Research thesis for Med (Mathematics) with course work

An experimental approach to the derivative using Geogebra

by

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Abstract

This qualitative research study was carried out with six Grade 11 pupils from a school for boys in Pietermaritzburg, South Africa. Its main intentions were to explore if the computer software *Geogebra* could aid pupils in discovering the differentiation rule for elementary polynomials and to assess the effectiveness of *Geogebra* in enhancing the development of concepts related to the derivative. *Geogebra* is a free dynamic computer software program that combines geometry and algebra into a user friendly mouse driven package.

The study was informed by constructivism and Kolb's (1984) experiential learning theory. One-to-one task-based interviews were the main data collection strategy. Analysis of the data, in an interpretive paradigm, suggested that *Geogebra* can indeed provide the necessary support required by the pupils to deduce inductively the differentiation rule for elementary polynomials. The results also suggested that the evolution of the general result follows a linear process and as such the strategic sequencing of the task-based activities is of paramount importance. Additionally, all the students who successfully deduced the result displayed high levels of conviction regarding its generality and they expressed a need for an explanation. This arguably sets the tone for the teaching of the formal proof, which in this case serves to explain why the empirically derived result is always true.

The study also found that the experience with the *Geogebra* applets might help students resolve conceptual difficulties associated with the derivative. In particular, it explored the effect of the *Geogebra* experience on the students' ability to solve non-routine graphing problems involving the first derivative. Analysis of the results produced some insights that may be important for instructional design. It found that students concentrate on either the degree of the polynomial or the derivative as a function of *x*. The ability to correlate the two aspects leads to the successful solution of the problem, and evidence from the interviews indicated that *Geogebra* has a role to play in helping pupils reconcile these concepts.

Lord grant me the serenity to

accept things that I cannot change;

the courage to change the things I can;

and the wisdom to know the difference.

Reinhold Niebuhr

I, Simakatso Ngwenya, declare that:

- the research reported in this thesis, except where acknowledgements indicate otherwise, is my original work;

- the thesis has not been submitted in any form for any degree or diploma to any tertiary institution;

- this thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

Signed: Date:

Simakatso Ngwenya: Student Number (212559572)

Approval by MEd Supervisor: Submission of thesis

I, Prof. Michael de Villiers, as the candidate's Supervisor, agree to the submission of this thesis by Simakatso Ngwenya, entitled *An experimental approach to the derivative using Geogebra*.

Signed: Date:

Prof. Michael de Villiers (MEd Supervisor)

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Chapter 1: Introduction

1.0 Focus and purpose of the study

The view that students are merely processors of information has long been replaced by the central tenet that perceives students as active participants in an evolving mathematical reality of their own making. Analyses of students' mathematical reasoning centre on inferring on the quality of their mathematical experience (Zbiek, Heid, Blume & Dick, 2007). The abstract nature of mathematics, in my view, has partly ensured that the teaching methods employed in many South African classrooms have continued to be traditional. Such traditional approaches still view the pupil as the recipient and the teacher as the fountain of knowledge. The teaching approaches do not appeal to the investigative nature of the pupils, instead they focus on ensuring that the pupils remodel what the teacher demonstrates in class. Proponents of mathematics education reforms have lamented the outcomes of these endeavours arguing that graduates of these courses demonstrate proficiency in the facts that mathematicians have developed without ever understanding how mathematicians think (Cuoco, Goldenberg & Mark, 1996).

The twenty first century has witnessed a marked increase in the number of students who have access to tablet computers and laptops that have capabilities of running software such as *Geogebra, Sketchpad, Cinderella* etc. These devices also enable access to powerful computational websites such as Wolfram Alpha at http://www.wolframalpha.com/examples/. *Geogebra* is a free dynamic computer software program that combines geometry and algebra into a user friendly mouse driven package. Such software could enable students to experiment and investigate certain mathematical concepts without having to be spoon-fed by the teacher. Cobb (1999) contends that a pupil's mathematical activity is not only confined to the classroom but extends within the broader systems of social and cultural aspects of mathematical activity.

This study sought to explore the possibility of using, experimentally, the dynamic graphing software *Geogebra* for concept acquisition and development in calculus. In particular, it investigated the following;

- can *Geogebra* aid students in discovering the power rule for differentiating elementary polynomials?
- having discovered the rule, are the pupils convinced about its truth and generality? Do they demonstrate a desire for an explanation for why the result works?

- does the use of dynamic graphing software such as *Geogebra* enhance conceptual understanding and resolve difficulties associated with the derivative as documented in the literature?

The "chalk and talk" approach to the teaching of differentiation is usually characterised by providing the students with the rule and following this by drill and practice questions. Discussions with colleagues have revealed that time constraints make them hesitant to teach differentiation using an investigative approach. I argue that such teaching strategies do not encourage pupils to make conjectures in unfamiliar situations. However, the use of software such as *Geogebra* could eliminate such perceived difficulties and facilitate the development of the habits of mind of mathematicians as encouraged by Cuoco et al. (1996). The focus of mathematics courses should not be to communicate established results and methods with a view of equipping students with a bag of facts that they will use after school (Cuoco et al., 1996). They contend that the emphasis should be on the mathematical habits of mind used by the mathematicians who created those results. The methods used to create those results and the final product should be given the same level of prominence in mathematics curricula.

> A curriculum organised around the habits of mind tries to close the gap between what the users and makers of mathematics *do* and what they *say.* Such a curriculum lets students in on the process of creating, inventing, conjecturing and experimenting; it lets them experience what goes on behind the study door before new results are polished and presented. It is a curriculum that encourages false starts, calculations, experiments, and special cases (Cuoco et al., 1996, p.376)

They further argue that teachers should encourage pupils to be pattern sniffers and should also foster within students a delight in finding hidden patterns. Additionally, they see experimentation as key in mathematical research and decry its rarity in mathematics classrooms. Simple ideas like recording results, keeping all but one variable fixed, trying very small or very large numbers and varying parameters in regular ways are missing from the backgrounds of many high school students (Cuoco et al., 1996 p.378).

Geogebra is a World Wide Web-based computer program that students have the option of downloading, free of charge, onto their personal computers at home. This negates the time-consuming argument that many teachers harp on because the students can do their investigations prior to the lesson. In addition, it could be used to model the thinking process in mathematics and the experimentation process that Cuoco et al. (1996) are agitating for in

mathematics education reform. The worksheet provided in Appendix 1 shows that pattern recognition, after experimenting with the *Geogebra* applets, is essential in arriving at the correct conclusion for the power rule in differentiation.

The study was informed by constructivism, in particular (and the learning activities follow) Kolb's (1984) experiential learning theory. Furthermore, it followed an action research design. It was carried out at an independent school in Pietermaritzburg. The data were collected using a worksheet that the students completed after working with *Geogebra* applets in the computer centre. The participants were also interviewed in a bid to understand the thinking processes that took place in completing the worksheet. It is my view that teaching differentiation this way may ensure conceptual understanding and simultaneously elucidate the link between the function and its derivative graphically. In this study the words pupil and student will be used synonymously to refer to high school learners.

<p align="center">***Chapter 2: Literature review***</p>

2.0 The computer in mathematics education

Throughout history mathematicians have used tools such as sliding rules, compasses and recently computers to simplify doing mathematics (Durmus & Karakirik, 2006). The ubiquitous nature of the computer in the twenty first century has been accompanied by the unrivalled enthusiasm for the potential of new technologies in the teaching and learning of mathematics (Fey et al., 1984 as cited in Zbiek et al., 2007). Education departments in different countries have embraced the idea of using computers to advance the teaching and learning process. In South Africa, information technology is one of the learning areas identified in the Curriculum and Assessment Policy Statement (CAPS) for grades 10 to 12. This is testament to the government's commitment to incorporating the use of computers not only in mathematics education but in the education process as a whole.

The availability of computers in the classroom has also resulted in the mushrooming of a multitude of software designed to address different needs of mathematics education. Parallel to these developments has been a growing interest in research aimed at understanding the impact of these technologies on the teaching, learning and curriculum (Zbiek et al., 2007).This current study also aims to contribute to this discourse by exploring the effect of *Geogebra* in the teaching and learning of the derivative in high school. The attributes of *Geogebra* as a teaching and learning software will be discussed in detail in the next section 2.3.

Mathematical activity in the classroom can be classified as either technical or conceptual. Zbiek et al. (2007) contend that technical activities centre on acting on mathematical objects or on representations of those objects. Examples of such activity include geometric constructions, numerical computation, graphing, algebraic manipulation, solving equations and so on. Thus technical activity is primarily concerned with tasks of mechanical or procedural performance. They further argue that conceptual mathematical activity pertains to tasks of inquiry, articulation and justification. Finding and describing patterns, defining, conjecturing and testing are activities that are associated with conceptual learning. Borwein (2005, p.2) in his description of experimental mathematics provides a case for which the computer can be used to support both forms of mathematical activities in the classroom. He suggests that the computer can be used for:

- gaining insight and intuition

- discovering new patterns and relationships

- graphing to expose mathematical principles

- testing and especially falsifying conjectures

- exploring a possible result to see if it merits formal proof

- suggesting approaches for formal proof

- replacing lengthy hand derivations and calculations

- confirming analytically derived results (Borwein 2005, p.2)

Replacing lengthy hand derivations, calculations and graphing to expose mathematical principles are points that are particularly pertinent to this study. Freeing students from the tedious exercise of hand plotting graphs of functions and then calculating the gradient at different values of x might enable them to focus on the common attributes of the gradient functions generated for different plots in the same family of curves. Kaput (1992) concurs with this observation, arguing that offloading the routine computations provides a learning efficiency in terms of compacting and enriching experiences. The compacted technical activity thus affords an opportunity for conceptual activity.

The affordability of computers and their pervasive nature has resulted in a shift in the way in which they are utilised in mathematics classrooms in the twenty first century. Initially the emphasis was on learning to use the computer to do mathematics but recent approaches dwell on using the computer as an aid in a mathematics lesson (Durmus & Karakirik, 2006). The difference in the two approaches is subtle but, in my view, significant. Earlier approaches expected pupils to master the technology while recent discourse centres on using the computer technology as a cognitive tool. A cognitive tool's role in mathematics education is that of externalising representations (Heid, 1988). Such tools afford the teacher and the student opportunities to expose cognitive conflicts (Zbiek et al., 2007). The role of the computer in the classroom has also been viewed by Salomon et al. (1991 as cited in Durmus & Karakirik, 2006) as an intellectual partnership. In this view, it is the intentional engagement of students in tasks afforded by the computer.

The shrewd use of the computer, according to de Villiers (2004), makes it possible to democratise the mathematical process. He argues that the computer simplifies quasi-

empirical exploration, in the process making widely accessible the type of playing around that was once the preserve of only the most persistent or imaginative. Quasi-empirical exploration in this context refers to all non-deductive methods involving experimental, intuitive, inductive or analogical reasoning and it is pivotal in the making of new mathematics (De Villiers, 2004). In its genesis, mathematics is often an experimental and inductive science (Lakatos, 1983 as cited in de Villiers, 2004). Thus the computer enables a researcher to formulate a great number of conjectures and to immediately test them by varying only a few parameters of a particular situation.

De Villiers (2004) concurs with Borwein (2005) that using computers in experimental environments will encourage pupils to make conjectures. Furthermore, he contends that a curriculum which emphasises quasi-empirical methods might encourage pupils to be good problem posers as opposed to simply being convergent problem solvers. Students might be more willing to try out new ideas and explore new avenues. This lays the foundation for the teacher to create an opportunity for the students to confront any misconceptions that may arise from such conjectures. Counter examples lead to the global refutation of students' incorrect understanding of mathematical concepts (De Villiers, 2004). The computer is ideal for generating such examples.

The use of computers in quasi-empirical exploration usually produces highly convincing results. In light of such convincing empirical evidence one could be excused for thinking that there is no need for formal and rigorous proofs. In his concluding remarks after using the computer program *Mathematica* to explore and verify some geometric results, Grünbaum (1993 as cited in de Villiers, 2004, p. 402) urges the mathematics community to open up to the new modes of investigation that have been made possible by computers. He goes further to suggest that his empirically discovered assertions should be referred to as theorems. However, the high level of conviction obtained through such quasi-empirical experimentation does not negate the need for proof but instead is a prerequisite for looking for one (De Villiers, 2004). The proof in this case serves to explain and clarify why a result is true (Horgan, 1993 as cited in de Villiers, 2004). While quasi-empirical methods generate a high level of conviction on their own they are not sufficient (Polya, 1954). For example Cauchy assumed that a continuous function implied its differentiability until Weistrass produced a continuous function that was not differentiable at any point (De Villiers, 2004).

To de Villiers (2004), a symbiotic relationship exists between the uses of quasi-empirical exploration and proof: one does not preclude the other. He further stresses that students must be made to realise that mathematicians are not just convinced by quasi-empirical evidence but are also motivated by it to search for a deductive proof. As already pointed out, the deductive proof may serve as an explanation for the empirically discovered result. De Villiers (1999) laments the lack of emphasis towards the teaching of proof in high school mathematics curricula. Mudaly (1998) found that although pupils attained very high levels of conviction after discovering a result following experimentation using the software *Sketchpad*, they still demonstrated a desire to understand why the result was true. He also established that pupils expressed surprise at their discovery and this evoked a strong need to obtain an explanation. Both de Villiers (1999) and Mudaly (1998) agree that such quasi-empirical explorations may lay a foundation for teaching proof as an explanation. Since the need for the explanation comes from the student it may make the learning process a lot more meaningful, providing a refreshing escape from the traditional way of teaching proof as a way of verifying a result.

Although de Villiers (1999) and Mudaly (1998) both worked within the context of geometry, their findings could be applied to other branches of mathematics. For instance, de Villiers (1999) argues that attempting to teach proof the traditional way (to verify the result) when the pupils are already convinced leads to the onset of negative attitudes and resentment amongst students. In this study the pupils were guided towards the discovery of the power rule for differentiating polynomials. In the follow-on interviews their levels of conviction were established (see Appendix 2). Furthermore, the researcher sought to establish whether they required an explanation for why the rule worked. The interview questions were adapted from Mudaly and de Villiers (2000).

2.1 Physical versus cognitive mathematical tools

In order to have a good understanding of the computer programs as cognitive tools, they will firstly be contrasted with their physical counterparts. Mathematical representations, whether physical or computer-based, could help students recognise connections among related concepts and help improve their communication skills in mathematics (Durmus & Karakirik, 2006). However, if representations are used as an end product rather than a tool to interpret the reality they will fail to serve their main purpose of making concepts easier to understand.

Pea (1987 as cited in Zbiek et al., 2007) describes a cognitive technology tool as one that helps to transcend the limitations in thinking, learning and problem-solving activities. These cognitive tools include simulations, computer algebra systems (CAS) and software such as *Geogebra*. Cognitive tools have the ability to react in response to the user by providing observable evidence of the consequences of the user's actions on the computer screen. Physical tools on the other hand do not automatically react to a user's action to give feedback. The student often has to work to extract the feedback from interactions with a physical tool. In most cases the teacher plays a greater role in guiding the pupil to attain the required conclusion (Zbiek et al., 2007). In addition, cognitive tools by design ensure that the actions on the external representations are essentially mathematically meaningful. They will respond to the student's actions and the immediate feedback affords the student an opportunity to evaluate the significance of each action. A greater chance of missing mathematical meaning exists when students are presented with a physical tool. The student may use a compass to poke holes while instructions are being given, thus with physical tools the extent to which meaning is derived is at the student's discretion (Zbiek et al., 2007). Durmus and Karakirik (2006) refer to physical mathematical tools as physical manipulatives. They define physical manipulatives as concrete models that involve mathematical concepts, appeal to several senses and can be touched and moved around by the learners. Their primary purpose is also to make abstract ideas accessible to pupils.

2.2 Virtual manipulatives

Moyer, Bolyard and Spikell (2002) view the equivalent of a cognitive tool as a virtual manipulative. Virtual manipulatives provide interactive environments in which students are able to pose their own questions and form connections between mathematical concepts and operations. Their main advantage is that they provide students with immediate feedback and in most cases will prompt the student to reflect on their conceptualisation. True virtual manipulatives

> " are visual images on the computer that are like pictures in books, drawings on an overhead projector, sketches on a chalkboard and so on. In addition these dynamic visual representations can be manipulated in the same way that a concrete manipulative can. Just as a student can flip, slide and turn a concrete manipulative by hand, he or she can use a computer mouse to actually flip, slide and turn the

dynamic visual representation as if it were a three dimensional object." (Moyer et al., 2002, p.372).

The above characterisation thus excludes static representations of mathematical objects on a computer screen. *Geogebra*, the software used in this study, qualifies to be called a virtual manipulative. It has the capability of producing dynamic applets and does not preclude user involvement. In defending classrooms that do not put an emphasis on investigative learning strategies, most teachers argue that they do not have enough time during the school day to make use of virtual manipulatives. Web-based and freely downloadable virtual manipulatives, such as *Geogebra*, may allow the pupils and the busy teachers who do not have time during the school day to make use of them after hours. Artigue (2002 as cited in Durmus & Karakirik, 2006) argues that current practice in mathematics education does not aim to promote efficient mathematical practices but is rather concerned with the transmission of the bases of mathematical culture. Computer manipulatives can be used to develop mathematical practices such as conjecturing and pattern recognition in certain mathematical concepts. Setting activities that pupils can complete at home can help engender these mathematical practices.

2.3 What is Geogebra and what does it offer?

The provision of computer technology to classroom teachers by stake holders in education does not necessarily translate to their fruitful utilisation. Despite the numerous benefits of using technology the process of incorporating it in the classroom is very slow and complex (Cuban, Kirkpatrick & Peck, 2001). Adapting teaching to strategies that encourage computer use requires a teacher to re-evaluate his or her traditional teaching approach. Jenson and Williams (1992 as cited in Ndlovu, Wessels & De Villiers, 2011) found in their study that technology initially complicates the teachers' life instead of simplifying it. Thus technology integration is, to some extent, dependent on the teacher's mastery of the software packages available.

Geogebra is dynamic computer software designed by Markus Hohenwarter in 2001 to alleviate the perceived difficulties associated with computer integration in the classroom. Its main feature is that it is a combination of Computer Algebra Systems (CAS) and Dynamic Geometry Systems (DGS) in one package. Unlike its predecessors, for example *Cabri* and *The Geometer's Sketchpad* which have the same capabilities, it is absolutely free while the

free versions of the former only have limited features. It is open-source software for mathematics teaching and learning that offers geometry, algebra and calculus features in a fully connected and easy to use software environment (Hohenwarter M, Hohenwarter J, Kreis & Lavicza, 2008). The open-source nature of the software implies that it is freely available on the internet or to download onto the computer's local hard drive for use both in the classroom and at home. The interactive and dynamic *Geogebra* worksheets can be used with any internet browser that supports Java (Hohenwarter & Fuchs, 2004). This study made use of this feature. Students used dynamic *Geogebra* applets that the researcher created and saved on the *Geogebra* tube (available at http://www.geogebratube.org/student/m94502). In the school's computer centre the pupils did not have to install the software but just accessed the Java applets using the link provided to them.

Research results suggest that *Geogebra* can be used to encourage discovery and experimentation and its visualisation features can be used advantageously to teach children how to generate conjectures (Lavicza 2006 as cited in Hohenwater et al., 2008). In *Geogebra*, geometric constructions may be altered dynamically by dragging free objects within the construction. In addition, it is possible to enter coordinates of points or vectors, equations of lines, conic sections or functions and numbers or angles directly (Hohenwarter & Fuchs, 2004). Furthermore the open-source nature gives teachers the opportunity to create interactive online learning environments and the ability to share them with other teachers worldwide (Hohenwarter et al., 2008).

The primary aim of this study was to explore the possibility of using *Geogebra* for the development of concepts in differentiation, in particular the discovery of the power rule and the relationship between the gradient function and the original function. Accordingly the affordances of *Geogebra* in this branch of mathematics will be discussed. Hohenwarter et al. (2008) point out that the teaching of calculus with *Geogebra* is still an extensive area of development. The study intends to contribute to this discourse. Two different ways of integrating *Geogebra* into calculus teaching and learning have been suggested by Hohenwarter et al. (2008) namely, presentation and mathematical experiments. The presentation strategy is a teacher-centred approach in which the teacher uses previously prepared *Geogebra* files to present concepts to the students. Ndlovu et al. (2011) contend that even if there is only one computer connected to a data projector, the teacher is afforded a wider range of teaching possibilities than with a static blackboard or an over head projector.

In the mathematical experiments approach the teacher may provide incomplete interactive sketches and the students then use these to explore and rediscover mathematical concepts (Hohenwarter et al., 2008). The latter approach is employed in this study.

2.4 Understanding student involvement in computer learning environments

The preceding section discussed the potential of computer technology in the mathematics classroom. Its visualisation capabilities imply that software such as *Geogebra* can enable students to extract what is common to a number of different situations and can arguably encourage conceptual understanding. As pointed out in the introductory chapter, computer aided teaching and learning is a relatively new phenomenon that has resulted from the pervasive nature of computers in the twenty first century. In a bid to understand and characterise student use of technology for research purposes, Zbiek et al. (2007) conceived exploratory activity and expressive activity as the main constructs of student involvement in computer aided learning environments.

In an exploratory activity the students work with a model that was created by someone else. The pupils follow a set of instructions in guided explorations and the ultimate goal is to discover a predetermined result that was set by the teacher. In such typical guided explorations the student is expected to drag and observe the properties of some dynamic figure. The explorations can also be less structured and the pupils have some form of freedom in the exercise. For instance in this study, although the pupils are expected to work towards a predetermined result using a model constructed by the teacher, they still have some form of freedom as they are able to input different functions into the *Geogebra* applet. The main advantage of exploratory activity is that it directs students' attention to the mathematical characteristics of the concept under investigation and facilitates symbolic descriptions (Clements & Battista 2001as cited in Zbiek et al., 2007 p.1182).

Expressive activities are more open-ended. The student is given a cognitive tool to answer a question of his or her own choosing using a method of his or her own choice (Zbiek et al., 2007). They further contend that unstructured play encourages pupils to work expressively and in the process they are able to determine the tool's capabilities and limitations. In addition, such activities help students to develop an intense, personal and purposeful relationship with the tool. While the expressive activity offers a fair share of advantages,

Zbiek et al. (2007) caution that such use of technology might undermine the teacher's and the curriculum's objectives. Allowing pupils to choose approaches to a problem might raise the possibility that some of the options available to them can enable them to bypass the ideas that their teacher intended them to encounter. Careful analysis of student actions during tasks can provide different insight into student learning.

This study carefully chose questions within the same family of functions and used variation theory (discussed in a later section) to guide pupils to discovering the differentiation rule of basic polynomials. The students used applets that were already prepared for them to attain a predetermined result as per syllabus requirement. In accordance with the Zbiek et al. (2007) characterisation, the students were engaged in an exploratory activity as they completed the tasks. The tasks consisted of dragging a point on the graph of a function. To further understand student activity in dragging tasks, Arzarello, Olivero, Paola and Robutti (2002) classify different dragging goals and attitudes demonstrated by students in such tasks. Although their classification was based on *Cabri* (a Dynamic Geometry Software) activities, it is my view that their findings can be used to understand the goals of students within any DGS environment. Furthermore they have pertinent implications for designing computer aided instruction. They argue that dragging dynamic figures supports the production of conjectures and also allows students to discover the invariant properties of a geometric figure (Arzarello et al., 2002 p.66). Dragging facilitates the cognitive transition from the perceptual level to the theoretical level. They identify "wander dragging" as that which is random. In the process the student is searching for regularities or some interesting result that occurs when some object is dragged. One might argue that this is synonymous with an expressive activity in the Zbiek et al. (2007) classification. For example the student might just drag a point in a secant-tangent applet with the intention of seeing what happens when the two points coincide. The second type of dragging that they identify which is relevant to this study is "dummy locus dragging". In this case the student moves a basic point so that the drawing keeps some discovered property. The point which is being moved follows a particular path whose locus may not be explicit to the student. The dummy locus dragging marks the construction of a conjecture. To confirm this conjecture the student then performs the dragging test (Arzarello et al., 2002). Thus the purpose of a dragging test involves a search, not for results but for confirmation (or disconfirmation) of some result (Zbiek et al., 2007).

Distinguishing between student activity and behaviour while they are using cognitive tools is important for researchers (Zbiek et al., 2007, Arzarello et al., 2002). They contend that a lack of distinction in student activity might prevent a researcher from effectively explaining potentially conflicting results.

2.5 Understanding teacher involvement in computer learning environments

Anyone attempting to describe the roles of technology in mathematics education faces challenges similar to describing a newly active volcano (Kaput, 1992). Kaput further adds that the mathematics education landscape has been changing due to forces acting on it and within it simultaneously. It is my view that the teacher is one of the key role players responsible for changes, or the lack of them, in mathematics classrooms. Zbiek et al. (2007) concur with this assessment, arguing that students' mathematical behaviour is influenced by the ways in which the teacher chooses to engage them in mathematical activity. It is therefore vitally important to understand how and why teachers choose to use a particular piece of technology in their work. Significant strides have been made in a bid to categorise teachers' use of technology since Kaput's (1992) assertion. Zbiek et al. (2007) have identified privileging and pedagogical fidelity as two lenses with which teacher involvement can be scrutinised.

Pedagogical fidelity is the degree of match between a particular cognitive technology and a teacher's practice and beliefs. After reviewing a number of studies Zbiek et al. (2007) concluded that pedagogical fidelity as a construct can be used to explain seemingly disconnected and complicated sets of findings from studies of teachers' thinking, planning and use of technology in mathematics classrooms. For instance, teachers evaluated the use of spreadsheets, graphing utilities and geometry programs as successful if they promoted some investigation (Ruthven & Hennesey, 2002 as cited in Zbiek et al., 2007). Thus the technology demonstrated high pedagogical fidelity because it supported the investigation component that the teachers deemed to be an important objective of the teaching process. In another study Zbiek (1995 as cited in Zbiek et al., 2007) found that the teacher adjusted the tasks in an attempt to negate the capabilities of the technology that they deemed to undermine some of the skills they intended to develop in their students. Thus in such cases it can be argued that the technology showed a low degree of pedagogical fidelity. It is worth noting that if the

technological tool displays a high degree of pedagogical fidelity the teacher will tend to stick with it instead of switching to traditional forms of instructions such as using the textbook.

As the students engage with technology related tasks the teacher assumes different roles depending on whether the task is exploratory or expressive in nature. In general the teachers' practices tend to be compatible with constructivist views on teaching and learning. If the activity is exploratory the students are involved primarily in doing and the teacher takes a back seat and plays the role of manager overseeing the task. If the task is of an expressive nature the teacher acts as the manager of reflection and devotes his or her time to ensuring that the students create something which is worth focusing their reflection on the mathematical concepts involved (Zbiek et al., 2007). The teacher's role in constructivist learning environments will be discussed further under constructivism (see section 3.1).

Privileging is a construct coined by Wertsch (1990 as cited in Zbiek et al., 2007) to describe how teachers intentionally or unintentionally place a priority on certain things in their practice. In an attempt to understand what aspect of the technology a teacher prioritises the black box/white box principle might be useful. Primarily black box usage of computer technology refers to cases where the students make use of the computer program without an understanding of the mathematical operations that they are asking of the computer. At face value such usage of computer software may be detrimental to students (Pimm 1995, Lagrange 1999 as cited in Zbiek et al., 2007). These scholars argue that it hampers technique development in certain aspects of symbolic representation. There is no consensus in the research fraternity as other authors argue that the black box use of technology may be productive. For instance, Heid (1988) found that students were able to develop conceptual understanding of calculus topics when manipulation details were assigned to the computer. These students were able to engage in investigations without first mastering by hand the routine calculations performed by the symbolic manipulation program. For instance, the students in the experimental group demonstrated a deeper understanding of the concept of the derivative. These students could give a broader array of appropriate associations when explaining the concept of the derivative than those who were compelled to first master the computational skills. Heid (1988) concludes that using the computer generated graphs with the experimental group enabled them to develop the more meaningful understanding.

Zangor (2000 as cited in Zbiek et al., 2007) contends that if black box technology use is privileged, then the teacher's questioning protocol is of importance. Through careful implementation of the computer program the teacher draws the students' attention away from the workings of the tool to the mathematical justifications of the work. This study showed that the black-box use supported justification activity. In the white box computer usage the students are actively aware of the operations that the computer program is performing. Cowell and Prosser (1991 as cited in Tall, 1992) reported that there was no clear benefit towards conceptual understanding when students took a computer programming course to complement their symbolic paper and pencil manipulations. The programming course enabled the pupils to learn by telling the computer how to carry out the required algorithms. This is equivalent to white box usage of the computer technology. The study found that students agreed that the computer assignments were well integrated but disagreed that the computer enhanced their interest in the course material. Furthermore, comparing the scores on examinations with scores on the previous non-computer course showed virtually identical median and quartile scores.

As a guiding principle towards computer usage in the classroom Buchberger (1989) suggests the following;

- if the area of study is new to the students, then black box usage of the computer technology should be discouraged. Students must be afforded the opportunity to explore the area thoroughly. They must be aware of, among other things, the basic concepts, theorems, proofs and hand calculations.

- If the area of study has been thoroughly dealt with, that is the hand calculations of simple examples have become routine and those of complex examples have become difficult to deal with, then the students must be encouraged to make use of the algorithms available in the symbolic software systems.

There is evidence that computer use in general is beneficial to mathematics learning. Tall (1992) argues that a student plus a manipulation tool can be more successful in conceptual and computational tasks than a student who works in a traditional manner.

In this study the participants had used the tangent method to calculate the gradient of a curve at a point $(x;y)$ in their Grade 10 year. It is, therefore, reasonable to assume that as the students dragged point A $(x;y)$ on a curve using the *Geogebra* applets they were aware of the

mathematics involved in producing the trace of the gradient at different values of *x*. The point I am making is that the use of *Geogebra* in this instance followed a white box principle. The hand calculations are of importance to me as the teacher, hence this played a significant role in the choice of how the software was used.

2.6 A brief history of the derivative

The derivative has a long and illustrious history spanning a period of over two hundred years. Its development was not always characterised by certainty but intuition played a significant role. Its origins can be traced back to the ancient Greeks who pioneered the study of curves, particularly that of conics and circles (Grabiner, 1983). This was followed by the invention of analytical geometry by René Descartes and Pierre Fermat, French mathematicians working independently in the 1630s. The analytical geometry meant that curves could be represented by equations (Grabiner, 1983). In the early 17th century there was a great deal of interest in maxima and minima problems and also finding the tangents of curves. For example, Fermat wanted to calculate the shortest path travelled by a ray of light as it moved from one medium to another (Grabiner, 1983), and so an effective method for this computation was required. Clearly the methods that had been devised by the Greeks up to this time were no longer sufficient to deal with the new curves that were being discovered and new tools were thus needed.

The evolution of the derivative was in response to such practical problems that mathematicians and scientists were grappling with during those times. In the 1630s Fermat devised a method for finding extrema and applied it to optics. He assumed that a ray of light which goes from one medium to another always takes the quickest path. His solution of such problems yielded Snell's law of refraction (Grabiner, 1983). Linked to the issue of extrema and the discovery of new curves was the tangent concept. Fermat's method of finding the tangent, in today's notation, described herein from Grabiner (1983, p.198) is as follows: given the equation of the curve $y = f(x)$ its tangent was considered to be a secant for which the two points come closer and closer until they coincide. The slope of the secant was computed from the expression $\dfrac{f(x+h) - f(x)}{h}$

The diagram (Figure 1) shows that when the quantity *h* vanishes the secant becomes the tangent. To find the minimum or maximum of the curve the expression of the tangent was asked to equal zero.
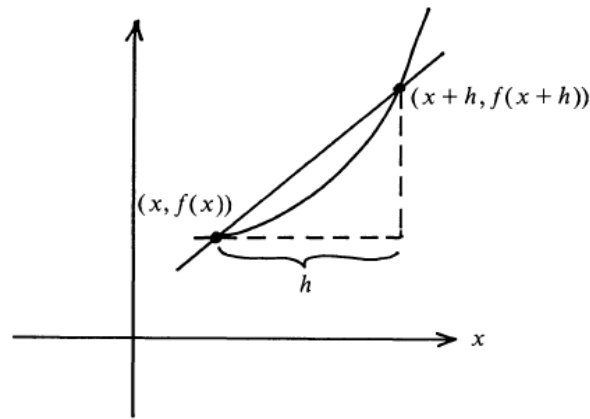
Figure 2.2: Secant to tangent

In his original expression Fermat used *E* instead of *h* and he faced a lot of criticism from his counterparts, particularly Descartes, for the vanishing terms containing the variable *E* (Kleiner, 2001). However, the method provided the required solutions and hence it withstood the criticism. In 1660 the relationship between the problem of extrema and that of the tangent was clearly understood Grabiner (1983).

It can be argued that Fermat had not realised the relationship between the area under the curve and his process of computing the derivative. This discovery is credited to Sir Isaac Newton, an Englishman, and Gottfried Leibniz, a German. The two, working independently of each other in the later third of the 17th century, are arguably responsible for the Calculus as we know it today. They created the symbolic and systematic method of analytic operations to be performed by strictly formal rules independent of geometric meaning (Rosenthal, 1951). To Newton the derivative, which he called the fluxion, had a great potential for application in the physical world. He argued that everything in the world changes as the time passes and this was pivotal in his formulation of the laws of motion (Schechter, 2006). Grabiner (1983, p.199) gives a detailed account of Newton's conception of the fundamental theorem of Calculus. From his work Newton concluded that derivatives are involved in areas as well as tangents. It can be argued that by inventing the Calculus Newton and Leibniz at this stage not only discovered the derivative, but the fundamental law of calculus, namely that differentiation and integration are inverse operations.

Like Fermat, Newton and Leibniz faced a lot criticism for the vanishing quantities (infinitesimals) because they too did not obey the Archimedian axiom. This axiom was the basis of the Greek theory of ratios, which was in turn the basis of algebra (Grabiner, 1983). It stated that, given any two real numbers *a* and *b*, there exists a positive integer *n* such that *na*

> *b*. But if *a* is an infinitesimal and *b* = 1, then *na* < 1 for every positive integer *n* (Kleiner, 2001, p.153). However Leibniz's notation and the fundamental theorem of calculus according to Grabiner (1983) had too much power and also easily withstood the criticism. Furthermore, Newton's laws of motion and Hooke's law of elasticity were practical examples of the effectiveness of the derivative.

In 1715 Brook Taylor invented the Taylor series and it became a powerful tool for solving differential equations. In particular, Joseph-Louis Lagrange used it to great effect to improve on Newton's limit concept in an attempt to quell disgruntlement with the fluxion. In his work Lagrange showed that the derivative was not a different being, but it was still a part of the original function (Grabiner, 1983, p.203). Lagrange worked with finite series and he erroneously assumed that his findings would be applicable to infinite ones. It was in the 19th century that Augustin Cauchy, a French mathematician, pointed out that this assumption was incorrect and he put forward his own definition of the derivative (Grabiner, 1983). He defined it as "....the limit, when it exists, of the quotient of differences $\dfrac{f(x+h)-f(x)}{h}$ as *h* goes to zero"(Cauchy, 1823 as cited in Grabiner, 1983, p.204). For Cauchy the notion of the limit was now the underlying concept of calculus. During this time there was also a realisation that inequalities had a greater role to play in giving definitions (Kleiner, 2001). Accordingly Cauchy, in collaboration with Karl Weierstrass, used the algebraic inequality characterisation every time he needed a limit. They introduced the $\varepsilon - \delta$ definition of a limit as we know it today in the 1840s (Dunham, 2005 p.14). After Cauchy and Weierstrass calculus was now viewed as a rigorous subject with good definitions and theorems. The proofs of these theorems were now based on the definitions rather than a set of powerful methods used by their predecessors (Grabiner, 1983).

The foregoing discussion chronicles the development of the derivative from Fermat to Weiestrass, a period stretching over two hundred years. The inability to define the limit rigorously prolonged the developmental process. The provision of proofs by Cauchy and Weierstrass and their limit definition resolved the problems that accompanied the developmental process. Grabiner (1983) sums this process nicely. Fermat used the derivative implicitly. Newton and Leibniz discovered it while Taylor and Euler developed it. Lagrange gave it its name and characterised it, ultimately Cauchy and Weierstrass rigorously defined it.

2.7 Contemporary issues in the teaching and learning of the derivative

Leading scholars in the research of teaching and learning of calculus concepts all seem to acknowledge that students create their own meanings. For instance, Dubinsky (2010) argues that an individual's mathematical knowledge is his/her ability to respond to perceived mathematical problem situations and their solutions by (re)constructing mental structures required to deal with each situation. Harel and Tall (1991 as cited in Biza, Christou & Zachariades, 2008) identify two ways in which students deal with new knowledge, namely expansive and reconstructive generalisation. In expansive generalisation students extend their existing cognitive structures without changing their current ideas. It is generally the default course of action as it is perceived to be easier in comparison to reconstructive generalisation. In this second generalisation the existing concept image has to be changed radically so that it can be applicable in a broader context. It is worth noting that these positions are similar to those held by earlier proponents of constructivist learning such as Piaget and Ausubel.

An individual does not learn mathematical concepts directly but must apply mental structures to a situation to make sense of it (Piaget, 1964 as cited in Maharaj, 2013). If the required mental structures are missing then understanding the mathematical concept is near impossible. It is within reason to argue that the goal of teaching should be to present students with opportunities to develop the relevant mental structures. Ausubel's (1963 as cited in Woolfolk, 2007) advance organisers may be useful in helping students develop the required mental structures. He argues that advance organisers, such as a broad introductory statement, direct students' attention to what is important in the coming material and also highlight the relationships in the ideas that will be presented. Furthermore, they serve to remind the students of what they already know. Such a teaching strategy may encourage the students to act on the new information and create new generalisations as pointed out by Harel and Tall (1991 as cited in Biza et al., 2008). This study intended to contribute to this discourse by making use of the computer software *Geogebra*. Its use in an exploratory manner may plant the seed for analytical thinking.

Research studies detail the difficulties that that students grapple with as they attempt to learn calculus concepts (for example Biza et al., 2008; Maharaj, 2013; Rivera-Figueroa & Ponce-Campuzano, 2012; Tall, 1993; Park, 2013; Baker, Cooley & Trigueros, 2000; Pillay, 2008). The difficulties are attributed to the curriculum, the teaching and the cognitive capabilities of the students. If one takes into account the history of the derivative and the vanishing

quantities associated with its early development, it is perhaps not surprising that students today battle with the limit concept. In addition, any discussion about the derivative inevitably brings the limit concept into play. Tall (1993) argues that calculus represents the first time in which the student is confronted with the limit concept involving calculations that are no longer performed by simple arithmetic and algebra. Indirect arguments come into play, in which the students deal with the infinite concept. He further argues that the language used in calculus may also compound student difficulties. Terms like "limit", "tends to" and "approaches" have powerful colloquial meanings that conflict with the formal concepts. Teachers tend to avoid reference to the language of limits in the initial stages and thus provide students with a simplistic view of the concept. However, Tall (1993) cautions that such approaches may result in students constructing simple long term representations of the concept under discussion and they may show reluctance to reconstruct these images in later stages. Furthermore, he contends that firmly held concept images can prove notoriously difficult to dislodge even when they conflict with formal definitions.

The alternative would be to present the language used in the early stages, for example in the Greek curriculum the formal $\varepsilon - \delta$ method is taught early. This presents its own difficulties as it may reduce incidence of the infinitesimal methods (Tall, 1992). An informal approach is likely to involve factors which have the potential to conflict with any formal approach whilst a formal approach may prove too difficult a starting point. A lack of the required mental structures may also be the reason that students find it difficult to conceive the limit concept (Maharaj, 2013). In his findings Maharaj tends to agree with Tall in that he argues that teaching should focus on verbal and graphical approaches to finding limits. He suggests that an equal emphasis may result in highly developed schemas that will stand the students in good stead in dealing with the limit concept.

The importance of language is further emphasised by Park (2013). She reports that word use plays an important role in students' understanding of the derivative. Students referred to the derivative as either a function, tangent line or a point specific object and did not seem to fully appreciate their relation (Park, 2013). Additionally, Park contends that the colloquial use of the derivative compounds the students' difficulties as they attempt to view it as a function. For instance, it is common practice to ask "is the derivative positive?" or to instruct the students to "take the derivative". Such indiscriminate use of terminology arguably further ensures that students battle to conceptualise the derivative as a function. Park also found that

this influences students' ability to solve problems that require an awareness of the relation between the three notions of the derivative.

To mitigate such situations and strive towards ensuring that students fully understand that the derivative is not just a tangent line or a point specific object, she advocates teaching that emphasises that each value of $f'(x)$ represents the slope of the tangent line. Teachers must graph $f'(x)$ and $f(x)$ on two transparent sheets and then overlay them to highlight the relationship between the two (Park, 2013). This study investigated whether such a teaching strategy, using Geogebra, develops the understanding of the derivative as a function thus moving away from just viewing it as a point specific object. Question five in Appendix 1 specifically deals with this issue.

High school curricula, such as the Cambridge International Exams (CIE) and the Independent Examinations Board (IEB), do not place a lot of emphasis on the limit concept. The students are expected to make use of differentiation rules to routinely find the derivative of a given function. Examinations are biased towards testing for proficiency in such skills to the detriment of conceptual understanding. The net result has been that on the few occasions that the pupils have been asked to demonstrate conceptual understanding they have been found wanting (for example Baker et al., 2000). Students' concentration on procedural aspects that are set in examinations may have a long term effect on their attitudes in future calculus courses. It is possible that procedural, technique-oriented secondary school courses in Calculus may predispose students to attend to more procedural aspects of the college courses (Ferrini-Munday & Gaudard, 1992 in Tall, 1993, p.4). Students use different arguments in situations that are technique-oriented as such arguments allow them to keep disconcerting conflicts in separate compartments (Tall, 1993) and this prevents conceptual understanding.

Mathematics educators have decried this emphasis on procedural understanding and there seems to be a consensus within the community that teaching should focus on encouraging conceptual understanding. Students who learn from reform curricula consistently outperform those from traditional curricula in tests of conceptual understanding and problem solving (Schoenfeld, 2002). Ideas in mathematics are characterised by deep structure rather than visible appearances or known functions like everyday objects (Dienes, 1963 as cited in Durmus & Karakirik, 2006). Hallet (1991 as cited in Tall, 1992) suggests that wherever possible topics should be taught graphically, numerically as well as analytically. He argues

that such a balanced curriculum will enable the students to see each major idea from several angles. This view is supported by Robert and Boschet (1984 as cited in Tall, 1993) who reported that successful students were those who could easily alternate between forms of representation. Additionally, dynamic computer graphics may be used to help students to see concepts such as local straightness (Tall, 1993). As previously mentioned, this study seeks to contribute to the pedagogical discussion of how a graphical approach, using *Geogebra*, can be used to introduce differentiation.

The derivative has a geometric origin and a deeper learning of its properties and applications for the study of functions may be achieved if teachers present lessons within a highly graphic context (Rivera-Figueroa & Ponce-Campuzano, 2012). According to the Van Hiele learning theory, visualisation and informal reasoning are prerequisites for developing higher and abstract reasoning (Burger & Shaughnessy, 1986). It is therefore appropriate that the concept of the derivative (gradient function) is introduced using visual mediators such as graphs. The differentiation process is usually presented in schools as one of obtaining a formula (the derivative function) from another source (the function). Such practices, Rivera-Figueroa and Ponce-Campuzano (2012) argue, will lead to misconceptions in students whereby they calculate incorrectly a formula for the derivative without due care to the differentiability of the function at the point of interest. The derivative instead should be taught as a process of obtaining the derivative of a function at each point. The derivative function is obtained as a result of this process rather than as a result of applying a set of algorithms to a formula (Rivera-Figueroa & Ponce-Campuzano, 2012 p.288). The *Geogebra* applets used in this study clearly demonstrate this concept. The coordinates of the dynamic point S (see Appendix 3) are (x ; slope of the function at x) and the trace then generates the corresponding path of the derived function.

While Rivera-Figueroa and Ponce-Campuzano (2012) advocate the use of graphics to develop the ideas and properties of the derivative, they also warn that carelessly interpreting these graphs may lead to the genesis of some misconceptions. The paper and pencil graphs that can be drawn or visualised by students are in most cases far from showing the generality of the geometric and analytic situations (Rivera-Figueroa & Ponce-Campuzano, 2012). For instance the graph of $y = x^3$ is usually used to demonstrate that the existence of a critical point (where the first derivative is zero) does not always imply that a maximum or minimum turning point exists. Associated with this graph is the conclusion that at the critical point there

exists an inflection point. This may lead students to believe, incorrectly, that inflection points exist at critical points only (Rivera-Figueroa & Ponce-Campuzano, 2012). Counter examples of functions that have critical points and have neither a maximum nor a minimum nor an inflection point must also be used with the illustrations often shown in textbooks.

Biza et al. (2008) found that students' early conceptions of the tangent have an impact on how well they fare in analysis courses. Students first encounter the tangent as a line that has one point of contact with the circle and new, often incorrect, structures are dominated by this circle tangent (Biza et al., 2008). They found that students' tangent perspectives may be classified into three categories, namely geometric global, intermediate global and analytical global. In the geometric global perspective the dominant thinking is that the tangent only has one common point of contact with the curve and a tangent can exist at an edge point (cusp). The thinking that characterises the intermediate local perspective is that a tangent line can have more than one common point with the curve, but there exists a neighbourhood around the tangency point where there is no other common point between the line and the curve. At the analytical global perspective level students are able to articulate that the curve could have more than one common point with the curve, exist at an inflection point, could coincide with the curve and does not exist at a cusp. They further argue that the teaching of the tangent in an analysis course could be facilitated through their model to ensure student progression from the geometric global to the analytical global perspective. It seems to me that the graphical context advocated by Rivera-Figueroa and Ponce-Campuzano (2012) may also help students immensely. Such representations will force them to confront their current perspectives and encourage them to reconstruct their mental images leading to a progression in thinking.

Pressing on with the notion of introducing differentiation within a graphical context, the Baker et al. (2000) project is of particular relevance to this study. Working with undergraduate students they found that the participants had difficulties coordinating information required to solve a non-routine calculus graphing problem. They sought to categorise the development of such a coherent calculus graphing schema. In the process they observed that students will at times rely heavily on one given condition and ignore others in an attempt to solve the graphing problem. The calculus graphing schema for a student is defined by a combination of the student's levels of development in understanding the derivative, limits, continuity as well as precalculus ideas (Baker et al., 2000). The ability to

coordinate given conditions across the intervals of the domain signifies the maturity of the schema (Baker et al., 2000). They further argue that the development of the graphing schema is best described by the interaction of the domain-interval schema and the condition-property schema. The condition-property schema involves understanding each analytical condition as it relates to a graphical property of the function and coordinating these conditions. Such conditions include information about the first and second derivatives, limits of the function and continuity of the function. The domain-interval schema involves understanding the interval notation, connecting contiguous intervals, and coordinating the overlap of intervals (Baker et al., 2000)

Could the interaction with the *Geogebra* applet in this study enable the students involved to successfully graph a function from its derivative graph and vice versa? In other words, will this interaction help students better coordinate the necessary conditions and properties thus overcoming the difficulties identified by Baker et al. (2000)?

2.8 The role of inductive (plausible) reasoning in mathematics

One of the main questions that were investigated in this study is; can *Geogebra* aid students in discovering the power rule for differentiating elementary polynomials? The students initially explored specific cases on the provided *Geogebra* applets and then attempted to use their experiences to formulate a rule for a general elementary polynomial of the form $f(x) = ax^n$. Such a process of mathematical thought is known as inductive reasoning. It is an argument that begins with specific cases leading to a general statement (De Villiers, 1992). The conclusion is often informed by experience or by experimental evidence (Borwein, 2012). In addition, Borwin points out that the conclusion goes beyond the information given in the premises and does not follow necessarily from them, for instance, the prevalence of white swans in varying places provides a strong basis for the incorrect conclusion that all swans are white.

Is there a place for such a seemingly misleading form of reasoning in mathematics? Polya (1954) goes as far as labelling the plausible reasoning hazardous, controversial and provisional. In the same breath he posits that it is via the plausible reasoning that we learn new things about the world. Our knowledge outside mathematics consists of conjectures and the standards of our plausible reasoning are fluid and there is no theory of such reasoning that could be contrasted to demonstrative logic (Polya, 1954). Demonstrative logic secures

mathematical knowledge: it is safe, beyond controversy and final. However, unlike plausible reasoning, it is incapable of producing new knowledge about the world around us. Polya further points out that demonstrative reasoning is characterised by rigid standards that are clarified by logic. A mathematical proof is an example of demonstrative reasoning. Demonstrative logic can be equated to deductive reasoning. It is reasoning that starts from an accepted generalisation to specific instances (De Villiers, 1992). According to Borwein (2012) deduction is a process of reasoning in which a conclusion is drawn from premises so that it cannot be false when the original premises are true.

So is there a place for inductive (plausible) reasoning in mathematics despite its short comings when pitted against deductive reasoning? It seems that the answer to this question is a definite yes. Polya (1954) laments the presentation of the finished mathematical product as purely demonstrative, consisting of proofs only. He argues that mathematics in the making resembles any other human knowledge in the making. The final products, namely proofs, are a result of guess work. One has to guess a mathematical theorem before proving it. The illustrious history of mathematics is filled with examples of discoveries made on the basis of inductive reasoning. The history of the derivative, discussed earlier, is a classic example of inductive reasoning at play. The properties of the numbers known today have been mostly discovered by observation, long before their truth has been confirmed by rigid demonstrations (Euler as cited in de Villiers, 1992). If mathematics describes an objective world just like physics, then there is no reason why inductive methods should not be applied to mathematics just as in physics (Gödel as cited in Borwein, 2012). In addressing the question posed in this section Polya (1981 as cited in Borwein, 2012) boldly states that intuition comes to us much earlier, and with very little outside influence, than formal arguments that we cannot really understand. In addition, if the learning of mathematics is to mirror its invention, it must have a place for guessing, a place for plausible inference. In teaching high school age students intuitive insight must be emphasised more than, and long before, deductive reasoning (Polya, 1954). He further argues that in such learning, teaching must concentrate on encouraging students to make the distinction between a more reasonable guess from a less reasonable one.

Results obtained through inductive reasoning may satisfy what Harel (2013) terms an intellectual need. Intellectual need has to do with disciplinary knowledge being created out of people's current knowledge through engagement in problematic situations conceived as such

by them (Harel, 2013 p.122). It is a necessary condition for the construction of knowledge, a yearning to attain equilibrium. He further identifies five categories of intellectual need, of which the need for certainty and the need for causality have a bearing on this study. The need for certainty is at the focal point of human endeavour culminating in the determination of whether a conjecture is a fact. Fulfilment of this need through whatever means perceived as appropriate by an individual results in him or her gaining new knowledge about the conjecture. Empirical proof schemes (Harel, 2013; Lin, Yang, Lee, Tabach & Stylianides, 2012) such as the use of graphing software like *Geogebra* may lead to conviction about the truth of a conjecture.

Having established the truth of a conjecture, an individual may want to know why the conjecture is fact. Thus, the need for causality is one's desire to explain, to determine a cause of a phenomenon (Harel, 2013). Attaining certainty through the use of undesirable proof schemes such authoritative teacher justifications, Harel (2013) argues, has dominated students' reasoning impeding, their ability to make deductive proofs. He accordingly agitates for pedagogical practices that will encourage students to focus more on cause of phenomena rather than certainty.

2.9 Teaching conjecturing in the classroom

The literary evidence in the foregoing section overwhelmingly favours a curriculum that encourages intuition amongst mathematics students. Conjecturing and proving have a symbiotic relationship in human activities, not only for discovering and verifying mathematical knowledge but also for other educational purposes such as initiating mathematical thinking (Lakatos, 1976 as cited in Lin et al., 2012). The challenge is to come up with the tasks that will tend to these requirements. Lin et al. (2012) note that teachers have not been able to include conjecturing tasks in their teaching partly because of a lack of clarity when it comes to task design principles. Harel (2013) concurs and laments the use of undesirable proof schemes employed by teachers and students to draw certainty in mathematics classrooms. Lin et al. (2012) take it a step further and identify that an important learning goal that could be served by clearly spelt out design principles is that of "proof schemes".

In response to a lack of design principles Lin et al. (2012) offer a framework for designing conjecturing tasks in which they point out three classes of proof schemes, of which the

empirical proof scheme is of relevance to this study. It includes inductive and perceptual proof schemes and is ideal for tasks that aim to teach conjecturing rather than formal proving. In this study the use of *Geogebra* was designed to enable students to come up with a conjecture for differentiating elementary polynomials on the basis of the perceptual evidence obtained from the applets. The main aim of all science is to first observe phenomena, then to explain them and finally to predict (Gale 1990 as cited in Lin et al., 2012). Accordingly, Lin et al. (2012) propose that any task designed to engender conjecturing and proving should provide opportunities for students to observe, construct, transform and reflect.

A conjecture is a result of constant observation (Harel & Sowder 1998). For Lin et al. (2012), observation refers to activities that involve the intentional analysis of specific cases in order to understand and/or make a generalisation about the cases. Opportunities for observation may include the assessment of finite examples in which a student is asked to systematically observe a particular example (Lin et al., 2012). For example, in the current study there were specific instructions that the pupils had to follow with a finite number of polynomials with a view to generalising the result.

Arzarello et al. (2002) further encourage the use of technology in classrooms by arguing that it may facilitate students' observation opportunities dynamically. Marton and Booth (1997 as cited in Watson & Mason 2006) affirm the importance of observation by arguing that learners cannot resist creating generalisations by imposing patterns on data. Additionally they point out that the starting point of any sense-making is the discernment of variations within it. Accordingly tasks that carefully display constrained variation are generally likely to result in progress in ways that unstructured sets of tasks do not (Watson & Mason 2006). The dimensions of variation will be the basis on which students make their conjectures. Keeping the coefficient of the terms in x constant in the linear equations in this study (see question 1 in appendix 1) was aimed at ensuring that not too many things were varying at the same time. The intention was to make sure that the pupils realised that the resulting graph for the gradient function was $y = m$ and get them to deliberate on whether this would also apply to quadratics.

Learning activities that promote conjecturing must also provide an opportunity for students to engage in construction (Lin et al., 2012). Primarily they propose that when teaching fundamental rules or structures, students must be given the opportunity to associate new

mathematical knowledge with prior knowledge and this may result in them formulating conjectures. One of the benefits of this approach is that it may expose misconceptions should students construct incorrect conjectures. Conjecturing is not an isolated occurrence but is mostly connected to a person's prior knowledge (Lin et al., 2012). Additionally they proclaim that the construction principle provides opportunities to make new conjectures based on the newly constructed knowledge. For instance, having explored the graphs of the different polynomials, in this study, students may be able to implicitly make a conjecture about the shape of the graph of the derivative before dragging the point *A*. Furthermore, such conjectures may be helpful in dealing with questions similar to item 5 and 6 (see Appendix 1)

Lin (et al., 2012) also argue that conjecturing can be promoted if a task provides an opportunity for reflection, they call this the reflection principle. The conjectures that the students may come up with after observing several examples can be incorrect and meaningless (Lin et al., 2012). Opportunities for reflection ought to be integrated into the task as they provide students with a chance to further explore mathematical problems and improve their conjectures. In this study, students will have the *Geogebra* applets at their disposal to check the validity of their generalisations.

Chapter 3: Theoretical framework

"Tell me, and I will forget. Show me, and I may remember. Involve me, and I will understand."

Confucius

3.0 Introduction

Educationists and psychologists over the years have sought to understand how pupils learn and their quest has yielded very vibrant debate. Although there has been no consensus, the theories that have been put forward have, to some extent, illuminated the penumbra of teaching and learning. Constructivism is one such learning theory that has been at the fore of education reforms, and it has rich and significant consequences for mathematics education (Lerman, 1989). Accordingly, this chapter is going to discuss constructivism with a view of positioning this study in the current literature. It will culminate in a discussion of Experiential Learning Theory (ETL), the theory that informed this study.

3.1 Constructivism

The early roots of constructivism are from the educational theories of John Dewey and Jean Piaget (Brown & Green, 2006). Dewey set the foundation for constructivism by identifying inquiry to be a fundamental part of learning. The Piagetian concepts of schema, assimilation and accommodation also contributed significantly in shaping constructivism. Constructivism is an educational philosophy that contends that learners are active in the construction of their own knowledge and that social interactions are important to knowledge construction (Bruning, Schraw, Norby & Roning as cited in Woolfolk, 2007). Slavin (1997) posits that for students to understand and be able to apply knowledge, they must work to solve problems, to discover things for themselves and to wrestle with ideas. Additionally, constructivism argues that students bring their own ideas to the learning situation and they continually refer to these ideas as they attempt to interpret the teacher's instructions (Cobb, 1988). The start of the learning process is signalled by a cognitive disequilibrium between the student's prior knowledge and the new knowledge; overcoming this contradiction results in new constructions (Piaget, 1975 as cited in Balacheff, 1991).

Mathematical meanings are socially constructed and culturally situated, hence the need for social interactions (Cobb, Yackel & Wood, 1992). A consideration of the major developments of mathematics, for example the history of the derivative discussed in section 2.6, reveals that they were in response to the needs of the particular generations and the ideas

were influenced by society. It is on this basis that mathematics teaching must contextualise, in every possible way, the mathematical concepts being taught. Mathematics cannot be understood outside its history (Ernest, 1985).

According to Lerman (1989, p.211) constructivism is defined by a widely accepted hypothesis which states that knowledge is actively constructed by the cognizing subject, not passively received from the environment. Constructivism also provides the mathematics education community with insights concerning how children learn mathematics. Furthermore, it guides us to use instructional strategies that begin with children rather than ourselves (Van de Walle, 2004). In describing constructivist compatible instruction, Ravitz et al. (1998 as cited in Zbiek et al., 2007) note that it emanates from the theory of learning that suggest that understanding arises only through prolonged engagement of the learner in relating new ideas and explanations to the learner's own prior beliefs. They further point out that a student's ability to utilise procedural knowledge comes only from experience in working with concrete problems. Such concrete problems should provide experience in deciding how and when to call upon each of a diverse set of skills.

The child's prior knowledge is important for understanding in a constructivist environment. It determines the type of knowledge that will be gained from an experience. This approach to teaching provides the teacher with an opportunity to look at learning from the child's perspective, a chance for the teacher to be in the child's shoes (Olivier, 1989). Misconceptions thus form an important starting point for teaching. Olivier (1989) reports that students learning about multiplication in a context that only involves whole numbers develop the misconception that multiplication will always result in a bigger value. Such misconceptions should be challenged by the use of counter examples (eg Almeida, 2010; Bell, 1993; Olivier, 1989). Almeida (2010) argues that teaching is more effective if it focuses on identifying, challenging and ameliorating the misconceptions. Essentially, according to constructivism, students should be given authentic tasks. Such tasks should allow them to experimentally explore, observe, make conjectures and construct generalisations, which they can be encouraged to support by providing a logical explanation or explaining why a generalisation is true (Slavin, 1997).

Referring specifically to mathematics teaching and learning, Cobb (1988, p.89), a constructivist, notes;

> "A fundamental goal of mathematics instruction is or should be to help students build structures that are more complex, powerful, and abstract than those that they possess when instruction commences. The teacher's role is not merely to convey to students information about mathematics. One of the teacher's primary responsibilities is to facilitate profound cognitive restructuring and conceptual reorganization."

One can argue that tools such a *Geogebra* can be utilised to facilitate such cognitive restructuring.

3.3 An overview of Experiential Learning theory

Experiential learning theory (ETL) views learning as:

> "the process whereby knowledge is created through the transformation of experience. Knowledge results from the combination of grasping and transforming experience". (Kolb, 1984 p.41)

Kolb further argues that it provides a holistic model of the learning process. The theory is called "Experiential Learning" to emphasise the central role that experience plays in the learning process (Kolb, Boyatzis & Mainemelis, 2001). It explicitly shares an important notion with constructivism; that the individual is actively involved in his or her construction of knowledge. Knowledge is continuously derived from and tested out in the experiences of the student (Kolb, 1984). ETL contends that learning is best conceived as a process, not in terms of outcomes (Kolb, 1984). In addition, Kolb (1984, p.26) points out that ideas are not fixed and immutable elements of thought but are formed and re-formed through experience. Everyone brings to a learning situation ideas that are less articulate about the topic of discussion. The teacher's job is not only to implant new ideas but to also dispose of or to modify the old ones (Kolb, 1984). Additionally he contends that the education process should begin by bringing out the student's beliefs and theories, testing them and then integrating the more refined ideas into the person's belief system. This is a central tenet in constructivist compatible instruction. Furthermore, the origins of ETL and constructivism can both be traced to the intellectual work of John Dewey and Jean Piaget (Kolb, 1984). Although it originally addressed adult development, it has also found its way into formal education (Kolb

et al., 2001). This is because the process of learning from experience is present in human activity everywhere all the time (Passarelli & Kolb, 2011).

3.4 Characteristics of Experiential Learning

As pointed out in the preceding section, ETL combines the intellectual work of several foundational experiential learning scholars. Taken together, Kolb (1984) argues that the learning theories of the different scholars form a unique perspective on learning and development that can be modelled around six propositions. The following propositions are taken from Kolb (1984, p.26) and Passarelli and Kolb (2011).

Learning is best conceived as a process, not in terms of outcomes. Learning is indeed punctuated by knowledge milestones but it does not end at an outcome, nor is it always shown by a performance. It is an emergent process whose outcomes represent only a historical record, not knowledge of the future. Learning occurs through the course of connected experiences. This study acknowledged the importance of connecting experiences. It associated the concrete experience of $f(x)$ and $f'(x)$ from the Geogebra applets with questions that required the pupils to predict the shape of the gradient function from that of the original and vice-versa (see Appendix 1, questions 5 and 6).

All learning is re-learning. Everyone enters a learning situation with more or less articulate ideas about the topic at hand (Kolb, 1984; Polya, 1954). Learning should therefore be a process that draws out the learners' beliefs and ideas about a topic so that they can be examined, tested and integrated with new, more refined ideas. According to Piaget this proposition is known as constructivism; individuals construct their knowledge of the world based on their experience as well as their present ideas (Passarelli & Kolb, 2011).

Learning requires the resolution of conflicts between dialectically opposed modes of adaptation to the world. Kolb (1984) contends that learning is a tension and conflict filled process. Additionally, he points out that new knowledge, skills or attitudes are achieved through four modes of experiential learning. These will be discussed in the next section. In the process of learning one is called upon to move back and forth between opposing modes of reflection and action, and feeling and thinking (Passarelli & Kolb, 2011). In learning situations the student moves in varying degrees from actor to observer and from specific involvement to general analytical detachment (Kolb, 1984).

Learning is a holistic process of adaptation. Learning draws on the total function of the individual in an integrated fashion. This involves thinking, feeling, perceiving and behaving (Kolb, 1984). Furthermore, it includes other specialised models of adaptation from the scientific method to problem solving, decision making and creativity. When learning is viewed as a holistic process, it provides conceptual bridges across life situations such as school and work, thus portraying learning as a continuous, lifelong process.

Learning involves transactions between the person and the environment. When a state of equilibrium has been attained between dialectically opposed processes, such as assimilation and accommodation in Piagetian terms, learning is said to have taken place. Experiential learning theory contends that the quality of the learning experience is dependent on the characteristics of the individual and the learning environment. In this study, providing a concrete experience through the *Geogebra* applets ensured that the learning environment was conducive for both experimentation and verification of proposed results by students.

Learning is the process of creating knowledge. The ETL rejects the transmission model of teaching that views ideas as fixed and transferable to the student. In contrast it agitates for a view that is in sync with the constructivist theory of learning. Knowledge is as a result of the transaction between social knowledge and personal knowledge (Kolb, 1984). The social knowledge is more refined and is a result of the accumulation of previous human cultural experience. It is created and recreated in the personal knowledge of the learner. Although the social knowledge is refined, partial scepticism from the pupil is essential for learning to take place.

3.5 The Experiential Learning cycle

The ETL identifies two modes of grasping experience, namely concrete experience (CE) and abstract conceptualisation (AB). Reflective observation (RO) and active experimentation (AO) are the two dialectically related modes of transforming experience (Kolb et al., 2001). Figure 3.1 summarises the ETL process.
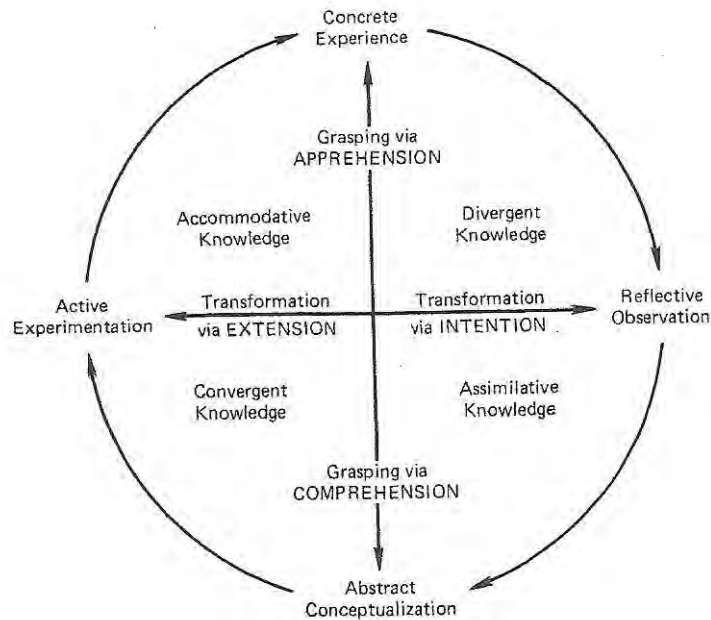
Figure 3.1: The Experiential Learning cycle

It is worth pointing out that immediate or concrete experience is the starting point for any observations and reflections (Kolb et al., 2001). The grasping of experience does not necessarily follow the use of senses and exposing the student to a concrete reality. It is also possible to take hold of new information through symbolic representation or abstract conceptualisation. Abstract conceptualisation involves thinking about, analyzing, or systematically planning, rather than using sensation as a guide.

Kolb (1984) identifies two dialectically opposed ways of processing the new experience. He classifies the students into either reflective observers or active experimenters. In reflective observation the student carefully watches others who are involved in the experience and makes sense of what happens. Those involved in the activity process the experience by active experimentation. Each dimension of the learning process affords the pupil an opportunity to choose the mode of incorporating the new experience (Kolb, 1984). The choice made between concrete and abstract, and between active and reflective exposes the learning style of

the student. Figure 3.1 shows the different learning styles namely diverging, assimilating, converging, and accommodating. See Kolb (1984, p.61) for an in-depth analysis of the learning styles.

3.6 Experiential Learning theory in this study

The four-stage cycle of the ETL (see figure 3.1) can be used to understand the learning activities in this study.

Concrete experience
Kolb's (1984) cycle starts with a concrete experience. The individual is assigned a task that has to be completed. For Kolb the individual cannot learn by simply watching or reading about the concept. It is important to actually do. In this study the students experimented with the *Geogebra* applets. This provided them with an opportunity to empirically construct the gradient function, an aspect which is missing in the traditional way of teaching the derivative.

Reflective observation
Reflective observation requires that the student takes a break from doing. It entails reviewing what has been done and experienced in the task. Questions are asked about the experience and vocabulary of the task plays an important part in the ensuing discussions. In this study the pupils took a break from experimenting with the applets and had to find the equation of the trace. The interviews revealed their thought processes as they reflected about the task. Additionally, the language they used was noted.

Abstract conceptualisation
Abstract conceptualisation is the process of making sense of things that have happened and understanding relationships between them. At this juncture, students have to make comparisons with what they have done and what they already know. They may draw upon theory from textbooks for framing and explaining events. In this study, this process culminated with the pupils' proposed general rule for finding the gradient function.

Active experimentation
At this stage of Kolb's learning cycle, the students consider how they are going to apply what they have learnt. The student has to make predictions as to what will happen next or what actions could be taken to refine the discovery. The pupils were asked to propose a rule for obtaining the gradient function from the equation of the original function in the study. Returning to the applets to test out their hypotheses represents active experimentation in

Kolb's cycle of learning. Furthermore, asking the pupils to sketch the graph of the gradient function from that of the original required them to apply the knowledge that they obtained from the concrete experimentation stage. This is important because Kolb (1984) argues that for learning to be useful the student has to be able to place it into a context.

3.7 Another voice in experiential learning

Experiential learning theory provides a general argument for learning, ranging from adult education to formal schooling. It does not refer explicitly to the teaching and learning of mathematical concepts. James (1992) bridges this gap by proposing a mathematics teaching model that has some striking similarities to ETL. Although James (1992) does not refer to his model as experiential learning it has a strong emphasis on investigation, a trait that also characterises ETL. He argues that if children are taught mathematics in an investigative way their performance in national curriculum tests should rise significantly.

In James' (1992) model investigation is pivotal in the development of mathematical concepts. Students are presented with different scenarios that have a common underlying concept. The students then investigate the scenarios with the hope of indentifying the underlying sameness that evolves with each particular case. This investigative stage, in my view, is synonymous with the concrete stage in Kolb's (1984) learning cycle. In this study the different functions that the pupils explored on the *Geogebra* applets represent the particular cases as argued for by James (1992). Gradually the student becomes able to articulate that sameness and comes to grips with the concepts that underpin it. The final stage is that of checking the generalisations and proving why the generalisations work. Figure 3.2 depicts the thinking processes involved in dealing with an investigative problem.
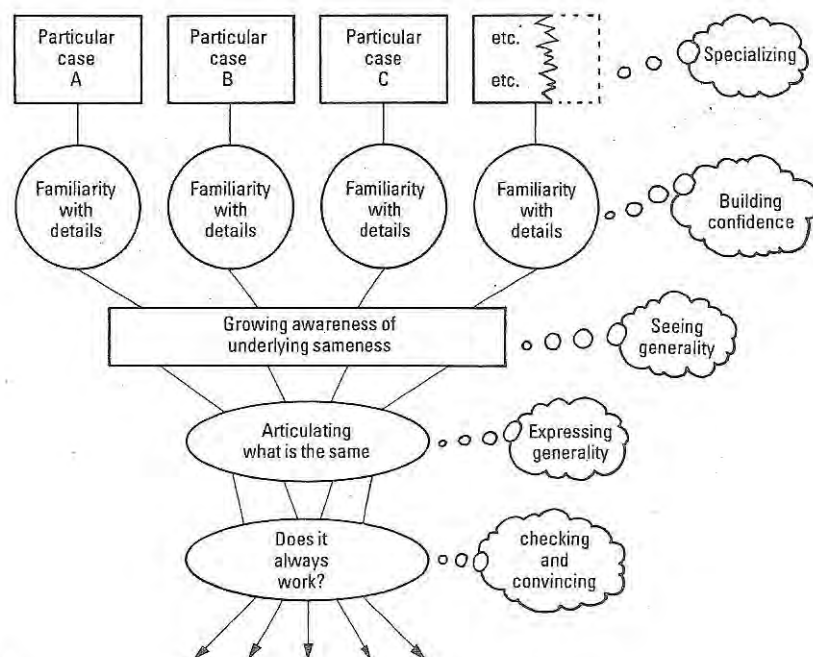
Figure 3.2: James' inductive process of generalisation

A supportive environment is important in ensuring that pupils have the time and space to explore particular cases towards an articulation of underlying sameness. In such an environment, students are free to discuss their conjectures without fear of being judged and modify their conjectures as a result of ensuing discussions with other students (James, 1992). In addition, he contends that in the supportive environment pupils should be at ease to reach out for some equipment to model each special case. In this study the use of *Geogebra* applets acknowledges the importance of this point.

3.9 Conclusion

This chapter has explored the meaning of constructivism as a learning theory. The intention was to show that it shares a bond with David Kolb's (1984) experiential learning theory. A detailed discussion of experiential learning theory was also necessary as it showed how it maps this current study. Concrete experience is a key component in Kolb's (1984) learning cycle and the use of *Geogebra* in this study is an acknowledgement of its importance. Furthermore, *Geogebra* provided the pupils with an opportunity to actively experiment and test out their hypotheses, this is also another stage identified in the learning cycle.

Chapter 4: Research design and methodology

"The constructivist is fully aware of the fact that an organism's conceptual constructions are not fancy-free. On the contrary, the process of constructing is constantly curbed and held in check by the constraints it runs into."

Von Glaserfeld, 1990

4.0 Introduction

Interplay between the research questions, the chosen learning theory and the research paradigm prescribes the overall design of a research study, particularly the method(s) of data collection. The preceding chapters have presented a literary review of theoretical frameworks and contemporary issues within mathematics education, with a bias towards the teaching and learning of calculus concepts. In part, the intention was to provide justification for the methodology used in this study to answer the research questions as outlined in section 1.0, repeated here for continuity:

- Can *Geogebra* aid students in discovering the power rule for differentiating elementary polynomials?

- Having discovered the rule, are the pupils convinced about its truth and generality? Do they demonstrate a desire for an explanation for why the result works?

- Does the use of dynamic graphing software such as *Geogebra* enhance conceptual understanding and resolve difficulties associated with the derivative as documented in the literature?

Accordingly, the focus of this chapter is to expose the research design and methodology of the study. It commences with a general discussion of the qualitative research methodology, the interpretive paradigm and the action research approach as applied in this study. This is followed by an outline of the research participants including the research setting and the sampling technique that was used. Additionally, it also covers the shortcomings of the research design.

4.1 Qualitative research methodology

Qualitative research designs emphasise gathering data on naturally occurring phenomena (Mcmillan & Schumacher, 2005). They further point out that these data are mostly in the form of words and are generated through the use of a variety of methods until a deep understanding is achieved. In the light of the key research questions and the need to

understand deeply how pupils construct concepts, the qualitative approach through an action research-based strategy situated within the interpretive research paradigm was chosen to answer the questions. Action research and the interpretive paradigm will be discussed in sections 4.2 and 4.3 respectively.

The qualitative research approach is a broad orientation that can be classified into either interactive or non-interactive (Mcmillan & Schumacher, 2005). The face-to-face techniques employed in interactive methods give prominence to the research participants and their natural setting. The actual words of the participants are crucial in conveying the meaning systems of the subjects which eventually become the results or the findings of the study (Filstead, 1979). It is worth pointing out that the researcher also plays an important role in interpreting the participants' responses and their interactions with any interventions. Consequently in this study the researcher drew inferences on how students made use of the applets to arrive at the differentiation rule, to sketch graphs of functions and to convince themselves that their rule will work at all times.

Essentially, the interactive methods allow the researcher through the use of interviews, conversations, field notes, recordings and photographs to observe, interpret or make sense of the participants' engagement or response towards a phenomenon under consideration in a natural setting such as a typical mathematics classroom (Denzin & Lincoln, 2005). As discussed previously, the active involvement of pupils is pivotal to constructing meaningful understanding. For this reason this study sought to develop an in-depth understanding of the reasoning that pupils undertook as they worked with the *Geogebra* applet in an attempt to deduce the differentiation rule for elementary polynomials. Furthermore, the interviews conducted also provided insight into whether working with the applets allowed them to transfer that experience to situations that required them to sketch graphs. If indeed the experience is transferable then one might argue that the use of *Geogebra* helps pupils resolve the conceptual problems associated with graphing the function and its derivative (see Park, 2013; Baker et al., 2000; Biza et al., 2008).

4.2 Action research

Action research is a specific strategy embedded within the qualitative research paradigm. It involves the use of research methods by practitioners to study current problems or issues (Mcmillan & Schumacher, 2005). They further point out that the research questions are

rooted in practice and it is a systematic approach aimed at helping professionals better their practice. It is also seen as a way to encourage the professional development of teachers as they acquire skills that allow them to be reflective and inquiring practitioners (Feldman & Minstrell, 2000). Its flexibility makes it accessible, among other possibilities, to the individual teacher or a group of teachers working cooperatively within one school (Holly & Whitehead, 1986 as cited in Cohen, Manion & Morrison, 2007). The involvement of teachers in action research is advantageous in that it helps to reduce the time lag between the generation of new knowledge and its application in the classroom (Feldman & Minstrell, 2000). The following definition aptly captures action research:

> "Action research is a small-scale intervention in the functioning of the real
>
> world and a close examination of the effects of such intervention"

<div align="right">(Cohen & Manion, 1994 in Cohen et al., 2007 p.297)</div>

Furthermore, they identify replacing a traditional teaching method by a discovery one as one of the specific objectives of action research. This assertion is pertinent to this study. The intervention that was investigated was the use of the software *Geogebra* in place of the traditional chalk and talk method to aid pupils towards discovering the power rule for differentiating elementary functions. The opening quote at the beginning of this chapter insinuates that interventions in a constructivist learning environment are important in overcoming the constraints that students come across. It is my view that action research may help determine which interventions are effective. Accordingly, the effectiveness of this intervention is discussed under the research findings.

4.2.1 Limitations of action research

Since action research is mostly conducted by teachers with the aim of improving practice, it means that the distance between the subject and the object of study has been reduced to zero. This places a large emphasis on the ability of the teacher(s) involved to subject their investigations to critique from within and from outside (Feldman, 1998 as cited in Feldman & Minstrell, 2000). Such a critique might not happen effectively since most teachers are not trained researchers and as a result this may affect the findings. Feldman and Minstrell (2000) further argue that action research is non reproducible. This is because teaching situations change continuously and no two classes are similar enough to reproduce the same teaching conditions.

Despite these weaknesses action research still provides a viable avenue for effecting change in classroom instruction. Its interpretive nature seeks understanding and teachers do not need to demonstrate that what they have learned is applicable to all cases. Instead, they just need to show that what they have learned is true in the particular case of their teaching in their classrooms (Feldman & Minstrell, 2000).

4.3 Interpretive paradigm

The interpretive nature of the data analysis also makes this study qualitative. Patton (2002) posits that a paradigm is a world view, a way of breaking down the complexities of the world and that paradigms are deeply embedded in the socialisation of adherents and practitioners. A researcher's worldview will determine what he/she chooses to place emphasis on during the data gathering and analysis stages. All researchers interpret the world through some conceptual lens formed by their beliefs, previous experiences, existing knowledge, assumptions about the world, theories about knowledge and how it is accrued (Carrol & Swatman, 2000).

In reaching a conclusion in an interpretive paradigm the researcher relies heavily on the participants' perspectives. The interpretive perspective is grounded on the premise that each person's way of making sense of the world is worthy of respect (Patton, 2002). From this point of view, the researcher in this study engaged the students in one-to-one task-based interviews to attempt to infer how they arrived at their conjectures and generalisations. In addition, the on-task worksheets also provided feedback key to arriving at the findings. This characterises the interpretive paradigm: the researcher tries to make sense of the world from the participants' point of view. The intention is to get inside the participant and to understand his/her world from within (Cohen et al., 2007).

4.4 The sample

The selection of participants in this study followed purposeful sampling. It is a sampling strategy in which the researcher handpicks the cases to be included in the sample on the basis of his/her judgement of their typicality or possession of the particular trait that is being investigated (Cohen et al., 2007 p.115). Purposeful sampling is also driven by the objectives of the study and is ideal if the intention of the study is to gain an in-depth understanding of a phenomenon and not to generalise over the entire population (Mcmillan & Schumacher, 2005). In this study six Grade 11 boys aged between 16 and 17 were chosen as participants.

They were the top-performing Grade 10 students at the school in the Cambridge International Examinations (CIE) written in the November/December session 2013. They were all, at the time of the study, doing their Advanced Subsidiary (AS) level. They had all dealt with the gradient of a curve as a point specific object in their Grade 10 year. It is worth noting that they had not all been exposed to *Geogebra* prior to the study. Additionally, they had not yet encountered the term "derivative" in the mathematics context.

In the South African schools Calculus topics are generally taught as an extension to the top students. In this study, the students were required to find the equation of the trace produced by a point and their algebra skills had to be reasonably good. Pillay (2008) found that students experienced a lot of difficulties in carrying out procedures flexibly and accurately. In this study if the pupils were unable to determine the equation of the trace they might have consequently found it impossible to discover the differentiation rule. This justified the selection of the top students. Furthermore, since the purpose of this study was not to generalise the sample size of six was ideal. All the participants were boys, four White, one Indian and one Black. They all attend the school that I as a researcher teach at and it was convenient to use this sample. It afforded me the opportunity to schedule interviews with the boys easily, and gave easy access to the computer laboratory at the school. The school is a cosmopolitan institution enrolling students from relatively affluent homes that provide good educational support. About 250 students out of about 500 boys are boarders.

4.5 Research procedure

The study was conducted over a period of four weeks. Each student was interviewed over two sessions. The following outline shows how the data collection was structured.

Stage 1: Pre task activity to teach boys how to interact with *Geogebra* and how to find the equation of a parabola from the graph.

Stage 2: (a) Completion of task 1 numbered 1 to 4 in the instrument (See Appendix 1). Video recordings were made.

(b) Levels of conviction. Interviews also explored if the participants required an explanation for why the rule works.

Stage 3: Completion of items 5 and 6 was followed by interviews. The opportunity was also utilised here to thank the pupils for their involvement.

4.6 Data collection

This study sought to gain an in-depth understanding of students' thinking processes as they tackled the research questions. Consequently, face-to-face data collection strategies were deemed appropriate. Task-based interviews and observations were the main data collection strategies. The interview sessions were video recorded and then transcribed to ensure that the pertinent information was adequately captured. Furthermore an analysis of the completed worksheets also formed part of the data collection. Different approaches were employed in a bid to corroborate the findings. The interviews comprised two sessions for each student. The first session dealt with research items 1 to 4 in Appendix 1 while the second addressed items five to six.

4.6.1 The task-based interview

An interview in qualitative research attempts to establish how individuals picture their world and how they explain or make sense of the important events in their lives (Mcmillan & Schumacher, 2005). Its use marks a move away from seeing research participants as simply controllable and data as somehow external to individuals, and towards regarding knowledge as generated between humans, often through conversations (Kvale, 1996 as cited in Cohen et al., 2007, p. 349). Furthermore, Goldin (1997) argues that the emphasis in mathematics education has evolved over the years to stress conceptual understanding, higher-level problem solving processes and children's internal constructions instead of procedural and algorithmic learning. In light of these developments he contends that the interview has found greater acceptance as a research method. Over the years various forms of the interview have been conceived to cater for differing purposes. Examples include the informal conversation interview, the key informant interview, focus group interviews, standardised open-ended interviews, narrative interviews and one-to-one task based interviews (Cohen et al., 2007; Mcmillan & Schumacher, 2005; Goldin, 1997).

In keeping with the research questions and the underpinnings of constructivism that guided this study, the one-to-one task-based interview was chosen as the primary data collection strategy. Davis (1984) views a task based interview as one in which a student is seated at a desk, paper and pens are provided and the student is asked to solve a specific mathematics problem; one or more adults are present collecting data. In addition to these basic tools Goldin (1997) advocates the use of various external representational capabilities that permit

interaction with a rich observable learning environment. In this study the *Geogebra* applet (see section 4.6.2) was the additional manipulative provided to each student. Pioneered by Piaget, the task-based interview requires that the research participant talks during or immediately after solving a problem (Koichu & Harel, 2007). They further claim that there is substantial evidence that the task-based interviews open a window into the research participant's knowledge, problem-solving behaviours and reasoning.

At the core of the interview design is that the student should engage in free problem solving to the maximum extent possible. This makes it feasible to observe of spontaneous behaviours and to formulate reasons for these spontaneous choices (Goldin, 1997). During the interview the participants are occasionally reminded to talk aloud about what they are doing and to describe what they are thinking. Additionally, a major task goal is the construction of representations by the students. In this study one of the objectives was to explore whether the students could deduce the differentiation rule in the environment supported by *Geogebra*. In the process of completing the task, hints and prompts, or new questions, should be offered only after the opportunity for free problem solving and allowing sufficient time to observe how the student responds. Goldin (1997) concedes that on some occasions this rule can be broken because of our desire, as researchers, to ensuring that a subsequent section of the interview is reached. However, he cautions that breaking this rule possibly leads to losing important information. As a result of literary considerations informing tasks-based interviews the following interview protocols were devised to answer the respective research questions.

Interview protocol for question 5: Identifying ƒ'(x) given ƒ (x)

Researcher: Explain your choice for corresponding gradient graph.

Satisfactory answer

Answer not satisfactory/correct

Proceed to the next item in the table.

Probe and redirect thinking making reference to the *Geogebra* applets used in item 1 of appendix one.

Answer not satisfactory/correct

Provide additional hints

Answer not satisfactory/correct. Stop
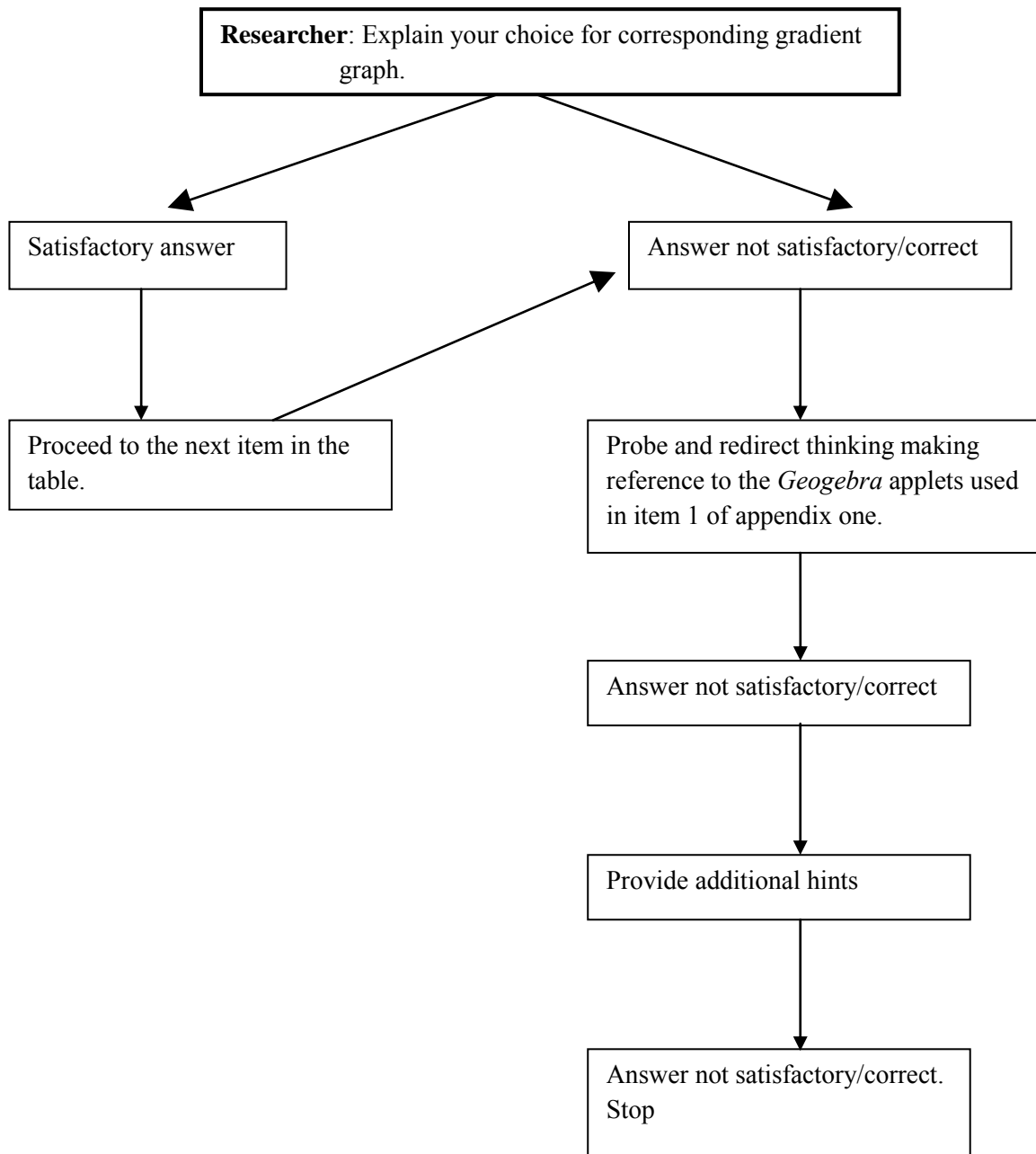
Figure 4.1:Interview protocol for question 5 (b)

Interview protocol for question 6: Identifying f (x) given f '(x)
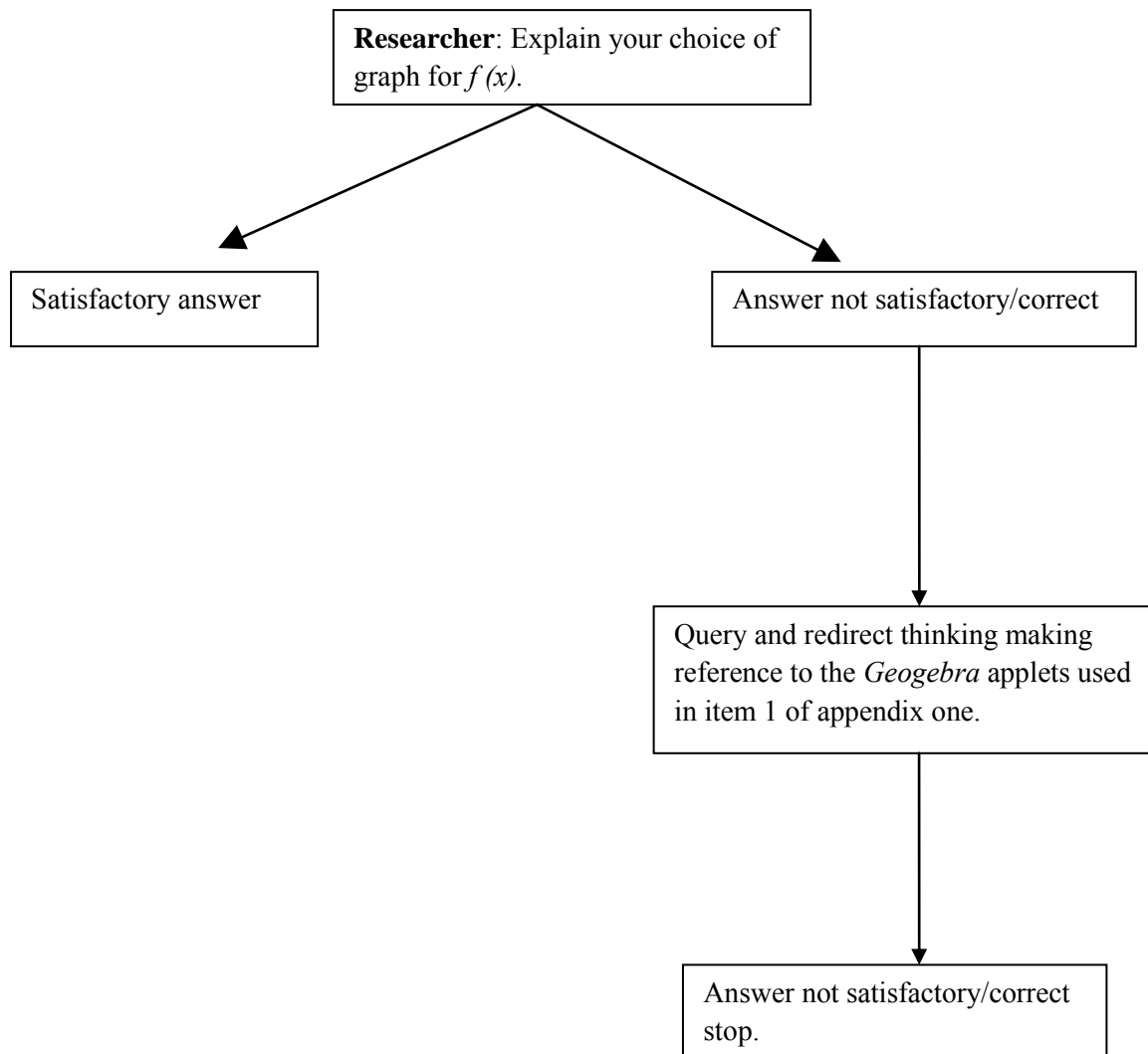
Figure 5: Interview protocol for question 6

It was envisaged that each interview was going to run for fifty five minutes but some of the pupils required more time, however, all the interviews were completed in less than ninety minutes.

4.6.2 The electronic environment (Geogebra Applet)

It has been noted that *Geogebra* was the software of choice for this study because it is freely available for download at http://www.geogebra.org/cms/download. More importantly, its choice was driven by the aims of this study. These aims implied that the appropriate software should facilitate the students in:

- their exploration of the function graph and its derivative graph
- their move from the geometrical context to the analytical context and the connection between these contexts.

Proficiency in the use of the computer software was not the main concern of this study. A user friendly interface that offered a combination of Computer Algebra Systems (CAS) and Dynamic Geometry Systems (DGS) in one package was ideal and *Geogebra* ticked all these boxes, hence it was chosen.

For the requirements of this study, a specific applet was created (Figure 4.3 below is adapted from Hohenwarter & Hohenwarter, 2013, p.39) with the following constructions:

- a function graph, labelled, in Figure 4.3 as $f(x)$

- a movable point $A(x; f(x))$

- a tangent to the curve at point A

- a point $S(x; \text{slope of curve at } x)$

The user can change the position of point A by dragging it, causing the dynamic changes to the tangent such as its orientation relative to the sign of the gradient. This also causes point S to move creating a trace of the gradient function in the process. The function graph can be changed easily without any consequence on the other parts of the construction. This made it easy for the pupils to alternate between different examples of functions as they worked through the worksheet. The design of the study hinged on the exploration and discussion of several examples and this facility proved to be very useful. The dynamic applet (Figure 4.3) can accessed at http://www.geogebratube.org/student/m94502
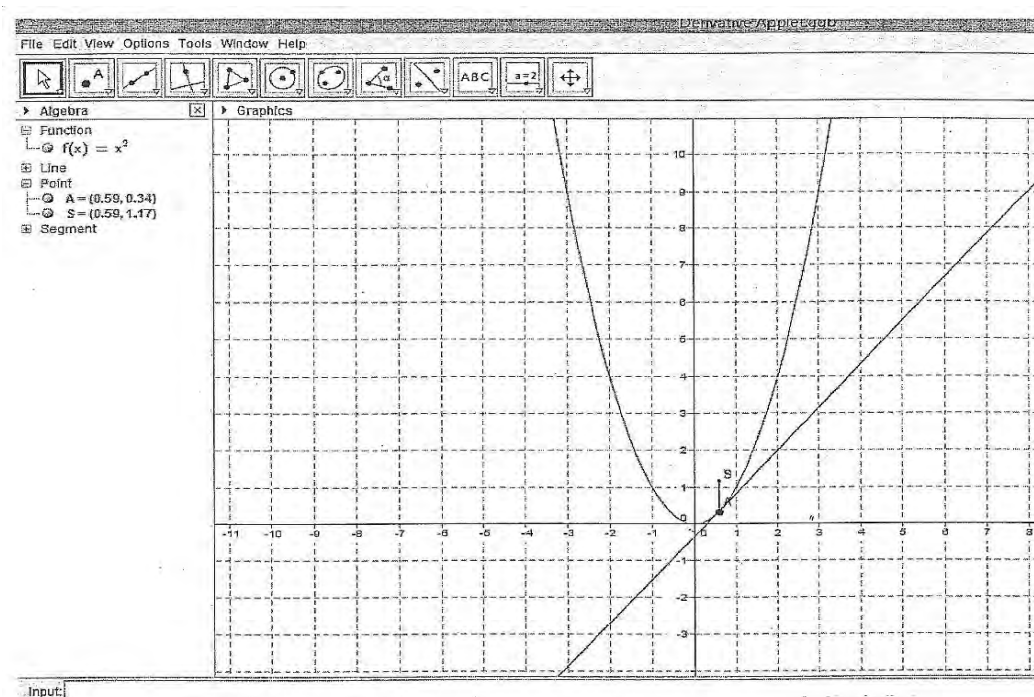

Figure 4.3: Applet used for data collection.

The task-based interview was preceded by the researcher demonstrating to the pupils how to manipulate the electronic construction. In particular the pupils were first shown how to drag point *A*, input a new function in the input bar, and then how to refresh the view to delete the trace created by a previous function. At the same time the students had to demonstrate the ability to check that their calculated equation also followed the trace made by point *S*. It must be pointed out that when the pupils experienced difficulties while working with the applet they were given the required aid by the researcher. This underscores the fact that the research design took into account that expertise in the use of the software was not a priority of the study. However, the students seemed to come to grips with the functionality of the applet with ease, further affirming the user friendliness of *Geogebra*. During the same session the pupils were also shown how to calculate the equation of a quadratic function given its graph.

4.6.3 Observation

All the interviews that were conducted with the students were video recorded for transcription purposes during data analysis. In addition the recordings provided me with an opportunity to relive the interview sessions and an opportunity to observe the students' reactions and other body language as they completed the worksheets. Observation is the fundamental base of all research methods in the social and behavioural sciences (Adler & Adler, 1994 in Angrosino & Rosenberg, 2013). It offers the researcher the opportunity to gather live data from naturally occurring social situations (Cohen et al., 2007). They further argue that the observation enables the investigator to use immediate awareness as the principal mode of research and this, they posit, may result in data that are more valid and authentic. Furthermore, observations in qualitative studies complement interviews and are usually unstructured (Cohen et al., 2007).

In this study I observed the students while I conducted the interviews, a role identified by Cresswell (2003) as that of a participant observer. The focus in this data gathering strategy was, among other things, specifically to (i) identify any difficulties and successes that the students experienced during task completion, (ii) note any body language as they interacted with the *Geogebra* applet and (iii) characterise the pupils' reactions to cues provided to them during the interview process.

Chapter 5: Data analysis and findings

5.0 Introduction

The focal points of this chapter are to simultaneously present and analyse the data generated in the study. The chapter commences with general theoretical considerations pertinent to data analysis in qualitative research studies. This is followed by an analysis, in section 5.2, of the findings for research questions 1 and 2;

- Can *Geogebra* aid students in discovering the power rule for differentiating elementary polynomials?

- Having discovered the rule, are the pupils convinced about its truth and generality? Do they demonstrate a desire for an explanation for why the result works?

The students' responses to questions 1 through to 4 (see Appendix 1) and the interview protocol (see appendix 2) were used to address these questions.

Section 5.3 is dedicated to the findings and analysis for research question 3;

- Does the use of dynamic graphing software such as *Geogebra* enhance conceptual understanding and resolve difficulties associated with the derivative as documented in literature?

Responses to question 5 and 6 (see Appendix 1) were the main sources of data in answering the above question.

5.1 Considerations in data analysis

Qualitative data analysis is usually based on an interpretive philosophy that is focused on making sense out of text and symbolic content of qualitative data (Nieuwenhuis, 2007, p.99; Cresswell, 2009, p.183). Mcmillan and Schumacher (2005) concur by claiming that qualitative data analysis is a relatively systematic process of categorising, and interpreting data to provide explanations of a single phenomenon.

A simplistic dichotomy splits qualitative data analysis into inductive and *a priori* coding. Coding is the process of cataloguing information into chunks or segments of text before bringing meaning to information (Rossman & Rallis, 1998 as cited in Cresswell, 2009, p.186).

"It involves taking text data or pictures gathered during data collection, segmenting sentences (or paragraphs) or images into categories, and labelling those categories

with a term, often a term based in the language of the participant (called an *in vivo* term)."

(Cresswell, 2009, p.183)

Inductive coding is when a qualitative researcher develops the codes by directly examining the data and flexibly lets the codes emerge from it (Nieuwenhuis, 2007).

In contrast, this study employs *a priori* coding whereby the researcher comes up with the codes before examining the collected data (Nieuwenhuis, 2007, p. 107; Mcmillan & Schumacher, 2005; Govender, 2013). During analysis the researcher then searches the data generated during collection for the preset topics. Accordingly the literature survey and the theoretical framework informed the *a priori* codes used in this study. Table 5.1 below, adapted from Govender (2013, p.231), shows how a student might proceed to inductively come up with the general rule for differentiating a polynomial. Of equal importance is to note that it takes into account aspects of Kolb's (1984) and James' (1992) models of using empirical evidence to arrive at a generalisation of a concept.

Formulating a conjecture/ experimentation	Student might observe particular cases, look for patterns and then make a tentative pronouncement about a general case
Validating the conjecture	Test the conjecture using a new particular case
Generalising the conjecture	Testing the conjecture with numerous cases and plotting the results on *Geogebra* may convince the pupils that the conjecture is generally true.

Table 5.1: Analytical framework for inductive generalisation

Justification for the proposed generalisation was required. During the interviews the pupils were pressed to articulate how they arrived at their rule and why they thought it would work at all times. Table 5.1 provides the types of justifications that the pupils could provide in an effort to explain their reasoning.

The coding and categorising process was preceded by a manual verbatim transcription of all the video recorded one-to-one task-based interviews. It was essential to do the transcription

manually so that the researcher could include some non-verbal cues in the transcript. Such non verbal cues may communicate embarrassment or emotional distress, or simply a pause for thought (Nieuwenhuis, 2007).

Justification Level	Descriptors
Level 0: No Justification	Responses do not address justification.
Level 1: Appeal to external authority	Reference is made to the correctness stated by some other individual or reference material.
Level 2: Empirical evidence	Justification is provided through the correctness of particular examples.
Level 3:Generic example	Deductive justification is expressed in a particular instance.
Level 4: Deductive Justification	Validity is given through a deductive argument that is independent of particular instances.

Table 5.2: Analytical framework for justification (Lannin, 2005, p. 236)

5.2 Using Geogebra to empirically arrive at a conjecture

This section will present the findings from the one-to-one task-based interviews that were conducted with a view of establishing if *Geogebra* could help pupils to empirically discover the differentiation rule for elementary polynomials. Each interview commenced with the researcher putting the student at ease. This was achieved by explaining once more to the pupils the objectives of the study and reiterating that no marks would be allocated at the end of the exercise. The components of the construction (see section 4.6.2) were also explained again to the pupils to ensure that they clearly understood what they were expected to do. It is worth pointing out that although I teach at the school none of the boys in the sample were in any of my mathematics classes. The following excerpt typifies the introductory remarks before each interview.

> *Researcher*: Gerald, thank you once again for agreeing to take part in this study. As I explained to you in the earlier briefing, you will be expected to work with the *Geogebra* applet that you are looking at to see if it might help us find an easier way of calculating the gradient equation of a function.

This was preceded by the researcher checking to see if the pupils still remembered the laborious tangent method that they employed in Grade 10 to calculate the gradient of a curve

at a given point. This was important because the applet also made use of the tangent to draw the corresponding gradient curve. This also ensured that the pupils' understanding of the concept to be developed was based on prior knowledge, a pertinent point in constructivist learning. All the boys were able to correctly describe the tangent method. Each pupil was then afforded the opportunity to familiarise himself with the construction. It has to be noted that some of the pupils initially found it difficult to grasp what the point S was actually tracing out. The construction, in particular the significance, of the y coordinate of point S had to be explained yet again. However, once the students got going they manipulated the applet with relative ease.

Once the pupils were satisfied that they could operate the software, they were asked to read aloud the task (see item 1 Appendix 1) and that they understood what was required of them. If they required more clarity the researcher provided it. If they answered in the affirmative they were then asked to proceed with the dragging exercise. Generally the students got comfortable with the software with ease, confirming its user friendliness.

The dragging exercise enabled the pupils to trace out gradient functions of particular cases. These results were then used to find a rule that could be used to differentiate a general case. All the six pupils were able to come up with a general rule for differentiating a function of the form $y = mx + c$ and they justified it by referring to their Grade 10 work. They simply stated that the gradient of the line would be equal to m because they were told so in the previous year. This is a level 1 justification in the Lannin (2005) classification. Two of the boys went on to calculate the gradient of the function graph using the y step divided by the x step. As they progressed through the quadratic functions, all six boys were quick to realise that the exponent of the function graph became the coefficient of the x term in the gradient function.

Finding the equation of the quadratic trace generated by the cubic expressions presented a challenge for some of the students. Since the focus of this study was to investigate if the pupils could inductively deduce the rule for differentiating the elementary polynomials, it was imperative that they obtained the correct algebraic expressions. As a result, assistance was provided to those pupils who had difficulties remembering the formula for finding the equation of the quadratic function.

Generally it was observed that as they worked through the specific cases they did not attempt to make any generalisations unless prompted. The process of pattern recognition began with the realisation that the equation of the gradient function in sloping straight line graphs was equal to the coefficient of *x*. A consideration of the other cases, exponents of two and three, resulted in the generalisation by all six boys that the exponent becomes the coefficient of the gradient function. At that at this stage no reference was made by the students to the constant term. When asked to explain what happened to it the common response was that it just disappears and it has no bearing on the new gradient function. Evidently the students could see that the derivative was zero but a lack of the correct terminology hindered their ability to put this into words.

Five of the students' generalisations began by noting that the exponent becomes the coefficient, while Allen first pointed out that the exponent drops by one. The power reduction was also often explained as: drop the exponent and make it the coefficient and then divide by *x* to get it to one less. The generalisation part of the one-to-one task-based interviews will now be presented and all names mentioned herein are pseudonyms. Additionally, scaffolding provided varied from one pupil to the other depending on the obstacles that each encountered. Ultimately five out of six students managed to deduce the correct generalisation.

The Case of Tinashe

As Tinashe worked through the straight lines he noticed that the coefficient of *x* becomes the equation of the gradient function. His general process of finding the equation of a gradient function is now presented;

> Tinashe: As I move on to the parabola I have noticed so far that the power of the *x* will give you the coefficient of *x* in the gradient function.
>
> Researcher: Ok. Carry on.

Having finished working with the quadratic functions he moved onto the cubic functions and his first observation after tracing out the gradient function of $f(x) = x^3 - x$ was that the resulting gradient graph was a parabola.

> Tinashe: With the x^3 the drawing of the gradient graph I notice that it results in a parabola. So I was right with this pattern thing! (After finding the equation of the curve)
>
> Researcher: What do you mean you were right?

Tinashe:	The power of x (pointing at x^3) is equivalent to the coefficient of this gradient function.
Researcher:	Ok it seems to work out. Carry on.
Tinashe:	Oh…haha (with a surprised look on his face)
Researcher:	What seems to be the problem?
Tinashe:	It doesn't seem to follow the pattern as I had said it does.
Researcher:	Are you sure? What does it mean?
Tinashe:	My pattern was incorrect (with a nervous laugh),
Researcher:	Pattern may not have been incorrect, it means that you just need to re-examine it, you are not far off. Try the last one before you attempt to generalise.

After completing the next item $f(x) = \dfrac{x^3}{3}$ and some pondering;

Tinashe:	I think I have got it.
Researcher:	Tell me, what is it that you have realised?
Tinashe:	As I said take the power, make it the coefficient of x and then divide by what you have (pointing at the 3)….that's what I see.
Reseacher:	But Tinashe you have not said anything about the resulting power.
Tinashe:	Umh…the resulting power is one less…its one power down. Here it is x^2 and there it's now x one.
Reseacher:	Ok will that work with the straight lines?
Tinashe:	Straight lines? (Pause) Yes I see it will work throughout. You actually divided by x.
Researcher:	Divide by x?
Tinashe:	Ja….well….to reduce the power, that's what I'm trying to say, sir.
Reseacher:	Ok that seems to make sense.

He then proceeded to complete the rest of the items and when given another specific example he was, although he took some time, able to apply his rule to arrive at the correct result. He also used the applet to confirm the result. Figure 5.1 summarises Tinashe's differentiation rule.

The case of Gerald

This dialogue commences when Gerald had finished calculating the equations for the gradient functions for the straight line and quadratic functions.

Researcher: You told me that with the straight line functions the gradient equation was determined by the coefficient of x in $y = mx + c$. Can you notice anything with the square functions?

Gerald: Yes….umh…with the x^2 thing….umh the equation of the gradient function is equal to $2x$, if you add c (referring to -1 in $x^2 - 1$) the graph shifts down.

Researcher: Ok. Remember that at the end we want to see if there is a way of finding the equation of the gradient function without using *Geogebra*. Keep that in mind.
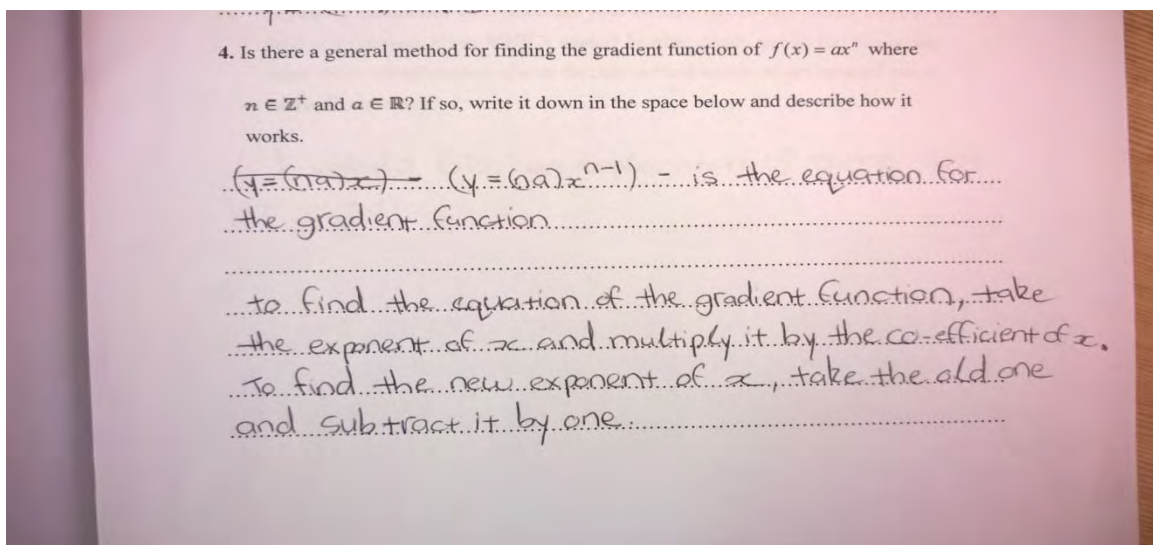
Figure 5.1: Tinashe's differentiation rule

After Gerald had completed all the items the interview continued as follows;

Researcher: Are you seeing a pattern or not in your answers?

Gerald: This one is over three (pointing at $\frac{x^3}{3}$)……umh…..I can't see a pattern.

Researcher: You said with the straight lines you get the equation by taking the coefficient of x, can you notice any similarity or a general rule for x^2 and x^3?

Gerald: Well for x^2 I see that you just add the two in front and I also see that you just add the three in front.

Researcher: What do you mean adding?

Gerald:	I mean times it by three (referring to the x^3 term). So for a squared you times it by two, for a normal x it's just the number in front of x. (Long pause) When it's over three the threes cancel out.
Researcher:	You have not taken into consideration what happens to the remaining power of x. How do you get to the new power?
Gerald:	Ja…..you divide by $1/x$....this one (pointing at x^2) you multiply by $2/x$ and one x will cancel out…..but it won't work here pointing at x^3.
Researcher:	Why will it not work? Consider again your starting powers and final powers?

After a pause he exclaimed that the power is always one back. I then asked him to recap his rule and he correctly applied it to $f(x) = \dfrac{x^3}{3} - x^2 + 2x$. At this point he did not attempt to explain what happens to the constant term. I also gave him another specific example and asked him to find the gradient function without using the *Geogebra* applet and he correctly applied his rule. When queried about how he could confirm that the derived function was correct he said that he was going to plot it on the applet and check it. This showed that he was using empirical evidence as justification for his method. This was a common response in all the participants. Even though some of them did not suggest on their own that the applet could be used to check the answer, they quickly accepted that the result was true when the researcher asked them to check it on *Geogebra*.

The case of Allen

Allen also worked quickly through the specific cases provided in the worksheet and made no attempt to generalise as he progressed. The scaffolding provided during the one-to-one task-based interview enabled him to come up with the general rule. In a bid to find the general rule, Allen began by first noticing that the exponent drops by one. All the other students first observed that the exponent becomes the coefficient.

Researcher:	Do you see a pattern emerging for obtaining the equation of the gradient function?
Allen:	Umh….well the index is one less than what it was in the function.
Researcher:	Which one?
Allen:	Here in an x^3 function (pointing at $f(x) = x^3 - x$) the gradient always has an x^2 as the greatest power.

Researcher:	Ok. What about the coefficients do you notice anything?
Allen:	Well for the first four functions (referring to the straight lines) the coefficient of x is the gradient function.
Researcher:	And the x^2 one?
Allen:	Its $2x$.
Researcher:	Ok how then do you get this one($f(x) = x^3 - x$)?
Allen:	Umh (long pause). It's a parabola and ….. because you have an x^2….(sigh) I'm not sure, it's easier with the graph (points at the applet).
Researcher:	Maybe you didn't get my question. I asked you if you could see a pattern emerging. You then mentioned something about the exponents. How did we get the 3 in $3x^2$ for the function $f(x) = x^3 - x$?
Allen:	Umh….in the ones where we divide by three, there is no three in the x^2, so here we must have a three.
Researcher:	Ok and how do we get the two here in x^2?
Allen:	The exponent is the coefficient.

He seemed to have obtained the result at this stage but was battling to put it in words. Before he proceeded to number 2 (see Appendix 1) I had to check that he had mastered the rule for the particular cases.

Researcher:	Before we move on what would be the gradient function of $3x^2 + 6x$?
Allen:	Umh .It will be $6x$ (long pause)
Researcher:	What are you thinking?
Allen:	I'm wondering if I can split them up to find the gradient function.
Researcher:	Why don't you compare it with this one $f(x) = \dfrac{x^3}{3} - x^2 + 2x$?
Allen:	Yeah they are actually sort of together (After some pondering). The negative x^2 becomes $-2x$.
Researcher:	Ok. Let's check this one $f(x) = x^3 - x$ is it consistent with what you are saying?
Allen:	Ok it seems consistent (after considering the expressions)
Researcher:	Ok are you happy with that?
Allen:	Yes.
Researcher:	Ok lets go back to this one $3x^2 + 6x$.
Allen:	It will be $6x + 6$.

Asked how he could confirm that his result was true, Allen described how he was going to use the applet. He proceeded to check it and then told me that his trace fits the equation of the gradient function he had calculated and hence he was convinced that the result was true. Figure 5.2 shows how Allen summarised his general rule.
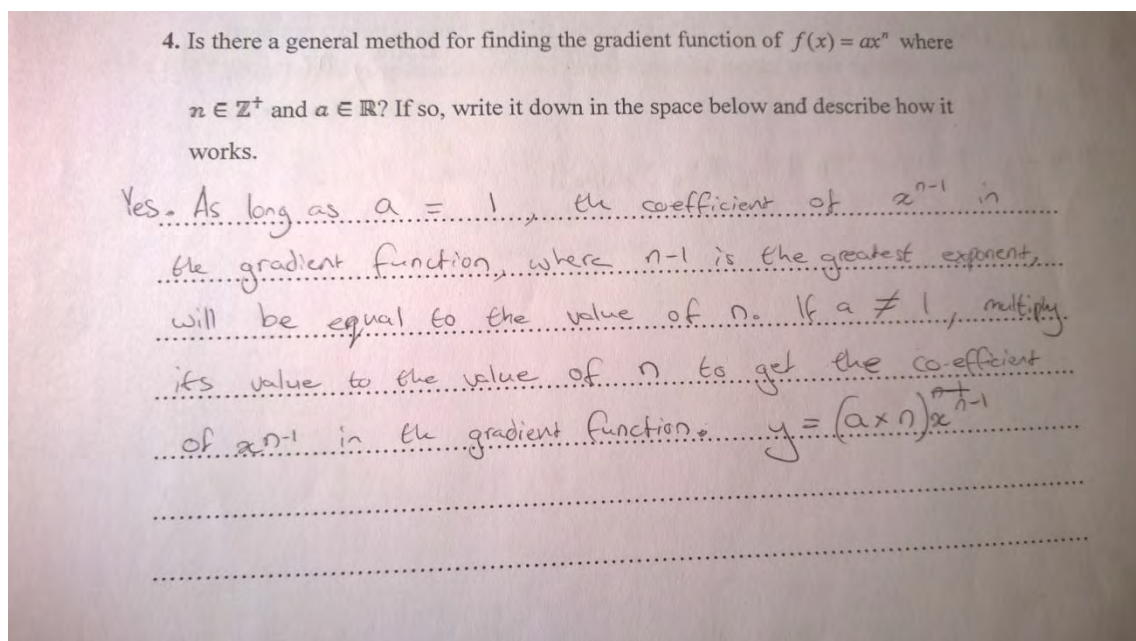
Figure 5.2:Allen's conceptualisation of the differentiation rule

The other two students, Wayne and Adam, both managed to arrive at the rule and their conceptions where similar to the ones that have been presented.

The Case of Mark

Mark is the only student who failed to come up with the general rule. It has to be said that he seemed to be very low in confidence during both sessions of data collection. However, he was able to generalise for the straight lines and he justified it by saying that he knew that in the form $y = mx + c$ the m represents the gradient. The justification was based on his prior knowledge learnt in Grade 10.

> Researcher: Having completed the table, can you see any pattern for determining the equation for the gradient function?
>
> Mark: Kind of up to here (referring to the straight lines) for $y = mx + c$ form umh…the number in front of the x would be the gradient.
>
> Researcher: Yes?
>
> Mark: Over here (referring to the quadratic terms) I'm not too sure.

Researcher:	Ok. Just compare the two equations. You said you had seen something what was it that you had seen?
Mark:	The exponent represents something but I'm not so sure….the exponent (pointing at $f(x) = x^2$) is the number in front of x (pointing at the gradient function) but I'm not sure.
Researcher:	How do you know that it is wrong?
Mark:	It doesn't seem logical.
Researcher:	Ok in the straight lines how come you didn't doubt that the number before x was going to give you the gradient function?
Mark:	Coz I know the $y = mx + c$ form, Sir.
Researcher:	Ok look at the x^2 functions again and their gradient functions, what can you see?
Mark:	Only thing that I can see is that the exponent is the number in front of x in the gradient function.
Researcher:	Ok you are right, you are on to something in terms of the exponents. Do you want to look at the next one ($f(x) = x^3 - x$)
Mark:	Ok, Sir. Also here the exponent (gradient function) is one less that of this equation (function graph). I'm not really sure.

The $\frac{x^3}{3}$ term seemed to discredit his hypothesis that the exponent becomes the coefficient in the gradient function. This caused him to abandon it and he just focused on the only observation that that seemed consistent to him; that the exponent drops by one in the gradient function. He gave up and refused to act on any other prompts designed to help him overcome the obstacle, additional hints would have given the rule away.

In this study the development of the general rule seemed to follow a linear process. In all the five one-to-one task-based interviews the students first noticed that the coefficient of the x term in the general equation of the straight line, $y = mx + c$, ended up giving the equation of the gradient function. Even though they proceeded to experiment with all the other linear functions using the Geogebra applet they seemed convinced of the result. The justification for this outcome was linked to their Grade 10 experience, thus it was based on their prior knowledge which was given to them by the teacher and Lannin (2005) assigns level 0 to such a justification.

A cognitive conflict seemed to force the students to revise the premature generalisation when they encountered firstly the quadratic functions and then the $\frac{x^3}{3}$ term in the cubic functions. In the quadratic functions the coefficient of the x^2 term was no longer the equation of the gradient function. The start of the learning process is signalled by a cognitive disequilibrium between the student's prior knowledge and the new knowledge; overcoming this contradiction results in new constructions (Piaget, 1975 as cited in Balacheff, 1991). This cognitive disequilibrium forced the students to re-examine the specific cases in the quadratic class of functions. The results of the reconsideration led them to observe that the exponent of the function graph becomes the new coefficient of the gradient function. This observation was carried over to the $x^3 - x$ function. In four of the cases this was then followed by the realisation that the exponent in the gradient function would be one less that of the original function. Allen is the only student who started by realising that the exponent would be one less. The empirical evidence that enabled them to calculate the equations of the gradient functions helped the pupils to come up with these observations.

Cognitive disequilibrium was also observed when the pupils encountered the $\frac{x^3}{3}$ term in the cubic functions. In fact, this term was the Achilles heel for Mark, the only student who failed to construct the general rule. At this point all the students (including Mark) had noticed that the exponent becomes the new coefficient and the absence of the three in the gradient function meant that the "general rule" had to be revisited once more. It did eventually become evident to the other five students that it cancelled out with the denominator but Mark gave up despite the prompts given to him in an attempt to help and direct his thinking. The last step in the generalisation process was this realisation that one just had to multiply the exponent with the coefficient of x term.

A counter argument could be made against this linear progression of the generalisation process because of the way the questions were structured in the worksheet. Granted, the questions moved from linear to cubic functions and so did their relative complexity, which in a way might have led the students to generalise in the observed linear manner. This on its own, in my view, is not enough to discredit this hypothesis as Watson and Mason (2006) found that the starting point of any sense-making is the discernment of any variation within the data. Consequently they argue that tasks that display controlled variation have the

potential to help pupils make meaningful constructions. These considerations helped craft the data collection instrument. Furthermore, James (1992) also argues that mathematical tasks should encourage investigation with a view of deducing a generalisation based on some underlying sameness. Additionally, he contends that a supportive environment is crucial for the discernment of this sameness. Presenting the functions in this study in progressive classes, from linear to cubic, was a way of ensuring that the pupils were provided with the vital scaffolding.

5.2.1 Findings based on Section 5.2: Using Geogebra to empirically arrive at a conjecture

1. Five out of the six students were able to inductively deduce the differentiation rule. The deduction was based on the empirical evidence generated through the use of the *Geogebra* applet, a level 2 justification in Lannin's (2005) classification.

2. The generalisation process seemed to follow a linear pattern. The students first came up with a rule for linear functions followed by a rule for quadratic functions, after which they proceeded to generalise for cubic functions. This was then successfully extended to the general exponent *n*.

3. Cognitive disequilibrium during the progression from one set of functions to the next, for example linear to quadratic, spurred the students on to revisit and refine their generalisations until they were consistent with all the given particular cases.

5.2.2 Levels of conviction and need for an explanation

The five students who successfully arrived at the differentiation rule were interviewed to establish how convinced they were that their rule would work at all times. The interview protocol in Appendix 2 was the primary instrument used to assess the pupils' levels of conviction. It is not surprising that all five students displayed very high levels of conviction. Other studies (see Mudaly & de Villiers, 2000; Govender, 2013) also found that students do exhibit very high levels of conviction after making discoveries based on empirical evidence. Harel (2013) refers to this as the satisfaction of the intellectual need for certainty. It is also interesting to note in comparison that the research mathematician Grünbaum (in de Villiers, 2004) argues that the level of conviction is so high in relation to his empirically discovered assertions that he urges the mathematics community to accept them as theorems. Likewise, before the students in this study moved on to item 2 in Appendix 1 they were given another

specific example and then asked to find the gradient function without using *Geogebra*. They all confidently applied their rule and then used the applet to check their results. Four of the students said they were one hundred percent convinced that the rule would work at all times while the fifth student said he was eighty percent convinced. The following interview with Allen typifies the responses of the students who had attained one hundred percent level of conviction.

The case of Allen: Levels of conviction and need for an explanation

Researcher: How sure are you that your method/rule above works for any *n*? Say *n* = 131? Are you 100% sure or do you have some doubt?

Allen: I am 100% sure.

Researcher: If you have some doubt can you provide some examples where your rule will not work? How would you become more convinced? Or what would convince you completely?

Allen: I have no doubt (He proceeded to formulate and differentiate a problem of his own with *n* = 131 to show that he had no doubts)

Researcher: If you are completely convinced that your method/rule always works, do you have any curiosity about WHY it works? In other words, would you like to see some form of explanation of why the rule/method works, or you are satisfied just to know that it works?

Allen: Yes (eagerly). I would like to see why it works and the mathematics behind the process involved in reaching the answer.

Wayne said that he was 80% convinced that the rule would work. He said that more examples would make him more certain, adding that he wanted to do more specific cases until the rule failed. He too said that he would like an explanation as to why the rule worked. Other studies have also found that pupils were keen for an explanation after discovering a rule or making a conjecture using empirical evidence (see Mudaly & de Villiers, 2000; Govender, 2013). Such a need sets the basis for introducing a formal proof for the result. The proof in such cases serves as an explanation for the result, satisfying an intellectual need identified by Harel (2013) as the need for causality.

5.2.3 Findings based on section 5.2.2: Levels of conviction and need for an explanation

All the students who arrived at the general rule displayed high levels of conviction. Four students were 100% convinced that the rule would work at all times. One student said that he was 80% convinced the rule would work and said that doing more examples will make up for the 20%. All the students said that they would like an explanation for why the rule worked.

5.3 Does the use of dynamic graphing software such as Geogebra enhance conceptual understanding?

It was envisaged that the *Geogebra* experience would help pupils overcome difficulties that relate to graphing the function and its derivative. Several studies reviewed in section 2.7 detail the difficulties that students encounter as they attempt to learn the derivative concept. Pillay (2008) for instance found that students favour questions that have a bias towards the application of algebraic rules. It is my view that the teaching strategies employed and the curriculum tests are in part responsible for this trend. Park (2013) recommends the use of a graphical context in introducing the derivative. She argues that such a strategy would help pupils realise that the derivative is not just a tangent line or a point specific object but that it is actually a function. One of the findings in the Ndlovu et al. (2011) study seems to concur with Park's (2013) assertion. They tentatively concluded that the use of the computer graphing software *Sketchpad* could help students achieve a reasonable understanding of mathematical concepts. In particular, one of the dragging exercises demonstrated that the gradient of a curve depends on the value of x by generating a table of values for the slope of the curve against x (see Ndlovu et al., 2011, p.13). This arguably helps students realise that the derivative is also a function.

In the same vein, the dragging activity in this study produced a graph obtained by plotting the value of the gradient of the curve (function) at corresponding values of x. However, the emphasis was on the path created by point S and the accompanying tangent as opposed to the limiting value of the difference quotient. It was hoped that the activity would draw the pupils' attention to other analytical properties of the function graph such as (i) the orientation of the tangent at the turning point, (ii) the value of the gradient, whether it is positive or negative in relation to the orientation of the tangent at the different x values across the domain explored. Ultimately the expectation, among other things, was that after the dragging experience the pupils would understand that the derivative too is a function.

The following comment was key to redirecting the students' thinking when they were experiencing difficulties while solving questions 5 and 6;

Researcher: I want you to cast your mind back to your experience with the *Geogebra* applet. As you dragged point *A* what was point *S* plotting? Try and recreate that trace as you move from left to right of the graph.

The study also sought to evaluate the impact of the *Geogebra* applet on the students' ability to solve a graphing problem. Its overt reference in the above statement was intentional with a view to establishing whether it would help the students resolve the cognitive conflict, leading to a correct graph.

The findings by Baker et al. (2000) also have a significant bearing in the analysis of the results of this study. In their study they analysed the students' different approaches as they attempted to solve a non-routine calculus graphing problem. They concluded that students found it difficult to coordinate information across different intervals and will sometimes rely heavily on one aspect, such as the first derivative, and ignore other aspects of the function because they present a cognitive conflict. This tendency was also observed in this study. As the students attempted to solve questions 5 to 6 in Appendix 1, more often than not they focused on the degree of the polynomial (equivalent to the condition-property aspect of the function in the Baker et al. (2000) categorisation. For instance, when considering the graph of a cubic function, they would immediately conclude that the associated gradient function would be quadratic. However, no mention or attention was given to the analytical aspects such as linking the turning points of the function with the *y* value being zero on the gradient function.

5.3.1 Results and findings for question 5 (a)

The following dialogues provide the findings from the one-to-one task-based interviews. The reader is reminded that the dialogues are not the complete interviews but provide only the salient points. Each commences after the researcher had checked if the student understood what was expected of him in the task. Five out of the six students attempted the questions. Mark is the only pupil who failed to engage with the questions despite all the attempts to guide him. He typically responded by saying he was not sure how to do any of the questions. It must be pointed out that he seemed very low in confidence and was afraid of being wrong despite being reassured that the focus of the exercise was to assess his thought process as he attempted the questions.

The Case of Wayne

Wayne: f_2 is a cubic function, then whatever is on this side (referring to the distracters) should be a parabola according to my theory I derived last week. The only parabola I see is (e)

He went on to explain that if it is cubic to obtain the derivative function one has to reduce the power by one. Essentially he applied the differentiation rule and then used elimination to discard the other graphs. In an attempt to get him to pay more attention to the other properties of the graph,

Researcher: You may want to cast your mind back to how we drew the graph of the gradient function using Geogebra, you may want to consider the orientation of the tangent as we move from one end of the graph to the other.

Wayne: Ok, over here ($x < 0$ for f_2) it (derivative) is going from negative to….to still negative. But it passes through the zero line (implying a horizontal tangent) therefore it has to be (e).

Researcher: Ok.

Wayne: f_3 goes from negative to positive then to zero.

Researcher: Where is it zero?

Wayne: There (tracing out a horizontal line at the maximum turning point)

Researcher: Ok, carry on.

Wayne: This graph, f_3, has three zero gradients (referring to the turning points) but this doesn't seem to be the complete expression on (b)

Researcher: Why?

Wayne: Because (b) only has two zero gradients (pointing at the turning points).

He took a long time trying to come up with an explanation and for the sake of progress I had to intervene.

Researcher: Ok, have a look at the other graph and you can come back to f_3.

Wayne: f_4 is a straight line, with a negative gradient. Therefore, the gradient function should also be a straight line leaving the only possible answers as (a) and (c).

Researcher: Ok.

Wayne: Wait (excitedly) maybe we just need to realise that if it is above the *x*-axis it is a positive *y* (pointing to the gradient function) and if it is below then it is a negative *y*.

Researcher: Right?

He compared this new piece of information to f_1 and after satisfying himself that it was consistent,

Wayne: Now let's try this for f_3. It has a negative gradient (before the first local minimum) which corresponds to a negative *y*. But then again it goes into a positive before the apex (looking very frustrated).

Researcher: But is that a problem?

Wayne: Well it throws out my theory, over here it is wholly negative (pointing to f_3 as it approached the first turning point from the left) before it turns into being a zero gradient, but here (graph b) it goes into the positive quadrant for *y* before the zero gradient.

This point clearly illustrates that Wayne expected the gradient graph to be below the *x*-axis until it attained a local maximum value as displayed by the function graph f_3. Evidently he did not seem to realise that the turning point on f_3 signified a change in the sign of the gradient function, thus where the graph cuts the *x*-axis.

No mention had been made so far about the value of the gradient at the local maximum and minimum values. Debatably the use *Geogebra* enabled Wayne to come up with the observation on his own. It took him a very long time, however, to match f_3 with its correct gradient function. Initially he tried to use elimination and when this failed he attempted to compare segments of f_3 (see dialogue above) with the graphs that he suspected would be correct for the gradient function. In the process, I observed that he was oblivious of what the *y* values represented on the f' graph. He was attempting to find a graph that had the same pattern as the starting function, for example a segment that had a negative slope on the function graph was expected to have a corresponding segment (negative slope) on the f' graph. Specifically he had observed that for f_3, as *x* approached zero from the left hand side, its gradient changed from negative to zero, positive to zero and then to negative. He then tried to find a function that would do the same thing from the distracters. Furthermore he did not seem to realise that the turning points on the function graph became the *x* intercepts on the f' graph.

In the context of the problem at hand, Wayne showed good understanding of the gradient property of the function graph across the given domain. However, he could not superimpose this knowledge onto the Cartesian plane to independently create the gradient graph. This was consistent with the Baker et al. (2000) findings. In their characterisation, Wayne was

operating at the trans property-intra interval level. His attempt to solve the problem relied heavily on the property (gradient) schema which was well coordinated hence I assigned it a trans level. His inability to competently transfer this information onto the Cartesian plane and draw it correctly justified my assignment of the intra level to his interval schema.

In the rest of the interviews the students typically attempted to classify the given function according to its degree (property). This was then followed by differentiating the resulting expression and then matching it with one of the graphs in Figure 2. The following interviews typify the students' reasoning.

The case of Adam

Adam: I have just written $y = mx + c$ as an idea for a straight line (referring to f_4)

He then differentiated the expression making use of the general result he had deduced in his previous session.

Adam: So it will be either (a) or (c) coz it's a flat line (tracing out a horizontal line with his finger)

Researcher: Ok.

Adam: (after some consideration) It will be (a)

Researcher: Ok, why?

Adam: If you drop the x (meaning differentiate) you multiply by a negative so we get $y = -m$. Although we do not know what m (value) is we know that it will be below the x-axis.

Researcher: Ok.

Adam: I gather that f_2 is a x^3 graph and because it slopes in this direction (gesturing) it has to be a negative x^3.

Researcher: OK?

Adam: So if you drop by one, the y function…..gradient…will be $y = -3x^2$ and that's a quadratic function. The answer will probably be (e) because of the negative three and (e) is a negative sloped parabola.

The same strategy was applied to f_3.

Adam: Because there are two turning points, it must be a x^4 graph.

Researcher: Ok

Adam: In $y = x^4$ if we drop the power we get $y = 4x^3$ it would make sense if (b) was the answer because of x to the 3 graph.

Adam's strategy relied heavily on his knowledge of the general forms of the function graphs. This is consistent with what Baker et al. (2000) observed; in a graphing problem pupils tend to focus on one aspect of the problem. In this case Adam focused on the properties (equations) of the function graphs and then differentiated them. Since he was given a selection of graphs to choose from he did not attempt to explain the generation of the gradient graph at the different intervals. When he was required to provide the sketch on his own (see section 5.3.2) Adam successfully produced the graph and his explanation was coherent, suggesting a relatively mature graphing schema, albeit in a less complicated problem than that considered by Baker et al. (2000).

The case of Gerald

According to Park (2013) pupils have difficulties in conceiving the derivative as a function and accordingly she argues for a graphical context to introduce the concept. In attempting to provide an explanation for his choice of gradient graph for f_4 Gerald commented that the appropriate graph would be a horizontal line because it had no gradient. Arguably he did not realise that this was a constant function whose value was independent of x. This also underscores an additional point raised by Park (2013) about the important role played by language in students' ability to solve problems in a derivative context. The following excerpt shows the difficulties Gerald encountered.

Gerald: I will do f_4 first because it's the easiest one. So it takes the form of $y = mx + c$ and last week from the *Geogebra* thing I learnt that the x will cancel out, and there won't be a gradient, it's either going to be (a) or (c).

Researcher: What do you mean there will be no gradient?

Gerald: It will be a straight line (tracing out a horizontal line with his finger)

Researcher: And that has no gradient?

Gerald: It will have a gradient of zero.

Researcher: Ok.

Gerald was able to correct himself and more importantly construct the concept that a horizontal line has a zero gradient.

Gerald: I think it (answer) will be (a), wait I think it will be (c).

Researcher: Why?

His explanation was not satisfactory and he tried, unsuccessfully, to reconstruct what he had observed during the dragging exercise and he neglected to relate this to the general form of the straight line that he had correctly identified. The inability to realise that the line had a negative value for the gradient in the general form was his downfall.

In attempting to match f_2 with its gradient function, he initially battled to correctly identify the given graph as a cubic, instead he said that the graph was that of a quadratic function. Evidently, basic knowledge of the equation of the function graph is important in correctly identifying the related gradient graph.

Gerald: f_2 will go with (e)

Researcher: Alright?

Gerald: Coz this one (f_2) has two turning points and this one (e) has one turning point. Now I know from last week that when its x^2 (referring to f_2) it has one turning point, wait…..(long pause)

Researcher: I will have you know that f_2 is not an x^2 graph, remember an x^2 graph results in a parabola.

Gerald: Yeah! Yeah! This (pointing at e) is an x^2 and this is an x^3 (f_2) so it is (graph e)

Researcher: Why?

Gerald: Because I remember…if it is x^3 you have to minus 1 x so it will be an x^2 graph.

His reasoning in this case was based on his experience with the *Geogebra* activity and the resulting generalisation. Thus he applied his differentiation rule that he had deduced in the previous session. This strategy was also applied to f_3 and he argued that the given graph looked like that of a polynomial of the fourth degree, hence its resulting gradient curve would be cubic. No attempt was made to explain why the gradient graph intercepted the axis where it did and why it was below the *x*-axis in the entire domain.

5.3.2 Results and findings for question 5 (b)
High levels of proficiency by students in routine calculus problems that require symbolic manipulation are well documented in literature (eg Rivera-Figueroa & Ponce-Campuzano 2012; Pillay, 2008; Baker et al., 2000). There seems to be a united voice in the research community bemoaning the obsession that curriculum tests appear to have about symbolic manipulation at the expense of conceptual understanding. Can students conceptualise these manipulation actions and work with them if they are not presented in equation form? (Baker

et al., 2000, p.557). Park (2013), Rivera-Figueroa & Ponce-Campuzano (2012) and Biza et al. (2008) all argue for a graphical context when teaching the derivative concepts. Questions 5 and 6 in Appendix 1 of this study sought to ascertain whether *Geogebra* could positively affect a student's ability to graph the gradient function of a curve given the function graph (question 5) and to graph antiderivative from the gradient graph (question 6). Question 5b is repeated here for the reader's convenience.

5 (b) The graph of another f_5 function is shown below. Sketch, on the same axis, the graph of the gradient function of f_5.
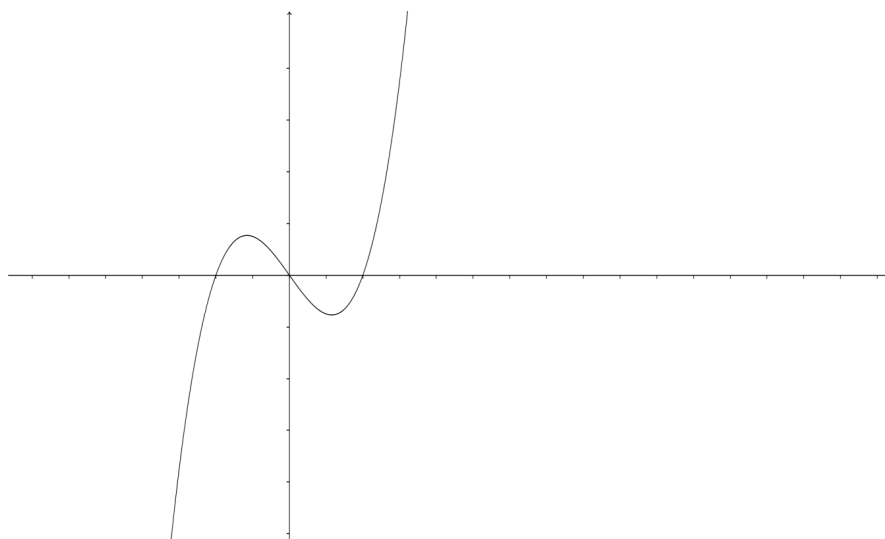
The question intended to assess the maturity of the students' graphing schema. Particularly it sought to scrutinise the students' thought processes and their ability to correlate information relating to the degree of the polynomial and their understanding of how the gradient function varied over the given domain.

Analyses of the findings produced results that were consistent with those of the Baker et al. (2000) study. Primarily it was observed that when students are confronted with a graphing problem, they will place more emphasis on one aspect of the problem at the expense of other properties. In this particular problem, the students often attempted to come up with an equation that satisfied the graph. This was then followed by an attempt to derive the resulting expression. The final step was to try to superimpose it on the given function graph. Due to insufficient information, the students were unable to deduce accurately the equation of the function graph. However, five out of the six pupils realised that the function was cubic. This observation was accompanied by the assertion that the resulting gradient function had to be quadratic.

Three of the students were not able to produce the correct graph. They only paid attention to the degree of the polynomial and the resulting sketch did not attend to the special relationship of the turning points on the function graph and the *x*-intercepts on the gradient function graph. When quizzed about why they chose the particular intercepts the explanation was often incoherent or they randomly altered their graphs and provided no satisfactory justification. Only one student, Adam, provided a satisfactory explanation for his graph. Responses from the one-to-one task-based interviews will now be presented. Once again the reader is reminded that these are summaries of the full transcripts and are intended to capture the significant points of the students' thought processes.

The case of Adam

After he had been working quietly for some time I had to engage him.

Researcher: So what are you thinking Adam?

Adam: I have just plugged the *x* and *y* intercepts here and I am now just trying to find other points that I can use.

Researcher: What do you need those points for?

Adam: I'm trying to think of the equation of this graph.

Researcher: Ok, if the graph did not have any graduations on the axes, that is if I told you that it was not to scale, would you not be able to sketch the gradient graph?

Adam: Ok, I know that it's a cubic function just from its shape. We know that the other function has to be a parabola.

Researcher: Right, what do you mean the other function?

Adam: Umh…the gradient function has got to be a squared coz it (power) must go down. My biggest problem is that I need to know what the value of *a* is (in $y = ax^2 + bx + c$) to determine whether it is a positive or negative parabola. But we know from the applet that the point *S* drops like this (gesturing).

Adam then tried to find some point on his imagined path traced by the point *S* in a bid to compute the equation of the graph. He had the sense to see that there existed different possibilities for the path for point *S* and with the given information it was impossible to precisely find the right path.

Researcher: Why don't you just choose any random points in that quadrant?

Adam: It might not lie on the actual path for *S*.

Researcher: Ok, but why did you not mark your S here? (pointing to a point in the third quadrant, trying to probe further)

Adam:	I remember from working with the applet.
Researcher:	But what does the point *S* actually measure?
Adam:	The turning point……the gradient at that point.
Researcher:	Ok, now if you used the same reasoning say the point *A* was here (choosing a point on the graph in the third quadrant) where would S be?
Adam:	Ok, that's a positive gradient here (tracing out a tangent)
Researcher:	Where would *S* be bearing in mind that its *y* coordinate measures the gradient?
Adam:	It will be somewhere up here, ok (smiles) this makes a lot of sense. Then *S* at that point (drawing a tangent at the maximum turning point) where its zero (gradient) will also be zero on the *S* point (marks a point on the *x* axis directly in line with the maximum turning point, see Figure 5.3).

He then drew the first half of the parabola and continued to reason as follows;

Adam:	I think I have the gist of it, since it is about the same distance away (referring to the turning points) this (*y*-axis) would be the axis of symmetry.
Researcher:	Why did you choose to make your graph cut at these points? I mean it could have been wider or narrower than it is? (I was referring to the *x* intercepts)
Adam:	Because on those points (maximum and minimum turning points) the gradient is zero so the *y* point on *S* would be equal to zero and it's the same *x* value so it's directly above it (tracing out a line from the *x*-axis to the maximum turning point).

This response showed that he did not randomly draw the graph but he was cognisant of the properties of the gradient graph, in particular what it meant when the tangent was horizontal. Figure 5.3 shows his sketch graph.
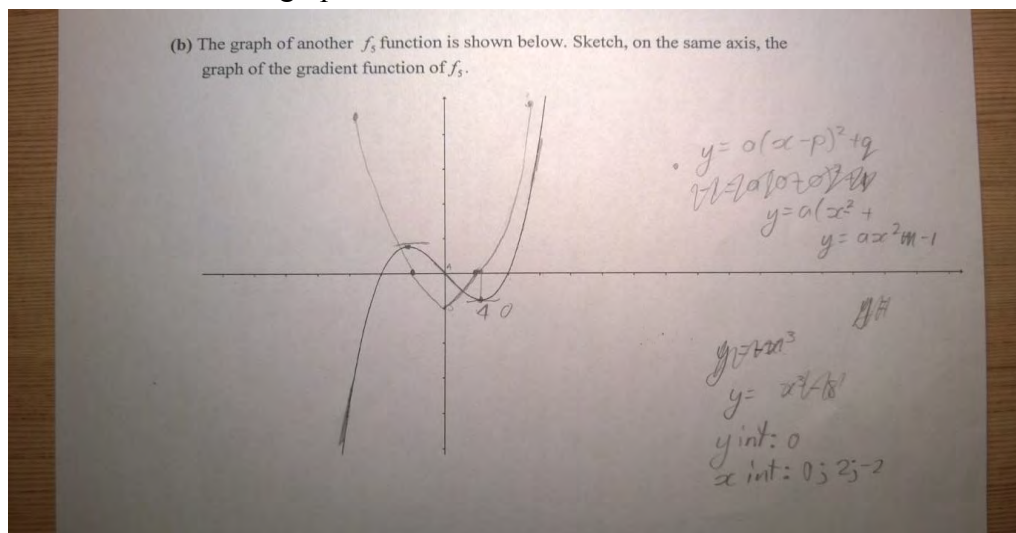

Figure 5.3:Adam's sketch graph for 5b

The above sketch suggests a reasonably mature graphing schema. Adam paid attention to all the relevant aspects of the graph. He was able to correlate the following, *(i)* his differentiation rule that he had deduced to state that the resulting graph should be that of a quadratic function, *(ii)* the shape of the graph (concave up) in relation to the values of the gradient along the function graph, *(iii)* the *x*-intercepts and their relation to the turning points on the function graph and *(iv)* the position of the turning point.

The case of Wayne

Like the other students Wayne initially attempted to find the equation of the graph. Having been told that the graph was not to scale and determination of the equation would not be easy he proceeded to consider segments of the graph at different intervals and analysed the sign of the gradient of the function.

Wayne: These two segments over here represent positive gradients, except for this line (segment) in between the turning points. It represents a negative gradient.

His progress was slow and the reasoning was incoherent and so for the sake of progress I had to engage him again.

Researcher: I want you to cast your mind back to your experience with the *Geogebra* applet. Earlier you told me that this segment of the graph represents a positive gradient right?

Wayne: Yes.

Researcher: Now where would point *S* be positioned since it calculated the value of the gradient? Will it be above or below the *x*-axis?

Wayne: It will be above the *x*-axis I would think.

Researcher: Ok, why?

Wayne: Because it is a positive gradient.

Researcher: Ok, now travel from left to right of that graph and try and recreate the path traced by *S*.

This hint was meant to help him realise that the graph he was to produce was a function of *x* and also to draw his attention to the value of the gradient at each *x* value.

Wayne: Ok, it is positive here and then there is a zero gradient.

Researcher: Right, where is that?

Wayne: At the turning point.

Researcher: Where is that positioned, I mean where would point *S* go?

Wayne: Somewhere on the *x*-axis.

Wayne went on to further demonstrate that he was aware of the fact that the gradient function is obtained by plotting different values of the gradient (y) against x. However, he could not coordinate all the information correctly as evidenced by where he chose to make his graph cut the x-axis. In justifying the x intercepts for his graph, he correctly reasoned that those points represented cases where the gradient of the function graph was zero. When quizzed further about other possible positions along the x-axis he showed that he was oblivious of the fact that the turning point on the function graph and the x intercept of the gradient graph shared the same x value. The inability to successfully correlate all the pertinent information showed that Wayne's graphing schema was not mature. He could for instance verbalise that the gradient of the curve is zero at the turning point but his inability to represent this on the Cartesian plane exposed the immaturity of the schema. Figure 5.4 shows Wayne's sketch.
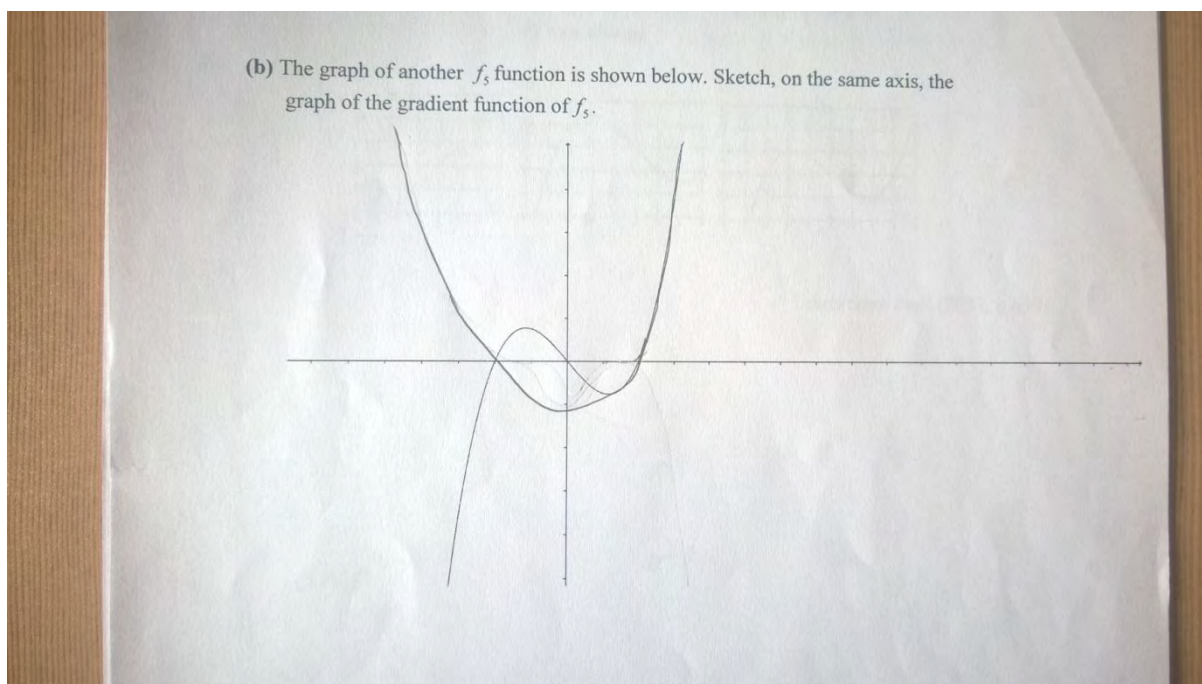

Figure 6: Wayne's sketch graph for 5b

The case of Allen

Figure 5.5 shows Allen's approach and his accompanying graph. He used the graduations on the x axis to come up with the equation of a cubic function. This was then differentiated, resulting in a quadratic function which he then sketched. He insisted on this approach despite being told that the graph was not to scale. Allen did not pay specific attention to the relationship between the x intercepts of his gradient graph and the turning points on the function graph. This behaviour is consistent with the development of a graphing schema as

observed by Baker et al. (2000). They found that a student will concentrate on one aspect of the problem and intentionally neglect other properties because they result in a cognitive conflict that they cannot resolve. The inability to correlate this information shows that the graphing schema was not fully developed.
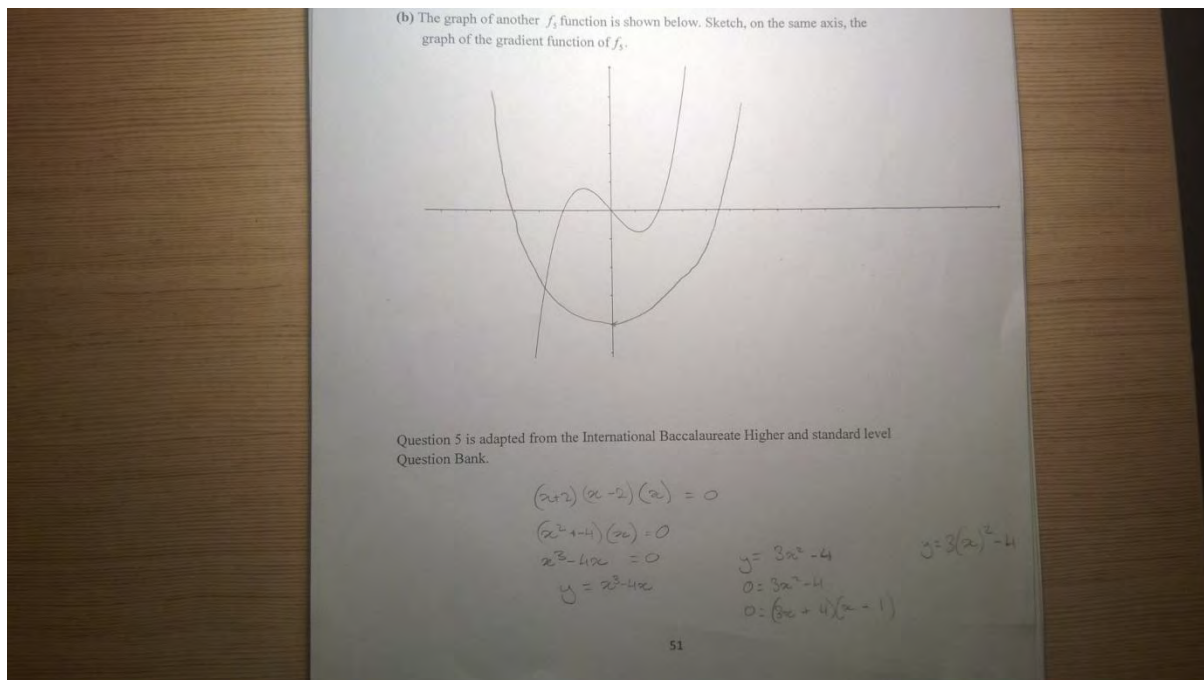

Figure 5.5: Allen's sketch graph for 5b

The Case of Gerald

Gerald also found it very challenging to coordinate all the information represented by the graph. Although he did not attempt to calculate the equation of the graph he also started by observing that the graph was that of a cubic function. He went on to confidently draw a parabola (concave down) and he reasoned as follows:

Gerald: This (function graph) is an x^3 graph and I know that the resulting function is going to be an x^2. It is looks like this (concave down) because the starting graph is that of a negative x^3.

In an attempt to draw his attention to the finer details of the problem and other properties of the graph he was asked to relate his sketch to the gradient of the curve.

Researcher: Try and relate your sketch to the gradient of the curve at different *x* values. How did we plot the gradient graph of any function using the *Geogebra* applet?

Gerald: We took a point *A* and moved it along the graph.

Researcher: Ok as we moved it what were we calculating?

Gerald: The gradient.

Researcher: Good, now if you had point *A* here (third quadrant of graph) and then calculated the value of the gradient is it going to be positive or negative? Where would point *S* be?

Gerald: The value of the gradient would be positive.

Researcher: Ok is that consistent with your sketch?

Gerald: No (erasing his sketch)

The resulting sketch that he drew was still concave down and did not attend to the important properties such as intercepts with the axis. The assumption that the graph represented a negative x^3 still dominated his thinking and all the other points raised during the interview presented a cognitive conflict but he did not attend to them. Having been led to the realisation that the gradient of the function was in fact positive in the third quadrant he then changed his sketch to that shown in Figure 5.6.
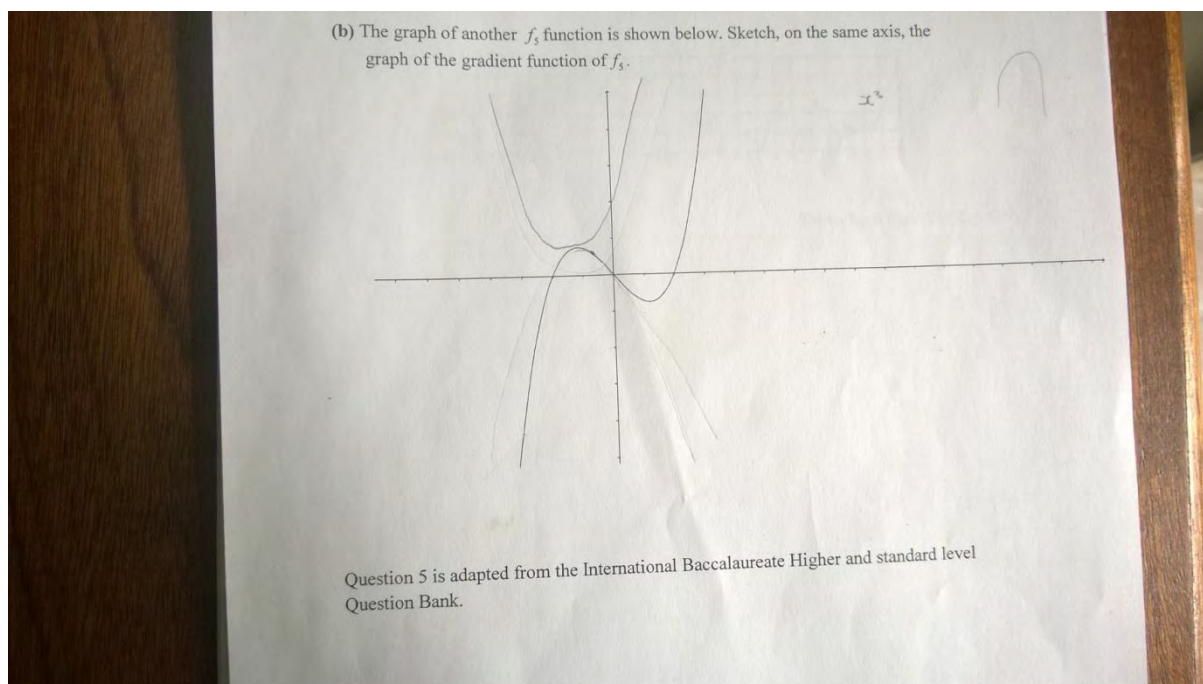

Figure 5.6: Gerald's sketch graph for 5b

Researcher: Are you happy with your final sketch?

Gerald: Yes, sir.

Researcher: Ok please explain to me how you arrived at it.

Gerald: Well, because the gradient is positive (pointing to the segment of the graph in the third quadrant) it has to be above the *x*-axis and also above the turning point.

Researcher: Are you saying that the gradient of this graph is always going to be positive?

Gerald: Yes.

Researcher: Ok, look at the graph are there any points along it where the tangent will have a negative slope?

Gerald: Over here it will be negative (correctly identifying the segment)

Researcher: Ok, that is correct but have you accounted for that segment in your sketch?

Gerald: No, Sir.

Researcher: Would you like to reconsider your sketch then?

Gerald: I am very confused now, Sir.

Gerald's approach relied on the degree of the polynomial. Although he finally deduced that the resulting gradient graph was concave up, he failed to incorporate other relevant information. For instance, he did acknowledge that the graph had a segment with a negative gradient but this was ignored when drawing the final sketch. This tendency to ignore relevant information because it poses a problem when it comes to sketching a graph was also observed by Baker et al. (2000).

5.3.3 Results and findings for question 6

Question 6 also sought to establish whether the *Geogebra* experience could help pupils understand a graphing problem. During the dragging exercise the students were given the function graph from which they were expected to get the gradient function. In this particular question, the students were now confronted with the graph of the first derivative and they had to choose the corresponding function graph. The question is repeated here for continuity.

6. *Below is the graph of the derivative (gradient function) f'(x) of a function f (x) . Which choice a) to e) could be a graph of the function f (x). Circle your choice.*
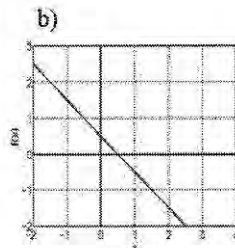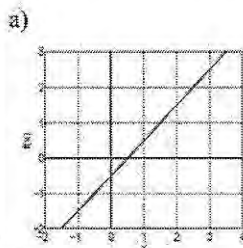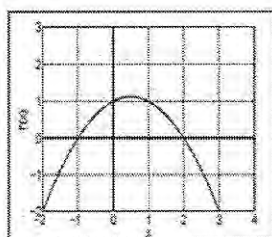
f): none of these.
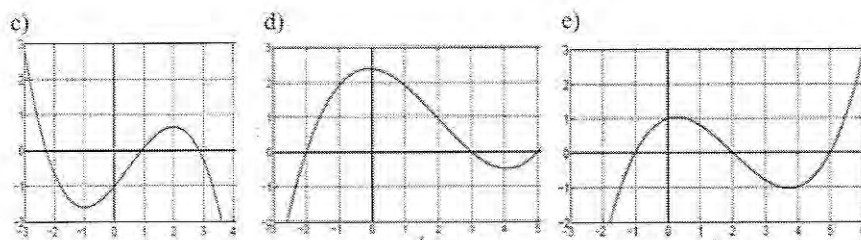
In the main, two strategies were observed as the pupils attempted to solve the problem. In total four students managed to make the correct choice. Two students, Allen and Gerald, first deduced that the given graph represented a quadratic function hence the required graph from the given distracters had to be cubic. They then used their knowledge of cubic graphs to make the correct choice. The other two students, Adam and Wayne, took it a step further and attempted to explain their choice of graph and made explicit reference to the gradient of the chosen function graph at different points. Gestures and traces made often mimicked the orientation of the tangent along the graph similar to what they observed during the dragging exercise. Arguably their success could be attributed to their experience with *Geogebra* experience. Mark did not engage with the problem while, Tinashe tried but eventually gave up after failing to coordinate the pertinent information required to make a choice.

The case of Gerald

Gerald eliminated (a) and (b) by saying that the required function had to have two turning points. These graphs, he further argued, were graphs of linear functions.

Gerald: The parabola that we are given represents a sad face (concave down) and so the required graph must have a negative gradient.

Researcher: What do you mean a negative gradient?

Gerald: I mean a graph with a negative x^3 and I think it is (c) because of the shape of the graph. It starts from the second quadrant going into the fourth quadrant and that's the graph of a negative x^3.

The case of Adam

Adam: We know that its (given graph) a parabola with a negative *a* value.

Researcher: Ok.

Adam: We know that it (required answer) has to be an x^3 graph so we can cancel out (a) and (b). We now look for the x intercepts on the parabola those will be the places on the cubic function where we have a zero gradient.

Researcher: Yes?

Adam: I am going to choose (c) because if you look at these points (turning points) they are at negative one and two and the gradient at these points is zero. That means the matching point (on gradient function) y has to be zero.

Just to further check his understanding,

Researcher: Ok, but why are you discarding this one (referring to d)? It also has a turning point.

Adam: Yes but not at 2, its gradient is not zero at two.

Adam's reasoning was consistent with what was observed as he solved question 5b. He considered not only the degree of the polynomial but also the value of the gradient along the chosen graph and compared it successfully with the given parabola.

The case of Wayne

Wayne: I am going to cross out (a) and (b) because they represent straight lines with x to the power of one and the corresponding gradient functions would have x to the power of zero. The given graph is a parabola so it is an x^2, and one of these to be the actual graph it has to be an x^3 function.

Researcher: Ok.

Wayne: I will go with (c) based on my earlier argument that if the gradient is negative it has to be below the x-axis, now both (d) and (e) have positive gradients (pointing to segments in the third quadrant) and they should be above the x-axis….(c) has a negative gradient and the parabola also starts below the x-axis. So I will go with (c).

He went on to match the segment of the graph that had a positive gradient and said that on the gradient function that segment had to be above the x-axis.

Wayne: This part has a positive gradient and it basically corresponds to that entire part of the parabola (tracing out the part of the parabola above the x-axis)

5.4 Findings based on Section 5.3: Does the use of dynamic graphing software such as Geogebra enhance conceptual understanding?

1. Five students successfully completed question 5a. Only one pupil was able to produce the correct sketch for question 5b and this sketch was accompanied by a satisfactory explanation. Although the adequacy of the explanation varied, four students correctly chose graph (c) for question 6.

2. Two key factors influence reasoning when students attempt to solve a graphing problem involving a function and its first derivative. These are the ability to recognise the degree of the polynomial and the value of the gradient of the function at special points.

3. When confronted with a graphing problem, students tend to rely heavily on aspects that they fully understand and they conveniently ignore aspects that create a cognitive conflict. The gradient at the turning point, often referred to as the zero gradient by students, presented difficulties and the inability to correctly identify these points on the x-axis resulted in incorrect conclusions or sketches.

4. Reference to the Geogebra applet, in particular the movement of point S as A was dragged, seemed to help some students progress when they were having difficulties while attempting to solve a problem.

5.5 Limitations of the research study

While every possible avenue was explored to counter any factors that might have influenced the findings of this study, it is still important to draw the reader's attention to some of the project's limitations. These relate to the sample, data collection interval and the data collection strategy. Accordingly these will be discussed in sections 5.5.1, 5.5.2 and 5.5.3 respectively.

5.5.1 The Sample

Six students took part in this study and they were all boys. The sampling itself was purposeful and the top students in the grade were selected. One of the main research questions was to determine if the use of the *Geogebra* applet could help students to inductively deduce the differentiation rule for polynomials. Consequently, proficiency in algebraic manipulation was a determining factor in the choice of students. The choice of the

boys only was based on convenience as the researcher teaches at the school, which enrols only boys. These factors alone imply that the findings of this study cannot be generalised to heterogeneous populations. However, as previously pointed out, qualitative research studies such as the current one do not aim to generalise findings. The detailed descriptions of students' thought processes as they engaged with the problems provide, in my view, guidelines for which educators can model learning activities. Merriam (2000) underscores this point by arguing that we can transfer what we learn in a particular situation to similar situations. It is therefore the reader's prerogative to determine the extent to which these findings would be applicable to a situation he/she encounters (Lincoln & Guba, 1985).

5.5.2 Data collection interval

The one-to-one task-based interviews were conducted on the days that the participants were not involved in the intensive sporting programme of the school. While it would have been ideal to have the two sessions conducted on consecutive days it was not possible. The implication is that the time lag between the first session and the second session might have impacted on the findings. Attempting to measure the impact of this time difference, whether positive or negative, would be an impossible task. Furthermore, after the first session each student was asked not discuss with other participants what had transpired during the interview sessions. It is hoped that this instruction was respected. There was no blatant evidence to suggest that the pupils had discussed their sessions.

5.5.3 Data collection strategy

One-to-one task-based interviews were the main data gathering tool in this study. This is not what typically transpires in a mathematics classroom. It would, therefore, be difficult to predict how the students might interact with the software in a traditional classroom set up.

Chapter 6: Discussion of findings

6.0 Introduction

The limitations discussed in the preceding section do not invalidate the study's findings. The intention of the study was to contribute to the ongoing discourse about the benefits of using technology in the mathematics classroom, in particular its use in introducing the derivative and the development of calculus related concepts. The shortcomings identified apply mainly when it comes to generalising the results and it has been acknowledged that this was not the concern of the project. This chapter will provide answers to the main research questions and also endeavour to relate the results of the current study to the literature reviewed.

6.1 Research questions:

- ***Can Geogebra aid students in discovering the power rule for differentiating elementary polynomials?***
- ***Having discovered the rule, are the pupils convinced about its truth and generality? Do they demonstrate a desire for an explanation for why the result works?***

It is my view that an affirmative answer is appropriate for both research questions. Five out of a total of six students were able to come up with the differentiation rule for elementary polynomials following their experience with the *Geogebra* applets. So what if the students were able to deduce the result? It is the process of their discovery that is of significance here. As a teacher/researcher I subscribe to the ideas of constructivist education that agitates for providing students with opportunities to "rediscover" and "recreate" mathematical ideas. Research suggests that once the students discover a result on their own they tend to retain it better than when it is presented to them as fact. The objective was to afford the pupils an opportunity to experience what mathematicians encounter during the discovery of new results. De Villiers (2004) encourages the use of computer software, arguing that it democratises the mathematical process. Furthermore, this study, by encouraging conjecturing and pattern recognition attempted to inculcate the habits of mind of mathematicians as conceived by Cuoco et al. (1996).

The way the questions were sequenced in this study ensured that they exhibited constrained variation as advised by Watson and Mason (2006) or some underlying sameness to be identified by the students (James, 1992). The progression from one family of polynomials to another (that is from linear to cubic) prompted students to make premature conjectures only

to review them when they failed. The opportunity to experiment and be able to make mistakes is at the core of Kolb's (1984) experiential learning theory, which also guided the design of tasks for this study.

A special by product of the study that was not overtly explored was that students got to observe on the applet how the derivative function is constructed, by plotting the gradient of a function at different *x* values. Consequently the resulting graph allowed them to see that the derivative of a function is also a function. Pillay (2008) found that the derivative to students is synonymous with applying the differentiation rule. In her study Park (2013) concluded that students battled to see the derivative as a function and she advocates the use of transparencies to demonstrate this concept. Teaching the derivative using dynamic software such as *Geogebra* shows how the function is obtained and that it is not just a formula obtained through the application of rules.

Educators have consistently argued that teaching concepts in an investigative way is time consuming. However, I contend that in this case the argument is invalid because the pupils can be given the worksheet to complete at home. The evidence from the one-to-one task-based interviews showed that the pupils had very few difficulties when they used the software, affirming its user friendliness. It is an unfortunate reality that in South Africa and in most African countries students still do not have access to computers at home. However, nowadays in most mathematics classrooms there is at least a computer for the teacher and this can be used to quickly generate graphs and rich discussions can focus on them.

My results agree with what other studies have revealed about students' levels of conviction after empirically discovering a mathematical result or making a conjecture (e.g Mudaly & de Villiers, 2000; Govender, 2013). All five students who correctly derived the rule showed high levels of conviction, four were at 100% while one was at 80% and he said that doing more examples would get him to the 100%. Interestingly the high levels of conviction were not enough, as all five students said they would like to know why the rule worked. In Adam's words, he wanted to see the mathematics behind it. Arguably this sets the tone for introducing proof as an explanation (De Villiers, 1999) and also to satisfy the students' intellectual need for causality (Harel, 2013).

6.2 Research question:

- ***Does the use of dynamic graphing software such as Geogebra enhance conceptual understanding and resolve difficulties associated with the derivative as documented in the literature?***

The study sought to find out if the experience with the *Geogebra* applets would enable students to successfully tackle non-routine calculus graphing problems that involve a function and its first derivative. The graphing questions presented some challenges for the students and their difficulties were observed by other scholars (eg Baker et al., 2000; Biza et al., 2008; Park, 2013). The students did fairly well in questions where they could choose a matching graph, namely question 5a and question 6. However, only one student was able to produce the correct graph for question 5b and this was accompanied by correct justification. Despite the difficulties exhibited by the students it is my belief that the *Geogebra* experience was beneficial to the students' advancement.

The ensuing discussions during the one-to-one task-based interviews yielded responses that could be invaluable for instructional design. Following on the students' dragging exercise with the *Geogebra* applets two main points were observed as they tackled the graphing problems. Firstly, the differentiation rule that they had derived enabled them to recognise that when given the function graph the resulting graph of the first derivative would be a polynomial of one degree less and vice-versa. Secondly, the presence of the tangent attached to the point *A* on the applet drew the students' attention to the fact that while the derivative is a function, it is also a point specific object. Park (2013) found that pupils had difficulties conceptualising these two aspects of the derivative. During the interviews students started using phrases and language that had not been formally introduced to them. For instance Wayne initially referred to the turning point as the apex of the graph and at a later stage called it the turning point. Furthermore, students often referred to a zero gradient at the turning point, further affirming their appreciation that the derivative is a point specific object. The use of phrases such as *it changes from a positive gradient to a zero gradient* suggests an awareness of the fact that the derivative is a function. Referring to the derivative as "it" and that it is positive arguably shows that they did appreciate that it was a collection of points; a function.

The use of the applet also helped students deduce the shape of the gradient graph. The question used to direct the pupils' thinking explicitly referred to the *Geogebra* applet and it

seemed to help the pupils. For instance, Adam exclaimed that the graphing question made a lot of sense following on from the researcher's hint. Wayne also made the generalisation that if a function has a positive gradient it must be wholly above the *x*-axis and vice-versa if the gradient is negative. Again, the said success could be attributed to the dragging exercise with the applet. Biza et al. (2008) reported that students' early conceptions of the tangent line influence their understanding in future reasoning when solving questions. The way the students spoke about the gradient function in this study and the dynamic tangent attached to the point *A* in the applet did elucidate and deepen their understanding of tangency.

The ability to produce the required graph for question 5b seemed to be dependent on correctly identifying the *x*-intercept as the one that coincides with the turning point on the function graph. The study intended to assess the students' intuitive reasoning following the *Geogebra* experience and pointing this out to the pupils would have given the answer away. Computer based teaching that aims to expose this connection could help the students' concept formulation.

Chapter 7: Conclusion, recommendations and directions

7.0 Concluding remarks

Advocates for mathematics education reform have campaigned for a mathematics curriculum that is, among other things, student centred and provides opportunities for experimentation and self-discovery. Ideally such a curriculum will inculcate the habits of mind of the mathematicians responsible for the discovery of the results to be taught (Cuoco et al., 1996). The abstract nature of mathematics has, in part, ensured that students continue to experience it as a readymade product in many classrooms. However, the ubiquity of computers could democratise the mathematics process (de Villiers, 2004).

This action research study sought to contribute to the ongoing discourse on how to integrate computer technology into the teaching of the derivative in high school. Specifically, it intended to find out if the experimental use of the computer software *Geogebra* could enable students to discover the rule for differentiating elementary polynomials. Results from Chapter 5 showed that it is suitable for this process. The study also categorised how students reasoned as they developed their rule. Furthermore, when students discovered the rule they wanted to know how and why it always worked. Other scholars working with a different software package in a geometry context, *Sketchpad*, (eg Mudaly & de Villiers, 2000; Govender, 2013) also drew similar conclusions. These high levels of conviction provide the spring board for the teaching of proof as an explanation.

The study also found that the use of the software could help students solve some non-routine calculus graphing problems. Students often associate the derivative with the manipulation of algebraic expressions (Pillay, 2008) and also find it difficult to conceive it as a function (Park, 2013). Student dialogues and their sketches from Chapter 5 revealed that *Geogebra* can help students develop the concept of a derivative as a function and thus help them realise that it is not just about manipulating symbols. Uddin (2011) also found that *Geogebra* can help students visualise and develop conceptual understanding pertaining to the transformation of functions. Based on the findings in Chapter 5 this study tentatively draws a similar conclusion in that *Geogebra* can aid students' conceptual understanding of the derivative and their ability to solve non-routine graphing problems. The conclusion is tentative because it acknowledges the difficulties that the students experienced while solving questions 5(b) and 6. It is possible that even the student who managed to produce the correct graph may have

failed if his thinking had not been redirected to the applet. In retrospect, more items requiring the students to produce a graph, similar to 5 (b), and removing the distracters from question 6 would have provided a stronger foundation for an assertive conclusion.

7.1 Recommendations for the teacher

The experience with the *Geogebra* applets highlighted the concepts that students rely on when they tackle a graphing problem. All the students either focused on determining the equation of the function only or they attempted to solve the problem by trying to recreate the trace of the gradient function. The successful student was able to move between these two concepts. Learners found it difficult to represent the turning point on the *x*-axis although they could verbalise that the gradient was zero at those points. It is my opinion that explicit reference to this by teachers as the students work with the applet would improve the students' performance in similar situations. Additionally, the interviews also provided a window into the learners' thought processes. The use of *Geogebra* could provide a favourable environment in which the teacher can sagely play a supporting role while the students construct their own knowledge of the derivative. Rich discussions emanating from such environments are not always possible with paper and pencil static graphs. The study can be adapted for a classroom situation by, where possible, giving the students a link to the applet and then they can complete the worksheet at home.

If the students do not have internet access, a single computer could still be used to introduce the concept. The teacher could carry out the dragging exercise while the students calculate the equation of the trace. Such an approach is advantageous in that it presents the student with an opportunity to create links within his/her prior knowledge and not learn the concept of the derivative as an isolated idea. For instance in the process of finding the equation of the trace, the student revises the concept of graphs as evidenced by most students in this study who began by classifying the type of function presented to them. It would be interesting to see the effect of such an approach on the retention of the constructed concepts. The benefits of computer technology in aiding self-discovery learning are well documented (eg Borwein, 2005; de Villiers, 2004) and as a general rule teachers ought to explore ways of incorporating technology in their teaching. *Geogebra* can be used to model the difference quotient and once the students have inductively deduced the differentiation rule, the software could then be used to introduce differentiation from first principles.

Furthermore, it has been shown that students exhibit very high levels of conviction and have a desire to satisfy their intellectual need for causality. It would therefore be appropriate for the teacher to guide pupils through a logical proof. An elementary proof, adapted from Finney and Thomas (1990, p.142), that makes use of the binomial theorem is suggested herein.

$$\text{If } f(x) = x^n, \text{ then for } n > 1$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\left(x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n\right) - x^n}{h}$$

$$= \lim_{h \to 0} \left(nx^{n-1} + \binom{n}{2}x^{n-2}h^1 + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1}\right)$$

$$= nx^{n-1}$$

The above proof relies on students' understanding of the binomial theorem and the limit concept. The sequencing of the teaching of these topics is essential in ensuring that the pupils understand the proof.

7.2 Directions for further research

A glaring limitation of this study is that the participants were all boys. This is not representative of most classroom situations and it would be interesting to see if similar conclusions will be drawn with a heterogeneous group of students. Furthermore, since this was an action research study with a view to improving classroom practice, reproducing the study with larger class sizes, a longer implementation time period and improved materials could be more informative. The data collection instruments could be improved by removing the scales on the axis of the given original graphs and also including more questions that require the pupils to produce a sketch as opposed to choosing from given distracters.

Satisfying students' intellectual needs identified by Harel (2013) should be the focal point of any mathematics education programme. The participants in this study clearly demonstrated a

need for an explanation for the differentiation rule. *Geogebra* can be used to demonstrate how a secant becomes the tangent, simultaneously generating a table of values to show the limiting value. This approach could be used to guide students towards determining the differentiation rule and also to introduce the difference quotient which is important for the proof presented in section 7.1 (p.88). Further research, in addition to the primary questions in this study, could be instituted to answer the following question:

Can students construct a guided logical proof (explanation) for the differentiation rule? If so, does the proof satisfy their need for causality?

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Appendix 1: Task-based interview questions

Name_____

1. (a) By using the Geogebra Applets provided, determine the equation of the path traced by the point S on each function. Note that the y-coordinate of S gives the gradient of the curve at point x

(b) Refresh the view under tools and then type your equation in the input bar. Does it match the path of the trace? If it does, enter the equation in the provided space in the table below.

Function	Equation of the gradient function
$f(x) = 2x$	
$f(x) = 2x + 3$	
$f(x) = 3x$	
$f(x) = 3x + 3$	
$f(x) = x^2$	
$f(x) = x^2 - 1$	
$f(x) = x^3 - x$	
$f(x) = \dfrac{x^3}{3}$	
$f(x) = \dfrac{x^3}{3} - x^2 + 2x$	

2. Is there a rule for finding the equation of the gradient function for a function of the form $f(x) = mx + c$ where m and $c \in \mathbb{R}$? Describe how you arrived at this rule.

....

....

....

....

3. Is there a rule for finding the gradient function for the equation of the form $f(x) = ax^2 + bx + c$ where a, b and $c \in \mathbb{R}$? Describe in your own words how you arrived at this rule.

....

....

....

....

....

4. Is there a general method for finding the gradient function of $f(x) = ax^n$ where $n \in \mathbb{Z}^+$ and $a \in \mathbb{R}$? If so, write it down in the space below and describe how it works.

....

....

....

....

....

....

....

5. (a) Figure 1 shows the graphs of the functions f_1, f_2, f_3, f_4.
   Figure 2 includes the graphs of the gradient of the functions shown in **Figure 1**,
   e.g. the gradient function of f_1 is shown in diagram (d).

Figure 1

f_1

f_2

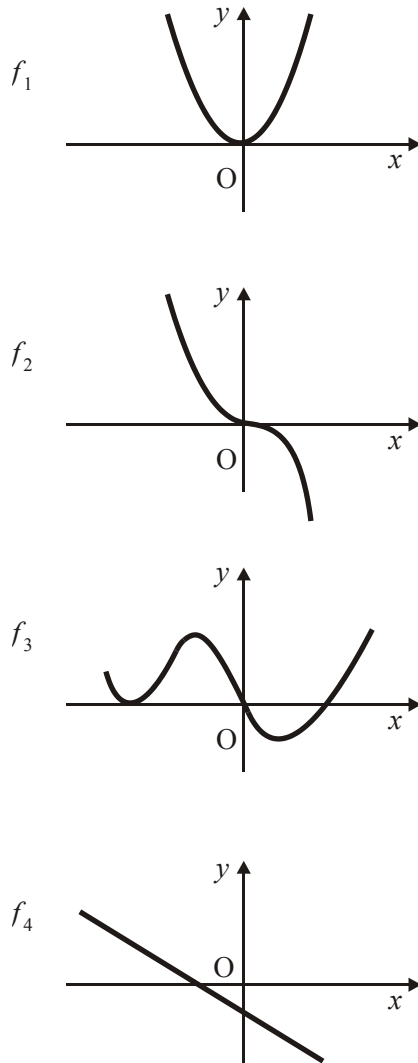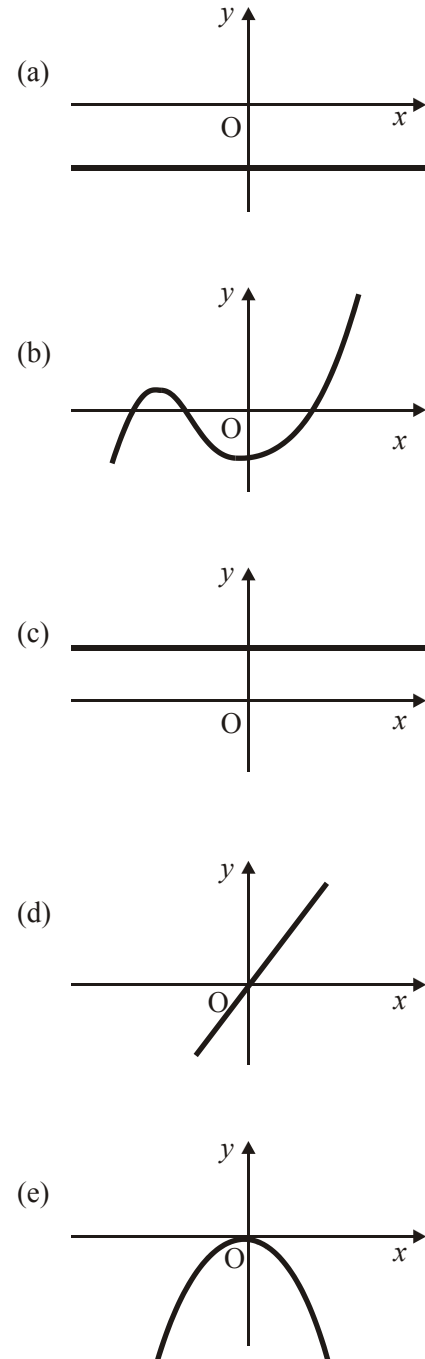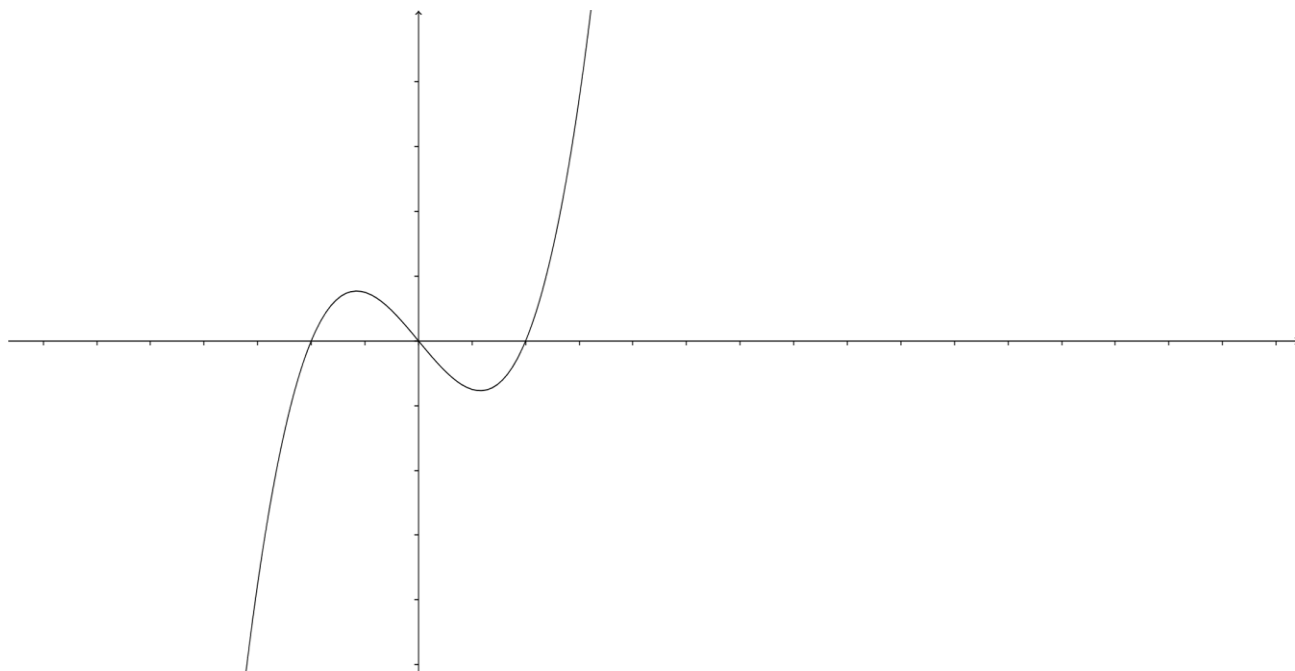
f_3

f_4

Figure 2

(a)

(b)

(c)

(d)

(e)

Complete the table below by matching each function in figure 1 with its gradient function in figure two.

Function	Gradient function
f_1	(d)
f_2	
f_3	
f_4	

(b) The graph of another f_5 function is shown below. Sketch, on the same axis, the graph of the gradient function of f_5.
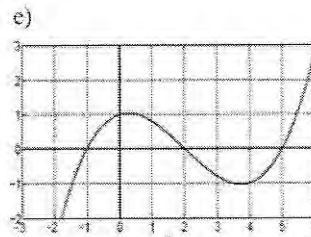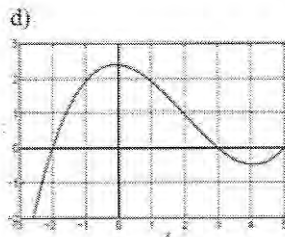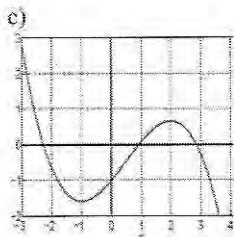

Question 5 is adapted from the International Baccalaureate Higher and standard level Question Bank.

6. Below is the graph of the derivative (gradient function) $f'(x)$ of a function $f(x)$. Which choice a) to e) could be a graph of the function $f(x)$. Circle your choice.
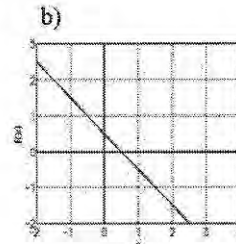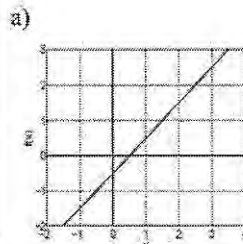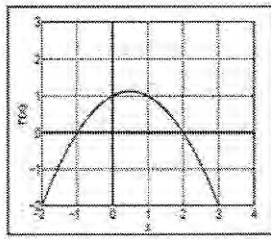
f): none of these.

Taken from Park (2013, p.639)

Appendix 2 : Interview protocol for levels of conviction

This interview schedule followed the completion of item 4 in Appendix 1.

Name

1a) How sure are you that your method/rule above always works for any *n*? Say n = 131? Are you 100% sure or do you have some doubt?

b) If you have some doubt can you provide some examples where your rule will not work? How would you become more convinced? Or what would convince you completely?

c) If you are completely convinced that your method/rule always works, do you have any curiosity about WHY it works? In other words, would you like to see some form of explanation of why the rule/method works, or are you satisfied just to know that it works?

Appendix 3: The electronic environment (Geogebra applet)