such that $0 < q = \frac{c}{d} < 1$, then there exist positive integers ℓ and r such that $8 \leq \ell \leq d^2(\lceil \frac{p}{2} \rceil + 4)$, $r < d^2(\lceil \frac{p}{2} \rceil + 4)$ and $q = \frac{p+r}{2p+r+\ell}$.*

Proof. Since $c < d$, we have $c \geq 1$, $d \geq 2$ and $d - c \geq 1$. Let $t = \lceil \frac{p}{2} \rceil + 4$. Then $dt(d-c) \geq 2t$ and $cdt \geq 2t$. However, $2t \geq p + 8$, whence $dt(d-c) \geq p + 8$ and $cdt > p$. Let t be the smallest positive integer such that $dt(d-c) \geq p + 8$ and $cdt > p$. It follows that $t \leq \lceil \frac{p}{2} \rceil + 4$. Let $r = cdt - p$ and $\ell = ddt - cdt - p$. Note that r and ℓ are both positive integers such that $r, \ell < ddt \leq d^2(\lceil \frac{p}{2} \rceil + 4)$. Furthermore, $\ell \geq 8$ and $q = \frac{p+r}{2p+r+\ell}$. $\diamondsuit$

Theorem 1 *The decision problem* **TSSF** *is* **NP**-*complete.*

Proof. Obviously, **TSSF** is in **NP**.

Let G be a 4-regular graph, $p = n(G)$ and k be an integer such that $k \leq p/2$. By Lemma 6.1, there exists positive integers r, ℓ such that $\ell \geq 8$ and $q = \frac{p+r}{2p+r+\ell}$. Let H be the graph constructed from G as follows: Take a complete graph F on $p + \ell$ vertices, a fixed subset $U \subseteq V(F)$ with $|U| = 3$ and an empty graph L on r vertices, and let H be obtained from the disjoint union of F, G, and L by joining each vertex of U to every vertex in $V(G) \cup V(L)$. Since $n(H) = 2p + r + \ell < 2(p + d^2(\lceil \frac{p}{2} \rceil + 4))$, the graph H can be constructed from G in polynomial time.

We will use the abbreviations TDS for a total dominating set and TSSF for a total signed dominating function. We start by showing that if S is a TDS of G of cardinality at most k, then there is a TSSF f of H of weight at most $2k-2p-r-\ell+6$. Define $f : V(H) \to \{-1,1\}$ by $f(v) = 1$ if $v \in S \cup U$, while $f(v) = -1$ otherwise.

Let $v \in V(G)$. Since S is a TDS of G, v is adjacent to some vertex $u \in S$ for which $f(u) = 1$. Since G is 4-regular and $f(U) = 3$, we have $f[v] \geq 1$. It is clear that $f[w] = 3$ for each vertex $w \in V(L)$, so that $f[v] \geq 1$ for at least $p+r = q(2p+r+\ell) = qn(H)$ vertices. This shows that f is a TSSF of H of weight $2|S| - 2p - r - \ell + 6 \leq 2k - 2p - r - \ell + 6$.

For the converse, assume that $\gamma_q(H) \leq 2k - 2p - r - \ell + 6$. Among all the minimum TSSF's of H, let f be one that assigns the value $+1$ to as many vertices of U as possible. Let P and M be the sets of vertices in H that are assigned the values $+1$ and -1, respectively, under f. Then $|P| + |M| = 2p + r + \ell$, and $|P| - |M| = \gamma_q(H)$. Before proceeding further we prove three claims.

Claim 1 $|P| \leq k + 3$.

Proof. Suppose $|P| \geq k + 4$. Then $|M| \leq 2p + r + \ell - k - 4$, so that $\gamma_q(H) = |P| - |M| \geq 2k - 2p - r - \ell + 8$, which contradicts the fact that $\gamma_q(H) \leq 2k - 2p - r - \ell + 6$. $\diamond$

Claim 2 $f[v] \leq 0$ for all $v \in V(F)$.

Proof. Suppose there exists a $v \in V(F)$ such that $f[v] \geq 1$. If $v \in U$, then, since v dominates H, it follows that $0 = 1 - 1 \leq f[v] + f(v) = f(V(H)) = \gamma_q(H) \leq 2k - 2p - r - \ell + 6$, whence $p + \frac{r}{2} < k$, which is a contradiction. Hence $v \in V(F) - U$. Since $N(v) = V(F) - \{v\}$ and $f[v] \geq 1$, it follows that more than half of the vertices of the set $V(F) - \{v\}$ have the value 1 assigned to them under f. This implies that $|P| \geq \frac{p+\ell}{2} = \frac{p}{2} + \frac{\ell}{2} \geq \frac{p}{2} + 4$. By Claim 1 and the fact that $k \leq \frac{p}{2}$, it follows that $|P| \leq \frac{p}{2} + 3$, which is a contradiction. $\diamond$

By Claim 2, it follows that $f[v] \geq 1$ for all $v \in V(G) \cup V(L)$.

Claim 3 $f(U) = 3$.

Proof. Suppose that $f(u) = -1$ for some $u \in U$. If $f(v) = -1$ for all $v \in V(G)$, then $f[v] \leq -3$ for all $v \in V(G)$, which is a contradiction. It follows that there exists a $v \in V(G)$ such that $f(v) = 1$. Define $g : V(H) \to \{-1, 1\}$ by $g(w) = f(w)$ if $w \in V(H) - \{u, v\}$, $g(v) = -1$ and $g(u) = 1$, and consider a vertex $x \in V(G) \cup V(L)$. Note that if $x \notin N(v)$ or $x = v$, then $g[x] = f[x] + 2$, while if $x \in N(v)$, then $g[x] = f[x]$. It follows that $g[v] \geq 1$ for at least q of the vertices of H while the weights of g and f are equal. Hence g is a TSSF of H of weight $\gamma_q(H)$ that assigns the value +1 to more vertices of U than does f, contradicting our choice of f. ◊

Let $S = P \cap V(G)$. Since $f[v] \geq 1$ for all $v \in V(G)$, it follows that every $v \in V(G)$ is adjacent to some vertex in S, which shows that S is a TDS of G. Since $f(U) = 3$, Claim 1 implies that $|S| \leq k$, which completes the proof. ◊

6.3 Computing $\gamma_{tks}^{-11}(T)$ for a tree T

In this section, we will present a cubic time algorithm to compute the total signed k-subdomination number of a tree.

The tree T will be rooted and represented by the resulting parent array parent[1 ... n]. We make use of the well-known fact that the tree T can be constructed recursively from the single vertex K_1 using only one rule of composition, which combines two trees (G, x) and (H, y), by adding an edge between x and y and calling x the root of the larger tree F. We express this as follows: $(F, x) = (G, x) \circ (H, y)$. With each such subtree (F, x), we associate the following data structure:

1. **table[x].numvertices:** the number of vertices in the subtree (F, x).

2. **table[x].degree:** $\deg_F(x)$.

3. **table[x].sum[$f(x), t, k$]:** the minimum weight of a function $f : V(F) \to \{-1, 1\}$ such that x is assigned $f(x)$, $|t| \leq \deg_T(x) - \deg_F(x)$ (representing all

possible sums of assignments of -1 and $+1$ to the vertices of $N_T(x) - N_F(x)$ and $|\{v \mid f(N_F(v)) + t \geq 1$ when $v = x$ and $f(N_F(v)) \geq 1$ when $v \neq x\}| \geq k$, where $1 \leq k \leq$ **table**$[x]$.**numvertices**.)

Our input consist of the order of the tree T, say **n**, and the **parent** array of the tree, rooted at a certain vertex. The root of the tree T is labeled with 1, the vertices on the next level are labeled with 2 through 2 plus the number of vertices on level 2, and so on. Using the **parent** array, we compute $\deg_T(x)$ for each vertex x, $x = 1, \ldots, n$. We then initialize the variable **table**$[x]$ for each vertex x, where $x = 1, \ldots, n$. Let x be an arbitrary vertex of T. Initially, $(F, x) = (K_1, x)$, whence **table**$[x]$.**numvertices**=1 and **table**$[x]$.**degree**=0. Suppose t is an integer such that $|t| \leq \deg_T(x) - \deg_F(x) = \deg_T(x)$, representing all possible sums of assignments of -1 and $+1$ to the vertices of $N_T(x) - N_F(x) = N_T(x)$. Then $t \in \{-\deg_T(x), -\deg_T(x)+2, \ldots, \deg_T(x)\}$. The only way for $f(N_F(x))+t = t \geq 1$, is for $t \geq 1$ if $\deg_T(x)$ is odd and for $t \geq 2$ if $\deg_T(x)$ is even. Thus, we have the following initializations:

Case 1: $\deg_T(x)$ is odd and $t \in \{1, 3, \ldots, \deg_T(x)\}$.
Then **table**$[x]$.**sum**$[f(x), t, 1]$ = **table**$[x]$.**sum**$[f(x), t, 0]$ = $f(x)$ where $f(x) \in \{-1, 1\}$.

Case 2: $\deg_T(x)$ is odd and $t \in \{-\deg_T(x), -\deg_T(x) + 2, \ldots, -1\}$.
Then **table**$[x]$.**sum**$[f(x), t, 1]$ is undefined, and **table**$[x]$.**sum**$[f(x), t, 0]$=$f(x)$where $f(x) \in \{-1, 1\}$.

Case 3: $\deg_T(x)$ is even and $t \in \{2, 4, \ldots, \deg_T(x)\}$.
Then **table**$[x]$.**sum**$[f(x), t, 1]$ = **table**$[x]$.**sum**$[f(x), t, 0]$ = $f(x)$ where $f(x) \in \{-1, 1\}$.

Case 4: $\deg_T(x)$ is even and $t \in \{-\deg_T(x), -\deg_T(x) + 2, \ldots, 0\}$.
Then **table**$[x]$.**sum**$[f(x), t, 1]$ is undefined, and **table**$[x]$.**sum**$[f(x), t, 0]$=$f(x)$

where $f(x) \in \{-1, 1\}$.

The following code implements the aforementioned discussion.

Algorithm: To compute $\gamma_{tks}^{-11}(T)$ for a tree T.

```
for vertex ← 1 to n do
    degree[vertex] ← 0
for vertex ← 2 to n do
begin
   degree[vertex] ← degree[vertex]+1
   degree[parent[vertex]] ← degree[parent[vertex]]+1
end
for vertex ← 1 to n do
begin
   table[vertex].numvertices ← 1
   table[vertex].degree ← 0

   if degree[vertex] is odd
   then startvalue ← 1
   else startvalue ← 0

   for excessvalue ← startvalue to degree[vertex] step 2 do
   begin
      table[vertex].sum[1,excessvalue,1] ← 1
      table[vertex].sum[-1,excessvalue,1] ← -1
      table[vertex].sum[1,excessvalue,0] ← 1
      table[vertex].sum[-1,excessvalue,0] ← -1
   end

   for excessvalue ← -degree[vertex] to startvalue-2 step 2 do
   begin
```

```
    table[vertex].sum[1,excessvalue,1] ← 10000
    table[vertex].sum[-1,excessvalue,1] ← 10000
    table[vertex].sum[1,excessvalue,0] ← 1
    table[vertex].sum[-1,excessvalue,0] ← -1
  end
end
```

Inputting the **parent** array takes $O(n)$ steps, while computing the **degree** array from the **parent** array also takes $O(n)$ steps. Initializing the array **table** takes

$$O(\sum_{\text{vertex}=1}^{n} \deg_T(\text{vertex})) = O(2m(T)) = O(2(n-1)) = O(n)$$

steps. Thus, the overall complexity here is $O(n^2)$.

Our next result shows that our algorithm is correct.

Theorem 6.2 *Suppose (G,x) and (H,y) are two disjoint rooted subtrees, and let $(F,x) = (G,x) \circ (H,y)$. Let $s \in \{-1,1\}$, t be an integer such that $|t| \leq \deg_T(x) - \deg_F(x)$ with $t \equiv \deg_T(x) - \deg_F(x) (mod\ 2)$, and k be an integer with $0 \leq k \leq |V(F)|$. Then*

table$[x]$.**sum**$[s,t,k]$= $\min\{$**table**$[x]$.**sum**$[s,t+s',j]$+**table**$[y]$.**sum**$[s',s,k-j]$ $\mid s' \in \{-1,1\}, 0 \leq j \leq k\}$ = $\min\{$**table**$[x]$.**sum**$[s,t+s',j]$+**table**$[y]$.**sum**$[s',s,k-j]$ $\mid s' \in \{-1,1\}, \max\{0, k-|V(H)|\} \leq j \leq \min\{k, |V(G)|\}\}$.

Moreover, $|t| \leq \deg_T(x) - \deg_F(x)$ if and only if $-(\deg_T(x) - \deg_G(x) - 1) \leq t \leq \deg_T(x) - \deg_G(x) - 1$.

Proof. Suppose $f : V(F) \to \{-1, 1\}$ such that

$$f(V(F)) = \mathbf{table[x].sum[s, t, k]}.$$

Let g (respectively, h) be the restriction of f on $V(G)$ (respectively, $V(H)$) and $s^* = h(y) = f(y)$. Note that $f(N_F(x)) + t = g(N_G(x)) + t + s^*$ and $f(N_F(v)) = g(N_G(v))$ for all $v \in V(G) - \{x\}$, while $f(N_F(y)) = h(N_H(y)) + s$ and $f(N_F(v)) = g(N_H(v))$ for all $v \in V(H) - \{y\}$. Thus, $k \leq |\{v \,|\, f(N_F(v)) + t \geq 1$ when $v = x$ and $f(N_F(v)) \geq 1$ when $v \neq x\}| = |\{v \,|\, g(N_G(v)) + t + s^* \geq 1$ when $v = x$ and $g(N_G(v)) \geq 1$ when $v \neq x\}| + |\{v \,|\, h(N_H(v)) + s \geq 1$ when $v = y$ and $h(N_H(v)) \geq 1$ when $v \neq y\}|$. If $j = |\{v \,|\, g(N_G(v)) + t + s^* \geq 1$ when $v = x$ and $g(N_G(v)) \geq 1$ when $v \neq x\}|$, then $k - j \leq |\{v \,|\, h(N_H(v)) + s \geq 1$ when $v = y$ and $h(N_H(v)) \geq 1$ when $v \neq y\}|$. It now follows that $\mathbf{table}[x]\mathbf{.sum}[s, t + s^*, j] + \mathbf{table}[y]\mathbf{.sum}[s^*, s, k - j] \leq g(V(G)) + h(V(H)) = \mathbf{table}[x]\mathbf{.sum}[s, t, k]$. Hence, $\min\{\mathbf{table}[x]\mathbf{.sum}[s, t + s', j] + \mathbf{table}[y]\mathbf{.sum}[s', s, k - j] \;|\, s' \in \{-1, 1\}, 0 \leq j \leq k\} \leq \mathbf{table}[x]\mathbf{.sum}[s, t, k]$.

On the other hand, suppose $g \colon V(G) \to \{-1, 1\}$ such that

$$g(V(G)) = \mathbf{table[x].sum[s, t + s', j]}$$

and $h : V(H) \to \{-1, 1\}$ such that

$$h(V(H)) = \mathbf{table[y].sum[s', s, k - j]}.$$

Define $f : V(F) \to \{-1, 1\}$ by $f(v) = g(v)$ if $v \in V(G)$ and $f(v) = h(v)$ for all $v \in V(H)$. As before, $f(N_F(x)) + t = g(N_G(x)) + t + s'$ and $f(N_F(v)) = g(N_G(v))$ for all $v \in V(G) - \{x\}$, while $f(N_F(y)) = h(N_H(y)) + s$ and $f(N_F(v)) = g(N_H(v))$ for all $v \in V(H) - \{y\}$. Thus, $|\{v \,|\, f(N_F(v)) + t \geq 1$ when $v = x$ and $f(N_F(v)) \geq 1$ when $v \neq x\}| = |\{v \,|\, g(N_G(v)) + t + s' \geq 1$ when $v = x$ and $g(N_G(v)) \geq 1$ when $v \neq x\}| + |\{v \,|\, h(N_H(v)) + s \geq 1$ when $v = y$ and $h(N_H(v)) \geq 1$ when $v \neq y\}| \geq j + (k - j) = k$. Hence, $\mathbf{table}[x]\mathbf{.sum}[s, t, k] \leq f(V(F)) =$

$g(V(G)) + h(V(H)) =$**table**$[x]$.**sum**$[s, t+s', j]$ + **table**$[y]$.**sum**$[s', s, k-j]$. Thus, **table**$[x]$.**sum**$[s,t,k] \leq \min\{$**table**$[x]$.**sum**$[s, t+s', j]$+**table**$[y]$.**sum**$[s', s, k-j]$ $\mid s' \in \{-1, 1\}, 0 \leq j \leq k\}$.

Since $0 \leq j \leq |V(G)|$ and $j \leq k$, we have $0 \leq k - j \leq |V(H)|$, so that $0 \geq j - k \geq -|V(H)|$, whence $j \geq k - |V(H)|$. We conclude that $\max\{0, k - |V(H)|\} \leq j \leq \min\{k, |V(G)|\}$.

Lastly, $|t| \leq \deg_T(x) - \deg_F(x)$ if and only if $-\deg_T(x) + \deg_G(x) + 1 \leq t \leq \deg_T(x) - \deg_G(x) - 1$, since $\deg_F(x) = \deg_G(x) + 1$. $\diamondsuit$

At the conclusion of our algorithm, $T = F$, and so $t = 0$. Clearly, $\gamma_{tks}^{-11}(T) =$ min{ **table**[1].**sum**$[1, 0, k]$,**table**[1].**sum**$[-1, 0, k]$ }.

We are now in a position to present the remainder of the algorithm.

<u>**Algorithm**</u>: **To compute $\gamma_{tks}^{-11}(T)$ for a tree T (continued).**

```
for oldroot ← n downto 2 do
begin
  resulttable.numvertices ← table[oldroot].numvertices +
                             table[parent[oldroot]].numvertices
  resulttable.degree ← table[parent[oldroot]].degree + 1
  range ← degree[parent[oldroot]] - resulttable.degree
  for newrootvalue ← -1 to 1 step 2 do
     for newrootexcess ← -range to range step 2 do
        for k ← 0 to resulttable.numvertices do
        begin
          minimum ← 1000
          startvalue ← max(0, k - table[oldroot].numvertices)
          stopvalue ← min(k, table[parent[oldroot]].numvertices)
          for j ← startvalue to stopvalue do
          begin
```

```
            for oldrootvalue ← -1 to 1 step 2 do
            begin
              number ← degree[parent[oldroot]] - table[parent[oldroot]].degree - 1
              if -number ≤ newrootexcess ≤ number then
              begin
                summand1 ← table[parent[oldroot]].
                              sum[newrootvalue, newrootexcess + oldrootvalue, j]

                summand2 ← table[oldroot].
                              sum[oldrootvalue, newrootvalue, k-j]

                temp ← summand1 + summand2
              end
              if (temp < minimum)
              then minimum ← temp
            end
          end
          resulttable.sum[newrootvalue, newrootexcess, k] ← minimum
        end
  table[parent[oldroot]] ← resulttable
end
for k ← 0 to n do
  output (k, min(table[1].sum[1, 0, k],table[1].sum[-1, 0, k]))
```

The complexity of the above part of the algorithm, excluding the output phase, is

$$
\begin{aligned}
& O(\textstyle\sum_{\text{n-oldroot}=0}^{(n-2)} 2 \times \deg_T[\text{parent[oldroot]}] \times n \times n \times 2) \\
= \; & O(4n^2 \textstyle\sum_{v \in V(T)} \deg_T(v)) \\
= \; & O(4n^2 2m(T)) = (4n^2 \times 2 \times (n-1)) \\
= \; & O(n^3),
\end{aligned}
$$

while the complexity of the output phase is $O(n)$. Thus, the overall complexity of the algorithm is $O(n^3)$.

6.4 A cubic algorithm to compute $\gamma_{ks}^{-11}(T)$ of a tree T

A "quadratic" time algorithm to compute the total signed k-subdomination number of a tree appears in [35]. Unfortunately, the initialization phase of the algorithm is omitted and other aspects of the algorithm are not clear either. Also, the complexity analysis of the algorithm seems to be incorrect. In this section, we present a cubic algorithm to compute $\gamma_{ks}^{-11}(T)$ of a tree T. The approach here is similar to what we described in the previous section. Here we have the following data structure, associated with the subtree (F, x).

1. **table**$[x]$**.numvertices:** the number of vertices in the subtree (F, x).
2. **table**$[x]$**.degree:** $\deg_F(x)$.
3. **table**$[x]$**.sum**$[f(x), t, k]$: the minimum weight of a function $f : V(F) \to \{-1, 1\}$ such that x is assigned $f(x)$, $|t| \leq \deg_T(x) - \deg_F(x)$ (representing all possible sums of assignments of -1 and $+1$ to the vertices of $N_T(x) - N_F(x)$ and $|\{v \mid f(N_F[v]) + t \geq 1$ when $v = x$ and $f(N_F[v]) \geq 1$ when $v \neq x\}| \geq k$, where $1 \leq k \leq$ **table**$[x]$**.numvertices**.

The initialization phase here proceeds as follows.

Let x be an arbitrary vertex of T. Initially, $(F, x) = (K_1, x)$, whence **table**$[x]$**.numvertices**=1 and **table**$[x]$**.degree**=0. Suppose t is an integer such that $|t| \leq \deg_T(x) - \deg_F(x) = \deg_T(x)$, representing all possible sums of assignments of -1 and $+1$ to the vertices of $N_T(x) - N_F(x) = N_T(x)$. Then

$t \in \{-\deg_T(x), -\deg_T(x)+2, \ldots, \deg_T(x)\}$. The only way for $f(N_F(x))+f(x)+t = f(x)+t \geq 1$, is for $t \geq 2-f(x)$ if $\deg_T(x)$ is odd and for $t \geq 1-f(x)$ if $\deg_T(x)$ is even. Thus, we have the following initializations:

Case 1: $\deg_T(x)$ is odd and $t \in \{2-f(x), 4-f(x), \ldots, \deg_T(x)\}$.
Then **table**[x]**.sum**[$f(x), t, 1$] = **table**[x]**.sum**[$f(x), t, 0$] = $f(x)$ where $f(x) \in \{-1, 1\}$.

Case 2: $\deg_T(x)$ is odd and $t \in \{-\deg_T(x), -\deg_T(x)+2, \ldots, -f(x)\}$.
Then **table**[x]**.sum**[$f(x), t, 1$] is undefined, and **table**[x]**.sum**[$f(x), t, 0$]=$f(x)$where $f(x) \in \{-1, 1\}$.

Case 3: $\deg_T(x)$ is even and $t \in \{1-f(x), 3-f(x), \ldots, \deg_T(x)\}$.
Then **table**[x]**.sum**[$f(x), t, 1$] = **table**[x]**.sum**[$f(x), t, 0$] = $f(x)$ where $f(x) \in \{-1, 1\}$.

Case 4: $\deg_T(x)$ is even and $t \in \{-\deg_T(x), -\deg_T(x)+2, \ldots, -1-f(x)\}$.
Then **table**[x]**.sum**[$f(x), t, 1$] is undefined, and **table**[x]**.sum**[$f(x), t, 0$]=$f(x)$ where $f(x) \in \{-1, 1\}$.

A result analogous to Theorem 6.2 appears in [35].

We are now in a position to state the algorithm. Note that the initialization phase of the algorithm has complexity

$$O(\sum v \in V(T) \times 2 \times \deg_T(v)) = O(2m(T)) = O(2(n-1)) = O(n).$$

Thus, the overall complexity of the algorithm is also $O(n^3)$.

Algorithm: To compute $\gamma_{ks}^{-11}(T)$ for a tree T.

```
for vertex ← 1 to n do
    degree[vertex] ← 0
for vertex ← 2 to n do
begin
```

```
  degree[vertex] ← degree[vertex]+1
  degree[parent[vertex]] ← degree[parent[vertex]]+1
end
for vertex ← 1 to n do
   if degree[vertex] is odd then
   begin
     for rootvalue ← -1 to 1 step 2 do
     begin
       for excessvalue ← 2 - rootvalue to degree[vertex] step 2 do
       begin
         table[vertex].sum[rootvalue,excessvalue,1] ← rootvalue
         table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
       end
       for excessvalue ← -degree[vertex] to -rootvalue step 2 do
       begin
         table[vertex].sum[rootvalue,excessvalue,1] ← 10000
         table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
       end
     end
   end
   else
   begin
     for rootvalue ← -1 to 1 step 2 do
     begin
       for excessvalue ← 1 - rootvalue to degree[vertex] step 2 do
       begin
         table[vertex].sum[rootvalue,excessvalue,1] ← rootvalue
         table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
       end
       for excessvalue ← -degree[vertex] to -1-rootvalue step 2 do
```

```
        begin
          table[vertex].sum[rootvalue,excessvalue,1] ← 10000
          table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
        end
      end
    end
end

for oldroot ← n downto 2 do
begin
  resulttable.numvertices ← table[oldroot].numvertices +
                            table[parent[oldroot]].numvertices
  resulttable.degree ← table[parent[oldroot]].degree + 1
  range ← degree[parent[oldroot]] - resulttable.degree
  for newrootvalue ← -1 to 1 step 2 do
     for newrootexcess ← -range to range step 2 do
        for k ← 0 to resulttable.numvertices do
        begin
          minimum ← 1000
          startvalue ← max(0, k - table[oldroot].numvertices)
          stopvalue ← min(k, table[parent[oldroot]].numvertices)
          for j ← startvalue to stopvalue do
          begin
            for oldrootvalue ← -1 to 1 step 2 do
            begin
              number ← degree[parent[oldroot]] - table[parent[oldroot]].degree - 1
              if -number ≤ newrootexcess ≤ number then
              begin
                summand1 ← table[parent[oldroot]].
                          sum[newrootvalue, newrootexcess + oldrootvalue, j]
```

```
          summand2 ← table[oldroot].
                          sum[oldrootvalue, newrootvalue, k-j]

          temp ← summand1 + summand2
        end
        if (temp < minimum)
        then minimum ← temp
      end
    end
    resulttable.sum[newrootvalue, newrootexcess, k] ← minimum
  end
 table[parent[oldroot]] ← resulttable
end
for k ← 0 to n do
 output (k, min(table[1].sum[-1, 0, k],table[1].sum[1, 0, k]))
```

6.5 Complexity result for total minus domination

In this section we will show that the decision problem corresponding to the computation of the total minus k-subdomination number is **NP**-complete by describing a polynomial transformation from the **NP**-complete problem **EXACT COVER BY 3-SETS**.

Let $r = \frac{a}{b} \leq 1$ be a fixed positive rational number (in lowest terms). Consider the decision problem

TOTAL MINUS SUBDOMINATING FUNCTION (TMSF)

INSTANCE: A graph G and an integer ℓ.

QUESTION: Is there a function $f : V(G) \to \{-1, 0, 1\}$ of weight ℓ or less for G such that $|C_f| \geq r|V(G)|$?

In this section we show that **TMSF** is **NP**-complete by describing a polynomial transformation from the following **NP**-complete problem (see [12]):

EXACT COVER BY 3-SETS (X3C)

INSTANCE: A set $X = \{x_1, \dots, x_{3q}\}$ and a set $\mathcal{C} = \{C_1, \dots, C_m\}$ where $C_j \subseteq X$ and $|C_j| = 3$ for $j = 1, \dots, m$.

QUESTION: Does $\mathcal{C}$ have a pairwise disjoint q-subset of $\mathcal{C}$ whose union is X (i.e. an exact cover)?

If $r = 1$, then **TMSF** is the **NP**-complete problem **TOTAL MINUS DOMINATING FUNCTION** (see Theorem 5.1). Hence, we also assume that $r < 1$. For two real numbers a and b, we say that a *divides* b if there is an integer k such that $b = ka$.

Theorem 6.3 **TMSF** *is* **NP**-*complete, even for bipartite graphs.*

Proof. It is obvious that **TMSF** is in **NP**. To show that **TMSF** is an **NP**-complete problem, we will establish a polynomial transformation from the **NP**-complete problem **X3C**. Let $X = \{x_1, \dots, x_{3q}\}$ and $\mathcal{C} = \{C_1, \dots, C_m\}$ be an arbitrary instance of **X3C** where $C_j \subseteq X$ and $|C_j| = 3$ for $1 \leq j \leq m$. We will construct a bipartite graph G and an integer ℓ such that this instance of **X3C** will have an exact cover if and only if

there is a function $f : V(G) \to \{-1, 0, 1\}$ of weight at most ℓ such that $|C_f| \geq r|V(G)|$.

Corresponding to each $x_i \in X$, associate the graph constructed from the path P_5, with vertices labeled x_i, y_i, z_i, v_i, w_i, and the path P_2, with vertices labeled u_i and t_i, by joining the vertices u_i and v_i. Corresponding to each C_j, associate the graph constructed from the path P_4, with vertices labeled c_j, d_j, e_j, f_j, and the path P_2, with vertices labeled g_j, h_j, by joining the vertices g_j and e_j. Add the edges $\{x_i c_j \mid x_i \in C_j\}$ and call the resulting graph H. Note that $n(H) = 6m + 21q$. Let

$$\mu = \begin{cases} 0 & \text{if } r \text{ divides } 6m + 21q \\ a - (6m + 21q) \bmod a & \text{otherwise} \end{cases}.$$

Then μ is the smallest nonnegative integer that may be added to $6m + 21q$ so that r divides $6m + 21q + \mu$ evenly. Construct the (bipartite) graph $G = (V, E)$ as follows. Take the disjoint union of two copies of H, say H_1 and H_2, add a set S of $\alpha := 2(\frac{6m+21q+\mu}{r} - (6m+21q))$ vertices, and, with $S = \{s_1, s_2, \ldots, s_\alpha\}$, add the edges $s_k s_{k+1}$, where $k = 1, 3, \ldots, \alpha - 1$. The graph G has order $2(\frac{6m+21q+\mu}{r})$, and, since $0 \leq \mu \leq a - 1$, G can be constructed from the input in polynomial time. Lastly, let $\ell = 2(8m + 28q + 2\mu - (\frac{6m+21q+\mu}{r}))$. We will denote a vertex ν_i or ν_j of H in H_β by $\nu_{i,\beta}$ or $\nu_{j,\beta}$, for $\beta = 1, 2$.

Suppose $C' \subseteq C$ is an exact cover for X. Let $P = \bigcup_{i=1}^{3q} \{u_{i,1}, u_{i,2}, v_{i,1}, v_{i,2}, z_{i,1}, z_{i,2}\} \cup \bigcup_{j=1}^{m} \{g_{j,1}, g_{j,2}, e_{j,1}, e_{j,2}, d_{j,1}, d_{j,2}\} \cup \bigcup_{k=1}^{2\mu} \{s_k\} \cup \{c_{j,1}, c_{j,2} \mid C_j \in C'\}$ and $M = \bigcup_{i=1}^{3q} \{w_{i,1}, w_{i,2}\} \cup \bigcup_{j=1}^{m} \{f_{j,1}, f_{j,2}\} \cup \bigcup_{k=2\mu+1}^{\alpha} \{s_k\}$.

Define $f : V \to \{-1, 0, 1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in P \\ -1 & \text{if } v \in M \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$\begin{aligned} f(V) &= \sum_{i=1}^{3q} 6 + \sum_{j=1}^{m} 6 + \sum_{k=1}^{2\mu} 1 + 2q - \left(\sum_{i=1}^{3q} 2 + \sum_{j=1}^{m} 2 + \sum_{k=2\mu+1}^{\alpha} 1 \right) \\ &= 14q + 4m + 4\mu - \alpha \\ &= 2(8m + 28q + 2\mu - \tfrac{6m+21q+\mu}{r}). \end{aligned}$$

Note that $x \in C_f$ for all $x \in V(H_1) \cup V(H_2) \cup \cup_{k=1}^{2\mu}\{s_k\}$. Thus, $|C_f| \geq 2(6m+21q)+2\mu = 2(6m+21q+\mu) = r2\frac{6m+21q+\mu}{r} = r|V|$.

We now prove the converse. Let L be the set of all leaves of G. Among all functions $f : V \to \{-1,0,1\}$ for which $f(V) \leq \ell$ and $|C_f| \geq r|V(G)|$, choose one, say f, for which $f(L)$ is as small as possible. This implies that $f(x) \in \{-1,1\}$ for all $x \in S$. Note that $|C_f| \geq r2\frac{6m+21q+\mu}{r} = 12m+42q+2\mu$.

The function $g_{r_1,\rho_1,r_2,\rho_2}$ is the function obtained from f by assigning some vertex r_1 in $V(H_1) \cup V(H_2)$ the value ρ_1, some vertex r_2 in S the value ρ_2, while all other vertices are assigned the same value as under f, where $\rho_1, \rho_2 \in \{-1,0,1\}$. In all cases, a neighbor of r_1 will become covered, while the neighbor of r_2 will no longer be covered, so $|C_g| \geq |C_f|$. Moreover, $g(V) \leq f(V) \leq \ell$.

Let $i \in \{1,\ldots,3q\}, j \in \{1,\ldots,m\}$ and $\beta \in \{1,2\}$.

Fact 1. $f(g_{j,\beta}) = 1$ (and, similarly, $f(e_{j,\beta}) = f(u_{i,\beta}) = f(v_{i,\beta}) = 1$).

Proof. For suppose, to the contrary, $f(g_{j,\beta}) \leq 0$. Then $h_{j,\beta} \notin C_f$, and since $|C_f| \geq 12m+42q$, there is $s_k \in S \cap C_f$. This implies $f(s_{k-1}) = 1$ or $f(s_{k+1}) = 1$ – assume the latter. Then $g = g_{g_{j,\beta},1,s_{k+1},f(g_{j,\beta})}$ has $g(L) < f(L)$, which is a contradiction. $\diamondsuit$

Fact 2. $f(h_{j,\beta}) = 0$ (and, similarly, $f(t_{i,\beta}) = 0$).

Proof. For suppose, to the contrary, $f(h_{j,\beta}) = -1$. Then $g_{j,\beta} \notin C_f$. Since $|C_f| \geq 12m+42q$, there is $s_k \in S \cap C_f$. This implies $f(s_{k-1}) = 1$ or $f(s_{k+1}) = 1$ – assume the latter. Then $g = g_{h_{j,\beta},0,s_{k+1},-1}$ has $g(L) < f(L)$, which is a contradiction. Furthermore, since $f(L)$ is a minimum, $f(h_{j,\beta}) = f(t_{i,\beta}) = 0$. $\diamondsuit$

Fact 3. $e_{j,\beta} \in C_f$ (and, similarly, $v_{i,\beta} \in C_f$).

Proof. For suppose, to the contrary, that $e_{j,\beta} \notin C_f$. Then $f(d_{j,\beta}) + f(f_{j,\beta}) \leq -1$. This implies that $f(d_{j,\beta}) \leq 0$. Since $|C_f| \geq 12m+42q$, there is $s_k \in S \cap C_f$. This implies $f(s_{k-1}) = 1$ or $f(s_{k+1}) = 1$ – assume the latter. Then $g = g_{d_{j,\beta},1,s_{k+1},f(d_{j,\beta})}$

has $g(L) < f(L)$, which is a contradiction. $\diamondsuit$

Fact 4. $f(f_{j,\beta}) = -1$ and $f(d_{j,\beta}) = 1$ (and, similarly, $f(z_{i,\beta}) = 1$ and $f(w_{i,\beta}) = -1$).

Proof. Since $e_{j,\beta} \in C_f$, $f(d_{j,\beta}) + f(g_{j,\beta}) + f(f_{j,\beta}) = f(d_{j,\beta}) + 1 + f(f_{j,\beta}) \geq 1$, which implies $f(d_{j,\beta}) + f(f_{j,\beta}) \geq 0$.

If $f(d_{j,\beta}) + f(f_{j,\beta}) \geq 1$, then $f(f_{j,\beta}) \geq 0$. Then $g : V \to \{-1, 0, 1\}$ defined by

$$g(x) = \begin{cases} f(x) - 1 & \text{if } x = f_{j,\beta} \\ f(x) & \text{otherwise} \end{cases}$$

is a function such that $|C_g| = |C_f| \geq r|V|$, $g(V) \leq \ell$ and $g(L) < f(L)$, which is a contradiction.

Hence, $f(d_{j,\beta}) + f(f_{j,\beta}) = 0$. If $f(f_{j,\beta}) \geq 0$, then the function $g : V \to \{-1, 0, 1\}$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \{d_{j,\beta}, f_{j,\beta}\} \\ f(x) + 1 & \text{if } x = d_{j,\beta} \\ f(x) - 1 & \text{otherwise} \end{cases}$$

is a function such that $|C_g| = |C_f| \geq r|V|$, $g(V) \leq \ell$ and $g(L) < f(L)$, which is a contradiction. $\diamondsuit$

Fact 5. $d_{j,\beta} \in C_f$ and $f(c_{j,\beta}) \geq 0$ (and, similarly, $z_{i,\beta} \in C_f$ and $f(y_{i,\beta}) \geq 0$).

Proof. If $d_{j,\beta} \notin C_f$, then, since $f(e_{j,\beta}) = 1$, $f(c_{j,\beta}) = -1$. Since $|C_f| \geq 12m + 42q$, there is $s_k \in S \cap C_f$. This implies $f(s_{k-1}) = 1$ or $f(s_{k+1}) = 1$ – assume the latter. Then $g = g_{c_{j,\beta},1,s_{k+1},-1}$ has $g(L) < f(L)$, which is a contradiction. $\diamondsuit$

In a similar way, one can show that

Fact 6. $y_{i,\beta} \in C_f$ and $f(c_{i,\beta}) \geq 0$. $\diamondsuit$

Fact 7. $c_{j,\beta} \in C_f$ and $x_{i,\beta} \in C_f$.

Proof. Since $c_{j,\beta}$ is adjacent to $d_{j,\beta}$, which is assigned $+1$ under f, and three vertices in $\{x_{1,\beta}, \ldots, x_{3q,\beta}\}$, all of which are assigned at least 0 under f, we have that $c_{j,\beta} \in C_f$.

Suppose, to the contrary, that $x_{i,\beta} \notin C_f$. Then all vertices adjacent to $x_{i,\beta}$ are assigned the value 0 under f – particularly $f(y_{i,\beta}) = 0$. Since $|C_f| \geq 12m + 42q$, there is $s_k \in S \cap C_f$. This implies $f(s_{k-1}) = 1$ or $f(s_{k+1}) = 1$ – assume the latter. Then $g = g_{y_{i,\beta},1,s_{k+1},-1}$ has $g(L) < f(L)$, which is a contradiction. $\diamondsuit$

Combining the facts above, we have $V(H_1) \cup V(H_2) \subseteq C_f$. Since $n(H_1) + n(H_2) = 12m + 42q$ and $|C_f| \geq 12m + 42q + 2\mu$, $|S \cap C_f| \geq 2\mu$.

Let $X_\beta = \{x_{1,\beta}, \ldots, x_{3q,\beta}\}$, $Y_\beta = \{y_{1,\beta}, \ldots, y_{3q,\beta}\}$, $C_\beta = \{y_{1,\beta}, \ldots, y_{m,\beta}\}$, $cx_\beta = |X_\beta \cap P_f|$, $cy_\beta = |Y_\beta \cap P_f|$ and $cc_\beta = |C_\beta \cap P_f|$.

Since $f(V(H_1 \cup H_2) - X_1 - X_2 - Y_1 - Y_2 - C_1 - C_2) = 2(2(3q) + 2m) = 12q + 4m$, $f(V) \geq 12q+4m+cx_1+cx_2+cy_1+cy_2+cc_1+cc_2+2\mu+(-1)(2\frac{6m+21q+\mu}{r}-2(6m+21q)-2\mu) = 54q + 16m + 4\mu + cx_1 + cx_2 + cy_1 + cy_2 + cc_1 + cc_2 - 2(\frac{6m+21q+\mu}{r})$. But $f(V) \leq 56q + 16m + 4\mu - 2(\frac{6m+21q+\mu}{r})$, and so $cx_1 + cx_2 + cy_1 + cy_2 + cc_1 + cc_2 \leq 2q$. Hence $cc_1+cc_2 \leq 2q-(cx_1+cx_2+cy_1+cy_2)$, and so at most $3(c_1+c_2) \leq 6q-3(x_1+x_2+y_1+y_2)$ vertices of $X_1 \cup X_2$ are adjacent to vertices of $(C_1 \cup C_2) \cap P_f$, cx_1 vertices in X_1 are assigned a $+1$ under f, cx_2 vertices in X_2 are assigned a $+1$ under f, cy_1 vertices in Y_1 are assigned a $+1$ under f, and cy_2 vertices in Y_2 are assigned a $+1$ under f. Thus, at most $6q-3(cx_1+cx_2+cy_1+cy_2)+cx_1+cx_2+cy_1+cy_2 = 6q-2(cx_1+cx_2+cy_1+cy_2)$ of $X_1 \cup X_2$ are either adjacent to a vertex of $(Y_1 \cup Y_2 \cup C_1 \cup C_2) \cap P_f$ or assigned a $+1$ under f. If $cx_1 + cx_2 + cy_1 + cy_2 > 0$, then there is a vertex in $X_1 \cup X_2$, say x, such that $x \notin C_f$, which is a contradiction. Thus, $cx_1 + cx_2 + cy_1 + cy_2 = 0$, and $c_1 + c_2 \leq 2q$. Since $x_{i,\beta} \in C_f$ for $i = 1, \ldots 3q$ and $\beta = 1, 2$, $c_1 = q$ and $c_2 = q$. It now follows that $C' = \{C_j \mid f(c_{j,1}) = 1\}$ is an exact three cover for X. $\diamondsuit$

6.6 A cubic algorithm to compute $\gamma_{tks}^{-101}(T)$ of a tree T

In this section, we will present a cubic time algorithm to compute the total minus k-subdomination number of a tree. The tree T will be rooted and represented by the resulting parent array parent[1 ... n]. We make use of the well-known fact that the tree T can be constructed recursively from the single vertex K_1 using only one rule of composition, which combines two trees (G, x) and (H, y), by adding an edge between x and y and calling x the root of the larger tree F. We express this as follows: $(F, x) = (G, x) \circ (H, y)$. With each such subtree (F, x), we associate the following data structure:

1. **table[x].numvertices:** the number of vertices in the subtree (F, x).
2. **table[x].degree:** $\deg_F(x)$.
3. **table[x].sum[$f(x), t, k$]:** the minimum weight of a function $f : V(F) \rightarrow \{-1, 0, 1\}$ such that x is assigned $f(x)$, $|t| \leq \deg_T(x) - \deg_F(x)$ (representing all possible sums of assignments of -1, 0 and $+1$ to the vertices of $N_T(x) - N_F(x)$ and $|\{v \mid f(N_F(v)) + t \geq 1$ when $v = x$ and $f(N_F(v)) \geq 1$ when $v \neq x\}| \geq k$, where $1 \leq k \leq$ **table[x].numvertices**.

Our input consist of the order of the tree T, say **n**, and the **parent** array of the tree, rooted at a certain vertex. The root of the tree T is labeled with 1, the vertices on the next level are labeled with 2 through 2 plus the number of vertices on level 2, and so on. Using the **parent** array, we compute $\deg_T(x)$ for each vertex x, $x = 1, \ldots, n$. We then initialize the variable **table[x]** for each vertex x, where $x = 1, \ldots, n$. Let x be an arbitrary vertex of T. Initially, $(F, x) = (K_1, x)$, whence **table[x].numvertices**=1 and **table[x].degree**=0. Suppose t is an integer such that $|t| \leq \deg_T(x) - \deg_F(x) = \deg_T(x)$, representing all possible sums of assignments of -1, 0 and $+1$ to the vertices of $N_T(x) - N_F(x) = N_T(x)$. Then $t \in \{-\deg_T(x), \ldots, \deg_T(x)\}$. The only way for $f(N_F(x)) + t = t \geq 1$, is for $t \geq 1$. Thus, we have the following initializations:

Case 1: $t \in \{1, \ldots, \deg_T(x)\}$.
Then **table**$[x]$.**sum**$[f(x), t, 1]$ $=$ **table**$[x]$.**sum**$[f(x), t, 0]$ $= f(x)$ where $f(x) \in \{-1, 0, 1\}$.

Case 2: $t \in \{-\deg_T(x), \ldots, 0\}$.
Then **table**$[x]$.**sum**$[f(x), t, 1]$ is undefined, and **table**$[x]$.**sum**$[f(x), t, 0]$$=f(x)$ where $f(x) \in \{-1, 0, 1\}$.

The following code implements the aforementioned discussion.

Algorithm: To compute $\gamma_{tks}^{-101}(T)$ for a tree T

```
for vertex ← 1 to n do
    degree[vertex] ← 0
for vertex ← 2 to n do
begin
   degree[vertex] ← degree[vertex]+1
   degree[parent[vertex]] ← degree[parent[vertex]]+1
end
for vertex ← 1 to n do
begin
   table[vertex].numvertices ← 1
   table[vertex].degree ← 0
   for excessvalue ← 1 to degree[vertex] do
       forrootvalue ← -1 to 1 do
          begin
             table[vertex].sum[rootvalue,excessvalue,1] ← rootvalue
             table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
```

```
        end
  for excessvalue ← -degree[vertex] to 0 do
      forrootvalue ← -1 to 1 do
        begin
          table[vertex].sum[rootvalue,excessvalue,1] ← 10000
          table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
        end
  end
```

Inputting the **parent** array takes $O(n)$ steps, while computing the **degree** array from the **parent** array also takes $O(n)$ steps. Initializing the array **table** takes

$$\begin{aligned} & O\left(\sum_{\text{vertex}=1}^{n} \times (2\deg_T(\text{vertex})+1)\times 3\right) \\ = \; & O(6\times 2m(T)) + O(3n) \\ = \; & O(12(n-1)) + O(n) \\ = \; & O(n) \end{aligned}$$

steps. Thus, the overall complexity here is $O(n^2)$.

Theorem 6.4 *Suppose (G,x) and (H,y) are two disjoint rooted subtrees, and let $(F,x) = (G,x)\circ(H,y)$. Let $s \in \{-1,0,1\}$, t be an integer such that $|t| \leq \deg_T(x) - \deg_F(x)$ and k be an integer with $0 \leq k \leq |V(F)|$. Then*

table$[x]$.**sum**$[s,t,k]$=
min{**table**$[x]$.**sum**$[s,t+s',j]$+**table**$[y]$.**sum**$[s',s,k-j]$ $\mid s' \in \{-1,0,1\}, 0 \leq j \leq k\}$ = min{**table**$[x]$.**sum**$[s,t+s',j]$+**table**$[y]$.**sum**$[s',s,k-j]$ $\mid s' \in \{-1,0,1\}$, $\max\{0, k-|V(H)|\} \leq j \leq \min\{k, |V(G)|\}\}$.

Moreover, $|t| \leq \deg_T(x) - \deg_F(x)$ *if and only if* $-(\deg_T(x) - \deg_G(x) - 1) \leq t \leq \deg_T(x) - \deg_G(x) - 1$.

Proof. Suppose $f : V(F) \to \{-1, 0, 1\}$ such that

$$f(V(F)) = \mathbf{table[x].sum[s,t,k]}.$$

Let g (respectively, h) be the restriction of f on $V(G)$ (respectively, $V(H)$) and $s^* = h(y) = f(y)$. Note that $f(N_F(x)) + t = g(N_G(x)) + t + s^*$ and $f(N_F(v)) = g(N_G(v))$ for all $v \in V(G) - \{x\}$, while $f(N_F(y)) = h(N_H(y)) + s$ and $f(N_F(v)) = g(N_H(v))$ for all $v \in V(H) - \{y\}$. Thus, $k \leq |\{v \,|\, f(N_F(v)) + t \geq 1$ when $v = x$ and $f(N_F(v)) \geq 1$ when $v \neq x\}| = |\{v \,|\, g(N_G(v)) + t + s^* \geq 1$ when $v = x$ and $g(N_G(v)) \geq 1$ when $v \neq x\}| + |\{v \,|\, h(N_H(v)) + s \geq 1$ when $v = y$ and $h(N_H(v)) \geq 1$ when $v \neq y\}|$. If $j = |\{v \,|\, g(N_G(v)) + t + s^* \geq 1$ when $v = x$ and $g(N_G(v)) \geq 1$ when $v \neq x\}|$, then $k - j \leq |\{v \,|\, h(N_H(v)) + s \geq 1$ when $v = y$ and $h(N_H(v)) \geq 1$ when $v \neq y\}|$. It now follows that $\mathbf{table}[x]\mathbf{.sum}[s, t + s^*, j] + \mathbf{table}[y]\mathbf{.sum}[s^*, s, k - j] \leq g(V(G)) + h(V(H)) = \mathbf{table}[x]\mathbf{.sum}[s, t, k]$. Hence, $\min\{\mathbf{table}[x]\mathbf{.sum}[s, t + s', j] + \mathbf{table}[y]\mathbf{.sum}[s', s, k - j] \ | s' \in \{-1, 1\}, 0 \leq j \leq k\} \leq \mathbf{table}[x]\mathbf{.sum}[s, t, k]$.

On the other hand, suppose $g \colon V(G) \to \{-1, 0, 1\}$ such that

$$g(V(G)) = \mathbf{table[x].sum[s,t+s',j]}$$

and $h \colon V(H) \to \{-1, 0, 1\}$ such that

$$h(V(H)) = \mathbf{table[y].sum[s',s,k-j]}.$$

Define $f : V(F) \to \{-1, 0, 1\}$ by $f(v) = g(v)$ if $v \in V(G)$ and $f(v) = h(v)$ for all $v \in V(H)$. As before, $f(N_F(x)) + t = g(N_G(x)) + t + s'$ and $f(N_F(v)) = g(N_G(v))$ for all $v \in V(G) - \{x\}$, while $f(N_F(y)) = h(N_H(y)) + s$ and $f(N_F(v)) = g(N_H(v))$ for all $v \in V(H) - \{y\}$. Thus, $|\{v \,|\, f(N_F(v)) + t \geq 1$ when $v = x$ and

$f(N_F(v)) \geq 1$ when $v \neq x\}| = |\{v \,|\, g(N_G(v)) + t + s' \geq 1$ when $v = x$ and $g(N_G(v)) \geq 1$ when $v \neq x\}| + |\{v \,|\, h(N_H(v)) + s \geq 1$ when $v = y$ and $h(N_H(v)) \geq 1$ when $v \neq y\}| \geq j + (k - j) = k$. Hence, **table**$[x]$.**sum**$[s,t,k] \leq f(V(F)) = g(V(G)) + h(V(H)) =$ **table**$[x]$.**sum**$[s, t+s', j]$ + **table**$[y]$.**sum**$[s', s, k-j]$. Thus, **table**$[x]$.**sum**$[s,t,k] \leq \min\{$**table**$[x]$.**sum**$[s, t+s', j]$+**table**$[y]$.**sum**$[s', s, k-j] \,|\, s' \in \{-1,1\}, 0 \leq j \leq k\}$.

Since $0 \leq j \leq |V(G)|$ and $j \leq k$, we have $0 \leq k - j \leq |V(H)|$, so that $0 \geq j - k \geq -|V(H)|$, whence $j \geq k - |V(H)|$. We conclude that $\max\{0, k - |V(H)|\} \leq j \leq \min\{k, |V(G)|\}$.

Lastly, $|t| \leq \deg_T(x) - \deg_F(x)$ if and only if $-\deg_T(x) + \deg_G(x) + 1 \leq t \leq \deg_T(x) - \deg_G(x) - 1$, since $\deg_F(x) = \deg_G(x) + 1$. $\diamondsuit$

At the conclusion of our algorithm, $T = F$, and so $t = 0$. Clearly, $\gamma_{tks}^{-101}(T) = \min\{$ **table**$[1]$.**sum**$[-1,0,k]$, **table**$[1]$.**sum**$[0,0,k]$ **table**$[1]$.**sum**$[1,0,k]$ $\}$.

We are now in a position to present the remainder of the algorithm.

<u>Algorithm</u>: To compute $\gamma_{tks}^{-101}(T)$ for a tree T (continued).

```
for oldroot ← n downto 2 do
begin
  resulttable.numvertices ← table[oldroot].numvertices +
                              table[parent[oldroot]].numvertices
  resulttable.degree ← table[parent[oldroot]].degree + 1
  range ← degree[parent[oldroot]] - resulttable.degree
  for newrootvalue ← -1 to 1 do
      for newrootexcess ← -range to range do
          for k ← 0 to resulttable.numvertices do
          begin
            minimum ← 1000
            startvalue ← max(0, k - table[oldroot].numvertices)
```

```
            stopvalue ← min(k, table[parent[oldroot]].numvertices)
            for j ← startvalue to stopvalue do
            begin
              for oldrootvalue ← -1 to 1 do
              begin
                number ← degree[parent[oldroot]] - table[parent[oldroot]].degree - 1
                if -number ≤ newrootexcess ≤ number then
                begin
                  summand1 ← table[parent[oldroot]].
                             sum[newrootvalue, newrootexcess + oldrootvalue, j]

                  summand2 ← table[oldroot].
                             sum[oldrootvalue, newrootvalue, k-j]

                  temp ← summand1 + summand2
                end
                if (temp < minimum)
                then minimum ← temp
              end
            end
            resulttable.sum[newrootvalue, newrootexcess, k] ← minimum
          end
  table[parent[oldroot]] ← resulttable
end
for k ← 0 to n do
  output (k, min(table[1].sum[1, 0, k], table[1].sum[0, 0, k], table[1].sum[-1, 0, k]))
```

The complexity of the above part of the algorithm, excluding the output phase, is

$$\begin{aligned} & O(\textstyle\sum_{\text{n-oldroot}=0}^{(n-2)} \times 3 \times (2 \times \deg_T[\text{parent}[\text{oldroot}]] + 1) \times n \times n \times 3) \\ = \; & O(18n^2 \textstyle\sum_{v \in V(T)} \deg_T(v)) + O(n \times 9n^2) \\ = \; & O(18n^2 2m(T)) + O(n^3) \\ = \; & (18n^2 \times 2 \times (n-1)) + O(n^3) \\ = \; & O(n^3), \end{aligned}$$

while the complexity of the output phase is $O(n)$. Thus, the overall complexity of the algorithm is $O(n^3)$.

6.7 A cubic algorithm to compute $\gamma_{ks}^{-101}(T)$ of a tree T

In this section, we present a cubic algorithm to compute $\gamma_{ks}^{-101}(T)$ of a tree T. The approach here is similar to what we described in the previous section. Here we have the following data structure, associated with the subtree (F, x).

1. **table**$[x]$**.numvertices:** the number of vertices in the subtree (F, x).
2. **table**$[x]$**.degree:** $\deg_F(x)$.
3. **table**$[x]$**.sum**$[f(x), t, k]$: the minimum weight of a function $f : V(F) \to \{-1, 0, 1\}$ such that x is assigned $f(x)$, $|t| \leq \deg_T(x) - \deg_F(x)$ (representing all possible sums of assignments of -1, 0 and $+1$ to the vertices of $N_T(x) - N_F(x)$ and $|\{v \mid f(N_F[v]) + t \geq 1 \text{ when } v = x \text{ and } f(N_F[v]) \geq 1 \text{ when } v \neq x\}| \geq k$, where $1 \leq k \leq$ **table**$[x]$**.numvertices**.

The initialization phase here proceeds as follows.

Let x be an arbitrary vertex of T. Initially, $(F, x) = (K_1, x)$, whence **table**$[x]$**.numvertices**=1 and **table**$[x]$**.degree**=0. Suppose t is an integer such that $|t| \leq \deg_T(x) - \deg_F(x) = \deg_T(x)$, representing all possible sums of assignments of -1, 0 and $+1$ to the vertices of $N_T(x) - N_F(x) = N_T(x)$. Then $t \in \{-\deg_T(x), \ldots, \deg_T(x)\}$. The only way for $f(N_F(x)) + f(x) + t = f(x) + t \geq 1$, is for $t \geq 1 - f(x)$. Thus, we have the following initializations:

Case 1: $t \in \{1 - f(x), \ldots, \deg_T(x)\}$.
Then **table**$[x]$**.sum**$[f(x), t, 1]$ = **table**$[x]$**.sum**$[f(x), t, 0]$ = $f(x)$ where $f(x) \in \{-1, 0, 1\}$.

Case 2: $t \in \{-\deg_T(x), \ldots, -f(x)\}$.
Then **table**$[x]$**.sum**$[f(x), t, 1]$ is undefined, and **table**$[x]$**.sum**$[f(x), t, 0]$=$f(x)$ where $f(x) \in \{-1, 0, 1\}$.

One may prove a result analogous to Theorem 6.2

We are now in a position to state the algorithm. Note that the initialization phase of the algorithm has complexity

$$\begin{aligned} & O(\textstyle\sum_{v \in V(T)} 3 \times (2\deg_T(v) + 1)) \\ = \; & O(6 \times 2m(T)) + O(3n) \\ = \; & O(12(n-1)) + O(n) \\ = \; & O(n). \end{aligned}$$

Thus, the overall complexity of the algorithm is also $O(n^3)$.

Algorithm: To compute $\gamma_{ks}^{-101}(T)$ for a tree T.

```
for vertex ← 1 to n do
   degree[vertex] ← 0
for vertex ← 2 to n do
begin
  degree[vertex] ← degree[vertex]+1
```

```
  degree[parent[vertex]] ← degree[parent[vertex]]+1
end
for vertex ← 1 to n do
    for rootvalue ← -1 to 1 do
    begin
      for excessvalue ← 1 - rootvalue to degree[vertex] do
      begin
        table[vertex].sum[rootvalue,excessvalue,1] ← rootvalue
        table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
      end
      for excessvalue ← -degree[vertex] to -rootvalue do
      begin
        table[vertex].sum[rootvalue,excessvalue,1] ← 10000
        table[vertex].sum[rootvalue,excessvalue,0] ← rootvalue
      end
    end
for oldroot ← n downto 2 do
begin
  resulttable.numvertices ← table[oldroot].numvertices +
                            table[parent[oldroot]].numvertices
  resulttable.degree ← table[parent[oldroot]].degree + 1
  range ← degree[parent[oldroot]] - resulttable.degree
  for newrootvalue ← -1 to 1 do
      for newrootexcess ← -range to range do
          for k ← 0 to resulttable.numvertices do
          begin
            minimum ← 1000
            startvalue ← max(0, k - table[oldroot].numvertices)
            stopvalue ← min(k, table[parent[oldroot]].numvertices)
            for j ← startvalue to stopvalue do
```

```
          begin
            for oldrootvalue ← -1 to 1 do
            begin
              number ← degree[parent[oldroot]] - table[parent[oldroot]].degree - 1
              if -number ≤ newrootexcess ≤ number then
              begin
                summand1 ← table[parent[oldroot]].
                          sum[newrootvalue, newrootexcess + oldrootvalue, j]

                summand2 ← table[oldroot].
                          sum[oldrootvalue, newrootvalue, k-j]

                temp ← summand1 + summand2
              end
              if (temp < minimum)
              then minimum ← temp
            end
          end
          resulttable.sum[newrootvalue, newrootexcess, k] ← minimum
        end
  table[parent[oldroot]] ← resulttable
end
for k ← 0 to n do
  output (k, min(table[1].sum[1, 0, k], table[1].sum[0, 0, k], table[1].sum[-1, 0, k]))
```

All the above algorithms are implemented in C++.

Bibliography

[1] L.W. Beineke and M.A. Henning, Opinion functions on trees. *Discrete Math.* **167** (1997), 167–178.

[2] C. Berge, *Theory of Graphs and its Applications*, Methuen, London (1962).

[3] I. Broere, J.E. Dunbar and J.H. Hattingh, Minus k-subdomination in graphs. *Ars Combin.* **50** (1998), 177–186.

[4] I. Broere, J.H. Hattingh, M.A. Henning and A.A. McRae, Majority domination in graphs. *Discrete Math.* **138** (1995), 125–135.

[5] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs. *Networks* **7** (1977), 247–261.

[6] G. Chang, S.C. Liaw and H.G. Yeh, k-Subdomination in graphs. *Discrete Appl. Math.* **120** (2002), 55–60.

[7] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Third Edition, Wadsworth and Brooks/Cole, Monterey, CA, 1996.

[8] E.J. Cockayne and C.M. Mynhardt, On a generalisation of signed dominating functions of graphs. *Ars Combin.* **43** (1996), 235–245.

[9] J. E. Dunbar, W. Goddard, S.T. Hedetniemi, M. A. Henning and A.A. McRae, The algorithmic complexity of minus domination in graphs. *Discrete Appl. Math.* **68** (1996), 73–84.

[10] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning and A.A. McRae, Minus domination in graphs. Minus domination in graphs. *Discrete Math.* **199** (1999), 35–47.

[11] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning and P.J. Slater, Signed domination in graphs. *Graph Theory, Combinatorics, and Applications*, John Wiley & Sons, Inc., **1** (1995), 311–322.

[12] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Freeman, New York (1979).

[13] L. Harris and J.H. Hattingh, The algorithmic complexity of certain functional variations of total domination in graphs. Submitted for publication.

[14] L. Harris, J.H. Hattingh and M.A. Henning, Total k-subdominating functions on graphs. Submitted for publication.

[15] L. Harris, J.H Hattingh, and M.A. Henning, Algorithmic aspects of total k-subdomination in graphs. Submitted for publication.

[16] L. Harris, J.H Hattingh, and M.A. Henning, Algorithmic aspects of minus total k-subdomination in graphs. Submitted for publication.

[17] L. Harris, J.H. Hattingh and E. Ungerer, The signed and minus k-subdomination number of certain multipartite graphs and their complements. *Util. Math.* **59** (2001), 77–95.

[18] J.H. Hattingh, M.A. Henning and P.J. Slater, The algorithmic complexity of signed domination in graphs. *Australas. J. Combin.* **12** (1995), 101–112.

[19] J.H. Hattingh, M.A. Henning and E. Ungerer, Partial signed domination in graphs. *Ars Combin.* **48** (1998), 33–42.

[20] J.H. Hattingh, A.A. McRae and E. Ungerer, Minus k-subdomination in graphs III. *Australas. J. Combin.* **17** (1998), 69–76.

[21] J.H. Hattingh and E. Ungerer, Minus k-subdomination in graphs II. *Discrete Math.* **171** (1997), 141–151.

[22] J.H. Hattingh and E. Ungerer, On the signed k-subdomination number of trees. *Proceedings of the Eighth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms, and Applications*, **II** (1998), 479–490.

[23] J.H. Hattingh and E. Ungerer, The signed and minus k-subdomination numbers of comets. *Discrete Math.* **183** (1998), 141–152.

[24] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[25] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

[26] M.A. Henning, Domination in regular graphs. *Ars Combin.* **43** (1996), 263–271.

[27] M.A. Henning, Signed total domination in graphs. To appear in *Discrete Math.*

[28] M.A. Henning and H.R. Hind, Strict majority functions on graphs. *J. Graph Theory* **28** (1998), 49–56.

[29] M.A. Henning and P.J. Slater, Inequalities relating domination parameters in cubic graphs. *Discrete Math.* **158** (1996), 87–98.

[30] T.S. Holm, On majority domination in graphs. *Discrete Math.* **239** (2001), 1–12.

[31] L. Kang, C. Dang, M. Cai and E. Shan, Upper bounds for k-subdomination number of graphs. *Discrete Math.***247** (2002), 229–234

[32] O. Ore, *Theory of Graphs* A.M.S. Colloq. Publ. **38**, A.M.S. Providence, RI (1962).

[33] J. Pfaff, *Algorithmic Complexities of Domination-related Graph Parameters.* PhD thesis, Clemson Univ., 1984.

[34] X. Yang, X. Huang and Q. Hou, On graphs with equal signed and majority domination number. Preprint.

[35] H. Yeh and G. J. Chang, Algorithmic aspects of majority domination. *Taiwanese J. Math.* **1** (1997), 343–350.

[36] B. Zelinka, Some remarks on domination in cubic graphs. *Discrete Math.* **158** (1996), 249–255.

[37] B. Zelinka, Signed total domination number of a graph. *Czechoslovak Math. J.* **51** (2001), 225–229.