A New Method With Regularization For Solving Split Variational Inequality Problem In Real Hilbert Spaces

by

Francis Akutsah

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UNIVERSITY OF
KWAZULU-NATAL
INYUVESI
YAKWAZULU-NATALI

School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal, South Africa.

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Francis Akutsah
B. Ed (University of Cape Coast) and MSc (Kwame Nkrumah University of Science and Technology)

As the candidate's supervisor, I have approved this thesis for submission.

Dr. O. K. Narain

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Dedication

This thesis is dedicated to God Almighty.

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Abstract

The concept of the optimization problem, fixed point theory and its application constitute the nucleus of nonlinear analysis, which is a major branch of mathematics. Optimization theory, fixed point theory and its applications have a wide range of application in practically every field of science, particularly mathematical sciences. The theory of optimization and fixed point have received great attention from authors around the world and these areas will continue to receive such great attention. The theory has been well developed by well-known researchers in these areas. However, there are still a lot of work to be done. The goal of this thesis is to advance the theory of optimization and fixed point in the framework of Hilbert and Banach spaces. The substance of this thesis is separated into two parts. The research efforts of the first part of this thesis (Chapter 3 to Chapter 6) has to do with introducing some new iterative methods for approximating the solution of a variational inequality problems, split variational inequality problems, equilibrium problems, split monotone variational inclusion problem, split generalized mixed equilibrium problem and fixed point problems in the framework of a Hilbert and Banach spaces. In addition, we introduce a new class of bilevel problem in the framework of real Hilbert spaces and a new regularization technique, and inertial terms for approximating solutions of split bilevel variational inequality problems. Furthermore, we establish that the proposed iterative methods converges strongly to the solution of the aforementioned problems as the case may be. Then, we present some numerical experiments to show the efficiency and applicability of our proposed methods in comparison with other state-of-the-art iterative methods in the literature. The second part (Chapter 7) of this thesis deals with developing iterative algorithms and introducing some nonlinear mappings in the framework of the Hilbert and Banach spaces. First, we present a modified (improved) generalized M-iteration with the inertial technique for three quasi-nonexpansive multivalued mappings in a real Hilbert space. In addition, we present some fixed point results for a general class of nonexpansive mappings in the framework of the Banach space and also proposed a new iterative scheme for approximating the fixed point of this class of mappings in the framework of uniformly convex Banach spaces. Finally, we apply our convergence results to certain optimization problems, integral equations, and we present some numerical experiments to show the efficiency and applicability of the proposed method in comparison with other existing methods in the literature.

Contents

Declaration

This thesis in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used, proper reference has been made.

Akutsah Francis

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Contributed papers from the thesis

The major part of this thesis has been published/accepted in the following Scopus, ISI and DHET accredited journals indexed by science citation.

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3. **F. AKUTSAH**, H. A. Abass, A. A. Mebawondu and O. K. Narain, On split generalized mixed equilibrium and fixed point problems of infinite family of quasi-nonexpansive multi-valued mappings in real hilbert spaces, *Asain Eur. J. of Math.,* (2022), DOI:10.1142/S1793557122500826.
4. **F. AKUTSAH**, H. A. Abass, A. A. Mebawondu and O. K. Narain, Shrinking approximation method for solution of split monotone variational inclusion and fixed point problems in Banach spaces, *Int. J. of Nonlin. Anal. and Appl.,* **12** (2), (2021), 825–842.
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8. **F. AKUTSAH**, A. A. Mebawondu, G. C. Ugwunnadi, P. Pillay and O. K. Narain, Inertial extrapolation method with regularization for solving a new class of bilevel problem in real Hilbert spaces, *SeMA Journal*, (2022), 1–22
9. **F. AKUTSAH**, A. A. Mebawondu, H. A. Abass, M. O. Aibinu and O. K. Narain, Inertial relaxed Tseng method for solving variational inequality problem in Hilbert spaces, *Advances in Mathematics: Scientific Journal* **10** (10), (2021), 3597–3623.

Chapter 1

General Introduction

1.1 Background of Study

Fixed point and optimization theory has become an invaluable area of study in mathematics as many problems in mathematical sciences, engineering, physics, economics, game theory, etc., can be transformed into a fixed point problem. Optimization problems which includes minimization problems, variational inequality problems, equilibrium problems, monotone inclusion problems, and so on can be referred to as the nucleus of fixed point theory and its application. It is well-known that solving a fixed point or optimization problems analytically is very difficult or almost impossible and thus the need to consider approximate methods of solution arises. Hence, researchers in this area have developed different methods for solving fixed point and optimization problems. To mention a few; proximal-like methods, fixed point methods, auxiliary principles, decomposition methods, extra-gradient methods, sub-gradient and extra-gradient methods, projection contraction methods, and normal map equations (see [21, 25, 169, 203, 204, 372] and the references therein). Over the years, optimization problems have been extensively studied in different abstract spaces (Hilbert spaces, Banach spaces, p-uniformly convex metric spaces, CAT(0) spaces, and Hadmard spaces). In the light of this, finding the solution of any optimization problem(s) implies finding the fixed point(s) of the nonlinear mapping(s) or nonlinear operator(s). For example, finding a solution of a minimization problem, monotone inclusion problem, variational inequality problem, equilibrium problem and so on, is equivalent to finding the fixed point of the re-solvent of the convex function associated with each of the problem. Due to this fact, a lot of research effort is going into developing different iterative techniques (methods or schemes) for finding solutions of optimization problems and fixed point problems.

Let X be any arbitrary space, a point $x \in X$ is called a fixed point of a mapping $T : X \to X$ if

$$Tx = x, \tag{1.1.1}$$

that is, a point $x \in X$ which remains invariant under the action of the mapping T.

Example 1.1.1. *Let* $T : \mathbb{R} \to \mathbb{R}$ *be defined by* $T(x) = x^2 + x - 1$. *It is easy to see that* $x = 1$ *and* $x = -1$ *are the fixed points of* T, *because* $T(1) = 1$ *and* $T(-1) = -1$.

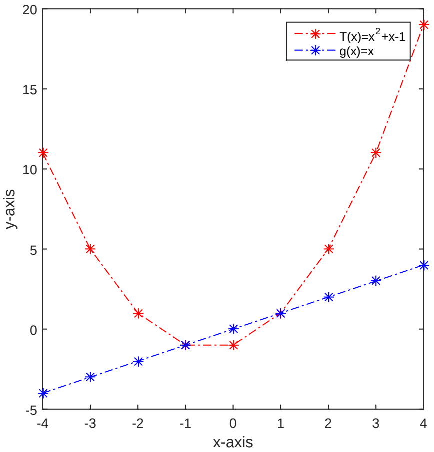

Figure 1.1: Graph of $T(x) = x^2 + x - 1$ and $g(x) = x$.

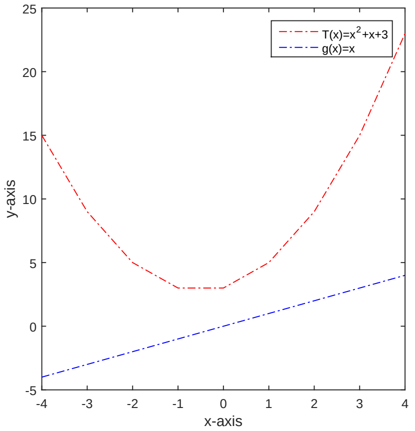

Figure 1.2: Graph of $T(x) = x^2 + x + 3$ and $g(x) = x$.

Example 1.1.2. *Let* $T : \mathbb{R} \to \mathbb{R}$ *be defined by* $T(x) = x^2 + x + 3$. *It is also easy to see that* T *has no fixed point.*

Remark 1.1.3. *Geometrically, a fixed point implies that the point* $(x, T(x))$ *is on the line* $y = x$.

For more than 50 years, the theory of fixed point is one of the most developed areas of research in the field of nonlinear analysis and its application, this development has continued to attract the attention of several researchers all over the globe due to its fruitful applications in almost all disciplines. For example, fixed point techniques have been greatly applied in fields such as fuzzy theory, signal processing, inverse problems, economics, mathematical sciences, optimal control, engineering, physics, biology, chemistry, game theory,and mathematical sciences (see [20, 30, 31, 32, 33, 226, 227, 229] and the references therein).

The Banach fixed point theorem is the most instrumental and applied result in nonlinear analysis. The only required condition for establishing the Banach contraction result is the completeness of the metric space. Furthermore, the Banach contraction result is easy to establish since it makes use of iterative algorithms, also, it can easily be implemented on a computer system to find the fixed point of the contractive mapping as it produces approximations of any required accuracy.

Definition 1.1.4 ([39]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a contraction if there exists a constant $\delta \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \delta d(x, y) \quad \forall x, y \in X. \tag{1.1.2}$$

Theorem 1.1.5 ([39]). *Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping. Then, T has a unique fixed point x^*, and for any $x \in X$ the sequence $\{T^n x\}$ converges to x^*.*

The Banach fixed point theorem has become an instrumental tool in establishing the existence and uniqueness of solution of the Volterra integral equations, dynamical programming, nonlinear integro-differential equations, game theory, random, ordinary and partial differential equations and so on. Let (X, d) be a metric space and the self mapping $T : X \to X$ for $x_1 \in X$, the Picard iterative method is defined as follows:

$$x_{n+1} = Tx_n, \ \forall n \in \mathbb{N}. \tag{1.1.3}$$

The Picard iterative method was one of the first iterative algorithms for determining the convergence of a contraction mapping. It has been established that even when the fixed point of the nonexpansive mapping (a mapping is nonexpansive if $\delta = 1$ in (1.1.2)) is known, the Picard iterative method fails to approximate it. The authors in [67] established that the nonexpansive mapping on a closed and bounded subset of a uniformly convex Banach space has a fixed point. This effort has opened a lot of research grounds for authors. Some well known iterative methods include the following; Mann iterative method introduced by Mann [220], is defined in a real Hilbert space H as follows:

$$\begin{cases} x_1 \in H \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1.4}$$

where α_n is a sequence in $(0, 1)$ satisfying

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$.

It has been established in the literature that, the Mann iterative process can only converge to the fixed point of a continuous mapping T. That is the Mann iterative method can only be used to approximate the fixed point of a nonexpansive mapping and some contractive mappings. More so, if T is not continuous, then the Mann iterative process might fail to converge to the fixed point of T even when the fixed point is known. In 1974, Ishikawa [166] introduced an iterative process named after him (Ishikawa iterative process). This iterative process is a generalization of the Mann iterative process. The Ishikawa iterative process is used for approximating the fixed points of continuous and none continuous mappings in Hilbert spaces. The Ishikawa iterative method fill the loop hole in the Mann iterative method. The Ishikawa iterative process is defined in a real Hilbert space H as follows:

$$\begin{cases} x_1 \in H \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying

(i) $\lim\limits_{n\to\infty} \beta_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

However, with the Ishikawa iterative process authors could only establish weak convergence.

In 2007, Agarwal [14] introduced and studied the S-iterative process in the framework of real Hilbert space H as follows:

$$\begin{cases} x_1 \in H, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \\ x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1.6}$$

where $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$. It is easy to see that the iterative process (1.1.6) is independent of (1.1.4) and (1.1.5). Furthermore, the Agarwal iterative process has a better rate of convergence compared to the Mann and Ishikawa iterative processes. Furthermore, the Picard, Mann, Ishikawa and S-iteration processes converge weakly.

Remark 1.1.6. *In this area of research, a strong convergence result is preferable to a weak convergence.*

In the light of Remark 1.1.6, Haugazeau in [155] introduced and studied the Haugazeau iterative process in the framework of a real Hilbert space H, the iterative process is defined as follows:

$$\begin{cases} x_1 \in H, \\ y_n = T x_n, \\ C_n = \{w \in H : \langle x_n - y_n, y_n - z\rangle \geq 0\}, \\ Q_n = \{w \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1), \quad n \in \mathbb{N}, \end{cases} \tag{1.1.7}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto the intersection of the convex sets C_n and Q_n. Using (1.1.7), Haugazeau [155] established a strong convergence result for approximating the fixed points of the mapping T (nonexpansive mapping). Halpern [154] introduced the well-known Halpern iterative scheme.

$$\begin{cases} x_1, u \in H, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1.8}$$

where $\{\alpha_n\}$ is a sequence in (0,1) such that

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

The Halpern scheme have been studied for more general mappings than the nonexpansive mapping (see [231, 344, 345] and the references therein). In 2004, Xu [368] introduced and studied the viscosity iterative scheme which is a generalization of the Halpern iterative algorithm. This scheme is used for approximating the fixed point of a nonexpansive mapping and other nonlinear mappings in different abstract spaces. The viscosity iterative scheme is defined as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1.9}$$

where $\{\alpha_n\}$ is a sequence in (0,1) and f is a contraction mapping on X such that

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Remark 1.1.7. *One of the major advantages of iterative process* (1.1.9) *on Halpern iterative process* (1.1.8) *is that it also converges strongly to a unique solution of some variational inequalities with the contraction mapping f. It has also been established that the viscosity iterative process has a better rate of convergence than that of the Halpern iterative scheme.*

In the study of optimization theory, fixed point theory, and its applications, the relevance of nonlinear mappings and iterative processes cannot be overstated. Many researchers have extended, generalized, and enhanced the class of contraction mappings, nonexpansive mappings, quasi-nonexpansive mappings, demicontrative mappings etc., and the aforementioned iterative techniques in many abstract spaces due to their importance (see [7, 8, 9, 37, 38, 157, 287, 288] and the reference therein). However, the open question remains, can we build an iterative method that converges faster, approximates better, is efficient, and successful in approximating the solutions of fixed point problems and optimization problems as compared to existing iterative methods in the literature? Can we also introduce a class of nonlinear mappings that improve, generalize, and unify previously published results on nonlinear mappings in the literature? As a result, the goal of this thesis is to provide an affirmative answer to these questions.

1.2 Research Problems and Motivation

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, C be a nonempty closed convex subset of H and $A : H \to H$ be an operator. The classical Variational Inequality Problem (VIP) is formulated as: Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \ \forall \ y \in C. \tag{1.2.1}$$

The notion of VIP was introduced independently by Stampacchia [308, 309] and Fichera [133, 132] for modeling problems arising from mechanics and for solving the Signorini problem. It is well-known that many problems in economics, mathematical sciences, mathematical physics can be formulated as VIP. Censor et al. in [85] extended the concept of VIP (1.2.1) to the following Split Variational Inequality Problem (SVIP): Find

$$x^* \in C \quad \text{that solves} \ \ \langle A_1 x^*, x - x^* \rangle \geq 0, \quad \forall \ x \in C, \tag{1.2.2}$$

such that $y^* = Tx^* \in Q$ solves

$$\langle A_2 y^*, y - y^* \rangle \geq 0, \quad \forall \ y \in Q, \tag{1.2.3}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A_1 : H_1 \to H_1, \ A_2 : H_2 \to H_2$ are two operators and $T : H_1 \to H_2$ is a bounded linear operator. When $A_1 = A_2 = 0$, the SVIP reduces to the Split Feasibility Problem (SFP). That is, find

$$x^* \in C \ \ \text{such that} \ \ y^* = Tx^* \in Q. \tag{1.2.4}$$

The concept of SFP was introduced by Censor and Elfving [84] in the framework of finite-dimensional Hilbert spaces. The SFP has found applications in many real-life problems such as image recovery, signal processing, control theory, data compression, computer tomography and so on (see [84, 85, 231] and the references therein). Researchers in this area have introduced different iterative methods to approximate the solution of VIPs, SVIPs and SFPs in the framework of Hilbert spaces. For example, Xu [364] introduced the iterative process

$$x_{n+1} = P_C(I - \lambda A)x_n. \tag{1.2.5}$$

It has been established that if A is strongly monotone and is Lipschitz continuous, then the iterative scheme (1.2.5) has strong convergence results under some suitable conditions. In addition, if A is inverse strongly monotone, the iterative scheme (1.2.5) has weak convergence results under some suitable conditions. The strict condition on the cost operator A becomes a very big challenge to a researcher in this area. An attempt to overcome these drawbacks (weaken the cost operation A) was made by Korpelevich [192]. The extragradient type method which is given by (1.2.6), was introduced. The convergence of the method was established for a monotone and Lipschitz continuous operator A in the finite-dimensional Euclidean spaces.

$$\begin{cases} x_1 \in C \\ y_n = P_C(x_n - \lambda A x_n) \\ x_{n+1} = P_C(y_n - \lambda A y_n) \ \ \forall \ \ n \in \mathbb{N}. \end{cases} \tag{1.2.6}$$

Under some suitable conditions, the sequence $\{x_n\}$ was shown to converge to the solution set of the problem (1.2.1). Since then, other authors have studied the VIP (1.2.1) in Hilbert spaces using different iterative algorithms, (see [142, 143, 156] and the references therein). However, in all of these approaches, the convergence of their methods was obtained under the inversely strongly monotone or strongly pseudomonotone or monotonicity and Lipschitz continuity or pseudomonotonicity and Lipschitz continuity assumption of the underlying cost operator A. The challenge about these methods is how to calculate the Lipschitz constant of the given monotone or pseudomonotone operator, which is difficult or even sometimes impossible. Thus, making their methods very difficult in applications.

Remark 1.2.1. *Considering the iterative processes* (1.2.5) *and* (1.2.6)*, it is natural to ask, if an iterative algorithm can be introduced to approximate the VIP* (1.2.1) *in which the underlining operator is just pseudomonotone, with the minimum metric projection.*

Motivated by the problem (1.2.1), Mainge introduced and studied the Bilevel Variational Inequality Problem (BVIP) of the form:

$$\text{Find } x^* \in VI(A,C) \ \text{ such that } \ \langle Fx^*, x - x^* \rangle \geq 0, \ \forall \ x \in VI(A,C), \tag{1.2.7}$$

where $F : H \to H$ is L-Lipschitz continuous and γ-strongly monotone. He proposed a hybrid extragradient scheme described as follows:

$$\begin{cases} u_0 \in C \\ v_n = P_C(u_n - \lambda_n A u_n) \\ t_n = P_C(u_n - \lambda_n A v_n) \\ u_{n+1} = t_n - \alpha_n F t_n, \end{cases} \tag{1.2.8}$$

where $\{\lambda_n\} \subset [a,b] \subset \left(0, \frac{1}{L}\right)$ and $\alpha_n \subset (0,1)$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$. It was established that the resulting sequence $\{x_n\}$ converges strongly to a unique solution of the problem (1.2.7). This BVIP has applications in mathematical programming with equilibrium constraints [210], bilevel convex programming model [346] and the minimum norm problem with the solution set of variational inequality [376, 381]. In addition, the notion of BVIP has been extended and generalized by researchers in this area. For example, the problem

$$\text{Find } x^* \in VI(A,C)\cap F(S) \ \text{ such that } \ \langle Fx^*, x-x^*\rangle \geq 0,\ \forall\ x \in VI(A,C)\cap F(S), \tag{1.2.9}$$

is a generalization of (1.2.7), where $F : H \to H$ is L-Lipschitz continuous, $S : H \to H$ is a nonlinear map and γ-strongly monotone. Furthermore, Minh, Van and Anh in [236], studied the following Split Bilevel Variational Inequality Problem (SBVIP): Find

$$x^* \in \Gamma \ \text{ such that } \ \langle F_2x^*, x-x^*\rangle \geq 0, \tag{1.2.10}$$

for any $x \in \Gamma$, where

$$\Gamma = \{x^* \in VI(F_1, C) : Ax^* \in F(S)\}.$$

They proposed an iterative method and established a strong convergence theorem for the proposed iterative method.

Remark 1.2.2. *The question still remains open, if it is possible to introduce a new type of bilevel problem that generalizes existing ones in the literature. In addition, can one further introduce new iterative methods that will provide an affirmative answers to various setbacks noted in the above-mentioned iterative methods?*

Another interesting optimization problem is the Equilibrium Problem (EP). Following the work of Fan [129], who established the well-known result about minimax inequality, Blum and Oetlli [58] factored out the word equilibrium. Consequently, Blum and Oetlli are referred to as the pioneer of the optimization problem called EP. Let C be a nonempty, closed and convex subset of a metric space (X, d) (where d is the metric) and $F : C \times C$ is a bifunction. The EP is defined as: Find $x \in C$ such that

$$F(x,y) \geq 0 \qquad \forall y \in C, \tag{1.2.11}$$

or equivalently, the EP is defined as: Find

$$x \in \operatorname{argmin}\{F(x,y) : y \in C\}. \tag{1.2.12}$$

The notion of EP has been applied in almost all disciplines, to mention a few, the notion has been applied in mathematical sciences, game theory, network analysis, and so on. The concept of Generalized Mixed Equilibrium Problem ($GMEP$) was introduced and studied by Zhang [387] and Yao et al. [373] independently. The concept of equilibrium and generalized equilibrium problems have applications in almost all areas of human endeavour. For example, equilibrium and generalized equilibrium problems have application in finance, economics, networking analysis, transportation elasticity among many others [58, 112, 129, 237]. The concept of GMEP include fixed-point problems, variational inequality problems, Nash equilibria and the equilibrium problem as special cases. Let $F : C \times C \to \mathbb{R}$ be a nonlinear bifuntion and $B : C \to H$ be a mapping. Let $\psi : C \to \mathbb{R}$ be a real-valued function, then the GMEP is to find $x^* \in C$ such that

$$F(x^*, x) + \langle Bx^*, y-x^*\rangle + \psi(y) - \psi(x^*) \geq 0,\ \forall\ x \in C. \tag{1.2.13}$$

For solving GMEP (1.2.13), the bifunction F is assumed to satisfy the following conditions:
(L1) $F(x,x)=0$ for all $x \in C$;
(L2) F is monotone, i.e $F(x,y)+F(y,x) \geq 0$, for all $x,y \in C$;
(L3) for each $x,y \in C, \lim_{t\to 0} F(tz+(1-t)x,y) \leq F(x,y)$;
(L4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.
In addition, let H_1 and H_2 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $F: C\times C \to \mathbb{R}, G: Q\times Q \to \mathbb{R}$ be bifunctions, $\psi_1 : C \to \mathbb{R}\cup\{+\infty\}, \psi_2 : Q \to \mathbb{R}\cup\{+\infty\}$ be functions and $B_1 : C \to H_1, B_2 : Q \to H_2$ be nonlinear mappings. Let $A: H_1 \to H_2$ be a bounded linear operator. Then the Split Generalized Mixed Equilibrium Problem (SGMEP) is to find $x^* \in C$ such that

$$F(x^*,x)+\langle B_1x^*, x-x^*\rangle + \psi_1(x)-\psi_1(x^*) \geq 0, \ \forall\ x \in C; \tag{1.2.14}$$

and $y^* = Ax^* \in Q$ solves

$$G(y^*,y)+\langle B_2y^*, y-y^*\rangle + \psi_2(y)-\psi_2(y^*) \geq 0, \ \forall\ y \in Q. \tag{1.2.15}$$

Many authors have studied the aforementioned problem in different abstract spaces using different iterative methods (see [50, 57, 110, 310] and references therein).

In this thesis, motivated by the above research problems and some existing results in the literature, we intend to provide an affirmative answers to the question raised above and further develop the theory of optimization and fixed point theory in the framework of Hilbert and Banach spaces. To achieve this, we introduce a new type of bilevel variational inequality problems, nonlinear mappings and some different iterative methods for approximating the solution of a VIP (1.2.1), an SVIP (1.2.2)-(1.2.3), an SFP (1.2.4), an EP (1.2.11), GEP (1.2.13), an SGMEP (1.2.14)- (1.2.15) and other optimization problems in the framework of Hilbert and Banach spaces. In addition, we present some numerical experiments and application where necessary to establish the applicability of our iterative methods.

1.3 Aims and Objectives

At the end of this study, we aim to achieve the following:

1. introduce a new class of bilevel problem, namely Modified Bilevel Variational Inequality (Regularized Variational Inequality) Problem in the frame work of Hilbert spaces, and propose an iterative method for approximating its solution;
2. further develop the study of MIPs using shrinking approximation iterative method in Banach spaces;
3. further develop the study of split generalized mixed equilibrium and fixed point problems in the frame work of Hilbert spaces;
4. introduce and study a new inertial extrapolation method with regularization for approximating solutions of split variational inequality problems in the frame work of real Hilbert spaces;

5. introduce and study some modified inertial-type iterative methods with self-adaptive step-size for solving variational inequality problems in Hilbert space;
6. introduce and study a new inertial relaxed Tseng extrapolation method with weaker conditions for approximating the solution of a variational inequality problems and fixed point problems.
7. introduce a generalized inertial extrapolation method with regularization term for approximating the solutions of a monotone and Lipschitz variational inequality and fixed point problems in a real Hilbert space.
8. introduce a generalized contractive mappings and an iterative method in Banach spaces which are more general than existing contractive mappings and iterative methods in the literature;
9. apply the results of our findings of the aforementioned optimization problems to other optimization problems, such as convex minimization and split feasibility;
10. we present an application and some numerical experiments to show the efficiency and applicability of our methods in comparison with other methods in the literature.

1.4 Organization of the Thesis

The thesis is organized as follows:

Chapter 1: In this chapter, we provide a quick overview of our research. We also discussed about the research problem and the motivation for our study. Finally, we give the objectives of the study and a comprehensive organization of the thesis.

Chapter 2: In this chapter, we give some basic definitions, discuss some concepts, terms, and results that are important to our study. We also provide a detailed literature review of some recent and important past works that are relevant to our study.

Chapter 3: The research efforts of this chapter is to present some iterative methods for approximating the solution of a variational inequality problems and a split variational inequality problems in the frame work of a Hilbert spaces. First, we present a new inertial relaxed Tseng extrapolation method with weaker conditions for approximating the solution of a variational inequality problem, where the underlying operator is only required to be pseudomonotone. We also introduce a generalized inertial extrapolation method with regularization term for approximating the solution of a monotone and Lipschitz variational inequality and fixed point problems in a real Hilbert space. In addition, we introduce a new inertial extrapolation method with regularization for approximating solutions of split variational inequality problems in the frame work of real Hilbert spaces. Lastly, we present some numerical experiments to show the efficiency and applicability of the proposed methods in comparison with other existing methods in the literature.

The findings of this chapter have been published / accepted in the following journals

1. **F. AKUTSAH**, and O. K. Narain, A new method with regularization for solving split variational inequality problems in real Hilbert spaces, *Aust. J. Math. Anal. Appl.,* **18** (2), (2021), 1–20.
2. **F. AKUTSAH**, A. A. Mebawondu, G. C. Ugwunnadi and O. K. Narain, Inertial extrapolation method with regularization for solving monotone bilevel variation inequalities and fixed point problems in real Hilbert space, *J. of Nonlin. Fun. Anal.,* (2022), 1–25.
3. **F. AKUTSAH**, A. A. Mebawondu, H. A. Abass, M. O. Aibinu and O. K. Narain, Inertial relaxed Tseng method for solving variational inequality problem in Hilbert spaces, *Advances in Mathematics: Scientific Journal* **10** (10), (2021), 3597–3623.

Chapter 4: In this Chapter, we introduce a new class of bilevel problem in the frame work of a real Hilbert spaces. In addition, we introduce a new inertial extrapolation method with regularization for approximating solutions of split bilevel variational inequality problems. Lastly, we present some numerical experiments to show the efficiency and applicability of our proposed methods in comparison with other iterative methods in the literature.

The findings of this chapter have been published/accepted in the following journals

1. **F. AKUTSAH**, A. A. Mebawondu, H. A. Abass and O. K. Narain, A self adaptive method for solving a class of bilevel variational inequalities with split variational inequlaity and composed fixed point problem constraints in Hilbert spaces, *Numer. Alg., Cont. and Opt.,* (2021), 1–22, DOI: 10.3934/naco.2021046.

Chapter 5: In this chapter, we introduce a shrinking algorithm for finding a solution of split monotone variational inclusion problem which is also a common fixed point problem of relative non-expansive mappings in uniformly convex real Banach spaces which are also uniformly smooth. More so, we apply our main result to the split convex minimization problem.

The findings of this chapter have been published in the following journal

1. **F. AKUTSAH**, H. A. Abass, A. A. Mebawondu and O. K. Narain, Shrinking approximation method for solution of split monotone variational inclusion and fixed point problems in Banach spaces, *Int. J. of Nonlin. Anal. and Appl.,* **12** (2), (2021), 825–842.

Chapter 6: In this chapter, we study split generalized mixed equilibrium problem and fixed point problem in real Hilbert spaces with a view to analyze an iterative method for approximating a common solution of split generalized mixed equilibrium problem and fixed point problem of an infinite family of a quasi-nonexpansive multi-valued mappings.

The findings of this chapter have been published in the following journal

1. **F. AKUTSAH**, H. A. Abass, A. A. Mebawondu and O. K. Narain, On split generalized mixed equilibrium and fixed point problems of infinite family of quasi-nonexpansive multi-valued mappings in real hilbert spaces, *Asain Eur. J. of Math.,* (2022), DOI:10.1142/S1793557122500826.

Chapter 7 In this chapter, we present a modified (improved) generalized M-iteration with the inertial technique for three quasi-nonexpansive multivalued mappings in a real Hilbert space. In addition, we present some fixed point results for a general class of nonexpansive mappings in the framework of the Banach space and also proposed a new iterative scheme for approximating the fixed point of this class of mappings in the framework of uniformly convex Banach spaces. Finally, we apply our convergence results to certain optimization problems, integral equations, and we present some numerical experiments to show the efficiency and applicability of the proposed method in comparison with other existing methods in the literature.

The findings of this chapter have been published/accepted in the following journals

1. **F. AKUTSAH**, O. K. Narain and J. K. Kim, On generalized (α, β)-nonexpansive mappings in Banach spaces with Application, *Nonlin. Fun. Anal. and Appl.,* **26** (4), 2021, 663–684.
2. **F. AKUTSAH**, O. K. Narain and J. K. Kim, Improve generalized M-iteration for quasinonexpansive multivalued mappings with application in real Hilbert spaces, *Nonlin. Fun. Anal. and Appl.,* **27**, (1), (2022), 59–82.

Chapter 8: In this chapter, we give the conclusion of our study and highlight the contributions of our study to existing knowledge. In addition, we also identify and discuss some possible future work.

Chapter 2

Preliminaries and Literature Review

In this chapter, we give definitions of concepts and discuss some important results that will be useful throughout this study. We also give detailed a literature review of previous works in line with the results considered in this study.

2.1 Preliminaries

In this section, we give definitions of concepts and discuss some important results that will be useful throughout this study.

2.1.1 Some Definitions and Important Results

In this section, we recall some basic definitions of functions, recall important lemmas and propositions that are relevant to the rest of this study.

Let H be a real Hilbert space. The set of fixed points of a nonlinear mapping $T : H \to H$ will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We denote strong and weak convergence by ”$\to$” and ”$\rightharpoonup$”, respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well-known that

Lemma 2.1.1. *Let H be a real Hilbert space. Then, for all $x, y \in H$ and $t \in \mathbb{R}$,*

1. $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2$;
2. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
3. $||tx + (1 - t)y||^2 = t||x||^2 + (1 - t)||y||^2 - t(1 - t)||x - y||^2$.

Lemma 2.1.2. *[97] Let H be a real Hilbert spaces, for $i \leq 1 \leq m$, $x_i \in H$ and $\alpha_i \in (0, 1)$*

such that $\sum\limits_{i=1}^{m} \alpha_i = 1$. *Then,*

$$\left\| \sum_{i=1}^{m} \alpha_i x_i \right\|^2 = \sum_{i=1}^{m} \alpha_i \left\| x_i \right\|^2 - \sum_{i,j=1,\ i\neq j}^{m} \alpha_i \alpha_j ||x_i - x_j||^2.$$

Definition 2.1.3. *[230, 314] Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. A mapping $T : C \to C$ is said to be demiclosed at 0, if for any sequence $\{x_n\} \subset C$ which converges weakly to x and $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tx = x$.*

Lemma 2.1.4. *[314]. Let C be a nonempty, closed and convex subset of a q-uniformly smooth real Banach space E which admits weakly continuous generalized duality mapping j_p from E into E^*. Let $T : C \to C$ be a mapping such that $F(T) \neq \emptyset$. Then, for all $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ as $n \to \infty$, then $x = Tx$.*

Definition 2.1.5. *[43]. Given two points $x, y \in \mathbb{R}^n$, we write $[x, y]$ for the line segments whose end points are x and y. This is a generalization of the notation $[a, b]$ used for the closed interval in $\mathbb{R}$ with end points a and b. The line segment $[x, y]$ has a convenient parametrization:*

$$[x, y] = \{\lambda x + (1-\lambda)y : \lambda \in [0, 1]\}.$$

The segments $(x, y]$, $[x, y)$ and (x, y) are defined analogously. A set $C \subset \mathbb{R}^n$ is called convex, if for $x, y \in C$, $[x, y] \subset C$. The sets $\emptyset$, singleton set $C = \{x\}$ and all of $\mathbb{R}^n$ are all convex sets.

Definition 2.1.6. *[43]. Let C be a nonempty, closed and convex subset of a real Hilbert H (resp, Banach space E) and $f : C \to \mathbb{R} \cup \{+\infty\}$ be a function.*

1. *The effective domain of f, denoted $domf$ is defined by*

$$domf := \{x \in H : f(x) < +\infty\};$$

2. *The epigraph of f, denoted $epif$ is defined by*

$$epif := \{(x, \alpha) \in H \times \mathbb{R} : f(x) < f(\alpha)\};$$

3. *f is called proper if the set $domf \neq \emptyset$;*
4. *f is said to be convex if for every $x, y \in H$ and $t \in (0, 1)$, we have*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y);$$

5. *f is lower semi-continuous at $x_0 \in domf$ if and only if*

$$f(x_0) \leq \liminf_{x\to x_0} f(x);$$

and f is upper semi-continuous at $x_0 \in domf$ if and only if

$$\liminf_{x\to x_0} f(x) \leq f(x_0).$$

We denote by *intdomf* the interior of the domain of f.

Definition 2.1.7. *[42]. Let $x \in intdomf$ and $y \in E$ be given, the right-hand derivative of f at x in the direction of y is evaluated as:*

$$f^{\circ}(x,y) := \lim_{t \to 0^+} \frac{f(x+ty)-f(y)}{t}. \tag{2.1.1}$$

The function f is said to be Gâteaux differentiable at x if the limit in (2.1.1) exists for any y. In this case, the gradient of f at x is the linear function $\nabla f(x)$ defined by $\langle y, \nabla f(x)\rangle := f^{\circ}(x,y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in intdomf$. When the limit in (2.1.1) is attained uniformly for any $y \in E$ with $||y|| = 1$ as t tends to zero, then we say that f is Fréchet differentiable at x. It is well known (see [42]) that f is Gâteaux (resp. Fréchet) differentiable at $x \in intdomf$ if and only if the gradient ∇f is norm-to-weak (resp. norm-to-norm) continuous at x.*

Definition 2.1.8. *Let f be a convex function. Then f is said to be subdifferentiable at a point $x \in E$ if the set*

$$\partial f(x) := \{w \in E : f(x) + \langle w, y-x\rangle \leq f(y), \forall\ y \in E\} \tag{2.1.2}$$

is nonempty. Each element $\partial f(x)$ is called a subgradient of f at x, $\partial f(x)$ is the subdifferential of f at x and (2.1.2) is called the subdifferential inequality. The function f is subdifferentiable on E, if f is subdifferentiable at each $x \in E$.

Proposition 2.1.9. *[108]. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function. Then*

- *(i) the function is subdifferentiable on intdomf,*
- *(ii) the function f is Gâteaux differentiable at $x \in intdomf$ if and only if its subgradient $\partial f(x) = \nabla f(x)$ is a singleton set.*

Definition 2.1.10. *[108]. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous function. The Fénchel conjugate of f is the convex function $f^* : E^* \to (-\infty, +\infty]$ given by*

$$f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x) : x \in E\}.$$

The function f is known to satisfy the Young-Fénchel inequality

$$\langle x^*, x\rangle \leq f^*(x^*) + f(x), \quad x \in E, \quad x^* \in E^*.$$

In the case where $x^* \in \partial f(x)$, then

$$\langle x^*, x\rangle = f^*(x^*) + f(x), \quad x \in E, \quad x^* \in E^*.$$

Definition 2.1.11. *[108]. The function f is called Legendre if it satisfies the following two conditions:*

1. *f is Gâteaux differentiable, $intdomf \neq \emptyset$ and $dom\nabla f = intdomf$,*

2. *f^* is Gâteaux differentiable, $intdomf^* \neq \emptyset$ and $dom\nabla f^* = intdomf^*$.*

Definition 2.1.12. *[43]. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. The proximal operator of f denoted $prox_f$ is defined by*

$$prox_f(x) = \arg\min_{y \in H}(f(y) + \frac{1}{2}||y - x||^2).$$

The proximal operator of the scaled function λf, where $\lambda > 0$, which is expressed as

$$prox_{\lambda f}(x) = \arg\min_{y}(f(y) + \frac{1}{2\lambda}||y - x||^2),$$

is called the proximal operator of f with order λ.

Proposition 2.1.13. *[43] Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinous function and $\lambda > 0$. Then, the following holds*

1. *For $x, p \in H$,*

$$p = prox_{\lambda f} \iff \langle p - y, p - x\rangle + f(p) \leq f(y), \quad \forall\ y \in H;$$

2. *$prox_{\lambda f}$ is firmly nonexpansive;*

3. *$F(prox_{\lambda f}) = \arg\min f$.*

Suppose f is the indicator function, that is

$$I_C(x) = \begin{cases} 0 & x \in C \\ +\infty & x \notin C, \end{cases} \tag{2.1.3}$$

where C is a closed nonempty convex set, the proximal operator reduces to the metric projection (details about metric projection will be seen in Section 2.2).

Definition 2.1.14. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. The mapping $T : C \to C$ is said to be:*

1. *nonexpansive, if*

$$||Tx - Ty|| \leq ||x - y||, \quad \forall\ x, y \in C,$$

2. *quasi nonexpansive, if $F(T) \neq \emptyset$ and*

$$||Tx - x^*|| \leq ||x - x^*||, \quad \forall\ x \in C \text{ and } x^* \in F(T),$$

3. *firmly nonexpansive, if*

$$||Tx - Ty|| \leq ||x - y||^2 - ||(x - y) - (Tx - Ty)||^2, \quad \forall\ x, y \in C,$$

   or alternatively

$$||Tx - Ty||^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall\ x, y \in C,$$

4. *k-strictly pseudo-contractive mapping if for $k \in (0,1)$, we have*

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \quad \forall\ x, y \in C$$

5. *k-demicontractive, if $F(T) \neq \emptyset$ and for $k \in (0,1)$, we have*

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + k||x - Tx||^2, \quad \forall\ x \in C \text{ and } x^* \in F(T),$$

or alternatively, T is k-demicontractive, if for all $x \in C$ and $x^* \in F(T)$

$$\langle x - Tx, x - x^* \rangle \geq \frac{1-k}{2}||x - Tx||^2,$$

6. *directed (also called to be firmly quasi-nonexpansive) if $F(T) \neq \emptyset$ and*

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - p\|^2 \quad \forall\ x \in H \quad \text{and} \quad p \in F(T).$$

Definition 2.1.15. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be*

1. *nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;*
2. *mean nonexpansive, if there exist $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ such that $\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$, for all $x, y \in C$;*
3. *satisfy condition (C), if $\frac{1}{2}\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$; for all $x, y \in C$;*
4. *satisfy condition (C_λ), if $\lambda\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;*
5. *generalized mean nonexpansive mapping if there exist $\alpha, \beta, \lambda \in [0,1)$, with $\alpha + \beta < 1$ such that for all $x, y \in C$, $\lambda\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$;*
6. *α-nonexpansive mapping if there exists $\alpha < 1$ such that for all $x, y \in C$, $\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2$;*

Definition 2.1.16. *Let $A : H \to H$ be a nonlinear mapping. Then A is called*

1. *monotone, if*

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall\ x, y \in H;$$

2. *α-strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle Ax - Ay, x - y \rangle \geq \alpha\|x - y\|^2, \quad \forall\ x, y \in H;$$

3. *β-inverse strongly monotone (β-ism, for short), if there exists a constant $\beta > 0$ such that*

$$\langle Ax - Ay, x - y \rangle \geq \beta||Ax - Ay||^2, \quad \forall\ x, y \in H;$$

4. *pseudomonotone, if*
$$\langle Ax, y-x\rangle \geq 0 \Rightarrow \langle Ay, y-x\rangle \geq 0 \quad \forall\ x,y \in H;$$
5. *quasi-monotone, if*
$$\langle Ay, x-y\rangle > 0 \Rightarrow \langle Ax, x-y\rangle \geq 0; \quad \forall\ x,y \in H;$$
6. *semistrictly quasi-monotone, if A is quasi-monotone and for all distinct points $x, y \in H$, we have*
$$\langle Ay, x-y\rangle > 0 \Rightarrow \langle Aw, x-y\rangle > 0,$$
for some $w \in (\frac{1}{2}(x+y), x)$.

Remark 2.1.17. *Every β-inverse strongly monotone A is $\frac{1}{\beta}$-Lipschitz , (i.e A is L=$\frac{1}{\beta}$-Lipschitz continuous).*

A subset $C \subset H$ is said to be proximinal if for each $x \in H$, there exists $y \in C$ such that
$$\|x-y\| = d(x,C) = \inf\{\|x-z\| : z \in C\}.$$
Let $CB(C), K(C)$ and $P(C)$ denote the families of nonempty closed bounded, compact and proximinal bounded subset of C, respectively. The Hausdorff metric on $CB(C)$ is defined by
$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$$
for all $A, B \in CB(C)$, where $d(x,B) = \inf\{\|x-b\|\}$.
A multivalued mapping $T : C \to CB(C)$is said to be nonexpansive if
$$H(Tx,Ty) \leq \|x-y\|$$
for all $x, y \in C$. If the fixed point of T is nonempty and
$$H(Tx,Tp) \leq \|x-p\|$$
for all $x \in C$ and $p \in F(T)$, then T is said to be quasinonexpansive mapping.
Condition (A). Let H be a Hilbert space and C be a subset of H. A multivalued mapping $T : C \to CB(C)$ is said to satisfy Condition (A) if $\|x-p\| = d(x,Tp)$ for all $x \in H$ and $p \in F(T)$.

Lemma 2.1.18. *[101] Let H be a real Hilbert space. Let $T : H \to CB(H)$ be a quasi-nonexpansive mapping with $F(T) = \emptyset$. Then, $F(T)$ is closed, and if T satisfies Condition (A), then $F(T)$ is convex.*

Lemma 2.1.19. *[311] Let X be a Banach space satisfying Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{n_m}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.1.20. *[101] Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a hybrid multivalued mapping. Let $\{x_n\}$ be a sequence in C, such that $x_n \rightharpoonup x^*$ and $\lim\limits_{n\to\infty} \|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then, $x^* \in Tx^*$.*

Lemma 2.1.21. *[101] Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a hybrid multivalued mapping with $F(T) \neq \emptyset$ and then $F(T)$ is closed.*

Lemma 2.1.22. *[222] Let C be a nonempty closed convex subset of a real Hilbert space H. For each $x, y \in H$ and $b \in R$, the set*

$$C = \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + b\},$$

is closed and convex.

Lemma 2.1.23. *Consider VI(A,C) with C being a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ being a pseudomonotone and continuous operator. Then $x^* \in VI(A, C)$ if and only if*

$$\langle Ax, x - x^*\rangle \geq 0$$

for all $x \in C$.

Definition 2.1.24. *Let $B : H \to 2^H$ be a multivalued mapping. The effective domain of B, $domB = \{x \in H : Bx \neq \emptyset\}$. A multivalued mapping B is said to be monotone on H if*

$$\langle x^* - y^*, x - y\rangle \geq 0, \quad \forall\ x, y \in domB,\ x^* \in Bx \text{ and } y^* \in By.$$

A monotone operator B is said to be maximal if its graph, that is $Gr(B) = \{(x, x^*) : x \in domB$ and $x^* \in Bx\}$ is not properly contained in the graph of any monotone mapping. In other word, B is maximal if and only if $(x, x^*) \in H \times H$, $\langle x^* - y^*, x - y\rangle \geq 0$ for every $(y, y^*) \in Gr(B)$ implies $x^* \in Bx$. Also, B is maximal monotone if and only if the range of $I + B$ is the whole of H. That is $ran(I + B) = H$ where I is the identity operator on H.

Lemma 2.1.25. *[354] Let $T : H \to H$ be an operator. Then the following statements are equivalent:*

1. *T is directed;*
2. *there holds the relation*

$$\|x - Tx\|^2 \leq \langle x - p, x - Tx\rangle, \quad \forall p \in F(T),\ x \in H; \tag{2.1.4}$$

3. *there holds the relation*

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \|x - Tx\|^2, \quad \forall p \in F(T),\ x \in H. \tag{2.1.5}$$

Lemma 2.1.26. *[22] Let H be a real Hilbert space and $F : H \to H$ a β-strongly monotone and L-Lipschitz continuous mapping on H with $L, \beta > 0$. If $\alpha \in (0, 1), \eta \in [0, 1 - \alpha]$ and $\mu \in (0, \frac{2\beta}{L^2})$, then for all $x, y \in H$, we have*

$$\|[(1 - \eta)x - \alpha\mu F(x)] - [(1 - \eta)y - \alpha\mu F(y)] \leq (1 - \eta - \alpha\tau)\|x - y\|,$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0, 1)$.

Definition 2.1.27. *[55]. A mapping $T : H \to H$ is said to be an averaged mapping if and only if T can be written as the average of the identity and a nonexpansive mapping, that is*

$$T = (1 - \alpha)I + \alpha S,$$

where $\alpha \in [0, 1]$ and $S : H \to H$ is a nonexpansive mapping. It is known that firmly nonexpansive mappings are $\frac{1}{2}$-averaged. The identity mapping I on H is also averaged for every $\alpha \in [0, 1]$.

Lemma 2.1.28. *Let C be a closed and convex subset of a Hilbert space H and $T : C \to C$ be nonexpansive mapping with $F(T) \neq \emptyset$. Then T is demiclosed at 0.*

Lemma 2.1.29. *[55]. Let the operators $S, T, U : H \to H$ be given.*

- *(i) Let $T = (1 - \alpha)S + \alpha U$ for some $\alpha \in [0, 1]$, if S is averaged and U is nonexpansive, then T is averaged.*
- *(ii) T is firmly nonexpansive, if and only if the complement $I - T$ is firmly nonexpansive.*
- *(iii) Let $T = (1 - \alpha)S + \alpha U$ for some $\alpha \in [0, 1]$, if S is firmly nonexpansive and U is nonexpansive, then T is averaged.*
- *(iv) If the mappings $\{T_i\}_i^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} F(T_i) = F(T_1 \cdots T_N).$$

Lemma 2.1.30. *[55, 224] Let $T : H \to H$ be a mapping. Then*

1. *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*
2. *If T is β-ism, then for $\gamma > 0$, γT is $\frac{\beta}{\gamma}$-ism.*
3. *T is averaged if and only if the complement $I - T$ is β-ism for some $\beta > \frac{1}{2}$. Indeed, for $\alpha \in [0, 1]$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*
4. *The composite of finitely many averaged mapping is averaged. That is, if each of the mappings $\{T_i\}_i^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_i is α_i averaged for $i = 1, 2$ with $\alpha_i \in [0, 1]$, then $T_1 \cdot T_2$ is α-averaged where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

Definition 2.1.31. *A bifunction $f : C \times C \to \mathbb{R}$ is called*

1. *strongly monotone on C, if there exists a constant $\gamma > 0$ such that*

$$f(x, y) + f(y, x) \leq -\gamma||x - y||^2, \quad \forall\ x, y \in C;$$

2. *monotone on C, if*

$$f(x, y) + f(y, x) \leq 0, \quad \forall\ x, y \in C;$$

3. *strongly pseudomonotone on C if there exists a constant $\alpha > 0$ such that*

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq -\gamma ||x-y||^2, \quad \forall\ x,y \in C;$$

4. *pseudomonotone on C, if*

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \quad \forall\ x,y \in C;$$

5. *the bifunction f is said to satisfy the Lipschitz-type condition, if there exist constants $\alpha, \beta > 0$ such that*

$$f(x,y) + f(y,z) \geq f(x,z) - \alpha ||x-y||^2 - \beta ||y-z||^2, \quad \forall\ x,y,z \in C.$$

Lemma 2.1.32. *[208].Let H be a real Hilbert space. Let $B : H \to 2^H$ be a maximal monotone operator and $A :\to H$ be an α-inverse strongly monotone mapping on H. Define $T_r := (I + rA)(I - rB)x$, $r > 0$, then*

$$F(T_r) = (A+B)^{-1}(0),$$

where $F(T_r)$ is the set of fixed points of T_r. The mapping T_r is nonexpansive and $F(T_r)$ is closed and convex.

Lemma 2.1.33. *[105]. Let H be a real Hilbert space and $B : H \to H$ be a multivalued maximal monotone mapping. For each $x \in H$, $\lambda > 0$ and $J_\lambda^B(x) = (I + \lambda B)^{-1}(x)$, then*

1. *J_λ^B is single-valued and firmly nonexpansive;*
2. *$dom(J_\lambda^B) = H$ and $F(J_\lambda^B) = \{x \in H : \ 0 \in Bx\}$;*
3. *$||x - J_\lambda^B x|| \leq ||x - J_\gamma^B||$ for all $0 \leq \lambda < \gamma$, $x \in H$;*
4. *suppose $B^{-1}(0) \neq \emptyset$. Then*

$$||x - J_\lambda^B x||^2 + ||J_\lambda^B x - y||^2 \leq ||x-y||^2,$$

for each $x \in H$ and $y \in B^{-1}(0)$;

5. *suppose $B^{-1}(0) \neq \emptyset$. Then $\langle x - J_\lambda^B x, J_\lambda^B x - y \rangle \geq 0$ for each $x \in H$ and $y \in B^{-1}(0)$.*

The normal cone N_C to C at a point $x \in C$ is defined by

$$N_C(x) = \{v \in E^* : \langle v, y - x \rangle \leq 0, \forall\ y \in C\}.$$

Lemma 2.1.34. *[343]. Let C be a nonempty closed and convex subset of a real Banach space E. Let $g : E \to (-\infty, \infty]$ be a Gáteaux differentiable and lower semicontinuous function on C. Then, $\hat{x}$ is a solution to the following convex problem*

$$\min\{g(x) : x \in C\}$$

if and only if $0 \in \partial g(\hat{x}) + N_C(\hat{x})$ where $\partial g(\cdot)$ is the subdifferential of g and $N_C(\cdot)$ is the normal cone of C at $\hat{x}$.

The following assumptions will be used in this thesis for approximating the solution of an Equilibrium Problem (EP):

Assumption A: The bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

1. $F(x, x) = 0$ for all $x \in C$;
2. F is monotone, i.e, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
3. $\limsup_{t \downarrow 0} F(x + t(z - x), y) \leq F(x, y), \quad \forall\ x, y, z \in C$;
4. the function $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Lemma 2.1.35. *[289] Let C be a nonempty, closed and convex subset of a real Banach space. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (1)-(4). For all $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall\ y \in C.$$

The mapping $T_\lambda : E \to 2^C$ defined by

$$T_\lambda^F(x) = F(z, y) + \frac{1}{\lambda}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall\ y \in C.$$

is called the resolvent of F and has the following properties:

1. *T_λ^F is single-valued,*
2. *T_λ^F is firmly nonexpansive-type, i.e*
$$\langle T_\lambda^F z - T_\lambda^F y, JT_\lambda^F z - JT_\lambda^F y\rangle \leq \langle T_\lambda^F z - T_\lambda^F y, Jz - Jy\rangle, \quad \forall\ z, y \in E,$$
3. *$F(T_\lambda^F) = EP(F)$,*
4. *$EP(F)$ is closed and convex.*

Lemma 2.1.36. *[365]. Let C be a nonempty, closed and convex subset of a real Banach space. For each $i \leq 1 \leq m$, let $T_i : C \to C$ be a λ_i-strict pseudocontraction for some $0 \leq \lambda_i \leq 1$. Assume $\{\eta_i\}$ is a sequence of positive numbers such that $\sum_{i=1}^{m} \eta_i = 1$ Then $\sum_{i=1}^{m} \eta_i T_i$ is a λ_i-strict pseudocontractive with $\lambda = \min\{\lambda_i : i \leq 1 \leq m\}$. If in addition $\{T_i\}_{i=1}^m$ has a common fixed point, then*

$$F\left(\sum_{i=1}^{m} \eta_i T_i\right) = \bigcap_{i=1}^{m} F(T_i).$$

Lemma 2.1.37. *[365]. Assume $\{a_n\}$ is a sequence of positive real numbers in $(0,1)$ such that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, and d_n be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n d_n, \quad n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$, for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim\limits_{n\to 0} a_n = 0$.

Lemma 2.1.38. *[367]. Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 1,$$

where

1. $\{\alpha_n\} \subset (0,1]$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,
2. $\limsup\limits_{n\to\infty} \sigma_n \leq 0$,
3. $\gamma_n \geq 0$, $(n \geq 1)$ *and* $\sum\limits_{n=1}^{\infty} \gamma_n < \infty$.

Then, $\lim\limits_{n\to 0} a_n = 0$.

Lemma 2.1.39. *[265, Lemma 3.1]. Let $\{\Gamma_n\}$ be a sequence of real numbers such that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ with $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n : \Gamma_n < \Gamma_{n+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a non-decreasing sequence verifying $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$ the following estimates hold:

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \text{ and } \Gamma_n \leq \Gamma_{n+1}.$$

Lemma 2.1.40. *[20] Let $\{a_n\}, \{\delta_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$, such that*

$$a_{n+1} \leq a_n + \beta_n(a_n - a_{n-1}) + \delta_n$$

for all $n \in \mathbb{N}, \sum_{n=1}^{\infty} \delta_n < \infty$ and there exists a real number β with $0 \leq \beta_n <$ for all $n \in \mathbb{N}$. Then the following hold:

1. *there exists $a^* \in [0,\infty)$ such that $\lim_{n\to\infty} a_n = a^*$;*
2. $\sum_{n\in\mathbb{N}}(a_n - a_{n-1}) < \infty$, *where* $[t]_+ = \max\{t, 0\}$.

Lemma 2.1.41. *[366] Let X be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty}\|x_n\| \leq c, \limsup_{n\to\infty}\|y_n\| \leq c$ and $\lim_{n\to\infty}\|t_nx_n + (1-t_n)y_n\| = c$ holds for some $c \geq 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

2.2 Metric Projection

Let C be a nonempty, closed and convex subset of H. For every $x \in H$, there exists the unique nearest point in C denoted by $P_C x$ such that

$$||x - P_C x|| \leq ||x - y||, \quad \forall\, y \in C.$$

The operator P_C is called the metric projection of H onto C. We present some examples of the metric projection. A more detailed example can be found in [74, 145].

1. Let C be a close ball, that is $C := \{x \in H : ||x - a|| < r\}$ centred at a with radius $r > 0$, then
$$P_C x = \begin{cases} a + \dfrac{r(x-a)}{||x-a||}, & \text{if } \ x \notin C, \\ x, & \text{if } \ x \in C. \end{cases}$$
2. Let $C = [a, b]$ be a closed ball in rectangle in $\mathbb{R}^n$ with $a = (\{a_i\}_i^n)^T$ and $b = (\{b_i\}_i^n)^T$, then for $1 \leq i \leq n$, $P_C x$ has the i^{th} coordinate given by
$$(P_C x)_i = \begin{cases} a_i, & \text{if } \ x_i \leq a_i, \\ x_i, & \text{if } \ x_i \in [a_i, b_i], \\ b_i, & \text{if } \ x_i > b_i. \end{cases}$$
3. Let $C = \{y \in H : \langle a, y\rangle = \alpha\}$, that is C a hyperplane with $a \neq 0$ and $\alpha \in \mathbb{R}$, then
$$P_C x = x - \frac{\langle a, x\rangle - \alpha}{||a||^2} a.$$
4. Let C be the closed halfspace, that is $C = \{y \in H : \langle a, y\rangle \leq \alpha\}$ with $a \neq 0$ and $\alpha \in \mathbb{R}$, then
$$P_C x = \begin{cases} x - \dfrac{x - \langle a, x\rangle}{||a||^2}, & \text{if } \ \langle a, x\rangle > \alpha, \\ x, & \text{if } \ \langle a, x\rangle \leq \alpha. \end{cases}$$
5. Let C be the range of a $m \times n$ matrix A with full column rank, then $P_C x = A(A^*A)^{-1}A^*x$ where A^* is the adjoint of A.

Proposition 2.2.1. *[145]. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $x \in H$. Then,*

1. $||P_C x - P_C y||^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall\, y \in H.$
2. $||x - P_C y||^2 \leq ||x - y||^2 - ||y - P_C y||^2, \quad \forall\, y \in C.$
3. $\|(I - P_C)x - (I - P_C)y\|^2 \leq \langle (I - P_C)x - (I - P_C)y, x - y\rangle \ \forall\, x \in H$ *and* $y \in C$.
4. $\langle x - P_C x, y - P_C x\rangle \leq 0, \ \forall\, x \in H$ *and* $y \in C$.
5. $z = P_C x \iff \langle x - z, y - z\rangle \leq 0, \quad \forall\, y \in C.$

2.2.1 Generalized Projection

Definition 2.2.2. *[16, 157]. The generalized projection is the map $\Pi_C : E \to C$ that assigns to an arbitrary point $x \in E$, the minimum point of the functional $\phi(x, y)$. That is $z = \Pi_C x$, where z is the solution of the minimization problem*

$$\phi(x, z) = \arg\min_{y \in C} \phi(x, y). \tag{2.2.1}$$

It is easy to see that $\Pi_C(x) \subset C \subset E$ and in Hilbert space $\Pi_C \equiv P_C$ (the metric projection defined in 2.2). The following are some well-known characteristics of the generalized projection Π_C :

1. *The operator Π_C is the identity on C.*
2. *Π_C is a d-accretive operator in E i.e,*

$$\langle \Pi_C x - \Pi_C y, x - y \rangle \geq 0, \quad \forall\ x, y \in C.$$

3. *The operator Π_C produces a best approximation of $x \in E$ relative to the functional $\phi(x, y)$, that is*

$$\phi(\Pi_C x, y) \leq \phi(x, y) - \phi(x, \Pi_C x), \quad \forall\ x \in C. \tag{2.2.2}$$

4. *The generalized projection Π_C is also characterized by the variational inequality:*

$$z = \Pi_C x \iff \langle z - y, Jz - Jx \rangle \leq 0, \quad x \in E, \ \forall\ y \in C. \tag{2.2.3}$$

We also use the functional $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^* \rangle + ||x^*||^2, \quad \forall\ x \in E \text{ and } x^* \in E^*. \tag{2.2.4}$$

Then, it is known that $V(y, x) = \phi(y, J^{-1}x)$ for all $x \in E$ and $y \in E^*$. The next lemma describes a property of the functional $V(\cdot, \cdot)$.

Lemma 2.2.3. *[286] $V(x, x^*) + 2\langle J^{-1}x^* - x, y \rangle \leq V(x, x^* + y^*), \quad \forall\ x \in E,\ x^*, y^* \in E^*$.*

Lemma 2.2.4. *[369] Let E be a 2-uniformly convex and smooth Banach space. Then, for all $x, y \in E$, we have*

$$||x - y|| \leq \frac{2}{c^2}||Jx - Jy|| \tag{2.2.5}$$

where $\frac{1}{c}$, $c \in (0, 1]$ is the 2-uniformly convex constant of E.

2.3 Some Geometric Properties of Banach Space

The content of this section was taken from the monographs of [95]. For further reading and details, we refer the reader to [95]. We recall that: A Banach space is a complete normed vector space.

2.3.1 Uniformly Convex Spaces

Let E be an arbitrary normed space. For a fixed $x \in E$, denote by S_r the sphere centred at x with radius $r > 0$, that is,

$$S_r := \{y \in E : ||y - x|| = r\}.$$

A normed space E is said to be uniform convex if for any $\epsilon \in (0, 2]$ there exists $\delta(\epsilon) > 0$ such that for any $x, y \in E$ with $||x|| = ||y|| = 1$ and $||x-y|| \geq \epsilon$, then $||x+y|| \leq 2(1-\delta(\epsilon))$. We note that in some textbooks, $||x|| \leq 1$ and $||y|| \leq 1$ are also used when defining uniform convexity.

Let E be of dimension greater than 2, that is $(dimE \geq 2)$. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ given by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}||x + y|| : ||x|| = ||y|| = 1, \quad ||x - y|| = \epsilon\}.$$

The space E is said to be uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$ and p-uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$. The space E is 2-uniformly convex if there exists $c > 0$ such that $\delta_E(\epsilon) > c\epsilon^2$ for all $\epsilon \in (0, 2]$. It is known that every 2-uniformly convex Banach space is uniformly convex.

Lemma 2.3.1. *[365]. Let E be a uniformly convex real Banach space. Let $r > 0$, then there exists a strictly continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$, then*

$$||\lambda x + (1 - \lambda)y||^2 \leq \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)g(||x - y||), \tag{2.3.1}$$

for all $x, y \in B_r$ where $B_r = \{w \in E, \ ||w|| \leq r\}$ and $\lambda \in [0, 1]$.

Lemma 2.3.2. *[164] Let E be a uniformly convex Banach space, $r > 0$ be positive number and $B_r(0)$ be a closed ball of E. Then, for any given $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and a given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive number with $\sum\limits_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous strictly increasing and convex function $g : [0, 2r] \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in \mathbb{N}$, with $i < j$*

$$||\sum_{n=1}^{\infty} \lambda_n x_n||^2 \leq \sum_{n=1}^{\infty} \lambda_n||x_n||^2 - \lambda_i\lambda_j g(||x_i - x_j||).$$

A Banach space E is called strictly convex if for every $x \neq y \in E$ and for all $t \in (0, 1)$, then

$$||tx + (1 - t)y|| < 1.$$

Theorem 2.3.3. *[95] Every uniformly convex Banach space is strictly convex.*

Example 2.3.4. *[95] The space l_1 is not strictly convex. Indeed, let $\epsilon = 1$ and choose $\bar{x} = (1, 0, 0, 0, \cdots)$, $\bar{y} = (0, -1, 0, 0, \cdots)$. Clearly $\bar{x}, \bar{y} \in l_1$ and $||x||_{l_1} = ||y||_{l_1}$, $||\bar{x} - \bar{y}||_{l_1} = 2 > \epsilon$. However, $||\bar{x} + \bar{y}|| = 2$, hence l_1 is not convex.*

2.3.2 Smooth Banach Spaces

Definition 2.3.5. *A Banach space E is called smooth if for every $x \in E$, with $||x|| = 1$, there exists a unique $x^* \in E^*$ such that $||x^*|| = 1$ and $\langle x, x^* \rangle = ||x||$. Also, the space E is said to be smooth if*

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} \tag{2.3.2}$$

exists for all $x, y \in S_r$. The space E is said to be uniformly smooth if (2.3.2) converges uniformly in $x, y \in S_r$.

Let E be a Banach space with $dim E \geq 2$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\begin{aligned} \rho_E(t) &:= \sup \left\{ \frac{||x + y|| - ||x - y||}{2} - 1 : \quad ||x|| = 1; \; ||y|| = t \right\} \\ &= \sup \left\{ \frac{||x + ty|| - ||x - ty||}{2} - 1 : \quad ||x|| = ||y|| = 1 \right\}. \end{aligned}$$

It is obvious that $\rho_E(0) = 0$. A Banach space is uniformly smooth q-uniformly smooth if there exists $c_q > 0$ such that $\rho_E(t) \leq c_q t^q$ for any $t > 0$. When $q = 2$, then E is called 2-uniformly smooth. It is established in [370] that there is no Banach space which is q-uniformly smooth with $q > 2$. Clearly, a q-uniformly smooth Banach space must be uniformly smooth. Hilbert space, L_p (or l_p) spaces, $1 < p < \infty$ and the Sobolev spaces, N_m^p, $1 < p < \infty$, are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while L_p (or l_p) or

$$N_m^p \text{ is } \begin{cases} \text{p-uniformly smooth if } 1 < p \leq 2 \\ \text{2-uniformly smooth if } p \geq 2 \end{cases}. \tag{2.3.3}$$

Proposition 2.3.6. *[95] For every Banach space E, the modulus of smoothness, ρ_E is a convex and continuous function.*

Lemma 2.3.7. *[36] Let $\frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 1$. The space E is q-uniformly smooth if and only if its dual E^* is p-uniformly convex.*

Lemma 2.3.8. *[164] Let E be a real Banach space. The following are equivalent:*

1. *E is 2-uniformly smooth;*
2. *there exists a constant $d > 0$ such that for all $x, y \in E$*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, Jx \rangle + 2||dy||^2, \tag{2.3.4}$$

where d is the 2-uniform smoothness constant.

2.4 Duality Mappings on Banach Space

Let E be a Banach space. The space of all linear continuous functions on E denoted E^* is referred to as the dual space of E. For $x^* \in E^*$ and $x \in E$, the value of x^* is denoted by $\langle x, x^* \rangle$. Here are some of the properties of the dual space E^*.

1. The dual space E^* is a Banach space with respect to the norm
$$||x^*||_{E^*} = sup\{\langle x, x^* \rangle : ||x|| \leq 1\}.$$
2. The dual space of E^* is E^{**}, the bidual of E. Since, in general $E \subset E^{**}$, we say that E is reflexive if $E = E^{**}$.
3. The concept of uniformly convex and strictly convex Banach spaces coincides in finite dimensional spaces.

2.4.1 Duality Pairing

Definition 2.4.1. *1. A convex and strictly continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = 0$ and $\lim\limits_{t \to 0^+} \varphi(t) = +\infty$ is called a weight function.*

2. The duality mapping of weight φ is the mapping $J_\varphi : E \to 2^{E^*}$ is defined by
$$J_\varphi(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||||x^*||,\ ||x^*|| = \varphi(||x||)\}.$$

Theorem 2.4.2. *[108]. If J_φ is a duality mapping of weight, then $J_\varphi(x) = \partial(||x||)$ for each $x \in E$.*

Let E be a Banach space with dual space E^*, for $p > 1$ define the weight function φ by $\varphi(t) = t^{p-1}$, then the generalized duality mapping is the set-valued mapping J_p defined by
$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||||x^*||,\ ||x^*|| = ||x||^{p-1}\}.$$
In particular, if $p = 2$, $J_2 = J$ is called normalized duality mapping.

1. If E is smooth, then J is single-valued denoted by j.
2. If E is strictly convex, then J is one-to-one and strictly monotone, i.e
$$\langle x - y, Jx - Jy \rangle \geq 0, \quad \forall\ x, y \in E. \tag{2.4.1}$$
3. If E is reflexive, then J is surjective.
4. If E is uniformly smooth, then J is a norm-to-norm continuous bounded subset of E.
5. If E^* is uniformly convex, then J is single-valued, one-to-one and uniformly continuous on bounded subsets of E.
6. If E is reflexive and strictly convex, then J^{-1} is norm-to-weak continuous.

2.5 Some Functions and Projections in Banach spaces

2.5.1 Lyapunnov Functional

Let E be a Banach space with dual E^*. The Lyapunnov functional $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x,y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \quad \forall\ x,y \in E. \tag{2.5.1}$$

Observe that $\phi(x,y) = ||x-y||^2$ in a real Hilbert space for all $x,y \in H$.

1. $\phi(x,y) = \phi(x,z) + \phi(y,z) + 2\langle x-z, Jy - Jz\rangle \ \ \forall\ x,y,z \in E$.
2. $\phi(x, J^{-1}(\alpha Jy + (1-\alpha)Jz)) \leq \alpha\phi(x,y) + (1-\alpha)\phi(x,z), \ \ \forall\ x,y,z \in E$ and $\alpha \in (0,1)$.
3. $\phi(x,y) \leq ||x||||Jx - Jy|| + ||y||||x-y||, \ \ \forall\ x,y \in E$.
4. $\phi(x,y) + \phi(y,x) = \langle x-y, Jx - Jy\rangle, \ \ \forall\ x,y \in E$.

For more details about the Lyapunnov functional see [16].

Lemma 2.5.1. *[179]. Suppose that E is 2-uniformly convex Banach space. Then, there exists $\nu \geq 1$ such that*

$$\phi(x,y) \geq \frac{1}{\nu}||x-y||^2, \ \forall\ x,y \in E.$$

Lemma 2.5.2. *[177]. Let E be a uniformly convex and uniformly smooth real Banach space. Let $\{x_n\}$ and $\{y_n\}$ be sequence sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $||x_n - y_n|| \to 0$ as $n \to \infty$.*

Lemma 2.5.3. *[389]. Let X be a smooth and uniformly convex real Banach space, let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in X. If $||x_n - y_n|| \to 0$ as $n \to \infty$, then $\phi(x_n, y_n) \to 0$ as $n \to \infty$.*

Definition 2.5.4. *[389]. Let C be a nonempty, closed and convex subset of a Banach space E. A mapping $T : C \to C$ is said to be:*

1. *firmly nonexpansive type mapping, if for all $x,y \in C$,*

   $$\phi(Tx,Ty) + \phi(Ty,Tx) \leq \phi(Tx,y) + \phi(Ty,x) - \phi(Tx,x) - \phi(Ty,y)$$

   or equivalently

   $$\langle Tx - Ty, Jx - JTx - (Jy - JTy)\rangle \geq 0; \tag{2.5.2}$$

2. *relatively nonexpansive, if $F(T) \neq \emptyset$ and $\hat{F}(T) = F(T)$ such that*

   $$\phi(p, Tx) \leq \phi(p,x), \quad \forall x \in C, p \in F(T);$$

3. *ϕ-nonexpansive if $\phi(Tx, Ty) \leq \phi(x, y)$ for all $x, y \in C$ and quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;*

4. *relatively asymptotically nonexpansive, if $F(T) \neq \emptyset$ with $\hat{F}(T) = F(T)$ and there exists a sequence $k_n \subset [1, +\infty)$ such that $k_n \to 1$ as $n \to \infty$ and*

$$\phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall\ x \in C, p \in F(T);$$

5. *quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \emptyset$ and there exists $k_n \subset [1, +\infty)$ with $k_n \to 1$ as $n \to \infty$ such that*

$$\phi(p, T^n x) \leq k_n \phi(p, x), \quad \forall\ x \in C, p \in F(T);$$

6. *totally quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n, \mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(0) = 0$ such that*

$$\phi(p, T^n x) \leq \phi(p, x) + \nu_n \psi(\phi(p, x)) + \mu_n, \quad \forall\ n \geq 1, \quad \forall x \in C, \quad p \in F(T).$$

Definition 2.5.5. *[165, 328] A mapping $T : C \to C$ is said to be:*

(i) nonspreading, if $\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$ for $x, y \in C$;

(ii) hybrid, if $2\phi(Tx, Ty) + \phi(Ty, x) \leq \phi(x, y) + \phi(Tx, y) + \phi(Ty, x)$;

(iii) (α, β)-generalized hybrid, if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y)$$

for all $x, y \in C$.

2.6 Literature Review

In this section, we review some recent and important past works on variational inequality problems, minimization problems, monotone inclusion problems, equilibrium problems and fixed point problems.

2.6.1 Variational Inequalities Problems and Bilevel Variational Problems

As mentioned in Chapter 1 of this work, the importance of VIP (1.2.1), SVIP (1.2.2)-(1.2.3) and SFP (1.2.4) can not be overemphasized in the study of fixed point theory and its application. Due to their fruitful applications in almost all areas of human endeavours, researchers have developed a series of iterative methods for approximating a solution of the aforementioned problems in different abstract spaces. For instance, Ceng et al. [76] proposed the following iterative method for solving the SFP:

$$\begin{cases} x_0 = x \in C \\ y_n = (1-\beta_n)x_n + \beta_n P_C(x_n - \lambda \nabla f_{\alpha_n}(x_n)) \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n) S P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)), \quad n \in \mathbb{N}, \end{cases} \tag{2.6.1}$$

where $\nabla f_{\alpha_n} = \alpha_n I + T^*(I - P_Q)T$, $S : C \to C$ is a nonexpansive mapping and the sequences of parameters $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$. The above iterative algorithm is a combination of the regularization method and extragradient method due to Nadezhkina and Takahashi [244]. Under some mild assumptions, they established that the sequence generated by the iterative method converges weakly to a common solution of the SFP and fixed point problem for nonexpansive mapping.

In 2020, Chuasuk and Kaewcharoen [106] proposed the following iterative scheme:

$$\begin{cases} x_0 = H_1 \\ y_n = P_C(x_n - \lambda_n(T^*(I - SP_Q))T + \alpha_n I)x_n) \\ z_n = P_C(x_n - \lambda_n(T^*(I - SP_Q))T + \alpha_n I)y_n) \\ w_n = (1-\sigma_n)z_n + \sigma_n U z_n \\ s_n = (1-\beta_n)z_n + \beta_n U w_n \\ x_{n+1} = (1-\gamma_n)z_n + \gamma_n U s_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.6.2}$$

where $S : Q \to Q$ is a nonexpansive mapping, $U : C \to C$ is a pseudo-contractive an L-Lipschitzian continuous mapping and the sequences of parameters $\{\sigma_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$. Under some mild assumptions, they established that the sequence generated by the iterative method converges weakly to a common solution of the SFP and a fixed point problem for nonexpansive mapping. The above iterative scheme is the combination of an extragradient method with regularization due to a generalized Ishikawa iterative scheme.

As already mentioned, Algorithms (2.6.1) and (2.6.2) are regularization-type methods with the regularization steps involving $\alpha I + T^*(I - P_Q)T$. Regularization-type methods have been employed in a number of problems, mainly due to its efficiency in solving these problems. For example, let $f : H \to \mathbb{R}$ be a continuous differentiable function, then the minimization problem

$$\min_{x \in C} f(x) := \frac{1}{2}\|Tx - P_Q Tx\|^2$$

is ill-posed (see [164]). To address this problem, Xu [164] considered the following Tikhonov regularized problem:

$$\min_{x \in C} f_\alpha(x) := \frac{1}{2}\|Tx - P_Q Tx\|^2 + \frac{1}{2}\alpha\|x\|,$$

where $\alpha > 0$ is the regularization parameter.

Remark 2.6.1. *Thus, the traditional Tikhonov regularization methods are usually used to solve ill-posed optimization problems. Moreover, one of the advantages of regularization methods is their possible strong convergence to minimum-norm solutions to optimization problems (see [78, 76, 77, 164] and the references therein).*

Remark 2.6.2. *It is natural to ask the following questions:*

Question 1: *If a more effective algorithm can be constructed to approximate VIP* (1.2.1)*?*

Question 2: *If Algorithms* (2.6.1) *and* (2.6.2) *can be modified to converge strongly to a minimum-norm solution of the SVIP* (1.2.2)-(1.2.3)*?*

Question 3: *If a new type of iterative method can be developed to approximate the SVIP* (1.2.2)-(1.2.3)*?*

Remark 2.6.3. *In the course of this thesis, we will provide an affirmative answer to the questions raised above. This will be done in Chapter 3 of this study.*

Another interesting generalization of VIP (1.2.1) is Split Variational Inequality and Fixed Point Problem (SVIFPP). That is:

Find

$$x^* \in C \quad \text{that solves} \;\; \langle A_1x^*, x - x^*\rangle \geq 0 \;\; \forall\, x \in C \tag{2.6.3}$$

such that

$$Ax^* \in F(T),$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A_1 : H_1 \to H_1$, $T : H_2 \to H_2$ are two operators and $A : H_1 \to H_2$ is a bounded linear operator.

The fixed point problem finds application in proving the existence of a solution to many nonlinear problems arising in many real-life situations. From the existence of solution of differential, partial differential, integral, random differential and random integral equations, and evolutionary equations. For details about fixed point problem see [156, 142, 295, 383] and the reference therein. Furthermore, a common solution of a VIP (1.2.1) and a fixed point problem find applications in real-life problems like network resource allocation, image recovery, signal processing, for further details, the reader should see, for instance,[85, 74, 84, 86, 340, 371] and the references therein.

Mainge [218] proposed and study a new type of optimization.

Find

$$x^* \in VI(A_1, C) \cap F(T) \;\; \text{such that} \quad \langle A_2x^*, x - x^*\rangle \geq 0, \; \forall\, x \in VI(A_1, C) \cap F(T), \tag{2.6.4}$$

where $A_1 : H \to H$ is monotone and L-Lipschitz continuous, $A_2 : H \to H$ is η-strongly monotone and k-Lispchitz continuous and $T : H \to H$ is a γ-demicontractive mapping and demiclosed at zero. He proposed the following iterative algorithm

$$\begin{cases} x_0 = H_1 \\ y_n = P_C(x_n - \lambda_n A_1 x_n) \\ z_n = P_C(x_n - \lambda_n A_1 y_n) \\ t_n = z_n - \alpha_n A_2(z_n) \\ x_{n+1} = (1-\omega)t_n + \omega T(t_n), \quad n \in \mathbb{N}, \end{cases} \tag{2.6.5}$$

where $\lambda_n \subset [a,b] \subset (0, \frac{1}{L})$, $\{\alpha_n\} \subset (0,1)$, $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ and $\omega \in (0, \frac{1-\gamma}{2})$. He established that the sequence generated by algorithm (2.6.5) converges strongly to the solution set.

Remark 2.6.4. *It is easy to see that in Algorithm (2.6.5), the projection P_C onto feasible set C is evaluated two times in each iteration and this may have adverse effect on the performance of the algorithm. In addition, the value of the Lipschitz constant is required which is very difficult or impossible to compute. Thus, the above iterative scheme is not easily applicable.*

Since the inception of Algorithm 2.6.5 and Problem 2.6.4, other authors have studied problem (2.6.4) in Hilbert spaces and introduced different iterative algorithms with minimum metric projection in the framework of Hilbert spaces. For example, the authors in [236], considered the problem (1.2.10), proposed the following iterative algorithm and established a strong convergence theorem.

Algorithm 2.6.5. Initialization:

Step 0: $\{\omega_n\} \subset [\underline{\omega}, \overline{\omega}] \subset \left(0, \frac{1-\gamma}{\|A\|+1}\right), \{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{L}), \{\alpha_n\} \subset (0,1)$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Step 1. *Let* $x_0 \in H_1$. *Set* $n := 0$.

Step 2. *Compute* $u_n = A(x_n)$ *and*

$$y_n = x_n + \omega_n A^*(S(u_n) - u_n). \tag{2.6.6}$$

Step 3.

$$z_n = P_C(y_n - \lambda_n F_1 y_n), \tag{2.6.7}$$

$$t_n = P_{T_n}(y_n - \lambda_n F_1 z_n), \tag{2.6.8}$$

where $T_n = \{\omega \in H : \langle y_n - \lambda_n F_1 y_n - z_n, \omega - z_n \rangle \le 0\}$.

Step 3. *Compute*

$$x_{n+1} = t_n - \alpha_n F_2 t_n, \tag{2.6.9}$$

---

where $A_2 : H_1 \to H_1$ is η-strongly monotone and k-Lipschitz continuous on H_1, $A_1 : H_1 \to H_1$ is pseudomonotone on C, L-Lipschitz continuous on H_1, $\limsup_{n\to\infty}\langle A_1 x_n, y - y_n\rangle \leq \langle A_1\overline{x}, y - \overline{y}\rangle$ for every sequence $\{x_n\}, \{y_n\}$ in H_1 converging weakly to $\overline{x}$ and $\overline{y}$ respectively and $S : H_2 \to H_2$ is a γ-demicontrative mapping.

Remark 2.6.6. *It is well-known that step sizes play essential roles in the convergence properties of iterative methods, since the efficiency of the methods depend heavily on it. When the step size depends on the knowledge of either the operator norm or the coefficient of an operator, it usually slows down the convergence rate of the method. Moreover, in many practical cases, the operator norm or the coefficient of a given operator may not be known or may be difficult to estimate, thus, making the applicability of such method to be questionable. Thus, iterative methods that does not depend on any of these, are more applicable in practice. From Algorithm* (2.6.5), *we have that*

$$\{\omega_n\} \subset [\underline{\omega}, \overline{\omega}] \subset \left(0, \frac{1-\gamma}{\|A\|+1}\right),$$

thus this condition makes the iterative algorithm not applicable to real life problems.

Remark 2.6.7. *One can now ask the following questions*

Question 4: *Can we can further modify and generalize problem* (1.2.10) *and problem* (2.6.4)*?*
Question 5: *Is it possible to develop a more effective iterative algorithm that approximate problem* (1.2.10) *and problem* (2.6.4) *better?*

Hieu, et al. in [161], introduced a regularization-projection method for solving the problem (2.6.11). They proposed the following iterative algorithm:

$$\begin{cases} u_0 \in H \\ v_n = P_C(u_n - \lambda_n(Au_n + \alpha_n u_n)) \\ T_n = \{z \in H : \langle u_n - \lambda_n(Au_n + \alpha_n F u_n) - v_n, z - v_n\rangle \leq 0\} \\ \quad \text{and} \\ u_{n+1} = P_{T_n}(u_n - \lambda_n(Av_n + \alpha_n u_n)) \\ \text{update } \lambda_{n+1} : \text{ if } \lambda_n\|Au_n - Av_n\| \leq \mu\|u_n - v_n\| \text{ then } \lambda_{n+1} = \lambda_n. \\ \text{Else } \lambda_{n+1} = \frac{\mu\|u_n - v_n\|}{\|Au_n - Av_n\|} \end{cases} \tag{2.6.10}$$

where $\lambda_0 \in (0,\infty), \mu \in (0,1)$ and $\{\alpha_n\} \subset (0,\infty)$. It was established that the sequence generated by $\{u_n\}$ converges strongly to the solution of the problem (2.6.11). They further established that the main idea of the regularization method for handling a monotone VIP is to add a strongly monotone operator depending on the so-called regularization parameter to the monotone cost operator for obtaining a strongly monotone VIP. The resulting regularized problem has a unique solution depending continuously on the regularization parameter. They associated the VIP with the following regularized variational inequality problem (RVIP):

$$\text{Find } x \in C \text{ such that } \langle Ax + \alpha Fx, y - x\rangle \geq 0 \;\forall\; y \in C, \tag{2.6.11}$$

where $\alpha > 0$ is a real parameter (regularization parameter) and $F : H \to H$ is L-Lipschitz continuous and γ-strongly monotone. Since A is monotone and Lipschitz continuous, $A + \alpha F$ is strongly monotone and Lipschitz continuous. Thus, the RVIP is uniquely solvable for each $\alpha > 0$, and this unique solution is denoted by p_α. They further study the relationship between the regularization solution p_α of the RVIP and the unique solution p^* of the problem (1.2.7). For details about the $RVIP$, the reader should see [162, 163, 161]. The following result establishes the relationship between p_α and p^*.

Lemma 2.6.8. *[161] Let L and γ be the Lipschitz constant and the modulus of strong monotonicity of the operator F. Then*

1. $\|p_\alpha\| \leq \|p^*\| + \frac{\|Fp^*\|}{\gamma}$.
2. $\|p_\alpha - p_\beta\| \leq \frac{\|\alpha-\beta\|}{\alpha} M$ *for all* $\alpha, \beta > 0$, *where* $M = \frac{1}{\gamma}[2L\|p^*\| + (1 + \frac{L}{\gamma})\|Fp^*\|]$.
3. $\lim_{\alpha \to 0} \|p_\alpha - p^*\| = 0$.

Question 6: Is it possible to apply the regularization technique to the problem (2.6.11)? In addition, can one construct a viscosity-type iterative method with the regularization and thus a more generalized inertial technique?

Remark 2.6.9. *We provide an affirmative answers to these questions raised above in Chapter four of this study.*

2.6.2 Monotone Inclusion Problems

The monotone inclusion problem is to find an element $x \in H$ such that $0 \in B(x)$, where $B : H \to 2^H$ is a multi-valued operator and H is a real Hilbert space. This problem is very important in many areas such as convex optimization and monotone variational inequalities. It is worth mentioning that every monotone operator on Hilbert spaces can be regularized into a single-valued, nonexpansive, Lipschitz continuous monotone operator by means of the Yosida approximation notion. The inclusion problem can also be defined in terms of a sum of two monotone operators f and B, where one of these operators is α-inverse strongly monotone which is $\frac{1}{\alpha}$-Lipschitz continuous.

Let E be a real Banach space with dual space E^* and $\langle f, x\rangle$ the value of $f \in E^*$ at $x \in E$. Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $f : E \to E^*$ be an α-inverse strongly monotone operator. The Monotone Variational Inclusion Problem (MVIP) is find $x \in E$ such that

$$0 \in (B + f)x. \tag{2.6.12}$$

We denote by $(f+B)^{-1}(0)$ the solution set of (2.6.12). Based on a series of studies in the past years, the splitting method has been known to be a popular method for solving (2.6.12). The splitting methods for linear equations was introduced by Peaceman and Rashford [258]. Extensions to nonlinear equations in Hilbert spaces were carried out by Lions and Mercier [206]. Since then, many authors have considered approximating solutions of MVIP (2.6.12) using this method, (see [1, 2, 119, 295] and the references contained in).
Recently, Zhang and Jiang [385] proved the following strong convergence theorem for approximating solutions for a common zero point of the sum of two monotone operators which is also a fixed point of a family of countable quasi-nonexpansive mapping in the framework of Hilbert spaces as follows:

Theorem 2.6.10. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, $A : C \to H$ be an α-inverse strongly monotone operator and B be a maximal monotone operator on H such that $Dom(B)$ is included in C. Let $\{S_n\} : C \to C$ be a family of countable quasi-nonexpansive mappings which are uniformly closed. Assume that $\Gamma := F(S_n) \cap (A+B)^{-1}(0) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence and $\{\alpha_n\}$ be a real number sequence in [0,1). Let $\{x_n\}$ be a sequence of C generated by*

$$\begin{cases} x_1 \in C_1 = C, \text{ chosen arbitrarily;} \\ z_n = J_{r_n}(x_n - r_n A x_n); \\ y_n = \alpha_n z_n + (1-\alpha_n) S_n z_n; \\ C_{n+1} = \{z \in C_n : ||z_n - z|| \leq ||y_n - z|| \leq ||x_n - z||\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \in \mathbb{N}; \end{cases}$$

where $J_{r_n} = (I + r_n B)^{-1}$, $\liminf_{n\to\infty} r_n > 0$, $r_n \leq 2\alpha$ and $\limsup_{n\to\infty} \alpha_n < 1$. Then the sequence $\{x_n\}$ converges strongly to $q = P_\Gamma x_0$.

Motivated by the work of Censor et al. [84], Moudafi [238] studied and introduced a new type of split problem called the Split Monotone Variational Inclusion Problem (SMVIP) which is to find

$$x^* \in H_1 \text{ such that } 0 \in f(x^*) + F(x^*), \tag{2.6.13}$$

and such that $y^* = Ax^* \in H_2$ solves

$$0 \in g(y^*) + G(y^*), \tag{2.6.14}$$

where $F : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ are multivalued mappings, $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are single-valued operators.

Remark 2.6.11. *As observed by Moudafi, setting $F = N_C$ and $G = N_Q$ in SMVIP (2.6.13)-(2.6.14), where N_C and N_Q are the normal cones of C and Q respectively, then we recover SVIP (1.2.2)-(1.2.3) (where $f = A_1$ and $g = A_2$). In summary, SMVIP can be seen as an important generalization of SFP, SVIP, MVIP and other related problems in the literature.*

Recently, Ezeora and Izuchukwu [126] introduced and studied the following problem:

$$\text{Find } z \in (F+f)^{-1}(0) \text{ such that } Az \in F(S). \tag{2.6.15}$$

They proposed the following iterative method for approximating the solution of (2.6.15). For arbitrary $x_1, u \in H_1$

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u \\ y_n = P_C(u_n - \gamma_n A^*(I-T_\gamma)Au_n) \\ x_{n+1} = J_\lambda^F(I-\lambda f)y_n, \ n \in \mathbb{N}, \end{cases} \tag{2.6.16}$$

where $T_\gamma = \gamma I + (1-\gamma)S$ with $\gamma \in [\mu, 1)$, $\{\gamma_n\} \subset [a,b]$ for some $a, b \in \left(0, \frac{1}{||A||^2}\right)$, $F : H_1 \to 2^{H_1}$ is a multivalued maximal monotone mapping, $f : H_1 \to H_1$ is an α-ism and $S : H_2 \to H_2$ being μ-strictly pseudocontractive mapping with T_γ being a nonexpansive mapping. They proved that the sequence $\{x_n\}$ converges strongly to a solution of the problem (2.6.15).

Remark 2.6.12. *It is well-known that step sizes play essential roles in the convergence properties of iterative methods, since the efficiency of the methods depends heavily on it. When the stepsize depends on the knowledge of either the operator norm or the coefficient of an operator, it usually slows down the convergence rate of the method. Moreover, in many practical cases, the operator norm or the coefficient of a given operator may not be known or may be difficult to estimate, thus, making the applicability of such method to be questionable. Therefore, iterative methods that do not depend on any of these, are more applicable in practice. It is easy to see in Algorithm 7.1.12 that*

$$a, b \in \left(0, \frac{1}{||A||^2}\right),$$

thus this condition makes the iterative algorithm not applicable to real life problems.

Question 7: It is natural to ask if we can further generalize the problem (2.6.15)? In addition, can we introduce an iterative algorithm that is more effective in terms of approximating the solution of (2.6.15)?

Remark 2.6.13. *We provide an affirmative answers to these questions in Chapter 5 of this study.*

2.6.3 Equilibrium Problem

Studying Equilibrium Problems (EPs) (1.2.11) from inception requires the nonlinear bifunction operator say F that is monotone. This bifunction plays a critical role in the approximation of the solution of EP (1.2.11) and in real-life applications. In the light of this, researchers have introduced and studied EP (1.2.11) under a more weaker bifunction such as the relaxed monotone operator, pseudomonotone operator, quasimonotone operator, relaxed monotone operator, relaxed semimonotone operator and so on (see

[117, 249, 343, 351, 353] and the references therein). Karamardian and Schaible in [181] were the first to establish some results in this direction. Thereafter, researchers tend to extend and generalize the notion of EP (1.2.11) to problems of common solution of EP (1.2.11) and fixed point problem (FFP) in different abstract spaces. To mention a few, Singthong and Suantai [305] introduced an iterative algorithm for finding a common element of the set of solutions of an equilibrium problem and fixed points set of a nonspreading-type mapping in the Hilbert space. They stated and proved the following strong convergence theorem

Theorem 2.6.14. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(L1)-(L2)$. Let $T : C \to C$ be a k-strictly pseudo nonspreading mapping with a nonempty fixed point set and $F(T) \cap EP(F) \neq \emptyset$. Let $\beta \in [k, 1)$ and $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1)$ and $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ satisfying the conditions:*
$\lim \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\liminf_{n\to\infty} r_n > 0$.
Let $u \in C$ and $\{x_n\}_{n=1}^{\infty}, \{u_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \ \forall \ y \in C; \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, \\ z_n = \frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m x_n, n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}_{n=1}^{\infty}, \{u_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $P_{F(T)\cap EP(F)}u$, where $P_{Fix(T)\cap EP(F)} : H \to F(T) \cap EP(F)$ is the metric projection of H onto $Fix(T) \cap EP(F)$.

Furthermore, iterative methods for approximating the solution of equilibrium problems in which the bifunction is pseudomonotone have been studied by authors (see [343, 353] and the references therein). In the above-listed works, we observe that approximating the solution of these classes of problems, the extragradient methods have been greatly used. However, the bifunctions are required to be Lipschitz continuous. It is well-known that the Lipschitz constants even when known are very difficult to compute, thus, the rate of convergence is greatly affected. In order to resolve this challenge, authors have proposed the use of the line-search technique for establishing strong and weak convergence results for different iterative methods in the different abstract spaces (see [74, 117, 195, 383] and the references therein).

2.6.4 Iterative Methods

Over the years researchers have developed several iterative schemes for solving fixed point problems for different operators but the research are still ongoing in order to develop faster and more efficient iterative algorithms. In 2011, Phuengrattana and Suantai [262] introduced SP-iterative process, as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n \\ x_{n+1} = (1-\gamma_n)y_n + \gamma_n T y_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.6.17}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. In 2017, Karakaya et al., in [179] introduce a new iterative process, as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = T x_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n \\ x_{n+1} = T y_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.6.18}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. They proved that their iterative process converges faster than all of Picard, Mann, Ishikawa, Noor, Abass et al., process and some existing ones in literature.

In 2018, Ullah et al., in [349] introduce a new iterative process called the M-iteration process, as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = T z_n \\ x_{n+1} = T y_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.6.19}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. They established that the M-iteration iterative process converges faster than the Picard, Mann, Ishikawa, Noor, Abass et al., SP, CR, Normal-S process, the above-listed iterative processes and some existing ones in literature.

Remark 2.6.15. *It was established in [3] that the iterative processes* (2.6.18) *and* (2.6.19) *have the same rate of convergence.*

In 2020, Chuadchawna et al., in [104] introduced a generalized M-iteration in the framework of hyperbolic spaces. We will give the corresponding definition of generalized M-iteration as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = \beta_n z_n + (1-\beta_n) T z_n \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n) T y_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.6.20}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. They established some fixed point results in the framework of hyperbolic spaces. They also stated it clearly that for $\beta_n = \gamma_n = 0$, then iterative process (2.6.20) becomes (2.6.19). More so, they claim the the generalized M-iteration converges faster than the M-iteration. They gave a numerical example to justify this claim.

Remark 2.6.16. 1. *If $\alpha = \beta_n = \gamma_n = \frac{1}{2}$, then iterative processes* (2.6.20) *and* (2.6.17) *are the same.*

2. *If $\alpha = \beta = \gamma = 1$, the SP iteration becomes the M-iteration.*

Remark 2.6.17. *It is natural to ask if one can:*

Question 8: *construct an iterative scheme that converges and approximates better than existing iterative schemes in the literature?*
Question 9: *modify iterative process* (2.6.20) *and obtain strong convergence for common fixed point of nonlinear mappings?*
Question 10: *modify iterative process* (2.6.20) *to approximate certain optimization problem?*

The inertial extrapolation method has proven to be an effective way for accelerating the rate of convergence of iterative algorithms. The technique was introduced in 1964 and is based on a discrete version of a second-order dissipative dynamical system [248, 265]. The inertial type algorithms use its two previous iterates to obtain its next iterate [20]. For details on the inertial extrapolation see [34, 35, 48] and the references therein. In 2001, Alvarez and Attouch [20] employed the inertial technique for maximal monotone operators by the proximal point algorithm. This scheme is called the inertial proximal point algorithm, it is define as follows:

For each $x_0, x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \delta_n B)^{-1} y_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.6.21}$$

where I is the identity mapping. They also established that if $\{\delta_n\}$ is nondecreasing and $\theta_n \subset [0, 1)$ with

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty, \tag{2.6.22}$$

the algorithm 2.6.21 converges weakly to a zero of B. In addition, condition 2.6.22 holds for $\theta_n < \frac{1}{3}$.

Remark 2.6.18. *We observe that in Algorithm 2.6.22 that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$ needs to be computed in every iteration and this will definitely affect the effectiveness of the scheme.*

Remark 2.6.19. *We provide an affirmative answers to these questions raised above in Chapter seven of this study.*

Chapter 3

Contributions to Variational Inequality and Split Variational Inequality Problems in Hilbert Spaces

The theory of split variational inequality and variational inequality problems have evolved as a fascinating concept with applications in game theory, mathematical sciences, finance, economics, engineering, mathematical programming, random differential and partial differential equations, minimization problems, optimum control problems, and equilibrium problems. Due to its useful applications, this field is active and growing in both theory and applications. When approximating the solution of a variational inequality problem, at least one projection onto the closed convex set must be computed per iteration. It has been established that projections onto a general closed convex set are difficult to carry out, which may have a negative impact on the rate of convergence and usability of the methods. To address this flaw and other flaws of iterative methods that have been introduced in the literature (see Chapter 2), we present in this chapter certain inertial iterative algorithms with regularization techniques, modified inertial technique, self-adaptive step sizes, in which every projection onto the closed convex set is substituted by a projection onto some half-space and the line-search technique, ensuring easy implementation. Furthermore, we will study some strong convergence results for approximating a common solution of variational inequality problem and fixed point problems in Hilbert spaces.

The results obtained in this chapter extend, generalize and improve several results in this direction.

3.1 Inertial Relaxed Tseng Method for Solving Variational Inequality Problem in Hilbert Space

The research efforts of this section are to present a new inertial relaxed Tseng extrapolation method with weaker conditions for approximating the solution of a variational inequality problem, where the underlying operator is only required to be pseudomonotone. The strongly pseudomonotonicity and inverse strongly monotonicity assumptions which the existing literature used are successfully weakened. The strong convergence of the proposed method to a minimum-norm solution of a variational inequality problem is established. Furthermore, we present an application and some numerical experiments to show the efficiency and applicability of our method in comparison with other methods in the literature.

3.1.1 Main Result

In this section, we present our proposed method and discuss some motivations for proposing it. We also establish strong convergence of the proposed method to a minimum-norm solution of the aforementioned problem. We begin with the following assumptions under which our strong convergence is obtained.

Assumption 3.1.1. *Suppose that the following conditions hold:*

1. *The set C is a nonempty closed and convex subset of the real Hilbert space H.*
2. *$A : H \to H$ is pseudomonotone, sequentially weakly continuous and uniformly continuous on bounded subsets of C.*
3. *The solution set $\Omega = \{x \in C : \langle Ax, y - x\rangle \geq 0 \ \forall \ y \in C\} \neq \emptyset$.*

Algorithm 3.1.2. Initialization: *Given $\gamma, \kappa > 0$, $\rho \in (0, 1]$ and $\theta_n, \beta_n, \alpha_n, \mu, l, \in (0, 1)$, for all $n \in \mathbb{N}$, let $x_0, x_1, \in H$ be arbitrary.*

Iterative step:
Step 1: *Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{\theta}{\kappa}, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \\ \frac{\theta}{\kappa}, & \text{otherwise}, \end{cases} \tag{3.1.1}$$

where $\theta > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$.
Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = P_C(w_n - \lambda_n A w_n), \tag{3.1.2}$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda \|Aw_n - Au_n\| \leq \mu \|w_n - u_n\|. \tag{3.1.3}$$

If $w_n = u_n$, then stop, u_n is a solution of (1.2.1).
Step 3. *Compute*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n((1-\rho)w_n + \rho u_n + \rho\lambda_n(Aw_n - Au_n)). \tag{3.1.4}$$

Stopping criterion: *If $w_n = u_n = x_n$, then stop, otherwise, set $n := n+1$ and go back to* **Step 1.**

The highlight of the motivation for the proposed algorithm.

Remark 3.1.3. 1. *A notable advantage of this method (Algorithm 3.1.2) is that the operator A is pseudomonotone unlike the inversely strongly monotone or strongly pseudomonotonicity assumptions used in other papers (see for example, [142, 156, 304, 292]). No extra projection is required under the setting. The use of the Armijo-line search rule in our algorithm stands as a local approximation of the Lipschitz constant of the operator A. The knowledge of the Lipschitz constant of A is not required.*

2. *The proof of the strong convergence of Algorithm 3.1.2 (that is, proof of Theorem 3.1.7) does not rely on the usual "Two cases approach (Case 1 and Case 2)" usually used in numerous papers for solving optimization problems (see [143, 304, 338, 335] and the reference therein). The techniques and ideas employed in the strong convergence analysis is new.*

3. *In Algorithm 3.1.2, it is easy to compute step 1 since the value of $\|x_n - x_{n-1}\|$ is a prior knowledge before choosing θ_n. It is easy to see from (3.1.1) that $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.*
Recall that, $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$, which means that $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$. Clearly, we have that $\theta_n\|x_n - x_{n-1}\| \leq \epsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, it follows that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$

It is worth mentioning that, we can take $\alpha_n = 1/(n+1)^p$ and $\epsilon_n = 1/(n+1)^{1-p}$, where $p \in [0, 1/2)$.

3.1.2 Convergence Analysis

Lemma 3.1.4. *Let A be an operator satisfying the Assumption 3.1.1. Then, for all $p \in \Omega$, we have the*

$$\|u_n - p\|^2 \leq \|w_n - p\|^2 - \|w_n - u_n\|^2 - 2\lambda_n \langle Aw_n - Au_n, u_n - p\rangle$$

Proof. Since $u_n = P_C(w_n - \lambda_n(Aw_n))$ and $p \in \Omega$, then by the characteristics of P_C, we have that

$$\langle w_n - u_n - \lambda_n Aw_n, u_n - p\rangle \geq 0,$$

which is equivalent to

$$2\langle w_n - u_n, u_n - p\rangle - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle - 2\lambda_n\langle Au_n, u_n - p\rangle \geq 0. \tag{3.1.5}$$

Since $2\langle w_n - u_n, u_n - p\rangle = \|w_n - p\|^2 - \|w_n - u\|^2 - \|u_n - p\|^2$, (3.1.5) becomes

$$\|w_n - p\|^2 - \|w_n - u\|^2 - \|u_n - p\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle - 2\lambda_n\langle Au_n, u_n - p\rangle \geq 0. \tag{3.1.6}$$

Using the fact that A is pseudomonotone, we have that $\langle Au_n, u_n - p\rangle \geq 0$. It follows that

$$\begin{aligned}\|u_n - p\|^2 \leq \|w_n - p\|^2 - \|w_n - u\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle - 2\lambda_n\langle Au_n, u_n - p\rangle \\ \leq \|w_n - p\|^2 - \|w_n - u\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle.\end{aligned}$$

This implies that

$$\|u_n - p\|^2 \leq \|w_n - p\|^2 - \|w_n - u_n\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle.$$

□

Lemma 3.1.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1.2. Then, under the Assumptions 3.1.1, we have that $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega$ and since $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_1$. Then from **Step** 2 of Algorithm 3.1.2, we have

$$\begin{aligned}&\|w_n - p\| \\ &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\leq \|x_n - p\| + \alpha_n N_1.\end{aligned} \tag{3.1.7}$$

Now, suppose that $v_n = (1-\rho)w_n + \rho u_n + \rho\lambda_n(Aw_n - Au_n)$. Then, using Lemma 3.1.4, we have that

$$\begin{aligned}
&\|v_n - p\|^2 \\
&= \|(1-\rho)w_n + \rho u_n + \rho\lambda_n(Aw_n - Au_n) - p\|^2 \\
&= \|(1-\rho)(w_n - p) + \rho(u_n - p) + \rho\lambda_n(Aw_n - Au_n)\|^2 \\
&= (1-\rho)^2\|w_n - p\|^2 + \rho^2\|u_n - p\|^2 + \rho^2\lambda_n^2\|Aw_n - Au_n\|^2 + 2\rho(1-\rho)\langle w_n - p, u_n - p\rangle \\
&+ 2\lambda_n\rho(1-\rho)\langle w_n - p, Aw_n - Au_n\rangle + 2\lambda_n\rho^2\langle u_n - p, Aw_n - Au_n\rangle \\
&= (1-\rho)^2\|w_n - p\|^2 + \rho^2\|u_n - p\|^2 + \rho^2\lambda_n^2\|Aw_n - Au_n\|^2 + \rho(1-\rho)[\|w_n - p\|^2 + \|u_n - p\|^2 \\
&- \|w_n - u_n\|^2] + 2\lambda_n\rho(1-\rho)\langle w_n - p, Aw_n - Au_n\rangle + 2\lambda_n\rho^2\langle u_n - p, Aw_n - Au_n\rangle \\
&= (1-\rho)\|w_n - p\|^2 + \rho\|u_n - p\|^2 - \rho(1-\rho)\|w_n - u_n\| + \rho^2\lambda_n^2\|Aw_n - Au_n\|^2 \\
&+ 2\lambda_n\rho(1-\rho)\langle w_n - p, Aw_n - Au_n\rangle + 2\lambda_n\rho^2\langle u_n - p, Aw_n - Au_n\rangle \\
&\le (1-\rho)\|w_n - p\|^2 + \rho[\|w_n - p\|^2 - \|w_n - u_n\|^2 - 2\lambda_n\langle Aw_n - Au_n, u_n - p\rangle] \\
&- \rho(1-\rho)\|w_n - u_n\| + \rho^2\lambda_n^2\|Aw_n - Au_n\|^2 + 2\lambda_n\rho(1-\rho)\langle w_n - p, Aw_n - Au_n\rangle \\
&+ 2\lambda_n\rho^2\langle u_n - p, Aw_n - Au_n\rangle \\
&= \|w_n - p\|^2 - \rho(2-\rho)\|w_n - u_n\|^2 + \rho^2\lambda_n^2\|Aw_n - Au_n\|^2 + 2\lambda_n\rho^2\langle w_n - u_n, Aw_n - Au_n\rangle \\
&\le \|w_n - p\|^2 - \rho[2 - \rho - \mu(2(1-\rho) + \mu)]\|w_n - u_n\|^2 \\
&\le \|w_n - p\|^2, \qquad (3.1.8)
\end{aligned}$$

which implies that

$$\|v_n - p\| \le \|w_n - p\|. \qquad (3.1.9)$$

Also, using Algorithm 3.1.2, (3.1.7) and (3.1.8), we have that

$$\begin{aligned}
&\|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(v_n - p)\|^2 \\
&= (1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|v_n - p\|^2 + 2(1-\alpha_n-\beta_n)\beta_n\langle x_n - p, v_n - p\rangle \\
&\le (1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + 2(1-\alpha_n-\beta_n)\beta_n\|x_n - p\|\|v_n - p\| \\
&\le (1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + (1-\alpha_n-\beta_n)\beta_n\|x_n - p\|^2 \\
&+ (1-\alpha_n-\beta_n)\beta_n\|v_n - p\|^2 \\
&\le (1-\alpha_n-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + (1-\alpha_n-\beta_n)\beta_n\|x_n - p\|^2 \\
&+ (1-\alpha_n-\beta_n)\beta_n\|w_n - p\|^2 \\
&= (1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|w_n - p\|^2 \\
&\le (1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)\beta_n(\|x_n - p\| + \alpha_n N_1)^2 \\
&= (1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|x_n - p\|^2 \\
&+ 2(1-\alpha_n)\beta_n\alpha_n\|x_n - p\|N_1 + (1-\alpha_n)\beta_n\alpha_n^2 N_1^2 \\
&\le (1-\alpha_n)(1-\alpha_n)\|x_n - p\|^2 + 2(1-\alpha_n)\alpha_n\|x_n - p\|N_1 + \alpha_n^2 N_1^2 \\
&= [(1-\alpha_n)\|x_n - p\| + \alpha_n N_1]^2, \qquad (3.1.10)
\end{aligned}$$

which implies that

$$\|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(v_n - p)\| \le (1-\alpha_n)\|x_n - p\| + \alpha_n N_1. \qquad (3.1.11)$$

We then have that

$$\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)-\alpha_n p\| \\
&\le \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)\|+\alpha_n\|p\| \\
&\le (1-\alpha_n)\|x_n-p\|+\alpha_n N_1+\alpha_n\|p\| \\
&= (1-\alpha_n)\|x_n-p\|+\alpha_n(N_1+\|p\|) \\
&\le \max\{\|x_n-p\|, N_1+\|p\|\} \\
&\ \vdots \\
&\le \max\{\|x_1-p\|, N_1+\|p\|\}. \qquad (3.1.12)
\end{aligned}$$

Thus, $\{x_n\}$ generated by Algorithm 3.1.2 is bounded. □

Lemma 3.1.6. *Let Assumption 3.1.1 hold and let $\{x_n\}$ be a sequence generated by Algorithm 3.1.2. Assume that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to a point x^*, and $\lim\limits_{k\to\infty}\|u_{n_k}-w_{n_k}\|=0$, then, $x^*\in\Omega$.*

Proof. By Lemma 2.2.1 we obtain

$$\langle w_{n_k}-\lambda_{n_k}A(w_{n_k})-u_{n_k},\quad x-u_{n_k}\rangle\le 0,\ \forall\ x\in C,$$

which implies that

$$\frac{1}{\lambda_{n_k}}\langle w_{n_k}-u_{n_k},\quad x-u_{n_k}\rangle\le\langle A(w_{n_k}),\quad x-u_{n_k}\rangle,\ \forall\ x\in C.$$

Consequently, we have

$$\frac{1}{\lambda_{n_k}}\langle w_{n_k}-u_{n_k},\quad x-u_{n_k}\rangle+\langle A(w_{n_k}),\quad u_{n_k}-w_{n_k}\rangle\le\langle A(w_{n_k}),\quad x-w_{n_k}\rangle,\ \forall x\in C. \qquad (3.1.13)$$

Suppose that $x\in C$ is fix and using the fact that $\lim\limits_{k\to\infty}\|w_{n_k}-u_{n_k}\|=0$, we have from (3.1.13) that

$$0\le\liminf_{k\to\infty}\langle A(w_{n_k}),x-w_{n_k}\rangle\quad\forall x\in C. \qquad (3.1.14)$$

Now, choose a sequence $\{\eta_k\}$ of positive numbers such that $\eta_{k+1}\le\eta_k,\ \forall\ k\in\mathbb{N}$ and $\eta_k\to 0$ as $k\to\infty$. Then, for each η_k, we denote by M_k the smallest positive integer such that

$$\langle A(u_{n_k}),x-u_{n_k}\rangle+\eta_k\ge 0\quad\forall k\ge M_k. \qquad (3.1.15)$$

Since $\{\eta_k\}$ is decreasing, it follows that $\{M_k\}$ is increasing. Now, we set for each $k\in\mathbb{N}$, $n_{M_k}=\frac{A(w_{M_k})}{\|A(w_{M_k})\|^2}$, provided $A(w_{M_k})\ne 0$. Then it is easy to see that $\langle A(w_{M_k}),n_{M_k}\rangle=1$ for each $k\in\mathbb{N}$. Using (3.1.15), we have that

$$\langle A(w_{M_k}),x+\eta_k n_{M_k}-w_{M_k}\rangle\ge 0,$$

by the pseudomonotonicity of A, we have that

$$\langle A(x+\eta_k n_{M_k}), x+\eta_k n_{M_k}-w_{N_k}\rangle \geq 0. \tag{3.1.16}$$

Since $\{x_{n_k}\}$ converges weakly to x^*, we obtain by our hypothesis that $\{u_{n_k}\}$ and $\{w_{n_k}\}$ also converge weakly to x^*. Thus, by the sequentially weakly continuity of A, we have that $\{A(w_{n_k})\}$ converges weakly to $A(x^*)$. If $A(x^*)=0$, then $x^*\in\Omega$. On the other hand, if we suppose that $A(x^*)\neq 0$, then by the weakly lower semicontinuity of $\|\cdot\|$, we obtain that

$$0<\|A(x^*)\|\leq \liminf_{k\to\infty}\|A(w_{n_k})\|.$$

Since $\{w_{M_k}\}\subset\{w_{n_k}\}$, we obtain that

$$0\leq \limsup_{k\to\infty}\|\eta_k n_{M_k}\| = \limsup_{k\to\infty}\left(\frac{\eta_k}{\|A(w_{n_k})\|}\right)\leq \frac{\limsup\limits_{k\to\infty}\eta_k}{\liminf\limits_{k\to\infty}\|A(w_{n_k})\|}=0,$$

which implies that, $\lim\limits_{k\to\infty}\|\eta_k n_{M_k}\|=0$. Thus, letting $k\to\infty$ in (3.1.16) yields

$$\langle A(x), x-x^*\rangle \geq 0 \quad \forall x\in C, \tag{3.1.17}$$

which implies by Lemma 2.2.1 that $x^*\in\Omega$. ☐

Theorem 3.1.7. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1.2. Then, under the Assumptions 3.1.1, if $\lim\limits_{n\to\infty}\alpha_n=0, \sum_{n=1}^{\infty}\alpha_n=\infty, 0\leq\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$ and $\lim\limits_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|=0$. Then, $\{x_n\}$ converges strongly to $p\in\Omega$, where $\|p\|=\min\{\|x^*\|:x^*\in\Omega\}$.*

Proof. Let $p\in\Omega$. To start with, observe that

$$\begin{aligned}
\|w_n-p\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-p\|^2\\
&= \|x_n-p\|^2+2\theta_n\langle x_n-p, x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2\\
&\leq \|x_n-p\|^2+2\theta_n\|x_n-x_{n-1}\|\|x_n-p\|+\theta_n^2\|x_n-x_{n-1}\|^2\\
&= \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|[2\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|]\\
&= \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|[2\|x_n-p\|+\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|]\\
&\leq \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|[2\|x_n-p\|+\alpha_n N_1]\\
&\leq \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2,
\end{aligned} \tag{3.1.18}$$

where $N_2 := 2\|x_n - x^*\| + \alpha_n N_1$. In addition, we have that

$$\begin{aligned}
&\|(1-\beta_n)x_n + \beta_n v_n - p\|^2 \\
&= \|(1-\beta_n)(x_n - p) + \beta_n(v_n - p)\|^2 \\
&= (1-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|v_n - p\|^2 + 2(1-\beta_n)\beta_n\langle x_n - p, v_n - p\rangle \\
&\leq (1-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + 2(1-\beta_n)\beta_n\|x_n - p\|\|v_n - p\| \\
&\leq (1-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + (1-\beta_n)\beta_n\|x_n - p\|^2 \\
&+ (1-\beta_n)\beta_n\|v_n - p\|^2 \\
&\leq (1-\beta_n)^2\|x_n - p\|^2 + \beta_n^2\|w_n - p\|^2 + (1-\beta_n)\beta_n\|x_n - p\|^2 \\
&+ (1-\beta_n)\beta_n\|w_n - p\|^2 \\
&= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|w_n - p\|^2 \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n[\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2] \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2. \qquad (3.1.19)
\end{aligned}$$

More so, we have that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|(1-\alpha_n)[(1-\beta_n)x_n + \beta_n v_n - p] - [\beta_n\alpha_n(x_n - v_n) + \alpha_n p]\|^2 \\
&\leq (1-\alpha_n)^2\|(1-\beta_n)x_n + \beta_n v_n - p\|^2 - 2\langle\beta_n\alpha_n(x_n - v_n) + \alpha_n p, x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)^2\|(1-\beta_n)x_n + \beta_n v_n - p\|^2 + 2\langle\beta_n\alpha_n(x_n - v_n), x_{n+1} - p\rangle + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\leq (1-\alpha_n)[\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2] + 2\alpha_n\beta_n\|x_n - v_n\|\|x_{n+1} - p\| + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\beta_n\|x_n - v_n\|\|x_{n+1} - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[2\beta_n\|x_n - v_n\|\|x_{n+1} - p\| + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\langle p, p - x_{n+1}\rangle] \\
&= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\delta_n, \qquad (3.1.20)
\end{aligned}$$

where $\delta_n := 2\beta_n\|x_n - v_n\|\|x_{n+1} - p\| + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\langle p, p - x_{n+1}\rangle$. According to Lemma 2.1.37, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\delta_{n_k} \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty}\{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \qquad (3.1.21)$$

To establish that $\limsup_{k\to\infty}\delta_{n_k} \leq 0$, we suppose that for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ such that (3.1.21) holds. Then,

$$\begin{aligned}
&\liminf_{k\to\infty}\{\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\} \\
&= \liminf_{k\to\infty}\{(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|)(\|x_{n_k+1} - p\| + \|x_{n_k} - p\|)\} \geq 0. \qquad (3.1.22)
\end{aligned}$$

Now, using Algorithm 3.1.2, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|(1-\alpha_n-\beta_n)x_n+\beta_n v_n-p\|^2\\
&=\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)-\alpha_n p\|^2\\
&\leq\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)\|^2+\alpha_n^2\|p\|^2-2\alpha_n\langle(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p),p\rangle\\
&\leq\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)\|^2+\alpha_n M\\
&\leq(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|v_n-p\|^2-(1-\alpha_n-\beta_n)\beta_n\|v_n-x_n\|^2+\alpha_n M\\
&\leq(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|w_n-p\|^2-(1-\alpha_n-\beta_n)\beta_n\|v_n-x_n\|^2+\alpha_n M\\
&\leq\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2-(1-\alpha_n-\beta_n)\beta_n\|v_n-x_n\|^2+\alpha_n M,
\end{aligned}\tag{3.1.23}$$

for some $M>0$. It implies from (3.1.22) that

$$\begin{aligned}
&\limsup_{k\to\infty}[(1-\alpha_{n_k}-\beta_{n_k})\beta_{n_k}\|v_{n_k}-x_{n_k}\|^2]\\
&\leq\limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\\
&+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2+\alpha_{n_k}M]\\
&\leq-\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0,
\end{aligned}\tag{3.1.24}$$

which gives

$$\lim_{k\to\infty}\|v_{n_k}-x_{n_k}\|=0.\tag{3.1.25}$$

Also, using Algorithm 3.1.2 and (3.1.8), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|(1-\alpha_n-\beta_n)x_n+\beta_n v_n-p\|^2\\
&=\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)-\alpha_n p\|^2\\
&\leq\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)\|^2+\alpha_n^2\|p\|^2-2\alpha_n\langle(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p),p\rangle\\
&\leq\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(v_n-p)\|^2+\alpha_n M\\
&\leq(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|v_n-p\|^2-(1-\alpha_n-\beta_n)\beta_n\|v_n-x_n\|^2+\alpha_n M\\
&\leq(1-\alpha_n-\beta_n)\|x_n-p\|^2+\|w_n-p\|^2-\rho[2-\rho-\mu(2(1-\rho)+\mu)]\|w_n-u_n\|^2\\
&-(1-\alpha_n-\beta_n)\beta_n\|v_n-x_n\|^2+\alpha_n M.
\end{aligned}\tag{3.1.26}$$

It can be deduced from (3.1.22) that

$$\begin{aligned}
&\limsup_{k\to\infty}[\rho[2-\rho-\mu(2(1-\rho)+\mu)]\|w_{n_k}-u_{n_k}\|^2]\\
&\leq\limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\\
&+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\\
&-(1-\alpha_{n_k}-\beta_{n_k})\beta_{n_k}\|v_n-x_n\|^2\\
&+\alpha_{n_k}M]\\
&\leq-\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0,
\end{aligned}\tag{3.1.27}$$

which gives

$$\lim_{k\to\infty} \|w_{n_k} - u_{n_k}\| = 0. \tag{3.1.28}$$

Notice that as $k \to \infty$, we have

$$\|w_{n_k} - x_{n_k}\| = \theta_{n_k}||x_{n_k} - x_{n_k-1}|| = \alpha_{n_k} \cdot \frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}|| \to 0. \tag{3.1.29}$$

In addition, we have the following

$$\|w_{n_k} - v_{n_k}\| \leq \|w_{n_k} - x_{n_k}\| + \|x_{n_k} - v_{n_k}\| \to 0 \quad \text{as} \quad k \to \infty, \tag{3.1.30}$$

$$\|u_{n_k} - x_{n_k}\| \leq \|u_{n_k} - v_{n_k}\| + \|v_{n_k} - x_{n_k}\| \to 0 \ \text{ as } k \to \infty. \tag{3.1.31}$$

From the Algorithm 3.1.2 and (3.1.25), observe that

$$\begin{aligned}\|x_{n_k+1} - v_{n_k}\| &= \|(1-\alpha_n-\beta_n)x_{n_k} + \beta_n v_{n_k} - v_{n_k}\| \\ &\leq (1-\alpha_{n_k}-\beta_{n_k})\|x_{n_k} - v_{n_k}\| + \beta_{n_k}\|v_{n_k} - v_{n_k}\| + \alpha_{n_k}\|v_{n_k}\| \to 0 \text{ as } k \to \infty.\end{aligned} \tag{3.1.32}$$

Using (3.1.32) and (3.1.25), it gives

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - v_{n_k}\| + \|v_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{3.1.33}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $x^* \in H_1$. By (3.1.25), (3.1.29) and (3.1.31), we have that the subsequences $\{w_{n_{k_j}}\}$ of $\{w_{n_k}\}$, $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ and $\{v_{n_{k_j}}\}$ of $\{v_{n_k}\}$, all converge weakly to x^* respectively. From (3.1.28) and Lemma 3.1.6, we have that $x^* \in \Omega$.

Since $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to x^* such that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k}\rangle = \lim_{j\to\infty}\langle p, p - x_{n_{k_j}}\rangle = \langle p, p - x^*\rangle. \tag{3.1.34}$$

Hence, since $p = P_\Omega 0$, we have obtain from (3.1.34) that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k}\rangle = \langle p, p - x^*\rangle \leq 0, \tag{3.1.35}$$

we have

$$\limsup_{k\to\infty}\langle p, p - x_{n_k+1}\rangle \leq 0, \tag{3.1.36}$$

Using our assumption, (3.1.25) and (3.1.36), we have that $\limsup_{k\to\infty}\delta_{n_k} := 2\beta_n\|x_n - v_n\|\|x_{n+1} - p\| + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\langle p, p - x_{n+1}\rangle \leq 0$. Thus, the last part of Lemma 2.1.37 is achieved. Hence, we have that $\lim\limits_{n\to\infty}\|x_n - p\| = 0$. Thus, $\{x_n\}$ converges strongly to $p \in \Omega$. $\square$

3.1.3 Application to Equilibrium Problem

In this section, we apply our results to the equilibrium problem.

The equilibrium problem is one of the interesting problems in this area of research. Equilibrium problems are special cases of monotone inclusion problems, saddle point problems, minimization problems, optimization problems, variational inequality problems, Nash equilibria in noncooperative games, and various forms of feasibility problems. Let C be a closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction, the equilibrium problem is defined as finding $x \in C$ such that

$$F(x, y) \geq 0 \quad \forall\ y \in C. \tag{3.1.37}$$

The solution set for x is denoted by $EP(F)$. It is well-known that to approximate the solution of problem (3.1.37), we assume the bifunction F satisfying the following well-known conditions:

1. $F(x, x) = 0 \quad \forall x \in C$,
2. F is monotone, that is $F(x, y) + F(y, x) \leq 0 \quad \forall \quad x, y \in C$,
3. for each $x, y, z \in C \qquad \lim_{t \to 0^+} F(\alpha z + (1 - \alpha)x, y) \leq F(x, y)$,
4. for each $x \in C, y \to F(x, y)$ is convex and lower semi-continuous.

Lemma 3.1.8. *[58] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(1) - (4)$. Suppose that $\lambda > 0$ and $x \in H$, thus, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{3.1.38}$$

In addition, if

$$J_\lambda^F x = \{x \in C : F(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}, \tag{3.1.39}$$

then the following hold:

1. *J_λ^F is single-valued and firmly nonexpansive,*
2. *$F(J_\lambda^F) = EP(F)$,*
3. *$EP(F)$ is closed and convex.*

We note that J_λ^F is the resolvent of F for $\lambda > 0$.

Lemma 3.1.9. *[319] Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(1)-(4)$. Let B_F be a set-valued mapping of H into H defined by*

$$B_F = \begin{cases} \{z \in H : F(x,y) + \langle y - x, z\rangle \geq 0 \quad \forall y \in C\} & \text{if } x \in C \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3.1.40}$$

Then $EP(F) = B_F^{-1}(0)$ and B_F is a maximal monotone operator with $Dom(B_F) \subset C$. Furthermore, for any $x \in H$ and $\lambda > 0$, the resolvent J_λ^F of F coincides with the resolvent of B_F, that is

$$J_\lambda^F(x) = (I + B_F)^{-1}(x).$$

Using the above results. Setting $A = 0$, and $J_\lambda^F(x) = (I + B_F)^{-1}(x)$ from Lemma 3.1.9, we obtain the following algorithm and result.

Assumption 3.1.10. *Suppose that the following conditions hold:*

1. *The set C is a nonempty closed and convex subset of the real Hilbert space H.*
2. *$F : C \times C \to \mathbb{R}$ be a function satisfying conditions $(1)-(4)$.*
3. *The solution set $\Omega = EP(F) \neq \emptyset$.*

Algorithm 3.1.11. Initialization: *Given $\gamma, \kappa > 0$, $\rho \in (0,1]$ and $\theta_n, \beta_n, \alpha_n, \mu, l, \in (0,1)$, for all $n \in \mathbb{N}$, let $x_0, x_1, \in H$ be arbitrary.*

Iterative step:

Step 1: *Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{\theta}{\kappa}, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \\ \frac{\theta}{\kappa}, & \text{otherwise,} \end{cases} \tag{3.1.41}$$

where $\theta > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$.

Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = J_{\lambda_n}^F w_n, \tag{3.1.42}$$

where λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying

$$\lambda\|Fw_n - Fu_n\| \leq \mu\|w_n - u_n\|. \tag{3.1.43}$$

Step 3. *Compute*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n((1-\rho)w_n + \rho u_n + \rho\lambda_n(Aw_n - Au_n)). \tag{3.1.44}$$

Stopping criterion: *Set $n := n + 1$ and go back to* **Step 1.**

Theorem 3.1.12. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1.11. Then, under the Assumptions 3.1.10, if* $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $0 \leq \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Then, $\{x_n\}$ converges strongly to $p \in \Omega$, where* $\|p\| = \min\{\|x^*\| : x^* \in \Omega\}$.

3.1.4 Numerical Example

In this section, we present some numerical experiments in finite and infinite dimensional Hilbert spaces and compare our proposed Algorithm 3.1.2 with Algorithm 3.1 of [103] and Algorithm 3.1 of [340].

Example 3.1.13. *Let $H = L^2([0,1])$ and norm $\|x\| = (\int_0^1 |x(t)|dt)^{\frac{1}{2}}$ and the inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$ for all $x, y \in L^2([0,1])$. Define the operator $A : L^2([0,1]) \to L^2([0,1])$ by*

$$Ax(t) = \max\{0, x(t)\}. \tag{3.1.45}$$

Suppose that $C = \{x \in H : \|x\| \leq 1\}$ is a unit ball, then

$$P_C(x) = \begin{cases} \frac{x}{\|x\|_{L^2}}, & \text{if } \|x\|_{L^2} > 1, \\ x, & \text{if } \|x\|_{L^2} \leq 1. \end{cases} \tag{3.1.46}$$

Choose $\gamma = 0.03, l = 1, \mu = 0.38, \kappa = 1, \theta = 0.001, \alpha_n = \frac{1}{n+1}, \epsilon_n = \frac{1}{(n+1)^2}, \beta_n = \frac{3n}{3n+5}, \rho = 0.38$. It is easy to verify that all hypotheses of Theorem 7.1.6 are satisfied and the set of solutions to the $VI(A, C)$ (1.2.1) is given by $\Omega = \{0\} \neq \emptyset$. We use different choices of x_0, x_1 and test the convergence of our algorithm with $\|x_{n+1} - x_n\| < 10^{-5}$ as a stopping criterion. We compare the performance of Algorithm 3.1.2 with the Algorithm 3.1 of Cholamjiak et al. [103].

1. *Case I:* $x_0(t) = \frac{-3te^{2t}}{5}, x_1(t) = e^{-2t}$.
2. *Case II:* $x_0(t) = \sin 5t, x_1(t) = \cos(-3t)$.
3. *Case III:* $x_0(t) = t^3 + 1, x_1(t) = e^{2t}$.
4. *Case IV:* $x_0(t) = e^{3t}, x_1(t) = -2\sin 2t$.

The computational results are shown in Table 3.1 and Figure 3.1.

Table 3.1: Computation result for Example 3.1.13.

		Algorithm 7.1.2	Algorithm 3.1 of [103]
Case I	No of Iter.	14	12
	CPU time (sec)	2.6427	4.0313
Case II	No of Iter.	13	12
	CPU time (sec)	2.5892	5.3755
Case III	No of Iter.	17	17
	CPU time (sec)	2.1036	8.2650
Case IV	No of Iter.	17	18
	CPU time (sec)	3.1534	7.1918

Example 3.1.14. *Let* $H = \mathbb{R}^N$, *with the Euclidean norm on* $\mathbb{R}^N$. *Suppose that* $C = \{x \in H : \|x\| \leq 1\}$ *is the unit ball, define the operator* $A : C \to \mathbb{R}^N$ *by*

$$Ax(t) = x. \tag{3.1.47}$$

We have

$$P_C(x) = \begin{cases} \frac{x}{\|x\|}, & \text{if } \|x\| > 1, \\ x, & \text{if } \|x\| \leq 1. \end{cases} \tag{3.1.48}$$

With these given C *and* A, *the set of solutions to the* $VI(A, C)$ (1.2.1) *is known to be* $\Omega = \{0\} \neq \emptyset$. *Choose* $\gamma = 0.05, l = 4, \mu = 0.38, \kappa = 0.01, \theta = 0.001, \alpha_n = \frac{1}{\sqrt{n+1}}, \epsilon_n = \frac{1}{(n+1)}, \beta_n = \frac{1}{2} + \frac{2}{(2n+4)}, \rho = 0.38$ *It is easy to verify that all hypotheses of Theorem 7.1.6 are satisfied. We use different choices of* x_0, x_1 *and test the convergence of our algorithm with* $\|x_{n+1} - x_n\| < 10^{-6}$ *as a stopping criterion. We compare the performance of the Algorithm 3.1.2 with the Algorithm 3.1 of Thong et al. [340].*

1. *Case I:* $N = 5$.
2. *Case II:* $N = 10$.
3. *Case III:* $N = 30$.
4. *Case IV:* $N = 50$.

The computational results are shown in Table 3.2 and Figure 3.2.

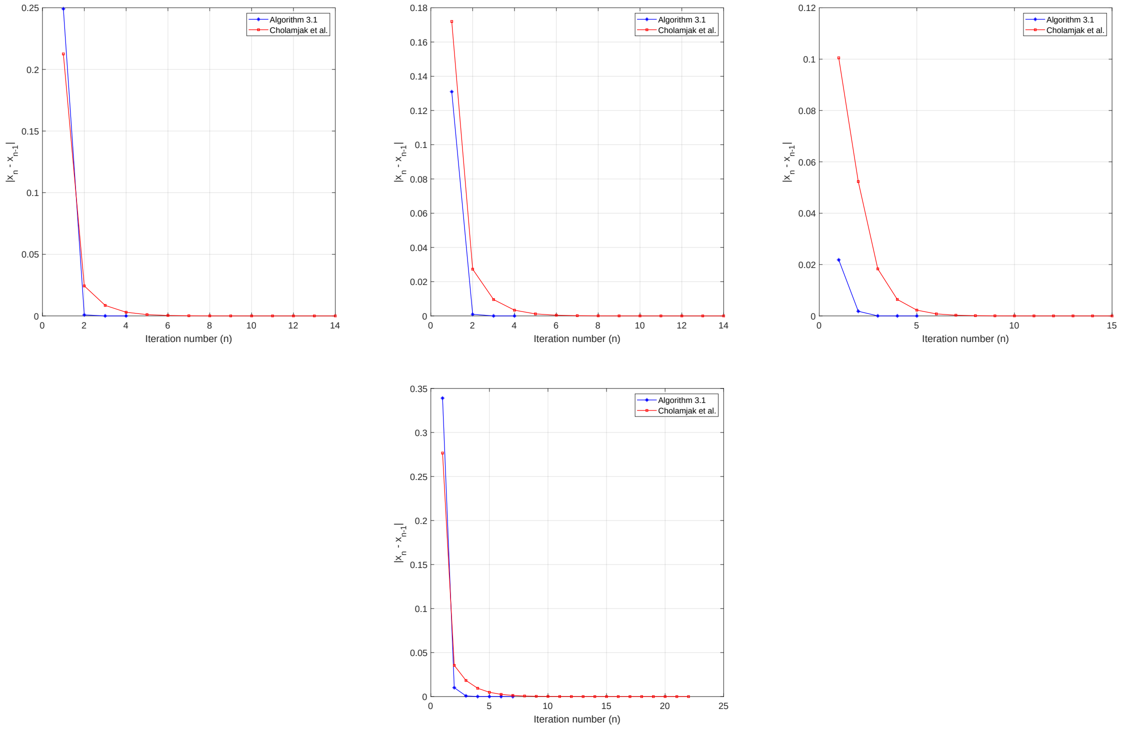

Figure 3.1: Example 3.1.13, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

3.2 Inertial Extrapolation Method with Regularization for Solving Monotone Bilevel Variation Inequalities and Fixed Point Problems in Real Hilbert Space

In this section, we introduce a generalized inertial extrapolation method with regularization term for approximating the solutions of a monotone and Lipschitz variational inequality and fixed point problems in a real Hilbert space. In addition, we establish the strong convergence of the resulting methods under certain conditions imposed on regularization parameters. Our method of proof work with or without knowing the Lipschitz constant of the cost operator. Finally, we present some numerical experiments to show the efficiency and applicability of the proposed method.

3.2.1 Main Result

In this section, we present our proposed method and highlight some of its important features. Also, we establish the strong convergence of the resulting methods under certain conditions imposed on regularization parameters. We begin with the following assumptions

Table 3.2: Computation result for Example 3.3.7.

		Algorithm 7.1.2	Algorithm 3.1 of [340]
Case I	No of Iter.	11	27
	CPU time (sec)	0.0022	0.0042
Case II	No of Iter.	12	28
	CPU time (sec)	0.0016	0.0027
Case III	No of Iter.	12	29
	CPU time (sec)	0.0014	0.0049
Case IV	No of Iter.	12	29
	CPU time (sec)	0.0014	0.0041

under which our strong convergence is obtained.

Assumption 3.2.1. *Suppose that the following conditions hold:*

Condition A.

1. *H is a Hilbert space and C is a nonempty closed and convex subset of H.*
2. *$\{S_n\}$ is a sequence of nonexpansive mapping on H.*
3. *$A : H \to H$ is monotone and L_1- Lipschitz continuous operator and $F : H \to H$ is γ-strongly monotone and L_2-Lipschitz continuous operator, where $L_1, L_2 > 0$ and $\gamma > 0$.*
4. *$S : H \to H$ is a nonexpansive mapping and $f : H \to H$ is a contraction mapping with coefficient $k \in (0, 1)$.*
5. *The solution set $\Gamma = \{p^* \in VI(A, C) \cap F(S)$ such that $\langle Fp^*, x - p^*\rangle \geq 0, \ \forall \ x \in VI(A, C)\} \neq \emptyset$.*
6. *The solution set $\Omega = \{p \in RVIP \cap F(S)$ such that $\langle Fp, x - p\rangle \geq 0, \ \forall \ x \in RVIP\} \neq \emptyset$.*

Condition B.

1. *$\beta_n \subset (0, 1), \lim\limits_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$.*
2. *$\alpha_n \in (0, 1), \lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.*
3. *$\{\delta_n\} \subset (0, \delta_0) \in (0, 1), \{\gamma_n\}, \{\eta_n\} \subset (0, 1)$ such that $\beta_n + \delta_n + \eta_n = 1$, $\lambda_0 > 0, \mu \in (0, 1)$ and choose the nonnegative real sequence $\{\zeta_n\}$ such that $\sum_{n=1}^{\infty} \zeta_n < \infty$.*

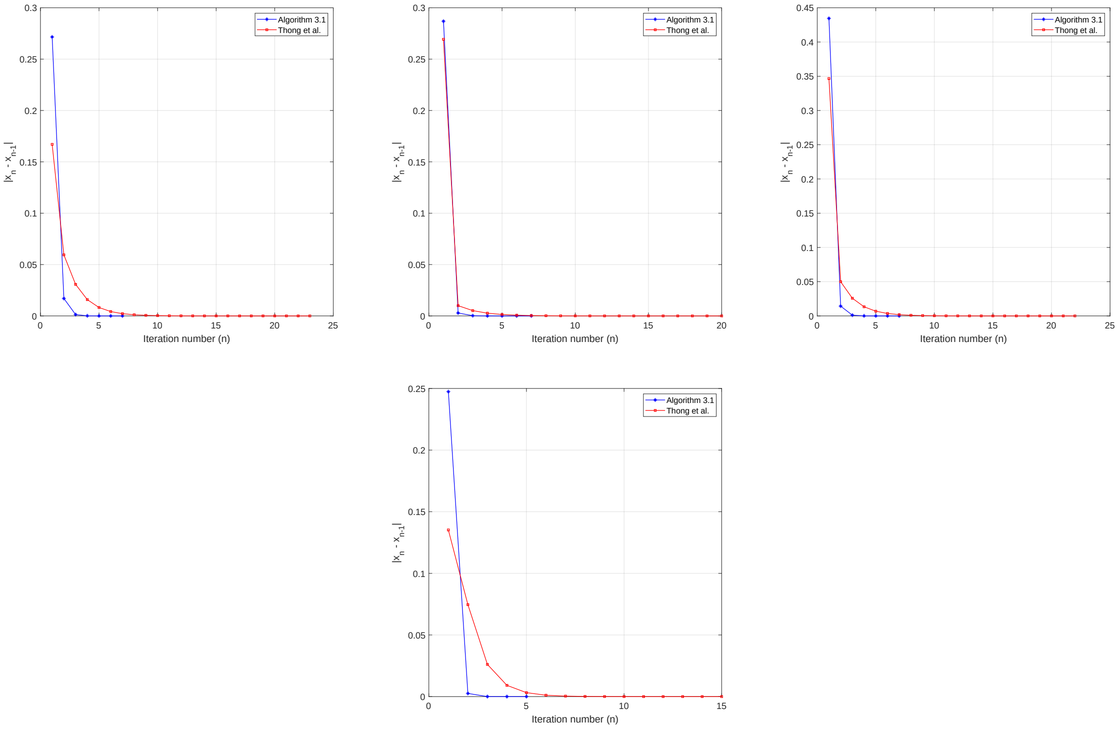

Figure 3.2: Example 3.1.14, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

We present the following iterative algorithm.

Algorithm 3.2.2. Iterative steps: *Given* $x_0, x_1 \in H$, *the parameters*λ_0, μ, *and sequences* $\gamma_n, \beta_n, \eta_n, \delta_n$ *satisfying the conditions above,* $L_2 \in (0,2)$ *and* $\theta_n \in (0,1)$.

Step 1: *Given the iterates* x_{n-1} *and* x_n *for all* $n \in \mathbb{N}$, *choose* θ_n *such that* $0 \le \theta_n \le \bar{\theta}_n$, *where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise} \end{cases} \tag{3.2.1}$$

with θ *being a positive constant and* $\{\epsilon_n\}$ *is a positive sequence such that* $\epsilon_n = \circ(\beta_n)$.
Step 2. *Set*

$$w_n = x_n + \theta_n(S_n x_n - S_n x_{n-1}).$$

Then, compute

$$z_n = P_C(w_n - \lambda_n(Aw_n + \alpha_n F w_n)) \tag{3.2.2}$$
$$u_n = \gamma_n w_n + (1 - \gamma_n) q_n, \tag{3.2.3}$$

where $q_n = P_{T_n}(w_n - \lambda_n(Az_n + \alpha_n F w_n)), T_n = \{w \in H : \langle w_n - \lambda_n(Aw_n + \alpha_n F w_n) - z_n, w -$

$z_n\rangle \le 0\}$ *and*

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu(\|w_n-z_n\|^2+\|q_n-z_n\|^2)}{2\langle Aw_n-Az_n,q_n-z_n\rangle},\ \lambda_n+\zeta_n\right\}, & \text{if } \langle Aw_n - Az_n, q_n - z_n\rangle > 0, \\ \lambda_n + \zeta_n, & \text{otherwise.} \end{cases} \tag{3.2.4}$$

Step 4. *Compute*

$$x_{n+1} = \beta_n f(x_n) + \eta_n x_n + \delta_n S u_n \tag{3.2.5}$$

Remark 3.2.3. *1. $C \subset T_n$ for all $n \in \mathbb{N}$. Indeed from the definition of z_n and the characteristic of the metric projection, that is Lemma 2.2.1 (3), we have that*

$$\langle w_n - \lambda_n(Aw_n + \alpha_n F w_n) - z_n, w - z_n\rangle \le 0$$

for all $w \in C$. Thus together with the definition of T_n, this implies that $C \subset T_n$ for all $n \in \mathbb{N}$.

2. *The step size $\{\lambda_n\}$ is self-adaptive and save computational time unlike the line search method that requires loop computations at each iteration, and thus increase computational time.*

3. *As we shall see in our convergence analysis, we do not use the popular two cases method usually used in numerous papers to guarantee strong convergence see [143, 304, 339, 338, 335]. Thus the techniques and ideas employed in our strong convergence analysis is new for solving the problem considered in this study.*

4. *In Algorithm 3.2.2, it is easy to compute step 1 since the value of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. It is also easy to see from (3.2.1) that $\lim\limits_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| = 0$.*

   Indeed, since, $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\beta_n)$, which means that $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$, we have that $\theta_n\|x_n - x_{n-1}\| \le \epsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$, it implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \le \lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0.$$

5. *The sequences of nonexpansive mapping $\{S_n\}$ help speed up the rate of convergence. See Section 3.2.3 for the comparison of our proposed iterative algorithm with the sequence $\{S_n\}$ and without the sequence $\{S_n\}$*

6. *We note that the result of the relationship between the regularization solution p_α of the problem (1.2.9) and the unique solution p^* of the problem (1.2.7), are the same with Lemma 2.6.8.*

3.2.2 Convergence Analysis

In this section, we establish strong convergence results using our proposed method.

Lemma 3.2.4. *Let $\{\lambda_n\}$ be the sequence generated by Algorithm* (3.2.2)*. Then we have* $\lim_{n\to\infty}\lambda_n = \lambda$ *and* $\lambda \in [\min\{\lambda_1, \frac{\mu}{L_1}\}, \lambda_1 + \zeta]$.

Proof. The proof follows a similar approach as in Lemma 3.1 of [209], thus we omit it. $\square$

Lemma 3.2.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2.2. Then, under the Assumption 3.2.1, we have that $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega$ and since $\lim\limits_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le N_1$, for all $n \in \mathbb{N}$.

$$\begin{aligned}\|w_n - p\| &= \|x_n + \theta_n(S_nx_n - S_nx_{n-1}) - p\| \\ &\le \|x_n - p\| + \theta_n\|S_nx_n - S_nx_{n-1}\| \\ &= \|x_n - p\| + \beta_n\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \\ &\le \|x_n - p\| + \beta_nN_1. \end{aligned} \tag{3.2.6}$$

Suppose that $q_n = P_{T_n}(w_n - \lambda_n(Az_n + \alpha_nFw_n))$, we have

$$\begin{aligned}&\|q_n - p\|^2 \\ &= \|P_{T_n}(w_n - \lambda_n(Az_n + \alpha_nFw_n)) - p\|^2 \\ &\le \|w_n - \lambda_n(Az_n + \alpha_nFw_n) - p\|^2 - \|w_n - \lambda_n(Az_n + \alpha_nFw_n) - q_n\|^2 \\ &= \|(w_n - p) - \lambda_n(Az_n + \alpha_nFw_n)\|^2 - \|(w_n - q_n) - \lambda_n(Az_n + \alpha_nFw_n)\|^2 \\ &= \|w_n - p\|^2 - 2\lambda_n\langle w_n - p, Az_n + \alpha_nFw_n\rangle - \|w_n - q_n\|^2 + 2\lambda_n\langle w_n - q_n, Az_n + \alpha_nFw_n\rangle \\ &= \|w_n - p\|^2 - \|w_n - q_n\|^2 - 2\lambda_n\langle q_n - p, Az_n + \alpha_nFw_n\rangle \\ &= \|w_n - p\|^2 - \|w_n - q_n\|^2 + 2\langle w_n - z_n, z_n - q_n\rangle + 2\lambda_n\langle Az_n + \alpha_nFw_n, p - z_n\rangle \\ &+ 2\lambda_n\langle Az_n - Aw_n, z_n - q_n\rangle + 2\langle w_n - \lambda_n(Aw_n + \alpha_nFw_n) - z_n, q_n - z_n\rangle. \end{aligned} \tag{3.2.7}$$

Since $q_n \in T_n$, from the definition of T_n, we have that

$$\langle w_n - \lambda_n(Aw_n + \alpha_nFw_n) - z_n, q_n - z_n\rangle \le 0$$

and that

$$2\langle w_n - z_n, z_n - q_n\rangle = \|w_n - q_n\|^2 - \|w_n - z_n\|^2 - \|z_n - q_n\|^2.$$

Thus from (3.2.7), we have

$$\begin{aligned}&\|q_n - p\|^2 \\ &\le 2\lambda_n\langle Az_n - Aw_n, z_n - q_n\rangle + \|w_n - p\|^2 - \|z_n - q_n\|^2 - \|w_n - z_n\|^2 \\ &+ 2\lambda_n\langle Az_n + \alpha_nFw_n, p - z_n\rangle. \end{aligned} \tag{3.2.8}$$

Now, using the Monotonicity of A, we have that $\langle Az_n - Ap, p - z_n\rangle \leq 0$, thus, we get

$$\begin{aligned}
&2\lambda_n\langle Az_n + \alpha_n Fw_n, p - z_n\rangle \\
&= 2\lambda_n\langle Az_n - Ap, p - z_n\rangle + 2\lambda_n\langle Ap + \alpha_n Fp, p - z_n\rangle + 2\lambda_n\alpha_n\langle Fw_n - Fp, p - z_n\rangle \\
&\leq 2\lambda_n\langle Ap + \alpha_n Fp, p - z_n\rangle + 2\lambda_n\alpha_n\langle Fw_n - Fp, p - z_n\rangle, \qquad (3.2.9)
\end{aligned}$$

since p is a solution of $RVIP$ and $z_n \in C$, we have that $\langle Ap + \alpha_n Fp, z_n - p\rangle \geq 0$ which implies $\langle Ap + \alpha_n Fp, p - z_n\rangle \leq 0$.

$$2\lambda_n\langle Az_n + \alpha_n Fw_n, p - z_n\rangle \leq 2\lambda_n\alpha_n\langle Fw_n - Fp, p - z_n\rangle, \qquad (3.2.10)$$

Thus, from (3.2.10) and using the γ-strongly monotonicity of F, we have that

$$\begin{aligned}
2\lambda_n\langle Az_n + \alpha_n Fw_n, p - z_n\rangle &\leq 2\lambda_n\alpha_n\langle Fw_n - Fp, p - w_n\rangle + 2\lambda_n\alpha_n\langle Fw_n - Fp, w_n - z_n\rangle \\
&\leq -2\lambda_n\alpha_n\gamma\|p - w_n\|^2 + 2\lambda_n\alpha_n\langle Fw_n - Fp, w_n - z_n\rangle.
\end{aligned} \qquad (3.2.11)$$

Thus, we have (3.2.8) becomes

$$\begin{aligned}
&\|q_n - p\|^2 \\
&\leq (1 - 2\lambda_n\alpha_n)\|w_n - p\|^2 - \|z_n - q_n\|^2 - \|w_n - z_n\|^2 + 2\lambda_n\langle Az_n - Aw_n, z_n - q_n\rangle \\
&+ 2\lambda_n\langle Fw_n - Fp, w_n - z_n\rangle \\
&\leq (1 - 2\lambda_n\alpha_n)\|w_n - p\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|z_n - w_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|z_n - q_n\|^2 \\
&+ 2\lambda_n\alpha_n L_2\|w_n - p\|\|w_n - z_n\| \\
&\leq (1 - 2\lambda_n\alpha_n)\|w_n - p\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|z_n - w_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|z_n - q_n\|^2 \\
&+ \lambda_n\alpha_n L_2\Big(\|w_n - p\|^2 + \|w_n - z_n\|^2\Big) \\
&= (1 - \lambda_n\alpha_n(2 - L_2))\|w_n - p\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \lambda_n\alpha_n L_2\right)\|z_n - w_n\|^2 \\
&- \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|z_n - q_n\|^2. \qquad (3.2.12)
\end{aligned}$$

Thus, considering the limit

$$\lim_{n\to\infty}(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}) = 1 - \mu > 0.$$

Hence, there exists $N \geq 0$, such that $n \geq N$, we have that $1 - \frac{\mu\lambda_n}{\lambda_{n+1}} > 0$. Thus, it follows that for all $n \geq N$, we have

$$\|q_n - p\|^2 = \|w_n - p\|^2 \Rightarrow \|q_n - p\| \leq \|w_n - p\|. \qquad (3.2.13)$$

Thus, we have that

$$\begin{aligned}
\|u_n - p\| &\leq \gamma_n\|w_n - p\| + (1 - \gamma_n)\|q_n - p\| \\
&\leq \gamma_n\|w_n - p\| + (1 - \gamma_n)\|w_n - p\| \\
&= \|w_n - p\|. \qquad (3.2.14)
\end{aligned}$$

$$
\begin{aligned}
\|x_{n+1}-p\| &\le \beta_n\|f(x_n)-f(p)\| + \beta_n\|f(p)-p\| + \eta_n\|x_n-p\| + \delta_n\|Su_n-p\| \\
&\le \beta_n k\|x_n-p\| + \alpha_n\|f(p)-p\| + \eta_n\|x_n-p\| + \delta_n\|u_n-p\| \\
&\le \beta_n k\|x_n-p\| + \alpha_n\|f(p)-p\| + \eta_n\|x_n-p\| + \delta_n\|w_n-p\| \\
&\le \beta_n k\|x_n-p\| + \alpha_n\|f(p)-p\| + \eta_n\|x_n-p\| + \delta_n\|x_n-p\| + \delta_n\beta_n N_1 \\
&= (1-\beta_n(1-k))\|x_n-p\| + \delta_n\beta_n N_1 + \beta_n\|f(p)-p\| \\
&\le (1-\beta_n(1-k))\|x_n-p\| + \beta_n(1-k)\left[\frac{\delta_n N_1 + \|f(p)-p\|}{(1-k)}\right]. \qquad (3.2.15)
\end{aligned}
$$

It follows from induction that

$$
\|x_{n+1}-p\| \le \max\{\|x_0-p\|, \frac{\delta_0 N_1 + \|f(p)-p\|}{(1-k)}\}. \qquad (3.2.16)
$$

Thus, we have that $\{x_n\}$ is bounded. $\square$

Theorem 3.2.6. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.2.2. Then, under the Assumption 3.2.1. Then, $\{x_n\}$ converges strongly to $p^* \in \Gamma$, where $p^* = P_\Gamma \circ f(p^*)$.*

Proof. Let $p \in \Omega$, observe that

$$
\begin{aligned}
\|w_n-p\|^2 &= \|x_n + \theta_n(S_n x_n - S_n x_{n-1}) - p\|^2 \\
&= \|x_n-p\|^2 + 2\theta_n\langle x_n-p, S_n x_n - S_n x_{n-1}\rangle + \theta_n^2\|S_n x_n - S_n x_{n-1}\|^2 \\
&\le \|x_n-p\|^2 + 2\theta_n\|x_n-p\|\|x_n-x_{n-1}\| + \theta_n^2\|x_n-x_{n-1}\|^2 \\
&= \|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|[2\|x_n-p\| + \theta_n\|x_n-x_{n-1}\|] \\
&= \|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|[2\|x_n-p\| + \beta_n\frac{\theta_n}{\beta_n}\|x_n-x_{n-1}\|] \\
&\le \|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|[2\|x_n-p\| + \beta_n N_1] \\
&\le \|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|N_2, \qquad (3.2.17)
\end{aligned}
$$

for some $N_2 > 0$.

More so, using Lemma (2.1.1) (3) and (3.2.13), we have

$$
\begin{aligned}
\|u_n-p\|^2 &= \gamma_n\|w_n-p\|^2 + (1-\gamma_n)\|q_n-p\|^2 - \gamma_n(1-\gamma_n)\|w_n-q_n\|^2 \\
&\le \gamma_n\|w_n-p\|^2 + (1-\gamma_n)\|w_n-p\|^2 - \gamma_n(1-\gamma_n)\|w_n-q_n\|^2 \\
&= \|w_n-p\|^2 - \gamma_n(1-\gamma_n)\|w_n-q_n\|^2 \\
&\le \|w_n-p\|^2. \qquad (3.2.18)
\end{aligned}
$$

Furthermore, using (3.2.18) and (3.2.17), we have

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&= \|\beta_n f(x_n)+\eta_n x_n+\delta_n Su_n - p\|^2\\
&\le \|\eta_n(x_n-p)+\delta_n(Su_n-p)\|^2 + 2\beta_n\langle f(x_n)-p, x_{n+1}-p\rangle\\
&\le \eta_n^2\|x_n-p\|^2+\delta_n^2\|Su_n-p\|^2+2\delta_n\eta_n\|x_n-p\|\|Su_n-p\|+2\beta_n\langle f(x_n)-p,x_{n+1}-p\rangle\\
&\le \eta_n^2\|x_n-p\|^2+\eta_n^2\|u_n-p\|^2+\delta_n\eta_n(\|x_n-p\|^2+\|u_n-p\|^2)\\
&+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \eta_n(\delta_n+\eta_n)\|x_n-p\|^2+\delta_n(\eta_n+\delta_n)\|u_n-p\|^2+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle\\
&+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \eta_n(\delta_n+\eta_n)\|x_n-p\|^2+\delta_n(\eta_n+\delta_n)\|w_n-p\|^2\\
&+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \eta_n(\delta_n+\eta_n)\|x_n-p\|^2+\delta_n(\eta_n+\delta_n)[\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2]\\
&+2\alpha_n k\|x_n-p\|\|x_{n+1}-p\|+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&=(\delta_n+\eta_n)^2\|x_n-p\|^2+\delta_n(\eta_n+\delta_n)\theta_n\|x_n-x_{n-1}\|N_2\\
&+\beta_n k\|x_n-p\|^2+\beta_n k\|x_{n+1}-p\|^2+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le ((1-\beta_n)^2+\beta_n k)\|x_n-p\|^2+\delta_n(1-\beta_n)\theta_n\|x_n-x_{n-1}\|N_2\\
&+\alpha_n k\|x_{n+1}-p\|^2+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\
&=(1-2\beta_n+\beta_n k)\|x_n-p\|^2+\beta_n^2\|x_n-p\|^2+\delta_n(1-\beta_n)\theta_n\|x_n-x_{n-1}\|N_2\\
&+\beta_n k\|x_{n+1}-p\|^2+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle,
\end{aligned}
\tag{3.2.19}
$$

this implies that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le \left(1-\frac{2\beta_n(1-k)}{1-\beta_n k}\right)\|x_n-p\|^2+\frac{2\beta_n(1-k)}{1-\beta_n k}\left[\frac{\delta_n(1-\beta_n)\theta_n}{2\beta_n(1-k)}\|x_n-x_{n-1}\|N_2+\frac{\beta_n N_3}{2(1-k)}\right.\\
&\left.+\frac{1}{((1-k)}\langle f(p)-p,x_{n+1}-p\rangle\right]\\
&=\left(1-\frac{2\beta_n(1-k)}{1-\beta_n k}\right)\|x_n-p\|^2+\frac{2\beta_n(1-k)}{1-\beta_n k}\Psi_n,
\end{aligned}
\tag{3.2.20}
$$

where $N_3=\sup_{n\in\mathbb{N}}\{\|x_n-p\|^2 : n\ge\mathbb{N}\}$ and $\Psi_n=\Big[\frac{\delta_n(1-\alpha_n)\theta_n}{2\alpha_n(1-k)}\|x_n-x_{n-1}\|N_2+\frac{\beta_n N_3}{2(1-k)}$ $+\frac{1}{((1-k)}\langle f(p)-p,x_{n+1}-p\rangle\Big]$. According to Lemma 7.1.13, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\Psi_n\le 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying the condition:

$$
\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\}\ge 0. \tag{3.2.21}
$$

To establish that $\limsup_{k\to\infty}\Psi_n\le 0$, we suppose that for every subsequence $\{\|x_{n_k}-p\|\}$

of $\{\|x_n - p\|\}$ such that (4.2) holds. Then,

$$\begin{aligned}&\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\}\\&=\liminf_{k\to\infty}\{(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|)\}\geq 0.\end{aligned}\tag{3.2.22}$$

Now, from (3.2.19) and some simple calculations, we obtain

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&\leq\left(1-\frac{2\beta_n(1-k)}{1-\beta_nk}\right)\|x_n-p\|^2+\frac{2\beta_n(1-k)}{1-\beta_nk}\left[\frac{\delta_n(1-\beta_n)\theta_n}{2\beta_n(1-k)}\|x_n-x_{n-1}\|N_2+\frac{\beta_nN_3}{2(1-k)}\right.\\&\left.-\frac{\gamma_n(1-\gamma_n)\delta_n(1-\beta_n)}{2\alpha_n(1-\alpha_nk)(1-k)}\|w_n-q_n\|^2+\frac{1}{((1-k)}\langle f(p)-p,x_{n+1}-p\rangle\right]\\&\leq\|x_n-p\|^2+\frac{2\beta_n(1-k)}{1-\beta_nk}\left[\frac{\delta_n(1-\beta_n)\theta_n}{2\beta_n(1-k)}\|x_n-x_{n-1}\|N_2+\frac{\beta_nN_3}{2(1-k)}\right.\\&\left.-\frac{\gamma_n(1-\gamma_n)\delta_n(1-\beta_n)}{2\beta_n(1-k)}\|w_n-q_n\|^2+\frac{1}{((1-k)}\langle f(p)-p,x_{n+1}-p\rangle\right],\end{aligned}\tag{3.2.23}$$

which implies that

$$\begin{aligned}&\limsup_{k\to\infty}\left(\gamma_{n_k}(1-\gamma_{n_k})\delta_{n_k}(1-\beta_{n_k})\|w_{n_k}-q_{n_k}\|^2\right)\\&\leq\limsup_{k\to\infty}\left[\|x_{n_k}-p\|^2+\frac{1}{1-\beta_{n_k}k}\left[\beta_{n_k}\delta_{n_k}(1-\beta_{n_k})\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\right.\right.\\&\left.\left.+\beta_{n_k}^2N_3+2\beta_n\langle f(p)-p,x_{n_k+1}-p\rangle\right]-\|x_{n_k+1}-p\|^2\right]\\&\leq-\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0.\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty}\|w_{n_k}-q_{n_k}\|=0.\tag{3.2.24}$$

In addition, using (3.2.18) and (3.2.17), we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|\alpha_n f(x_n)+\eta_n x_n+\delta_n Su_n-p\|^2\\
&\le \|\eta_n(x_n-p)+\delta_n(Su_n-p)\|^2+2\beta_n\langle f(x_n)-p,x_{n+1}-p\rangle\\
&\le \eta_n^2\|x_n-p\|^2+\delta_n^2\|Su_n-p\|^2+2\delta_n\eta_n\|x_n-p\|\|Su_n-p\|+2\beta_n\langle f(x_n)-p,x_{n+1}-p\rangle\\
&\le \eta_n^2\|x_n-p\|^2+\delta_n^2\|u_n-p\|^2+\delta_n\eta_n(\|x_n-p\|^2+\|u_n-p\|^2)+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle\\
&+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \eta_n\|x_n-p\|^2+\delta_n\|u_n-p\|^2+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \eta_n\|x_n-p\|^2+\delta_n\gamma_n\|w_n-p\|^2+(1-\gamma_n)\|q_n-p\|^2-\gamma_n(1-\gamma_n)\|w_n-q_n\|^2\\
&+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \eta_n\|x_n-p\|^2+\delta_n\|w_n-p\|^2+\delta_n(1-\gamma_n)[(1-\lambda_n\alpha_n(2-L))\|w_n-p\|^2-\\
&\left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}-\lambda_n\alpha_n L\right)\|z_n-w_n\|]\\
&-\gamma_n(1-\gamma_n)\|w_n-q_n\|^2+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \eta_n\|x_n-p\|^2+\delta_n[\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2]-\left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}-\lambda_n\alpha_n L\right)(1-\gamma_n)\|z_n-w_n\|^2]\\
&-\gamma_n(1-\gamma_n)\|w_n-q_n\|^2+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\le \|x_n-p\|^2+\delta_n\theta_n\|x_n-x_{n-1}\|N_2-\left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}-\delta_n\lambda_n\alpha_n L\right)(1-\gamma_n)\|z_n-w_n\|^2\\
&-\gamma_n(1-\gamma_n)\|w_n-q_n\|^2+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle,
\end{aligned} \tag{3.2.25}$$

which implies that

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\left(1-\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}}-\delta_{n_k}\lambda_{n_k}\alpha_{n_k}L\right)(1-\gamma_{n_k})\|z_{n_k}-w_{n_k}\|^2\right)\\
&\le \limsup_{k\to\infty}\Bigg[\|x_{n_k}-p\|^2+\delta_{n_k}\beta_{n_k}\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2-\gamma_{n_k}(1-\gamma_{n_k})\|w_{n_k}-q_{n_k}\|^2\\
&+2\beta_{n_k}\langle f(x_{n_k})-f(p),x_{n_k+1}-p\rangle+2\beta_{n_k}\langle f(p)-p,x_{n_k+1}-p\rangle-\|x_{n_k+1}-p\|^2\Bigg]\\
&\le -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\le 0.
\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty}\|z_{n_k}-w_{n_k}\|=0. \tag{3.2.26}$$

Using a similar approach as in (3.2.19) and using (3.2.26), we have that

$$\lim_{k\to\infty}\|z_{n_k}-q_{n_k}\|=0. \tag{3.2.27}$$

In addition, we have that

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&= \beta_n\|f(x_n)-p\|^2+\eta_n\|x_n-p\|^2+\delta_n\|Su_n-p\|^2-\eta_n\delta_n\|x_n-Su_n\|^2\\
&\leq \beta_n\|f(x_n)-p\|^2+\eta_n\|x_n-p\|^2+\delta_n\|u_n-p\|^2-\eta_n\delta_n\|x_n-Su_n\|^2\\
&\leq \beta_n\|f(x_n)-p\|^2+\eta_n\|x_n-p\|^2+\delta_n\|w_n-p\|^2-\eta_n\delta_n\|x_n-Su_n\|^2\\
&\leq \beta_n\|f(x_n)-p\|^2+\eta_n\|x_n-p\|^2+\delta_n\|x_n-p\|^2+\delta\theta_n\|x_n-x_{n-1}\|N_2-\eta_n\delta_n\|x_n-Su_n\|^2\\
&= (\eta_n+\delta_n)\|x_n-p\|^2+\beta_n\|f(x_n)-p\|^2+\delta\theta_n\|x_n-x_{n-1}\|N_2-\eta_n\delta_n\|x_n-Su_n\|^2\\
&\leq \|x_n-p\|^2+\beta_n\|f(x_n)-p\|^2+\delta\theta_n\|x_n-x_{n-1}\|N_2-\eta_n\delta_n\|x_n-Su_n\|^2,
\end{aligned}\tag{3.2.28}$$

which implies that

$$\limsup_{k\to\infty}\left(\eta_n\delta_n\|x_{n_k}-Su_{n_k}\|^2\right) \leq \limsup_{k\to\infty}\left[\|x_{n_k}-p\|^2+\delta\beta_{n_k}\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\right.\tag{3.2.29}$$

$$\left.+\beta_n\|f(x_n)-p\|^2-\|x_{n_k+1}-p\|^2\right]$$

$$\leq -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0.\tag{3.2.30}$$

Using (3.2.20), we have that

$$\lim_{k\to\infty}\|x_{n_k}-Su_{n_k}\|=0.\tag{3.2.31}$$

It is easy to see that, as $k\to\infty$, we have

$$\|w_{n_k}-x_{n_k}\|=\theta_{n_k}\|x_{n_k}-x_{n_k-1}\|=\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|\to 0.\tag{3.2.32}$$

In addition, we have that

$$\|u_{n_k}-w_{n_k}\|\leq\gamma_{n_k}\|w_{n_k}-w_{n_k}\|+(1-\gamma_{n_k})\|q_{n_k}-w_{n_k}\|\to 0 \;\text{ as }\; k\to\infty.\tag{3.2.33}$$

$$\|u_{n_k}-x_{n_k}\|\leq\|u_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\to 0 \;\text{ as }\; k\to\infty.\tag{3.2.34}$$

$$\|u_{n_k}-Su_{n_k}\|\leq\|u_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|+\|x_{n_k}-Su_{n_k}\|\to 0 \;\text{ as }\; k\to\infty.\tag{3.2.35}$$

Thus, we have

$$\|x_{n_k+1}-x_{n_k}\|\leq\beta_n\|f(x_{n_k})-x_{n_k})\|+\eta_n\|x_n-x_{n_k}\|+\delta_{n_k}\|Su_{n_k}-x_{n_k}\|\to 0 \;\text{ as }\; k\to\infty.\tag{3.2.36}$$

Now, since $\{x_{n_k}\}$ is bounded, then there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $x^*\in H$. In addition, using (3.2.34) and the boundedness of $\{u_{n_k}\}$, there exists there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ such that $\{u_{n_{k_j}}\}$ converges weakly to $x^* \in H$ and since S is demiclosed with (3.2.35), we have that $x^* \in F(S)$. Furthermore, we obtain that

$$\limsup_{k\to\infty} \langle f(p) - p, x_{n_k} - p\rangle = \lim_{j\to\infty} \langle f(p) - p, x_{n_{k_j}} - p\rangle = \langle f(p) - p, x^* - p\rangle. \tag{3.2.37}$$

Hence, since p is a unique solution of $RVIP$, we have obtain from (3.2.37) that

$$\limsup_{k\to\infty} \langle f(p) - p, x_{n_k} - p\rangle = \langle f(p) - p, x^* - p\rangle \leq 0, \tag{3.2.38}$$

which implies that

$$\limsup_{k\to\infty} \langle f(p) - p, x_{n_k+1} - p\rangle \leq 0. \tag{3.2.39}$$

Using our assumption and (3.2.39), we have that $\Psi_n = \left[\frac{\delta_n(1-\beta_n)\theta_n}{2\beta_n(1-k)}\|x_n - x_{n-1}\|N_2 + \frac{\beta_n N_3}{2(1-k)} + \frac{1}{((1-k)}\langle f(p) - p, x_{n+1} - p\rangle\right] \leq 0$. Thus, From Lemma 2.1.37, we have that $\lim_{n\to\infty} \|x_n - p\| = 0$. From Lemma 2.6.8 (iii), we obtain that $\|p - p^*\| \to 0$ as $n \to \infty$, thus, we have that

$$\|x_n - p^*\| \leq \|x_n - p\| + \|p - p^*\| \to 0 \quad \text{as } n \to \infty. \tag{3.2.40}$$

Thus, $\{x_n\}$ converges strongly to $p^* \in \Gamma$. □

3.2.3 Numerical Experiments

In this section, we present some numerical experiments to show the efficiency and applicability of our method in comparison with our type of Algorithm without the sequence $\{S_n\}$ in the inertial term in the framework of infinite and finite dimensional Hilbert spaces.

Example 3.2.7. *Let $H = L_2([0,1])$ be equipped with the inner product*

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \ \ \forall \ \ x, y \in L_2([0,1]) \ \ \textit{and} \ \ \|x\|^2 := \int_0^1 |x(t)|^2 dt \ \ \forall x, y, \in L_2([0,1]).$$

Let $F; A; f : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Ax(t) = \max\{0, x(t)\}, \quad t \in [0,1], Fx(t) = fx(t) = \frac{x(t)}{2}.$$

It is easy to see that A is 1-Lipschitz continuous and monotone, F γ-strongly monotone and f is a contraction on $L_2([0,1])$. Let $S_n; S : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Sx(s) = \int_0^1 t^n x(s)ds \ \ \forall \ \ t \in L_2([0,1])$$

and

$$S_n x(t) = \int_0^1 \sin x(t).$$

Let C *be defined by* $C = \{x \in L_2 : \langle a, x\rangle = b\}$ *where* $a \neq 0$ *and* $b = 2$*. Thus, we have*

$$P_C(\bar{x}) = \max\left\{0, \frac{b - \langle a, \bar{x}\rangle}{\|a\|^2}\right\} a + \bar{x}.$$

We choose $\zeta_n = 0.25, \mu = 0.5, \theta_n = \overline{\theta}, \alpha_n = \frac{1}{n+6}, \beta_n = \frac{1}{5n+6}, \eta_n = \frac{2}{3n+2}, \delta_n = 1-\eta_n-\beta_n, \epsilon_n = \frac{10^{20}}{n^2}$*, for all*

$n \in \mathbb{N}$*. It is easy to verify that all hypotheses of Theorem 3.2.6 are satisfied. We implement our algorithm for different values of* x_0, x_1 *as follows.*

Case I: $x_0(t) = 2t^2 + t + 2, \quad x_1(t) = t + 2;$

Case II: $x_0(t) = 2t^2, \quad x_1(t) = -5t + 2;$

Case III: $x_0(t) = t, \quad x_1(t) = \log(t);$

Case IV: $x_0(t) = 5t + 1, \; x_1(t) = 3t^2.$

Example 3.2.8. *Let* $H = \mathbb{R}^2$*, consider a nonlinear operator* $A : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$A(x_1, x_2) = (x_1 + x_2 + \cos(x_1), -x_1 + x_2 + \cos(x_2)),$$

$f(x) = \frac{x}{2}, F(x) = \sin x$ *and* C *be defined as* $C = [-1, 1] \times [-1, 1]$*. It is easy to see that* A *is* 3*-Lipschitz continuous and monotone,* F γ*-strongly monotone and* f *is a contraction on* $\mathbb{R}^2$*. Let* Y *be a* 2×2 *matrix defined by*

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

We define the mapping $S : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $Sx = \|Y\|^{-1}Yx$*, where* $x = (x_1, x_2)^T$*. It is easily see that* S *is a nonexpansive mapping. We choose* $\zeta_n = \frac{1}{(n+1)^2}, \mu = 0.5, \theta_n = \overline{\theta} = \frac{1}{3}, \alpha_n = \frac{1}{50n+13}, \beta_n = \frac{1}{5n+6}, \eta_n = \frac{2n}{3n+2}, \delta_n = 1 - \eta_n - \beta_n, \epsilon_n = \frac{1}{n^2}$*, and* $\gamma_n = \frac{1}{2n+1}$*. For, Algorithm 3.2.2 and Dang et al. [161], we choose* $\lambda_1 = 0.75$*, HEG of [218], we choose* $\lambda_n = \frac{1}{4.5}$*. It is easy to verify that all hypothesis of Theorem 3.2.6 are satisfied. We implement our algorithm for different values of* x_0, x_1 *as follows.*

Case I: $x_0 = (1, 2)', \qquad x_1 = (1.2, 0.5);$

Case II: $x_0 = (1, 0)', \quad x_1 = (0, 1)';$

Case III: $x_0 = (0.98, 1.02)', \quad x_1 = (1.50, 2.36)';$

Case IV: $x_0 = (-2, -4)', \quad x_1 = (1, 0.5)';$

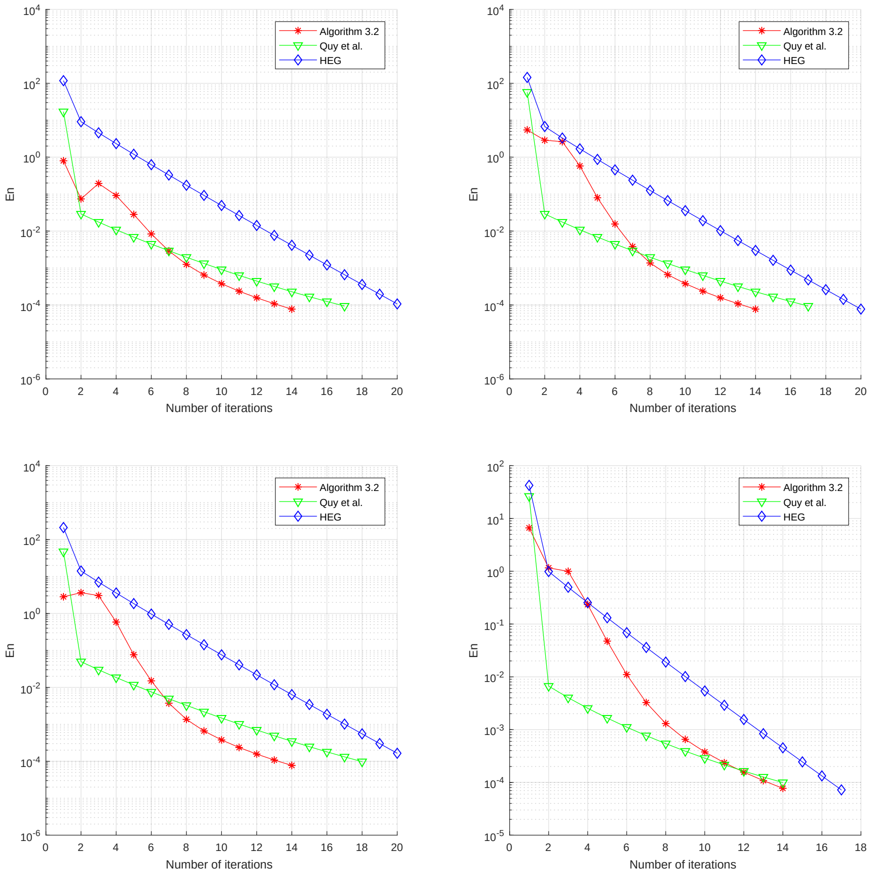

Figure 3.3: Example 3.2.7, Top Left: Case I; Top Right: Case II; Bottom left: Case III; Bottom right: Case IV.

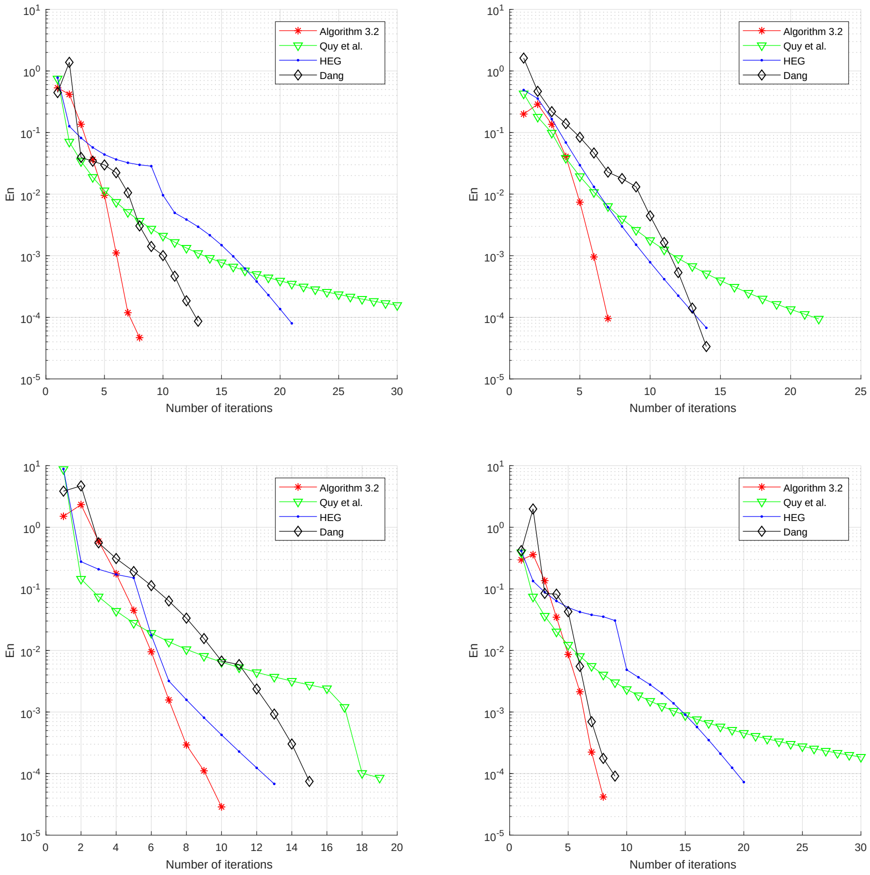

Figure 3.4: Example 4.1.11, Top Left: Case I; Top Right: Case II; Bottom left: Case III; Bottom right: Case IV

3.3 A New Method with Regularization for Solving Split Variation Inequality Problems in Real Hilbert Spaces

In this section, we provide an affirmative answer to Questions raised in Chapter 2 of this study by introducing a new inertial extrapolation method with regularization for approximating solutions of split variational inequality problems in the framework of real Hilbert spaces. We prove that the proposed method converges strongly to a minimum-norm solution of the problem without using the conventional two cases approach. In addition, we present some numerical experiments to show the efficiency and applicability of the proposed method. The results obtained in this study extend, generalize and improve several results in this direction.

3.3.1 Main Result

In this section, we present our proposed method and highlight some of its important features. We begin with the following assumptions under which our strong convergence is obtained.

Assumption 3.3.1. *Suppose that the following conditions hold:*

1. *The sets C and Q are nonempty closed and convex subsets of the real Hilbert spaces H_1 and H_2 respectively.*
2. *$A_1 : H_1 \to H_1$ is monotone and Lipschitz continuous operator and $A_2 : H_2 \to H_2$ is α-inverse strongly monotone operator.*
3. *$T : H_1 \to H_2$ is a bounded linear operator.*
4. *The solution set $\Gamma = \{x \in VI(A_1, C) : Tx \in VI(A_2, Q)\} \neq \emptyset$, where $VI(A_1, C)$ is the solution set for the classical VIP (1.2.1).*

We present the following iterative algorithm.

Algorithm 3.3.2. Initialization: *Given $\lambda, \gamma_n > 0$, $\theta_n, \alpha_n, \mu \in (0,1)$, and $\beta_n \subset (b, 1-\alpha_n)$ for some $b > 0$, for all $n \in \mathbb{N}$. Let $x_0, x_1, \in H$ be arbitrary.*

Iterative steps:
Step 1: *Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\max\{n^2\|x_n - x_{n-1}\|^2, n^2\|x_n - x_{n-1}\|\}}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise} \end{cases} \tag{3.3.1}$$

with θ being a positive constant and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$.

Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$u_n = w_n - \gamma_n(T^*(I - P_Q(I - \eta A_2))T + \alpha_n I)w_n, \tag{3.3.2}$$

where $\eta \in (0, 2\alpha)$ and

$$\gamma_n \in \left(\epsilon, \frac{\|P_Q(I - \lambda_n A_2) - I)Tw_n\|^2}{\|T^*(P_Q(I - \lambda_n A_2) - I)Tw_n\|^2} - \epsilon\right), \text{ if } P_Q(I - \lambda_n A_2)Tw_n \neq Tw_n$$

otherwise $\gamma_n = \epsilon, .$

Step 3. *Compute*

$$v_n = P_C(u_n - \lambda_n A_1 u_n) \tag{3.3.3}$$

$$y_n = u_n - \tau_n b_n, \tag{3.3.4}$$

where $b_n = u_n - v_n - \lambda_n(A_1 u_n - A_1 v_n)$; $\tau_n = \frac{\langle u_n - v_n, b_n\rangle}{\|b_n\|^2}$ if $b_n \neq 0$; otherwise $\tau_n = 0$; and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|u_n - v_n\|}{\|A_1 u_n - A_1 v_n\|},\ \lambda_n\right\}, & \text{if } A_1 u_n \neq A_1 v_n, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{3.3.5}$$

Step 4. *Compute*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n y_n. \tag{3.3.6}$$

Remark 3.3.3.

1. *A notable advantage of this method (Algorithm 3.3.2) is that $\{x_n\}$ converges strongly to a minimum-norm solution of the SVIP. This is very desirable in optimization theory.*

2. *The choice of the stepsize $\{\gamma_n\}$ used in Algorithm 3.3.2 does not require the prior knowledge of the operator norm $\|T\|$ which is very difficult to find in practice. In addition, the stepsize $\{\lambda_n\}$ is self adaptive.*

3. *As we shall see in our convergence analysis, we do not use the popular two cases method usually used in numerous papers to guarantee strong convergence. Thus the techniques and ideas employed in our strong convergence analysis are new for solving the problem considered in this paper.*

4. *In Algorithm 3.3.2, it is easy to compute step 1 since the value of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. It is also easy to see from (3.3.1) that $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$.*

   Indeed, since, $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n)$, which means that $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, we have that $\theta_n\|x_n - x_{n-1}\| \leq \epsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, it implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$

5. *It is easy to see in (3.3.5) that* $\lambda_{n+1} \leq \lambda_n$ *for all* $n \in \mathbb{N}$. *Furthermore, since* A_1 *is* L*-Lipschitz continuous, we obtain in the case when* $Au_n \neq Av_n$ *that*

$$\frac{\mu\|u_n - v_n\|}{\|A_1u_n - A_1v_n\|} \geq \frac{\mu\|u_n - v_n\|}{L\|u_n - v_n\|} = \frac{\mu}{L},$$

which follows that $\lambda_n \geq \min\{\lambda_1, \frac{\mu}{L}\}$ *for all* $n \in \mathbb{N}$. *This gives that the limit of* $\{\lambda_n\}$ *exists and* $\lim\limits_{n\to\infty} \lambda_n \geq \min\{\lambda_1, \frac{\mu}{L}\} > 0$.

3.3.2 Convergence Analysis

In this section, we establish a strong convergence result of our proposed method.

Lemma 3.3.4. *Let* $\{x_n\}$ *be a sequence generated by Algorithm 3.3.2. Then, under Assumption 3.3.1, we have that* $\{x_n\}$ *is bounded.*

Proof. Let $p \in \Gamma$ and since $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_1$, for all $n \in \mathbb{N}$. Then from **Step** 2, we have

$$\begin{aligned}\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\ &\leq \|x_n - p\| + \alpha_n N_1. \end{aligned} \tag{3.3.7}$$

Using Lemma 2.1.1, we obtain

$$\begin{aligned}&\|u_n - p\|^2 \\ &= \|w_n + \gamma_n(T^*(P_Q(I - \eta A_2) - I)T - \alpha_n I)w_n - p\|^2 \\ &= \|w_n - p\|^2 + \|\gamma_n(T^*(P_Q(I - \eta A_2) - I)T - \alpha_n I)w_n\|^2 \\ &+ 2\langle w_n - p, \gamma_n(T^*(P_Q(I - \eta A_2) - I)T - \alpha_n I)w_n\rangle \\ &\leq \|w_n - p\|^2 + \gamma_n^2\|T^*P_Q(I - \eta A_2) - I)Tw_n\|^2 + 2\langle \gamma_n\alpha_n w_n, \gamma_n(T^*(P_Q(I - \eta A_2) - I)T - \alpha_n I)w_n\rangle \\ &+ 2\langle w_n - p, \gamma_n T^*(P_Q(I - \eta A_2) - I)Tw_n\rangle + 2\langle w_n - p, -\gamma_n\alpha_n w_n\rangle \\ &= \|w_n - p\|^2 + \gamma_n^2\|T^*(P_Q(I - \eta A_2) - I)Tw_n\|^2 + 2\gamma_n\langle w_n - p, T^*(P_Q(I - \eta A_2) - I)Tw_n\rangle \\ &- \gamma_n\alpha_n\langle 2(u_n - p) + \gamma_n\alpha_n w_n, w_n\rangle. \end{aligned} \tag{3.3.8}$$

Now, observe that

$$
\begin{aligned}
&\langle w_n - p, T^*(P_Q(I-\eta A_2)-I)Tw_n\rangle\\
&= \langle Tw_n - Tp, P_Q(I-\eta A_2)Tw_n - Tw_n\rangle\\
&= \langle Tw_n + P_Q(I-\eta A_2)Tw_n - P_Q(I-\eta A_2)Tw_n - Tw_n + Tw_n - Tp,\\
&P_Q(I-\eta A_2)Tw_n - Tw_n\rangle\\
&= \langle P_Q(I-\eta A_2)Tw_n - Tp, P_Q(I-\eta A_2)Tw_n - Tw_n\rangle\\
&- \|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2\\
&= \frac{1}{2}[\|P_Q(I-\eta A_2)Tw_n - Tp\|^2 + \|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2 - \|Tw_n - Tp\|^2]\\
&- \|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2\\
&\leq \frac{1}{2}\|Tw_n - Tp\|^2 - \frac{1}{2}\|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2 - \frac{1}{2}\|Tw_n - Tp\|^2\\
&= -\frac{1}{2}\|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2. \qquad (3.3.9)
\end{aligned}
$$

Substituting (3.3.9) into (3.3.8), we have

$$
\begin{aligned}
&\|u_n - p\|^2\\
&\leq \|w_n - p\|^2 + \gamma_n^2\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2 + -\gamma_n\|P_Q(I-\eta A_2)Tw_n - Tw_n\|^2\\
&- \gamma_n\alpha_n\langle 2(u_n - p) + \gamma_n\alpha_n w_n, w_n\rangle\\
&\leq \|w_n - p\|^2 + \gamma_n^2\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2 - \gamma_n(\gamma_n + \epsilon)\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2\\
&- \gamma_n\alpha_n\langle 2(u_n - p) + \gamma_n\alpha_n w_n, w_n\rangle\\
&= \|w_n - p\|^2 - \gamma_n[\epsilon\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2 + \alpha_n\langle 2(u_n - p) + \gamma_n\alpha_n w_n, w_n\rangle] \qquad (3.3.10)\\
&\leq \|w_n - p\|^2,
\end{aligned}
$$

and this implies that

$$
\|u_n - p\| \leq \|w_n - p\|. \qquad (3.3.11)
$$

Since $v_n = P_C(u_n - \lambda_n A_1 u_n)$ and $p \in VI(A_1, C) \subset C$, then by the characterization of P_C, we have

$$
\langle v_n - p, v_n - u_n + \lambda_n A_1 u_n\rangle \leq 0.
$$

Using the monotonicity of A_1, we obtain

$$
\begin{aligned}
\langle v_n - p, b_n\rangle &= \langle v_n - p, u_n - v_n - \lambda_n A_1 u_n\rangle + \lambda_n\langle v_n - p, A_1 v_n\rangle\\
&\geq \lambda_n\langle v_n - p, A_1 v_n\rangle\\
&= \lambda_n\langle v_n - p, A_1 v_n - A_1 p\rangle + \lambda_n\langle v_n - p, A_1 p\rangle \geq 0.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\langle u_n - p, b_n\rangle &= \langle u_n - v_n, b_n\rangle + \langle v_n - p, b_n\rangle\\
&\geq \langle u_n - v_n, b_n\rangle. \qquad (3.3.12)
\end{aligned}
$$

Hence, from **Step** 3 and (3.3.12), we have

$$\begin{aligned}\|y_n-p\|^2 &= \|u_n-\tau_n b_n-p\|^2\\ &= \|u_n-p\|^2+\tau_n^2\|b_n\|^2-2\tau_n\langle u_n-p,b_n\rangle\\ &\leq \|u_n-p\|^2+\tau_n^2\|b_n\|^2-2\tau_n\langle u_n-v_n,b_n\rangle\\ &\leq \|u_n-p\|^2+\tau_n^2\|b_n\|^2-2\tau_n^2\|b_n\|^2\\ &= \|u_n-p\|^2-\|\tau_n b_n\|^2\\ &\leq \|u_n-p\|^2, \end{aligned}\tag{3.3.13}$$

and this implies that

$$\|y_n-p\|\leq\|w_n-p\|.\tag{3.3.14}$$

In addition, we observe that

$$\begin{aligned}&\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)\|^2\\ &=(1-\alpha_n-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|y_n-p\|^2+2(1-\alpha_n-\beta_n)\beta_n\langle x_n-p,y_n-p\rangle\\ &\leq(1-\alpha_n-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|y_n-p\|^2+2(1-\alpha_n-\beta_n)\beta_n\|x_n-p\|\|y_n-p\|\\ &\leq(1-\alpha_n-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|y_n-p\|^2\\ &+(1-\alpha_n-\beta_n)\beta_n\|x_n-p\|^2+(1-\alpha_n-\beta_n)\beta_n\|y_n-p\|^2\\ &=(1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)\beta_n\|y_n-p\|^2\\ &\leq(1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)\beta_n\|w_n-p\|^2\\ &\leq(1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)\beta_n[\|x_n-p\|+\alpha_nN_1]^2\\ &=(1-\alpha_n-\beta_n)(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)\beta_n\|x_n-p\|^2\\ &+2(1-\alpha_n)\beta_n\alpha_n\|x_n-p\|N_1+(1-\alpha_n)\beta_n\alpha_n^2N_1^2\\ &\leq(1-\alpha_n)^2\|x_n-p\|^2+2(1-\alpha_n)\alpha_n\|x_n-p\|N_1+\alpha_n^2N_1^2\\ &=[(1-\alpha_n)\|x_n-p\|+\alpha_nN_1]^2,\end{aligned}\tag{3.3.15}$$

This implies that

$$\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)\|\leq(1-\alpha_n)\|x_n-p\|+\alpha_nN_1.\tag{3.3.16}$$

Lastly, we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)-\alpha_np\|\\ &\leq \|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)\|+\alpha_n\|p\|\\ &\leq (1-\alpha_n)\|x_n-p\|+\alpha_nN_1+\alpha_n\|p\|\\ &= (1-\alpha_n)\|x_n-p\|+\alpha_n(N_1+\|p\|)\\ &\leq \max\{\|x_n-p\|,N_1+\|p\|\}\\ &\vdots\\ &\leq \max\{\|x_1-p\|,N_1+\|p\|\}.\end{aligned}\tag{3.3.17}$$

Thus, $\{x_n\}$ is bounded. □

Lemma 3.3.5. *Let Assumption 3.3.1 hold and let $\{x_n\}$ be a sequence generated by Algorithm 3.3.2. Assume that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to a point x^*, and $\lim\limits_{k\to\infty}\|u_{n_k}-w_{n_k}\| = \lim\limits_{k\to\infty}\|u_{n_k}-v_{n_k}\| = 0$, then, $x^*\in\Gamma$.*

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $x^*\in H_1$. It is easy to see that

$$\|w_{n_k}-x_{n_k}\| = \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_n-1}\| \to 0 \text{ as } k\to\infty. \tag{3.3.18}$$

It follows that

$$\|u_{n_k}-x_{n_k}\| \le \|u_{n_k}-w_{n_k}\| + \|w_{n_k}-x_{n_k}\| \to 0 \text{ as } k\to\infty. \tag{3.3.19}$$

Since T is a bounded linear operator, it follows from (3.3.18) that $\{Tw_{n_k}\}$ converges weakly to $Tx^*\in Q\subset H_2$. Also, by (3.3.19), we obtain that u_{n_k} converges weakly to x^*. In addition, we have

$$\|v_{n_k}-x_{n_k}\| \le \|v_{n_k}-u_{n_k}\| + \|u_{n_k}-x_{n_k}\| \to 0 \text{ as } k\to\infty. \tag{3.3.20}$$

From (3.3.10), we have that

$$\begin{aligned}\|u_n-p\|^2 &\le \|w_n-p\|^2 - \gamma_n\epsilon\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2\\ &\le \|w_n-p\|^2 - \epsilon^2\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2,\end{aligned} \tag{3.3.21}$$

which implies that

$$\begin{aligned}\epsilon^2\|T^*(P_Q(I-\eta A_2)-I)Tw_{n_k}\|^2 &\le \|w_{n_k}-p\|^2 - \|u_{n_k}-p\|^2\\ &\le \|w_{n_k}-u_{n_k}\|^2 + 2\|u_{n_k}-p\|\|w_{n_k}-u_{n_k}\|,\end{aligned} \tag{3.3.22}$$

thus, we have that

$$\lim_{k\to\infty}\|T^*(P_Q(I-\eta A_2)-I)Tw_{n_k}\| = 0. \tag{3.3.23}$$

More so, from (3.3.10), we have

$$\begin{aligned}\|u_n-p\| &\le \|w_n-p\|^2 + \gamma_n^2\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2 - \gamma_n\|P_Q(I-\eta A_2)Tw_n-Tw_n\|^2\\ &\le \|w_n-p\|^2 + \gamma_n^2\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2 - \epsilon\|P_Q(I-\eta A_2)Tw_n-Tw_n\|^2,\end{aligned} \tag{3.3.24}$$

which implies that

$$\begin{aligned}&\epsilon\|P_Q(I-\eta A_2)Tw_{n_k}-Tw_{n_k}\|^2\\ &\le \|w_{n_k}-p\|^2 - \|u_{n_k}-p\|^2 + \gamma_n^2\|T^*(P_Q(I-\eta A_2)-I)Tw_{n_k}\|^2\\ &\le \|w_{n_k}-u_{n_k}\|^2 + 2\|u_{n_k}-p\|\|w_{n_k}-u_{n_k}\| + \gamma_n^2\|T^*(P_Q(I-\eta A_2)-I)Tw_{n_k}\|^2,\end{aligned} \tag{3.3.25}$$

which implies that

$$\lim_{k\to\infty}\|P_Q(I-\eta A_2)Tw_{n_k}-Tw_{n_k}\| = 0. \tag{3.3.26}$$

Using Lemma 2.1.28 and (3.3.26), we have that

$$Tx^* \in F(P_Q(I-\eta A_2)) \Rightarrow Tx^* \in VI(A_2,Q). \tag{3.3.27}$$

In addition, since $v_{n_k} = P_C(u_{n_k} - \lambda_{n_k} A_1 u_{n_k})$, we obtain

$$\langle u_{n_k} - \lambda_{n_k} A_1 u_{n_k} - v_{n_k}, v - v_{n_k}\rangle \leq 0 \ \forall v \in C. \tag{3.3.28}$$

Then,

$$\begin{aligned}\langle u_{n_k} - v_{n_k}, v - v_{n_k}\rangle &\leq \lambda_{n_k}\langle A_1 u_{n_k}, v - v_{n_k}\rangle \\ &\leq \lambda_{n_k}\langle A_1 u_{n_k}, u_{n_k} - v_{n_k}\rangle + \lambda_{n_k}\langle A_1 u_{n_k}, v - u_{n_k}\rangle \quad \forall \quad v \in C.\end{aligned} \tag{3.3.29}$$

Now, fix $v \in C$ and take limit as $n \to \infty$ in (3.3.29), since $\|u_{n_k} - v_{n_k}\| \to 0$ and $\liminf \lambda_{n_k} > 0$, we have

$$0 \leq \liminf_{k\to\infty}\langle A_1 u_{n_k}, v - u_{n_k}\rangle \quad \forall v \in C. \tag{3.3.30}$$

Since A_1 is monotone, we then have

$$\langle A_1 v, v - u_{n_k}\rangle \geq \langle A_1 u_{n_k}, v - u_{n_k}\rangle \quad \forall \quad v \in C. \tag{3.3.31}$$

Taking liminf of both sides, we have

$$\liminf_{k\to\infty}\langle A_1 v, v - u_{n_k}\rangle \geq \liminf_{k\to\infty}\langle A_1 u_{n_k}, v - u_{n_k}\rangle \quad \forall \quad v \in C. \tag{3.3.32}$$

So, since $\{u_{n_k}\}$ converges weakly to x^*, it then follows from (3.3.30) and (3.3.32) that

$$\langle A_1 v, v - x^*\rangle = \liminf_{k\to\infty}\langle A_1 v, v - u_{n_k}\rangle \geq 0. \tag{3.3.33}$$

Thus, using Lemma 2.2.1, we have that $x^* \in VI(A_1, C)$. from this fact and (3.3.27), we have that $x^* \in \Gamma$. □

Theorem 3.3.6. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.3.2. Then, under the Assumption 3.3.1, if $\lim\limits_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$ and $0 \leq \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Then, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $\|p\| = \min\{\|x^*\| : x^* \in \Gamma\}$.*

Proof. Let $p \in \Gamma$. To start with, observe that

$$\begin{aligned}\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\ &= \|x_n - p\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - p\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|] \\ &= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|] \\ &\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \alpha_n N_1] \\ &\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2,\end{aligned} \tag{3.3.34}$$

for some $N_2 > 0$. □

Also,

$$\begin{aligned}
&\|(1-\beta_n)x_n+\beta_n y_n-p\|^2\\
&=\|(1-\beta_n)(x_n-p)+\beta_n(y_n-p)\|^2\\
&=(1-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|y_n-p\|^2+2(1-\beta_n)\beta_n\langle x_n-p,y_n-p\rangle\\
&\leq(1-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|w_n-p\|^2+2(1-\beta_n)\beta_n\|x_n-p\|\|y_n-p\|\\
&\leq(1-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|w_n-p\|^2+(1-\beta_n)\beta_n\|x_n-p\|^2\\
&+(1-\beta_n)\beta_n\|y_n-p\|^2\\
&\leq(1-\beta_n)^2\|x_n-p\|^2+\beta_n^2\|w_n-p\|^2+(1-\beta_n)\beta_n\|x_n-p\|^2\\
&+(1-\beta_n)\beta_n\|w_n-p\|^2\\
&=(1-\beta_n)\|x_n-p\|^2+\beta_n\|w_n-p\|^2\\
&\leq(1-\beta_n)\|x_n-p\|^2+\beta_n[\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2]\\
&\leq\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2. \qquad (3.3.35)
\end{aligned}$$

We also have that

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|(1-\alpha_n)[(1-\beta_n)x_n+\beta_n y_n-p]-[\beta_n\alpha_n(x_n-y_n)+\alpha_n p]\|^2\\
&\leq(1-\alpha_n)^2\|(1-\beta_n)x_n+\beta_n y_n-p\|^2-2\langle\beta_n\alpha_n(x_n-y_n)+\alpha_n p,x_{n+1}-p\rangle\\
&\leq(1-\alpha_n)^2\|(1-\beta_n)x_n+\beta_n y_n-p\|^2+2\langle\beta_n\alpha_n(x_n-y_n),p-x_{n+1}\rangle+2\alpha_n\langle p,p-x_{n+1}\rangle\\
&\leq(1-\alpha_n)[\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2]+2\alpha_n\beta_n\|x_n-y_n\|\|x_{n+1}-p\|+2\alpha_n\langle p,p-x_{n+1}\rangle\\
&\leq(1-\alpha_n)\|x_n-p\|^2+2\alpha_n\beta_n\|x_n-y_n\|\|x_{n+1}-p\|+\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|N_2+2\alpha_n\langle p,p-x_{n+1}\rangle\\
&=(1-\alpha_n)\|x_n-p\|^2+\alpha_n[2\beta_n\|x_n-y_n\|\|x_{n+1}-p\|+\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|N_2+2\langle p,p-x_{n+1}\rangle]\\
&=(1-\alpha_n)\|x_n-p\|^2+\alpha_n\delta_n, \qquad (3.3.36)
\end{aligned}$$

where $\delta_n:=2\beta_n\|x_n-y_n\|\|x_{n+1}-p\|+\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|N_2+2\langle p,p-x_{n+1}\rangle$. According to Lemma 2.1.37, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\delta_{n_k}\leq 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\}\geq 0. \qquad (3.3.37)$$

To establish that $\limsup_{k\to\infty}\delta_{n_k}\leq 0$, we suppose that for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ such that (3.3.37) holds. Then,

$$\begin{aligned}
&\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\}\\
&=\liminf_{k\to\infty}\{(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|)\}\geq 0. \qquad (3.3.38)
\end{aligned}$$

Now, using **Step 4** , we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|(1-\alpha_n-\beta_n)x_n+\beta_n y_n-p\|^2\\
&=\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)-\alpha_n p\|^2\\
&\leq\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)\|^2+\alpha_n^2\|p\|^2-2\alpha_n\langle(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p),p\rangle\\
&\leq\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(y_n-p)\|^2+\alpha_n M\\
&\leq(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|y_n-p\|^2-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M\\
&\leq(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|w_n-p\|^2-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M\\
&\leq\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M, \qquad (3.3.39)
\end{aligned}$$

for some $M>0$. This implies from (3.3.38) that

$$\begin{aligned}
&\limsup_{k\to\infty}[(1-\alpha_{n_k}-\beta_{n_k})\beta_{n_k}\|y_{n_k}-x_{n_k}\|^2]\\
&\leq\limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2+\alpha_{n_k}M]\\
&\leq-\liminf_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2]\leq 0, \qquad (3.3.40)
\end{aligned}$$

which gives

$$\lim_{k\to\infty}\|y_{n_k}-x_{n_k}\|=0. \qquad (3.3.41)$$

Similarly, using **Step 4,** (3.3.10), (3.3.13), (3.3.34), (3.3.41) and (3.3.39), we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\leq(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n[\|u_n-p\|^2-\|\tau_n b\|^2]-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M\\
&\leq(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|w_n-p\|^2-\beta_n\|u_n-y_n\|^2-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M\\
&\leq\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2-\beta_n\|u_n-y_n\|^2-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_n M. \qquad (3.3.42)
\end{aligned}$$

This implies from (3.3.38) that

$$\begin{aligned}
\limsup_{k\to\infty}[\beta_{n_k}\|u_{n_k}-y_{n_k}\|^2]&\leq\limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\\
&\quad-(1-\alpha_{n_k}-\beta_{n_k})\beta_n\|y_{n_k}-x_{n_k}\|^2+\alpha_{n_k}M]\\
&\leq-\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0, \qquad (3.3.43)
\end{aligned}$$

which gives

$$\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\|=0. \qquad (3.3.44)$$

Now, observe that

$$\begin{aligned}\langle u_{n_k}-v_{n_k}, b_{n_k}\rangle &= \langle u_{n_k}-v_{n_k}, u_{n_k}-v_{n_k}-\lambda_{n_k}(A_1u_{n_k}-A_1v_{n_k})\rangle\\ &= \|u_{n_k}-v_{n_k}\|^2-\langle u_{n_k}-v_{n_k}, \lambda_{n_k}(A_1u_{n_k}-A_1v_{n_k})\rangle\\ &\geq \|u_{n_k}-v_{n_k}\|^2-\lambda_{n_k}\|u_{n_k}-v_{n_k}\|\|A_1u_{n_k}-A_1v_{n_k}\|\\ &\geq \|u_{n_k}-v_{n_k}\|^2-\frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}}\|u_{n_k}-v_{n_k}\|^2\\ &= (1-\frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}})\|u_{n_k}-v_{n_k}\|^2, \end{aligned} \tag{3.3.45}$$

This implies that

$$\begin{aligned}\|u_{n_k}-v_{n_k}\|^2 &\leq \frac{\lambda_{n_k+1}}{\lambda_{n_k+1}-\lambda_{n_k}\mu}\langle u_{n_k}-v_{n_k}, b_{n_k}\rangle\\ &= \frac{\lambda_{n_k+1}}{\lambda_{n_k+1}-\lambda_{n_k}\mu}\tau_{n_k}\|b_{n_k}\|^2\\ &= \frac{\lambda_{n_k+1}}{\lambda_{n_k+1}-\lambda_{n_k}\mu}\tau_{n_k}\|b_{n_k}\|\|u_{n_k}-v_{n_k}-\lambda_{n_k}(A_1u_{n_k}-A_1v_{n_k})\|\\ &\leq \frac{\lambda_{n_k+1}}{\lambda_{n_k+1}-\lambda_{n_k}\mu}\|u_{n_k}-y_{n_k}\|[\|u_{n_k}-v_{n_k}\|+\lambda_{n_k}\|A_1v_{n_k}-A_1u_{n_k}\|]\\ &= \frac{\lambda_{n_k+1}}{\lambda_{n_k+1}-\lambda_{n_k}\mu}(1+\frac{\lambda_{n_k}\mu}{\lambda_{n_k+1}})\|u_{n_k}-y_{n_k}\|\|u_{n_k}-v_{n_k}\|. \end{aligned} \tag{3.3.46}$$

Using (3.3.44), we have that

$$\lim_{k\to\infty}\|u_{n_k}-v_{n_k}\| = 0. \tag{3.3.47}$$

Using **Step 4,** (3.3.2), (3.3.10), (3.3.41) and (3.3.39), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|u_n-p\|^2-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_nM\\ &\leq (1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|w_n-p\|^2-\epsilon^2\beta_n\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2\\ &-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_nM\\ &\leq \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2-\epsilon^2\beta_n\|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2\\ &-(1-\alpha_n-\beta_n)\beta_n\|y_n-x_n\|^2+\alpha_nM, \end{aligned} \tag{3.3.48}$$

for some $M>0$. This implies from (3.3.38)

$$\begin{aligned}&\limsup_{k\to\infty}[\epsilon^2\beta_{n_k}\|T^*(P_Q(I-\eta A_2)-I)Tw_{n_k}\|^2]\\ &\leq \limsup_{k\to\infty}[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\\ &-(1-\alpha_{n_k}-\beta_{n_k})\beta_{n_k}\|y_{n_k}-x_{n_k}\|^2+\alpha_{n_k}M]\\ &\leq -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0, \end{aligned} \tag{3.3.49}$$

which gives

$$\lim_{k\to\infty} \|T^*(P_Q(I-\eta A_2)-I)Tw_n\|^2 = 0. \tag{3.3.50}$$

Using a similar approach as in (3.3.48) and (3.3.24), we have that

$$\lim_{k\to\infty} \|(P_Q(I-\eta A_2)-I)Tw_n\|^2 = 0. \tag{3.3.51}$$

Using (3.3.50) and our hypothesis, we have

$$\|u_{n_k} - w_{n_k}\| = \|w_{n_k} + \gamma_n T^*(P_Q(I-\eta A_2 - I)Tw_n - \alpha_n I)w_{n_k} - w_{n_k}\| \to 0 \text{ as } k\to\infty. \tag{3.3.52}$$

It is easy to see that, as $k \to \infty$, we have

$$\|w_{n_k} - x_{n_k}\| = \theta_{n_k}||x_{n_k} - x_{n_k-1}|| = \alpha_{n_k} \cdot \frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}|| \to 0. \tag{3.3.53}$$

In addition, we have that

$$\|w_{n_k} - y_{n_k}\| \le \|w_{n_k} - x_{n_k}\| + \|x_{n_k} - y_{n_k}\| \to 0 \quad \text{as} \quad k\to\infty. \tag{3.3.54}$$

$$\|u_{n_k} - x_{n_k}\| \le \|u_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0 \quad \text{as } k\to\infty. \tag{3.3.55}$$

$$\|v_{n_k} - x_{n_k}\| \le \|v_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0 \quad \text{as } k\to\infty. \tag{3.3.56}$$

And from the Algorithm 3.3.2 and (3.3.41), it is clear that

$$\begin{aligned} &\|x_{n_k+1} - y_{n_k}\| = \|(1-\alpha_n-\beta_n)x_{n_k} + \beta_n y_{n_k} - y_{n_k}\| \\ &\le (1-\alpha_{n_k}-\beta_{n_k})\|x_{n_k} - y_{n_k}\| + \beta_{n_k}\|y_{n_k} - y_{n_k}\| + \alpha_{n_k}\|y_{n_k}\| \to 0 \text{ as } k\to\infty. \end{aligned} \tag{3.3.57}$$

Using (3.3.57) and (3.3.41), it is easy to see that

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0 \text{ as } k\to\infty. \tag{3.3.58}$$

Since $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to x^* such that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k}\rangle = \lim_{j\to\infty}\langle p, p - x_{n_{k_j}}\rangle = \langle p, p - x^*\rangle. \tag{3.3.59}$$

Also, we obtain from (3.3.52), (3.3.47) and Lemma 3.3.5 that $x^* \in \Gamma$. Hence, from $p = P_\Omega 0$, we obtain from (3.3.59) that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k}\rangle = \langle p, p - x^*\rangle \le 0 \tag{3.3.60}$$

This implies that

$$\limsup_{k\to\infty}\langle p, p - x_{n_k+1}\rangle \le 0. \tag{3.3.61}$$

Using our assumption, (3.3.41) and (3.3.61), we have that $\limsup_{k\to\infty}\delta_{n_k} := 2\beta_n\|x_n - y_n\|\|x_{n+1} - p\| + \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|N_2 + 2\langle p, p - x_{n+1}\rangle \le 0$. Thus, the last part of Lemma 2.1.37 is achieved. Hence, we have that $\lim\limits_{n\to\infty}\|x_n - p\| = 0$. Thus, $\{x_n\}$ converges strongly to $p \in \Gamma$.

3.3.3 Numerical Examples

In this section, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite dimensional Hilbert spaces.

Example 3.3.7. *Let $H_1 = H_2 = \ell_2$ be the linear space whose elements consists of all 2-summable sequence of scalars $(x_1, x_2, \dots, x_j, \dots)$, i.e.,*

$$\ell_2 = \left\{ \bar{x} = (x_1, x_2, \dots, x_j, \dots) \quad \textit{and} \quad \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}$$

with inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle \bar{x}, \bar{y} \rangle = \sum_{j=1}^{\infty} x_j y_j$ and norm $\|x\|_2 := \left(\sum_{j=1}^{\infty} |x_j|^2 \right)^{\frac{1}{2}}$, where $\bar{x} = \{x_j\} \in \ell_2$ and $\bar{y} = \{y_j\} \in \ell_2$. Let C be defined by $C = \{x \in \ell_2 : \langle a, x \rangle = b\}$ where $a = (3, 5, 3, 0, \dots, 0, \dots)$ and $b = 4$ and $Q := \{x \in \ell_2 : \langle c, x \rangle \geq d\}$ where $c = (3, 1, 0, 0, ;0, \dots)$ and $d = 3$. Thus, we have

$$P_C(\bar{x}) = \max \left\{ 0, \frac{b - \langle a, \bar{x} \rangle}{\|a\|_2^2} \right\} a + \bar{x},$$

and

$$P_Q(\bar{x}) = \frac{d - \langle c, \bar{x} \rangle}{\|c\|_2^2} c + \bar{x}.$$

Let $T : \ell_2 \to \ell_2$ be defined by $T\bar{x} = 5\bar{x}$, thus T is a bounded linear operator. Suppose $A_1 : \ell_2 \to \ell_2$ be defined by $A_1 \bar{x} = (3x_1, 3x_2, \dots, 3x_j, \dots)$ and $A_2 : \ell_2 \to \ell_2$ be defined by $A_2 \bar{x} = \left(\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_j}{2}, \dots \right)$. It is easy to see that A_1 and monotone and Lipschitz continuous and A_2 is inverse strongly monotone. We choose $\gamma_n = 2, \lambda_1 = 1, \mu = 0.5, \theta_n = \bar{\theta}, \alpha_n = \frac{1}{5n+2}, \epsilon_n = \frac{\alpha_n}{n^{0.01}}, \beta_n = \frac{1}{2} - \alpha_n$, for all $n \in \mathbb{N}$. It is easy to verify that all hypothesis of Theorem 7.1.6 are satisfied. We implement our algorithm for different values of x_0, x_1 as follows.

Case I: $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \dots)$, $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10} \dots)$;

Case II: $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10} \dots)$; $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \dots)$,

Case III: $x_1 = (1, \frac{1}{4}, \frac{1}{8}, \dots)$, $x_0 = (2, 1, \frac{1}{8}, \dots)$;

Case IV: $x_1 = x_0 = (2, 1, \frac{1}{8}, \dots)$; $x_0 = (1, \frac{1}{4}, \frac{1}{9}, \dots)$.

Example 3.3.8. *Let $H_1 = H_2 = L_2([0, 1])$ be equipped with the inner product*

$$\langle x, y \rangle = \int_0^1 x(t) y(t) dt \quad \forall \quad x, y \in L_2([0,1]) \quad \textit{and} \quad \|x\|^2 := \int_0^1 |x(t)|^2 dt \quad \forall x, y, \in L_2([0,1]).$$

Let $A_1, A_2 : L_2([0,1]) \to L_2([0,1])$ be defined by

$$A_1 x(t) = \int_0^1 \left(x(t) - \left(\frac{2tse^{t+s}}{e\sqrt{e^2 - 1}} \right) \cos x(s) \right) ds + \frac{2te^t}{e\sqrt{e^2 - 1}}, \quad x \in L_2([0,1])$$

$$\textit{and} \quad A_2 x(t) = \max\{0, \frac{x(t)}{2}\}, \quad t \in [0,1].$$

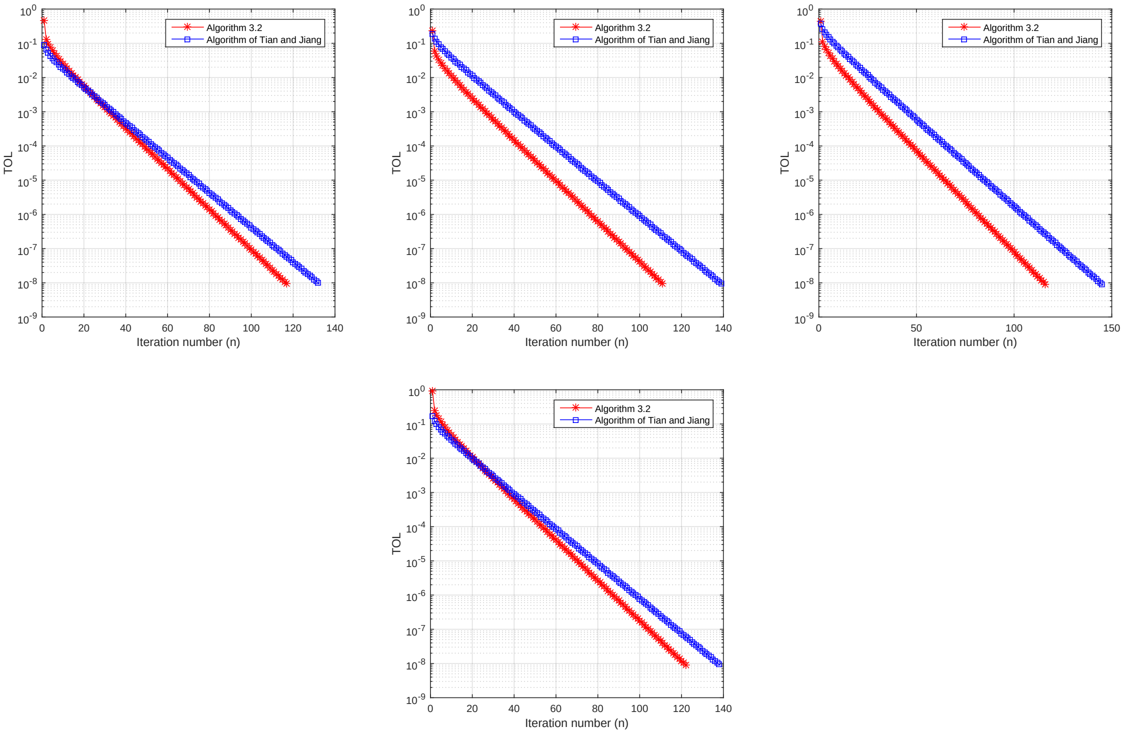

Figure 3.5: Example 3.3.7, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

It is easy to see that A_1 *is Lipschitz continuous and monotone and* A_2 *is inverse strongly monotone on* $L_2([0,1])$*. Let* $T : L_2([0,1]) \to L_2([0,1])$ *be defined by*

$$Tx(s) = \int_0^1 K(s,t)x(t)dt \quad \forall \quad x \in L_2([0,1]),$$

where K *is a continuous real-valued function defined on* $[0,1]\times[0,1]$*. Thus,* T *is a bounded linear operator with adjoint*

$$T^*x(s) = \int_0^1 K(t,s)x(t)dt \quad \forall \quad x \in L_2([0,1]).$$

Let C *be defined by* $C = \{x \in L_2 : \langle a, x\rangle = b\}$ *where* $a \neq 0$ *and* $b = 2$ *and* $Q := \{x \in L_2 : \langle c, x\rangle \geq d\}$ *where* $c \neq 0$ *and* $d = 4$*. Thus, we have*

$$P_C(\bar{x}) = \max\left\{0, \frac{b - \langle a, \bar{x}\rangle}{\|a\|^2}\right\} a + \bar{x},$$

and

$$P_Q(\bar{x}) = \frac{d - \langle c, \bar{x}\rangle}{\|c\|^2} c + \bar{x}.$$

We choose $\gamma_n = 2, \lambda_1 = 1, \mu = 0.5, \theta_n = \overline{\theta}, \alpha_n = \frac{1}{5n+2}, \epsilon_n = \frac{\alpha_n}{n^{0.01}}, \beta_n = \frac{1}{2} - \alpha_n,$ *for all* $n \in \mathbb{N}$*. It is easy to verify that all hypotheses of Theorem 7.1.6 are satisfied. We implement our algorithm for different values of* x_0, x_1 *as follows.*

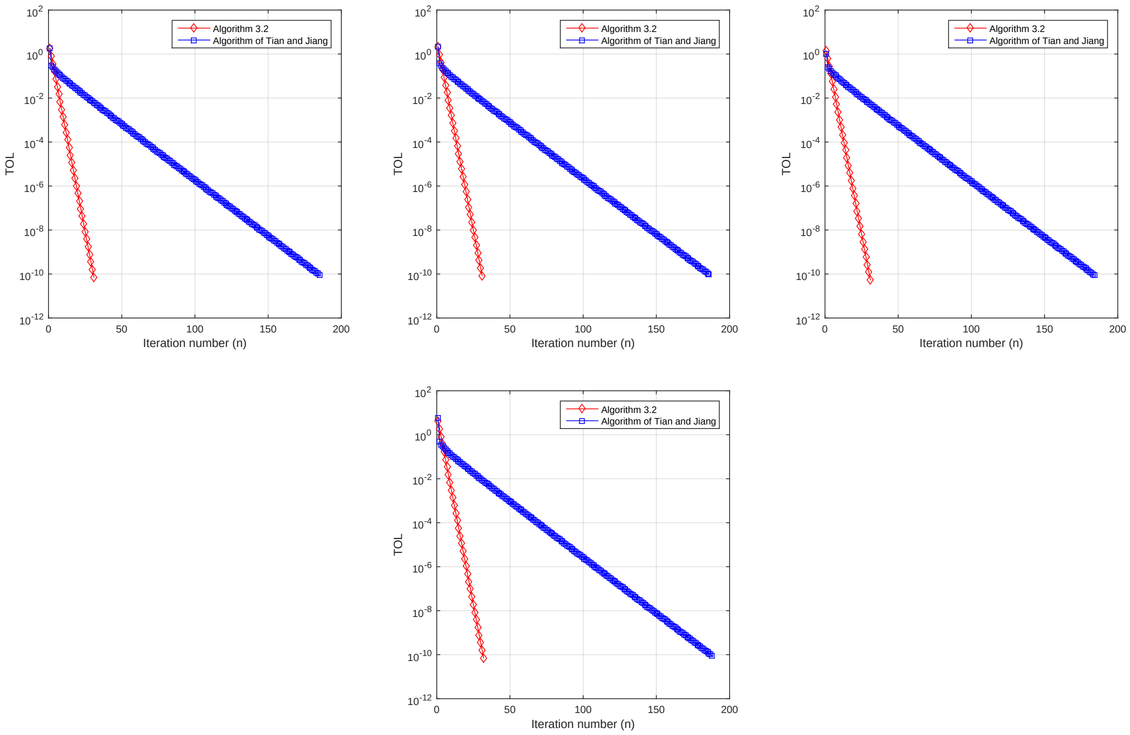

Figure 3.6: Example 3.3.7, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

Case I: $x_0(t) = 2t^2 + t + 2, \quad x_1(t) = t;$

Case II: $x_0(t) = 2t^2 + e^{2t} + 1, \quad x_1(t) = 3t^3 + 3;$

Case III: $x_0(t) = t + 2, \quad x_1(t) = \cos(t);$

Case IV: $x_0(t) = \cos(t) + 2t^2 + 4, \quad x_1(t) = 2t + 2 + e^t.$

3.4 Conclusion

In the first section of this chapter, we introduce a new inertial relaxed Tseng extrapolation method for approximating the solution of a variational inequality problem in which the underlying operator is pseudomonotone in the framework of Hilbert space. The main advantage of this method is the fact that the sequence $\{x_n\}$ generated by Algorithm 3.1.2 converges strongly to the minimum-norm of the solution set Ω. In addition, the proposed iterative algorithm is the combination of both the inertial extrapolation step and relaxation parameter, which is known to help speed up the rate of convergence. Furthermore, we present some examples and numerical experiments to show the efficiency and applicability of our method in the framework of infinite and finite dimensional Hilbert spaces. The results obtained in this work extend, generalize and improve several results in this direction.

In addition, we introduced a modified inertial type viscosity iterative scheme with regularization methods for solving a fixed point problem and variational inequality problem involving a monotone and Lipschitz continuous operator in framework of real Hilbert space. Our method uses step sizes that are generated at each iteration by some simple computations, which allows it to be easily implemented without the prior knowledge of the operator norm or the coefficient of an underlying operator. Furthermore, we prove that the proposed method converges strongly to a solution of the problem (1.2.9) in real Hilbert spaces. In addition, we present some examples and numerical experiments to show the efficiency and implementation of our method in the framework of infinite and finite dimensional Hilbert spaces. Our comparison shows that our modified inertial type algorithm helps speed up the rate of convergence as compared to the one without the sequence $\{S_n\}$. We emphasize that one of the novelty of this work is in the use of the regularization approach, the generalized inertial introduced and the method of proof of the strong convergence of our iterative algorithm to the solution of problems (1.2.9).

Finally, in this chapter, we introduced a new inertial regularization method for solving the SVIP (1.2.2)-(1.2.3) is proposed, we establish strong convergence to a minimum-norm solution of the problem in two real Hilbert spaces. The main advantage of this method is the combination of both the inertial extrapolation step and the regularization method, which has not been used to solve the SVIP (1.2.2)-(1.2.3). In addition, our method uses a simple self-adaptive step size that is generated at each iteration, which allows it to be easily implemented without the prior knowledge of the operator norm as well as the Lipschitz constant. Finally, we present some numerical experiments to establish the applicability and efficiency of our method. The results obtained in this chapter are new in solving the SVIP (1.2.2)-(1.2.3).

3.5 Open Problem

The results obtained in Section 3.3 of this chapter are quite new in the framework of Hilbert space. It is natural to ask if the concept can be extended to other abstract spaces like Banach, Hadamard, p-uniformly convex spaces.

Chapter 4

Contributions to Bilevel Problems in Hilbert Spaces

The theory of Bilevel Variational Inequality Problem (BVIP) is one of the generalizations of variational inequality problem in this area of research as introduced by Mainge [218]. This problem has applications in mathematical programming with equilibrium constraints [210], bilevel convex programming model [346] and the minimum norm problem with the solution set of variational inequality [376, 381]. Due to the fruitful application of the BVIP, many researchers have developed different iterative methods to solve the BVIP. As mentioned in Chapter 2 of this thesis some of the iterative methods introduced have their drawbacks as such not making them good for applicability in real life. To address this drawbacks of iterative methods that have been introduced in the literature (see Chapter 2), we present in this chapter certain inertial iterative algorithms with regularization techniques, modified inertial technique, self-adaptive step sizes, in which every projection onto the closed convex set is substituted by a projection onto some half-space and the line-search technique, ensuring easy implementation. Furthermore, we will establish strong convergence results for these iterative methods introduced in the framework of Hilbert spaces.

The results obtained in this chapter extend, generalize and improve several results in this direction.

4.1 A Self Adaptive Method for Solving a Class of Bilevel Variational Inequalities with Split Variational Inequality and Composed Fixed Point Problem Constraints in Hilbert Spaces

In this section, we provide an affirmative answer to Questions raised in Chapter 2 of this study by introducing a new inertial extrapolation method with regularization for approximating solutions of split bilevel variational inequality problems in the framework of real Hilbert spaces. We prove that the proposed method converges strongly to a minimum-norm solution to the problem without using the conventional two cases approach. In addition, we present some numerical experiments to show the efficiency and applicability of the proposed method.

4.1.1 Main Result

In this section, we propose an inertial extrapolation method for solving the following problem. Find

$$x^* \in \Gamma \;\text{ such that }\; \langle F_2 x^*, x - x^* \rangle \geq 0 \;\forall\; x \in \Gamma, \tag{4.1.1}$$

where

$$\Gamma = \{x^* \in VI(F_1, C) : Ax^* \in F(T_1 \circ T_2)\}.$$

Lemma 4.1.1. *Let H be a real Hilbert space, $T_1 : H \to H$ be a δ-demicontractive mapping and $T_2 : H \to H$ be a directed mapping such that $F(T_1) \cap F(T_2) \neq \emptyset$. Then $F(T_1 \circ T_2) = F(T_1) \cap F(T_2)$.*

Proof. We need to establish that $F(T_1 \circ T_2) \subseteq F(T_1) \cap F(T_2)$ and $F(T_1) \cap F(T_2) \subseteq F(T_1 \circ T_2)$. It is easy to see that $F(T_1) \cap F(T_2) \subseteq F(T_1 \circ T_2)$. We now establish that $F(T_1 \circ T_2) \subseteq F(T_1) \cap F(T_2)$. Let $y \in F(T_1 \circ T_2)$ and $x \in F(T_1) \cap F(T_2)$, we have

$$\begin{aligned}
\|y - x\|^2 &= \|T_1(T_2 y) - x\|^2 \\
&\leq \|T_2 y - x\|^2 + \delta \|T_2 y - T_1(T_2 y)\|^2 \\
&= \|T_2 y - x\|^2 + \delta \|T_2 y - (T_1 \circ T_2) y\|^2 \\
&= \|T_2 y - x\|^2 + \delta \|T_2 y - y\|^2.
\end{aligned} \tag{4.1.2}$$

Also, using (4.1.2), we have

$$\begin{aligned}
\|T_2 y - x\|^2 &\leq \|y - x\|^2 - \|y - T_2 y\|^2 \\
&\leq \|T_2 y - x\|^2 + \delta \|T_2 y - y\|^2 - \|y - T_2 y\|^2 \\
&= \|T_2 y - x\|^2 - (1 - \delta) \|T_2 y - y\|^2,
\end{aligned} \tag{4.1.3}$$

which implies that $\|T_2 y - y\|^2 = 0 \Rightarrow \|T_2 y - y\| = 0 \Rightarrow T_2 y = y$. Using this fact, we have that

$$y = (T_1 \circ T_2) y = T_1(T_2 y) = T_1 y \Rightarrow y \in F(T_1) \cap F(T_2). \tag{4.1.4}$$

Hence, $F(T_1 \circ T_2) \subseteq F(T_1) \cap F(T_2)$, and so $F(T_1 \circ T_2) = F(T_1) \cap F(T_2)$. □

Lemma 4.1.2. *Let H be a real Hilbert space, $T_1 : H \to H$ be a δ-demicontractive mapping and $T_2 : H \to H$ be a directed mapping. Then, $T_1 \circ T_2$ is a ν-demicontractive type mapping, where $\nu = \frac{\delta}{1-\delta}$.*

Proof. Let $x \in H$ and $y \in F(T_1 \circ T_2)$, using Lemma 4.1.1, we have that $y \in F(T_1) \cap F(T_2)$, which implies that $y = T_1 y$ and $y = T_2 y$. Now, observe that

$$\begin{aligned}\|T_2x - y\|^2 &\leq \|x - y\|^2 - \|x - T_2x\|^2 \\ &= \|x - y\|^2 - [\|x - y\|^2 + \|T_2x - y\| - 2\langle x - y, T_2x - y\rangle] \\ &= -\|T_2x - y\| + 2\langle x - y, T_2x - y\rangle, \qquad (4.1.5)\end{aligned}$$

which implies that

$$\langle x - y, T_2x - y\rangle \geq \|T_2x - y\|. \qquad (4.1.6)$$

Also, we have

$$\begin{aligned}\|(T_1 \circ T_2)x - y\|^2 &= \|T_1(T_2x) - y\|^2 \\ &\leq \|T_2x - y\|^2 + \delta\|T_2x - T_1(T_2x)\|^2 \\ &= \|T_2x - y\|^2 + \delta[\|T_2x - y\|^2 + \|(T_1 \circ T_2)x - y\|^2 \\ &- 2\langle T_2x - y, (T_1 \circ T_2)x - y\rangle], \qquad (4.1.7)\end{aligned}$$

which implies that

$$2\delta\langle T_2x - y, (T_1 \circ T_2)x - y\rangle \leq (1 + \delta)\|T_2x - y\|^2 - (1 - \delta)\|(T_1 \circ T_2)x - y\|^2. \qquad (4.1.8)$$

Now, using (4.1.6) and (4.1.8), we have

$$\begin{aligned}0 &\leq \|(x - y) - (1 - \delta)(T_2x - y) - \delta(T_1 \circ T_2)x - y\|^2 \\ &= \|x - y\|^2 + (1 - \delta)^2\|T_2x - y\|^2 + \delta^2\|(T_1 \circ T_2)x - y\|^2 \\ &- 2(1 - \delta)\langle x - y, T_2 - y\rangle + 2\delta(1 - \delta)\langle T_2x - y, (T_1 \circ T_2)x - y\rangle - 2\delta\langle x - y, (T_1 \circ T_2)x - y\rangle \\ &\leq \|x - y\|^2 + (1 - \delta)^2\|T_2x - y\|^2 + \delta^2\|(T_1 \circ T_2)x - y\|^2 - 2\delta\langle x - y, (T_1 \circ T_2)x - y\rangle \\ &+ (1 - \delta)[-2\|T_2x - y\|^2] + (1 - \delta)[(1 + \delta)\|T_2x - y\|^2 - (1 - \delta)\|(T_1 \circ T_2)x - y\|^2] \\ &= \|x - y\|^2 - (1 - 2\delta)\|(T_1 \circ T_2)x - y\|^2 - 2\delta\langle x - y, (T_1 \circ T_2)x - y\rangle \\ &= \|x - y\|^2 - (1 - 2\delta)\|(T_1 \circ T_2)x - y\|^2 - \delta[\|x - y\|^2 + \|(T_1 \circ T_2)x - y\|^2 - \|x - (T_1 \circ T_2)x\|^2] \\ &= (1 - \delta)\|x - y\|^2 - (1 - \delta)\|(T_1 \circ T_2)x - y\|^2 + \delta\|x - (T_1 \circ T_2)x\|^2,\end{aligned}$$

which implies that

$$(1 - \delta)\|(T_1 \circ T_2)x - y\|^2 \leq (1 - \delta)\|x - y\|^2 + \delta\|x - (T_1 \circ T_2)x\|^2,$$

thus, we have

$$\|(T_1 \circ T_2)x - y\|^2 \leq \|x - y\|^2 + \nu\|x - (T_1 \circ T_2)x\|^2,$$

where $\nu = \frac{\delta}{1-\delta}$. Hence, $T_1 \circ T_2$ is a ν-demicontractive type mapping. □

Remark 4.1.3. *We note that*

1. *if* $\delta \in (0, \frac{1}{2})$, *we have that* $\nu < 1$.

2. *if* $\delta \in [\frac{1}{2}, 1)$, *we have that* $\nu \geq 1$.

Thus, going forward, we suppose that $\nu \in (0, \frac{1}{2})$.

We now proposed an iterative method and highlight some of its importance in comparison with existing ones in the literature.

Assumption 4.1.4. *Suppose that the following conditions hold:*

1. *The set* C *is nonempty closed and convex subsets of the real Hilbert spaces* H_1.
2. $F_1 : H_1 \to H_1$ *is pseudomonotone,* L_1*-Lipschitz (Lipschitz constant need not be known) and sequentially weakly continuous on* C*, and* $F_2 : H_1 \to H_1$ *is* α*-strongly monotone operator and* L_2*- Lipschitz continuous on* H.
3. $T_1 : H_2 \to H_2$ *is a* δ*-demicontractive type mapping, demiclosed at zero, and* $T_2 : H_2 \to H_2$ *is a directed mapping and* $T_1 \circ T_2$ *is demiclosed at zero.*
4. $A : H_1 \to H_2$ *is a bounded linear operator.*
5. *Given* $\gamma_n(\epsilon - \nu(\gamma_n + \epsilon)), \lambda_1, \gamma_n > 0$, $\theta_n \in (0, 1], \mu, \beta_n \in (0, 1)$*, and* $\sum_{n=1}^{\infty} \zeta_n < \infty$ *for all* $n \in \mathbb{N}$*, with* θ *been a positive constant and* $\{\epsilon_n\}$ *is a positive such that* $\epsilon_n = \circ(\beta_n)$.
6. Ω *is the solution set for the problem* (4.1.1).

We present the following iterative algorithm.

Algorithm 4.1.5. Iterative steps:

Step 1: *Given the iterates* x_{n-1} *and* x_n *for all* $n \in \mathbb{N}$*, choose* θ_n *such that* $0 \leq \theta_n \leq \bar{\theta}_n$*, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\max\{n^2\|x_n - x_{n-1}\|^2, n\|x_n - x_{n-1}\|\}}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise.} \end{cases} \tag{4.1.9}$$

Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$y_n = w_n + \gamma_n A^*(U(Aw_n) - Aw_n), \tag{4.1.10}$$
$$z_n = P_C(y_n - \lambda_n F_1 y_n), \tag{4.1.11}$$
$$q_n = P_{T_n}(y_n - \lambda_n F_1 z_n), \tag{4.1.12}$$

where $T_n = \{y \in H : \langle y_n - \lambda_n F_1 y_n - z_n, y - z_n\rangle \leq 0\}$, $T_1 \circ T_2 = U$,

$$\gamma_n = \left(\epsilon, \frac{\|U(Aw_n) - Aw_n\|}{\|A^*(U(Aw_n) - Aw_n)\|} - \epsilon\right) \tag{4.1.13}$$

and

$$\lambda_{n+1}=\begin{cases}\min\left\{\frac{\mu\|y_n-z_n\|^2+\|q_n-z_n\|^2}{2\langle F_1y_n-F_1z_n,q_n-z_n\rangle},\ \lambda_n+\zeta_n\right\}, & \textit{if } \langle F_1y_n-F_1z_n,q_n-z_n\rangle>0\\ \lambda_n+\zeta_n, & \textit{otherwise}\end{cases} \tag{4.1.14}$$

Step 3. *Compute*

$$x_{n+1}=q_n-\beta_nF_2q_n. \tag{4.1.15}$$

Remark 4.1.6. *1. Our method uses simple self-adaptive stepsizes that are generated at each iteration by some simple computations. Thus, the implementation of our method does not depend on the knowledge of the bounded linear operator $\|A\|$ or that of the Lipschitz constant. This feature is very important as algorithms whose implementation depends on the operator norm require the computation of the norm of the bounded linear operator, which in general is a very difficult and sometimes impossible to compute.*

2. The sequence generated by the proposed method converges strongly to a minimum-norm solution of the problem (4.1.1) *in real Hilbert spaces.*

3. Furthermore, the strong convergence analysis of our proposed method does not rely on the usual "Two Cases Approach" widely used in many articles to guarantee strong convergence.

4. Our computational experiments show that our proposed method is efficient and better than the iterative scheme in [236].

4.1.2 Convergence Analysis

In this section, we establish a strong convergence result of our proposed method.

Lemma 4.1.7. *Let $\{\lambda_n\}$ be the sequence generated by Algorithm* (4.1.5)*. Then we have* $\lim_{n\to\infty}\lambda_n=\lambda$ *and* $\lambda\in[\min\{\lambda_1,\frac{\mu}{L_1}\},\lambda_1+\zeta]$.

Proof. The proof follows a similar approach as in Lemma 3.1 of [209], as such we omit it. ☐

Lemma 4.1.8. *Let $\{x_n\}$ be a sequence generated by Algorithm 4.1.5. Then, under Assumption 4.1.4, we have that $\{x_n\}$ is bounded.*

Proof. Let $p\in\Omega$ and since $\lim\limits_{n\to\infty}\frac{\theta_n}{\beta_n}\|x_n-x_{n-1}\|=0$, there exists $N_1>0$ such that $\frac{\theta_n}{\beta_n}\|x_n-x_{n-1}\|\leq N_1$, for all $n\in\mathbb{N}$. Then from **Step** 2, we have

$$\begin{aligned}\|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - p\| + \beta_n\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \\ &\leq \|x_n - p\| + \beta_n N_1. \end{aligned} \tag{4.1.16}$$

Also, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|w_n + \gamma_n A^*(U(Aw_n) - Aw_n) - p\|^2 \\ &= \|(w_n - p) + \gamma_n A^*(U(Aw_n) - Aw_n)\|^2 \\ &= \|w_n - p\|^2 + \gamma_n^2\|A^*(U(Aw_n) - Aw_n)\|^2 + 2\gamma_n\langle w_n - p, A^*(U(Aw_n) - Aw_n)\rangle \\ &= \|w_n - p\|^2 + \gamma_n^2\|A^*(U(Aw_n) - Aw_n)\|^2 + 2\gamma_n\langle A(w_n - p), U(Aw_n) - Aw_n\rangle. \end{aligned} \tag{4.1.17}$$

Now, observe that

$$\begin{aligned}&\langle A(w_n - p), U(Aw_n) - Aw_n\rangle \\ &= \langle A(w_n - p) + U(Aw_n) - U(Aw_n) - Aw_n + Aw_n, U(Aw_n) - Aw_n\rangle \\ &= \langle U(Aw_n) - Ap, U(Aw_n) - Aw_n\rangle - \|U(Aw_n) - Aw_n\|^2 \\ &= \frac{1}{2}\left[\|U(Aw_n) - Ap\|^2 + \|U(Aw_n) - Aw_n\|^2 - \|Aw_n - Ap\|^2\right] - \|U(Aw_n) - Aw_n\|^2 \\ &= \frac{1}{2}\|U(Aw_n) - Ap\|^2 - \frac{1}{2}\|U(Aw_n) - Aw_n\|^2 - \frac{1}{2}\|Aw_n - Ap\|^2 \\ &= \frac{1}{2}\|U(Aw_n) - p\|^2 - \frac{1}{2}\|U(Aw_n) - Aw_n\|^2 - \frac{1}{2}\|Aw_n - Ap\|^2 \\ &\leq \frac{1}{2}[\|Aw_n - p\|^2 + \nu\|Aw_n - U(Aw_n)\|^2] - \frac{1}{2}\|U(Aw_n) - Aw_n\|^2 - \frac{1}{2}\|Aw_n - Ap\|^2 \\ &= \frac{1}{2}[\|Aw_n - Ap\|^2 + \nu\|Aw_n - U(Aw_n)\|^2] - \frac{1}{2}\|U(Aw_n) - Aw_n\|^2 - \frac{1}{2}\|Aw_n - Ap\|^2 \\ &= -\frac{1}{2}(1 - \nu)\|U(Aw_n) - Aw_n\|^2. \end{aligned} \tag{4.1.18}$$

Substituting (4.1.18) into (4.1.17), we have

$$\begin{aligned}&\|y_n - p\|^2 \\ &\leq \|w_n - p\|^2 + \gamma_n^2\|A^*(U(Aw_n) - Aw_n)\|^2 - \gamma_n(1 - \nu)\|U(Aw_n) - Aw_n\|^2 \\ &= \|w_n - p\|^2 + \gamma_n^2\|A^*(U(Aw_n) - Aw_n)\|^2 - \gamma_n(1 - \nu)(\gamma_n + \epsilon)\|A^*(U(Aw_n) - Aw_n)\|^2 \\ &= \|w_n - p\|^2 - \gamma_n(\epsilon - \nu(\gamma_n + \epsilon))\|A^*(U(Aw_n) - Aw_n)\|^2 \\ &\leq \|w_n - p\|^2. \end{aligned} \tag{4.1.19}$$

In addition, using Lemma 2.2.1, we have

$$\begin{aligned}
&\|q_n - p\|^2 \\
&= \|P_{T_n}(y_n - \lambda_n F_1 z_n) - p\|^2 \\
&\leq \|y_n - \lambda_n F_1 z_n - p\|^2 - \|y_n - \lambda_n F_1 z_n - q_n\|^2 \\
&= \|y_n - p\|^2 - 2\lambda_n\langle y_n - p, F_1 z_n\rangle - \|y_n - q_n\|^2 + 2\lambda_n\langle y_n - q_n, F_1 z_n\rangle \\
&= \|y_n - p\|^2 - \|y_n - q_n\|^2 - 2\lambda_n\langle q_n - p, F_1 z_n\rangle \\
&= \|y_n - p\|^2 - \|y_n - q_n\|^2 - 2\lambda_n\langle q_n - z_n, F_1 z_n\rangle - 2\lambda_n\langle z_n - p, F_1 z_n\rangle \\
&= \|y_n - p\|^2 - \|y_n - z_n\|^2 - \|z_n - q_n\|^2 + \langle q_n - z_n, y_n - z_n\rangle \\
&- 2\lambda_n\langle q_n - z_n, F_1 z_n\rangle - 2\lambda_n\langle z_n - p, F_1 z_n\rangle \\
&= \|y_n - p\|^2 - \|y_n - z_n\|^2 - \|z_n - q_n\|^2 - 2\lambda_n\langle z_n - q_n, y_n - \lambda_n F_1 z_n - z_n\rangle \\
&- 2\lambda_n\langle z_n - p, F_1 z_n\rangle \\
&= \|y_n - p\|^2 - \|y_n - z_n\|^2 - \|z_n - q_n\|^2 - 2\langle z_n - q_n, y_n - \lambda_n F_1 y_n - z_n\rangle \\
&+ 2\lambda_n\langle F_1 y_n - F_1 z_n, q_n - z_n\rangle - 2\lambda_n\langle z_n - p, F_1 z_n\rangle. \qquad (4.1.20)
\end{aligned}$$

Since $p \in \Omega$, $z_n \in C$ and the fact that F_1 is pseudomonotone we have that $\langle z_n - p, F_1 p\rangle \geq 0$ which implies $\langle z_n - p, F_1 z_n\rangle \geq 0$. Also from $q_n \in T_n$, we get that $\langle z_n - q_n, y_n - \lambda_n F_1 y_n - z_n\rangle \geq 0$. Therefore, using (4.1.14), we have (4.1.20) becomes

$$\begin{aligned}
\|q_n - p\|^2 &\leq \|y_n - p\|^2 - \|y_n - z_n\|^2 - \|z_n - q_n\|^2 + 2\lambda_n\langle F_1 y_n - F_1 z_n, q_n - z_n\rangle \\
&= \|y_n - p\|^2 - \|y_n - z_n\|^2 - \|z_n - q_n\|^2 + \frac{\mu\lambda_n}{\lambda_{n+1}}\|y_n - z_n\|^2 + \frac{\mu\lambda_n}{\lambda_{n+1}}\|q_n - z_n\|^2 \\
&= \|y_n - p\|^2 - (1 - \frac{\mu\lambda_n}{\lambda_{n+1}})\|y_n - z_n\|^2 - (1 - \frac{\mu\lambda_n}{\lambda_{n+1}})\|z_n - q_n\|^2. \qquad (4.1.21)
\end{aligned}$$

Thus, considering the limit

$$\lim_{n\to\infty}(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}) = 1 - \mu > 0.$$

Hence, there exists $N \geq 0$, such that $n \geq N$, we have that $1 - \frac{\mu\lambda_n}{\lambda_{n+1}} > 0$. Thus, it follows that for all $n \geq N$, we have

$$\|q_n - p\|^2 = \|y_n - p\|^2, \qquad (4.1.22)$$

which implies that

$$\|q_n - p\| = \|y_n - p\| \leq \|w_n - p\|. \qquad (4.1.23)$$

Using the fact that F_2 is L_2-Lipschitz continuous and α strongly monotone on H_1, we have that

$$\|F_2 q_n\| \leq \|F_2 q_n - F_2 p\| + \|F_2 p\| \leq L_2\|q_n - p\| + \|F_2 p\| \qquad (4.1.24)$$

and

$$\begin{aligned}
&\|q_n - p - \mu_1(F_2 q_n - Fp)\|^2 \\
&= \|q_n - p\|^2 - 2\mu_1\langle q_n - p, F_2 q_n - F_2 p\rangle + \mu_1^2\|Fq_n - Fp\|^2 \\
&\leq \|q_n - p\|^2 - 2\mu_1\alpha\|q_n - p\|^2 + \mu_1^2 L_2^2\|q_n - p\|^2 \\
&= (1 - \mu_1(2\alpha - \mu_1 L_2^2))\|q_n - p\|^2. \qquad (4.1.25)
\end{aligned}$$

Using (4.1.25), we have

$$\begin{aligned}
\|q_n - \beta_n F_2 q_n - (p - \beta_n F_2 p)\| &= \|(1 - \frac{\beta_n}{\mu_1})(q_n - p) + \frac{\beta_n}{\mu_1}[q_n - p - \mu_1(F_2 q_n - F_2 p)] \\
&\leq (1 - \frac{\beta_n}{\mu_1})\|q_n - p\| + \frac{\beta_n}{\mu_1}\|q_n - p - \mu_1(F_2 q_n - F_2 p)\| \\
&\leq (1 - \frac{\beta_n}{\mu_1})\|q_n - p\| + \frac{\beta_n}{\mu_1}\sqrt{(1 - \mu_1(2\alpha - \mu_1 L_2^2))}\|q_n - p\| \\
&= [1 - \frac{\beta_n}{\mu_1}(1 - \sqrt{(1 - \mu_1(2\alpha - \mu_1 L_2^2))})]\|q_n - p\| \\
&= (1 - \frac{\beta_n \tau_1}{\mu_1})\|q_n - p\|, \qquad (4.1.26)
\end{aligned}$$

where $\tau_1 = 1 - \sqrt{(1 - \mu_1(2\alpha - \mu_1 L_2^2))} \in (0, 1]$.
More so, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - \frac{\beta_n \tau_1}{\mu_1})\|q_n - p\| + \beta_n\|F_2 p\| \\
&\leq (1 - \frac{\beta_n \tau_1}{\mu_1})\|w_n - p\| + \beta_n\|F_2 p\| \\
&= (1 - \frac{\beta_n \tau_1}{\mu_1})\|x_n - p\| + \mu_1\frac{\beta_n \tau_1}{\mu_1 \tau_1}N_1 + \mu_1\frac{\beta_n \tau_1}{\mu_1 \tau_1}\|F_2 p\| \\
&= (1 - \frac{\beta_n \tau_1}{\mu_1})\|x_n - p\| + \frac{\beta_n \tau_1}{\mu_1}(\frac{\mu_1}{\tau_1}N_1. + \frac{\mu_1}{\tau_1}\|F_2 p\|) \\
&\leq \max\{\|x_n - p\|, \frac{\mu_1}{\tau_1}N_1 + \frac{\mu_1}{\tau_1}\|F_2 p\|\} \\
&\vdots \\
&\leq \max\{\|x_1 - p\|, \frac{\mu_1}{\tau_1}(N_1 + \|F_2 p\|)\}. \qquad (4.1.27)
\end{aligned}$$

Thus, $\{x_n\}$ is bounded. □

Lemma 4.1.9. *Let Assumption 4.1.4 hold and let $\{x_n\}$ be a sequence generated by Algorithm 4.1.5. Assume that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to a point x^*, and $\lim_{k\to\infty} \|y_{n_k} - w_{n_k}\| = \lim_{k\to\infty} \|y_{n_k} - z_{n_k}\| = 0$, then, $x^* \in \Gamma$.*

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $x^* \in H$. It is easy to see that

$$\|w_{n_k} - x_{n_k}\| = \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_n-1}\| \to 0 \text{ as } k \to \infty. \qquad (4.1.28)$$

It follows that

$$\|y_{n_k} - x_{n_k}\| \leq \|y_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \qquad (4.1.29)$$

Since A is a bounded linear operator, it follows from (4.1.28) that $\{Aw_{n_k}\}$ converges weakly to $Ax^* \in H_2$. Also, by (4.1.29), we obtain that y_{n_k} converges weakly to x^*. In addition, we have

$$\|z_{n_k} - x_{n_k}\| \leq \|z_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{4.1.30}$$

From (4.1.19), we have that

$$\begin{aligned}\|y_n - p\|^2 &\leq \|w_n - p\|^2 - \gamma_n(\epsilon - \nu(\gamma_n + \epsilon))\|A^*(U(Aw_n) - Aw_n)\|^2 \\ &\leq \|w_n - p\|^2 - \epsilon^2(1 - 2\nu)\|A^*(U(Aw_n) - Aw_n)\|^2 \end{aligned} \tag{4.1.31}$$

which implies that

$$\begin{aligned}\epsilon^2(1 - 2\nu)\|A^*(U(Aw_{n_k}) - Aw_{n_k})\|^2 &\leq \|w_{n_k} - p\|^2 - \|y_{n_k} - p\|^2 \\ &\leq \|w_{n_k} - y_{n_k}\|^2 + 2\|y_{n_k} - p\|\|w_{n_k} - y_{n_k}\|, \end{aligned} \tag{4.1.32}$$

thus, we have that

$$\lim_{k\to\infty} \|A^*(U(Aw_{n_k}) - Aw_{n_k})\| = 0. \tag{4.1.33}$$

Also from (4.1.19), we have

$$\begin{aligned}\|y_n - p\|^2 &\leq \|w_n - p\|^2 + \gamma_n^2\|A^*(U(Aw_n) - Aw_n)\|^2 - \gamma_n(1 - \nu)\|U(Aw_n) - Aw_n\|^2 \\ &\leq \|w_n - p\|^2 + \epsilon^2\|A^*(U(Aw_n) - Aw_n)\|^2 - \epsilon(1 - \nu)\|U(Aw_n) - Aw_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}&\epsilon(1 - \nu)\|U(Aw_{n_k}) - Aw_{n_k}\|^2 \\ &\leq \|w_{n_k} - p\|^2 - \|y_{n_k} - p\|^2 + \epsilon^2\|A^*(U(Aw_{n_k}) - Aw_{n_k})\|^2 \\ &\leq |w_{n_k} - y_{n_k}\|^2 + 2\|y_{n_k} - p\|\|w_{n_k} - y_{n_k}\| + \epsilon^2\|A^*(U(Aw_{n_k}) - Aw_{n_k})\|^2,\end{aligned}$$

thus, using (4.1.33), we have that

$$\lim_{k\to\infty} \|U(Aw_{n_k}) - Aw_{n_k}\| = 0. \tag{4.1.34}$$

Thus, using our assumption (demicloseness), Lemma 4.1.1 and (4.1.34), we have

$$Ax^* \in F(U) \tag{4.1.35}$$

In addition, by the definition of $\{z_n\}$ and Lemma 2.2.1, that

$$\langle y_{n_j} - \lambda_{n_j} F_1 y_{n_j} - z_{n_j}, v - z_{n_j}\rangle \leq 0, \quad \forall\ v \in C,$$

which implies

$$\frac{1}{\lambda_{n_j}}\langle y_{n_j} - z_{n_j}, v - z_{n_j}\rangle \leq \langle F_1 y_{n_j}, v - z_{n_j}\rangle \quad \forall\ v \in C$$

As such, we have

$$\frac{1}{\lambda_{n_j}}\langle y_{n_j} - z_{n_j}\rangle + \langle F_1 y_{n_j}, z_{n_j} - y_{n_j}\rangle \leq \langle F_1 y_{n_j}, v - y_{n_j}\rangle, \quad \forall\ v \in C. \tag{4.1.36}$$

Since $\{y_{n_j}\}$ is convergent, it is bounded. Then, since F_1 is Lipschitz continuous, $\{F_1 y_{n_j}\}$ is bounded. In addition, we have that $\{z_{n_j}\}$ is bounded since $\|y_{n_j} - z_{n_j}\| \to 0$ as $j \to \infty$ and $\lambda_{n_j} \in [\min\left\{\lambda_1, \frac{\mu}{L}\right\}, \lambda_1 + \zeta]$. Taking the limit as $j \to \infty$ in (4.1.36) we obtain

$$\liminf_{j\to\infty} \langle F_1 y_{n_j}, v - y_{n_j}\rangle \geq 0.$$

Now, note that

$$\langle F_1 y_{n_j}, v - v_{n_j}\rangle = \langle F_1 z_{n_j} - F_1 y_{n_j}, v - z_{n_j}\rangle + \langle F_1 z_{n_j}, v - y_{k_j}\rangle + \langle F_1 z_{n_j}, y_{n_j} - z_{n_j}\rangle. \tag{4.1.37}$$

Using $\lim\limits_{j\to\infty} \|y_{n_j} - z_{n_j}\| = 0$ and the Lipschitz continuity of F_1, we have $\lim\limits_{j\to\infty} \|F_1 y_{n_j} - F_1 z_{n_j}\| = 0$. Thus, from (4.1.37), we have $\liminf\limits_{j\to\infty}\langle F_1 y_{k_j}, v - y_{k_j}\rangle \geq 0$. We choose a subsequence $\{\epsilon_j\}$ of positive number decreasing such that $\epsilon_j \to 0$ as $j \to \infty$. For each j, let N_j be the smallest nonnegative integer such that

$$\langle F_1 z_{n_i}, v - z_{n_i}\rangle + \epsilon_j \geq 0, \quad \forall\, i \geq N_j. \tag{4.1.38}$$

Since $\{\epsilon_j\}$ is decreasing, it is obvious that N_j is increasing. Further, for each $j \in \mathbb{N}$, $\{z_{N_j}\} \subset C$. Suppose $F_1 z_{N_j} \neq 0$ so that z_{N_j} is not a solution of the $VIP(C, F_1)$, set

$$\nu_{N_j} = \frac{F_1 z_{N_j}}{\|F_1 z_{N_j}\|^2},$$

so that $\langle F_1 z_{N_j}, \nu_{N_j}\rangle = 1$ for each j. It follows from this and (4.1.38), that $\langle F_1 z_{N_j}, v + \epsilon_j \nu_{N_j} - z_{N_j}\rangle \geq 0$. Since F_1 is pseudomonotone, we have $F_1(v + \epsilon_j \nu_{N_j}), v + \epsilon_j \nu_{N_j} - z_{N_j}\rangle \geq 0$ and thus

$$\langle F_1 v, v - z_{N_j}\rangle \geq \langle F_1 v - F_1(v + \epsilon_j \nu_{N_j}), v + \epsilon_j \nu_{N_j} - z_{N_j}\rangle - \epsilon_j \langle F v, \nu_{N_j}\rangle. \tag{4.1.39}$$

Next, we show that $\epsilon_j \nu_{N_j} \to 0$ as $j \to \infty$. To see this, using our hypothesis we have $z_{N_j} \rightharpoonup x^*$ as $j \to \infty$. By $\{z_n\} \subset C$, we have that $x^* \in C$. Since F_1 is sequentially weakly continuous on C, we have $F_1 z_{N_j} \rightharpoonup F_1 x^*$. Suppose that $F x^* \neq 0$ so that $x^* \in VI(F_1, C)$. Using our assumption of the sequentially weakly continuous, we obtain

$$0 < \|F x^*\| \leq \liminf_{j\to\infty} \|F z_{N_j}\|.$$

From $\{z_{N_j}\} \subset \{z_{n_j}\}$ and $\epsilon_j \to 0$ as $j \to \infty$, we have

$$0 \leq \lim_{j\to\infty} \|\epsilon_j \nu_{N_j}\| = \lim_{j\to\infty}\left(\frac{\epsilon_j}{\|F_1 z_{n_j}\|}\right) \leq \frac{0}{\|F_1 x^*\|} = 0,$$

that is

$$\lim_{j\to\infty} \|\epsilon_j \nu_{N_j}\| = 0.$$

Now letting $j \to \infty$, and using the continuity of F_1 we have $\{y_{N_j}\}$, $\{\nu_{N_j}\}$ and bounded $\lim\limits_{j\to\infty} \|\epsilon_j \nu_{N_j}\| = 0$. Thus,

$$\liminf_{j\to\infty} \langle F_1 v, v - z_{N_j}\rangle \geq 0.$$

Therefore, for all $v \in C$, we have

$$\langle F_1 v, v - x^* \rangle = \lim_{j\to\infty} \langle F_1 v, v - z_{N_j} \rangle = \liminf_{j\to\infty} \langle F_1 v, v - z_{N_j} \rangle \geq 0.$$

Hence, by Lemma 2.1.23 we have $x^* \in VI(F_1, C)$. The proof is thus complete. $\square$

Theorem 4.1.10. *Let $\{x_n\}$ be the sequence generated by Algorithm 4.1.5. Then, under the Assumption 4.1.4, if $\lim\limits_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty$. Then, $\{x_n\}$ converges strongly to $p \in \Omega$, where $\|p\| = \min\{\|x^*\| : x^* \in \Omega\}$.*

Proof. Let $p \in \Omega$, observe that

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1} \rangle + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + 2\theta_n \|x_{n-1} - p\| \|x_n - p\| + \theta_n^2 \|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \theta_n \|x_n - x_{n-1}\|] \\
&= \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \beta_n \frac{\theta_n}{\beta_n} \|x_n - x_{n-1}\|] \\
&\leq \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| [2\|x_n - p\| + \alpha_n N_1] \\
&\leq \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| N_2, \qquad (4.1.40)
\end{aligned}$$

for some $N_2 > 0$. Furthermore, we have that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|q_n - \beta_n F_2 q_n - p\|^2 \\
&= \|q_n - \beta_n F_2 q_n - (p - \beta_n F p) - \beta_n F p\|^2 \\
&\leq \|q_n - \beta_n F_2 q_n - (p - \beta_n F p)\|^2 + 2\beta_n \langle F_2 p, p - x_{n+1} \rangle \\
&\leq (1 - \frac{\beta_n \tau_1}{\mu_1})^2 \|q_n - p\|^2 + 2\beta_n \langle F_2 p, p - x_{n+1} \rangle \\
&\leq (1 - \frac{\beta_n \tau_1}{\mu_1}) \|q_n - p\|^2 + 2\beta_n \langle F_2 p, p - x_{n+1} \rangle \\
&\leq (1 - \frac{\beta_n \tau_1}{\mu_1}) \|y_n - p\|^2 + 2\beta_n \langle F_2 p, p - x_{n+1} \rangle \\
&\leq (1 - \frac{\beta_n \tau_1}{\mu_1}) [\|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| N_2] + 2\beta_n \langle F_2 p, p - x_{n+1} \rangle \\
&= (1 - \frac{\beta_n \tau_1}{\mu_1}) \|x_n - p\|^2 + \beta_n [\beta_n (1 - \frac{\beta_n \tau_1}{\mu_1}) \frac{\theta_n}{\beta_n^2} \|x_n - x_{n-1}\| N_2 + 2\langle F_2 p, p - x_{n+1} \rangle]. \\
&= (1 - \beta_n \iota) \|x_n - p\|^2 + \beta_n \Psi_n, \qquad (4.1.41)
\end{aligned}$$

where $\Psi = \beta_n(1 - \frac{\beta_n \tau_1}{\mu_1}) \frac{\theta_n}{\beta_n^2} \|x_n - x_{n-1}\| N_2 + 2\langle F_2 p, p - x_{n+1} \rangle$. According to Lemma 2.1.37, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty} \Psi_n \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty} \{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \qquad (4.1.42)$$

To establish that $\limsup_{k\to\infty}\Psi_n \leq 0$, we suppose that for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ such that (4.1.42) holds. Then,

$$\begin{aligned}
&\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\}\\
&=\liminf_{k\to\infty}\{(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|)\}\geq 0. \qquad (4.1.43)
\end{aligned}$$

In addition, using (4.1.41) and (4.1.21), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\leq (1-\frac{\beta_n\tau_1}{\mu_1})^2\|q_n-p\|^2+2\beta_n\langle F_2p,p-x_{n+1}\rangle\\
&\leq \|q_n-p\|^2+2\beta_n\langle F_2p,p-x_{n+1}\rangle\\
&\leq \|y_n-p\|^2-(1-\frac{\mu\lambda_n}{\lambda_{n+1}})\|y_n-z_n\|^2-(1-\frac{\mu\lambda_n}{\lambda_{n+1}})\|z_n-q_n\|^2+2\beta_n\eta\langle F_2p,p-x_{n+1}\rangle\\
&\leq \|w_n-p\|^2-(1-\frac{\mu\lambda_n}{\lambda_{n+1}})\|y_n-z_n\|^2-(1-\frac{\mu\lambda_n}{\lambda_{n+1}})\|z_n-q_n\|^2+2\beta_n\eta\langle F_2p,p-x_{n+1}\rangle,
\end{aligned} \qquad (4.1.44)$$

this implies that

$$\begin{aligned}
&(1-\frac{\mu\lambda_n}{\lambda_{n+1}})\|y_n-z_n\|^2+(1-\frac{\mu\lambda_n}{\lambda_{n+1}})\|z_n-q_n\|^2\\
&\leq \|w_n-p\|^2+2\beta_n\eta\langle F_2p,p-x_{n+1}\rangle-\|x_{n+1}-p\|^2\\
&\leq \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2+2\beta_n\eta\langle F_2p,p-x_{n+1}\rangle-\|x_{n+1}-p\|^2. \qquad (4.1.45)
\end{aligned}$$

$$\begin{aligned}
&\limsup_{k\to\infty}\left((1-\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})\|y_{n_k}-z_{n_k}\|^2+(1-\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})\|z_{n_k}-q_{n_k}\|^2\right)\\
&\leq \limsup_{k\to\infty}\Bigg[\|x_{n_k}-p\|^2+\beta_{n_k}\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2 \qquad (4.1.46)\\
&+2\beta_{n_k}\eta\langle F_2p,p-x_{n_k+1}\rangle-\|x_{n_k+1}-p\|^2\Bigg] \qquad (4.1.47)\\
&\leq -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0. \qquad (4.1.48)
\end{aligned}$$

It follows that

$$\lim_{k\to\infty}\left((1-\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})\|y_{n_k}-z_{n_k}\|^2+(1-\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}})\|z_{n_k}-q_{n_k}\|^2\right)=0,$$

as such, we have that

$$lim_{k\to\infty}\|y_{n_k}-z_{n_k}\|^2=0=lim_{k\to\infty}\|z_{n_k}-q_{n_k}\|^2.$$

Hence, we have

$$\lim_{k\to\infty}\|y_{n_k}-z_{n_k}\|=0 \text{ and } \lim_{k\to\infty}\|z_{n_k}-q_{n_k}\|=0. \qquad (4.1.49)$$

From (4.1.41) and (4.1.19), we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\leq \|y_n-p\|^2+2\beta_n\eta\langle F_2p, p-x_{n+1}\rangle\\
&\leq \|w_n-p\|^2-\gamma_n(\epsilon-\nu(\gamma_n+\epsilon))\|A^*(U(Aw_n)-Aw_n)\|^2+2\beta_n\eta\langle F_2p, p-x_{n+1}\rangle\\
&\leq \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2-\epsilon(\epsilon-\nu(\gamma_n+\epsilon))\|A^*(U(Aw_n)-Aw_n)\|^2\\
&+2\beta_n\eta\langle F_2p, p-x_{n+1}\rangle, \qquad (4.1.50)
\end{aligned}$$

it follows that

$$\begin{aligned}
&\limsup_{k\to\infty}\left[\epsilon(\epsilon-\nu(\gamma_{n_k}+\epsilon))\|A^*(U(Aw_{n_k})-Aw_{n_k})\|^2\right]\\
&\leq \limsup_{k\to\infty}\Bigg[\|x_{n_k}-p\|^2+\beta_{n_k}\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\\
&+2\beta_{n_k}\eta\langle F_2p, p-x_{n_k+1}\rangle-\|x_{n_k+1}-p\|^2\Bigg]\\
&\leq -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0.
\end{aligned}$$

As such, we obtain

$$\lim_{k\to\infty}\|A^*(U(Aw_{n_k})-Aw_{n_k})\|=0. \qquad (4.1.51)$$

Using a similar approach as in (4.1.50), we obtain that

$$\lim_{k\to\infty}\|U(Aw_{n_k})-Aw_{n_k}\|=0. \qquad (4.1.52)$$

Using (4.1.52), we have that

$$\|y_{n_k}-w_{n_k}\|=\|\gamma_nA^*(U(Aw_{n_k})-Aw_{n_k})\|\to 0 \quad \text{as} \quad k\to\infty. \qquad (4.1.53)$$

It is easy to see that, as $k\to\infty$, we have

$$\|w_{n_k}-x_{n_k}\|=\theta_{n_k}||x_{n_k}-x_{n_k-1}||=\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||\to 0. \qquad (4.1.54)$$

In addition, we have that

$$\|y_{n_k}-q_{n_k}\|\leq\|y_{n_k}-z_{n_k}\|+\|z_{n_k}-q_{n_k}\|\to 0 \quad \text{as} \quad k\to\infty. \qquad (4.1.55)$$

$$\|w_{n_k}-q_{n_k}\|\leq\|w_{n_k}-y_{n_k}\|+\|y_{n_k}-q_{n_k}\|\to 0 \quad \text{as } k\to\infty. \qquad (4.1.56)$$

$$\|y_{n_k}-x_{n_k}\|\leq\|y_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\to 0 \quad \text{as } k\to\infty. \qquad (4.1.57)$$

$$\|z_{n_k}-x_{n_k}\|\leq\|z_{n_k}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\|\to 0 \quad \text{as } k\to\infty. \qquad (4.1.58)$$

$$\|q_{n_k} - x_{n_k}\| \leq \|q_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \to 0 \ \text{ as } k \to \infty. \tag{4.1.59}$$

$$\|x_{n_k+1} - q_{n_k}\| \leq \beta_n \eta \|F_2 q_n\| \to 0 \ \text{ as } k \to \infty. \tag{4.1.60}$$

Thus, we have

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - q_{n_k}\| + \|q_{n_k} - x_{n_k}\| \to 0 \ \text{ as } k \to \infty. \tag{4.1.61}$$

Since $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to x^* such that

$$\limsup_{k\to\infty} \langle F_2 p, p - x_{n_k}\rangle = \lim_{j\to\infty} \langle F_2 p, p - x_{n_{k_j}}\rangle = \langle F_2 p, p - x^*\rangle. \tag{4.1.62}$$

Also, we obtain from (4.1.49), (4.1.53) and Lemma 4.1.9 that $x^* \in \Gamma$. Hence, since p is a unique solution of Ω, we have obtain from (4.1.62) that

$$\limsup_{k\to\infty} \langle F_2 p, p - x_{n_k}\rangle = \langle F_2 p, p - x^*\rangle \leq 0, \tag{4.1.63}$$

which implies that

$$\limsup_{k\to\infty} \langle F_2 p, p - x_{n_k+1}\rangle \leq 0. \tag{4.1.64}$$

Using our assumption and (4.1.64), we have that $\limsup_{k\to\infty} \Psi_{n_k} := \beta_n(1 - \frac{\beta_n \tau_1}{\mu_1})\frac{\theta_n}{\beta_n^2}\|x_n - x_{n-1}\|N_2 + 2\langle F_2 p, p - x_{n+1}\rangle$. Thus, the last part of Lemma 2.1.37 is achieved. Hence, we have that $\lim\limits_{n\to\infty} \|x_n - p\| = 0$. Thus, $\{x_n\}$ converges strongly to $p \in \Omega$. □

4.1.3 Numerical Examples

In this section, we present some numerical examples to show the efficiency and applicability of our method in comparison with Algorithm 2.6.5 in the framework of infinite dimensional Hilbert spaces.

Example 4.1.11. *[236] Let $H_1 = \mathbb{R}^4$ with norm $\|x\| = \sqrt{(x_1^2 + x_2^2 + x_3^2 + x_4^2)}$ for $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ and $H_2 = \mathbb{R}^2$ with the norm $\|y\| = \sqrt{y_1^2 + y_2^2}$ for $y = (y_1, y_2)^T \in \mathbb{R}^2$. Consider the mapping $F_2 : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $F_2 x = x$ for all $x \in \mathbb{R}^4$. It is easy to see that F_2 is strongly monotone with $\alpha = 1$ and Lipschitz continuous with $L = 1$ on $\mathbb{R}^4$. Let $Ax = (x_1 + x_3 + x_4, x_2 + x_3 - x_4)^T$ for all $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ then A is a bounded linear operator with $\|A\| = \sqrt{3}$. For $y = (y_1, y_2)^T \in \mathbb{R}^2$, let $B(y) = (y_1, y_2, y_1 + y_2, y_1 - y_2)^T$, then B is a bounded linear operator with $\|B\| = \sqrt{3}$. It is easy to see that $B = A^*$ is the adjoint of A. Let $C = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : 12x_1 - 4x_2 + 4x_3 - 4x_4 \geq 9\}$ and define $F_1 : \mathbb{R}^4 \to \mathbb{R}^4$ by $F_1(x) = (\sin\|x\| + 2)a^0$ for all $x \in \mathbb{R}^4$, where $a^0 = (12, -4, 4, -4)^T \in \mathbb{R}^4$. It is easy to see that F_1 is pseudomonotone, also, define $S; U : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$S(y) = U(y) = \begin{cases} (y_1, y_2)^T, & \text{if } y_1 \leq 0 \\ (-2y_1, y_2)^T, & \text{if } y_1 > 0. \end{cases} \tag{4.1.65}$$

It is easy to see that $S(y) = U(y)$ *is* $\frac{1}{3}$*-demicontractive. We choose* $\omega_n = \gamma_n = 0.02, \lambda_0 = 0.03, \mu = 0.05, \theta_n = \overline{\theta} = 0.05, \alpha_n = \frac{1}{n+2} = \beta_n, \epsilon_n = \beta_n^2, \zeta_n = \frac{1}{n+1}$ *for all* $n \in \mathbb{N}$*. It is easy to verify that all hypotheses of Theorem 4.1.10 are satisfied. We implement our algorithm for different values of* x_0, x_1 *as follows.*

Case I: $x_0(t) = (-2, 3, 5, -4)^T, \quad x_1(t) = (1, 3, 2, 1)^T;$

Case II: $x_0(t) = (12, 3, 15, -4)^T, \quad x_1(t) = (19, 3, -2, 11)^T;$

Case III: $x_0(t) = (1, 3, 2, 1)^T, \quad x_1(t) = (-2, 3, 5, -4)^T;$

Case IV: $x_0(t) = (-2.5, 0.3, 5.7, 4.8)^T, \quad x_1(t) = (1.6, 3.9, -2, 10)^T.$

The computational results are shown in Table 4.1 and Figure 4.1.

Table 4.1: Computation result for Example 4.1.11.

		Algorithm 4.1.5	Algorithm 2.6.5
Case I	No of Iter.	11	26
	CPU time (sec)	0.0049	0.0092
Case II	No of Iter.	12	13
	CPU time (sec)	0.0063	0.0099
Case III	No of Iter.	16	29
	CPU time (sec)	0.0087	0.0112
Case IV	No of Iter.	10	29
	CPU time (sec)	0.0040	0.0102

Example 4.1.12. *Let* $H_1 = H_2 = L_2([0,1])$ *be equipped with the inner product*

$$\langle x, y \rangle = \int_0^1 x(t)y(t)dt \quad \forall \quad x, y \in L_2([0,1]) \quad \textit{and} \quad \|x\|^2 := \int_0^1 |x(t)|^2 dt \quad \forall x, y, \in L_2([0,1]).$$

Let $F_1; F_2 : L_2([0,1]) \to L_2([0,1])$ *be defined by*

$$F_1 x(t) = \int_0^1 \left(x(t) - \left(\frac{2tse^{t+s}}{e\sqrt{e^2-1}} \right) \cos x(s) \right) ds + \frac{2te^t}{e\sqrt{e^2-1}}, \quad x \in L_2([0,1])$$

$$\textit{and} \quad F_2 x(t) = \max\{0, \frac{x(t)}{2}\}, \quad t \in [0,1].$$

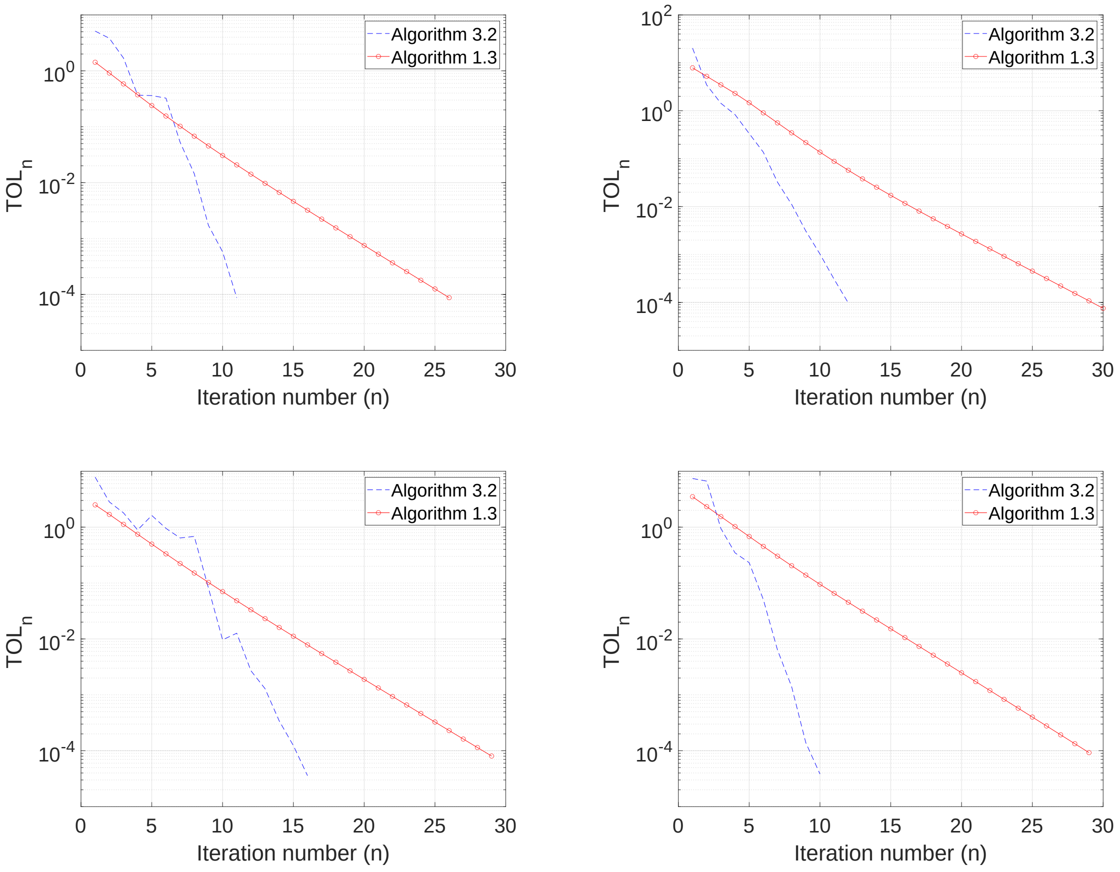

Figure 4.1: Example 4.1.11, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

It is easy to see that F_1 *is Lipschitz continuous and pseudomonotone and* F_2 *is* α*-strongly monotone on* $L_2([0,1])$*. Let* $U;S : L_2([0,1]) \to L_2([0,1])$ *be defined by*

$$U(x) = S(x) = \frac{-3x}{2} \quad \forall \quad x \in L_2([0,1])$$

and $A : L_2([0,1]) \to L_2([0,1])$ *be defined by*

$$Ax(s) = \int_0^1 K(s,t)x(t)dt \quad \forall \quad x \in L_2([0,1]),$$

where K *is a continuous real-valued function defined on* $[0,1] \times [0,1]$*. Thus,* T *is a bounded linear operator with adjoint*

$$A^*x(s) = \int_0^1 K(t,s)x(t)dt \quad \forall \quad x \in L_2([0,1]).$$

Let C *be defined by* $C = \{x \in L_2 : \langle a, x\rangle = b\}$ *where* $a \neq 0$ *and* $b = 2$*. Thus, we have*

$$P_C(\bar{x}) = \max\left\{0, \frac{b - \langle a, \bar{x}\rangle}{\|a\|^2}\right\} a + \bar{x}.$$

We choose $\omega_n = \gamma_n = 2, \lambda_1 = 1, \zeta_n = \frac{100}{(n+1)^{1.1}}, \mu = 0.5, \theta_n = \overline{\theta}, \alpha_n = \beta_n = \frac{1}{5n+2}, \epsilon_n = \frac{\beta_n}{n^{0.01}},$ *for all* $n \in \mathbb{N}$*. It is easy to verify that all hypotheses of Theorem 4.1.10 are satisfied. We implement our algorithm for different values of* x_0, x_1 *as follows.*

Case I: $x_0(t) = 2t^2 + t + 2, \quad x_1(t) = t;$

Case II: $x_0(t) = 2t^2 + e^{2t} + 1, \quad x_1(t) = 3t^3 + 3;$

Case III: $x_0(t) = t + 2, \quad x_1(t) = \cos(t);$

Case IV: $x_0(t) = \cos(t) + 2t^2 + 4, \quad x_1(t) = 2t + 2 + e^t.$

4.2 Conclusion

In this chapter, we introduced a new inertial method for solving strongly monotone variational inequality problems with split variational inequality and composed fixed point problem constraints are proposed, we establish strong converge to a minimum-norm solution of the problem in two real Hilbert spaces. The main advantage of this method is that our inertial term can take the value 1 which has not been explored by researchers in this area. In addition, our method uses a simple self-adaptive step size that is generated at each iteration, which allows it to be easily implemented without the prior knowledge of the operator norm as well as the Lipschitz constant. Finally, we present some numerical experiments to establish the applicability and efficiency of our method.

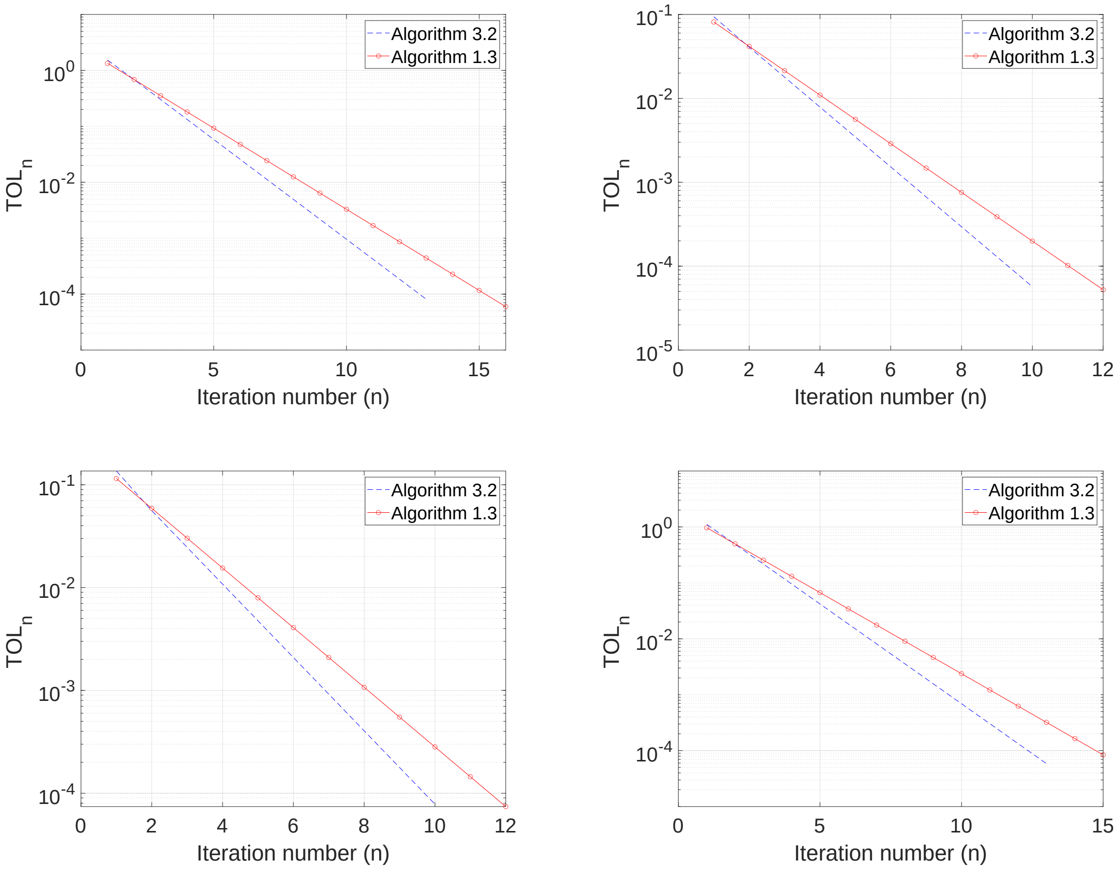

Figure 4.2: Example 4.1.12, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

Table 4.2: Computation result for Example 4.1.12.

		Algorithm 7.1.2	Algorithm 2.6.5
Case I	No of Iter. CPU time (sec)	13 0.7291	16 3.8738
Case II	No of Iter. CPU time (sec)	10 0.5459	11 1.1119
Case III	No of Iter. CPU time (sec)	10 1.9603	12 6.2247
Case IV	No of Iter. CPU time (sec)	13 1.9259	15 6.2476

4.3 Open Problem

The results obtained in this chapter is based on approximating the solution of bilevel type of optimization problems in the framework of Hilbert spaces. To the best of our knowledge, the concept is yet to be extended to abstract spaces like Hadamard and p-uniformly convex spaces. Is it possible to extend the concept to these spaces?

Chapter 5

Contribution to Monotone Inclusion Problems in Banach Spaces

The study of Monotone Inclusion Problems (MIPs) in Banach spaces, as described in Chapter 2, is extremely helpful in almost all disciplines. It has been applied to computing medians and means of trees, which are particularly significant in computer science and mathematical sciences. One of the very important application of MIPs is the application to phylogenetics, diffusion tensor imaging, consensus methods, and modeling of human lungs' airway systems. In this chapter, we further develop the concept of MIPs in the framework of Banach space.

The results obtained in this chapter extend, generalize and improve several results in this direction.

5.1 Shrinking Approximation Method for Solution of Split Monotone Variational Inclusion and Fixed Point Problems in Banach Space

In this section, we introduce and study a shrinking algorithm for finding a solution of split monotone variational inclusion problem which is also a common fixed point problem of a relatively nonexpansive mapping in uniformly convex real Banach spaces which are also uniformly smooth. The iterative algorithm employed in this chapter is design in such a way that it does not require prior knowledge of operator norms. We prove a strong convergence result for approximating the solutions of the aforementioned problems and give applications of our main result to the split convex minimization problem.

5.1.1 Main Result

Lemma 5.1.1. *Suppose $F : E \to 2^{E^*}$ is a maximal monotone operator and $f : E \to E^*$ is a λ-ism mapping with $\lambda > 0$ such that $(F + f)^{-1}(0) \neq \emptyset$. Then*

$$\phi(u, L_\lambda^F \circ A_\lambda^f(x)) + \phi(L_\lambda^F \circ A_\lambda^f(x), x) \leq \phi(u, x),$$

for any $u \in (F + f)^{-1}(0)$ and $x \in E$.

The proof of the Lemma stated above is similar to the one in [251].

Lemma 5.1.2. *Let E be a real Banach space, $T : E \to E$ be a relatively nonexpansive mapping and $F : E \to 2^{E^*}$ be a maximal monotone operator. Suppose $f : E \to E^*$ is a λ-ism mapping for $\lambda > 0$ and $(f + F)^{-1}(0) \neq \emptyset$, then*

$$Fix(T(L_\lambda^F \circ A_\lambda^f)) = Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f).$$

Proof. Clearly, $Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f) \subseteq Fix(T(L_\lambda^F \circ A_\lambda^f))$. We only need to prove that $Fix(T(L_\lambda^F \circ A_\lambda^f)) \subseteq Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f)$. Let $p \in Fix(T(L_\lambda^F \circ A_\lambda^f))$ and $q \in Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f)$, then

$$\begin{aligned}\phi(q, p) &= \phi(q, T(L_\lambda^F \circ A_\lambda^f)x)\\ &\leq \phi(q, (L_\lambda^F \circ A_\lambda^f)x). \end{aligned} \tag{5.1.1}$$

Now by applying Lemma 5.1.2 and (5.1.1), we get

$$\begin{aligned}\phi(p, (L_\lambda^F \circ A_\lambda^f)) &= \phi(q, p) - \phi(q, (L_\lambda^F \circ A_\lambda^f)x)\\ &\leq \phi(q, p) - \phi(q, p)\\ &= 0.\end{aligned}$$

Hence, $p \in Fix(L_\lambda^F \circ A_\lambda^f)$.
Next, we show that $p \in Fix(T)$ since $p \in Fix(T(L_\lambda^F \circ A_\lambda^f))$, we obtain

$$\begin{aligned}\phi(p, Tp) &= \phi(p, (T(L_\lambda^F \circ A_\lambda^f)p))\\ &= \phi(p, Tp)\\ &= 0.\end{aligned}$$

Hence $p \in Fix(T)$. This implies that $p \in Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f)$. Therefore, we conclude that $Fix(T(L_\lambda^F \circ A_\lambda^f)) = Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f)$. □

Theorem 5.1.3. *Let E_1, E_2 be 2-uniformly convex and uniformly smooth real Banach spaces with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$ and duals E_1^*, E_2^*, respectively. Let Q be a nonempty, closed and convex subset of E_2, $T : E_1 \to E_1$ and $S : E_2 \to E_2$ be relatively nonexpansive mappings respectively. Suppose that $A : E_1 \to E_2$ is a bounded linear operator with adjoint A^*, $F : E_1 \to 2^{E_1^*}$ and $G : E_2 \to 2^{E_2^*}$ are maximal monotone operators. Let $f : E_1 \to E_1^*$ and $g : E_2 \to E_2^*$ be single-valued λ, μ-ism operators with $R_\lambda^F \circ B_\lambda^f := (J + \lambda F)^{-1} \circ (J - \lambda f) : E_1 \to domF$ for $\lambda > 0$ and $R_\mu^G \circ B_\mu^g := (J + \mu) G^{-1} \circ (J -$*

$\mu g) : E-2 \to domG$ for $\mu > 0$, respectively. Assume that $\Gamma := \{x^* \in Fix(T) \cap (F+f)^{-1}(0)$ and $Ax^* \in Fix(S) \cap (G+g)^{-1}(0)\} \neq \emptyset$, then $\{x_n\}_{n=0}^{\infty}$ is generated iteratively by $x_1 \in E_1$ and $C_1 = E_1$ with

$$\begin{cases} w_n = J_1^{-1}(J_1x_n - \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n); \\ u_n = J_1^{-1}[(1-\beta_n)J_1w_n + \beta_n J_1(T(R_\lambda^F \circ B_\lambda^f))w_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}}x_1;\ n \geq 1; \end{cases} \tag{5.1.2}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of E_1 onto C_{n+1}. Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence in (0,1) such that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, and the step size γ_n is chosen in such a way that $\gamma_n = \frac{\rho_n||(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n||^2}{||A^*J_2(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n||^2}$, for $Ax_n \neq (S(R_\mu^G \circ B_\mu^g))Ax_n$, where $0 < d \leq \rho_n \leq e < 1$ for $d, e \in \mathbb{R}$, otherwise $\gamma_n = \gamma$ (γ being any nonnegative real number). Then, the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$, where $\overline{x} = \Pi_\Gamma x_1$.

We divide our proof into several steps:
Step 1: We prove using Theorem 5.1.3 that C_n is a closed and convex for each $n \geq 1$.

Proof. We obtain from Theorem 5.1.3 that $C_1 = E_1$, therefore C_1 is closed and convex. Now assume that C_n is closed and convex, then

$$\begin{aligned} \phi(v, u_n) \leq \phi(v, x_n) & \\ &\Leftrightarrow ||v||^2 - 2\langle v, J_1u_n\rangle + ||u_n||^2 \\ &\leq ||v||^2 - 2\langle v, J_1x_n\rangle + ||x_n||^2 \\ &\Leftrightarrow 2\langle v, J_1x_n - J_1u_n\rangle \leq ||x_n||^2 - ||u_n||^2. \end{aligned} \tag{5.1.3}$$

We have from (5.1.3) that C_{n+1} is closed and convex subset of E_1. Therefore, $\Pi_{C_{n+1}}$ is well defined. □

Step 2: We show that $\Gamma \subseteq C_n$ for all $n \geq 1$.

Proof. Let $x^* \in \Gamma \subseteq C_n$, for $n \geq 1$ then we have from (5.1.2) and Lemma 2.3.1 that

$$\begin{aligned} &\phi(x^*, u_n) \\ &= \phi(x^*, J_1^{-1}((1-\beta_n)J_1w_n + \beta_n J_1(T(R_\lambda^F \circ B_\lambda^f))w_n)) \\ &= ||x^*||^2 - 2\langle x^*, (1-\beta_n)J_1w_n + \beta_n J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\rangle \\ &+ ||(1-\beta_n)J_1w_n + \beta_n J_1(T(R_\lambda^F \circ B_\lambda^f))w_n||^2 \\ &\leq ||x^*||^2 - 2(1-\beta_n)\langle x^*, J_1w_n\rangle - 2\beta_n\langle x^*, J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\rangle \\ &+ (1-\beta_n)||w_n||^2 + \beta_n||(T(R_\lambda^F \circ B_\lambda^f))w_n||^2 - \beta_n(1-\beta_n)g(||J_1w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n||) \\ &= (1-\beta_n)\phi(x^*, w_n) + \beta_n\phi(x^*, (T(R_\lambda^F \circ B_\lambda^f))w_n) - \beta_n(1-\beta_n)g(||J_1w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n||) \\ &\leq (1-\beta_n)\phi(x^*, w_n) + \beta_n\phi(x^*, w_n) - \beta_n(1-\beta_n)g(||J_1w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n||) \\ &= \phi(x^*, w_n) - \beta_n(1-\beta_n)g(||J_1w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n||) \\ &\leq \phi(x^*, w_n). \end{aligned} \tag{5.1.4}$$

Also, we obtain from (5.1.2) that

$$
\begin{aligned}
&\phi(x^*, w_n) \\
&= \phi(x^*, J_1^{-1}(J_1 x_n - \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n))) \\
&= ||x^*||^2 - 2\langle x^*, J_1 x_n - \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n\rangle \\
&+ ||J_1 x_n - \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n||^2 \\
&= ||x^*||^2 - 2\langle x^*, J_1 x_n - \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n\rangle \\
&+ ||x_n - J_1^{-1}\gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n||^2 \\
&\leq ||x^*||^2 - 2\langle x^*, J_1 x_n\rangle + 2\gamma_n\langle x^*, A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\rangle \\
&+ 2||kx_n||^2 - 2\langle x_n, \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\rangle + \gamma_n^2||A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2 \\
&\leq ||x^*||^2 - 2\langle x^*, J_1 x_n\rangle + 2\gamma_n\langle x^*, A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\rangle \\
&+ ||x_n||^2 - 2\langle x_n, \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\rangle + \gamma_n^2||A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2 \\
&= \phi(x^*, x_n) - 2\gamma_n\langle x_n - x^*, A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\rangle + \gamma_n^2||A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2 \\
&= \phi(x^*, x_n) - 2\gamma_n\langle Ax_n - Ax^*, J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\rangle + \gamma_n^2||A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2.
\end{aligned} \tag{5.1.5}
$$

Applying Lemma 2.3.4, we get

$$
\begin{aligned}
&\langle Ax_n - Ax^*, J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\rangle \\
&= \langle Ax_n - (S(R_\mu^G \circ B_\mu^g))Ax_n + (S(R_\mu^G \circ B_\mu^g))Ax_n - Ax^*, J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\rangle \\
&= ||(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2 + \langle S(R_\mu^G \circ B_\mu^g)Ax_n - Ax^*, J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\rangle \\
&\geq ||(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2 + \frac{1}{2}\big(||Ax_n - Ax^*||^2 - ||(I - (S(R_\mu^G \circ B_\mu^g)))||^2 \\
&- ||(S(R_\mu^G \circ B_\mu^g))Ax_n - Ax^*||^2\big) \\
&\geq \frac{1}{2}||(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2.
\end{aligned} \tag{5.1.6}
$$

On substituting (5.1.6) into (5.1.5), we obtain

$$
\begin{aligned}
&\phi(x^*, w_n) \\
&\leq \phi(x^*, x_n) - \gamma_n\big(||(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2 + \gamma_n||A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2\big) &(5.1.7) \\
&\leq \phi(x^*, x_n). &(5.1.8)
\end{aligned}
$$

Hence, we conclude from (5.1.4) and (5.1.7) that

$$\phi(x^*, u_n) \leq \phi(x^*, w_n) \leq \phi(x^*, x_n). \tag{5.1.9}$$

Therefore, we conclude that $x^* \in C_{n+1}$. This implies that $\Gamma \subseteq C_n$ for all $n \geq 1$. Hence, (5.1.2) is well-defined. □

Step 3: We show that $\{x_n\}$ is a Cauchy sequence.

Proof. Let $x^* \in \Gamma$, by using the definition of C_n, we have that $x_n = \Pi_{C_n}x_1$ for all $n \geq 1$. It follows from (2.2.2), we have that

$$\begin{aligned}\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) &\leq \phi(x^*, x_1) - \phi(x^*, \Pi_{C_n}x_1)\\ &\leq \phi(x^*, x_1), \ \forall\ n \geq 1.\end{aligned}$$

This implies that $\{\phi(x_n, x_1)\}$ is bounded. More so, since $x_n = \Pi_{C_n}x_1$ and $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subseteq C_n$, we have that

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \ \forall\ n \geq 1. \tag{5.1.10}$$

Therefore, $\{\phi(x_n, x_1)\}$ is non-decreasing and hence bounded. So, the limit also exists. From Lemma 2.2.2, we obtain that

$$\begin{aligned}\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n}x_1) &\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n}x_1, x_1)\\ &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1),\end{aligned} \tag{5.1.11}$$

thus, we have that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{5.1.12}$$

Applying Lemma 2.5.2, we obtain that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{5.1.13}$$

Suppose $x_n = \Pi_{C_n}x_1 \subseteq C_m$, for some positive integers m, n with $m \leq n$, then applying Lemma 2.2.2 and using the same approach as in (5.1.11), we obtain that

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n}x_1)\\ &\leq \phi(x_m, x_1) - \phi(\Pi_{C_n}x_1, x_1)\\ &= \phi(x_m, x_1) - \phi(x_n, x_1).\end{aligned} \tag{5.1.14}$$

Since $\lim_{n\to\infty} \phi(x_n, x_1)$ exists, it follows from (5.1.14) and Lemma 2.5.2 that $\lim_{n\to\infty} ||x_n - x_m|| = 0$. Hence, we conclude that $\{x_n\}$ is a Cauchy sequence. ☐

Step 4: Let $\{x_n\}$ be a sequence generated by (5.1.2), then (i) $\lim_{n\to\infty} ||T(R_\lambda^F \circ B_\lambda^f)w_n - w_n|| = 0$.
(ii) $\lim_{n\to\infty} ||(I - S(R_\mu^G \circ B_\mu^g))Ax_n|| = 0$.
(iii) $\lim_{n\to\infty} ||A^*J_2(I - S(R_\mu^G \circ B_\mu^g))Ax_n|| = 0$.

Proof. Since $x_{n+1} = \Pi_{C_{n+1}} \in C_{n+1} \subseteq C_n$, by the definition of C_{n+1}, (5.1.10) and (5.1.12), we have that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \to 0, \ n \to \infty. \tag{5.1.15}$$

We have from Lemma 2.5.2 that

$$\lim_{n\to\infty} ||x_{n+1} - u_n|| = 0. \tag{5.1.16}$$

Also, from (5.1.13) and (5.1.16), we have that

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{5.1.17}$$

From (5.1.4) and (5.1.7), we have that

$$\begin{aligned}\phi(x^*, u_n) &\leq \phi(x^*, x_n) - \beta_n(1-\beta_n)g(||J_1 w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n||)\\ &\quad - \gamma_n\big(||(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2 + \gamma_n||A^* J_2(S(R_\mu^G \circ B_\mu^g))Ax_n||^2\big).\end{aligned} \tag{5.1.18}$$

It then follows that

$$\begin{aligned}&\beta_n(1-\beta_n)g(||J_1 w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n||)\\ &\leq \phi(x^*, x_n) - \phi(x^*, u_n)\\ &= ||x^*||^2 - 2\langle x^*, J_1 x_n\rangle + ||x_n||^2 - ||x^*||^2 + 2\langle x^*, J_1 u_n\rangle - ||u_n||^2\\ &= 2\langle x^*, J_1 u_n - J_1 x_n\rangle + ||x_n||^2 - ||u_n||^2\\ &\leq 2||x^*||\ ||J_1 u_n - J_1 x_n|| + ||x_n - u_n||\ (||x_n|| + ||u_n||).\end{aligned} \tag{5.1.19}$$

Since E_1 is 2-uniformly convex and uniformly smooth Banach space, J_1 is uniformly continuous from norm-to-norm. Then, we obtain from (5.1.17) that

$$\lim_{n\to\infty} ||J_1 u_n - J_1 x_n|| = 0. \tag{5.1.20}$$

By applying the condition $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and (5.1.20) in (5.1.19), we obtain that

$$\lim_{n\to\infty} g(||J_1 w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n||) = 0. \tag{5.1.21}$$

Using the property of g in Lemma 2.3.1, we have that

$$\lim_{n\to\infty} ||J_1 w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n|| = 0. \tag{5.1.22}$$

Since J_1^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} ||w_n - (T(R_\lambda^F \circ B_\lambda^f))w_n|| = 0. \tag{5.1.23}$$

Also, from (5.1.18) and following the same approach in (5.1.19), we have that

$$\begin{aligned}&\gamma_n\big(||(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2 - \gamma_n||A^* J_2(I - S(R_\mu^G \circ B_\mu^g))Ax_n||^2\big)\\ &\leq \phi(x^*, x_n) - \phi(x^*, u_n)\\ &= ||x^*||^2 - 2\langle x^*, J_1 x_n\rangle + ||x_n||^2 - ||x^*||^2 + 2\langle x^*, J_1 u_n\rangle - ||u_n||^2\\ &= 2\langle x^*, J_1 u_n - J_1 x_n\rangle + ||x_n||^2 - ||u_n||^2\\ &\leq 2||x^*||\ ||J_1 u_n - J_1 x_n|| + ||x_n - u_n||\ (||x_n|| + ||u_n||).\end{aligned}$$

Using (5.1.17) and (5.1.20), we have that

$$\lim_{n\to\infty} \gamma_n\big(||(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n||^2 - \gamma_n||A^* J_2(I - S(R_\mu^G \circ B_\mu^g))Ax_n||^2\big) = 0. \tag{5.1.24}$$

Applying the definition on γ_n and the fact that ρ_n is bounded from above and away from zero, (5.1.24) gives

$$\lim_{n\to\infty} \frac{||(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n||^4}{||A^*J_2(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n||^2} = 0. \tag{5.1.25}$$

Observe that

$$\begin{aligned} ||A^*J_2(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n|| &\leq ||A^*||\ ||J_2(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n|| \\ &= ||A||\ ||(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n||. \end{aligned} \tag{5.1.26}$$

Therefore, from (5.1.25), we get

$$\lim_{n\to\infty} ||(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n|| \leq ||A|| \lim_{n\to\infty} \frac{||(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n||^4}{||A^*J_2(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n||^2} = 0. \tag{5.1.27}$$

It follows from (5.1.26) and (5.1.27) that

$$\lim_{n\to\infty} ||A^*J_2(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n|| = 0. \tag{5.1.28}$$

From (5.1.2) and (5.1.25), we have that

$$\begin{aligned} ||J_1w_n - J_1x_n|| &= \gamma_n||A^*J_2(I-S(R_\mu^G \circ B_\mu^g))Ax_n|| \\ &\leq \frac{\rho_n||(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n||^2}{||A^*J_2(I-(S(R_\mu^G \circ B_\mu^g)))Ax_n||}. \end{aligned} \tag{5.1.29}$$

From (5.1.2), (5.1.29) and by uniform continuity of J_1 and J_1^{-1} on bounded subset, we obtain that

$$\lim_{n\to\infty} ||w_n - x_n|| = 0. \tag{5.1.30}$$

Also, from (5.1.2) and (5.1.22), we get

$$||J_1u_n - J_1w_n|| \leq \beta_n||J_1(T(R_\lambda^F \circ B_\lambda^f))w_n - J_1w_n|| \to 0, \text{ as } n\to\infty. \tag{5.1.31}$$

More so, from (5.1.31) and by uniform continuity of J_1 and J_1^* on bounded subset, we obtain that

$$\lim_{n\to\infty} ||u_n - w_n|| = 0. \tag{5.1.32}$$

From (5.1.30) and (5.1.32), we get that

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{5.1.33}$$

□

Step 4: We show that $\overline{x} \in \Gamma$.

Proof. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\overline{x} \in E_1$ such that $x_{n_k} \rightharpoonup \overline{x}$. Now using (5.1.30) and (5.1.33), there exist subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{w_n\}$ and $\{u_n\}$ converges weakly to $\overline{x}$. From (5.1.23), the fact that T is relatively nonexpansive mapping and Lemma 5.1.1, we obtain that $\overline{x} \in Fix(T(R_\lambda^F \circ B_\lambda^f)) = Fix(T) \cap (F+f)^{-1}(0)$. Also, since A is a bounded linear operator, we have that $A x_{n_k} \rightharpoonup A\overline{x}$. Thus from (5.1.27), the fact that S is a relatively nonexpansive mapping, the demiclosedness principle and Lemma 5.1.1, we have that $A\overline{x} \in Fix(S(R_\lambda^G \circ B_\lambda^g)) = Fix(S) \cap (G+g)^{-1}(0)$. Hence, we therefore conclude that $\overline{x} \in \Gamma$. □

Step 5: We prove that $\{x_n\} \to \overline{x}$

Proof. Let $\overline{x} = \Pi_\Gamma x_1$, $\overline{x} \in \Gamma$, from $x_n = \Pi_{C_n} x_1$ and $\overline{x} \in \Gamma \subseteq C_n$, we have

$$\phi(x_n, x_1) \leq \phi(\overline{x}, x_1), \tag{5.1.34}$$

which implies that

$$\phi(\overline{x}, x_1) \leq \liminf_{n\to\infty} \phi(x_n, x_1) \leq \phi(\overline{x}, x_1). \tag{5.1.35}$$

From the definition of $\overline{x} = \Pi_\Gamma x_1$, we have that $x^* = \overline{x}$. Hence $\liminf_{n\to\infty} x_n = \overline{x} = \Pi_C x_1$. We therefore conclude that $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$, where $\overline{x} = \Pi_\Gamma x_1$. □

In the following result, we considered only the SMVIP without the fixed point problems.

Corollary 5.1.4. *Let E_1, E_2 be 2-uniformly convex and uniformly smooth real Banach spaces with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$ and duals E_1^*, E_2^*, respectively. Let Q be a nonempty, closed and convex subset of E_2. Suppose that $A : E_1 \to E_2$ is a bounded linear operator with adjoint A^*, $F : E_1 \to 2^{E_1^*}$ and $G : E_2 \to 2^{E_2^*}$ are maximal monotone operators. Let $f : E_1 \to E_1^*$ and $g : E_2 \to E_2^*$ be single-valued λ, μ-ism operators with $R_\lambda^F \circ B_\lambda^f := (J+\lambda F)^{-1} \circ (J - \lambda f) : E_1 \to domF$ for $\lambda > 0$ and $R_\mu^G \circ B_\mu^g := (J+\mu)G^{-1} \circ (J - \mu g) : E-2 \to domG$ for $\mu > 0$, respectively. Assume that $\Gamma := \{x^* \in (F+f)^{-1}(0)$ and $Ax^* \in (G+g)^{-1}(0)\} \neq \emptyset$, then $\{x_n\}_{n=0}^\infty$ is generated iteratively by $x_1 \in E_1$ and $C_1 = E_1$ with*

$$\begin{cases} w_n = J_1^{-1}(J_1 x_n - \gamma_n A^* J_2(I - (R_\mu^G \circ B_\mu^g))Ax_n); \\ u_n = J_1^{-1}[(1-\beta_n)J_1 w_n + \beta_n J_1(R_\lambda^F \circ B_\lambda^f)w_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1; \ n \geq 1; \end{cases} \tag{5.1.36}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of E_1 onto C_{n+1}. Suppose $\{\beta_n\}_{n=1}^\infty$ is a sequence in (0,1) such that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, and the step size γ_n is chosen in such a way that $\gamma_n = \frac{\rho_n ||(I-(R_\mu^G \circ B_\mu^g))Ax_n||^2}{||A^* J_2(I-(R_\mu^G \circ B_\mu^g))Ax_n||^2}$, for $Ax_n \neq (R_\mu^G \circ B_\mu^g)Ax_n$, where $0 < d \leq \rho_n \leq e < 1$ for $d, e \in \mathbb{R}$, otherwise $\gamma_n = \gamma$ (γ being any nonnegative real number). Then, the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$, where $\overline{x} = \Pi_\Gamma x_1$.

Also, in the result discussed below, we considered the split common fixed point problem.

Corollary 5.1.5. *Let E_1, E_2 be 2-uniformly convex and uniformly smooth real Banach spaces with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$ and duals E_1^*, E_2^*, respectively. Let Q be a nonempty, closed and convex subset of E_2, $T : E_1 \to E_1$ and $S : E_2 \to E_2$ be relatively nonexpansive mappings respectively. Suppose that $A : E_1 \to E_2$ is a bounded linear operator with adjoint A^*. Assume that $\Gamma := \{x^* \in Fix(T) \text{ and } Ax^* \in Fix(S)\} \neq \emptyset$, then $\{x_n\}_{n=0}^{\infty}$ is generated iteratively by $x_1 \in E_1$ and $C_1 = E_1$ with*

$$\begin{cases} w_n = J_1^{-1}(J_1 x_n - \gamma_n A^* J_2(I-S)Ax_n); \\ u_n = J_1^{-1}[(1-\beta_n)J_1 w_n + \beta_n J_1(T)w_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1;\ n \geq 1; \end{cases} \tag{5.1.37}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of E_1 onto C_{n+1}. Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence in (0,1) such that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, and the step size γ_n is chosen in such a way that $\gamma_n = \frac{\rho_n ||(I-S)Ax_n||^2}{||A^* J_2(I-S)Ax_n||^2}$, for $Ax_n \neq (S)Ax_n$, where $0 < d \leq \rho_n \leq e < 1$ for $d, e \in \mathbb{R}$, otherwise $\gamma_n = \gamma$ (γ being any nonnegative real number). Then, the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$, where $\overline{x} = \Pi_\Gamma x_1$.

Remark 5.1.6. *The result discussed in this article generalizes many related results, most especially, results where SVIP and SMVIP were discussed in the framework of real Hilbert spaces. Our results hold for the classes of nonexpansive and pseudocontractive mappings in the framework of real Hilbert spaces.*

5.1.2 Applications

In this section, we apply our result to a split convex minimization problem.

Let E_1 and E_2 be real Banach spaces. Let $M : E_1 \to \mathbb{R}$ and $N : E_2 \to \mathbb{R}$ are convex and differentiable functions and $F : E_1 \to (-\infty, +\infty]$ and $G : E_2 \to (-\infty, +\infty]$ are proper, convex and lower semi-continuous functions. It is clear that if $\bigtriangledown M$ and $\bigtriangledown N$ is $\frac{1}{\alpha}$ and $\frac{1}{\theta}-$ Lipschitz continuous, then it is α, θ-ism, where $\bigtriangledown M$ and $\bigtriangledown N$ are the gradients of M and N respectively. It is also known that the subdifferential ∂F and ∂G are maximal monotone (see [**?**]). Furthermore,

$$M(x^*) + F(x^*) = \min_{x \in E_1} \left[M(x) + F(x)\right] \Leftrightarrow 0 \in M(x^*) + \partial F(x^*)$$

and

$$N(x^*) + G(x^*) = \min_{x \in E_2} \left[N(x) + G(x)\right] \Leftrightarrow 0 \in N(x^*) + \partial G(x^*).$$

Our aim is to solve the following Split Convex Minimization and Fixed Point Problem, (in short, SCMFPP): find $x^* \in E_1$ such that

$$x^* \in Fix(T) \cap argmin_{x \in E_1} M(x) + F(x) \text{ and } y^* = Ax^* \in Fix(S) \cap argim_{y \in E_2} N(y) + G(y). \tag{5.1.38}$$

Suppose the solution set of (5.1.38) is denoted by $\varTheta$, then by setting $F = \partial F, G = \partial G, f = \bigtriangledown M$ and $g = \bigtriangledown N$, (5.1.2) becomes

$$\begin{cases} w_n = J_1^{-1}(J_1 x_n - \gamma_n A^* J_2(I - (S(R_\mu^{\partial G} \circ B_\mu^{\bigtriangledown N})))Ax_n); \\ u_n = J_1^{-1}[(1-\beta_n)J_1 w_n + \beta_n J_1(T(R_\lambda^{\partial F} \circ B_\lambda^{\bigtriangledown M}))w_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1;\ n \geq 1. \end{cases} \tag{5.1.39}$$

Assume that the conditions in (5.1.2) holds, then $\{x_n\}$ converges strongly to an element in $\varTheta$.

5.2 Conclusion

In this chapter, we introduced and studied a new type of shrinking algorithm for finding a solution of the split monotone variational inclusion problem which is also a common fixed point problem of a relatively nonexpansive mapping in uniformly convex real Banach spaces which are also uniformly smooth. We proved a strong convergence result for approximating the solutions to the aforementioned problems and gave applications of our main result to split convex minimization problem.

Chapter 6

Contribution to Equilibrium Problems in Hilbert Spaces

The study of Equilibrium Problems (EPs) in the framework of Hilbert spaces, as described in Chapter 2, has been proven to be highly applicable in all facets of human endeavours. It has been applied to various problems arising in finance, network analysis, transportation, economics, the elasticity of demand and so on. In this chapter, we further develop the concept of EPs in the framework of Hilbert space.

The results obtained in this chapter extend, generalize and improve several results in this direction.

6.1 On Split Generalized Mixed Equilibrium and Fixed Point Problems of an Infinite Family of Quasinonexpansive Mult-valued Mappings in Real Hilbert Space

In this section, we study split generalized mixed equilibrium problems and fixed point problem in the framework of real Hilbert spaces to analyzing an iterative method for approximating a common solution of a split generalized mixed equilibrium problem and fixed point problem of an infinite family of a quasi-nonexpansive multi-valued mapping. The iterative algorithm introduced in this chapter is designed in such a way that it does not require the knowledge of the operator norm. This makes our iterative method more applicable compared to some of the iterative methods discussed in Chapter 2 of this study and many more in the literature. In addition, we state and prove a strong convergence result of the aforementioned problems and also give an application of our main result to a split variational inequality problem.

6.1.1 Main Result

In this section, we introduce our iterative method and establish the strong convergence result.

Theorem 6.1.1. *Let H_1 and H_2 be two real Hilbert spaces, let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(L1)-(L4)$ and G is upper semicontinuous in the first argument. Let $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ be continuous and monotone mappings, $\psi_1 : C \to \mathbb{R} \cup \{+\infty\}$ and $\psi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let $T_i : C \to K(C)$, for $i = 1, 2, 3, ...$ be a countable family of quasi-nonexpansive multi-valued mapping for $T_i p = \{p\}$ and $S : C \to C$ be a quasi nonexpansive mapping respectively such that $\Omega := \cap_{i=1}^{\infty} Fix(T_i) \cap Fix(S) \cap \Gamma \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ and $\{t_n\}$ be a sequence in $(0, 1-a)$ for some $a > 0$. Let the step size γ_n be chosen in such a way that for some $\varepsilon > 0$,*

$$\gamma_n \in \left(\varepsilon, \frac{||T_{r_n}^G - I)Aw_n||^2}{||A^*(T_{r_n}^G - I)Aw_n||^2} - \varepsilon\right),$$

for $T_{r_n}^G Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then, the sequences $\{w_n\}, \{u_n\}$ and $\{x_n\}$ are generated iteratively for an arbitrary $x_0 \in C$ and a fixed point $u \in C$

$$\begin{cases} w_n = (1 - \alpha_n - t_n)x_n + \alpha_n Sx_n + t_n u; \\ u_n = T_{r_n}^F(w_n + \gamma_n A^*(T_{r_n}^G - I)Aw_n); \\ x_{n+1} = \beta_{n,0}u_n + \sum_{i=1}^{\infty} \beta_{n,i} z_n^i, \ \ n \geq 1; \end{cases} \tag{6.1.1}$$

where $z_n^i \in T_i u_n$ and $r_n \subset (0, \infty)$ satisfies the following conditions:
(i) $\beta_{n,0}, \beta_{n,i} \in (0, 1), \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *such that* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$;
(ii) $\liminf_{n\to\infty} r_n > 0$;
(iii) $\lim_{n\to\infty} t_n = 0, \sum_{n=0}^{\infty} t_n = \infty$ *and* $\alpha_n + t_n < 1$;
(iv) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.
Then the sequences $\{x_n\}, \{u_n\}$ and $\{w_n\}$ converges strongly to an element in Ω.

Proof. Let $p \in \cap_{i=1}^{\infty} Fix(T_i) \cap Fix(S) \cap \Gamma$, then from (6.1.1), we have

$$\begin{aligned} ||u_n - p||^2 &= ||T_{r_n}^F(w_n + \gamma_n A^*(T_{r_n}^G - I)Aw_n - p||^2 \\ &\leq ||w_n + \gamma_n A^*(T_{r_n}^G - I)Aw_n - p||^2 \\ &= ||w_n - p||^2 + \gamma_n^2||A^*(T_{r_n}^G - I)Aw_n||^2 \\ &+ 2\gamma_n\langle w_n - p, A^*(T_{r_n}^G - I)Aw_n\rangle. \end{aligned} \tag{6.1.2}$$

From Lemma 2.3.4, we have that

$$\begin{aligned}
&2\gamma_n\langle w_n-p, A^*(T^G_{r_n}-I)Aw_n\rangle\\
&=2\gamma_n\langle A(w_n-p)+(T^G_{r_n}-I)Aw_n-(T^G_{r_n}-I)Aw_n,(T^G_{r_n}-I)Aw_n\rangle\\
&=2\gamma_n\big[\langle T^G_{r_n}Aw_n-Ap,(T^G_{r_n}-I)Aw_n\rangle-||(T^G_{r_n}-I)Aw_n||^2\\
&\leq 2\gamma_n[\frac{1}{2}||(T^G_{r_n}-I)Aw_n||^2-||(T^G_{r_n}-I)Aw_n||^2]\\
&=-\gamma_n||(T^G_{r_n}-I)Aw_n||^2
\end{aligned}\tag{6.1.3}$$

Therefore, from (6.1.2), (6.1.3) and condition $\gamma_n\in\left(\varepsilon,\frac{||T^G_{r_n}-I)Aw_n||^2}{||A^*(T^G_{r_n}-I)Aw_n||^2}-\varepsilon\right)$, we have that

$$\begin{aligned}
||u_n-p||^2&\leq||w_n-p||^2+\gamma_n^2||A^*(T^G_{r_n}-I)Aw_n||^2-\gamma_n||(T^G_{r_n}-I)Aw_n||^2\\
&=||w_n-p||^2+\gamma_n\big[\gamma_n||A^*(T^G_{r_n}-I)Aw_n||^2-||(T^G_{r_n}-I)Aw_n||^2\big]\\
&\leq||w_n-p||^2.
\end{aligned}\tag{6.1.4}$$

Since T_i is a quasi-nonexpansive multi-valued mapping, then we have

$$\begin{aligned}
||x_{n+1}-p||^2&=||\beta_{n,0}(u_n-p)+\sum_{i=1}^{\infty}(z_n^i-p)||^2\\
&\leq\beta_{n,0}||u_n-p||^2+\sum_{i=1}^{\infty}\beta_{n,i}||z_n^i-p||^2-\beta_{n,0}\beta_{n,i}||u_n-z_n^i||^2\\
&\leq\beta_{n,0}||u_n-p||^2+\sum_{i=1}^{\infty}\beta_{n,i}d(z_n^i,p)^2\\
&\leq\beta_{n,0}||u_n-p||^2+\sum_{i=1}^{\infty}\beta_{n,i}\mathcal{H}(T_i^nu_n,T_i^np)^2\\
&\leq\beta_{n,0}||u_n-p||^2+\sum_{i=1}^{\infty}\beta_{n,i}||u_n-p||^2\\
&=||u_n-p||^2\\
&\leq||w_n-p||^2.
\end{aligned}\tag{6.1.5}$$

From (6.1.1), the convexity of $||.||^2$, and the fact that $S:C\to C$ is a quasi-nonexpansive mapping, we have

$$\begin{aligned}
||w_n-p||^2&=||(1-\alpha_n-t_n)x_n+\alpha_nSx_n+t_nu-p||^2\\
&=||(1-\alpha_n-t_n)(x_n-p)+\alpha_n(Sx_n-p)+t_n(u-p)||^2\\
&\leq(1-\alpha_n-t_n)||x_n-p||^2+\alpha_n||Sx_n-p||^2+t_n||u-p||^2\\
&\leq(1-\alpha_n-t_n)||x_n-p||^2+\alpha_n||x_n-p||^2+t_n||u-p||^2\\
&=(1-t_n)||x_n-p||^2+t_n||u-p||^2.
\end{aligned}\tag{6.1.6}$$

Hence from (6.1.6), we have that

$$\begin{aligned}||x_{n+1}-p||^2 &\leq (1-t_n)||x_n-p||^2+t_n||u-p||^2\\ &\leq \max[||x_n-p||^2,||u-p||^2]\\ &\vdots\\ &\leq \max[||x_0-p||,||u-p||].\end{aligned}$$

Therefore $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$ and $\{w_n\}$ are bounded. Also from (6.1.2), (6.1.6), Lemma 2.5.2 and the fact that T_i is a quasi-nonexpansive multi-valued mapping, we have

$$\begin{aligned}||x_{n+1}-p||^2 &= ||\beta_{n,0}(u_n-p)+\sum_{i=1}^{\infty}(z_n^i-p)||^2\\ &\leq \beta_{n,0}||u_n-p||^2+\sum_{i=1}^{\infty}\beta_{n,i}||z_n^i-p||^2-\beta_{n,0}\beta_{n,i}g(||u_n-z_n^i||)\\ &\leq \beta_{n,0}||u_n-p||^2+\sum_{i=1}^{\infty}\beta_{n,i}(d(T_iu_n,p))^2-\beta_{n,0}\beta_{n,i}g(||u_n-z_n^i||)\\ &\leq \beta_{n,0}||u_n-p||^2+\sum_{i=1}^{\infty}\beta_{n,i}(\mathcal{H}(T_iu_n,T_ip))^2-\beta_{n,0}\beta_{n,i}g(||u_n-z_n^i||)\\ &\leq \beta_{n,0}||u_n-p||^2+\sum_{i=1}^{\infty}\beta_{n,i}||u_n-p||^2-\beta_{n,0}\beta_{n,i}g(||u_n-z_n^i||)\\ &\leq \beta_{n,0}||u_n-p||^2+\sum_{i=1}^{\infty}\beta_{n,i}||u_n-p||^2-\beta_{n,0}\beta_{n,i}g(||u_n-z_n^i||)\\ &= ||u_n-p||^2-\beta_{n,0}\beta_{n,i}g(||u_n-z_n^i||)\\ &\leq ||w_n-p||^2-\beta_{n,0}\beta_{n,i}g(||u_n-z_n^i||)\\ &= ||(1-\alpha_n)(x_n-p)+\alpha_n(Sx_n-p)+t_n(u-x_n)||^2-\beta_{n,0}\beta_{n,i}g(||u_n-z_n^i||)\\ &= ||(1-\alpha_n)(x_n-p)+\alpha_n(Sx_n-p)||^2+t_n^2||x_n-u||^2\\ &+2t_n\langle u-x_n,(1-\alpha_n)(x_n-p)+\alpha_n(Sx_n-p)\rangle-\beta_{n,0}\beta_{n,i}g(||u_n-z_n^i||)\\ &\leq ||x_n-p||^2-\alpha_n(1-\alpha_n)||Sx_n-x_n||^2+t_n^2||x_n-u||^2\\ &+2t_n\langle u-x_n,(1-\alpha_n)(x_n-p)+\alpha_n(Sx_n-p)\rangle-\beta_{n,0}\beta_{n,i}g(||u_n-z_n^i||).\end{aligned}\tag{6.1.7}$$

We now consider two cases to establish strong convergence of $\{x_n\}$ to p.
Case I: Assume that $\{||x_n-p||\}$ is a monotonically non-increasing sequence. Then $\{x_n\}$ is convergent and clearly

$$\lim_{n\to\infty}||x_n-p||=\lim_{n\to\infty}||x_{n+1}-p||.\tag{6.1.8}$$

Thus from (6.1.7), condition (iii) and (iv) of (6.1.1), we have that

$$\begin{aligned}0\leq \varepsilon^2g(||u_n-z_n^i||)&\leq ||x_n-p||^2-||x_{n+1}-p||^2-\alpha_n(1-\alpha_n)||Sx_n-x_n||^2+t_n^2||x_n-u||^2\\ &+2t_n\langle u-x_n,(1-\alpha_n)(x_n-p)+\alpha_n(Sx_n-p)\rangle\to 0,\ \text{as } \infty.\end{aligned}$$

Hence, we have that

$$\lim_{n\to\infty} g(||u_n - z_n^i||) = 0,$$

and by property of g in Lemma 2.3.1, we have that $\lim_{n\to\infty} ||u_n - z_n^i|| = 0$. Since $\{u_n\}$ and $\{x_n\}$ are bounded, we have that

$$\lim_{n\to\infty} d(u_n, T_i u_n) \leq \lim_{n\to} ||u_n - z_n^i|| = 0. \tag{6.1.9}$$

Also,(6.1.7), we have that

$$\begin{aligned} \alpha_n(1-\alpha_n)||Sx_n - x_n||^2 \leq{}& ||x_n - p||^2 - ||x_{n+1} - p||^2 + t_n^2 \\ &+ 2t_n\langle u - x_n, (1-\alpha_)(x_n - p) + \alpha_n(Sx_n - p), \end{aligned}$$

hence from condition (iii) and (iv)of (6.1.1), we have that

$$||Sx_n - x_n|| = 0 \tag{6.1.10}$$

From (6.1.1), we have that

$$\begin{aligned} &||x_{n+1} - p||^2 \\ &= ||\beta_{n,0}(u_n - p) + \sum_{i=1}^{\infty} \beta_{n,i}(z_n^i - p)||^2 \\ &= \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} ||z_n^i - p||^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}||u_n - z_n^i||^2 - \sum_{i,j=1, i\neq j}^{\infty} \beta_{n,i}\beta_{n,k}||z_n^i - z_n^k||^2 \\ &\leq \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} ||z_n^i - p||^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}||u_n - z_n^i||^2 \\ &\leq \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}(d(T_i u_n, p))^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}||u_n - z_n^i||^2 \\ &\leq \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}(\mathcal{H}(T_i u_n, T_i p))^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}||u_n - z_n^i||^2 \\ &\leq \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}||u_n - p||^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}||u_n - z_n^i||^2 \\ &\leq \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} ||u_n - p||^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}||u_n - z_n^i||^2 \\ &= ||u_n - p||^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}||u_n - z_n^i||^2 \\ &\leq ||w_n - p||^2 + \gamma_n^2||A^*(T_{r_n}^G - I)Aw_n||^2 - \gamma_n||(T_{r_n}^G - I)Aw_n||^2 \\ &\leq ||x_n - p||^2 - \alpha_n(1-\alpha_n)||Sx_n - x_n||^2 + t_n^2||x_n - u||^2 + 2t_n\langle u - x_n, (1-\alpha)(x_n - p) \\ &+ \alpha_n(Sx_n - p)\rangle + \gamma_n^2||A^*(T_{r_n}^G - I)Aw_n||^2 - \gamma_n||T_{r_n}^G - I)Aw_n||^2 \\ &\leq ||x_n - p||^2 - \alpha_n(1-\alpha_n)||Sx_n - x_n||^2 + t_n^2||x_n - u||^2 + 2t_n\langle u - x_n, (1-\alpha)(x_n - p) \\ &+ \alpha_n(Sx_n - p)\rangle + \gamma_n[\gamma_n||A^*(T_{r_n}^G - I)Aw_n||^2 - ||(T_{r_n}^G - I)Aw_n||^2]. \end{aligned} \tag{6.1.11}$$

It then follows from condition (i) of (6.1.1) and the condition $\gamma_n \in \left(\varepsilon, \frac{||T^G_{r_n}-I)Aw_n||^2}{||A^*(T^G_{r_n}-I)Aw_n||^2}-\varepsilon\right)$, that

$$\begin{aligned}||x_{n+1}-p||^2 \leq{}& ||x_n-p||^2-\alpha_n(1-\alpha_n)||Sx_n-x_n||^2+t_n^2||x_n-u||^2\\ &+2t_n\langle u-x_n,(1-\alpha_n)(x_n-p)+\alpha_n(Sx_n-p)\rangle-\varepsilon||A^*(T^G_{r_n}-I)Aw_n||^2,\end{aligned} \tag{6.1.12}$$

which implies that

$$\begin{aligned}\varepsilon||A^*(T^G_{r_n}-I)Aw_n||^2 \leq{}& ||x_n-p||^2-||x_{n+1}-p||^2+t_n^2||x_n-u||^2\\ &+2t_n\langle u-x_n,(1-\alpha_n)(x_n-p)+\alpha_n(Sx_n-p)\rangle\end{aligned} \tag{6.1.13}$$

Hence,

$$\lim_{n\to\infty}||A^*(T^G_{r_n}-I)Aw_n||^2=0. \tag{6.1.14}$$

From condition (i) of (6.1.1) and (6.1.14), we obtain that

$$\begin{aligned}&\gamma_n||(T^G_{r_n}-I)Aw_n||^2\\ &\leq ||x_n-p||^2-||x_{n+1}-p||^2+t_n^2||x_n-u||^2\\ \\ &+2t_n\langle u-x_n,(1-\alpha_n)(x_p)+\alpha_n(Sx_n-p)\rangle+\gamma_n^2||A^*(T^G_{r_n}-I)Aw_n||^2.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}||(T^G_{r_n}-I)Aw_n||^2=0. \tag{6.1.15}$$

Also,

$$\begin{aligned}&||u_n-p||^2\\ &=||T^F_{r_n}(w_n+\gamma_nA^*(T^G_{r_n}-I)Aw_n-p||^2\\ &\leq\langle u_n-p,w_n+\gamma_nA^*(T^G_{r_n}-I)Aw_n-p\rangle\\ &=\frac{1}{2}[||u_n-p||^2+||w_n+\gamma_nA^*(T^G_{r_n}-I)Aw_n-p||^2-||u_n-p-(w_n+\gamma_nA^*(T^G_{r_n}-I)Aw_n-p)||^2]\\ &\leq[\frac{1}{2}||u_n-p||^2+||w_n-p||^2+\gamma_n\big(\gamma_n||A^*(T^G_{r_n}-I)Aw_n||^2-||(T^G_{r_n}-I)Aw_n||^2\big)\\ &-||u_n-p-(w_n+\gamma_nA^*(T^G_{r_n}-I)Aw_n-p)||^2]\\ &\leq\frac{1}{2}[||u_n-p||^2+||w_n-p||^2-(||u_n-w_n||^2+\gamma_n^2||A^*(T^G_{r_n}-I)Aw_n||\\ &-2\gamma_n\langle u-w_n,A^*(T^G_{r_n}-I)Aw_n\rangle)]\\ &\leq\frac{1}{2}[||u_n-p||^2+||w_n-p||^2-||u_n-w_n||^2+\gamma_n^2||A^*(T^G_{r_n}-I)Aw_n||\\ &+2\gamma_n\langle u-w_n,A^*(T^G_{r_n}-I)Aw_n\rangle)]\end{aligned} \tag{6.1.16}$$

That is

$$||u_n-p||^2\leq||w_n-p||^2-||u_n-w_n||^2+2\gamma_n||u_n-w_n||\;||A^*(T^G_{r_n}-I)Aw_n||. \tag{6.1.17}$$

It follows from condition (i) of (6.1.1) and (6.1.17) that

$$||x_{n+1}-p||^2 \leq ||w_n-p||^2 - ||u_n-w_n||^{+}2\gamma_n||u_n-w_n||\ ||A^*(T_{r_n}^G-I)Aw_n|| \tag{6.1.18}$$

and this implies that

$$\begin{aligned}
&||u_n-w_n||^2\\
&\leq ||w_n-p||^2 - ||x_{n+1}-p||^2 + 2\gamma_n||u_n-w_n||\ ||A^*(T_{r_n}^G-I)Aw_n||\\
&= ||(1-\alpha_n-t_n)x_n+\alpha_nSx_n+t_nu-p||^2 - ||x_{n+1}-p||^2 + 2\gamma_n||u_n-w_n||\ ||A^*(T_{r_n}^G-I)Aw_n||\\
&\leq ||x_n-p||^2 - ||x_{n+1}-p||^2 - \alpha_n(1-\alpha_n)||Sx_n-x_n||^2 + t_n^2||x_n-u||^2\\
&+ 2t_n\langle u-x_n,(1-\alpha_n)(x_n-p)+\alpha_n(Sx_n-p)\rangle + 2\gamma_n||u_n-w_n||\ ||A^*(T_{r_n}^G-I)Aw_n||\\
&\to 0, \text{ as } n\to\infty.
\end{aligned} \tag{6.1.19}$$

Also, from (6.1.10) and condition (iv) of (6.1.1) , we have that

$$||w_n-x_n|| = ||(1-\alpha_n-t_n)(x_n-x_n)+\alpha_n(Sx_n-x_n)+t_n(u-x_n)|| \to 0, \text{ as } n\to\infty. \tag{6.1.20}$$

From (6.1.19) and (6.1.20), we have that

$$||u_n-x_n|| \leq ||x_n-w_n|| + ||w_n-u_n|| \to 0, \text{ as } n\to\ \infty. \tag{6.1.21}$$

From (6.1.1) and (6.1.9), we have that

$$\begin{aligned}
||x_{n+1}-u_n|| &= ||\beta_{n,0}u_n + \sum_{i=1}^{\infty}\beta_{n,i}z_n^i - u - u_n||\\
&= ||\beta_{n,0}(u_n-u_n) + \sum_{i=1}^{\infty}(z_n^i-u_n)||\\
&\leq \sum_{i=1}^{\infty}\beta_{n,i}||z_n^i-u_n|| \to 0, \text{ as } n\to\infty.
\end{aligned} \tag{6.1.22}$$

Hence, from (6.1.21) and (6.1.22) we have that

$$||x_{n+1}-x_n|| \leq ||x_{n+1}-u_n|| + ||u_n-x_n|| \to 0, \text{ as } n\to\infty. \tag{6.1.23}$$

It follows from (6.1.9), (6.1.21) and the demiclosedness principle that $\{u_n\}$ converges weakly to $p\in\cap_{i=1}^{\infty}Fix(T_i)\cap Fix(S)$ and consequently $\{x_n\}$ and $\{w_n\}$ converges weakly to p.
Next we show that $p\in GMEP(F,B_1,\psi_1)$. Since $u_n = T_{r_n}^F(w_n+\gamma_nA^*(T_{r_n}^G-I)Aw_n)$, we have

$$\begin{aligned}
&F(u_n,u) + \langle B_1u_n,u-u_n\rangle + \psi_1(u) - \psi(u_n)\\
&+\frac{1}{r_n}\langle u-u_n,u_n-w_n\rangle - \frac{1}{r_n}\langle u-u_n,\gamma_nA^*(T_{r_n}^G-I)Aw_n\rangle \geq 0, \forall\ u\in C.
\end{aligned} \tag{6.1.24}$$

Hence, from the monotonicity of $\phi_1(x,u) := F(x,u) + \langle B_1 x, u-x\rangle + \psi(u) - \psi(x)$, we have

$$\begin{aligned}
&\frac{1}{r_n}\langle u-u_n, u_n-w_n\rangle - \frac{1}{r_n}\langle u-u_n, \gamma_n A^*(T_{r_n}^G - I)Aw_n\rangle \\
&\quad \geq F(u,u_n) + \langle B_1 u, u_n - u\rangle + \psi(u_n) - \psi(u), \qquad (6.1.25)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{r_{n_k}}\langle u-u_{n_k}, u_{n_k}-w_{n_k}\rangle - \frac{1}{r_{n_k}}\langle u-u_{n_k}, \gamma_n A^*(T_{r_{n_k}}^G - I)Aw_{n_k}\rangle \geq F(u,u_{n_k}) \\
&+ \langle B_1 u, u_{n_k} - u\rangle + \psi(u_{n_k}) - \psi(u). \qquad (6.1.26)
\end{aligned}$$

Since $u_n \rightharpoonup p$, then it follows from (6.1.15), (6.1.18), (6.1.20), (6.1.21) and $(L4)$ that

$$F(u,p) + \langle B_1 u, p-u\rangle + \psi_1(p) - \psi_1(u) \leq 0, \ \forall \ u \in C. \qquad (6.1.27)$$

Now for fixed $u \in C$, let $u_t = tu + (1-t)p$ for all $t \in (0,1)$. This implies that $u_t \in C$. Thus from $(L1)$ and $(L4)$, we obtain

$$\begin{aligned}
0 &= F(u_t,u_t) + \langle B_1 u_t, u_t - u_t\rangle + \psi_1(u_t) - \psi_1(u_t) \\
&\leq t\big[F(u_t,u) + \langle B_1 u_t, u - u_t\rangle + \psi_1(u) - \psi_1(u_t)\big] \\
&+ (1-t)[F(u_t,p) + \langle B_1 u_t, p - u_t\rangle + \psi_1(p) - \psi_1(u_t)] \\
&\leq t[F(u_t,u) + \langle B_1 u_t, u - u_t\rangle + \psi_1(u) - \psi_1(u_t)]. \qquad (6.1.28)
\end{aligned}$$

Therefore

$$F(u_t,u) + \langle B_1 u_t, u-u_t\rangle + \psi_1(u) - \psi_1(u_t) \geq 0 \qquad (6.1.29)$$

Furthermore, from (L4), we have that

$$F(p,u) + \langle B_1 p, u-p\rangle + \psi_1(u) - \psi_1(p) \geq 0, \qquad (6.1.30)$$

which implies that $p \in GMEP(F, B_1, \psi_1)$. Next, we show that $Ap \in GMEP(G, B_2, \psi_2)$. since $\{w_n\}$ is bounded and $w_n \rightharpoonup p$ and since A is a bounded linear operator, $Aw_{n_k} \to Ap$. Set $v_{n_k} = Aw_{n_k} - T_{r_{n_k}}^G Aw_{n_k}$. Then we have that $Aw_{n_k} - v_{n_k} = T_{r_{n_k}}^G Aw_{n_k}$, and from (6.1.15), we have that

$$\lim_{n\to\infty} v_{n_k} = 0. \qquad (6.1.31)$$

Therefore, from the definition of $T_{r_{n_k}}^G$, we observe that

$$\begin{aligned}
&G(Aw_{n_k} - v_{n_k}, u) + \langle B_2 w_{n_k} - v_{n_k}, u - w_{n_k} + v_{n_k}\rangle + \psi_2(u) - \psi_2(u)(w_{n_k} - v_{n_k}) \\
&+ \frac{1}{r_{n_k}}\langle u - (w_{n_k} - v_{n_k}), (w - n_k - v_{n_k}) - w_{n_k}\rangle \geq 0, \ \forall \ u \in C. \qquad (6.1.32)
\end{aligned}$$

Since G is upper semicontinuous in the first argument, then G is defined as

$$\phi_2(x,y) := G(x,u) + \langle B_2 x, u-x\rangle + \psi_2(u) - \psi_2(x). \qquad (6.1.33)$$

Thus, taking $\limsup$ to the inequality (6.1.32) as $k \to \infty$ and using the assumption ($L3$), we have

$$G(Ap, u) + \langle B_2Ap, u - Ap\rangle + \psi_2(u) - \psi_2(Ap) \geq 0, \ \forall \ u \in C; \tag{6.1.34}$$

which implies that $Ap \in GMEP(G, B_2, \psi_2)$. Hence $p \in \cap_{i=1}^{\infty} Fix(T_i) \cap Fix(S) \cap \Gamma$. We now show that $\{x_n\}$ converges strongly to p.
From (6.1.1), we have

$$\begin{aligned}
&||x_{n+1} - p||^2 \\
&= ||\beta_{n,0}u_n + \sum_{i=1}^{\infty} \beta_{n,i}z_n^i - p||^2 \\
&= ||\beta_{n,0}(u_n - p) + \sum_{i=1}^{\infty} \beta_{n,i}(z_n^i - p)||^2 \\
&\leq \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}||z_n^i - p||^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}||u_n - z_n^i||^2 - \sum_{i,j=1, i\neq j} \beta_{n,i}\beta_{n,j}||z_n^i - z_n^j||^2 \\
&\leq \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}d(T_iu_n, p)^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}||u_n - z_n^i||^2 \\
&\leq \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}\mathcal{H}(T_iu_n, T_ip) \\
&\leq \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}||u_n - p||^2 \\
&\leq \beta_{n,0}||u_n - p||^2 + \sum_{i=1}^{\infty} \beta_{n,i}||u_n - p||^2 \\
&= ||u_n - p||^2 \\
&\leq ||w_n - p||^2 \\
&= ||(1 - \alpha_n - t_n)x_n + \alpha_nSx_n + t_nu - p||^2 \\
&= ||(1 - \alpha_n - t_n)(x_n - p) + \alpha_n(Sx_n - p) + t_n(u - p)||^2 \\
&\leq ||(1 - \alpha_n - t_n)(x_n - p) + \alpha_n(Sx_n - p)||^2 + 2t_n\langle x_{n+1} - p, u - p\rangle \\
&\leq [(1 - \alpha_n - t_n)||x_n - p|| + \alpha_n||Sx_n - p||]^2 + 2t_n\langle x_{n+1} - p, u - p\rangle \\
&\leq (1 - t_n)||x_n - p||^2 + 2t_n\langle x_{n+1} - p, u - p\rangle.
\end{aligned}$$

Since $x_n \rightharpoonup p$, then $x_{n+1} - p \to 0$. Therefore, by Lemma (2.1.38), we have that $x_n \to p$, as $n \to \infty$.
CASE II: Assume that $\{||x_n - p||\}$ is a monotonically increasing sequence. Set $\Upsilon_n = ||x_n - p||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large enough n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Upsilon_k \leq \Upsilon_{k+1}\} \tag{6.1.35}$$

Obviously, $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Upsilon_{\tau}(n) \leq \Upsilon_{\tau(n)+1}, \text{ for } n \geq n_0.$$

Following the case argument as in case 1, we can show that

$$\lim_{\tau(n)\to\infty} ||(T^G_{r_{\tau(n)}} - I)Aw_{\tau(n)}|| = 0.$$

By the same argument as (6.1.7) and (6.1.23) in Case 1, we conclude that $\{x_{\tau(n)}\}, \{y_{\tau(n)}\}$ and $\{w_{\tau(n)}\}$ converge weakly to $p \in Fix(T_i) \cap Fix(S) \cap \Gamma$. Now for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2 \\ &\leq (1 - t_{\tau(n)})||x_{\tau(n)} - p||^2 + 2t_n\langle x_{\tau(n)+1} - p, u - p\rangle - ||x_{\tau(n)} - p||^2 \\ &= t_{\tau(n)}[2t_{\tau(n)}\langle x_{\tau(n)+1} - p, u - p\rangle - ||x_{\tau(n)} - p||^2] \end{aligned}$$

Therefore,

$$||x_{\tau(n)} - p||^2 \leq t_{\tau(n)}[2t_{\tau(n)}\langle x_{\tau(n)+1} - p, u - p\rangle] \to 0, \text{ as } n \to \infty. \tag{6.1.36}$$

Hence,

$$\lim_{n\to\infty} ||x_{\tau(n)} - p|| = 0; \tag{6.1.37}$$

and

$$\lim_{n\to\infty} \Upsilon_{\tau(n)} = \lim_{n\to\infty} \Upsilon_{\tau(n)+1} \tag{6.1.38}$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) since $\Upsilon_j > \Upsilon_{j+1}$ for $\tau(n) + 1 \leq j \leq n$.
Consequently, for all $n \geq n_0$,

$$0 < \Upsilon_n \leq \max\{\Upsilon_\tau(n), \Upsilon_{\tau(n)+1}\} = \Upsilon_{\tau(n)+1}.$$

So $\lim_{n\to\infty} \Upsilon_n = 0$, that is $\{x_n\}, \{u_n\}$ and $\{w_n\}$ converge strongly to $p \in F(T_i) \cap F(S) \cap \Gamma, \forall\ n > 0$. □

Remark 6.1.2. *(i) If* $B_1 = 0$ *and* $B_2 = 0$*, then SGMEP* (1.2.14)-(1.2.15) *reduces to the following Split Mixed Equilibrium Problem (SMEP), find* $x^* \in C$ *such that*

$$F(x^*, x) + \psi_1(x) - \psi - 1(x^*) \geq 0,\ \forall\ x \in C; \tag{6.1.39}$$

and $y^* = Ax^* \in Q$ *solves*

$$G(y^*, y) + \psi_2(y) - \psi(y^*) \geq 0,\ \forall! y \in Q; \tag{6.1.40}$$

with solution set $\Theta_1 := \{x^* \in MEP(F, \psi_1) : Ax^* \in MP(G, \psi_2)\}$.
(ii) If $\psi_1 = \psi_2 = 0$ *in SGMEP* (1.2.14)- (1.2.15)*, then we have the following Split Generalized Equilibrium Problem, find* $x^* \in C$ *such that*

$$F(x^*, x) + \langle B_1x^*, x - x^*\rangle \geq 0,\ \forall\ x \in C; \tag{6.1.41}$$

and $y^* = Ax^* \in Q$ *solves*

$$G(y^*, y) + \langle B_2y^*, y - y^*\rangle \geq 0,\ \forall\ y \in Q; \tag{6.1.42}$$

with solution set $\Theta_2 := \{x^* \in GEP(F, B_1) : Ax^* \in GEP(G, B_2)\}$.
(iii) If $B_1 = B_2$ *and* $\psi_1 = \psi_2 = 0$, *we have the following Split Equilibrium Problem studied by Kazmi and Rizvi [183] in 2013 which is to find* $x^* \in C$ *such that*

$$F(x^*, x) \geq 0, \ \forall \ x \in C; \tag{6.1.43}$$

and $y^* = Ax^* \in Q$ *solves*

$$G(y^*, y) \geq 0, \ \forall \ y \in Q \tag{6.1.44}$$

with the solution set $\Theta_3 := \{x^* \in EP(F) : Ax^* \in EP(G)\}$.
(iv) If $F = G = 0$ *and* $\psi_1 = \psi_2 = 0$, *then SGMEP* (1.2.14)-(1.2.15) *becomes Split Variational Inequality Problem (in shoert SVIP), which is to find* $x^* \in C$ *such that*

$$\langle B_1 x^*, x - x^* \rangle \geq 0, \forall \ x \in C, \tag{6.1.45}$$

and $y^* = Ax^* \in Q$ *solves*

$$\langle B_2 y^*, y - y^* \rangle \geq 0, \forall \ y \in Q. \tag{6.1.46}$$

We denote by $SVIP(B_1, B_2)$ *the solution set of* (6.1.45)-(6.1.46).

Corollary 6.1.3. *Let* H_1 *and* H_2 *be two real Hilbert spaces, let* $C \subset H_1$ *and* $Q \subset H_2$ *be nonempty, closed and convex subsets of* H_1 *and* H_2 *respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator and* A^* *the adjoint of* A. *Let* $F : C \times C \to \mathbb{R}$ *and* $G : Q \times Q \to \mathbb{R}$ *be bifunctions satisfying conditions* $(L1) - (L4)$ *and* G *is upper semicontinuous in the first argument. Let* $B_1 : C \to H_1$ *and* $B_2 : Q \to H_2$ *be continuous and monotone mappings,* $\psi_1 : C \to \mathbb{R} \cup \{+\infty\}$ *and* $\psi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ *be proper lower semicontinuous and convex functions. Let* $T_i : C \to K(C)$, *for* $i = 1, 2, 3, \ldots$ *be a countable family of quasi-nonexpansive multi-valued mappings for* $T_i p = \{p\}$ *and* $S : C \to C$ *be a quasi nonexpansive mapping respectively such that* $\Omega := \cap_{i=1}^{\infty} Fix(T_i) \cap Fix(S) \cap \Gamma \neq \emptyset$. *Let* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$, *then the step size* γ_n *is chosen in such a way that for some* $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{||T_{r_n}^G - I)Aw_n||^2}{||A^*(T_{r_n}^G - I)Aw_n||^2} - \varepsilon \right),$$

for $T_{r_n}^G Aw_n \neq Aw_n$ *and* $\gamma_n = \gamma$, *otherwise* (γ *being any nonnegative real number). Then, the sequences* $\{w_n\}, \{u_n\}$ *and* $\{x_n\}$ *are generated iteratively for an arbitrary* $x_0 \in C$ *and a fixed point* $u \in C$

$$\begin{cases} w_n = (1 - \alpha_n)x_n + \alpha_n x_n; \\ u_n = T_{r_n}^F (w_n + \gamma_n A^*(T_{r_n}^G - I)Aw_n); \\ x_{n+1} = \beta_{n,0} u_n + \sum_{i=1}^{N} \beta_{n,i} z_n^i, \ n \geq 1; \end{cases} \tag{6.1.47}$$

where $z_n^i \in T_i u_n$ *and* $r_n \subset (0, \infty)$ *satisfies the following conditions:*
(i) $\beta_{n,0}, \beta_{n,i} \in (0, 1), \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *such that* $\sum_{i=1}^{N} \beta_{n,i} = 1$;
(ii) $\liminf_{n\to\infty} r_n > 0$;
(iii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.
Then the sequences $\{x_n\}, \{u_n\}$ *and* $\{w_n\}$ *converges strongly to an element in* Ω.

Corollary 6.1.4. *Let H_1 and H_2 be two real Hilbert spaces, let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(L1) - (L4)$ and G is upper semicontinuous in the first argument. Let $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ be continuous and monotone mappings, $\psi_1 : C \to \mathbb{R} \cup \{+\infty\}$ and $\psi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let $T_i : C \to K(C)$, for $i = 1, 2, 3, ...$ be a countable family of quasi-nonexpansive multi-valued mappings for $T_i p = \{p\}$ and $S : C \to C$ be a nonexpansive mapping respectively such that $\Omega := \cap_{i=1}^{\infty} Fix(T_i) \cap Fix(S) \cap \Gamma \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ be a sequence in $(0, 1-a)$ for some $a > 0$. Let the step size γ_n be chosen in such a way that for some $\varepsilon > 0$,*

$$\gamma_n \in \left(\varepsilon, \frac{||T_{r_n}^G - I)Aw_n||^2}{||A^*(T_{r_n}^G - I)Aw_n||^2} - \varepsilon \right),$$

for $T_{r_n}^G Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then, the sequences $\{w_n\}, \{u_n\}$ and $\{x_n\}$ are generated iteratively for an arbitrary $x_0 \in C$ and a fixed point $u \in C$

$$\begin{cases} w_n = (1 - \alpha_n - t_n)x_n + \alpha_n S x_n + t_n u; \\ u_n = T_{r_n}^F (w_n + \gamma_n A^*(T_{r_n}^G - I)Aw_n); \\ x_{n+1} = \beta_{n,0} u_n + \sum_{i=1}^{\infty} \beta_{n,i} z_n^i, \ n \geq 1; \end{cases} \tag{6.1.48}$$

where $z_n^i \in T_i u_n$ and $r_n \subset (0, \infty)$ satisfies the following conditions:
(i) $\beta_{n,0}, \beta_{n,i} \in (0,1), \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *such that* $\sum_{i=1}^{\infty} \beta_{n,i} = 1$;
(ii) $\liminf_{n\to\infty} r_n > 0$;
(iii) $\lim_{n\to\infty} t_n = 0, \sum_{n=0}^{\infty} t_n = \infty$ *and* $\alpha_n + t_n < 1$;
(iv) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.
Then the sequences $\{x_n\}, \{u_n\}$ and $\{w_n\}$ converges strongly to an element in $p \in \cap_{i=1}^{\infty} Fix(T_i) \cap Fix(S) \cap \Gamma$.

We highlight some of our contributions as follows:

1. The class of mappings considered in this article generalizes the ones in [26, 100].
2. A strong convergence result was proved in our article which is desirable to the weak convergence proved in [100].
3. The problems discussed in our thesis generalizes the ones considered in [26, 100, 183, 312].
4. The result discussed in this article holds for the class of nonexpansive mappings.

6.1.2 Application to SVIP

Variational inequality problem is one of the most important problems in optimization as it is used in studying differential equations, minimax problems and has certain applications to mechanics and economic theory. Also the SVIP is known the include certain optimization problems such as split feasibility problems, split zero problems and split minimization problems as special cases, (see [2, 1, 113, 197]). We now state a result in solving $SVIP(B_1, B_2)$ as discussed in (6.1.45)-(6.1.46).

Theorem 6.1.5. *Let H_1 and H_2 be two real Hilbert spaces, let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ be continuous and monotone mappings, and $T_i : C \to K(C)$, for $i = 1, 2, 3, \ldots$ be a countable family of quasi-nonexpansive multi-valued mappings for $T_i p = \{p\}$, and $S : C \to C$ be a nonexpansive mapping respectively. Assume $\Omega := \cap_{i=1}^{\infty} Fix(T_i) \cap Fix(S) \cap SVIP(B_1, B_2) \neq \emptyset$ with $\{\alpha_n\}$ being a sequence in $(0, 1)$ and $\{t_n\}$ being a sequence in $(0, 1 - a)$ for some $a > 0$. Let the step size γ_n be chosen in such a way that for some $\varepsilon > 0$,*

$$\gamma_n \in \left(\varepsilon, \frac{||P_Q(I - r_n B_2) - I)Aw_n||^2}{||A^*(P_Q(I - r_n B_2) - I)Aw_n||^2} - \varepsilon\right),$$

for $P_Q(I - r_n B_2)Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then, the sequences $\{w_n\}, \{u_n\}$ and $\{x_n\}$ are generated iteratively for an arbitrary $x_0 \in C$ and a fixed point $u \in C$

$$\begin{cases} w_n = (1 - \alpha_n - t_n)x_n + \alpha_n S x_n + t_n u; \\ u_n = P_C(I - r_n B_1)(w_n + \gamma_n A^*(P_Q(I - r_n B_2))Aw_n); \\ x_{n+1} = \beta_{n,0} u_n + \sum_{i=1}^{\infty} \beta_{n,i} z_n^i, \ n \geq 1; \end{cases} \tag{6.1.49}$$

where $z_n^i \in T_i u_n$ and $r_n \subset (0, \infty)$ satisfies the following conditions:
(i) $\beta_{n,0}, \beta_{n,i} \in (0, 1), \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ such that $\sum_{i=1}^{\infty} \beta_{n,i} = 1$;
(ii) $\liminf_{n\to\infty} r_n > 0$;
(iii) $\lim_{n\to\infty} t_n = 0, \sum_{n=0}^{\infty} t_n = \infty$ and $\alpha_n + t_n < 1$;
(iv) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.
Then the sequences $\{x_n\}, \{u_n\}$ and $\{w_n\}$ converges strongly to an element in $p \in \cap_{i=1}^{\infty} Fix(T_i) \cap Fix(S) \cap SVIP(B_1, B_2)$.

6.2 Conclusion

In this chapter, we introduced and studied a new iterative method for approximating a common solution of the split generalized mixed equilibrium problem and the fixed point problem of an infinite family of a quasi-nonexpansive multi-valued mappings in the framework of the Hilbert space. In addition, we stated and proved a strong convergence result of the aforementioned problems and also gave applications of our main result to the split variational inequality problem.

6.3 Open Problem

In this Chapter, we established our main result using the strict fixed point condition ($T_i p = \{p\}$). Is it possible to achieve the same result by replacing the strict fixed point condition with end point, gate condition or the demicontractive type mappings. Also, is it possible to establish that $F(T_i) \circ F(S) = F(T_i) \cap F(S)$?

Chapter 7

Iterative Schemes and Nonlinear Mappings

For the past 50 years, researchers have paid very good attention to finding an analytical solution to (1.1.1), but this has been almost practically impossible. In view of this, iterative method has been adopted in finding an approximate solution to (1.1.1). A good number of iterative processes (explicit, implicit, Jungck-type and so on) have been introduced and studied by many authors, (see [166, 176, 193, 220, 250] and the reference there in). Iterative methods can produce numerical solutions to certain classes of problems of nonlinear analysis, that can be thought of in terms of fixed point theory, where analytical methods may fail:

1. studying general variational inequalities;
2. finding solutions to constrained optimization problems;
3. designing algorithms for signal and image processing;
4. approximating the solution of a Legendre Equation;
5. approximating the zeros of complex polynomials.

Developing faster and more effective iterative techniques for approximating fixed points of nonlinear mappings is still an open problem in this area of research. Consequently, the purpose of this chapter is to further develop the concept of iterative methods and nonlinear mappings in the framework of Hilbert and Banach spaces respectively.

7.1 Improved Generalized M-Iteration for Quasinonexpansive Multivalued Mappings with Application in Real Hilbert Spaces

In this section, we present a modified (improved) generalized M-iteration with the inertial technique for three quasinonexpansive multivalued mappings in a real Hilbert space. In addition, we obtain a weak convergence result under suitable conditions and the strong convergence result is achieved using the CQ projection method with our modified generalized M-iteration. Finally, we apply our convergence results to certain optimization problems, and present some numerical experiments to show the efficiency and applicability of the proposed method in comparison with other existing methods (Modified NOOR iterative scheme and Modified SP-iterative scheme) in the literature. The results obtained in this chapter extends, generalizes and improves several results in the literature.

7.1.1 Main Results

In this section, we prove a weak convergence theorem for a modified generalized M-iterative scheme with the inertial technique term for three quasi-nonexpansive multivalued mappings. In addition, we established a strong convergence result using the hybrid projection method with our modified generalized M-iteration.

Assumption 7.1.1. *Suppose that the following conditions hold:*

1. *The set C is a nonempty closed and convex subset of the real Hilbert space H.*
2. *$P, Q, R : C \to CB(C)$ a quasi-nonexpansive multivalued mappings with $F(P) \cap F(Q) \cap F(R) \neq \emptyset$ and $I - Q, I - P, I - R$ are demiclosed at 0.*
3. *P, Q, R satisfying condition (A).*
4. $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1.$
5. $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1.$
6. $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1.$

Algorithm 7.1.2. Initialization: *Given $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\epsilon_n\} \subset (0, 1)$ for all $n \in \mathbb{N}$. Let $x_0, x_1 \in C$ be arbitrary.*

Iterative step:
Step 1: *Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n < \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\epsilon_n}{||x_n - x_{n-1}||}\}, & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise} \end{cases} \tag{7.1.1}$$

where $\theta > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n) \Rightarrow \lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n}$
Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$\begin{aligned} z_n &\in (1-\alpha_n)w_n + \alpha_n P w_n, \\ y_n &\in \beta_n z_n + (1-\beta_n)Q z_n \\ x_{n+1} &\in \gamma_n y_n + (1-\gamma_n)R y_n, \quad n \geq 1. \end{aligned} \tag{7.1.2}$$

From step (7.1.1), It is easy to see that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$. Indeed, we have that $\theta_n\|x_n - x_{n-1}\| \leq \epsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$

Theorem 7.1.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 7.1.2. Then, under the Assumptions 7.1.10 and the Opial's condition, thus, $\{x_n\}$ converges weakly to a common fixed point of P, Q, and R.*

Proof. Let $p \in F(P) \cap F(Q) \cap F(R)$, $a_n \in Pw_n, b_n \in Qz_n, c_n \in Ry_n$ and using the Algorithm 7.1.2, we have

$$\begin{aligned} \|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{7.1.3}$$

Also, using Algorithm 7.1.2 and (7.1.3), we have

$$\begin{aligned} \|z_n - p\| &\leq (1-\alpha_n)\|w_n - p\| + \alpha_n\|a_n - p\| \\ &= (1-\alpha_n)\|w_n - p\| + \alpha_n d(a_n, Pp) \\ &\leq (1-\alpha_n)\|w_n - p\| + \alpha_n H(Pw_n, Pp) \\ &\leq (1-\alpha_n)\|w_n - p\| + \alpha_n\|w_n - p\| \\ &= \|w_n - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{7.1.4}$$

In addition, using Algorithm 7.1.2, (7.1.3) and (7.1.4), we have

$$\begin{aligned} \|y_n - p\| &\leq \beta_n\|z_n - p\| + (1-\beta_n)\|b_n - p\| \\ &= \beta_n\|z_n - p\| + (1-\beta_n)d(b_n, Qp) \\ &\leq \beta_n\|z_n - p\| + (1-\beta_n)H(Qz_n, Qp) \\ &\leq \beta_n\|z_n - p\| + (1-\beta_n)\|z_n - p\| \\ &= \|z_n - p\| \\ &\leq \|w_n - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{7.1.5}$$

Lastly, using Algorithm 7.1.2, (7.1.3),(7.1.4) and (7.1.5), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \gamma_n\|y_n-p\|+(1-\gamma_n)\|c_n-p\| \\
&= \gamma_n\|y_n-p\|+(1-\gamma_n)d(c_n,Rp) \\
&\leq \gamma_n\|y_n-p\|+(1-\gamma_n)H(Ry_n,Rp) \\
&\leq \gamma_n\|y_n-p\|+(1-\gamma_n)\|y_n-p\| \\
&= \|y_n-p\| \\
&\leq \|z_n-p\| \\
&\leq \|w_n-p\| \\
&\leq \|x_n-p\|+\theta_n\|x_n-x_{n-1}\|. \qquad (7.1.6)
\end{aligned}$$

It follows from Lemma 2.1.40 that $\lim_{n\to\infty}\|x_n-p\|$ exists and thus $\{x_n\}$ is bounded. Furthermore, using Algorithm 7.1.2 and Lemma 2.1.1, we have

$$\begin{aligned}
\|w_n-p\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-p\|^2 \\
&= \|x_n-p\|^2+2\theta_n\langle x_n-p,x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2. \qquad (7.1.7)
\end{aligned}$$

In addition, using Algorithm 7.1.2 and Lemma 2.1.1, we obtain

$$\begin{aligned}
\|z_n-p\|^2 &= (1-\alpha_n)\|w_n-p\|^2+\alpha_n\|a_n-p\|-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 \\
&= (1-\alpha_n)\|w_n-p\|^2+\alpha_n d(a_n,Pp)^2-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 \\
&\leq (1-\alpha_n)\|w_n-p\|^2+\alpha_n H(Pw_n,Pp)^2-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 \\
&\leq (1-\alpha_n)\|w_n-p\|^2+\alpha_n\|w_n-p\|^2-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 \\
&= \|w_n-p\|^2-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 \\
&\leq \|x_n-p\|^2+2\theta_n\langle x_n-p,x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2. \\
& \qquad (7.1.8)
\end{aligned}$$

Using Lemma 2.1.1 and (7.1.8), we obtain

$$\begin{aligned}
\|y_n-p\|^2 &= \beta_n\|z_n-p\|^2+(1-\beta_n)\|b_n-p\|^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2 \\
&= \beta_n\|z_n-p\|^2+(1-\beta_n)d(b_n,Qp)^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2 \\
&\leq \beta_n\|z_n-p\|^2+(1-\beta_n)H(Qz_n,Qp)^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2 \\
&\leq \beta_n\|z_n-p\|^2+(1-\beta_n)\|z_n-p\|^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2 \\
&= \|z_n-p\|^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2 \\
&\leq \|x_n-p\|^2+2\theta_n\langle x_n-p,x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2 \\
&-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2. \qquad (7.1.9)
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \gamma_n\|y_n-p\|^2+(1-\gamma_n)\|c_n-p\|^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2\\ &= \gamma_n\|y_n-p\|^2+(1-\gamma_n)d(c_n,Rp)^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2\\ &\leq \gamma_n\|y_n-p\|^2+(1-\gamma_n)H(Ry_n,Rp)^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2\\ &\leq \gamma_n\|y_n-p\|^2+(1-\gamma_n)\|y_n-p\|^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2\\ &= \|y_n-p\|^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2\\ &\leq \|x_n-p\|^2+2\theta_n\langle x_n-p,x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2\\ &-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2.\end{aligned} \tag{7.1.10}$$

This implies that

$$\begin{aligned}\gamma_n(1-\gamma_n)\|y_n-c_n\|^2+\beta_n(1-\beta_n)\|z_n-b_n\|^2+\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2\\ +\frac{\theta_n}{\alpha_n}2\alpha_n\langle x_n-p,x_n-x_{n-1}\rangle+\frac{\theta_n}{\alpha_n}\theta_n\alpha_n\|x_n-x_{n-1}\|^2 &\to 0 \text{ as } n\to\infty,\end{aligned} \tag{7.1.11}$$

and using our assumptions and the fact that $\lim_{n\to\infty}\|x_n-p\|$ exists, we obtain

$$\lim_{n\to\infty}\|y_n-c_n\|=\lim_{n\to\infty}\|z_n-b_n\|=\lim_{n\to\infty}\|w_n-a_n\|=0. \tag{7.1.12}$$

Using (7.1.12), we have

$$\|w_n-x_n\|=\|x_n+\theta_n(x_n-x_{n-1})-x_n\|=\frac{\theta_n}{\alpha_n}\alpha_n\|x_n-x_{n-1}\|\to 0 \quad\text{as}\quad n\to\infty. \tag{7.1.13}$$

$$\|z_n-w_n\|=\|(1-\alpha_n)w_n+\alpha_na_n-w_n\|=\alpha_n\|w_n-a_n\|\to 0 \quad\text{as}\quad n\to\infty, \tag{7.1.14}$$

$$\|z_n-x_n\|=\|z_n-w_n\|+\|w_n-x_n\|\to 0 \quad\text{as}\quad n\to\infty, \tag{7.1.15}$$

$$\|y_n-z_n\|=\|\beta_nz_n+(1-\beta_n)b_n-z_n\|\leq\|b_n-z_n\|+\beta_n\|z_n-b_n\|\to 0 \quad\text{as}\quad n\to\infty, \tag{7.1.16}$$

$$\|y_n-x_n\|=\|y_n-z_n\|+\|z_n-x_n\|\to 0 \quad\text{as}\quad n\to\infty. \tag{7.1.17}$$

Since $\{x_n\}$ is bounded, there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$ for some $x^*\in C$. By using (7.1.13), we obtain that $w_{n_k}\rightharpoonup x^*$ and since $I-P$ is demiclosed at 0 and using (7.1.12), we have that $x^*\in Px^*$. In addition, using (7.1.15), we obtain that $z_{n_k}\rightharpoonup x^*$ and since $I-Q$ is demiclosed at 0 and using (7.1.12), we have that $x^*\in Qx^*$. Lastly, using (7.1.17), we obtain that $y_{n_k}\rightharpoonup x^*$ and since $I-P$ is demiclosed at 0 and using (7.1.12), we have that $x^*\in Rx^*$. Thus, we have that $x^*\in F(P)\cap F(Q)\cap F(R)$. Furthermore, suppose that $\{x_n\}$ converges weakly to some y^* and let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ converging weakly to some $y^* \in F(P) \cap F(Q) \cap F(R)$. Now, suppose that $x^* \neq y^*$, then by Opial's condition and Lemma 2.1.19, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\| &= \lim_{k\to\infty} \|x_{n_k} - x^*\| \\
&< \lim_{k\to\infty} \|x_{n_k} - y^*\| \\
&= \lim_{n\to\infty} \|x_n - y^*\| \\
&= \lim_{j\to\infty} \|x_{n_j} - y\| \\
&< \lim_{j\to\infty} \|x_{n_j} - x\| \\
&= \lim_{n\to\infty} \|x_n - x\|.
\end{aligned}$$

This is a contradiction. So $x^* = y^*$. Hence, $\{x_n\}$ converges weakly to a common fixed point of P, Q, and R. ☐

In what follows, we present an algorithm for the strong convergence of our modified iteration.

Assumption 7.1.4. *Suppose that the following conditions hold:*

1. *The set C is a nonempty closed and convex subset of the real Hilbert space H.*
2. *$P, Q, R : C \to CB(C)$ a quasi-nonexpansive multivalued mappings with $F(P) \cap F(Q) \cap F(R) \neq \emptyset$ and $I - Q, I - P, I - R$ are demiclosed at 0.*
3. *P, Q, R satisfies condition (A).*
4. $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$.
5. $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.
6. $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

Algorithm 7.1.5. Initialization: *Given $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\epsilon_n\} \subset (0, 1)$ for all $n \in \mathbb{N}$. Let $x_0, x_1 \in C$, be arbitrary and $C = C_1$.*

Iterative step:

Step 1: *Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n < \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\} & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise} \end{cases} \tag{7.1.18}$$

where $\theta > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n) \Rightarrow \lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n}$.

Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$\begin{aligned} z_n &\in (1-\alpha_n)w_n + \alpha_n P w_n,\\ y_n &\in \beta_n z_n + (1-\beta_n)Qz_n\\ q_n &\in \gamma_n y_n + (1-\gamma_n)Ry_n,\\ C_{n+1} &= \{z \in C_n : \|q_n - z\|^2 \leq \|x_n - z\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2\\ &\quad - 2\theta_n\langle x_n - z, x_{n-1} - x_n\rangle\}\\ x_{n+1} &= P_{C_{n+1}}x_1, \forall n \geq 1. \end{aligned} \tag{7.1.19}$$

Theorem 7.1.6. *Let $\{x_n\}$ be the sequence generated by Algorithm 7.1.5. Then, under the Assumptions 7.1.4, thus, $\{x_n\}$ converges strongly to a common fixed point of P, Q, and R.*

Proof. For clarity, we divide our proofs into 4 steps.

Step 1. We will establish that $\{x_n\}$ is well defined.

Let $a_n \in Pw_n, b_n \in Qz_n$ and $c_n \in Ry_n$. Since P, Q, and R satisfy condition (A), using Lemma 2.1.18, we obtain that $F(P) \cap F(Q) \cap F(R)$ is closed and convex. In addition, using the usual routine, it is easy to show that C_n is closed and convex. More so, using the definition of C_{n+1} and Lemma 2.1.22, we obtain that C_{n+1} is also closed and convex. Thus, C_n is closed and convex for all $n \in \mathbb{N}$. Now, for all $p \in F(P) \cap F(Q) \cap F(R)$, we have that

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2\\ &\leq \|x_n - p\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle. \end{aligned} \tag{7.1.20}$$

From (7.1.20), we have

$$\begin{aligned} \|z_n - p\|^2 &= (1-\alpha_n)\|w_n - p\|^2 + \alpha_n\|a_n - p\| - \alpha_n(1-\alpha_n)\|w_n - a_n\|^2\\ &\leq (1-\alpha_n)\|w_n - p\|^2 + \alpha_n d(a_n, Pp)^2\\ &\leq (1-\alpha_n)\|w_n - p\|^2 + \alpha_n H(Pw_n, Pp)^2\\ &\leq (1-\alpha_n)\|w_n - p\|^2 + \alpha_n\|w_n - p\|^2\\ &= \|w_n - p\|^2\\ &\leq \|x_n - p\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle. \end{aligned} \tag{7.1.21}$$

Again, using (7.1.21), we obtain

$$\begin{aligned} \|y_n - p\|^2 &= \beta_n\|z_n - p\|^2 + (1-\beta_n)\|b_n - p\|^2 - \beta_n(1-\beta_n)\|z_n - b_n\|^2\\ &\leq \beta_n\|z_n - p\|^2 + (1-\beta_n)d(b_n, Qp)^2\\ &\leq \beta_n\|z_n - p\|^2 + (1-\beta_n)H(Qz_n, Qp)^2\\ &\leq \beta_n\|z_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2\\ &= \|z_n - p\|^2\\ &\leq \|w_n - p\|^2 \qquad (7.1.22)\\ &\leq \|x_n - p\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle. \end{aligned}$$

Lastly, using (7.1.22), we have that

$$\begin{aligned}\|q_n - p\|^2 &= \gamma_n\|y_n - p\|^2 + (1-\gamma_n)\|c_n - p\|^2 - \gamma_n(1-\gamma_n)\|y_n - c_n\|^2\\ &= \gamma_n\|y_n - p\|^2 + (1-\gamma_n)d(c_n, Rp)^2\\ &\leq \gamma_n\|y_n - p\|^2 + (1-\gamma_n)H(Ry_n, Rp)^2\\ &\leq \gamma_n\|y_n - p\|^2 + (1-\gamma_n)\|y_n - p\|^2\\ &= \|y_n - p\|^2\\ &\leq \|x_n - p\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle. \end{aligned} \tag{7.1.23}$$

Thus, using (7.1.23), we have that $p \in C_n$ for all $n \in \mathbb{N}$. It follows that

$$F(P) \cap F(Q) \cap F(R) \subseteq C_n,$$

for all $n \in \mathbb{N}$ as such $C_n \neq \emptyset$. Hence, $\{x_n\}$ is well-defined.

Step 2. We will establish that $\{x_n\}$ is a Cauchy sequence in C and that $x \to x^* \in C$ as $n \to \infty$.

Since $x_n \in P_{C_n}x_1, C_{n+1} \subseteq C_n$ and $x_{n+1} \in C_n$, we have

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\| \tag{7.1.24}$$

for all $n \in \mathbb{N}$. In addition, since $F(P) \cap F(Q) \cap F(R) \subseteq C_n$, we have that

$$\|x_n - x_1\| \leq \|z - x_1\| \tag{7.1.25}$$

for all $n \in \mathbb{N}$ and $z \in F(P) \cap F(Q) \cap F(R)$. It follows from (7.1.24) and (7.1.25) that $\{\|x_n - x_1\|\}$ is bounded and nondecreasing. Hence, we obtain that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. More so, for $m > n$ and by the definition of C_n, we have that $x_m \in P_{C_m}x_1 \in C_m \subseteq C_n$. Using Lemma 2.2.1, we have that

$$\|x_n - x_m\|^2 + \|x_n - x_1\|^2 \leq \|x_m - x_1\|^2. \tag{7.1.26}$$

It follows from (7.1.26) that $\lim_{n\to\infty}\|x_n - x_m\| = 0$, since $\lim_{n\to\infty}\|x_n - x_1\|$ exists. As such, we have that $\{x_n\}$ is a Cauchy sequence in C, hence $x_n \to x^* \in C$ as $n \to \infty$.

Step 3. We will establish that $\lim_{n\to\infty}\|y_n - c_n\| = \lim_{n\to\infty}\|z_n - b_n\| = \lim_{n\to\infty}\|w_n - a_n\| = 0$.

From step 2, it is easy to see that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since, $x_{n+1} \in C_n$, using the fact that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have that

$$\begin{aligned}\|q_n - x_n\| &= \|q_n - x_{n+1} + x_{n+1} - x_n\|\\ &\leq \|q_n - x_{n+1}\| + \|x_{n+1} - x_n\|\\ &\leq \sqrt{\|x_n - x_{n+1}\|^2 + 2\theta_n\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - x_{n+1}, x_{n-1} - x_n\rangle}\\ &\quad + \|x_{n+1} - x_n\| \to 0 \;\text{ as }\; n \to \infty. \end{aligned} \tag{7.1.27}$$

Since R satisfies condition (A) and using (7.1.8) and (7.1.9), we have that

$$\begin{aligned}
\|q_n - p\|^2 &= \gamma_n\|y_n - p\|^2 + (1-\gamma_n)\|c_n - p\|^2 - \gamma_n(1-\gamma_n)\|y_n - c_n\|^2 \\
&\leq \|y_n - p\|^2 - \gamma_n(1-\gamma_n)\|y_n - c_n\|^2 \\
&\leq \|x_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, w_n - p\rangle - (1-\alpha_n)\alpha_n\|w_n - a_n\|^2 \\
&- \beta_n(1-\beta_n)\|z_n - b_n\| - \gamma(1-\gamma_n))\|y_n - c_n\|^2,
\end{aligned} \tag{7.1.28}$$

it implies

$$\begin{aligned}
&(1-\alpha_n)\alpha_n\|w_n - a_n\|^2 + \beta_n(1-\beta_n)\|z_n - b_n\| + \gamma(1-\gamma_n))\|y_n - c_n\|^2 \\
&\qquad \leq \|x_n - p\|^2 - \|q_n - p\|^2 + 2\frac{\theta_n}{\alpha_n}\alpha_n\langle x_n - x_{n-1}, w_n - p\rangle.
\end{aligned} \tag{7.1.29}$$

Using (7.1.27) and our assumption, we have that

$$\lim_{n\to\infty}\|y_n - c_n\| = \lim_{n\to\infty}\|z_n - b_n\| = \lim_{n\to\infty}\|w_n - a_n\| = 0. \tag{7.1.30}$$

Using (7.1.30), we have

$$\|w_n - x_n\| = \|x_n + \theta_n(x_n - x_{n-1}) - x_n\| = \frac{\theta_n}{\alpha_n}\alpha_n\|x_n - x_{n-1}\| \to 0 \quad \text{as} \quad n \to \infty, \tag{7.1.31}$$

$$\|z_n - w_n\| = \|(1-\alpha_n)w_n + \alpha_n a_n - w_n\| = \alpha_n\|w_n - a_n\| \to 0 \quad \text{as} \quad n \to \infty, \tag{7.1.32}$$

$$\|z_n - x_n\| = \|z_n - w_n\| + \|w_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty, \tag{7.1.33}$$

$$\|y_n - z_n\| = \|\beta_n z_n + (1-\beta_n)b_n - z_n\| \leq \|b_n - z_n\| + \beta_n\|z_n - b_n\| \to 0 \quad \text{as} \quad n \to \infty, \tag{7.1.34}$$

$$\|y_n - x_n\| = \|y_n - z_n\| + \|z_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{7.1.35}$$

We have established that $x_n \to x^* \in C$, it follows from (7.1.31), we obtain that $w_{n_k} \to x^*$ and since $I - P$ is demiclosed at 0 and using (7.1.30), we have that $x^* \in Px^*$. In addition, using a similar approach, we obtain that $x^* \in F(Q)$ and $x^* \in F(R)$. Thus, we have that $x^* \in F(P) \cap F(Q) \cap F(R)$.

Step 4. Finally, we have to show that $x^* \in P_{F(P)\cap F(Q)\cap F(R)}x_1$.

It follows from (7.1.25), we have that

$$\|x^* - x_1\| \leq \|z - x_1\|$$

for all $z \in F(P) \cap F(Q) \cap F(R)$. Thus, by the definition of projection operator (P_C) we have that $x^* = P_{F(P)\cap F(Q)\cap F(R)}x_1$. Thus, the proof is complete.

□

7.1.2 Application and Numerical Examples

In this section, we present an application and a numerical example in finite dimensional Hilbert spaces and compare our proposed Algorithm 7.1.2 and Algorithm 7.1.5 with modified NOOR and modified SP-iteration (see that appendix for these algorithms).

7.1.3 Application to Common Inclusion Problem

In this section, we apply our results to the common inclusion problem.

The common inclusion problem is one of the interesting problems in this area of research. This problem has received great attention over the years due to its fruitful applications in almost all areas of sciences. In particular, it is applied to some problems in image processing, machine learning, signal processing and linear inverse problem. The inclusion problem is defined as find $x \in H$, such that

$$0 \in Ax + Bx, \tag{7.1.36}$$

where $A : H \to H$ is an α-inversely strongly monotone operator and $B : H \to 2^H$ is a maximal monotone operator. It is well-known that the resolvent $J_\lambda^B(I - \lambda A)$ is nonexpansive if $\lambda \in (0, 2\alpha)$. Consequently, our Algorithms take the form:

Assumption 7.1.7. *Suppose that the following conditions hold:*

1. *The set C is a nonempty closed and convex subset of the real Hilbert space H.*
2. *Let $A_i : H \to H$ is an α-inversely strongly monotone operator and $B_i : H \to 2^H$ is a maximal monotone operator, where $i = 1, 2, 3$.*
3. *The solution set $\Omega = \{\cap_{i=1}^3 (A_i + B_i)^{-1}(0)\} \neq \emptyset$.*
4. $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$.
5. $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.
6. $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

Algorithm 7.1.8. Initialization: *Given $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\epsilon_n\} \subset (0, 1)$ for all $n \in \mathbb{N}$. Let $x_0, x_1 \in C$, be arbitrary and $C = C_1$.*

Iterative step:
Step 1: *Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n < \bar{\theta}_n$, where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \\ \theta, & \text{otherwise}, \end{cases} \tag{7.1.37}$$

where $\theta > 0$ *and* $\{\epsilon_n\}$ *is a positive sequence such that* $\epsilon_n = \circ(\alpha_n) \Rightarrow \lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.
Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$\begin{aligned} z_n &= (1-\alpha_n)w_n + \alpha_n J_\lambda^{B_1}(I - \lambda A_1)w_n, \\ y_n &= \beta_n z_n + (1-\beta_n)J_\lambda^{B_2}(I - \lambda A_2)z_n \\ q_n &= \gamma_n y_n + (1-\gamma_n)J_\lambda^{B_3}(I - \lambda A_3)y_n, \\ C_{n+1} &= \{z \in C_n : \|q_n - z\|^2 \le \|x_n - z\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 \\ &\quad - 2\theta_n\langle x_n - z, x_{n-1} - x_n\rangle\} \\ x_{n+1} &= P_{C_{n+1}}x_1, \forall n \ge 1. \end{aligned} \tag{7.1.38}$$

Theorem 7.1.9. *Let* $\{x_n\}$ *be the sequence generated by Algorithm 7.1.8. Then, under the Assumptions 7.1.7, thus,* $\{x_n\}$ *converges strongly to a* Ω.

Modified SP-Iterative Scheme

Assumption 7.1.10. *Suppose that the following conditions hold:*

1. *The set* C *is a nonempty closed and convex subset of the real Hilbert space* H.
2. $T_1, T_2, T_3 : C \to CB(C)$ *are quasi-nonexpansive multivalued mappings with* $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ *and* $I - T_1, I - T_2, I - T_3$ *are demiclosed at* 0.
3. T_1, T_2, T_3 *satisfies condition (A).*
4. $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$.
5. $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.
6. $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

Algorithm 7.1.11. Initialization: *Given* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\epsilon_n\} \subset (0,1)$ *for all* $n \in \mathbb{N}$. *Let* $x_0, x_1 \in C$ *be arbitrary.*

Iterative step:
Step 1: *Given the iterates* x_{n-1} *and* x_n *for all* $n \in \mathbb{N}$, *choose* θ_n *such that* $0 \le \theta_n < \bar{\theta}_n$, *where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise} \end{cases} \tag{7.1.39}$$

where $\theta > 0$ *and* $\{\epsilon_n\}$ *is a positive sequence such that* $\epsilon_n = \circ(\alpha_n) \Rightarrow \lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n}$.
Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$\begin{aligned} z_n &\in (1-\alpha_n)w_n + \alpha_n T_1 w_n, \\ y_n &\in (1-\beta_n)z_n + \beta_n T_2 z_n \\ x_{n+1} &\in (1-\gamma_n)y_n + \gamma_n T_3 y_n, \quad n \geq 1. \end{aligned} \tag{7.1.40}$$

Assumption 7.1.12. *Suppose that the following conditions hold:*

1. *The set C is a nonempty closed and convex subset of the real Hilbert space H.*
2. $T_1, T_2, T_3 : C \to CB(C)$ *are quasi-nonexpansive multivalued mappings with* $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ *and* $I - T_1, I - T_2, I - T_3$ *are demiclosed at* 0.
3. T_1, T_2, T_3 *satisfies condition (A).*
4. $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$.
5. $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.
6. $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

Algorithm 7.1.13. Initialization: *Given* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\epsilon_n\} \subset (0,1)$ *for all* $n \in \mathbb{N}$. *Let* $x_0, x_1 \in C$, *be arbitrary and* $C = C_1$.

Iterative step:
Step 1: *Given the iterates* x_{n-1} *and* x_n *for all* $n \in \mathbb{N}$, *choose* θ_n *such that* $0 \leq \theta_n < \bar{\theta}_n$, *where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\} & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise} \end{cases} \tag{7.1.41}$$

where $\theta > 0$ *and* $\{\epsilon_n\}$ *is a positive sequence such that* $\epsilon_n = \circ(\alpha_n) \Rightarrow \lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$
Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$\begin{aligned} z_n &\in (1-\alpha_n)w_n + \alpha_n T_1 w_n, \\ y_n &\in (1-\beta_n)z_n + \beta_n T_2 z_n \\ q_n &\in (1-\gamma_n)y_n + \gamma_n T_3 y_n, \\ C_{n+1} &= \{z \in C_n : \|q_n - z\|^2 \leq \|x_n - z\|^2 + 2\theta_n^2 \|x_n - x_{n-1}\|^2 \\ &\quad - 2\theta_n \langle x_n - z, x_{n-1} - x_n \rangle\} \\ x_{n+1} &= P_{C_{n+1}} x_1, \forall n \geq 1. \end{aligned} \tag{7.1.42}$$

7.1.4 Numerical Examples

Example 7.1.14. *Define a mapping* $P, Q, R : [0,1] \to [0,1]$ *as*

$$Px = \begin{cases} [0, \frac{x}{2}] & if \;\; x \leq 0.5, \\ \{1\} & if \;\; x > 0.5, \end{cases} \tag{7.1.43}$$

$$Qx = \begin{cases} [0, \frac{x}{4}] & if \;\; x \leq 0.5, \\ \{1\} & if \;\; x > 0.5, \end{cases} \tag{7.1.44}$$

and

$$Rx = \begin{cases} [0, \frac{x}{10}] & if \;\; x \leq 0.5, \\ \{1\} & if \;\; x > 0.5. \end{cases} \tag{7.1.45}$$

It is easy to see that P, Q *and* R *are quasinonexpansive and satisfies condition* (A)*, and* $F(P) \cap F(Q) \cap F(R) = \{0, 1\}$*. We choose the following parameter* $\theta = 0.01, \epsilon_n = \frac{1}{(n+1)^2}, \alpha_n = \frac{4n+2}{5n+2}, \beta_n = \frac{n+1}{5n+4}, \gamma_n = \frac{2n}{3n+5}$ *We make different choices of the initial values* x_0 *and* x_1 *as follows:*
Ex 4.4a: $x_0 = 0.5, x_1 = 0.3;$
Ex 4.4b: $x_0 = 0.9, x_1 = 0.4;$
Ex 4.4c: $x_0 = 0.75, x_1 = 0.12;$
Ex 4.4d: $x_0 = 0.29, x_1 = 0.49.$

Table 7.1: Numerical results.

		Alg. 3.2	Alg 5.2
Ex 4.4a	CPU time (sec)	0.0012	0.0016
	No of Iter.	10	15
Ex 4.4b	CPU time (sec)	0.0013	0.0019
	No of Iter.	11	20
Ex 4.4c	CPU time (sec)	0.0011	0.0012
	No of Iter.	9	18
Ex 4.4d	CPU time (sec)	0.0011	0.0012
	No of Iter.	10	16

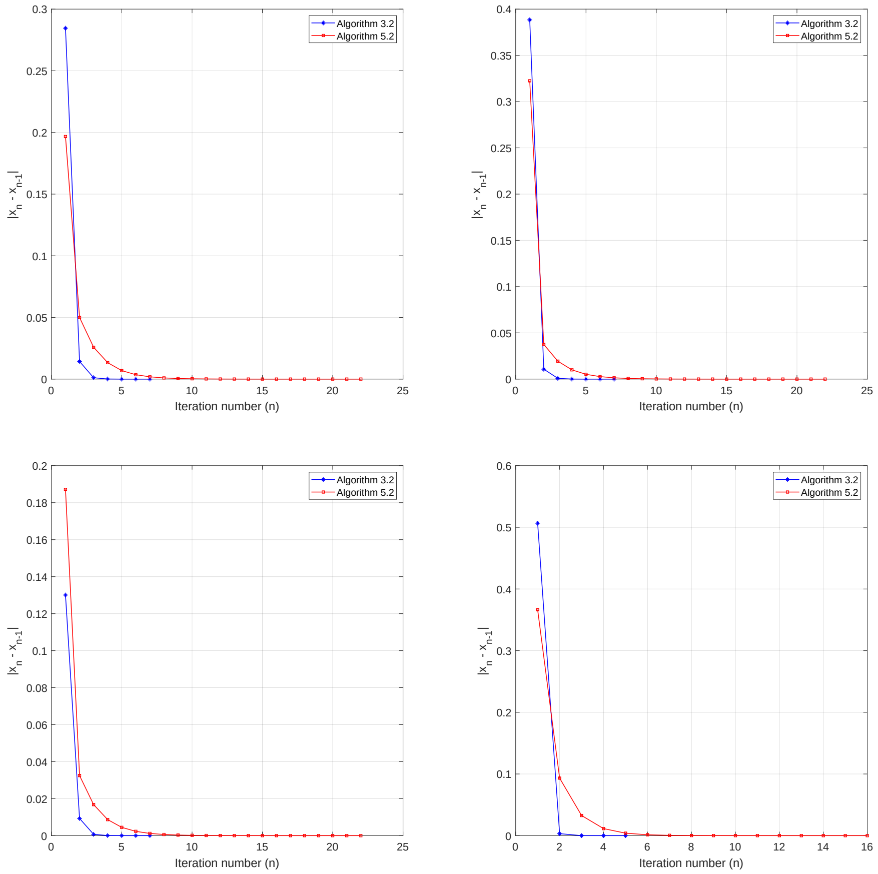

Figure 7.1: Example 7.1.14, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

Example 7.1.15. *Let* $H=\mathbb{R}^3, C=[3,6]^3$ *and* $C_1=\{x=(x_1,x_2,x_3)\in\mathbb{R}^3 : \sqrt{(x_1-6)^2+(x_2-6)^2+(x_3-6)^2}\leq 3\}$. *We defined* $P,Q,R:\mathbb{R}^3\to CB(\mathbb{R}^3)$ *as*

$$Px=\begin{cases}(6,6,6), & \textit{if } x_1\in C_1\\ \\ \{y=(y_1,y_2,y_3)\in C:\sqrt{(y_1-6)^2+(y_2-6)^2+(y_3-6)^2}\leq\frac{1}{\|x\|_1}\} & \textit{otherwise}\end{cases} \tag{7.1.46}$$

$$Qx=\begin{cases}(6,6,6), & \textit{if } x_1\in C_1\\ \\ \{y=(6,y,6)\in C:y\in[(x_2+6)(\frac{\arcsin(19x_2-76)}{2})+x_2,6]\} & \textit{otherwise}\end{cases} \tag{7.1.47}$$

and

$$Rx=\begin{cases}(6,6,6), & \textit{if } x_1\in C_1\\ \\ \{y=(6,6,y)\in C:y\in[(x_2-6)(\frac{\arccos(15x_2-60)}{5})+x_2,6]\} & \textit{otherwise,}\end{cases} \tag{7.1.48}$$

Choose $\theta=0.001, \alpha_n=\frac{1}{n+1}, \epsilon_n=\frac{1}{(n+1)^2}, \beta_n=\frac{3n}{3n+5}, \beta_n=\frac{2}{n^2+5}$. *It is easy to verify that all hypotheses of Theorem 7.1.6 and Theorem 7.1.3 are satisfied and* $F(P)\cap F(Q)\cap F(R)=(6,6,6)\neq\emptyset$. *We use different choices of* x_0, x_1 *and test the convergence of our algorithm with* $\|x_{n+1}-x_n\|<10^{-7}$ *as a stopping criterion. We choose the following parameter* $\theta=0.01, \epsilon_n=\frac{1}{(n+1)^2}, \alpha_n=\frac{4n+2}{5n+2}, \beta_n=\frac{n+1}{5n+4}, \gamma_n=\frac{2n}{3n+5}$ *We make different choices of the initial values* x_0 *and* x_1 *as follows:*
Ex 4.5a: $x_0=(4.1,4.7,5), x_1=(4.893,5.77,5)$.
Ex 4.5b: $x_0=(4.98,4.3,4), x_1=(4.33,4.42,4.42)$.
Ex 4.5c: $x_0=(4.2,4.3,4.2), x_1=(5.3,5.2,5.42)$.
Ex 4.5d: $x_0=(4.59,5.23,4.89), x_1=(5.98,5,5.24)$.

7.2 On Generalized (α,β)-Nonexpansive Mappings in Banach Space with Application

In this section, we present some fixed point results for a general class of nonexpansive mappings in the framework of Banach space and also proposed a new iterative scheme for approximating the fixed point of this class of mappings in the framework of uniformly convex Banach spaces. Furthermore, we establish some basic properties and convergence results for our new class of mappings in uniformly convex Banach spaces. Finally, we present an application to a nonlinear integral equation and also, a numerical example to illustrate our main result and then display the efficiency of the proposed algorithm compared to different iterative algorithms in the literature with different choices of parameters and initial guesses.

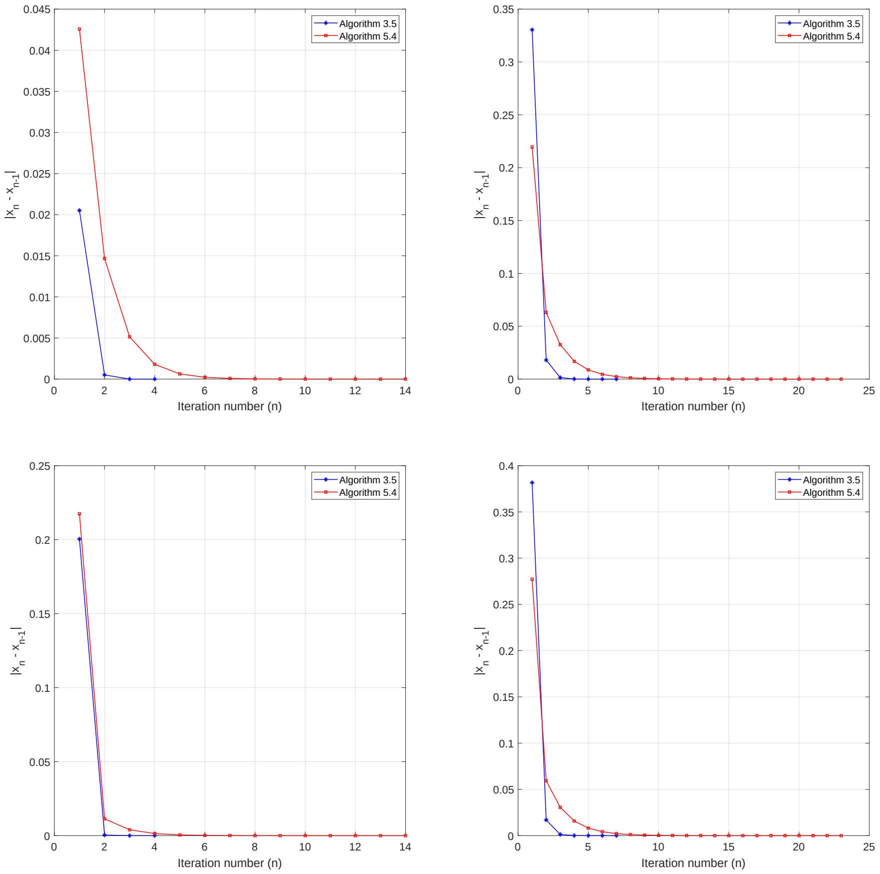

Figure 7.2: Example 7.1.15, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

Table 7.2: Numerical results.

		Alg. 3.5	Alg 5.4
Ex 4.4a	CPU time (sec)	0.0035	0.0040
	No of Iter.	23	25
Ex 4.4b	CPU time (sec)	0.0048	0.0056
	No of Iter.	21	27
Ex 4.4c	CPU time (sec)	0.0045	0.0060
	No of Iter.	20	26
Ex 4.4d	CPU time (sec)	0.0045	0.0062
	No of Iter.	21	29

7.2.1 Main Result

In this section, we introduced the notion of generalized (α, β)-nonexpansive mappings and establish some basic properties for this class of mapping. In addition, we establish some convergence results of a new three steps iterative method generated by generalized (α, β)-nonexpansive type 1 mapping in a uniformly convex Banach space.

Definition 7.2.1. *Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ will be called generalized (α, β)-nonexpansive type 1 mapping if there exist $\alpha, \beta, \lambda \in [0, 1)$, with $\alpha \leq \beta$ and $\alpha + \beta < 1$ such that for all $x, y \in C$,*

$$\lambda\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\|. \tag{7.2.1}$$

Remark 7.2.2. *It is easy to see that if*

1. *$\alpha = \beta = 0$ and $\lambda = \frac{1}{2}$, we obtain mapping satisfying condition (C).*
2. *$\alpha = \beta = 0$ and $\lambda \in [0, 1)$, we obtain mapping satisfying condition (C_λ).*

Definition 7.2.3. *Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ will be called generalized (α, β)-nonexpansive type 2 mapping if there exist $\alpha, \beta, \lambda \in [0, 1)$, with $\alpha + \beta < 1$ such that for all $x, y \in C$,*

$$\lambda\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \max\Big\{P(x, y), Q(x, y)\Big\}, \tag{7.2.2}$$

where $P(x, y) = \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\|$ and $Q(x, y) = \alpha\|x - Tx\| + \beta\|y - Ty\| + (1 - (\alpha + \beta))\|x - y\|$.

Proposition 7.2.4.

1. *Every nonexpansive mapping is a generalized (α, β)-nonexpansive type 1 mapping.*
2. *Every mean nonexpansive mapping is a generalized (α, β)-nonexpansive type 1 mapping.*
3. *All mappings satisfying condition (C) is a (α, β)-nonexpansive type 1 mapping.*
4. *All mappings satisfying condition (C_λ) is a (α, β)-nonexpansive type 1 mapping.*

The following example shows that the converse of these statements are not always true.

Example 7.2.5. *Let $C = \{(0,0), (1,0), (3,0)\}$ be a subset of $\mathbb{R}^2$ with norm $\|\cdot\|$ on C defined $\|(x_1, x_2)\| = |x_1| + |x_2|$. Then $(C, \|\cdot\|)$ is a Banach space. Define a mapping $T : C \to C$ by*

$$T(x) = \begin{cases} (0,0), & \text{if } x \in \{(0,0), (1,0)\}, \\ (1,0), & \text{if } x = (3,0). \end{cases} \tag{7.2.3}$$

Let $C = \{(0,0), (1,0), (3,0)\}$ be a subset of $\mathbb{R}^2$ with norm $\|\cdot\|$ on C defined $\|(x_1, x_2)\| = |x_1| + |x_2|$. Then $(C, \|\cdot\|)$ is a Banach space. Define a mapping $T : C \to C$ by

$$T(x) = \begin{cases} (0,0), & \text{if } x \in \{(0,0), (1,0)\}, \\ (1,0), & \text{if } x = (3,0). \end{cases} \tag{7.2.4}$$

For $\lambda = \frac{1}{10}, \alpha = \frac{1}{2}$, and $\beta = \frac{1}{3}$, we consider the following cases.

Case I: *For $x = (0,0)$ and $y = (0,0)$. It is easy to see that T is a generalized $(\frac{1}{2}, \frac{1}{3})$-nonexpansive type 1 mapping.*

Case IIa: *For $x = (0,0)$ and $y = (1,0)$. We have that*

$$\frac{1}{10}\|(0,0) - (0,0)\| = 0 < 1 = \|x - y\|$$

and

$$\|Tx - Ty\| = 0 < \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.$$

Case IIb:

For $x = (1,0)$ and $y = (0,0)$. We have that

$$\frac{1}{10}\|(0,0) - (0,0)\| = 0 < 1 = \|x - y\|$$

and

$$\|Tx - Ty\| = 0 < \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.$$

Case IIIa: *For* $x = (0,0)$ *and* $y = (3,0)$. *We have that*

$$\frac{1}{10}\|(0,0) - (0,0)\| = 0 < 3 = \|x - y\|$$

and

$$\begin{aligned}\|Tx - Ty\| &= |(0,0) - (1,0)| = 1\\ &< \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.\end{aligned}$$

Case IIIb:

For $x = (3,0)$ *and* $y = (0,0)$. *We have that*

$$\frac{1}{10}\|(3,0) - (1,0)\| = \frac{1}{5} < 3 = \|x - y\|$$

and

$$\begin{aligned}\|Tx - Ty\| &= |(1,0) - (0,0)| = 1\\ &< \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.\end{aligned}$$

Case IVa: *For* $x = (1,0)$ *and* $y = (3,0)$. *We have that*

$$\frac{1}{10}\|(1,0) - (0,0)\| = \frac{1}{10} < 2 = \|x - y\|,$$

and

$$\begin{aligned}\|Tx - Ty\| &= |(0,0) - (1,0)| = 1\\ &< \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.\end{aligned}$$

Case IVb:

For $x = (3,0)$ *and* $y = (1,0)$. *We have that*

$$\frac{1}{10}\|(3,0) - (1,0)\| = \frac{1}{5} < 2 = \|x - y\|,$$

and

$$\begin{aligned}\|Tx - Ty\| &= |(1,0) - (0,0)| = 1\\ &< \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.\end{aligned}$$

Case V: *For $x = y = (3, 0)$. We have*

$$\frac{1}{10}\|(3,0) - (1,0)\| = \frac{1}{5} > 0 = \|x - y\|.$$

Also, $x = y = (1, 0)$. We have

$$\frac{1}{10}\|(1,0) - (0,0)\| = \frac{1}{10} > 0 = \|x - y\|,$$

so, we have nothing to show. Thus, we have that T is a generalized $(\frac{1}{2}, \frac{1}{3})$-nonexpansive type 1 mapping.

Now, we establish that T is not a mean nonexpansive, generalized mean nonexpansive, mappings satisfying condition (C), condition (C_λ) and α-nonexpansive mappings. Indeed, we suppose that T is a mean nonexpansive mapping, so, therefore, there exists nonnegative real numbers α and β, with $\alpha + \beta \leq 1$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$$

for all $x, y \in C$. Now, consider $x = (0, 0)$ and $y = (1, 0)$, we then have that

$$\begin{aligned} \|Tx - Ty\| &= 0 \\ &\leq \alpha\|x - y\| + \beta\|x - Ty\| \\ &= \alpha. \end{aligned}$$

Thus, we obtain that $\alpha \leq 1$ and $\beta = 0$. So, therefore, T is a nonexpansive mapping, which is a contradiction.

Proposition 7.2.6. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping with $F(T) \neq \emptyset$. Then T is quasi-nonexapansive.*

Proof. Let $x \in F(T)$ and $y \in C$,

$$\lambda\|Tx - x\| = 0 \leq \|x - y\|.$$

So, we have

$$\begin{aligned} \|x - Ty\| = \|Tx - Ty\| &\leq \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\| \\ &= \alpha\|y - x\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\| \\ \Rightarrow (1 - \beta)\|x - Ty\| &\leq (1 - \beta)\|x - y\| \\ \Rightarrow \|x - Ty\| &\leq \|x - y\|. \end{aligned}$$

Hence, T is quasi-nonexpanisve. ☐

Theorem 7.2.7. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive mapping type 1 mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to some $y \in C$. We show that $y \in F(T)$. Since

$$\lambda\|Tx_n - x_n\| = 0 \leq \|x_n - y\|,$$

so, we have

$$\begin{aligned} \|x_n - Ty\| &= \|Tx_n - Ty\| \\ &\leq \alpha\|y - Tx_n\| + \beta\|x_n - Ty\| + (1 - (\alpha + \beta))\|x_n - y\| \\ \Rightarrow \|x_n - Ty\| &\leq \|x_n - y\|. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - y\| = 0$, we obtain

$$\lim_{n\to\infty} \|x_n - Ty\| = 0.$$

and

$$Ty = y,$$

Hence, $F(T)$ is closed.

Now suppose that X is strictly convex and C is convex. We show that $F(T)$ is convex. Let $x, y \in F(T), z \in C$ with $x \neq y$. Since

$$\lambda\|x - Tx\| = 0 \leq \|x - z\|,$$

we obtain

$$\begin{aligned} \|x - Tz\| = \|Tx - Tz\| &\leq \alpha\|z - Tx\| + \beta\|x - Tz\| + (1 - (\alpha + \beta))\|x - z\| \\ \Rightarrow \|x - Tz\| &\leq \|x - z\|. \end{aligned} \tag{7.2.5}$$

Using similar argument, we have

$$\|y - Tz\| \leq \|y - z\|. \tag{7.2.6}$$

Let $z = \gamma x + (1 - \gamma)y \in C$, for $\gamma \in [0, 1]$, then from (7.2.5) and (7.2.6), we obtain

$$\begin{aligned} \|x - y\| &\leq \|x - Tz\| + \|Tz - y\| \\ &\leq \|x - z\| + \|z - y\| \\ &= \|x - (\gamma x + (1 - \gamma)y)\| + \|(\gamma x + (1 - \beta)y - y\| \\ &\leq (1 - \gamma)\|x - x\| + \gamma\|x - y\| + (1 - \gamma)\|x - y\| + \gamma\|y - y\| \\ &= \|x - y\|. \end{aligned} \tag{7.2.7}$$

Using the fact that X is strictly convex, there exists $\mu \in [0, 1]$ such that $Tz = \mu x + (1 - \mu)y$. Now

$$(1 - \mu)\|x - y\| = \|Tx - Tz\| \leq \|x - z\| = (1 - \gamma)\|x - y\| \tag{7.2.8}$$

and

$$\mu\|x - y\| = \|Ty - Tz\| \leq \|x - z\| = \gamma\|x - y\|. \tag{7.2.9}$$

From the above inequalities, having that $1 - \mu \leq 1 - \gamma$ and $\mu \leq \gamma$, this implies that $\mu = \gamma$. Thus, $z \in F(T)$ implies that $F(T)$ is convex. □

In view of Proposition 7.2.4, we have the following corollaries.

Corollary 7.2.8. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 7.2.9. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mean nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 7.2.10. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mapping satisfying condition (C). Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 7.2.11. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mapping satisfying condition (C_λ). Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 7.2.12. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized mean nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Lemma 7.2.13. *Let C be a nonempty subset of a Banach space X. Suppose that $T : C \to C$ is a generalized (α, β)-nonexpansive mapping type 1 mapping on C. Then for all $x, y \in C$ and for $\gamma \in [0, 1)$, we have the following*

1. $\|T^2x - Tx\| < \|Tx - x\|$,
2. *either* $\frac{\gamma}{2}\|x - Tx\| \leq \|x - y\|$ *or* $\frac{\gamma}{2}\|Tx - T^2x\| \leq \|Tx - y\|$,
3. *either* $\|Tx - Ty\| \leq \alpha\|Tx - y\| + \beta\|Ty - x\| + (1 - (\alpha + \beta))\|x - y\|$ *or* $\|T^2x - Ty\| \leq \alpha\|T^2x - y\| + \beta\|Ty - Tx\| + (1 - (\alpha + \beta))\|Tx - y\|$.

Proof. 1. For all $x \in C$, we have that $\lambda\|Tx - x\| \leq \|Tx - x\|$, which implies that

$$\begin{aligned}\|T^2x - Tx\| = \|T(Tx) - Tx\| &\leq \alpha\|T(Tx) - x\| + \beta\|Tx - Tx\| + (1 - (\alpha + \beta))\|Tx - x\| \\ &= \alpha\|T(Tx) - x\| + (1 - (\alpha + \beta))\|Tx - x\| \\ &\leq \alpha[\|T(Tx) - Tx\| + \|Tx - x\|] + (1 - (\alpha + \beta))\|Tx - x\| \\ &= \alpha\|T^2x - Tx\| + (1 - \beta))\|Tx - x\|,\end{aligned}$$

this implies that

$$\|T^2x - Tx\| \leq \frac{1 - \beta}{1 - \alpha}\|Tx - x\| < \|Tx - x\|.$$

2. Suppose, on the contrary $\frac{\gamma}{2}\|x - Tx\| > \|x - y\|$ or $\frac{\gamma}{2}\|Tx - T^2x\| > \|Tx - y\|$, for some $x, y \in C$. Now, using (1), observe that

$$\begin{aligned}\|x - Tx\| &\leq \|x - y\| + \|y - Tx\| \\ &< \frac{\gamma}{2}\|x - Tx\| + \frac{\gamma}{2}\|Tx - T^2x\| \\ &< \frac{\gamma}{2}\|x - Tx\| + \frac{\gamma}{2}\|x - Tx\| \\ &= \gamma\|x - Tx\| \\ &< \|x - Tx\|,\end{aligned}$$

   which is a contradiction. Thus, we obtain the desired result.

3. The proof of (3) follows from (2). Thus, we omit it.

□

Lemma 7.2.14. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ a generalized (α, β)-nonexpansive mapping type 1 mapping. Then for all $x, y \in C$,*

$$\|x - Ty\| \leq \frac{(2 + \alpha + \beta)}{(1 - \beta)}\|x - Tx\| + \|x - y\|.$$

Proof. From Lemma 7.2.13, we have that for all $x, y \in C$, $\|Tx - Ty\| \leq \alpha\|Tx - y\| + \beta\|Ty - x\| + (1 - (\alpha + \beta))\|x - y\|$ or $\|T^2x - Ty\| \leq \alpha\|T^2x - y\| + \beta\|Ty - Tx\| + (1 - (\alpha + \beta))\|Tx - y\|$.

Considering $\|Tx - Ty\| \leq \alpha\|Tx - y\| + \beta\|Ty - x\| + (1 - (\alpha + \beta))\|x - y\|$, we obtain that

$$\begin{aligned}\|x - Ty\| &\leq \|x - Tx\| + \|Tx - Ty\| \\ &\leq \|x - Tx\| + \alpha\|Tx - y\| + \beta\|Ty - x\| + (1 - (\alpha + \beta))\|x - y\| \\ &\leq \|x - Tx\| + \alpha\|Tx - x\| + \alpha\|x - y\| + \beta\|Ty - x\| + (1 - (\alpha + \beta))\|x - y\| \\ &= (1 + \alpha)\|x - Tx\| + \beta\|Ty - x\| + (1 - \beta)\|x - y\| \\ \Rightarrow \|x - Ty\| &\leq \frac{(1 + \alpha)}{(1 - \beta)}\|x - Tx\| + \|x - y\| \leq \frac{(2 + \alpha + \beta)}{(1 - \beta)}\|x - Tx\| + \|x - y\|.\end{aligned}$$

Also, considering $\|T^2x - Ty\| \leq \alpha\|T^2x - y\| + \beta\|Ty - Tx\| + (1 - (\alpha + \beta))\|Tx - y\|$, using (1) of Lemma 7.2.13, we obtain that

$$\begin{aligned}\|x - Ty\| &\leq \|x - Tx\| + \|Tx - T^2x\| + \|T^2x - Ty\| \\ &< \|x - Tx\| + \|x - Tx\| + \alpha\|T^2x - y\| + \beta\|Ty - Tx\| + (1 - (\alpha + \beta))\|Tx - y\| \\ &\leq 2\|x - Tx\| + \alpha\|T^2x - Tx\| + \alpha\|Tx - y\| + \beta\|Ty - x\| + \beta\|x - Tx\| + (1 - (\alpha + \beta))\|Tx - \\ &< 2\|x - Tx\| + \alpha\|x - Tx\| + \alpha\|Tx - y\| + \beta\|Ty - x\| + \beta\|x - Tx\| + (1 - (\alpha + \beta))\|Tx - y \\ &= (2 + \alpha + \beta)\|x - Tx\| + \beta\|Ty - x\| + (1 - \beta)\|x - y\| \\ \Rightarrow \|x - Ty\| &\leq \frac{(2 + \alpha + \beta)}{(1 - \beta)}\|x - Tx\| + \|x - y\|.\end{aligned}$$

Thus in both cases, we obtain the desired result. □

Theorem 7.2.15. *Let C be a nonempty closed subset of a Banach space X with Opial property and $T : C \to C$ be a generalized (α, β)-nonexpansive mapping type 1 mapping with $\lambda = \frac{\gamma}{2}, \gamma \in [0, 1)$. If $\{x_n\}$ converges weakly to x and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tx = x$. That is $I - T$ is demiclosed at zero, where I is the identity mapping on X.*

Proof. By Lemma 7.2.13

$$\lambda\|x_n - Tx_n\| \leq \|x_n - x\|.$$

Thus by definition

$$\|Tx_n - Tx\| \leq \alpha\|Tx_n - x\| + \beta\|Tx - x_n\| + (1 - (\alpha + \beta))\|x_n - x\|.$$

Now, observe that

$$\begin{aligned}
\|x_n - Tx\| &\leq \|x_n - Tx_n\| + \|Tx_n - Tx\| \\
&\leq \|x_n - Tx_n\| + \alpha\|Tx_n - x\| + \beta\|Tx - x_n\| + (1 - (\alpha + \beta))\|x_n - x\| \\
&\leq \|x_n - Tx_n\| + \alpha\|Tx_n - x_n\| + \alpha\|x_n - x\| + \beta\|Tx - x_n\| + (1 - (\alpha + \beta))\|x_n - x\| \\
&= (1 + \alpha)\|x_n - Tx_n\| + \beta\|Tx - x_n\| + (1 - \beta))\|x_n - x\| \\
\Rightarrow \|x_n - Tx\| &\leq \frac{1 + \alpha}{(1 - \beta)}\|x_n - Tx_n\| + \|x_n - x\|.
\end{aligned}$$

Using our hypothesis, we have that

$$\liminf_{n\to\infty} \|x_n - Tx\| \leq \liminf_{n\to\infty} \|x_n - x\|. \tag{7.2.10}$$

Using our hypothesis that $\{x_n\}$ converges weakly to x and Opial property, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - Tx\|,$$

which contradicts (7.2.10). Thus, we have that $Tx = x$. □

Theorem 7.2.16. *Let C be a nonempty compact subset Banach space X and $T : C \to C$ is a generalized (α, β)-nonexpansive mapping type 1 mapping with $\lambda = \frac{\gamma}{2}, \gamma \in [0, 1)$. Then T has a fixed point in C if and only if T admits an a.f.p.s.*

Proof. The proof follows a similar approach as in Theorem 7.2.15, and thus, we omit it. □

7.2.2 Convergence Results

In this section, we establish some convergence results for generalized (α, β)-nonexpansive mapping type 1 mapping via a new three steps iterative algorithm in the framework of uniformly convex Banach space. We define our iterative process as follows: For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n, \\ y_n = (1 - \alpha_n)Tz_n + \alpha_n T^2 z_n, \\ x_{n+1} = T[(1 - \beta_n)T^2 z_n + \beta_n T^2 y_n], \quad n \geq 0, \end{cases} \tag{7.2.11}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$.

Lemma 7.2.17. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive mapping type 1 mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (7.2.11), then, the following hold:*

(i) $\{x_n\}$ *is bounded.*

(ii) $\lim_{n\to\infty} \|x_n - x^*\|$ *exists for all* $x^* \in F(T)$.

Proof. Let $x^* \in F(T)$, using (7.2.11) and Proposition 7.2.6, we obtain

$$\begin{aligned} \|z_n - x^*\| &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|Tx_n - x^*\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{7.2.12}$$

Also, using (7.2.11), (7.2.12) and Proposition 7.2.6, we obtain

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\alpha_n)Tz_n + \alpha_n T^2 z_n - x^*\| \\ &\leq (1-\alpha_n)\|Tz_n - x^*\| + \alpha_n\|T(Tz_n) - x^*\| \\ &\leq (1-\alpha_n)\|z_n - x^*\| + \alpha_n\|Tz_n - x^*\| \\ &\leq (1-\alpha_n)\|z_n - x^*\| + \alpha_n\|z_n - x^*\| \\ &= \|z_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{7.2.13}$$

Lastly, using (7.2.11), (7.2.13) and Proposition 7.2.6, we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|T[(1-\beta_n)T^2 z_n + \beta_n T^2 y_n] - x^*\| \\ &\leq (1-\beta_n)\|T^2 z_n - x^*\| + \beta_n\|T^2 y_n - x^*\| \\ &= (1-\beta_n)\|T(Tz_n) - x^*\| + \beta_n\|T(Ty_n) - x^*\| \\ &\leq (1-\beta_n)\|Tz_n - x^*\| + \beta_n\|Ty_n - x^*\| \\ &\leq (1-\beta_n)\|z_n - x^*\| + \beta_n\|y_n - x^*\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{7.2.14}$$

This shows that $\{\|x_n - x^*\|\}$ is bounded and non-increasing for all $x^* \in F(T)$. Thus, $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists. ☐

Lemma 7.2.18. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive mapping type 1 mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (7.2.11), then* $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Proof. Since $F(T) \neq \emptyset$, suppose that $x^* \in F(T)$. It follows from Lemma 7.2.17 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F(T)$. Suppose that $\lim_{n\to\infty} \|x_n - x^*\| = c$. From (7.2.12), we obtain that $\|z_n - x^*\| \leq \|x_n - x^*\|$. Taking limsup of both sides, we have

$$\limsup_{n\to\infty} \|z_n - x^*\| \leq c. \tag{7.2.15}$$

In addition, using Proposition 7.2.6, we obtain that $\|Tx_n - x^*\| \leq \|x_n - x^*\|$, and that

$$\limsup_{n\to\infty} \|Tx_n - x^*\| \leq c. \tag{7.2.16}$$

From (7.2.14), we have

$$\|x_{n+1} - x^*\| \leq (1-\beta_n)\|z_n - x^*\| + \beta_n\|x_n - x^*\|.$$

Taking the $\liminf_{n\to\infty}$ of both sides and rearranging the inequalities, we have

$$\begin{aligned} c &\leq (1-\beta_n)\limsup_{n\to\infty}\|z_n - c\| + \beta_n c \\ c &\leq \liminf_{n\to\infty}\|z_n - x^*\|. \end{aligned} \tag{7.2.17}$$

From (7.2.15) and (7.2.17), we obtain that $\lim_{n\to\infty}\|z_n - x^*\| = c$. That is,

$$\lim_{n\to\infty}\|(1-\gamma_n)x_n + \gamma_n Tx_n - x^*\| = c.$$

Thus, by Lemma 2.1.41, we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

□

Theorem 7.2.19. *Let X be a uniformly convex Banach space which satisfies the Opial's condition and C a nonempty closed convex subset of X. Let $T : C \to C$ be a generalized (α,β)-nonexpansive mapping type 1 mapping such that $\lambda = \frac{\gamma}{2} \in [0, \frac{1}{2}]$ with $F(T) \neq \emptyset$ and $\{x_n\}$ be a sequence defined by iteration (7.2.11). Then, $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. It has been established in Lemma 7.2.17 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists and that $\{x_n\}$ is bounded. Now, since X is uniformly convex, we can find a subsequence say $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly in C. We now establish that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Let x and y be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Theorem 7.2.18, we have that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero by Theorem 7.2.15, we therefore have that $Tx = x$. Using similar approach, we can show that $y = Ty$. It follows from Lemma 7.2.17 that $\lim_{n\to\infty}\|x_n - y\|$ exists. Now, suppose that $x \neq y$, then by Opial's condition,

$$\begin{aligned} \lim_{n\to\infty}\|x_n - x\| &= \lim_{k\to\infty}\|x_{n_k} - x\| \\ &< \lim_{k\to\infty}\|x_{n_k} - y\| \\ &= \lim_{n\to\infty}\|x_n - y\| \\ &= \lim_{j\to\infty}\|x_{n_j} - y\| \\ &< \lim_{j\to\infty}\|x_{n_j} - x\| \\ &= \lim_{n\to\infty}\|x_n - x\|. \end{aligned}$$

This is a contradiction. So $x = y$. Hence, $\{x_n\}$ converges weakly to a fixed point of $F(T)$ and this completes the proof. □

Theorem 7.2.20. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a generalized (α, β)-nonexpansive mapping type 1 mapping on C, $\{x_n\}$ defined by (7.2.11) and $F(T) \neq \emptyset$. Then, $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.*

Proof. Let $\{x_n\}$ converges to x^* a fixed point of T. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$, and since $0 \leq d(x_n, F(T)) \leq d(x_n, x^*)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. It follows from Lemma 7.2.17 that $\lim_{n\to\infty} \|x_n - x^*\|$ exists and that $\lim_{n\to\infty} d(x_n, F(T))$ exists for all $x^* \in F(T)$. By our hypothesis, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Suppose $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and $\{u_k\}$ is a sequence in $F(T)$ such that for all $n \in \mathbb{N}$,

$$\|x_{n_k} - u_k\| < \frac{1}{2^k}$$

it follows from (7.2.14) that $\|x_{n+1} - u_k\| \leq \|x_n - u_k\| < \frac{1}{2^k}$, hence

$$\begin{aligned} \|u_{k+1} - u_k\| &\leq \|u_{k+1} - x_{n+1}\| + \|x_{n+1} - u_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

Thus, we have that $\{u_k\}$ is a Cauchy sequence in $F(T)$. Also, by Theorem 7.2.7, we have that $F(T)$ is closed. Thus $\{u_k\}$ is a convergent sequence in $F(T)$. Now, suppose that $\{u_k\}$ converges to $p \in F(T)$. Therefore, since

$$\|x_{n_k} - p\| \leq \|x_{n_k} - u_k\| + \|u_k - p\| \to 0 \text{ as } k \to \infty,$$

we obtain that $\lim\limits_{k\to\infty} \|x_{n_k} - p\| = 0$ and so $\{x_{n_k}\}$ converges strongly to $p \in F(T)$. Since $\lim\limits_{n\to\infty} \|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p. ☐

Theorem 7.2.21. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a generalized (α, β)-nonexpansive mapping type 1 mapping, $\{x_n\}$ defined by (7.2.11) and $F(T) \neq \emptyset$. Let T satisfy condition (I), then, $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Using Lemma 7.2.17 and Theorem 7.2.18, we obtain that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Using the fact that

$$0 \leq \lim_{n\to\infty} f(d(x_n, F(T)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0, \ \forall x \in C,$$

and that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$, since, f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$, it then follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus using Theorem 7.2.20, we obtain that $\{x_n\}$ converges strongly to $p \in F(T)$. ☐

7.2.3 Application to a Nonlinear Integral Equation

In this section, we present an application of our result to nonlinear integral equation of the form:

$$x(t) = g(t) + \gamma \int_a^b M(t,s)h(t,x(s))ds, \tag{7.2.18}$$

where $\gamma \in (0,1], M : [a,b] \times [a,b] \to \mathbb{R}^+$, $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$ with ordered relation $\leq$ in X defined as for $x,y \in X, x \leq y$ if and only if $x(s) \leq y(s)$ for all $s \in [a,b]$. We defined $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ by $\|x-y\| = \sup_{s\in[a,b]} |x(s)-y(s)|$.

Theorem 7.2.22. *Let* $X = C([a,b],\mathbb{R})$ *and* $T : X \to X$ *the operator given by*

$$Tx(t) = g(t) + \gamma \int_a^b M(t,s)h(t,x(s))ds$$

for all $t,s \in [a,b]$, *where* $\gamma \in [0,1], M : [a,b] \times [a,b] \to \mathbb{R}^+$, $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ *and* $h : [a,b] \to \mathbb{R}$ *are continuous functions. Let* $X = C([a,b],\mathbb{R})$ *be the space of all continuous real valued functions defined on* $[a,b]$. *Furthermore, suppose the following condition hold:*

1. *there exists a continuous mapping* $\upsilon : X \times X \to [0,\infty)$ *such that*

$$|h(s,x(s)) - h(s,y(s))| \leq \upsilon(x,y)|x(s)-y(s)|$$

   for all $s \in [a,b]$ *and* $x,y \in X$.

2. *there exists* $\omega \in [0,1]$, *such that*

$$\int_a^b M(t,s)\upsilon(x,y) \leq \omega.$$

Then the integral equation (7.2.18) *has a solution.*

Proof. Without loss of generality, we suppose that $x \leq y$, so that

$$\sup\{|y(s)-x(s)| : s \in [a,b]\} \geq \sup\{|Tx(s)-x(s)| : s \in [a,b]\},$$

which implies that

$$\lambda\|Tx-x\| \leq \|Tx-x\| \leq \|y-x\|,$$

where $\lambda \in [0,1)$. Thus, we have that

$$\begin{aligned}
|Ty(s) - Tx(s)| &= \left\| g(t) + \gamma \int_a^b M(t,s)h(t,y(s)) - g(t) - \gamma \int_a^b M(t,s)h(t,x(s))ds \right\| \\
&\leq \gamma \int_a^b |M(t,s)[h(t,y(s)) - h(t,x(s))]|ds \\
&\leq \gamma \int_a^b M(t,s)\upsilon(x,y)|y(s) - x(s)|ds \\
&\leq \sup_{s\in[a,b]} |y(s) - x(s)|\gamma \int_a^b M(t,s)\mu(x,y)ds \\
&\leq \gamma\omega\|y - x\| \\
&\leq \|y - x\|.
\end{aligned}$$

Thus, we have that

$$\lambda\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

Clearly, T satisfies condition (C_λ) and by Proposition 7.2.4, T is a generalized (α,β)-nonexpansive mapping and all the conditions in Theorem 7.2.16 are satisfied, as such T has a fixed point, that is the integral equation (7.2.18) has a solution. □

7.2.4 Numerical Examples

Example 7.2.23. *Define a mapping* $T : [0,1] \to [0,1]$ *as*

$$Tx = \begin{cases} 1 - x & \text{if } x \in [0, \frac{1}{8}), \\ \frac{x+7}{8} & \text{if } x \in [\frac{1}{8}, 1]. \end{cases} \tag{7.2.19}$$

It is easy to see that T satisfies condition (C), thus it is a generalized (α,β)-nonexpansive mapping.

In what follows, we numerically compare our new iteration process with some existing iterative processes.

Case I: *Taking,* $\alpha_n = \frac{1}{\sqrt{n^3+4}}, \gamma_n = \frac{3}{(n^3+200)}, \beta_n = \frac{2}{\sqrt{n^3+5}}$ *and* $x_0 = 0.5$.

Case II: *Taking,* $\alpha_n = \frac{1}{202}, \gamma_n = \frac{1}{1000}, \beta_n = \frac{1}{300}$ *and* $x_0 = 0.8$.

Case III: *Taking,* $\alpha_n = \frac{1}{\sqrt{n^{30}+40}}, \gamma_n = \frac{3}{300n^3}, \beta_n = \frac{1}{\sqrt{n^{10}+50}}$ *and* $x_0 = 0.3$.

Case IV: *Taking,* $\alpha_n = \frac{5}{300n^{30}}, \gamma_n = \frac{8}{1000n^{34}}, \beta_n = \frac{7}{200n^{20}}$ *and* $x_0 = 0.6$.

The comparison shows that the iterative processes (7.2.11) converges faster than the iterative processes (2.6.17), (2.6.20) and consequently converges faster than some exiting iterative schemes in the literature.

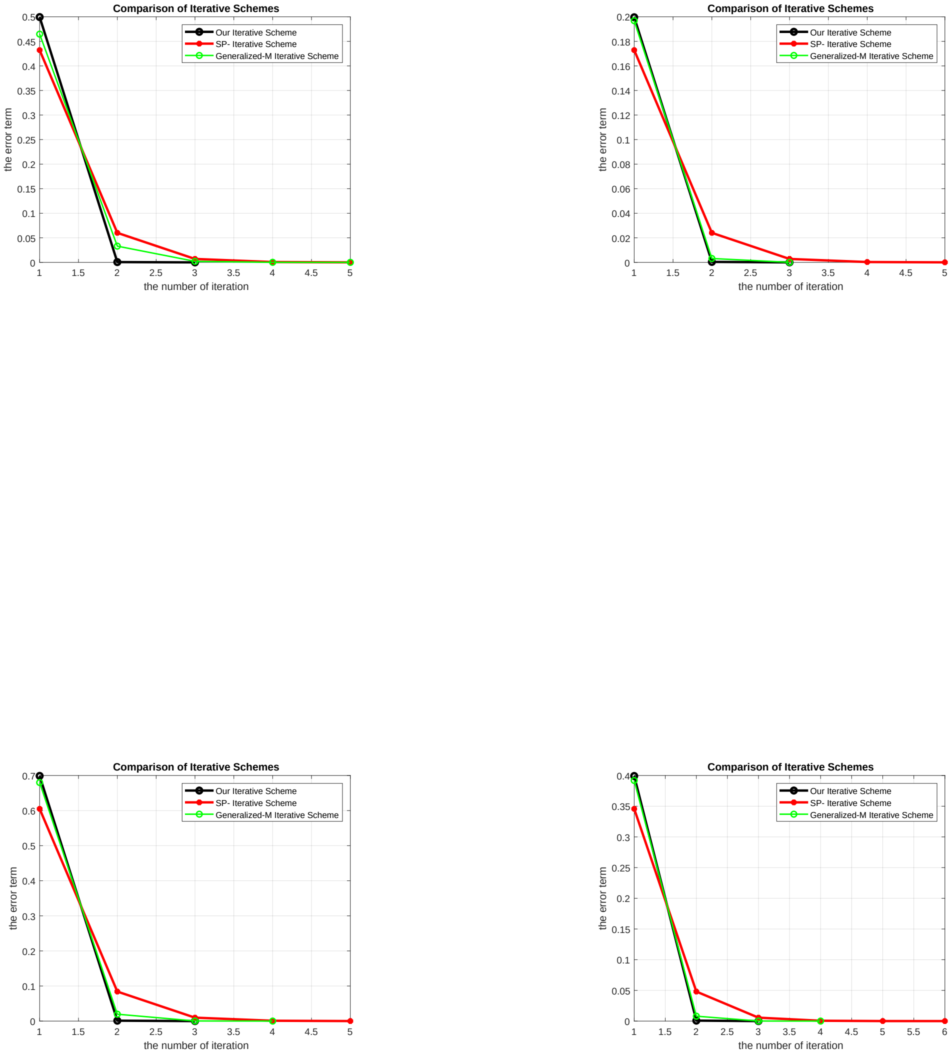

Figure 7.3: Example 7.2.23, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

7.3 Conclusion

In this section, we presented a modified (improved) generalized M-iteration with the inertial technique for three quasinonexpansive multivalued mappings in a real Hilbert space. In addition, we obtained a weak convergence result under suitable conditions and the strong convergence result is achieved using the CQ projection method with our modified generalized M-iteration. In addition, we presented some fixed point results for a general class of nonexpansive mappings and also proposed a new iterative scheme for approximating the fixed point of this class of mappings in the framework of uniformly convex Banach spaces. Finally, we applied our convergence results to certain optimization problems, and present some numerical experiments to show the efficiency and applicability of the proposed method in comparison with other existing methods (Modified NOOR iterative scheme, Modified SP-iterative scheme) in the literature.

7.4 Open Problem

In this chapter, we gave the definition of a generalized (α, β)-nonexpansive type 1 mapping, and generalized (α, β)-nonexpansive type 2 mapping, we established some results for the type 1 mapping but not for type 2. More so, it is possible to modify iterative method 7.2.11 to approximate the solution of a variational inequality problem (VIP) (1.2.1) and other optimization problems discussed in this study. In light of this, we leave this as an open problem for interested researchers in this area to explore.

Chapter 8

Conclusion, Contribution to Knowledge and Future Research

In this chapter, we conclude the study of this thesis and highlight the contributions of our study to existing knowledge. We also identify and discuss possible areas of future research.

8.1 Conclusion

This thesis presents a comprehensive study of nonlinear optimization and fixed point problems in the framework of Hilbert and Banach spaces. Some of these studies are extensions, modifications of existing results in the literature, and others are entirely new results in Hilbert and Banach spaces. We organized our research into 8 chapters and presented it in a logical manner. In Chapter 1, we gave a brief background of our research, discussed the motivation and research problems studied in this thesis, mainly the ones considered in Chapters 3 to Chapter 8 of this thesis. Thereafter, we highlighted the objectives and organization of our study. In Chapter 2, we defined several basic terminology and phrases that are of great importance throughout our research. In addition, we provided a detailed assessment of previous publications that are relevant to our research and lastly, we recalled numerous key findings from our study. Furthermore, Chapter 3 to Chapter 7 of this study was devoted to the study of nonlinear optimization and fixed point problems in the framework of Hilbert and Banach spaces. These chapters are made up of the major findings of this study. Furthermore, several numerical experiments and applications of our major findings are described in each of these chapters, along with comparisons to other results in the literature. The conclusions gained in these chapters are novel, essential generalizations, and insights into our contribution to the study of fixed point and its applications, which are the main results of this thesis. Lastly, some open problems concerning our established results were also presented in Section 3.5, Section 4.3, Section 6.3 and Section 7.4.

8.2 Contribution to Knowledge

In general, we have introduced some new nonlinear optimization problems, nonlinear mappings and new iterative methods which generalize, extend, improve, and unify existing results in the literature. Other researchers in this area may be motivated by the results in this thesis to extend, generalize, and unify various findings in the literature. Among other things, we made the following contributions:

In Chapter 3, the results obtained in Section 3.1 extends, improves and unifies the works of Thong and Hieu [336] and Thong, Vinh and Cho in [340]. Also, the results obtained in Section 3.2 a new in their own right and also extend, improve and unify the results obtained by the authors in [162, 163, 161, 218]. Lastly, the results obtained in Section 3.3 are new.

In Chapter 4, the results obtained in Section 4.1 extend, improve and unify the works of Mainge [218], Minh, Van and Anh [236], problem (2.6.4) and a host of others in the literature. Also, the results obtained in this section is new in its own right and also extend, improve and unify the results obtained by the authors in [162, 163, 161, 218, 336, 340]. Lastly, the results obtained in Section 3.3 are new.

The results obtained in Chapter 5 of this study extend, improve and unify the works of Moudafi [238], Censor et al. [84], Ezeora and Izuchukwu [126] and Zhang and Jiang [385].

The results obtained in Chapter 6 of this study extend, improve and unify the works of the author Abass, Ogbusi and Mewomo [5], Cholamjiak and Cholamjiak [100], Kazmi and Rizvi [183] and Suantai and Cholamjiak [312].

In Chapter 7, the results obtained in Section 7.1 extend, improve and unify the works authors in [179, 250, 166, 262, 349, 104]. Lastly, the results obtained in Section 7.2 extend, improve and unify the results obtained by the authors in [8, 3, 39, 128, 166, 153, 230, 179].

8.3 Future research

In regards to Chapter 3 through to Chapter 7 of this thesis, the following are some possible directions for future research;

1. provide and affirmative answer to the open problems in Section 3.5;
2. provide and affirmative answer to the open problems in Section 4.3;
3. provide and affirmative answer to the open problems in Section 6.3;
4. provide and affirmative answer to the open problems in Section 7.4.

We established a nonlinear mapping and a three-step iterative process in Chapter 7, Section 7.2, and shown that our newly proposed iterative schemes can approximate nonlinear mappings better, faster, and more precisely than certain existing iterative schemes in the literature. It will be interesting to investigate the following:

1. extending these iterative schemes to multi-valued type iterative schemes and obtain corresponding results obtained in these chapters;

2. extending the iterative algorithms to a more general space like complex valued control metric spaces, complex valued double controlled metric spaces, control type metric spaces, double control type metric spaces, complex valued Banach spaces, modular spaces, modular function spaces, hyperbolic spaces, and CAT(0) spaces and obtain corresponding results obtained in this chapters;

3. introduce an implicit type of the following iterative schemes in more general spaces;

4. using these iterative schemes to solve some optimization problems (Variational inequality, minimization problem, split feasibility problems and so on).

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